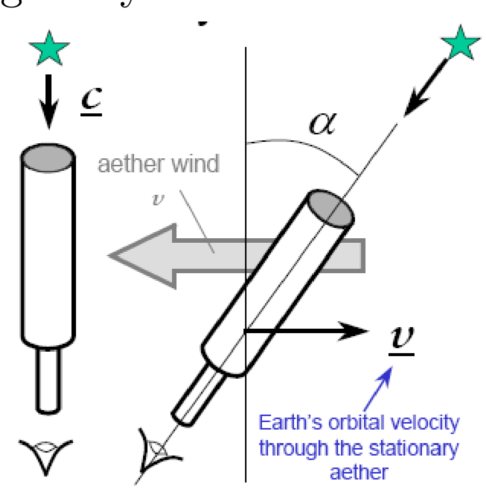
Special Relativity

- The **Galilean Transformations** between a frame S and a frame S', **moving** at a speed $\boldsymbol{v}$ relative to it in the **horizontal** direction are

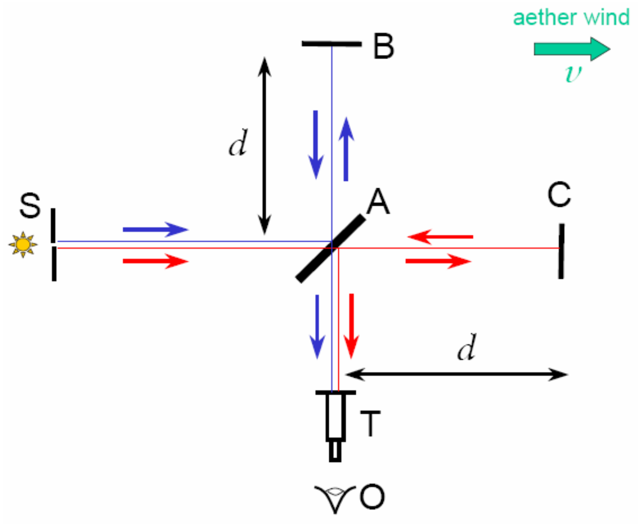
$$\Delta x' = \Delta x - v\Delta t$$
$$\Delta y' = \Delta y$$
$$\Delta z' = \Delta z$$
$$\Delta t' = \Delta t$$

- **Problems with classical mechanics**
  - **Maxwell's Laws of Electromagnetism** did *not* conform to the **Galilean Transformations** – they predicted that the speed of light in a vacuum was $(\varepsilon_o \mu_0)^{-1/2}$ – this expression involves only constants, and does not take into account the **relative speed of the inertial frame**. There were therefore two possibilities:
    - There was a given **universal rest frame** which contained a **medium** in which light travelled at $(\varepsilon_o \mu_0)^{-1/2}$ (the **aether**), and **this** is what Maxwell's Equations were coming out with. This is very similar in the way sound travels in air.
    - The speed of light is $(\varepsilon_o \mu_0)^{-1/2}$ in ***all*** inertial frames.
  - Early beliefs were in the *first* theory; there was one observation that tended to *support* that theory, while another was at odds with it. Both were really attempts to find **our velocity** with respect to the **aether's**.
    - ***Stellar Aberration*** – the light from **celestial objects** descended to us **at an angle**, as opposed to straight down. This was taken as evidence that we were **moving relative** to the **aether**, and that since light was travelling through it, the "**aether wind**" made light look like it was coming diagonally down at us:



Bradley observed this in 1725, and was able to find $\alpha \approx v/c \approx 10^{-4}$ radians. [This is explained in terms of special relativity using the relativistic addition of speeds – see later].

- *The Michelson-Morley Experiment* used an interferometer like this one:



  Light from S splits at A and travels to B and C and back to A, where it is **recombined**. If the earth was moving through the **aether**, and an **aether wind** existed as shown, the path times ABA and ACA would be **different**, by approximately dv^2/c. Using light of period about 10^{-15} s and plugging in numbers, this should correspond to a shift of about **half a fringe** (very easily observable). But such a shift was **never observed**.

