8.223: Classical Mechanics II

Lecturer: Professor Mike Williams

Notes by: Andrew Lin

IAP 2019

Notes have been slightly edited; references to the problem sets have been deleted.

1 January 7, 2019 (Lecture)

It is 10:05, so it is 10 o'clock in MIT time. (Most people are on time! This occurrence will likely decay exponentially as time goes on; we should use the back door of 4-270 if we are late to class.)

Professor Williams taught 8.012 a few times, and this IAP class is kind of the continuation of that. The main textbook, Landau and Lifshitz's Mechanics (3rd ed.), is very concise, because it was written for physicists, not physics students. (In other words, there is a lot of "this is evident," which is annoying.)

1.1 Course information

Lectures meet every day from 10 to 11:30 in 4-270, and recitations are 12 to 1 (also in the same room). We'll likely not have lecture until 11:30 every day; this is partially because projecting a voice five days a week is hard.

The prereq for this class is 8.01 or 8.012; we have a lot to cover, so we won't be "babying through this." Office hours are Thursday 2-3 in building 26, and TA office hours are TBD (the recitations will be taught by grad students).

There are no exams in this class. Grading will primarily come from $4 \times 20\%$ problem sets, due every Friday at 10am (so that people do actually go to class). There is also a **project**: the goal is to optimize a medieval siege engine called a trebuchet. This project is due in two parts – the first part is due next Friday, and the optimization of the design (numerically or physically) is due Wednesday, January 30. If we do only the physical version, we can work with a partner, and if we do both, we can work in a group of 3.

A Stellar site is up now, but here's the general syllabus for the course:

- Week 1: Lagrangian formalism.
- Week 2: Oscillatory systems: an extension of 8.01, where we can take problems from 8.01 and crank them up a lot.
- Week 3: Hamiltonian formalism.
- Week 4: Various stuff, including scattering theory.

The purpose of this class is to help us "think like a physicist:" for example, we'll derive Newton's laws as properties of spacetime. More practically, the goal is to teach us some formalism to do quantum mechanics and field theory. Plus, if we stay in academia, we'll likely need to go through qualifying and entrance exams; those will include a lot of questions like the ones on our problem sets.

Fact 1

It's useful at this point to explain what we mean by "classical mechanics." Our models apply to systems not moving too fast (not comparable to the speed of light, so slower than about 10^6 meters per second), and not too small (at least a few hundred atoms). However, the formalism we develop will be useful in general.

1.2 Estimating things in two minutes

We'll start looking at Lagrangians tomorrow, but for now, we're doing something a bit lighter.

Fact 2

Here is a short list of different dimensions, with the notation we use to denote each unit:

mass	[M]
length	[L]
time	[T]

We **always need three dimensions** (or dimensionful quantities) to figure out enough about our system, but sometimes we'll use slightly different quantities than $[M], [L], [T]$: for example, we often use energy and momentum in quantum mechanics. In this class, square brackets refer to a quantity of that given dimension: the idea is that whenever we have three dimensionful quantities in a problem, all other quantities can be expressed as some product of those three.

Example 3

The units of energy are

$$E = \frac{[M][L]^2}{[T^2]}.$$

(This can be checked by looking at the units of any energy quantity.)

Example 4

We have a pendulum with massless string of length ℓ, mass m, and gravitational acceleration g. How do we find ω, the angular frequency?

We'll use dimensional analysis: we can say that $\omega = O(1)\ell^\alpha m^\beta g^\gamma$, where $O(1)$ is some constant (like π and α, β, γ are constants.) ω has units of $[T^{-1}]$, and ℓ, m, g have dimensions of $[L], [M], \left[\frac{L}{T^2}\right]$, respectively. So if we consider the units, we just want to solve the equation

$$[T^{-1}] = [L]^\alpha [M]^\beta \left[\frac{L}{T^2}\right]^\gamma.$$

We can now set dimensions equal on both sides. There is no mass on the left side, so we must have $\beta = 0$. Looking at time (the $[T]$ exponents), $-1 = -2\gamma$, so $\gamma = \frac{1}{2}$, and looking at length (the $[L]$ exponents), we must have $\alpha = -\frac{1}{2}$. Thus,

$$\omega = O(1)\sqrt{\frac{g}{\ell}}.$$

We're lucky that the constant $O(1)$ actually turns out to be 1 in this case, but this is usually not true. For example, if we had asked for the period, we would be off by a factor of 2π, since we would have found $\sqrt{\frac{\ell}{g}}$ with this method.

Fact 5

"Order 1" means a constant that is between 0.1 and 10. Most constants we find with dimensional analysis should be in this range, unless we're missing some fundamental principle.

Example 6

Let's estimate the temperature of the sun using dimensional analysis.

First of all, we can represent temperature in terms of kinetic energy of molecules, so we must have

$$O(1) \cdot KE = Nk_BT,$$

where N is the number of protons and k_B is the Boltzmann constant. (We're kind of defining temperature this way, so it's okay to use a random constant k_B that is known.) Then this kinetic energy can depend on a few **parameters**: it depends on how much hydrogen is in the sun, which depends on the radius $R_\odot$ and mass $M_\odot$ of the sun. But we can think of the sun as a large clump of hydrogen gas; this has a lot of gravitational potential energy, which turns into kinetic energy as the star collapses. So the universal gravitational constant G matters as well, and therefore we can write our equation as

$$O(1) \cdot KE = O(1) \cdot M_\odot^\alpha R_\odot^\beta G^\gamma.$$

Remark 7. *We're ignoring nuclear fusion in this first approximation, but fusion is actually the reason the star doesn't collapse, not what contributes to the heat or temperature itself. Basically, gravitational potential energy is converted into heat, but fusion prevents the star from collapsing into a black hole, so we can ignore it up to a constant.*

Converting this equation to units,

$$\left[\frac{ML^2}{T^2}\right] = [M]^\alpha [L]^\beta \left[\frac{L^3}{MT^2}\right]^\gamma.$$

Looking at length, $2 = \beta + 3\gamma$. Meanwhile, mass gives $1 = \alpha - \gamma$, and time gives $-2 = -2\gamma$. This system of equations is easy to solve: $\gamma = 1$, so $\alpha = 2$ and $\beta = -1$, yielding

$$Nk_BT = O(1) \cdot \frac{GM_\odot^2}{R_\odot}.$$

The sun is mostly protons, so N is approximately the mass of the sun divided by the mass of a proton. This gives us

$$T = O(1) \cdot 23 \text{ million Kelvin}.$$

The actual temperature is around 17 million Kelvin, so we've gotten a pretty good answer given that we've done basically no work.

Fact 8

If we didn't know about nuclear fusion, we could estimate the lifetime of the sun by the luminosity, and studying this gives a lifetime approximation of 300 million years. This caused some problems for Darwin's theory of evolution, but nuclear fusion corrects this number to a few billion years.

Example 9

Now, let's estimate the explosive yield of the atomic bomb.

This was top-secret until a few years after World War II; there weren't even images of atomic explosions available to the public. Eventually, photos from the July 16, 1945 "Trinity" test were declassified. It's usually really hard to find the yield even if we know a lot of nuclear physics, but the pictures gave a length and time scale for a few hundred pictures taken a few milliseconds after the initial blast.

Fact 10

In this case, we care about the radius R of the sphere (which is quite easy to measure), the time t since the initial explosion, the density of the air ρ, and the energy E released.

Since the radius is the easiest thing to see in the declassified pictures, we'll write our equation as

$$R = O(1)\rho^\alpha t^\beta E^\gamma \implies [L] = \left[\frac{M}{L^3}\right]^\alpha [T]^\beta \left[\frac{ML^2}{T^2}\right]^\gamma$$

We can also try including other constants like the density of air ρ inside the blast, but this can be absorbed up to a constant into the density outside the blast, so we won't do that. The resulting system of equations is

$$\begin{cases} 1 = -3\alpha + 2\gamma \\ 0 = \alpha + \gamma \\ 0 = \beta - 2\gamma \end{cases}$$

(for length, mass, and time respectively), and this is solved with $\alpha = -\frac{1}{5}, \beta = \frac{2}{5}, \gamma = \frac{1}{5}$. So

$$R = O(1)\frac{E^{1/5}T^{2/5}}{\rho^{1/5}}.$$

Now since we have so many images, we can check that we have the ratio $\frac{R(t_2)}{R(t_1)} = \left(\frac{t_2}{t_1}\right)^{2/5}$. This helps us check that the $O(1)$ term is actually a constant, and this does hold pretty accurately! Finally, rearranging the equation,

$$E = O(1)\frac{R^5\rho}{t^2}.$$

For example, a picture taken at 0.025 seconds has blast radius of about 130 meters. The precision of the radius is about 2 percent (by measurement error), and the precision of time is around 0.1 percent, since the camera takes pictures at 10000 frames per second. The density of the air is $1.0 \pm 0.2\frac{kg}{m^3}$ at the location of the blast, at the given time on the day of the blast. So most of the constants are pretty well-known: we're really limited in precision mostly by the $O(1)$ number, since we're not doing any physics. This gives a final answer of

$$E = O(1) \times 6.3 \times 10^{13} J = O(1) \times 15 \text{ kT}$$

(kT refers to a kiloton of TNT), with maybe 20 percent uncertainty. And the official number is 15 to 20 kilotons of TNT, so we've done a very good job.

Fact 11

British physicist G.I. Taylor tried this calculation as well, except he actually tried to calculate the $O(1)$ constant. Taylor argues that it's a function of the ratio $\gamma = \frac{C_V}{C_P}$, which is between 1.2 and 1.7. He then finds that γ is between around 0.6 and 1.7, and the actual value turns out to be around 1.

See the following xkcds for some comic relief: https://xkcd.com/793/ and https://xkcd.com/687/. Our goal for the next few days is to derive most of the basic laws of 8.01 in a more formal way.

2 January 7, 2019 (Recitation)

The grad student teaching the first two weeks of recitation is **Kaitlyn Shin**.

It is very hard to find a good time for office hours that everyone is happy with.

Fact 12

Somewhat democratically, office hours will be Wednesdays 3–4 and 7–8 in the Solarium on the 6th floor of building 37.

2.1 Total vs. partial derivatives

Let's start with a bit of 18.02 review: total derivatives are denoted $\frac{d}{dx}$, while partial derivatives are denoted $\frac{\partial}{\partial y}$. If we have a function $f(x, y)$, $\frac{df}{dx}$ assumes that all other variables (in this case y) depend on x. Basically, we will need to use a combination of the chain rule and product rule to compute total derivatives.

Example 13

Consider the function $f(x, y) = x^2 y + y^2 \sin(x)$.

We can think of this as a function $f(x, y(x))$ that depends on x to find that

$$\frac{df}{dx} = x^2 \frac{dy}{dx} + 2xy + 2y \frac{dy}{dx} \sin x + y^2 \cos x,$$

and similarly we can check that

$$\frac{df}{dy} = 2xy \frac{dx}{dy} + x^2 + 2y \sin x + y^2 \cos x \frac{dx}{dy}.$$

But if we're taking a partial derivative, we don't assume y depends on x; other variables are just treated as constants. So we can check that

$$\frac{\partial f}{\partial x} = 2xy + y^2 \cos x, \quad \frac{\partial f}{\partial y} = x^2 + 2y \sin(x).$$

Notice that this expression is the same as the one for the total derivative, except with **no dependence between the variables**: if we set $\frac{dx}{dy} = \frac{dy}{dx} = 0$, then the total derivatives collapse to the partial derivatives. One other way to write this is that the total and partial derivatives are related via

$$\frac{df}{dx} = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} \frac{dy}{dx} + \cdots,$$

which is secretly just the multivariable chain rule because $\frac{dx}{dx} = 1$.

2.2 Variational calculus: the basics

This is in Landau's textbook, chapter 2, but we'll do it a bit more slowly.

Let's say we have a **functional** $S[q(t)]$, which means we have a function of a function. Our goal is to find the function $q(t)$ that minimizes the value of $S[q(t)]$. (Technically, it's just an extremum, but we'll just go with it.) Specifically, the problem we're trying to solve is to minimize

$$S[q(t)] = \int_{t_1}^{t_2} L(q(t), \dot{q}(t), t)dt$$

given the boundary conditions $q(t_1) = q_1, q(t_2) = q_2$. Here, note that S depends on the entire function q, and L is an "ordinary" function. Here, q is a **generalized coordinate**, so we can think of q as just x if it is easier to think about.

Since we want to find a function $q(t)$ that minimizes S, we know that S is increased if we replace $q(t)$ with any slightly perturbed function $q(t) + \delta q(t)$. (Here, $\delta q(t)$ is called the **variation** of q.) In order for our perturbed function $q + \delta q$ to still be a valid function, our boundary conditions must still hold, so $\delta q(t_1) = \delta q(t_2) = 0$. So now (notice that t does not change, since we're varying q instead of t),

$$S[q + \delta q] = \int_{t_1}^{t_2} L(q + \delta q, \dot{q} + \delta\dot{q}, t)dt$$

so the variation of S, denoted δS, is

$$\delta S = S[q + \delta q] - S[q] = \boxed{\int_{t_1}^{t_2} L(q + \delta q, \dot{q} + \delta\dot{q}, t)dt - \int_{t_1}^{t_2} L(q, \dot{q}, t)dt}.$$

From here, we can expand the difference to first order in powers of δq and $\delta\dot{q}$, and note that we should have

$$\delta S = \delta \int_{t_1}^{t_2} L(q, \dot{q}, t)dt = 0,$$

since we essentially are at a "critical point." So the chain rule applied on this expressoin tells us that

$$\int_{t_1}^{t_2} \left(\frac{\partial L}{\partial q}\delta q + \frac{\partial L}{\partial \dot{q}}\delta\dot{q}\right) = 0$$

(this is called **effecting the variation**). Now, plug in $\delta\dot{q} = \frac{d(\delta q(t)}{dt}$ (by definition) to find that

$$0 = \int_{t_1}^{t_2} \left(\frac{\partial L}{\partial q}\delta q + \frac{\partial L}{\partial \dot{q}}\frac{d\delta q}{dt}\right) dt.$$

By integration by parts, the second term $\int_{t_1}^{t_2} \left(\frac{\partial L}{\partial \dot{q}}\frac{d\delta q}{dt}\right) dt$ simplifies to $\left[\frac{\partial L}{\partial \dot{q}}\delta q(t)\right]_{t_1}^{t_2} - \int_{t_1}^{t_2} \frac{d}{dt}\frac{\partial L}{\partial \dot{q}}\delta q dt$. But our boundary conditions force the first term to be zero, so we instead just have

$$0 = \int_{t_1}^{t_2} \left(\frac{\partial L}{\partial q}\delta q - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}}\delta q\right) dt = \int_{t_1}^{t_2} \left(\frac{\partial L}{\partial q} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}}\right) \delta q\, dt = 0.$$

Since this must hold true for all δq, the expression in parentheses must always be 0, so we have the **Euler-Lagrange equations**

$$\boxed{\frac{\partial L}{\partial q} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) = 0}$$

for each coordinate q. (We technically only showed this is a critical point, but Landau ignores this.)

Example 14 (Brachistocrone problem)
We can use these EL equations to show that the shortest path under gravity is not a line but an ellipsoid. We'll see much more of this in action soon!

3 January 8, 2019 (Lecture)

3.1 Introduction

In 8.01, we were just told Newton's laws without any explanation. (He was a pretty smart guy.) To derive them, we'll start by assuming **only** the following **first principles**:

1. There exist inertial reference frames; in any given frame, the center of mass of an isolated mechanical system experiences no acceleration.
2. Space-time is **homogeneous** and **isotropic**. In other words, the fundamental laws are space and time-independent. In particular, there is no preferred direction, so things like rotations and translations preserve the properties of spacetime.

Isolated here means that there is nothing else acting on the system (no external forces or other actions). It's a bit harder to define **mechanical**: we'll say that the inherent properties of the objects of the system, like mass, do not change.

We shouldn't really assume these principles are inherently true – they should hold up to experimental evidence. Indeed, scientists test them, and as best as we can tell, there haven't been any counterexamples. So let's assume that the principles are reasonably verified, and now we'll look for some equations of motions and conserved quantities that follow from them.

Remark 15. *Newton did not derive his laws this way – plus, he was wrong about the spinning bucket and a few other things.*

3.2 Working in one dimension

Let's say we have two masses (labeled 1 and 2), with masses m_1 and m_2 as well as positions x_1 and x_2. We denote their velocities as $\dot{x}_1$ and $\dot{x}_2$.

A good first step is to use the first postulate, and we want to work in the **center-of-mass frame**: we pick the frame where the center of mass starts at the origin and never moves. And that's why we often start with the initial conditions of x_1 and x_2's position and velocity at time $t = 0$ – usually, those four quantities tell us the position and velocity of the center of mass, which means we can establish the desired. frame. Unfortunately, we want to derive things from first principles here, so we can't just evolve the system like this yet.

(Note that we'll always need 4 inputs; normally we use the initial position and velocity.)

Example 16
In this case, we're going to start with $x_1(t_a), x_2(t_a), x_1(t_b), x_2(t_b)$: suppose we know the positions of the two masses at two different times.

We want to find the system's determined "physical path" from time t_a to t_b. (Quantum mechanics complains, but we're ignoring that.)

Fact 17

In a one-dimensional system, we have a function

$$S = \int_{t_a}^{t_b} L(x_1, \dot{x}_1, x_2, \dot{x}_2)dt,$$

where L assigns a value to any given selection of the coordinates that we have, and S maps any path we take into a number (by integrating the value of L). L is called the **Lagrangian**, and S is called the **action**.

Our goal is to use S to find the physical path, and note that we need an integral because there are always infinitely many possible paths that will satisfy a finite set of initial conditions. Therefore, if we want to pick the "correct physical path" everywhere, we must consider our entire path from time t_a to t_b. The central idea here is that in classical mechanics, **there is a path** that is taken, so there must be some property S has (directly relating to the path) that we want.

Question 18. *Why does this function L only depend on x_1, $\dot{x}_1$, x_2, and $\dot{x}_2$?*

(We'll get back to this later.)

Proposition 19

If we replace S with $S + \pi$, or πS, this doesn't actually change the relevant property of S that we care about. In fact, if we replace S with

$$S + \int_{t_a}^{t_b} \frac{d}{dt}[f(x_1, x_2)]dt,$$

this also doesn't change the properties of S.

The central idea here is that the integral term above simplifies (by the fundamental theorem of calculus) to

$$f(x_1(t_b), x_2(t_b)) - f(x_1(t_a), x_2(t_a))$$

which is some constant because we've already specified the endpoints of our path. So this is an important idea: **anything in our function L that can be written as a total time derivative is irrelevant for finding the path**, and we can discard it in our consideration of S. We'll say that $\delta S = 0$ if a transformation doesn't change the important properties of S.

Fact 20 (From postulate 1)

If we take all velocities $\dot{x}_i$ and replace them with $\dot{x}_i + v$ (for some fixed v), then $\delta S = 0$.

Fact 21 (From postulate 2)

We have a few consequences from space and time-invariance.

- If we replace $t \to t + \delta_t$ (for a constant δ_x), then $\delta S = 0$.
- If we take all x_i and replace them with $x_i + \delta_x$ (for a constant δ_x), $\delta S = 0$.
- S is invariant under rotations, so in the one-dimensional case, $x_i \to -x_i$ also yields $\delta S = 0$.

Suppose that we have two **isolated free particles** (under zero potential). If there are no interactions, then we don't have quantities like $x_1 - x_2$.

Proposition 22

For a system of two isolated free particles, we can write $L = K(\dot{x}_1) + K(\dot{x}_2)$ for some arbitrary function K.

This is because the properties of the two masses are independent of each other, and the absolute positions x_1 and x_2 cannot matter by space-invariance. Also, by rotation, we have $K(-\dot{x}) = K(\dot{x})$, so K must be even. Therefore, its Taylor series can be written as

$$K(\dot{x}) = c_0 + c_2\dot{x}^2 + c_4\dot{x}^4 + \cdots.$$

Constants don't matter, so we can toss c_0 out of the picture. To understand more, we must consider what happens when we add $+v$ to both velocities in our system.

- The $\dot{x}^2$ term will become
$$(\dot{x} + v)^2 = \dot{x}^2 + v^2 + 2v\dot{x}.$$
v^2 is a constant that can be ignored, so this is equivalent to $\dot{x}^2 + \frac{d}{dt}[2vx]$, and we found earlier that we can toss total time derivatives. So changing velocities does not change the inherent properties of S, and we are okay.
- But $\dot{x}^4$ cannot appear in K, because it transforms like
$$(\dot{x} + v)^4 = \dot{x}^4 + v^4 + 4v^3\dot{x} + 6v^2\dot{x}^2 + 4v\dot{x}^3.$$
The v^4 is a constant that disappears, and $4v^3\dot{x}$ is a time derivative of $4v^3x$. However, we still have the $6v^2\dot{x}^2 + 4v\dot{x}^3$ left over, and there is no way to write this as a total time derivative. So we can't use this term.
- In fact, this just gets worse as we get larger and larger terms, so those are out. (Alternatively, remember that dimensions need to match up, so it becomes more and more difficult to balance out the units with higher powers of $\dot{x}$.) A similar argument shows that no second-order derivative ($\ddot{x}$) terms can appear in the function K.

