# Domino Tilings and the Arctic Circle Theorem

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May 11, 2021

# Outline:

Based on Jockusch – Propp – Shor '95, Rost '81

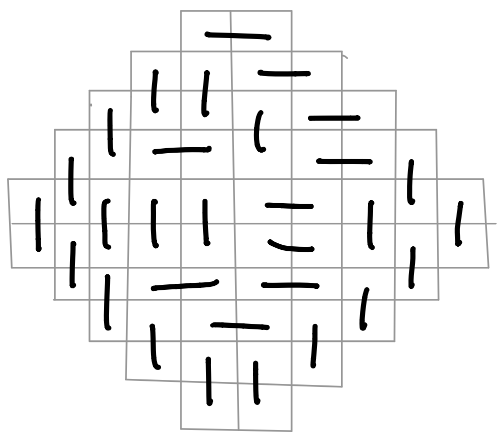
- Review of Aztec diamond and random tilings; simplification to partitions and TASEP

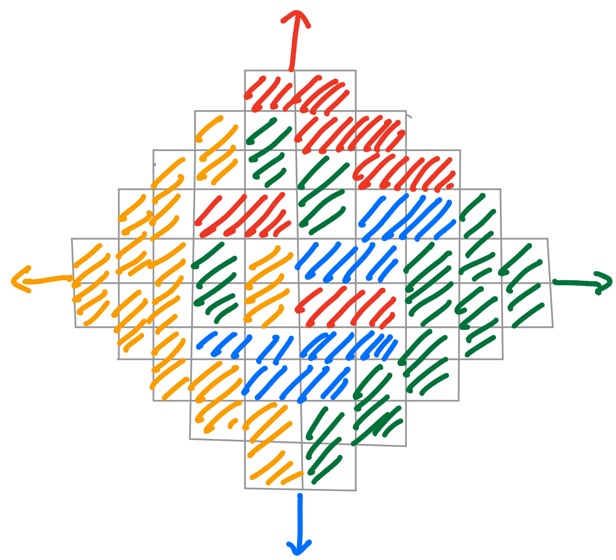
- Calculating the density profile: convergence and inequalities
  → Doing continuous-time for simplicity

- Highlighting differences between discrete & continuous case.

# ① Review and problem statement

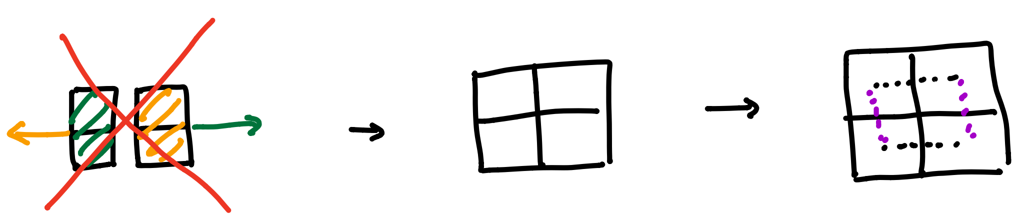
Aztec diamond: tiled with four kinds of dominos.



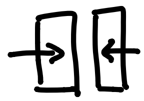


**Thm** The size-$n$ Aztec diamond has $2^{n(n+1)/2}$ different tilings.

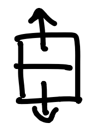
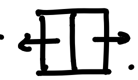
Proved by shuffling algorithm, which randomly generates uniform tiling recursively.



More formally:

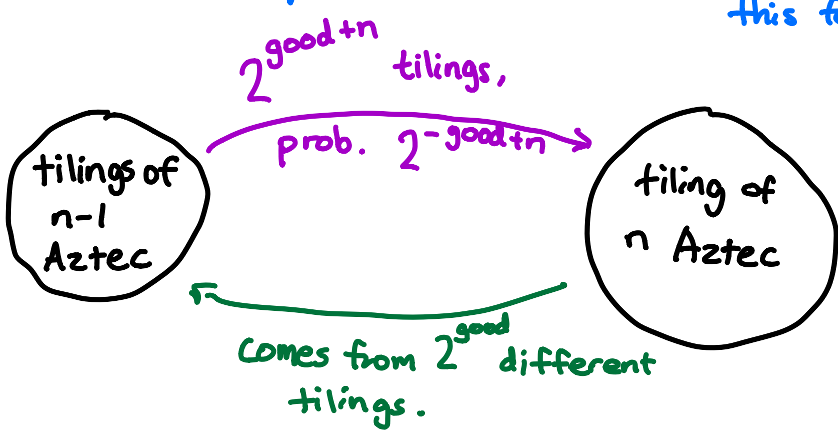
- Find all  s and remove them.