- **Einstein's basic Postulates** were that:
  - **All the laws of physics are the *same* in every inertial frame** [ie: all laws of physics must be able to be written in terms of **4-vectors**]. This implies that there is **no way to distinguish** between the frame one is in and any *other* frame by **experiment** (without comparison with another frame). It also implies that **empty space** is **homogenous** and **isotropic**.
  - The speed of light **in a vacuum** [note: can be different in other media!] is the same for **all observers** in **any inertial frame**. This really is two statements:
    - When light is **emitted** in **one frame**, it travels at c **in that frame** [fairly obvious from the relativity postulate].
    - That **same beam**, when viewed in **another frame**, also travels at c. (ie: light doesn't behave like a baseball!)
- The **consequences** of these postulates are as follows:
  - ***Simultaneity*** – events simultaneous in one frame are not necessarily so in another. *[Consider someone in the middle of a carriage beaming light to the two ends].*
  - ***Time dilation*** – $\Delta t = \gamma \Delta t'$ (for two events at the same point in space in S'). If something takes a certain time in S', it takes **longer** in our frame. Note that for this result to occur, it is **essential** that the two events occur in the **same place** in S'! Otherwise, we get <u>the</u> paradox! Thus, when S sees

S''s time go slowly and vice versa, it's not a contradiction – it's just that they're using **completely different scales** (S' is using a scale in which $\Delta x' = 0$, and S is using a scale in which $\Delta x = 0$) *[Consider the weird photon clock in a train].*

- ***Length contraction parallel to motion*** – $L' = L/\gamma \Rightarrow L = \gamma L'$ (where L is the length in a frame where the distance is **at rest**). In other words, a ruler is **longest in its rest frame**, and **moving things appear smaller**. *[Consider A standing next to a stick on the ground and B flying by, and consider the time it takes for B to get from one end of the stick to the other according to both or consider the time it takes for light to go back and forth in a train]. To derive from the Lorentz Transformations, realise that length is a measurement of both ends of the stick at the same time (ie: a Δx with a $\Delta t = 0$), even in a frame in which the length is moving. Then, in a frame in which the length is stationary, even if $\Delta t \neq 0$, Δx will be the length in that frame. The key point is that in a frame in which the length is moving, only measurements for which $\Delta t = 0$ can be called lengths.*
- ***No length contraction perpendicular to motion*** *[use the neat argument of a train on tracks – if there was perpendicular length contraction, then we can tell whether the frame we are in is moving by seeing which train comes off the tracks!].*

- Note: synchronisation of clocks can be done by placing a **light beam** *exactly* in the **middle of them**, shining it, and setting both clocks to the same time when the light beam arrives to them.
- We therefore need to **modify** the Galilean Transformations to get the **Lorentz Transformations**, which are consistent with these observations:

$$\Delta t' = \gamma\left[\Delta t - \frac{v\Delta x}{c^2}\right]$$

$$\Delta x' = \gamma[\Delta x - v\Delta t]$$

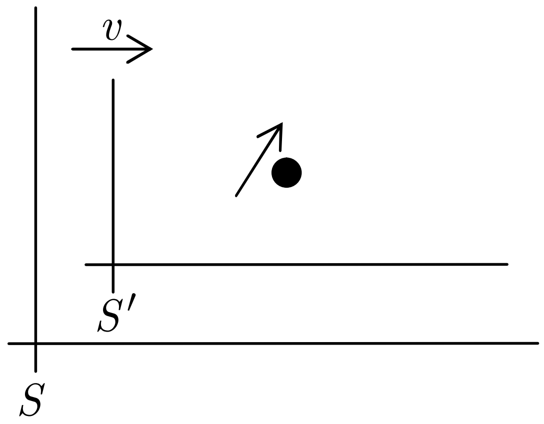
$$\Delta y' = \Delta y$$

$$\Delta z' = \Delta z$$

The inverse transform is obtained by substituting $v = -v$

- We now define two quantities:
  - The **time** between two events that occur at the **same place** is the **shortest time** between those two events in **any frame**. This is the **proper time**, τ.
  - The **distance** between two events that occur at the **same time** is the **shortest distance** between those two events in **any frame**. This is the **proper distance**, ℓ.
  - Clearly, two events **cannot** have *both* a τ and an ℓ, because if two events have proper distance ℓ, then they can hardly occur at the **same place** at which we would find the proper time.

