18.905: Algebraic Topology I

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Fall 2020

Introduction

18.905 is the graduate algebraic topology class at MIT, and it'll be a weird semester to take it because we're all over the world, but hopefully the class can still be a good time. The goal is for it to not feel like a "completely virtual experience."

Fact 1

The students of the class shared where we are currently located, as well as what we want to get out of the class and why we want to learn algebraic topology.

Because a lot of students are interested in category theory, the class will put a "categorical spin" on the topic – since we are following Professor Haynes Miller's notes, there will be a fair amount of that.

In terms of logistics, we'll be using the Canvas website for 18.905. Professor Miller's notes are posted on the welcome message, and a general syllabus is also posted in its corresponding location. We will not use the Hatcher book extensively (it's mroe geometric than Dr. Hahn prefers), but it can be an alternative source.

The point of this class is to cover homology, cohomology, and Poincaré duality – if we read the 100 pages corresponding to that material, we should understand the class, and the "finiteness" of content makes this class hopefully approachable. Homology is a tool in algebraic topology which (along with cohomology) is used everywhere in mathematics, especially in algebraic geometry and differential geometry. And we'll see how it can show up in surprising places through category theory and applied mathematics, too. However, this class will not cover point-set topology and the fundamental group – we can read Hatcher or the end of Munkres for that.

Zoom lectures will be recorded, so those of us attending in inconvenient time zones can still learn the material. Logistics-wise, we should **display the name we want to go by**, as well as pronouns optionally. And we should unmute ourselves whenever we want to talk or send questions into the chat in order to keep the class interactive.

There will be biweekly problem sets – we should submit our homework only if we are taking the class for credit, and if so, we should **make sure to meet in groups**. That's why the website https://psetpartners.mit.edu has been developed: this makes it easier for us to work on problem sets together, and we should just remember to write down all sources and other people that we consulted at the top of each pset. Problem sets will be worth 85 percent of our grade, and the last 15 percent is a (significantly easier) exam that should take only about an hour to complete.

1 September 2, 2020

We'll start with a small introduction of "what homology is" and where we're going to go with it during this semester. The basic concept is that we want to **study spaces by studying how many holes they have** – this is a very coarse invariant, but it can still be useful.

Example 2

The shape O has "one hole," while the shape Y has "no holes" and the shape 8 has "two holes."

We need to be able to rigorously define what a "hole" means, and we need to have a rigorous way to calculate how many such holes there are, so that we can distinguish different kinds of spaces. The **fundamental group** gives one way for us to measure the number of holes: the fundamental group of the circle is $\mathbb{Z}$, while the fundamental group of Y is the trivial group, and the fundamental group of 8 is the free group on two generators.

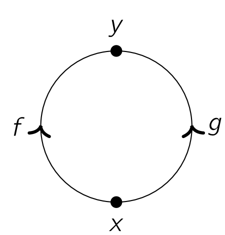
But there's no "surface" to any of these three shapes, and the fundamental group is inherently one-dimensional. Since we'll want to talk about higher-dimensional spaces, we will use a different method to understand these "holes," called the **first homology group** H_1 (called such because it measures "one-dimensional holes" or loops, but generalizing better to higher dimensions).

We'll now describe an algorithm for calculating the first homology group, and we'll spend some of the class explaining why this is rigorous:

Example 3

Let's compute the homology group H_1 of the circle.

First of all, we'll think of the circle has a discrete graph: take two points on the circle, and imagine that they are connected by two segments. Then for every edge in the graph, we can draw an edge:



These edge directions can be **assigned arbitrarily** – it turns out that it won't matter where the arrows point. In the diagram above, the edges f and g have source x and target y.

Now, let $\mathbb{Z}\{f, g\}$ denote the free abelian group generated by f and g: such elements can be written as formal sums $7f + 4g$, or 0, or $3f - 2g$. Similarly, let $\mathbb{Z}\{x, y\}$ denote the free abelian group generated by the two vertices x and y. We can consider the **group homomorphism**

$$d : \mathbb{Z}\{f, g\} \to \mathbb{Z}\{x, y\}$$

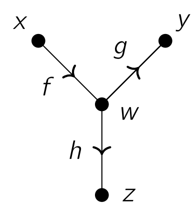
where f and g both map to $y - x$ (that is, the target vertex minus the source vertex). The kernel of this homomorphism is then the free abelian group generated by $f - g$ – **since there is one element generating this kernel, there is one hole**. This kernel is called the first homology group of the space, and the number of generators is also the **first Betti number**.

We'll spend the rest of today doing similar types of examples to help us gain some intuition.

Example 4

Let's compute the homology group H_1 for the Y shape, which we expect to have zero holes.

We can imagine Y as being embedded in $\mathbb{R}^2$ with the subspace topology, and the natural way to turn this into a graph is as follows:



Like before, we can consider a group homomorphism

$$d : \mathbb{Z}\{f, g, h\} \to \mathbb{Z}\{x, y, z, w\}$$

where f, g, h are sent to $w - x, y - w, z - w$, respectively. And this time the kernel is **trivial** (which is the free abelian group with no generators), and therefore we indeed find that there are **zero holes** with this method.

Remark 5. *Notice that we can cut up the two shapes above in different ways as well: for example, we can add extra vertices and create more edges. We'll soon see that we can add an extra vertex in the middle of some of our edges, and this will also not change our final answer.*

We'll soon see how this concept generalizes to higher dimensional spaces, as well as why this invariant exists for topological spaces.

2 September 4, 2020

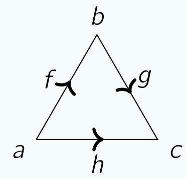
Our first problem set has been posted – it's due in two weeks (September 18 at class time) on Canvas. We can be matched with other students as "pset partners" if we enter our available times on the psetpartners website soon! It's recommended that we start on the problem set early, so that we can absorb the material a little at a time (some of the topics will make sense to us very soon, while other topics will take more time). Also, we won't have class on Monday because of Labor Day.

Remark 6. *Starting next week, to accommodate different time zones, office hours will be held the hour before class on Wednesdays and Fridays.*

Recall that last time, we learned how to compute the first homology group $H_1(G)$ when G is a directed graph (an object with vertices and some arrows between them). This homology group basically let us count the number of loops in our graph – let's do another example to illustrate the concept.

Example 7

Suppose that our graph looks like a triangle:



To compute the first homology group, we form the map of free abelian groups

$$\mathbb{Z}\{f, g, h\} \xrightarrow{\partial} \mathbb{Z}\{a, b, c\}$$

that sends f to $b-a$, g to $c-b$, and h to $c-a$. Then the kernel of the map ∂, also called a **boundary homomorphism**, is generated by $g-h+f$, so

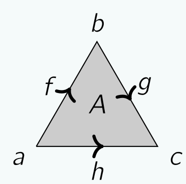
$$H_1(G) = \ker(\partial) = \text{ free abelian group generated by } g-h+f.$$

And the fact that we have one generator means that our object has "one hole."

But we want to be able to talk about higher-dimensional objects, not just one-dimensional graphs, and we want to be able to deal with topological spaces (like subspaces of $\mathbb{R}^n$), not just combinatorial objects. We'll tackle the first problem first: we'll understand how to deal with finding a **combinatorial representation in higher dimensions**.

Example 8

Say that we glue in a two-dimensional triangle into our shape from above:



This "solid triangle" should now have zero holes, so we expect the group H_1 to be trivial. Here, $g-h+f$ is a **cycle**, so it'll still be in the kernel of our boundary operator, but it'll also be in the image of the boundary operator

$$\partial A = g - h + f,$$

because we can imagine going along the boundary of our triangle A, counting edges with direction! So **homology** will record cycles **modulo the boundaries**, and we'll rigorize this now.

Definition 9

A **semisimplicial set** X is a sequence of sets $X_0, X_1, X_2, \cdots,$, and functions $d_0, d_1 : X_1 \to X_0$, $d_0, d_1, d_2 : X_2 \to X_1$, and so on (in general, we have $n+1$ functions from $X_n \to X_{n-1}$ for every $n \geq 1$), such that the **simplicial identities** are satisfied:

$$d_i d_j = d_{j-1} d_i$$

whenever $i < j$ and those equations make sense.

The idea is that we're building arbitrary shapes out of points, lines, triangles, tetrahedra, and so on.

Example 10

Suppose we have a semisimplicial set where $X_2 = \varnothing, X_3 = \varnothing, \cdots$.

Then we actually just have a directed graph: X_0 can be thought of as the set of vertices, and X_1 can be thought of as the set of edges. Here, $d_0, d_1 : X_1 \to X_0$ record the "target" and "source" of our edges, and it turns out the simplicial identities are always automatically satisfied.

Example 11

Let's return to the two-dimensional solid triangle from above.

In this case, we have $X_3, X_4, \cdots$ all empty, and we have

$$X_2 = \{A\}, \quad X_1 = \{f, g, h\}, \quad X_0 = \{a, b, c\}$$

corresponding to the faces, edges, and vertices. Now we need to specify a bunch of functions. The functions $d_0, d_1 : X_1 \to X_0$ are defined similarly to before: we have

$$d_0 f = b, \quad d_0 g = c, \quad d_0 h = c,$$

and

$$d_1 f = a, \quad d_1 g = b, \quad d_1 h = a.$$

It turns out that we can finish our description by setting

$$d_0 A = g, \quad d_1 A = h, \quad d_2 A = f.$$

We could have arrived at these equation by using the simplicial identity

$$d_0 d_2 A = d_1 d_0 A,$$

and it turns out that there is only one way to assign $\{0, 1, 2\}$ to $\{f, g, h\}$ so that A "maps to all three edges" and also satisfies the simplicial identities. (But we'll also see a better explanation soon.)

We can now explain how to compute the homology of a semisimplicial set X. Let $S_n(X)$ denote the free abelian group generated by the set X_n (this is also called the **set of n-simplices**).

Definition 12

For all $n \geq 1$, the **boundary operators** are group homomorphisms

$$\partial_n : S_n(X) \to S_{n-1}(X)$$

are defined by sending $\sigma \in X_n$ to

$$\sum_{k=0}^{n} (-1)^k d_k \sigma.$$

We also define $\partial_0 : S_0(X) \to 0$ to be the zero homomorphism.

Definition 13

Let X be a semisimplicial set. The **group of n-cycles** in X, denoted $Z_n(X)$, is the kernel of the boundary operator ∂_n, and the **group of n-boundaries** in X, denoted $B_n(X)$, is the image of ∂_{n+1}.

Basically, the idea is to "get rid of higher-dimensional cycles." We'll prove on our homework that $B_n(X)$ is a subgroup of $Z_n(X)$, so every boundary is a cycle (if we take the boundary of a boundary, we get zero), which is in turn a subgroup of $S_n(X)$.

Because our boundaries are the "boring cycles" (which are actually just part of higher-dimensional behavior), it makes sense to define the quotient abelian group

$$H_n(X) = Z_n(X)/B_n(X),$$

which intuitively measures the number of n-dimensional holes in our object.

We can now turn to the second question: how can we turn our attention from semisimplicial (combinatorial) sets to topological spaces? Specifically, how can we do the same process as we did with the circle and Y and other shapes during the first class, **cutting up our topological spaces to form a semisimplicial set**?

Showing that all methods of cutting give us the same answer will take up some time, so we'll dodge the question by defining a "maximal" or "canonical" method of cutting up a topological space.

Definition 14

For any $n \geq 0$, the **standard n-simplex** Δ^n is a subspace of $\mathbb{R}^{n+1}$, defined as the convex hull of the standard basis $\{e_0, e_1, \cdots, e_n\}$. In other words,

$$\Delta^n = \left\{ \sum_i t_i e_i : \sum t_i = 1,\ t_i \geq 0 \right\}.$$

For example, Δ^1 is basically a line segment in $\mathbb{R}^2$, and Δ^2 is a triangle in $\mathbb{R}^3$.

Definition 15

Let X be a topological space. Define $\mathrm{Sing}_n(X)$ to be the set of all continuous maps $\Delta^n \to X$.

These are extremely infinite sets – for example, $\mathrm{Sing}_0(X)$ is the set of maps from a point to X, which means we can think of it as the set of points in X. But we can still talk about these sets in useful ways: specifically, we have maps

$$d_i : \mathrm{Sing}_n(X) \to \mathrm{Sing}_{n-1}(X),$$

which are defined by "forgetting" the basis element e_i.

Example 16

In the map $d_1 : \mathrm{Sing}_2(X) \to \mathrm{Sing}_1(X)$, we take the image of the triangle with vertices e_0, e_1, e_2, and we only keep the part which is the image of the line segment from e_0 to e_2.

These d_i maps turn out to make $\mathrm{Sing}(X)$ into a semisimplicial set! And this way, we don't need to cut a space into simplices, because we're doing the "maximal" construction where we consider all continuous functions.

Remark 17. *We can look into Hatcher's treatment of "Δ-complexes" (which is an alternate name for a "semisimplicial set") – there are some good pictures.*

Definition 18

If X is a topological space, define $S_n(X)$, $Z_n(X)$, $B_n(X)$, and $H_n(X)$ to be $S_n(\text{Sing}\, X)$, $Z_n(\text{Sing}\, X)$, $B_n(\text{Sing}\, X)$, and $H_n(\text{Sing}\, X)$, respectively.

We can think of a certain hierarchy being used here, going from geometric to combinatorial to algebraic objects:

$$\text{Topological Spaces} \xrightarrow{\text{Sing}} \text{Semisimplicial sets} \xrightarrow{H_n} \text{Abelian groups.}$$

Here, "Sing" is short for "singular," because we have some continuous but very ugly maps that can be created from this process. And now we can understand why we chose $d_1 A = h$ and so on in the solid triangle above: labeling our vertices of the triangle as e_0, e_1, e_2, the edge $d_1 A$ is the one "opposite from" vertex e_1, and so on.

3 September 9, 2020

A lot of us have been assigned to pset groups by the psetpartners website, but if we're not satisfied with our groups or had trouble with the process, we should email Dr. Hahn or go to office hours (which will now take place on Wednesdays and Fridays before class, as previously mentioned). Our first problem set will be due next Friday.

Last time, we set up the basic idea of singular homology, and we discussed certain transformations described in the diagram below:

$$\text{Topological Spaces} \xrightarrow{\text{Sing}} \text{Semisimplicial sets} \xrightarrow{S_n, Z_n, B_n, H_n} \text{Abelian groups.}$$

That means that if we're given a topological space, we should be able to extract a semisimplicial set, and then subsequently extract an abelian group. But the arrows in the above picture have a more specific meaning – the maps Sing, S_n, Z_n, B_n, H_n are compatible with "maps between objects:"

Fact 19

If $X \to Y$ is a continuous map of topological spaces, and $\sigma : \Delta^n \to X$ is a continuous function, then the composite map $\Delta^n \to X \to Y$ is also continuous, so a map $X \to Y$ induces a map $\text{Sing}_n(X) \to \text{Sing}_n(Y)$.

To be more specific with the underlying ideas here, we'll introduce a lot of **category theory** notation which will encode what we care about here:

Definition 20

A **category** $\mathcal{C}$ consists of the following:

- A class $\text{ob}(\mathcal{C})$ of **objects** in $\mathcal{C}$,
- For every pair of objects $X, Y \in \text{ob}(\mathcal{C})$, a set of **morphisms** denoted $\text{Hom}_{\mathcal{C}}(X, Y)$,
- For every $X \in \text{ob}(\mathcal{C})$, an **identity morphism** $1_X \in \text{Hom}_{\mathcal{C}}(X, X)$,
- For every (X, Y, Z) (with $X, Y, Z \in \text{Hom}_{\mathcal{C}}(X, X)$), a **composition operation**

$$\text{Hom}_{\mathcal{C}}(X, Y) \times \text{Hom}_{\mathcal{C}}(Y, Z) \to \text{Hom}_{\mathcal{C}}(X, Z),$$

written as (f, g) mapping to $g \circ f$. This composition must satisfy $(h \circ g) \circ f = h \circ (g \circ f)$ and

$$1_Y \circ f = f \quad \forall f \in \text{Hom}_{\mathcal{C}}(X, Y), \quad f \circ 1_X = f \quad \forall f \in \text{Hom}_{\mathcal{C}}(Y, X).$$

Remark 21. *Note that categories have a* **class** *of objects instead of a* **set***, which means the collection sits in a higher Grothendieck universe. But we won't need to worry about these issues for now. (This is really just to avoid needing to worry about things like "the set of all sets.")*

We have perhaps already seen many examples of categories from other math classes:

Example 22

There is a category Set of sets, which means that ob(Set) is the class of all sets, and morphisms are functions from one set to another.

In particular, if X, Y are two sets, then $\text{Hom}_{\text{Set}}(X, Y)$ is the set of all functions from X to Y. The composition operation is just the composition of functions, and the identity morphism is the identity function.

Example 23

There is a category Ab of abelian groups, which means that the objects in ob(Ab) are abelian groups, and the morphisms $\text{Hom}_{\text{Ab}}(A, B)$ refers to the set of all **group homomorphisms** $A \to B$.

The idea here is that the composition of two group homomorphisms is always a homomorphism, and we always have identity homomorphisms.

Example 24

There is a category $\text{Vect}_{\mathbb{R}}$ of real vector spaces, where the objects in $\text{ob}(\text{Vect}_{\mathbb{R}})$ are vector spaces (with scalar field $\mathbb{R}$) and the morphisms are linear transformations.

In general, "identity morphism is a morphism" and "composition of morphisms is a morphism" are really the two facts that we get out of saying that something is a category.

Example 25

There is a category Top of topological spaces, where the objects are topological spaces, and $\text{Hom}_{\text{Top}}(X, Y)$ denotes the set of all continuous maps $X \to Y$.

Remark 26. *When $\mathcal{C}$ is a category, we'll often use $X \in \mathcal{C}$ as shorthand for $X \in ob(\mathcal{C})$. (So $X \in \text{Top}$ often means X is a topological space.) Similarly, we'll often write $f : X \to Y$ when we mean $f \in \text{Hom}_{\mathcal{C}}(X, Y)$, and we'll use the word "map" instead of "morphism" frequently.*

Definition 27

A morphism $f : X \to Y$ in a category $\mathcal{C}$ is called an **isomorphism** if there exists a map $g : Y \to X$ such that $g \circ f = 1_X$ and $f \circ g = 1_Y$.

Example 28

A morphism in Set is an isomorphism if and only if it is a bijection, and a morphism in Ab is an isomorphism if and only if it is a group isomorphism. Similarly, an isomorphism in Top is a homeomorphism (a continuous map with a continuous inverse).

Proposition 29

Suppose $f : X \to Y$ is an isomorphism in a category $\mathcal{C}$. Then the inverse $g : Y \to X$ is unique, so we can talk about "the inverse" rather than "an inverse."

Proof. Suppose that $g' : Y \to X$ were some other inverse of f. Since $g \circ f = 1_X$, we know that

$$(g \circ f) \circ g' = 1_X \circ g' = g',$$

but also (by associativity) this is equal to

$$g \circ (f \circ g') = g \circ 1_Y = g,$$

so $g = g'$. □

While category theory does unify a lot of mathematical concepts we've already talked about, there's another reason we care about it:

Definition 30

A **functor** $F : \mathcal{C} \to \mathcal{D}$ of categories consists of

- An **assignment** $F : \text{ob}(\mathcal{C}) \to \text{ob}(\mathcal{D})$ from objects to objects, and
- for all $X, Y \in \text{ob}(\mathcal{C})$, there is a function $F : \text{Hom}_{\mathcal{C}}(X, Y) \to \text{Hom}_{\mathcal{D}}(F(X), F(Y))$.

Furthermore, we must have $F(1_X) = 1_{F(X)}$ for all $X \in \text{ob}(\mathcal{C})$, and for all composable pairs of morphisms $f, g \in \mathcal{C}$,

$$F(g \circ f) = F(g) \circ F(f).$$

Example 31

For each $n \geq 0$, there is a functor $\text{Sing}_n : \text{Top} \to \text{Set}$, and there is a functor $S_n : \text{Top} \to \text{Ab}$.

Let's unpack this data: every topological space is associated to a set, and if we have a function between topological spaces, we get a canonical function between sets by applying Sing_n. Similarly, we can get a homomorphism between abelian groups from a function between topological spaces by applying S_n.

And now we can do some "Inception-level stuff:"

Example 32

There is a (huge) category Cat of categories, where objects in ob(Cat) are categories and morphisms in Cat are functors.

The idea is that if $\mathcal{C}$ and $\mathcal{D}$ are two categories, then $\text{Hom}_{\text{Cat}}(\mathcal{C}, \mathcal{D})$ might be a class and not a set. But (again) we won't worry about this distinction too much.

The statement that "Cat is a category" means that we can compose two functors to get another functor:

Example 33

There is a functor Free: Set $\to$ Ab, taking a set to the free abelian group generated by that set. Then S_n : Top $\to$ Ab is the composite functor Free $\circ\,\text{Sing}_n$.

Note now that if $X \to Y$ is a map in Top (a continuous map between topological spaces), then we can form the following diagram for each $0 \le i \le n$, corresponding to our "degeneracy" maps d_i from last time:

$$\begin{array}{ccc} \text{Sing}_n(X) & \xrightarrow{d_i} & \text{Sing}_{n-1}(X) \\ {\scriptstyle \text{Sing}_n(f)}\downarrow & & \downarrow{\scriptstyle \text{Sing}_{n-1}(f)} \\ \text{Sing}_n(Y) & \xrightarrow{d_i} & \text{Sing}_{n-1}(Y) \end{array}$$

This diagram **commutes**, meaning that the two different composite functions we can form from $\text{Sing}_n(X) \to \text{Sing}_{n-1}(Y)$ are the same. We can formalize this idea:

Definition 34

Let $F, G : \mathcal{C} \to \mathcal{D}$ be two functors. A **natural transformation** $\Theta : F \to G$ consists of maps $\Theta_X : F(X) \to G(X)$ for all $X \in \text{ob}(\mathcal{C})$, such that for all maps $f : X \to Y$ in $\mathcal{C}$, the diagram below commutes:

$$\begin{array}{ccc} F(X) & \xrightarrow{\Theta_X} & G(X) \\ {\scriptstyle F(f)}\downarrow & & \downarrow{\scriptstyle G(f)} \\ F(Y) & \xrightarrow{\Theta_Y} & G(Y) \end{array}$$

Example 35

Suppose $n \ge 1$ and $0 \le i \le n$. Then there is a natural transformation $d_i : \text{Sing}_n \to \text{Sing}_{n-1}$, where Sing_n and Sing_{n-1} are functors from Top to Set.

Definition 36

A natural transformation $\Theta : F \to G$ is a **natural isomorphism** if the map Θ_X is an isomorphism for all $X \in \text{ob}(\mathcal{C})$.

We'll finish with a construction: suppose that $\mathcal{C}$ and $\mathcal{D}$ are categories, and $\mathcal{C}$ has a set of objects (instead of a class). Then there is another category $\text{Fun}(\mathcal{C}, \mathcal{D})$, whose objects are functors from $\mathcal{C}$ to $\mathcal{D}$, and morphisms are natural transformations of those functors. When we say that this is a valid construction, we're saying that a composition of two natural transformations is a natural transformation. We'll continue with this and do some more examples next time!

4 September 11, 2020

We'll continue to discuss the formal language of category theory today. We looked at **functors** $\text{Sing}_n : \text{Top} \to \text{Set}$, which record the set of continuous functions from a standard n-simplex into X, as well as **natural transformations** $d_i : \text{Sing}_n \to \text{Sing}_{n-1}$. But we haven't put them together in the semisimplicial set construction, and we haven't really motivated the semisimplicial identities either. So we'll try to put all of that into a categorical context today.

To do this, we're going to need to construct new categories out of old categories.

Definition 37

Let $\mathcal{C}$ be a category. Then the **opposite category** $\mathcal{C}^{\text{op}}$ is a category so that $\text{ob}(\mathcal{C}^{\text{op}}) = \text{ob}(\mathcal{C})$, but for any $X, Y \in \text{ob}(\mathcal{C}^{\text{op}})$, we define

$$\text{Hom}_{\mathcal{C}^{\text{op}}}(X, Y) = \text{Hom}_{\mathcal{C}}(Y, X).$$

If $f \in \text{Hom}_{\mathcal{C}(Y,X)}$ is a morphism, we denote the corresponding morphism in $\text{Hom}_{\mathcal{C}^{\text{op}}}$ by f^{op}. The composition law we use is

$$(f \circ g)^{\text{op}} = g^{\text{op}} \circ f^{\text{op}}.$$

In other words, all of our morphisms get "flipped around."

Example 38

Recall the category of real vector spaces $\text{Vect}_{\mathbb{R}}$ from last time: the objects of this category are vector spaces, and the morphisms are linear transformations. Since every V has a dual $V^* = \underline{\text{Hom}}_{\text{Vect}_{\mathbb{R}}}(V, \mathbb{R})$ (where the underline refers to the vector space of such transformations, rather than just the set), note that if $W \to V$ is a linear map, and $V \to \mathbb{R}$ is a linear map, their composite is also linear. Thus, a linear map $W \to V$ induces a map $V^* \to W^*$ in the opposite direction after dualization.

This structure can be described concisely by saying that **there exists a functor**

$$(\quad)^* : \text{Vect}_{\mathbb{R}}^{\text{op}} \to \text{Vect}_{\mathbb{R}}.$$

Remark 39. *We can also understand opposite categories in terms of functors. If $\mathcal{C}$ and $\mathcal{D}$ are categories, then a functor $F : \mathcal{C} \to \mathcal{D}$ takes objects to objects, and maps $c \to c'$ to maps $F(c) \to F(c')$. But sometimes we want to go in reverse, which is why we have functors $F : \mathcal{C}^{op} \to \mathcal{D}$, which still take objects to objects but send maps $c \to c'$ to maps $F(c') \to F(c)$.*

Last time, we also described the following construction. Let $\mathcal{C}$ and $\mathcal{D}$ are categories, such that $\mathcal{C}$ has a set of objects. Then $\text{Fun}(\mathcal{C}, \mathcal{D})$ is a category, such that the objects are the set of functors from $\mathcal{C}$ to $\mathcal{D}$, and the morphisms are **natural transformations** of functors. What this is really hiding is a claim that we can compose natural transformations properly, which we should check on our own.

So now we're ready to return to our semisimplicial sets:

Definition 40

Let Δ_{inj} denote the category with objects

$$\text{ob}(\Delta_{\text{inj}}) = \{[0], [1], [2], \cdots\},$$

and morphisms between objects given by

$$\text{Hom}_{\Delta_{\text{inj}}}([a], [b]) = \{\text{injective functions } f : \{0, 1, \cdots, a\} \to \{0, 1, \cdots, b\} \text{ that preserve order}\}.$$

(In other words, $f(x) < f(y)$ whenever $x < y$.) This means that there are no morphisms from $[a]$ to $[b]$ when $b < a$, and we're also implicitly assuming that the composition of injective functions is injective, and that the composition of order-preserving functions is order-preserving.

Example 41

There are three maps in Δ_{inj} from [1] to [2]: such (injective order-preserving) functions $\{0, 1\} \to \{0, 1, 2\}$ are just determined by the image, which is either $\{0, 1\}$, $\{0, 2\}$, or $\{1, 2\}$.

Proposition 42

A semisimplicial set is a functor $\Delta_{\text{inj}}^{\text{op}} \to \text{Set}$.

There's a lot of notation to unpack here, and we're encouraged to think through this ourselves and see how the precise correspondence works. But remember that a semisimplicial set was defined as a sequence of sets $X_0, X_1, X_2, \cdots$ together with maps $d_0, d_1 : X_1 \to X_0$, $d_0, d_1, d_2 : X_2 \to X_1$, and so on. From this, we should be able to **extract a functor** $F : \Delta_{\text{inj}}^{\text{op}} \to \text{Set}$.

Let's try to understand that: the set of objects of $\Delta_{\text{inj}}^{\text{op}}$ is the same as the set of objects of Δ_{inj}, which is $\{[0], [1], [2], \cdots\}$. So we'll have our functor send $F([i])$ to X_i. But we need to see where the d_i maps fit in, too:

Example 43

By definition of the opposite construction,

$$\text{Hom}_{\Delta_{\text{inj}}^{\text{op}}}([2], [1]) = \text{Hom}_{\Delta_{\text{inj}}}([1], [2]),$$

which consists of the three maps in Example 41.

We'll label these three maps: let f_0 be the map with image $\{1, 2\}$, f_1 be the map with image $\{0, 2\}$, and f_2 be the map with image $\{0, 1\}$. So now we'll define the d_i map to be $F(f_i^{\text{op}}) : X_2 \to X_1$, and it's good for us to think about what this really means.

Note that this abstraction tells us that $\text{Fun}(\Delta_{\text{inj}}^{\text{op}}, \text{Set})$ is a category (with natural transformations as morphisms). This allows us to assert the following theorem that we've been working towards, which explains the "arrows" from the first class more clearly:

Theorem 44

There is a functor

$$\text{Sing} : \text{Top} \to \text{Fun}(\Delta_{\text{inj}}^{\text{op}}, \text{Set}).$$

Theorem 45

There are functors

$$S_n, Z_n, B_n, H_n : \mathrm{Fun}(\Delta_{\mathrm{inj}}^{\mathrm{op}}, \mathrm{Set}) \to \mathrm{Ab}.$$

Both of these results tell us that there are correspondences both in the objects and in the maps of the two categories, and we can put those results together as well (because of the "huge" category that tells us functors compose):

Corollary 46

We have (composite) functors

$$S_n, Z_n, B_n, H_n : \mathrm{Top} \to \mathrm{Ab}.$$

Definition 47

Let Fil denote the category with one object for each non-negative integer, no morphisms from a to b if $a < b$, and a unique morphism otherwise.

It's worth thinking why this is well-defined: this has the same objects as Δ_{inj}, but very different morphisms.

Proposition 48

A functor Fil $\to$ Ab can be thought of as a sequence of abelian groups with maps between them:

$$A_0 \xleftarrow{\partial_1} A_1 \xleftarrow{\partial_2} A_2 \xleftarrow{\partial_3} \cdots.$$

Definition 49

A **chain complex** of abelian groups is a functor Fil $\to$ Ab with the property that

$$\partial_{i-1} \circ \partial_i = 0 \quad \forall\, i \geq 2.$$

There is indeed a category of chain complexes of abelian groups, denoted chAb, and the morphisms are then natural transformations of functors Fil $\to$ Ab. If we unwind the definition of a natural transformation, we find that a map of chain complexes is a diagram as below:

$$\begin{array}{ccccccc}
A_0 & \xleftarrow{\partial_1} & A_1 & \xleftarrow{\partial_2} & A_2 & \longleftarrow & \cdots \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
B_0 & \xleftarrow[\partial_1]{} & B_1 & \xleftarrow[\partial_2]{} & B_2 & \longleftarrow & \cdots
\end{array}$$

In other words, we map A_0 to B_0, A_1 to B_1, and so on, so that all squares commute.

We'll continue to restate results from earlier on in class:

Theorem 50

There is a functor from semisimplicial sets to chain complexes of abelian groups:

$$S_* : \mathrm{Fun}(\Delta_{\mathrm{inj}}^{\mathrm{op}}, \mathrm{Set}) \to \mathrm{chAb},$$

mapping X to a sequence $S_0(X) \xleftarrow{\partial_1} S_1(X) \xleftarrow{\partial_2} S_2(X) \xleftarrow{\partial_3} \cdots$.

Theorem 51

There are functors Z_n, B_n, H_n : chAb $\to$ Ab from chain complexes to abelian groups.

We can put all of this together by saying that the homology group functor H_n : Top $\to$ Ab is a composite of the following functors:

- The Sing construction, which takes us from a topological space to a semisimplicial set,
- The S_* construction, which takes us to a chain complex of abelian groups,
- The $H_n = \ker(\partial_n)/\text{im}(\partial_{n+1})$ construction, which takes us to abelian groups.

5 September 14, 2020

Some more details are now known about the exam – it'll be held on Monday, November 8th, and it should be much easier than the problem sets (taking about an hour or less to complete). We'll be given a 24 hour period to do it, and we can consult books and the internet as we wish (but not other people). And as a reminder, our first problem set is due at class time on Friday.

Last lecture, we defined a category called chAb, but we'll quickly **change our conventions** to follow the official course notes: from now on, we'll call this category of **nonnegatively graded chain complexes** $\text{chAb}_{\geq 0}$. Recall that an object of this category is a sequence of abelian groups

$$\cdots \xrightarrow{\partial_3} A_2 \xrightarrow{\partial_2} A_1 \xrightarrow{\partial_1} A_0,$$

and a morphism is a diagram between two of these objects with all squares commuting:

$$\begin{array}{ccccccc} \cdots & \xrightarrow{\partial_3} & A_2 & \xrightarrow{\partial_2} & A_1 & \xrightarrow{\partial_1} & A_0 \\ & & \downarrow & & \downarrow & & \downarrow \\ \cdots & \xrightarrow{\partial_3} & B_2 & \xrightarrow{\partial_2} & B_1 & \xrightarrow{\partial_1} & B_0 \end{array}$$

Definition 52

A (not necessarily nonnegative) **chain complex** is a sequence of abelian groups with maps

$$\cdots \xrightarrow{\partial_3} A_2 \xrightarrow{\partial_2} A_1 \xrightarrow{\partial_1} A_0 \xrightarrow{\partial_0} A_{-1} \xrightarrow{\partial_{-1}} A_{-2} \xrightarrow{\partial_{-2}} \cdots.$$

These are objects of the category chAb, with morphisms forming similar diagrams as above.

We mentioned last time that we can think about $\text{chAb}_{\geq 0}$ as a **functor category** from Fil to Ab, and it's worth thinking about how that extends here. And to relate the two categories, we do have a functor $\text{chAb}_{\geq 0} \to \text{chAb}$, where any nonnegatively graded chain complex is just sent to $\cdots \to A_1 \to A_0 \to 0 \to 0 \to \cdots$.

Fact 53

We also have functors Z_n, B_n, H_n : chAb $\to$ Ab for **any** integer n, defined analogously as before, so we can still define the nth homology group functor for chAb instead of $\text{chAb}_{\geq 0}$, with $H_n(X) = 0$ for $n < 0$.

That's enough category theory for now, and we'll now spend some time looking at how to rigorously compute the homology groups for topological spaces.

Example 54

Let $X = p$ be the one-point topological space (that is, $\mathbb{R}^0$). Can we compute the homology groups here?

Since $\text{Sing}_n(X)$ is the set of continuous functions from $\Delta^n \to p$, where Δ^n is the standard n-simplex, there is only **one unique continuous map** for each n, which we can denote a_n. This then gives us the nonnegatively graded chain complex

$$S_*(X) = \mathbb{Z}\{a_0\} \xleftarrow{\partial_1} \mathbb{Z}\{a_1\} \xleftarrow{\partial_2} \mathbb{Z}\{a_2\} \leftarrow \cdots,$$

so each group here is abstractly **isomorphic to the integers**.

So now we want to explicitly write down the nth boundary map ∂_n. It suffices to show where the generator a_n goes, and that's defined via

$$\partial_n(a_n) = \sum_{k=0}^{n} (-1)^k (d_k a_n),$$

where d_k is a function of sets from $\text{Sing}_k(X) \to \text{Sing}_{k-1}(X)$. But there's only one $(n-1)$-simplex image that exists in X, so this expression will just be

$$= \sum_{k=0}^{n} (-1)^k a_{n-1} = \begin{cases} a_{n-1} & k \text{ even,} \\ 0 & k \text{ odd.} \end{cases}$$

So $S_*(X)$ is isomorphic in $\text{chAb}_{\geq 0}$ to the sequence of maps

$$\mathbb{Z} \xleftarrow{0} \mathbb{Z} \xleftarrow{1} \mathbb{Z} \xleftarrow{0} \mathbb{Z} \xleftarrow{1} \cdots,$$

and now we're ready to compute the homology groups. Remembering that ∂_0 is defined to be the zero map, we find that

$$H_0(X) \cong \ker(\partial_0)/\text{im}(\partial_1) \cong \mathbb{Z}/0 \cong \mathbb{Z}.$$

(We could have probably predicted this if we've already worked on the problem set a bit.) Moving on,

$$H_1(X) \cong \ker(\partial_1)/\text{im}(\partial_2) \cong \mathbb{Z}/\mathbb{Z} \cong 0.$$

And we can continue in this way:

$$H_2(X) \cong 0/0 \cong 0,$$

and all subsequent homology groups $H_3(X), H_4(X), \cdots$, will also be 0. Putting this all together,

$$\boxed{H_n(p) = \begin{cases} \mathbb{Z} & n = 0 \\ 0 & \text{otherwise.} \end{cases}}$$

We can also compute the homology of the empty set, but this is basically all of the possible direct computations we can do from the definition! So we'll move into a slightly less boring class of spaces:

Definition 55

A subset $X \subseteq \mathbb{R}^n$ is **star-shaped** with respect to a point $b \in X$ if for every point $x \in X$, the interval

$$\{tb + (1-t)x : t \in [0,1]\}$$

lies entirely in X.

In other words, we can draw a straight line from b to any point in subset X. Such shapes should be "completely filled in:" we expect that they are basically disks but with some deformations.

Theorem 56

Let X be star-shaped. Then $H_n(X) \cong H_n(p)$ for all n.

In order to approach this, we'll need to do some algebra in chAb (and we'll see as time progresses that this class is basically one-third algebra, one-third geometry, and one-third category theory).

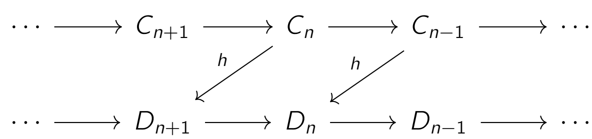
Definition 57

Let C_* and D_* be chain complexes, and let $f_0, f_1 : C_* \to D_*$ be two maps (called chain maps). A **chain homotopy** $h : f_0 \simeq f_1$ is a collection of homomorphisms $h : C_n \to D_{n+1}$, such that

$$\partial h + h\partial = f_1 - f_0.$$

If there exists a chain homotopy $h : f_0 \sim f_1$, then f_0, f_1 are **chain homotopic**.

The slightly relevant diagram is below:



This is **not a commutative diagram**, but it's a way for us to visualize what's going on. This kind of structure tends to arise out of geometric constructions, and it's useful because of the following result:

Lemma 58

Suppose $f_0, f_1 : C_* \to D_*$ are chain homotopic. Then the two maps $H_n(f_0), H_n(f_1) : H_n(C_*) \to H_n(D_*)$ are the same.

Recall that H_n is a functor, so a map like f_0 or f_1 gives us a map of abelian groups. And this result is saying that the induced maps of abelian groups are identical.

Proof. Let $c \in Z_n(C_*)$ be an n-cycle. We must show that

$$f_1 c - f_0 c \in Z_n(D_*) \in B_n(D_*).$$

Picking some chain homotopy h, we have that

$$f_1 c - f_0 c = (\partial h + h\partial)c = \partial hc + h\partial c,$$

but $\partial c = 0$ because c is a cycle, so this is just ∂hc. So we've written $f_1 c - f_0 c$ as the boundary of hc. □

Remark 59. *We're using a convention where we* **are omitting indices** *(some of the partials here should be ∂_n versus ∂_{n+1}). This makes it more practical to write things down, and we should feel free to do the same on our homework.*

Proof of Theorem 56. Suppose that X is starshaped with respect to $b \in X$. A map $X \to p$ induces a chain map $\varepsilon : S_*(X) \to S_*(p)$, and the map $b : p \to X$ induces a map $\eta : S_*(p) \to S_*(X)$ (in both cases, because S_* is a functor). We wish to show that for all n, $H_n(\varepsilon)$ and $H_n(\eta)$ are **inverse isomorphisms of abelian groups**. To do that, we need to show that both compositions are identity maps. First, note that

$$H_n(\varepsilon) \circ H_n(\eta) = H_n(\varepsilon \circ \eta)$$

by definition of functoriality, and because $\varepsilon \circ \eta$ is the identity map, this gives us

$$= H_n(1_{S_*(p)}) = 1_{H_n(S_*(p))}.$$

We need to show that the other order of composition also gives us something nice: specifically, that

$$\eta \circ \varepsilon : S_*(X) \to S_*(X)$$

is chain homotopic to the identity map, which is good enough by Lemma 58. We'll use the chain homotopy

$$h : S_q(X) \to S_{q+1}(X)$$

that sends simplices to simplices (so we send generators to generators), meaning that for any $\sigma \in \mathrm{Sing}_q(X)$, we define the map $h(\sigma) : \Delta^{q+1} \to X$ via

$$h(\sigma)(t_0, \cdots, t_{q+1}) = \begin{cases} b & t_0 = 1, \\ t_0 b + (1 - t_0)\sigma\left(\frac{t_1}{1-t_0}, \cdots, \frac{t_{q+1}}{1-t_0}\right) & \text{otherwise.} \end{cases}$$

Geometrically, what is happening here is that given a simplex σ in our star-shaped region, we're connecting it with straight lines to everything in b (which adds an extra dimension). **Because we're star-shaped**, this higher-dimensional simplex will stay inside the region. Then $d_0(h(\sigma)) = \sigma$, and for all $i > 0$, we have

$$d_i(h(\sigma)) = h(d_{i-1}(\sigma)).$$

This means that

$$\partial(h(\sigma)) = d_0 h\sigma - d_1 h\sigma + d_2 h\sigma - \cdots,$$

which simplifies to

$$\sigma - h(d_0\sigma - d_1\sigma + \cdots) = \sigma - h\partial\sigma.$$

This lets us conclude that $\partial h + h\partial = 1 - \eta\varepsilon$, so we indeed have a chain homotopy, and 1 and $\eta\varepsilon$ are chain homotopic. $\square$

6 September 16, 2020

The recording from last class had some technical issues, so it needs to be re-recorded before it can be posted on the class website. (But it won't be necessary for the problem set due on Friday.)

Last class, we defined the notion of **chain homotopy**. Basically, given two maps of chain complexes $f_0 : C_* \to D_*$ and $f_1 : C_* \to D_*$, we define a collection of group homomorphisms $h : C_n \to D_{n+1}$ such that

$$\partial h + h\partial = f_1 - f_0.$$

This is useful because chain homotopic maps f_0, f_1 give us the same homology groups $H_n(f_0), H_n(f_1)$, and we used this to show that star-shaped regions in $\mathbb{R}^n$ have identical homology groups to that of a single point. (We did this by taking k-dimensional simplices and connecting them to $(k+1)$-dimensional simplices.)

A valuable question to ask, then, is the following: if we have two maps $f, g : X \to Y$ in Top, when are $S_*(f)$ and $S_*(g)$ chain homotopic? In such a case, we have $H_n(f) = H_n(g)$, and the idea is that working with topological spaces can tell us something algebraic.

Definition 60

A **homotopy** h between maps of topological spaces $f, g : X \to Y$ is a continuous map

$$h : X \times [0, 1] \to Y$$

such that $h(x, 0) = f(x)$ and $h(x, 1) = g(x)$. (And f and g are **homotopic** if such an h exists.)

The second variable here in x is often thought of as "time:" we can imagine that we have a continuous movement (or family of interpolating maps) from the map f to the map g.

Theorem 61

If f and g are homotopic maps of topological spaces, then $S_*(f)$ and $S_*(g)$ are chain homotopic (meaning that $H_n(f) = H_n(g)$ as maps of abelian groups).

Proof. This is on our homework assignment, since it'll help us get more familiar with the concepts but isn't a particularly memorable proof. □

Fact 62

Suppose that $h_{12} : X \times [0, 1] \to Y$ is a homotopy from f_1 to f_2, and $h_{23} : X \times [0, 1] \to Y$ is a homotopy from f_2 to f_3. Then we can compose them together, in the sense that

$$h_{13}(x, y) = \begin{cases} h_{12}(x, 2t) & 0 \le t \le 1/2, \\ h_{23}(x, 2t - 1) & 1/2 \le t \le 1 \end{cases}$$

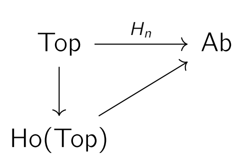
is a homotopy from f_1 to f_3.

This means that there is a transitivity property for being homotopic, and this is the argument we need to say that $f \simeq g$ (meaning f and g are homotopic) is an **equivalence relation** on the set of continuous maps $\mathrm{Hom}_{\mathrm{Top}}(X, Y)$. (We also need to verify reflexivity and symmetry, but those are easier to do.)

This also means we have a **homotopy category** Ho(Top) with ob(Ho(Top)) = ob(Top) and $\mathrm{Hom}_{\mathrm{Ho(Top)}} = \mathrm{Hom}_{\mathrm{Top}}(X, Y)/\simeq$, meaning that maps in the homotopy category are homotopy classes of maps in Top. (And in this assertion, we're saying things like "composition behaves well with this equivalence relation." This then gives us a canonical functor

$$\mathrm{Top} \to \mathrm{Ho(Top)}$$

which sends maps to their equivalence classes – we can think of this functor as some kind of "quotient functor." Theorem 61 implies that we have a diagram



which commutes – specifically, there exists a unique functor Ho(Top) $\to$ Ab. (This is a "first isomorphism" type construction for categories: we're saying that the functor H_n defined on Top has the property that it sends homotopic maps to identical maps in Ab, so it "uniquely factors through" this homotopy category construction.) But if we look at the functor S_* : Top $\to$ chAb instead and try to construct a similar diagram, there isn't a functor from Ho(Top) to chAb which makes the diagram commute, because homotopic maps of topological spaces are not sent to identical maps of chain complexes – only chain homotopic ones.

Remark 63. *There's a category* Top$_*$ *of "***pointed** *topological spaces" (topological spaces with a base point), and we also get a similar looking diagram out of the* π_1 *fundamental group functor:*

$$\begin{array}{ccc} \mathrm{Top}_* & \xrightarrow{\pi_1} & Ab \\ \downarrow & \nearrow & \\ Ho(\mathrm{Top}_*) & & \end{array}$$

Here, $\pi_1(X)$ *is the set of functions from* $S^1 \to X$ *which send base points to* X*, modded out by the homotopy equivalence* $\simeq$*.*

Definition 64

A continuous map $f : X \to Y$ of topological spaces is called a **homotopy equivalence** if it maps to an isomorphism under the functor Top $\to$ Ho(Top). Two spaces X, Y are **homotopy equivalent** if there exists a homotopy equivalence $f : X \to Y$.