So our function $K(\dot{x}_i)$ must be **quadratic**, and it can be written as $\alpha m_i\dot{x}_i^2$ (to fix our units) for some constant α.

Corollary 23

The Lagrangian for a system of two isolated free particles is

$$\alpha[m_1\dot{x}_1^2 + m_2\dot{x}_2^2].$$

If this is sketchy (we may wonder why α is the same for both terms after we add in the masses), we can just keep an index on the α and look at it later, but it will turn out not to matter.

Now, let's add an interaction that is **conservative**: our Lagrangian L gain have an additional term $I(|x_1 - x_2|)$ for some function I. This doesn't violate the space-invariance properties, and it doesn't violate rotation, so now let's try to see what properties I must have. We need to have the same units as the rest of the function L (energy): let's write the conservative force as $\beta U(|x_1 - x_2|)$ for a potential function U.

Remark 24. *By the way, note that we ignore propagation time (all forces should have a delay of interaction dictated by the speed of light). We can fix all of this with relativity, and for now we assume that all interactions are instantaneous.*

So now let's see if we can take our Lagrangian L and find the physical path. Fundamentally, remember that the value of S can change between reference frames, so the constant term doesn't matter (or we'd violate our postulates).

In other words, anything we extract from S about our physical path **can only depend on the differences of S between paths**.

Proposition 25 (Hamilton's principle)
At the physical path, $\delta S = 0$. In other words, nature will try to **minimize the action** (or make it an extremum).

A pretty okay explanation for this is "there aren't any other constraints that could work." Here's a better explanation: the dimension of S is energy times time, but the physical path shouldn't depend on the units that we're using, so 0 is the only reasonable thing to consider. (On the other hand, there is a fundamental quantity in quantum mechanics with those units, $\hbar$, so the problem becomes more complicated when we start working in the quantum realm.)

So now let's consider a path defined by the arbitrary functions $x_i(t)$ and $\dot{x}_i(t)$. We wish to vary the path by a small amount and see what happens to the value of S. But if $x_i(t)$ is replaced with $x_i(t) + \delta_i(t)$, which means $\dot{x}_i(t)$ is replaced with $\dot{x}_i(t) + \dot{\delta}_i(t)$, then by the calculus of variations (or a Taylor expansion), we must have

$$0 = \delta S = \int_{t_a}^{t_b} \sum_i \left(\frac{\partial L}{\partial x_i} \delta_i + \frac{\partial L}{\partial \dot{x}_i} \dot{\delta}_i \right) dt.$$

But luckily, we can write part of this expression as a total time derivative. By the product rule,

$$\frac{d}{dt}\left[\frac{\partial L}{\partial \dot{x}} \delta\right] = \frac{\partial L}{\partial \dot{x}} \dot{\delta} + \left(\frac{d}{dt}\frac{\partial L}{\partial \dot{x}}\right) \delta,$$

so plugging this in to the expression above, we can factor out a δ from the integral:

$$\delta S = \int_{t_a}^{t_b} \sum_i \left(\frac{\partial L}{\partial x_i} - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}_i} \right) \delta_i dt + \sum_i \left[\frac{\partial L}{\partial \dot{x}_i} \delta_i \right]_{t_a}^{t_b},$$

where we've used integration by parts on the blue term. And now the second sum is zero (because we know the values of the functions at the endpoints – we must have $\delta = 0$), so because **this is true for arbitrary** δ, we must have

$$\boxed{\frac{\partial L}{\partial x_i} - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}_i} = 0}$$

for each x_i. Let's plug in the Lagrangian from above into this equation now:

$$\frac{\partial L}{\partial x_1} - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}_1} = 0 \implies \beta \frac{\partial U}{\partial x_1} - \frac{d}{dt}(2\alpha m_1 \dot{x}_1) = 0.$$

Rearranging, we find that

$$m_1 \ddot{x}_1 = -\frac{\beta}{2\alpha} F(|x_1 - x_2|).$$

The same thing holds for x_2, except that we get an extra negative sign (from the chain rule). But then the force on x_1 is equal and opposite to the force on x_2, and now we've derived **Newton's third law.**

Fact 26
By convention, we say that $\alpha = \frac{1}{2}, \beta = -1$. Therefore, the Lagrangian for a system of two particles is

$$L = \frac{1}{2}[m_1 \dot{x}_1^2 + m_2 \dot{x}_2^2] - U(|x_1 - x_2|),$$

and if we change the constants α, β, we just change the definition of the force.

Now we see that it is just the ratio β to α that matters, and also why the αs must be the same for x_1 and x_2!

And notice that we have also just derived **Newton's second law**: $F = ma$, possibly with some arbitrary multiplicative constant.

4 January 8, 2019 (Recitation)

Office hours have changed to **8-320**. Our weekly problem sets can be submitted in the 3rd floor physics box.

Fact 27 (Stated by three people in different ways)
There is (probably) no important physical significance of the action S.

We may have questions regarding the argument for excluding second order terms in the Lagrangian. Basically, if we add a $\ddot{q}$ term to our Lagrangian L, we end up with fourth order equations of motion, which introduce extra degrees of freedom and result in unphysical behavior. (We'll also discuss this more next lecture.) So it's reasonable to assume (throughout this class) that for two point particles in one dimension, the Lagrangian takes the form

$$L = \frac{1}{2}[m_1\dot{x}_1^2 + m_2\dot{x}_2^2] - U(|x_1 - x_2|).$$

In general, we can write this function as $L = T - U$, where T is the **kinetic energy** and U is the **potential energy** of the system.

Example 28 (Simple harmonic oscillator)
Consider a mass on a spring in one dimension with mass m and spring constant k.

The Lagrangian here is

$$L = T - U = \frac{1}{2}m\dot{x}^2 - \frac{1}{2}kx^2.$$

The Euler-Lagrange equation is therefore

$$\frac{\partial L}{\partial x} - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}} = 0 \implies -kx - \frac{d}{dt}(m\dot{x}) = 0 \implies m\ddot{x} = -kx.$$

Notice that we didn't need to set up Newton's second law at all! To solve the ODE, we can try the ansatz $x = e^{\lambda t}$. Then

$$m\ddot{x} = -kx \implies -k = m\lambda^2$$

so $\lambda = i\sqrt{\frac{k}{m}}$. The resulting exponential solutions $e^{\pm i\sqrt{\frac{k}{m}}}$ can then be linearly combined to give oscillating solutions (cosine and sine by Euler's identity), so we end up with the general solution

$$x(t) = c_1 \cos\left(\sqrt{\frac{k}{m}}t\right) + c_2 \sin\left(\sqrt{\frac{k}{m}}t\right).$$

And if we're given initial conditions $x(0) = A, \dot{x}(0) = B$, we can find that $c_1 = A$ and $c_2 = B\sqrt{\frac{m}{k}}$.

5 January 9, 2019 (Lecture)

5.1 Review

Recall that we started with two fundamental principles: there exist inertial reference frames (where the center of mass experiences no acceleration) and spacetime is homogeneous and isotropic, (so most laws and physical constants are space and time-independent). Basically, we can think of ourselves being in an enclosed box in space – we have no idea which way we're facing outside that box.

Yesterday we considered the case of two particles in one dimension, interacting via a conservative potential. We gave ourselves the position boundary conditions $x_i(t_a), x_i(t_b)$ for $i = 1, 2$, and we basically wanted to find the classical path that the x_i particles must take. To do this, we considered the **action**

$$S = \int_{t_a}^{t_b} L(x_1, \dot{x}_1, x_2, \dot{x}_2) dt.$$

Remember here that S is a functional which takes in a function L, the **Lagrangian**, which quantifies the system at each given point in time, and spits out a number which has no direct significance. There were two important points in yesterday's argument:

- We do need to sample our data at infinitely many points, and we do this using an integral.
- There are some constraints on our function L. If we have a conservative interaction in one dimension, our Lagrangian must take on the form

$$L = \frac{1}{2} m_1 \dot{x}_1^2 + \frac{1}{2} m_2 \dot{x}_2^2 - U(|x_1 - x_2|) = K - U.$$

Suppose that we try graphing S as a function of the path (while the function space does has infinitely many dimensions, we can imagine putting paths that are closer together nearby on the graph). Then we want to know how nature picks the "best path." Remember that we can add constants and total time derivatives to the Lagrangian by shifting inertial reference frames; this just shifts the value of S up or down. Plus, we can always arbitrarily scale our Lagrangian by a constant factor, which also scales S, so any nonzero quantities won't work. So the only unambiguous way of distinguishing a point on this curve is if we find the **extremum** – this is Hamilton's principle, or the principle of least action.

And to find the minimum of a functional, we use calculus. In the case of an ordinary function, we find the point where the first derivative is 0, and similarly, with a functional, we want to find the path where if we move in any direction by a small amount, the leading (first-order) term is 0. This gives the Euler-Lagrange equation

$$\frac{\partial L}{\partial x_i} - \frac{d}{dt} \frac{\partial L}{\partial \dot{x}_i} = 0 \quad \forall i.$$

Notice that this equation does not depend on which coordinates we're using – it holds whether we're using Cartesian, spherical, or other systems! That means that this method of solving problems is convenient, especially because we can write our Lagrangian in the simplest possible coordinates.

5.2 Moving into three dimensions

Now, we'll generalize a bit and replace our coordinate x_i with a vector $\vec{x}_i$ (so that our velocities are now also vectors). We'll find that our Lagrangian must be of the form

$$L = K(\dot{\vec{x}}_1) + K(\dot{\vec{x}}_2) - U(|\vec{x}_1 - \vec{x}_2|).$$

It's interesting to go back and go through the whole arguments again, but we'll find that

$$K(\dot{x}_i) = \frac{1}{2} m_i (\dot{\vec{x}}_i \cdot \dot{\vec{x}}_i) = \frac{1}{2} m_i (\dot{x}_i^2 + \dot{y}_i^2 + \dot{z}_i^2).$$

(for $i = 1, 2$). The important thing in this 3-D case is that everything should be rotationally-invariant, and indeed the expression above only extracts the magnitude of our vector $\vec{x}$ (and not its direction). And carrying out all of the relevant calculations, it turns out the Euler-Lagrange equations work out again, and it holds **in each dimension for all coordinates**. Plugging in the form of our Lagrangian in the x-direction, we find that

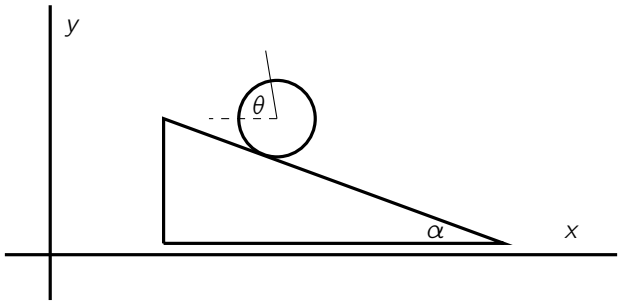
$$-\frac{\partial U}{\partial x_i} - \frac{d}{dt}[m_i \dot{x}_i] \implies m_i \ddot{x}_i = -\frac{\partial U}{\partial x_i}.$$

This gives Newton's law in the x-direction; repeating for the y- and z-directions means that Newton's second law holds in the way we expect.

Remark 29. *Let's go back briefly to why $\ddot{x}$ terms are not okay in our Lagrangian. It's easy to prove this fact in quantum mechanics, because including them violates* **unitarity**. *It's harder in classical mechanics, though: the main idea is that we have fourth-order equations of motion instead of second-order. This requires adding in some intrinsic properties of the material beyond just its position and velocity, which is starting to get out of the range of "classical systems."*

Example 30

Consider a spherical block on top of a right triangular wedge of angle α, which sits on a frictionless surface (so the wedge can move in the x-direction). The sphere rolls without slipping on the wedge. Suppose that the sphere has radius R and mass m_s, while the wedge has mass m_w.



Notice that "rotational energy" can also be thought of as the combined kinetic energy of a very large number of particles, so it's okay to include rotational kinetic energy just like "normal kinetic energy" in our Lagrangian.

We'll keep track of the (total) angle of rotation of the sphere with θ, where θ is chosen to be positive as the sphere rolls down. Trying to do this problem in 8.01 can be annoying, since we have rotated axes and have to keep track of a lot of forces. But here, we'll solve this with the Lagrangian.

We need the velocities to write down the Lagrangian: first of all, note that the velocity of the wedge is

$$\vec{v}_w = \dot{x}\hat{x},$$

where x is the coordinate that keeps track of the x-position of the wedge. (Conveniently, we don't even need to define the origin, since we only care about derivatives.) Similarly, we can find that

$$\vec{v}_s = (\dot{x} + R\dot{\theta}\cos\alpha)\hat{x} - R\dot{\theta}\sin\alpha\hat{y} :$$

the $\dot{x}$ term comes from the wedge moving, and the other terms come from the rotational motion of the sphere. And now we can write down the potential energy

$$U = -m_s g R\theta\sin\alpha + c$$

(where we don't care about the extra constant c), and the kinetic energy

$$K = \frac{1}{2}m_w\dot{x}^2 + \frac{1}{2}m_s\left[(\dot{x} + R\dot{\theta}\cos\alpha)^2 + R^2\dot{\theta}^2\sin^2\alpha\right] + \frac{1}{2}\left(\frac{2}{5}m_s R^2\right)\dot{\theta}^2.$$

Notice the expression has weird coordinates (x and θ, which aren't normally compatible with each other), but we don't care about that because Lagrangians don't mind! Expanding out K and writing $L = K - U$, we find that our Lagrangian is

$$L = \frac{1}{2}m_w\dot{x}^2 + \frac{1}{2}m_s\left(\dot{x}^2 + R^2\dot{\theta}^2 + 2R\dot{x}\dot{\theta}\cos\alpha\right) + \frac{1}{5}m_s R^2\dot{\theta}^2 + m_s g R\theta\sin\alpha.$$

The EL equation for x then yields

$$\frac{\partial L}{\partial x} - \frac{d}{dt}\frac{\partial L}{\partial\dot{x}} = 0 \implies 0 = \frac{d}{dt}\left[m_w\dot{x} + m_s\dot{x} + m_s R\dot{\theta}\cos\alpha\right].$$

Each term only has one term that varies by time, so the derivatives are easy to compute, and we have an equation of motion

$$\boxed{(m_w + m_s)\ddot{x} + m_s R\ddot{\theta}\cos\alpha = 0}.$$

Similarly, the EL equation for θ is

$$\frac{\partial L}{\partial\theta} - \frac{d}{dt}\frac{\partial L}{\partial\dot{\theta}} = 0 \implies m_s g R\sin\alpha - \frac{d}{dt}\left[m_s R^2\dot{\theta} + m_s R\dot{x}\cos\alpha + \frac{2}{5}m_s R^2\dot{\theta}\right] = 0.$$

This simplifies further to (canceling out the $m_s R$)

$$\boxed{\frac{7}{5}R\ddot{\theta} + \ddot{x}\cos\alpha = g\sin\alpha}.$$

In this case, we have a linear system of equations in terms of $\ddot{\theta}$ and $\ddot{x}$. By substitution,

$$\frac{7}{5}R\ddot{\theta} + \cos\alpha\left[\frac{-m_s R\ddot{\theta}\cos\alpha}{m_w + m_s}\right] = g\sin\alpha.$$

We can then solve for $\ddot{\theta}$ to find

$$\ddot{\theta} = \frac{g\sin\alpha}{\frac{7}{5}R - \frac{m_s R\cos^2\alpha}{m_w + m_s}},$$

and now we can find θ by integrating twice:

$$\boxed{\theta(t) = \frac{1}{2}\ddot{\theta}t^2 = \frac{g\sin\alpha}{2R\left(\frac{7}{5} - \frac{m_s\cos^2\alpha}{m_w+m_s}\right)}t^2},$$

and similarly we can also find

$$\boxed{x(t) = \frac{-m_s R\cos\alpha}{m_s+m_w}\theta(t)},$$

and we've completed the problem.

To recap, we realized we were in a situation with a conservative force (so it makes sense to write down a potential). Then, we wrote our Lagrangian in whatever sensible coordinates we wanted and chugged through the resulting algebra. Note that this problem would have been a lot harder in 8.01: the easiest way would be to use conservation of momentum in the x-direction and then differentiating the equation for conservation of energy.

If we had friction, or if there is an external force on our system, the Lagrangian does change, and we'll talk about this later.

5.3 Conservation and conserved quantities

Remember that the properties of spacetime do not depend on time t, and we can use this to derive an interesting relation. By the multivariable chain rule,

$$\frac{dL}{dt} = \frac{\partial L}{\partial t} + \sum_i \left[\frac{\partial L}{\partial q_i}\frac{dq_i}{dt} + \frac{\partial L}{\partial \dot{q}_i}\frac{d\dot{q}_i}{dt}\right]$$

where q_i is an arbitrary generalized coordinate which shows up in the expression for L. But there can't be an explicit time term in the Lagrangian, so $\frac{\partial L}{\partial t} = 0$, which means we can further simplify to

$$\frac{dL}{dt} = \sum_i \left[\frac{\partial L}{\partial q_i}\dot{q}_i + \frac{\partial L}{\partial \dot{q}_i}\ddot{q}_i\right].$$

And now we can plug in the Euler-Lagrange equation: we find that

$$\frac{\partial L}{\partial q_i} = \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} \implies \frac{dL}{dt} = \sum_i \left[\left(\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i}\right)\dot{q}_i + \frac{\partial L}{\partial \dot{q}_i}\ddot{q}_i\right],$$

but the term inside the sum is just $\frac{d}{dt}\left[\frac{\partial L}{\partial \dot{q}_i}\dot{q}_i\right]$ (expand out using the product rule)! So we actually have a total time derivative on both sides, and therefore

$$\boxed{\frac{d}{dt}\left[\sum_i \frac{\partial L}{\partial \dot{q}_i}\dot{q}_i - L\right] = 0}.$$

In other words, the time-independence has lead us to a **quantity that is conserved in time**.

Example 31

Suppose our Lagrangian L is of the form $\frac{1}{2}m\dot{x}^2 - U(|x|)$ for one particle.

Then the conserved quantity we have is

$$\frac{d}{dt}\left[\frac{\partial L}{\partial \dot{x}}\dot{x} - L\right] = 0 \implies \frac{d}{dt}\left[m\dot{x}\dot{x} - \frac{1}{2}m\dot{x}^2 + U\right] = 0.$$

In other words, $\frac{1}{2}m\dot{x}^2 + U$ is constant, and we generally refer to this quantity as the total **energy** of the system! (It's important to remember that energy is only conserved when $\frac{\partial L}{\partial t} = 0$, though – if there is an external force, we may be able to add or subtract energy from the system.)

Fact 32

This is a special case of **Noether's theorem**: any continuous symmetry of a Lagrangian (for example, sending t to $t + \delta t$) gives a conserved quantity (in this case, the energy).

This line of reasoning is the way we really should define energy in general. It turns out we can derive Maxwell's equations from this method as well (though we won't really talk about this). Tomorrow, we'll derive the properties of angular momentum.

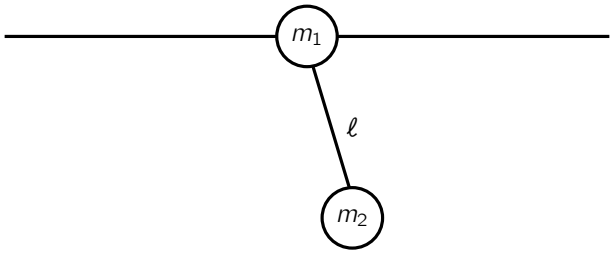
6 January 9, 2019 (Recitation)

One thing that's important to bring up, at least in passing: actions can be at saddle points for the physical path (so it's not true that we always have a minimum or maximum when $\delta S = 0$). In some other areas of physics, the stationary point is indeed not an extremum, but this probably won't come up in this class (since it only happens in weird systems).

Today, we'll do some more examples with a Lagrangian.

Example 33 (Moving pendulum)

We have a mass m_1 attached by a string of length ℓ to another mass m_2. The first mass is fixed to move only horizontally, while the other mass is free to move.



Let x_1 be the (horizontal) coordinate of the first mass, and let ϕ be the angle of the string and second mass from the vertical. Then the x-coordinate of the second mass is $x_1 + \ell \sin\phi$, while the y-coordinate is $-\ell\cos\phi$.