- Another consequence of the transformations is the **relativistic addition of speeds** *[To derive, consider the distance the particle travels in a given time in both frames, or the inner-product of the velocity 4-vector].*



In such a case:

$$u_x = \frac{u_x' + v}{1 + \frac{u_x' v}{c^2}}$$

$$u_y = \frac{u_y'}{\gamma\left(1 + \frac{u_x' v}{c^2}\right)}$$

$$u_z = \frac{u_z'}{\gamma\left(1 + \frac{u_x' v}{c^2}\right)}$$

NOTE: these formulae only need be used when transforming speeds **between frames**. If, for example, we have the speed of two things in *one* frame and we want to find their speeds relative to each other in that *same* frame, then we just add the speeds!

- This allows us to explain **stellar aberration** in terms of relativity. We simply consider the photon travelling in S' and find the angle it must therefore be travelling at in S, using the addition of speeds. The result is <u>approximately</u> v/c.
- A final consequence worthy of interest is the **Relativistic Doppler Effect**. If a source is moving **away from us** at a speed $\boldsymbol{v}$ in the x direction, and **emitting EM radiation** as a frequency f', then we **observe it** at a frequency f, where

$$f = f'\sqrt{\frac{1-\beta}{1+\beta}}$$

This basically the **everyday Doppler effect**, but with a slight modification to take **time-dilation** into account:

  - **Everyday Doppler effect** – let the time between photon-emissions in **our** frame be Δt. Then, in between two emissions, the **photon** will have moved **towards us** a distance $c\Delta t$, and the **source** will have moved **away from us** a distance $v\Delta t$. Thus, the distance between emitted photons is $(c+v)\Delta t$.
  - **Time contraction** – in the **source's frame**, $\Delta t' = 1/f'$ (this is the time between each emission). In a **Newtonian world**, we would have $\Delta t = \Delta t'$. However, under the Lorentz Transformations, $\Delta t = \gamma \Delta t'$ (since the flashes occur in the **same place** in the **source's frame**).

We then note that, therefore, the time ΔT between the <u>**arrivals**</u> of these flashes to our eye is the distance we found above divided by the speed (c). So:

$$\Delta T = \frac{1}{c}(c+v)\gamma\Delta t' = (1+\beta)\gamma\Delta t' = \frac{1+\beta}{\sqrt{1-\beta^2}}\Delta t' = \frac{1+\beta}{\sqrt{(1-\beta)(1+\beta)}}\Delta t' = \sqrt{\frac{1+\beta}{1-\beta}}\Delta t'$$

And since the frequency is $1/$*Time period*:

$$f = \frac{1}{\Delta T} = \frac{1}{\Delta t'}\sqrt{\frac{1-\beta}{1+\beta}} = f'\sqrt{\frac{1-\beta}{1+\beta}}$$

- And of course, another consequence is that the speed of light is the **ultimate speed**. This, however, only applies to **mass** (energy, actually). Something that has neither mass nor energy (eg: the point of intersection of two rulers) can move arbitrarily fast up to infinity.
- **Experimental evidence** for relativity includes:
  - ***Time dilation in the decay of muons*** – **cosmic rays** produce **shower of muons** at the **top** of the atmosphere. These have a **lifetime** of only about **2 μs**, and so should travel only a **few hundred metres** before **decaying** (their speeds are close to c). However, in practise, we measure **most of them** at **ground level**, after travelling through **many km of atmosphere**. This is because the "muon clocks" measure <u>*proper time*</u> between events, whereas we measure <u>*longer times*</u> on earth.
  - ***The Michelson-Morley experiment*** is **direct evidence** for the **absence of aether** and therefore of the **constancy** of the **speed of light in a vacuum**. More recent work has shown that there is no effect greater than on thousandth of that expected under the aether hypothesis.
  - ***Atomic clocks on jet aircrafts*** which were sent round the world **ran slow** compared with **identical clocks** kept **stationary on the ground**. [This was affected by **gravitational potential** (due to the **general theory**), but modifications were applied.
  - ***Magnetic forces*** between two current-carrying wires can be calculated from **relativistic modification** of the **electrostatic forces** between the **charges** in the wires. This demonstrates the consistency between electromagnetism and mechanics brought about by Einstein's postulates.
  - ***Clocks in GPS satellites*** need to be adjusted both to take into account their **less negative gravitational potential** and the effects of **special relativity**, given that they are moving. Otherwise, earth-base clocks are found to drift.