- Push all dominos in their corresponding directions.

- For each remaining $2\times 2$ square, fill it  or .

Generates <u>uniform</u> tiling because:

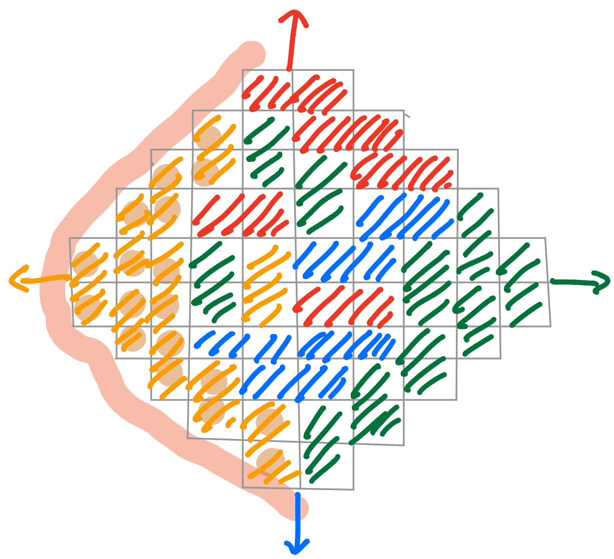
Think about process in reverse. Only blocks of this form. "good"




$$\Rightarrow \quad 2^{-(n-1)n/2} \cdot 2^{-n} = 2^{-n(n+1)/2}.$$

✓

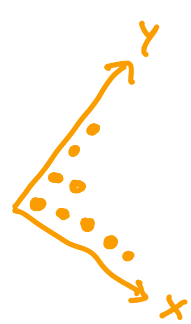
Looking more closely at frozen region:



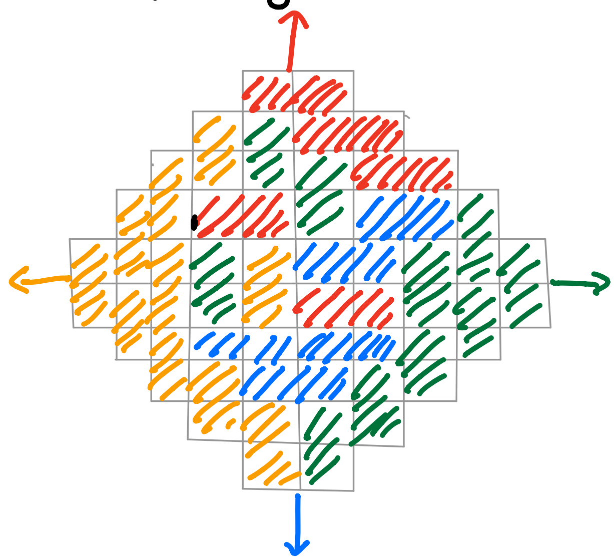
**Prop** Each frozen region is the set of dominos that "sit" on the marked wall above dominos/wall of that color.

Specifically, no other orange domino touches region above.

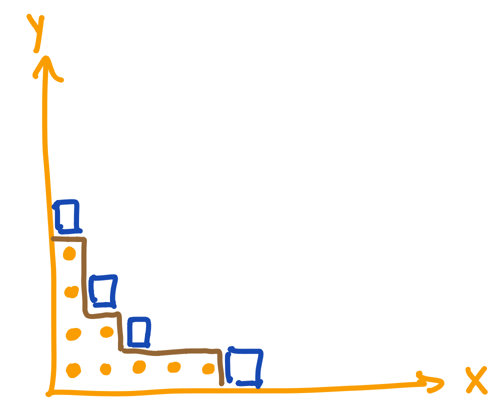
Forms a partition diagram:



Dynamics of shuffling:



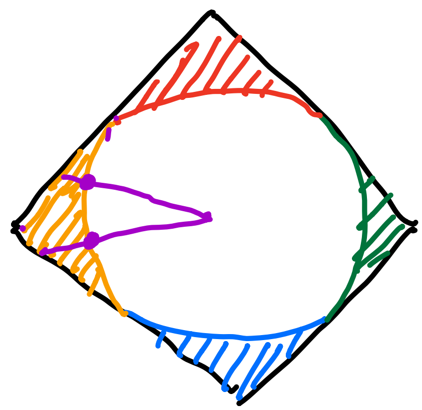
- Where can new orange dominos be added to the frozen region?
- How likely are those to be added?