If we unpack this a little, this means that f is a homotopy equivalence if and only if there exists an inverse map $g : Y \to X$ such that $f \circ g$ and $g \circ f$ are **homotopic** to the identity map. And the "slogan" to remember here is that **homology cannot tell between homotopy-equivalent spaces, or between homotopic maps**. So it's useful to be able to tell whether two spaces are homotopy equivalent quickly, and we'll show one simple way this often comes up:

Definition 65

An inclusion $A \hookrightarrow X$ is a **deformation retraction** if there is a map $h : X \times [0, 1] \to X$ such that

- $h(x, 0) = x$ for all $x \in X$,
- $h(x, 1) \in A$ for all $x \in X$,
- $h(a, t) = a$ for all $a \in A$ and $t \in [0, 1]$.

In other words, we start with X and "pull everything back" to A over time, such that everything inside A is untouched. We can check that this is indeed a homotopy equivalence between A and X, and often we can show that two spaces X and Y are homotopy equivalent if we can **exhibit a common deformation retraction** to some subspace A that sits inside both X and Y.

Example 66

We can check that $S^{n-1} \subset \mathbb{R}^n - \{0\}$ is a deformation retraction, even though the dimensions of the two spaces are different. And an annulus and a circle with a small "stick" attached to it are homotopy equivalent, because both deform into a circle.

Last class, we proved that star-shaped regions have the same homology as a point – in fact, the inclusion of that central point $b \in X$ to X is a deformation retraction. But this hasn't gotten us to proving things which are not homotopy equivalent to a point, and we'll need to understand some other simple spaces.

Specifically, we want to understand how inclusions $A \subseteq X$ of subspaces relate the homology $H_n(A)$ to $H_n(X)$. Notice that $S_*(A) \to S_*(X)$ is a **subcomplex** of chain complexes, meaning that we have a map of chain complexes (as usual, with commuting squares) except with vertical arrows being inclusions $C_i \subseteq D_i$ rather than just any maps $C_i \to D_i$.

This leads us to the following construction: let $C_* \subset D_*$ be a subcomplex. Then the **quotient complex** D_*/C_* has groups

$$(D_*/C_*)_n = D_n/C_n$$

and differential maps

$$\partial_n : D_n/C_n \to D_{n-1}/C_{n-1}$$

that are well-defined, because two classes in D_n that differ by a class in C_n will have boundaries that only differ by a class in C_{n-1}. (**This is exactly the assertion that the squares in the map of chain complexes commute**.)

Definition 67

Let $A \subseteq X$ be a subspace of a topological space. Then we denote $H_n(X, A)$ to be the nth homology group of the quotient complex $S_*(X)/S_*(A)$.

We'll discuss the relation between $H_n(A)$, $H_n(X)$, and $H_n(X, A)$ next time!

7 September 18, 2020

We'll continue to talk about the **relative homology group** $H_m(X, A)$ today. Recall that if we have a subspace $A \subseteq X$, we define $H_m(X, A)$ is defined to be the homology group of the quotient chain complex $S_*(X)/S_*(A)$. We know that we have a general strategy for computing these homology groups: look for a slightly simpler subspace A of X, and then try to relate the three groups

$$H_m(A), H_m(X), H_m(X, A).$$

To compute that third quantity $H_m(X, A)$, we'll learn today how to relate it to $H_m(X/A)$. And the geometric picture here is that we're "breaking up a space into smaller pieces and putting the computation together."

Definition 68

Let Top_2 denote the category with objects

$$\text{ob}(\text{Top}_2) = \{(X, A) : X \text{ is a topological space and } A \text{ is a subspace of } X\},$$

with morphisms

$$\text{Hom}_{\text{Top}_2}((X, A), (Y, B)) = \{\text{continuous maps } f : X \to Y \text{ with } f(A) \subset B\}.$$

This means we've basically extended homology from Top onto these pairs in Top_2: for each nonnegative integer m, we have a functor $H_m : \text{Top}_2 \to \text{Ab}$, which we'll use to understand our homology gruops. And also note that we have a functor $\text{Top} \to \text{Top}_2$ which sends X to $(X, \varnothing)$, so we can view Top as a **subcategory** of Top_2 if we want.

To understand the quotient chaim complexes a bit better, we'll need to do some algebra.

Definition 69
Let $A \xrightarrow{f} B \xrightarrow{g} C$ be a sequence of abelian groups with maps between them. This sequence is **exact** (at B) if $\ker g = \operatorname{im} f$.

If this sequence of groups is part of a chain complex, then "this sequence has no homology," since the homology groups are $\ker g / \operatorname{im} f$.

Example 70
A sequence $0 \to A \xrightarrow{f} B$ is exact if and only if f is injective (since the kernel of f should be the image of the zero map), and $A \xrightarrow{f} B \to 0$ is exact if and only if f is surjective (since the kernel of B, which is all of B, should be the image of f).

Definition 71
A longer sequence of abelian groups is **exact** if it is exact on every three-term (adjacent) subsequence.

Example 72
The sequence of maps

$$0 \to \mathbb{Z} \xrightarrow{2} \mathbb{Z} \xrightarrow{1} \mathbb{Z}/2\mathbb{Z} \to 0$$

is exact (we can check the condition on all three three-term subsequences). In general, an exact sequence of the form $0 \to A \to B \to C \to 0$ is called a **short exact sequence**.

Example 73
The chain complex for the emptyset $S_*(\varnothing)$ is exact, because all groups are just 0. On the other hand, the chain complex for a point $S_*(p)$ is not exact, because the zeroth homology of a point is $\mathbb{Z}$, though it is exact away from $S_0(p)$.

This kind of language is useful because it's often going to make some technical algebra machinery easier to deal with:

Theorem 74 (Five lemma)
Suppose we have maps of abelian groups as in the diagram below, such that that all squares commute:

$$\begin{array}{ccccccccc}
A_4 & \xrightarrow{d} & A_3 & \xrightarrow{d} & A_2 & \xrightarrow{d} & A_1 & \xrightarrow{d} & A_0 \\
\downarrow{\scriptstyle f_4} & & \downarrow{\scriptstyle f_3} & & \downarrow{\scriptstyle f_2} & & \downarrow{\scriptstyle f_1} & & \downarrow{\scriptstyle f_0} \\
B_4 & \xrightarrow{d} & B_3 & \xrightarrow{d} & B_2 & \xrightarrow{d} & B_1 & \xrightarrow{d} & B_0
\end{array}$$

If both rows are exact, and f_0, f_1, f_3, f_4 are isomorphisms of abelian groups, then so is f_2.

Even if this statement looks technical, it'll turn out to be very useful.

Proof. We need to check that f_2 is both injective and surjective. We'll show the latter here, and we'll get a chance to try injectivity on our own.

Pick some $b_2 \in B_2$ – we wish to write it as the image of some element of A_2. Because f_1 is an isomorphism, we know that $db_2 = f_1(a_1)$ for some a_1, and then $da_1 = 0$ because $f_0 da_1 = df_1 a_1 = ddb_2 = 0$ (last step because of exactness, and using the isomorphism property of f_0). Now by exactness, we know that $a_1 = da_2$ for some $a_2 \in A_2$, and now we know that

$$df_2 a_2 = f_1 da_2 = f_1 a_1 = db_2.$$

And now $b_2 - f_2(a_2)$ goes to 0 under d, so there exists some $b_3 \in B_3$ such that $db_3 = b_2 - f_2(a_2)$ by exactness. And now because f_3 is an isomorphism, we can "lift" this to an a_3 such that

$$f_3 a_3 = b_3, \implies f_2 da_3 = df_3 a_3 = b_2 - f a_2 a_2.$$

This means that $b_2 = f_2(a_2 + da_3)$, which means b_2 is in the image of f_2 as desired. $\square$

Remark 75. *Proofs like this are called* ***diagram chasing****: they're harder to do over Zoom because we're supposed to physically point at diagrams. So we should go through and verify that this kind of argument makes sense to us.*

Definition 76

A **short exact sequence of chain complexes** is a diagram $0_* \to A_* \xrightarrow{f} B_* \xrightarrow{g} C_* \to 0_*$.

We can expand out the definition to look like this:

$$\begin{array}{ccccccccc}
\cdots & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & \cdots \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
\cdots & \longrightarrow & A_{n+1} & \longrightarrow & A_n & \longrightarrow & A_{n-1} & \longrightarrow & \cdots \\
\downarrow & & \downarrow f_{n+1} & & \downarrow f_n & & \downarrow f_{n-1} & & \downarrow \\
\cdots & \longrightarrow & B_{n+1} & \longrightarrow & B_n & \longrightarrow & B_{n-1} & \longrightarrow & \cdots \\
\downarrow & & \downarrow g_{n+1} & & \downarrow g_n & & \downarrow g_{n-1} & & \downarrow \\
\cdots & \longrightarrow & C_{n+1} & \longrightarrow & C_n & \longrightarrow & C_{n-1} & \longrightarrow & \cdots \\
\downarrow & & \downarrow & & \downarrow f_2 & & \downarrow f_1 & & \downarrow f_0 \\
\cdots & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & \cdots
\end{array}$$

In this diagram, rows must be chain complexes (so the composite of two maps must be zero), and the vertical maps must be short exact sequences of abelian groups.

Example 77

Let $A \subseteq X$ be an inclusion of topological sequence. Then $0 \to S_*(A) \to S_*(X) \to S_*(X)/S_*(A) \to 0$ is a short exact sequence of chain complexes.

Theorem 78

Let $0 \to A_* \to B_* \to C_*$ be a short exact sequence of chain complexes. Then we have a long exact sequence of homology groups

$$\cdots \to H_{n+1}(B) \xrightarrow{H_{n+1}(g)} H_{n+1}(C) \to H_n(A) \xrightarrow{H_n(f)} H_n(B) \xrightarrow{H_n(g)} H_n(C) \to H_{n-1}(A) \xrightarrow{H_{n-1}(f)} H_{n-1}(B) \to \cdots$$

We know that the H_n maps between A, B, C for a fixed n exist by functoriality, but it's surprising that we have a degree-lowering map that relates the homology between A and C. It's not very easy to describe what these maps do, but we can try to understand it by playing with some of the algebra.

This is a version of the **Snake Lemma**, and we'll prove this algebraic result in digestible chunks on our homework as well.

Fact 79

It seems people like to mention that the Snake Lemma was proved in a Hollywood movie, but the time between the Snake Lemma being established and the movie coming out is now shorter than the time between the movie's release and the present day. We can see such a proof in the following Youtube video: https://www.youtube.com/watch?v=etbcKWEKnvg.

Corollary 80

Suppose $A \subseteq X$ is an inclusion of spaces. Then there is a long exact sequence

$$\cdots \to H_{n+1}(X) \to H_{n+1}(X, A) \to H_n(A) \to H_n(X) \to H_n(X, A) \to H_{n-1}(A) \to H_{n-1}(X) \to \cdots .$$

These "degree-lowering maps" are supposed to be pretty mysterious – in practice, we shouldn't need to dig into their definition when using this corollary. The point is to summarize the relation between our homology groups using the fact that "there exists these maps that make the sequence exact," and we'll see why this is true as we start computing homology groups in the next week or so. It's particularly powerful because it relates all of the homology groups H_n at once.

And now we can return to the question of computing $H_n(X, A)$, and we'll start with a simple case:

Example 81

Let b be a point of X, meaning that we have a pair $(X, \{b\})$ in Top_2. Let's compute the relative homology group $H_m(X, \{b\})$.

We'll be able to do things from first principles in this simple example. First of all, remember the definition: if we consider the map $S_*(\{b\}) \to S_*(X) \to S_*(X)/S_*(\{b\})$, we can note that $S_m(X)/S_m(\{b\})$ is just the quotient of the free abelian group generated by the m-simplices of X by the group generated by the single m-simplex $\Delta^n \to p \xrightarrow{b} X$. (In other words, we're just killing a single m-simplex.) We can check that for all $m > 0$, this implies that

$$H_m(X) \cong H_m(X, \{b\}),$$

because this is a "boring simplex" that is easy to calculate the boundary for. And as we showed on our homework, $H_0(X) \cong \mathbb{Z}\pi_0(X)$, where $\pi_0(X)$ is the set of path components of X. Then there is a direct sum decomposition

$$H_0(X) \cong \mathbb{Z} \oplus H_0(X, \{b\}),$$

where the $\mathbb{Z}$ corresponds to the path component containing b. This means we have a free abelian group of one lower rank when we go from $H_0(X)$ to $H_0(X, \{b\})$.

Theorem 82 (Excision)
Let (X, A) be a pair of spaces, and suppose that there is a subspace B of X so that $\overline{A} \subseteq \text{Int}(B)$, and $A \to B$ is a deformation retraction. Then for all m, there is an isomorphism

$$H_m(X, A) \cong H_m(X/A, p) \cong \begin{cases} H_m(X/A) & m > 0 \\ \ker(\mathbb{Z}\pi_0 X \to \mathbb{Z}) & m = 0 \end{cases}.$$

The condition on B is not really that restrictive here ($\text{Int}(B)$ refers to the interior of B) – it just makes sure we're not doing something horrible bad topologically. So we've shown some connection between relative homology and absolute homology, via the Snake Lemma. Next time, we'll start to gather things together so we can start doing real computations.

8 September 21, 2020

Our second homework assignment will be posted tonight – it will be due in a little under two weeks.

Last time, we discussed the functor $H_m : \text{Top}_2 \to \text{Ab}$ from pairs of topological spaces to abelian groups, and we started discussing the **excision theorem**: this stated that for some pair (X, A) of spaces and a subspace B of X satisfying some not-very-difficult conditions (that is, the closure of A is contained in the interior of B, and $A \to B$ is a deformation retraction), then

$$H_m(X, A) \cong H_m(X/A, p) \cong \tilde{H}_m(X/A).$$

This is the most subtle of the theorems in homology theory, and we'll spend a few days going over the proof of it. In fact, we're going to state and prove a more general version:

Definition 83
Suppose $U \subseteq A \subseteq X$ are three topological spaces. Then if $\overline{U} \subseteq \text{Int}(A)$, then the triple is called **excisive**, and the inclusion $(X - U, A - U) \subseteq (X, A)$ is called an excision.

Theorem 84
An excision induces an isomorphism of abelian groups

$$H_m(X - U, A - U) \cong H_m(X, A).$$

(Remember that we should think of $H_m(X, A)$ as basically looking at the homology group of a quotient space, and U is contained in both A and X, so $(X - U)/(A - U)$ should look a lot like X/A.)

Definition 85
Two maps $f, g : (X, A) \to (Y, B)$ are **homotopic** if there exists a function $h : X \times [0, 1] \to Y$ such that

- $h(x, 0) = f$ and $h(x, 1) = g$ for all $x \in X$,
- $h(a, t) \in B$ for all $a \in A$ and $t \in [0, 1]$ (that is, the subspace A is sent to the subspace B at all times).

This allows us to define the $\text{Ho}(\text{Top}_2)$ category, which behaves under the homology functors as we want.

Giving a quick summary of where we are right now, we've constructed the following ideas:

1. We have a sequence of functors $H_n : \text{Top}_2 \to \text{Ab}$ for all integers n,
2. We have a sequence of natural transformations $\partial : H_n(X, A) \to H_{n-1}(A) \equiv H_{n-1}(A, \varnothing)$ (via our embedding of Top into Top_2) from the Snake Lemma, such that
   (a) For any pair (X, A) of topological spaces, the sequence
   $$\cdots \to H_{q+1}(X, A) \xrightarrow{\partial} H_q(A) \to H_q(X) \to H_q(X, A) \xrightarrow{\partial} \cdots$$
   is exact (where all other maps besides the ∂ Snake Lemma ones exist because of naturality),
   (b) If $f_0, f_1 : (X, A) \to (Y, B)$ are homotopic, then $H_n(f_0) = H_n(f_1)$.
3. The excisions we described earlier in the lecture induce a homology isomorphism.
4. We've constructed the "dimension axiom:" the homology groups $H_n(p)$ of a point are $\mathbb{Z}$ for $n = 0$ and 0 otherwise.
5. Finally, we've checked some version of this result on our homework: if X_i are a collection of spaces indexed by a set I, then
   $$H_m\left(\bigsqcup_{i \in I} X_i\right) \cong \oplus_{i \in I} H_m(X_i).$$

Fact 86

These are the "basic facts of homology," and this is meant in a pretty strong sense: a theorem by Eilenberg and Steenrod says that these facts **completely characterize** the H_m functors. In other words, we can start distinguishing topological spaces with algebraic invariants if we just know the facts above!

Remark 87. *If we have a set of functors $E_n : \text{Top}_2 \to Ab$ which satisfy all axioms above except the fourth (dimension axiom), we get something called a* **extraordinary homology theory**. *We'll see a few examples of these extraordinary homology theories later on in the class, and there's a lot of active research in this area (see K-theory, bordism theory, topological modular forms, and so on).*

Before we jump into excision, though, we'll take a break and do a few calculations from these axioms that we've listed above.

Example 88

Let's calculate the homology groups $H_m(S^1)$ for a circle.

We know that
$$S^1 \cong [0, 1]/\{0, 1\}$$
(we take an interval and identify the two endpoints together). So we can use the following long exact sequence associated to the pair $([0, 1], \{0, 1\})$:

$$
\begin{array}{ccccc}
H_2(\{0,1\}) & \longrightarrow & H_2([0,1]) & \longrightarrow & H_2([0,1],\{0,1\}) \\
& & \xrightarrow{\partial} & & \\
H_1(\{0,1\}) & \longrightarrow & H_1([0,1]) & \longrightarrow & H_1([0,1],\{0,1\}) \\
& & \xrightarrow{\partial} & & \\
H_0(\{0,1\}) & \longrightarrow & H_0([0,1]) & \longrightarrow & H_0([0,1],\{0,1\})
\end{array}
$$

Now we can fill in the necessary information piece by piece:

- $H_2(\{0,1\}) = H_1(\{0,1\}) = 0$ by the dimension axiom and disjoint union.
- $H_2([0,1]) = H_1([0,1]) = 0$ because we have a deformation retraction to a point. Therefore, the maps $H_2(\{0,1\}) \to H_2([0,1])$ and $H_1(\{0,1\} \to H_1([0,1])$ are the zero maps.
- Therefore, by the definition of an exact sequence, $H_2([0,1],\{0,1\})$ is the zero group as well. (The same argument works for all $m > 2$ too, which is why it's not included here.)
- Now, $H_0(\{0,1\}) \cong \mathbb{Z} \oplus \mathbb{Z}$, and $H_0([0,1]) \cong \mathbb{Z}$. (Remember that we're creating free abelian groups of **path components** here.) Since the points 0 and 1 are both being "included" into the interval, the map $f : \mathbb{Z} \oplus \mathbb{Z} \to \mathbb{Z}$ should send both $(1,0)$ and $(0,1)$ to 1. (Another way to understand this is to consider the composite map $\{0\} \to \{0,1\} \to [0,1]$, which is a deformation retraction. Since $1 \mapsto (1,0) \mapsto 1$, we should indeed define our map as shown.)
- Now, we want to look at $H_1([0,1],\{0,1\})$. By exactness, this relative homology group is $\mathbb{Z}$, corresponding to $\ker f$ of the map defined in the last bullet point (which is the multiples of $(1,-1)$.
- Finally, we can look at $H_0([0,1],\{0,1\})$. The image of f is all of $\mathbb{Z}$, so the kernel of the map $H_0([0,1]) \to H_0([0,1],\{0,1\})$ is $\mathbb{Z}$, which is everything. But the next boundary map is also zero, so this group is 0.

Other than calculating the map f, we didn't have to do very many "geometric things," and we didn't need to know much about the boundary maps, either! So we know that

$$H_2([0,1],\{0,1\}) \cong 0, \quad H_1([0,1],\{0,1\}) \cong \mathbb{Z}, \quad H_0([0,1],\{0,1\}) \cong 0,$$

and now we can use the **excision theorem** to say that

$$H_m([0,1],\{0,1\}) \cong H_m(S^1, *) \cong \tilde{H}_m(S^1),$$

where $\tilde{H}_m(S^1)$ is the same as $H_m(S^1)$ except for $m = 0$. So putting everything together,

$$H_2(S_1) \cong 0, \quad H_1(S^1) \cong \mathbb{Z}, \quad H_0(S^1) \cong \mathbb{Z} \oplus \tilde{H}_0(S^1) \cong \mathbb{Z}.$$

So the homology of a circle tells us that we have **one path component and one one-dimensional hole**:

$$H_m(S^1) = \begin{cases} \mathbb{Z} & m = 0, 1 \\ 0 & \text{otherwise.} \end{cases}$$

We can extend this to calculate (on our homework) that

$$H_m(S^q) = \begin{cases} \mathbb{Z} & m = 0, q \\ 0 & \text{otherwise.} \end{cases}$$

And now we'll end this class with a few consequences of our calculations:

Corollary 89

If $q \neq r$, then S^q and S^r are not homotopy equivalent. Furthermore, $\mathbb{R}^q$ and $\mathbb{R}^r$ are not homeomorphic.

Proof. The first part follows from having different homology groups. For the second part, suppose that $\mathbb{R}^q$ and $\mathbb{R}^r$ are homeomorphic. Removing a point and its image tells us that $\mathbb{R}^q - \{b_1\}$ is isomorphic to $\mathbb{R}^r - \{b_2\}$ for two points $b_1 \in \mathbb{R}^q$, $B_2 \in \mathbb{R}^r$. But there is a deformation from S^{q-1} into $\mathbb{R}^q - \{b_1\}$, and also S^{r-1} into $\mathbb{R}^r - \{b_2\}$, which is a contradiction. □

Theorem 90 (Brouwer fixed point theorem)

Let $f : D^n \to D^n$ be a continuous map. Then there is a point $x \in D^n$ such that $f(x) = x$.

(In other words, if we "stir our coffee cup," some point of coffee will be where it started during the process.)

Proof. Suppose otherwise. Then we can define the map $g : D^n \to S^{n-1}$, such that

$$g(x) = \text{the point where the ray from } f(x) \text{ to } x \text{ hits the boundary of the disk.}$$

Notice also that for any $x \in S^{n-1} \subseteq D^n$ on the boundary, $g(x) = x$, so we have the identity map under the composition

$$S^{n-1} \hookrightarrow D^n \to S^{n-1}.$$

For $n \geq 2$, we can apply the functor H_{n-1} to get the identity map $\mathbb{Z} \to 0 \to \mathbb{Z}$ under composition, which is a contradiction. (And $n = 1$ is a case we can address on its own.) □

9 September 23, 2020

Today, we'll continue some arguments from last class and justify a few statements we didn't fully prove. Last time, claimed that if we have three spaces $U \subseteq A \subseteq X$ such that $\overline{U} \subseteq \text{Int}(A)$, then the **excision**

$$(X - U, A - U) \subseteq (X, A)$$

leads us to the final Eilenberg-Steenrod axiom that we haven't proved yet and won't prove on our homework, which is that **excisions induce homology equivalences**. So the next few classes will cover a proof of this, and then we'll be able to return to calculations and discover interesting facts about topological spaces. (In particular, we'll use this to justify why the homology of a pair $H_m(X, A)$ is essentially just $H_m(X/A, p)$ as long as we satisfy some mild point-set conditions.)

The key fact we'll need to use is the **locality principle**, which we'll set up now:

Definition 91

Let X be a topological space. A family $\mathcal{A}$ of subsets of X is a **cover** if X is the union of the interiors of all $A \in \mathcal{A}$.

(This is basically like an open covering, but we want to allow our subsets themselves to be not open as long as the interiors do the job.)

Definition 92

Let $\mathcal{A}$ be a cover of X. An n-simplex $\sigma : \Delta^n \to X$ is **$\mathcal{A}$-small** if the image of σ is entirely contained in a single element $A \in \mathcal{A}$.

In other words, a "small" simplex is one that lies entirely within one subset without needing to cross between different ones. Note that if $\sigma : \Delta^n \to X$ is $\mathcal{A}$-small, then so is $d_i\sigma$, which is entirely contained within σ itself (since we're just "forgetting some vertices"). But that means **we can form a semisimplicial set,** denoted $\text{Sing}^{\mathcal{A}}(X)$, where we choose our n-simplices to be the $\mathcal{A}$-small n-simplices in $\mathcal{X}$. And now via the usual functor, we know that we can create a chain complex $S_*^{\mathcal{A}}(X)$ (with elements of abelian groups just combinations of small simplices), which we can say something useful about:

Theorem 93 (Locality principle)

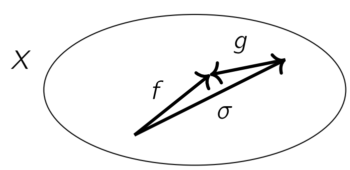
The inclusion $S_*^{\mathcal{A}}(X) \subseteq S_*(X)$ induces an isomorphism on homology groups.

So the key to excision is that we can make our simplices into "very small" ($\mathcal{A}$-small) simplices without changing the homology groups. Intuitively, we need to change a **cycle** in $Z_n(X)$ into a "sum of smaller simplices," and we can do this by adding more boundaries to our simplex. Here's some intuition first:

Example 94

Suppose X is a topological space, and $\sigma : \Delta^1 \to X$ is some (directed) path.

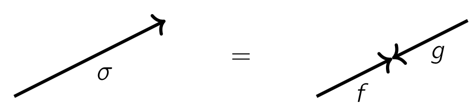
Suppose that we have a small triangle in X with σ as one of the edges, as shown below:



Looking at this diagram, we see that

$$f - g + \sigma = 0$$

in the homology group $H_1(X)$, because it's a "boundary" of the triangle. But now if we shrink this triangle , bringing the top vertex down so that σ points basically along the same axis as f and g, as shown below, we can say that σ is the same as $f - g$, mod our boundary map ∂. And we can always do this "breaking up," regardless of what our space X is, because g and f's images are contained inside σ's. The hope is that making lengths smaller will help us make "smaller simplices" in some sense.



To make this more precise, our goal is to construct a **natural transformation**, denoted \$, between the functor $S_n : \text{Top} \to \text{Ab}$ and itself. (This means that for any topological space X, we have to construct an abelian group map \$ from $S_n(X)$ to itself, which should chop up our n-simplices into combinations of smaller n-simplices.)

To specify such a map, we need to say what it does to each generator $\sigma : \Delta^n \to X$. The trick here (which will come up on our problem set, too) is to consider the following naturality square:

$$\begin{array}{ccc} S_n(\Delta^n) & \xrightarrow{\$} & S_n(\Delta^n) \\ {\scriptstyle S_n(\sigma)}\downarrow & & \downarrow{\scriptstyle S_n(\sigma)} \\ S_n(X) & \xrightarrow{\$} & S_n(X) \end{array}$$

This square should commute, and we have the special n-simplex

$$1_{\Delta^n} : \Delta^n \to \Delta^n$$

which is one of the generators of the free abelian group $S_n(\Delta^n)$. So let's start with that element in the top left corner and follow it around the square. Applying the vertical $S_n(\sigma)$ map to 1_{Δ^n} means we're composing the identity map $\Delta^n \to \Delta^n$ with the map $\sigma : \Delta^n \to X$, so

$$S_n(\sigma)(1_{\Delta^n}) = \sigma.$$

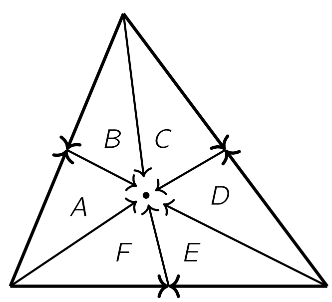
so now naturality (across the bottom and top paths of the square) forces

$$\$(\sigma) = S_n(\sigma)\$(1_{\Delta^n}).$$

So the **naturality square lets us specify a functor \$ described on every topological space, as long as we say what it does to** 1_{Δ^n}. Let's now \$ cuts our simplices into smaller pieces:

Fact 95
We define \$ by letting $\$(1_{\Delta^0}) = 1_{\Delta^0}$, which tells us that \$ is the identity on points. Also, we set $\$(1_{\Delta^1}) = f - g$, where f and g are oppositely oriented line segments inside Δ^1 as in the diagram above.

Going further, defining $\$(1_{\Delta^2})$ will come from the following cutting up of a triangle:



In the diagram here, we define $\$(1_{\Delta^2}) = A - B + C - D + E - F$. The important point is that this linaer combination is the same (at the level of homology) as the original 2-simplex, because they look the same modulo boundaries.

In general, to define $\$(1_{\Delta^n})$, we can let b denote the center of mass of Δ^n. We'll subdivide the boundary of Δ^n according to \$ of one lower dimension, and then we can connect each of those to the central point b:

$$\$1_{\Delta^n} = b * \$(\partial 1_{\Delta^n}),$$

where $b * (\text{arg}) : S_{n-1}(\Delta^n) \to S_n(\Delta^n)$ is the "cone construction" we previously saw in the star-shaped region proof. (In the triangle above, this means we take each of the six directed edges we've drawn along the boundary, and we connect them to the center point to form A, B, C, D, E, F.) And we're saying this doesn't change the homology of our simplex:

Theorem 96

For any topological space X, $\$: S_*(X) \to S_*(X)$ is a chain map, and it is chain homotopic to the identity (so in homology, it's not doing anything) via a natural chain homotopy.

Proof. We can check that

$$\partial \$ 1_{\Delta^n} = \$ \partial 1_{\Delta^n},$$

which is saying that the boundary of the subidivision – an expression like $A - B + C - D + E - F$ in the triangle above – is equivalent to the \$ expression obtained by taking the boundary of the original triangle.

So then naturality tells us for any n-simplex $\sigma : \Delta^n \to X$ that

$$\partial \$ \sigma = \partial \$ (S_n(\sigma)(1_{\Delta^n})) = S_n(\sigma)(\partial \$ 1_{\Delta^n}) = S_n(\sigma)(\$ \partial 1_{\Delta^n}) = \$ \partial \sigma,$$

(it commutes with the differential, which makes it a chain map). Now to show that we have actually have \$ chain homotopic to the identity, we need to define a chain homotopy $T : S_n(X) \to S_{n+1}(X)$ from \$ to $1_{S_*(X)}$. But the same naturality trick means that it suffices to define what T does to 1_{Δ^n}: we'll define $T(1_{\Delta^0})$ to be 0, and for any $n > 0$, we define

$$T(1_{\Delta^n}) = b * (\$ 1_{\Delta^n} - 1_{\Delta^n} - T \partial 1_{\Delta^n}).$$

This is an inductive construction, and it may appear to be a bit obscure, but we can check that it satisfies the condition for being chain homotopic to the identity with some symbol-pushing. For some geometric intuition, $T(1_{\Delta^1})$ is exactly the "squashed" 2-simplex living in $S_2(\Delta^1)$ that we described in the picture above, where f, g, σ all live on top of each other. □

So now if $\mathcal{A}$ is a cover of a space X, our goal is to show that we have a homology isomorphism by shrinking a chain via \$ repeatedly.

Lemma 97

Suppose $\mathcal{A}$ is a cover of Δ^n. Then for any simplex $\sigma \in S_n(\Delta^n)$, there exists an integer k such that $\$^k \sigma$ is $\mathcal{A}$-small.

Proof. This is a consequence of the following fact from point-set topology:

Lemma 98 (Lebesgue covering lemma)

Let M be a compact metric space, and let $\mathcal{U}$ be an open cover. Then there is some $\varepsilon > 0$ such that **for all** $x \in M$, $B_\varepsilon(x) \subseteq U$ for some $U \subseteq \mathcal{U}$.

Since Δ^n is a compact metric space, we can apply this lemma with the open cover $\mathcal{U} = \{\text{Int}(A) : A \in \mathcal{A}\}$. Subdividing via \$ will make our diameter smaller and smaller, until we get below this specified value of ε. □

And now we can bootstrap this to get the result for any space:

Lemma 99

Let $\mathcal{A}$ be a cover of a space X, and let $\sigma \in S_n(X)$. Then there exists an integer k such that $\$^k \sigma$ is $\mathcal{A}$-small.

Proof. This comes from thinking about the definitions of singular homology. Assume without loss of generality that σ is a single n-simplex. Then consider the inverse image $\sigma^{-1}(\text{Int}(A))$ for all $A \in \mathcal{A}$ these subsets form an open covering of Δ^n (as opposed to X), so Lemma 97 gives us what we want. □

Next time, we'll use this to prove the excision theorem.

10 September 25, 2020

We'll continue the proof of the excision theorem today, working off of the "locality principle" discussed yesterday, but we'll start by giving some geometric intuition for the "connecting map" in the homology long exact sequence. Suppose that we have a pair of spaces $A \subseteq X$ such that

$$H_m(X, A) \cong H_m(X/A, p),$$

which is just $H_m(X/A)$ for $m > 0$. (Remember that such a pair just needs to satisfy some mild point-set conditions.) Then the Eilenberg-Steenrod axiom tells us about the existence of a long exact sequence

$$\cdots \to H_m(A) \to H_m(X) \to H_m(X, A) \xrightarrow{\partial} H_{m-1}(A) \to H_{m-1}(X) \to \cdots .$$

We're supposed to think of the connecting map ∂ as generally very hard to compute, but there are some situations where that map is possible to understand. Suppose we pick a class $c \in S_m(X)$ which is not in $Z_m(X)$, meaning in particular that $\partial c \neq 0$ (we can't be a boundary because we're not even a cycle). However, if the image of c under the quotient map $S_m(X) \to S_m(X/A)$ is a cycle, then this means $\partial c \in S_{m-1}(X)$ must be in the image of the inclusion $S_{m-1}(A) \to S_{m-1}(X)$. (In fact, ∂c must be within $Z_{m-1}(A)$, because the boundary of a boundary is always zero, even when we don't start with a cycle.) So in this case, ∂c represents some element of $H_{m-1}(A)$, and it's possible to understand the boundary map.

Example 100

Let's return to one of last time's main examples: suppose that $X = [0, 1]$ and $A = \{0, 1\}$, so that X/A identifies the endpoints of the interval and gives us something homeomorphic to a circle.

If we consider $\sigma : \Delta^1 \to X$ to be the element of $S_1(X)$ that is a homeomorphism sending e_0 to 0 and e_1 to 1 (that is, the standard identification with the interval), then in the chain complex $S_*(X)$, $\partial\sigma = \{1\} - \{0\}$. This is not a trivial element in $S_0(X)$, but $\partial\sigma$ is sent to 0 under the quotient $X \to X/A$: in particular, σ is sent to an element in $Z_1(X/A)$. So σ represents a class in $H_1(X, A) \cong H_1(S^1)$, and the boundary is a class in $H_0(A)$ (the free abelian group on $\{0\}$ and $\{1\}$). Therefore, the boundary map in the diagram

$$\cdots \to H_1(S^1) \xrightarrow{\partial} H_0(\{0, 1\}) \to \cdots$$

maps $\mathbb{Z}$ to $\mathbb{Z} \oplus \mathbb{Z}$ by sending σ to the element $(1, -1)$. (And perhaps this helps us understand why the "boundary" map is called that.)

So now we'll return to excision. Last time, we showed that if we have a cover $\mathcal{A}$ of X (a set of subsets whose interiors form an open cover) and an element $\sigma \in S_n(X)$, then there exists an integer k such that the k-fold subdivision $\$^k\sigma$ is $\mathcal{A}$-small.

Before we jump further, recall that we found a natural chain homotopy T from \$ to the identity $1_{S_*}(X)$, which means that there is a natural chain homotopy T^k from $\$^k$ to $1_{S_*(X)}$ (this is saying that chain homotopies compose).

As a corollary of that, if we're given any $\sigma \in Z_n(X)$, then $\$^k\sigma$ must be equivalent to σ modulo boundaries. We won't need to use T^k very much, except that it's natural – we want to suppress the actual formula for T^k because it is a bit messy.

We'll restate the locality principle again here:

Theorem 101 (Locality principle)
Let $\mathcal{A}$ be a cover of a space X. Then the inclusion of chain complexes $S_*^{\mathcal{A}}(X) \subseteq S_*(X)$ induces isomorphisms $H_n^{\mathcal{A}}(X) \to H_n(X)$ for all n.

Proof. To show surjectivity, suppose that we have some cycle $c \in Z_n(X)$. Then our goal is to show there is some $\mathcal{A}$-small element of $Z_n(X)$ that is equivalent to c modulo boundaries, but this can be done using $\$^k c$ (by the argument from last class). Injectivity is a bit more tricky: here's the result we need to show. Suppose that we have $c \in Z_n^{\mathcal{A}}(X)$ is an $\mathcal{A}$-small cycle that is "zero in homology," such that $c = \partial b$ for some $b \in Z_{n+1}(X)$. We need to show that this is also "zero in homology" in the $\mathcal{A}$-small case: that is, we must find some $b' \in S_{n+1}^{\mathcal{A}}$ so that $\partial b' = c$ as well.

To do that, we choose k so that the subdivision $\$^k b$ is $\mathcal{A}$-small. If we try to calculate the boundary of the subdivision, we find that

$$\partial(\$^k b) - c = \partial((\$^k - 1_{S_{n+1}(X)})(b))$$

by definition, and now we can use the chain homotopy definition to rewrite as

$$\partial((\partial T_k + T_k\partial)(b)) = \partial\partial T_k b + \partial(T_k \partial b).$$

Boundaries of boundaries are always zero, so this leaves us with $\partial(T_k c)$. So this means that

$$c = \partial(\$^k b - T_k c),$$

and now we just need one more fact to finish:

Lemma 102
If $c \in S_n(X)$ is $\mathcal{A}$-small, then $T_k c \in S_{n+1}(X)$ is also $\mathcal{A}$-small.

Proof of lemma. Because $\mathcal{A}$-smallness is closed under sums and differences, we can assume without loss of generality that $c = \sigma$ is a single $\mathcal{A}$-small n-simplex. We can represent σ as the composite map $\Delta^n \to A \to X$, where $A \in \mathcal{A}$ is one of the subspaces in our cover; let σ' denote the map $\Delta^n \to A$ and ι denote the inclusion $A \to X$. In particular, this means we can write

$$\sigma = S_n(\iota)(\sigma').$$

But now T_k is natural, so we can push what it does to σ' along the inclusion:

$$T_k\sigma = (S_{n+1}(\iota))(T_k\sigma'),$$

and $T_k\sigma$ is the image of some $T_k\sigma' \in S_{n+1}(\mathcal{A})$. Therefore, $T_k\sigma$ is indeed $\mathcal{A}$-small, since it's the image of something going on in A itself. □

And that's enough for us, because $\$^k b$ and $T_k c$ are both $\mathcal{A}$-small, and we've shown injectivity. □

We now state the excision theorem again:

Theorem 103 (Excision)

Suppose that $U \subseteq A \subseteq X$ is excisive, so that $\overline{U} \subseteq \text{Int}(A)$. Then the map of pairs $(X - U, A - U) \to (X, A)$ induces homology isomorphisms.

Proof. The statement $\overline{U} \subseteq \text{Int}(A)$ is equivalent to the statement

$$\text{Int}(A) \cup \text{Int}(X - U) = X.$$

But this now looks more like an open covering! Letting B denote $X - U$, we have that $\{A, B\}$ is a cover of X, so now $(X - U, A - U)$ is the pair $(B, A \cap B)$. So this means we're trying to show that the inclusion of chain complexes $S_*(B, A \cap B) \to S_*(X, A)$ induces homology isomorphisms.

So here is the key diagram to look at:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & S_*(A) & \longrightarrow & S_*^{\mathcal{A}}(X) & \longrightarrow & S_*^{\mathcal{A}}(X)/S_*(A) & \longrightarrow & 0 \\ & & \downarrow{=} & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & S_*(A) & \longrightarrow & S_*(X) & \longrightarrow & S_*(X, A) & \longrightarrow & 0 \end{array}$$

The second and third arrows are inclusions, and we can note that there are actually short exact sequences of chain complexes (basically by definition, because $S_*(X, A)$ is just $S_*(X)/S_*(A)$). But for every short exact sequence of chain complexes, we know that we have a long exact sequence on homology groups. So for every map of short exact sequences, we get a map of long exact sequences of homology groups:

$$\begin{array}{ccccccccccccc} \cdots & \longrightarrow & H_m(A) & \longrightarrow & H_m^{\mathcal{A}}(X) & \longrightarrow & H_m(S_*^{\mathcal{A}}(X)/S_*(X)) & \xrightarrow{\partial} & H_{m-1}(A) & \longrightarrow & H_{m-1}^{\mathcal{A}}(X) & \longrightarrow & \cdots \\ & & \downarrow\cong & & \downarrow & & \downarrow & & \downarrow\cong & & \downarrow & & \\ \cdots & \longrightarrow & H_m(A) & \longrightarrow & H_m(X) & \longrightarrow & H_m(X, A) & \xrightarrow{\partial} & H_{m-1}(A) & \longrightarrow & H_{m-1}(X) & \longrightarrow & \cdots \end{array}$$

We have the isomorphisms in the diagram above because of the equalities of the $S_*(A)$s originally. In addition, we've just been proving in the last few classes (by locality) that $H_m^{\mathcal{A}}(X) \to H_m(X)$ is also an isomorphism, and so is $H_{m-1}^{\mathcal{A}}(X) \to H_{m-1}(X)$. **And now we use the five lemma**: we must have an isomorphism

$$H_m(S_*^{\mathcal{A}}(X)/S_*(X)) \to H_m(X, A),$$

and therefore $S_*^{\mathcal{A}}(X)/S_*(A) \to S_*(X)/S_*(A)$ must induce homology isomorphisms, where the homology satisfies

$$H_m(S_*(X)/S_*(A)) \cong H_m(X, A).$$

We can now observe that

$$S_n^{\mathcal{A}}(X)/S_n(A) = (S_n(A) + S_n(B))/S_n(A)$$

(because the $\mathcal{A}$-small chains are just contained in one of the two pieces of the cover), and because we have abelian groups, we can rewrite this as

$$\cong S_n(B)/(S_n(A) \cap S_n(B)) = S_n(B)/S_n(A \cap B) = S_n(X - U)/S_n(A - U),$$

which means the homology groups behave as desired. □

While we don't need to use the entire proof, since excision is an axiom for our purposes, some elements of the proof will be useful as a tool to solve other problems.

Corollary 104

Let (X, A) be a pair of spaces, such that there is a subspace B of X with $\overline{A} \subseteq \text{Int}(B)$ and $A \to B$ is a deformation retraction. Then $H_m(X, A) \to H_m(X/A, p)$ is an isomorphism for all m.

Proof. Consider the following diagram in pairs of topological spaces Top_2:

$$\begin{array}{ccccc} (X, A) & \xrightarrow{i} & (X, B) & \xleftarrow[j]{} & (X - A, B - A) \\ \downarrow & & \downarrow & & \downarrow k \\ (X/A, p) & \xrightarrow{\bar{i}} & (X/A, B/A) & \xleftarrow[\bar{j}]{} & (X/A - p, B/A - p) \end{array}$$

We claim that every map labeled with a letter induces a homology isomorphism, and then commuting squares tell us that the leftmost arrow does so too. This is true because:

- k is a homeomorphism in Top_2,
- j is an excision,
- i is a homology isomorphism because of the deformation retraction that induces a homotopy invariance of homology,
- $\bar{j}$ is an excision,
- $\bar{i}$ is also a deformation retraction obtained from the retraction $B \times I \to B$ by quotienting out by A.

$\square$

11 September 28, 2020

We'll start with a quick summary of what we've done so far: we have functors $H_n : \text{Top}_2 \to \text{Ab}$, where $H_n(X) = H_n(X, \varnothing)$, and there are natural transformations $\partial : H_n(X, A) \to H_{n-1}(A)$. These functors and transformations then need to satisfy the Eilenberg-Steenrod axioms. Last time, we talked about the excision theorem, and one application is that is that whenever $n > 0$, we have

$$H_n(X, A) \cong H_n(X/A, p) \cong H_n(X/A).$$

We'll now leverage these axioms to build some computationally practical tools to help us compute homology for more interesting spaces. First of all, we'll introduce another tool for computation besides the long exact sequence.

Theorem 105 (Mayer-Vietoris)

Let X be a space, and suppose $\{A, B\}$ is a cover of X, meaning that $A, B \subset X$ and $\text{Int}(A) \cup \text{Int}(B) = X$. We'll fix names $i : A \cap B \hookrightarrow A$, $j : A \cap B \hookrightarrow B$, $k : A \hookrightarrow X$, and $\ell : B \hookrightarrow X$. There is a long exact sequence

$$\cdots H_{n+1}(X) \xrightarrow{\partial} H_n(A \cap B) \to H_n(A) \oplus H_n(B) \to H_n(X) \xrightarrow{\partial} H_{n-1}(A \cap B),$$

with other maps being given (geometrically) by

$$H_n(A \cap B) \xrightarrow{H_n(i) \oplus H_n(j)} H_n(A) \oplus H_n(B) \xrightarrow{H_n(k) - H_n(\ell)} H_n(X).$$

Proof. We can consider the following short exact sequence of chain complexes, where $\mathcal{A} = \{A, B\}$:

$$0 \to S_*(A \cap B) \xrightarrow{S_*(i) \oplus S_*(j)} S_*(A) \oplus S_*(B) \xrightarrow{S_*(k) - S_*(\ell)} S_*^{\mathcal{A}} \to 0.$$

The kernel of the difference map is indeed the image of the inclusion from the intersection, and we also have surjectivity of the last map by "smallness" and injectivity of the first map easily. But now by the locality principle,

$$H_n(S_*^{\mathcal{A}}(X)) = H_n(X),$$

so we obtain the Mayer-Vietoris sequence by taking the long exact sequence from this short exact sequence of chain complexes. □

Remark 106. *Vietoris published his last mathematical paper at the age of 103; he was actually the oldest person alive at one point. But he's most famous for this theorem.*

Let's see some applications of this now:

Example 107

Let's calculate part of the homology of S^2 using Mayer-Vietoris. We'll use A to be the "upper hemisphere" plus a little bit, and B to be the "lower hemisphere" plus a little bit: then $A \cap B$ is a collar around the equator.