- The vector $\begin{pmatrix} c\Delta t & \Delta x & \Delta y & \Delta z \end{pmatrix}$ is a **4-vector** [a 4-dimensional vector that transforms between frames according to the Lorentz Transformations] In matrix form:

$$\begin{pmatrix} c\Delta t' \\ \Delta x' \\ \Delta y' \\ \Delta z' \end{pmatrix} = \begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} c\Delta t \\ \Delta x \\ \Delta y \\ \Delta z \end{pmatrix}$$

$$Where\ \beta = v_x / c$$

Notes:

- The **inner product** of two such **spacetime 4-vectors** is an **invariant**, **irrespective of the frame**. Thus:

$$c^2 (\Delta t)^2 - (\Delta x)^2 = (\Delta s)^2$$

Where the quantity Δs is called the **interval** and is an **invariant** under the **Lorentz Transformations**.

Note that this statement expressed with $\Delta s = 0$ is simply the speed of light postulate! This, however, is more general. It is exactly analogous to the fact that the distance between two points in 3D space is invariant in any frame.

- We distinguish between three kinds of intervals:
  - $(\Delta s)^2 > 0$ – **Timelike separated events**

    If $(\Delta s)^2 > 0$, then $\left|\frac{x}{t}\right| < c$. This means that there exists an inertial frame with $v < c$ in which the two events occur at the **same place** (this is because it is possible for a particle to get from one event to the other, and in the frame of that particle, they both occur in the same place).

    The **time between the events** in **this frame** is, of course, the **proper time**, $\tau = \frac{\Delta s}{c}$. For timelike separated, the invariant takes the form:

$$(\Delta t)^2 - \left(\tfrac{\Delta x}{c}\right)^2 = \tau^2$$

  - $(\Delta s)^2 < 0$ – **Spacelike separated events**

    If $(\Delta s)^2 < 0$, then $\left|\frac{t}{x}\right| < \frac{1}{c}$. This means that there exists an inertial frame with $v < c$ in which the two events occur at the **same time** (to see that this is true, use the Lorentz Transformations or a Minkowski Diagram).

The **distance between the events** in **this frame** is, of course, the **proper distance**, $\ell = |s|$. For spacelike separated events, the invariant takes the form:

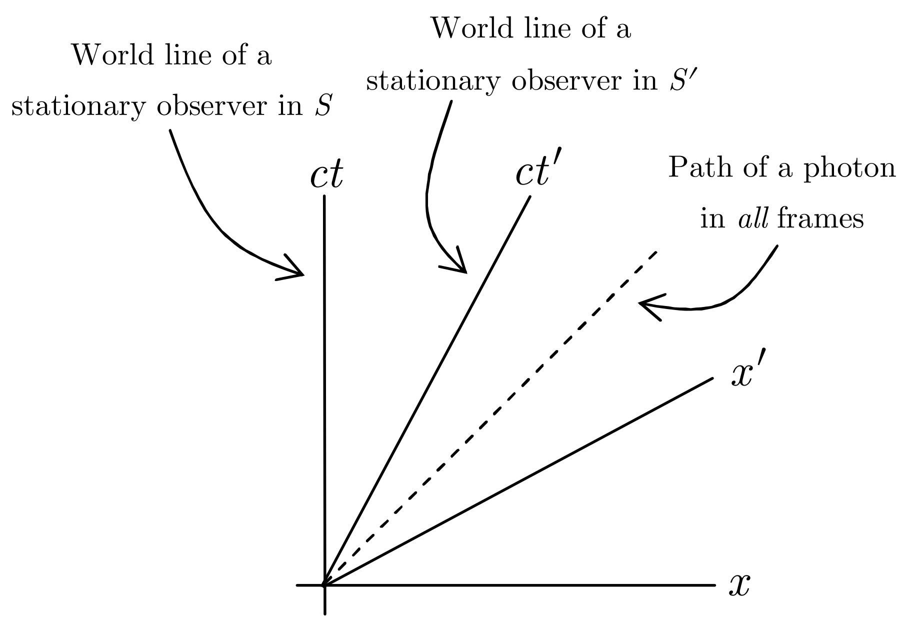
$$c^2 (\Delta t)^2 - (\Delta x)^2 = \ell^2$$

NOTE: The quantity Δx in any frame where $\Delta t \neq 0$ is not a length!