This is the discrete-time TASEP

We want to show:

**Arctic Circle Thm** The frozen regions trace out quarter-circle arcs in the limit $n \to \infty$ (that is, we're within $o(n)$ distance of this limit shape).



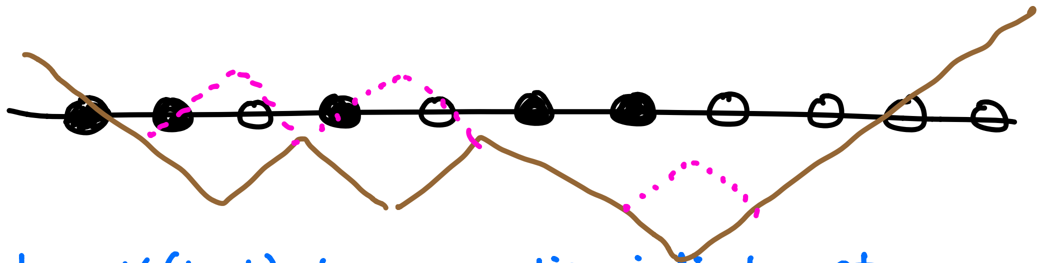
Equivalent to prove:

"Thm" Discrete-time TASEP has a limiting density profile along each line of constant slope (as above).

Idea is that path can only move down and right.

## ② TASEP and extracting the density profile

More formally setting up notation:

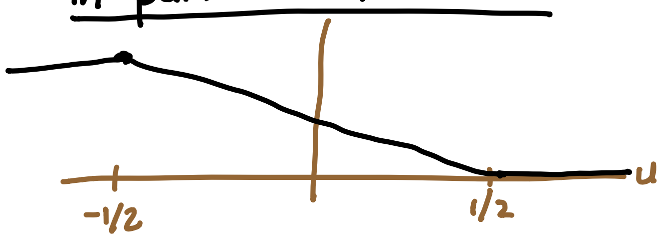


Let $X(k,t)$ be occupation indicator at position $k$, time $t$. Process is $X(t)$.

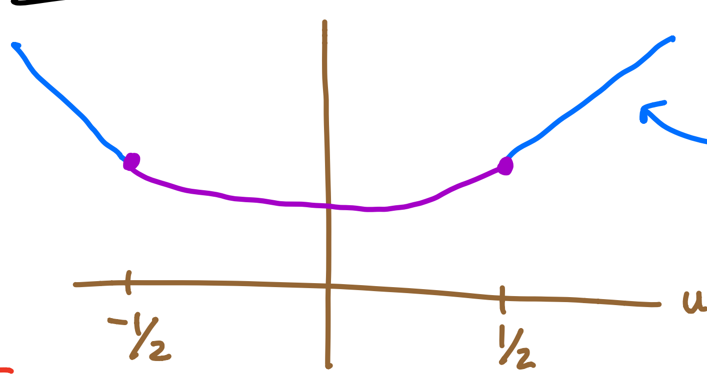
Interested in $\mathbb{E}\, X(\lfloor ut \rfloor, t)$.

Should behave like:

in particle expectation,




in TASEP visualization,




less interesting (binomial/Poisson).

## Stochastic ordering

Let $x_1, x_2$ be two points in TASEP state space.
Say $x_1 \leqq x_2$ if coordinate-wise $\leq$.

↳ Induces order on prob. measures:

$\mu \leqq \nu$ if there is coupling $\sigma$ s.t. $\sigma(x_1 \leqq x_2) = 1$
and $\mu, \nu$ are coordinate projections

**Lemma** Stochastic ordering is preserved under TASEP update.

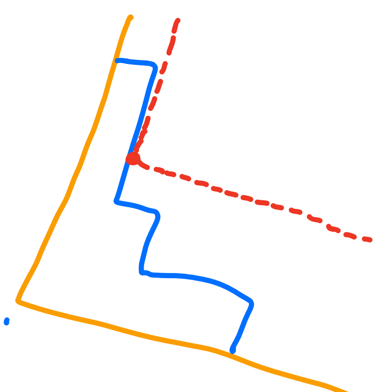
**Prop** Let $S(k,t) = \sum_{i \geq k} X(i,t)$. Then for all $k, \ell, r, t$,

$$\mathcal{L}(S(k,r)) * \mathcal{L}(S(\ell,t)) \geq \mathcal{L}(S(k+\ell, r+t)).$$