From here, we just need to identify the kinetic and potential energy of the system. The first mass has kinetic energy $\frac{1}{2}m_1\dot{x}_1^2$, while the second mass has kinetic energy (take the x and y-direction's kinetic energy and add them together)

$$\frac{1}{2}m_2\left((\dot{x}_1 + \ell\cos(\phi)\dot{\phi})^2 + (\ell\sin(\phi)\dot{\phi})^2\right) = \frac{1}{2}m_2\left(\dot{x}_1^2 + \ell^2\dot{\phi}^2 + 2\dot{x}_1\ell\cos(\phi)\dot{\phi}\right),$$

where we've used the fact that $\cos^2\phi + \sin^2\phi = 1$. Finally, since the potential energy is 0 (or some other constant) for the first mass and $-m_2 g\ell\cos\phi$ for the second mass, our Lagrangian is

$$L = K - U = \frac{1}{2}(m_1 + m_2)\dot{x}_1^2 + \frac{1}{2}m_2\left(2L\cos(\phi)\dot{\phi}\dot{x}_1 + L^2\dot{\phi}^2\right) + m_2 gL\cos\phi.$$

We can plug this in with the Euler-Lagrange equation, but we'll introduce the idea of the **generalized momentum**

$$p_i = \frac{\partial L}{\partial \dot{q}_i}$$

for each coordinate q_i. We set $q_1 = x_1$ and $q_2 = \phi$: then the momentum corresponding to x_1 is

$$p_{x_1} = \frac{\partial L}{\partial \dot{x}_1} = (m_1 + m_2)\dot{x}_1 + m_2 L \cos\phi\dot{\phi}.$$

If we want to check if this is conserved, the Euler-Lagrange equation tells us that it is sufficient to check if $\frac{\partial L}{\partial x_1} = 0$. This is true, so the generalized momentum for x_1 is conserved. Meanwhile, the generalized momentum for ϕ is

$$p_\phi = \frac{\partial L}{\partial \dot{\phi}} = m_2 L \cos\phi\dot{x} + m_2 L^2\dot{\phi}.$$

However, $\frac{\partial L}{\partial \phi} = 0$ is not always true, so the generalized momentum for ϕ is not always conserved.

Let's finish by considering a different system:

Example 34

A pendulum of length ℓ with a mass m at one end is attached (at the other end) to a point which moves around a circle with frequency γ and radius a.

We'll just write down the Lagrangian for this system. Let ϕ be the angle from the vertical for the pendulum (just like in the previous example). The x-coordinate of the mass is $x_m = a\cos(\gamma t) + \ell\sin(\phi)$, while the y-coordinate is $y_m = a\sin(\gamma t) - \ell\cos(\phi)$, so

$$\dot{x}_m = -a\gamma\sin(\gamma t) + \ell\cos(\phi)\dot{\phi}$$

while

$$\dot{y}_m = a\gamma\cos(\gamma t)\gamma + \ell\sin(\phi)\dot{\phi}.$$

Therefore, the kinetic energy of the system is

$$K = \frac{1}{2}m^2\left(\dot{x}_m^2 + \dot{y}_m^2\right) = \frac{1}{2}m^2\left((-a\gamma\sin(\gamma t) + \ell\cos(\phi)\dot{\phi})^2 + (a\gamma\cos(\gamma t)\gamma + \ell\sin(\phi)\dot{\phi})^2\right),$$

and expanding out the squares yields

$$= \frac{1}{2}m\left(a^2\gamma^2 + \ell^2\dot{\phi}^2 + 2a\ell\gamma\dot{\phi}[\sin\phi\cos(\gamma t) - \sin(\gamma t)\cos(\phi)]\right).$$

Meanwhile, the potential energy is

$$U = mgy_m = mga\sin(\gamma t) - mg\ell\cos\phi,$$

and we get the Lagrangian $L = K - U$ by subtracting these two parts.

7 January 10, 2019 (Lecture)

The first problem set is due tomorrow at 10am (before class), and we can turn it in in the drop box between buildings 8 and 16 on the 3rd floor. (The boxes are labeled by last name for convenience.)

Fact 35 (Hmm)

If we want to get paid for taking notes for this class, we can email SDS.

7.1 Quick review of previous material

To find the physical path of a system, we derived an **action**

$$S = \int L(q_i, \dot{q}_i)dt$$

which is determined based on a **Lagrangian** L. That Lagrangian is based on some generalized coordinates, and it was derived to be of the form $L = K - U$ for a conservative mechanical system, where K is the kinetic energy and U is the potential energy. (For a non-isolated system, we may have different Lagrangians.)

Then at $\delta S = 0$, a point of **stationary action** occurs - usually this happens at a local minimum. Expanding out the Lagrangian, we get the EL equations

$$\frac{\partial L}{\partial q_i} = \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i}$$

for all coordinates q_i, whether they are Cartesian, polar, or anything else.

Last time, we found that time-independence yields $\frac{\partial L}{\partial t} = 0$ (since the laws of physics should be the same if we pick any specific value of t). Then some calculations told us that

$$\frac{d}{dt}\boxed{\left(\sum_i \left(\frac{\partial L}{\partial \dot{q}_i}\dot{q}_i\right) - L\right)} = 0.$$

This means that the boxed term is **conserved** (since it doesn't change over time). We'll define it to be the **Hamiltonian** of the system, and we'll come back to this point around the third week of class.

Example 36

Let's say we have some number of simple particles in three dimensions with a conservative interaction: that is,

$$L = \sum_i \left(\frac{1}{2}m_i(\dot{x}_i^2 + \dot{y}_i^2 + \dot{z}_i^2)\right) - U,$$

where U only depends on the positions x_i of the particle.

Then the Hamiltonian is equal to

$$\sum_i (m_i(\dot{x}_i\dot{x}_i + \dot{y}_i\dot{y}_i + \dot{z}_i\dot{z}_i)) - L = \sum_i \left(\frac{1}{2}m_i(\dot{x}_i^2 + \dot{y}_i^2 + \dot{z}_i^2)\right) + U = E.$$

This is quite general, since the potential U can be anything we want (in a variety of different contexts)!

A more general theorem is **Noether's theorem**, which states that for any continuous symmetry of our system (like time-dependence), there exists some conserved quantity.

Fact 37

Emmy Noether has many fellowships named after her, so we may end up being Noether fellows or something.

7.2 More conservation laws

Let's try to derive another conserved quantity by translating in space in one direction: we apply the transformation $x_i \to x_i + \delta x$ for some fixed δx, and we'll do this for all particles i. By our starting postulates, the laws of physics

should still look the same after this transformation, so the change in the Lagrangian here is

$$0 = \delta L = \sum_i \frac{\partial L}{\partial x_i} \delta x.$$

(Notice we don't have any terms corresponding to $\dot{x}_i$, like we did for the time-independence case, since we are not changing any velocities with this transformation.) Now by the Euler-Lagrange equation, we can replace $\frac{\partial L}{\partial x_i}$ with $\frac{d}{dt}\frac{\partial L}{\partial \dot{x}_i}$, and δx is arbitrary (and equal in all terms), so we can divide it out. This gives us

$$\sum_i \frac{d}{dt}\frac{\partial L}{\partial \dot{x}_i} = \boxed{\frac{d}{dt}\left[\sum_i \frac{\partial L}{\partial \dot{x}_i}\right] = 0},$$

because sums and derivatives commute.

Remark 38. *Looking at our example Lagrangian from Example 36 again, $\frac{\partial L}{\partial \dot{x}_i} = m_i \dot{x}_i$, which is (sort of defined as) the momentum in the x-direction. So this is actually a statement about* **conservation of momentum.**

We can do the exact same thing in the y and z-directions, and this tells us that we have conservation of momentum in all directions. Again, we should note that this conservation holds regardless of the actual system we're dealing with – we only require that the behavior must hold independent of time and space translations!

Example 39

Let's try another isometry: we'll rotate about $\hat{z}$ by a small angle $\delta\phi$. (Recall that spacetime is **isotropic**, which means that it doesn't depend on the direction of our axes.)

$\delta\phi$ can be infinitesimally small, since we can just do an infinite number of infinitesimal rotations to build up to any finite rotation. Under this rotation, the coordinates x_i, y_i change to

$$x_i \to x_i \cos(\delta\phi) - y_i \sin(\delta\phi) \approx x_i - y_i \delta\phi$$

and

$$y_i \to y_i \cos(\delta\phi) + x_i \sin(\delta\phi) \approx y_i + x_i \delta\phi.$$

(z_i stays the same.) So the (infinitesimal) transformation we're applying is $\boxed{\delta x_i = -y_i\delta\phi, \delta y_i = x_i\delta\phi, \delta z_i = 0}$, which means the infinitesimal velocity changes are $\boxed{\delta\dot{x}_i = -\dot{y}_i\delta\phi,\ \delta\dot{y}_i = \dot{x}_i\delta\phi, \delta\dot{z}_i = 0}$. We can now plug everything into our Lagrangian again: we must have $\delta L = 0$ under this transformation as well, so

$$\delta L = \sum_i \left[\frac{\partial L}{\partial x_i}\delta x_i + \frac{\partial L}{\partial \dot{x}_i}\delta\dot{x}_i + \frac{\partial L}{\partial y_i}\delta y_i + \frac{\partial L}{\partial \dot{y}_i}\delta\dot{y}_i\right] = 0.$$

(The z terms can be ignored, since δz_i and $\delta\dot{z}_i$ are 0.) Now plugging in the expressions that we know and substituting using the EL equations like we did before, we find that

$$\sum_i \left(\left(\frac{d}{dt}\frac{\partial L}{\partial \dot{x}_i}\right)(-y_i\delta\phi) + \frac{\partial L}{\partial \dot{x}_i}(-\dot{y}_i\delta\phi) + \left(\frac{d}{dt}\frac{\partial L}{\partial \dot{y}_i}\right)(x_i\delta\phi) + \frac{\partial L}{\partial \dot{y}_i}(\dot{x}_i\delta\phi)\right) = 0.$$

Now the $\delta\phi$ appears in every term, so it drops out, and we can also factor the first and last two terms by the product rule. We then end up with

$$\boxed{\frac{d}{dt}\left(\sum_i \left(\frac{\partial L}{\partial \dot{x}_i}(-y_i) + \frac{\partial L}{\partial \dot{y}_i}x_i\right)\right) = 0.}$$

So we have another conserved quantity, and we can analyze it by looking again at the example Lagrangian in Example 36. We have

$$\sum_i \left(\frac{\partial L}{\partial \dot{x}_i}(-y_i) + \frac{\partial L}{\partial \dot{y}_i} x_i \right) = \sum_i \left(-m_i \dot{x}_i y_i + m_i \dot{y}_i x_i \right),$$

and this is actually just the z-component of the cross product between the position and momentum:

$$= \sum_i (\vec{r}_i \times \vec{p}_i)_z.$$

This is the **angular momentum** in the z-direction! We can repeat this argument for rotations around any axis, and this means that

$$\sum_i \vec{\ell}_i = \vec{\ell},$$

the angular momentum of the system, is conserved.

7.3 A physical demonstration

Fact 40

If a person is connected to a spring, obviously the "laws of physics" are different for them compared to if they are not connected to a spring. So it's interesting to think about how systems behave under external influences.

Professor Williams has brought a bicycle wheel to the front of the classroom.

Problem 41

Imagine we (and the wheel) are inside a sphere. Is there any way to identify the directions of this room with just that wheel? In other words, are there preferred directions of rotations?

To understand this, we can think about rotating the system about the x-axis. In other words, we can ask whether it's possible to distinguish between the wheel spinning with axis in the y-direction versus the z-direction.

And because of gravity, these two rotations can be distinguished from each other! So a rotation about the x-axis is not a symmetry of our system, meaning angular momentum in the x-direction is not conserved. Similarly, angular momentum is not conserved in the y-direction, since the wheel spinning in x- and z-directions are distinguishable from each other. However, angular momentum **is conserved** in the z-direction.

So if a person starts spinning the bicycle wheel with its axis parallel to the ground, the wheel will want to fall (because of gravity), and to preserve angular momentum in the z-direction, the center of the wheel will start rotating (around that person) in the xy-plane, despite there being no initial rotational motion! (This is incredibly baffling when shown to someone who doesn't understand the physics.)

7.4 The Brachistochrone problem

This problem originated from Johann Bernoulli in 1696. He "hoped to gain the gratitude of the whole scientific community by placing before the finest mathematicians of his time a problem which will test their methods and the strength of their intellect."

Example 42

Given two points $(0,0)$ and (x_1, y_1) in the xy-plane, with gravity acting in the y-direction, draw a path $x(y)$ from one point to the other. Which path do we choose so that the particle can get to the destination in the shortest amount of time?

We'll begin with an 8.01 argument – say the particle starts with zero potential energy. Then conservation of energy tells us

$$\frac{1}{2}mv^2 = -mgy \implies -2gy = \dot{x}^2 + \dot{y}^2,$$

and now we'll start to massage this into something we can optimize: the chain rule tells us that

$$-2gy = \left(\frac{dx}{dy}\frac{dy}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = \left(\frac{dy}{dt}\right)^2\left(\left(\frac{dx}{dy}\right)^2 + 1\right),$$

and now multiplying by dt^2 and taking the square root yields

$$dt = \sqrt{\frac{1+\left(\frac{dx}{dy}\right)^2}{-2gy}}dy.$$

We want to **minimize the total time**, so we can integrate through our entire path to find that

$$\tau = \int_0^{y_1}\sqrt{\frac{1+\left(\frac{dx}{dy}\right)^2}{-2gy}}dy.$$

And now we minimize this quantity over all possible paths $x(y)$. Notice that τ is very similar to the action S! It's an integral over a path of something that is the "Lagrangian", where y takes the role of time. So we can write the Euler-Lagrange equation

$$\frac{\partial L}{\partial x} - \frac{d}{dy}\frac{\partial L}{\partial x'} = 0$$

where we define x' to be $\frac{dx}{dy}$.

The first term is 0 (conveniently), so $\frac{\partial L}{\partial x'}$ should be a constant regardless of where we are on the path. Let's define it to be the constant $\frac{1}{\sqrt{-4ga}}$ for some real number a: then

$$\frac{\partial L}{\partial x'} = \frac{1}{\sqrt{-2gy}}\frac{1}{2}\left(1+(x')^2\right)^{-1/2}2x' = \frac{1}{\sqrt{-4ga}}.$$

Simplifying this further and squaring both sides,

$$\frac{y}{2a} = \frac{(x')^2}{1+(x')^2} = \frac{dx^2}{dy^2+dx^2}.$$

Rearranging,

$$dx^2 + dy^2 = \frac{2a}{y}dx^2 \implies dx = \frac{dy}{\sqrt{\frac{2a}{y}-1}}.$$

We can now integrate by using the trig substitution $y = a(1-\cos\theta)$; skipping some of the steps, we end up with our answer of

$$\boxed{x = a(\theta - \sin\theta), \quad y = a(1-\cos\theta)}.$$

This is the definition of a **cycloid**!

Fact 43

Bernoulli solved this problem within a few weeks. Legend has it that Newton was unaware for a while and apparently solved it within twelve hours. He also apparently did not explain his solution or write his name, but Bernoulli recognized the "claw mark." Newton did not like being "pestered and teased by foreigners about mathematical things."

8 January 10, 2019 (Recitation)

The first part of the project is due next Friday, at the same time as problem set 2.

Recall the small-angle approximation: if $\theta \ll 1$, then we can say that $\sin\theta \approx \theta$, because

$$\sin\theta = \theta - \frac{\theta^3}{6} + \frac{\theta^5}{120} + \cdots.$$

Remark 44. *There was an argument made that $\dot\theta \ll \ddot\theta$ if $\theta \ll 1$, but this isn't quite true. If we have a pendulum oscillating at frequency ω, we will need to make additional assumptions about the comparison between ω and θ if we want to make further terms more negligible.*

We're going to talk about Noether's theorem today in slightly more generality, which basically states the following:

Theorem 45

There's an associated conservation law for any continuous symmetry of the Lagrangian.

Remember that the EL equations are of the form (for any generalized coordinate $q_i(t)$)

$$\frac{\partial L}{\partial q_i} - \frac{d}{dt}\frac{\partial L}{\partial \dot q_i} = 0,$$

and a function $f(q, \dot q, \cdots)$ is **conserved** (in time) if $\dot f = 0$. Consider an **infinitesimal symmetry** of the form

$$q_i \to q_i' = q_i + \delta q_i,$$

and suppose that the Lagrangian stays constant under this symmetry: that is,

$$L(q, \dot q) = L(q', \dot q').$$

This perturbation δq_i can be a function of the q_i in general, so we'll write it as a function ε:

$$q_i \to q_i + \varepsilon_i(q).$$

To first order, the change in the Lagrangian is

$$0 = \delta L = \sum_i \frac{\partial L}{\partial q_i}\varepsilon_i(q) + \frac{\partial L}{\partial \dot q_i}\varepsilon_i(\dot q).$$

We can further simplify this expression in general to rewrite the two terms here as a derivative of a product:

$$\sum_i \left(\frac{d}{dt}\frac{\partial L}{\partial \dot q_i}\right)\varepsilon_i(q) + \frac{\partial L}{\partial \dot q_i}\varepsilon_i(\dot q) = 0 \implies \sum_i \left(\frac{d}{dt}\left(\frac{\partial L}{\partial \dot q_i}\varepsilon_i(q)\right)\right) = 0.$$

Sums and derivatives commute, so this means that we have the conserved quantity boxed below:

$$\frac{d}{dt}\boxed{\sum_i \left(\frac{\partial L}{\partial \dot{q}_i}\varepsilon_i(q)\right)} = 0.$$

Remember that all perturbations ε being equal to each other gives us conservation of linear momentum, and if we had used time as our coordinate instead, we would get the Hamiltonian (which is also conserved).

Let's finish by looking at another Lagrangian:

Example 46

Consider two massless springs in series, with rest length ℓ_1, ℓ_2 and spring constants k_1, k_2 respectively, and suppose the first spring is attached to the ceiling. There is a mass of m_1 between the springs and a mass of m_2 on the free end of the springs.

Let y_1 and y_2 be the displacements of the two masses (from equilibrium). Then the kinetic energy of the system is

$$KE = \frac{1}{2}m_1\dot{y}_1^2 + \frac{1}{2}m_2(\dot{y}_1+\dot{y}_2)^2,$$

while the potential energy is

$$U = \frac{1}{2}k_1(y_1-\ell_1)^2 + \frac{1}{2}k_2(y_2-\ell_2)^2 - m_1gy_1 - m_2g(y_1+y_2).$$

If we subtract these two expressions, we can find the Lagrangian

$$L = \frac{1}{2}m_1\dot{y}_1^2 + \frac{1}{2}m_2(\dot{y}_1+\dot{y}_2)^2 - \frac{1}{2}k_1(y_1-\ell_1)^2 - \frac{1}{2}k_2(y_2-\ell_2)^2 + m_1gy_1 + m_2g(y_1+y_2).$$

Now the Euler-Lagrange equation for y_1 yields

$$-k_1(y_1-\ell_1) + m_1g + m_2g - \frac{d}{dt}\left(m_1\dot{y}_1 + m_2(\dot{y}_1+\dot{y}_2)\right) = 0,$$

and EL for y_2 yields

$$-k_2(y_2-\ell_2) + m_2g - \frac{d}{dt}\left[m_2(\dot{y}_1+\dot{y}_2\right] = 0.$$

We can solve this system of equations like we do with coupled oscillators, or we can subtract the second equation from the first and do a further analysis.

9 January 11, 2019 (Lecture)

The first part of the project is not very hard, and it's probably equivalent to a hard pset problem.

Fact 47

If we want to teach ourselves how to use tikz, we can make a tikz picture and submit it with our project for extra credit.

Problem set 2 is now posted, and we should be able to do most of it by the end of today.

9.1 Newton's law of gravitation

Recall that we learned about Kepler's three (empirical) laws in 8.01:

- The orbit of every planet is an ellipse, and the Sun is at one of the two foci of that ellipse.
- If we draw a line between a planet and the sun, the area of the wedge swept out during some constant-length interval of time is constant.
- The square of the orbital period T is proportional to the cube of the semi-major axis a of the orbit

(Kepler didn't really have any theory that explains why any of this must be true, though.) Remember that we also have

$$\vec{F} = -\frac{Gm_1m_2}{r^2}\hat{r},$$

which states that there is an **attractive force** proportional to mass and inversely proportional to the radius squared along the line joining the two masses.

To understand why we should expect an inverse square law here, imagine having a light source that emits photons. If we take a panel of area A on the surface of a sphere of radius r, and we look at the projection of that solid angle onto a sphere of radius $3r$, we have an area of $9A$. But in both cases, we'll collect the same number of photons from the center light source! So force must be proportional to $\frac{1}{r^2}$ for consistency.

Remark 48. *If we lived in four dimensions, we'd have $\frac{1}{r^3}$-type laws instead of $\frac{1}{r^2}$. But experimentally, we haven't found anything of the sort.*

Let's see if there's any way we can verify or explain Newton's argument. Near the earth's surface, we have that the force on an object of mass m is

$$mg \approx F_g = \frac{Gm_em}{R_e^2}.$$

It's pretty hard to test this, though. It's easy to calculate g even without a precise timing device: we can just take a pendulum and measure the average period, which depends only on g and ℓ. But it's harder to calculate the mass m_e and radius R_e of the earth, especially in 1680. So we want to try something else instead if possible.

So we'll pick a simple orbit, like the moon orbiting the Earth (this is almost circular). Let m_m and m_e be the masses of the moon and earth, respecively. Then equating centripetal forces, we have that

$$m_m\frac{v_m^2}{R_{e-m}} = \frac{Gm_em_m}{R_{e-m}^2},$$

where R_{e-m} is the distance from the earth to the moon. This then simplifies to

$$v_m^2 = \frac{Gm_e}{R_{e-m}}.$$

We can find v_m because the orbit is moving in a circle: this orbital speed is $\frac{2\pi R_{e-m}}{\tau_m}$, where $\tau \approx 1$ month is the orbital period of the moon. Putting everything together, we can remove the mass of the earth from the picture, and this means that

$$\tau_m = \frac{2\pi R_{e-m}}{R_e\sqrt{g}}.$$

Fact 49

Eratosthenes in 240 BC found the radius of the earth to about 15% accuracy. He used the fact that Syene was on the Tropic of Cancer, then measured the angle of the sun at Alexandria (on the solstice, when the tropics are in line with the sun) and made a few distance calculations.