- $(\Delta s)^2 = 0$ – **Lightlike separated events**

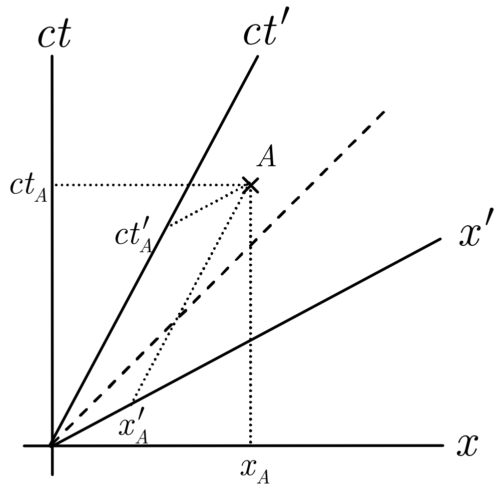
  If $(\Delta s)^2 = 0$, then $\left|\frac{x}{t}\right| = c$, and it is **impossible** to find a frame in which the two events occur at the **same time** or the **same place**. This is because the frame would have to travel at the **speed of light**. As we watch these event in a frame going successively faster and faster, the distance between then *and* the time between them **tend to 0**, but never quite get there.

- It turns out that here is also a **Velocity 4-vector** that exists; if we simply divide every component of the **spacetime 4-vector** by $\mathrm{d}\tau$, the **proper time** of the event which is **invariant in any frame**, and then note that $\mathrm{d}\tau = \mathrm{d}t / \gamma$, we find that the vector $\gamma\begin{pmatrix} c & v_x & v_y & v_z \end{pmatrix}$ is needed a **4-vector**. The **velocity addition formulae** can be derived from this by using the **invariance of the inner product** of two **4-vectors** (it gets much too messy with the Lorentz Matrix above).
- Note that it often helps to use units where $c = 1$. This can be done either by:
  - Using **lengths** in which **one unit** is equal to c metres. At the end, **divide by *c*** for every time unit.
  - Units **times** in which **one unit** is equal to $1/c$ seconds. At the end, **multiply by *c*** for every **distance** unit.
- A **Minkowski Diagram** is one in which we plot **distance** against ***ct***. The first thing to note is that a **photon** simply appears on the diagram as a **straight line** at a 45° angle to both the axes. Now, let frame S' move at a speed v (along the x-axis) with respect to frame S. What do the x' and ct' axes look like when **superimposed** onto the x and ct axes?

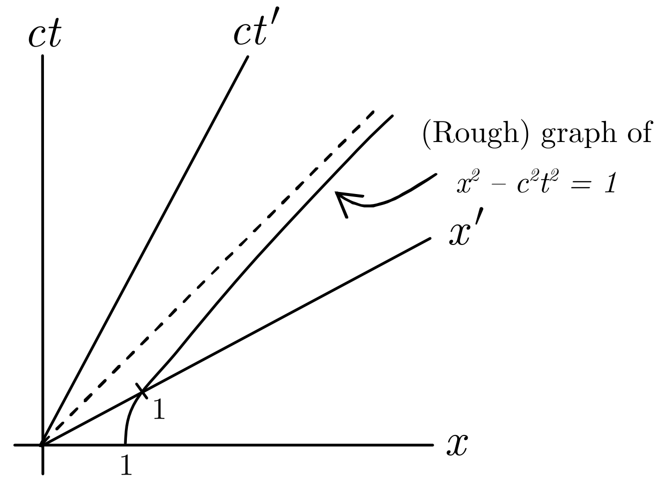


The two world lines in S' get **closer and closer** to the **photon world line**, but **never quite get there**, since nothing can travel at the speed of light.

