18.952: Theory of Differential Forms

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Introduction

This class will follow the textbook "Differential Forms," published by World Scientific and authored by Professor Guillemin and Peter Haine. We can reach out at **vwg@math.mit.edu** or stop by room 2-270 with any questions – regular office hours will be Mondays and Wednesdays from 1-2pm.

The prerequisites for the course are 18.100B or 18.901, and (not necessary but helpful) 18.101. And background in linear algebra (18.700 or helpfully 18.702) will be very essential – many of the concepts in the theory of differential forms are basically linear algebra in various aspects. For now, we won't have any exams, and grading will be done using biweekly problem sets. Hopefully, a grader will be recruited for the course who will also hold regular office hours.

1 January 31, 2022

We'll begin with a simplistic description of what differential forms look like, looking at a few concrete examples that occur in multivariable calculus (though they aren't called differential forms in that context):

Example 1

Line integrals of the form $\int_\gamma f_1 dx_1 + f_2 dx_2 + f_3 dx_3$ for functions f_1, f_2, f_3 in the three variables x_1, x_2, x_3, where γ is a curve in 3D space, are differential forms. Similarly, **surface integrals** of the form $\int_S f_1 dx_2 dx_3 + f_2 dx_1 dx_3 + f_3 dx_1 dx_3$ with respect to some compact surface S are differential forms, and so are **volume integrals** $\int_D f dx_1 dx_2 dx_3$ over some domain D.

We should be familiar already with how we compute such integrals, but one question we can ask is **what the integrands in these expressions mean intrinsically**. In a similar vein, we may remember expressions like grad, curl, and $\div$ from vector calculus – we may want to ask how to naturally generalize such operations to n-dimensional space, without having to explicitly write down complicated expressions. And that's going to lead us to the study of differential forms, but we'll start by thinking about **multilinear algebra** explicitly. (We'll want to generalize these notions when we extend our definitions later, so it's good to have everything in one place.)

Definition 2

A **vector space** V over $\mathbb{R}$ is a set with two basic operations $(v, w) \mapsto v + w$ (vector addition) and $(x, v) \mapsto xv$ (scalar multiplication) for any $v, w \in V$ and $x \in \mathbb{R}$, containing a zero vector, additive inverses, and also satisfying commutativity and associativity of addition, distributivity, $x_1(x_2 v) = (x_1 x_2)v$, and $1v = v$.

Definition 3

A set of vectors $v_1, \cdots, v_n$ **span** a vector space V if every $v \in V$ can be written as $v = c_1 v_1 + \cdots + c_n v_n$ for some $c_1, \cdots, c_n \in \mathbb{R}$, and it is **linearly independent** if whenever $c_1 v_1 + \cdots + c_n v_n = 0$, we have $c_1, \cdots, c_n = 0$. If $v_1, \cdots, v_n$ is both linearly independent and spanning, then we call the set a **basis**.

Fact 4

For any finite-dimensional vector space, there always exists a basis (in other words, the definitions above are legitimate).

Remark 5. *The first week or two of the class will revolve around a lot of linear algebra, and we'll be exclusively discussing finite-dimensional vector spaces (so we don't need to worry about any nuances with infinite-dimensional spaces).*

Definition 6

A subset $W \subseteq V$ is a **subspace** of V if it is closed under addition and scalar multiplication.

Definition 7

Let V and W be two vector spaces. A map $A : V_1 \to W$ is a **linear mapping** if for all $v_1, v_2 \in V_1$ and $c_1, c_2 \in \mathbb{R}$, we have $A(c_1 v_1 + c_2 v_2) = c_1 A(v_1) + c_2 A(v_2)$. The **kernel** of A is the set $\ker A = \{v \in V_1 : Av = 0\}$, and the **image** of A is the set $\operatorname{Im} A = \{w \in W : w = Av \text{ for some } v \in V_1\}$.

We can verify that $\ker A$ and $\operatorname{Im} A$ are subspaces of V_1 and W, respectively, and we also have the elementary result $\dim \ker A + \dim \operatorname{Im} A = \dim V_1$. And we also have a more concrete way of writing down what A looks like when V and W are finite-dimensional: if we let $v_1, \cdots, v_n$ be a basis of V and let $w_1, \cdots, w_m$ be a basis of W, then we can write $Av_i = \sum a_{ji} w_j$ for some constants $a_{ij} \in \mathbb{R}$, and then the matrix formed by these a_{ij}s gives us a **matrix representation** of the linear map A.

Remark 8. *The convention is often to use $Av_i = \sum a_{ij} w_j$ instead to define the matrix of A, but we'll use this definition for consistency with the book.*

Fact 9

The map between linear maps and matrices is a bijective correspondence, so we can use matricial identities to learn a lot of facts about linear maps. In particular, any map defined by a matrix $A = [a_{ij}]$ is a linear map.

Definition 10

Let V be a vector space. A **bilinear form** is a map $B : V \times V \to \mathbb{R}$ which is bilinear in each variable, meaning that $B(c_1 v_1 + c_2 v_2, v) = c_1 B(v_1, v) + c_2 B(v_2, v)$ and similar in the other variable.

We can refer to Chapter 1 of the textbook for more linear algebra details if we'd like, but we'll finish this lecture with a few other useful definitions:

Definition 11

Let W be a subspace of a vector space V. A subset of the form $v + W = \{v + w : w \in W\}$ is called a ***W*-coset** of V.

In particular, note that two W-cosets $v_1 + W$ and $v_2 + W$ are either the same set or completely disjoint, so that V is a disjoint union of its W-cosets. Thus, we can make the following definition:

Definition 12

Let W be a subspace of a vector V. Then V/W is the set of W-cosets of V.

It's left as an exercise for us to check that V/W is indeed a vector space (meaning that it satisfies the axioms of Definition 2), under the addition operation $(v_1 + W) + (v_2 + W) = (v_1 + v_2) + W$.

2 February 2, 2022

Last time, we did a review of linear algebra – in particular, for a subspace W of a vector space V, we defined the **W-coset** $v + W = \{v + w : w \in W\}$ for any $v \in V$, and we mentioned that $v_1 + W$ and $v_2 + W$ are always disjoint or identical, so that V can always be written as a disjoint union of its W-cosets.

We also mentioned that this set of cosets itself has a vector space structure: we can define a quotient space V/W of the set of W-cosets, with addition given by $(v_1 + W) + (v_2 + W) = (v_1 + v_2) + W$ and scalar multiplication given by $c(v + W) = cv + W$. (Then the zero vector is $0 + W = W$ itself, and we can check that all of the axioms of a vector space are satisfied.) We then have $\dim V/W = \dim V - \dim W$ whenever V is finite-dimensional.

Finally, we also discussed the concept of **functoriality**: suppose V and U are two finite-dimensional vector spaces, and $A : V \to U$ is a linear map. If $W = \ker A$, then we get an injective linear map $V/W \to U$, given by $v + W \mapsto Av$. Also, for any subspace W of V, the projection map $\pi : V \to V/W$ maps any $v \in V$ to the corresponding coset $v + W$.

We can now turn to the topic of today's lecture, **tensors**. Before we get to those definitions, we'll need to discuss a few more important linear algebra notions:

Definition 13

Let V be an n-dimensional vector space. The **dual space** of V, denoted V^*, is the set of linear maps $\{\ell : V \to \mathbb{R}, \ell \text{ linear}\}$.

If we let $e_1, \cdots, e_n$ be a basis of V, then we can consider the corresponding **dual basis** $e_1^*, \cdots, e_n^*$, where e_i^* is the linear map that sends e_i to 1 and all of the other e_js to 0: in other words,

$$v = a_1 e_1 + \cdots + a_n e_n \implies e_i^* v = a_i.$$

Proposition 14

These maps e_i^* form a basis of V^*, so choosing a basis for V automatically gives a basis for V^*.

Proof. To show that the e_i^*s span V^*, suppose we have some linear map $\ell \in V^*$ in the dual space. Then we can define $a_u = \ell(e_u)$, and we can write our map as the linear combination $\ell = \sum a_u e_u^*$, since

$$\ell(e_i) = \left(\sum a_u e_u^*\right)(e_i) = \sum a_u e_u^* e_i = a_i.$$

To show independence, suppose some map $\sum a_u e_u^*$ is the zero map. Then $0 = 0e_i = \left(\sum a_u e_u^*\right) e_i = a_i$, so all a_i must be zero. $\square$

With this, if we have a linear map $A : V \to W$, we can always define a linear map $A^* : W^* \to V^*$ in the following way: for any $\ell \in W^*$ (meaning that ℓ is a linear map $W \to \mathbb{R}$), we define $A^*\ell$ to be the map $V \to \mathbb{R}$ which applies A and then ℓ to any vector in v.

To understand what this looks like more concretely, we can think about this in terms of coordinates: let $e_1, \cdots, e_n$ be a basis of V, and let $f_1, \cdots, f_m$ be a basis of W. Then we can characterize A with the numbers a_{ij} given by

$$Ae_i = \sum a_{ji} f_j.$$

If we define $e_1^*, \cdots, e_n^*$ and f_1^*, f_m^* to be the dual bases of V^* and W^*, we now wish to check that

$$A^* f_i^* = \sum a_{ij} e_j^*.$$

This is left as an exercise to us, but essentially we should use the fact that $A^* f_i^*(e_j) = f_i^* Ae_j$. And what this tells us is that if $[a_{ij}]$ is the matrix for the map A, then its transpose $[a_{ji}]$ is the matrix for the map A^*.

Problem 15

Show that the **double dual** of V satisfies $(V^*)^* = V$. (As a hint, for any $v \in V$, we can define the map $u_v(\ell) = \ell(v)$, and that will give us a way to map the two spaces to each other.

Problem 16

Let W be a subspace of the vector space V, and let $W^\perp$ be the set of $\ell \in V^*$ such that $\ell(w) = 0$ for all $w \in W$. Show that $W^\perp$ is a subspace of V^* and that $(V/W)^* = W^\perp$.

We can now think about functoriality in this context: if $A : V \to W$ is a linear map, and $A^* : W^* \to V^*$ is its dual map, then we claim that

$$\ker A^* = (\operatorname{Im} A)^\perp, \quad \operatorname{Im} A^* = (\ker A)^\perp.$$

Definition 17

Let V^k denote the k-fold product $V \times V \times \cdots \times V$. A map $T : V^k \to \mathbb{R}$ is **linear in its ith slot** if for any fixed vectors $v_1, \cdots, v_{i-1}, v_{i+1}, \cdots, v_n$, the map $v \to T(v_1, \cdots, v_{i-1}, v, v_{i+1}, \cdots, v_n)$ is a linear map. T is a ***k*-tensor** if it is linear in all k slots. Let $\mathcal{L}^k(V)$ denote the set of k-tensors.

We can define a vector space structure on $\mathcal{L}^k(V)$ as follows: if we have two k-tensors $T_1 : V^k \to \mathbb{R}$ and $T_2 : V^k \to \mathbb{R}$, then we can check that $c_1 T_1 + c_2 T_2$ also gives us a valid k-tensor (by checking linearity in each slot).

Example 18

A 1-tensor is a map from $V \to \mathbb{R}$, so $\mathcal{L}^1(V) = V^*$. Meanwhile, a 2-tensor is a map $V \times V \to \mathbb{R}$, so $\mathcal{L}^2(V)$ is the set of bilinear forms on V. We'll use the convention that $\mathcal{L}^0(V) = \mathbb{R}$ for convenience.

Definition 19

A **multi-index of length *k*** is a sequence of integers $I = (i_1, \cdots, i_k)$, where $1 \le i_1, \cdots, i_k \le n$.

In particular, for a basis $e_1, \cdots, e_n$ of an n-dimensional vector space V, a k-tensor T, and a multi-index $I = (i_1, \cdots, i_k)$, we can define

$$T_I = T(e_{i_1}, \cdots, e_{i_k}).$$

As an exercise, we can check that these numbers (across all multi-indices I) determine the k-tensor T.

Problem 20

Let $\mathcal{I}$ denote the set of multi-indices, and let $\mathcal{L}^k(\mathcal{I})$ be the set of maps from $\mathcal{I}$ to $\mathbb{R}$. Show that $\dim \mathcal{L}^k(\mathcal{I}) = |\mathcal{I}| = n^k$, and show that the map from to $\tilde{L}(I)$, mapping $T \mapsto T_I$, is bijective. Thus, conclude that the dimension of the k-tensors is $\dim \mathcal{L}^k = n^k$.

We'll finish by defining the **tensor product operation**:

Definition 21

If $T_1 \in \mathcal{L}^k(V)$ and $T_2 \in \mathcal{L}^\ell(V)$ are two tensors, we can define the **tensor product** $T_1 \otimes T_2$, a $(k+\ell)$-tensor, via

$$(T_1 \otimes T_2)(v_1, \cdots, v_{k+\ell}) = T_1(v_1, \ldots, v_k) T_2(v_{k+1}, \cdots, v_{k+\ell}).$$

As an exercise, we should check that this is indeed a valid tensor – we'll be using it throughout this class.

3 February 4, 2022

Last lecture, we introduced the concept of a ***k*-tensor**, which is a map $T : V^k \to \mathbb{R}$ which is linear in each of the k copies of V. (In other words, if we fix vectors $v_1, \cdots, v_{i-1}, v_{i+1}, \cdots, v_k$, then for any $v \in V$, the map $v \mapsto T(v_1, \cdots, v_{i-1}, v, v_{i+1}, \cdots, v_k)$ is linear in v.) Recall that $\mathcal{L}^k(V)$ denote the set of k-tensors; this is in fact a vector space because any linear combination of k-tensors is itself a k-tensor.

In our definitions last time, we also introduced **multi-index notation**: a multi-index of length k is some sequence $I = (u_1, \cdots, u_k)$ of integers $1 \le u_i \le n$, and we can define a k-tensor using a multi-index description by defining the numbers $T_I = T(e_{u_1}, e_{u_2}, \cdots, e_{u_k})$ for each I. Those numbers then give us, by linearity, a concrete description of the k-tensor: as we'll describe later, if $v_i = \sum_{j=1}^k a_{ij} e_j$, then we have

$$\boxed{T(v_1, \cdots, v_k) = \sum_I a_I T_I}.$$

We'll spend today discussing some more algebraic properties of these k-tensors. Recall that given two tensors $T_1 \in \mathcal{L}^{k_1}(V)$ and $T_2 \in \mathcal{L}^{k_2}(V)$, we can define the **tensor product** $T_1 \otimes T_2 \in \mathcal{L}^{k_1+k_2}(V)$ via

$$T_1 \otimes T_2(v_1, v_2, \cdots, v_{k_1+k_2}) = T_1(v_1, \cdots, v_{k_1}) T_2(v_{k_1+1}, \cdots, v_{k_1+k_2}).$$

It's left as an exercise to use to check this is a valid tensor, and we can also verify some other properties of this product operation:

1. (Associativity) For any three tensors T_1, T_2, T_3, we have $(T_1 \otimes T_2) \otimes T_3 = T_1 \otimes (T_2 \otimes T_3)$, so we do not need to worry about parentheses when doing tensor product computations.
2. (Left and right distributivity) For any two tensors $T_1, T_2 \in \mathcal{L}^{k_1}$ of the same order and any $T_3 \in \mathcal{L}^{k_2}$, we have $(T_1 + T_2) \otimes T_3 = T_1 \otimes T_3 + T_2 \otimes T_3$, as well as $T_3 \otimes (T_1 + T_2) = T_3 \otimes T_1 + T_3 \otimes T_2$.

Note that we do not have commutativity (that is, $T_1 \otimes T_2 = T_2 \otimes T_1$) in general. But still, being able to take these kinds of products allows us to consider an important special class of tensors:

Definition 22

For all $1 \le i \le k$, let $\ell_i \in V^*$ be linear maps. Then $\ell_1 \otimes \cdots \otimes \ell_k$ is a **decomposable k-tensor**.

Theorem 23

Let $e_1, \cdots, e_n$ be a basis of a (finite-dimensional) vector space V, and let $e_1^*, \cdots, e_n^*$ be the corresponding dual basis. Let $I = (u_1, \cdots, u_k)$ be an arbitrary multi-index of indices. Then the (decomposable) k-tensors $e_I^* = e_{u_1}^* \otimes \cdots \otimes e_{u_k}^*$ form a basis of $\mathcal{L}^k(V)$.

Proof sketch. This is essentially a reformulation of the boxed equation $T(v_1, \cdots, v_k) = \sum_I a_I T_I$ from above. For any k-tensor V and any $v_1, \cdots, v_k$ satisfying $v_i = \sum_{j=1}^k a_{ij} e_j$, if we define $a_I = a_{1u_1} \cdots a_{ku_k}$, then plugging in the v_i expressions and using linearity of the tensor gives us

$$T(v_1, \cdots, v_k) = \sum a_I T_I,$$

so we've written our tensor T as a linear combination of these T_Is. From here, it just remains to show that the T_Is are linearly independent, which will be left for us to show. $\square$

Being able to uniquely express T as linear combinations of tensors of the form $e_{u_1}^* \otimes \cdots \otimes e_{u_k}^*$ will be useful for computations in the future!

4 February 7, 2022

We'll discuss the **theory of permutations** today as they relate to some of the linear algebra objects that we've been studying in this class:

Definition 24

A **permutation of order k** is a bijective map $\sigma : \{1, 2, \cdots, k\} \to \{1, 2, \cdots, k\}$. The set of all permutations of order k is denoted S_k.

This set of permutations has a natural group structure, because bijective maps can be composed and inverted: if σ, τ are elements of S_k, then $\sigma\tau$ is the permutation that sends i to $\sigma(\tau(i))$. (We can then check the group axioms ourselves from here as an exercise.)

Definition 25

Let $1 \le i, j \le k$. The **transposition permutation** τ_{ij} is the map sending i to j, j to i, and fixing all other integers.

It is a fact (that we might learn in an abstract algebra class) that every permutation is a finite product of transpositions – this can be proved by induction.

Definition 26

The **elementary transpositions** are the transposition permutations of the form $\tau_{i,i+1}$.

It turns out that we can also write every transposition as a finite product of **elementary** permutations – this is again proved by induction. Essentially, notice that $\tau_{i,i+2} = \tau_{i+1,i+2}\tau_{i,i+1}\tau_{i+1,i+2}$, and then $\tau_{i,i+3} = \tau_{i+2,i+3}\tau_{i,i+2}\tau_{i+2,i+3}$ (from which we can substitute in the expression for $\tau_{i,i+2}$), and so on. Thus, combining these two facts gives us the important result:

Proposition 27

Every transposition is a finite product of elementary transpositions.

Definition 28

Let $x_1, \cdots, x_n$ be coordinate functions on $\mathbb{R}^n$, and let $\sigma \in S_n$ be a permutation. The **sign** of the permutation σ, denoted $(-1)^\sigma$, is given by

$$(-1)^\sigma = \frac{\prod_{i<j} x_{\sigma(i)} - x_{\sigma(j)}}{\prod_{i<j}(x_i - x_j)}.$$

Notice that each set of indices $\{i, j\}$ will show up in both the numerator and denominator in some order, so the right-hand side will just be one of ± 1.

Lemma 29

If τ is any transposition permutation, then $(-1)^\tau = -1$.

(This is easily verified by plugging the form of τ back into the formula.)

Lemma 30

If $\sigma, \tau \in S_n$ are two permutations, then $(-1)^{\sigma\tau} = (-1)^\sigma(-1)^\tau$.

Proof. Plugging into the formula, we have

$$(-1)^{\sigma\tau} = \frac{\prod_{i<j} x_{\sigma\tau(i)} - x_{\sigma\tau(j)}}{\prod_{i<j}(x_i - x_j)}.$$

We can rewrite this product as

$$= \frac{\prod_{i<j} x_{\sigma\tau(i)} - x_{\sigma\tau(j)}}{\prod_{i<j}(x_{\tau(i)} - x_{\tau(j)})} \cdot \frac{\prod_{i<j}(x_{\tau(i)} - x_{\tau(j)})}{\prod_{i<j}(x_i - x_j)},$$

so that the second factor is $(-1)^\tau$ by definition. But now replacing i and j with $\tau(i)$ and $\tau(j)$ in the numerator and denominator is essentially a relabeling of the indices (alternatively, a reordering of the terms in the product). So replacing $i^\# = \tau(i)$ will make the first term $\frac{\prod_{i<j} x_{\sigma(i^\#)} - x_{\sigma(j^\#)}}{\prod_{i<j}(x_{i^\#} - x_{j^\#})}$, and these products each sum over each unordered pair $i^\#, j^\#$ exactly once, so this fraction is exactly the definition of $(-1)^\sigma$. This finishes the proof. □

Corollary 31

If $\sigma \in S_n$ is a product of k transpositions, then $(-1)^\sigma = (-1)^k$.

Definition 32

Let $T \in \mathcal{L}^k(V)$ be a k-tensor, and let $\sigma \in S_k$. We define T^σ to be the k-tensor such that

$$T^\sigma(v_1, \cdots, v_k) = T(v_{\sigma^{-1}(1)}, \cdots, v_{\sigma^{-1}(k)}).$$

(The reasons for using the inverse permutation will become clear soon.)

Example 33
Let $\ell_1, \cdots, \ell_k \in V^*$ be linear maps, and let T be the decomposable k-tensor $\ell_1 \otimes \cdots \otimes \ell_k$. Then $T^\sigma = \ell_{\sigma(1)}\ell_{\sigma(2)}\cdots\ell_{\sigma(k)}$.

Proposition 34
Let T be a k-tensor, and let $\sigma, \tau \in S_k$. Then $T^{\sigma\tau} = (T^\sigma)^\tau$ (meaning that we apply the σ permutation first to T, and then τ to that result).

Proof. By linearity, it suffices to show that this result holds for decomposable k-tensors, so we can apply Example 33 and verify that the result holds there. $\square$

We'll finish this lecture by introducing two important objects that we'll be using for the rest of this course:

Definition 35
A k-tensor $T \in \mathcal{L}^k(V)$ is a **symmetric k-tensor** if for every permutation $\sigma \in S_k$, $T^\sigma = T$.

But for the theory of differential forms, the even more important object is the following:

Definition 36
A k-tensor $T \in \mathcal{L}^k(V)$ is an **alternating k-tensor** if for every permutation $\sigma \in S_k$, $T^\sigma = (-1)^\sigma T$.

These alternating k-tensors will turn out to be the basic building blocks of differential forms, and we'll be studying them a lot in the coming lectures.

5 February 9, 2022

Last lecture, we introduced some properties of permutations, which are bijective maps $\sigma : \{1, \cdots, k\} \to \{1, \cdots, k\}$. Treating composition of these maps as multiplication, the set of permutations S_k has a group structure (where the inverse σ^{-1} of a permutation σ is the inverse map).

A very relevant property of permutations is their **sign**, which is either 1 or -1 and is given by the definition $(-1)^\sigma = \prod_{i<j} \frac{x_{\sigma(i)} - x_{\sigma(j)}}{x_i - x_j}$. In particular, if we take a k-tensor $T \in \mathcal{L}^k(V)$, we defined the k-tensor T^σ given by

$$T^\sigma(v_1, v_2, \cdots, v_k) = T(v_{\sigma^{-1}(1)}, v_{\sigma^{-1}(2)}, \cdots, v_{\sigma^{-1}(k)}),$$

and we said that a k-tensor is **alternating** if $T^\sigma = (-1)^\sigma T$ (and **symmetric** if $T^\sigma = T$) for all permutations $\sigma \in S^k$. These definitions become more clear if we look at the decomposable tensors of the form $T = \ell_1 \otimes \ell_2 \otimes \cdots \otimes \ell_k$ (where ℓ_is are linear maps) – for such tensors, we have $T^\sigma = \ell_{\sigma(1)} \otimes \ell_{\sigma(2)} \otimes \cdots \otimes \ell_{\sigma(k)}$. Looking at such decomposable k-tensors, which form a basis of all k-tensors, allows us to prove that $T^{(\sigma\tau)} = (T^\sigma)^\tau$.

We'll let $\mathcal{S}^k(V)$ denote the space of symmetric k-tensors, and we'll let $\mathcal{A}^k(V)$ denote the space of alternating k-tensors. The latter set will be essential for the theory of differential forms, and here we'll describe a way of constructing such alternating tensors:

Definition 37

Let $T \in \mathcal{L}^k(V)$ be an arbitrary tensor. The **alternation operation** is defined via

$$\mathrm{Alt}(T) = \sum_{\tau \in S_k} (-1)^\tau T^\tau.$$

Proposition 38

The facts below follow from definitions and properties of the sign of a permutation:

1. For any $T \in \mathcal{L}^k(V)$, we have $\mathrm{Alt}(T) \in \mathcal{A}^k(V)$.
2. For any $\sigma \in S_k$, we have $\mathrm{Alt}(T^\sigma) = (-1)^\sigma \mathrm{Alt}(T)$.
3. If $T \in \mathrm{Alt}(T)$, then $\mathrm{Alt}(T) = k!T$.

We will now construct a basis for $\mathcal{A}^k(V)$. First, let $e_1, \cdots, e_n$ be a basis of V, and let $e_1^*, \cdots, e_n^*$ be the corresponding dual basis of V^*. For any multi-index $I = (i_1, \dots, i_k)$, we define $e_I^* = e_{i_1}^* \otimes \cdots \otimes e_{i_k}^*$, and we may recall from previous lectures that these e_I^* form a basis of $\mathcal{L}^k(V)$. Given this fact, we now define

$$\phi_I = \mathrm{Alt}(e_I^*)$$

for all multi-indices I; because the e_I^* form a basis and Alt is surjective, we know that the ϕ_Is **span** $\mathcal{A}^k(V)$. But to avoid having linear dependence, we need to restrict the set of Is that we use:

Definition 39

A multi-index I is **strictly increasing** if $i_1 < i_2 < \cdots < i_r$.

Theorem 40

The set $\{\phi_I : I \text{ strictly increasing}\}$ is a basis for the set of alternating k-tensors $\mathcal{A}^k(V)$.