Use a coupling argument:

let $\tilde{S}$ evolve like $S$ until time $r$, then jump to

$$\tilde{S}(j, r) = \begin{cases} S(k,r), & j \geq k \\ S(k,r) + (k-j), & \text{otherwise.} \end{cases}$$



Then $\tilde{S}(k+\ell, r+t) - S(k,r)$ identical to $S(\ell, t)$,
but $\mathcal{L}(\tilde{S}) \geqq \mathcal{L}(S)$.

Specializing previous result,

$$\underbrace{\mathcal{L}(S(\lfloor ur \rfloor, r))}_{\mathcal{L}_r} * \underbrace{\mathcal{L}(S(\lfloor ut \rfloor, t))}_{\mathcal{L}_t} \geq \mathcal{L}(S(\lfloor ur \rfloor + \lfloor ut \rfloor, r+t))$$
$$\geq \underbrace{\mathcal{L}(S(\lfloor u(r+t) \rfloor, r+t))}_{\mathcal{L}_{r+t}}.$$

By Kesten–Hammersley, this means $\frac{S(\lfloor ut \rfloor, t)}{t}$ converges to some $h(u)$ almost surely and in $L^1$.

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Notice that $h$ is convex:

$$\mathbb{E}\Big(S(\lfloor c_1 ut \rfloor, c_1 t) + S(\lfloor c_2 vt \rfloor, c_2 t)\Big) \geq \mathbb{E}\Big(S(\lfloor (c_1 u + c_2 v)t \rfloor, (c_1+c_2)t)\Big)$$

so $\quad c_1 h(u) + c_2 h(v) \geq (c_1 + c_2)\, h(c_1 u + c_2 v).$

**Prop** As long as $\frac{k}{t} \to u$, we have

$$\lim_{t\to\infty} \mathbb{E}\, X(k,t) = -h'(u),$$ as long as $h$ is diff. at $u$.

(Consider $\quad h_n(v) = \int_v^\infty \mathbb{E}X(\lfloor xn \rfloor, n)\, dx. \approx \frac{1}{n} \sum_{k=\lfloor vn \rfloor}^{\infty} \mathbb{E}\, X(k,n).$

We have $h_n(v) \to h(v)$ as $n \to \infty$, and each $h_n$ is convex, so we can indeed differentiate.)

(exponential clocks)

From here, we do continuous-time for ease of calculation.

## Limiting "independence":

**Prop** The measures $\mu(\lfloor ut \rfloor, t)$ as $t \to \infty$ converge to an exchangeable measure

$$\mu^* = \int_0^1 \mathcal{B}_a \, \rho(da)$$

Bernoulli of parameter $a$.

- Similar coupling fact: if $\pi(s, \ell) = \mathbb{P}(\mathrm{Pois}(s) = \ell)$,

$$\boxed{\begin{gathered} \mu(k,t) \geqq \mu(k+1, t), \\ \mu(k, t+s) \;\cancel{\geqq}\; \leqq \sum_{\ell} \pi(s, \ell)\, \mu(k \cancel{+} - \ell, t) \end{gathered}}.$$

- $\mu^*$ and its shifted image must be identical, because one-point correlations are the same
- This implies invariance under TASEP semigroup.

these imply exchangability.

Also use di Finetti's theorem (conditionally independent given the value of $a$).

## Identifying the density

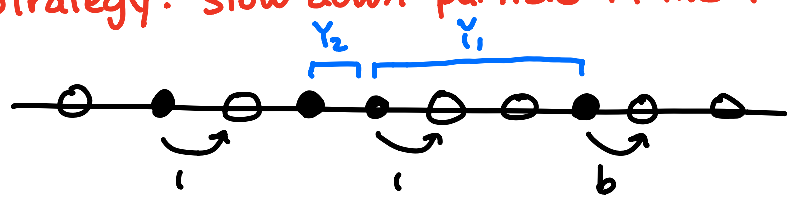
We know

$$h(u) = \lim_{t\to\infty} \frac{\mathbb{E}\, S(\lfloor ut\rfloor, t)}{t} = \int_u^\infty f(w)\, dw.$$

(Want to show that $f(u) = \begin{cases} 1 & u<-1 \\ \frac{1}{2}(1-u) & -1\le u\le 1 \\ 0 & u>1 \end{cases}$.)