Events can then just be plotted on the diagram and the values of x and t read off. For example:

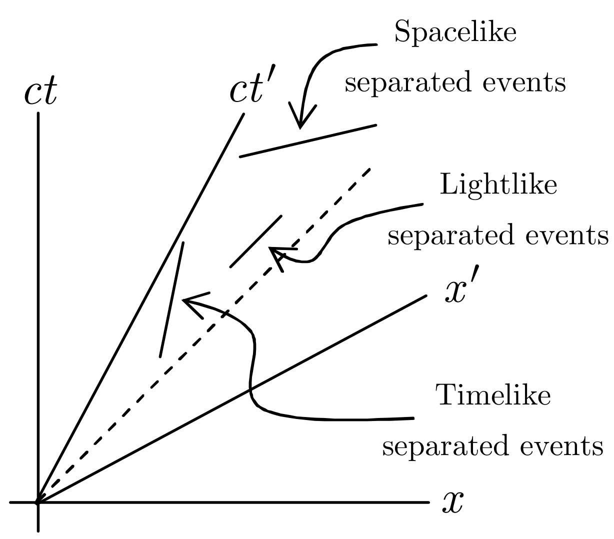


It is important, however, to realise that **the scale on each of the axes is different**. It can be obtained by **calibration with an invariant**:



We then note that the scales on the two axis *must* be the same, because the ray line bisects one set of axes (and therefore *every* set) and $x = ct$.

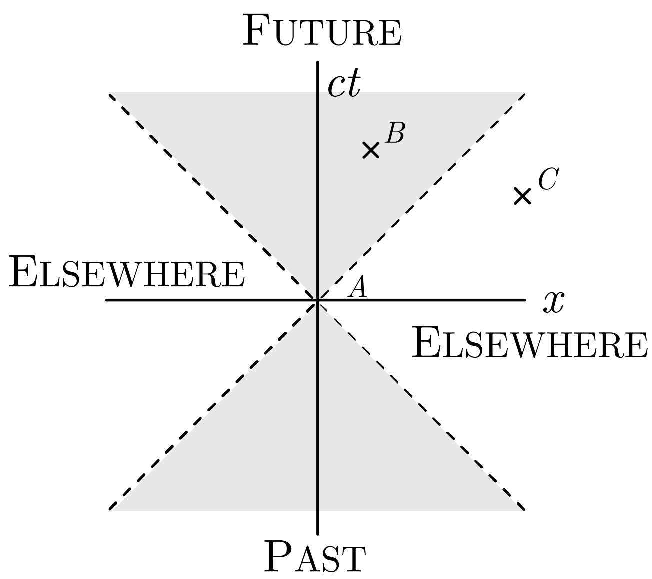
Finally, we note that Minkowski Diagrams are very useful to view the three kinds of interval we described above. We note that since $(\Delta s)^2 = (c\Delta t)^2 - (\Delta x^2)$, $(\Delta s)^2$ is a measure of the **inclination** of the line between two events in a Minkowski diagram; for example, if $(\Delta s)^2 > 0$, $(c\Delta t)^2 > (\Delta x^2)$, and the line between the two events is inclined at **more than 45°**. Thus:



Notice how:

- It is possible to find a velocity such that the x' axis becomes **parallel** to the **spacelike interval**. At that point, the two events occur **at the same time**. It is even possible to find a velocity in which the **order of the two events is reversed**. However, it is **never** possible to find a velocity at which the **time** axis is parallel to the spacelike interval. Thus, there is **no frame** in which the two events occur at the **same place**.
- Similar observations apply for the timelike interval.
- It is **never** possible to find a velocity such that **either** of the axes is parallel to the **lightlike interval**. This means that there is **no frame** in which the events occur at the same time, and **no frame** in which the events occur at the same place.

Now, assuming that we have an event A that occurs at the origin of a Minkowski diagram, we can make some interesting conclusions:



We note that if A **causes** an event, then the event must be in the "future" cone of A (for example, B above). Similarly, any event that **causes** A must line in the "past" cone of A. This is because the line from A to another event is basically the path of the "information" from A that causes the other event, and that just can't travel faster than the speed of light. If, instead, the event is in the "elsewhere" of A, then it is possible to find a frame in which both events occur at the **same time** at in which **one occurs before the other**. This is impossible, of course.

Dynamics

- The **total momentum** of a system is $\mathbf{p} = \sum \gamma_i m_i \mathbf{v}_i$ and it is **conserved**.
- The **total energy** of a system is $\mathbf{p} = \sum \gamma_i m_i c^2$ and is also **conserved**.