Essentially, the idea behind his argument is that we can treat the sun is infinitely far away, so the rays are all parallel to each other. When the sun is shining straight down at Syene, suppose that it is shining at an angle θ at some other city a distance D away from Syene. Then

$$\tan\theta \approx \frac{D}{R_e} \implies R_e \approx \frac{D}{\theta}.$$

This yields $R \approx 6400$ km, and now we can verify our equation above if we can find the distance between the earth and the moon.

Proposition 50

We can use the **lunar parallax** to estimate the earth-moon distance. Basically, look at the moon from different points on the earth; if we observe it from the south pole and north pole, its position will move relative to distant stars.

For example, we can try putting two fingers in front of our face, one close and one far away. When we close one eye versus the other, the closer objects move more, and that's the idea that we're going for here.

Fact 51

The first precise measurement of G was made about a hundred years after Newton died.

9.2 The two-body problem

Let's now try to work out some of the consequences of Newton's law. Suppose we have two masses m_1 and m_2 at positions $\vec{r}_1$ and $\vec{r}_2$. Then the displacement vector between the masses is

$$\vec{r} = \vec{r}_1 - \vec{r}_2.$$

Define $M = m_1 + m_2$ and $\vec{R} = \frac{1}{M}(m_1\vec{r}_1 + m_2\vec{r}_2)$ (this is the position of the **center of mass** of the two particles). Then we can write the Newton's gravitational equations as

$$m_1\ddot{\vec{r}}_1 = -\frac{Gm_1m_2}{r^2}\hat{r}$$

and

$$m_2\ddot{\vec{r}}_2 = \frac{Gm_1m_2}{r^2}\hat{r}.$$

So now we can get an equation for the displacement vector:

$$\ddot{\vec{r}} = \ddot{\vec{r}}_1 - \ddot{\vec{r}}_2 = -\left[\frac{1}{m_1} + \frac{1}{m_2}\right]\frac{Gm_1m_2}{r^2}\hat{r}.$$

Define the **reduced mass** of the system $\mu = \frac{m_1m_2}{m_1+m_2}$, so that $\frac{1}{m_1} + \frac{1}{m_2} = \frac{1}{\mu}$. We can plug this in, and now the equation

$$\mu\ddot{\vec{r}} = -\frac{Gm_1m_2}{r^2}\hat{r}$$

looks like a one-body problem with the reduced mass! And notice also that

$$M\ddot{\vec{R}} = m_1\ddot{\vec{r}}_1 + m_2\ddot{\vec{r}}_2 = 0,$$

so the center of mass always moves at a constant velocity. The idea here is that we can **factorize** gravity problems: solve for the center of mass position, and solve the problem in the center-of-mass frame.

Remark 52. *Notice that if $m_1 \ll m_2$, then the reduced mass can be approximated as $\mu \to m_1$, while the combined mass can be approximated as $M \to m_2$.*

So if we have two masses m_1 and m_2, notice that the center of mass is always on the segment connecting the two masses. **Choose coordinates** so that this point is the origin. Now if the center of mass $\vec{R}$ is stationary, then $0 = \dot{\vec{R}} = m_1\dot{\vec{r}}_1 + m_2\dot{\vec{r}}_2$, so the momentum of the two particles must be equal in magnitude and pointing in opposite directions. And if the distance between the two masses is r, the distances from m_1 and m_2 to the origin must be $\frac{m_2 r}{M}$ and $\frac{m_1 r}{M}$ respectively.

So here's the key point: at any given time, we can look at a plane defined by the position and momentum vectors. There are no external forces (the only force affecting the motion of the particles is gravity), so the angular momentum must be conserved, so the **motion for each particle is confined to a plane**. And looking more closely, this plane is the same for both particles, and the two particles have the same θ in polar coordinates by symmetry.

With all of this in mind, we can now write down the Lagrangian. Since the kinetic energy of a particle in (two-dimensional) polar coordinates is $\dot{r}^2 + (r\dot{\theta})^2$, we have that

$$L = K - U = \frac{1}{2}m_1\left(\left(\frac{m_2}{M}\dot{r}\right)^2 + \left(\frac{m_2}{M}r\dot{\theta}\right)^2\right) + \frac{1}{2}m_2\left(\left(\frac{m_1}{M}\dot{r}\right)^2 + \left(\frac{m_1}{M}r\dot{\theta}\right)^2\right) + \frac{Gm_1m_2}{r}.$$

Collecting terms, this all simplifies into

$$= \frac{1}{2M^2}\left(\dot{r}^2(m_1m_2^2 + m_2m_1^2) + r^2\dot{\theta}^2(m_1m_2^2 + m_2m_1^2)\right) + \frac{Gm_1m_2}{r},$$

and since $m_1m_2^2 + m_2m_1^2 = m_1m_2M$, we can write our Lagrangian in terms of the reduces mass as

$$\boxed{L = \frac{1}{2}\mu(\dot{r}^2 + r^2\dot{\theta}^2) + \frac{Gm_1m_2}{r}}.$$

Notice that this Lagrangian behaves like one-body motion, just like we were deriving with 8.01 arguments above! And now by the Euler-Lagrange equation,

$$\frac{\partial L}{\partial \theta} - \frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} = 0 \implies \mu r^2\dot{\theta} = C,$$

and this constant C is exactly the orbital angular momentum ℓ.

Example 53

Consider a particle on an orbital path, and suppose that at some instance in time, it is a distance r away from the object it's orbiting.

If this particle travels through an angle $\Delta\theta$, it will travel an approximate distance $r\Delta\theta$, and its new distance from the center is $r + \Delta r$. So the area it sweeps out is

$$\Delta A \approx \frac{1}{2}(r + \delta r)(r)\sin(\delta\theta) \approx \frac{1}{2}r^2\Delta\theta.$$

Taking the limit as $\Delta\theta \to 0$, we find that

$$\frac{dA}{d\theta} = \frac{1}{2}r^2\dot{\theta} = \frac{\ell}{2\mu}.$$

In such a two-body system, this is a **constant**, which means we've proved Kepler's second law!

We can learn a bit more if we use the Euler-Lagrange equation for r on the boxed Lagrangian above:

$$\frac{\partial L}{\partial r} - \frac{d}{dt}\frac{\partial L}{\partial \dot{r}} = 0 \implies \mu r\dot{\theta}^2 - \frac{Gm_1m_2}{r^2} - \frac{d}{dt}(\mu\dot{r}) = 0.$$

Since $\dot{\theta}^2$ is related to our angular momentum ℓ, we can substitute to get a differential equation that only involves r and constants:

$$\boxed{\mu\ddot{r} = \frac{\ell^2}{\mu r^3} - \frac{Gm_1m_2}{r^2}}.$$

This almost looks like Newton's second law for reduced mass! We just have the extra first term on the right hand side, and this is called the (ficticious) **centrifugal force**; it keeps the particles from getting too close to each other.

To proceed, notice that the energy of this system is conserved (because $\frac{\partial L}{\partial t} = 0$), and specifically, this means that the Hamiltonian

$$E = \frac{\partial L}{\partial \dot{r}}\dot{r} + \frac{\partial L}{\partial \dot{\theta}}\dot{\theta} - L$$

is a constant. Going through the algebra, we find that

$$E = \frac{1}{2}\mu\dot{r}^2 + \frac{1}{2}\mu r^2\dot{\theta}^2 - \frac{Gm_1m_2}{r}.$$

We can again replace the $\dot{\theta}$ term with things involving ℓ:

$$= \frac{1}{2}\mu\dot{r}^2 + \frac{\ell^2}{2\mu r^2} - \frac{Gm_1m_2}{r},$$

and now we're just left with a function of r! (The second two terms here are known as the **effective potential** $U_{\text{eff}}(r)$.) Solving for $\dot{r}$, we find that

$$\dot{r} = \sqrt{\frac{2}{\mu}(E - U_{\text{eff}}(r))},$$

and since $d\theta = \frac{d\theta}{dt}\frac{dt}{dr}dr = \frac{\dot{\theta}}{\dot{r}}dr$, we find that

$$\theta = \int d\theta = \int F(r)dr$$

for some function r. This will yield

$$r = \frac{\alpha}{1 + \varepsilon\cos\theta},$$

which yields an ellipse in polar coordinates when the system is bound ($\varepsilon < 1$), meaning we've shown **Kepler's first law**. Finally, the area of the ellipse can be related to the orbital period using the second law. This yields

$$\tau^2 = \frac{4\pi^2 a^3}{GM},$$

which is **Kepler's third law**.

Let's apply this last result to a practical example: if we can measure the period of an orbit, and we can measure that orbit's semi-major axis, we can work out the mass M of the sum of the two masses. And if $m_1 \ll m_2$, this means we have a good estimate of the mass of the larger object.

Example 54

There is a large object in the middle of our galaxy, called S0-2, which flings stars around it. How can we estimate its mass?

We can work this out as a ratio:

$$\frac{M}{M_{\text{sun}}} = \left(\frac{a_{S0-2}}{a_{\text{earth}}}\right)^3 \left(\frac{\tau_{\text{earth}}}{\tau_{S0-2}}\right)^2 = \left(\frac{1000AU}{1AU}\right)^3 \left(\frac{1 \text{ year}}{16 \text{ years}}\right)^2,$$

which is about **4 million times** the mass of our sun. Because this is so large, the best explanation is that this object at the center of our galaxy is actually a **black hole**.

10 January 11, 2019 (Recitation)

Fact 55

Problem set 1 solutions were discussed during recitation – this has been omitted.

Example 56

Consider a massless circular hoop of radius a rotating around the z-axis at some angular frequency ϕ. Our goal is to analyze the behavior of a particle of mass m that sits on this hoop.

We use spherical coordinates. The only varying coordinate that is relevant to us is θ (the relative angular position of the mass on the hoop), so our Lagrangian should only depend on θ and $\dot{\theta}$. Let's verify this: the Cartesian coordinates are $x = a\sin\theta\cos\phi$, $y = a\sin\theta\sin\phi$, $z = a\cos\theta$, and we'll use these to calculate the kinetic energy. Taking a derivative and using the product rule yields

$$\dot{x} = a(-\sin(\theta)\sin(\phi)\dot{\phi} + \cos(\phi)\cos(\theta)\dot{\theta}),$$

$$\dot{y} = a(\sin(\theta)\cos(\phi)\dot{\phi} + \sin(\phi)\cos(\theta)\dot{\theta}),$$

$$\dot{z} = -a\sin(\theta)\dot{\theta}.$$

Plugging all of these in and doing a bit of algebra, the kinetic energy is $KE = \frac{1}{2}ma^2(\dot{\theta}^2 + \omega^2\sin^2(\theta))$. Since we also know the potential energy, $U = mga\cos\theta$, we can find the Lagrangian and the EL equation for θ: this yields

$$\ddot{\theta} - \omega^2\sin\theta\cos\theta - \frac{g}{a}\sin\theta = 0.$$

The Hamiltonian (or energy) is of the form

$$\frac{1}{2}ma^2\dot{\theta}^2 - \frac{1}{2}ma^2\omega^2\sin^2\theta + mga\cos\theta,$$

which is notably **not** $K + E$, since K is not just of the form $\frac{1}{2}m\dot{x}^2$! (The reason for this difference is that there must be some external force to keep the hoop always spinning at the same angular frequency at all times.) So now we can see the **minimum spinning frequency** needed to keep the particle stationary at a point other than the bottom: we need to have $\dot{\theta} = \ddot{\theta} = 0$. Plugging this in,

$$\omega^2\cos\theta = -\frac{g}{a},$$

and the minimum ω happens when $\cos\theta = -1$, which happens at $\boxed{\omega^2 = \frac{g}{a}}$.

11 January 14, 2019 (Lecture)

11.1 Review of last lecture

Last week, we discussed gravity and solved the two-body problem. We had two masses in 3-dimensional space, which we described to be at positions $\vec{r}_1$ and $\vec{r}_2$ with masses m_1 and m_2, respectively, and we needed to use $\vec{r} = \vec{r}_1 - \vec{r}_2$ to characterize the gravitational force.

The central idea was that we can reduce this six-dimensional problem into a one-dimensional problem. We first split up the problem into the center of mass movement and relative motion: we let $M = m_1 + m_2$, and we defined $\vec{R} = \frac{1}{M}(m_1\vec{r}_1 + m_2\vec{r}_2)$ to be the position of the center of mass. With this, we could switch reference frames so that the center of mass stayed fixed at the origin throughout the motion.

From there, we defined a **reduced mass** $\mu = \frac{m_1 m_2}{m_1+m_2}$, and Newton's second law reduced to

$$\mu\ddot{\vec{r}} = -\frac{GM_1M_2}{r^2}\hat{r}.$$

The Lagrangian then turns out to be

$$L = \frac{1}{2}\mu\left(\dot{r}^2 + r^2\dot{\theta}^2\right) + \frac{GM_1M_2}{r}$$

in the center-of-mass frame, and conservation of angular momentum showed us that the resulting motion must be confined to a plane. (This was because the Euler-Lagrange equation for θ yields that $\ell = \mu r^2\dot{\theta}$, the angular momentum for the **pseudo-particle** corresponding to the reduced mass, is conserved.)

We also found a force equation (through the Euler-Lagrange equation for r) which simplifies to

$$\boxed{\mu\ddot{r} = \frac{\ell^2}{\mu r^3} - \frac{GM_1M_2}{r^2}}.$$

This has no dependencies on θ, since ℓ is constant, and that $\frac{\ell^2}{\mu r^3}$ term is a **fictitious force**, essentially coming from the fact that we combined our two masses into an effective reduced mass, while ignoring that they can't be too close to each other.

The last quantity that we considered last time was the Hamiltonian:

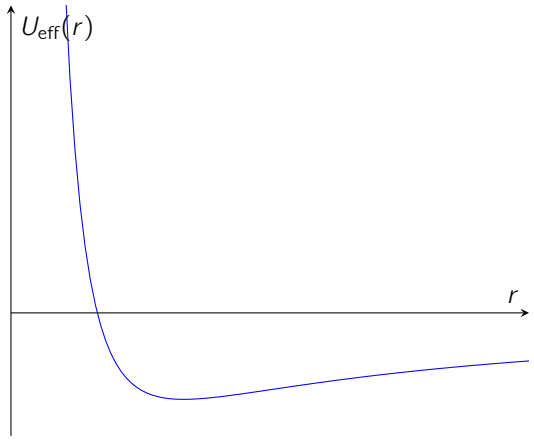
$$\frac{\partial L}{\partial \dot{r}} + \frac{\partial L}{\partial \dot{\theta}}\dot{\theta} - L$$

is conserved in our system, and in this specific case it told us that

$$\frac{1}{2}\mu\dot{r}^2 + \frac{\ell^2}{2\mu r^2} - \frac{GM_1M_2}{r}$$

is conserved. The first term here is the kinetic energy of the reduced mass, and the second two terms are the **effective potential** (the extra term coming from the fact that we're simplifying two bodies into one).

Let's now expand on this discussion a little bit. First of all, let's try to **graph the potential** $U_{\text{eff}} = \frac{\ell^2}{2\mu r^2} - \frac{Gm_1m_2}{r}$. Notice that the first term of the potential, $\frac{\ell^2}{2\mu r^2}$, dominates for small r, while the second term, $-\frac{GM_1M_2}{r}$, dominates for large r. That means that the graph will look something like this:



Now if we pick the value of r that minimizes $U_{\text{eff}}(r)$, we're in a minimum-energy state, and this will correspond to a **circular orbit**. But we can say something more general about the orbit – if we're at some energy $E < 0$, we're at a bounded range of allowed r, and this corresponds to an **ellipse**! In general, the orbit is always a conic: at $E = 0$, we get a **parabola**, and at $E > 0$, we get a **hyperbola**. This last case means that we can have $r \to \infty$, and the system can be unbounded.

This tells us that if we have two masses of equal mass orbiting each other, they each follow an elliptical orbit. Remember that $\vec{r}$ is the difference between the positions of the two masses, so the elliptical orbit for the reduced mass corresponds to the difference of the two positions.

Fact 57

Peri-(something) is the term for which the orbit is closest to the (something) being orbited, so **perihelion** refers to the closest point in an orbit to the sun. Similarly, ap-(something) refers to being furthest away, so **aphelion** is the farthest point from the sun.

Fact 58

If a spaceship gets close enough to a planet, the ship can "whip around" the planet to gain speed. Voyager 1 and 2 did this when launched, because the outer planets' orbits were lined up in a convenient way. (Those spacecraft have now exited the solar system.)

11.2 Other properties of orbits

We're going to simplify the algebra by starting with a circular orbit of radius R, in which case we know that $\dot{r} = \ddot{r} = 0$. Then the (boxed) force equation from above tells us that

$$\frac{GM_1M_2}{R^2} = \frac{\ell^2}{\mu R^3} \implies R = \frac{\ell^2}{\mu G m_1 m_2}.$$

Also, in a circular orbit with r constant, $\dot{\theta}$ is constant (since $\mu r^2\dot{\theta}$ is constant), so we can define the constant $\Omega = \frac{\ell}{\mu R^2} = \dot{\theta}$. Suppose the orbit is **slightly perturbed** (for example, because it interacts with another object in orbit). Then $r = R + \delta r$ for some perturbation δr, so $\dot{r} = \dot{\delta r}, \ddot{r} = \ddot{\delta r}$. Now we can use the Taylor expansion

$$\frac{1}{(R+\delta r)^n} = \frac{1}{R^n\left(1+\frac{\delta r}{R}\right)^n} \approx \frac{1}{R^n}\left(1 - n\frac{\delta r}{R}\right).$$

Plugging this into our force equation,

$$\mu\ddot{r} = \boxed{\mu \cdot \ddot{\delta r} \approx \frac{\ell^2}{\mu R^3}\left(1 - 3\frac{\delta r}{R}\right) - \frac{GM_1M_2}{R^2}\left(1 - 2\frac{\delta r}{R}\right)}.$$

Grouping terms,

$$\mu \cdot \ddot{\delta r} \approx \frac{\ell^2}{\mu R^3} - \frac{GM_1M_2}{r^2} - \delta r\left[\frac{3\ell^2}{\mu R^4} - \frac{2GM_1M_2}{R^3}\right].$$

But now the first two terms (that is, the ones not depending on δr) must cancel out, since we defined R to be the initial orbit radius. And we can also simplify the δr terms in terms of Ω and μ:

$$\mu \cdot \ddot{\delta r} \approx -\delta r\left[3\Omega^2\mu - 2\Omega^2\mu\right] \implies \ddot{\delta r} \approx -\Omega^2\delta r.$$

For small perturbations, this is a simple harmonic oscillator with $\omega_r = \Omega$. So the frequency of oscillations in the orbital direction is the same as the frequency in the radial direction. This actually just means our orbit will become an ellipse, meaning our path will still be periodic.

Corollary 59

Newtonian orbits are **closed**: they return to their initial location and repeat their path indefinitely.

It's interesting to note that there are only two potentials for which this is true regardless of the perturbation: it works for Newtonian orbits, as well as for harmonic oscillator potentials.

11.3 Making corrections to the Newtonian approximation

Despite all of the theory we've been establishing so far, it turns out that Mercury's orbit is **not closed** – the supposed ellipse is actually rotating around the Sun! This is partially happening because we're only considering the ordinary two-body problem (and not considering the gravitational effect of other planets, for example), but even with this factor, there's still an error of about 5×10^{-7} radians per orbit that is unaccounted for.

To explain this, let's take a look at the effective potential

$$U_{\text{eff}}(r) = \left(\frac{\ell^2}{2mr^2} - \frac{GM_\odot m}{r}\right).$$

General relativity actually gives us another factor in this potential! To try to get a sense of what that correction is, note that because $m \ll M_\odot$, we want some small correction (to leading order) of the form

$$U_{\text{eff}}(r) = \left(\frac{\ell^2}{2mr^2} - \frac{GM_\odot m}{r}\right) \times (1 + O(1)(\text{dimensionless quantity})).$$

We also have the **Schwarzschild radius** $S_\odot$ as a natural length scale for this problem, so it makes sense that we'd have a correction of the form

$$U_{\text{eff}}(r) = \left(\frac{\ell^2}{2mr^2} - \frac{GM_\odot m}{r}\right) \times \left(1 + O(1)\frac{S_\odot}{r}\right).$$

In this particular problem (of Mercury orbiting the Sun), $S_\odot = \frac{2GM_\odot}{c^2}$ is about 3 kilometers, while r is a few million kilometers. And it turns out $O(1)$ ends up being -1, because the effective potential is supposed to be 0 at the Schwarzschild barrier. We can further simplify the problem at first order: note that

$$\frac{GM_\odot m}{r} = \frac{1}{2}mc^2\frac{S_\odot}{r},$$

so multiplying it with the additional $O(1)\frac{S_\odot}{r}$ term gives something second order, which we can ignore. So in the end, we have an effective potential

$$\boxed{U_{\text{eff}}(r) = \frac{\ell^2}{2mr^2}\left(1 - \frac{S_\odot}{r}\right) - \frac{GM_\odot m}{r}}.$$

(If we know some relativity, we can also work this out by accounting for **time dilation** when we consider that ℓ^2 is no longer constant.) Regardless, we now have a (small) $\frac{1}{r^3}$ term in our potential, which means we get a small force of order $\frac{1}{r^4}$. And we should try to understand what happens to the orbit now when we perturb slightly away from a circular orbit.