**Prop** $h(u) \geq \frac{1}{4}(1-u)^2$ for all $u$ in $[-1,1]$.

Strategy: slow down particle in the front.




Study gaps $Y_i$. Invariant measure $(\gamma^b)$: all $Y_i$ iid with $\mathbb{P}(Y_i > m) = b^m$.

★ Our initial measure $\leq$ invariant measure, so

$$\mathbb{E}_{\text{slow}}\left(\sum_{i=1}^k Y_i(t)\right) \leq \int \left(\sum_{i=1}^k \overset{(\text{invariant})}{Y_i}\right) \gamma^b(dy).$$

$$= k \sum_{m\geq 0} \mathbb{P}^{\text{slow}}(Y_i > m) = \frac{k}{1-b}.$$

Take $k = \lfloor at\rfloor$, $t\to\infty$. Dividing through by $t$,

$$\limsup_{t\to\infty} \frac{1}{t} \mathbb{E}_{\text{slow}} \sum_{i=1}^{\lfloor at \rfloor} Y_i(t) \le \frac{a}{1-b},$$

- Use Law of Large Numbers on

$$\text{LHS} = \frac{1}{t}\Big( \underbrace{\text{Poisson}(bt)}_{\text{position of front particle}} - (\text{position of } \lfloor at \rfloor\text{th particle}) \Big),$$

to get (letting Z be pos. of particle)

$$\boxed{\lim_{t\to\infty} \mathbb{P}^{\text{slow}}\left( \frac{Z(\lfloor at \rfloor, t)}{t} < b - \frac{a}{1-b} - \epsilon \right) = 0 \quad \text{for any } \epsilon > 0.}$$

Also true for $\mathbb{P}^{\text{regular}}$ instead of $\mathbb{P}^{\text{slow}}$, also maximize RHS at $b = 1-\sqrt{a}$ to find

$$\lim_{t\to\infty} \mathbb{P}\left( \frac{Z(\lfloor at \rfloor, t)}{t} < 1 - 2\sqrt{a} - \epsilon \right) = 0,$$

so $$\mathbb{P}\left( \frac{S((1-2\sqrt{a}-\epsilon)t,\, t)}{t} > a \right) = 1.$$

$$\Rightarrow \quad h(1-2\sqrt{a}) \ge a$$

$$\Rightarrow \quad h(u) \ge \frac{1}{4}(1-u)^2. \qquad \square$$

**Prop** $h(u) \le \frac{1}{4}(1-u)^2$ for all $|u| \le 1$.

Strategy: .. clever resumming?

Idea: $S(\lfloor ut \rfloor, t)$ is # particles faster than a person traveling at speed $u$.

$$\mathbb{E}\, S\left(k, \tfrac{k}{u}\right) = \sum_{i=1}^{k} \mathbb{E}\left(S\left(i, \tfrac{i}{u}\right) - S\left(i-1, \tfrac{i}{u}\right)\right)$$
$$+\ \mathbb{E}\left(S\left(i-1, \tfrac{i}{u}\right) - S\left(i-1, \tfrac{i-1}{u}\right)\right)$$

$$= -\sum_{i=1}^{k} \mathbb{E}\left(X\left(i, \tfrac{i}{u}\right)\right) + \sum_{i=1}^{k} \mathbb{E}\left(\text{jumps from } i-1 \text{ to } i \text{ in time } \tfrac{1}{u}.\right)$$

$\downarrow$ scale by $\frac{u}{k}$, $k \to \infty$

$$\star\ h(u) = -u h'(u) + \lim_{T\to\infty} \frac{1}{T} \int_0^T \mu(\lfloor ut \rfloor, t) \cdot 1\left\{ (X_0, X_1) = (1,0) \right\}$$

Now Jensen's inequality: integrand is limiting to $\int a(1-a)\, p(da)$, so

because $f(u) = \int a\, p(da)$, we have

$$\star\ f(u)\,(1 - f(u)) \ge \boxed{\phantom{x}}.$$

$$\star:\ h(u) \le -u f(u) + f(u)(1 - f(u)) \le \frac{1}{4}(u-1)^2.$$

minimized at $f(u) = \frac{1-u}{2}$

$\square$

## ③ Differences in the discrete-time case

- Markov measures $\mu_d$ are more complicated.