Beginning of the proof. First of all, call a multi-index I **repeating** if $i_r = i_s$ for some $r \neq s$. Notice that $\phi_I = 0$ for any repeating multi-index I – we can see this by breaking S_k up into the two cosets formed by the subgroup $\{1, \sigma_{rs}\}$, or equivalently saying that $T_I = T_I^{\sigma_{rs}}$, so that

$$\mathrm{Alt}(T_I) = \mathrm{Alt}(T_I^{\sigma_{rs}}) = -\mathrm{Alt}(T_I) \implies \mathrm{Alt}(T_I) = 0.$$

Thus we do not want to include ϕ_I for repeating I. Now if we consider a non-repeating multi-index I, let $(i_{r_1}, i_{r_2}, \cdots, i_{r_k})$ be the reordering of the indices of I so that $r_1 < r_2 < \cdots < r_k$, and let $\sigma \in S_k$ be the permutation that takes i_ℓ to i_{r_ℓ}, then I^σ is strictly increasing, and $\mathrm{Alt}(e_{I^\sigma}^*) = (-1)^\sigma \mathrm{Alt}(e_I^*)$ (because $(e_I^*)^\sigma = e_{I^\sigma}^*$). Thus any ϕ_I is $\pm\phi_{I^\sigma}$ for some increasing multi-index I^σ, so the set of ϕ_I formed by just the increasing multi-indices I also spans the whole set $\mathcal{A}^k(V^*)$. The rest of the proof will be shown next time! $\square$

6 February 11, 2022

Our first homework assignment will be on Canvas today, and it will be due two weeks from today.

First, let's do some review. We've been studying the "permuted" versions of tensors in the last few lectures: for any tensor $T \in \mathcal{L}^k(V)$ and any permutation $\sigma \in S_k$, we define T^σ via $T^\sigma(v_1, \cdots, v_k) = T(v_{\sigma^{-1}(1)}, \cdots, v_{\sigma^{-1}(k)})$. (This definition is motivated by the fact that a decomposable tensor $T = \ell_1 \otimes \cdots \otimes \ell_k$ becomes $T^\sigma = \ell_{\sigma(1)} \otimes \cdots \otimes \ell_{\sigma(k)}$.) The fundamental objects important for the theory of differential forms are the **alternating tensors** satisfying $T = (-1)^\sigma T^\sigma$ (or equivalently $T^\sigma = (-1)^\sigma T$) – we can construct the tensor

$$\mathrm{Alt}(T^\sigma) = \sum_{\sigma \in S_k} (-1)^\sigma T^\sigma.$$

It turns out that Alt is a surjective map from $\mathcal{L}^k(V)$ to $\mathcal{A}^k(V)$, and $\mathrm{Alt}(T) = k!T$ if T is an alternating tensor. It then makes sense to ask about the kernel of the map Alt:

Definition 41

A decomposable k-tensor $T = \ell_1 \otimes \cdots \otimes \ell_k$ is **redundant** if $\ell_r = \ell_{r+1}$. The linear span of all redundant k-tensors is denoted $\mathcal{I}^k$ (elements of $\mathcal{I}^k$ will be called redundant as well).

Proposition 42

If $T \in \mathcal{I}^k$, then $\mathrm{Alt}(T) = 0$.

Proof. Suppose $T = \ell_1 \otimes \cdots \otimes \ell_k$ is decomposable with $\ell_r = \ell_{r+1}$. Let σ be the transposition permutation swapping r and $r+1$, so that

$$\mathrm{Alt}(T) = \mathrm{Alt}(T^\sigma) = (-1)^\sigma \mathrm{Alt}(T) = -\mathrm{Alt}(T),$$

so that $\mathrm{Alt}(T) = 0$. Since these redundant tensors span $\mathcal{I}^k$, $\mathrm{Alt}(T) = 0$ for any element of $\mathcal{I}^k$. □

We aim to show that these redundant k-tensors span the kernel of Alt, which is the converse result. This is essentially showing the other part of the theorem from last lecture, but first, we mention some important preliminary results:

Lemma 43

If T_1 and T_2 are elements of $\mathcal{I}^r(V)$ and $\mathcal{I}^s(V)$, respectively, then $T_1 \otimes T_2$ and $T_2 \otimes T_1$ are element sof $\mathcal{I}^{r+s}(V)$.

Proof. It suffices to consider the case where T_1 and T_2 are both decomposable redundant k-tensors, so that T_1 has some repeat $\ell_i = \ell_{i+1}$. Then whether we are tensoring $T_1 \otimes T_2$ or $T_2 \otimes T_1$, we will have two adjacent slots where the linear maps are the same (either $\ell_i = \ell_{i+1}$ or $\ell_{i+s} = \ell_{i+s+1}$, respectively). □

Lemma 44

If $T \in \mathcal{L}^k(V)$ and $\sigma \in S_k$, then $T = (-1)^\sigma T^\sigma + T'$ for some $T' \in \mathcal{I}^k$.

Proof. By linearity, it suffices to show the result when T is decomposable. Let $T = \ell_1 \otimes \cdots \otimes \ell_k$. We know that any permutation σ can be written as a product of elementary transpositions of the form $\tau_i = \sigma_{i,i+1}$; if we consider the case where σ is a single transposition τ_i, then

$$T + T^\sigma = \ell_1 \otimes \cdots \otimes (\ell_i \otimes \ell_{i+1} + \ell_{i+1} \otimes \ell_i) \otimes \cdots \otimes \ell_k.$$

But this right-hand side is in the linear span of redundant decomposable tensors, because

$$(\ell_i \otimes \ell_{i+1} + \ell_{i+1} \otimes \ell_i) = (\ell_i + \ell_{i+1}) \otimes (\ell_i + \ell_{i+1}) - \ell_i \otimes \ell_i - \ell_{i+1} \otimes \ell_{i+1}.$$

Thus $T + T^\sigma$ is in $\mathcal{I}^k$, and because $(-1)^\sigma = -1$ in this case that's exactly what we want to show.

From here, we use induction: suppose σ is a product of the form $\tau_1\tau_2 \ldots \tau_{m-1}$, and we know that $T - (-1)^\sigma T^\sigma$ is in $\mathcal{I}^k$. Then by the inductive hypothesis we have (abusing notation a little here)

$$T^{\sigma\tau} = (T^\sigma)^\tau = (-1)^\tau T^\sigma + \mathcal{I}^k,$$

and again applying the inductive hypothesis we get

$$= (-1)^\tau (-1)^\sigma T + \mathcal{I}^k = (-1)^{\sigma\tau} T + \mathcal{I}^k,$$

showing the desired result. $\square$

Corollary 45
For all $T \in \mathcal{L}^k(V)$, we have

$$T = \frac{1}{k!}\operatorname{Alt}(T) + I$$

for some $I \in \mathcal{I}^k(V)$.

Proof. By Lemma 44, we have

$$\operatorname{Alt}(T) = \sum_\sigma (-1)^\sigma T^\sigma = \sum_\sigma (-1)^\sigma (-1)^\sigma T + S_\sigma$$

for some $S_\sigma \in \mathcal{I}^k$ for each $\sigma \in S_k$. Since $(-1)^\sigma(-1)^\sigma = 1$, this simplifies to

$$\operatorname{Alt}(T) = k!T + \sum_\sigma S_\sigma.$$

Dividing through by $k!$ and rearranging gives the result – because each S_σ is in $\mathcal{I}^*$, so is their average. $\square$

That finally gives us the result about the kernel of the map Alt:

Corollary 46
For any k-tensor T, if $\operatorname{Alt}(T) = 0$, then $T \in \mathcal{I}^k$. Also, any k-tensor T can be uniquely written as $T = T' + S'$, where T' is alternating and S' is redundant.

Proof. The first part follows by setting $\operatorname{Alt}(T) = 0$ in Corollary 45. For the second part, suppose we have $T = T' + S' = T_1' + S_1'$, where T', T_1' are alternating and S', S_1' are redundant. Taking Alt of both sides, we find that $\operatorname{Alt}(T) = k!T' = k!T_1'$, so $T' = T_1'$ and thus $S' = S_1'$. $\square$

We thus can think of alternating k-tensors as a quotient space, and that will be foundational for our future study.

7 February 14, 2022

Last time, we introduced the set of redundant k-tensors $\mathcal{I}^k(V)$, which are the linear span of the decomposable k-tensors $\ell_1 \otimes \cdots \otimes \ell_k$ with $\ell_i = \ell_{i+1}$ for some i. We then proved that if T is redundant, then $\operatorname{Alt}(T) = 0$, and in fact

$\mathcal{I}^k(V)$ is the kernel of the map $\text{Alt} : \mathcal{L}^k(V) \to \mathcal{A}^k(V)$.

This allows us to make the definition of the most important object of the class:

Definition 47

The **space of exterior k-forms** on a vector space V is the quotient space $\Lambda^k(V^*) = \mathcal{L}^k(V)/\mathcal{I}^k(V)$.

In particular, here we're making use of the **exact sequence**

$$0 \to \mathcal{I}^k(V) \to \mathcal{L}^k(V) \stackrel{\text{Alt}}{\to} \mathcal{A}^k(V) \to 0$$

(where exactness follows from the result we showed last lecture). And furthermore, we now get a **natural bijection**

$$\Lambda^k(V^*) \to \mathcal{A}^k(V),$$

where the idea is that the two descriptions of the same object (either as a quotient space, or as a subspace) can be useful in conjunction. (Since we're looking at the space of tensors over V^*, we can imagine these spaces as being spanned by $v_{i_1} \otimes \cdots \otimes v_{i_k}$, where the v_{i_j}s are now basis elements of the original space V.)

Definition 48

Let $\pi : \mathcal{L}^k(V) \to \mathcal{L}^k(V)/\mathcal{I}^k(V) = \Lambda^k(V^*)$ be the natural projection map. The **wedge product operation** is defined as follows: for $\omega_1 \in \Lambda^{k_1}(V^*)$ and $\omega_2 \in \Lambda^{k_2}(V^*)$, pick some $T_i \in \mathcal{L}^{k_i}(V)$ so that $\pi(T_i) = \omega_i$. Then define $\omega_1 \wedge \omega_2 = \pi(T_1 \otimes T_2)$.

This definition basically tells us to go back into the full space of k-tensors and do a tensor product there, so that the wedge product is a "factored version" of the tensor product. We can check that this is well-defined – in particular, if we have T_1 or T_2 redundant, then $T_1 \otimes T_2$ is redundant, so that after quotienting out we'll still have 0. Thus our choice of T_1 (which is up to a redundant tensor defined) will not change the end result $\pi(T_1 \otimes T_2)$.

Definition 49

Let V be an n-dimensional vector space, and let $T \in \mathcal{L}^k(V)$. The **interior product operation** is defined as follows: for $v \in V$, $\iota_v(T)$ is the $(k-1)$-tensor given by

$$\iota_v(T)(v_1, \cdots, v_{k-1}) = \sum_{r=1}^{k} (-1)^{r-1} T(v_1, \cdots, v_{r-1}, v, v_{r+1}, \cdots, v_{k-1}).$$

We'll talk more about this next time!

8 February 16, 2022

Last time, we wrote down the short exact sequence

$$0 \to \mathcal{I}^k(V) \to \mathcal{L}^k(V) \stackrel{\text{Alt}}{\to} \mathcal{A}^k(V) \to 0,$$

which is basically a cleaner way of explaining that $\mathcal{I}^*$ is the kernel of the Alt map $\mathcal{L}^k(V) \to \mathcal{A}^k(V)$. This enabled us to define the space of exterior k-forms $\Lambda^k(V^*) = \mathcal{L}^k(V)/\mathcal{I}^k(V)$, in such a way that we have a bijective map between $\Lambda^k(V^*)$ and $\mathcal{A}^k(V)$. From there, the projection operation $\mathcal{L}^k(V) \to \mathcal{L}^k(V)/\mathcal{I}^k(V) = \Lambda^k(V^*)$ (**which we can think**

about as basically applying Alt) allows us to define the **wedge product**: if $\omega_1 \in \Lambda^{k_1}(V^*)$ and $\omega_2 \in \Lambda^{k_2}(V^*)$, then we can pick $T_1 \in \mathcal{L}^{k_1}(V)$ and $T_2 \in \mathcal{L}^{k_2}(V)$ such that $\pi(T_1) = \omega_1$ and $\pi(T_2) = \omega_2$ (this is always possible because the projection map is onto). Then $\omega_1 \wedge \omega_2 = \pi(T_1 \otimes T_2)$, and we mentioned last time that this indeed well-defined.

We can now turn to considerations of functoriality:

Definition 50

If we have a linear map $A : V \to W$ between vector spaces, then we also get a linear map $A^* : \mathcal{L}^k(W) \to \mathcal{L}^k(V)$, given by the **pullback operation**: for any k-tensor $T \in \mathcal{L}^k(W)$, we have

$$A^*T(v_1, \cdots, v_k) = T(Av_1, \cdots, Av_k).$$

In particular, we can check that if $T \in \mathcal{I}^k(W)$, then $A^*T \in \mathcal{I}^k(V)$ (start with a decomposable k-tensor, noticing that if $\ell_r = \ell_{r+1}$, then $A^*\ell_r = A^*\ell_{r+1}$), and we can also verify the relation

$$A^*(T_1 \otimes T_2) = A^*(T_1) \otimes A^*(T_2).$$

Thus, we can also define an induced map on the quotient spaces

$$A^* : \mathcal{L}^k(W)/\mathcal{I}^k(W) \to \mathcal{L}^k(V)/\mathcal{I}^k(V),$$

and therefore **this pullback operation takes a linear map** $A : V \to W$ **and gives us a map**

$$A^* : \Lambda^k(W^*) \to \mathcal{A}^k(V).$$

(The reason we write $\mathcal{A}^k(V)$ instead of $\Lambda^k(V^*)$ here is that it's more in line with the naturality of quotienting by the kernel and ending up with the image, but it's essentially the same thing.) In particular, bringing our definitions together, we find that for $\omega_1 \in \Lambda^{k_1}(V^*)$ and $\omega_2 \in \Lambda^{k_2}(V^*)$, we have

$$A^*(\omega_1 \wedge \omega_2) = A\omega_1 \wedge A\omega_2,$$

and thus in the special case where we're wedging together linear maps, we get

$$\boxed{A^*(\ell_1 \wedge \cdots \wedge \ell_k) = A^*\ell_1 \wedge \cdots \wedge A^*\ell_k}$$

(we'll be using this a lot throughout the rest of the course!). With this in mind, we can apply our discussion to the notion of a **determinant** from linear algebra:

Proposition 51

Let V, W be two n-dimensional vector spaces, and let $e_1, \cdots, e_n$ and $f_1, \cdots, f_n$ be bases of V and W, respectively with corresponding dual bases $e_1^*, \cdots, e_n^*$ and $f_1^*, \cdots, f_n^*$. Let $A : V \to W$ be a linear map with corresponding matrix $[a_{ij}]$ with respect to these bases, so that $Ae_j = \sum a_{ij} f_i$ and $A^*f_i^* = \sum a_{ij} e_j^*$. Then

$$A^*(f_1^* \wedge \cdots \wedge f_n^*) = A^*f_1^* \wedge \cdots \wedge A^*f_n^* = \left(\sum a_{1j_1} e_{j_1}^*\right) \wedge \cdots \wedge \left(\sum a_{nj_n} e_{j_n}^*\right) = \sum a_{1j_1} a_{2j_2} \cdots a_{nj_n} e_J^*,$$

where $e_J^* = e_{j_1}^* e_{j_2}^* \cdots e_{j_n}^*$.

In particular, for any repeating multi-index J, $e_J^* = 0$, and for any non-repeating multi-index, there is a permutation $\sigma \in S_n$ such that $\sigma(i) = j_i$ for all i, meaning that $e_J^* = (-1)^\sigma e_1^* \wedge \cdots \wedge e_n^*$. And thus, our familiar above reduces to something more familiar, since we have a summation of the form

$$\sum (-1)^\sigma a_{1j_1} \cdots a_{nj_n} e_1^* \wedge e_2^* \wedge \cdots \wedge e_n^* :$$

Corollary 52

For any linear map $A : V \to W$, we have

$$A^*(f_1^* \wedge \cdots \wedge f_n^*) = (\det A) e_1^* \wedge \cdots \wedge e_n^*.$$

This functoriality allows us to prove some nice results from linear algebra without needing too much computation: for example, we can see quickly that $\det(AB) = \det(A)\det(B)$, using that

$$\det(AB)\omega = (AB)^*\omega = B^*(A^*\omega) = B^*(\det A \cdot \omega) = \det AB^*(\omega) = \det(A)\det(B)\omega.$$

We can also see that the identity map $A : V \to V$ has $\det(A) = 1$, and that whenever $A : V \to V$ is not onto, $\det(A) = 0$

9 February 18, 2022

Fact 53

We're assigned to read Section 1.9 and Section 2.1 of the textbook on our own over the weekend, which essentially covers the concepts of orientations, vector fields, and 1-forms. But we'll mention a few of the facts here:

Definition 54

Let ℓ be a line through the origin in $\mathbb{R}^2$. Then $\ell - \{0\}$ has two connected components, and an **orientation** of ℓ is a choice of one of these two components.

(This is equivalent to essentially choosing a direction for the line.) We also have a natural generalization of this which will connect back to the material in the class:

Definition 55

Let L be a one-dimensional vector space. Then $L - \{0\}$ has two components; specifically, if we fix some $v \in L - \{0\}$, we have the component $L^+ = \{\lambda v : \lambda > 0\}$ and $L^- = \{\lambda v : \lambda < 0\}$. An **orientation** of L is a choice of either L^+ or L^-.

Even more generally, if V is an n-dimensional vector space, an **orientation** of V is an orientation of the space $\Lambda^n(V)$. If $(e_1, \cdots, e_n)$ form an ordered basis of V, then the basis is said to be **positively oriented** if $e_1 \wedge \cdots \wedge e_n$ is in the positive part of $\Lambda^n(V)$.

(Note that the dimension of the space $\Lambda^k(V)$ is $\binom{n}{k}$, so $\Lambda^n(V)$ indeed has dimension $\binom{n}{n} = 1$.)

Proposition 56
Let V be an n-dimensional vector space, and let $W \subseteq V$ be a k-dimensional subspace. If we are given orientations on V and W, then there is a natural orientation of the quotient space V/W given as follows: choose an **oriented basis** $v_1, \cdots, v_n$ of V such that the first k of these vectors form an oriented basis of W. Then we can orient V/W by projecting the remaining $n-k$ vectors onto V/W.

With that, we'll start making a few remarks that will start us on the next section of the class next week (Chapter 2 of the book). The idea is that Section 2.1 looks at simple but pivotal objects in the study of differential forms.

Definition 57
Let $p \in \mathbb{R}^n$ be a point. The **tangent space** at p, denoted $T_p(\mathbb{R}^n)$, is the set

$$T_p(\mathbb{R}^n) = \{(p, v) : v \in \mathbb{R}^n\}.$$

The point p can be called a **base point**.

We can form the obvious vector space structure on $T_p(\mathbb{R}^n)$, keeping the same base point, via

$$(p, v_1) + (p, v_2) = (p, v_1 + v_2), \quad \lambda(p, v) = (p, \lambda v).$$

The idea is to think of v as an arrow originating from p and pointing in the direction of v.

Definition 58
Let U be an open subset of $\mathbb{R}^n$. A **vector field** v on U is a function which assigns to each point $p \in U$ a corresponding $v(p) \in T_p(U) = T_p(\mathbb{R}^n)$. A **one-form** on U is a function which assigns to each point $p \in U$ an element $\omega(p) \in T_p^*(\mathbb{R}^n)$.

Starting next week, we'll talk more about these definitions, generalize to k-forms, and get into the main topic of this course!

10 February 22, 2022

Today, we'll fully begin discussing the concept of **differential forms**. Last lecture, we introduced the concept of a **tangent space** (the set of pairs (p, v) for some fixed $p \in \mathbb{R}^n$) – we convert this to a vector space by using the usual vector space structure on $\mathbb{R}^n$ through v, keeping the base point p fixed:

$$(p, v_1) + (p, v_2) = (p, v_1 + v_2), \quad \lambda(p, v) = (p, \lambda v).$$

We can now think about **functoriality** in the following way: suppose we have open sets $U, V \in \mathbb{R}^n, \mathbb{R}^m$ and we have a C^∞ (smooth) map $\phi : U \to V$. Then we can define the **derivative map** $D\phi(r) : \mathbb{R}^n \to \mathbb{R}^m$ encoded by the matrix $\left[\frac{\partial \phi_i}{\partial x_j}(r)\right]$. This leads us to the base pointed version of the derivative definition:

Definition 59

Let $\phi : U \to V$ be a C^∞ map, and let $q = \phi(p)$. The map $(d\phi)_p$ between tangent spaces $T_pU \to T_qV$ is given by

$$(d\phi)_p(p, v) = (q, D\phi(p)(v)).$$

In other words, we map the base point to the new base point, and we apply the derivative map to the vector v. One result from calculus is the familiar chain rule: if W is additionally an open set in $\mathbb{R}^\ell$ and $\psi : V \to W$ is a C^∞ map, then we know that

$$(d\psi \circ \phi)_p = d\psi_q \circ d\phi_p.$$

Definition 60

The **cotangent space** to U at p is the vector space dual of the corresponding tangent space:

$$T_p^*U = (T_pU)^*.$$

In particular, if $f \in C^\infty(U)$ is a smooth real-valued function, $p \in U$ is some point, and $u = f(p)$. Then we can think of f as a map $(U, p) \to (\mathbb{R}, u)$. Taking its derivative, we then know that df_p is a map $T_pU \to T_u\mathbb{R} \cong \mathbb{R}$, so it's fundamentally an element of the cotangent space: $\boxed{df_p \in T_p^*U}$.

Problem 61

Let $x_1, \cdots, x_n$ be coordinate functions on $U \subseteq \mathbb{R}^n$. Then $(dx_1)_p, \cdots, (dx_n)_p$ form a basis of the cotangent space T_p^*U.

This basis can be described in an alternative way as well: if we let e_i be the standard basis vector with a 1 in the ith spot and 0s in the others, and we use the notation

$$\left(\frac{\partial}{\partial x_i}\right)_p = (p, e_i) \in T_pU,$$

then these elements form a basis of T_p, and then $(dx_i)_p$ is the corresponding dual basis on T_p^*U.

We can now return to the definitions of vector fields and one-forms that we started last lecture: recall that a **vector field** is a map $v : U \to T_pU$, and we can write that as

$$v(p) = (p, \mathbf{v}(p))$$

for some map $\mathbf{v} : U \to \mathbb{R}^n$.

Example 62

Let e_i again be the standard basis vectors. Then the vector field $p \mapsto (p, e_i)$ is denoted $\frac{\partial}{\partial x_i}$. Since $\left(\frac{\partial}{\partial x_i}\right)_p = (p, e_i)$ form a basis for T_pU, we can write every vector field as

$$v(p) = \sum f_i(p) \left(\frac{\partial}{\partial x_i}\right)_p \implies v = \sum f_i \frac{\partial}{\partial x_i}.$$

for real-valued functions (the coefficients for our basis vectors) f_i.

Definition 63
A vector field v is C^∞ if $\mathbf{v}$ is C^∞, or equivalently if the corresponding f_is are functions in $C^\infty(U)$.

The duals to these vector fields are the **one-forms**: as we defined last time, a one-form is functions u which send points $p \in U$ to elements of T_p^*U. In particular, if $f \in C^\infty(U)$ is a smooth function, then df is the one-form on U mapping

$$df : p \mapsto df_p.$$

The coordinate functions $x_1, \cdots, x_n$ on U then give us one-forms $dx_1, \cdots, dx_n$. Since we've shown that these are basis elements of T_p^*U, we find that **every one-form u on U can be written uniquely as**

$$u = f_1 dx_1 + \cdots + f_n dx_n$$

for some real-valued functions $f_i : U \to \mathbb{R}$.

Definition 64
A one-form is C^∞ if the corresponding f_is are C^∞ functions. The space of C^∞ one-forms is denoted $\Omega^1(U)$.

Example 65
If $f \in C^\infty(U)$ is any smooth function, then the one-form df defined above is C^∞. Indeed, as we might expect, $df = \sum \frac{\partial f}{\partial x_i} dx_i$.

We'll finish this lecture by defining the pullback operation on one-forms:

Definition 66
Let $U \in \mathbb{R}^n$ and $V \in \mathbb{R}^m$ be open sets, and let $f : U \to V$ be a C^∞ map. Then given a one-form ν on V, we can define the one-form f_ν^* via

$$f_\nu^*(p) = (df_p)^* \nu(f(p)),$$

where $(df_p)^* : T_q^*V \to T_p^*U$ is the map we defined earlier as being given by the transpose of $df_p : T_pU \to T_qV$.

Problem 67
Suppose $y_1, \cdots, y_m$ are the standard coordinates on V, and $f_i = f_{y_i}^*$. Then any C^∞ one-form on V of the form $\nu = \sum a_i dy_i$ has corresponding pullback $f^*\nu = \sum f^* a_i df_i$, so that $f^*\nu$ is a C^∞ one-form if ν is.

In particular, f^* is a map $\Omega^1(V) \to \Omega^1(U)$.

11 February 23, 2022

Fact 68
Our last lecture was just yesterday, and today's lecture is a lot of review of that material (because of the new definitions).

Last lecture, we made a lot of important definitions relevant to our eventual introduction of differential forms. Specifically, we discussed the tangent space T_pU of a point p in an open set U, and we mentioned that when we have a smooth map $f : U \to W$ between open sets, we can also define a base-pointed version of the differential map $Df(p)$, which we call df_p (this just maps base point p to the new base point $f(p)$ and applies Df to the "tangent" vector v). We then have a chain rule (which is essentially a matrix multiplication statement) for these base-pointed differential maps, given by

$$(dg \circ f)_p = dg_q \circ df_p.$$

We next defined the dual space T_p^*U; specifically, base-pointed differential maps df_p can be thought of as maps $(U, p) \to (\mathbb{R}, q) \cong \mathbb{R}$, which are linear maps on the tangent space and thus elements of that dual space. And with all of these new vector spaces, we can construct associated linear bases: for example, the $(dx_i)_p$ for $1 \le i \le n$ form a basis of T_p^*U if we're in n-dimensional space, and the dual basis of T_pU consists of the elements $\left(\frac{\partial}{\partial x_i}\right)_p = (p, e_i)$ for $1 \le i \le n$.