(Remember that we need the **interiors** of A and B to cover X, so we can't just use the hemispheres.) Then one part of our long Mayer-Vietoris sequence becomes

$$H_2(A) \oplus H_2(B) \to H_2(S^2) \xrightarrow{\partial} H_1(A \cap B) \to H_1(A) \oplus H_1(B).$$

Here, A and B are **homeomorphic to the disk**, and a disk is homotopy equivalent to a point (because, for example, it's star-shaped). Therefore,

$$H_2(A) = H_2(D^2) = H_2(p) = 0,$$

and the same holds for B. Thus, the direct sum $H_2(A) \oplus H_2(B)$ is just the zero group, and the same is true for $H_1(A) \oplus H_1(B)$. Next, notice that $H_1(A \cap B)$ is homeomorphic to the annulus, which is homotopy equivalent to S^1 (by a deformation retraction), so $H_1(A \cap B) = \mathbb{Z}$.

Remark 108. *We'll always have to do this kind of "induction" argument, where we know the homology of S^1 to get the one for S^2.*

Thus, we have an exact sequence

$$0 \to H_2(S^2) \xrightarrow{f} \mathbb{Z} \to 0,$$

such that f is injective and also surjective, so it must be a bijection. Thus $\boxed{H_2(S^2) = \mathbb{Z}}$.

Example 109

Consider the torus $T \cong S^1 \times S^1$, which we can draw as a hollow donut or as a filled-in square with opposite edges identified (in the same orientation). Our goal is to cover this torus with simpler spaces as well.

We'll do something similar to the sphere – we'll take the "left part" and "right part" of the donuts, which are "macaronis" that intersect in the middle. So A and B are homeomorphic to cylinders, and $A \cap B$ is the intersection of two (smaller) cylinders.

This time, part of the Mayer-Vietoris sequence is

$$H_2(A) \oplus H_2(B) \to H_2(T) \xrightarrow{\partial} H_1(A \cap B) \to H_1(A) \oplus H_1(B) \to H_1(T) \xrightarrow{\partial} H_0(A \cap B) \to H_0(A) \oplus H_0(B).$$

We'll still go with the philosophy that the values of the individual groups, and some of the non-boundary maps, are easier to compute than the boundary ones. First of all, A and B are both homoeomorphic to the annulus, which is homotopy equivalent to a circle (by deformation retraction again). Thus,

$$H_2(A) \oplus H_2(B) \cong 0 \oplus 0 = 0$$

(remember that $H_q(S^n)$ is $\mathbb{Z}$ for $q = 0, n$ and 0 otherwise), while

$$H_1(A) \oplus H_1(B) \cong \mathbb{Z} \oplus \mathbb{Z}, \quad H_0(A) \oplus H_0(B) \cong \mathbb{Z} \oplus \mathbb{Z}.$$

Next, $A \cap B$ is the equivalent to the disjoint union of two cylinders, so $H_0(A \cap B) = \mathbb{Z} \oplus \mathbb{Z}$ (we have two path components), and $H_1(A \cap B) = \mathbb{Z} \oplus \mathbb{Z}$. So now we can rewrite our Mayer-Vietoris sequence as

$$0 \to H_2(T) \to \mathbb{Z} \oplus \mathbb{Z} \xrightarrow{f} \mathbb{Z} \oplus \mathbb{Z} \to H_1(T) \to \mathbb{Z} \oplus \mathbb{Z} \xrightarrow{g} \mathbb{Z} \oplus \mathbb{Z}.$$

This isn't enough on its own to determine $H_2(T)$ and $H_1(T)$, so we need to also understand a few of the maps. Because we don't know anything about the homology of the torus, we can't do much with the Snake Lemma boundary maps, but we can at least understand the maps labeled f, g above.

Let's start with g, which sends $H_0(A \cap B) \to H_0(A) \oplus H_0(B)$. To understand this, we can look at the path components: we'll send $(1, 0)$ to $(1, 1)$, because any element of $A \cap B$ is in both A and B, and similarly $(0, 1)$ is sent to $(1, 1)$. So in general, **an element** $(x, y) \in H_0(A \cap B)$ **will be sent to** $(x + y, x + y)$**.** On the other hand, the map $f : H_1(A \cap B) \to H_1(A) \oplus H_1(B)$ is a bit harder to understand, but A "provides a homotopy equivalence of one cylinder Q of $A \cap B$ to the other one: looking at topological spaces, the sequence of maps of spaces

$$Q \to A \cap B \to A \sqcup B \to A$$

give us a map of homology groups via the functor H_1:

$$\mathbb{Z} \to \mathbb{Z} \oplus \mathbb{Z} \to \mathbb{Z} \oplus \mathbb{Z} \to \mathbb{Z}.$$

We can follow the maps and find that $1 \mapsto (1, 0) \mapsto f(1, 0) \mapsto ?$, but if we look at the composite map from Q to A, we're including a "small tip" of the macaroni into the whole thing, so we have a deformation retraction. Thus 1 must end up being sent to 1. Since the map $A \sqcup B \to A$ is a projection, and so is $A \sqcup \to B$, we know that $f(1, 0) = (a, b)$ must have $a = 1$. The same argument tells us that $b = 1$, and thus we have

$$1 \mapsto (1, 0) \mapsto (1, 1) \mapsto 1.$$

This therefore tells us that the map f is also $(x, y) \mapsto (x + y, x + y)$, and now we have enough information to compute $H_2(T)$ and $H_1(T)$. First of all, consider

$$0 \to H_2(T) \to \mathbb{Z} \oplus \mathbb{Z} \xrightarrow{f} \mathbb{Z} \oplus \mathbb{Z}.$$

Because the kernel of f is $\mathbb{Z}$ (it's all multiples of $(1, -1)$), and the map $H_2(T) \to \mathbb{Z} \oplus \mathbb{Z}$ must be injective, exactness implies that $\boxed{H_2(T) \cong \ker(f) \cong \mathbb{Z}\{(1, -1)\} \cong \mathbb{Z}}$. And now, we can look at

$$\mathbb{Z} \oplus \mathbb{Z} \xrightarrow{f} \mathbb{Z} \oplus \mathbb{Z} \xrightarrow{r} H_1(T) \to \mathbb{Z} \oplus \mathbb{Z} \xrightarrow{g} \mathbb{Z} \oplus \mathbb{Z}.$$

We have

$$\ker(r) = \text{im}(f) = \text{diagonal } \mathbb{Z} \hookrightarrow \mathbb{Z} \oplus \mathbb{Z},$$

and the image of r is a copy of $\mathbb{Z}$ in $H_1(T)$. The rest of the argument will be left as an exercise to us, but we can find that $\boxed{H_1(T) \cong \mathbb{Z} \oplus \mathbb{Z}}$.

Soon, we'll introduce tools for more combinatorial arguments for calculating homology groups, like the graph drawing we did on the first day of class. For example, if we take our square with opposite edges identified (call them f, g) and cut along the diagonal h, we'll have two faces A and B. This will form a semisimplicial set for the torus, which gives us the chain complex $\mathbb{Z}\{A, B\} \to \mathbb{Z}\{f, g, h\} \to \mathbb{Z}\{x\}$, sending $A \mapsto f + g - h$, $B \mapsto g + f - h$, and f, g, h all to 0. Our aim will be to prove that no matter how we cut up our space, this kind of construction is indeed valid!

12 September 30, 2020

Our second problem set is due at class time on Friday – there was a sign error in the last part, which has been fixed now.

As mentioned last time, we're going to return to talking about calculating homology using a finite semisimplicial set and proving that this construction is valid. It'll take us a bit of time to get there, though:

Definition 110

Let $\mathcal{C}$ be a category, and consider a diagram in $\mathcal{C}$:

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ \downarrow g & & \\ C & & \end{array}$$

A **pushout** of such a diagram is a commuting square

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ \downarrow g & & \downarrow p_B \\ C & \xrightarrow{p_C} & P \end{array}$$

with the following **(universal) property** that for any other square with P replaced with P', there is a unique map $p : P \to P'$ such that

$$p \circ p_C = p'_C, \quad p \circ p_B = p'_B.$$

This definition should remind us of that of a product, which is also fundamentally about a unique map coming from a diagram. (We'll get some practice on the next problem set looking at specific examples of what these look like.)

Example 111

Let's describe (without proof) what pushouts look like in Top.

If we have three topological spaces and two continuous maps $A \to B$, $A \to C$, then the pushout has

$$P = (B \sqcup C)/(f(a) = g(a) \, \forall a \in A).$$

In other words, we use the quotient topology where we identify some points in the disjoint union $B \sqcup C$, by making any point $f(a)$ equal to the corresponding point $g(a)$. So we have a sequence of maps

$$B \to B \sqcup C \to P,$$

and then the composite of these two maps $B \to P$ is the map p_B.

Example 112

There is a unique function from the emptyset to any other topological space, so the pushout of the diagram

$$\begin{array}{ccc} \varnothing & \longrightarrow & B \\ \downarrow & & \\ C & & \end{array}$$

is just the disjoint union $B \sqcup C$.

Example 113

There is a unique map from any topological space to the one-point topological space p, so the pushout of the diagram

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ \downarrow{\scriptstyle g} & & \\ p & & \end{array}$$

is $P = B/\text{im}(f)$, because all points under f are identified with each other.

There's also a particularly important kind of pushout that we should keep an eye out for:

Definition 114

Suppose we have a pushout square as shown:

$$\begin{array}{ccc} \bigsqcup_{i \in I} S^{n-1} & \longrightarrow & B \\ \big\downarrow & & \big\downarrow \\ \bigsqcup_{i \in I} D^n & \longrightarrow & P \end{array}$$

We then say that P is obtained from B by **attaching n-cells**.

Here, I is some index set, so $\bigsqcup_{i \in I} S^{n-1}$ is a disjoint union of spheres. We then apply the inclusion map into the corresponding boundaries of the disks D^n, and the idea is that we've identified some $(n-1)$-spheres inside of B to get P.

Example 115

Consider the following diagram:

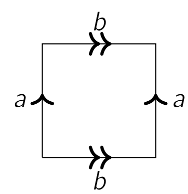
$$\begin{array}{ccc} S^1 \sqcup S^1 & \longrightarrow & p \\ \big\downarrow & & \big\downarrow \\ D^2 \sqcup D^2 & \longrightarrow & P \end{array}$$

Then we obtain P from the one-point space by attaching two 2-cells, which we can think of as "sticking two hollow spheres onto a point." That's because we're taking the disjoint union of two disks, but we're taking both boundaries and identifying all points there together. So the disks "curl up," and we're left with our two hollow spheres attached at a central point.

Similarly, the **figure-8 graph** is obtained from the same attaching process in one lower dimension:

$$\begin{array}{ccc} S^0 \sqcup S^0 & \longrightarrow & p \\ \big\downarrow & & \big\downarrow \\ D^1 \sqcup D^1 & \longrightarrow & P \end{array}$$

This space is homeomorphic to the perimeter of a square with opposite edges identified in the same direction:



Specifically, this means there is a continuous map $r : S^1 \to \text{(figure-8)}$, called $aba^{-1}b^{-1}$, corresponding to going along a, then b, then a backwards, then b backwards in the square. And then we can glue a 2-cell to this figure-8 shape by using r as the boundary:

$$\begin{array}{ccc} S^1 & \xrightarrow{r} & \text{(figure-8)} \\ \big\downarrow & & \big\downarrow \\ D^2 & \longrightarrow & P \end{array}$$

Then P is homeomorphic to the disjoint union of a disk and the hollow square, but then we identify the two boundaries, so **this is actually homeomorphic to the torus**. In other words, if we take a solid square and identify two opposite edges a, we get a cylinder, and then identifying the other two gives us a torus. Rewording this claim, we say that T is obtained from the figure-8 by attaching a 2-cell.

Definition 116

A **CW complex** X is a space together with a sequence of subgroups

$$\varnothing = Sk_{-1}X \subseteq Sk_0X \subseteq Sk_1X \subseteq Sk_2X \subseteq \cdots \subseteq X,$$

such that X is the union of the Sk_nX (the "n-skeleta" of X), and we have a pushout diagram:

$$\begin{array}{ccc} \bigsqcup_{i \in I_n} S^{n-1} & \longrightarrow & Sk_{n-1}X \\ \downarrow & & \downarrow \\ \bigsqcup_{i \in I_n} D^n & \longrightarrow & Sk_nX \end{array}$$

In other words, the n-skeleton is obtained from the $(n-1)$-skeleton by attaching n-cells.

(The "C" here stands for "cell," and the "W" stands for "weak," because we're using the **weak topology** – a set is open if it is open when restricted to each skeleton.)

Example 117

The torus T can be given the structure of a CW complex by letting

$$Sk_0T = p, \quad Sk_1T = (\text{figure-8}), \quad Sk_2T = Sk_3T = \cdots = T.$$

Definition 118

A CW complex is **finite-dimensional** if $Sk_nX = X$ for some n, and the **dimension** of X is the smallest n for which this is true. A CW complex is of **finite type** if I_n is finite for each n.

It's possible that CW complexes can have cells in larger and larger dimensions, and it's also possible that we can attach an infinite number of n-cells at each step. So the descriptions of "finite-dimensional" and "finite-type" help us keep the shapes more manageable.

Definition 119

A CW complex is **finite** if it is finite-dimensional and of finite type.

Equivalently, a CW complex is finite if it has finitely many cells. (For instance, the torus has one 0-cell, two 1-cells, and one 2-cells with the specific structure above.)

Theorem 120 (from point-set topology)

Any CW complex is Hausdorff, and a CW complex is compact if and only if it is finite. In fact, any compact smooth manifold can be given some finite CW complex structure.

We can organize CW complexes into a category which we call CWcomp, with morphisms being diagrams where all squares commute:

$$
\begin{array}{ccc}
\vdots & & \vdots \\
\cup\!\mid & & \cup\!\mid \\
Sk_{n+1}X & \longrightarrow & Sk_{n+1}Y \\
\cup\!\mid & & \cup\!\mid \\
Sk_nX & \longrightarrow & Sk_nY \\
\cup\!\mid & & \cup\!\mid \\
Sk_{n-1}X & \longrightarrow & Sk_{n-1}Y \\
\cup\!\mid & & \cup\!\mid \\
\vdots & & \vdots
\end{array}
$$

We then have a functor

$$U : \text{CWcomp} \to \text{Top},$$

which records only the union of the skeleta (which "forgets" the subspace structure and only keeps the underlying topological space).

Fact 121

There is a functor

$$\text{Fun}(\Delta_{\text{inj}}^{\text{op}}, \text{Set}) \to \text{CWcomp}$$

from semisimplicial sets to CW complexes, where we view the sequences $X_0, X_1, \cdots$ as collections of points, edges (attached to points via the d_i maps), triangles, and so on.

Specifically, if $X.$ is a semisimplicial set, then X_n is the set of n-cells in the corresponding CW complex.

Remark 122. *There are CW complexes that don't arise from the image of the functor here – we can think of these CW complexes as slightly more combinatorial than topological spaces, but still somewhere between semisimplicial sets and topological spaces.*

Then the composite map

$$\text{Fun}(\Delta_{\text{inj}}^{\text{op}}, \text{Set}) \to \text{CWcomp} \to \text{Top}$$

is sometimes called a **geometric realization** of a semisimplicial set.

Our goal from here is to understand homology groups H_n for a geometric realization, and we want to prove that they can be obtained by taking the homology of the semisimplicial set (using the method from the first few lectures). We'll do this by more generally understanding the homology of CW complexes.

Example 123

The n-sphere S^n can be given a CW structure with one 0-cell and one n-cell.

This basically tells us that we have a pushout diagram as shown:

$$
\begin{array}{ccc}
S^{n-1} & \longrightarrow & p \\
\big\downarrow & & \big\downarrow \\
D^n & \longrightarrow & S^n
\end{array}
$$

This is a very compact structure, and it's a reason we should think of CW complexes as being "more flexible" than our semisimplicial sets. But there is a different CW structure on S^n which we'll occasionally make use of, by setting

$$Sk_{-1}S^n = \varnothing, \quad Sk_0 S^n = p \sqcup p = S^0,$$

and then we take

$$Sk_1 S^n = S^1, \quad Sk_2 S^n = S^2, \cdots, Sk_n S^n = Sk_{n+1} S^n = S^n.$$

Basically, we **keep gluing in a top and bottom hemisphere** to our k-dimensional sphere, until we get to an n-dimensional one. So if we want to get from $Sk_{k-1}S^n$ to $Sk_k S^n$, we glue two k-cells as hemispheres. And this means that we have a CW complex S^∞, defined via $Sk_n S^\infty = S_n$: while this CW complex is finite type, it is not finite, because it describes an infinite-dimensional space. We can try thinking about what homology S^∞ has, and we'll continue to explore CW complexes next time.

13 October 2, 2020

We'll continue talking about CW complexes and related notions today. Last time, we discussed the specific CW complex

$$S^\infty = \bigcup S^n$$

with $Sk_{-1}S^\infty = 0$, $Sk_0 S^\infty \cong S^0$, $Sk_1 S^\infty \cong S^1$, and so on. (So passing from one skeleton to the next, we attach two cells that correspond to "hemispheres" in the next dimension.)

We know that

$$H_q(S^n) = \begin{cases} \mathbb{Z} & q = 0, n \\ 0 & \text{otherwise.} \end{cases}$$

So as $n \to \infty$, we should guess that the homology just reduces to

$$H_q(S^\infty) = \begin{cases} \mathbb{Z} & q = 0 \\ 0 & \text{otherwise.} \end{cases}$$

This can actually be made precise: we can prove this because homology is made out of simplices, and each individual simplex is compact, so mapping it into a union of spheres must map it into a particular sphere. And each homotopy also comes from a compact space, so everything is going on in some particular finite-dimensional sphere.

Proposition 124

S^∞ is contractible; that is, it is homotopy equivalent to a point.

Proof. We can write a point $x \in S^\infty$ as a sequence

$$x = (x_0, x_1, x_2, \cdots)$$

such that $x_n = 0$ for all sufficiently large n, and $\sum_i x_i^2 = 1$. (Remember that by definition, S^∞ is the union of S^ns, so any element x must sit in a **specific** S^n, and that makes our sequence finite.) So now we can look at the map $f : p \to S^\infty$ which picks out some point, say $(1, 0, 0, \cdots)$, and we can also look at the unique map $g : S^\infty \to p$ which sends everything to p.

We want to show that f and g are inverse homotopy equivalences, meaning that

$$f \circ g \cong \mathrm{id}_{S^\infty}, \quad g \circ f \cong 1_p.$$

The latter is clear because $g \circ f$ is exactly the 1_p map, but we need to show that the map $f \circ g$ (which sends every point to $(1, 0, 0, \cdots)$ is homotopic to the identity. To do that, consider the map

$$T : S^\infty \to S^\infty, \quad T(x_0, x_1, \cdots) \mapsto (0, x_0, x_1, \cdots)$$

which shifts the sequence to the right by one spot (this is indeed a continuous function). We'll show that $T \cong 1_{S^\infty}$ and also that $T \cong f \circ g$, which is enough to prove what we want.

For the first of these homotopies, we can consider the map $h : S^\infty \times [0, 1] \to S^\infty$.

$$h(x, t) = \frac{tx + (1-t)Tx}{||tx + (1-t)Tx||}.$$

This is a well-defined, continuous map because $tx + (1-t)Tx$ is never the origin – the first nonzero coordinate in Tx occurs after the first nonzero coordinate in x, and even in the $t = 0$ case we're okay because Tx isn't the origin either. We can check that this satisfies the properties that we want, and now we turn to a homotopy $T \cong f \circ g$: we use

$$h(x, t) = \frac{tTx + (1-t)(1, 0, \cdots)}{||tTx + (1-t)(1, 0, \cdots)||}.$$

Similarly this is well-defined because the denominator has a nonzero first coordinate except when $t = 1$, and even in that case we're fine. ☐

This "shifting" process, allowed only because our sequences are infinite, is called a **swindle**. And we've proved a fun result, even if it won't be theoretically imporant for us going forward. It's just important to keep in mind that a CW complex can have infinitely many cells, but we still end up with something that is just homotopy equivalent to a point.

We can talk about another CW complex that is extremely important:

Example 125

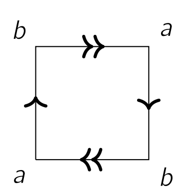
The **real projective k-space** $\mathbb{RP}^k$ is defined as the quotient

$$\mathbb{RP}^k = S^k/(x \sim -x)$$

which identifies opposite (**antipodal**) points on the sphere.

For example, $\mathbb{RP}^0 = S^0/(x \sim -x)$, where S^0 consists of two points that are antipodal (because S^0 is the subspace of $\mathbb{R}^1$ of distance 1 away from the origin, which is $\{(-1), (1)\}$). So that means that $\mathbb{RP}^0$ is a single point.

Next, $\mathbb{RP}^1$ is a circle with opposite points identified, which can be thought of as identifying opposite edges of a hollow square (both in opposite orientations). But this turns out to just be **another circle**. If we're not super sure how to visualize this, we can write it out as shown:



Then the double-headed arrow goes from b to a, and the single-headed arrow goes from a to b. So this means we indeed have a circle, where a and b are two of the points.

But now going on to $\mathbb{RP}^2$ and higher, we're identifying antipodal points of a 2-sphere, and we can't draw this inside a 3D space (much like we can't draw a Klein bottle inside 3D space).

Fact 126

We can notice that the inclusion of the equator $S^k \to S^{k+1}$ is compatible with the map $x \mapsto -x$ (antipodal points are still antipodal). So we have inclusions

$$\varnothing \subseteq \mathbb{RP}^0 \subseteq \mathbb{RP}^1 \subseteq \mathbb{RP}^2,$$

and this gives us a CW complex.

We can write out the pushout diagram more explicitly:

$$\begin{array}{ccc} S^{k-1} & \longrightarrow & \mathbb{RP}^{k-1} \\ \downarrow & & \downarrow \\ D^k & \longrightarrow & \mathbb{RP}^k \end{array}$$

Here, $S^{k-1} \to \mathbb{RP}^{k-1}$ is the quotient map, and the inclusion $S^{k-1} \hookrightarrow D^k$ is the boundary of the **upper** hemisphere. (It's worth thinking about this a bit more so that we understand this point.) So $\mathbb{RP}^k$ can be thought of as a CW complex with exactly one cell in each dimension up to k, meaning that

$$\mathbb{RP}^\infty = \bigcup \mathbb{RP}^n$$

is finite type but not a finite CW complex, just like S^∞. We'll prove, though, that unlike S^∞, this is **not** contractible, and it will turn out that there's a lot of structure encoded in this one space!

Now that we have some fundamental examples of CW complexes, we can examine their homology to look at the geometric realization of a semisimplicial set. First, we'll need to do a calculation:

Definition 127

A **wedge of k-spheres** is a space, denoted $\bigvee_{i \in I} S^k$, consisting of $|I|$ different k-spheres all meeting (only) at a single point.

For example, the figure-8 graph that we drew last class is a wedge of two 1-spheres, and we can add another circle to the central point to get a wedge of three 1-spheres, and so on. We can write this more formally as

$$\bigvee_{i \in I} S^k = \bigsqcup_{i \in I} S^k / \bigsqcup_{i \in I} p.$$

Then the **reduced homology group** is

$$\tilde{H}_q\left(\bigvee_{i \in I} S^k\right) = H_q\left(\bigvee_{i \in I} S^k, p\right)$$

(remember that when $q > 0$, this is the same as ordinary homology, and we have one less $\mathbb{Z}$ when $q = 0$). Excision tells us that this is the same as

$$= H_q\left(\bigsqcup_{i \in I} S^k, \bigsqcup_{i \in I} p\right).$$

And we can calculate these relative homology groups using the long exact sequence of the pair $\left(\coprod_{i\in I} S^k, \coprod_{i\in I} p\right)$: this looks like

$$\cdots \to H_q\left(\coprod_{i\in I} p\right) \to H_q\left(\coprod_{i\in I} S^k\right) \to H_q\left(\coprod_{i\in I} S^k, \coprod_{i\in I} p\right) \xrightarrow{\partial} H_{q-1}\left(\coprod_{i\in I} p\right) \to H_{q-1}\left(\coprod_{i\in I} S^k\right) \to \cdots.$$

And we understand many of these groups already: using the Eilenberg-Steenrod axiom for disjoint union tells us that

$$H_q\left(\coprod_{i\in I} S^k\right) \cong \begin{cases} \bigoplus_{i\in I} \mathbb{Z} & q = 0, k \\ 0 & \text{otherwise,} \end{cases}$$

and also that

$$H_q\left(\coprod_{i\in I} p\right) = \begin{cases} \bigoplus_{i\in I} \mathbb{Z} & q = 0 \\ 0 & \text{otherwise.} \end{cases}$$

So the groups $H_q\left(\coprod_{i\in I} S^k, \coprod_{i\in I} p\right)$ will be zero almost everywhere: we can check (in a straightforward way) that

$$\boxed{\tilde{H}_q\left(\bigvee_{i\in I} S^k\right) \cong \begin{cases} \bigoplus_{i\in I} \mathbb{Z} & q = k \\ 0 & \text{otherwise.} \end{cases}}$$

In particular, when k is positive, the zeroth relative homology group makes sense: we only have one path component, so H_0 of the wedge is indeed $\mathbb{Z}$, and thus $\tilde{H}_0$ of the wedge should be 0.

This calculation is important to us because of the following idea: if we let X be a CW complex, we can consider the relation between homology groups of skeleta

$$H_q(Sk_{k-1}X), \quad H_q(Sk_kX).$$

But if we want to do something like this, we need to consider the long exact sequence of the pair $(Sk_kX, Sk_{k-1}X)$, which requires us to understand the maps

$$H_{q+1}(Sk_kX, Sk_{k-1}X) \xrightarrow{\partial} H_q(Sk_{k-1}X) \to H_q(Sk_kX) \to H_q(Sk_kX, Sk_{k-1}X).$$

The first and last maps here are isomorphic to $\tilde{H}_{q+1}(Sk_kX/Sk_{k-1}X)$ and $\tilde{H}_q(Sk_kX/Sk_{k-1}X)$, respectively, and there is a pushout square

$$\begin{array}{ccc} \coprod_{i\in I_k} S^{k-1} & \longrightarrow & Sk_{k-1}X \\ \downarrow & & \downarrow \\ \coprod_{i\in I_k} D^k & \longrightarrow & Sk_kX \end{array}$$

which tells us that

$$Sk_kX/Sk_{k-1}X \cong \coprod_{i\in I_k} D^k / \coprod_{i\in I_k} S^{k-1}$$

is really just identifying a single point on the boundaries of all of our disks, because the pushout construction for Sk_kX first puts $Sk_{k-1}X$ together with a bunch of disks, and then the quotient then collapses all of sk_{k-1}. And taking a disjoint union of disks and identifying their boundaries indeed gives us a wedge $\bigvee_{i\in I_k} S^k$. (For visualization, we can consider $k = 1$ and putting two line segments together into the figure-8 graph.)

So the relevant parts of the long exact sequence are exactly given by the reduced homology of the wedge of spheres, which we've just calculated. And now we've basically spelled out the following result, which we'll cover next time:

Proposition 128

Suppose X is a CW complex, and say that $k, q \geq 0$ are integers. Then

$$H_q(Sk_k X) = \begin{cases} 0 & k < q \\ H_q(X) & k > q \end{cases}.$$

In other words, for a fixed q, nothing really happens except around $q = k$. We'll see with some examples next time that the homology will indeed change at the qth place, and then at the $(q+1)$th place, and never again.

14 October 5, 2020

Our third problem set is now posted – it'll be due two weeks from last Friday, and it'll be a bit easier than the previous one (where the last problem had a lot to absorb).

We'll start by discussing homology for a CW complex: recall that the result we stated at the end of class last time was that for a CW complex X and $k, q \geq 0$,

$$H_q(Sk_k X) \cong \begin{cases} 0 & k < q \\ H_q(X) & k > q. \end{cases}$$

Let's look at some small values of q first to understand what's going on:

- When $q = 0$, we're saying that $H_0(Sk_k X) \cong H_0(X)$ for all $k > 0$. Here, we should be thinking of $Sk_0 X$ as a bunch of points, $Sk_1 X$ as gluing edges to points in the 0-skeleton that already exist, $Sk_2 X$ as gluing disks to the 1-skeleton, and so on. And the point is that attaching a cell to objects that are already present does not change the number of path components that we have.
- When $q = 1$, we're saying that $H_1(Sk_0 X) \cong 0$, $H_1(Sk_1 X)$ is something that we aren't explicitly describing, and $H_1(Sk_2 X) \cong H_1(X)$ (along with all subsequent$H_1(Sk_k X)$). The first result here says that the disjoint union of a bunch of points has trivial first homology group, which makes sense. Beyond that, we're saying that we're adding loops and cycles when we add the 1-skeleton (specifically, we're "adding to" $Z_1(X)$), but then we need to mod out by boundaries, which come from 2-cells. So attaching the 2-cells will help us do that "modding out" of $B_1(X)$.

In general, the idea is that $H_q(Sk_{q-1} X) \cong 0$, then $H_q(Sk_q X)$ will surject onto $H_q(X)$, and then we'll finally mod out appropriately to get $H_q(Sk_{q+1} X) \cong H_q(X)$. But let's prove this rigorously for CW complexes.

Proof. We can compare $H_q(Sk_{k-1} X)$ and $H_q(Sk_k X)$ by using the long exact sequence of the pair $(Sk_k X, Sk_{k-1} X)$. We know that (by excision)

$$H_q(Sk_k X, Sk_{k-1} X) \cong \tilde{H}_q(Sk_k X / Sk_{k-1} X),$$

which we indicated at the end of last class is just a wedge of spheres

$$\tilde{H}_q\left(\bigvee_{i \in I_k} S^k\right) \cong \bigoplus_{i \in I_k} \mathbb{Z} \text{ only when } q = k \text{ and } 0 \text{ otherwise.}$$

(Geometrically, we can imagine that we're "filling in some triangles" when we go from $Sk_1 X$ to $Sk_2 X$, and then each of those triangles becomes a hollow 2-sphere when we collapse the whole skeleton $Sk_1 X$.) Then the long exact sequence tells us that the homology of the skeleton can only change in a few places: the qth homology of the k-skeleton is 0 if $k < q$, because the groups $H_q(Sk_kX)$ and $H_q(Sk_{k-1}X)$ are isomorphic for small k.

And for $k > q$, we find that $H_q(Sk_kX) \cong H_q(Sk_{k+1}X)$, so we will be done if we can show that the homology of the k-skeleton is actually the homology of X itself for $k > q$. (That is, X is the infinite union of the k-skeleta – we need to make sure that nothing weird happens like with the S^∞ construction.) To do that, note that Δ^q and Δ^{q+1} are compact, and anything in H_q has to be represented by a sum or difference of finitely many simplices. So each given sum of simplices lives in some specific k-skeleton; knowing Δ^q is compact tells us surjectivity from H_q to H_{q+1}, and then the boundary relations are also present because Δ^{q+1} is compact as well. □

Our next step is to look at **cellular homology**, which is a method to construct homology groups using these CW complexes:

Definition 129

Suppose X is a CW complex. Let $\boldsymbol{C_n(X) = C_n^{\text{cell}}(X)}$ denote the relative homology group

$$H_n(Sk_kX, Sk_{n-1}X) \cong \tilde{H}_n\left(\bigvee_{i \in I_n} S^n\right),$$

which is the free abelian group on the set of n-cells of X.

Definition 130

For each $n \geq 0$, define the map

$$d : C_{n+1}(X) \to C_n(X)$$

to be the composite map $C_{n+1}(X) = H_{n+1}(Sk_{n+1}X, Sk_n(X)) \xrightarrow{\partial} H_n(Sk_nX) \to H_n(Sk_nX, Sk_{n-1}X) = C_n(X)$, where ∂ is the boundary map that comes from the long exact sequence described earlier.

Theorem 131

The maps $d : C_{n+1}(X) \to C_n(X)$ make $C_*(X)$ into a chain complex (that is, $d \circ d = 0$). Furthermore, the homology of this chain complex is isomorphic to the homology of X.

For some intuition, we'll look at a few examples now to see how cellular homology works.

Example 132

Consider a sequence of functors

$$\text{Fun}(\Delta_{\text{inj}}^{\text{op}}, \text{Set}) \xrightarrow{F} \text{CWcomp} \xrightarrow{U} \text{Top},$$

where the composite functor is called the **geometric realization** of semisimplicial sets.

Then we know that

$$C_n^{\text{cell}}(F(X)) = \mathbb{Z}\text{Sing}_n X = S_n(X)$$

is the free abelian group on the n-simplices, and then

$$C_*^{\text{cell}}(F(X)) = S_*(X)$$

is the standard chain complex we've been working with. So the point of this theorem is that the semisimplicial homology of X agrees with the singular homology of the geometric realization of X, which means we finally have the justifications for the maneuvers from the beginning of class – it indeed doesn't matter how we chop up our simplices. In fact, the singular homology is not just an invariant of the topological space – it's an invariant up to homotopy equivalence, too. (And this emphasis on homotopy types rather than topological spaces is why this is an "algebraic topology" course.)

But cellular homology works even when our CW complexes don't come from semisimplicial sets:

Example 133

Let's compute the homology of the sphere S^n again, using cellular homology. We'll assume $n \geq 2$.

We know that S^n has a very simple CW complex structure, which doesn't come from a semisimplicial set. (Basically, we have one 0-cell and one n-cell.) If we use this CW structure, then $C_\ell^{\text{cell}}(X)$ is the free abelian group on the set of ℓ-cells, meaning that

$$C_0^{\text{cell}}(S^n) \cong \mathbb{Z}, \quad C_n^{\text{cell}}(S^n) \cong \mathbb{Z},$$

and all other $C_\ell^{\text{cell}}(S^n) \cong 0$. So $C_*(S^2)$ is isomorphic to the chain complex

$$\mathbb{Z} \xleftarrow{d} 0 \xleftarrow{d} \mathbb{Z} \xleftarrow{d} 0 \xleftarrow{d} 0 \cdots,$$

but all of the d maps are just trivial because there's always a 0 somewhere! So then we can compute

$$H_2(S^2) = \ker(d)/\text{im}(d) = \mathbb{Z}/0 \cong \mathbb{Z},$$

and in general doing this "kernel mod image" calculation will give us $\mathbb{Z}$ for H_0 and H_2 and 0 everywhere else, which is the expected answer. And notice that we arrived at this answer directly, as opposed to using an inductive method with our previous methods, so this is another sign that CW complexes are more flexible.

In particular, if a CW complex X has only even-dimensional cells (2-cells, 4-cells, and so on), then the same chain complex calculations will tell us that

$$H_q(X) \cong \begin{cases} 0 & q \text{ odd} \\ \bigoplus_{i \in I_q} \mathbb{Z} & q \text{ even.} \end{cases}$$

And in geometric situations, for example for varieties defined over $\mathbb{C}$, we sometimes are able to construct CW complexes with only even cell structure, while we can't get "even semisimplicial sets" in the same way.

In most examples, computing those boundary maps can be difficult or tedious. But let's also do an example where the boundary maps are easier to deal with:

Example 134

Let's compute the homology of the torus T^2, using the CW structure

$$Sk_0 T = p, \quad Sk_1 T = (\text{figure-8}), \quad Sk_2 T = T.$$

Remember that we draw the figure-8 often as the boundary of a square with opposite edges identified (so that all vertices collapse to a single one). This has one 2-cell, which is attached to the 1-skeleton via the map $b^{-1}a^{-1}ba : S^1 \to Sk_1 T$.

We can now compute the homology group from the cellular chain complex

$$\mathbb{Z}\{x\} \xleftarrow{d_1} \mathbb{Z}\{a, b\} \xleftarrow{d_2} \mathbb{Z} \leftarrow 0 \cdots,$$

We know that d_1 is the zero map (thinking about the "target minus source," or using the fact that we know $H_0(T) \cong \mathbb{Z}$ a priori), and we can also understand how to compute the map d_2: since the boundary of our 2-cell is glued with $b^{-1}a^{-1}ba$, we must have

$$d_2 u = -b - a + b + a = 0.$$

So both the d_1 and d_2 maps are zero, and this recovers the homology group we've already calculated.

Proof of Theorem 131. Consider the following diagram with many long exact sequences of pairs (the d maps have been labeled as d_x and d_y just for ease of reference):

$$\begin{array}{ccccccc}
 & & C^{\text{cell}}_{n+1}(X) = H_{n+1}(Sk_{n+1}X, Sk_n(X)) & \overset{d_x}{\searrow} & & & 0 = H_{n-1}(Sk_{n-2}X) \\
 & & \downarrow \partial_n & & & & \downarrow \\
0 = H_{n+1}(Sk_{n-1}X) & \longrightarrow & H_n(Sk_n)X & \xrightarrow{j_n} & H_n(Sk_nX, Sk_{n-1}X) & \xrightarrow{\partial_{n-1}} & H_{n-1}(Sk_{n-1}X) \\
 & & \downarrow & & & \overset{d_y}{\searrow} & \downarrow j_{n-1} \\
 & & H_n(Sk_{n+1}(X)) & & & & H_{n-1}(Sk_{n-1}X, Sk_{n-2}X) \\
 & & \downarrow & & & & \\
 & & 0 = H_n(Sk_{n+1}X, Sk_nX) & & & &
\end{array}$$

All rows and columns here are exact, and each d is defined as the composite arrows described here. But now we know that $d_y \circ d_x = 0$, because it's a sequence that can be computed by taking the vertical and horizontal arrows, and the composition of the two horizontal maps is zero by exactness.

From here, j_{n-1} is injective by exactness of the rightmost column, and therefore

$$\ker(d_y) = \ker(\partial_{n-1})$$

(injection doesn't increase the kernel), and then this is $\text{im}(j_n)$ by horizontal exactness. Furthermore, j_n is injective by exactness of the horizontal sequence, and therefore we can identify $\text{im}(j_n)$ with $H_n(Sk_nX)$. Putting this all together,

$$\ker(d_y)/\text{im}(d_x) = H_n(Sk_nX)/\text{im}(\partial_n),$$

which is exactly the homology group $H_n(Sk_{n+1}X)$ by the left vertical exact sequence, and that's $H_n(X)$. □

15 October 7, 2020

We'll continue the discussion of CW homology today. Last time, we saw that the cellular chain complex $C^{\text{cell}}_*(X)$ of a CW complex X indeed computes the singular homology groups for the union of the skeleta, and that's good because this is more "compact" than the very infinite Sing complex. And we saw a few examples of this: for instance, we can compute the homology group of S^2 easily with the minimal cell structure (one 0-cell and one 2-cell), without needing to worry about boundary maps or the long exact sequence. (And we combined this with the functor from semisimplicial sets to chain complexes to verify that the homology calculations we did on the first day of class do indeed work.)

As we move along further in math, we'll get a few more tools to help us deal with some of the differential maps, but the cellular differentials are a bit subtle to compute if we don't have anything combinatorial. Remember that we computed last time that the differentials in the chain complex $C^{\text{cell}}_*(T)$ of a torus are both 0, but we discovered this by drawing the torus as a square with opposite edges identified, so we did have some combinatorial or geometric intuition there.

Example 135

Consider S^2 with a different (less minimal) cell structure

$$Sk_0 S^2 = S^0, \quad Sk_1 S^2 = S^1, \quad Sk_2 S^2 = S^2.$$

We do this by starting with two points (a 0-sphere) x and y, and then we attach two paths u and v connecting x to y, and then we attach two "hemispheres" A and B to the top and bottom of the 1-sphere we've just formed. So our cellular chain complex looks like

$$C_*^{\text{cell}}(S^2) = \mathbb{Z}\{x, y\} \xleftarrow{d} \mathbb{Z}\{u, v\} \xleftarrow{d} \mathbb{Z}\{A, B\} \leftarrow 0 \cdots .$$

(There's an ambiguous notation here – we're suppressing the fact that the object on the left side depends on the CW structure.) To calculate the maps d, we can make a choice between the orientations of the edges: for example, we can pick

$$du = y - x, \quad dv = y - x.$$

If we're just trying to compute the 0th homology group, we only care about the image of the d map, and indeed changing the orientation will never do anything to that (we'll just change u to $-u$). So this tells us already that the zeroth homology group is

$$H_0(S^2) = \mathbb{Z}\{x, y\}/\mathbb{Z}\{y - x\} \cong \mathbb{Z}.$$

(As always, the numerator is the kernel of the d_0 map, which is the whole group, and the image is the kernel of the d_1 map.) And now we need to figure out what dA and dB are – to do that, it's worth going back to some of the definitions. We should remember that we are doing an attachment of cells when we go from the 1-cell to the 2-cell, which gives us the following pushout square:

$$\begin{array}{ccc} S^1 \sqcup S^1 & \xrightarrow{f} & Sk_1 S^2 = S^1 \\ \downarrow & & \downarrow \\ D^2 \sqcup D^2 & \longrightarrow & Sk_2 S^2 = S^2 \end{array}$$

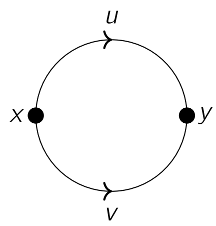
This pushout square governs how A and B are attached, and we care about understanding the **top map** f here (because once we know the map from $S^1 \sqcup S^1$ to $Sk_1 S^2 = S^1$ – in other words, "how the boundary of our disks are attached" – we determine the 2-skeleton). And we have a way of determining the cellular boundary d map just from this map f, using the composite map

$$S^1 \sqcup S^1 \to Sk_1 S^2 \to Sk_1 S^2 / Sk_0 S^2.$$

Here, the last topological space $Sk_1 S^2 / Sk_0 S^2$ is a wedge of two 1-spheres (corresponding to the loops u This gives us a map on H_1 of the form

$$\mathbb{Z} \oplus \mathbb{Z} \to \mathbb{Z} \oplus \mathbb{Z},$$

where we're associating the $\mathbb{Z}$s to A, B, u, v, respectively, and this map records the value of dA and dB (if we look at how things go into the quotient space). Remember that the boundary of A looks like this:



Then the map $S^1 \to Sk_1 S^2$ (the gluing map that determines A) is the identity map, but we don't even need to understand how things map onto the 1-skeleton of the sphere – it's enough to know how S^1 maps onto the wedge of two 1-spheres obtained by identifying x and y. The same holds for B.

In more generality, if we want to compute the boundary of an n-cell, we know that this n-cell is determined by some map

$$S^{n-1} \to Sk_{n-1}(X)$$

(which tell us "where we attach the n-cell"). And then the value of the composite map

$$S^{n-1} \to Sk_{n-1}(X) \to Sk_{n-1}(X)/Sk_{n-2}(X),$$

where the last group is a wedge $\bigvee_{i \in I_{n-1}} S^{n-1}$, gives us the boundary d of the n-cell. And now we have a geometric way of understanding what the cellular boundary map is doing!

So finishing up our example, we can now characterize the d maps of

$$C_*^{\text{cell}}(S^2) = \mathbb{Z}\{x, y\} \stackrel{d}{\leftarrow} \mathbb{Z}\{u, v\} \stackrel{d}{\leftarrow} \mathbb{Z}\{A, B\} \leftarrow 0 :$$

we have

$$du = y - x, \quad dv = y - x, \quad dA = v - u, \quad dB = v - u.$$

And we can confirm by calculation that this gives us homology groups $\{\mathbb{Z}, 0, \mathbb{Z}, 0, \cdots\}$, just like before.

Fact 136

Note that a generator of $H_2(S^2)$ in this particular CW structure is given by $A - B$ (the difference of the two hemispheres). And we can see that this is true without a lot of work, because the attaching maps from $S^1 \to Sk_1 S^1$ that define A and B are identical, so $dA = dB$.

We've now seen that homology can be used to prove that topological spaces are not homotopy equivalent (if we have two spaces and they have non-isomorphic homology groups, they can't be homotopy equivalent, because homology is an invariant of homotopy type). But another thing we can do is prove that continuous maps are not homotopic. Homology is a functor, and if we have two continuous maps of topological spaces which are sent to different maps of abelian groups, that tells us important information.

Definition 137

Let $f : S^n \to S^n$ be a continuous map. The **degree** of f is the value of 1 under the group homomorphism $H_n(f) : \mathbb{Z} \to \mathbb{Z}$.

So if $f, g : S^n \to S^n$ are two maps of different degrees, then they cannot be homotopic – otherwise we'd have an equivalence of maps in homology.

Lemma 138

Suppose that $f, g : S^n \to S^n$ are two maps. Then

$$\deg(g \circ f) = \deg(g)\deg(f).$$

Proof. Consider the sequence of maps

$$S^n \xrightarrow{f} S^n \xrightarrow{g} S^n,$$

which gives us (under H_n) a composite map of groups

$$\mathbb{Z} \xrightarrow{H_n(f)} \mathbb{Z} \xrightarrow{H_n(g)} \mathbb{Z}.$$

Since $H_n(f)$ sends 1 to $\deg(f)$, and $H_n(g)$ sends 1 to $\deg(g)$, we must indeed send 1 to $\deg(f)\deg(g)$ under the composite map (because we have a homomorphism). ☐

Corollary 139

Suppose $f : S^n \to S^n$ is a homeomorphism. Then $\deg(f)$ is either 1 or -1.

Proof. A homeomorphism is an isomorphism in Top, so there must exist a map $g : S^n \to S^n$ such that the composite is the identity in Top. But the degree of the the identity map is 1 (this follows by functoriality, since $H_n(1_{S^n})$ must be $1_{\mathbb{Z}}$), so

$$1 = \deg(g)\deg(f),$$

and the result follows. ☐

Example 140

Let's consider the degree of the map $f : S^n \to S^n$ which reflects across a hyperplane. (For instance, we could imagine reflecting around the equator of S^2, so A and B swap places.)

We can note that f is not just a map from a 2-sphere to itself – it's a map of CW complexes! And on C_*^{cell}, it sends $A - B$ to $B - A$ and doesn't do anything else. So looking at the cellular chain complexes (rather than the homology groups) tells us that the degree must be -1, and this argument holds for any $f : S^n \to S^n$.

Corollary 141

A reflection of S^n along an equator is not homotopic to the identity.

Example 142

Now let's consider the degree of a rotation of S^n (for example, spinning along a globe).

We know that any rotation of S^n along an axis is homotopic to the identity by writing out an explicit homotopy: we continuously "raise the amount of rotation" from 0 to the final amount. Therefore, the degree of a rotation must be $\boxed{1}$. (We may know that the composition of two reflections is a rotation, and that's indeed compatible with the fact that $(-1)(-1) = 1$.)