Well, the radius of the circular orbit does change slightly: our force equation is now

$$m\ddot{r} = \frac{\ell^2}{mr^3} - \frac{GM_\odot m}{r^2} - \frac{3S_\odot \ell^2}{2mr^4} \implies \frac{GM_\odot m}{R^2} + \frac{3S_\odot \ell^2}{2mR^4} = \frac{\ell^2}{mR^3} = mR\Omega^2.$$

This gives us a quadratic in r, and we can solve it to find that R changes by a factor of approximately $1 + \frac{S_\odot}{r}$ from the original expression. Now we do our perturbation: again by the Taylor expansion trick,

$$m\ddot{\delta r} = \frac{\ell^2}{mR^3}\left(1 - 3\frac{\delta r}{R}\right) - \frac{GM_\odot m}{R^2}\left(1 - 2\frac{\delta r}{R}\right) - \frac{3S_\odot \ell^2}{2mR^4}\left(1 - 4\frac{\delta r}{R}\right).$$

The terms that don't depend on δr cancel out again by definition of R, and we are left with

$$\ddot{\delta r} = -\delta r\left(\frac{3\ell^2}{m^2R^4} - \frac{2GM_\odot m}{mR^3} - 6\frac{S_\odot \ell^2}{m^2R^5}\right) = -\delta r\left(3\Omega^2 - \frac{2}{mR}\left(\frac{GM_\odot m}{R^2} + \frac{3}{2}\frac{S_\odot \ell^2}{mR^4} + \frac{3}{2}\frac{S_\odot \ell^2}{mR^4}\right)\right)$$

(where we've added and then subtracted the last term), and now the two blue terms together are equal to $MR\Omega^2$, so

$$\ddot{\delta r} = -\delta r\left(3\Omega^2 - 2\Omega^2 - 3\frac{S_\odot}{R}\frac{\ell^2}{m^2R^4}\right) = -\delta r\left(\Omega^2\left(1 - 3\frac{S_\odot}{R}\right)\right),$$

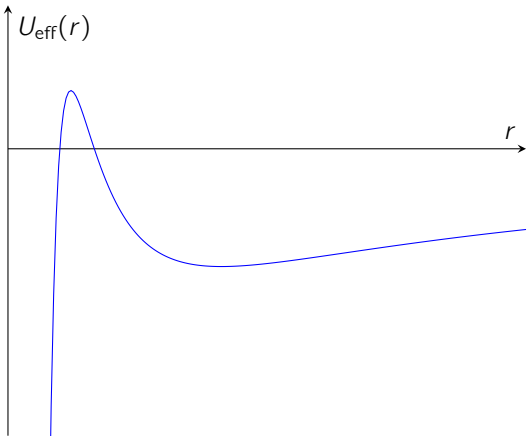
which means our simple harmonic motion has angular frequency (again by a Taylor approximation)

$$\omega = \Omega\sqrt{1 - 3\frac{S_\odot}{R}} \approx \boxed{\Omega\left(1 - \frac{3}{2}\frac{S_\odot}{R}\right)} \neq \Omega.$$

So the oscillations from the perturbation will not be in sync with the initial circular orbit – we'll end up with a **precession** per orbit of

$$2\pi\left(1 - \frac{\omega}{\Omega}\right) = 3\pi\frac{S_\odot}{R} = 3\pi \cdot \frac{3\text{ km}}{58\text{ million km}} \approx 5 \times 10^{-7}\text{ rad/orbit}.$$

Finally, we can draw the effective potential for this general relativity correction, which has an additional $-\frac{1}{r^3}$ term:



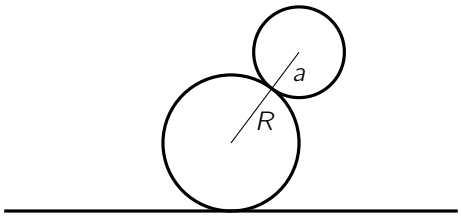
This extra cubic term now dominates at very small r, so the effective potential now has two zeros instead of one. And there are now two extrema for the potential – the minima corresponds to a circular orbit, but so does the maxima (although it is an unstable orbit)! The maxima is actually the point where we need to be at the speed of light to maintain the orbit, and if we further decrease r, we will essentially be inside a black hole. (We can take 8.033 to learn more about this!)

12 January 14, 2019 (Recitation)

We'll do a few more practice problems today:

Example 60

A hoop of radius a and mass m is rolling on a stationary cylinder of radius R. The angle of $\vec{r}$ (the line connecting the centers) is defined to be at an angle θ to the horizontal, and the angle of rotation of the hoop is tracked by ϕ.



Our goal is to find the Lagrangian for this system. Because the cylinder is stationary (its potential and kinetic energy are constant), we don't need to include it in our Lagrangian, which means that we just have

$$L = K - U = \frac{1}{2}mv^2 + \frac{1}{2}I\omega^2 - mgr\sin\theta.$$

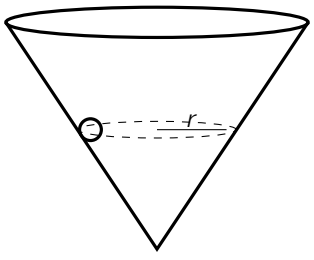
The first term is the translational kinetic energy $\frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2)$ of the hoop, and then the rotational inertia of the hoop is ma^2. Since $\omega = \dot{\theta}$, our final answer is simply

$$\boxed{L = \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\theta}^2 + a^2\dot{\phi}^2\right) - mgr\sin\theta}.$$

It is possible to forget components of the Lagrangian if we're not careful, so we should always use Cartesian coordinates if we're not sure what we're doing.

Example 61

We have an upside-down cone (vertex at the bottom) with slopes at an angle α from the vertical. There is a particle of mass m rolling around (without friction) inside the cone, with coordinates defined so that the horizontal cross-section has radius r and angle of rotation along the cross-section is θ.



To find this Lagrangian, we first write it in terms of Cartesian coordinates: we know that

$$L = K - U = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) - mgz.$$

But we can use cylindrical coordinates here: the angle of the cone tells us that we can parameterize via $x = r\cos\theta$, $y = r\sin\theta$, $z = \frac{r}{\tan\alpha}$. This yields

$$\dot{x} = \dot{r}\sin\theta + r\dot{\theta}\cos\theta,\ \dot{y} = \dot{r}\cos\theta - r\dot{\theta}\sin\theta,\ \dot{z} = \frac{\dot{r}}{\tan\alpha},$$

and now we can plug in as usual:

$$K = \frac{1}{2}m\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right) = \frac{1}{2}m\left(\left(\dot{r}\sin\theta + r\dot{\theta}\cos\theta\right)^2 + \left(\dot{r}\cos\theta - r\dot{\theta}\sin\theta\right)^2 + \left(\frac{\dot{r}}{\tan\alpha}\right)^2\right).$$

which simplifies to

$$= \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\theta}^2 + \frac{\dot{r}^2}{\tan^2\alpha}\right) = \frac{1}{2}m\left(\dot{r}^2\csc^2\alpha + r^2\dot{\theta}^2\right)$$

(by using the fact that $1 + \cot^2\alpha = \csc^2\alpha$). Meanwhile, the potential energy is just $mgz = \frac{mgr}{\tan\alpha}$, so our final Lagrangian is

$$L = K - U = \boxed{\frac{1}{2}m\left(\dot{r}^2\csc^2\alpha + r^2\dot{\theta}^2\right) - \frac{mgr}{\tan\alpha}}.$$

We can now find the Euler-Lagrange equations: the one for r gives us

$$\frac{\partial L}{\partial r} - \frac{d}{dt}\frac{\partial L}{\partial \dot{r}} = 0 \implies mr\dot{\theta}^2 - \frac{mg}{\tan\alpha} - \frac{d}{dt}\left(\frac{m\dot{r}}{\sin^2\alpha}\right) = 0.$$

The $\frac{d}{dt}$ term just adds a dot to the $\dot{r}$ (because everything else is constant inside the derivative), and we can cancel out an m to get

$$r\dot{\theta}^2 - \frac{g}{\tan\alpha} - \frac{\ddot{r}}{\sin^2\alpha} = 0,$$

and multiplying through by $\sin^2\alpha$ yields our equation of motion

$$\ddot{r} - r\dot{\theta}^2\sin^2\alpha + g\sin\alpha\cos\alpha = 0.$$

Similarly, we can find the EL equation for θ:

$$\frac{\partial L}{\partial \theta} - \frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} = 0 \implies -\frac{d}{dt}\left(mr^2\dot{\theta}\right) = 0$$

which is conservation of angular momentum in the xy-plane, which means that p_θ is conserved. (Note that in this problem, the generalized momentum p_r is not conserved because $\frac{\partial L}{\partial r} \neq 0$.) This simplifies to

$$2mr\dot{r}\dot{\theta} + mr^2\ddot{\theta} = 0 \implies 2\dot{r}\dot{\theta} + r\ddot{\theta} = 0.$$

13 January 15, 2019 (Lecture)

The Virial Theorem is used for a number of important physical examples, and we'll do one famous example today. As a reminder, part 1 of our project, as well as problem set 2, are due on Friday. (By the end of today, we will be able to do everything but part 5c.)

13.1 Deriving the Virial Theorem

Suppose we have a collection of particles, and say that the total kinetic energy of this collection is

$$K = \frac{1}{2}\sum_i m_i v_i^2 = \frac{1}{2}\sum_i \vec{p}_i \cdot \vec{v}_i.$$

By the product rule, we can also write this equivalently as

$$= \frac{1}{2}\left(\frac{d}{dt}\left(\sum_i \vec{p}_i \cdot \vec{r}_i\right) - \sum_i \frac{d\vec{p}_i}{dt} \cdot \vec{r}_i\right).$$

Definition 62

The **time-average** of a function f, denoted $\langle f \rangle$, is

$$\langle f \rangle = \lim_{\tau\to\infty} \frac{1}{\tau}\int_0^\tau f(t)dt.$$

For example, the average of a constant is just that constant itself, and the average of $\sin^2 t$ is $\frac{1}{2}$. We'll take the time-average of the kinetic energy expression above, and we can multiply by a factor of 2 to remove the $\frac{1}{2}$:

$$2\langle K \rangle = \left\langle \frac{d}{dt}\sum_i \vec{p}_i \cdot \vec{r}_i \right\rangle - \left\langle \sum_i \frac{d\vec{p}_i}{dt} \cdot \vec{r}_i \right\rangle.$$

The first term here is the time average of a total derivative, so it evaluates to

$$\lim_{\tau\to\infty} \frac{1}{\tau}\left(\sum_i \vec{p}_i \cdot \vec{r}_i\right)_0^\tau$$

by the second fundamental theorem of calculus. But if we accept that the value of $(\sum \vec{p}_i \cdot \vec{r}_i)$ is finite for a bounded system in the center-of-mass frame, and $\frac{1}{\tau}$ goes to 0, then **this first term evaluates to** 0.

That last assumption we made might be a bit hard to believe, so let's see a few examples. (Notably, $\left(\sum_i \vec{p}_i \cdot \vec{r}_i\right)$ will not be bounded if we aren't in the center-of-mass frame.)

Example 63

All oscillatory systems can't give infinite values of $\vec{p}\cdot\vec{r}$ (since they have to return back to the starting point after a finite amount of time, and $\vec{p}\cdot\vec{r}$ is continuous). Let's verify tnis for an elliptical orbit.

We can write the momentum of our orbiting object as

$$\vec{p} = m(\dot{r}\hat{r} + r\dot{\theta}\hat{\theta}),$$

where $\hat{r}$ and $\hat{\theta}$ are perpendicular. Then $\vec{p}\cdot\vec{r} = m\dot{r}r$, and

$$\left\langle \frac{d}{dt}(m\dot{r}r) \right\rangle = \left\langle m\ddot{r}r + m\dot{r}^2 \right\rangle.$$

For the two-body problem, we can substitute in some results from earlier in the class to find that this is indeed

$$\left\langle 2E + \frac{GMm}{r} \right\rangle = 0.$$

So going back to our initial derivation, we've now justified why we can toss the total derivative term, and we're left with

$$2\langle K\rangle = -\left\langle \sum_i \frac{d\vec{p}_i}{dt}\cdot\vec{r}_i \right\rangle = -\left\langle \sum_i \vec{r}_i\cdot\vec{F}_i \right\rangle,$$

and this yields the following generic expression:

Proposition 64 (Virial Theorem)
The average value of the kinetic energy of a system satisfies the relation

$$2\langle K\rangle = -\left\langle \sum_i \vec{r}_i\cdot\vec{F}_i \right\rangle.$$

This is the general form of the Virial Theorem, and it is not particularly enlightening in this expression. To make it more useful, we'll restrict ourselves to specific cases and see how the equation simplifies.

13.2 A useful statement of the Virial Theorem

We'll restrict ourselves to the case where we can write our force as a derivative of a potential. Let's consider the case where we have a **central force**, so we can write

$$\vec{F}_i = -\sum_{j\neq i}\frac{\partial U}{\partial r_{ij}}\hat{r}_{ij}.$$

Plugging this in to our result, the negative signs from the force and the right-hand side of the Virial Theorem cancel, and we're left with

$$2\langle K\rangle = \left\langle \sum_{i\neq j}\vec{r}_i\cdot\frac{\partial U}{\partial r_{ij}}\hat{r}_{ij} \right\rangle.$$

But the $\hat{r}_{ij}$ vectors refer to relative distances, while the $\vec{r}_i$ vectors refer to absolute locations. To help with this, let's consider two specific particles i and j. $\hat{r}_{ij}$ is defined as $\vec{r}_i - \vec{r}_j$, so the two particles contribute a combined

$$\vec{r}_i\cdot\frac{\partial U}{\partial r_{ij}}\hat{r}_{ij} + \vec{r}_j\cdot\frac{\partial U}{\partial r_{ij}}(-\hat{r}_{ij})$$

(the negative sign in the second term coming since we flipped the direction of the vector) to the right-hand side, which further simplifies to

$$= (\vec{r}_i - \vec{r}_j)\frac{\partial U}{\partial r_{ij}}\hat{r}_{ij} = r_{ij}\frac{\partial U}{\partial r_{ij}}.$$

So by rewriting the sum in a slightly different way, we've removed the vectors and replaced them with lengths! And now we have that

$$2\langle K\rangle = \left\langle \sum_{i<j} r_{ij}\frac{\partial U}{\partial r_{ij}} \right\rangle$$

(we write $i<j$ to avoid double-counting, so we only count each pair of particles once). Let's look at this in the special case where the potential follows some **power law** $U(r) = \alpha r^\beta$ for constants α,β. (This encompasses gravity, most mechanical laws, and so on, so it is still fairly general.) Then we have

$$2\langle K\rangle = \langle\sum_{i<j} r_{ij}\alpha\beta r_{ij}^{\beta-1}\rangle = \langle\sum_{i<j}\beta\cdot\alpha r_{ij}^\beta\rangle \implies \boxed{2\langle K\rangle = \beta\langle U\rangle}.$$

In words, **twice the average value of the kinetic energy is β times the average value of the potential energy.**

Example 65

Gravity's potential decays like $\frac{1}{r}$, so we have that $2\langle K\rangle = -\langle U\rangle$. Meanwhile, a spring-mass system's potential grows proportional to r^2, so we have that $\langle K\rangle = \langle U\rangle$.

13.3 An application

We'll now use this result to analyze the total mass of a galaxy cluster. Start with the kinetic energy

$$2\langle K\rangle = \left\langle \sum_i m_i v_i^2 \right\rangle.$$

For the sake of the argument, let's say that all the individual parts have approximately the same mass, and say that there are n particles with total mass M. Then this simplifies to

$$\frac{M}{n}\left\langle \sum v_i^2 \right\rangle \approx \frac{M}{n}\left\langle n\overline{v^2}\right\rangle,$$

which means that $2\langle K\rangle \approx M\overline{v^2}$. (We drop the time-average because we don't get enough time to go through a full period of motion for the galaxy cluster; we just assume that there's enough statistical variance that the averages work out.) Now, we can find the value of our potential energy U: the virial theorem gives us

$$M\overline{v^2} \approx -U = G\sum_{i<j}\frac{M_i M_j}{r_{ij}}.$$

This is $\approx \frac{3}{5}\frac{GM^2}{R}$ if we assume there is a uniform mass distribution in a sphere, or we can assume that all distances are R to get $\approx \frac{G(M/n)^2}{R}\frac{n(n-1)}{2} \approx \frac{1}{2}\frac{GM^2}{R}$. Either way, we have some relationship of the form

$$M\overline{v^2} \approx O(1)\cdot\frac{GM^2}{R} \implies M = O(1)\frac{R\overline{v^2}}{G}.$$

We can easily observe R, and we know G, so the only thing we need to estimate the total mass M is the average value $\overline{v^2}$. For this, we can use the **Doppler shift**: the frequency of light changes based on whether a particle is moving closer to or away from us. Since we know that cosmic dust usually absorbs some specific wavelengths, we can measure the frequency of light from our galaxy cluster and see what happens to those wavelengths! This gives us the relative velocity along the line of sight, and then we can bring in some information about transverse motion to refine our estimate.

Fact 66

This process was carried out in 1937 for the Coma Cluster. The result was interesting, because when we analyzed the light that was coming from that galaxy cluster, a large amount of mass was not coming from the stars. This led to the discovery of dark matter!

Another way to estimate the mass of a cluster is use the idea of **gravitational lensing.** Due to general relativity, light actually bends when it goes through spacetime, so we often see **Einstein rings**: if one galaxy is behind another one, the light from the back galaxy will be seen as a ring as it bends around the front galaxy.

This gives another way to determine the amount of mass in a cluster: how much bending is happening? And experimentally, the results that we get basically agree with the previous argument.

13.4 Further intricacies of this argument

Suppose now that we have a **spiral galaxy**, which we can approximate as a sphere with radius R and mass M, plus some spiral arms that have much less mass. Consider an individual star that lives on an arm a distance r from the center of the galaxy, and say that it has mass m_s.

For the sake of simplicity, we'll assume this star is in a circular orbit, so we have the equation

$$m_s \frac{v^2}{r} = \frac{GMm_s}{r^2} \implies v = \sqrt{\frac{GM}{r}}.$$

But on the other hand, consider some star inside the sphere of radius R. By Gauss's law (or the shell theorem), we only need to consider the mass $M(r)$ that is at a radius r or less away from the center, meaning

$$m_s \frac{v^2}{r} = \frac{GM(r)m_s}{r^2} \implies v^2 = \frac{G}{r} M(r) = \frac{G}{r} \left(\frac{r}{R}\right)^3 M \implies v = r\sqrt{\frac{GM}{R^3}}.$$

So the velocity should be linear as we go from the center to the edge of the sphere, and then it decays as an inverse square root.

Fact 67

This predicted behavior does not happen physically! It's generally observed that the velocity of the orbit is independent of the distance past R.

So stars are rotating too fast; they really shouldn't be bound under normal circumstances. So there's some mass that we can't see in the universe, and it is distributed very differently from how normal matter is distributed.

Fact 68

From our observations, this unknown matter probably has density $\rho = \frac{M}{4\pi R r^2}$, and it seems to stay in a spherical distribution, unlike normal matter (which collapses into a disk).

One last point: after two galaxy clusters collide, we often see x-rays emitted from some areas (known as **excited areas**), plus some other areas of gravitational lensing (where the mass is supposed to be). Observations of these areas seem to indicate that dark matter does not get "slowed down" in the same way that normal matter does!

14 January 15, 2019 (Recitation)

We'll talk about two special problems today.

14.1 Proving the ideal gas law from the Virial Theorem

First of all, let's write the Virial Theorem in the form

$$2\langle K \rangle = -\left\langle \sum_i \vec{r}_i \cdot \vec{F}_i \right\rangle.$$

We will use the following fact from statistical mechanics:

Proposition 69 (Equipartition Theorem for 3D ideal gases)

Using standard constants,

$$\langle K \rangle = \frac{3}{2} N k_b T,$$

where N is the number of particles in our gas, k_b is the Boltzmann constant, and T is the temperature.

Suppose now that we have an ideal gas inside an enclosed surface. Then the force on the surface is the pressure (which is constant everywhere inside the surface) times the unit area:

$$d\vec{F} = -P\hat{n}\, dA.$$

So now if we integrate over all particles (in the continuum), we want to consider all forces acting on the surface, and we can say that

$$\left\langle \sum_i \vec{r}_i \cdot \vec{F}_i \right\rangle \to \oiint \vec{r} \cdot d\vec{F} = -P \oiint \hat{n} \cdot \vec{r}\, dA.$$

Now by the divergence theorem, this is equal to

$$-P \int (\vec{\nabla} \cdot \vec{r}) dV = -3P \int dV = -3PV,$$

since the divergence of $\langle x, y, z \rangle$ is just $1 + 1 + 1 = 3$. And now combining the two sides of the Virial Theorem,

$$2\left(\frac{3}{2} N k_b T\right) = -(-3PV) \implies PV = 3N k_b T,$$

as desired.