Remark 69. *The reason for the definition $\left(\frac{\partial}{\partial x_i}\right)_p = (p, e_i)$ is essentially that an element (p, v) of the tangent space encodes a vector v rooted at p, so a vector pointed in the e_i direction is inherently connected to the notion of taking the derivative along the x_i coordinate.*

Additionally, we defined one-forms, which are maps from U to T_p^*U (assigning to each point p a corresponding element $u_p \in T_p^*U$). Today, we'll start by discussing operations that we can do on these one-forms:

1. If u_1, u_2 are one-forms on U, then $u_1 + u_2$ is the one-form u such that $u(p) = u_1(p) + u_2(p)$ for all $p \in U$.
2. If u is a one-form on U and $\phi \in C^\infty(U)$ is a smooth function, then ϕu is the one-form such that $(\ell u)(p) = \phi(p)u(p)$ for all p. (But we cannot just multiply two one-forms together because $u(p)$ is not a number.)
3. If $\rho \in C^\infty(U)$ is a smooth function, then we can define the one-form $d\rho$ which sends $p \mapsto d\rho_p$. In particular, each coordinate function x_i is a smooth function, so dx_i (for $1 \le i \le n$) are all one-forms, and in fact every one-form is "locally" a linear combination of this nature: we have a class of C^∞ one-forms, denoted $\Omega^1(U)$, given by

$$u = \phi_1 dx_1 + \cdots + \phi_n dx_n$$

for some $\phi_i \in C^\infty(U)$ (and more generally, every one-form is of this form but for arbitrary real-valued functions ϕ_i). In particular, $\Omega^1(U)$ is closed under the first two operations of addition and multiplication by ϕ.

The last object that we defined last lecture is the pullback operation: if we have a smooth map $U \to V$, then we get a corresponding map $f^* : \Omega^1(V) \to \Omega^1(U)$ mapping one-forms via $\nu \mapsto f^*\nu$. Specifically (changing the notation slightly from last lecture), we have

$$(f^*\nu)_p = (df_p)^*\nu_q,$$

where (as usual) $q = f(p)$. As mentioned last time, one way to understand why this **pullback operation** is defined in this way is that if $\nu = dx_i$ and our smooth function is $f = (f_1, \cdots, f_m)$, then $f^*dx_i = df_i$. (And more generally, if ν is a C^∞ one-form and f is a smooth mapping, then $f^*\nu$ is also a C^∞ one-form.)

We'll finish this lecture by reviewing a few operations on vector fields, analogous to the ones on one-forms before: if v_1, v_2 are vector fields (maps $U \to T_pU$), then we can define $v_1 + v_2$ by pointwise addition

$$(v_1 + v_2)(p) = v_1(p) + v_2(p).$$

We can also define multiplication by a smooth function $\phi : U \to \mathbb{R}$

$$\phi v(p) = \phi(p)v(p).$$

Finally, the objects $\frac{\partial}{\partial x_i}$ are vector fields on U which send p to $\left(\frac{\partial}{\partial x_i}\right)_p$, and all vector fields are linear combinations of these fundamental vector fields of the form

$$v = \sum \phi_i \frac{\partial}{\partial x_i}.$$

In particular, if ϕ_i are C^∞ functions, then we call v a C^∞ vector field.

12 February 28, 2022

(Friday's class did not occur because of a snow day, so we're having that lecture instead.) We'll discuss the theory of **integral curves of vector fields** today.

Definition 70

Let U be an open set in $\mathbb{R}^n$, and let v be a vector field on U. A function $\gamma : (a, b) \to U$ is an **integral curve** of v if for all $a < t < b$, if we define $p = \gamma(t)$, we have

$$v(p) = \left(p, \frac{d\gamma}{dt}(t)\right).$$

More explicitly, if our curve is written out as $\gamma(t) = (x_1(t), \cdots, x_n(t))$, then this equation has the more explicit form

$$\frac{dx_i}{dt}(t) = v_i(x(t)),$$

where we're writing out our vector field explicitly as $v = \sum v_i \frac{\partial}{\partial x_i}$. With this, we can do some basic ODE theory. We'll start by citing some relevant local results:

Fact 71 (Existence of integral curves)

Let v be a vector field, and let $p_0 \in U$ and $a \in \mathbb{R}$. Then there exists an interval $I = (a-\varepsilon, a+\varepsilon)$ for some $\varepsilon > 0$, an open set $U_0 \subset U$ containing p_0, and a C^∞ map $F : U_0 \times I \to U$, such that $\gamma_p(t)$ is an integral curve of v.

Fact 72 (Uniqueness of integral curves)

Suppose $\gamma_1 : I_1 \to U$ and $\gamma_2 : I_2 \to U$ are two integral curves of v. Then if $\gamma_1(t_0) = \gamma_2(t_0)$ for some $t \in I_1 \cap I_2$, then $\gamma_1(t) = \gamma_2(t)$ for all $t \in I_1 \cap I_2$, and patching the two curves

$$\gamma(t) = \begin{cases} \gamma_1(t) & t \in I_1, \\ \gamma_2(t) & t \in I_2, \end{cases}$$

also results in an integral curve of v.

Fact 73

If we let $I = (a, b)$ and $I_c = (a - c, b - c)$, then given an integral curve $\gamma(t) : I \to U$, we also have the integral curve $\gamma(t + c) : I_c \to U$.

There are also some relevant global results:

Definition 74

A vector field v on U is **complete** if for every $p \in U$, there exists an integral curve $\gamma_p(t) : \mathbb{R} \to U$ such that $\gamma_p(0) = p$, and the map $F : U \times (-\infty, \infty) \to U$ defined by $F(p,t) = \gamma_p(t)$ is a C^∞ map.

In other words, there is an integral curve of v going through any point p in our open set which exists for all time. (And these notions will come up when we generalize from Euclidean space to manifolds as well.) This means that the map $f_t : U \to U$ defined by $f_t(p) = F(t,p)$ is a C^∞ map. Furthermore, f_0 is the identity map on U, and $f_t \circ f_a = f_{t+a}$ (so that f_t and f_{-t} are inverses). While this condition looks strong, it turns out that there is a large collection of vector fields which satisfy this property:

Definition 75

A vector field $v = \sum v_i \frac{\partial}{\partial x_i}$ on U is **compactly supported** if $v_i \in C_0^\infty(U)$ for all i.

Proposition 76

If a vector field v is compactly supported, then v is complete.

Proof sketch. Suppose we have $v(p_0) = 0$. Then the curve $\gamma_0(t) = p_0$ for $t \in \mathbb{R}$ (not to be confused with the $\gamma_p(t)$ maps above) satisfies

$$0 = \frac{d}{dt}\gamma_0(t) = v(p_0),$$

so γ_0 is an integral curve for v and is the unique integral curve through the point p_0. Now defining the set $A = \{p \in U : v(p) = 0\}$, for any $p \in A$, the ODE uniqueness tells us that the unique integral curve of v through p is the constant curve $\gamma_0(t) = p_0$.

Definition 77

If an integral curve γ on $[0, T)$ cannot be extended to a larger interval $[0, T_1)$, then it is **maximal**.

We claim that if $\gamma_p(t)$ is maximal, then $T = \infty$. Indeed, if $p \in A$ and $\gamma_p(t) \in A$ for all $0 \le t < T$, then $\gamma_p(t) \to q \in A$ as t approaches T because of compactness of A. Then using local existence at uniqueness at q, we can show that $\gamma_p(t)$ can be extended to an interval $0 \le t \le T + \varepsilon$ (so we can always extend if T is finite). In particular, $\gamma_p(t)$ is well-defined for $-\infty < t \le 0$ and $0 \le t < \infty$, and we've shown completeness. □

Definition 78

A function $\phi \in C^\infty(U)$ is an **integral of motion** for the dynamical system generated by a vector field v if for every integral curve $\gamma(t)$, we have $\frac{d}{dt}\phi(\gamma(t)) = 0$.

Theorem 79

A necessary and sufficient condition for ϕ to have this property is that the **Lie derivative** of ϕ is $L_v\phi = 0$.

Proof. We have

$$\frac{d}{dt}\phi(\gamma(t))\bigg|_{t=0} = (d\phi)_p\left(\frac{d}{dt}\gamma_p(0)\right) = (d\phi)_p v(p) = 0$$

by the definition of an integral of motion. □

Proposition 80

Suppose ϕ is **proper** (meaning that preimages of compact sets are compact). Then $\phi^{-1}([-a, a])$ is compact for all $a \in \mathbb{R}$.

We can prove this by noting that if $\phi(\gamma(t))$ is constant for $a < t < b$, then $\gamma(t)$ can't go off to ∞ as $t \to a$ or $t \to b$.

Example 81

Let $U = \mathbb{R}^2$, and let $v = x_1 \frac{\partial}{\partial x_2} - x_{\frac{\partial}{\partial x_1}}$ be our integral curve. We can check that the function $\phi(x_1, x_2) = x_1^2 + x_2^2$ is an integral of motion for v, so v is complete.

Next lecture, we'll connect this back to our definition of differential forms and the definition of the space $\Lambda^k(V)$.

13 March 2, 2022

We'll start by reviewing some material from previous lectures about the **exterior algebra**: recall that if $\mathcal{L}^k(V)$ is the space of k-tensors on V, and $\mathcal{I}^k(V)$ is the space of redundant k-tensors, spanned by elements $\ell_1 \otimes \cdots \otimes \ell_k$ where $\ell_i = \ell_{i+1}$, we can define the space $\Lambda^k(V^*) = \mathcal{L}^k(V)/\mathcal{I}^k(V)$. Then because the tensor product of redundant k-tensors is itself redundant, we can also define the **wedge product operation** as follows: letting π_k be the projection $\mathcal{L}^k \mapsto \mathcal{L}^k(V)/\mathcal{I}^k(V)$, it is well-defined to let the wedge product of $\omega_1 = \pi_{k_1} T_1$ and $\omega_2 = \pi_{k_2} T_2$ be

$$\omega_1 \wedge \omega_2 = \pi_{k_1+k_2}(T_1 \otimes T_2) = \Lambda^k(V^*).$$

In particular, if $\ell_1, \cdots, \ell_n$ form a basis of V^*, then the wedge products $\omega_I = \ell_{i_1} \wedge \ell_{i_2} \cdots \wedge \ell_{i_k}$ with increasing indices $i_1 < i_2 < \cdots < i_k$ form a basis of $\Lambda^k(V^*)$, and the dimension of $\Lambda^k(V^*)$ is $\binom{n}{k}$ (so $\Lambda^k(V^*) = 0$ for $k > n$).

Fact 82

We will set $\Lambda^0(V^*) = \mathbb{R}$ for convention, and this is consistent with having an $\binom{n}{0} = 1$-dimensional vector space.

Today, we'll now connect this back to the definition of k-forms and differential forms. If we let $U \subseteq \mathbb{R}^n$ be an open subset and $p \in U$ be a point, recall that the tangent space of U is the set of points $\{(p, v) : v \in \mathbb{R}^n\}$.

Definition 83

A ***k*-form** on an open set $U \subseteq \mathbb{R}^n$ is a "function" A, which assigns to each $p \in U$ an element $\omega_p \in \Lambda^k(T_p^* U)$.

Example 84

If $\omega_1, \cdots, \omega_k$ are 1-forms on U, then for any $p \in U$, $\omega_1(p) \in T_p^* U$, so $\omega_1(p) \wedge \cdots \wedge \omega_k(p) \in \Lambda^k(T_p^* U)$. Therefore, $\omega_1 \wedge \cdots \wedge \omega_k$ is a k-form on U, assigning $p \in U$ to $\omega_1(p) \wedge \cdots \wedge \omega_k(p)$.

Example 85

If $f_i \in C^\infty(U)$ are smooth functions, then df_i is a 1-form for each $1 \le i \le k$, so $df_1 \wedge \cdots \wedge df_k$ is a k-form. More specifically, letting $I = (i_1, \cdots, i_k)$ be a multi-index of length k, we can define $dx_I = dx_{i_1} \wedge \cdots \wedge dx_{i_k}$.

Since $(dx_1)_p, \cdots, (dx_n)_p$ form a basis of the cotangent space T_p^*U, we find that the elements

$$(dx_I)_p = (dx_{i_1})_p \wedge \cdots \wedge (dx_{i_k})_p, \quad i_1 < i_2 < \cdots < i_k$$

form a basis for the kth exterior power of the cotangent space, $\Lambda^k(T_p^*U)$. Therefore, **given any k-form ω on U, we can write**

$$\omega_p = \sum_I f_I(p) dx_I,$$

where the sum I goes over increasing multi-indices I and $f_I(p)$ are each real numbers. Therefore, we are really saying that the k-form can be represented as

$$\omega = \sum_I f_I dx_I,$$

where $f_I : U \to \mathbb{R}$ is a function mapping each $p \in U$ to $f_I(p)$.

Remark 86. *As a check, notice that if $I = (i_1, \cdots, i_k)$ is a repeating multi-index with $i_r = i_{r+1}$, then $(dx_{i_1})_p \wedge \cdots \wedge (dx_{i_k})_p = 0$ for all p, and thus $dx_I = 0$. On the other hand, for any non-repeating multi-index I, I^σ is strictly increasing for some permutation σ, and then $(dx_I)_p = (-1)^\sigma (dx_{I^\sigma})_p$ so we do not need to include the non-increasing dx_Is in our sum for ω. (This is why our sum only needs to go over increasing multi-indices.)*

Definition 87

With the notation above, a k-form ω is C^∞ if the f_I's are each in $C^\infty(U)$, and we let $\Omega^k(U)$ denote the space (linear span) of C^∞ k-forms on U.

We can now take our discussion above about the wedge product into consideration:

Definition 88

Let $\omega_1 \in \Omega^{k_1}(U)$ and $\omega_2 \in \Omega^{k_2}(U)$. The **wedge product** $\omega_1 \wedge \wedge_2$ is the $(k_1 + k_2)$-form which sends $p \in U$ to $\omega_1(p) \wedge \omega_2(p) \in \Omega_p^{k_1+k_2}(U)$.

We can verify from the definition directly that the wedge product is indeed C^∞. This is one of the two fundamental operations that we'll be using for k-forms, and the other we'll now discuss (the $d-$ operation). We define $\Omega^0(U) = C^\infty(U)$, and we'll start by considering a 0-form f (which is just a function). Motivated by the differential statement

$$df = \sum_{i=1}^n \frac{\partial f_i}{\partial x_i} dx_i,$$

we make the following definition:

Definition 89

For a general k-form uniquely written as $\omega = \sum_I f_I dx_I$ (with sum over increasing multi-indices I), we define

$$d\omega = \sum df_I \wedge dx_I.$$

We are now ready to discuss a few important properties based on these definitions:

Theorem 90

We have the following properties of k-forms:

1. For $\omega_1, \omega_2 \in \Omega^k(U)$, we have $d(\omega_1 + \omega_2) = d\omega_1 + d\omega_2$.

2. For $\omega_1 \in \Omega^1(U)$ and $\omega_2 \in \Omega^2(U)$, $d(\omega_1 \wedge \omega_2) = d\omega_1 \wedge \omega_2 + (-1)^{k_1}\omega_1 \wedge d\omega_2$.

3. For any k-form ω, we have $d(d\omega) = 0$.

Looking back at Remark 86, we're saying that permutations do not really cause us issues up to a sign, and in fact for all multi-indices I we have

$$d(f_I \wedge dx_I) = df_I \wedge dx_I.$$

To prove (2), notice that for $f, g \in C^\infty(U)$, we can use the product rule to find the special result for one-forms

$$d(fg) = g\sum \frac{\partial f}{\partial x_i}dx_i + f\sum \frac{\partial g}{\partial x_i}dx_i \implies d(fg) = gdf + fdg.$$

More generally, if we take $\omega_1 = f_I dx_I \in \Omega^{k_1}(U)$ and $\omega_2 = g_J dx_J \in \Omega^{k_2}(U)$, we find that

$$d(\omega_1 \wedge \omega_2) = d(f_I g_J) dx_I \wedge dx_J,$$

and then using that product rule above gives us point (2) from the theorem.

14 March 4, 2022

Fact 91

Because Wednesday's lecture covered a lot of material, most of this class is review of that content.

We'll start today's lecture by reviewing the **interior product operation** from a few lectures ago in the context of the exterior algebra: recall that for any $v \in V$ and any $\omega = \ell_1 \wedge \cdots \wedge \ell_k \in \Lambda^k(V^*)$, we can define the interior product $\iota(v)\omega$, an element of $\Lambda^{k-1}(V^*)$, via

$$\iota(v)\omega = \sum_{r=1}^{n} (-1)^{r-1}\ell_r(v)\ell_1 \wedge \cdots \wedge \ell_{r-1} \wedge \ell_{r+1} \cdots \wedge \ell_k,$$

and one fact we can prove as an exercise is that $\iota(v)\iota(v)\omega = 0$ for any v.

Last lecture, we also defined k-forms, which take in a point $p \in U \subseteq \mathbb{R}^n$ and output an element of $\Lambda^k(T_p^*U)$. (In particular, the wedge product of k 1-forms on U is a k-form, and those are easy to think about because 1-forms assign to each point $p \in U$ an element $\omega_p \in T_p^*U$). Specifically, if $\omega_i = df_i$ for C^∞ functions f_i, we get k-forms that look like $df_1 \wedge \cdots \wedge df_k$, and more specifically, we can define the k-forms $dx_I = dx_{i_1} \wedge \cdots \wedge dx_{i_k}$ for any multi-index $I = (i_1, \cdots, i_k)$. And last time, we described that because $(dx_i)_p$ form a basis for the cotangent space T_p^*U, the $(dx_I)_p$ for increasing I form a basis for $\Lambda^k(T_p^*)$, meaning any k-form sends p to some $\omega_p = \sum f_I(p)dx_I$, and thus $\omega = \sum f_I dx_I$ for some functions $f_I : U \to \mathbb{R}$.

From here, we can add k-forms ω_1, ω_2 together to get another k-form $\omega_1 + \omega_2$, and we can wedge any k_1-form ω_1 and k_2-form ω_2 together (by sending p to $\omega_1(p) \wedge \omega_2(p)$ – this gives us a C^∞ $(k_1 + k_2)$-form as long as ω_1, ω_2 were C^∞). And we also have the important d operation, motivated by $df = \sum_i \frac{\partial f}{\partial x_i} x_i$ from calculus (taking a 0-form f to a

1-form): we have

$$\omega = \sum_I f_I dx_I \implies d\omega = \sum_I df_I \wedge dx_I.$$

The important properties of this d operation are then that $d(\omega_1 + \omega_2) = d\omega_1 + d\omega_2$, $d(d\omega) = 0$, and also that

$$d(\omega_1 \wedge \omega_2) = d\omega_1 \wedge \omega_2 + (-1)^{k_1}\omega_1 \wedge d\omega_2.$$

In particular, because $dx_{I^\sigma} = (-1)^\sigma dx_I$, combining this fact with the "product rule" above tells us that $d(f_I dx_I) = df_I \wedge dx_I$ for **all** multi-indices I, which will make later calculations easier. And we can check that $dd = 0$ by looking at k-forms of the form $\omega f_I dx_I$ and plugging in

$$dd\omega = d(df_I \wedge dx_I) = d(df_I) \wedge dx_I - df_I \wedge d(dx_I),$$

and noticing that $d(df_I) = 0$ and $d(dx_I) = 0$ (left as an exercise by writing out the double sum and thinking about reversing the indices in the sum).

15 March 7, 2022

Last time, we started by talking about the interior product operation, sending a k-form $\omega \in \Lambda^k(V^*)$ to a $(k-1)$-form $\iota(v)\omega \in \Lambda^{k-1}(V^*)$: by linearity we can just consider the action on $\omega = \ell_1 \wedge \cdots \wedge \ell_k$ for $\ell_i \in V^*$, and we have

$$\iota(v)\omega = \sum_{r=1}^{k} \left((-1)^{r-1}\ell_r(v)\right) \ell_1 \wedge \cdots \wedge \ell_{r-1} \wedge \ell_{r+1} \wedge \cdots \wedge \ell_k.$$

We'll mention a few more properties of that operation today: notice from the definition that $\iota(v)\omega$ is linear in both v and in ω, and also for any one-form ω, we have $\iota(v)\omega = \omega(v) \in \mathbb{R}$. Additionally, connected to the fact that $\iota(v)\iota(v)\omega = 0$ for any v, we also have $\iota(v_1)\iota(v_2)\omega = -\iota(v_2)\iota(v_1)\omega$. Finally, combining the properties of the interior and wedge product gives us (again being careful about the sign change)

$$\iota(v)(\omega_1 \wedge \omega_2) = \iota(v)\omega_1 \wedge \omega_2 + (-1)^k \omega_1 \wedge \iota(v)\omega_2.$$

We'll now generalize this definition to vector fields:

Definition 92

Let $U \subseteq \mathbb{R}^n$ be an open set, v a **vector field** on U, and $\omega \in \Omega^k(U)$ a differential k-form. Define $\iota(v)\omega$ to be the differential $(k-1)$-form given by

$$(\iota(v)\omega)_p = \iota(v_p)\omega_p.$$

(It's left as an exercise for us to check that if v is a C^∞ vector field, and $\omega \in \Omega^k(U)$, then $\iota(v)\omega$ is indeed in $\Omega^{k-1}(U)$.) Our properties from before now generalize – in particular, linearity in v and ω of the interior product operation imply that for any k-forms ω_1, ω_2

$$\iota(v)(\omega_1 + \omega_2) = \iota(v)\omega_1 + \iota(v)\omega_2, \quad \iota(v_1 + v_2)\omega = \iota(v_1)\omega + \iota(v_2)\omega,$$

and the wedge product identity also gives us (for any k_1-form ω_1 and k_2-form ω_2)

$$\iota(v)(\omega_1 \wedge \omega_2) = \iota(v)\omega_1 \wedge \omega_2 + (-1)^{k_1}\omega_1 \wedge \iota(v)\omega_2.$$

Also, we still have $\iota(v)\iota(v)\omega = 0$ and thus $\iota(v)\iota(w)\omega = -\iota(w)\iota(v)\omega$ like before. And finally, we can write out the interior product operation more explicitly as

$$\iota(v)(dx_{i_1} \wedge \cdots \wedge dx_{i_k}) = \sum_{\ell=1}^{k} (-1)^{\ell-1} v_\ell dx_{i_1} \wedge \cdots \wedge dx_{i_{\ell-1}} \wedge dx_{i_{\ell+1}} \wedge dx_{i_k}$$

(where v_ℓ is the ℓth coordinate of the vector field, and it outputs a number if we evaluate it at a point $p \in U$).

We're now ready to talk about a more complicated operation on forms:

Definition 93

Let $\omega \in \Omega^k(U)$, and let v be a C^∞ vector field. The **Lie differentiation** operation is defined via

$$\mathcal{L}_v \omega = \iota(v) d\omega + d(\iota(v)\omega).$$

Proposition 94

This Lie differentiation operation commutes with the d operation:

$$\mathcal{L}_v d\omega = d\mathcal{L}_v \omega.$$

Additionally, when interacting with the wedge product, we have

$$\mathcal{L}_v(\omega_1 \wedge \omega_2) = \mathcal{L}_v \omega_1 \wedge \omega_2 + \omega_1 \wedge \mathcal{L}_v \omega_2.$$

Proof. For the first property, note that (by definition)

$$\mathcal{L}_v d\omega = \iota(v) dd\omega + d\iota(v) d\omega,$$

while

$$d(\mathcal{L}_v \omega) = d\iota(v) d\omega + dd\iota(v)\omega.$$

But the terms with dd now vanish (because applying the d operation twice gives us 0), and thus the two terms are indeed both equal (to $d\iota(v)d\omega$). The second property is left as an exercise to us, but we can use the fact that by definition,

$$\mathcal{L}_v \omega_1 \wedge \omega_2 = d\iota(v)(\omega_1 \wedge \omega_2) + \iota(v) d(\omega_1 \wedge \omega_2)$$

and now simplifying with properties of d and ι gives us

$$= (d\iota(v)\omega_1) \wedge \omega_2) + (-1)^k \omega_1 \wedge d\iota(v)\omega_2 + \iota(v)(d\omega_1 \wedge \omega_2 + (-1)^k \omega_1 \wedge d\omega_2).$$

□

We're now going to turn to the **divergence formula**. Notice that for any vector field which we write in the form $v = \sum v_i \frac{\partial}{\partial x_i}$, we have

$$\mathcal{L}_v dx_i = d\iota(v) dx_i + \iota(v) ddx_i = dv_i.$$

In other words, if we have specifically an n-form $\omega = dx_1 \wedge \cdots \wedge dx_n$, we find that

$$\mathcal{L}_v \omega = \sum_i dx_1 \wedge \cdots \wedge dx_{i-1} \wedge \mathcal{L}_v dx_i \wedge dx_{i+1} \wedge \cdots \wedge dx_n.$$

Now because $\mathcal{L}_v dx_i = dv_i$ and $dv_i = \sum_i \frac{\partial v_i}{\partial x_j} dx_j$, and notice that $dx_i \wedge dx_j = 0$ for $i \neq j$, meaning all of the contribution to the blue term that is not from dx_i goes away. So in fact our expression simplifies to

$$\mathcal{L}_v \omega = \sum_i dx_1 \wedge \cdots \wedge dx_{i-1} \wedge \frac{\partial v_i}{\partial x_i} dx_i \wedge dx_{i+1} \wedge \cdots \wedge dx_n,$$

and thus we can understand the action of the Lie differentiation operation as it connects to calculus:

$$\boxed{\mathcal{L}_v \omega = \left(\sum_i \frac{\partial v_i}{\partial x_i} \right) \omega},$$

and this term in parentheses is the **divergence** of the vector field v which we might remember from calculus.