So now we can think about the antipodal map $S^1 \to S^1$ sending $x \mapsto -x$: because this is a 180 degree rotation, it has degree 1. In contrast, the antipodal map $S^2 \to S^2$ is not a rotation, because there's no axis being fixed. To understand this, we should think of a point of S^2 as a triple $(a, b, c) \in \mathbb{R}^3$ with $a^2 + b^2 + c^2 = 1$, and then this map sends $(a, b, c) \mapsto (-a, -b, -c)$. But this is a composite of three maps, where we swap the sign of a, b, and then finally c. So we actually have a composite of **three reflections**, each with degree -1, so our final answer is $(-1)^3 = \boxed{-1}$. This generalizes to n dimensions – in general, the degree of the antipodal map $S^n \to S^n$ is $\boxed{(-1)^{n+1}}$. We'll see next time how this can help us compute the homology of the real projective spaces, which are the quotients under this antipodal map.

16 October 9, 2020

We'll start with a summary of the last few classes, repeating the main points of **cellular homology**. Recall that if we have a CW complex X, we define a chain complex $C_*^{\text{cell}}(X)$ (which is "fairly finite" compared to the singular chain complex), whose homology groups compute the singular homology groups. Then the nth group $C_n^{\text{cell}}(X)$ is the free abelian group on the set of n-cells I_n, and then we can compute the differential $d : C_n^{\text{cell}}(X) \to C_{n-1}^{\text{cell}}(X)$ by considering the diagram below (an extension of the pushout square):

$$\begin{array}{ccccc} \coprod_{i \in I_n} S^{n-1} & \longrightarrow & Sk_{n-1}(X) & \longrightarrow & Sk_{n-1}(X)/Sk_{n-2}(X) \cong \bigvee_{i \in I_{n-1}} S^{n-1} \\ \downarrow & & \downarrow & & \\ \coprod_{i \in I_n} D^n & \longrightarrow & Sk_n(X) & & \end{array}$$

If we apply H_{n-1} to the top row of this diagram, we go from a map of topological spaces to a map of abelian groups. The top left corner of this square tracks (or parameterizes) the boundary of our n-cells, while the top right corner is an easier-to-understand version of the $(n-1)$-cells, so this characterizes $d : C_n^{\text{cell}}(X) \to C_{n-1}^{\text{cell}}(X)$ completely.

Note that we definitely need to know the homology of spheres and wedges of spheres – either using Mayer-Vietoris or other methods – for this to be a viable strategy for computing homology groups. And these differential maps d are sometimes difficult to compute, but if we manage to write down a semisimplicial set (so that everything is in triangles instead of cells), we have a combinatorial explicit way to write down all of the relevant maps.

From here, we started discussing the homology of maps $H_q(f) : H_q(X) \to H_q(Y)$, and to figure out what that is, we often put some CW structure on X and Y (semisimplicial sets, or just some generic CW structure), so that f induces a map of $C_*^{\text{cell}}(X) \to C_*^{\text{cell}}(Y)$, not just from $H_q(X) \to H_q(Y)$. This was helpful when we looked at the **degree** of a reflection $f : S^2 \to S^2$ around the equator, and the idea was to treat the top and bottom hemispheres as 2-cells (that is, pick a useful CW structure for the problem). Then the reflection swaps A and B, both of which have the same boundary, because we're attaching the top and bottom hemisphere in the same way. So $dA = dB$ and therefore $A - B$ is in the kernel of d, meaning it's an element of $H_2^{\text{cell}}(S^2)$. And since we know that the homology $H_2(S^2)$ is isomorphic to $\mathbb{Z}$, $A - B$ must be a generator of $H_2^{\text{cell}}(S^2)$, and this means we can think about the following map of cellular chain complexes:

$$\begin{array}{ccccc} \mathbb{Z}\{A, B\} & \longrightarrow & \mathbb{Z}\{u, v\} & \longrightarrow & \mathbb{Z}\{x, y\} \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle h} \\ \mathbb{Z}\{A, B\} & \longrightarrow & \mathbb{Z}\{u, v\} & \longrightarrow & \mathbb{Z}\{x, y\} \end{array}$$

Here, f maps A, B to B, A respectively, while g and h are the identity map. Since $A - B$ maps to $B - A$, this tells us that the degree of the reflection must be -1, because $H_2(f) : \mathbb{Z} \to \mathbb{Z}$ is the multiplication by -1.

We'll get lots of practice choosing the right CW structure and working through calculations on this problem set and the next one!

At the end of last class, we generalized this to show that for any sphere S^n, a reflection along an $S^{n-1} \subset S^n$ has degree -1, and therefore the antipodal map $S^n \to S^n$ (secretly a composite of reflections) has degree $(-1)^{n+1}$. So today, we'll compute one more important example of cellular homology right now, and then the rest of the computations will be mostly left for our homework.

> **Example 143**
> Let's compute the homology for the real projective spaces, using the tools we've developed in cellular homology.

Recall that we define the real projective spaces via

$$\mathbb{RP}^n = S^n/(x \sim -x);$$

that is, we identify opposite points on the sphere with each other. If we were asked to compute the homology, we could try to construct a semisimplicial model, but that can be very challenging – in fact, a lot of algebraic topology concerns combinatorially describe a space that's more naturally presented geometrically.

Instead, we can use a CW structure on $\mathbb{RP}^n$, which contains one k-cell in each dimension $0 \le k \le n$. Let's check this in some low dimensions:

- S^0 is two points (a CW complex with two 0-cells), and $\mathbb{RP}^0$ is what we get when we identify those two points together, which indeed gives us a single 0-cell.
- S^1 looks like a circle, and we can describe the cell decomposition as a CW complex with two points and two (half-circle) edges connecting them. Then $\mathbb{RP}^1$ has the two points identified, as well as the two edges, so we just have one 0-cell and one 1-cell. (In other words, $\mathbb{RP}^1$ is homeomorphic to a single point with a loop, which is just another circle.)
- If we want to understand $\mathbb{RP}^2$, we start with S^2, which we can describe (as usual) with two points x, y, two edges u, v connecting them, and two hemispheres A and B above and below the equator. Then $\mathbb{RP}^2$ identifies each of those pairs together. But this isn't a space that we're super familiar with – in particular, this isn't homeomorphic to anything that embeds in 3-dimensional space (much like a Klein bottle)!

So instead of drawing on prior knowledge of simple spaces, we'll think about the pushout square that defines $\mathbb{RP}^2$, where (loop) refers to the 1-skeleton $\mathbb{RP}^1$.

$$\begin{array}{ccccc} S^1 & \longrightarrow & \text{(loop)} & \longrightarrow & \mathbb{RP}^1/\mathbb{RP}^0 \cong S^1 \\ \downarrow & & \downarrow & & \\ D^2 & \longrightarrow & \mathbb{RP}^2. & & \end{array}$$

Then the horizontal composite map on H_1 gives us a map $\mathbb{Z} \to \mathbb{Z}$, and when we go once around the original circle S^1, we go through u twice.

So that means that we can write down the cellular chain complex:

$$C_*^{\text{cell}}(\mathbb{RP}^2) \cong \mathbb{Z}\{A\} \to \mathbb{Z}\{u\} \to \mathbb{Z}\{x\},$$

where A is sent to $2u$. And then we know that u is sent to 0 for a variety of reasons, and this allows us to compute $H_q(\mathbb{RP}^2)$:

$$H_0(\mathbb{RP}^2) = \mathbb{Z}, \quad H_1(\mathbb{RP}^2) = \mathbb{Z}/2\mathbb{Z}, \quad H_2(\mathbb{RP}^2) = H_3(\mathbb{RP}^2) = \cdots = 0.$$

In other words, we have a loop that's not a boundary, but if we go around it twice, it becomes a boundary! (This is similar to some ideas with the Mobius strip.) Ad even though we have some two-dimensional information, $\mathbb{RP}^2$ doesn't have any H_2 in it, so somehow the 2-cell doesn't give us a 2-dimensional hole – it just tells us about how going around a circle twice gives us a boundary.

So now let's move on to $C_*^{\text{cell}}(\mathbb{RP}^n)$ in general. We'll do this inductively: assume that we understand $\mathbb{RP}^{n-1}$. Then we can draw the diagram from attaching our n-cell:

$$\begin{array}{ccccc} S^{n-1} & \longrightarrow & \mathbb{RP}^{n-1} & \longrightarrow & \mathbb{RP}^{n-1}/\mathbb{RP}^{n-2} \cong S^{n-1} \\ \downarrow & & \downarrow & & \\ D^n & \longrightarrow & \mathbb{RP}^n. & & \end{array}$$

To make progress here, we need to understand the maps geometrically, and using the same cell diagrams as above, we can think of the quotient sphere as two $(n-1)$-spheres wedged together and also identified together using the antipodal map.

One way to understand this is that the map $S^{n-1} \to \mathbb{RP}^{n-1}/\mathbb{RP}^{n-2}$ is the action of first **pinching** the equator together (giving us a wedge of two $(n-1)$-spheres), and then quotienting under antipodal identification. So the differential $d : C_{n+1}^{\text{cell}}(\mathbb{RP}^{n+1}) \to C_n^{\text{cell}}(\mathbb{RP}^n)$ is calculated by looking at the homology of the "pinch + identify" map. Therefore, we have a composite map

$$\mathbb{Z} \xrightarrow{\text{pinch}} \to \mathbb{Z} \oplus \mathbb{Z} \xrightarrow{\text{quotient}} \to \mathbb{Z},$$

where the pinch map sends 1 to $(1,1)$, and then the quotient map sends $(1,0)$ to 1 and $(0,1)$ to the degree of the antipodal map. And $d : C_{n+1}^{\text{cell}}(\mathbb{RP}^{n+1}) \to C_n^{\text{cell}}(\mathbb{RP}^n)$ is the multiplication map $\mathbb{Z} \to \mathbb{Z}$ by $(1 + (-1)^n)$.

Example 144

We can compute the homology groups of $\mathbb{RP}^4$ by saying that

$$C_*^{\text{cell}}(\mathbb{RP}^4) \cong \mathbb{Z} \xrightarrow{2} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{2} \mathbb{Z} \xrightarrow{0} \mathbb{Z}.$$

Then computing the homology groups directly gives us the more sophisticated answer of

$$H_q(\mathbb{RP}^4) = \begin{cases} \mathbb{Z} & q = 0 \\ \mathbb{Z}/2\mathbb{Z} & q = 1 \\ 0 & q = 2 \\ \mathbb{Z}/2\mathbb{Z} & q = 3 \\ 0 & \text{otherwise.} \end{cases}$$

From here, we'll move in a different direction, looking at **invariants beyond homology**. In algebraic topology, some invariants are easier to compute than homology, while others are hard, but they "see" different information about a topological space. We'll spend some time here talking about "easier" invariants (meaning that they distinguish fewer spaces than homology, but they're easier to compute).

Definition 145

Let X be a finite CW complex. The **Euler characteristic** of X is

$$\chi(X) = \sum_k (-1)^k |I_k|,$$

where $|I_k|$ is the number of k-cells.

Example 146

The 2-sphere has a minimal CW structure with one 0-cell and one 2-cell, so its Euler characteristic is $1 + 1 = 2$. Alternatively, we can equip it with an alternative CW structuer with two 0-cells, two 1-cells, and two 2-cells, and then the Euler characteristic is $2 - 2 + 2 = 2$.

Getting the same answer in both ways is not an accident:

Theorem 147

The Euler characteristic $\chi(X)$ depends only on the homotopy type of X (the union of the skeleta, up to homotopy equivalence), not the specific choice of CW structure.

Example 148

The Euler characteristic of the torus can be calculated by picking the CW composition with $Sk_0(T)$ being a single point, $Sk_1(T)$ being a figure-8, and $Sk_2(T)$ being the whole torus. Then the Euler characteristic is $1-2+1 = 0$.

And because this is more straightforward to compute than homology, we can use the above theorem to get the following result easily without needing to do much computation:

Corollary 149

The torus and the sphere are not homotopy equivalent.

We're going to be working with the following more refined result:

Theorem 150

If X is a finite CW complex, then

$$\chi(X) = \sum_k (-1)^k \text{rank}(H_k(X)).$$

Since we can write this in terms of only homology groups, $\chi(X)$ is indeed "strictly weaker" than homology. And we should explain what **rank** means in the theorem above. Since $\mathbb{Z}$ is a PID, and abelian groups are $\mathbb{Z}$-modules, there is a classification of finitely generated $\mathbb{Z}$-modules – every finitely generated abelian group is isomorphic to $\mathbb{Z}^r \oplus \mathbb{Z}/n_1\mathbb{Z} \oplus \mathbb{Z}/n_2\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}/n_t\mathbb{Z}$, where $r \geq 0$ and $n_1, \cdots, n_t \geq 2$, and we call r the **rank** of A. (So something like $\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$ has rank 1.) And if X is a finite CW complex, we can compute the homology groups of X with cellular homology, and that finiteness tells us that $H_k(X)$ is indeed finitely generated for each k.

17 October 13, 2020

Last time, we started discussing topological invariants that are easier to compute than homology, and our primary example was the Euler characteristic (of a CW complex X)

$$\chi(X) = \sum_k (-1)^k |I_k|.$$

The fact that this is indeed an invariant was an important theorem that we stated – we claimed that the Euler characteristic doesn't depend on the CW structure of X, and it's also a strictly easier invariant than homology:

$$\chi(X) = \sum_k (-1)^k \text{rk}(H_k(X)),$$

where the rank of the finitely generated abelian group $H_k(X)$ is the "number of free copies of $\mathbb{Z}$." (Because we have a finite cellular chain complex, we do know that all H_k are finitely generated.) Let's prove this now, starting with an algebraic fact:

Lemma 151

Let $0 \to A \to B \to C \to 0$ be a short exact sequence of finitely generated abelian groups. Then $\text{rk}(B) = \text{rk}(A) + \text{rk}(C)$.

We'll prove a variant of this on problem set 4 (which will be posted this Friday when we turn in problem set 3), so the proof is omitted for now.

Proof of invariance of $\chi(X)$. For all $k \geq 0$, we have short exact sequences of finitely generated abelian groups

$$0 \to Z_k^{\text{cell}}(X) \to C_k^{\text{cell}}(X) \xrightarrow{\partial} B_{k-1}^{\text{cell}}(X) \to 0.$$

Let's look at

- Exactness at $Z_k^{\text{cell}}(X)$ tells us that the map from cycles into chains is injective, but indeed the cycles are just a subset of the chains.
- The boundaries are defined to be the image of the boundary map ∂, so we have exactness as $B_{k-1}^{\text{cell}}(X)$.
- The above two facts together give exactness at the middle, too.

We also have a short exact sequence

$$0 \to B_k^{\text{cell}}(X) \to Z_k^{\text{cell}}(X) \xrightarrow{\partial} H_k^{\text{cell}}(X) \to 0.$$

(To verify this, we need to check that every boundary is a cycle, and that the homology is the quotient of the cycles by the boundaries.) So now

$$\sum_k (-1)^k |I_k| = \sum_k (-1)^k \text{rk}(C_k^{\text{cell}}(X)),$$

since the rank of the free abelian group on the k-cells is the number of k-cells. And now we can write this as

$$= \sum_k (-1)^k \left(\text{rk}(Z_k^{\text{cell}}(X)) + \text{rk}(B_{k-1}^{\text{cell}}(X))\right)$$

with the first exact sequence, and then use the second exact sequence to rewrite this again as

$$== \sum_k (-1)^k \left(\text{rk}(B_k^{\text{cell}}(X)) + \text{rk}(H_k^{\text{cell}}(X)) + \text{rk}(B_{k-1}^{\text{cell}}(X)) \right),$$

and the two terms with B_ks cancel out with each other, leaving the result. □

The Euler characteristic is not a very powerful invariant, because it doesn't distinguish quite as many spaces, so let's look at another one now which is still easier to compute than homology, called **homology with coefficients**. The idea is that we don't necessarily need to use the free abelian group functor:

Definition 152

Let R be a commutative ring and X be a semisimplicial set. Then for any $k \geq 0$, $\boldsymbol{S_k(X; R)}$ is the **free R-module** generated by X_k.

For example, $S_k(X; \mathbb{Z}) = S_k(X)$ (because $\mathbb{Z}$-modules are just abelian groups). And we can do a similar thing for topological spaces:

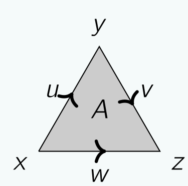
Definition 153

Let R be a commutative ring and X be a topological space. Then for any $k \geq 0$, $S_k(X; R)$ is the free R-module generated by $\text{Sing}_k(X)$.

In both cases, the alternating sum of the semisimplicial face maps d_i will create the differential maps $S_k(X; R) \to S_{k-1}(X; R)$ for a chain complex $S_*(X; R)$ of R-modules. (So this is a direct clean generalization.) And $S_*(X; \mathbb{Z}) = S_*(X)$ as well.

Example 154

Consider the semisimplicial set shown below:



We'll consider $S_*(X; \mathbb{Q})$, which is a chain complex of rational vector spaces with linear maps between them – it'll be isomorphic to

$$0 \to \mathbb{Q} \to \mathbb{Q} \oplus \mathbb{Q} \oplus \mathbb{Q} \to \mathbb{Q} \oplus \mathbb{Q} \oplus \mathbb{Q} \to 0 \to \cdots.$$

(Here, the first $\mathbb{Q}$ is the vector space with basis A, the first direct sum is the space with basis $\{u, v, w\}$, and the second direct sum is the one with basis $\{x, y, z\}$.) Then the linear maps are determined in the usual way: we send u, v, w to $y - x, z - y, z - x$, respectively, and knowing how the basis elements behave under the maps tells us how the whole map behaves. Therefore, we get exactly the same thing as before, just with groups replaced with vector spaces.

So now the homology R-modules of $S_*(X; R)$, denoted $H_q(X; R)$, can be calculated with a similar method as with $H_q(X; \mathbb{Z})$. We can check that all of our (Eilenberg-Steenrod) homology proofs still work for homology with coefficients in any commutative ring R, except that

$$H_q(*; R) = \begin{cases} R & q = 0 \\ 0 & \text{otherwise.} \end{cases}$$

In particular, if X is a CW complex, we can always compute $H_q(X; R)$ by looking at the cellular chain complex $C_*^{\text{cell}}(X; R)$.

Example 155

Let's look again at the homology of $\mathbb{RP}^2$, which has a cell structure with one 0-cell, one 1-cell, and one 2-cell.

Earlier, we calculated $H_q(\mathbb{RP}^2) = H_q(\mathbb{RP}^2; \mathbb{Z})$ with a cellular chain complex

$$\cdots \to 0 \to \mathbb{Z} \xrightarrow{2} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \to 0 \to \cdots,$$

telling us that

$$H_q(\mathbb{RP}^2) = \begin{cases} \mathbb{Z} & q = 0 \\ \mathbb{Z}/2\mathbb{Z} & q = 1 \\ 0 & \text{otherwise.} \end{cases}$$

So now let's move from "integral homology" to "rational homology:" we can calculate using the similar-looking cellular chain complex

$$\cdots \to 0 \to \mathbb{Q} \xrightarrow{2} \mathbb{Q} \xrightarrow{0} \mathbb{Q} \to 0 \to \cdots.$$

Therefore, we can

$$H_q(\mathbb{RP}^2; Q) = \begin{cases} \mathbb{Q} & q = 0 \\ 0 & q = 1 \\ 0 & \text{otherwise} \end{cases}$$

because this time, $\mathbb{Q}/2\mathbb{Q}$ is the trivial group – every rational number can be written as twice another rational number.

And now let's calculate $H_q(\mathbb{RP}^2, \mathbb{F}_2)$, where $\mathbb{F}_2$ denotes the field with two elements. This time, if we look at

$$\cdots \to 0 \to \mathbb{F}_2 \xrightarrow{2} \mathbb{F}_2 \xrightarrow{0} \mathbb{F}_2 \to 0 \to \cdots.$$

And the point is that the map 2 is the same as the map 0 in this case, so

$$H_q(\mathbb{RP}^2; \mathbb{F}_2) = \begin{cases} \mathbb{F}_2 & q = 0 \\ \mathbb{F}_2 & q = 1 \\ \mathbb{F}_2 & q = 2 \\ 0 & \text{otherwise.} \end{cases}$$

We can tell that our answers do indeed change a lot depending on the choice of commutative ring R.

Fact 156

All subsequent material won't be tested until after November 9 – it's important for us to have a good understanding of cellular homology, so pset 4, due October 30, will give us lots of practice with that. After the election, we'll have our brief exam on November 9.

In the next few weeks, we'll be talking about material for problem sets 5 and later. Here are some of the key questions we'll be answering:

- In what sense can $H_q(X; R)$ be easier than $H_q(X; \mathbb{Z})$? In particular, can we compute $H_q(X; R)$, homology with coefficients, in terms of the integral (standard) homology? (The answer turns out to be yes, but the translation process is a bit subtle.) In some sense, the integral homology groups are the best ones – for example, the rational homology groups will "pick out the free parts" of them.

Remark 157. *In applied topology (for instance persistent homology in computer science), we may be given a giant collection of points in $\mathbb{R}^{1000}$. Then if we fix a radius R and connect any two points that are less than R apart, then we can draw a 2-simplex for points, all of which are in some diameter R circle, and keep building up a semisimplicial set in this way. And there are theorems that say that homology with $\mathbb{Q}$ coefficients has much lower computational complexity, and provably so!*

- How can we compute $H_q(X \times Y)$ in terms of $H_q(X)$ and $H_q(Y)$?
- Extending that thought, any topological space has a diagonal map $\Delta : X \to X \times X$, mapping x to (x, x). What can we say about $H_q(\Delta)$? In particular, there will be spaces where $H_q(\Delta)$ is not the same as $H_q(X)$ – effective homology on the diagonal will give us some information about X that the pure homology groups don't see.
- What special features do manifolds have with respect to homology (compared to generic topological spaces)?

In the process of answering these questions, we'll introduce the (purely algebraic) functors Tor, Ext, and also introduce the relevant ideas of **cohomology**.

18 October 14, 2020

Yesterday, we introduced "homology with coefficients" in a general commutative ring as a generalization of $H_q(X)$ (which is homology "in the integers"). Today, we'll start looking at how to compute $H_q(X; R)$ from $H_q(X) = H_q(X; \mathbb{Z})$, which will turn out to be related to computing the homology of a product space $X \times Y$ in terms of the homologies of X and Y.

All of these tools will be purely algebraic, so we'll need to set up some tools from category theory and algebra. Recall that for two abelian groups A and B, we defined $\text{Hom}_{\text{Ab}}(A, B)$ to be the set of functors (group homomorphisms) from A to B. But this set has extra structure – if $f, g : A \to B$ are two homomorphisms, then $f + g$ and $f - g$ are also homomorphisms. (For example, $f - g : A \to B$ is defined by the formula $(f - g)(a) = f(a) - g(a)$.) That means the set $\text{Hom}_{\text{Ab}}(A, B)$ **is an abelian group** with this addition law, and we will denote this object of Ab by $\underline{\text{Hom}}_{\text{Ab}}(A, B)$. Let's generalize this:

Definition 158

Suppose R is a commutative ring, and let R-mod be the category of R-modules. Then $\text{Hom}_{\text{R-mod}}(M, N)$ can be given the structure of an R-module, and we denote the resulting object in R-mod by $\underline{\text{Hom}}_{\text{R-mod}}(M, N)$.

(For example, the set of linear maps between vector spaces over a field form another vector space.) This construction turns out to be **functorial**: mapping $(M, N) \mapsto \underline{\text{Hom}}_{\text{R-mod}}(M, N)$ is functorial in its inputs. Here's what we need to check to verify that statement:

- Suppose that $N \to N'$ is some map of R-modules. Then any map $M \to N$ gives a composite map $M \to N \to N'$, so there will be an R-module map

$$\underline{\text{Hom}}_{\text{R-mod}}(M, N) \to \underline{\text{Hom}}_{\text{R-mod}}(M, N').$$

- Now suppose that $M \to M'$ is some fixed map of R-modules. Then there will be a corresponding map

$$\underline{\mathrm{Hom}}_{\text{R-mod}}(M', N) \to \underline{\mathrm{Hom}}_{\text{R-mod}}(M, N),$$

where the map $f : M' \to N$ gets sent to the composite map $M \to M' \to N$ (notice that the direction of maps is being flipped here).

In other words, we have a functor

$$\underline{\mathrm{Hom}}_{\text{R-mod}} : (\text{R-mod})^{\text{op}} \times (\text{R-mod}) \to (\text{R-mod}),$$

sending (M, N) to $\underline{\mathrm{Hom}}_{\text{R-mod}}$. And the more general structure here is that some categories $\mathcal{C}$ (such as R-mod) have **internal Homs**

$$\underline{\mathrm{Hom}}_{\mathcal{C}} : \mathcal{C}^{\text{op}} \times \mathcal{C} \to \mathcal{C},$$

where $\mathcal{C}^{\text{op}} \times \mathcal{C}$ is the product category in the category of categories Cat.

Example 159

The category Set of all sets also has an internal Hom, where we can just set

$$\underline{\mathrm{Hom}}_{\text{Set}}(A, B) = \mathrm{Hom}_{\text{Set}}(A, B).$$

It turns out that there is a **currying isomorphism** here: if A, B, C are sets, then we have a bijection

$$\mathrm{Hom}_{\text{Set}}(A \times B, C) \cong \mathrm{Hom}_{\text{Set}}(A, \mathrm{Hom}_{\text{Set}}(B, C)).$$

For example, if we have a function $f : A \to \mathrm{Hom}_{\text{Set}}(B, C)$, we can send it to a function $g : A \times B \to C$, given by

$$g(a, b) = (f(a))(b).$$

Internal Homs will usually give something of this flavor, so we should ask if we can find the analog of this currying isomorphism in R-mod.

Theorem 160

Let R be a commutative ring. There is a functor (called the **tensor product**)

$$\otimes_R : (\text{R-mod}) \times (\text{R-mod}) \to (\text{R-mod}),$$

with the property that

$$\underline{\mathrm{Hom}}_{\text{R-mod}}(A \otimes_R B, C) \cong \underline{\mathrm{Hom}}_{\text{R-mod}}(A, \underline{\mathrm{Hom}}_{\text{R-mod}}(B, C)).$$

In addition, this isomorphism is natural in A, B, C, and this uniquely determines $\otimes_R$.

We typically denote $\otimes_R(A, B)$ as $A \otimes_R B$ instead. Note that $\otimes_R$ and $\underline{\mathrm{Hom}}_{\text{R-mod}}$ are actually **adjoint functors**, which is an important concept in category theory that we might see later. The important thought is that both functors determine internal structure, and they actually determine each other.

The first step is to define a functor $\otimes_R$: R-mod $\times$ R-mod, and then we need to check the isomorphism and naturality conditions.

Definition 161

If A and B are two R-modules, then the **tensor product** $A \otimes_R B$ is the R-module generated by symbols $a \otimes b$ with $a \in A$ and $b \in B$, with the following (quotienting) relations:

- Distributivity: $a \otimes (b + b') = a \otimes b + a \otimes b'$, and $(a + a') \otimes b = a \otimes b + a' \otimes b$.
- Scalar multiplication: $(ra \otimes b) = r(a \otimes b)$ and $a \otimes (rb) = r(a \otimes b)$.

We can check ourselves that this actually defines a functor $\otimes_R$, but once we have our tensor product more concretely, we're ready to sketch the proof of the above theorem.

Proof sketch of Theorem 160. With our definition, a map of R-modules $A \otimes_R B \to C$ is determined by where it sends the generators $a \otimes b$. So given a map $f : A \otimes_R B \to C$, we can define a map $g : A \to \underline{\text{Hom}}_{\text{R-mod}}(B, C)$ via $(g(a))(b) = f(a \otimes b)$, and analogously given g, we can define $f(a \otimes b) = (g(a))(b)$. What we need to check is that the relations on $A \otimes_R B$ are defined to ensure that g is indeed an R-module map if and only if f is, and this is a basic algebraic check. □

Let's look at a few examples that are most relevant for us:

Example 162

If $R = \mathbb{Z}$, what does $A \otimes_{\mathbb{Z}} B$ look like for $\mathbb{Z}$-modules A and B?

Here, A and B are abelian groups, and $A \otimes_{\mathbb{Z}} B$ will be some other abelian group.

- We'll do one example in full detail. If $A = \mathbb{Z}/2\mathbb{Z}$ and $B = \mathbb{Z}/4\mathbb{Z}$, then the tensor product has eight generators

$$0 \otimes 0, \quad 1 \otimes 0, \quad 0 \otimes 1, \quad 1 \otimes 1, \quad 0 \otimes 2, \quad 1 \otimes 2, \quad 0 \otimes 3, \quad 1 \otimes 3,$$

  with various relations. For example,

$$0 \otimes 2 = (0 \cdot 0) \otimes 2 = 0(0 \otimes 2) = 0$$

  by scalar multiplication, and this holds for all other generators with a 0 appearing anywhere. So our tensor product is generated just by $1 \otimes 1, 1 \otimes 2, 1 \otimes 3$, but

$$1 \otimes 1 + 1 \otimes 1 = 1 \otimes (1 + 1) = 1 \otimes 2,$$

  and similarly $1 \otimes 1 + 1 \otimes 1 + 1 \otimes 1 = 1 \otimes 3$, so the other generators can just be written in terms of $1 \otimes 1$. Since $1 \otimes 1 + 1 \otimes 1 + 1 \otimes 1 + 1 \otimes 1 = 1 \otimes 2 + 1 \otimes 2 = 1 \otimes (2 + 2) = 1 \otimes 4 = 0$, our tensor product group is just generated by $1 \otimes 1$, and we have a cyclic group. But in fact

$$1 \otimes 1 + 1 \otimes 1 = (1 + 1) \otimes 1 = 2 \otimes 1 = 0,$$

  so $1 \otimes 1$ has order 2, meaning that

$$\boxed{\mathbb{Z}/2\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/4\mathbb{Z} = \mathbb{Z}/2\mathbb{Z}}.$$

  (We do need to check that $1 \otimes 1$ is nonzero.)

- Next, let's check an example of Theorem 160: let's verify that

$$\boxed{\text{Hom}_{\text{Ab}}(\mathbb{Z}/4\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/2\mathbb{Z}, \mathbb{Z}/2\mathbb{Z})} \cong \text{Hom}_{\text{Ab}}(\mathbb{Z}/2\mathbb{Z}, \mathbb{Z}/2\mathbb{Z})$$

is actually isomorphic to

$$\boxed{\mathrm{Hom}_{\mathrm{Ab}}(\mathbb{Z}/4\mathbb{Z}, \underline{\mathrm{Hom}}_{\mathrm{Ab}}(\mathbb{Z}/2\mathbb{Z}, \mathbb{Z}/2\mathbb{Z}))} \cong \mathrm{Hom}_{\mathrm{Ab}}(\mathbb{Z}/4\mathbb{Z}, \mathbb{Z}/2\mathbb{Z}).$$

(The boxed groups are the ones actually stated in Theorem 160.) So we should have the same number of group homomorphisms from $\mathbb{Z}/2\mathbb{Z}$ to $\mathbb{Z}/2\mathbb{Z}$ as from $\mathbb{Z}/4\mathbb{Z}$ to $\mathbb{Z}/2\mathbb{Z}$, which is indeed true: either we send 1 to 0 or 1 to 1 in both cases.

We'll look at some more example computations next time!

19 October 16, 2020

We calculated the homology groups of $\mathbb{RP}^\infty$ in a combinatorial way, through the nerve $N(BC_2)$, on our homework – the idea was to point out that we have a geometric and a combinatorial-algebraic way to compute the homology groups, but the latter isn't necessarily easier to do by brute force.

Last class, we started discussing the internal Hom and tensor product in the category of R-modules: these were related by a natural isomorphism (which can be used as the definition of a tensor product)

$$\mathrm{Hom}_{\text{R-mod}}(A \otimes_R B, C) \cong \mathrm{Hom}_{\text{R-mod}}(A, \underline{\mathrm{Hom}}_{\text{R-mod}}(B, C)).$$

Concretely, we also wrote down that $A \otimes_R B$ is an R-module generated by symbols $a \otimes b$ with $a \in A, b \in B$, along with certain relations (regarding distributivity and scalar multiplication). One thing we used implicitly is that $A \otimes_R B = B \otimes_R A$, and this is clear in the concrete construction because all of the relations pair up in a symmetric way. But from the abstract internal Hom description, this is perhaps more surprising.

We started calculating some tensor products over $\mathbb{Z}$ last lecture as well: for example, we found that

$$\mathbb{Z}/2\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/4\mathbb{Z} \cong \mathbb{Z}/4\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/2\mathbb{Z} \cong \mathbb{Z}/2\mathbb{Z}.$$

Let's try looking at another example:

Example 163

What is the tensor product $\mathbb{Z}/2\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/3\mathbb{Z}$?

At first glance, we have the six generators $0 \otimes 0, 0 \otimes 1, 0 \otimes 2, 1 \otimes 0, 1 \otimes 1, 1 \otimes 2$. But we know that anything with a 0 is just the zero element, so the only generators that remain are $1 \otimes 1$ and $1 \otimes 2$, and now

$$1 \otimes 1 + 1 \otimes 1 = 1 \otimes (1 + 1) = 1 \otimes 2.$$

So this means our tensor product is generated by $1 \otimes 1$ alone, but now

$$1 \otimes 1 + 1 \otimes 1 + 1 \otimes 1 = 1 \otimes 3 = 0, \quad 1 \otimes 1 + 1 \otimes 1 = 2 \otimes 1 = 0.$$

So the order of this element must divide both 2 and 3, meaning $1 \otimes 1 = 0$. Therefore, this group is the trivial group.

Proposition 164

If A is an abelian group (that is, any $\mathbb{Z}$-module), then $\mathbb{Z}/2\mathbb{Z} \otimes_{\mathbb{Z}} A \cong A/2A$. More generally, we have

$$\mathbb{Z}/n\mathbb{Z} \otimes_{\mathbb{Z}} A \cong A/nA.$$

Proof. This tensor product is generated by elements $1 \otimes a$ for $a \in A$, with the relations that $2(1 \otimes a) = 2 \otimes a = 0$. A similar argument works for general n. $\square$

From here, the next step is to look at the group $\mathbb{Z} \otimes_{\mathbb{Z}} A$. If we have an abelian group B, we can use the abstract characterization to say that (all Homs below are in the Ab category)

$$\boxed{\text{Hom}(A \otimes_{\mathbb{Z}} \mathbb{Z}, B)} \cong \text{Hom}(A, \underline{\text{Hom}}(\mathbb{Z}, B)).$$

Since B is abelian, the set of group homomorphisms from $\mathbb{Z}$ to B is just B itself – it's determined by where 1 is sent. So this is the same as $\boxed{\text{Hom}(A, B)}$, and therefore it makes sense that $A \cong A \otimes_{\mathbb{Z}} \mathbb{Z}$, which we can check by showing the isomorphism of groups $a \mapsto 1 \otimes a$.

Proposition 165

Let R be any ring, and let M be an R-module. Then

$$R \otimes_R M \cong M.$$

A similar argument here relies on the fact that

$$\underline{\text{Hom}}_{\text{R-mod}}(R, N) \cong N,$$

and the rest is similar. So now we can tensor with cyclic abelian groups, as well as with the ring itself, and we can do a bit more theory for understanding now.

Definition 166

Let R be a ring, and let A, B, C be R-modules. A **bilinear map** $f : A \times B \to C$ is a function of sets with the following relations:

- $f(a + a', b) = f(a, b) + f(a', b)$ and $f(a, b + b') = f(a, b) + f(a, b')$ for all $a, a' \in A$ and $b, b' \in B$,
- $f(ra, b) = rf(a, b)$ and $f(a, rb) = rf(a, b)$ for all $a \in A, b \in B, r \in R$.

Theorem 167

Bilinear maps from $A \times B \to C$ are in bijection with R-module maps from $A \otimes_R B \to C$.

Proof. This is basically just the concrete description of a tensor product – specifying a bilinear map $A \times B \to C$ requires us to satisfy certain conditions, which are exactly the same as the tensor product constraints. $\square$

We won't use this way of thinking very much, but it can be useful in certain contexts.

Next, we'll turn our attention to the tensor product of direct sums:

Theorem 168

Let A, B, C be three R-modules. Then

$$(A \oplus B) \otimes_R C \cong (A \otimes_R C) \oplus (B \otimes_R C),$$

with isomorphism given by $(a, b) \otimes c \mapsto (a \otimes c, b \otimes c)$.

We can check on our own that this is an isomorphism by constructing the reverse map. One way of interpreting this is that the operations $(\oplus_R, \otimes_R)$ make R-mod (also denoted Mod_R) into a **categorical ring**. We can check things like the associativity of our tensor product: there is a natural isomorphism

$$(A \otimes_R B) \otimes_R C \cong A \otimes_R (B \otimes_R C).$$

So R-mod form a commutative ring, but in the sense of categories instead of classical algebra. And we can actually completely understand a ring R in terms of its R-modules, which we can look into if we're interested.

And this helps us understand basically all tensor products that we care about in this class: we know how to compute $\otimes_{\mathbb{Z}}$ of finitely generated abelian groups.

Example 169

Suppose we wanted to compute something like

$$(\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/3\mathbb{Z}) \otimes_{\mathbb{Z}} (\mathbb{Z}/6\mathbb{Z} \otimes \mathbb{Z}).$$

We can then distribute the tensor product as

$$= ((\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/3\mathbb{Z}) \otimes \mathbb{Z}/6\mathbb{Z}) \oplus ((\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/3\mathbb{Z}) \otimes \mathbb{Z}),$$

which further simplifies (like in ordinary algebra) to

$$\mathbb{Z}/2\mathbb{Z} \otimes \mathbb{Z}/6\mathbb{Z} \oplus \mathbb{Z}/3\mathbb{Z} \otimes \mathbb{Z}/6\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \otimes \mathbb{Z} \oplus \mathbb{Z}/3\mathbb{Z} \otimes \mathbb{Z},$$

where we use the tensor product as the primary operation in the order of operations, and simplification from the work we've done before yields

$$\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/3\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/3\mathbb{Z}.$$

And certainly with finite type CW complexes, we should only expect to see this kind of finitely generated abelian group. We are interested in homology with coefficients, but often the rings R that we use are $\mathbb{Z}, \mathbb{Q}$, and finite fields like $\mathbb{F}_3$. And modules over a field are just vector space, and now we can figure out how those work too!

Example 170

In $\mathbb{Q}$-modules (rational vector spaces), let's compute

$$(\mathbb{Q} \oplus \mathbb{Q}) \otimes_{\mathbb{Q}} (\mathbb{Q} \otimes \mathbb{Q} \otimes \mathbb{Q}).$$

(This is basically the generic example that we might see with finite-dimensional vector spaces.) If we expand this out with the distributive law, we end up with six copies of $\mathbb{Q} \otimes \mathbb{Q} \cong \mathbb{Q}$, so we end up with a 6-dimensional $\mathbb{Q}$-vector space $\mathbb{Q} \oplus \mathbb{Q} \oplus \mathbb{Q} \oplus \mathbb{Q} \oplus \mathbb{Q} \oplus \mathbb{Q}$. So we basically **multiply the dimensions** when we're working over a field.

This is all we'll say about tensor products of abelian groups for now, and we'll start tackling a slightly fancier question now – tensor products of **chain complexes**.

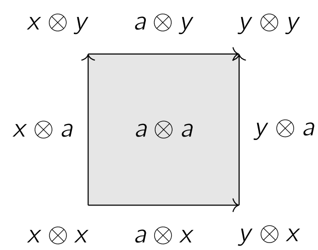
Example 171

Consider the interval $I = [0, 1] = D^1$ as a CW complex with two 0-cells and one 1-cell. We're interested in looking at $I \times I$.

Then our cellular chain complex looks like

$$\mathbb{Z}\{a\} \to \mathbb{Z}\{x, y\},$$

sending a to $y - x$. We can calculate the homology of an interval, which is the same as the homology of a point. So now we can get a product cell decomposition for $I \times I$:



Basically, every cell here comes from taking the product of some cell in the first I and some cell in the second I. And this naming might help us understand the differential map: we can write down equations like

$$\partial(a \otimes x) = y \otimes x - x \otimes x = \text{“}(y - x) \otimes x\text{''} = \text{“}\partial a \otimes x,\text{''}$$

but also

$$\partial(a \otimes a) = a \otimes x + y \otimes a - a \otimes y - x \otimes a = \text{“}(y - x) \otimes a + a \otimes (x - y),\text{''}$$

which looks a lot like $\partial a \otimes a - a \otimes \partial a$. This leads us to the following definition:

Definition 172

Let C_*, D_* be two chain complexes of R-modules. The **tensor product** $C_* \otimes_R D_*$ is a chain complex with groups given by

$$(C_* \otimes_R D_*)_n = \bigoplus_{p+q=n} C_p \otimes D_q$$

and differential map

$$\partial(c_p \otimes d_q) = (\partial c_p \otimes d_q) + (-1)^p (c_p \otimes \partial d_q).$$

This sign coefficient $(-1)^p$ can be obtained by thinking about geometry, but it's important for us in showing that the composition of two ∂ maps is the zero map. And we'll eventually prove that this kind of tensor product chain complex is what we want for computing the homology of a product space:

$$C_*^{\text{cell}}(X) \otimes C_*^{\text{cell}}(Y) \cong C_*^{\text{cell}}(X \times Y).$$

The idea is that the homology groups of $C_*^{\text{cell}}(X) \otimes C_*^{\text{cell}}(Y)$ are computable purely in terms of $H_q(X)$ and $H_q(Y)$, so that will help us look at $X \times Y$. (We don't need to understand the entire cellular chain complexes for X and Y – just the homology groups are sufficient!)

Unfortunately, this construction has some bad properties as well. The main issue is that if C_* and D_* are two chain complexes, the homology of $C_* \otimes D_*$ can't be determined in general from the homology of C_* and the homology of D_*. So it's a very special property of those cellular chain complexes that come up in our study – chain complexes from other sources may not have homology groups coming from the homology group of the two pieces.

In addition, there isn't an obvious "internal Hom" in chain complexes: if we look at all the maps between two chain complexes, there isn't an obvious way to make that into a chain complex. So that might make it more confusing to deal with a tensor product description, especially with the abstract definition! This property is actually related to the last one – next class, we'll talk about why the chain complexes arising from CW complexes behave better than other ones.

20 October 19, 2020

Problem set 4, giving us more practice with cellular homology, is now posted – as always, it will be due in about two weeks.

We're going to continue discussing the algebra of tensor products today. Recall that we were motivated last time by the product CW structure on a product of two CW complexes to define a tensor product of two chain complexes of R-modules, with nth group given by

$$(C_* \otimes_R D_*)_n = \bigoplus_{p+q=n} C_p \oplus_R D_q,$$

and boundary maps given by

$$\partial(c_p \otimes d_q) = (\partial c_p) \otimes d_q + (-1)^p (c_p \otimes (\partial d_q)).$$

Recall that the term $(-1)^p$ is necessary for the partial maps to form a chain complex. But there are still some confusing features – it turns out that the homology of a tensor product of this form is only simply related to the homology of the individual chain complexes C_* and D_* when the origin is from CW complexes. In addition, the set of maps $\hom(C_*, D_*)$ is not obviously a chain complex, so we haven't quite figured out how to relate tensor products to our internal Homs.

So we'll jump into some more algebraic theory today. Let's start with a short exact sequence

$$0 \to A \to B \to C \to 0,$$

and consider some other R-module. It makes sense to then consider the sequence

$$0 \to A \otimes_R M \to B \otimes_R M \to C \otimes_R M \to 0$$

and ask whether it is exact. The answer turns out to be **no**, and we can see this concretely:

Example 173

Take $R = \mathbb{Z}$, and consider the short exact sequence

$$0 \to \mathbb{Z} \xrightarrow{2} \mathbb{Z} \xrightarrow{q} \mathbb{Z}/2\mathbb{Z} \to 0$$

where q denotes the quotient map.

If we tensor this short exact sequence with $\mathbb{Z}/2\mathbb{Z}$, we find a sequence

$$0 \to \mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/2\mathbb{Z} \to \mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/2\mathbb{Z} \to \mathbb{Z}/2\mathbb{Z} \otimes \mathbb{Z}?2\mathbb{Z} \to 0,$$

which is isomorphic to

$$0 \to \mathbb{Z}/2\mathbb{Z} \to \mathbb{Z}/2\mathbb{Z} \to \mathbb{Z}/2\mathbb{Z} \to 0.$$

(For the first two groups, this was because tensoring $\mathbb{Z} \otimes_{\mathbb{Z}} A$ always gives us back A, and for the third, recall that $\mathbb{Z}/2\mathbb{Z} \otimes_{\mathbb{Z}} A \cong A/2A$.) But now the multiplication by 2 map becomes the 0 map, which is not injective, so we cannot have exactness.

However, there is an important result that helps our tensor product behave a little bit nicer with exactness:

Theorem 174

Let M be an R-module. Then the functor

$$\cdot \otimes_R M : \text{R-mod} \to \text{R-mod}$$

is **right exact**, meaning that if $A \to B \to C \to 0$ is an exact sequence of R-modules, so is $A \otimes_R M \to B \otimes_R M \to C \otimes_R M \to 0$.

Remember that exactness failed at the first place in our example above, so we've gone around this issue by only look at the "right" part of the sequence. (In fact, $A \to B \to C \to 0$ is exact if and only if $A \otimes_R M \to B \otimes_R M \to C \otimes_R M \to 0$ is exact for every R-module M, because we can just take $M = R$.)

Fact 175

It's important to remember that whenever we have a theorem or result about tensor products, we can also restate it using internal Homs – they can often be easier to prove that way.

Theorem 176

A sequence of R-modules $A \to B \to C \to 0$ is exact if and only if

$$0 \to \underline{\text{Hom}}_{\text{R-mod}}(C, N) \to \underline{\text{Hom}}_{\text{R-mod}}(B, N) \to \underline{\text{Hom}}_{\text{R-mod}}(A, N)$$

is exact for all R-modules N.