→ Shift-invariant Markov measure depends on $p_1 = \mathbb{P}(X_0 = 1)$, also $q_{ij} = \mathbb{P}(i \to j)$.

if $d$, turns out $\mathbb{P}(X_1 = 0 \mid X_0 = 0)$ is $\dfrac{-d+\sqrt{d^2+(1-d)^2}}{1-d}$.

Requirement : $q_{01} q_{10} = 2 q_{00} q_{11}$

(think about stationary measure on a cycle).

→ Verify invariant under TASEP evolution.

- "All stationary, translation-invariant measures are convex combinations of the $\mu_d$ s."

↳ Clever coupling makes this easier to prove than "exchangeable measures" argument above.

- Lower bound (LLN, etc):

$\gamma^b$ invariant measure is now

$$\mathbb{P}(y_i \geq m) = \begin{cases} 1 & m=0 \\ b\left(\frac{b}{2-b}\right)^{m-1} & \text{otherwise.} \end{cases}$$

- Upper bound (Jensen):

Instead of $\mathbb{P}((X_0, X_1) = (1,0)) = a(1-a)$,

we have $1-\sqrt{a^2+(1-a)^2}$.

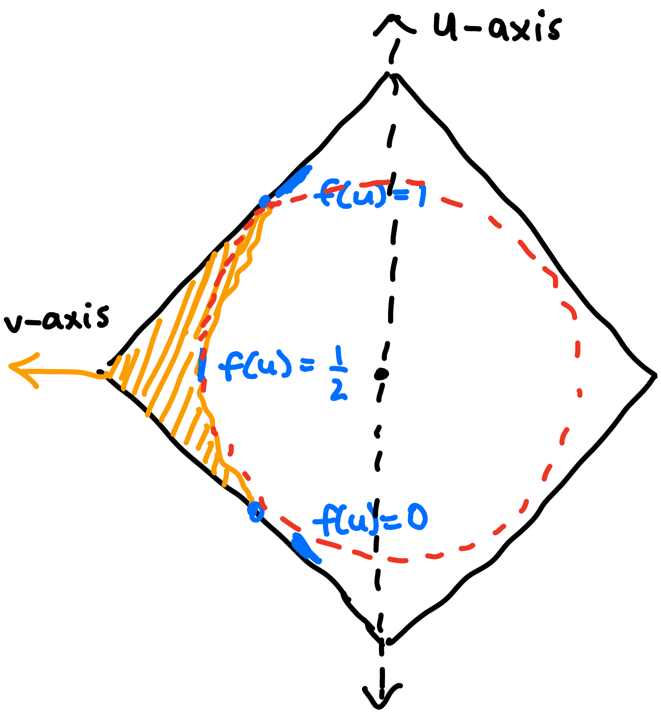
But the general analytic techniques remain the same.

Verifying the final calculation:

(for discrete-time TASEP, )

$$h(u) = \begin{cases} -u & u < -1/2 \\ \frac{1-u}{2} - \frac{1}{2}\sqrt{\frac{1}{2} - u^2} & -\frac{1}{2} \le u \le \frac{1}{2} \\ 0 & u > 1/2 \end{cases},$$

so $$\boxed{f(u) = \begin{cases} 1 & u < -1/2 \\ \frac{1}{2} - \frac{u}{\sqrt{2-4u^2}} & |u| \le \frac{1}{2} \\ 0 & u > 1/2. \end{cases}}$$




Indeed, for $u^2 + v^2 = \frac{1}{2}$

slope at $u$ is

$$m = \frac{dv}{du} = \frac{\pm u}{v} = \frac{\pm u}{\sqrt{\frac{1}{2} - u^2}}.$$

Corresponds to

$$f(u) = \frac{1-m}{2} = \frac{1}{2} - \frac{u}{\sqrt{2-4u^2}},$$

as predicted.

$\square$

## References:


[1] Rost, H. Non-equilibrium behavior of a many-particle process: Density profile and local equilibria. https://doi.org/10.1007/BF00536194

[2] Jockush, W., Propp, J., and Shor, P. Random domino tilings and the arctic circle theorem. arxiv.org/abs/math.CO/9801068


Any questions?