We'll finish this lecture by starting to look again at functorial properties of differential forms (and continue this next time): recall from early on in 18.952 that for any linear map $A : W_1 \to W_2$ of vector spaces, we get the dual map $A^* : W_2^* \to W_1^*$. Recall that we get a linear map $A^* : \Lambda^k(W_2^*) \to \Lambda^k(W_1^*)$ as well, with the additional properties that (this is how we can define the map)

$$A^*(\omega_1 \wedge \omega_2) = A^*\omega_1 \wedge A^*\omega_2,$$

and for any linear map $B : W_2 \to W_3$ we also get

$$(BA)^*\omega = A^* B^* \omega.$$

We can now generalize this by letting U be an open set in $\mathbb{R}^n$ and V an open set in $\mathbb{R}^m$: for any map $f : U \to V$ such that $p \mapsto q$, we get the map $df_p : T_p U \to T_q V$ and also the corresponding dual map $df_p^* : \Lambda^k(T_q^* V) \to \Lambda^k(T_p^* U)$. The assertion is now that for any $\omega \in \Omega^k(V)$, we have an element $\omega_q \in \Lambda^k(T_q^* V)$, and then we can define the "pullback"

$$(f^*\omega)_p = (df_p)^* \omega_q,$$

allowing us to define the k-form $f^*\omega$. In particular, we have $f^* d\phi = df^*\phi$ for any $\phi \in C^\infty(V)$, and for coordinates $x_1, \cdots, x_n$ and a function $f = (f_1, \cdots, f_n)$, we have $f^* dx_i = df_i$ (so that the pullback of a general k-form $\sum a_I dx_{i_1} \wedge \cdots \wedge dx_{i_k}$ is obtained by replacing a_I with $f^* a_I$ and dx_{i_1} with df_{i_1}). But we'll talk more about this next time!

16 March 9, 2022

Last lecture, we generalized the interior product operation to vector fields: for a C^∞ vector field on U and a differential k-form $\omega \in \Omega^k(U)$, we get the differential $(k-1)$-form $\iota(v)\omega$ defined as

$$(\iota(v)\omega)_p = \iota(v_p)\omega_p,$$

where recall that ω_p is an element of $\Lambda^k(T_p^* U)$ and v_p is an element of $T_p U$ (so that this interior product makes sense). We also introduced the Lie differentiation operation

$$\mathcal{L}_v \omega = \iota(v) d\omega + d\iota(v)\omega,$$

and we can also consider the following alternate definition:

Definition 95

Suppose $f_t : U \to U$ is a one-parameter group of diffeomorphisms generated by a complete vector field v, meaning that we have paths γ_p for each $p \in U$ satisfying $\gamma_p(t) = f_t(p)$. (Recall that then in fact γ_p is the unique integral curve of v satisfying $\gamma_0(p) = p$.) Then we can also define Lie differentiation via

$$\mathcal{L}_v \omega = \frac{d}{dt} f_t^* \omega \bigg|_{t=0}.$$

Proposition 96

With the definition above, we also have (for all time t)

$$\frac{d}{dt} f_t^* \omega = f_t^a st \mathcal{L}_v \omega.$$

Proof. We have (by change of variables in ordinary calculus)

$$\frac{d}{dt} f_t^* \omega = \frac{d}{ds} f_{s+t}^*(\omega) \bigg|_{s=0},$$

and because we have a one-parameter group, we can rewrite this as

$$= \frac{d}{ds} \left(f_t^* \omega \circ f_s^* \omega \right)_{s=0},$$

and pulling out the f_t^* and applying the definition gives us what we want. □

Thus, the relation for the Lie derivative can be written

$$\frac{d}{dt} f_t^* \omega = f_t^* \left(\iota(v) d\omega + d\iota(v)\omega \right) = f_t^* \iota(v) d\omega + d f_t^* \omega,$$

which we will define to be

$$dQ_t \omega + Q_t d\omega, \quad Q_t = f_t^* \iota(v) \omega.$$

If we then integrate this equation from a to b, we find that

$$f_b^* \omega = f_a^* \omega = \tilde{Q}_{a,b} d\omega + d\tilde{Q}_{ab} \omega,$$

where $\tilde{Q}_{a,b}$ is the integral of Q_t from a to b. This motivates the following definition:

Definition 97

Suppose $\omega_1, \omega_2 \in \Omega^k(U)$ are differential k-forms on U, and suppose they are **closed** (meaning that $d\omega_1 = 0$ and $d\omega_2 = 0$). Then ω_1 and ω_2 are **cohomologous**, denoted $\omega_1 \sim \omega_2$, if $\omega_1 - \omega_2 = du$ for some $u \in \Omega^{k-1}(U)$.

In particular, the calculation above for $f_b^* \omega - f_a^* \omega$ shows the following result:

Theorem 98

Suppose $\omega \in \Omega^k(U)$ is a closed k-form. Then $f_b^* \omega \sim f_a^* \omega$ for all a, b.

Since f_0 is always the identity map in a one-parameter family, this tells us that

$$f_b^* \omega - \omega = d\tilde{Q}_b \omega$$

for $\tilde{Q}_b = \tilde{Q}_{b,0}$. In other words, the pullback operation does not change the "cohomology class" – the difference between ω and $f_b^*\omega$ is d of some differential form.

We'll now move on to the **integration operation** in the theory of differential forms. Recall that for any n-dimensional vector space, the nth exterior power of V, $\Lambda^n(V^*)$, is a one-dimensional space, so $\Lambda^n(V^*) - \{0\}$ has two components, and an **orientation** of V is the choice of one of these two components which we denote $\Lambda^n(V^*)_+$ (the other is then denoted $\Lambda^n(V^*)_-$). Furthermore, if we let $v_1, \cdots, v_n$ be a basis of V and $v_1^*, \cdots, v_n^*$ be the dual basis, then $(v_1, \cdots, v_n)$ is **positively oriented** if $v_1^* \wedge \cdots \wedge v_n^*$ is in $\Lambda^n(V^*)_+$ and **negatively oriented** otherwise.

Definition 99

Let $A : V \to W$ be a bijective linear map between n-dimensional vector spaces. Then A is **orientation-preserving** if $A^* : \Lambda^n(W^*) \to \Lambda^n(V^*)$ maps $\Lambda^n(W^*)_+$ onto $\Lambda^n(V^*)_+$.

This can be written more explicitly using coordinates: if $(v_1, \cdots, v_n)$ is a basis of V and $(w_1, \cdots, w_n)$ is our basis of W, then we can represent A with the matrix $[a_{ij}]$ where $Av_j = \sum_i a_{ij} w_i$, and furthermore

$$A^*(w_1^* \wedge \cdots \wedge w_n^*) = \det[a_{ij}] v_1^* \wedge \cdots \wedge v_n^*.$$

We can thus say that A is orientation-preserving if $\det[a_{ij}] > 0$ (which might look familiar from other linear algebra classes we've taken).

We can generalize these concepts to open subsets of $\mathbb{R}^n$ as well:

Definition 100

Let U be an open set in $\mathbb{R}^n$, and let $p \in U$. We can orient the space T_pU by requiring that

$$(dx_1 \wedge \cdots \wedge dx_n)_p \in \Lambda^n(T_p^*U)_+.$$

In other words, we can alternatively choose that $\left(\frac{\partial}{\partial x_1}\right)_p, \cdots, \left(\frac{\partial}{\partial x_n}\right)_p$ form a positively oriented basis for T_pU.

Definition 101

Suppose $f : U \to V$ is a C^∞ diffeomorphism between open sets U, V of $\mathbb{R}^n$. Then f is **orientation-preserving** if for all $p \in U$ and $q = f(p)$, $df_p : T_pU \to T_qV$ is orientation-preserving.

We can apply this to the change of variables formula from calculus: recall that for any C^∞ diffeomorphism $f : U \to V$, we can make a "u-substitution"

$$\int_V \phi dx = \int_U f^*\phi \, |\det J_s| \, dy,$$

where J_s denotes the Jacobian matrix. In other words, we have $\int_V \phi = \int_U f^* \det J_s$ if f is orientation-preserving, and we have a negative sign in that equality otherwise. We can now rewrite this standard calculus result in a nicer way: let $f : U \to V$ be a diffeomorphism. Then if $p \in U$ and $q = f(p)$, then

$$(df_p)\left(\frac{\partial}{\partial x_j}\right)_p = \sum_i \frac{\partial f_i}{\partial x_j}(p)\left(\frac{\partial}{\partial x_i}\right)_p.$$

We then get the dual result

$$df_p^*(dx_1 \wedge \cdots \wedge dx_n)_p = \det(J_f)_p (dx_1)_p \wedge \cdots \wedge (dx_n)_p$$

for the Jacobian matrix $(J_f)_p = \left[\frac{\partial f_i}{\partial x_j}(p)\right]$, so that our change-of-variables formula now reads

$$\boxed{f^*(dx_1 \wedge \cdots \wedge dx_n) = \det(J_f) dx_1 \wedge \cdots \wedge dx_n}.$$

And next lecture, we'll see a more intrinsic version of this formula that doesn't involve matrices at all!

17 March 11, 2022

Remark 102. *I was not able to attend this lecture in person, so these notes are transcribed from Paige Bright and Jeffery Yu's class notes.*

Last lecture, we mentioned the change of variables formula from integral calculus, which states that for an orientation-preserving diffeomorphism $f : U \to V$ between open subsets of $\mathbb{R}^n$ and a compactly supported differential form $\omega \in \Omega_c^k(V)$, we have

$$\int_V \omega = \int_U f^*\omega.$$

We'll be proving that result next week, but we'll work on some other results towards that goal for now:

Theorem 103 (Poincaré lemma for rectangles)
Let $U = I_1 \times \cdots \times I_n \subset \mathbb{R}^n$ be the open rectangle with $I_r = (a_r, b_r)$ for all r. Suppose $\omega \in \Omega_c^k(U)$. Then ω is **exact**, meaning that $\omega = d\mu$ for $\mu \in \Omega_c^{k-1}(U)$, if and only if $\int_{\mathbb{R}^n} \omega = 0$.

Proof of forward direction. By the definition of the d operation, we know that if we write $\mu = \sum_{i=1}^k f_i dx_i \wedge \cdots \wedge dx_{i-1} \wedge dx_{i+1} \wedge \cdots \wedge dx_n$, then

$$\omega = d\mu = \sum_i (-1)^{i-1} \frac{\partial f_i}{\partial x_i} dx_1 \wedge \cdots \wedge dx_n.$$

In particular, this means that if $\omega = d\mu$, we have

$$\begin{aligned}\int_{\mathbb{R}^n} \omega = \int_U \omega &= \int_U (-1)^{i-1} \frac{\partial f_i}{\partial x_i} dx_1 \wedge \cdots \wedge dx_n \\ &= \sum \left(\int_{I_r} \frac{\partial f_i}{\partial x_i} dx_i \right) dx_1 dx_i \wedge \cdots \wedge dx_{i-1} \wedge dx_{i+1} \wedge \cdots \wedge dx_n \\ &= \sum \left(f_i(b_r) - f_i(a_r) \right) dx_1 dx_i \wedge \cdots \wedge dx_{i-1} \wedge dx_{i+1} \wedge \cdots \wedge dx_n\end{aligned}$$

(slightly abusing notation here because $f_i(b_r)$ and $f_i(a_r)$ depend on the other x_i values too, but the logic still works) because all f_is are compactly supported and thus f_i vanishes on the boundary. □

For the other direction, we'll need to do some more work, but it follows directly from the following result:

Theorem 104
Let $U \subseteq \mathbb{R}^m$ be open, and let $A \subset \mathbb{R}$ be an open interval. Suppose U satisfies that for all $\omega \in \Omega_c^m(U)$ with $\int \omega = 0$, we have $\omega \in d\Omega_c^{m-1}(U)$. Then $A \times U$ also satisfies this condition.

Proof. We proceed by induction on m.The base case $m = 1$ is where $U = (a, b) \subset \mathbb{R}$. Indeed, if $f \in C_0^\infty(A)$ and $\int_a^b f(t)dt = 0$, then we can use the function $g(x) = \int_0^x f(t)dt \in C_0^\infty(U)$, and we indeed have $dg = f$ (this is the ordinary fundamental theorem of calculus).

Say that an open set that satisfies the condition in this theorem satisfies CPL. We then want to prove that if U is CPL, then $A \times U$ is also CPL. For this, let $x_1, \cdots, x_n$ be coordinates on U and $t, x_1, \cdots, x_n$ be the corresponding coordinates on $A \times U$. We can then write any $\omega \in \Omega^{m+1}(A \times U)$ as

$$\omega = f(x,t)dt \wedge dx_1 \wedge \cdots \wedge dx_n,$$

with corresponding integral $\int \omega = \int_{A \times U} f(x,t)dt \wedge d\vec{x}$. Define

$$\theta = \left(\int_A f(x,t)dt \right) d\vec{x},$$

so that $\int_{\mathbb{R}^n} \omega = \int_{\mathbb{R}^n} f(x,t)dt = \int_{\mathbb{R}^{n-1}} dx$ (in other words, integrate out the variable t in advance). Then if $\int \omega = 0$, then $\int \theta = 0$ as well. So now suppose U is CPL (as we have in our assumption), so that $\theta = d\nu$ for some $\nu \in \Omega_c^{n-1}(U)$. Let $\rho \in C^\infty(\mathbb{R})$ be a bump function (in particular, compactly supported) on A, such that $\int_A \rho = 1$. Defining $K = \rho dt \wedge \nu$, we then have

$$dK = \rho dt \wedge d\nu = \rho(t)dt \wedge \theta,$$

so that

$$\omega - d\nu = dt \wedge (\nu - \rho(t)\theta) = dt \wedge u(x,t)dx_1 \wedge \cdots \wedge dx_{n-1},$$

where we define the function

$$u(x,t) = f(x,t) - \rho(t) \int_A f(x,t)dt.$$

In particular, because ρ integrates out to 1, we find that $\int u(x,t)dt = 0$. So for our interval $A = (a,b)$, if we define

$$v(x,t) = \int_a^t u(x,s)ds,$$

we'll have $v \in C_0^\infty(U \times A)$ and $\partial_t v = u$ by the ordinary fundamental theorem of calculus. We're now ready to finish: letting $\phi = v(x,t)dx_1 \wedge \cdots \wedge dx_{n-1}$, we then have that

$$d\phi = dt \wedge u(x,t)dx_1 \wedge \cdots \wedge dx_{n-1} = \omega - dK,$$

so that $\omega = d(\phi - K)$, and we've proved CPL for $A \times U$. $\square$

We've now proved both directions of the Poincaré lemma for rectangles, and next we'll generalize this to compactly supported forms on open subset of $\mathbb{R}^n$.

Theorem 105 (Poincaré lemma)

Let U be a connected open subset of $\mathbb{R}^n$, and let $\omega \in \Omega_c^n(U)$ (so that the support of ω is contained in U). Then $\int \omega = 0$ if and only if there is some $\mu \in \Omega_c^{n-1}(U)$ with $\omega = d\mu$.

The backwards direction follows from the Poincaré lemma for rectangles. Indeed, if $\mu \in \Omega_c^{n-1}(U)$, then the support of μ is contained within some rectangle and thus $\int d\mu = 0$.

For the forwards direction, suppose $\omega_1, \omega_2 \in \Omega_c^n(U)$. Recall that we write $\omega_1 \sim \omega_2$ if $\omega_1 - \omega_2 = d\mu$ for some compactly supported $(n-1)$-form μ. It's equivalent to prove the following:

Theorem 106

Let $Q_0 \subset U$ be a rectangle, and let ω_0 be an n-form with support contained in Q_0 and total integral 1. Let $\omega \in \Omega_c^n(U)$ be a compactly supported n-form with support contained in U and total integral $c = \int \omega$. Then we have $\omega \sim c\omega_0$.

In particular, setting $c = 0$ proves the forward direction of the Poincaré lemma, since it implies that $\omega \sim 0$ and thus $\omega = d\mu$ for some μ.

Proof. Construct a collection of rectangles $Q_i \subset U$ such that U is the union of the interiors of Q_i. Let ϕ_i be a partition of unity (meaning we have compactly supported functions which in total add to the function 1 on U) such that $\text{supp}(\phi_i)$ is contained within the interior of Q_i for each i. Then for large enough m, $\omega = \sum_{i=1}^m \phi_i\omega$ by compactness, so we can reduce the statement to the case where we prove this result for each $\phi_i\omega$. In other words, it suffices to prove this result when the support of ω is contained within the interior of some open rectangle Q_i, which we call Q.

We claim that we can connect Q_0 and Q with a sequence of rectangles $Q_0 = R_0, \cdots, R_{N+1} = R$ such that the interior of any two consecutive rectangles is non-empty. Indeed, the set

$$A = \{x \in U : \text{there exists a sequence } \{R_i\} \text{ with } x \in \text{int}(R_{N+1})\}$$

is an open set, and its complement is also open. Thus because U is connected, $U = A$. Now to finish the proof, for each i, choose a compactly supported n-form v_i contained within $\text{int}(R_i) \cap \text{int}(R_{i+1})$ such that $\int v_i = 0$. Since $v_i - v_{i+1}$ is supported in $\text{int}(R_{i+1})$, by the Poincaré lemma for rectangles we have $v_i \sim v_{i+1}$. The same logic says that because we also have $\omega_0 \sim v_0$, we have $c\omega_) \sim cv_0$, and finally $\omega \sim cv_N$. Thus this chain of equivalences tells us that $\omega \sim c\omega_0$, as desired, finishing the proof. □

18 March 14, 2022

We'll start today by going over yesterday's proof of the Poincaré lemma in more detail. Recall that we started by proving the rectangle version of the lemma, which states for any open rectangle $Q = I_1 \times \cdots \times I_n$ and any $\omega \in \Omega_c^k(Q)$ (that is, ω is a compactly-supported k-form), if $d\omega = 0$, then $\omega \in d\Omega_c^{k-1}(Q)$ (because $d^2 = 0$). And a more involved result is that we have $\int_Q \omega = 0$ if and only if ω is exact (meaning that $\omega = d\mu$ for some μ).

Generalizing this to arbitrary connected open sets $U \subseteq \mathbb{R}^n$ then involved a lemma which proved that for any $p, q \in U$, we have a sequence of rectangles Q_i such that $Q_i \cap Q_{i+1} \neq \varnothing$, and p is in the first rectangle and q is in the last rectangle. We showed this by constructing the set of points q for which this is true, mentioning that it is open (because there is always an neighborhood of q within a rectangle it's contained in) and so is its complement (for basically the same reason – otherwise we could find arbitrarily close points to q which can be reached, meaning q can also be reached), so connectivity of U implies that this set of points must be all of U (since it is nonempty).

That lemma was useful because it then allows us to repeatedly use the rectangle Poincaré lemma. Specifically, we are then able to prove a slight generalization of the Poincaré lemma, which is that if ω_0 is a compactly supported n-form on an open rectangle $Q_0 \subseteq U$, then $\omega \sim c\omega_0$ for $c = \int_U \omega$. The argument is to use compactness to break up ω into a finite sum of forms supported on rectangles, and to note that the result we are trying to prove holds for finite sums if it holds for the individual parts. From there, we use our connected set of rectangles $\{Q_i\}$, showing repeated equivalence of our n-forms by looking at compactly supported forms on the intersections $Q_i \cap Q_{i+1}$.

As of this result, we can now talk about the change of variables formula. The version that we've already talked about states that if $f : U \to V$ is a diffeomorphism between connected subsets of $\mathbb{R}^n$, then $\int_U f^*\omega = \int_V \omega$ if f is orientation-preserving, and similar with a negative sign if f is orientation-reversing. Today, we'll replace "diffeomorphism" with a much weaker assumption, but first we'll review a bit of point-set topology:

Proposition 107

Let X, Y be topological spaces, and let $f : X \to Y$ be an arbitrary continuous map. Then if C is a compact subset of X, then $f(C)$ is compact.

The converse is not true as stated (that is, the preimage of a compact set is not always compact), but there are many maps for which the converse does hold:

Definition 108

A continuous map $f : X \to Y$ is **proper** if for every compact subset C, $f^{-1}(C)$ is compact.

We can check that if $f : X \to Y$ and $g : Y \to Z$ are proper maps, then $g \circ f$ is also proper. We'll discuss proper maps more next lecture, but we'll connect this back to differential forms for now:

Proposition 109

Let $f : U \to V$ be a proper map between open subsets U, V of $\mathbb{R}^n$. Then if $\omega \in \Omega_c^k(V)$, then $f^*\omega \in \Omega_c^k(U)$.

Proof. Let C be the support of ω. Then the support of the pullback form, $f^*\omega$, is contained in the preimage $f^{-1}(C)$, which is compact. □

Theorem 110

Let U, V be open subsets of $\mathbb{R}^n$, and let $f : U \to V$ be a proper C^∞ map. Then for all $\omega \in \Omega_c^n(V)$, we have

$$\int_U f^*\omega = \deg(f) \int_V \omega.$$

Here, $\deg(f)$ denotes the **degree** of the map f. We'll give a full definition of what it is in the proof, but provisionally we'll use the following definition: if we fix (forever) an $\omega_0 \in \Omega_c^n(V)$ with $\int_V \omega_0 = 1$, then we set

$$\deg(f) = \int_U f^*\omega_0.$$

(This integral indeed makes sense because $f^*\omega_0$ is compactly supported.)

Proof. For any $\omega \in \Omega_c^n(V)$, define $c = \int_V \omega$. Then

$$\int_V \omega - c\omega_0 = \int_V \omega - c \int_V \omega_0 = c - c = 0,$$

so by the Poincaré lemma, $\omega - c\omega_0 = d\nu$ for some $\nu \in \Omega_c^{n-1}(V)$. Applying the pullback map, we find that

$$f^*\omega - cf^*\omega_0 = f^*d\nu = df^*\nu,$$

and because $f^*\nu$ is compactly supported integrating it out over U gives us 0. Thus

$$\int_U f^*\omega = \int_U cf^*\omega_0 = c\deg(f) = \deg(f) \int_V \omega,$$

as desired. □

We'll convert this wacky definition of degree into something more fundamental next time (which is essentially about "counting the number of points in the preimage"), instead of needing to rely on this choice of ω_0

19 March 16, 2022

Last time, we discussed the Poincaré lemma, which states that any compactly supported n-form $\omega \in \Omega_c^n(U)$ on a connected open subset U of $\mathbb{R}^n$ has $\int_U \omega = 0$ if and only if $\omega \in d\Omega_c^{n-1}(U)$. We then started analyzing the connections of this result to the change of variables formula – in particular, if $f : U \to V$ is a **proper** map (meaning that preimages of compact sets are compact), then $\omega \in \Omega_c^k(V)$ implies that $f^*\omega \in \Omega_c^k(U)$, and this allows us to use the Poincaré lemma to prove that

$$\int_U f^*\omega = \deg(f) \int_V \omega.$$

However, last time, we gave a sketchy definition of the degree $\deg(f)$, where we fix a compactly supported n-form ω_0 with $\int_V \omega_0 = 1$ and defined $\deg(f) = \int_U f^*\omega_0$. Specifically, with this definition and setting $c = \int_V \omega$, we found that $\int_V \omega - c\omega_0 = 0$, so that $\omega - c\omega_0 = d\nu$ for some compactly supported $(n-1)$-form ν. Applying the pullback map and integrating then gave us

$$\int_U f^*\omega - \int cf^*\omega_0 = \int df^*\nu = 0 \implies \int_U f^*\omega = c \int f^*\omega_0 = \deg(f) \int_V \omega.$$

We can now talk more about the properties of this degree map and start doing some applications:

Proposition 111

Suppose U, V, W are connected open subsets of $\mathbb{R}^n$, and suppose $f : U \to V$ and $g : V \to W$ are proper C^∞ maps. Then $\deg(g \circ f) = \deg(f)\deg(g)$.

Proof. By functoriality, we have

$$(g \circ f)^*\omega = f^*g^*\omega,$$

so that

$$\deg(g \circ f) \int_U \omega \int_W (g \circ f)^*\omega = \int_W f^*g^*\omega = \deg(f) \int_V g^*\omega = \deg f \deg g \int_U \omega.$$

Equating the first and last expressions in this chain of equalities gives us the result. ☐

We'll now get this result to look more like the usual language of calculus: suppose U and V are connected open subsets of $\mathbb{R}^n$, and let $f : U \to V$ be a diffeomorphism and ϕ be a scalar function. Then we know that $\int_V \phi = \int_U f^*\phi |\det J_f|$. In the language of this course, for a differential form of the form

$$\omega = \phi(x) dx_1 \wedge \cdots \wedge dx_n,$$

we can then show that $\int_V \omega = \int_U f^*\omega = \pm \int_V \omega$, depending on whether $f : U \to V$ is orientation-preserving or orientation-reversing. Here's that statement in alternative words:

Proposition 112

Let U and V be connected open subsets of $\mathbb{R}^n$, and let $f : U \to V$ be a C^∞ diffeomorphism. Then $\deg(f) = 1$ if f is orientation-preserving and -1 if orientation-reversing.

We'll prove this topologically, but first we need preliminary results:

Lemma 113

Suppose U, V, W are connected open subsets of $\mathbb{R}^n$, and suppose $f : U \to V$ and $g : V \to W$ are C^∞ diffeomorphisms. Suppose that Proposition 112 holds for both f and g. Then it also holds for $g \circ f$.

(As a hint toward proving this, fix some $\omega \in \Omega_c^n(W)$. Then because $g \circ f$ is also a diffeomorphism, we know that

$$(g \circ f)^* \omega = f^* g^* \omega \implies \int (g \circ f)^* \omega = \pm \int g^* \omega$$

because the degrees of f and g are both ± 1.) We will also need the following (which we will prove as exercises):

Lemma 114

Let $a \in \mathbb{R}^n$, and consider the translation map $T_a : x \mapsto x + a$. Then *Proposition* 112 does indeed hold for the map T_a. Also, if $A : \mathbb{R}^n \to \mathbb{R}^n$ is a bijective linear map, then *Proposition* 112 does hold for A.