In particular, we should notice the **opposite functoriality** here – the arrows have been flipped.

Proof that Theorem 176 implies Theorem 174. Assume that $A \to B \to C \to 0$ is exact. To check that the tensor product sequence $A \otimes_R M \to B \otimes_R M \to C \otimes_R M \to 0$ is exact, it suffices to check (by Theorem 176) that

$$0 \to \underline{\text{Hom}}_{\text{R-mod}}(C \otimes_R M, N) \to \underline{\text{Hom}}_{\text{R-mod}}(B \otimes_R M, N) \to \underline{\text{Hom}}_{\text{R-mod}}(A \otimes_R M, N)$$

is exact for every N. And this is equivalent to wanting

$$0 \to \underline{\text{Hom}}(C, \underline{\text{Hom}}(M, N)) \to \underline{\text{Hom}}(B, \underline{\text{Hom}}(M, N)) \to \underline{\text{Hom}}(A, \underline{\text{Hom}}(M, N))$$

to be exact for every N, but now we can apply Theorem 176) with N replaced with $\underline{\text{Hom}}(M, N)$. ☐

Proof of Theorem 176. We'll prove one direction (the other is left as an exercise). Exactness at $\underline{\text{Hom}}(C, N)$ is equivalent to the map $\underline{\text{Hom}}(C, N) \to \underline{\text{Hom}}(B, N)$ being injective. In other words, by functoriality of our internal Hom, we want to know whether a map $C \to N$ is determined by the composite map $B \to C \to N$. But that's true because $B \to C$ is surjective by exactness (at C in our original sequence), meaning that we can look at the image of the composite map of some b that maps to c.

So now it remains to check exactness at $\underline{\text{Hom}}(B, N)$, which can be restated as follows: $f : B \to N$ restricts to the zero map $A \to B \to N$ if and only if we can write f as a composite map $B \to C \to N$.

- If we have a composite map $B \to C \to N$, then we can look at $A \to B \to C \to N$, and the map $A \to C$ (and thus also the map $A \to N$) is the zero map by exactness of the original sequence at B.

- Next, if $f : B \to N$ restricts to the zero map $A \to B \to N$, we want to find a map $g : B \to C$ so that f is the composite $B \to C \xrightarrow{g} N$. Recall that C is the quotient of B by the image of A, so we just need to make sure that f's kernel contains the image of A to have a well-defined composite map. And this is true because $A \to B \to N$ is the zero map, so $B \to N$ sends everything in $\text{im}(A)$ to zero.

□

We've now proved that if $0 \to A \to B \to C \to 0$ is exact, then $A \otimes_R M \to B \otimes_R M \to C \otimes_R M \to 0$ is exact, but we don't necessarily have exactness $0 \to A \otimes_R M \to B \otimes_R M$. But notice that this sequence will stay exact if we use $M = R$ or $M = R \oplus R$ – in the latter case, we end up with a sequence $0 \to A \oplus A \to B \oplus B \to C \oplus C \to 0$. In general, if M is a free module (such as a vector space over a field), we preserve exactness (not just right exactness).

And to connect this back to topology, note that if we have a topological space X, we have a chain complex of **free** R-modules $C_*^{\text{cell}}(X; R)$, and perhaps that will start giving us a sense of what's special about these particular chain complexes.

21 October 21, 2020

Last time, we talked about the way tensor products interact with exactness (which encodes a lot of useful algebraic facts). Specifically, if M is an R-module, then the tenso r product functor

$$\cdot \otimes_R M : \text{R-mod} \to \text{R-mod}$$

is right exact but not necessarily exact. But we mentioned that for **free** R-modules M (which always happens when R is a field, for example), then tensoring with M is exact.

We primarily care about chain complexes because they help us encode homology, and that plays into the next definition:

Definition 177

Suppose C_* and D_* are two chain complexes of R-modules. A chain map $f : C_* \to D_*$ is a **quasi-isomorphism** if $H_q(f)$ is an isomorphism for each integer q.

In other words, the map is an isomorphism at the level of homology groups. In particular, having this quasi-isomorphism tells us that C and D have the same homology, but there isn't always a quasi-isomorphism between two such chain complexes (so quasi-isomorphism is stronger).

Example 178

Suppose $R = \mathbb{Z}$, and we have a two-term chain complex with $\mathbb{Z}$s in degree 1 and 0:

$$C_* = \cdots \to 0 \to 0 \to \mathbb{Z} \xrightarrow{5} \mathbb{Z} \to 0 \to 0 \to \cdots .$$

We also have another chain complex

$$D_* = \cdots \to 0 \to 0 \to \mathbb{Z}/5\mathbb{Z} \to 0 \to 0 \to \cdots .$$

Then C_* and D_* have the same homology groups (we can check this ourselves), and in fact we have the following chain map which is a quais-isomorphism:

$$\begin{array}{ccccccccccc}
\cdots & \longrightarrow & 0 & \longrightarrow & \mathbb{Z} & \xrightarrow{5} & \mathbb{Z} & \longrightarrow & 0 & \longrightarrow & \cdots \\
 & & \downarrow & & \downarrow & & \downarrow{\scriptstyle q} & & \downarrow & & \\
\cdots & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & \mathbb{Z}/5\mathbb{Z} & \longrightarrow & 0 & \longrightarrow & \cdots
\end{array}$$

We can check that this diagram does commute, so C_* and D_* are **quasi-isomorphic**. And this construction motivates the next idea:

Definition 179

Let M be an R-module. A **free resolution** of M is a chain complex C_* of free R-modules and a quasi-isomorphism $C_* \to M$, where we think of M as a chain complex concentrated in degree 0 (that is, we have M at degree 0 and 0 everywhere else).

The example above shows us that C_* is a free resolution of $M = \mathbb{Z}/5\mathbb{Z}$ for $R = \mathbb{Z}$, and we'll work a bit more with this idea now.

Example 180

If R is a field, then every module is itself a free resolution, because all R-modules are free.

Example 181

Suppose $R = \mathbb{Z}$ (this is basically the only ring that's not a field that we'll need to worry about for now), and take $M = \mathbb{Z}/3\mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$.

We can write down a free resolution in the folloiwng diagram:

$$\begin{array}{ccccccccccc}
\cdots & \longrightarrow & 0 & \longrightarrow & \mathbb{Z}\oplus\mathbb{Z} & \xrightarrow{f} & \mathbb{Z}\oplus\mathbb{Z}\oplus\mathbb{Z} & \longrightarrow & 0 & \longrightarrow & \cdots \\
 & & \downarrow & & \downarrow & & \downarrow{\scriptstyle q} & & \downarrow & & \\
\cdots & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & \mathbb{Z}/3\mathbb{Z}\oplus\mathbb{Z}\oplus\mathbb{Z}/2\mathbb{Z} & \longrightarrow & 0 & \longrightarrow & \cdots
\end{array}$$

Here, in the map f, we send $(1, 0)$ to $(3, 0, 0)$, and we send $(0, 1)$ to $(0, 0, 2)$. We can check that the homology of the top row and bottom row agree, and that the square commutes, so we do have a quasi-isomorphism. And this example is basically the generic one – we can always find a two-term free resolution by taking a surjection from a free module and adding some relations with the map f.

Remark 182. *Note that free resolutions aren't unique: consider the following diagram.*

$$\begin{array}{ccccccccccc}
\cdots & \longrightarrow & \mathbb{Z} & \xrightarrow{f} & \mathbb{Z}\oplus\mathbb{Z}\oplus\mathbb{Z} & \xrightarrow{g} & \mathbb{Z}\oplus\mathbb{Z}\oplus\mathbb{Z} & \longrightarrow & 0 & \longrightarrow & \cdots \\
 & & \downarrow & & \downarrow & & \downarrow{\scriptstyle q} & & \downarrow & & \\
\cdots & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & \mathbb{Z}/3\mathbb{Z}\oplus\mathbb{Z}\oplus\mathbb{Z}/2\mathbb{Z} & \longrightarrow & 0 & \longrightarrow & \cdots
\end{array}$$

If f sends 1 to $(0, 0, 1)$, and g sends $(1, 0, 0), (0, 1, 0), (0, 0, 1)$ to $(3, 0, 0), (0, 0, 2), (0, 0, 0)$ respectively, then we also have a free resolution of the same $M = \mathbb{Z}/3\mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$. This isn't a minimal resolution – there's an extra relation in the map from degree 1 to degree 0 – but it is valid.

However, there won't always be such a compact solution for more complicated rings:

Fact 183

Suppose $R = \mathbb{Q}[t]/t^2$, and let M be the module $\mathbb{Q}$ where t acts by (multiplication by) 0. This also has a free resolution, but even the "smallest" one is infinite. But we won't really need to worry about those kinds of rings.

Theorem 184 (Fundamental theorem of homological algebra)

Let N and M be R-modules. Let

$$\begin{array}{ccccccc} \cdots & \longrightarrow & F_2 & \longrightarrow & F_1 & \longrightarrow & F_0 \\ & & \downarrow & & \downarrow & & \downarrow \\ \cdots & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & N \end{array} \quad \text{and} \quad \begin{array}{ccccccc} \cdots & \longrightarrow & E_2 & \longrightarrow & E_1 & \longrightarrow & E_0 \\ & & \downarrow & & \downarrow & & \downarrow \\ \cdots & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & M \end{array}$$

be free resolutions of N and M, respectively. Then any R-module map $f : N \to M$ lifts to a chain map $f_* : F_* \to E_*$, and f_* is unique up to chain homotopy.

Let's see what this theorem means in practice:

Example 185

Take $R = \mathbb{Z}$, and consider the abelian group map $f : \mathbb{Z}/2\mathbb{Z} \to \mathbb{Z}/6\mathbb{Z}$ (so $N = \mathbb{Z}/2\mathbb{Z}$ and $M = \mathbb{Z}/6\mathbb{Z}$) that takes 1 to 3.

A free resolution of $\mathbb{Z}/2\mathbb{Z}$ has top row

$$F_* = \cdots \to 0 \to \mathbb{Z} \xrightarrow{2} \mathbb{Z} \to 0 \to \cdots,$$

and we have a similar chain complex E_* for $\mathbb{Z}/6\mathbb{Z}$. So then we can get a chain map $F_* \to E_*$ as shown:

$$\begin{array}{ccccccccccc} \cdots & \longrightarrow & 0 & \longrightarrow & \mathbb{Z} & \xrightarrow{2} & \mathbb{Z} & \longrightarrow & 0 & \longrightarrow & \cdots \\ & & \downarrow{\scriptstyle 0} & & \downarrow{\scriptstyle 1} & & \downarrow{\scriptstyle 3} & & \downarrow{\scriptstyle 0} & & \\ \cdots & \longrightarrow & 0 & \longrightarrow & \mathbb{Z} & \xrightarrow{6} & \mathbb{Z} & \longrightarrow & 0 & \longrightarrow & \cdots \end{array}$$

The squares all commute, so this chain map gives us a map $H_0(f_*) : H_0(F_*) \to H_0(E_*)$. And $H_0(F_*)$ is isomorphic to $\mathbb{Z}/2\mathbb{Z}$, $H_0(E_*)$ is isomorphic to $\mathbb{Z}/6\mathbb{Z}$, and thus $H_0(f_*)$ is exactly the map f that we started with!So understanding maps between R-modules is the same as understanding maps between free resolutions of R-modules, and furthermore, we only need to understand maps up to chain homotopy.

Proof sketch of Theorem 184. We can build up a map between the free resolutions inductively, showing that there is only one choice up to chain homotopy at every stage.

The first step is to create the dashed arrow in the following square:

$$\begin{array}{ccc} F_0 & \overset{f_0}{\dashrightarrow} & E_0 \\ \downarrow{\scriptstyle \varepsilon_N} & & \downarrow{\scriptstyle \varepsilon_M} \\ N & \xrightarrow{f} & M \end{array}$$

We know that F_0 is free on some set S_0 by definition, so for each $s_0 \in S_0$, we can define $f_0(s_0)$ to be any arbitrary element of E_0 such that

$$\varepsilon_M(f_0(s_0)) = f(\varepsilon_N(s_0)).$$

(This can always be done because ε_M is surjective, since $H_0(F_*) = F_0/\text{im}(F_1 \to F_0)$ is isomorphic to N. Similarly, this tells us that ε_N is also surjective.) So now we can extend our diagram as shown:

$$\begin{array}{ccc} F_1 & \overset{f_1}{\dashrightarrow} & E_1 \\ \downarrow & & \downarrow \\ \ker(\varepsilon_N) & \xrightarrow{g_0} & \ker(\varepsilon_M) \\ \downarrow & & \downarrow \\ F_0 & \xrightarrow{f_0} & E_0 \\ \downarrow \varepsilon_N & & \downarrow \varepsilon_M \\ N & \xrightarrow{f} & M \end{array}$$

The top two vertical maps are surjective, so this helps us create $F_1 \to E_1$, and so on. The rest of the proof is left for us to think about – it may show up in our homework as well. $\square$

In practice, this theorem is really easy to use – we are guaranteed a chain map inducing in homology from the module that we started with. (We'll see this in action soon.)

Definition 186 (Sketchy for now)

Let R be a commutative ring, and let ch(R-mod) be the category of chain complexes of R-modules. The **derived category** of R, denoted $D(R)$, is a category obtained from ch(R-mod) by formally inverting all quasi-isomorphisms.

Quasi-isomorphisms aren't actually isomorphisms, but we're changing the category we're talking about. So the objects of $D(R)$ are chain complexes, but there are **many more morphisms** than before. Specifically, for any quasi-isomorphism $f_* : C_* \to D_*$, we have a formal inverse $g : D_* \to C_*$ in $D(R)$. (This is kind of like rigorously going from the natural numbers to the integers – we just "add in inverses" to get the negative numbers.) So this means we also add morphisms that are the composition of a quasi-isomorphism and a chain map, and so on – the goal is to **force quasi-isomorphisms to become isomorphisms** in the minimal way.

Fact 187

The actual construction of $D(R)$ is beyond the scope of this class, because there are set-theory technicalities (such as having a set of maps versus a class of maps). But the point is that ch(R-mod) doesn't matter to us – $D(R)$ does, because we're working with homology.

Example 188

In $D(R)$, every object is isomorphic to a chain complex of free R-modules, and free resolutions give us an example of this.

Our next steps will be to learn some general facts about $D(R)$ and then prove a few consequences of these facts which can be stated without reference to $D(R)$ itself. (This "derived category" doesn't actually show up in Hatcher or in Miller's notes because it's more technical, but taking this path will help motivate us to understand why these consequences exist.)

For example, consider the following construction of **tensor products in** $D(R)$**.** If C_* and D_* are two chain complexes, the **derived tensor product** $C_* \otimes^{\mathbb{L}} D_*$ is defined as follows:

- Replace C_*, D_* with quasi-isomorphic chain complexes of free modules C'_*, D'_*.

- Take the ordinary tensor product $C'_* \otimes_R D'_*$.
- Check that the result is well-defined up to quasi-isomorphism.

Since we're avoiding the mention of quasi-isomorphism, we're going to prove a consequence of this claim instead. Next time, we'll show that if M and N are R-modules, then the groups $H_q(M \otimes^{\mathbb{L}} N)$ is well-defined for every q. (These groups are called **Tor groups**.) And we'll do this without explicit mention to the derived category.

22 October 23, 2020

Last time, we introduced the **derived category** $D(R)$ for a commutative ring R. Since quasiisomorphisms induce isomorphisms in homology, the main point of the construction is that two chain complexes that are isomorphic in $D(R)$ must have isomorphic homology R-modules.

We then started seeing how the tensor product plays into this: the **derived tensor product** is denoted $\otimes^{\mathbb{L}}$ or $\otimes^{\mathbb{L}}_R$, and we described the construction for it last time. (The reason for the symbol $\mathbb{L}$ is historical – it has to do with the "**left** adjoint" in the currying isomorphism.) One useful fact, in the case where at least one of C_* and D_* is a chain complex of free R-modules, is that

$$C_* \otimes^{\mathbb{L}}_R D_* \cong C_* \otimes_R D_*.$$

And as we mentioned last time, we won't try to go too far into this statement directly, since we aren't talking about $D(R)$ directly. But we'll prove some consequences of this fact:

Definition 189

Let M and N be two R-modules. The group $\mathbf{Tor}_i(M, N)$ is defined as the homology group $H_i(M \otimes^{\mathbb{L}}_R N)$, where M and N are chain complexes concentrated at degree 0.

We'll prove soon that $\mathrm{Tor}_i(M, N)$ is well-defined rigorously in this lecture (that is, it doesn't take on different forms if we pick different free resolutions), but for now it's worth getting used to the algorithmic practices of computation:

Example 190

Suppose $R = \mathbb{Z}$. What is $\mathbb{Z}/2\mathbb{Z} \otimes^{\mathbb{L}}_{\mathbb{Z}} \mathbb{Z}/4\mathbb{Z}$?

There are different ways we can compute this, based on which module we choose to make a free resolution for and also how we do it. So we'll do this in a few different ways – one idea is to write

$$\mathbb{Z}/2\mathbb{Z} \otimes^{\mathbb{L}} \mathbb{Z}/4\mathbb{Z} \cong (\mathbb{Z} \xrightarrow{2} \mathbb{Z}) \otimes \mathbb{Z}/4\mathbb{Z},$$

and then we can take the tensor product from here to get nonzero things in degree 1 and 0:

$$\cong \to 0 \to \mathbb{Z}/4\mathbb{Z} \xrightarrow{2} \mathbb{Z}/4\mathbb{Z} \to 0 \to \cdots.$$

So now $\mathrm{Tor}_0(\mathbb{Z}/2\mathbb{Z}, \mathbb{Z}/4\mathbb{Z})$ is $\mathbb{Z}/4\mathbb{Z}$ mod the image of the multiplication-by-2 map, which is $\mathbb{Z}/2\mathbb{Z}$. Similarly, we find that $\mathrm{Tor}_1(\mathbb{Z}/2\mathbb{Z}, \mathbb{Z}/4\mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z}$, and there is no homology anywhere else.

Alternatively, we could have computed this tensor product in another way by resolving the $\mathbb{Z}/4\mathbb{Z}$, so that

$$\mathbb{Z}/2\mathbb{Z} \otimes^{\mathbb{L}} \mathbb{Z}/4\mathbb{Z} \cong \mathbb{Z}/2\mathbb{Z} \otimes (\mathbb{Z} \xrightarrow{4} \mathbb{Z}),$$

and then computing this ordinary tensor product gives us

$$\cong \cdots \to \mathbb{Z}/2\mathbb{Z} \xrightarrow{4} \mathbb{Z}/2\mathbb{Z} \to \cdots,$$

again with zeros beyond here. And again Tor_0 and Tor_1 are $\mathbb{Z}/2\mathbb{Z}$, because the multiplication-by-4 map is just the zero map on $\mathbb{Z}/2\mathbb{Z}$. So indeed the homology groups are the same, and that's what we should believe if we believe in the derived category $D(\mathbb{Z})$.

Finally, let's compute this in a third, "less efficient way" by resolving both sides:

$$\mathbb{Z}/2\mathbb{Z} \otimes^{\mathbb{L}} \mathbb{Z}/4\mathbb{Z} \cong (\mathbb{Z}\{a\} \xrightarrow{2} \mathbb{Z}\{b\}) \otimes (\mathbb{Z}\{c\} \xrightarrow{4} \mathbb{Z}\{d\}),$$

(the a, b, c, d just to make our notation easier, but those groups are all isomorphic to $\mathbb{Z}$) and the tensor product of these chain complexes is

$$\mathbb{Z}\{a \otimes c\} \to \mathbb{Z}\{b \otimes c, a \otimes d\} \to \mathbb{Z}\{b \otimes d\}$$

(with these groups being in degree $2, 1, 0$), with one of the differentials given by

$$\partial(a \otimes c) = (\partial a \otimes c) + (-1)(a \otimes \partial c) = (2b \otimes c) + (-1)(a \otimes 4c) = 2(b \otimes c) - 4(a \otimes d),$$

where we've used the formulas ∂ from the resolutions of our two modules. (Remember the $\pm$ sign comes from the original degree of the left term.) We can also calculate

$$\partial(b \otimes c) = \partial b \otimes c + b \otimes \partial c = 0 \otimes c + b \otimes 4d = 4(b \otimes d),$$

$$\partial(a \otimes d) = \partial a \otimes d - a \otimes \partial d = 2b \otimes d - a \otimes 0 = 2(b \otimes d),$$

and now we can confirm that this is a chain complex (that is, we haven't made any calculation errors), because

$$\partial\partial(a \otimes c) = \partial(2(b \otimes c) - 4(a \otimes d)) = 2 \cdot 4(b \otimes d) - 4 \cdot 2(b \otimes d) = 0.$$

And now we're ready to compute the homology of the chain complex: H_0 is $\mathbb{Z}\{b \otimes d\}/\text{im}(\partial_1) = \mathbb{Z}\{b \otimes d\}/\mathbb{Z}\{2(b \otimes d)\} \cong \mathbb{Z}/2\mathbb{Z}$, H_1 is the kernel of ∂_1 mod the image of ∂_2, which is $\mathbb{Z}\{b \otimes c - 2(a \otimes d)\}/\mathbb{Z}\{2(b \otimes c) - 4(a \otimes d) \cong \mathbb{Z}/2\mathbb{Z}$, and H_2 is zero because ∂_2 is injective. So we've recovered the same result as before with more calculation!

Example 191

Next, let's compute the derived tensor product $\mathbb{Z}/2\mathbb{Z} \otimes^{\mathbb{L}} \mathbb{Z}/3\mathbb{Z}$.

The most efficient way is to write it as

$$\cong (\mathbb{Z} \xrightarrow{2} \mathbb{Z}) \otimes \mathbb{Z}/3\mathbb{Z} \cong \mathbb{Z}/3\mathbb{Z} \xrightarrow{2} \mathbb{Z}/3\mathbb{Z},$$

and because the multiplication-by-2 map is surjective, we find that the Tor groups are all just zero in this case.

Example 192

The next idea to look at is $\mathbb{Z}/3\mathbb{Z} \otimes^{\mathbb{L}} \mathbb{Z}$.

This time, we already have a free resolution, and tensoring by $\mathbb{Z}$ does nothing in this case. So this is a chain complex with $\mathbb{Z}/3\mathbb{Z}$ concentratred at degree 0, and thus the Tor groups are $\mathbb{Z}/3\mathbb{Z}$ in degree 0 and 0 otherwise.

Example 193

If R is a field, and V, W are vector spaces over R, then $\text{Tor}_0(V, W) \cong V \otimes_R W$ (a vector space with dimension $\dim(V)\dim(W)$) and $\text{Tor}_i(V, W) \cong 0$ for all $i \neq 0$.

In the logic above, we're using the fact that every module over a field is its own free resolution.

Proof that the Tor *groups are well-defined.* Suppose that M and N are two R-modules, and F_* and F'_* are two free resolutions of M. We need to prove that (here we use the regular tensor products) $F_* \otimes_R N$ and $F'_* \otimes_R N$ have the same homology (Tor) groups.

We can in fact show that we have a chain homotopy between the two chain complexes, which is enough to show that they have the same homology groups. And this goes back to the **fundamental theorem of homological algebra**: up to chain homotopy, there exists a (unique) map $f : F_* \to F'_*$ that lifts 1_M, and there also exists a (unique) map $g : F'_* \to F_*$ that lifts 1_M as well. So $f \circ g$ is a map from $F'_* \to F'_*$ that lifts 1_M, so the fundamental theorem of homological algebra tells us that $f \circ g$ must be chain homotopic to the identity (which also lifts 1_M). So that means that

$$(f \circ g) \otimes_R N = (f \otimes N) \circ (g \otimes N)$$

is chain homotopic to $1_{F'_* \otimes N}$, and the same argument works for showing that $(g \circ f) \otimes_R N$ is chain homotopic to $1_{F_* \otimes N}$. This means $g \otimes_R N$ and $f \otimes_R N$ are inverses up to chain homotopy, meaning $F_* \otimes_R N$ and $F'_* \otimes_R N$ are chain homotopy equivalent. $\square$

Let's now take a look at a related construction of Tor by studying a bit about internal Homs. If F_* is a chain complex of free R-modules, and N is an R-module, we can form a new chain complex,

$$\underline{\text{Hom}}_{D(R)}(F_*, N),$$

which is basically the internal Hom in the derived category, but notice the negative sign:

$$\underline{\text{Hom}}_{D(R)}(F_*, N)_n = \underline{\text{Hom}}_{\text{R-mod}}(F_{-n}, N).$$

Let's illustrate this more clearly:

Example 194

Suppose $R = \mathbb{Z}$, and our F_* chain complex has only terms in degree 1 and 0:

$$F_* = \cdots \to 0 \to \mathbb{Z} \xrightarrow{4} \mathbb{Z} \to 0 \to \cdots.$$

Take $N = \mathbb{Z}/2\mathbb{Z}$. Then

$$\underline{\text{Hom}}_{D(\mathbb{Z})}(F_*, N) \cong \cdots \to 0 \to \mathbb{Z}/2\mathbb{Z} \xrightarrow{\partial} \mathbb{Z}/2\mathbb{Z} \to 0 \to \cdots,$$

where we have terms in **degrees** 0 **and** -1, instead of 1 and 0. And the differential map ∂ exists here, because a map from the degree-0 $\mathbb{Z}$ to $\mathbb{Z}/2\mathbb{Z}$ can be composed with the multiplication-by-4 map to get us a map from the degree-1 $\mathbb{Z}$ to $\mathbb{Z}/2\mathbb{Z}$ – this construction comes from the boundary map in F_*. (This is related to the **contravariance** of the Hom functor.) So if M and N are R-modules, then we can compute $\underline{\text{Hom}}_{D(R)}(M, N)$ by replacing M with a free resolution and then doing the kind of calculations above.

Definition 195

Let M and N be R-modules. The group $\text{Ext}^i_R(M, N)$ is the $-i$th homology group $H_{-i}(\underline{\text{Hom}}_{D(R)}(M, N))$.

In other words, this is the homology group $H_{-i}(\underline{\text{Hom}}_{D(R)}(F_*, N))$. We'll need to prove on our homework that, similar to Tor, this construction doesn't depend on the free resolution F_* of M, and this comes from the fundamental theorem of homological algebra again. It turns out that these Ext groups is more natural than a lot of the other things going on here, and that's because we do have an internal Hom here. We'll be able to use these Ext and Tor groups to compute homology with coefficients, and we'll see that next time!

23 October 26, 2020

Last class, we defined and worked a bit with the Ext and Tor functors. Now that we've done a bunch of algebra, we can answer some topological questions that motivated that discussion, particularly dealing with homology with coefficients.

Recall that the nth group of the chain complex of R-modules $S_*(X; R)$ is the free R-module generated by $\text{Sing}_n(X)$, with boundary map given by the usual alternating sum of the d_i maps, which means that $S_*(X) = S_*(X; \mathbb{Z})$ by definition.

Note that this chain complex $S_*(X; R)$ can be alternatively described as $S_*(X) \otimes_{\mathbb{Z}} R$ because S_* is made out of free $\mathbb{Z}$-modules, and tensoring with $\mathbb{Z}$ turns them into Rs. And this ordinary tensor product also calculates the derived tensor product $S_*(X) \otimes^{\mathbb{L}}_{\mathbb{Z}} R$ because S_* is free. But in fact, we can make a more general definition:

Definition 196

Let M be an abelian group, and let X be a topological space. Then we can define

$$S_*(X; M) = S_*(X) \otimes_{\mathbb{Z}} M, \quad H_q(X; M) = H_q(S_*(X; M)).$$

In other words, we replace the ring R with an arbitrary abelian group – phrasing in terms of tensor products mean we never use the ring structure at all. We still have $M = \mathbb{Z}$ or a field as the most interesting cases, though.

Fact 197

If R is a commutative ring and M is an R-module, then $H_q(X; M)$ will acquire the structure of an R-module. In particular, if $M = R$ itself is a ring, then $H_q(X; R)$ is itself an R-module.

At this point, let's also introduce the new, related idea of **cohomology**, which will come up soon. Suppose that we have an abelian group M and an topological space X. Then we can make a new chain complex concentrated in **non-positive** degrees, where the $(-n)$th term is $\underline{\text{Hom}}_{\mathbb{Z}\text{-mod}}(S_n(X), M)$ (itself an abelian group). This in fact helps us calculate groups related to the derived category, since this calculates $\underline{\text{Hom}}_{D(\mathbb{Z})}(S_*(X), M)$.

Definition 198

The **qth cohomology group** $H^q(X; M)$ of a topological space X is the $(-q)$th homology group of the chain complex $\underline{\text{Hom}}_{D(\mathbb{Z})}(S_*(X), M)$.

It's interesting to then think about homology with coefficients and cohomology with coefficients, and ask whether these tell us more than homology with integers. And it turns out the answer is no – homology with integer coefficients still determines everything we're talking about, but it's sometimes easier to compute (so still useful).

To understand how this determination happens, we'll talk about the **universal coefficient theorems**. We'll start with the **cohomology version**:

Theorem 199

Let X be a topological space, and let M be any abelian group. Then for any integer q, there is an isomorphism

$$H^q(X; M) \cong \underline{\text{Hom}}_{\text{Ab}}(H_q(X), M) \oplus \text{Ext}^1_{\mathbb{Z}}(H_{q-1}(X), M)$$

The point is that the right side only has to do with integer homology groups, but the left side has coefficients in M. Somehow, cohomology comes from mapping homology into M, but we also have an extra error term (the Ext one).

Let's now state the homology version in more generality and spend more time talking about it:

Theorem 200

Let C_* be a chain complex of free $\mathbb{Z}$-modules (such as $S_*(X)$ or $C_*^{\text{cell}}(X)$), and let M be an abelian group. Then for any integer q,

$$H_q(C_* \otimes_{\mathbb{Z}} M) \cong H_q(C_*) \otimes_{\mathbb{Z}} M \oplus \text{Tor}_1^{\mathbb{Z}}(H_{q-1}(C_*), M).$$

Remark 201. *The isomorphism above* **is not natural***, though – even though we have a natural* **short** *exact sequence*

$$0 \to H_q(C_*) \otimes_{\mathbb{Z}} M \to H_q(C_* \otimes_{\mathbb{Z}} M) \to \text{Tor}_1^{\mathbb{Z}}(H_{q-1}(C_*), M),$$

and in fact the term in the middle is the direct sum of the outside terms, that doesn't mean the direct sum decomposition is natural in the isomorphism above. (For an example where this fails, we can see example 24.2 of Miller's notes.)

Example 202

Consider the real projective plane $X = \mathbb{RP}^2$ – recall that we have a standard cell structure $C_*^{\text{cell}}(\mathbb{RP}^2) = C_*^{\text{cell}}(\mathbb{RP}^2; \mathbb{Z})$, which is $\mathbb{Z} \xrightarrow{2} \mathbb{Z} \xrightarrow{0} \mathbb{Z}$.

As a reminder, this cellular chain complex allows us to compute

$$\boxed{H_q(\mathbb{RP}^2) = \begin{cases} \mathbb{Z} & q = 0 \\ \mathbb{Z}/2\mathbb{Z} & q = 1 \\ 0 & \text{otherwise.} \end{cases}}$$

Let's look again at how to compute $H_q(\mathbb{RP}^2, M)$ for some other group M. For example, recall that we calculated $H_q(\mathbb{RP}^2, \mathbb{F}_2)$ by calculating $C_*^{\text{cell}}(\mathbb{RP}^2) \otimes_{\mathbb{Z}} \mathbb{F}_2$ directly, and that gives us

$$\mathbb{F}_2 \xrightarrow{2} \mathbb{F}_2 \xrightarrow{0} \mathbb{F}_2 \implies H_q(\mathbb{RP}^2; \mathbb{F}_2) = \begin{cases} \mathbb{F}_2 & q = 0, 1, 2 \\ 0 & \text{otherwise.} \end{cases}$$

But now let's try another method, which is to use the universal coefficients method above. The advantage here is that we don't need to know the cellular chain complex to do the tensor product – this time, we **only need to assume the boxed homology groups above**. For example,

$$H_2(\mathbb{RP}^2; \mathbb{F}_2) \cong \left(H_2(\mathbb{RP}^2) \otimes_{\mathbb{Z}} \mathbb{F}_2\right) \oplus \text{Tor}_1(H_1(\mathbb{RP}^2), \mathbb{F}_2).$$

(So the homology with $\mathbb{F}_2$ coefficients can be determined in terms of the ordinary H_2 and H_1 homology groups.) Plugging this in and remembering that $\mathbb{F}_2$ is the same as $\mathbb{Z}/2\mathbb{Z}$,

$$H_2(\mathbb{RP}^2; \mathbb{F}_2) \cong (0 \otimes_{\mathbb{Z}} \mathbb{F}_2) \oplus \text{Tor}_1(\mathbb{F}_2, \mathbb{F}_2) \cong \text{Tor}_1(\mathbb{F}_2, \mathbb{F}_2).$$

This Tor group is defined to be the derived tensor product

$$\cong H_1(\mathbb{F}_2 \otimes^{\mathbb{L}}_{\mathbb{Z}} \mathbb{F}_2),$$

and we compute this by doing a free resolution:

$$= H_1((\mathbb{Z} \xrightarrow{2} \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{F}_2) = H_1(\mathbb{F}_2 \xrightarrow{2} \mathbb{F}_2).$$

This two-term chain complex has first homology group $\mathbb{F}_2$ (multiplication by 2 is the zero map), and thus we've arrived at $\boxed{H_2(\mathbb{RP}^2; \mathbb{F}_2) = \mathbb{F}_2}$, just like before but with less assumptions!

Similarly, we can compute using the universal coefficients formula again:

$$H_1(\mathbb{RP}^2; \mathbb{F}_2) \cong H_1(\mathbb{RP}^2) \otimes_{\mathbb{Z}} \mathbb{F}_2 \otimes \text{Tor}_1^{\mathbb{Z}}(H_0(\mathbb{RP}^2), \mathbb{F}_2),$$

and reading off things from above yields

$$\cong (\mathbb{F}_2 \otimes_{\mathbb{Z}} \mathbb{F}_2) \oplus \text{Tor}_1^{\mathbb{Z}}(\mathbb{Z}, \mathbb{F}_2) \cong \mathbb{F}_2 \oplus \text{Tor}_1^{\mathbb{Z}}(\mathbb{Z}, \mathbb{F}_2).$$

But now $\mathbb{Z}$ is its own free resolution, so

$$\text{Tor}_1^{\mathbb{Z}}(\mathbb{Z}, \mathbb{F}_2) = H_1(\mathbb{Z} \otimes^{\mathbb{L}} \mathbb{F}_2) = H_1(\to 0 \to \mathbb{F}_2 \to \cdots),$$

which is the zero group. So indeed, $\boxed{H_1(\mathbb{RP}^2; \mathbb{F}_2) = \mathbb{F}_2}$, and the Tor group contributes nothing to this one.

Example 203

Let's compute $H_2(S^2; \mathbb{F}_3)$ (we can just equip it with the $\mathbb{Z}/3\mathbb{Z}$ structure, but it doesn't really matter).

We have

$$H_2(S^2; \mathbb{F}_3) \cong (H_2(S^2) \otimes_{\mathbb{Z}} \mathbb{F}_3) \oplus \text{Tor}_1(H_1(S^2), \mathbb{F}_3),$$

and now because $H_2(S^2) \cong \mathbb{Z}$ and $H_1(S^2) \cong 0$, we have

$$\cong \mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{F}_3 \oplus \text{Tor}_1(0, \mathbb{F}_3).$$

But 0 is its own free resolution, so

$$H_1(0 \otimes^{\mathbb{L}} \mathbb{F}_3) = H_1(0 \to 0) \cong 0,$$

and thus $\boxed{H_2(S^2; \mathbb{F}_3) \cong \mathbb{F}_3}$. This is indeed what we expect, because we can also compute this by using the minimal cell structure on S^2:

$$H_2(C_*^{\text{cell}}(S^2) \otimes_{\mathbb{Z}} \mathbb{F}_3) \cong H_2((\mathbb{Z} \to 0 \to \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{F}_3),$$

which simplifies to

$$\cong H_2(\mathbb{F}_3 \to 0 \to \mathbb{F}_3) \cong \mathbb{F}_3.$$

Again, the point is that we didn't need to know this cellular structure to get our final answer.

Proof outline of the universal coefficients theorem. Suppose that we have a chain complex C_* of free $\mathbb{Z}$-modules and an abelian group M. We need to calculate $C_* \otimes_{\mathbb{Z}} M$, and we can always do this by finding a two-term free resolution

$$\cdots \to F_1 \to F_0 \to 0 \to \cdots$$

over the integers (this happens because the integers are a PID). So the short exact sequence of chain complexes that we end up with is

$$0 \to C_* \otimes_{\mathbb{Z}} F_1 \to C_* \otimes_{\mathbb{Z}} F_0 \to C_* \otimes_{\mathbb{Z}} M \to 0,$$

and now we can use the Snake Lemma to get a long exact sequence in homology (all tensor products here are over $\mathbb{Z}$)

$$H_q(C_* \otimes F_1) \to H_q(C_* \otimes F_0) \to H_q(C_* \otimes M) \xrightarrow{\partial} H_{q-1}(C_* \otimes F_1) \to H_{q-1}(C_* \otimes F_0).$$

Notice though that this exact sequence is the same as if we take out the F_0, F_1 (because the tensor product is exact when we tensor with a free module). So this is isomorphic to

$$H_q(C_*) \otimes F_1 \to H_q(C_*) \otimes F_0 \to H_q(C_* \otimes M) \xrightarrow{\partial} H_{q-1}(C_*) \otimes F_1 \to H_{q-1}(C_*) \otimes F_0,$$

and this implies that we have a short exact sequence

$$0 \to H_q(C_*) \otimes F_0/F_1 \to H_q(C_* \otimes M) \to \ker\left(H_{q-1}(C_q) \otimes F_1 \to H_{q-1}(C_*) \otimes F_0\right) \to 0.$$

But F_0/F_1 is exactly M, meaning that the kernel is exactly the Tor group $\text{Tor}_1(H_{q-1}(C_*), M)$ that we want. □

24 October 28, 2020

Last time, we discussed the universal coefficient theorems, which let us define and compute homology with coefficients in an arbitrary abelian group. We'll continue developing some consequences of the existence of the Tor and Ext functors, using the fact that we can calculate Tor with any free resolution and end up with the same final answer.

The first question we'll think about is how to compute the homology of a tensor product of chain complexes:

Theorem 204

Suppose that C_*, D_* are chain complexes of $\mathbb{Z}$-modules, and suppose that C_* is a complex of **free** $\mathbb{Z}$-modules. Then

$$H_n(C_* \otimes_{\mathbb{Z}} D_*) \cong \left(\bigoplus_{p+q=n} H_p(C_*) \otimes_{\mathbb{Z}} H_q(D_*)\right) \oplus \left(\bigoplus_{p+q=n-1} \text{Tor}_1^{\mathbb{Z}}(H_p(C_*), H_q(D_*))\right).$$

Note that if we tensor together two chain complexes, there isn't a way of computing the homology of the result in terms of the homology of the individual pieces, except when the tensor product agrees with the derived tensor product. That's why C_* having free $\mathbb{Z}$-modules is important here!

In fact, this is actually a generalization of our results from yesterday: when D_* is concentrated in degree 0, meaning there is only a nonzero group in the 0th spot, then this reduces to the universal coefficients theorem. And the proof from last time can show this theorem whenever D_* is concentrated in **any single degree.**

First proof idea. The tensor product $C_* \otimes D_*$ that we're trying to compute actually calculates the derived product $C_* \otimes^{\mathbb{L}} D_*$ in $D(\mathbb{Z})$, meaning that the homology will not change if we replace D_* with any chain complex isomorphic to it in $D(\mathbb{Z})$. (In other words, we can change D_* by a quasi-isomorphism without changing the homology of the tensor product.)

A reasonable choice to make is to replace D_* with the chain complex D'_* satisfying $(D'_*)_q = H_q(D_*)$ and all boundary maps being zero (this has the same homology by construction). Then D'_* becomes a direct sum of chain complexes each concentrated in a single degree, so we can use the proof of the universal coefficients theorem to finish. ☐

Let's now give a proof that doesn't involve the derived category, but is still similar in spirit:

Second proof idea. Consider the chain complex denoted $Z(D_*)$, where the nth group

$$Z(D_*)_n = Z_n(D_*) = \ker(\partial : D_n \to D_{n-1})$$

is the set of cycles in the original chain complex, and all boundary maps are zero. We have an inclusion of chain complexes $Z(D_*) \to D_*$, and this gives us a valid chain map. Furthermore, because the boundary maps of $Z(D_*)$ are zero, we know that it is a direct sum of chain complexes each concentrated in a single degree.

From here, we can think about the quotient chain complex $D_*/Z(D_*)$: this also has all boundary maps zero, so it is also a direct sum of chain complexes concentrated in a single degree. So we have the short exact sequence of chain complexes

$$0 \to C_* \otimes_{\mathbb{Z}} Z(D_*) \to C_* \otimes_{\mathbb{Z}} D_* \to C_* \otimes_{\mathbb{Z}} D_*/Z(D_*) \to 0,$$

where we're tensoring by a free module C_* so we do preserve exactness. Any short exact sequence of chain complexes can then be expanded to a long exact sequence in homology, meaning $H_n(C_* \otimes_{\mathbb{Z}} D_*)$ can be written in terms of the homology of $H_n(C_* \otimes_{\mathbb{Z}} Z(D_*))$ and $H_{n-1}(C_* \otimes_{\mathbb{Z}} D_*/Z(D_*))$, and now we reduce to the UCT because of the "direct sum of concentrated chain complexes" idea. ☐

The above theorem holds more generally: if R is a PID, and C_*, D_* are chain complexes of R-modules with C_* consisting of free modules, then we have

$$H_n(C_* \otimes_R D_*) \cong \left(\bigoplus_{p+q=n} H_p(C_* \otimes_R H_q(D_*)\right) \oplus \left(\bigoplus_{p+q=n-1} \mathrm{Tor}_1^R(H_p(C_*), H_q(D_*))\right).$$

Being a PID is the relevant assumption, because the universal coefficients theorem relied on having the very short two-term free resolutions. In fact, if R is a field, things further simplify to

$$H_n(C_* \otimes_R D_*) \cong \bigoplus_{p+q=n} H_p(C_*) \otimes_R H_q(D_*).$$

All of this is relevant to algebraic topology because of the following result:

Theorem 205 (Eilenberg-Zilber)
Suppose that X and Y are topological spaces, and R is a commutative ring. Then $S_*(X \times Y; R)$ is quasi-isomorphic (in particular, has the same homology) to $S_*(X; R) \otimes_R S_*(Y; R)$.

And this works nicely with our previous work: we can now compute the homology of a product in terms of the homology of the individual spaces. One way to think about this is to examine the product CW structure on $X \times Y$, which we sketched a bit a few classes ago. (After all, the tensor product is actually designed to match the cellular chain complex on a product.) But not every topological space comes from a CW complex, and we haven't really developed the product CW structure idea yet. So we'll prove this theorem directly next time, and for now we'll explore how this theorem works in practice.

Example 206

How can we compute the homology groups $H_q(\mathbb{RP}^2 \times \mathbb{RP}^2; \mathbb{F}_2)$ (notice that we're using field coefficients)?

Recall that

$$H_q(\mathbb{RP}^2; \mathbb{F}_2) \cong \begin{cases} F_2 & q = 0, 1, 2 \\ 0 & \text{otherwise.} \end{cases}$$

We can compute

$$H_0(\mathbb{RP}^2 \times \mathbb{RP}^2; \mathbb{F}_2) \cong \bigoplus_{p+q=0} H_p(\mathbb{RP}^2; \mathbb{F}_2) \otimes_{\mathbb{F}_2} H_q(\mathbb{RP}^2; \mathbb{F}_2) \cong H_0(\mathbb{RP}^2; \mathbb{F}_2) \otimes_{\mathbb{F}_2} H_0(\mathbb{RP}^2; \mathbb{F}_2)$$

(there's no other way to get nonzero groups p, q), and this is just $\mathbb{F}_2 \otimes_{\mathbb{F}_2} \mathbb{F}_2 = \boxed{\mathbb{F}_2}$ (we just multiply the dimensions of the vector spaces). Next,

$$H_1(\mathbb{RP}^2 \times \mathbb{RP}^2; \mathbb{F}_2) \cong \left(H_0(\mathbb{RP}^2; \mathbb{F}_2) \otimes_{\mathbb{F}_2} H_1(\mathbb{RP}^2; \mathbb{F}_2)\right) \oplus \left(H_1(\mathbb{RP}^2; \mathbb{F}_2) \otimes_{\mathbb{F}_2} H_0(\mathbb{RP}^2; \mathbb{F}_2)\right)$$

then simplifies to $(FF_2 \otimes_{\mathbb{F}_2} \mathbb{F}_2) \oplus (FF_2 \otimes_{\mathbb{F}_2} \mathbb{F}_2) = \boxed{\mathbb{F}_2 \oplus \mathbb{F}_2}$. We can do a similar computation to find that H_2 has three copies of $\mathbb{F}_2$ (from each of $(p, q) = (0, 2), (1, 1), (2, 0)$), H_3 has two copies (from $(p, q) = (1, 2)$ and $(2, 1)$, since the $(0, 3)$ and $(3, 0)$ have zero groups being tensored), and H_4 has one copy. This gives us our final answer:

$$H_q(\mathbb{RP}^2 \times \mathbb{RP}^2; \mathbb{F}_2) \cong \begin{cases} \mathbb{F}_2 & q = 0, 4, \\ \mathbb{F}_2 \oplus \mathbb{F}_2 & q = 1, 3, \\ \mathbb{F}_2 \oplus \mathbb{F}_2 \oplus \mathbb{F}_2 & q = 2, \\ 0 & \text{otherwise.} \end{cases}$$

It's difficult to explicitly describe the space $\mathbb{RP}^2 \otimes \mathbb{RP}^2$ – for example, it's hard to triangulate or make a semisimplicial model for it. But computing its homology in this way is not too difficult in the $\mathbb{F}_2$ case, and the $\mathbb{Z}$ case just has us doing some more computation.