14.2 An example of a central potential

Example 70

Consider a particle orbiting in a central potential $V(r)$, meaning the acceleration of the particle is $-\nabla V$ (assume the particle has a mass of 1). We wish to analyze the behavior of circular orbits.

Let's start by trying to determine the **period** of such an orbit. We know that $T = \frac{2\pi}{\omega}$, where $\omega = \dot{\phi}$ is the angular frequency, so our goal is to find $\dot{\phi}$ in terms of $r, V, \frac{dV}{dr}$. First of all, the total energy of our system is

$$E = K + U = \frac{1}{2}(\dot{r}^2 + r^2\dot{\phi}^2) + V(r).$$

We can simplify this to

$$\frac{1}{2}\dot{r}^2 + V_{\text{eff}}(r),$$

where $V_{\text{eff}} = \frac{1}{2} r^2 \dot{\phi}^2 + V(r)$, to match the usual form of our Hamiltonians. Notice that $\ddot{r}$ is 0 for a circular orbit, so by Newton's second law, $\frac{dV_{\text{eff}}}{dr} = 0$, which means that

$$0 = \frac{dV_{\text{eff}}}{dr} = \frac{d}{dr}\left(\frac{1}{2} r^2 \dot{\phi}^2\right) + \frac{dV}{dr}.$$

Rewriting $\dot{\phi}$ in terms of the angular momentum ℓ (here, $\ell = r^2\dot{\phi}$), the above equation simplifies to

$$-\frac{\ell^2}{r^3} + \frac{dV}{dr} = 0 \implies \frac{dV}{dr} = \frac{\ell^2}{r^3} = r\dot{\phi}^2.$$

So we can write $\dot{\phi}$ in terms of r and $\frac{dV}{dr}$, and plugging everything in yields a period of

$$\boxed{T = 2\pi \left(\frac{1}{r}\frac{dV}{dr}\right)^{-1/2}}.$$

Now let's perturb this circular orbit: how do we know if a small perturbation will change the path dramatically? Using similar arguments as we've already done, we find that the frequency of radial oscillation for a noncircular orbit $r(t) = r_0 + \varepsilon(t)$ (where $\varepsilon^2 \ll r_0^2$) is

$$\omega_r = \left(\frac{1}{r^3}\frac{d}{dr}\left(r^3\frac{dV}{dr}\right)\right)^{1/2}.$$

Example 71

Consider the **Yukawa potential** (which is important in particle physics)

$$V(r) = -\frac{GM}{r}e^{-kr}.$$

(Here $k = \alpha m$; for the Newtonian potential, we have $\alpha = 0$.)

Let's try to determine the maximum possible r for stable circular orbits. Stability is determined by the sign of ω_r^2, which is proportional to $\frac{d^2V}{dr^2}$. (If $\omega_r^2 > 0$, the orbit is stable, since we have a point of concavity.) Plugging in the Yukawa potential into our expression for ω_r above, we find that

$$\omega_r^2 = \frac{1}{r^3}\frac{d}{dr}\left(r^3\frac{d}{dr}\left(-\frac{GM}{r}e^{-kr}\right)\right),$$

and with some work, this ends up simplifying to

$$= \frac{GM}{r^3}e^{-kr}(1 + kr - (kr)^2).$$

This means we want $1 + kr - (kr)^2 > 0$, which happens as long as $r < \frac{1+\sqrt{5}}{2k}$. (Larger r than that will give us an unstable circular orbit.)

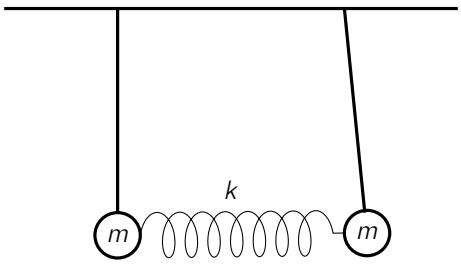
15 January 16, 2019 (Lecture)

Today we're going to be talking about normal modes, which have some connection to analogous quantum systems.

15.1 A coupled problem

Example 72

Consider two pendulums: both have length ℓ and mass m, and they are connected by a spring of spring constant k. Let θ_1 and θ_2 be the angular displacements of the two pendulums – suppose that the spring is in its equilibrium length at $\theta_1 = \theta_2 = 0$.



As always, we'll take $\theta_1, \theta_2 \ll 1$. We'll solve this problem by first writing down the Lagrangian for the system:

$$L = K - U = \frac{1}{2}m\ell^2(\dot{\theta}_1^2 + \dot{\theta}_2^2) - mg\ell(1 - \cos\theta_1) - mg\ell(1 - \cos\theta_2) - \frac{1}{2}k(\ell\sin\theta_1 - \ell\sin\theta_2)^2.$$

Let's go through this part by part. The first term here is kinetic energy, since $v = r\omega = \ell\dot{\theta}$ for the pendulums. The next two terms are gravitational potential energy, and the last term comes from spring potential energy, since $\ell\sin\theta_1$ and $\ell\sin\theta_2$ are the horizontal displacements of the two pendulums. Using the small angle approximation, $1-\cos\theta \approx \frac{\theta^2}{2}$ and $\sin\theta \approx \theta$, which yields a polynomial Lagrangian

$$L = \frac{1}{2}m\ell^2(\dot{\theta}_1^2 + \dot{\theta}_2^2) - \frac{mg\ell}{2}(\theta_1^2 + \theta_2^2) - \frac{k\ell^2}{2}(\theta_1 - \theta_2)^2.$$

Notice that all terms in the Lagrangian are squared at leading order, so we can't make an approximation of $\cos\theta = 1$. (Alternatively, we can be more careful and approximate after we use the Euler-Lagrange equation.) Now the EL equation for θ_1 is

$$\frac{\partial L}{\partial\theta_1} - \frac{d}{dt}\frac{\partial L}{\partial\dot{\theta}_1} = 0 \implies -mg\ell\theta_1 - k\ell^2(\theta_1 - \theta_2) - \frac{d}{dt}\left(m\ell^2\dot{\theta}_1\right),$$

which yields an equation of motion (after dividing through by $-m\ell^2$) of

$$\boxed{\ddot{\theta}_1 + \frac{g}{\ell}\theta_1 - \frac{k}{m}(\theta_2 - \theta_1) = 0.}$$

The EL equation for θ_2 looks basically the same: the Lagrangian is completely symmetric, so we will get the same equation by changing all the indices:

$$\boxed{\ddot{\theta}_2 + \frac{g}{\ell}\theta_2 - \frac{k}{m}(\theta_1 - \theta_2) = 0}.$$

These are **coupled differential equations**: how θ_1 behaves is affected by how θ_2 behaves, and vice versa.

15.2 Eigenvalues and normal modes

Whenever we have oscillators that are coupled like this, the standard solution is to look for specific nice behavior of our system:

Definition 73

A **normal mode** is a solution to our equations of motion, where all the elements of the system are oscillating at the same frequency.

Writing these out algebraically, we start with the trial solution

$$\theta_1(t) = c_1 e^{i\omega t}, \quad \theta_2(t) = c_2 e^{i\omega t}$$

for some frequency ω to be determined. (This is the complex-valued expression for the solution – we could also just

use cosines and sines, but writing the trial solutions this way makes it easier to take derivatives.) Notice that the amplitudes of the two masses can be different, but ω, the angular frequency, is the same. Then $\ddot{\theta}_1 = -\omega^2\theta_1$, so

$$-\omega^2 c_1 e^{i\omega t} + \frac{g}{\ell} c_1 e^{i\omega t} - \frac{k}{m}(c_2 - c_1)e^{i\omega t} = 0 \implies \boxed{c_1\left(-\omega^2 + \frac{g}{\ell} + \frac{k}{m}\right) - \frac{k}{m}c_2 = 0}.$$

By symmetry, we also get the analogous solution for θ_2:

$$\boxed{c_2\left(-\omega^2 + \frac{g}{\ell} + \frac{k}{m}\right) - \frac{k}{m}c_1 = 0}.$$

Our equations are now linear in c_1 and c_2, meaning that we can write the two equations together in the matrix form

$$\begin{bmatrix} \frac{g}{\ell} + \frac{k}{m} & -\frac{k}{m} \\ -\frac{k}{m} & \frac{g}{\ell} + \frac{k}{m} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \omega^2 \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}.$$

Our goal is to solve for c_1 and c_2, and this is an **eigenvalue problem** – the values of ω^2 are the eigenvalues, and the amplitude ratios are the eigenvectors. There are many ways to solve this: usually we'll be dealing with 2×2 or a 3×3 matrix, so the numbers won't be too ugly.

Proposition 74 (The linear algebra way to do this)

To find the eigenvalues λ and eigenvectors v of a matrix A, we want to pick out the values of λ such that

$$Av = \lambda v \implies Av = \lambda I v = 0 \implies (A - \lambda I)v = 0,$$

meaning the **determinant of $A - \lambda I$ is 0.**

So in this specific problem, we want to find the values of ω where

$$\det \begin{bmatrix} \frac{g}{\ell} + \frac{k}{m} - \omega^2 & -\frac{k}{m} \\ -\frac{k}{m} & \frac{g}{\ell} + \frac{k}{m} - \omega^2 \end{bmatrix} = 0.$$

This simplifies to

$$\left(\frac{g}{\ell} + \frac{k}{m} - \omega^2\right)^2 - \left(\frac{k}{m}\right)^2 = 0,$$

which means $\frac{g}{\ell} + \frac{k}{m} - \omega^2 = \pm\frac{k}{m}$, so the possible frequencies are $\boxed{\omega^2 = \frac{g}{\ell}}$ and $\boxed{\frac{g}{\ell} + \frac{2k}{m}}$. Let's analyze each of these cases separately.

For $\omega^2 = \frac{g}{\ell}$,

$$c_1\frac{k}{m} - \frac{k}{m}c_2 = 0 \implies c_1 = c_2.$$

This corresponds to the case where the two masses are **in phase**: they move parallel to each other, so the spring doesn't apply any force. We can also define the angle $\boxed{\eta_a = \frac{1}{2}(\theta_1 + \theta_2)}$, where the $\frac{1}{2}$ is just a normalization factor and will become clear soon.

On the other hand, when $\omega^2 = \frac{g}{\ell} + \frac{2k}{m}$, we have that

$$-\frac{k}{m}c_1 - \frac{k}{m}c_2 = 0 \implies c_1 = -c_2.$$

So the two masses are **out of phase** this time, and here we'll use the defining angle $\boxed{\eta_b = \frac{1}{2}(\theta_1 - \theta_2)}$.

Fact 75 (18.03 aside)

These ηs are important, because they are the change of coordinates needed to **decouple** the motion.

(Decoupling basically means that we can solve differential equations of the variables independently of each other.) Let's go back to the two initial equations of motion: if we add them together, we'll find that

$$\ddot{\theta}_1 + \ddot{\theta}_2 + \frac{g}{\ell}(\theta_1 + \theta_2) = 0.$$

Notice that this means $\theta_1 + \theta_2$, and therefore η_a, oscillates in a simple harmonic motion with

$$\ddot{\eta}_a + \omega_a^2 \eta_a = 0,$$

and this is easy to describe with our usual methods. Similarly, we can also subtract the two equations of motion to find

$$(\ddot{\theta}_1 - \ddot{\theta}_2) + \frac{g}{\ell}(\theta_1 - \theta_2) + \frac{2k}{m}(\theta_1 - \theta_2) = 0,$$

and now $\theta_1 - \theta_2$ (and therefore η_b as well) will oscillate in simple harmonic motion with

$$\ddot{\eta}_b + \left(\frac{g}{\ell} + \frac{2k}{m}\right)\eta_b = 0.$$

Therefore, we can write

$$\eta_a(t) = A\cos(\omega_a t) + B\sin(\omega_a t)$$

and

$$\eta_b(t) = C\cos(\omega_b t) + D\sin(\omega_b t),$$

and now η_a and η_b are linear combinations of θ_1 and θ_2, so it's easy for us to go backwards and solve for θ_1 and θ_2 from here.

The idea is that if we set up the problem with normal mode initial conditions, it will remain in that normal mode! The generic motion is pretty complicated, but it will always be a **linear combination** of the normal modes (which often form a basis of the allowed solutions).

Fact 76

In general, if a system has N degrees of freedom, then there are N normal modes (which are the simplest patterns of movement for the system).

Although we won't talk about it here, quantum mechanics uses normal modes fundamentally to construct superpositions of states (similarly to what we are discussing here).

15.3 Using initial conditions

Example 77

Suppose we know that our system satisfies

$$\theta_1(0) = 0, \quad \theta_2(0) = 0, \quad \dot{\theta}_1(0) = \Omega, \quad \dot{\theta}_2 = 0$$

(so the system starts at rest, but we just give one of the pendulums a kick at $t = 0$).

The easiest way to account for these initial conditions is to solve for the analogous initial conditions for η_a and η_b. Using the A, B, C, D coefficients for our expressions η_a and η_b from above, we find that

$$\eta_a(0) = A = 0, \quad \eta_b(0) = C = 0.$$

This means that some of the terms drop out, and we can take derivatives more easily:

$$\dot{\eta}_a(0) = B\omega_a = \frac{1}{2}(\dot{\theta}_1(0) + \dot{\theta}_2(0)) = \frac{\Omega}{2} \implies B = \frac{\Omega}{2\omega_a}.$$

Similarly,

$$\dot{\eta}_b(0) = D\omega_b = \frac{\Omega}{2} \implies D = \frac{\Omega}{2\omega_b}.$$

So we've fully determined the equations of motion for

$$\eta_a(t) = \frac{\Omega}{2\omega_a}\sin(\omega_a t), \quad \eta_b(t) = \frac{\Omega}{2\omega_b}\sin(\omega_b t),$$

and now we know the exact form of θ_1 and θ_2 by finding the appropriate linear combinations:

$$\boxed{\theta_1(t) = \eta_a(t) + \eta_b(t) = \frac{\Omega}{2}\left[\frac{\sin(\omega_a t)}{\omega_a} + \frac{\sin(\omega_b t)}{\omega_b}\right]},$$

$$\boxed{\theta_2(t) = \eta_a(t) - \eta_b(t) = \frac{\Omega}{2}\left[\frac{\sin(\omega_a t)}{\omega_a} - \frac{\sin(\omega_b t)}{\omega_b}\right]}.$$

Notice that this system is not in either of the two pure normal modes, which means that the energy in the system will evolve and shift between the normal modes. In particular, this also means that there will be times when the first mass is moving but the second isn't, and vice versa.

15.4 A real-world example

Neutrinos are like having coupled pendulums with different spring constants. Just like in the example above, we can only physically create them by "hitting one pendulum," so they will never be in a pure normal mode. And the oscillation probabilities will depend on the time when we look at the electron neutrino!

Fact 78

Quoted from Wikipedia: "Today it is accepted that the neutrinos produced in the Sun are not massless particles as predicted by the Standard Model but rather mixed quantum states made up of defined-mass eigenstates in different (complex) proportions." This has to do with the **solar neutrino problem**.

(A good point to keep in mind for these quantum particles is that frequency in quantum mechanics correspond to energy.) This kind of "mixing of normal modes" happens in quarks as well – generally, nature will "screw things up" to get mixing of a lot of different states, which is part of the reason why it is so hard to keep track of particles.

16 January 16, 2018 (Recitation)

Problem set 1 has been graded and passed back; we should check the Stellar grade to make sure everything is consistent. Today, we're going to do a double vertical spring as an example of a normal mode problem.

Example 79

Consider a system where a mass of $m_1 = 2m$ is hanging from the ceiling by a vertical spring of constant k. Another mass of $m_2 = m$ is hanging from the first mass with a spring also of spring constant k.

Define x_1, x_2 to be the displacements of the masses from their equilibrium position. Then the total kinetic energy is

$$K = \frac{1}{2}m_1\dot{x}_1^2 + \frac{1}{2}m_2\dot{x}_2^2,$$

and the potential energy (relative to equilibrium) is

$$U = -2mgx_1 - mgx_2 + \frac{1}{2}kx_1^2 + \frac{1}{2}k(x_2 - x_1)^2,$$

where the first two terms are from the gravitational potential and the last two terms are from the spring potential. So our total Lagrangian for this problem is

$$L = \frac{1}{2}m_1\dot{x}_1^2 + \frac{1}{2}m_2\dot{x}_2^2 + 2mgx_1 + mgx_2 - \frac{1}{2}kx_1^2 - \frac{1}{2}k(x_2 - x_1)^2.$$

The Euler-Lagrange equation for x_1 is

$$\frac{\partial L}{\partial x_1} - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}_1} = 0 \implies 2mg - kx_1 + k(x_2 - x_1) - 2m\ddot{x}_1 = 0,$$

which can be written as

$$\boxed{2m\ddot{x}_1 - 2mg + k(2x_1 - x_2) = 0}.$$

Similarly, Euler-Lagrange for x_2 yields

$$\frac{\partial L}{\partial x_2} - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}_2} = 0 \implies \boxed{m\ddot{x}_2 - mg + k(x_2 - x_1) = 0}.$$

There is a contribution due to gravity in both equations, but with an appropriate change of variables, we can get rid of it: with a specific linear translation $y_1 = x_1 + a, y_2 = x_2 + b$ (proportional to $\frac{mg}{k}$), we will remove the mg terms, and this will create new equations of motion

$$2m\ddot{y}_1 + k(2y_1 - y_2) = 0,$$

$$m\ddot{y}_2 + k(y_2 - y_1) = 0.$$

These can be written in the more suggestive form

$$\left(\frac{d^2}{dt^2} + \frac{k}{m}\right)y_1 - \left(\frac{k}{2m}\right)y_2 = 0,$$

$$-\left(\frac{k}{m}\right)y_1 + \left(\frac{d^2}{dt^2} + \frac{k}{m}\right)y_2 = 0,$$

which can be combined into a single matrix equation

$$\begin{bmatrix} \frac{d^2}{dt^2} + \frac{k}{m} & -\frac{k}{2m} \\ -\frac{k}{m} & \frac{d^2}{dt^2} + \frac{k}{m} \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Adopting the trial solution $\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} e^{i\omega t}$, taking a second derivative will yield $-\omega^2$ times the trial solution, so the

above matrix equation can be written as

$$\begin{bmatrix} -\omega^2 + \frac{k}{m} & -\frac{k}{2m} \\ -\frac{k}{m} & -\omega^2 + \frac{k}{m} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} e^{i\omega t} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

This is now an eigenvalue problem for $\lambda = \omega^2$, so setting the determinant equal to 0 gives us the equation

$$\left(\frac{k}{m} - \omega^2\right)^2 - \frac{1}{2}\frac{k^2}{m^2} = 0.$$

This is a quadratic in ω^2: we find that

$$\omega_{\pm}^2 = \frac{1}{2}\left(\frac{2k}{m} \pm \sqrt{\frac{4k^2}{m^2} - \frac{4k^2}{2m^2}}\right) = \frac{k}{m}\left(1 \pm \frac{1}{\sqrt{2}}\right).$$

We can now solve for c_1 and c_2 (or more specifically, their ratio) by plugging in values of ω into the matrix form. For $\omega^2 = \frac{k}{m}\left(1 + \frac{1}{\sqrt{2}}\right)$,

$$\left(\frac{k}{m} - \frac{k}{m}\left(1 + \frac{1}{\sqrt{2}}\right)\right) c_1 - \frac{k}{2m} c_2 = 0 \implies c_1 = -\frac{1}{\sqrt{2}} c_2,$$

which implies that

$$y_1(t) = -\frac{1}{\sqrt{2}} y_2(t).$$

In words, one of our normal modes has the two masses moving in **opposite** (or **out-of-phase**) motion. Similarly, the eigenvalue $\omega^2 = \frac{k}{m}\left(1 - \frac{1}{\sqrt{2}}\right)$ yields

$$c_1 = \frac{1}{\sqrt{2}} c_2 \implies y_1(t) = \frac{1}{\sqrt{2}} y_2(t),$$

which is the normal mode where the two masses are moving in **similar** (or **in-phase**) motion.

17 January 17, 2019 (Lecture)

We'll continue talking about normal modes today.

17.1 Effective spring constants

Consider some arbitrary potential $U(x)$. If we Taylor expand the function about a minimum point x_0, we have that

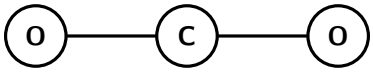
$$U(x_0 + x) \approx U(x_0) + \left.\frac{dU}{dx}\right|_{x_0} x + \frac{1}{2}\left.\frac{d^2U}{dx^2}\right|_{x_0} x^2 + \cdots.$$

The first term is some arbitrary constant, which we can ignore. The second term is zero because x_0 is a minimum, and the second derivative $\frac{d^2U}{dx^2}$ must be positive by the second derivative test (again, because we are at a minimum). This is essentially our **spring constant** for small oscillations about the equilibrium x_0! So we can say that as long as we are sufficiently close to the equilibrium point, $U(x) \approx \frac{1}{2}kx^2$.

Example 80

Consider a carbon dioxide molecule, and graph the potential energy coming from the C–O bonds. Although the energy in this bond is not quadratic, it does hit a minimum of around -0.21 eV at about 1.1 angstroms, so the energy is quadratic near that minimum.