Beginning of proof of Proposition 112. From our preliminary results, we may assume by applying translation maps (before and after applying f) that $0 \in U \circ V$ and that $f(0) = 0$. This means that df_0 is the identity map, so we can then write $f(x) = x + g(x)$, where $g(0) = 0$ and $g'(0) = 0$ so that $g(x) \le C|x|^2$ for some C. If we now choose $\delta > 0$ so that $C\delta \le \frac{1}{2}$, and we pick a function $\rho(x) \in C_0^\infty(\mathbb{R}^n)$ such that $0 \le \rho(x) \le 1$ and also satisfying

$$\rho(x) = \begin{cases} 1 & |x| \le \frac{\delta}{2}, \\ 0 & |x| \ge \delta, \end{cases}$$

we then have $\rho(x)g(x) = 0$ for $|x| \ge \delta$, so that

$$|\rho(x)g(x)| \le \rho(x) C\delta |x| \le \frac{1}{2}|x|$$

and $\rho(x) = g(x)$ for $|x| \le \frac{\delta}{2}$. From here, we can look at the function $\tilde{f}(x) = f(x) + \rho(x)g(x)$ and show that $\deg(f) = \deg(\tilde{f})$, and the remainder of the proof is topological. $\square$

20 March 18, 2022

After spring break, we'll be shifting gears from differential forms on $\mathbb{R}^n$ to differential forms on manifolds. But for now, we'll focus on the concepts we've been discussing so far – recall that last time, we related the differential forms ω and $f^*\omega$ via the change of variables formula, which made use of the notion of **degree**. Specifically, if we fix some compactly supported n-form $\omega_0 \in \Omega_c^n(V)$ with $\int_V \omega_0 = 1$, then for any proper map $f : U \to V$, we defined $\deg(f) = \int_U f^* \omega_0$. The idea is that this is actually an **intrinsic** definition which doesn't depend on the choice of ω_0: specifically, for all $\omega \in \Omega_c^n(V)$, we have that

$$\int_U f^* \omega = \deg(f) \int_V \omega,$$

so the degree of f is indeed going to yield the same answer no matter which ω_0 we choose.

Theorem 115

Let U, V be connected open subsets of $\mathbb{R}^n$, and let $f : U \to V$ be a proper C^∞ map. If f is not onto, then $\deg(f) = 0$.

Proof. To prove this result, we'll need the following topological fact:

Fact 116

Let $f : U \to V$ be a proper C^∞ map between open subsets of $\mathbb{R}^n$. Then $f(U)$ is a closed subset of V, so if f is not onto and $q \in V \setminus f(U)$, there is an open neighborhood W of q in V, such that $W \cap f(U) = \varnothing$.

We can then pick a compactly supported form $\omega_0 \in \Omega_c^n(W)$ satisfying $\int_W \omega_0 = 1$. Applying the pullback map, we know that $\omega_q = 0$ for all $q \in f(U)$, so $0 = \int_U f^*\omega_0$ and thus the degree of f is zero. □

Our main goals for today's lecture are to discuss how to **compute degree geometrically** and **show homotopy invariance of degree**, again showing that the degree of a map f is very topological. First, the following is an exercise in our textbook:

Proposition 117

Let U and V be open sets of $\mathbb{R}^n$, and let $f : U \to V$ be a proper C^∞ map. Suppose we have some $q \in V$ and $f^{-1}(q) \subseteq U_0$ for some open set U_0 of U. Then there exists a neighborhood V_0 of q in V, such that $f^{-1}(V_0) \subseteq U_0$.

In other words, we can expand around a given point's preimage by an "arbitrarily small amount."

Definition 118

Let $f : U \to V$ be a proper C^∞ map. A point $p \in U$ is a **critical point** if the map $df_p : T_p \to T_q$ (where $q = f(p)$) is not bijective. Let C_f be the set of critical points, also known as the **critical set**; we call $f(C_f)$ the set of **critical values**.

We may notice that C_f is closed, and if f is proper, then $f(C_f)$ is also closed. Furthermore, the set of **regular values** $V_{\text{reg}} = V \setminus f(C)$ is open.

Lemma 119

For any regular value q of f, $f^{-1}(q)$ is a finite set.

Proof. Since q is a regular value, any $p \in f^{-1}(q)$ is a regular point, meaning that $df_p : T_p \to T_q$ is bijective. Thus, by the inverse function theorem, f maps a neighborhood U_p of p diffeomorphically to a neighborhood V_q of q.

We thus know that the U_ps cover $f^{-1}(q)$, and since f is proper, $f^{-1}(q)$ is compact. By compactness, we can thus find a finite subcover $U_{p_1}, \cdots, U_{p_N}$ of $f^{-1}(q)$. Because we have a bijective mapping from these neighborhoods U_{p_i}, the only point in each neighborhood U_{p_i} mapping to the point q is p_i itself. Thus the preimage of q is the finite set $\{q_1, \cdots, q_N\}$. □

We can then choose the neighborhoods to be disjoint (by choosing small enough neighborhoods), so that we have a neighborhood V_q of q and a neighborhood U_{p_i} of each p_i satisfying

$$f^{-1}(V_q) = \bigcup_{i=1}^N U_{p_i}, \quad U_{p_i} \cap U_{p_j} = \varnothing \quad \forall i,j, \quad f : U_{p_i} \to V_q \text{ diffeomorphism.}$$

(This is known as the **stack of records theorem**.) We can then finally understand another way to define the degree:

Theorem 120 (Degree theorem)

Let q be a regular value of f, and let $\{p_1, \cdots, p_N\}$ be its preimage. Then we have

$$\deg(f) = \sum_{i=1}^{N} \sigma_{p_i},$$

where each σ_{p_i} is 1 or -1, depending on whether $df_{p_i} : T_{p_i} \to T_q$ is orientation-preserving or orientation-reversing, respectively.

Proof. Choose some $\omega_0 \in \Omega_c^n(V)$ with $\int_V \omega_0 = 1$. By the definition of degree, we know that

$$\deg(f) = \int_V f^*\omega_0 = \sum_i \int_{U_{p_i}} f^*\omega_0.$$

And by our work last lecture, because $f : U_{p_i} \to V$ is a diffeomorphism, each term in this sum will be 1 if f is orientation-preserving and -1 otherwise. $\square$

This then connects back to Theorem 115: if $f : U \to V$ is a proper C^∞ map and q is not in the image of f, then $q \in C_f$ must be a critical point. Additionally, $f^{-1}(q)$ will be empty and the degree of f is zero.

We'll now move on quickly to the topic of homotopy:

Definition 121

Let $f_0, f_1 : U \to V$ be proper C^∞ maps. A **homotopy** between f_0 and f_1 is a C^∞ map $F : U \times [0, 1] \to V$ such that $F(x, 0) = f_0(x)$ and $F(x, 1) = f_1(x)$. If F is proper, then we call it a **proper homotopy**.

(In other words, a homotopy involves continuously deforming f_0 into f_1 with a smooth map.)

Theorem 122

If f_0 and f_1 are properly homotopic, then they have the same degree.

Proof. Fix a compactly supported $\omega_0 \in \Omega_c^n(V)$ such that $\int_V \omega_0 = 1$. Then for each $t \in [0, 1]$, $\deg(f_t) = \int f_t^*\omega$ depends smoothly on t, but the degree is always an integer by Theorem 120. Thus $\deg(f_t)$ must be constant in t and $\deg(f_0) = \deg(f_1)$. $\square$

21 March 28, 2022

We'll start by reviewing the **degree theorem** from before spring break, as well as the related definitions. If U and V are open subsets of $\mathbb{R}^n$ and $f : U \to V$ is a proper C^∞ map, then the **degree** is an invariant of the map f satisfying

$$\int_U f^*\omega = \deg(f) \int_V \omega$$

whenever $\omega \in \Omega_c^n(V)$ (so that both $f^*\omega$ and ω are compactly supported and we can have both integrals make sense). For another perspective on this degree, recall that a point $q \in V$ is a **regular value** of f if $df_p : T_pU \to T_qV$ is bijective for any $p \in f^{-1}(q)$. (We previously mentioned Sard's theorem, wihch states that the set of regular values forms an open dense subset of U.) Then for any regular value q, we proved that $f^{-1}(q)$ is a finite set $\{p_1, \cdots, p_k\}$, and then we found the formula

$$\deg(f) = \sum_{i=1}^{k} \sigma_{p_i},$$

where σ_{p_i} is 1 if $df_{p_i} : T_{p_i} \to T_q$ is orientation-preserving and -1 if it is orientation-reversing (this makes sense because the map is basically $\mathbb{R}^n \to \mathbb{R}^n$). So if f is not surjective, then $\deg(f) = 0$. Finally, we discussed the homotopy invariance of degree, which tells us that two proper C^∞ maps $f_0, f_1 : U \to V$ that are homotopic also have the same degree (because the degree must be continuous and integer-valued).

Today, we'll talk about some applications, starting with a familiar result:

Theorem 123 (Fundamental theorem of algebra)

Consider the monic polynomial $p(z) = z^n + a_{n-1}z^{n-1} + \cdots + a_0$ (where $a_0, \cdots, a_{n-1} \in \mathbb{C}$). The equation $p(z) = 0$ has at least one complex solution z.

Proof. Notice that p is a C^∞ map $\mathbb{C} \to \mathbb{C}$. Identify $\mathbb{C}$ with $\mathbb{R}^2$ (by sending $x + iy$ to (x, y)) and think of p as a map from $\mathbb{R}^2$ to $\mathbb{R}^2$. Then notice that

$$|p(z)| \geq |z|^n + \sum_{i=0}^{n-1} |a_i||z^i|,$$

so $|p(z)| \to \infty$ as $|z| \to \infty$ because the leading term here, and thus $p(z)$ is a proper map. Now define

$$p(z, t) = (1 - t)z^n + tp(z)$$

(we can think of this as a homotopy between $p_0(z) = z^n$ and $p(z)$). Plugging in the expression for $p(z)$, we know that $|p(z, t)| \geq |z|^n - t\sum_{i=0}^{n-1} |a_i||z^i|$, so again $|p(z, t)|$ is unbounded as $|z| \to \infty$ and we have a proper homotopy between z^n and $p(z)$. Since the degree of $p_0(z)$ is n, so is the degree of $p(z)$. (This is left as an exercise for us: as a hint, let ϕ be a $C_0^\infty(\mathbb{R})$ function, and consider $\omega = \phi(x^2 + y^2)dxdy$ for calculation purposes. In polar coordinates, this is $\phi(r^2)rdrd\theta$, so we can check that $p_0^*\omega = n^2\phi(r^{2n})r^{2n-1}drd\theta$ and $\int_{\mathbb{R}^2} p_0^*\omega = n\int_{\mathbb{R}^2} \omega$.) Thus the degree of $p(z)$ is nonzero and p is surjective; in particular, there is at least one point where $p(z) = 0$. □

Theorem 124 (Brouwer fixed point theorem)

Let B^n be the unit ball $\{x \in \mathbb{R}^n : |x| \leq 1\}$, and let $S^{n-1} = \text{Bd}(B^n)$ be the unit $(n-1)$-sphere which bounds that ball. If $f : B^n \to B^n$ is a C^∞ map, then there exists some $x_0 \in B^n$ such that $f(x_0) = x_0$ (that is, we always have a fixed point).

Proof. Suppose for the sake of contradiction that there is no fixed point. Consider the map $\gamma : B^n \to S^{n-1}$ defined in the following way: for each $x \in B^n$, draw a ray originating at $f(x)$ and pointing in the direction of x (because $f(x) \neq x$), and let $\gamma(x)$ be the point on the boundary S^{n-1} that the ray intersects.

Notice that $\gamma(x) = x$ if $|x| = 1$, so $|\gamma(x) - x| = 0$ for all $x \in S^{n-1}$. But because γ is a C^∞ map, it extends to an open subset U of B^n, and for every $\varepsilon > 0$, there is some $\delta > 0$ such that $B^n_{1+\delta} \supseteq U$ and $|\gamma(x) - x| \leq \varepsilon$ for all $1 \leq |x| \leq 1 + \delta$. If we now pick some $\rho \in C_0^\infty(\text{Int}(B^n_{1+\delta})$ such that ρ is identically 1 for $|x| \geq 1$, we can define the map $\tilde{\gamma} : \mathbb{R}^n \to \mathbb{R}^n$ such that

$$\tilde{\gamma}(x) = \begin{cases} \rho(x)\gamma(x) + (1 - \rho(x))x & |x| \leq 1 + \delta, \\ x & |x| \geq 1 + \delta, \end{cases}$$

and consider the homotopy

$$g(x, t) = t\tilde{\gamma}(x) + (1 - t)x.$$

This is a proper homotopy, so $\tilde{\gamma}(x)$ has the same degree as the identity map, which is 1. But $\tilde{\gamma}(x) \geq 1 - \varepsilon$ everywhere, so for any $|x_0| \leq 1 - \varepsilon$, $\tilde{\gamma}^{-1}$ is empty, which is a contradiction with the degree being nonzero. $\square$

22 March 30, 2022

Fact 125

As a reading assignment, we should read about the implicit function theorem in Appendix B of the textbook, because it will have applications to what we'll do in the coming weeks.

We're currently transitioning from talking about differential forms on open subsets of $\mathbb{R}^n$ to differential forms on **manifolds** – in particular, we'll soon explain what a differentiable manifold is and start exploring relevant properties. But today, we'll first go back to some previous topics that we've discussed, starting with some multivariable calculus results:

Proposition 126 (Inverse function theorem)

Let U and V be open subsets of $\mathbb{R}^n$, and suppose $f : U \to V$ is a C^∞ map such that $f(p) = q$. If $df_p : T_pU \to T_qV$ is bijective, then it maps a neighborhood U_p of p diffeomorphically to a neighborhood V_q of q.

This result is actually the special case of two more general results:

Proposition 127 (Canonical submersion theorem)

Let $k < n$, and define the canonical submersion $\pi : \mathbb{R}^n \to \mathbb{R}^k$ to be the map $(x_1, \cdots, x_n) \mapsto (x_1, \cdots, x_k)$. Let U be an open subset of $\mathbb{R}^n$, and suppose $f : U \to \mathbb{R}^k$ is a C^∞ map with $f(p) = 0$. If $df_p : T_pU \to T_0\mathbb{R}^k$ is surjective, then there exists a neighborhood W of 0 in $\mathbb{R}^k$, a neighborhood U_p of U, and a diffeomorphism $\phi : W \to U_p$ sending 0 to p such that $\pi = f \circ \phi$.

Proposition 128 (Canonical immersion theorem)

Let $k < n$, and define the canonical immersion $\iota : \mathbb{R}^k \to \mathbb{R}^n$ to be the map $(x_1, \cdots, x_k) \mapsto (x_1, \cdots, x_n)$. Let U be an open neighborhood of 0 in $\mathbb{R}^k$, and suppose $f : U \to \mathbb{R}^n$ is a $C^\infty map$ with $f(0) = p$. If $df_p : T_0U \to T_p\mathbb{R}^n$ is injective, then there exists a neighborhood W of 0 in $\mathbb{R}^n$, a neighborhood V_p of p in $\mathbb{R}^n$, and a diffeomorphism $\phi : V_p \to W$ sending p to 0 such that $\iota = \phi \circ f$.

We can now turn to a study of manifolds, starting with a preliminary definition:

Definition 129

Let X be a subset of $\mathbb{R}^M$, Y be a subset of $\mathbb{R}^N$, and $f : X \to Y$ be a continuous map. Then f is a C^∞ map if for every $p \in X$, there exists a neighborhood U of p in $\mathbb{R}^M$ and a C^∞ map $\tilde{f} : U \to \mathbb{R}^N$ such that $\tilde{f}|_{U \cap X} = f|_{U \cap X}$.

Definition 130

Let $N \geq n$. A set $X \subset \mathbb{R}^N$ is an ***n*-dimensional manifold** if for every $p \in X$, there exists an open set U of $\mathbb{R}^n$, an open neighborhood V of p in $\mathbb{R}^N$, and a diffeomorphism $\phi : U \to V \cap X$. (We call ϕ a **parameterization** of X at p.)

In other words, neighborhoods of p look locally like neighborhoods of $\mathbb{R}^n$, and we'll now try to define differential forms on manifolds. In particular, if we let $a = \phi^{-1}(p)$, and we let $T_a : \mathbb{R}^n \to \mathbb{R}^n$ be the translation map $x \mapsto x + a$, we can compose ϕ with T_a and always assume that $\phi^{-1}(p) = 0$. (In this case, we say that the parameterization is **centered at zero**.) Manifolds may seem like an abstract concept, but it's worthwhile to think about them because they come up as solutions to systems of equations:

Proposition 131
Let U be an open subset of $\mathbb{R}^N$, and let $f : U \to \mathbb{R}^k$ be a C^∞ mapping. (Recall that a point a is a **regular value** if for all $p \in f^{-1}(a)$, $df_p : T_pU \to T_a\mathbb{R}^k$ is surjective. If a is a regular value, then $X = f^{-1}(a)$ is an n-dimensional manifold, where $n = N - k$.

In fact, many of the examples of manifolds we'll be seeing from now on are of this form $X = f^{-1}(a)$.

Proof. We can replace f with $f - a$, so that we can assume $a = 0$. If 0 is a regular value of f, that means that for all $p \in f^{-1}(0)$, $df_p : T_p\mathbb{R}^N \to T_0\mathbb{R}^k$ is surjective. By the canonical submersion theorem, there are open neighborhoods U of 0 and V of P in $\mathbb{R}^N$, as well as a diffeomorphism $\phi : U \to V$ satisfying $\pi = f \circ \phi$ (for the canonical submersion π). But any $x \in \pi^{-1}(0)$ is of the form $(0, \cdots, 0, x_{k+1}, \cdots, x_N)$, so ϕ maps a neighborhood of 0 in $\mathbb{R}^n$ diffeomorphically onto a neighborhood of $p \in f^{-1}(0)$, and thus $f^{-1}(0)$ is an n-dimensional manifold. □

23 April 1, 2022

Last lecture, we introduced the concept of a **manifold** by explaining what it means to be a C^∞ map between subsets X and Y of $\mathbb{R}^M$ and $\mathbb{R}^N$, namely that for every point $p \in X$, there is a neighborhood U of p and a C^∞ map $\tilde{f} : U \to \mathbb{R}^N$ which agrees with f on $U \cap X$. We say that f is a **diffeomorphism** if it is bijective and both f and f^{-1} is C^∞, and that allows us to define a subset $X \subseteq \mathbb{R}^N$ to be an n-**dimensional manifold** if there is a neighborhood V_p around any point $p \in X$ such that we have a diffeomorphism ϕ (called a **parameterization** at p, **centered** at p if $\phi(0) = p$) from $U \to V_p \cap X$ for some open set U of $\mathbb{R}^n$ (so that X looks locally like an open subset of $\mathbb{R}^n$).

Example 132
The n-sphere $S^n \subseteq \mathbb{R}^{n+1}$, defined as

$$\{x \in \mathbb{R}^{n+1} : x_1^2 + \cdots + x_{n+1}^2 = 1\}$$

is an n-dimensional manifold. To check this, let $H_i^\pm$ be the set of all points $(x_1, \cdots, x_{n+1}) \in S^n$ with $x_i > 0$ or $x_i < 0$, respectively. We can verify that these are open sets which cover S^n, and furthermore the map $H_i^\pm \to \mathbb{R}^n$ sending $(x_1, \cdots, x_{n+1})$ to $(x_1, \cdots, x_{i-1}, x_{i+1}, \cdots, x_{n+1})$ is a diffeomorphism. So any point in S^n has a neighborhood (one of the $H_i^\pm$s that contains it) which looks like an open set of $\mathbb{R}^n$, and we can choose our parameterization ϕ to be the inverse of one of the $H_i^\pm$s.

Example 133
Let $X_i \subseteq \mathbb{R}^{N_i}$ be C^∞ manifolds of dimension n_i for $1 \le i \le k$. Then the **product manifold** $X_1 \times \cdots \times X_k$ is a manifold of dimension $n = n_1 + \cdots + n_k$, because any point in the manifold is of the form $p = (p_1, \cdots, p_k)$ where $p_i \in \mathbb{R}^{N_i}$. Each of these p_is come along with a neighborhood V_{p_i}, a subset U_i of $\mathbb{R}^{n_i}$, and a map $\phi_i : U_i \to V_{p_i} \cap X_i$, and then we can check that the product map $(\phi_1, \cdots, \phi_k)$ is a parameterization of the product manifold at p.

Last time, we mentioned that one important source of manifolds is the following: if U is an open subset of $\mathbb{R}^N$, $f : U \to \mathbb{R}^k$ is a C^∞ map, and $a \in \mathbb{R}^k$ is a regular value, then $X = f^{-1}(a)$ is an n-dimensional manifold. (As an example of this, if we let $f : \mathbb{R}^{n+1} \to \mathbb{R}$ be the map $f(x_1, \cdots, x_{n+1}) = x_1^2 + \cdots + x_{n+1}^2$, then $Df(x) = 2(x_1, \cdots, x_{n+1})$. 1 is then a regular value, and $f^{-1}(1)$ is an n-dimensional manifold, namely S^n. We'll now see a more substantial example:

Example 134

Let $\mathcal{M}_{n\times n}$ be the set of all $n \times n$ real-valued matrices $[a_{ij}]$, and let $\mathcal{S}_{n\times n}$ be the set of symmetric $n \times n$ real-valued matrices (satisfying $a_{ij} = a_{ji}$ for all i, j). We can verify that $\mathcal{M}_{n\times n} \cong \mathbb{R}^{n^2}$ and that there is a natural map $\phi : \mathcal{M}_{n\times n} \to \mathcal{S}_{n\times n}$ given by $\phi(A) = A^T A$. Then because I is a regular value of ϕ, $\phi^{-1}(I_n)$ is a manifold, namely the set of **orthogonal** $n \times n$ **matrices** (satisfying $A^T A = I$).

Example 135

Let $G(k, n)$ be the set of all symmetric matrices $P \in \mathcal{S}_{n\times n}$ such that the rank of P is k (the image of the linear mapping has dimension k) and $P^2 = P$. We can show that $G(n, k)$ is a $k(n - k)$-dimensional manifold using the methods from above, known as the **Grassmannian** (we can think of it as the set of k-dimensional linear subspaces of $\mathbb{R}^n$).

(This will likely be on our next, and possibly last, homework assignment.) From here, we can now move on and generalize the notion of a **tangent space** that we introduced originally on $\mathbb{R}^n$:

Proposition 136

Let $X \subseteq \mathbb{R}^n$ be a k-dimensional manifold, let p be a point in X, and let W be a neighborhood of p in $\mathbb{R}^N$. Suppose $\phi : U \to V = W \cap X$ is a parameterization of X centered at p (for some open set U of $\mathbb{R}^k$). If we think of ϕ as a C^∞ map of U into W, then its derivative $d\phi_0 : T_0U \to T_pW$ is injective.

Proof. By definition, there is a C^∞ extension $\psi : W \to U$ of ϕ^{-1}, such that $d\psi_p \circ d\phi_0$ is the identity map, which can only happen if $d\phi_0$ is injective. $\square$

Definition 137

With the notation in the result above, we let the **tangent space of p at X** be $T_pX = \mathrm{Im}(d\phi_0)$.

Proposition 138

The tangent space T_pX has an intrinsic definition not dependent on the particular choice of ϕ.

Proof. If $\phi_1 : U_1 \to V_1$ and $\phi_2 : U_2 \to V_2$ are two parameterizations centered at p, replacing V_1, V_2 by $V = V_1 \cap V_2$ and replacing U_1, U_2 by $\phi_i^{-1}(V)$ allows us to assume that $\phi_1 : U_1 \to V$ and $\phi_2 : U_2 \to V$ are both diffeomorphisms. Then we have a diffeomorphism $\psi = \phi_2^{-1} \circ \phi_1$. Since $(d\phi_1)_0$ and $(d\phi_2)_0$ are injective and $(d\psi)_0$ is bijective, the images of $(d\phi_1)_0$ and $(d\phi_2)_0$ must be the same (since they have the same dimension, and if $(d\phi_1)_0 = (d\phi_2)_0 \circ (d\psi)_0$ then the image of $(d\phi_1)_0$ is contained in the image of $(d\phi_2)_0$). $\square$

We'll finish by mentioning a functoriality property for T_pX: if $X \subseteq \mathbb{R}^m$ and $Y \subseteq \mathbb{R}^\ell$ are C^∞ manifolds, and $f : X \to Y$ is a C^∞ map sending p to q, then $df_p : T_pX \to T_qY$ is well-defined. But we'll talk more about this next time!

24 April 4, 2022

Last time, we introduced the concept of a tangent space on a manifold. Specifically, recall that a manifold $X \subseteq \mathbb{R}^N$ is a set that comes with a parameterization centered at p (for any $p \in X$), in which we take a neighborhood U of 0 in $\mathbb{R}^n$ and a neighborhood W of p in $\mathbb{R}^N$, and we have a diffeomorphism $\phi : U \to W \cap X$ sending 0 to p. In particular, if we think of ϕ actually as a C^∞ map from U to W (which contains $W \cap X$), the map $d\phi_0 : T_0U \to T_pW$ (this maps an n-dimensional space to an N-dimensional space) is injective. (The proof of this was that because ϕ was originally a diffeomorphism from U to $W \cap X$, it has a C^∞ inverse map ψ which extends to some open neighborhood W' of p. Then using $W' \cap W$ instead of W, the chain rule on Euclidean space tells us $d\psi_p \circ d\phi_0 = \mathrm{id}$, which implies that $d\phi_0$ is injective.) We then proved that this gives us an intrinsic definition $T_pX = \mathrm{Im}(d\phi_0)$.

We'll now explore the functoriality concepts that we started discussing at the very end of last lecture. In particular, let X be a submanifold of $\mathbb{R}^{N_1}$ (that is, a manifold in $\mathbb{R}^{N_1}$) and Y be a submanifold of $\mathbb{R}^{N_2}$. If we have a C^∞ map $X \to Y$ sending p to q, then we can define the derivative of f at p, $df_p : T_pX \to T_qY$ in two different ways:

Definition 139

Taking the notation above, suppose W is a neighborhood of p in $\mathbb{R}^{N_1}$, and $g : W \to \mathbb{R}^{N_2}$ is a C^∞ map such that $f = g$ restricted to $X \cap W$. Then df_p is defined to be dg_p restricted to T_pX.