- The **kinetic energy** of a particle is $K = (\gamma - 1)mc^2$ – it is **<u>not</u>** conserved, and almost **never used**.
- The **Energy-momentum 4-vector** is (can be obtained by multiplying the **Velocity 4-vector** by m, the rest mass, which is an invariant)

$$\mathbf{P} = \left(\frac{E}{c}, p_x, p_y, p_z\right)$$

The following are important properties of **P**:

  - **Both** the **conservation laws** can be expressed as the fact that $\sum \mathbf{P}_i$ is **conserved**.
  - This **4-vector** is transformed between frames in the same way the **space-time 4-vector** is (ie: using the **Lorentz Transformations**).
  - The **inner-product** of **any two 4-vectors** in **Minkowski space** is an **invariant** – ie: it has the **same value** in **all frames**. Since **linear combinations** of **4-vectors** are themselves **4-vectors**, we can talk of the **4-vector of a system**, **P**, which is $\mathbf{P} = \sum \mathbf{P}_i$. Taking the **inner product** of that vector with itself:

$$\mathbf{P} \cdot \mathbf{P} = \frac{E^2}{c^2} - p_x^2 - p_y^2 - p_z^2$$

    Thus:

$$E^2 - c^2\mathbf{p}^2 = K^2c^4$$

    Now, in a frame where every particle in the system is at rest $\gamma_i = 1$ and the **total energy** of the system is $E = \sum m_i c^2$. However, in such a frame, $\mathbf{p}_i = \mathbf{0}$, and the expression above reduces to $E = Kc^2$. Thus, the K above is the **total mass of the system** in a frame in which **all the particles are at rest** (if there is such a frame). This frame, however, sometimes doesn't exist. In such a case, we simply describe m as the **energy in the ZMF frame**.
  - For a **single particle**, K is equal to the **rest mass** of the particle. So:

$$E^2 - c^2\mathbf{p}^2 = \mathbf{P} \cdot \mathbf{P} = m^2c^4$$

    NOTE: Some want to call m the **rest mass** in contrast to γm, the "**relativistic mass**", in the hope that if we take the mass of a particle to be γm, the particle then behaves in a Newtonian way. This, however is **doomed to failure** and **senseless**. For example, even though it **is** true that $\mathbf{F} = \gamma m\mathbf{a}$ for **transverse forces**, it is **not true** for **longitudinal forces**.
  - In a **ZMF**, this invariant is simply equal to $\boldsymbol{E^2}$.
  - The (rest) **mass** of a **photon** is 0. Thus, $\mathbf{P} \cdot \mathbf{P} = 0$ for a **photon**. Furthermore, we can use the invariant to show that $E^2 = \mathbf{p}^2c^2$, which is rather useful since neither the usual expressions for momentum and energy yield anything useful [note that we're not using the fact it's an invariant

quantity – that's only needed when we **change frames**. We're just using the expression]

- Another useful relation, which holds for particles of any mass [from the **definition** of ***p*** and E], is:

$$\frac{\mathbf{p}}{E} = \frac{\mathbf{v}}{c^2}$$

This is certainly the fastest may to get **v** if **p** and E are known.

- In questions, the following tips are useful:
    - When finding a **4-vector** for a given particle, the relation $E^2 = m^2c^4 + c^2\mathbf{p}^2$ is useful if only the energy or momentum are known.
    - To **eliminate ugly 4-vectors**, it often helps to **re-arrange** the **conservation law** and then take **square both sides** (ie: inner products). This means that we can put **one 4-vector** on one side of the equation **by itself**, so that it reduces to m when it is squared.

Miscellaneous Points

- The **relativistic effects** in a **given frame** depend only on the **instantaneous velocity** of the **frame** – ***not*** its **acceleration** [this is why in the twin paradox, the twin left on earth can freely apply special relativity to the twin travelling, even though she accelerates at some point]. However, if ***our*** frame is **accelerating**, then special relativity does **not** apply to us [this is one of the solutions of the twin paradox – the travelling twin can't just look back and apply special relativity with the earthbound twin, because he's accelerating]. This is a fact that is **well supported** by **experimental evidence**.