Remark 207. *We can notice that the answer "reflects" around $q = 2$, and this has to do with Poincaré duality – there is some interesting geometry that explains why we have this symmetry, and we'll prove some results like this near the end of the semester.*

Remark 208. *We have a diagonal map $\mathbb{RP}^2 \to \mathbb{RP}^2 \times \mathbb{RP}^2$ mapping $x \mapsto (x, x)$, and thus we have a map $H_2(\Delta)$: $H_2(\mathbb{RP}^2; \mathbb{F}_2) \to H_2(\mathbb{RP}^2 \times \mathbb{RP}^2; \mathbb{F}_2)$, which is a map from $\mathbb{F}_2$ to $\mathbb{F}_2 \oplus \mathbb{F}_2 \oplus \mathbb{F}_2$. It turns out that it's very interesting to answer questions about what this map actually is (given the source and target), and that will also come up in the last part of the course.*

We'll finish today's class with a bit of setup for the Eilenberg-Zilber theorem. Since we're trying to show that $S_*(X \times Y; R)$ is quasi-isomorphic to $S_*(X; R) \otimes_R S_*(Y; R)$, we're actually going to construct an explicit quasi-isomorphism here:

Definition 209

The **Alexander-Whitney map** $A : S_*(X \times Y; R) \to S_*(X; R) \otimes_R S_*(Y; R)$ is a chain map defined by constructing a map $A : S_n(X \times Y; R) \to S_p(X; R) \otimes S_q(Y; R)$ for all integers n, p, q with $p + q = n$. It suffices to define $A(\sigma)$ for $\sigma : \Delta^n \to X \times Y \in \text{Sing}_n(X \times Y)$, and this is determined by the maps $\alpha : \Delta^n \to X$ and $\beta : \Delta^n \to Y$. We define

$$A(\sigma) = (\alpha|_{\Delta^p}) \otimes (\beta|_{\Delta^q}),$$

with restriction to Δ^p given by the $\Delta^p \to \Delta^n$ map sending $(e_0, \cdots, e_p)$ to $(e_0, \cdots, e_p, 0, \cdots, 0)$, and restriction to Δ^q given by the map $\Delta^q \to \Delta^n$ sending $(e_0, \cdots, e_q)$ to $(0, \cdots, 0, e_0, \cdots, e_q)$.

Next time, we'll indeed show this is a quasi-isomorphism, and that will conclude our unit on tensor products and homology.

25 October 30, 2020

We're now getting into the "dog days" of the semester, but we'll have a break for the next few days (no assignments until the exam on the 9th). Today, we'll finish discussing the Eilenberg-Zilber theorem that we've been working towards, which tells us that the Alexander-Whitney map $A : S_*(X \times Y) \to S_*(X) \otimes_{\mathbb{Z}} S_*(Y)$ for topological spaces X, Y, given by the maps

$$A(\sigma) = \sum_{p+q=n} \alpha|_{\Delta^p} \otimes \beta|_{\Delta^q}$$

for all $\sigma : \Delta^n \to X \times Y$, is a quasi-isomorphism. Here, $\alpha|_{\Delta^p}$ is the composite map

$$\Delta^p \xrightarrow{\text{first coords}} \Delta^n \xrightarrow{\sigma} (X \times Y) \xrightarrow{p_X} X,$$

while β is the same composite but with $\Delta^q \xrightarrow{\text{last coords}} \Delta^n$ and a p_Y projection map. Perhaps surprisingly, though, the exact map is not super important for the proof of the theorem! It's left as an exercise for us to check that this is a chain map – that is, it commutes with the boundary maps of the two chain complexes.

In order to prove the Eilenberg-Zilber theorem, we'll basically elaborate (in a technical way) on two main ideas we've been developing: naturality and the fundamental theorem of homological algebra.

To do that, consider the following construction: let $\mathcal{C}$ be a category, and let $\mathbb{M}$ be a set of objects of $\mathcal{C}$ (called the **set of models**). For any collection $\{m_1, \cdots, m_r\} \subset \mathbb{M}$, consider the functor $F : \mathcal{C} \to \text{Ab}$, given by

$$F(c) = \mathbb{Z}\left[\coprod_i \text{Hom}(m_i, c)\right].$$

This is called an **$\mathbb{M}$-free** functor, and it is determined by a special set of objects (and checking naturality is left for us).

Example 210

Take $\mathcal{C} = \text{Top} \times \text{Top}$ (pairs of topological spaces, with no assumptions about one being a subset of the other), and let $\mathbb{M}$ be the set of pairs $\{(\Delta^p, \Delta^q) : p, q \geq 0\}$.

These are functors that are "controlled by what they do on simplices," and we can recall that we made a similar type of naturality argument in the past. So now the functor corresponding to the single element $\{\Delta^n \times \Delta^n)$ is

$$F_{\{\Delta^n \times \Delta^n\}}(X, Y) = \mathbb{Z}\left[\text{Hom}_{\text{Top} \times \text{Top}}(\Delta^n, \Delta^n), (X, Y)\right] = \mathbb{Z}\left[\text{Hom}_{\text{Top}}(\Delta^{\cdot} X) \times \text{Hom}_{\text{Top}}(\Delta^n, Y)\right]$$

(by the definition of the product category and how we compute Homs). But now picking an n-simplex in each of X and Y is equivalent to picking an n-simplex for the product:

$$= \mathbb{Z}\left[\mathrm{Hom}_{\mathrm{Top}}(\Delta^n, X \times Y)\right] = S_n(X \times Y).$$

So we have some language for describing our S_n maps now! Similarly, if we make the functor for the set $M = \{(\Delta^0, \Delta^n), (\Delta^1, \Delta^{n-1}, \cdots, (\Delta^n, \Delta^0)\}$, then

$$F_M(X, Y) = \mathbb{Z}\left[\coprod_{p+q=n} \mathrm{Hom}_{\mathrm{Top}\times\mathrm{Top}}((\Delta^p, \Delta^q), (X, Y))\right] = \mathbb{Z}\left[\coprod_{p+q=n} \mathrm{Hom}_{\mathrm{Top}}(\Delta^p, X) \times \mathrm{Hom}(\Delta^q, Y)\right].$$

But this is exactly

$$= \bigoplus_{p+q=n} \mathbb{Z}\left[\mathrm{Hom}_{\mathrm{Top}}(\Delta^p, X) \times \mathrm{Hom}_{\mathrm{Top}}(\Delta^q, Y)\right] \cong \bigoplus_{p+q=n} S_p(X) \otimes_{\mathbb{Z}} S_q(Y),$$

because both $S_p(X)$ and $S_p(Y)$ are free, and this is exactly $S_*(X) \otimes_{\mathbb{Z}} S_*(Y)$. So the point of this example was to describe that the nth group of $S_*(X \times Y)$, as well as $S_*(X) \otimes_{\mathbb{Z}} S_*(Y)$, are controlled by maps from simplices.

So now, we'll return to the general setup:

Definition 211

Let $\mathcal{C}$ be a category, and let $\mathbb{M}$ be a set of models in $\mathcal{C}$. If $F : \mathcal{C} \to \mathrm{Ab}$ is any functor, an **$\mathbb{M}$-free resolution** of F is a functor $F_* : \mathcal{C} \to \mathrm{chAb}$ with a natural transformation $\varepsilon_F : H_0(F_*) \to F$, satisfying the following two conditions:

- Each F_n is an $\mathbb{M}$-free functor,
- If we evaluate F_* on an object of $\mathbb{M}$, then it is a free resolution of F, meaning that $H_i(F_*(m)) = \begin{cases} 0 & i \neq 0 \\ F(m) & i = 0 \end{cases}$ and that $\varepsilon_F : H_0(F_*(m)) \to F(m)$ is an isomorphism.

This is a pretty involved generalization of a free resolution – the idea is that functors are controlled by what they do on the model set, so it can be useful to talk about being a free resolution only on those elements.

Theorem 212

Let $\Theta : F \to G$ be a natural transformation of functors $F, G : \mathcal{C} \to \mathrm{Ab}$. If F_* and G_* are $\mathbb{M}$-free resolutions of F and G, then there is a natural transformation $\Theta_* : F_* \to G_*$ lifting Θ, unique up to natural chain homotopy.

This is probably the most elaborate construction we'll do in this class, and we won't get too deeply into it, but a small part will be on our homework. The language is set up to have a strong analogy between this result and the fundamental theorem of homological algebra, though.

As a corollary of this, we actually find that the Eilenberg-Zilber theorem is true, and that the Alexander-Whitney map A is unique up to natural chain homotopy – let's show how this is proved:

Proof of Eilenberg-Zilber. Let $\mathcal{C} = \mathrm{Top} \times \mathrm{Top}$, and let

$$F(X, Y) = H_0(X \times Y), \quad G(X, Y) = H_0(S_*(X) \otimes_{\mathbb{Z}} S_*(Y)).$$

Then we can let

$$F_*(X, Y) = S_*(X \times Y), \quad G_*(X, Y) = S_*(X) \otimes_{\mathbb{Z}} S_*(Y).$$

It's not true that F_* and G_* are always free resolutions, but when X and Y are simplices, F_* and G_* indeed record chain complexes with homology only found in degree 0. So F_* and G_* are free resolutions at least on the model objects, and in general they are indeed $\mathbb{M}$-free resolutions. So now we can specify a natural map $H_0(X\times Y) \to H_0(S_*(X)\otimes_{\mathbb{Z}} S_*(Y))$, we have a unique lift up to chain homotopy to this Alexander-Whitney map $A : S_*(X \times Y) \to S_*(X) \otimes_{\mathbb{Z}} S_*(Y)$!

So we do need to say what this map does on zeroth homology, but that's just counting connected components. We basically pick a natural isomorphism $H_0(X \times Y) \to H_0(S_*(X) \otimes_{\mathbb{Z}} S_*(Y))$ along with the natural inverse $H_0(S_*(X) \otimes_{\mathbb{Z}} S_*(Y)) \to H_0(X \times Y)$, and going around and composing these maps gives the identity on either $H_0(X \times Y)$ or $H_0(S_*(X) \otimes_{\mathbb{Z}} S_*(Y))$. Thus, $S_*(X \times Y)$ and $S_*(X) \otimes_{\mathbb{Z}} S_*(Y)$ must be chain homotopy equivalent. The point is that inducing an isomorphism on H_0 means that we are chain homotopy equivalent to the identity! So quasi-isomorphism has been shown. □

So now we can turn to asking why homology of product spaces are actually interesting to us, and we'll understand this some more by looking at the diagonal map

$$X \xrightarrow{\Delta} X \times X,$$

which exists for any topological space X. If we look at such a space X which has free abelian groups for homology groups (e.g. not Klein bottle), Then we have that

$$H_n(X \times X) = \bigoplus_{p+q=n} H_p(X) \otimes_{\mathbb{Z}} H_q(X),$$

because the Tor terms in the Künneth theorem go away, and now we let $H_*(X)$ denote the direct sum $\bigoplus_n H_n(X)$. (For example, we have four copies of $\mathbb{Z}$ for a torus, coming from $1 + 2 + 1$ copies in H_0, H_1, H_2.) Then

$$\boxed{H_*(X) \otimes_{\mathbb{Z}} H_*(X)} = \left(\bigoplus_p H_p(X)\right) \otimes_{\mathbb{Z}} \left(\bigoplus_q H_q(X)\right),$$

and now distributing the tensor product over the direct sum yields

$$\cong \bigoplus_{p,q} H_p(X) \otimes_{\mathbb{Z}} H_q(X) = \bigoplus_n \bigoplus_{p+q=n} H_p(X) \otimes_{\mathbb{Z}} H_q(X) \cong \bigoplus_n H_n(X \times X) = \boxed{H_*(X \times X)}.$$

Therefore, we have a map

$$H_*(X) \to H_*(X \times X) \cong H_*(X) \otimes_{\mathbb{Z}} H_*(X)$$

induced by the diagonal map $\Delta^n \to \Delta^n$. And let's think about what's going on in algebra here: if we have an abelian group A and a map $A \to A \otimes_{\mathbb{Z}} A$, this is what we actually call **comultiplication**, which is the (categorical) opposite of a **multiplication**:

Definition 213

Let A be an abelian group. A **multiplication** is a map of abelian groups $A \otimes_{\mathbb{Z}} A \xrightarrow{m} A$, taking $a \otimes a'$ to an element $a'' = m(a \otimes a')$, which satisfies

$$m((a_1 + a_2), a') = m(a_1 \otimes a') + m(a_2 \otimes a'), \quad m(a \otimes (a_1' + a_2')) = m(a \otimes a_1') + m(a \otimes a_2').$$

In other words, this multiplication distributes over addition (from our abelian group). For example, a ring is an abelian group A with a multiplication satisfying certain properties, so ring multiplication is a valid example here. But comultiplications are a bit weirder, and that means the inducing of a comultiplication by Δ (rather than a multiplication) means that this is not as related to familiar mathematical structures for us. For that reason, we'll look into **cohomology groups** (with any coefficients) next time, and we'll find that Δ induces a multiplication there. So we don't need to talk about comultiplication directly, which will be convenient! (Basically, studying "corings" in homology is less good than studying rings in cohomology.)

26 November 2, 2020

We'll talk a lot about cohomology and the diagonal map over the next few classes, but we'll first do a bit more discussion about homology with coefficients (because there have been a lot of questions).

One of the main questions we might ask is "why we care about homology with coefficients if we can calculate them from the ordinary homology groups anyway," and applied algebraic topology in theoretical computer science was used as an example. But let's see some instances in pure math too:

Example 214

On our problem set, we looked a **group homology**: given a group G, we can form a category BG and a simplicial set (the "nerve" of that category) $N(BG)$, and we may care about the homology of $H_q(N(BG); M)$ for various $q \in \mathbb{Z}$, groups G, and coefficient groups M.

We know that there's a very specific combinatorial gadget we can use, but we should remember that the number of simplices grows exponentially, so it's hard to actually compute the groups in closed form with any algorithm.

For instance, if $G = GL_n(\mathbb{F}_q)$ for some finite field q, we know the answer for $M = \mathbb{Q}$, but it's an open problem to determine the answer for $M = \mathbb{Z}$ (and this is an active area of research right now).

Example 215

If X is a topological space and we have an integer k, we can define a **configuration space** $\text{Config}_k(X)$, which is the subspace of $X^{\times k}$ given by

$$\{(x_1, \cdots, x_k) \in X^{\times k} : x_i \neq x_j \text{ when } i \neq j\}.$$

Picking a point of our "configuration space" basically means we pick k distinct points of our topological space X, and then we can also define the **unordered configuration space**

$$B_k(X) = \text{Config}_k(X)/S_k,$$

where S_k is the symmetric group. (In the unordered configruation space, we don't care about the order that the k points are chosen.) It turns out that we know $H_q(B_k(\mathbb{R}^n); \mathbb{Z})$, but we don't know $H_q(B_k(T); \mathbb{Z})$ (where T denotes the torus). On the other hand, we learned about $H_q(B_k(T); \mathbb{F}_2)$ in the 1980s and $H_q(B_k(T); \mathbb{Q})$ in the 2010s, and many people have tried to understand the answer for $\mathbb{F}_3$ or $\mathbb{Z}$ without success.

So now let's turn our attention to cohomology, starting with a quick review of the definitions. If we have a topological space X, and M is an abelian group, we can form the internal Hom

$$\underline{\text{Hom}}_{D(\mathbb{Z})}(S_*(X), M),$$

which is an object of $D(\mathbb{Z})$, meaning it has well-defined homology groups. Then we defined the qth cohomology group of X in terms of the homology groups of this object:

$$H^q(X; M) = H_{-q}(\underline{\text{Hom}}_{D(\mathbb{Z})}(S_*(X), M)).$$

This internal Hom is easy to compute, because we don't need to resolve anything when S_* is already free. Our first step is to define

$$S^q(X; M) = \underline{\text{Hom}}_{\text{Ab}}(S_q(X), M),$$

and because $\underline{\text{Hom}}_{\text{Ab}}(\cdot, M)$ is a functor from Ab^{op} to Ab, we know that the map $\partial : S_q(X) \to S_{q-1}(X)$ must induce a corresponding map $\partial : S^{q-1}(X; M) \to S^q(X; M)$ (precomposing with ∂). This gives us the **cochain complex**

$$S^*_{\text{Hatcher}}(X; M) = \cdots \to 0 \to S^0(X; M) \xrightarrow{\partial} S^1(X; M) \xrightarrow{\partial} S^2(X; M) \to \cdots,$$

which can be thought of as a chain complex concentrated in nonpositive degrees. The cohomology groups can then be found via

$$H^q(X; M) = \ker(S^q(X; M) \to S^{q+1}(X; M))/\text{im}(S^{q-1}(X; M) \to S^q(X; M)).$$

Fact 216

As a warning, note that we can also form a cochain complex

$$S^*_{\text{Miller}}(X; M) = \cdots \to 0 \to S^0(X; M) \xrightarrow{-\partial} S^1(X; M) \xrightarrow{\partial} S^2(X; M) \xrightarrow{-\partial} \cdots,$$

but with the slightly different boundary maps $S^q(X; M) \to S^{q+1}(X; M)$ of $(-1)^{q+1}\partial$. These two objects $S^*_{\text{Hatcher}}(X; M)$ and $S^*_{\text{Miller}}(X; M)$ are isomorphic in $D(\mathbb{Z})$, but they aren't literally the same.

In lecture, we'll have $S^*(X; M)$ denote the Miller convention, but we can use either one. (So unlike the signs in the tensor product, these are just a matter of convention.) And a useful fact is that $S^*(X; \mathbb{F}_2)$ is the same in both conventions, because $-1 = 1$.

Let's quickly recall the universal coefficients theorem for cohomology:

$$H^q(X; M) \cong \text{Ext}^1_{\mathbb{Z}}(H_{q-1}(X); M) \oplus \underline{\text{Hom}}_{\text{Ab}}(H_q(X), M).$$

This result shows us that cohomology with coefficients in M, can always be computed in terms of coefficients with coefficients in $\mathbb{Z}$, just like for homology.

Example 217

Let's compute the cohomology group $H^2(\mathbb{RP}^2; \mathbb{F}_2)$.

The first method we can try is to use the cellular cochain complex: recall that we have the chain complex

$$C^{\text{cell}}_*(\mathbb{RP}^2, \mathbb{Z}) \cong \cdots \to 0 \to \mathbb{Z} \xrightarrow{2} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \to 0 \to \cdots.$$

So the cochain complex will look like

$$C^*_{\text{cell}}(\mathbb{RP}^2; \mathbb{F}_2) = \cdots \to 0 \to \underline{\text{Hom}}_{\text{Ab}}(\mathbb{Z}, \mathbb{F}_2) \to \underline{\text{Hom}}_{\text{Ab}}(\mathbb{Z}, \mathbb{F}_2) \to \underline{\text{Hom}}_{\text{Ab}}(\mathbb{Z}, \mathbb{F}_2) \to 0 \to \cdots,$$

and $\underline{\text{Hom}}_{\text{Ab}}(\mathbb{Z}, \mathbb{F}_2)$ is just $\mathbb{F}_2$ because the map is determined by where 1 is sent. And now the two maps are ± 2 and ± 0, but since we're working with $\mathbb{F}_2$ coefficients, all of the maps are just zero. So we're left with the cochain complex

$$\cdots \to 0 \to \mathbb{F}_2 \xrightarrow{0} \mathbb{F}_2 \xrightarrow{0} \mathbb{F}_2 \to 0 \to \cdots,$$

and this tells us that

$$\boxed{H^q(\mathbb{RP}^2; \mathbb{F}_2) = \begin{cases} \mathbb{F}_2 & q = 0, 1, 2, \\ 0 & \text{otherwise.} \end{cases}}$$

The alternative method, though, is to use the UCT directly: for example,

$$H^2(\mathbb{RP}^2; \mathbb{F}_2) \cong \text{Ext}^1_{\mathbb{Z}}(H_1(\mathbb{RP}^2), \mathbb{F}_2) \oplus \underline{\text{Hom}}_{\text{Ab}}(H_2(\mathbb{RP}^2); \mathbb{F}_2),$$

and we know the integral **homology** groups of $\mathbb{RP}^2$ already:

$$\cong \text{Ext}^1_{\mathbb{Z}}(\mathbb{Z}/2\mathbb{Z}, \mathbb{F}_2) \oplus \underline{\text{Hom}}_{\text{Ab}}(0; \mathbb{F}_2) \cong \text{Ext}^1_{\mathbb{Z}}(\mathbb{F}_2; \mathbb{F}_2).$$

In order to compute this Ext group, we need to compute

$$H_{-1}\left(\underline{\text{Hom}}_{D(\mathbb{Z})}(\mathbb{F}_2, \mathbb{F}_2)\right),$$

and this requires us to find a free resolution. We'll replace the first $\mathbb{F}_2$ with the isomorphic chain complex $\mathbb{Z} \xrightarrow{2} \mathbb{Z}$, and thus we need to find

$$H_{-1}(\mathbb{F}_2 \xrightarrow{2} \mathbb{F}_2),$$

where the two groups are concentrated in degrees $0, -1$, and this recovers the same answer of $\mathbb{F}_2$ as above.

So now we're ready to think about how cohomology (specifically how the functor $H^*(\cdot; M)$) interacts with the diagonal map:

Definition 218

We use the notation (analogous to the one for homology)

$$H^*(X; M) \cong \bigoplus_q H^q(X; M).$$

The idea is that if $X \to Y$ is a continuous map of topological spaces, and M is an abelian group, then there is a natural map

$$H^q(Y; M) \to H^q(X; M)$$

(cohomology induces a map in the **opposite** direction, because internal Homs reverse arrows – they're contravariant functors). So if we set $Y = X \times X$, we always have a natural map

$$H^q(X \times X; M) \to H^q(X; M)$$

induced from the continuous diagonal map $X \to X \times X$. If we add this up for all q, we end up with a map

$$H^*(X \times X; M) \to H^*(X; M),$$

and we want to view this as a "multiplication" (rather than as a "comultiplication" that we had to work with last time). We know the homology of a product space in terms of the homology of the factors (using the Künneth theorem), and we know how to relate that to the cohomology of our product space using the cohomology version of the UCT:

$$H^*(X \times Y; M) \cong \underline{\text{Hom}}(H_*(X \times Y; \mathbb{Z}), M) \oplus (\text{Ext terms}).$$

So together, we get a "cohomology Künneth theorem" from these two isomorphisms.

Remark 219. *If R is a ring, and if $H_*(X; R)$ is a free R-module, then we know (by the UCT) that*

$$H^*(X \times X; R) = H^*(X; R) \otimes_R H^*(X; R).$$

So again, things look a lot simpler if we have free modules.

In general, if we have two spaces X and Y and a ring R, we can always construct a map

$$H^*(X; R) \otimes_R H^*(Y; R) \to H^*(X \times Y; R).$$

However, if $H_*(X; R)$ and $H_*(Y; R)$ are free modules, then this will be an isomorphism.

Corollary 220

If X is a topological space and R is a ring, then we have a natural multiplication

$$H^*(X; R) \otimes_R H^*(X; R) \to H^*(X; R)$$

defined by composing the maps

$$H^*(X; R) \otimes_R H^*(X; R) \to H^*(X \times X; R) \xrightarrow{H^q(\Delta; R)} H^q(X; R)$$

In other words, there are individual cohomology groups for each space, but direct summing all of the cohomology groups together allows us to multiply them too! After we do a bit more work, we'll soon be able to talk about the "cohomology rings" for individual spaces.

27 November 4, 2020

Today, we'll discuss a cool fact: if X is a topological space and R is a (always commutative) ring, then

$$H^*(X; R) = \bigoplus_{q \geq 0} H^q(X; R)$$

is a **graded-commutative ring** (which we'll define later). To be a **graded abelian group**, we must satisfy the following conditions:

- Each class (element) $x \in H^q(X; R) \subseteq H^*(X; R)$ is **homogeneous of degree q**.
- Every class $x \in H^*(X; R)$ can be written as a sum of finitely many homogeneous elements.

So if we want to think about how to go from a group to a ring, our next step is to figure out how our multiplication looks.

Definition 221

Let X, Y be topological spaces and R a ring. Then there is a natural sequence of maps

$$H^*(X; R) \otimes_R H^*(Y; R) \xrightarrow{f_1} H^*(S^*(X; R) \otimes_R S^*(Y; R)) \xrightarrow{f_2} H^*(\underline{\text{Hom}}_{D(\mathbb{Z})}(S_*(X) \otimes_{\mathbb{Z}} S_*(Y), R)) \xrightarrow{f_3} H^*(X \times Y; R),$$

and the composite map is called the **cohomology cross product** $\times : H^*(X; R) \otimes_R H^*(Y; R) \to H^*(X \times Y; R)$.

This cross product is an isomorphism if either $H_q(X; R)$ or $H_q(Y; R)$ is a finitely-generated free R-module for all integers q. (So when R is a field, it's true if all homology groups are finite-dimensional vector spaces.) But the cross product always exists, whether or not it is an isomorphism.

There are two different assumptions going on here: freeness is asking f_1 to be an isomorphism, which has to do with the Künneth theorem, and the finite generation is asking about f_2. So let's look at these f_1, f_2, f_3 maps more carefully.

- The map f_3 is always an isomorphism, because by definition we compute the cohomology as

$$H^*(X \times Y; R) = H^*(\underline{\mathrm{Hom}}_{D(\mathbb{Z})}(S_*(X \times Y), R)),$$

  and then the Eilenberg-Zilber theorem tells us that $S_*(X \times Y)$ is the same, in the derived category, as $S_*(X) \otimes_{\mathbb{Z}} S_*(Y)$.

- The map f_2 is induced from a chain map

$$\underline{\mathrm{Hom}}_{D(\mathbb{Z})}(S_*(X), R) \otimes_{\mathbb{Z}} \underline{\mathrm{Hom}}_{D(\mathbb{Z})}(S_*(Y), R) \to \underline{\mathrm{Hom}}_{D(\mathbb{Z})}(S_*(X) \otimes_{\mathbb{Z}} S_*(Y), R),$$

  where the actual map is given by

$$f \otimes g \mapsto \begin{cases} x \otimes y \mapsto (-1)^{pq} f(x) g(y) & |x| = |f| = p, |y| = |g| = q, \\ 0 & \text{otherwise.} \end{cases}$$

- The map f_1 is something we've already seen – if R is a PID, then f_1 is the map from the Künneth theorem. Recall that the short exact sequence looks like

$$0 \to H_*(C_*) \otimes_R H_*(D_*) \xrightarrow{f_1} H_*(C_* \otimes_R D_*) \to \text{Tor terms} \to 0,$$

  and the f_1 map is the one labeled. (If R is not a PID, it still exists and is natural, but it's just not part of the short exact sequence.)

Fact 222

In general, we should think of the finite generation hypothesis as an **artifact of switching between homology and cohomology**.

We can look at $H^0(X; R)$, the zeroth cohomology group, for more understanding as well. Recall that the zeroth homology group $H_0(X; R)$ counts the number of path components, so we expect something similarly simple for $H^0(X; R)$. Writing $\pi_0 X$ for the set of path components, we know that $H^0(X; R)$ is the set $\mathrm{Hom}_{\mathrm{Set}}(\pi_0(X), R)$, additionally equipped with a natural R-module structure. For example, if X has two path components, $H^0(X; R)$ is the set of maps from two points into R, which is something like $R \oplus R$, and indeed whenever $\pi_0 X$ is finite,

$$\mathrm{Hom}_{\mathrm{Set}}(\pi_0 X, R) \cong \bigoplus_{\pi_0 X} R.$$

But if $\pi_0 X$ is infinite, this is not a direct sum, because every object of a direct sum must be a **finite** sum of the individual pieces. Meanwhile, functions from $\pi_0 X$ to R pick out infinite sequences! (This is similar to the distinction between the product of abelian groups versus the direct sum of abelian groups.) Again, this has to do with switching between homology and cohomology – $H_0(X; R)$ is always a direct sum.

So now we'll return to the cross product and the cohomology ring. Again, recall that for any map $X \to Y$ of topological spaces, we have a map $S_*(X) \to S_*(Y)$, and thus we have a map

$$\underline{\text{Hom}}_{D(\mathbb{Z})}(S_*(Y), R) \to \underline{\text{Hom}}_{D(\mathbb{Z})}(S_*(X), R)$$

(contravariance switches the direction of the arrows), and thus we have a map $H^*(Y; R) \to H^*(X; R)$. Using the diagonal map $X \to X \times X$, we get a map

$$H^*(X \times X; R) \to H^*(X; R),$$

and now we use the cross product to extend this to

$$H^*(X; R) \otimes_R H^*(X; R) \xrightarrow{\times} H^*(X \times X; R) \xrightarrow{H^*(\Delta; R)} H^*(X; R).$$

The composite map now gives us the multiplication of the "cohomology ring," called the **cup product** in cohomology (with coefficients in R). We do want to show that we have the desired ring structure:

Theorem 223

The cup product makes $H^*(X; R)$ into an associative and **graded-commutative** ring.

That means we need to check the following conditions:

- There is an element $1 \in H^0(X; R)$ such that the cup product $1x = x1 = x$ for all $x \in H^*(X; R)$. (It's the thing that takes every path component and sends it to the multiplicative identity 1 in R.)
- If $x, y, z \in H^*(X; R)$, then
$$(xy)z = x(yz),$$
so we can talk about xyz without any ambiguity.
- Being **graded-commutative** is not the same as being commutative: if $x \in H^p(X; R)$ and $y \in H^q(X; R)$, then
$$xy = (-1)^{pq} yx \in H^{p+q}(X; R).$$
So if we want to multiply two elements in $H^*(X; R)$, we need to break it up into its direct sum components first.

As always, if we want to do calculations with our constructions, we need to build from previous calculations, so we must figure out how this product works in simple spaces first. For now, let's try to get some geometric understanding for this product, which we'll eventually justify.

Example 224

If we calculate the cohomology groups of the torus T, we'll find that

$$H^q(T; \mathbb{F}_2) = \begin{cases} \mathbb{F}_2 & q = 0, 2 \\ \mathbb{F}_2 \oplus \mathbb{F}_2 & q = 1 \\ 0 & \text{otherwise.} \end{cases}$$

We can see that the homology and cohomology are identical here, and there is indeed some kind of duality that relates the two. We can draw two loops a, b around the torus, one "horizontal" and one "cross-section," representing the two generators of $H_1(T, \mathbb{F}_2) \cong H^1(T, \mathbb{F}_2)$, and now we can ask about the cup product $ab \in H^2(T; \mathbb{F}_2)$. It turns out that $ab = 1$, because a, b intersect! So **the cup product has to do with intersections**, and we're claiming that no matter how we wiggle a (deform by homotopy), it'll always intersect b at least once (and always an odd number of times). Being able to justify this idea will take us a while, though.

For now, we'll think about how to compute cohomology rings H^* of more complicated spaces from the cohomology rings of simple ones – something like the Mayer-Vietoris theorem, or trying to compute the cohomology of a product space from the cohomology of the individual parts. We'll prove next time that not only is H^* a graded-commutative, associative, unital ring, we can compute $H^*(X \times Y; R)$ by thinking about $H^*(X; R)$ and $H^*(Y; R)$ as rings.

28 November 6, 2020

For our test on Monday, we need to know how to use cellular or semisimplicial structures to calculate homology with coefficients in R. (So there won't be any Ext or Tor groups, and the Künneth theorem won't come up.) Further announcements about the test will be posted on Canvas, and we'll still have a (relaxed) lecture on Monday.

Last class, we discussed the multiplication structure of $H^*(X; R)$, the direct sum of the cohomology groups for a topological space X (coming from a combination of the cross product and the diagonal map). We'll look more at this structure today, proving that it is unital, associative, and graded-commutative. (Recall that this last condition means that if $x \in H^p(X; R)$ and $y \in H^q(X; R)$, then $xy = (-1)^{pq} yx \in H^{p+q}(X; R)$. This kind of thing also shows up in places like supersymmetry in physics!) Additionally, if we have a continuous map of topological spaces $X \to Y$ (a morphism in Top), then there is an induced map $H^*(Y; R) \to H^*(X; R)$ which is a **map of rings** (so addition and multiplication are respected by the induced map).

Fact 225

Even if we don't remember the proofs from today, the above facts are important for us to use in practice. If we take a look at Sections 28 and 29 of Miller's notes, or Section 3.2 of Hatcher, we can see the proofs in more detail (but we should be aware of the different sign conventions).

We want to make our cup product more explicit, meaning that we need to understand how products work for explicit elements in the cohomology group $H^p(X; R)$. Note that any element of $H^p(X; R)$ is represented by a class

$$f \in S^p(X; R) = \underline{\mathrm{Hom}}_{\mathrm{Ab}}(S_p(X), R).$$

Furthermore, f must be a **cocycle**, and it must not change if we add a **coboundary** to f. (**Cocycles** and **coboundaries** have the same definition as cycles and boundaries, except that our chain complex is going in the opposite direction.) So we notice that some of the arguments will depend on the specific chain complex that we choose.

Explicitly, an element of $\underline{\mathrm{Hom}}_{\mathrm{Ab}}(S_p(X), R)$ is a function from $S_p(X)$ to R that respects addition (it's an abelian group homomorphism), so it's determined by a map $f : \mathrm{Sing}_p(X) \to R$ from the simplices $\sigma : \Delta^p \to X$ to the ring R. So if we have two such functions, we want to construct the multiplication: if $f \in S^p(X; R)$, and $g \in S^q(X; R)$ are two cocycles (meaning they do represent something in cohomology), we should have $fg \in S^{p+q}(X; R)$ spit out a map from $(p+q)$-simplices to R. And in fact, if we go through the definition, we'll have

$$(fg)(\sigma) = (-1)^{pq} f(\sigma|_{\Delta^p}) g(\sigma|_{\Delta^q}),$$

where the f term is the "front p-face" of Δ^{p+q}, while the g term is the "back q-face."

Proof of associativity. Suppose we have $f \in S^p(X; R), g \in S^q(X; R), h \in S^r(X; R)$ that are all cocycles. Then we can compute $(fg)h$ and $f(gh)$ by applying them both to $\sigma \in \text{Sing}_{p+q+r}(X)$: we find that

$$((fg)h)\sigma = (-1)^{(p+q)r}(fg)(\sigma|_{\Delta^{p+q}})h(\sigma|_{\Delta^r}),$$

which further simplifies to

$$= (-1)^{pr+qr}(-1)^{pq}f(\sigma|_{\Delta^p})g(\sigma|_{\Delta^q})h(\sigma_{\Delta^r}).$$

So f accounts for the front, g for the middle, and h for the back coordinates. Expanding this out gives us

$$= \boxed{(-1)^{pr+pq+qr}f(\sigma|_{\Delta^p})g(\sigma|_{\Delta^q})h(\sigma_{\Delta^r})}.$$

Similarly,

$$(f(gh))(\sigma) = (-1)^{p(q+r)}f(\sigma|_{\Delta^p})(gh)(\sigma|_{\Delta^{q+r}})$$

simplifies to

$$= (-1)^{pq+pr}f(\sigma|_{\Delta^p})(-1)^{qr}g(\sigma|_{\Delta^q})h(\sigma_{\Delta^r}),$$

and rearranging gives us the same thing as the boxed expression above. □

Proof of unitality. We want to show that there is an element $1 \in H^0(X; R)$ which serves as a multiplicative identity. We can use the function $1 : \text{Sing}_0(X) \to R$, which sends every 0-simplex to the identity element $1 \in R$. So now for every cocycle $f \in S^p(X; R)$,

$$(f \cdot 1)(\sigma) = (-1)^{p \cdot 0}f(\sigma|_{\Delta^p})1(\sigma|_{\Delta^0}) = 1 \cdot f(\sigma) \cdot 1 = f(\sigma),$$

because σ is a p-simplex to start with. Indeed, this means that $f \cdot 1 = f$, and a similar computation shows that $1 \cdot f = f$. □

The next proof is Theorem 3.11 in Hatcher, and it's a bit trickier than the first two. To see why, suppose we have two cocycles $f \in S^p(X; R)$ and $g \in S^q(X; R)$. Then for a $(p+q)$-simplex $\sigma : \Delta^{p+q} \to X$, we have

$$(fg)(\sigma) = (-1)^{pq}f(\sigma|_{\Delta^p})g(\sigma|_{\Delta^q}),$$

but direct evaluation of

$$(gf)(\sigma) = (-1)^{pq}g(\sigma|_{\Delta^q})f(\sigma|_{\Delta^p}),$$

where the g term now refers to the front q-face instead of the back q-face, which don't necessarily have anything to do with each other! So we don't get any obvious relations from the definitions.

What we want to do is show that $(fg) - (-1)^{pq}(gf)$ is a **coboundary**, which is all we need for the cohomology to work out properly.

Proof sketch of graded-commutativity. We construct the following chain map $S_*(X) \to S_*(X)$ (we're back to our ordinary chain complexes now), which takes a degree-p simplex $\sigma : \Delta^p \to X$ to $(-1)^{p(p+1)/2}\tilde{\sigma}$, where $\tilde{\sigma}$ is the composite map

$$\Delta^p \xrightarrow{f} \Delta^p \xrightarrow{\sigma} X,$$

and f reverses all of the orientations, sending $(v_0, v_1, \cdots, v_p)$ to $(v_p, v_{p-1}, \cdots, v_0)$. (When we reverse all of the coordinates, we do $\frac{p(p+1)}{2}$ transpositions.) This map turns out to be chain homotopic to the identity, because flipping the orientations picks up a negative sign. So we can apply this map to our $fg - (-1)^{pq}gf$ map, and that will show the desired result. □

Fact 226

If X is a topological space and R is a ring, $H^*(X;R)$ is not just a graded ring – it's also a **graded R-algebra**, which means that $H^*(X;R)$ is an R-module (not just an abelian group) and that the multiplication is R-linear (meaning the cup product is a map $H^*(X;R) \otimes_R H^*(X;R) \to H^*(X;R)$, with a tensor-product on the left side).

Next, we'll start talking about the cohomology ring of a product space, and we'll end this class with a construction. If A^* and B^* are two graded R-algebras, we can define their tensor product by ignoring the gradings and multiplication (we just think about everything as R-modules). This result $A^* \otimes_R B^*$ has a **tensor product graded R-algebra structure**, defined so that if a, b, a', b' are homogeneous, then

$$(a' \otimes b')(a \otimes b) = (-1)^{|b'||a|}(a'a \otimes b'b).$$

In a future class, we'll explain the sign convention, and we'll also how it allows us to have the following result:

Theorem 227

Let X, Y be two topological spaces. Then the cross product

$$\times : H^*(X;R) \otimes_R H^*(Y;R) \to H^*(X \times Y;R)$$

is a map of graded R-algebras.

In other words, the cohomology ring of a product has a lot of constraints coming from the cohomology ring of its individual spaces.

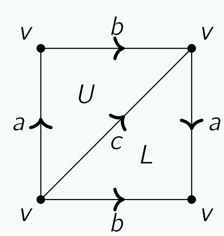
29 November 9, 2020

Today, we'll talk through some examples and do some calculations, but we should feel free to take class time to work on our test. We'll have one more group homework assignment after this, which will be a little bit longer than the first four.

Let's try calculating some cup products by brute force, going back to the definition. (This is not generally recommended – we usually use known answers of small spaces, plus some theorems that help us assemble the answer – but we want to show that it's possible to calculate directly.)

Example 228

Let's calculate $H^*(\text{Klein bottle}; \mathbb{F}_2)$, using the following semisimplicial decomposition with one 0-simplex, three 1-simplices, and two 2-simplices:



To calculate the chain complex $S_*(K)$, we take the free abelian groups generated by the simplices, and also look at the boundary maps:

$$\cdots \to S_*(K) \cong \mathbb{Z}\{U, L\} \to \mathbb{Z}\{a, b, c\} \to \mathbb{Z}\{v\} \to 0 \to \cdots$$

If we want to find the chain complex $S^*(K; \mathbb{F}_2)$ from this, we care about the free abelian group of homomorphisms from $\mathbb{Z}\{v\}, \mathbb{Z}\{a, b, c\}$, and $\mathbb{Z}\{U, L\}$ (respectively) to $\mathbb{F}_2$, so we have

$$\cdots \to \mathbb{F}_2\{\delta_v\} \xrightarrow{\partial} \mathbb{F}_2\{\delta_a, \delta_b, \delta_c\} \xrightarrow{\partial} \mathbb{F}_2\{\delta_U, \delta_L\} \to 0 \to \cdots,$$

where $\delta_v : \mathbb{Z}\{v\} \to \mathbb{F}_2$ is the map that sends $v \mapsto 1$, while $\delta_a : \mathbb{Z}\{a, b, c\} \to \mathbb{F}_2$ takes a to 1 and b, c to 0 (and other maps are defined similarly). Essentially, we're using the dual $\mathbb{F}_2$-vector spaces.

Next, we need to compute the cohomological boundary maps without invoking any theorems. To compute $\partial(\delta_v)$, we need to come up with a specific function on the free abelian group $\mathbb{Z}\{a, b, c\}$ to $\mathbb{F}_2$, so we need to evaluate it on a, b, c. And we know that because the **homological** boundary map ∂ sends a, b, c to 0, we

$$\partial(\delta_v)(a) = \partial(\delta_v)(b) = \partial(\delta_v)(c) = 0,$$

and indeed we want our zeroth cohomology group to be $\mathbb{F}_2$, so this is what we expect. Turning our attention now to the next boundary map, we need to compute $\partial(\delta_a)$, a map from $\mathbb{Z}\{U, L\}$ to $\mathbb{F}_2$, so we need to figure out what it does to U and L. Since our **homological** boundary map takes U to $a + b - c$ and L to $c + a - b$, we know that

$$\partial(\delta_a)(U) = \delta_a(a + b - c) = 1,$$

and similarly $\partial(\delta_a)(L) = 1$. (If we were doing things with $\mathbb{Z}$-coefficients, we'd have to worry about signs for these boundary maps, but not here because we use $\mathbb{F}_2$-coefficients.) We can also find that $\partial(\delta_b)(U) = 1, \partial(\delta_b)(L) = -1, \partial(\delta_c)(U) = -1, \partial(\delta_c)(U) = 1$. But every -1 is also just a 1, again because we're working in $\mathbb{F}_2$. So because each of $\delta_a, \delta_b, \delta_c$ get sent to functions which take both δ_U and δ_L to 1, we have

$$\delta_a, \delta_b, \delta_c \mapsto \delta_U + \delta_L.$$

We can now fill in our chain complex for our Klein bottle:

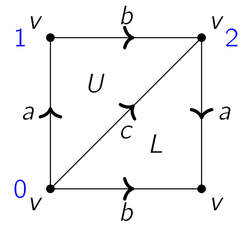
$$\cdots \to \mathbb{F}_2\{\delta_v\} \xrightarrow{0} \mathbb{F}_2\{\delta_a, \delta_b, \delta_c\} \xrightarrow{\partial} [\delta_a, \delta_b, \delta_c \mapsto \delta_U + \delta_L]\mathbb{F}_2\{\delta_U, \delta_L\} \to 0 \to \cdots,$$

so the zeroth cohomology group is $H^0(K; \mathbb{F}_2) \cong \mathbb{F}_2\{\delta_v\}$, a **one-dimensional vector space**, the first cohomology group is $H^1(K; \mathbb{F}_2) \cong \ker(\partial)$, which can be written in terms of various generators: two examples are $\mathbb{F}_2\{\delta_a + \delta_b, \delta_b + \delta_c\}$ or $\mathbb{F}_2\{\delta_a + \delta_b, \delta_a + \delta_c\}$ (we'll use the first one), which is a **two-dimensional vector space**. Finally, $H^2(K; \mathbb{F}_2) = \mathbb{F}_2\{\delta_U, \delta_L\}/(\delta_U + \delta_L)$, which is a **one-dimensional vector space**.

And now we should be able to compute the cup products too: we should be able to take two elements of $H^*(K; \mathbb{F}_2)$ and multiply them. For example,

$$(\delta_a + \delta_b)(\delta_a + \delta_b)$$

should be something that lives in the $H^2(K; \mathbb{F}_2)$ vector space, and in order to compute it, we need to look back at our geometric picture again, and we do this by figuring out what it does to U and L. We can order our vertices of U as shown, so that **edges always point from smaller to larger numbers**:



Here, the front 1-face (the first two coordinates) of U is a, and the back 1-face is b. So up to a sign that doesn't matter for us,

$$(\delta_a + \delta_b)(\delta_a + \delta_b)(U) = ((\delta_a + \delta_b)(a))((\delta_a + \delta_b)(b)) = 1 \cdot 1 = 1,$$

and similarly because the front face of L is c and the back face is a,

$$(\delta_a + \delta_b)(\delta_a + \delta_b)(L) = ((\delta_a + \delta_b)(c))((\delta_a + \delta_b)(a)) = 0 \cdot 1 = 0.$$

Therefore, the cup product result is δ_U (it's the map that sends U to 1 and L to 0), which is the nonzero element of our vector space $H^2(K; \mathbb{F}_2)$.

So if we say that $\alpha = \delta_a + \delta_b, \beta = \delta_b + \delta_c$ are the generators of the first cohomology group, and we also say that k is the generator of the second cohomology group, we've just found that $\boxed{\alpha^2 = k}$. Similarly,

$$(\delta_b + \delta_c)(\delta_b + \delta_c)(U) = ((\delta_b + \delta_c)(a))((\delta_b + \delta_c)(b)) = 0,$$

$$(\delta_b + \delta_c)(\delta_b + \delta_c)(L) = ((\delta_b + \delta_c)(c))((\delta_b + \delta_c)(a)) = 0,$$

so $\boxed{\beta^2 = 0}$. Finally,

$$(\delta_a + \delta_b)(\delta_b + \delta_c)(U) = ((\delta_a + \delta_b)(a))((\delta_b + \delta_c)(b)) = 1,$$

and

$$(\delta_a + \delta_b)(\delta_b + \delta_c)(L) = ((\delta_a + \delta_b)(c))((\delta_b + \delta_c)(a)) = 0,$$

so $\boxed{\alpha\beta = k}$. What we've found now is that

$$H^*(K; \mathbb{F}_2) \cong \mathbb{F}_2\{1\} \oplus \mathbb{F}_2\{\alpha\} \oplus \mathbb{F}_2\{\beta\} \oplus \mathbb{F}_2\{k\},$$

which are homogeneous in degree $0, 1, 1, 2$, respectively, with the boxed relations above, and also $k\alpha = k\beta = k^2 = 0$ (because there's nothing with degree larger than 2).