This definition makes sense intuitively (we look at derivatives on Euclidean space instead of on the manifold), but it depends on the choice of the function g. So we can try something else:

Definition 140

Again with the notation above, let $\phi_1 : U_1 \to V_1$ and $\phi_2 : U_2 \to V_2$ be parameterizations of X and Y centered at p and q, respectively. Without loss of generality, assume $V_1 \subseteq f^{-1}(V_2)$ (otherwise we can just make V_1 smaller because ϕ_1 is a diffeomorphism). Then df_p is the the map that completes the following diagram, where $\psi_0 = \phi_2^{-1} \circ f \circ \phi_1$:

$$\begin{array}{ccc} T_pX & \xrightarrow{df_p} & T_qY \\ {\scriptstyle (d\phi_1)_0}\uparrow & & \uparrow{\scriptstyle (d\phi_2)_0} \\ T_0U_1 & \xrightarrow{d\psi_0} & T_0U_2 \end{array}$$

Unfortunately, this definition may look like it instead depends on the extension mappings ϕ_1 and ϕ_2. So we'll reconcile that with the following diagram and the use of the 18.101 chain rule on $\tilde{f} \circ \phi_1 = \phi_2 \circ \psi_0$:

$$\begin{array}{ccc} T_pW & \xrightarrow{d\tilde{f}_p} & T_q\mathbb{R}^{N_2} \\ {\scriptstyle (d\phi_1)_0}\uparrow & & \uparrow{\scriptstyle (d\phi_2)_0} \\ T_0U_1 & \xrightarrow{d\psi_0} & T_0U_2 \end{array}$$

In particular, the tangent space T_pX of an n-dimensional manifold X is n-dimensional. As an exercise, we can now prove the **chain rule on manifolds**, which says that if $f_1 : X_1 \to X_2$ and $f_2 : X_2 \to X_3$ are C^∞ maps on manifolds and $f_1(p_1) = p_2$, then

$$(df_2 \circ f_1)_{p_1} = (df_2)_{p_2} \circ (df_1)_{p_1}.$$

An application of the chain rule is the following:

Proposition 141
Recall that for a map $U \to V$ between open sets of $\mathbb{R}^N$ and $\mathbb{R}^k$, and for a regular value $q \in V$ of f, $X = f^{-1}(q)$ is an n-dimensional manifold, where $N = N - k$. Then T_pX is the kernel of the map $df_p : T_pU \to T_qV$.

Proof. Let W be a neighborhood of p in $\mathbb{R}^N$, such that $\phi : U \to W \cap X$ is a parameterization of X centered at p. Then $f \circ \phi$ is the constant map $U \to X \to q$, so $df_p \circ d\phi_0 = 0$. But because $d\phi_0$ maps T_0U bijectively to T_pX, T_pX must be a subset of the kernel of df_p. On the other hand, $\text{Im}(df_p)$ is k-dimensional, so the kernel of df_p must be $(N - k = n)$-dimensional. Since T_pX is n-dimensional, this means the kernel of df_p must coincide with it. □

Next time, we'll talk more about the canonical immersion and submersion theorems and generalize them to manifolds!

25 April 6, 2022

We've now spent a few lectures understanding the notation and relevant definitions behind manifolds X, maps $X \to Y$ between them, and their tangent spaces T_pX. Last time, for a C^∞ map $f : X \to Y$ sending p to q, we defined the derivative map df_p at p between tangent spaces $T_pX \to T_qY$ in two ways, either extending f to an open subset of $\mathbb{R}^{N_1}$ (the Euclidean space that X lives in) and using the Euclidean space definition of df_p, or by making use of the parameterizations $U_1 \to W_1 \cap X$ and $U_2 \to W_2 \cap Y$ that come along with the manifolds X and Y and defining df_p in terms of the corresponding $d\psi_0$ map from the map on the Euclidean subsets $\psi : U_1 \to U_2$. It can be shown that these definitions are both legitimate and coincide with each other (though as a sidenote, the second definition works even if we have manifolds that aren't subsets of $\mathbb{R}^n$ for some n), so that we have a valid definition of the derivative df_p and thus of the map df. In particular, we also have a chain rule just like for ordinary Euclidean space.

Today, we'll show that our results about differential forms on Euclidean space apply for manifolds too.

Theorem 142 (Inverse function theorem for manifolds)
Let X and Y be n-dimensional manifolds, and let $f : X \to Y$ be a C^∞ map sending p to q. If $df_p : T_pX \to T_qY$ is bijective, then f maps a neighborhood U_p of p diffeomorphically onto a neighborhood V_q of q in Y.

We may recall from a few lectures ago that we also stated the canonical submersion and canonical immersion theorem for open subsets of $\mathbb{R}^n$, and there are analogous versions of those too. But because the canonical submersion (for example) is defined from $\mathbb{R}^n \to \mathbb{R}^k$ (for $k < n$) as sending $(x_1, \cdots, x_n)$ to $(x_1, \cdots, x_k)$, we're going to need to do that "submersion" on the open sets U coming with the parameterizations:

Theorem 143 (Canonical submersion theorem for manifolds)
Let X and Y be manifolds such that $f : X \to Y$ is a C^∞ map sending p to q. If $df_p : T_pX \to T_qY$ is **surjective**, then there exist neighborhoods U_p and V_q of p and q in X and Y, respectively, and parameterizations $\phi : U \to U_p$ sending 0 to p and $\psi : V \to V_q$ sending 0 to q, such that $f(U_p) \subseteq V_q$ and the following commutative diagram is followed, with bottom arrow being the canonical submersion:

$$\begin{array}{ccc} U_p & \xrightarrow{f} & V_q \\ \phi\uparrow & & \uparrow\psi \\ U & \xrightarrow{\pi} & V \end{array}$$

In other words, every submersion looks locally like the canonical submersion. A similar property holds for the canonical immersion $\iota : (x_1, \cdots, x_k \mapsto (x_1, \cdots, x_k, 0, \cdots, 0)$:

Theorem 144 (Canonical immersion theorem for manifolds)
Let X and Y be manifolds such that $f : X \to Y$ is a C^∞ map sending p to q. If $df_p : T_pX \to T_qY$ is **injective**, then there exist neighborhoods U_p and V_q of p and q in X and Y, respectively, and parameterizations $\phi : U \to U_p$ sending 0 to p and $\psi : V \to V_q$ sending 0 to q, such that $f(U_p) \subseteq V_q$ and the following commutative diagram is followed, with bottom arrow being the canonical immersion:

$$\begin{tikzcd} U_p \arrow[r, "f"] & V_q \\ U \arrow[u, "\phi"] \arrow[r, "\iota"] & V \arrow[u, "\psi"'] \end{tikzcd}$$

We may read the proofs on our own, but the idea is to choose coordinates so that the manifolds actually become locally open subsets of Euclidean space and applying the canonical submersion and immersion theorems that we've already seen. (And we should note that these are **local** results – everything here has to do with picking a particular point p and q.)

26 April 8, 2022

We'll start to talk about differential forms on manifolds today – just like on Euclidean space, if X is an n-dimensional manifold, recall that we can define the tangent space T_pX at any point $p \in X$. We can then define the analogous cotangent (dual) space T_p^*X, and we can define $\Lambda^k(T_p^*X)$ in the same way using the standard linear algebra techniques.

Definition 145
Let X be a manifold. A ***k*-form** on X is a "function" ω which takes in a point $p \in X$ and produces an element $\omega_p \in \Lambda^k(T_p^*X)$.

Many of the properties of k-forms that we've already studied still hold – we'll talk first about functoriality. If X_1 and X_2 are manifolds, and $f : X_1 \to X_2$ is a C^∞ map sending p_1 to p_2, then we have a mapping $df : T_{p_1}X_1 \to T_{p_2}X_2$, and accordingly we get a transpose (pullback) map $(df_p)^* : \Lambda^k(T_{p_2}^*X_2) \to \Lambda^k(T_{p_1}^*X_1)$. Extending this pullback operation to all points p, we find that given any k-form ω on X_2, we can define a k-form $f^*\omega_2$ on X_1 via

$$(f^*\omega)_{p_1} = df_{p_1}^*\omega_{p_2}.$$

We also have the usual "chain rule:" if $X_1 \xrightarrow{f_1} X_2 \xrightarrow{f_2} X_3$ is a sequence of maps between manifolds, we then have

$$(f_2 \circ f_1)^*\omega = f_1^* f_2^* \omega.$$

We may recall that we like to restrict our attention to C^∞ k-forms when we first studied everything over Euclidean space, and that will be relevant here as well:

Definition 146

Let ω be a k-form on X. Then ω is C^∞ if for every $p \in X$, there exists a parameterization $\phi : U \to V \cap X$ (for U, V open subsets of Euclidean space) such that $\phi^*\omega$ restricted to X is a C^∞ k-form on U. The space of C^∞ k-forms on X is denoted $\Omega^k(X)$.

In particular, if ω is a C^∞ k-form, then $f^*\omega$ is also a C^∞ k-form. We do need to show that this definition is independent of ϕ, but this is true because if we have diffeomorphisms $\phi_1 : U_1 \to V \cap X$ and $\phi_2 : U_2 \to V \cap X$, then $\psi = \phi_2^{-1} \circ \phi_1$ is a diffeomorphism as well, and $\phi_1^*\omega = \psi^*\phi_2^*\omega_2$ (so $\phi_1^*\omega$ is C^∞ if and only if $\phi_2^*\omega$ is).

Returning to operations on k-forms, just like over Euclidean space, if we have $\omega_1, \omega_2 \in \Omega^k(X)$, then their sum $\omega_1 + \omega_2$ (defined pointwise) is also in $\Omega^k(X)$. Also, we have a d operation just like over Euclidean space, defined in the following way:

Definition 147

For any $\omega \in \Omega^k(X)$ and $p \in X$, let $\phi : U \to V \cap X$ be a parameterization of X at p (sending 0 to p). Then we define

$$d\omega|_X = (\phi^{-1})^* d(\phi^*\omega).$$

In other words, we use the definition of the d operation on Euclidean space, and this is a legitimate definition because of the functoriality argument we just made (left as an exercise to us).

Proposition 148

Let $f : X_1 \to X_2$ be a C^∞ map. Then for any $\omega \in \Omega^k(X_2)$, we have $f^*d\omega = df^*\omega$.

We showed this result for open subsets of Euclidean space, but they also hold for differentiable manifolds.

Proof. Let $f(p_1) = p_2$ for $p_1 \in X_1$ and $p_2 \in X_2$. Suppose we have parameterizations $\phi_1 : U_1 \to V_1$ and $\phi_2 : U_2 \to V_2$ centered at p_1 and p_2 (from now on we'll suppress the "intersection with X_1" and so on). If we pick our neighborhoods so that $f^{-1}(V_2) \supset V_1$, then we have the following diagram:

$$\begin{tikzcd} V_1 \arrow[r, "f"] & V_2 \\ U_1 \arrow[u, "\phi_1"] \arrow[r, "\psi"] & U_2 \arrow[u, "\phi_2"'] \end{tikzcd}$$

By the chain rule and the commutativity of the diagram, we know that $\phi_1^* f^* d\omega = \psi^*\phi_2^* d\omega$, which leads us to $d\psi^*\phi_2^*\omega = d\phi_1^* f^*\omega$. Some more manipulation (using the definition of d and the pullback operation) gives us the result. ☐

From the definition of d, we can essentially read off the usual properties

$$d(\omega_1 + \omega_2) = d\omega_1 + d\omega_2, \quad d(\omega_1 \wedge \omega_2) = d\omega_1 \wedge d\omega_2 + (-1)^{k_1}\omega_1 \wedge d\omega_2, \quad d(d\omega) = 0$$

(where ω_1 is a k_1-form), because of the versions that we proved earlier on in the class.

27 April 11, 2022

Last time, we discussed the definition of a (C^∞) differential k-form on an n-dimensional manifold (by looking at the tangent space T_pX, the corresponding exterior algebra $\Lambda^k(T_p^*X)$, and then having ω assign to each $p \in X$ an element $\omega_p \in \Lambda^k(T_p^*)$ in a way such that the pullback $\phi^*\omega$ of the parameterization at any p is a C^∞ k-form on the corresponding domain, an open subset of $\mathbb{R}^n$). And as a reminder, in the Euclidean case, a k-form $\omega = \sum \phi_I dx_I$ is said to be C^∞ if each ϕ_I is a C^∞ function on U.

We also described the pullback operation, which gives us a differential form $f^*\omega$ given a differential form ω and a C^∞ map. More explicitly, recall that on Euclidean space, we did this by saying that if $\omega = \sum \phi_I dx_I$, then $f^*\omega = \sum f^*\phi_I df_I$, where $f^*\phi_I$ is still a C^∞ function and $df_I = f^*dx_I$. Parameterizations then let us generalize that definition (as we mentioned last time). Similarly, we saw that the d operation, which takes $\omega = \sum f_I dx_I$ to $d\omega = \sum df_I \wedge dx_I$, can be extended to manifolds as well. And while these definitions appear to depend on the diffeomorphism ϕ that comes with the manifold X, we can verify that the objects we define are in fact intrinsic. Furthermore, we verified that the usual functorial properties of these operations that we proved over Euclidean space still hold more generally.

We'll now talk about **vector fields on manifolds**, generalizing our discussion on Euclidean space. the pullback

Definition 149

Let X be a manifold. A **vector field** v on X assigns to each $p \in X$ an element $v(p) \in T_pX$, and a vector field is C^∞ if its pullback is C^∞.

Definition 150

Let X be a manifold, $\omega \in \Omega^k(X)$ be a differential k-form, and v be a C^∞ vector field on X. Like over Euclidean space, let the **interior product** $\iota(v)\omega$ be the differential $(k-1)$-form given by $(\iota(v)\omega)_p = \iota(v_p)\omega_p$.

The only thing we must check is that this indeed gives us a C^∞ k-form:

Proposition 151

If ω, v are C^∞, then so is $\iota(v)\omega$.

Proof. Let $\phi : U \to V_p$ be a parameterization of our manifold X at p. Define $v^\#$ to be the pullback $\phi^*(v)$ restricted to X, and similarly let $\omega^\# = \phi^*\omega|_V$. Then we can check that $\phi^*(\iota(v)\omega|_V) = \iota(v^\sharp)\omega^\sharp$. $\square$

With this verification, we are now permitted to extend the Lie differentiation to differential forms on manifolds:

Definition 152

Let v be a C^∞ vector field on X, and let $\omega \in \Omega^k(X)$. Then the **Lie differentiation** operation is defined (just like on Euclidean space) as

$$\mathcal{L}_v\omega = d\iota(v)\omega + \iota(v)d\omega$$

Proposition 153

Just like over Euclidean space, we have

$$d\mathcal{L}_v\omega = \mathcal{L}_v d\omega, \quad \mathcal{L}_v(\omega_1 \wedge \omega_2) = \mathcal{L}_v\omega_1 \wedge \omega_2 + \omega_1 \wedge \mathcal{L}_v\omega_2.$$

(The same proofs carry over verbatim.) We also have an alternative definition of the Lie derivative.

Proposition 154
Suppose that a vector field v is **complete** (recall that this means there is a one-parameter family of diffeomorphisms f_t such that for each $p \in X$, $f_t(p)$ is an integral curve of v with $f_0(p) = p$). Then

$$\mathcal{L}_v \omega = \frac{d}{dt} (f_t^* \omega)_{t=0}.$$

This proof is left as an exercise to us, but the idea is that for any $\phi \in C^\infty(X)$, we have

$$\boxed{f_t^* \phi(p) = \phi(\gamma_t(p))}$$

for an integral curve $\gamma_t(p)$ with $\gamma_0(p) = p$. Then because $\boxed{f_t^* d\omega = f_t^* \omega}$ and $\boxed{f_t^*(\omega_1 \wedge \omega_2) = f_t^* \omega \wedge f_2^* \omega_2}$ (discussed previously in the class), we can differentiate the first boxed equation at $t = 0$ to conclude the result for differential 0-forms (that is, functions), differentiate the second boxed equation at $t = 0$ to conclude the result for $d\omega$ given the result for ω, and differentiate the third boxed equation at $t = 0$ to conclude the result for wedge products. Since every k-form is a sum of products of 1-forms, this proves the result.

28 April 13, 2022

We'll be talking about integration of differential forms on manifolds in the next section of this course. Our first result is one about "partitions of unity:"

Theorem 155
Let $X \subseteq \mathbb{R}^N$ be an n-dimensional manifold, and let $\mathcal{U} = \{U_\alpha : \alpha \in \mathcal{I}\}$ be an open covering of X (where $\mathcal{I}$ is some index set). Then there exists a family of functions $\rho_i \in C_0^\infty(X)$, such that

- each ρ_i is nonnegative on X,
- for every compact subset $C \subseteq X$, there is some $M > 0$ such that $\text{supp}\phi_i \cap C = \varnothing$for all $i > M$,
- $\sum \rho_i = 1$ on all of X,
- for every $i \in \alpha$, there is some α such that $\text{supp}(\rho_i) \subseteq U_\alpha$.

Proof. For every $p \in X$, there is some U_α with $p \in U_\alpha$, so that we can choose an open set O_p in $\mathbb{R}^N$ containing p such that $O_p \cap X \subseteq U_\alpha$. If we define $\mathcal{O} = \bigcup_p O_p$, and we let $\tilde{\rho}_i \in C_0^\infty(\mathcal{O})$ be a partition of unity subordinate to the covering by the O_ps (here we're using the partition of unity result on Euclidean space), we can let ρ_i be $\tilde{\rho}_i$ restricted to X. Then the properties in the theorem statement can be readily verified. $\square$

We'll also define the concept of an "orientation" on a manifold, just like over Euclidean space. Recall that for any one-dimensional vector space V, an orientation is a labeling of the connected components of $V - \{0\}$ as V_+ and V_-, and thus for any n-dimensional vector space V, an orientation is a labeling of the connected components of $\Lambda^n(V) - \{0\}$ as $\Lambda^n(V)_+$ and $\Lambda^n(V)_-$.

Definition 156

Let X be an n-dimensional manifold. An **orientation** of X is a function which assigns to each $p \in X$ an orientation of the tangent space T_pX. The orientation is **smooth** if for every $p \in X$, there is an open neighborhood U of p in X and a C^∞ n-form $\omega \in \Omega^n(U)$, such that $\omega_q \in \Lambda^n(T_p^*)_+$ for all $q \in U$.

Example 157

Let U be an open subset of $\mathbb{R}^n$. The **standard orientation** of U is the canonical orientation defined by the Euclidean volume element $dx_1 \wedge \cdots \wedge dx_n$.

If we let X and Y be oriented n-dimensional manifolds and let $f : X \to Y$ be a diffeomorphism, we can again make a functoriality argument: we again call f **orientation-preserving** if for every $p \in X$ (and corresponding $q = f(p)$), the map $(df_p)^*$ sends $\Lambda^n(T_q^*)_+$ to $\Lambda^n(T_p^*)_+$ (and otherwise orientation-reversing).

Definition 158

Let X be an oriented manifold, U be an open subset of $\mathbb{R}^n$, V an open subset of X, and $\phi : U \to V$ a diffeomorphism. ϕ is an **oriented parameterization** if ϕ is orientation-preserving.

One important note is that if U is connected but ϕ is not orientation-preserving (that is, it is orientation-reserving), we can replace the open set U with $U' = \{(x_1, \cdots, x_n) : (x_1, \cdots, x_{n-1}, -x_n) \in U\}$. Then replacing ϕ with ϕ', defined via $\phi'(x_1, \cdots, x_n) = \phi(x_1, \cdots, x_{n-1}, -x_n)$, gives us an orientation-preserving map ϕ'.

Our goal is now to define the integral $\int_X \omega$ for any compactly supported $\omega \in \Omega_c^n(X)$, and we'll be doing so under the assumption that X is oriented. **We'll first do the case where** $\omega \in \Omega_c^n(V)$, where V is a parameterizable open set of X. In that case, we choose an oriented parameterization $\phi : U \to V$, we can then define

$$\int_X \omega = \int_V \omega = \int_U \phi^*\omega,$$

where we can use the definition of differentiation of forms over Euclidean space by multivariable calculus.

Lemma 159

The definition of $\int_V \omega$ is intrinsic (not dependent on the choice of parameterization ϕ).

Proof. Suppose $\phi_1 : U_1 \to V$ and $\phi_2 : U_2 \to V$ be two oriented parameterizations. Because parameterizations are diffeomorphisms, we get a diffeomorphism $g = \phi_2^{-1} \circ \phi_1$, and then

$$\int_{U_1} \phi_1^*\omega = \int_{U_1} (\phi_2 \circ g)^*\omega = \int_U g^*\phi_2^*\omega.$$

But because g^* is orientation-preserving (since ϕ_1, ϕ_2 are orientation-preserving), this right-hand side is equal to $\int_{U_2} \phi_2^*\omega$. So the integral is indeed independent of our choice of parameterization. $\square$

We can now define integration of forms in the general case by using Theorem 155:

Definition 160

Let $\{V_\alpha : \alpha \in U\}$ be an open covering of X by parameterizable open sets, and let $\rho_i \in C^\infty(X)$ be a partition of unity subordinate to the V_α. Then for any compactly supported n-form $\omega \in \Omega_c^n(X)$, we may define

$$\int_X \omega = \sum \int_X \rho_i \omega,$$

where the right-hand side is well-defined because each $\rho_i \omega$ is supported within a parameterizable open set.

By the last point of Theorem 155, because ω is compactly supported, this sum eventually vanishes, and thus there are no convergence issues. And furthermore, we can check that this definition is independent of our choice of partition of unity – to see this, suppose $U' = \{U'_\beta\}$ is a different open covering of X by parameterizable open sets and ρ'_i is a corresponding partition of unity. Then by swapping the order of summation,

$$\sum_i \int_X \rho_i \omega = \sum_{i,j} \int_X \rho_i \rho_j \omega = \sum_{j,i} \int_X \rho_j \rho_i \omega = \sum_j \int_X \rho_j \omega,$$

showing that the definitions agree. And we can show functoriality as well: if X_1 and X_2 are n-dimensional manifolds, and $f : X_1 \to X_2$ is an orientation-preserving diffeomorphism, then for any $\omega \in \Omega_c^n(X_2)$, we have $\int_{X_1} f^*\omega = \int_{X_2} \omega$.

29 April 15, 2022

Last lecture, we started discussing integration of forms on a manifold X. Specifically, we said that if V is a parameterizable open set in X, $\omega \in \Omega_c(V)$ is compactly supported, and $\phi : U \to V$ is an oriented parameterization of V, then we can define

$$\int_X \omega = \int_U \phi^*\omega$$

via the definition of integration of forms on Euclidean space and the definition of the pullback map. More generally, for an arbitrary form ω, we can choose a covering of X by parameterizable open sets and then forming a partition of unity subordinate to that covering, which then allows us to define

$$\int_X \omega = \sum_i \int_X \rho_i \omega,$$

in which we justified why this sum is well-defined last lecture. Finally, we mentioned that the functoriality properties of integration still hold over oriented manifolds: for any orientation-preserving diffeomorphism $f : X_1 \to X_2$ between n-dimensional manifolds, we have $\int_{X_1} f^*\omega = \int_{X_2} \omega$ for any compactly supported n-form $\omega \in \Omega_c^n(X_2)$.

Today, we'll look some fundamental results in multivariable calculus, namely **Stokes' theorem and the divergence theorem**, and generalize them to manifold versions. We'll start with a "model case" of Stokes' theorem on Euclidean space:

Proposition 161

Let $\mathbb{H}^n = \{(x_1, \cdots, x_n) \subseteq \mathbb{R}^n : x_1 < 0\}$ be a half-space of $\mathbb{R}^n$, then $\mathbb{R}^{n-1} = \{(x_1, \cdots, x_n) : x_1 = 0\}$ (identified with the inclusion map $\iota_{\mathbb{R}^{n-1}} : (x_1, \cdots, x_{n-1}) \to (x_1, \cdots, x_{n-1}, 0)$ is its **boundary** $\mathrm{bd}(\mathbb{H}^n)$. Then for a compactly supported $(n-1)$-form $\omega \in \Omega_c^{n-1}(\mathbb{R}^n)$, we have

$$\int_{\mathbb{H}^n} d\omega = \int_{\mathbb{R}^{n-1}} \iota_{\mathbb{R}^{n-1}}^* \omega$$

This proof is left as an exercise for us – the idea is that for any $i > 1$, $\int_{\mathbb{H}^n} \frac{\partial f}{\partial x_i} dx_1 \cdots dx_n = 0$ by integration by parts, while for $i = 1$ we have

$$\int_{\mathbb{H}^n} \frac{\partial f}{\partial x_1} dx_1 \cdots dx_n = f(0, x_2, \cdots, x_n) dx_2 \cdots dx_n.$$

We'll now generalize this as promised:

Definition 162

Let X be an n-dimensional manifold. An open subset D of X is a **smooth domain** if for every $p \in \text{bd}(D)$, there exists a parameterization $\phi : U \to V$ centered at p such that ϕ maps $U \cap \mathbb{H}^n$ diffeomorphically onto $V \cap D$. (This is called a ***D*-adapted parameterization**.)

In other words, every point on the boundary of D looks locally like a point on the boundary of the half-space $\mathbb{H}^n$. We can notice a few important facts:

- If D is a smooth domain and $Z = \text{bd}(D)$, then because $\text{Bd}(\mathbb{H}^n)$ is isomorphic to $\mathbb{H}^{n-1}$, from the parameterization ϕ we have a diffeomorphism $U \cap \mathbb{R}^{n-1}$ onto $V \cap Z$. In other words, Z is a submanifold of X, and $\phi : U \cap \mathbb{R}^{n-1} \to V \cap Z$ is a parameterization of Z at p.
- Assuming now that X is an **oriented** manifold, we can assume that ϕ is an oriented parameterization (otherwise do the trick from last time, where we replace U with $U' = \{(x_1, \cdots, x_n) : (x_1, \cdots, -x_n) \in U\}$ and use the map $\phi'(x_1, \cdots, x_n) = \phi(x_1, \cdots, x_{n-1}, -x_n)$ in place of ϕ). We then find that the map $\phi : U \cap \mathbb{R}^{n-1} \to V \cap Z$ is **orientation-preserving**. This is in fact an intrisic property: if $\phi_1 : U_1 \to V$ and $\phi_2 : U_2 \to V$ are two oriented D-adapted parameterizations, and $f = \phi_2^{-1} \circ \phi_1$, then we get an orientation-preserving map of $U_1 \cap \mathbb{H}^n$ onto $U_2 \cap \mathbb{H}^n$, so if $\phi_1 : U_1 \cap \mathbb{H}^n$ is orientation-preserving then so is $\phi_2 : U_2 \cap \mathbb{H}^n$.