As a quick reminder, here's what carbon dioxide looks like:



The central idea is that if we slightly excite the molecule out of its ground state, it should oscillate like three masses connected by springs. So let's find the **normal modes** of such a system.

Our molecule is made up of three masses (two oxygen atoms and one carbon atom). We're in 3-dimensional space, so there are $3 \times 3 = 9$ degrees of freedom. In principle, then, there should be 9 normal modes. Three of the nine modes are just **translational modes**: basically, all the masses are moving in some direction with a constant velocity. An additional two modes are just rotations (around the axes that are not the axis of the molecule), so we should expect **four normal modes** that actually involve compressing the springs in some form.

17.2 Solving the normal mode problem

Let the masses of the oxygen be m_o and the mass of the carbon be m_c. (Remember that we're treating the C–O bond as a spring with some fixed effective spring constant k.) First, let's find the modes that only involve x-direction motion, because those are easier on our notation.

Suppose the displacements of the three atoms are x_1, x_2, x_3 from their equilibrium positions. The Lagrangian is then

$$L = K - U = \frac{1}{2}m_o(\dot{x}_1^2 + \dot{x}_3^2) + \frac{1}{2}m_c\dot{x}_2^2 - \frac{1}{2}k\left[(x_1 - x_2)^2 + (x_3 - x_2)^2\right].$$

Now by the Euler-Lagrange equations, we can find that

$$\frac{\partial L}{\partial x_1} = \frac{d}{dt}\frac{\partial L}{\partial \dot{x}_1} \implies -k(x_1 - x_2) = \frac{d}{dt}\left[m_o\dot{x}_1\right] \implies \boxed{\ddot{x}_1 + \frac{k}{m_o}(x_1 - x_2) = 0},$$

and similarly

$$\frac{\partial L}{\partial x_2} = \frac{d}{dt}\frac{\partial L}{\partial \dot{x}_2} \implies -k[-(x_1 - x_2) - (x_3 - x_2)] = \frac{d}{dt}[m_c\dot{x}_2] \implies \boxed{\ddot{x}_2 + \frac{k}{m_c}[2x_2 - x_1 - x_3] = 0}.$$

By symmetry, the Lagrangian is identical for masses 1 and 3, so we'll also have an equation of motion

$$\frac{\partial L}{\partial x_3} = \frac{d}{dt}\frac{\partial L}{\partial \dot{x}_3} \implies \boxed{\ddot{x}_3 + \frac{k}{m_o}(x_3 - x_2) = 0}.$$

We now have a set of three coupled differential equations. As always, we find the normal modes by starting with a trial solution

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} e^{i\omega t}.$$

Again, the second derivatives of each of x_1, x_2, x_3 are just $-\omega^2$ times the functions themselves, so our equations become

$$-\omega^2 c_1 + \frac{k}{m_o}(c_1 - c_2) = 0 \quad -\omega^2 c_2 + \frac{k}{m_c}[2c_2 - c_1 - c_3] \quad = 0 \quad -\omega^2 c_3 + \frac{k}{m_o}(c_3 - c_2) = 0.$$

Now we want to find the eigenvalues: we can write these equations in the matrix form

$$\begin{bmatrix} \frac{k}{m_o} - \omega^2 & -\frac{k}{m_o} & 0 \\ -\frac{k}{m_c} & \frac{2k}{m_c} - \omega^2 & -\frac{k}{m_c} \\ 0 & -\frac{k}{m_o} & \frac{k}{m_o} - \omega^2 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = 0,$$

and we want to set the determinant of the 3 by 3 matrix equal to 0. This yields (after a bit of calculation)

$$0 = \omega^2 \left(\frac{k}{m_o} - \omega^2 \right) \left(\omega^2 - \frac{2k}{m_c} - \frac{k}{m_o} \right) = 0.$$

The first root here is $\omega^2 = 0$, which corresponds to **translational motion**. More specifically, this is a non-oscillating normal mode, since it gives $c_1 = c_2 = c_3$ and means the center of mass is moving, but there aren't really any oscillations. (This happens pretty often; it's a degenerate case, and we've already predicted this kind of behavior above.)

On the other hand, the second root is $\omega^2 = \frac{k}{m_o}$, which does correspond to oscillatory motion. The matrix simplifies to

$$\begin{bmatrix} 0 & -\frac{k}{m_o} & 0 \\ -\frac{k}{m_c} & \frac{2k}{m_c} - \frac{k}{m_o} & -\frac{k}{m_c} \\ 0 & -\frac{k}{m_o} & 0 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = 0.$$

Solving the resulting system of equations, we find that $c_2 = 0$ and $c_1 = -c_3$. This means the two oxygen molecules are oscillating in opposite motion, while the middle mass does nothing (since the carbon atom has equal and opposite forces from the two oxygen atoms acting on it at all times).

Finally, the last root is $\omega^2 = \frac{2k}{m_c} + \frac{k}{m_o}$. Plugging this in,

$$\begin{bmatrix} -\frac{2k}{m_c} & -\frac{k}{m_o} & 0 \\ -\frac{k}{m_c} & -\frac{k}{m_o} & -\frac{k}{m_c} \\ 0 & -\frac{k}{m_o} & -\frac{2k}{m_c} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = 0$$

This yields $c_2 = -\frac{2m_o}{m_c} c_1 = -\frac{2m_o}{m_c} c_3$, so our normal mode amplitudes are proportional to $\begin{bmatrix} m_c \\ -2m_o \\ m_c \end{bmatrix}$.

Remark 81. *We could have also used the fact that the eigenvectors of different eigenvalues should always be* **orthogonal**. *Therefore, when trying to find the final normal mode above,* $\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$ *and* $\begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}$ *must be orthogonal, meaning we already know that* $c_1 = c_3$ *without much work. However, we need to make sure we're using the same frame for all of our eigenvectors! This means that if we write our kinetic energy as*

$$K = \frac{1}{2} \sum_{ij} \dot{x}_i m_{ij} \dot{x}_j,$$

we will get a matrix that defines a symmetric form, and the eigenvectors must be orthogonal **under this form**.

17.3 Finding the missing modes

We've now found two of our four oscillating normal modes, and we want to find the other two. Let's assume the original system stood on the x-axis.

We'll now allow motion in the y-axis as well. But we have to be careful: if we just move the first mass up and the third mass down, we just rotate our molecule, and we are still essentially in a one-dimensional normal mode. Similarly, we can't just move all three particles in the positive y-direction, as this is just a translational mode. So that lets us simplify our situation: we must want the oxygens to move the same amount (to avoid rotation) and the carbon to move in a contrary manner (to avoid translation). More explicitly, we want the center of mass to stay stationary, so

$$m_o(y_1 + y_3) + m_c y_2 = 0.$$

As we just described, we should also have $y_1 = y_3$, so this tells us that our amplitudes are

$$\boxed{y_1 = y, \quad y_2 = -\frac{2m_o}{m_c}y, \quad y_3 = y}.$$

Following this line of reasoning is much less annoying than trying to solve for a normal mode generically. (It's also a bit less rigorous, but we have our motion constrained in terms of one variable, and this is a believable normal mode.) We just need the frequency of oscillation, and for that we'll find the equations of motion.

Specifically, let's look at the Lagrangian in the y-direction as a result of this type of motion. We have

$$L = K - U = 2 \cdot \frac{1}{2}m_o\dot{y}^2 + \frac{1}{2}m_c\left(\frac{2m_o}{m_c}\dot{y}\right)^2 - \frac{1}{2}k_\perp\left[2\left(y + \frac{2m_o}{m_c}\right)\right]^2,$$

where this final term comes from **deformations** from being linear. $k_\perp$ now has nothing to do with the usual spring constant: it's some other parameter of how well the molecule responds to being bent.

Now by the EL equation for y,

$$\frac{\partial L}{\partial y} = \frac{d}{dt}\frac{\partial L}{\partial \dot{y}} \implies -4k_\perp\left(1 + \frac{2m_o}{m_c}\right)^2 - \frac{d}{dt}\left(2m_o\dot{y}(1 + 2\frac{m_o}{m_c})\right) \implies \ddot{y} + 2k_\perp\left(\frac{1 + 2\frac{m_o}{m_c}}{m_o}\right)y = 0.$$

And this is just a simple harmonic motion in the y-direction with

$$\omega_\perp^2 = 2k_\perp\left(\frac{1 + 2\frac{m_o}{m_c}}{m_o}\right),$$

and there is also a corresponding oscillation in the z-direction. This gives us 4 normal modes, as desired, and we're done!

17.4 Quantum mechanics and the environment

We know that the wavelength of a photon with frequency ω is

$$\lambda = \frac{2\pi c}{\omega}.$$

Let's use this to analyze the frequencies of oscillation (along the axis of the molecule) that we just found, by plugging in our numbers ($m_o = 16$ amu, $m_c = 12$ amu, and $k = 140$ eV per squared angstrom). CO_2 does not have a dipole moment (by symmetry), so we can't actually get into the $\begin{bmatrix}1\\0\\-1\end{bmatrix}$ normal mode with a photon. However, the $\begin{bmatrix}1\\-2\\1\end{bmatrix}$ mode does induce a dipole moment! And doing some calculations, we find that the wavelength of the corresponding photon is $\lambda \approx 3.4\mu$m – actual measurements yield vibrational modes at 4.2μm and 15μm. Such photons are in the infrared range, and now we can analyze some real-world implications of our calculations.

Question 82. *Why do humans see only visible light?*

Answer: This is the range of light that has low absorption from the atmosphere. Everything above blue light is blocked by the upper atmosphere, while infrared radiation is absorbed by the oceans.

Question 83. *What color is the Sun?*

Answer: It's white! It looks yellow on earth because blue light is more preferentially scattered away from us.

Question 84. *Why is the sun white?*

Answer: This has to do with black-body radiation – the hotter an object is, the lower the wavelength of the light it emits. Some stars are blue and some are red, but around a surface temperature of 5000 to 8000 Kelvin (the Sun is in this range), we'll get a pretty white star. On the other hand, the Earth has a temperature of about 300 Kelvin, so it radiates infrared light.

And it turns out this is a problem for us. Carbon dioxide likes to absorb light in the infrared range (as we solved above) because of these normal modes, and this makes it a **greenhouse gas** (it traps heat)! Most of the atmosphere is Nitrogen, which is a symmetric molecule; such molecules can't really absorb infrared light because they don't develop dipoles at all. It's unfortunate that carbon dioxide is slightly too complicated (and has three molecules instead of two).

18 January 17, 2019 (Recitation)

The second problem set and first part of the project are due tomorrow.

Example 85

Consider a triple coupled pendulum. All three masses have mass m, and all pendulums have length ℓ. The first and second mass are connected by a spring, as are the second and third.

Let ϕ_1, ϕ_2, ϕ_3 be the angular displacements of the three pendulums. Then the velocity of a particle is $\dot{x}_i = \ell\dot{\phi}_i$, so our total kinetic energy is

$$K = \frac{1}{2}m\ell^2(\dot{\phi}_1^2 + \dot{\phi}_2^2 + \dot{\phi}_3^2).$$

Meanwhile, potential energy from gravity is

$$U_{\text{grav}} = \frac{1}{2}mg\ell((1-\cos\phi_1) + (1-\cos\phi_2) + (1-\cos\phi_3)) \approx \frac{1}{2}mg\ell(\phi_1^2 + \phi_2^2 + \phi_3^2),$$

and we also have potential energy from the springs:

$$U_{\text{spr}} = \frac{1}{2}k\ell^2\left((\phi_2-\phi_1)^2 + (\phi_3-\phi_2)^2\right) = \frac{1}{2}k\ell^2\left(\phi_1^2 + 2\phi_2^2 + \phi_3^2 - 2\phi_1\phi_2 - 2\phi_2\phi_3\right).$$

Thus, the total Lagrangian of our system is

$$\boxed{L = K - U = \frac{1}{2}m\ell^2(\dot{\phi}_1^2 + \dot{\phi}_2^2 + \dot{\phi}_3^2) - \frac{1}{2}mg\ell(\phi_1^2 + \phi_2^2 + \phi_3^2) - \frac{1}{2}k\ell^2\left(\phi_1^2 + 2\phi_2^2 + \phi_3^2 - 2\phi_1\phi_2 - 2\phi_2\phi_3\right)}.$$

Let's now extract the Euler-Lagrange equations: the one for θ_1 is

$$\frac{\partial L}{\partial \theta_1} = \frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}_1} \implies m\ell^2\ddot{\phi}_1 = -mg\ell\phi_1 - k\ell^2\phi_1 + k\ell^2\phi_2.$$

For convenience (and insight), define $\omega_p^2 = \frac{g}{\ell}$ and $\omega_s^2 = \frac{k}{m}$. Then dividing through by $m\ell^2$, our equation of motion simplifies to

$$\boxed{\ddot{\phi}_1 + \omega_p^2\phi_1 + \omega_s^2\phi_1 - \omega_s^2\phi_2 = 0}.$$

Similarly,

$$\frac{\partial L}{\partial \theta_2} = \frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}_2} \implies m\ell^2\ddot{\phi}_2 = -mg\ell\phi_2 - 2k\ell^2\phi_2 + k\ell^2\phi_1 + k\ell^2\phi_3.$$

Again dividing by $m\ell^2$,

$$\boxed{\ddot{\phi}_2 + \omega_p^2\phi_2 + 2\omega_s^2\phi_2 - \omega_s^2\phi_1 - \omega_s^2\phi_3 = 0}.$$

Since masses 1 and 3 are symmetric, our third equation of motion here will just be

$$\boxed{\ddot{\phi}_3 + \omega_p^2\phi_3 + \omega_s^2\phi_3 - \omega_s^2\phi_2 = 0}.$$

And now we can put everything in matrix form:

$$\begin{bmatrix} \frac{d^2}{dt^2} + \omega_p^2 + \omega_s^2 & -\omega_s^2 & 0 \\ -\omega_s^2 & \frac{d^2}{dt^2} + \omega_p^2 + 2\omega_s^2 & -\omega_s^2 \\ 0 & -\omega_s^2 & \frac{d^2}{dt^2} + \omega_p^2 + \omega_s^2 \end{bmatrix} \begin{bmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \end{bmatrix} = 0,$$

and now our usual ansatz of $\begin{bmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \end{bmatrix} = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} e^{i\omega t}$ yields

$$\begin{bmatrix} -\omega^2 + \omega_p^2 + \omega_s^2 & -\omega_s^2 & 0 \\ -\omega_s^2 & -\omega^2 + \omega_p^2 + 2\omega_s^2 & -\omega_s^2 \\ 0 & -\omega_s^2 & -\omega^2 + \omega_p^2 + \omega_s^2 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = 0.$$

This is an eigenvalue problem, so we want the determinant of the 3×3 matrix here to be zero: expanding the determinant along a column yields

$$(-\omega^2 + \omega_p^2 + \omega_s^2)\left((-\omega^2 + \omega_p^2 + 2\omega_s^2)(-\omega^2 + \omega_p^2 + \omega_s^2) - \omega_s^4\right) + \omega_s^2\left((-\omega_s^2)(-\omega^2 + \omega_p^2 + \omega_s^2)\right) = 0.$$

After some algebra, this simplifies to

$$(\omega_p^2 - \omega^2)(\omega_p^2 + \omega_s^2 - \omega^2)(\omega_p^2 + 3\omega_s^2 - \omega^2) = 0.$$

In other words, $\boxed{\omega_A^2 = \omega_p^2}$, $\boxed{\omega_B^2 = \omega_p^2 + \omega_s^2}$, $\boxed{\omega_C^2 = \omega_p^2 + 3\omega_s^2}$ are our three normal mode frequencies.

- Plugging in ω_A, we find that $c_1 = c_2 = c_3$ – in other words, all three pendulums are in phase. This means all pendulums are swinging together, and the springs are never stretched. The frequency of this motion is $\omega = \omega_p = \sqrt{\frac{g}{\ell}}$.
- Plugging in ω_B, we find that $c_1 = -c_3, c_2 = 0$. In other words, the middle pendulum is never moving, and the other two pendulums move in opposite directions (completely out of phase). This has frequency $\omega = \sqrt{\omega_p^2 + \omega_s^2} = \sqrt{\frac{g}{\ell} + \frac{k}{m}}$.
- Finally, ω_c yields $\omega_c^2 = \omega_p^2 + 3\omega_s^2$, which means $c_1 = c_3 = -\frac{1}{2}c_2$. The idea here is that the outside two pendulums move together, and the middle pendulum moves at twice the amplitude in the opposite direction. This has frequency $\omega = \sqrt{\omega_p^2 + 3\omega_s^2} = \sqrt{\frac{g}{\ell} + \frac{3k}{m}}$.

Notice these are the same ones as in the lecture example! If we'd like, we can see 8.03 for a discussion of the symmetry matrix and a deeper understanding of why this occurs.

So let's summarize the general approach here. To solve normal mode problems, first write down the Lagrangian L. Obtain the Lagrangian equations of motion, and write them in matrix form. Now plug in the ansatz for simple harmonic motion, which yields an eigenvalue problem for ω. Finally, given our eigenvalues, we can also find the (relative) normal mode amplitudes by solving for the eigenvectors.

19 January 18, 2019 (Lecture)

Our second problem set was due five minutes ago. The next problem set will be posted about an hour after class – it'll be due next Friday.

19.1 More oscillators: adding a driving force

We're now probably very familiar with the simple harmonic oscillator equation of motion

$$\ddot{x} + \omega_0^2 x = 0,$$

where $\omega_0^2 = \frac{k}{m}$. There's two ways we can write the general solution:

$$x(t) = A\cos(\omega_0 t + \phi) = B\cos(\omega_0 t) + C\sin(\omega_0 t).$$

The nice thing about the first solution form is that we can immediately see the maximum amplitude: it's just A, achieved when $\omega_0 t + \phi$ is a multiple of 2π, while that maximum is a bit less obvious in the other formulation. And the first form also allows us to write the total energy in the oscillator, $\frac{1}{2}kA^2$, in a simple form.

But now, let's try applying some other force to the mass besides the force from a simple spring. Our equation of motion then takes on the form

$$\ddot{x} + \omega_0^2 x = \frac{F(t)}{m},$$

where $F(t)$ is some varying force based on time. In principle, this can be any force, and we get a differential equation. The most interesting case happens when the force itself oscillates, though – let's take $F(t) = F\cos(\omega_f t)$.

To solve this, we use the ansatz $x(t) = A\cos(\omega_0 t + \phi) + B\cos(\omega_f t)$. Notice that this has a component that is oscillating at the natural frequency (corresponding to the **particular** term), as well as one at the driven frequency (corresponding to the **homogeneous** term). Plugging this in, the $A\cos(\omega_0 t + \phi)$ term evaluates to zero on the left side, so

$$B\left(-\omega_f^2\cos(\omega_f t) + \omega_0^2\cos(\omega_f t)\right) = \frac{F}{m}\cos(\omega_f t).$$

The $\cos(\omega_f t)$ terms all cancel (this is the same idea as all $e^{i\omega_f t}$ terms cancelling), so we can solve for B to find

$$B = \frac{F}{m(\omega_0^2 - \omega_f^2)}.$$

So in general, our general solution will be

$$x(t) = A\cos(\omega_0 t + \phi) + \frac{F}{m(\omega_0^2 - \omega_f^2)}\cos(\omega_f t),$$

where A and ϕ can be any constants (two degrees of freedom to satisfy the two initial conditions). However, all other terms are fixed by the magnitude of force and other constants.

There's one thing that's not nice about this: as $\omega_f \to \omega_0$, this solution goes to infinity! We have a divergent situation, and the resolution is that when we plugged in our ansatz, we got 0 on the left hand side (and something nonzero on the right hand side), which meant **our original ansatz must have been incorrect**. It can be verified instead that the actual correct form of the solution is (for $\omega = \omega_0 = \omega_f$)

$$x(t) = A\cos(\omega t + \phi) + \frac{Ft}{2m\omega}\sin(\omega t).$$

We can sanity check that this is equivalent to taking $\omega_f = \omega_0 + \varepsilon$ in the general solution. Indeed, as $t \to \infty$, this $x(t)$ does diverge – it just doesn't do so immediately. The frequency $\omega_0 = \omega_f$ for which this happens is called the **resonant frequency** of the system.

19.2 Adding a damping term

To further complicate the setup, let's add a resistance-type property, like friction or viscosity. First, consider the situation where there is no driving force. Then the equation of motion takes the form

$$\ddot{x} + \omega_0^2 x = -\gamma\dot{x},$$

where γ is a positive constant. This time, our trial solution is of the form $x(t) = \mathrm{Re}(e^{\beta t})$, for some **complex** number β, and the equation simplifies to

$$\beta^2 x + \omega_0^2 x + \gamma\beta x = 0.$$

The x's cancel, and this is a quadratic equation in β with roots

$$\beta = \frac{1}{2}\left(-\gamma \pm \sqrt{\gamma^2 - 4\omega_0^2}\right),$$

so our general solution is

$$\boxed{x(t) = e^{-\frac{\gamma}{2}t}\,\mathrm{Re}\left(Ae^{t\sqrt{\frac{\gamma^2}{4}-\omega_0^2}} + Be^{-t\sqrt{\frac{\Gamma^2}{4}-\omega_0^2}}\right)}.$$

There are three regimes of behavior, which depends on whether the square root term $\sqrt{\frac{\Gamma^2}{4} - \omega_0^2}$ is real or imaginary.