Fact 229

In our problem set, we may have used a CW structure instead of a semisimplicial set for the Klein bottle. But the cup product is only defined in terms of front and back faces, so that definition really only makes sense for semisimplicial sets.

30 November 13, 2020

Our final problem set will be posted on Monday, and it will be due on the last day of the semester that due dates can be assigned. It'll cover homological algebra, Ext, Tor, cohomology groups, and a bit of Poincare duality.

We're going to start computing cohomology rings for a variety of spaces so that we can use them for geometric purposes, and we should build up our toolbox for doing so. Sometimes, it's pretty easy to do this, so let's start with some "silly" rings:

Example 230

Let's compute the cohomology ring $H^*(S^2; \mathbb{Z})$.

We know that

$$H^m(S^2; \mathbb{Z}) = \underline{\text{Hom}}_{\text{Ab}}(H_m(S^2), \mathbb{Z}) \oplus (\text{Ext terms}),$$

but everything is free so the Ext terms will vanish. This means that because the homology groups are $\mathbb{Z}, 0, \mathbb{Z}$, the cohomology groups will also be $\boxed{\mathbb{Z}, 0, \mathbb{Z}}$. (Alternatively, we can use the minimal cell decomposition $0 \to \mathbb{Z} \to 0 \to \mathbb{Z} \to 0$ in degree 0 and 2, and dualize it to get $\mathbb{Z} \to 0 \to \mathbb{Z} \to$ in degree 0 and -2. This will give us the same answer.) Therefore,

$$H^*(S^2; \mathbb{Z}) = \mathbb{Z}\{1\} \oplus \mathbb{Z}\{x\},$$

where 1 is the degree-0 part and x is the degree-2 part. And we also need to describe the multiplicative structure, which follows

$$1 \cdot 1 = 1, \quad 1 \cdot x = x, \quad x \cdot 1 = x$$

because 1 is the multiplicative identity. (Indeed, recalling the graded-commutatve condition, we do have

$$x \cdot 1 = (-1)^{|x||1|} 1 \cdot x,$$

meaning that we don't need to talk about a left versus right identity.) Finally, because of degree reasons, we have

$$x \cdot x = 0.$$

Putting this all together, we have

$$H^*(S^2; \mathbb{Z}) \cong \mathbb{Z}[x]/x^2,$$

where x is homogeneous of degree 2. And we understood everything here using formal properties – we never had to dig into the geometry or topology of the sphere.

Similarly, for any ring R and any integer $n \geq 1$, we have

$$H^*(S^n; R) \cong R[x]/x^2,$$

where $|x| = n$ (the element x is homogeneous in degree n), and all products are forced. (This will always be isomorphic to $R \oplus R$ as an R-module.)

Example 231

Last time, we found that if K is the Klein bottle,

$$H^*(K; \mathbb{F}_2) \cong \mathbb{F}_2\{1\} \oplus \mathbb{F}_2\{\alpha\} \oplus \mathbb{F}_2\{\beta\} \oplus \mathbb{F}_2\{k\},$$

with $1, \alpha, \beta, k$ of homogeneous degree $0, 1, 1, 2$.

Furthermore, we found that $\alpha^2 = \alpha\beta = k$ and $\beta^2 = 0$, meaning that

$$\beta\alpha = (-1)^{1 \cdot 1} \alpha\beta = -k = k$$

by graded-commutativity and using that we're working in $\mathbb{F}_2$. And figuring out other multiplications can be done with just degree reasons and the identity element 1.

We can also look at a product like

$$(\alpha+\beta)(\beta) = \alpha\beta + \beta^2 = k + 0 = k,$$

because distributivity holds for our ring. The point is that we can present this ring with other bases as well, and at the end of the day we have

$$H^*(K; \mathbb{F}_2) \cong \mathbb{F}_2[\alpha, \beta, k]/(\alpha^2 - k, \alpha\beta - k, \alpha k, \beta k, k^2).$$

Remark 232. *We found this by using a particular finite semisimplicial set, and recall that we worked hard earlier on in the class to show that we can pick any semisimplicial set when we're computing* **homology** *groups. In order to understand why we can do something similar for computing our cup products and cohomology rings here, note that there is a map of seimsimplicial sets from our finite semisimplicial square diagram into* Sing(K)*. When we defined the cup product, we can think of everything we did last class as going on in the enormous semisimplicial set* Sing(K)*, and everything will still work.*

We'll get a glimpse now of how this ring structure can help us answer questions that aren't just directly about the computations:

Problem 233

Is there a continuous map $f : S^2 \to K$, such that $H^2(f; \mathbb{F}_2) : H^2(K; \mathbb{F}_2) \to H^2(S^2; \mathbb{F}_2)$ (cohomology reverses the direction of arrows) is nontrivial?

This is a map $\mathbb{F}_2 \to \mathbb{F}_2$, and we want to know whether there is a continuous map f such that the map is not the zero map. For example, sending all of S^2 to a single point wouldn't work.

To answer this, note that $f : S^2 \to K$ induces a map of rings, not just a map of cohomology groups. Specifically, we can look at the map

$$H^*(K; \mathbb{F}_2) \to H^*(S^2; \mathbb{F}_2),$$

which is actually a map

$$\mathbb{F}_2[\alpha, \beta, k]/(\alpha^2 - k, \alpha\beta - k, \alpha k, \beta k, k^2) \to \mathbb{F}_2[x]/x^2.$$

We're asking whether it's okay for this map to send k to x, and we know that because this is a map of **graded** rings, α, β must both be sent to 0 (there's no degree 1 part in the image). But then that means $\alpha^2 = k$ must be sent to 0 as well, so the answer is $\boxed{\text{no}}$.

Next, let's examine a construction that will help us start putting together cohomology rings. Suppose R is a ring, and A^* and B^* are two graded R-algebras. Then remember that $A^* \otimes_R B^*$ has a canonical graded R-algebra structure:

- If we forget the multiplication and grading, we're just tensoring two R-modules together, so this is indeed an R-module.
- If $a \in A^*$ is homogeneous of degree p, and $b \in B^*$ is homogeneous of degree q, then $a \otimes b$ is homogeneous of degree $p + q$. The multiplication is given by

  $$(a' \otimes b')(a \otimes b) = (-1)^{|b'||a|}(a'a \otimes b'b),$$

  where we're doing multiplications on A^* and B^* individually.

Two lectures ago, we stated the following result, and the sign $(-1)^{|b'||a|}$ is essential for it to work:

Theorem 234

If X and Y are topological spaces, then the cohomology cross product $\times : H^*(X;R) \otimes_R H^*(Y;R) \to H^*(X \times Y;R)$ is a homomorphism of graded R-algebras.

This is particularly useful because this is often an **isomorphism**, as long as the homology of one of the groups $H_q(X;R)$ is a finitely generated free R-module for all integers q.

Proof. We can see Miller's lecture 29 for more details, but our goal here is to not unravel the definition of the cross product too much. If $\alpha_1, \alpha_2 \in H^*(X;R)$ and $\beta_1, \beta_2 \in H^*(Y;R)$ are all homogeneous elements in their respective rings, we need to prove that the cross product interacts properly with the cup product:

$$(\alpha_1 \times \beta_1)(\alpha_2 \times \beta_2) = (-1)^{|\alpha_2||\beta_1|}((\alpha_1\alpha_2) \times (\beta_1\beta_2)).$$

We can now consider the difference between the maps

$$X \times Y \xrightarrow{\Delta_{X \times Y}} X \times Y \times X \times Y$$

and

$$X \times Y \xrightarrow{\Delta_X \times \Delta_Y} X \times X \times Y \times Y,$$

where there is an isomorphism from the two images given by $1 \times \text{Swap} \times 1$. Note that

$$(\alpha_1 \times \beta_1)(\alpha_2 \times \beta_2) = H^*(\Delta_{X \times Y})(\alpha_1 \times \beta_1 \times \alpha_2 \times \beta_2)$$

where we're using the associativity of the cross product. And now **cohomology is a functor**, so this can be written as

$$= (H^*(\Delta_x \times \Delta_y) \circ H^*(1 \times \text{Swap} \times 1))\,(\alpha_1 \times \beta_2 \times \alpha_2 \times \beta_2),$$

which can be re-parenthesized as

$$= H^*(\Delta_x \times \Delta_y)\,(H^*(1 \times \text{Swap} \times 1)(\alpha_1 \times \beta_2 \times \alpha_2 \times \beta_2))\,,$$

which is

$$= H^*(\Delta_x \times \Delta_y)\left((-1)^{|\beta_1||\alpha_2|}\alpha_1 \times \alpha_2 \times \beta_2 \times \beta_2)\right),$$

and this sign shows up because we're evaluating this Swap isomorphism. This indeed leaves us with

$$(-1)^{|\beta_1||\alpha_2|}H^*(\Delta_x \times \Delta_y)\,(\alpha_1 \times \alpha_2 \times \beta_2 \times \beta_2)) = (-1)^{|\beta_1||\alpha_2|}(\alpha_1\alpha_2) \times (\beta_1 \times \beta_2),$$

as desired. □

Example 235

How can we compute the cohomology ring $H^*(T;\mathbb{Z})$ for a torus T?

Noting that

$$H^*(T;\mathbb{Z}) = H^*(S^1 \times S^1;\mathbb{Z}),$$

we have a cross product

$$H^*(S^1;\mathbb{Z}) \otimes_{\mathbb{Z}} H^*(S^1;\mathbb{Z}) \to H^*(X^1 \times S^1;\mathbb{Z})$$

which is an isomorphism because we have free modules for the S^1s. Because $H^*(S^1;\mathbb{Z}) \cong \mathbb{Z}\{1\} \oplus \mathbb{Z}\{a\}$, where the degree of a is 1 and $a^2 = 0$, we can take a copy $\mathbb{Z}\{1\} \oplus \mathbb{Z}\{a\}$ and another copy $\mathbb{Z}\{1\} \oplus \mathbb{Z}\{b\}$, and their tensor product

$$H^*(S^1;\mathbb{Z}) \otimes_{\mathbb{Z}} H^*(S^1;\mathbb{Z}) \cong \mathbb{Z}\{1 \otimes 1\} \oplus \mathbb{Z}\{a \otimes 1\} \oplus \mathbb{Z}\{1 \otimes b\} \oplus \mathbb{Z}\{a \otimes b\},$$

where the individual summands have degree $0, 1, 1, 2$. (Indeed, a rank 2 free $\mathbb{Z}$-module tensored with a rank 2 free $\mathbb{Z}$-module should give us a rank 4 free $\mathbb{Z}$-module.) We now just need to figure out the multiplications – $1 \otimes 1$ needs to be the identity, and degree reasons mean that the only interesting thing left is products of degree-1 things. We know that

$$(a \otimes 1)(1 \otimes b) = (-1)^{0 \cdot 0}(a1 \otimes 1b) = a \otimes b,$$

$$(1 \otimes b)(a \otimes 1) = (-1)^{1 \cdot 1}(1a \otimes b1) = -a \otimes b,$$

so indeed if we define $a \otimes 1 = x, 1 \otimes b = y, a \otimes b = z$, we have $xy = z, yx = -z$ (which is what we want for graded-commutativity). Next,

$$(a \otimes 1)(a \otimes 1) = (-1)^{0 \cdot 1}(a^2 \otimes 1^2) = (-1)^0(0 \otimes 1) = 0,$$

so $x^2 = 0$, and similarly $y^2 = 0$. What we end up with is

$$H^*(T;\mathbb{Z}) \cong \mathbb{Z}[x, y, z]/(xy - z, x^2, y^2, xz, zy),$$

and we don't need to write down the $yx + z$ relation because it follows from graded-commutativity. (Note that we also know that $x^2 = x \cdot x = (-1)^{1 \cdot 1} x \cdot x = -x^2$, so we could have already learned that $x^2 = 0$ by graded-commutativity alone.) And we should be careful – if we saw this in algebra class, we would think that $yx = z$ as well, but that's not the case.

31 November 16, 2020

Our final homework assignment will be posted tonight (as discussed last time), and we'll have about three weeks, including Thanksgiving break, to complete it. We'll probably know everything we need to solve the problems by the end of this week – doing the homework problems is the most important part of this class! Also, Professor Haynes Miller is requesting comments on the notes, and there is a publishing deadline in December, so we can email any comments that we have.

We'll continue discussing basic techniques for assembling cohomology rings, and we'll start with an algebraic comment. If A^* and B^* are graded R-algebras, for example coming from cohomology with coefficients in R, then $A^* \oplus B^*$ (the direct sum of the underlying abelian groups) is also a graded R-algebra. This is easier to think about than the tensor product we discussed previously: we get a pointwise multiplication

$$(a, b)(a', b') = (aa', bb')$$

with unit $(1, 1)$. (And the homogeneous degree-p pieces are generated by $(a, 0)$ with $|a| = p$, as well as $(0, b)$ with $|b| = p$, so they end up being a homogeneous degree-p thing in each component.)

In category theory language, this object is the product of A^* and B^* in the category of graded R-modules. Let's see how we can use this for topology:

Proposition 236

Suppose X and Y are topological spaces. Then $H^*(X \sqcup Y; R) \cong H^*(X; R) \oplus H^*(Y; R)$ as R-algebras.

Proof. The inclusion $X \hookrightarrow X \sqcup Y$ induces a map of graded R-algebras $H^*(X \sqcup Y; R) \to H^*(X; R)$, and similarly $Y \hookrightarrow X \sqcup Y$ gives us a map $H^*(X \sqcup Y; R) \to H^*(Y; R)$. But by the universal property of the categorical product, this induces a map

$$H^*(X \sqcup Y; R) \to H^*(X; R) \sqcup H^*(Y; R).$$

Because of the way we've constructed this map, this is already a map of graded R-algebras, and we just need to show this is an isomorphism by showing it is a bijection (forgetting about the ring structure). But then the cohomology **groups** of a disjoint union can indeed be written this way by unpacking the definitions, and this is left as an exercise (using the universal coefficients theorem). □

We can talk now about wedges – we briefly looked at this when defining the figure-8 graph $S^1 \vee S^1$ or when working with cellular homology.

Definition 237

Suppose (X, x) is a topological space along with a point $x \in X$, and let (Y, y) be a topological space along with $y \in Y$. Then

$$X \vee Y = (X \sqcup Y)/(x \sim y).$$

Even though we don't explicitly say which point we're using to "glue" the two spaces together, the resulting wedge does depend on our choice of x and y.

Recall that we discovered that wedges don't change homology groups except in degree 0 (since we're just quotienting out by a single point, there's a long exact sequence which is only "interesting" in degree 0). Something similar turns out to happen for cohomology: if $r : X \sqcup Y \to X \vee Y$ denotes the quotient map, we can use the long exact sequence of a pair of spaces, which tells us (as we just said) that

$$H_q(X \sqcup Y) \to H_q(X \vee Y)$$

is an isomorphism for all integers $q > 0$. Thus, the universal coefficients theorem tells us that $H^q(X \vee Y; R) \to H^q(X \sqcup Y; R)$ is also an isomorphism for all $q > 0$:

Proposition 238

The quotient map $r : X \sqcup Y \to X \vee Y)$ induces a map of graded R-algebras

$$H^*(X \vee Y; R) \to H^*(X \sqcup Y; R)$$

which is an isomorphism in positive degrees and an injection in degree 0.

In other words, we can look at the larger ring and compute multiplications in the disjoint union instead of the wedge. And we often know what degree-0 groups looks explicitly, because it can be expressed in terms of path components.

Example 239

Suppose we have a topological space $S^2 \vee S^1 \vee S^1 = (S^2 \vee S^1) \vee S^1$. Then $H^*(S^2 \vee S^1 \vee S^1; \mathbb{Z})$ injects into $H^*(S^2; \mathbb{Z}) \oplus H^*(S^1; \mathbb{Z}) \oplus H^*(X^1; \mathbb{Z})$.

The direct sum of cohomology rings (that is, the image of the injection) looks like

$$(\mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}) \oplus (\mathbb{Z} \oplus \mathbb{Z}) \oplus (\mathbb{Z}),$$

where the first three $\mathbb{Z}$s are degree 0, generated by $(1_{S^2}, 0, 0), (0, 1_{S^1}, 0), (0, 0, 1_{S^1})$ (where the 1s denote the units in the individual cohomology rings for the spheres), the next two $\mathbb{Z}$s are degree 1, generated by $(0, x, 0)$ and $(0, 0, x)$, where the x's are generators of the corresponding groups $H^1(S^1; \mathbb{Z})$, and the last $\mathbb{Z}$ is of degree 2, generated by $(y, 0, 0)$, where y is a generator of $H^2(S^2; \mathbb{Z})$.

So to find the cohomology of the wedge $S^2 \vee S^2 \vee S^1$, we need to take the subring of this generated as an abelian group where we take all of the generators in positive degrees, $(0, x, 0), (0, 0, x)$, and $(y, 0, 0)$. But in degree 0, we only have one generator, because the wedge only has one path-component. Thus,

$$H^*(S^2 \vee S^1 \vee S^1; \mathbb{Z}) \cong \mathbb{Z} \oplus (\mathbb{Z} \oplus \mathbb{Z}) \oplus \mathbb{Z},$$

where the $\mathbb{Z}$s are in degree $0, 1, 1, 2$, and all products of positive-degree classes are zero (because each one comes from a different sphere). And this allows us to answer some other questions:

Example 240

Is $S^2 \vee S^1 \vee S^1$ homotopy equivalent to the torus T?

We can more easily prove that these two maps are not homeomorphic – for example, we can remove a point from the wedge and make it not path-connected anymore. So these two shapes may not look very similar to us, but we haven't developed any tools until now for proving things about why they're not homotopy equivalent (because it has the same homology groups as the torus)!

In other words, none of the algebraic invariants connected with homology have been able to distinguish these two spaces so far, but the one exception is the ring structure in cohomology (or the coring structure in homology). And indeed, $H^*(S^2 \vee S^1 \vee S^1; \mathbb{Z})$ has trivial positive-degree products, while the cohomology ring of the torus $H^*(T; \mathbb{Z}) \cong \mathbb{Z}[x, y, z]/(x^2, y^2, xy - z, z^2, xz, yz)$, where x, y are in degree 1 and z is in degree 2. In particular, this means $xy = z$, so there is a product of two positive-degree elements which is not zero. So $S^2 \vee S^1 \vee S^1 \not\cong T$ in Ho(Top), and one way to think about that is that the diagonal maps

$$(S^2 \vee S^1 \vee S^1) \to (S^2 \vee S^1 \vee S^1) \times (S^2 \vee S^1 \vee S^1), \quad T \to T \times T$$

induce different maps in homology (meaning their cohomology rings are different).

With this, we now have all of the most basic tools for assembling cohomology rings, and everything from here will be less elementary. Our focus will be on **Poincaré duality**, but we'll learn further tools for computing cohomology rings if we take 18.906 or go further into algebraic topology. We'll present the main ideas and statements this week so that we can work on the homework, and then we'll spend some time afterward doing the relevant proofs.

First of all, suppose A is an abelian group and R is a ring. Then there is a map

$$f : \underline{\text{Hom}}_{\text{Ab}}(A, R) \otimes_{\mathbb{Z}} A \to R,$$

which is adjoint to the identity $g : \underline{\text{Hom}}_{\text{Ab}}(A, R) \to \underline{\text{Hom}}_{\text{Ab}}(A, R)$ under the currying isomorphism. (This is because the tensor product was specifically designed to give us a bijection between maps of the forms f and g above.) So now if M is an R-module, we have a mp

$$f : \underline{\text{Hom}}_{\text{R-mod}}(M, R) \otimes_R M \to R,$$

similarly adjoint to the identity $g : \underline{\text{Hom}}_{\text{R-mod}}(M, R) \to \underline{\text{Hom}}_{\text{R-mod}}(M, R)$. Describing f explicitly is not too hard: if we have a map $M \to R$ and an element $m \in M$, we define $f(\phi \otimes m) = \phi(m)$, so this is an "evaluation map."

There's a variant of this for cohomology, caled the **Kronecker pairing**. If X is a topological space and R is a ring, then we have a pairing map

$$\langle \cdot, \cdot \rangle : H^q(X; R) \otimes_R H_q(X; R) \to R.$$

To define this pairing, note that a class in the **cohomology** group $H^q(X; R)$ can be represented by a cocycle in $S^q(X; R)$, modulo coboundaries, and we have our favorite explicit cochain complex designed to compute cohomology groups for us. And an object in $S^q(X; R)$ is a function on $\text{Sing}_q(X)$, or equivalently an abelian group map defined on the simplices $S_q(X)$. On the other hand, a class in the **homology** group $H_q(X; R)$ is a formal R-linear combination of classes in $\text{Sing}_q(X)$ that happens to form a cycle, modulo boundaries. So now we can take any function $\text{Sing}_q(X) \to R$ and evaluate it on a formal combination of classes in $\text{Sing}_q(X)$, and that's exactly what the pairing map does.

So this gives us a map

$$S^q(X; R) \otimes_R S_q(X; R) \to R,$$

which extends to a chain map $S^*(X; R) \otimes_R S_*(X : R) \to R$, where the left-hand side is a tensor product of chain complexes and the right-hand side is concentrated in degree 0. That finally gives us our desired map $H^q(X; R) \otimes_R H_q(X; R) \to R$, and we're claiming that this is well-defined – if we modify our cocycle by a coboundary, it doesn't change the result of the Kronecker pairing. We'll use this pairing next time to state Poincaré duality.

32 November 18, 2020

Last time, we discussed the Kronecker pairing (also called the "cap product"), the last product that we'll be needing in this class. This pairing is a map $H^q(X; R) \otimes_R H_q(X; R) \to R$, computed by evaluating the sum of maps on simplices (in $H^q(X; R)$) on the particular formal sums of simplices (in $H_q(X; R)$). We'll use this to state Poincaré duality, but we'll need to do some setup first.

Proposition 241

Let X be a finite type CW complex. Then the map $H^q(X; \mathbb{F}_2) \to \underline{\text{Hom}}_{\mathbb{F}_2\text{-mod}}(H_q(X, \mathbb{F}_2), \mathbb{F}_2)$ is an isomorphism, where the map is adjoint to the Kronecker paiirng.

We're basically claiming that if we have finiteness assumptions (the CW structure only has finitely many cells in each dimension, giving us finite-dimensional vector spaces) and we're working with coefficients in $\mathbb{F}_2$, cohomology and homology are **linear duals** in a canonical way involving the Kronecker product. We'll prove this on our homework – the result is also true if we replace $\mathbb{F}_2$ with another field, but we really only need it for $\mathbb{F}_2$ in this class.

The above proposition shows that the Kronecker pairing satisfies a particular condition:

Definition 242

A **perfect pairing** of two finitely-generated free R-modules V and W is an R-linear map $V \otimes_R W \to R$, such that the adjoint map $V \to \underline{\text{Hom}}_R(W, R)$ is an isomorphism of R-modules.

(This is basically a repackaging of a specific chosen isomorphism between V and the dual of W through the currying isomorphism.) And with this, we can start stating Poincaré duality, which is a useful fact about cup products in manifolds:

Definition 243

An ***n*-dimensional manifold** M is a Hausdorff topological space, where every point has an open neighborhood homeomorphic to $\mathbb{R}^n$.

Example 244

A 2-dimensional manifold is called a **surface**, and examples of 2-dimensional manifolds include $S^2, T, K, \mathbb{RP}^2$ and $\mathbb{R}^2$ itself (even though this last one is not compact). A non-example of a manifold is $S^2 \wedge S^2$, because there is no neighborhood around the wedge point which looks like $\mathbb{R}^2$.

Example 245

Examples of 3-dimensional manifolds include $\mathbb{R}^3$, S^3, $S^1 \times S^1 \times S^1$, and $\mathbb{RP}^3$.

Many people spend time trying to classify manifolds, because they're applicable in other fields as well – smooth algebraic varieties over $\mathbb{R}$ or $\mathbb{C}$ and configuration spaces in physics systems are both often manifolds.

Fact 246 (from point-set topology)

Any **compact** manifold is homotopy equivalent to a finite type CW complex.

This, in particular, implies that the homology and cohomology groups of any compact manifold are finitely generated.

Theorem 247 (Poincaré duality, version 1)

Let M be a compact n-dimensional manifold. Then there exists a unique class (element) $[m] \in H_n(M; \mathbb{F}_2)$, called the **fundamental class**, such that for all integers p, q with $p + q = n$, the map

$$H^p(M; \mathbb{F}_2) \otimes_{\mathbb{F}_2} H^q(M; \mathbb{F}_2) \xrightarrow{\cup} H^n(M; \mathbb{F}_2) \xrightarrow{\langle \cdot, [m] \rangle} \mathbb{F}_2$$

(the composition of the cup product and the Kronecker pairing) is a perfect pairing.

(Another way to say this is that we take the cup product, and then we evaluate that result on the fundamental class.) This can be restated as saying that $H^p(M; \mathbb{F}_2)$ is canonically isomorphic to the $\mathbb{F}_2$-linear dual of $H^q(M; \mathbb{F}_2)$ whenever $p + q = n$ (the adjoint map of the perfect pairing is an isomorphism).

Even if we don't talk about cup products at all and we're only interested in homology or cohomology groups, this is still a very strong constraint:

Example 248

Suppose M is a compact 3-dimensional manifold with $H^0(M; \mathbb{F}_2) \cong \mathbb{F}_2 \oplus \mathbb{F}_2$ (meaning it has two path components) and $H^1(M; \mathbb{F}_2) \cong \mathbb{F}_2 \oplus \mathbb{F}_2 \oplus \mathbb{F}_2$.

Now we can actually ask ourselves for **all** of the homology and cohomology groups of M (with $\mathbb{F}_2$ coefficients). For example, because $1 + 2 = 3$,

$$H^2(M; \mathbb{F}_2) \cong \underline{\text{Hom}}(H^1(M; \mathbb{F}_2), \mathbb{F}_2) \cong \mathbb{F}_2 \oplus \mathbb{F}_2 \oplus \mathbb{F}_2,$$

because the dual of a three-dimensional vector space is still a three-dimensional vector space. Similarly, because $0+3=3$, we also know that $H^3(M;\mathbb{F}_2) \cong \mathbb{F}_2 \oplus \mathbb{F}_2$. (And all other cohomology groups vanish, and we can actually prove this because $-1+4=3$ and the negative cohomology groups are all 0.) And furthermore, by Proposition 241, we know that homology is adjoint to cohomology, so $H_1(M;\mathbb{F}_2)$ is isomorphic to $H^1(M;\mathbb{F}_2)$ and so on. Therefore,

$$H_q(M;\mathbb{F}_2) = \begin{cases} \mathbb{F}_2 \oplus \mathbb{F}_2 & q = 0, 3 \\ \mathbb{F}_2 \oplus \mathbb{F}_2 \oplus \mathbb{F}_2 & q = 1, 2 \\ 0 & \text{otherwise.} \end{cases}$$

But in fact, Poincaré duality is saying something not just about the cohomology groups but also cup products – let's see some examples to get a feeling for that.

Example 249

Recall that we've previously computed $H^*(T;\mathbb{F}_2) \cong \mathbb{F}_2[x,y,z]/(xy-z, x^2, y^2, xz, yz, z^2) \cong \mathbb{F}_2\{1\} \oplus \mathbb{F}_2\{x,y\} \oplus \mathbb{F}_2\{z\}$, and this is a commutative ring because we're working with $\mathbb{F}_2$-coefficients.

Let's see what Poincaré duality is claiming for us here. First of all, there should be some fundamental class in T, which is an element of the second homology group $H_2(T;\mathbb{F}_2) \cong \mathbb{F}_2\{\delta_z\}$ (we're using again that the second homology group can be viewed as the dual of the second cohomology group). This fundamental class turns out to be δ_z, and now we can think about how this works out in practice: we're supposed to get a Poincaré duality perfect pairing

$$P : H^1(T;\mathbb{F}_2) \otimes_{\mathbb{F}_2} H^1(T;\mathbb{F}_2) \to \mathbb{F}_2,$$

which first takes the cup product and then pairs against the fundamental class: for instance,

$$P(x \otimes y) = \langle xy, \delta_z \rangle.$$

But we've computed that $xy = z$, so this is just

$$\langle z, \delta_z \rangle = \delta_z(z) = 1.$$

We can also find that

$$P(x \otimes x) = \langle x^2, \delta_z \rangle = \langle 0, \delta_z \rangle = \delta_z(0) = 0,$$

and similarly $P(y \otimes y) = 0$. But we also get a Poincaré duality for $H^0(T;\mathbb{F}_2) \otimes_{\mathbb{F}_2} H^2(T;\mathbb{F}_2) \to \mathbb{F}_2$: the only nonzero result comes from taking the nonzero elements in each of H^0 and H^2, which yields

$$P(1 \otimes z) = \langle 1z, \delta_z \rangle = \delta_z(z) = 1.$$

So we've checked how the pairings work in the torus, but now we want to think about how this pairing being perfect can help us compute the cohomology rings that we haven't previously computed.

Example 250

Let's compute the cohomology ring $H^*(\mathbb{RP}^2;\mathbb{F}_2)$ (using that $\mathbb{RP}^2$ is a two-dimensional manifold, because it has a canonical CW structure coming from a square).

We know that $H^0(\mathbb{RP}^2;\mathbb{F}_2) \cong \mathbb{F}_2$, because there's one path component and H^0 is the set of maps from path components into $\mathbb{F}_2$ (or alternatively it's the dual of the zeroth homology group). Also, $H^1(\mathbb{RP}^2;\mathbb{F}_2) \cong \mathbb{F}_2$ as well

– this is something we know from the cell decomposition of $\mathbb{RP}^2$, and it's not something Poincaré duality gives us directly. But now we are able to say that because $0 + 2 = 2$,

$$H^2(\mathbb{RP}^2; \mathbb{F}_2) \cong H^0(\mathbb{RP}^2; \mathbb{F}_2)$$

(which avoids needing to use the specifics of the cell decomposition), and also everything in higher dimensions is just 0. So now we can say that our cohomology ring (as a graded $\mathbb{F}_2$-vector space) is

$$H^*(\mathbb{RP}^2; \mathbb{F}_2) \cong \mathbb{F}_2\{1\} \oplus \mathbb{F}_2\{a\} \oplus \mathbb{F}_2\{b\}$$

where these $\mathbb{F}_2$s are in dimensions $0, 1, 2$ respectively. And now we just need to compute the cup products: multiplying anything with 1 gives us the original element, and $b^2 = ab = 0$ for degree reasons, and thus we only need to compute a^2 and see whether it's 0 or b. But Poincaré duality says that we have a perfect pairing

$$P : H^1(\mathbb{RP}^2) \otimes H^1(\mathbb{RP}^2) \to \mathbb{F}_2,$$

and a pairing from $\mathbb{F}_2 \otimes_{\mathbb{F}_2} \to \mathbb{F}_2$ is a **perfect** pairing only if it's a nontrivial map (a trivial map doesn't induce an isomorphism for the adjoint map after we apply currying). That means that

$$\langle a^2, [\mathbb{RP}^2] \rangle \neq 0,$$

where $[\mathbb{RP}^2]$ denotes the fundamental class of $\mathbb{RP}^2$. But pairing with 0 gives us 0, so indeed we must have $a^2 = b$, and we've finished describing the cohomology ring:

$$H^*(\mathbb{RP}^2; \mathbb{F}_2) \cong \mathbb{F}_2[a, b]/(a^2 - b, ab, b^2),$$

where the degree of a is 1 and the degree of b is 2.

Remark 251. *If we look at the cohomology ring $H^*(S^2 \vee S^1; \mathbb{F}_2)$ instead, that looks like $\mathbb{F}_2[a, b]/(a^2, ab, b^2)$. So the cohomology groups (as $\mathbb{F}_2$-vector spaces) are the same for $S^2 \vee S^1$ and $\mathbb{RP}^2$, but the ring structures are different – $a^2 = b$ in one case and $a^2 = 0$ in the other. And we can verify that $S^2 \vee S^1$ is not a manifold.*

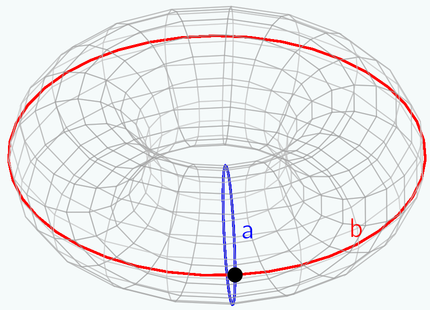
Next time, we'll talk more about the intuition behind Poincaré duality and how it works beyond $\mathbb{F}_2$ coefficients.

33 November 20, 2020

Today's the last day before Thanksgiving break, so we'll discuss a bit about geometric intuition behind Poincaré duality. After break, we'll do a proof of Poincaré duality that doesn't have very much to do with the geometry, but it's still helpful to have an image in mind.

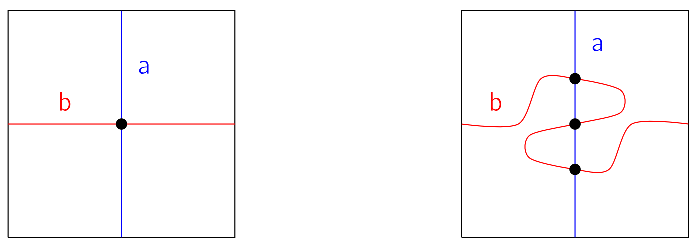
Example 252

Consider a torus T, and suppose we have the two standard loops a and b shown:



Even though a and b are geometric pictures of cycles, they also represent generators of $H_1(T; \mathbb{F}_2) \cong \mathbb{F}_2\{a, b\}$. We want to ask how many times a and b intersect, modulo 2 – the claim is that generically, no matter how we deform a and b, the two will intersect an odd number of times.

To see this, let's start with the standard a and b and look around a neighborhood of their unique intersection. a and b can then either look like straight lines, or we could have some small deformation:



The two ways of drawing b are homotopic, but the number of intersections is indeed different. It is odd in both cases, though, and it's worth convincing ourselves why this always happens generically.

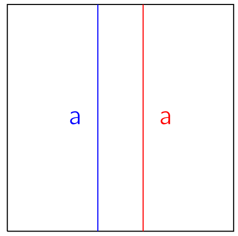
There are indeed special cases, like tangent intersections, where the total number of intersections is even. But those cases are always unstable, because deforming a or b slightly will return us to an odd number of intersections. If we take a class on smooth manifolds, the word for "generic" becomes "transverse," and thus the unstable situation comes up because we don't have transverse intersections. (So that's not really a topology question.) This is why it's nice that the cup product, a purely topological (in fact, homotopy-related) quantity, can help us analyze this intersection number – we don't have to do any of the smooth geometry that we otherwise would!

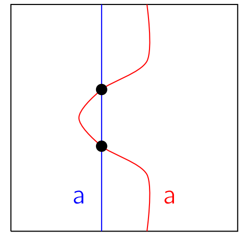
From here, we can try to ask stranger questions:

Example 253

How many times does a curve a intersect a? Here, we take two generic representations of the class a (rather than taking the same curve twice).

This time, if we draw some more pictures, we'll find that the intersection number is always even, though (again) it can be hard to make this intuition rigorous:





So now we can define an $\mathbb{F}_2$-module homomorphism

$$f_a : H_1(T; \mathbb{F}_2) \to \mathbb{F}_2,$$

which is a map $\mathbb{F}_2\{a, b\} \to \mathbb{F}_2$, and we'll define it to send a to 0 and b to 1 (because this function counts the **number of intersections with a, modulo** 2). But as we'll show on our homework, the first cohomology group $H^1(T; \mathbb{F}_2)$ should be isomorphic to $\underline{\text{Hom}}_{\mathbb{F}_2\text{-mod}}(H_1(T; \mathbb{F}_2), \mathbb{F}_2)$ the linear dual of the first homology group, so f_a represents a class in the cohomology group $H^1(T; \mathbb{F}_2)$ of the torus. This class is exactly the **Poincaré dual** of a.

In general, suppose we have a compact n-dimensional manifold M, and let p and q be two integers with $p + q = n$. If we fix a cycle $a \in H_p(M; \mathbb{F}_2)$, then for any $b \in H_q(M; \mathbb{F}_2)$, we can look at how many times a and b intersect.

Fact 254

The point is that whenever $p + q = n$, generic representatives of a and b will intersect at some number of points, rather than in a higher dimension. For example, a plane and line intersect at a point generically in $\mathbb{R}^3$, but two planes intersect at a line.

And it turns out that the number of points mod 2 is independent of the generic choice of geometric representative for a and b, so we can fix $a \in H_p(M; \mathbb{F}_2)$ and think about varying $b \in H_q(M; \mathbb{F}_2)$. Then we get a well-defined function f_a, which takes in possible q-cycles and spits out the number of intersections with a. And again, duality tells us that this f_a can be viewed as a class in $H^q(M; \mathbb{F}_2)$.

So now we can explain that the Poincaré duality theorem, geometrically, tells us that this definition of $a \mapsto f_a$ gives us an isomorphism $H_p(M; \mathbb{F}_2) \cong H^q(M; \mathbb{F}_2)$. In other words, **a cycle, modulo boundaries, is exactly determined by the parity of how many times it intersects all other cycles**! Remember that we had the pairing

$$P : H^q(X; \mathbb{F}_2) \otimes H^p(X; \mathbb{F}_2) \to \mathbb{F}_2$$

is a map that sends $u \otimes v \mapsto \langle uv, [M] \rangle$ for the fundamental class $[M]$. Furthermore, we said that this pairing $\langle \cdot, \cdot \rangle$ is perfect, meaning that we induce an isomorphism

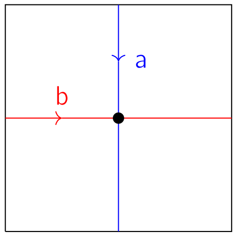
$$H^q(M; \mathbb{F}_2) \cong \underline{\text{Hom}}(H^p(X; \mathbb{F}_2), \mathbb{F}_2) \cong H_p(X; \mathbb{F}_2).$$

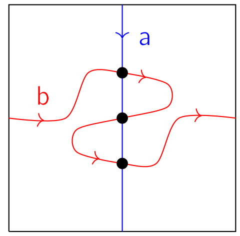
The claim, then, is that these two ways of constructing an isomorphism – one rigorously, and one more geometric – from H_p to H^q are the same. We won't actually show the connection to geometry rigorously in this class (it's beyond the scope), but even the formal existence of such a structure on the cup product has many applications of its own.

Remark 255. *The fundamental class $[M] \in H_n(M; \mathbb{F}_2)$ is supposed to be a canonical object Poincaré dual to a class in $H^0(M; \mathbb{F}_2)$. Remembering that $H^0(M; \mathbb{F}_2)$ sits inside $H^*(M; \mathbb{F}_2)$, the canonical class to use here is the function* 1.

It's also worth asking whether there is a version of Poincaré duality for integer coefficients, since we've been spending the whole class developing theory for arbitrary coefficient rings. The answer is yes, but it's more complicated and only works for **oriented manifolds**, and we'll try to explain it geometrically now. (But this won't come up on our homework.)

If we return to our torus, we do have an oriented manifold, and we can consider the standard picture of a and b intersecting again. As we've seen with semisimplicial sets, homology classes come with a direction: this time, we can draw arrows on a and b. If we now consider the three-intersection picture, we can zoom in on each of the three intersections individually.





The first and third intersections look similar to the original one we had, but the second one has b going to the left instead of the right! So our pictures differ by a reflection (a matrix with determinant -1), and thus we can count the number of intersections between our classes by taking sign into consideration. And now because the torus is oriented, our signed count will always be 1. (With something like the Klein bottle, we'll get issues arising from the torsion in the first homology group. But we'll be more rigorous about this next class.)

In the last few minutes of class, we'll see a few more applications of Poincaré duality with $\mathbb{F}_2$-coefficients. We computed last time that $H^*(\mathbb{RP}^2, \mathbb{F}_2) \cong \mathbb{F}_2[a]/a^3$, and we can more generally $H^*(\mathbb{RP}^n; \mathbb{F}_2)$ for any n.

Example 256

Let's compute the ring $H^*(\mathbb{RP}^3; \mathbb{F}_2)$: as a graded vector space, we know that

$$H^*(\mathbb{RP}^3; \mathbb{F}_2) \cong \mathbb{F}_2\{1\} \oplus \mathbb{F}_2\{a\} \oplus \mathbb{F}_2\{b\} \oplus \mathbb{F}_2\{c\},$$

where $1, a, b, c$ are in degree $0, 1, 2, 3$.

(Notice that we indeed see the characteristic "symmetry" around the middle dimension.) We have a pairing $H_1(\mathbb{RP}^2) \otimes_{\mathbb{F}_2} H^2(\mathbb{RP}^2) \to \mathbb{F}_2$, which can be viewed a map $\mathbb{F}_2\{a \otimes b\} \to \mathbb{F}_2$ sending $P(a \otimes b) = \langle ab, [\mathbb{RP}^3] \rangle$. But for this pairing to be perfect, it must be the nontrivial map from $\mathbb{F}_2$ to $\mathbb{F}_2$, so we must have $ab = c$ (the nontrivial element in degree 3). And continuing with these kinds of arguments, we can completely determine that

$$H^*(\mathbb{RP}^n) \cong \mathbb{F}_2[a]/a^{n+1},$$

where the degree of a is 1. (So it's always going to be a "truncated polynomial algebra" in a.)

Theorem 257 (Borsak-Ulam)

Let $f : S^n \to \mathbb{R}^n$ be any continuous function. Then there exists an $x \in S^n$ such that $f(x) = f(-x)$.

If we set $n = 2$, this tells us that any function from the two-dimensional sphere to the plane intersects itself at two antipodal points. So some antipodal set of points on the earth will share the same temperature and also the some air pressure (because we can encode that information in an ordered pair)!

Proof. Suppose for the sake of contradiction that no such x exists. Then we can consider the continuous function $g : S^n \to S^{n-1}$ with

$$x \mapsto \frac{f(x) - f(-x)}{|f(x) - f(-x)|}.$$

(This is well-defined exactly because of our assumption.) Now $g(-x) = -g(x)$ (antipodal points are sent to antipodal points), so there is a continuous map $\overline{g} : \mathbb{RP}^n \to \mathbb{RP}^{n-1}$ that is induced by g. (That is, we can quotient out by antipodal points, and g respects that.)

We claim now that $H_1(\overline{g}; \mathbb{F}_2)$ is nontrivial – this is a fact about paths, and part of the idea is that a path from the north pole to the south pole becomes a loop (cycle) when we identify those two points. If we believe this fact, then $H^1(\overline{g}; \mathbb{F}_2)$ is also nontrivial, because it's dual to a nontrivial element in homology. So that gives us a ring map

$$H^*(\overline{g}; \mathbb{F}_2) : H^*(\mathbb{RP}^{n-1}; \mathbb{F}_2) \to H^*(\mathbb{RP}^n; \mathbb{F}_2),$$

which is a map $\mathbb{F}_2[a]/a^n \to \mathbb{F}_2[a]/a^{n+1}$. But $H_1(\overline{g}; \mathbb{F}_2)$ being nontrivial means $a \mapsto a$, which means $0 = a^n \mapsto a^n$, which is a contradiction. $\square$

34 November 30, 2020

This week, we'll be looking at the proof of Poincaré duality and formulating similar results for coefficients besides $\mathbb{F}_2$. The reason we're being a bit relaxed about whether we understand the proof (instead focusing on how to apply the result on our homework) is that this one is concrete and more "low-tech:" we'll learn some new tools which lead to different and perhaps easier proofs of Poincaré duality if we continue studying algebraic topology.

The version of Poincaré duality that we're going to formulate is going to hold for manifolds with some extra structure: specifically, the theorem will hold for manifolds equipped with an R-**orientation**, which doesn't necessarily exist for every manifold (and isn't necessarily unique). But all manifolds are indeed $\mathbb{F}_2$-orientable, and that $\mathbb{F}_2$-orientation is indeed unique. So the main goal for today will be to discuss this notion of orientation, which will be required for the precise statement of Poincaré duality in more generality.

Example 258

For some geometric intuition, topological spaces like the Klein bottle, Möbius strip, or $\mathbb{RP}^2$ are not $\mathbb{Z}$-orientable, but the torus is $\mathbb{Z}$-orientable.

Before discussing orientation in general, we need to look locally near a point (where things look like $\mathbb{R}^n$):

Definition 259

Let M be an n-dimensional manifold, and let $x \in M$ be a point. The **local homology** of M at x is $H_*(M, M - \{x\})$.

The pair $(M, M - \{x\})$ is not one of the "nicely-behaved" pairs where we can just look at the homology of the quotient. But by excision, we do know that

$$H_*(M, M - \{x\}) \cong H_*(\mathbb{R}^n, \mathbb{R}^n - \{x\}),$$

and the **long exact sequence** of the pair $(\mathbb{R}^n, \mathbb{R}^n - \{x\})$ tells us that

$$H_*(\mathbb{R}^n, \mathbb{R}^n - \{x\}) \cong \begin{cases} \mathbb{Z} & \text{degree } n \\ 0 & \text{otherwise.} \end{cases}$$

(because we know the homology of $\mathbb{R}^n$ and also $\mathbb{R}^n - \{x\}$).

Definition 260

A **local $\mathbb{Z}$-orientation** of a manifold M near $x \in M$ is a choice of generator of $H_*(M, M - \{x\})$.

Since we can either pick 1 or -1 as a generator, there are two choices for the $\mathbb{Z}$-orientation at each point. And for a general ring R, note that

$$H_*(M, M - \{x\}) \otimes_{\mathbb{Z}} R \cong \mathbb{Z} \otimes_{\mathbb{Z}} R = R,$$

which gives us the following definition:

Definition 261

A **local R-orientation** of a manifold M near $x \in M$ is a choice of (R-module) generator the rank-1 free R-module $H_*(M, M - \{x\})$.

Local R-orientations of M near x are then in bijection with $R^\times$, the group of units in R. And since $\mathbb{F}_2^\times$ contains only the element 1, there is indeed a unique choice for the local orientation near any point $x \in M$.