Theorem 163 (Stokes' theorem on manifolds)

Let X be an n-dimensional manifold, and let $\omega \in \Omega_c^{n-1}(X)$. Let D be a smooth domain, and let Z be the boundary of D. Then

$$\int_Z \iota_Z^* \omega = \int_D d\omega.$$

Proof. Recalling the definition of integration of forms on manifolds, by a partition of unity we may assume that $\omega \in \Omega_c^{n-1}(W)$, where W is a parameterizable open set satisfying one of the following three properties:

- $W \cap \overline{D} = \varnothing$,
- $W \subseteq \text{Int}(D)$,
- there exists a D-adapted parameterization $\phi : U \to W$, in particular meaning that $\phi(U \cap \mathbb{H}^{n-1}) \to W \cap Z$.

In the first case, both sides of Stokes' theorem are zero. In the second case, if $\phi : U \to W$ is an oriented parameterization, then because d commutes with the pullback map,

$$\int_D d\omega = \int_{\text{Int}(D)} \omega = \int_U \phi^* d\omega = \int_U d\phi^* \omega = 0.$$

Finally, in the third case, we have

$$\int_D d\omega = \int_{\mathbb{H}^n} \phi^* d\omega = \int_{\mathbb{H}^n} d\phi^* \omega,$$

and on the other hand we have

$$\int_Z \iota_Z^* \omega = \int_{\mathbb{R}^{n-1}} \iota_{\mathbb{R}^{n-1}}^* \phi^* \omega.$$

So now we can just apply the Euclidean version of Stokes' theorem to get the desired result. $\square$

We also get a simple but important corollary:

Theorem 164 (Divergence theorem)

Let X be an n-dimensional manifold, and let v be a C^∞ vector field on X. If $\omega \in \Omega_c^n(X)$, then $\iota(v)\omega \in \Omega_c^{n-1}(X)$ and $\mathcal{L}_v\omega \in \Omega_c^n(X)$ are well-defined. Then for any smooth domain D, we have

$$\int_D \mathcal{L}_v \omega = \int_Z \iota(v)\omega.$$

Proof. Recall the definition of $\mathcal{L}_v\omega = \iota(v)d\omega + d\iota(v)\omega$. Applying Stokes' theorem gives us $\int_D d\iota(v)\omega = \int_Z \iota(v)\omega$, which implies the result. $\square$

In particular, formulating this result over Euclidean space gives us the ordinary divergence theorem.

30 April 20, 2022

Today, we'll extend the discussion of **degree theory** to oriented manifolds, starting with an important generalization:

Theorem 165 (Poincaré lemma for manifolds)

Let X be an oriented, connected n-dimensional manifold, and let $\omega \in \Omega_c^n(X)$. Then the following are equivalent:

1. $\int_X \omega = 0$,
2. $\omega = d\mu$ for some $\mu \in \Omega_c^{n-1}(X)$.

The above result can be restated in an alternate formulation: if ω_1, ω_2 are both compactly supported n-forms on X, say that $\omega_1 \sim \omega_2$ if $\omega_1 - \omega_2 \in d\Omega_c^{n-1}(X)$. Then Theorem 165 is equivalent to saying that $\omega_1 \sim \omega_2$ if and only if $\int_X \omega_1 = \int_X \omega_2$.

Notice that Stokes' theorem on manifolds already tells us that the second property implies the first, so we just need to prove that if $\int_X \omega = 0$, then we can write $\omega = d\mu$. Also, we can fix some parameterizable open set $U_0 \subseteq X$, and let $\omega_0 \in \Omega_c^n(U_0)$ be an n-form compactly supported on U_0. Then we may equivalently prove the following:

Theorem 166

Let X be an oriented manifold, let $\omega \in \Omega_c^n(X)$, and let $\omega_0 \in \Omega_c^n(U_0)$ be compactly supported on a parameterizable open set U_0 of X. If $\omega \in \Omega_c^n(X)$ and $\int_X \omega = c$, then $\omega \sim c\omega_0$.

Proof. By the definition of integration of forms, we can (using a partition of unity) assume that $\omega \in \Omega_c^n(U)$ for some parameterizable open set. Additionally, if $c = 0$, then $\int_U \omega = 0$ implies that $\omega \in d\Omega_c^{n-1}(U)$ (**here we're using the Poincaré lemma on $\mathbb{R}^n$ by applying the pullback map**), meaning that $d\omega \sim 0$. Otherwise, if $c \neq 0$, replacing ω with $\frac{\omega}{c}$ allows us to assume without loss of generality that $\int_U \omega = 1$.

The main step of the proof from here (which should look familiar) is to choose a sequence of parameterizable open sets $U_1, \cdots, U_N$, so that $U_N = U$ and $U_i \cap U_{i-1} = \varnothing$ for all i (including $i = 1$). Then for each i, pick an

$\omega_i \in \Omega_c^n(U_{i-1} \cap U_i)$ such that $\int \omega_i = 1$. Since U_{i-1} is always a parameterizable open set, $\omega_{i-1} \sim \omega_i$ for all i (since they have the same integral), and additionally $\omega \sim \omega_N$. Thus $\omega_0 \sim \omega_1 \sim \omega_2 \sim \cdots \sim \omega_N \sim \omega$ as desired. □

Definition 167

Let X and Y be oriented connected n-dimensional manifolds, and let $f : X \to Y$ be a proper C^∞ map. Define the **degree** of f to be the topological invariant such that for every $\omega \in \Omega_c^n(Y)$, we have $\int_X f^*\omega = \deg(f) \int_Y \omega$.

To check why such a topological invariant exists, suppose we choose some $\omega_0 \in \Omega_c^n(Y)$ such that $\int_Y \omega_0 = 1$, and define $\deg(f)$ to be $\int_X f^*\omega_0$. Now suppose that $\int_Y \omega = c$. Then by the variant of the Poincaré lemma above, $\omega \sim c\omega_0$, meaning that $\omega - c\omega_0 = d\mu$ for some $\mu \in \Omega_c^{n-1}(Y)$. Applying the pullback map, we find that

$$\int f^*\omega = c \int f^*\omega_0,$$

which leads us to the assertion $\int f^*\omega = \deg f \int \omega$.

Proposition 168

Let X, Y, Z be oriented n-dimensional manifolds, and let $f : X \to Y$ and $g : Y \to Z$ be proper maps. Then $\deg(g \circ f) = \deg(f)\deg(g)$.

Proof. We've seen this result before for Euclidean space, and the same proof works: for any $\omega \in \Omega_c^n(Z)$, we have $(g \circ f)^*\omega = f^*g^*\omega$, so $\deg(g \circ f) \int_Z \omega = \int_X (g \circ f)^*\omega = \int_X f^*g^*\omega = \deg(f) \int_Y g^*\omega = \deg(f)\deg(g) \int_Z \omega$. □

We'll now explain the **stack of records** theorem, which should again look familiar. Let $f : X \to Y$ be a proper map between oriented manifolds, and let $q \in Y$ be a regular value of f, meaning that the preimage $f^{-1}(q)$ is a set $\{p_1, \cdots, p_N\}$ and there exist disjoint neighborhoods U_i of p_i and V of q such that each $f_i : U_i \to V$ is a diffeomorphism and $f^{-1}(V) = \bigcup_{i=1}^N U_i$. Choose $\omega \in \Omega_c^n(V)$ with $\int_V \omega = 1$. Then

$$\int_X f^*\omega = \sum_i \int_{U_i} f^ast_i\omega,$$

and because each $f_i : U_i \to V$ is a diffeomorphism, $\int_{U_i} f_i^*\omega$ will be either 1 or -1 depending on whether f is orientation-preserving or orientation-reversing. Thus we again get a formula for the degree:

Theorem 169

Let $f : X \to Y$ be a proper map between oriented manifolds, and let q be a regular value of f with preimage $\{p_1, \cdots, p_N\}$. Then $\deg(f) = \sum_i \sigma_{p_i}$, where σ_{p_i} is 1 (resp. -1) if the map f_i above is orientation-preserving (resp. orientation-reversing).

Furthermore, we have the same homotopy invariance that we previously discussed (which explains that we indeed have a topological invariant):

Proposition 170

Let $f_0, f_1 : X \to Y$ be two C^∞ proper maps, and let $F : X \times [0, 1] \to Y$ be a proper homotopy between f_0 and f_1 (meaning that $F(x, 0) = f_0(x)$, $F(x, 1) = f_1(x)$ and we have a proper C^∞ map). Then if the map f_t is defined via $f_t(x) = F(x, t)$ for all x, then f_t is proper, and for a compact set $A \subseteq Y$, there exists a compact set $B \subseteq X$ with $f_t^{-1}(A) \subset B$.

(The relevant ideas here are the following: let $\pi : X \times [0,1] \to X$ be the projection $(x,t) \mapsto x$. Then there exists a compact set B in X such that $f_t^{-1}(A) \subseteq B$, because the f_ts are proper maps.) And because the result above tells us that the degree of any proper map f_t is always an integer, and the degree is continuous, we have the following result:

Theorem 171

If f_0 and f_1 are properly homotopic, then $\deg(f_0) = \deg(f_1)$.

31 April 22, 2022

We reviewed and generalized the definition of **degree** last lecture – for a proper C^∞ map $f : X \to Y$ between oriented connected n-dimensional manifolds, we found that there was an invariant $\deg(f)$ such that $\int_X f^*\omega = \deg(f) \int_Y \omega$ for all compactly supported n-forms $\omega \in \Omega_c^n(Y)$. We also established some familiar properties of this degree, namely that $\deg(g \circ f) = \deg(f)\deg(g)$, that the degree is homotopy invariant, and that we can calculate the degree of a map f via the "stack of records theorem." The latter result basically states that if q is a regular value of the map f with preimage $\{p_1, \cdots, p_N\}$, then we can associate to it diffeomorphisms $f_i : U_i \to V$ between (disjoint) neighborhoods of p_i and q for each i (with the condition that $f^{-1}(V) = \bigcup_{i=1}^N U_i$ for the purposes of the proof). We then have

$$\deg(f) = \sum_i \sigma_{p_i},$$

where each σ_{p_i} is 1 or -1 depending on whether f_i is orientation-preserving or orientation-reversing (and recall that this proof came from the fact that for any $\omega \in \Omega_c^n(V)$ with $\int_V \omega = 1$, we have $\deg(f) = \int_X f^*\omega = \sum_i \int_{U_i} f^*\omega$). We can apply these discussions of degree to the **extendability problem** (left as an exercise to us):

Proposition 172

Suppose M is an oriented n-dimensional manifold, $D \subseteq M$ be a smooth domain with compact closure, and $X = \mathrm{Bd}(D)$. Then if $f : X \to X$ is a proper map, then f **cannot** be extended to a proper map on D, because any extendable map has $\deg(f) = 0$.

We'll discuss another application of degree theory today, **index theory** for vector fields.

Definition 173

Let D be a smooth domain in $\mathbb{R}^n$ such that D is connected with compact closure. Let $X = \mathrm{Bd}(D)$, and let v be a C^∞ vector field on $\mathbb{R}^n$ such that $v(p) \neq 0$ for all $p \in X$. From this map v, we get a map $X \to S^{n-1}$ sending p to $\frac{v(p)}{|v(p)|}$. (Notice that X and S^{n-1} are both compact $(n-1)$-dimensional manifolds). The **index** of v over D, denoted $\mathrm{ind}(v, D)$, is then the degree of the map g.

Theorem 174

Let D_1 be a smooth domain in $\mathbb{R}^n$ whose closure is contained in D. Then if v has no zeros in $D \setminus D_1$, then $\mathrm{ind}(v, D) = \mathrm{ind}(v, D_1)$.

Proof. Let $W = D \setminus D_1$, and notice that the map $f_v : \partial D \to S^{n-1}$ extends to a map $F : W \to S^{n-1}$ (given by $p \mapsto \frac{v(p)}{|v(p)|}$. We may write $\partial W = X \cup X_1$, where $X = \mathrm{Bd}(D)$ and $X_1 = \mathrm{Bd}(D_1)$ with its orientation reversed (it's good to draw a diagram for this). Now choose a differential $(n-1)$ form $\omega \in \Omega^{n-1}(S^{n-1})$ such that $\int_{S^{n-1}} \omega = 1$. Define the maps $f_0 = f_v$ and $f_1 = (f_1)_v$ to be the restrictions of F to the two boundaries X and X_1. By Stokes' theorem (in the last step below), and the fact that $d\omega = 0$ (it's an n-form on an $(n-1)$-dimensional space), we find that

$$0 = \int_W F^* d\omega = \int_W dF^*\omega = \int_X f\omega - \int_{X_1} f\omega = \deg(f) - \deg(f_1).$$

Thus the degrees of the maps f and f_1 are equal, meaning that the index of v over D and over D_1 are identical. $\square$

Using this result, if we now assume that v only has a finite set of zeros $\{p_1, \cdots, p_k\}$ over its domain D, then we may let $B(p, \varepsilon)$ denote an ε-ball around p, and we can let $D^0 = D \setminus \bigcup_{i=1}^k B(p_i, \varepsilon)$. Our map g above will then extend to a map $G : \overline{D}^0 \to S^{n-1}$ (on which p still maps to $\frac{v(p)}{|v(p)|}$, and the boundary of $\overline{D}^0$ is the union X and the boundaries of the balls $B(p_i, \varepsilon)$. Thus, putting everything together, Theorem 174 gives us the following **index theorem**:

$$\deg(g) = \sum_i \text{ind}(v, B(p_i, \varepsilon)) \implies \boxed{\text{ind}(v, D) = \sum_i \text{ind}(v, p_i)}.$$

32 April 25, 2022

We'll spend the last few lectures of the class on the last chapter of the textbook, discussing **de Rham theory**. We'll start today with some basic definitions – recall that for a manifold X, $\Omega^k(X)$ is the set of C^∞ k-forms on X, and $\Omega_c^k(X)$ is the set of compactly supported C^∞ k-forms on X.

Definition 175

Let X be an n-dimensional manifold. Define the vector spaces

$$Z^k(X) = \{\omega \in \Omega^k(X) : d\omega = 0\}, \quad B^k(X) = \{\omega \in \Omega^k(X) : \mu \in d\Omega^{k-1}(X).$$

Analogously, we may define $Z_c^k(X) = \{\omega \in \Omega_c^k(X) : d\omega = 0\}$ and $B_c^k(X) = \{\omega \in \Omega_c^k(X) : \mu \in d\Omega_c^{k-1}(X)\}$.

In particular, we can note that $B^k(X) \subseteq Z^k(X)$ (because of exactness, coming from the fact that $d^2 = 0$), so the next definition also makes sense:

Definition 176

The ***k*th de Rham cohomology group of *X*** is given by $H^k(X) = Z^k(X)/B^k(X)$, and analogously we also define $H_c^k(X) = Z_c^k(X)/B_c^k(X)$.

Our goal will be to understand these objects and to understand methods for computing the cohomology groups. We'll use the notation that for any $\omega \in Z^k(X)$, $[\omega]$ is the image of ω in $H^k(X)$ (and similarly for the same statement for compactly supported forms).

Proposition 177

We have the following facts:

1. If X is connected, then $H^0(X) = \mathbb{R}$.
2. If X is n-dimensional, then $H^k(X) = H_c^k(X) = 0$ if $k > n$ or $k < 0$.
3. If X is connected and oriented, then $Z_c^n(\Omega) = \Omega_c^n(X)$.

Proof. For (1), because a 0-form on X is a C^∞ function and d of a function on a connected X is only zero if it is constant, $H^0(X)$ must consist of the set of constant functions. (2) follows from the fact that we only have nontrivial k-forms for $0 \le k \le n$. Finally, (3) holds because $d\omega$ of any n-form is zero, so all n-forms work. $\square$

If we now think about the integration operation $\int : \Omega_c^n(X) \to \mathbb{R}$, sending a form ω to $\int_X \omega$, then recall that we've previously proved that

$$\omega \in d\Omega_c^{n-1}(X) \iff \int_X \omega = 0.$$

Quotienting out by $B_c^n(X)$, we thus get a bijective map between $H_c^n(X)$ and $\mathbb{R}$ in this case (by associating to each form its integral on X).

Proposition 178

Let $U \subseteq \mathbb{R}^n$ be an open rectangle. Then for all $k > 0$, $H^k(U) = \{0\}$.

(This was on our homework, in which we proved that every form $\omega \in Z^k(U)$ is exact.)

Proposition 179

Let $U \subseteq \mathbb{R}^n$ be an open rectangle. Then for all $k < n$, $H_c^k(U) = \{0\}$.

(This was actually also on our homework – we proved that if ω is closed, then $\omega = d\mu$ for some $\mu \in \Omega_c^{k-1}(U)$, so in fact $Z^k(U) = B^k(U)$.) We'll next discuss the **Poincaré lemma for manifolds**: let X be an n-dimensional manifold, and at a point $p \in X$ let $\phi : U \to V$ be a parameterization sending 0 to p. We may assume (by restriction of ϕ) that U is an open rectangle. Then if $\alpha \in Z^k(X)$ for some $k > 0$, and α_0 is α restricted to V, then

$$d\alpha = 0 \implies d\alpha_0 = 0 \implies d\phi^*\alpha_0 = 0 \implies \alpha_0 = (\phi^{-1})^* d\beta = d(\phi_1^{-1})^*\beta,$$

and thus α restricted to V is an element of $B^k(X)$, because it's d of a $(k-1)$-form. Thus $H^k(V) = 0$ for all $k > 0$, and thus **every closed k-form, restricted to an open set, is exact**. Now that we've discussed some local results, we'll turn to global ones:

Theorem 180

For the sphere S^n, we have $H^0(S^n) = H^n(S^n) = \mathbb{R}$ and $H^k(S^n) = 0$ for $0 < k < n$.

Proof. Because S^n is compact and connected, we've already proved that $H^0(S^n) = H^n(S^n) = \mathbb{R}$ from our earlier remarks. It now suffices to consider the situation for $0 < k < n$. Let p_0 be the point $(0, 0, \cdots, 0, 1) \in S^n$, and let $\gamma : S^n - \{p_0\} \to \mathbb{R}^n$ be the **stereographic projection**, meaning that for any point $q \in S^n$, we draw a line between p_0 and q and consider where it intersects the hyperplane $\mathbb{R}^n \times \{0\}$ in $\mathbb{R}^{n+1}$. This is a bijective map (since for any point in $\mathbb{R}^n$ we can draw a line between it and p_0, and it will intersect the sphere at one other point q besides p_0).

Now for any $\alpha \in Z^k(S^n)$, there exists a neighborhood V of p and some $\beta \in \Omega^{k-1}(V)$ such that $\alpha = d\beta$ on V (by our discussion above for parameterizable open sets V). Letting $\rho \in C_0^\infty(V)$ be a "bump function" such that $\rho = 1$ on a neighborhood of p, we can then replace α with $\tilde{\alpha} = \alpha - d\rho\beta$, meaning that

$$\tilde{\alpha} \in \Omega_c^k(S^n - \{p_0\}) = \Omega_0^k(\mathbb{R}^n).$$

In particular, $\tilde{\alpha} = d\tilde{\beta}$ for some $\tilde{\beta} \in \Omega_c^{k-1}(S^n - \{p_0\})$. If we then write $\tilde{\tilde{\beta}} = \rho\beta + \tilde{\beta}$, we find that $d\tilde{\tilde{\beta}} = \alpha$, so $\alpha \in B^k(S^n)$. Therefore we mod out $Z^k(S^n)$ by itself and thus $H^k(X) = 0$. $\square$

We'll close by thinking about **functoriality** of these groups: if X and Y are C^∞ manifolds, and $f : X \to Y$ is a C^∞ map, then we have a corresponding map $f^* : \Omega^k(Y) \to \Omega^k(X)$. Since $f^* d\omega = df^*\omega<$ we find that we also get maps $f^* : Z^k(Y) \to Z^k(X)$ and $B^k(Y) \to B^k(X)$. We thus get an induced map

$$f^\sharp : H^k(Y) \to H^k(X)$$

between the kth de Rham cohomology groups of X and Y: specifically, the map sends $f^\sharp[\omega]$ to $[f^*\omega]$. In addition, if $f : X \to Y$ is a proper map, then preimages of compact sets are compact, so $\omega \in \Omega^k_c(Y) \implies f^*\omega \in \Omega^k_c(X)$. Thus, even within the compactly supported forms, we also get a map $f^\sharp : H^k_c(Y) \to H^k_c(X)$, again sending $[\omega]$ to $[f^*\omega]$. We'll be making use of these facts in the coming lectures, and next time we'll show that these cohomology groups are finite-dimensional as long as X is comacp.

33 April 27, 2022

Last lecture, we started introducing de Rham theory: recall that the de Rham cohomology groups of X are defined in terms of the d map $d : \Omega^k(X) \to \Omega^{k+1}(X)$ on k-forms of X. Specifically, if $Z^k(X)$ is the kernel of the d map, and $B^{k+1}(X)$ is the image, then we define $H^k(X) = Z^k(X)/B^k(X)$ (and also similarly define a version of this for compactly supported forms). The notation we'll use, going forward, is that $[\omega]$ denotes the image of a form $\omega \in Z^k(X)$ (resp. $Z^k_c(X)$) in $H^k(X)$ (resp $H^k_c(X)$).

We'll start to understand some techniques that can be used for computing these groups, starting with the **Mayer–Vietoris theorem**.

Definition 181

Let $C^1, C^2, \cdots$ be vector spaces, and let $d_i : C^i \to C^{i+1}$ be maps between those sequences (we will often just denote these maps as d to simplify notation). We call the sequence $C^0 \xrightarrow{d_0} C^1 \xrightarrow{d_1} C^2 \cdots$ is called a **complex** if $d_{i+1} \circ d_i = 0$ for all i.

Example 182

Because $d^2 = 0$ for the d map we've been discussing, we have the **de Rham complex**

$$0 \to \Omega^0(X) \xrightarrow{d} \Omega^1(X) \xrightarrow{d} \Omega^2(X) \cdots,$$

as well as a similar complex for compactly supported forms $0 \to \Omega^0_c(X) \xrightarrow{d} \Omega^1_c(X) \xrightarrow{d} \Omega^2_c(X) \cdots$.

We may also discuss **morphisms between complexes**:

Definition 183

If C^k_1, d and C^k_2, d form two different complexes, then a **morphism** between the complexes is a map $\alpha : C^k_1 \xrightarrow{\alpha} C^k_2$ such that $d\alpha = \alpha d$.

In other words, we can imagine writing out our complexes in two rows and having α be maps from the top to the bottom. Then the "squares" in the diagram will commute:

$$\begin{array}{ccccccc}
C_1^0 & \xrightarrow{d} & C_1^1 & \xrightarrow{d} & C_1^2 & \xrightarrow{d} & \cdots \\
\downarrow \alpha & & \downarrow \alpha & & \downarrow \alpha & & \\
C_2^0 & \xrightarrow{d} & C_2^1 & \xrightarrow{d} & C_2^2 & \xrightarrow{d} & \cdots
\end{array}$$

We'll now explain the general abstract definition of a cohomology group:

Definition 184

If C, d is a complex, then the **kth cohomology group** $H^k(C)$ of the complex C is given by defining $Z^i = \{c \in C^i : dc = 0\}$ and $B_{i+1} = dC^i$ and setting $H^k(C) = Z^i/B^i$.

In particular, a morphism of complexes will induce a map $\alpha : H^k(C_1) \to H^k(C_2)$ (we can check that this is indeed well-defined even though we have quotient groups because d and α commute).

Definition 185

Let $V_0, V_1, \cdots$ be a sequence of vector spaces, and let $\alpha_i : V_i \to V_{i+1}$ be maps between those spaces. The sequence $V_0 \xrightarrow{\alpha_0} V_1 \xrightarrow{\alpha_1} V_2 \to \cdots$ is **exact** if $\ker \alpha_i = \text{im } \alpha_{i-1}$ for all i. A **short exact sequence** is an exact sequence of the form $\{0\} \to V_1 \xrightarrow{\alpha_1} V_2 \xrightarrow{\alpha_2} V_3 \to 0$ (meaning that α_1 is injective, α_2 is surjective, and $\ker(\alpha_2) = \text{im }(\alpha_1)$).

With all of this notation, we're now ready to discuss the main result of today's lecture. Here's the setup: let $\{0\} \xrightarrow{d} C_r^1 \xrightarrow{d} C_r^2 \xrightarrow{d} \cdots$ be a complex for $r = 1, 2, 3$ (meaning that $d^2 = 0$), and suppose that $0 \to C_1^k \xrightarrow{i} C_2^k \xrightarrow{j} C_3^k \to 0$ is a short exact sequence for each k, meaning that the image of each i map is the kernel of the j map. Then we may draw the following commutative diagram (which extends further both up and down):

$$\begin{array}{ccccccccc}
0 & \longrightarrow & C_1^{k+1} & \xrightarrow{i} & C_2^{k+1} & \xrightarrow{j} & C_3^{k+1} & \longrightarrow & 0 \\
 & & \uparrow d & & \uparrow d & & \uparrow d & & \\
0 & \longrightarrow & C_1^{k} & \xrightarrow{i} & C_2^{k} & \xrightarrow{j} & C_3^{k} & \longrightarrow & 0 \\
 & & \uparrow d & & \uparrow d & & \uparrow d & & \\
0 & \longrightarrow & C_1^{k-1} & \xrightarrow{i} & C_2^{k-1} & \xrightarrow{j} & C_3^{k-1} & \longrightarrow & 0
\end{array}$$

Then defining $Z_r^k = \{c \in C_r^k : dc = 0\}$ and $B_r^k = dC_r^{k-1}$ for each k, r, we find that $i(B_1^k) \subseteq B_2^k$ and $i(Z_1^k) \subseteq Z_1^k$ (because i and d commute), giving us an induced map $i_\sharp : H^k(C_1) \to H^k(C_2)$. We similarly get an induced map $j_\sharp : H^k(C_2) \to H^k(C_3)$. We may check that $\ker j_\sharp = \text{im } i_\sharp$, meaning that $H^k(C_1) \xrightarrow{i_\sharp} H^k(C_2) \xrightarrow{j_\sharp} H^k(C_3)$ is exact. It turns out to **not** be short exact (we can't append a 0 to the beginning and end of it and still have exactness), but we instead have that the following diagram commutes:

$$\begin{array}{ccccc}
0 & & 0 & & \\
\uparrow d & & \uparrow d & & \\
C_1^{k+1} & \xrightarrow{i} & C_2^{k+1} & \xrightarrow{j} & C_3^{k+1} \\
 & & \uparrow d & & \uparrow d \\
 & & C_2^{k} & \xrightarrow{i} & C_3^{k}
\end{array}$$

We may check (by exploring the implications of this diagram) that we then get a **correspondence** between an element $c_3^k \in Z_3^k$ and an element $c_1^{k+1} \in Z_1^{k+1}$, so we induce a map $\delta : H^k(C_3) \to H^{k+1}(C_1)$, known as the **coboundary map**. We then get our result:

Theorem 186 (Mayer–Vietoris)
With the notation above, we have an exact sequence of the form

$$\cdots \to H^{k-1}(C_1) \xrightarrow{i_\sharp} H^{k-1}(C_2) \xrightarrow{j_\sharp} H^{k-1}(C_3) \xrightarrow{\delta} H^k(C_1) \xrightarrow{i_\sharp} H^k(C_2) \xrightarrow{j_\sharp} H^k(C_3) \xrightarrow{\delta} H^{k+1}(C_1) \to \cdots .$$

The proof of this requires a lot of commutative diagram chasing (keeping track of exactness and images and kernels), but we can read about it on our own. And while this sequence does, in principle, extend infinitely, recall that we've demonstrated that the H^k cohomology groups are eventually zero for finite-dimensional manifolds, so the sequence is only interesting within a finite range.