To go from local orientation to something more global, we need to pick these units in a consistent way:

Definition 262

Let M be an n-dimensional manifold. Define

$$o_M = \coprod_{x \in M} H_n(M, M - \{x\},$$

and similarly

$$o_M \otimes R = \coprod_{x \in M} H_n(M, M - \{x\} \otimes_{\mathbb{Z}} R.$$

We can make o_M (and similarly $o_M \otimes R$) into a topological space if we put a topology on it (it's a disjoint sum of Rs, with one copy per point in the manifold), and we want the topology to make the projection map $p : o_M \otimes R \to M$ (returning to the "point in the manifold that we're over") into a continuous map.

Definition 263

If $A \subseteq M$ is a closed subset of a manifold M, and $x \in A$, then define

$$j_{A,x} : H_n(M, M - A) \to H_n(M, M - \{x\})$$

to be the map on H_n induced by the inclusion $(M, M - A) \hookrightarrow (M, M - \{x\})$.

(In the Eilenberg-Steenrod axioms, we said that homology should be a functor of pairs, so we do indeed have this map.) So it turns out that **a basis of open sets for our topological space** o_M will be given by $\{V_{U,\alpha}\}$, where $U \subseteq M$ is an open set, $\alpha \in H_n(M, M - \overline{U})$, and

$$V_{U,\alpha} = \{j_{\overline{U},x}(\alpha) : x \in U\}.$$

And in fact, the map $p : o_M \otimes R \to M$ is an $|R|$-sheeted **covering space** (in the point-set topology sense).

Definition 264

A **section** of the projection map $p : o_M \otimes R \to M$ is a continuous function $f : M \to o_M \otimes R$ such that $p \circ f$ is the identity map. The set of all sections of p is denoted $\Gamma(M; o_M \otimes R)$.

(This is secretly an algebraic geometry notation.) For Poincaré duality, we care about special kinds of sections:

Definition 265

A section $f \in \Gamma(M; o_M \otimes R)$ is an ***R*-orientation** of M if at each point $x \in M$, the class $f(x) \in H_n(M, M - \{x\}) \otimes_{\mathbb{Z}} R$ in the fibre of the covering space is a local R-orientation.

Fact 266

There is a nice condition (from covering space theory) showing the existence of R-orientations: if M is an n-dimensional manifold where $\pi_1(M, x) = 0$ for every $x \in M$ (in other words, if M is simply connected), then the projection map $p : o_M \otimes R \to M$ is the trivial $|R|$-sheeted covering space, because of the Galois correspondence between covering spaces and subspaces of π_1. So sections are then easy to understand, and there are $|R^\times|$ total R-orientations.

This fact tells us that some simple spaces are $\mathbb{Z}$-orientable, and it is the most useful criterion if we want to use Poincaré duality for coefficients other than $\mathbb{F}_2$. (And while the torus is not simply connected, it still has understandable covering spaces.) But checking whether orientations don't exist is a bit more subtle:

Fact 267

If $M \subset N$ is an inclusion of manifolds, and M is not R-orientable, then N is also not R-orientable.

Because there's a copy of the Möbius strip inside the Klein bottle, we can use tricks like that to show non-orientability.

To see where the compactness comes in for Poincaré duality, we need another construction. If M is an n-dimensional manifold, then we have a map

$$j : H_n(M; R) \to \Gamma(M; o_M \otimes R)$$

defined by taking a class $a \in H_n(M; R)$ and $x \in M$ and evaluating $j(a)(x)$. This should give us a class in the local homology $H_n(M, M - \{x\}) \otimes_{\mathbb{Z}} R$, and we'll define it to be the restriction of $a \in H_n(M; R)$ to $H_n(M; M - \{x\}) \otimes_{\mathbb{Z}} R$. And this gets us to a big theorem in orientation theory:

Theorem 268

If M is a **compact** manifold, then the map j is an isomorphism.

This means that sections of the covering space (functions from the manifolds back to the disjoint union of copies of R) are exactly in bijection with the top homology group $H_n(M; R)$, given by the explicit function above. And to see how this all connects to Poincaré duality, we finally get the origin of the fundamental class mentioned in Poincaré duality:

Definition 269

Suppose that M is a compact, n-dimensional, R-oriented manifold (meaning that it's oriented by a particular section $f \in \Gamma(M; o_M \otimes R)$, which restricts at every point to a local orientation). Then there is a corresponding class $H_n(M; R)$, which we call the **fundamental class** $[M]$ of the R-oriented manifold.

Proof of Theorem 268 for coefficients in $\mathbb{Z}$. We'll show a slightly more general fact: for any n-dimensional manifold M, suppose we're thinking about a specific compact subset $A \subseteq M$. Then for each $x \in A$, we have a map $j_{A,x}$: $H_n(M, M - A) \to H_n(M, M - \{x\})$ coming from the H_n functor, and these assemble into a map (combining all $x \in A$)

$$j_A : H_n(M, M - A) \to \Gamma(A; o_M)$$

(take the disjoint union of all points $x \in A$, which is a subset of the sections of M using the subspace topology). So what we'll prove is the following:

Theorem 270

Let M be an n-dimensional manifold, and let A be a compact subset of M. Then $H_q(M, M - A) = 0$ for $q > n$, and the map $j_A : H_n(M, M - A) \to \Gamma(A; o_M)$ is an isomorphism.

(This theorem is true even if A is something like the Cantor set inside of Euclidean space!)

Lemma 271

Let A and B be two compact subsets of M. Then if the theorem is true for $A, B, A \cap B$, then it's true for $A \cup B$.

Proof of lemma. Consider the diagram

$$\begin{array}{ccccccc}
H_{n+1}(M, M - A \cap B) & \longrightarrow & H_n(M, M - A \cup B) & \longrightarrow & H_n(M, M - A) \oplus H_n(M, M - B) & \longrightarrow & H_n(M, M - A \cap B) \\
\downarrow{\scriptstyle 0} & & \downarrow{\scriptstyle j_{A \cup B}} & & \downarrow{\scriptstyle j_A \oplus j_B} & & \downarrow{\scriptstyle j_{A \cap B}} \\
0 & \longrightarrow & \Gamma(A \cup B; o_M) & \longrightarrow & \Gamma(A; o_M \oplus \Gamma(B; o_M) & \longrightarrow & \Gamma(A \cap B; o_M)
\end{array}$$

We want to show that the map in the second column $j_{A \cup B}$ is an isomorphism, given that the $j_A \oplus j_B$ and $j_{A \cap B}$ maps are isomorphisms, and we can do that using the **five lemma** as long as know that the top and bottom rows are exact. We can check on our own that (by definition of the section construction) the bottom row is exact, and the top row is exact by Mayer-Vietoris. ☐

And to finish the proof, we should read Miller's notes if we'd like to look more at the details – we use this lemma repeatedly to reduce ourselves to simple compact subsets inside $\mathbb{R}^n$, where we can check the result by hand.

☐

35 December 2, 2020

We'll sketch the (low-tech) proof of Poincaré duality today and Friday, and we'll talk about some more modern developments next week.

We've discussed a lot of structures on the homology and cohomology of a space: the **cross product** through the Alexander-Whitney map

$$\times : H^p(X; R) \otimes_R H^q(Y; R) \to H^{p+q}(X \times Y; R),$$

the **cup product**, which is the cross product combined with the diagonal map

$$\cup : H^p(X; R) \otimes_R H^q(X; R) \to H^{p+q}(X; R),$$

and the Kronecker pairing, coming from the concept that homology and cohomology are linearly dual:

$$H^p(X;R) \otimes_R H_p(X;R) \to R.$$

But today we'll introduce a final operation, along the same lines as the three above:

Definition 272

The **cap product**

$$\cap : H^p(X;R) \otimes_R H_n(X;R) \to H_{n-p}(X;R)$$

is a "fancier Kronecker pairing" defined by applying homology to the chain map

$$S^p(X;R) \otimes_R S_n(X;R) \xrightarrow{AW} S^p(X;R) \otimes_R S_p(X;R) \otimes_R S_{n-p}(X;R) \xrightarrow{\langle \cdot,\cdot \rangle} R \otimes_R S_{n-p}(X;R) \cong S_{n-p}(X;R),$$

where AW is the Alexander-Whitney map applied to $S_n(X;R)$, and $\langle \cdot, \cdot \rangle$ is the Kronecker pairing applied to $S^p(X;R) \otimes_R S_p(X;R)$.

We should note that all of these constructions are coming with the cross product, the diagonal map on a topological space, and the Kronecker pairing. And the most important of them is the cup product, but the others are relevant as well.

If we unwind the definitions, there are a few properties that are straightforward to show:

Proposition 273

The cap product satisfies the following properties:

- $(a \cup b) \cap x = a \cap (b \cap x)$, and $1 \cap x = x$. In other words, $H_*(X;R)$ is a module for $H^*(X;R)$ – **homology is a module over the cohomology ring using the cap product as a module multiplication**.
- $\langle a \cup b, x \rangle = \langle a, b \cap x \rangle$. So as long as we know about the Kronecker pairing, the cap and cup products are basically the same information.
- If $\varepsilon : H_0(X;R) \to R$ denotes the element adjoint to $1 \in H^0(X;R)$ (meaning it's the function from the free module on path components to R, sending each component to 1), and $b \in H^p(X;R), x \in H_p(X;R)$, then $\varepsilon(b \cap x) = \langle b, x \rangle$. So the cap product is a "souped-up" version of the Kronecker pairing.
- If we have a map $f : X \to Y$ of topological spaces, and we have $b \in H^p(Y)$ and $x \in H_n(X)$, then $H_*(f)\,[H^*(f)(b) \cap x] = b \cap [H_*(f)(x)]$ (this is called the **projection formula**).

And now we're ready to state the general Poincaré duality result:

Theorem 274 (Poincaré duality, version 2)

Let M be a compact n-dimensional manifold, equipped with an R-orientation for a PID R (so that we have things like the Künneth formula and short free resolutions). Then there is a unique class $[M] \in H_n(M;R)$ that restricts to the local orientation in $H_n(M, M - \{x\}) \otimes_{\mathbb{Z}} R$ for each point $x \in M$, with the property that the map $H^p(M;R) \to H_{n-p}(M;R)$ sending a to $a \cap [M]$ is an isomorphism for all p.

Geometrically, this isomorphism in reverse taking homology to cohomology takes a homology class and records the intersection number with every complementary-dimension homology class. It's hard to justify geometrically what's going on without smooth manifold theory, and we'll prove this result here for integer coefficients.

We're actually going to prove this by formulating a more general statement, which works for compact subsets of arbitrary manifolds (instead of just compact manifolds). The hope is that we'll be able to break down the uglier statement into pieces.

To do this, we'll need to introduce **relative cap products**. If $A \subseteq X$ is a subspace of X, then we have a **relative cap product**

$$H^p(X) \otimes H_n(X, A) \to H_{n-p}(X, A),$$

which makes the relative homology $H_*(X; A)$ also into a module over the ring $H^*(X)$. This structure exists because of the following diagram:

$$\begin{array}{ccccc}
0 & & & & 0 \\
\downarrow & & & & \downarrow \\
S^p(X) \otimes S_n(A) & \longrightarrow & S^p(A) \otimes S_n(A) & \xrightarrow{\cap} & S_{n-p}(A) \\
\downarrow & & & & \downarrow \\
S^p(X) \otimes S_n(X) & & \longrightarrow & & S_{n-p}(X) \\
\downarrow & & & & \downarrow \\
S^p(X) \otimes S_n(X, A) & & \dashrightarrow & & S_{n-p}(X, A) \\
\downarrow & & & & \downarrow \\
0 & & & & 0
\end{array}$$

The left column is the short exact sequence of chain complexes coming from relative homology – for reasonably nice spaces, everything is free, so the tensor product with $S^p(X)$ gets to be exact. (And another way to think about it is to use the fact that there's a noncanonical direct sum involved in the Künneth formula.) Then the map $S^p(X) \otimes S_n(A) \to S^p(A) \otimes S_n(A)$ in the second row is induced by the inclusion $A \hookrightarrow X$, and then we have a cap product following that. Then the square we have **commutes by the projection formula**, and then the right column is an actual short exact sequence.

So now the dashed blue map (the relative cap product we're interested in) exists by the **universal property of the quotient**. Specifically, remember that if we have a map of abelian groups $B \to A \to A/B$, and we have a map $A \to C$, we can show that a map $A/B \to C$ exists by checking that the composite map $B \to A \to C$ is zero. So here, the left column plays the role of $B \to A \to A/B$, and the bottom right corner $S_{n-p}(X, A)$ plays the role of C. We must check that the composite of the three red maps is zero, and we do this by using the commuting square to go through $S_{n-p}(A)$ in the top right corner instead. Then note that the composite map $S_{n-p}(A) \to S_{n-p}(X) \to S_{n-p}(X, A)$ is zero because the composite of any two maps in a short exact sequence is zero. Therefore, taking homology of this dashed blue arrow does indeed yield the relative cap product.

Next, we need to introduce **Čech cohomology.** If $K \subseteq X$ is a closed subset of a topological space X, then by excision, we know that $H_n(X, X - K) \cong H_n(U, U - K)$ for any open set U that contains K (that is, locally around K). So now we have a cap product

$$H^p(U) \otimes H_n(U, U - K) \to H_{n-p}(U, U - K),$$

meaning that $H_*(X, X - K)$ is a module over the cohomology ring $H^*(U)$ for **any** open set U that contains K, and our goal is to package all of that into a single module over a "germ" (in the sense of differential geometry). We're basically going to look at the limit of all open subsets over K and discuss what module structure we end up with.

Definition 275

Let K be a closed subset of X. The **Čech cohomology** of K is denoted $\check{H}^p(K)$, and an element $x \in \check{H}^p(K)$ is an element of $H^p(U)$ for some open U containing K. Furthermore, if $K \subseteq U \subseteq V$ for open sets U, V, we set $x \in H^p(V) \subseteq \check{H}^p(X)$ equal to $H^*(\iota)(x) \in H^p(U) \subseteq \check{H}^p(X)$, where ι is the inclusion of U into V.

We should note that $\check{H}^p(K)$ depends on how K sits in X – for reasonable closed sets, we have $\check{H}^*(K) \cong H^*(K)$, but this fails when we have things like a Cantor set. (So a deformation retract of an open neighborhood is okay.)

Fact 276

The Čech cohomology $\check{H}^*(K)$ is a ring, using the fact that the intersection of two open sets is open (so we can use the usual cohomology cup product). Furthermore, the point is that when we glue all the module structures together, $H_n(X, X-K)$ is a module over $\check{H}^p(K)$ (the universal thing that carries all the cohomology groups of open sets).

What we'll see next time is that if X is a space with closed subspace K, we've now constructed

$$\cap : \check{H}^p(K) \otimes H_n(X, X-K) \to H_{n-p}(X, X-K),$$

which is reminiscent of the kind of things that happen in Poincaré duality. And indeed, we'll be showing that this map is an isomorphism, perfect pairing, and so on, under various conditions on X and K. We'll do this by breaking up X and K into smaller pieces, where we already understand the result, and then assembling everything into the result we want using Mayer-Vietoris and the five lemma.

We'll end today with an example where Čech cohomology behaves differently from ordinary cohomology:

Example 277

Consider the **topologist's sine curve**, which is the graph of $y = \sin\left(\frac{2\pi}{x}\right)$ for $0 < x < 1$ (and gets more periodic as we approach 0). Let $K \subseteq \mathbb{R}^2$ be the union of this curve with $\{0\} \times [-1, 1]$, as well as some curve γ that connects $(0, -1)$ back to $(1, 0)$.

This curve looks like a "messed-up circle," where things get very complicated around the y-axis. Then

$$H^*(K) \cong \begin{cases} \mathbb{Z} & \text{degree } 0 \\ 0 & \text{otherwise,} \end{cases}$$

so singular cohomology thinks this weird space is a point. But

$$\check{H}^*(K) \cong H^*(S^1) \cong \begin{cases} \mathbb{Z} & \text{degree } 0, 1 \\ 0 & \text{otherwise.} \end{cases}$$

Basically, it's not clear why we want to study the topology of pathological spaces like this, but there are various things like Čech cohomology which do help us deal with them!

36 December 4, 2020

We'll finish our "bird's-eye tour" general overview of a low-tech proof of Poincaré duality today. Last time, we introduced Čech cohomology: if X is a topological space and $K \subseteq X$ is a closed subset, then (restating what defined last time in more detail) we defined

$$\check{H}^*(K) = \bigsqcup_{K \subseteq U} H^*(U)/\sim,$$

where the equivalence class $\sim$ tells us about how elements interact if one open set is contained within another. And we'll describe yet another variation on thi, thinking about Čech cohomology on a pair.

Definition 278

Let $L \subseteq K$ be a pair of closed subset of a space X. Then we define

$$\check{H}^*(K, L) = \bigsqcup_{L \subseteq V, K \subseteq U} H^*(U, V)/\sim,$$

where (U, V) ranges over all pairs of open sets $V \subseteq U$, V is an open neighborhood of L, and U is an open neighborhood of K. And the equivalence relation considers two classes to be the same if they're related along a restriction given by these pairs.

Theorem 279

If (K, L) is a closed pair in a space X, then there is a natural long exact sequence

$$\cdots \to \check{H}^p(K, L) \to \check{H}^p(K) \to \check{H}(L) \xrightarrow{\delta} \check{H}^{p+1} \to \cdots,$$

where the Snake Lemma map raises the degree.

Basically, we take the long exact sequences from pairs of open sets and check that they're compatible when we put them together.

Theorem 280

Let A and B be compact subsets of a Hausdorff space X. Then the inclusion $(B, A \cap B) \subseteq (A \cup B, A)$ induces an isomorphism of

$$\check{H}^p(A \cup B, A) \cong \check{H}^p(B, A \cap B)$$

for all integers p.

The hypotheses here are a bit more subtle – ordinary homology doesn't require things to be compact or Hausdorff. It's just that the excision statements interact in a particular way when we put them together for different open sets.

We also discussed the **cap product** for a closed subspace $K \subseteq X$:

$$\cap : \check{H}^p(K) \otimes_{\mathbb{Z}} H_n(X, X - K) \to H_{n-p}(X, X - K).$$

Basically, we could replace $(X, X - K)$ terms by "local" open sets U (by classical excision). And now we can write down a relative version of this to the Čech cohomology of a pair:

Definition 281

Let $L \subseteq K$ be a pair of closed subspaces of a space X. The **fully relative cap product** is a map

$$\cap : \check{H}^p(K, L) \otimes_{\mathbb{Z}} H_n(X, X - K) \to H_{n-p}(X - L, X - K).$$

This map in fact commutes with Mayer-Vietoris sequences, and we'll spell that out here:

Theorem 282

Let A and B be compact subsets of a Hausdorff space X. Fix a class $x_{A\cup B} \in H_n(X, X - A \cup B)$ – this class automatically gives us canonical classes $x_A \in H_n(X, X - A), x_B \in H_n(X, X - B), x_{A\cap B} \in H_n(X, X - A \cap B)$ using the induced maps from the inclusions. Then there is a map of long exact sequences

$$\begin{array}{ccccccccc} \cdots \to & \check{H}^p(A\cup B) & \to & \check{H}^p(A) \oplus \check{H}^p(B) & \to & \check{H}^p(A\cap B) & \xrightarrow{\delta} & \check{H}^{p+1}(A\cup B) & \to \cdots \\ & \downarrow{\cdot\cap x_{A\cup B}} & & \downarrow{(\cdot\cap x_A)\oplus(\cdot\cap x_B)} & & \downarrow{\cdot\cap x_{A\cap B}} & & \downarrow{\cdot\cap x_{A\cup B}} & \\ \cdots \to & H_{n-p}(X, X - A\cup B) & \to & H_{n-p}(X, X-A) \oplus H_{n-p}(X, X-B) & \to & H_{n-p}(X, X - A\cap B) & \xrightarrow{\partial} & H_{n-p-1}(X, X - A\cup B) & \to \cdots \end{array}$$

And now we'll come up with the maximally general version, which will make an "inductive" proof easier. Let M be an n-dimensional manifold (not necessarily compact) and K be a compact subset. We have a map $H_n(M, M - K) \to \Gamma(K; o_M)$ (recall the latter is the set of functions from K to the disjoint union of homology groups o_M, so that the composite map $K \xrightarrow{f} o_M \xrightarrow{p} K$ is the identity) which is an isomorphism. Then a $\mathbb{Z}$-orientation along K is a section of o_M over K (an element in the set of sections $\Gamma(K; o_M)$ which restricted to a generator of the local homology $H_n(M, M - \{x\})$ for every $x \in K$. This particularly nice section ($\mathbb{Z}$-orientation) then corresponds to a homology class $[M]_K \in H_n(M, M - K)$, called the **fundamental class along K**.

If $L \subseteq K$ is an inclusion of compact subsets of M, then the map coming from functoriality

$$H_n(M, M - K) \to H_n(M, M - L)$$

sends $[M]_K$ to $[M]_L$. (Another way of thinking of this is that a choice of local orientation that is compatible for every point in K will restrict to L and give us a choice of local orientation there.) And we have also just introduced the cap product

$$\cap : \check{H}^p(K, L) \otimes_{\mathbb{Z}} H_n(M, M - K) \to H_{n-p}(M - L, M - K),$$

so we're ready to put everything together:

Theorem 283 (Poincaré duality, version 3)

Let M be an n-dimensional manifold, and let $L \subseteq K$ be a pair of compact subspaces. Suppose we are given a $\mathbb{Z}$-orientation along K, giving us a fundamental class $[M]_K$. Then the map

$$\cdot \cap [M]_K : \check{H}^p(K, L) \to H_{n-p}(M - L, M - K)$$

is an isomorphism.

We can read Miller's notes for more details, but here are the key ideas of the induction:

Proof. First, we prove this for the special case where $M = \mathbb{R}^n$ and K and L are compact, convex subsets. This is a bit of analysis – we need to use the point-set topology of $\mathbb{R}^n$.

From there, we can do the case where $M = \mathbb{R}^n$ and K, L are a finite union of such compact, convex subsets – this requires writing down the Mayer-Vietoris sequence for a union, looking at the long exact sequences and using the five lemma. (This step is relatively easy.)

The next step is to do $M = \mathbb{R}^n$ and K, L any compact subsets of $\mathbb{R}^n$. This again requires some analysis: we use the fact that an arbitrary compact subset of $\mathbb{R}^n$ can be arbitrarily well-approximated by a finite disjoint union of compact, convex subsets (this is similar to the construction of the Lebesgue measure).

And from there, we're ready to switch to the case where M is an arbitrary manifold, but K and L are finite unions of compact Euclidean subspaces of M (here, Euclidean means that we take subsets of "local $\mathbb{R}^n$s" in neighborhoods of points). This comes from writing M as a union of $\mathbb{R}^n$s – we can pick finitely many if we're only interested in what's happening around K and L, because those subspaces are compact.

And finally, we get the case where M is arbitrary, and K and L are arbitrary compact spaces, using the fact that M is a manifold, and thus Hausdorff, to do the appropriate approximations. □

We'll slightly restrict the above statement for some particular applications:

Theorem 284
Let M be an n-dimensional manifold with compact subset K. A $\mathbb{Z}$-orientation along K determines a fundamental class $[M]_K \in H_n(M, M - K)$, and capping with it gives us an isomorphism

$$\check{H}^{n-p}(K) \to H_p(M, M - K).$$

And now we can get a result that we couldn't have with the original form of Poincaré duality, because $\mathbb{R}^n$ isn't compact:

Corollary 285 (Alexander duality)
Let K be a compact subset of $\mathbb{R}^n$. Then the composite map

$$\check{H}^{n-p}(K) \xrightarrow{\cong} H_p(\mathbb{R}^n, \mathbb{R}^n - K) \xrightarrow{\partial} \tilde{H}_{p-1}(\mathbb{R}^n - K)$$

is an isomorphism, where ∂ comes from the homology of a pair.

(Here, the first map is an isomorphism from the above result, and the second map is an isomorphism because $\mathbb{R}^n$ has a very boring homology.) So we're relating the Cech cohomology of a subspace to the homology of its complement.

Example 286
Let K be a closed loop in $\mathbb{R}^2$ (such as the image of a continuous function $S^1 \to \mathbb{R}^2$), which is nice enough that its Čech cohomology $\check{H}^1(K) \cong \mathbb{Z}$.

Basically, for a nice enough loop, Čech cohomology will be the same as ordinary homology. But there are some pathological loops (like those made of the topologist's sine curve) that we can deal with too! So Čech cohomology covers a much wider set of curves, though we won't prove that.

Then the reduced homology

$$\tilde{H}_0(\mathbb{R}^2 - K) \cong \hat{H}^1(K) \cong \mathbb{Z},$$

so if we replace the $\tilde{H}$ with the corresponding homology group, we find that

$$H_0(\mathbb{R}^2 - K) \cong \mathbb{Z} \oplus \mathbb{Z}.$$

Therefore, $\mathbb{R}^2 - K$ has exactly two path-components, and this is exactly the **Jordan curve theorem**!

Fact 287

On the other hand, the analog of the Jordan curve theorem is false in $\mathbb{R}^3$: there are injective continuous maps $f : S^2 \to \mathbb{R}^3$ such that $\mathbb{R}^3 - \text{im}(f)$ does not have two path components (a counterexample is the **Alexander horned sphere**). So there's something special about having two dimensions.

37 December 7, 2020

Today and Wednesday, we'll be talking about "algebraic topology after 18.905," exploring a bit of 18.906 and what's out in the world of topology afterwards. We've talked about four main topics in this course:

- Category theory (categories, functors, natural transformations, products, pushouts, internal Homs),
- Homological algebra (chain complexes of abelian groups, the derived category $D(\mathbb{Z})$, the Tor and Ext groups),
- Homotopy theory (study of the homotopy category Ho(Top)),
- Algebraic topology (study of Top itself, and in particular manifolds).

One thing that's good for us to notice is that homotopy theory and algebraic topology are two somewhat different topics – on Wednesday, we'll look at the former in more detail, but today we'll think about what comes next in each of the four topics.

Category theory: Products and pushouts are special cases of objects called **limits** and **colimits**, which are things defined by the universal property. We defined the currying isomorphism, which is an example of an **adjunction**. (So 18.906 will start with this if it's similar to how it's been taught in the past.) Starting in the early 2000s, people like to study **higher categories**, which have objects and morphisms like regular categories, but also **2-morphisms** (which are morphisms between morphisms), and possibly 3-morphisms and so on.

Example 288

"$H\mathbb{Z}$-modules" is the higher category (or equivalently, ∞-category) of chain complexes. An object of H$\mathbb{Z}$-modules is a chain complex of abelian groups, a morphism is a chain map, and 2-morphisms are chain homotopies.

Example 289

$\mathcal{S}$ is the ∞-category of homotopy theory, where objects are topological spaces, morphisms are continuous maps, and 2-morphisms are homotopies.

And since homotopy theory should not necessarily be associated with algebraic topology, we can also make an equivalent ∞-category using just the combinatorial definition (so we don't need to know what a topological space is if we just work with simplicial sets). We introduced simplicial sets in the homework because ∞-categories are becoming increasingly essential tools in different fields of math.

Example 290

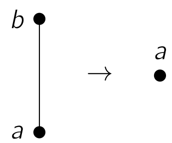
Cat_2 is the ∞-category of categories, where objects are categories, morphisms are functors, and 2-morphisms are natural transformations.

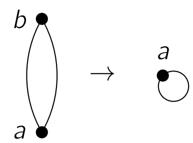
(Kerodon is a good source for learning more about this topic if we search it up online.)

Homological algebra: This topic is developed much more in algebraic geometry (such as in 18.725-726), where we can learn about the sheaf Hom, sheaf Ext, sheaf cohomology, and so on. If we consider a set like $\{x^2 + y^2 - 1 = 0\}$ in $\mathbb{R}^2$, we're using an algebraic way (the set of solutions to the equation) of describing a geometric object (the circle). We may want to ask whether we can use an algebraic algorithm to calculate the homology of the solution set for a given set of polynomials with real coefficients: in other words, if we were given the polynomial $x^2 + y^2 - 1$, could we find the homology without needing to know much about topology? And doing this is due to Grothendieck.

This is valuable, because we can then think about applying this algorithm to a polynomial with coefficients in a number field (in which it's not clear how to extract topology). Then we can talk about things like the cohomology of Diophantine equations – cohomology can be an invariant of polynomials over number fields or more algebraic fields! The Galois action on the number field then gives us a Galois representation, which is an object in the Langlands program.

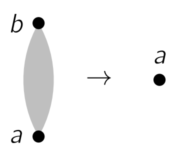
Homotopy theory: One way of thinking about this topic to understand why it's a subject separate from algebraic topology is that it's the "study of the equals sign," or more specifically the isomorphism sign. Suppose we have a graph of two vertices a and b connected by an edge: perhaps this edge means that a and b are "equal," so this is isomorphic to just having a single vertex a. But now if we have a graph where there are two edges between a and b, that means they are equal in two ways – there are two different equalities. Then we could use one of the equalities to identify the other point, but we still end up with a self-loop on the one vertex a.





If we also say that b is connected to c through a single edge, that means that b and c are equivalent in just one way, and we can contract that edge in our simplified diagram. So notice that we're really drawing homotopy equivalences here: we look at systems of objects that are equal to each other when connected by paths, remembering how many ways each connection can happen.

And if a and b are equal in two different ways, it's possible that we can have an equality between equalities, and drawing lines between the existing edges "fills in" the hole in the middle of the picture. So contracting this diagram gives us a single point again, which can be thought of as telling us why D^2 is equivalent to a point.



So homotopy theory is about objects, equalities between objects, but also equalities between equalities and so on. And a CW complex or semisimplicial set is then a combinatorial system which gives us this kind of thing: the 2-cells record equalities between equalities, the 3-cells record equalities between 2-equalities, and so on.

Algebraic topology: We can take a topological space and come up with certain invariants, and objects like homology really come from homotopy theory but are still useful for studying topological spaces. Beyond this, an example of a fundamental question we might want to ask in algebraic topology is "can we classify all n-dimensional compact manifolds up to homeomorphism," or alternatively do the same for smooth manifolds up to diffeomorphism.

Example 291

Surfaces (2-dimensional compact manifolds) have been completely classified – it's at the end of Munkres' topology book. (And Miller's lecture notes have a lecture devoted to this classification, too.)

To get a sense for this, we'll focus on the $\mathbb{Z}$-oriented surfaces, which are classified entirely by the **genus** of the surface (a genus g surface is a torus, but with g holes instead of 1). So in particular, we can determine a $\mathbb{Z}$-oriented surface by the first homology group, so if we have a collection of polynomial equations, we can see whether it'll produce a surface (this is something we can do in algebraic geometry), and to find out which surface, we just need to find its first homology group.

But we'll probably never classify all n-dimensional compact manifolds for large n, because the problem gets wildly complicated. So instead the question becomes how to classify manifolds with particularly simple homology. In 1961, Kervaire and Milnor classified all compact, simply connected, n-dimensional manifolds M that satisfy

$$H_q(M) \cong \begin{cases} \mathbb{Z} & q = 0, n \\ 0 & \text{otherwise,} \end{cases}$$

meaning they look like a sphere, for all $n \geq 5$. (The lower dimensions in topology can often be tricky, which has to do with how going up in dimension makes it easier to untangle knots.) We also know the answer for $n = 2$ (known since around 1900), and in 2003 we know the answer for $n = 3$ due to Perelman (and the work on the Poincaré conjecture). But $n = 4$ is still unknown to us right now.

Milnor asked (after completing his work) whether it's possible to classify simply connected $(2n)$-dimensional manifolds with $H_0(M) \cong \mathbb{Z}$ and $H_1(M) \cong H_2(M) \cong \cdots \cong H_{n-1}(M) \cong 0$. (So a lot of the homology groups vanish.) Note that anything which is simply connected is $\mathbb{Z}$-orientable, so Poincaré duality also tells us that the homology groups $H_{n+1} \cong H_{n+2}(M) \cong \cdots \cong H_{2n-1} \cong 0$ and $H_{2n}(M) \cong \mathbb{Z}$, but $H_n(M)$ is unconstrained. (So it's similar to the sphere where we have the top and bottom dimension, but we also allow for some freedom of one homology group.) And this is now done for all n **except** $2n = 4$ (we can't even do the simpler version of the problem here yet), $2n = 24$, and $2n = 126$. So there are some cool connections to sequences that stop after a finite amount of time – what can go wrong is that we have strange issues in dimensions larger than 24, which have to do with the Leech lattice or monster group.

Let's go through some history on the progress made on this question: Wall solved $n \equiv 6 \bmod 8$ in 1962, Brown and Peterson solved $n \equiv 5 \bmod 8$ in 1966, and Browder solved $n \equiv 3 \bmod 8$ in 1969. After that, Schultz solved $n \equiv 2 \bmod 8$ in 1972, and Stolz solved $n \equiv 1$ in 1985. In 2009, the case $n \neq 63$ and $n \equiv 7 \bmod 8$ was solved by Hill, Hopkins, and Ravenel. Hopkins was a professor at MIT of algebraic topology, and Hill was a graduate student (now at UCLA and the head of the Association for LGBT Mathematicians). And the final case, $n \neq 12$ and $n \equiv 0 \bmod 4$, was done by Dr. Hahn, along with Burklund and Senger (two grad students) last year! And Adela Zhang, another grad student at MIT, is thinking about extending this to two unconstrained homology groups.

From here on all we'll look at homotopy theory and the study of Ho(Top) (or some combinatorial system of equalities, avoiding the topological setup). The basic question we'll focus on is as follows:

Problem 292

If m and n are positive integers, how many maps are there from $S^m \to S^n$ up to homotopy?

The set of such maps, up to homotopy, is denoted $\pi_m S^n$, and we want to understand the answer for various m, n. We proved on our homework that $\pi_3 S^2$ has more than one element, because we showed that not every pair of maps from S^3 to S^2 is homotopic. Here are some facts we'll see in 18.906:

- If $m < n$, then all maps $S^m \to S^n$ are homotopic, so $\pi_m S^n$ has only one element.
- If $m = n$, then $\pi_m S^m \equiv \mathbb{Z}$: such maps are determined, up to homotopy, by their degree (the integer invariant we've explored a lot on our homework).
- If $m > n$, then unless we're in a special case where $m = 2n - 1$, $\pi_m S^n$ is a finite set.

We may want to ask for the cardinality of this set in the $m > n$ case, and a more refined question is to think about the **group structure** of this set. If we have two maps f, g from $S^m \to S^n$, we can define $f + g$ by first ensuring that the south pole of f is sent to the same point as the north pole of g (we can find some homotopic maps to do a deformation). So now we take the sphere S^m, pinch the equator using a quotient map, and we have two copies of S^m wedged together. So now we have a map $S^m \to S^m \vee S^m$, and then we'll send the top half by f and the bottom half by g.

Wikipedia has a table of all of these groups: for example, it turns out that $\pi_{14}(S^4) \equiv \mathbb{Z}/120\mathbb{Z} \times \mathbb{Z}/12\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$. We'll study these in more detail on Wednesday's class!

38 December 9, 2020

We'll talk some more about homotopy theory today, particularly looking at homotopy groups of spheres and what patterns arise. Recall that $\pi_m S^n$ is the set of continuous maps from S^m to S^n, up to homotopy, and we mentioned last time that this is a group. Specifically, we can add two maps $f : S^m \to S^n$ and $g : S^m \to S^n$ by pinching the equator of S^m to get $S^m \wedge S^m$, and then sending the top half of the sphere via f and the bottom half via g. (And we just need to use homotopy to adjust the maps f and g so that the south pole of f is sent to the north pole of g.)

We want to say as much as we can about these groups $\pi_m S^n$, and we'll learn the following theorem in 18.906:

Theorem 293 (Freudenthal)
For $n \geq k + 2$, $\pi_{n+k} S^n$ is independent of n.

The group $\pi_{n+k} S^n$ is denoted $\pi_k \mathbb{S}$ – basically, for sufficiently large n, these groups become "stable." Let's see what this looks like for $k = 1$:

Example 294
It is known that $\pi_3 S^2 \cong \mathbb{Z}, \pi_4 S^3 \cong \mathbb{Z}/2\mathbb{Z}, \pi_5 S^4 \cong \mathbb{Z}/2\mathbb{Z}$, and in general $\pi_{n+1} S^n$ is $\mathbb{Z}/2\mathbb{Z}$ for all larger n. So $\pi_1 \mathbb{S} \cong \mathbb{Z}/2\mathbb{Z}$, but we saw on our homework that there's a particular interesting map in $\pi_3 S^2$, and that's in fact a free generator of the group.

We'll focus on the "stable" groups $\pi_k \mathbb{S}$ in today's discussion: it turns out that the first eight groups for $k = 1, 2, 3, 4, 5, 6, 7, 8$ are $\mathbb{Z}/2\mathbb{Z}, \mathbb{Z}/2\mathbb{Z}, \mathbb{Z}/8\mathbb{Z} \oplus \mathbb{Z}/3\mathbb{Z}, 0, 0, \mathbb{Z}/2\mathbb{Z}, \mathbb{Z}/16\mathbb{Z} \oplus \mathbb{Z}/3\mathbb{Z} \oplus \mathbb{Z}/5\mathbb{Z}$, and $\mathbb{Z}/2\mathbb{Z}$, respectively. These first few examples exhibit some chaotic-looking behavior, and it's not clear what the pattern immediately is. In fact, in the 1950s and 1960s, finding each of these groups was a major theorem. So it was a bit disappointing when the results started coming in, because it wasn't clear what to conjecture in general.

So we'll use the one tool we've developed – homology – to understand these groups as best we can. An element of $\pi_7 \mathbb{S}$, for example, is a map $f : S^{7+n} \to S^n$ for $n \gg 0$, and it's difficult to distinguish these maps because the homology $H_*(f) : H_*(S^{7+n}) \to H_*(S^n)$ is going to be trivial no matter what our map f is, just for degree reasons. So it's a bit unclear how to see information about homotopy spheres directly.

Definition 295

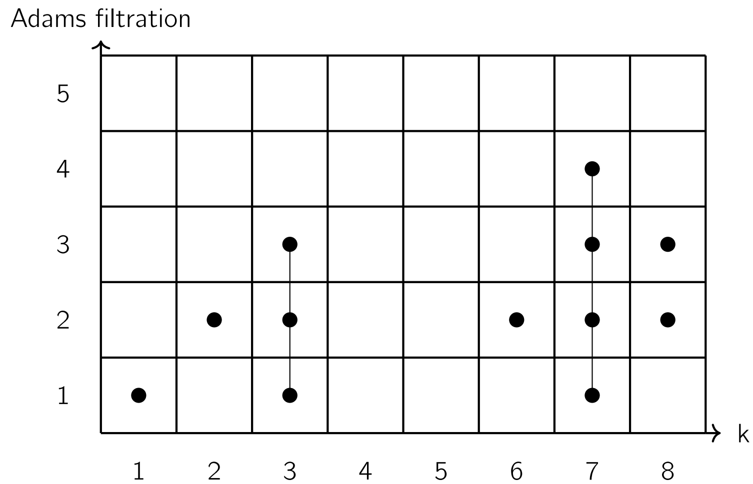
A map of spheres $f : S^m \to S^n$ has **$\mathbb{F}_p$-Adams filtration** at least k if we can find some factorization of f as a composite $S^m \xrightarrow{f_1} X_1 \xrightarrow{f_2} X_2 \xrightarrow{f_3} \cdots \xrightarrow{f_k} X_k = S^n$, such that each $H_*(f_i; \mathbb{F}_p)$ is trivial.

In other words, if a map has $\mathbb{F}_p$-Adams filtration at least k, we can factor through k different topological spaces, and we never know what any of those do through homology (so f gets more and more difficult to understand using homology as k grows). This notion turns out to be robust – every map has a specific $\mathbb{F}_p$-Adams filtration for every prime p.

Fact 296

We'll learn in 18.906 that $\pi_k \mathbb{S}$ is always a finite abelian group, and we can write such groups as direct sums by prime. So we can study $\pi_k \mathbb{S}$ one prime at a time, and for each prime we can draw a relevant diagram that helps us understand the structure better.

For example, let's take $p = 2$. As mentioned above, $\pi_1 \mathbb{S} \cong \mathbb{Z}/2\mathbb{Z}$, so there's only one interesting map $S^{n+1} \to S^n$, which has Adams filtration 1 (so it's visible to homology if we factor it through at least one other space). But π_2's interesting map has Adams filtration 2, so a map $S^{n+2} \to S^n$ can be factored through some space that's still invisible to homology. If we next look at π_3, it turns out a fraction 1/2, 1/4, and 1/4 of the interesting maps have Adams filtration $1, 2, 3$, respectively, which reflects the $\mathbb{Z}/8\mathbb{Z}$ structure. And in the diagram below, each connected component of m dots basically corresponds to a term $\mathbb{Z}/2^m\mathbb{Z}$ in the group $\pi_k \mathbb{S}$.



A supercomputer in China generated this kind of graph two years ago (work due to Isaksen, Wang, and Xu) – we've made a lot of progress recently, but we're at the limit of current knowledge with $k = 90$. Specifically, we're getting to the point where it takes a few months to compute! But we might hope that we can see some patterns in the columns to understand what we can say about the groups, and that's still a bit difficult to do. Organizing by Adams filtration does give us some interesting patterns: at the top of the picture, we have some interesting periodicity (those groups are called the **v_1-periodic homotopy groups of spheres**, and they're completely understood), and then there's a gap between those and the rest of the groups in the diagram. In other words, there's a simple collection of homotopy groups that are extremely invisible to homology, and then nothing for a while, and then a messy collection that are a lot easier to detect with ordinary homology.

One of the biggest open questions in this subject is whether this gap continues forever, and how big it actually is (between the v_1-periodic and other groups), and the answers would have some geometric applications to the study of manifolds as well. And another natural question is whether we can understand the non-v_1-periodic groups in the messy part of the picture.

Definition 297

An **extraordinary (co)homology theory** E_* is a functor $E_* : \text{Ho(Top)} \to$ (graded abelian groups), satisfying all of the Eilenberg-Steenrod axioms except the dimension axiom.

(In other words, we have Mayer-Vietoris sequences and excision and so on, but the homology or cohomology groups for a point are allowed to be complicated.) And what's nice about these homology theories is that they can sometimes tell differences between maps $f : S^m \to S^n$ apart. Specifically, the $E_*(f) : E_*(S^m) \to E_*(S^n)$ may be nontrivial, because we don't necessarily have concentration in a single degree like we do for H_*.

Example 298

The most important example of an extraordinary homology theory is $E_* = KO_*$ (known as **topological K-theory**), and this was heavily studied in the 1970s and 1980s. One notable feature is that we can use this theory to see the v_1-periodic part of $\pi_*\mathbb{S}$.

In particular, topological K-theory has a geometric definition in terms of **vector bundles** (which we'll also start to learn about in 18.906), and there's also some algebraic or combinatorial definitions that we can work with. But developments have been made since the 1980s, and we can try to create extraordinary cohomology theories that detect other elements in $\pi_*\mathbb{S}$ too. To understand that, we should talk a bit about homology theories in the abstract sense.

Definition 299

An **$\mathbb{E}_\infty$-ring** is a cohomology theory E^* taking values in graded commutative rings, instead of graded abelian groups.

In particular, this means we have a product (analogous to the cup product), so an example would be $E^*(X) = H^*(X; R)$ for some commutative ring R. But we can also take values in graded commutative rings by taking $E^*(X) = KO^*(X)$ (this is called **K-theory cohomology**). It turns out that if we have an E_∞-ring E^*, we can **extract a (canonical) classical ring** by taking E^* of a single point.

Example 300

The classical ring underlying $H^*(\cdot\,; R)$ is the cohomology of a point with coefficients in R, which is R in degree 0 (and nothing else). Meanwhile, the classical ring underlying topological K-theory $KO^*(\cdot)$ turns out to be 8-periodic in the degree (this is called **Bott periodicity**).

And the idea is that we can develop all of commutative algebra, replacing rings with $\mathbb{E}_\infty$-rings, and finding analogous statements for results like the Nakayama Lemma. Basically, cohomology theories should be the same in many ways as ordinary rings, and this field of study is what is called "higher algebra" or "derived algebra" today. It turns out that even though we get a lot of the same results, some differences do come up:

Example 301

In classical algebra and number theory, we like to study elliptic curves. Imitating the theory of elliptic curves in $\mathbb{E}_\infty$-rings gives us **elliptic cohomology theories**, which have interesting patterns that look like (for instance) Artin reciprocity, and this can help us see some of the groups in the Adams filtration picture.

But something interesting that doesn't happen in classical number theory is that there is a **universal elliptic cohomology theory**, which doesn't have a lower algebra analog, which sees all information in elliptic curves! This is known as **TMF** (topological modular forms), and if we take the TMF cohomology of a point, it turns out that $\text{TMF}^*(\text{point}) \otimes_{\mathbb{Z}} \mathbb{Q}$ is the classical ring of modular forms. So modular forms come up in topology, but in the context of a more complicated object.

Thus, we can study things like abelian varieties and automorphic forms by constructing analogs of them in the world of $\mathbb{E}_\infty$-rings. And we can also talk about cohomology theories only valued in associative rings instead of commutative ones (which is the notion of an $\mathbb{E}_1$-ring), and in general lots of (maximally complicated but universal) algebra can be constructed in this "higher algebra" world than we could have classically. The study of **chromatic homotopy theory** then assembles the homotopy groups $\pi_* \mathbb{S}$ out of things detected by $\mathbb{E}_\infty$-rings: each $\mathbb{E}_\infty$-ring has a chromatic height, where height 0 looks like ordinary homology, height 1 is topological K-theory, height 2 is TMF, and there's a convergence theorem (the **chromatic convergence theorem**) which says that each element in $\pi_* \mathbb{S}$ is detected at some height (some spot in the realm of the complexity of $\mathbb{E}_\infty$-rings).

So this filtration of very complicated groups slowly reveals more and more information to us, and some ongoing questions include how to compute in height 2 and how to connect number theory to height 3. So topologists sometimes conjecture things in number theory, and that kind of work has been fruitful for progress. And a final open question to consider is whether there is a geometric construction for EMF (rather than abstract higher algebra), just like we have cycles or vector bundles for homology or K-theory. According to physicists, $\text{TMF}^*(X)$ should be related to the Dirac operator on the space of loops in X, but no one has been able to make that mathematically well-defined yet.