Corollary 187
Let X be a manifold, and let U_1 and U_2 be open sets in X. If we let $C_1^k = \Omega^k(U_1 \cup U_2)$, $C_2^k = \Omega^k(U_1) \oplus \Omega^k(U_2)$, and $C_3^k = \Omega^k(U_1 \cap U_2)$, then we (may check that we) get short exact sequences $0 \to C_1^k \xrightarrow{i} C_2^k \xrightarrow{j} C_3^k \to 0$, where i is essentially restriction of a k-form on $U_1 \cup U_2$ to U_1 and U_2 and j sends $\omega_1 \oplus \omega_2$ to the difference of ω_1 and ω_2 on $U_1 \cap U_2$. Then we get along exact sequence of cohomology groups

$$\cdots \xrightarrow{\delta} H^k(U_1 \cup U_2) \xrightarrow{i_\sharp} H^k(U_1) \oplus H^k(U_2) \xrightarrow{j_\sharp} H^k(U_1 \cap U_2) \xrightarrow{\delta} H^{k+1}(U_1 \cup U_2) \xrightarrow{i_\sharp} \cdots$$

and also a similar result for compactly supported cohomology groups.

In particular, this often allows us to calculate the cohomology groups for $U_1 \cup U_2$ in terms of the cohomology groups of U_1, U_2, and $U_1 \cap U_2$.

34 April 29, 2022

Last lecture, we described the Mayer-Vietoris theorem, which in particular gives us an exact sequence involving de Rham cohomology groups of $U_1, U_2, U_1 \cap U_2$, and $U_1 \cup U_2$ (where U_1 and U_2 are open subsets of an n-dimensional manifold X). More explicitly, if we let $i : \Omega^k(U_1 \cup U_2) \to \Omega^k(U_1) \oplus \Omega^k(U_2)$ be the map which sends a k-form ω on $U_1 \cup U_2$ to the direct sum of the ω restricted to U_1 and U_2, and we let $j : \Omega^k(U_1) \oplus \Omega^k(U_2) \to \Omega^k(U_1) \cap \Omega^k(U_2)$ be the map that sends (ω_1, ω_2) to $\omega_1|_{U_1 \cap u_2} - \omega_2|_{U_1 \cap U_2}$, then we satisfy the requirements on the large commutative diagram that we drew last time, namely that $dj(\omega_1, \omega_2) = j(d\omega_1, d\omega_2)$ (that is, d and j commute), $di(\omega) = id\omega$ (that is, d and i commute), and composing the i and j maps gives us a short exact sequence. We'll reproduce that diagram here, but now in terms of the Ω_i spaces instead of vector spaces C^k from general complexes:

$$\begin{tikzcd}
0 \arrow[r] & \Omega^{k+1}(U_1 \cup U_2) \arrow[r, "i"] & \Omega^{k+1}(U_1) \oplus \Omega^{k+1}(U_2) \arrow[r, "j"] & \Omega^{k+1}(U_1 \cap U_2) \arrow[r] & 0 \\
0 \arrow[r] & \Omega^{k}(U_1 \cup U_2) \arrow[r, "i"] \arrow[u, "d"] & \Omega^{k}(U_1) \oplus \Omega^{k}(U_2) \arrow[r, "j"] \arrow[u, "d"] & \Omega^{k}(U_1 \cap U_2) \arrow[r] \arrow[u, "d"] & 0 \\
0 \arrow[r] & \Omega^{k-1}(U_1 \cup U_2) \arrow[r, "i"] \arrow[u, "d"] & \Omega^{k-1}(U_1) \oplus \Omega^{k-1}(U_2) \arrow[r, "j"] \arrow[u, "d"] & \Omega^{k-1}(U_1 \cap U_2) \arrow[r] \arrow[u, "d"] & 0
\end{tikzcd}$$

From here, "diagram chasing" lets us convert this commutative diagram into a statement about the H^k cohomology groups, namely that we have a long exact sequence

$$\cdots \xrightarrow{\delta} H^k(U_1 \cup U_2) \xrightarrow{i_\sharp} H^k(U_1) \oplus H^k(U_2) \xrightarrow{j_\sharp} H^k(U_1 \cap U_2) \xrightarrow{\delta} H^{k+1}(U_1 \cup U_2) \xrightarrow{i_\sharp} \cdots ,$$

where the sequence continues on between cohomology groups of $U_1 \cup U_2$, then of the direct sum of the groups for U_1 and U_2, then of the groups of $U_1 \cap U_2$, incrementing the index k.

Today, we'll see some applications of the Mayer-Vietoris theorem:

Theorem 188

Suppose U is a convex open set in $\mathbb{R}^n$. Then $H^0(U) = \mathbb{R}$ and $H^k(U) = 0$ for all $k > 0$.

Proof. Fix some point $p_0 \in U$, and for each $t \in [0, 1]$, let $f_t : U \to U$ be the map

$$f_t(p) = (1 - t)p + tp_0.$$

This map is well-defined because U is convex, and we may notice that $f_0(p) = p$ is the identity map, while $f_1(p) = p_0$ is the constant map. This gives us a homotopy between the two maps, and thus by homotopy invariance, we get an induced map $f_t^\sharp : H^k(U) \to H^k(U)$ independent of t. Thus the cohomology groups of a convex set must be the same as those of a point, which can be checked easily to be $H^0(U) = \mathbb{R}$ and $H^k(U) = 0$ for all $k > 0$. $\square$

Definition 189

Let X be an n-dimensional manifold, and suppose $\mathbb{U} = \{U_\alpha : \alpha \in \mathcal{I}\}$ is a covering of X by open sets (for some index set $\mathcal{I}$). Call $\mathbb{U}$ a **good cover** if for every finite subset $\alpha_1, \cdots, \alpha_k \in \mathcal{I}$, $U_{\alpha_1} \cap \cdots \cap U_{\alpha_k}$ is either empty or diffeomorphic to a convex subset of $\mathbb{R}^n$.

This is basically a "niceness" condition, and it in particular requires the U_αs to each be diffeomorphic to a convex subset of $\mathbb{R}^n$.

Example 190

Suppose X is an open subset of $\mathbb{R}^n$, and for every $p \in X$, let U_p be a convex neighborhood of p. Then $\{U_p : p \in X\}$ is a good cover (because the intersection of convex sets is convex).

We'll now generalize the notion of convexity on Euclidean space to an analogous version for manifolds, remembering that we always assume in this class our manifolds are subsets of $\mathbb{R}^N$ for some large N:

Definition 191

Let $X \subseteq \mathbb{R}^N$ be an n-dimensional manifold. For any $p \in X$, let T_pX be the tangent space of X at p, where we view T_pX as a subset of $T_p\mathbb{R}^N = \mathbb{R}^N$. Then let L_p be the set

$$L_p = \{p + v : v \in T_pX\}.$$

Let $\pi_p : X \to L_p$ be the orthogonal projection of X onto the (affine) hyperplane L_p. Then an open set U in X is **convex** if for every $p \in U$, π_p maps U bijectively onto a convex open set in L_p (which we can view as equivalent to Euclidean space).

(In particular, in the definition above, if X is a one-dimensional manifold, we can think of L_p as the literal tangent line to X at p, and more generally we can imagine the hyperplane formed by all tangent vectors at p in $\mathbb{R}^N$.)

Proposition 192

Let X be a manifold as in the above definition, and for each $p \in X$, let $B_\varepsilon(p)$ be an open ball of radius ε in L_p. Then for any $p \in X$, we may always pick a small enough ε such that $\pi(B_\varepsilon(p))$ is convex.

This result is left as an exercise to us, but the idea is to let $U_p = \pi_p^{-1}(B_\varepsilon(p))$ and notice that $\pi_q(U_p)$ is very close (for small ε) to the original copy $B_\varepsilon(p)$. And the idea is that a small enough perturbation of a convex set is still convex, but we won't write out the details. We also have the following result:

Proposition 193

If U_1 and U_2 are two convex open sets of X, then $U_1 \cap U_2$ is also convex.

(This basically follows from the fact that if $\pi_p(U_1)$ and $\pi_p(U_2)$ are convex in L_p, then so is their intersection.) This discussion of convexity can then lead us to the following result (by taking the $B_\varepsilon(p)$s for each point p of small enough ε so that each one is convex in X):

Corollary 194

Every manifold admits a good cover.

Definition 195

A manifold has **finite topology** if it admits a finite good cover $\{U_1, \cdots, U_m\}$.

Our next few lectures will be to explore the following idea: suppose we have a finite good cover $\{U_1, \cdots, U_m\}$ of a manifold X. We'll see that the cohomology groups of X can be read off from **intersection properties** of the U_is, and that will give us a concrete way to compute the groups $H^k(X)$. (This is known as Čech cohomology, and it will turn out that that theory will be isomorphic to de Rham cohomology theory.)

Theorem 196

If X admits a finite good cover, then $\dim H^k(X) < \infty$ for all k (all cohomology groups have finite dimension).

To show this, we first mention a lemma which we can prove as an exercise:

Lemma 197

Suppose $V_1 \xrightarrow{\alpha} V_2 \xrightarrow{\beta} V_3$ are linear maps such that $\operatorname{im} \alpha = \ker \beta$. Then if V_1 and V_3 are finite-dimensional, then so is V_2.

Proof of Theorem 196. We induct on the size of the good cover. The base case $m = 1$ is clear, because then we can use our convexity results from above. For the inductive step, if we have a good cover consisting of $\{U_1, \cdots, U_{m-1}\}$ and U_m, the inductive hypothesis tells us that the cohomology groups of $U_1 \cup \cdots \cup U_{m-1}$, U_m, and their intersection are all finite. Now we can apply Mayer-Vietoris and Lemma 197 to show the result (each cohomology group of $U_1 \cup \cdots \cup U_n$ is surrounded by two finite-dimensional spaces, and we have exactness at that group because it's part of a long exact sequence). More explicitly, we have exactness

$$H^{k-1}((U_1 \cup \cdots \cup U_{n-1}) \cap U_n) \to H^k(U_1 \cup \cdots \cup U_n) \to H^k(U_1 \cup \cdots \cup U_{n-1}) \oplus H^k(U_n),$$

and now because $U_1\cup\cdots\cup U_{n-1}$, U_n, and $(U_1\cup\cdots\cup U_{n-1})\cap U_n$ all admit good covers by at most $(n-1)$ open sets (for the latter using the open sets $U_i\cap U_n$ for $1\le i\le n-1$), all of the other cohomology groups except $H^k(U_1\cup\cdots\cup U_n)$ are finite-dimensional and thus the middle one is as well. □

In the coming lectures, we'll see how to use Mayer-Vietoris more explicitly and get a formula for these cohomology groups!

35 May 2, 2022

Today's lecture will work towards connecting de Rham cohomology (involving the exterior d operation) to the new theory that we started exploring last time, **Čech cohomology**. Recall from last time that this connection was introduced in the following way: a **good cover** of an n-dimensional manifold X by a family of open sets $\mathbb{U}=\{U_\alpha:\alpha\in\mathcal{I}\}$ is a covering where any finite intersection of the U_αs is either empty or diffeomorphic to a convex open set of $\mathbb{R}^n$. We showed last time (by Mayer-Vietoris) that if X admits a good cover by a finite collection of open sets, then all cohomology groups $H^k(X)$ are finite-dimensional. This then motivated us to think about studying cohomology via intersection properties of the U_αs, and we'll start that construction today.

Definition 198

Let $\mathbb{U}=\{U_1,U_2,\cdots,U_N\}$ be a good cover of X, and let N^k be the set of all multi-indices $I=(i_0,\cdots,i_k)\in\{1,\cdots,N\}^k$ **such that** $U_I=U_{i_1}\cap\cdots\cap U_{i_k}$ is nonempty. The **nerve** of a cover $\mathbb{U}$ is the collection of sets $N^0,N^1,N^2,\cdots$.

Definition 199

The set of **Čech cochains** $\check{C}^k(\mathbb{U},\mathbb{R})$ is the set of maps $c:N^k\to\mathbb{R}$, and the **Čech coboundary map** is a map $\delta:\check{C}^{k-1}(\mathbb{U},\mathbb{R})\to\check{C}^k(\mathbb{U},\mathbb{R})$ defined as follows: for any $c\in\check{C}^{k-1}(\mathbb{U},\mathbb{R})$ and any $I\in N^k$, we have

$$\delta c(I)=\sum_{r=0}^{k}(-1)^r c(I_r),$$

where I_r is the multi-index obtained from I by deleting the index i_r (that is, the $(r+1)$th index).

Proposition 200

The map $\delta\delta:\check{C}^{k-1}\to\check{C}^{k+1}$ is the zero map.

Proof. By applying the definition twice, we have (carefully keeping track of indices, having $c\in\check{C}^{k-1}$ and $I\in N^{k+1}$)

$$\delta\delta c(I)=\sum_{r=0}^{k+1}(-1)^r\delta c(I_r)=\sum_{r=0}^{k+1}\left(\sum_{s<r}(-1)^r(-1)^s c(I_{r,s})+\sum_{s>r}(-1)^r(-1)^{s-1}c(I_{r,s})\right),$$

with the two cases coming from the fact that indices after r get shifted by one in I_r compared to I. But then each $I_{r,s}$ shows up once in each subsum with opposite $(-1)^m$ prefactors, meaning that everything cancels and we indeed get 0. □

This implies that we have a complex

$$0\to\check{C}^0(\mathbb{U},\mathbb{R})\to\check{C}^1(\mathbb{U},\mathbb{R})\to\check{C}^2(\mathbb{U},\mathbb{R})\to\cdots,$$

and we can now finally define Čech cohomology groups:

Definition 201

The ***kth* Čech cohomology group** of an open covering $\mathbb{U}$ of X is given by

$$\check{H}^k(\mathbb{U}, \mathbb{R}) = \left(\ker \delta : \check{C}^k \to \check{C}^{k+1}\right) / \left(\text{im } \delta : \check{C}^{k-1} \to \check{C}^k\right).$$

Our goal in this last section of 18.952 is essentially to prove that

$$\check{H}(\mathbb{U}, \mathbb{R}) \cong H^k(X)$$

(in other words, that the Čech cohomology group and the de Rham cohmology groups are isomorphic). Before that, we'll need to make a generalization of these cochains:

Definition 202

A **Čech cochain of degree k with values in Ω^ℓ** is a map c which assigns to each $I \in N^k$ a ℓ-form $c(I) \in \Omega^\ell(U_I)$.

The set of Čech cochains of degree k with values in Ω^ℓ is the vector space $\check{C}^k(\mathbb{U}, \Omega^\ell)$, with addition given by pointwise addition: for any $c_1, c_2 \in \check{C}^k(\mathbb{U}, \Omega^\ell)$, $c_1 + c_2$ is the cochain which sends I to $c_1(I) + c_2(I)$. We also get a generalized coboundary map:

Definition 203

Let $I \in N^k$. For any $0 \le r \le k$, define the restriction map

$$\gamma_r : \Omega^\ell(U_{I_r}) \to \Omega^\ell(U_I)$$

(this makes sense because U_I is contained in U_{I_r}). Then define the coboundary map (mimicking the definition above) by setting

$$\delta c(I) = \sum_{r=0}^{k} (-1)^r \gamma_r c(I_r)$$

for any $c \in \check{C}^{k-1}(\mathbb{U}, \Omega^\ell)$ and $I \in N^k$.

We may check that $\delta\delta = 0$ for this more general definition, using basically the same proof as Proposition 200. We can now start computing cohomology groups by looking at small k: a $c \in \check{C}^0(\mathbb{U}, \Omega^\ell)$ is a map that assigns to each $1 \le i \le N$ an element $\omega_i \in \Omega^\ell(U_i)$. From the definition, we see that $\delta c = 0$ if and only if

$$\omega_i|_{U_i \cap U_j} - \omega_j|_{U_i \cap U_j} = 0 \quad \forall (i, j) \in N^1$$

In other words, the kernel of the map $\delta : \check{C}^0(\mathbb{U}, \Omega^\ell) \to \check{C}^1(\mathbb{U}, \Omega^\ell)$ is $\Omega^\ell(X)$ itself (the set of forms that can be defined consistently on all of the U_is at once). Next time, we'll extend this to see that we in fact get an exact sequence of the form

$$0 \to \Omega^\ell(X) \to \check{C}^0(U, \Omega^\ell) \to \check{C}^1(\mathbb{U}, \Omega^\ell) \to \cdots$$

and see how that leads us to the isomorphism we're after.

36 May 4, 2022

Last lecture, we defined the Čech cohomology groups $\check{H}^k(\mathbb{U}, \mathbb{R})$ (where $\mathbb{U} = \{U_0, \cdots, U_k\}$ is a finite open cover of a manifold X). The motivation for this cohomology theory is that it is easier to compute the Čech cohomology groups from the definition than the de Rham cohomology groups – recall that if N^k is the set of all multi-indices $(i_0, \cdots, i_k) \in \{1, \cdots, N\}^k$ such that $U_{i_0} \cap \cdots \cap U_{i_k}$ is nonempty, then Čech cohomology is defined in terms of the Čech cochains $\check{C}^k(\mathbb{U}, \mathbb{R})$ (the maps from N^k to $\mathbb{R}$) and the Čech coboundary map $\delta c(I) = \sum_{r=0}^k (-1)^r c(I_r)$. We showed that $\delta\delta = 0$, forming a complex $0 \to \check{C}^0(\mathbb{U}, \mathbb{R}) \to \check{C}^1(\mathbb{U}, \mathbb{R}) \to \cdots$, and that gives us the Čech cohomology groups $\check{H}^k(\mathbb{U}, \mathbb{R}) = (\ker \delta : \check{C}^k \to \check{C}^{k+1})/(\text{im } \delta : \check{C}^{k-1} \to \check{C}^k)$.

Today, we'll explain why the H^k and $\check{H}^k$ cohomology groups are in fact the same. Last time, we generalized Čech cochains to take values in Ω^ℓ (for some integer ℓ) – such cochains are maps c that assign to each multi-index $I \in N^k$ an element $c(I) \in \Omega^\ell(U_I)$. The sets $\check{C}^k(\mathbb{U}, \Omega^\ell)$ are vector spaces, and we defined a generalized δ map $\delta c(I) = \sum_{r=0}^k (-1)^r \gamma_r c(I_r)$ (where γ_r is the restriction map of a form from U_{I_r} to U_I). Since $\delta^2 = 0$ here as well, we are now motivated to consider the cohomology groups associated to this complex (of cochains with values in Ω^ℓ).

Last time, we mentioned that an element $c \in \check{C}^0(\mathbb{U}, \Omega^\ell)$ is a map that assigns to each index $1 \le i \le N$ an ℓ-form $\omega_\ell \in \Omega^\ell(U_i)$. Furthermore, $\delta c = 0$ for such a map if and only if for every $(i, j) \in N^1$, $\gamma_i c(j) - \gamma_j c(i) = 0$, meaning that there is some unique ℓ-form $\omega \in \Omega^\ell(X)$ such that ω restricted to each U_i is ω_i. Thus the kernel of the map from $\check{C}^0(\mathbb{U}, \Omega^\ell) \to \check{C}^1(\mathbb{U}, \Omega^\ell)$ is the set of all differential k-forms on X, $\Omega^\ell(X)$. Inserting this into the complex, we now arrive at the result from the end of last lecture, which is that we have a sequence

$$0 \to \Omega^\ell(X) \to \check{C}^0(\mathbb{U}, \Omega^\ell) \xrightarrow{\delta} \cdots .$$

Theorem 204

The sequence above is exact for any ℓ.

Proof. Let $\phi_i \in C_0^\infty(X)$ be a partition of unity, such that the support of ϕ_i is contained in U_i for each i. Define the map $Q : \check{C}^{k+1}(\mathbb{U}, \Omega^\ell) \to \check{C}^k(\mathbb{U}, \Omega^\ell)$ (notice that this lowers the order k instead of raising it like δ does) via

$$Qc(I) = \sum_{i=1}^N \phi_i c(i, I),$$

where we extend each $\phi_i c$ to be zero outside $U_i \cap U_I$. We may check that $(Q\delta + \delta Q)c(I) = c(I)$ for any I, and now if we fix k, we can define the map $d : C^k(\mathbb{U}, \Omega^\ell) \to C^k(\Omega^{\ell+1})$ via $dc(I) = d(c(I))$. Since $d^2 = 0$ on forms, we thus get a valid complex

$$\check{C}^k(\mathbb{U}, \Omega^0) \to \check{C}^k(\mathbb{U}, \Omega^1) \to \check{C}^k(\mathbb{U}, \Omega^2) \to \cdots$$

via the d map. Additionally, because 0-forms $c(I)$ are elements of $C^\infty(U_I)$, $dc(I) = 0$ if and only if $c(I)$ is a constant, and thus $dc = 0$ if and only if $c \in \check{C}^k(\mathbb{U}, \mathbb{R})$. Thus we get an **exact sequence**

$$0 \to \check{C}^k(\mathbb{U}, \mathbb{R}) \to \check{C}^k(\mathbb{U}, \Omega^0) \to \check{C}^k(\mathbb{U}, \Omega^1) \to \cdots ,$$

because if $dc = 0$ for some $c \in \check{C}^k(\mathbb{U}, \Omega^\ell)$, then $c(I) \in \Omega^\ell(U_I)$ must be closed for any I, and because U_I is diffeomorphic to a convex open set (by definition) we have $c(I) = dc(I)'$ for some $c(I)' \in \Omega^{\ell-1}(U_I)$. (This implies that the image of one d map is the kernel of the next.) □

It is now left as an exercise to check that the d and the δ maps in fact commute, meaning that for any $c \in \check{C}^k(\mathbb{U}, \Omega^\ell)$

we have $d\delta c = \delta dc$ (this can be checked from the definitions). From all of the properties we've verified, we thus get a commutative diagram of the following form (where empty arrows are basically inclusion maps):

$$\begin{tikzcd}
 & \vdots & \vdots & \vdots & \vdots & \\
0 \arrow[r] & \Omega^2(X) \arrow[r] \arrow[u, "d"] & \check{C}^0(\mathbb{U}, \Omega^2) \arrow[r, "\delta"] \arrow[u, "d"] & \check{C}^1(\mathbb{U}, \Omega^2) \arrow[r, "\delta"] \arrow[u, "d"] & \check{C}^2(\mathbb{U}, \Omega^2) \arrow[r, "\delta"] \arrow[u, "d"] & \cdots \\
0 \arrow[r] & \Omega^1(X) \arrow[r] \arrow[u, "d"] & \check{C}^0(\mathbb{U}, \Omega^1) \arrow[r, "\delta"] \arrow[u, "d"] & \check{C}^1(\mathbb{U}, \Omega^1) \arrow[r, "\delta"] \arrow[u, "d"] & \check{C}^2(\mathbb{U}, \Omega^1) \arrow[r, "\delta"] \arrow[u, "d"] & \cdots \\
0 \arrow[r] & \Omega^0(X) \arrow[r] \arrow[u, "d"] & \check{C}^0(\mathbb{U}, \Omega^0) \arrow[r, "\delta"] \arrow[u, "d"] & \check{C}^1(\mathbb{U}, \Omega^0) \arrow[r, "\delta"] \arrow[u, "d"] & \check{C}^2(\mathbb{U}, \Omega^0) \arrow[r, "\delta"] \arrow[u, "d"] & \cdots \\
 & 0 \arrow[r] \arrow[u] & \check{C}^0(\mathbb{U}, \mathbb{R}) \arrow[r, "\delta"] \arrow[u] & \check{C}^1(\mathbb{U}, \mathbb{R}) \arrow[r, "\delta"] \arrow[u] & \check{C}^2(\mathbb{U}, \mathbb{R}) \arrow[r, "\delta"] \arrow[u] & \cdots \\
 & & 0 \arrow[u] & 0 \arrow[u] & 0 \arrow[u] &
\end{tikzcd}$$

In this picture, everywhere except the blue column and row is exact, and those columns and rows are the de Rham and Čech complexes that we've defined in the last few lectures. Such a diagram therefore allows us to convert a form $c \in \Omega^{k+1}(X)$ such that $dc = 0$ to a cochain $\check{c} \in \check{C}^{k+1}(\mathbb{U}, \mathbb{R})$ such that $\delta\check{c} = 0$, by basically making a sequence of right-down-right-down moves between the left row of Ω^ks and the bottom column of $\check{C}^k$s. This shows that whenever X admits a good cover, H^k and $\check{H}^k$ indeed agree.