- Underdamping or light damping: if $\frac{\gamma}{2} < \omega_0$, then we can define a new frequency ω with the equation $\sqrt{\frac{\Gamma^2}{4} - \omega_0^2} = i\omega$. Then the solution simplifies to

  $$x(t) = Ae^{-\frac{\Gamma}{2}t}\cos(\omega t + \phi).$$

  As time goes on, the maximum amplitude of the oscillation gets smaller and smaller, but there is still a back-and-forth motion.

  One detail is that when $\gamma \ll \omega_0$ – that is, motion over one period is essentially unaffected by damping – the energy of the system is approximately

  $$E \approx \frac{1}{2}k\left(Ae^{-\frac{\gamma}{2}t}\right)^2.$$

- Overdamping: if $\frac{\gamma}{2} > \omega_0$, then the square root term is real, so the form of the solution will be

  $$x(t) = e^{-\frac{\gamma}{2}t}\left(Ae^{t\sqrt{\frac{\gamma^2}{4}-\omega_0^2}} + Be^{-t\sqrt{\frac{\Gamma^2}{4}-\omega_0^2}}\right).$$

  In this case, we don't have oscillation at all: the mass drags itself back to the origin, only crossing the equilibrium position at most once.

- Critical damping: if $\frac{\gamma}{2} = \omega_0$, we can't actually use both roots of the quadratic, since they're the same. We still need two independent solutions and unknown parameters to characterize the full solution, though. Doing some limiting arguments, the solution turns out to take the form

$$x(t) = e^{-\frac{\gamma}{2}t}(A + Bt).$$

This solution actually decays to zero faster than the overdamped case (since a very damped oscillator just means the mass doesn't want to move at all).

19.3 Combining the two together

If we have a damped driven oscillator corresponding to the equation of motion,

$$\ddot{x} + \gamma\dot{x} + \omega_0^2 x = \frac{F}{m}\cos(\omega_f t),$$

we will again have a **particular** as well as a **homogeneous** solution. If we denote $x_{F=0}(t)$ to be the homogeneous solution – that is, the solution to the damped oscillator if there were no driving force – our solution becomes

$$\boxed{x(t) = x_{F=0}(t) + \frac{F\cos(\omega_f t + \alpha)}{m\sqrt{(\omega_0^2 - \omega_f^2)^2 + (\omega_f\gamma)^2}}}$$

where the phase of the particular solution α is fixed to be

$$\alpha = \tan^{-1}\left(\frac{\omega_f\gamma}{\omega_f^2 - \omega_0^2}\right).$$

We can check this by using our ansatz and doing more nasty algebra. Notice that we've removed the divergence issues of the ordinary driven oscillator: even at resonance frequency, we can't have infinite solutions anymore. Also, the phase α means that the motion will lag behind the force, and the two will no longer be in phase.

To find the frequency ω_f where the **particular solution attains a maximum amplitude**, we want to set the denominator to a minimum. Thus,

$$\frac{d}{d\omega_f}\left((\omega_0^2 - \omega_f^2) + (\omega_f\gamma)^2\right) = 2(\omega_0^2 - \omega_f^2)(-2\omega_f) + 2\gamma^2\omega_f = 0,$$

which occurs at $\boxed{\omega_f^2 = \omega_0^2 - \frac{\gamma^2}{2}}$. In other words, damping induces a lag effect, so that means we actually want to drive the system at a slightly different frequency from the natural frequency to get maximum efficiency!

19.4 Application to architecture

Example 86

Consider a mass connected to a spring on a wall, and add a damping mechanism with damping factor γ_1 (this is often a piston with oil inhibiting the motion). Define the mass's position x_1 relative to its equilibrium position. Suppose we have a driving force $F\cos(\omega t)$, where the natural frequency is equal to the driving frequency for simplicity.

When the force F is very large, the particular solution dominates as

$$x_1(t) \approx \frac{F}{m_1\omega\gamma_1}\cos\left(\omega t + \frac{\pi}{2}\right) = \frac{F}{m_1\omega\gamma_1}\sin(\omega t).$$

This can be thought of the **effect of an earthquake**: a mass is just sitting somewhere, and then suddenly everything starts shaking with some very large force. In other words, we can imagine our spring-mass system as being a tall skyscraper, which has some natural rocking frequency.

Fact 87

0.1 to 0.5 hertz is the typical frequency of a skyscraper, which lines up very well with the frequencies of natural disasters like earthquakes and hurricanes.

Because of this, it's natural to ask how to reduce the amplitude of motion. Theoretically, we can increase γ, but unfortunately, we can't just "damp the building really hard" in real life. This is hard to do, because the top of the building is moving and we can't really just put a viscous fluid everywhere around it. We also can't just make the mass of the skyscraper very big.

Instead, the idea is to **add another mass** on top of m_1, where $m_2 \ll m_1$: we connect it to m_1 by a spring and also have a damping factor γ_2 between the two masses. The goal is to have this mass damp out the motion if an external force is applied.

We can pick $\omega_2 = \omega$ as well. (This is almost optimal.) Then if x is the displacement of the small mass relative to m_1, we can apply Newton's second law to find

$$m_2(\ddot{x}_1 + \ddot{x}) = -m_2\omega^2 x - m_2\gamma_2\dot{x} \implies \boxed{\ddot{x}_1 + \ddot{x} + \omega^2 x + \gamma_2\dot{x} = 0}.$$

By Newton's third law, a lot of the forces on mass 2 will be applied to mass 1, just in the opposite direction. This yields

$$m_1\ddot{x}_1 = -m_2(\ddot{x}_1 + \ddot{x}) - m_1\omega^2 x_1 - m_1\gamma_1\dot{x}_1 + F(t),$$

which simplifies to

$$\boxed{\ddot{x}_1\left(1 + \frac{m_2}{m_1}\right) + \omega^2 x_1 + \gamma_1\dot{x}_1 = \frac{F(t)}{m_1} - \frac{m_2}{m_1}\ddot{x}}.$$

We can neglect $\frac{m_2}{m_1}$ since it is small, and this means the left hand side is the standard form of the damped oscillator for x_1. The only difference is that the motion of the smaller mass now works against the driving force, which is what we want!

To solve for the actual path, we guess (correctly) that both masses will still oscillate. Then we can use the trial solution $x_1(t) \approx A\cos(\omega t + \beta)$. Substituting this in to the first boxed equation,

$$\ddot{x} + \gamma_2\dot{x} + \omega^2 x = -\ddot{x}_1 = \omega^2 A\cos(\omega t + \beta).$$

This is almost the standard form of the damped driven oscillator, and we find that

$$x(t) = \frac{\omega^2 A}{\omega\gamma_2}\cos\left(\omega t + \beta + \frac{\pi}{2}\right) = \frac{\omega A}{\gamma_2}\sin(\omega t + \beta).$$

We're almost done! We don't actually care about the motion of x: we are trying to make sure x_1 has small oscillations, not x. So now we return to the second boxed equation: now

$$\ddot{x} = -\frac{\omega^3 A}{\gamma_2}\sin(\omega t + \beta) = \frac{\omega^2}{\gamma_2}\dot{x}_1(t).$$

Therefore, our second boxed equation of motion is now

$$\ddot{x}_1 + \omega^2 x_1 + \gamma_1 \dot{x}_1 = \frac{F(t)}{m_1} - \frac{m_2}{m_1}\frac{\omega^2}{\gamma_2}\dot{x}_1 \implies \ddot{x}_1 + \omega^2 x_1 = \dot{x}_1\left(\gamma_1 + \frac{m_2}{m_1}\frac{\omega^2}{\gamma_2}\right) = \frac{F}{m_1}\cos(\omega t).$$

Since the amplitude of x_1 was inversely proportional to γ in the previous damped oscillator calculations, this means we've reduced the amplitude by a factor of

$$\frac{\gamma_1 + \frac{m_2}{m_1}\frac{\omega^2}{\gamma_2}}{\gamma_1} = 1 + \frac{m_2}{m_1}\frac{\omega^2}{\gamma_1\gamma_2}.$$

The ratio $\frac{\omega}{\gamma_2}$ is actually related to how much the second mass is allowed to move, so there are ways to set up the system and strongly damp the motion due to earthquakes! These types of solutions are called **tuned-mass dampers**; many large buildings have these, and they do a pretty good job.

(By the way, there's no class next Monday because of Martin Luther King Day.)

20 January 18, 2019 (Recitation)

Fact 88

Problem set 2 solutions were discussed during recitation – this has been omitted.

21 January 22, 2019 (Lecture)

Today we're going to start talking about the Hamiltonian formulation of physics.

21.1 Review

Recall that in the first two weeks of this class, we derived a Lagrangian formulation of physics: we found that we could define an action

$$S = \int L(q, \dot{q})dt,$$

where L is the difference between the kinetic and potential energies, and the only way to unambiguously describe the correct physical path is by solving the Euler-Lagrange equations

$$\delta S = 0 \implies \frac{\partial L}{\partial q} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}} = 0.$$

We also found that because of time-independence, we have a conserved quantity

$$H \equiv \sum_i \frac{\partial L}{\partial \dot{q}}\dot{q} - L.$$

It turns out this quantity is very important, and we'll be using it more in the next part of this class.

21.2 A particle in two dimensions

Let's say we have a single particle with mass m constrained to the xy-plane. Then our Lagrangian is

$$L = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) - U(x, y),$$

where U is the potential energy of the particle. Notice that

$$\frac{\partial L}{\partial \dot{x}} = m\dot{x} = p_x, \frac{\partial L}{\partial \dot{y}} = m\dot{y} = p_y$$

are the momenta of the particle in the x and y directions, respectively. This implies that our Hamiltonian is

$$H = p_x\dot{x} + p_y\dot{y} - \frac{1}{2}m\dot{x}^2 - \frac{1}{2}m\dot{y}^2 + U(x, y) = \boxed{\frac{p_x^2}{2m} + \frac{p_y^2}{2m} + U(x, y)}.$$

Therefore, we can write everything as a function of **momentum and position** (which we often prefer), rather than velocity and position.

Similarly, we can write the same Lagrangian in polar coordinates: if $\vec{v} = \dot{r}\hat{r} + r\dot{\theta}\hat{\theta}$,

$$L = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) - U(r, \theta).$$

Remember that the Lagrangian doesn't depend on what coordinates we're using, and we always get the same Lagrange equation. In the Cartesian case, our Hamiltonian lines up nicely with our usual definition of momentum, but our momenta will look a little bit more complicated:

$$\frac{\partial L}{\partial \dot{r}} = m\dot{r} = p_r, \frac{\partial L}{\partial \dot{\theta}} = mr^2\dot{\theta} \equiv p_\theta.$$

Definition 89

The **generalized momentum** for a coordinate q_i is

$$p_i = \frac{\partial L}{\partial \dot{q}_i}.$$

This then gives us a Hamiltonian of

$$H = p_r\dot{R} + p_\theta\dot{\theta} - \frac{1}{2}m\dot{r}^2 - \frac{1}{2}mr^2\dot{\theta}^2 + U(r, \theta) = \frac{p_r^2}{2m} + \frac{p_\theta^2}{2mr^2} + U(r, \theta),$$

and though the expression looks a bit different, many of the general takeaways are the same.

Fact 90

We generally want our Hamiltonian to be defined as $H(q, p)$, meaning it is written in terms of generalized coordinates and momenta. There are fundamental reasons to choose this, mostly coming from quantum mechanics.

21.3 Deriving the Hamiltonian formulation

We can now rewrite our action in terms of the definition of the Hamiltonian as

$$S = \int \left(\sum_i p_i\dot{q}_i - H(q_i, p_i) \right) dt.$$

We still want $\delta S = 0$, and this tells us that

$$\delta S = \int \sum_i \left(p_i(\delta \dot{q}_i) + \dot{q}_i(\delta p_i) - \frac{\partial H}{\partial q_i}(\delta q_i) - \frac{\partial H}{\partial p_i}(\delta p_i) \right) dt = 0.$$

By the same product rule trick we've been using, we can get rid of the $\delta \dot{q}_i$ term: because

$$\frac{d}{dt}\left(p_i\, \delta q_i\right) = p_i\, \delta \dot{q}_i + \dot{p}_i\, \delta q_i,$$

integrating both sides yields

$$\int p_i\, \delta \dot{q}_i\, dt = (p_i \delta q_i) - \int \dot{p}_i\, \delta q_i.$$

The term in parentheses is a constant, since δq is always 0 at the endpoints (we aren't allowed to vary our start and end locations). So now we can replace and find that, after a bit of factoring,

$$\delta S = \int \sum_i \left(\delta p_i \left(\dot{q}_i - \frac{\partial H}{\partial p_i} \right) - \delta q_i \left(\dot{p}_i + \frac{\partial H}{\partial q_i} \right) \right) dt = 0.$$

This is very similar to the derivation we did for Lagrangian formulation, except with different variables and a slightly different form of the function determining our system. And the above equation must be true for any variations δp_i and δq_i, which leads us to the following result:

Proposition 91 (Hamilton's equations)
Let H be a Hamiltonian. Then for any generalized coordinate q_i and conjugate momentum p_i, we have

$$\frac{\partial H}{\partial p_i} = \dot{q}_i, \frac{\partial H}{\partial q_i} = -\dot{p}_i.$$

Instead of one second-order equation, we now have two first-order equations, and it's nice that position and momenta are treated almost identically. For that reason, these are often called the **canonical equations of motion.** One fundamental idea is that there's very few problems that are easier to do with Hamiltonians at the small scale, but the Hamiltonian becomes more important as we have **more particles** in our system.

Finally, it can also be verified that

$$\frac{\partial H}{\partial t} = -\frac{\partial L}{\partial t},$$

so the two formulations we've discussed are connected.

21.4 A simple pendulum

Remember that the equation of motion for a pendulum of length ℓ is

$$\ddot{\theta} = -\frac{g}{\ell} \sin\theta.$$

The classical way of finding this was annoying, because we had to worry about signs and directions of forces. With the Lagrangian formulation, the problem became easier: we just wrote down the Lagrangian

$$L = \frac{1}{2} m\ell^2 \dot{\theta}^2 + mg\ell(\cos\theta - 1)$$

and used the EL equation $\frac{\partial L}{\partial \theta} = \frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}}$. This was particularly nice because we didn't have to worry about using good coordinates.

For contrast, we'll now write the Hamiltonian equations of motion. First of all, we have our generalized momentum

$$p_\theta = \frac{\partial L}{\partial \dot{\theta}} = m\ell^2\dot{\theta},$$

so from here, we can define the Hamiltonian

$$H = p_\theta\dot{\theta} - L = \frac{p_\theta^2}{m\ell^2} - \frac{p_\theta^2}{2m\ell^2} - mg\ell(\cos\theta - 1) = \frac{p_\theta^2}{2m\ell^2} + mg\ell(1-\cos\theta) = K + U.$$

The next step is to write out Hamilton's equations– the first equation says that

$$\frac{\partial H}{\partial p_\theta} = \frac{p_\theta}{m\ell^2} = \dot{\theta},$$

which tells us nothing new.

Fact 92

Usually **the first equation doesn't tell us anything immediately**, because of the way we found the Hamiltonian.

On the other hand, we also have

$$\frac{\partial H}{\partial \theta} = mg\ell\sin\theta = -\dot{p}_\theta.$$

From here, we generally get an equation of motion by combining with the first equation. Substituting in for $\dot{p}_\theta$ from the first equation, we have

$$mg\ell\sin\theta = -m\ell^2\ddot{\theta} \implies \boxed{\ddot{\theta} = -\frac{g}{\ell}\sin\theta},$$

which is the same equation of motion as before.

Remark 93. *Note that the Hamiltonian becomes more interesting when we start getting time-dependent forces or velocity-dependent potentials – that information can all be encoded in H*

By the way, if we go back to our Hamiltonian and take $\theta \ll 1$, we find that

$$H \approx \frac{p_\theta^2}{2m\ell^2} + \frac{1}{2}mg\ell\theta^2 = E$$

is conserved. Then we can represent the evolution of our system using **phase space**: use a coordinate system with θ and p_θ as the two axes, so that the trajectory of the particle is an ellipse. Specifically, when $p_\theta = 0$, we have $\theta_0 = \sqrt{\frac{2E}{mg\ell}}$, and when $\theta = 0$, we have $p_{\theta 0} = \ell\sqrt{2mE}$. (These give us the major and minor axes of the ellipse, respectively.) In general, lots of properties of phase space that are useful when we have more particles, and this comes up in statistical mechanics as well.

Looking at the form of the Hamiltonian, we can find that

$$\frac{\partial}{\partial p}\left(H\left(x, \frac{\partial L}{\partial \dot{x}}\right)\right) = \frac{\partial}{\partial p}(H(x,p)) = \dot{x}.$$

But we can reverse the roles of certain variables and find that

$$\frac{\partial}{\partial \dot{x}}\left(L\left(x, \frac{\partial H}{\partial p}\right)\right) = \frac{\partial}{\partial \dot{x}}(L(x,\dot{x})) = p.$$

This is pretty amazing: the Lagrangian and Hamiltonian are not inverses of each other, but their derivatives are inverse functions! It turns out that this has to do with the **Legendre transform**, which is important for some geometric treatments of physics.

In quantum mechanics, there isn't always a well-defined position or momentum for a particle. This connects to the Heisenberg uncertainty principle – in words, we can't have a well-defined peak for a (wave)function without adding together a lot of different wavelengths.

Fact 94

The Hamiltonian is the H from the Schrodinger equation

$$\hat{H}|\psi\rangle = i\hbar\frac{\partial L}{\partial t}|\psi\rangle.$$

We'll also be able to say a bit more about the right side of this equation tomorrow.

22 January 22, 2019 (Recitation)

The recitation instructor for the last two weeks of this class is **Seth Musser**. Office hours will continue to be 3– and 7–8pm on Wednesdays.

22.1 A normal mode problem

Example 95

Consider four masses of mass m restricted to move on a ring, with coordinates along the circumference of the ring. We want to know how the x-coordinates x_1, x_2, x_3, x_4 behave if all 4 adjacent pairs of masses are connected with springs with constant k (also stretched along the circumference).

To be more precise, we could replace x_i with $R\theta_i$, but the basic idea is the same. Let's start with the Lagrangian: we have that

$$L = K - U = \frac{1}{2}m(\dot{x}_1^2 + \dot{x}_2^2 + \dot{x}_3^2 + \dot{x}_4^2) - \frac{1}{2}k((x_1 - x_2)^2 + (x_2 - x_3)^2 + (x_3 - x_4)^2 + (x_4 - x_1)^2),$$

which means we can actually write the Lagrangian as

$$L = \frac{1}{2}m\sum_n \dot{x}_n^2 - \frac{1}{2}k\sum_n (x_n - x_{n+1})^2$$

where all the indices are taken mod 4 (that is, $x_1 = x_5$ and $x_0 = x_4$). We're doing this here to impose a periodicity, since we want to replace 4 with a larger number later.

Now we can write out our Euler-Lagenage equations all at once: for all $1 \le n \le 4$, we have

$$\frac{\partial L}{\partial x_n} = \frac{d}{dt}\frac{\partial L}{\partial \dot{x}_n} \implies -k(-x_{n-1} + 2x_n - x_{n+1}) = \frac{d}{dt}(m\dot{x}_n),$$

since each mass is connected to its adjacent two masses. If we're interested in finding the normal modes, we can start by writing our equations in matrix form:

$$m\begin{bmatrix}\ddot{x}_1\\ \ddot{x}_2\\ \ddot{x}_3\\ \ddot{x}_4\end{bmatrix} = k\begin{bmatrix}-2 & 1 & 0 & 1\\ 1 & -2 & 1 & 0\\ 0 & 1 & -2 & 1\\ 1 & 0 & 1 & -2\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\\ x_4\end{bmatrix}.$$

We now we have something that looks a lot like $M\ddot{X} = -KX$, but now M and K are matrices and X is a vector. Our goal is to find eigenvalues: normal modes are of the form $A\vec{x} = \omega^2 x$, where $x_n \propto e^{i\omega t}$ is oscillating at a frequency of ω. For convenience, set $\omega_0^2 = \frac{k}{m}$.

- One eigenvector we see immediately is $\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$: this is the degenerate case with $\boxed{\omega^2 = 0}$, and this corresponds to translational motion.

- If we fix masses 1 and 3, and we allow masses 2 and 4 to move in contrary motion, we get an eigenvector of $\begin{bmatrix} 0 \\ 1 \\ 0 \\ -1 \end{bmatrix}$. This corresponds to $\boxed{\omega^2 = 2\omega_0^2 \frac{2k}{m}}$, since the effective spring constant on both moving masses is $2k$.

- We can also fix masses 2 and 4 to get another eigenvector $\begin{bmatrix} 1 & 0 & -1 & 0 \end{bmatrix}$ with $\boxed{\omega^2 = 2\omega_0^2 \frac{2k}{m}}$.

- The last eigenvector is $\begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix}$ with eigenvalue $\boxed{\omega^2 = 4\omega_0^2}$.

Remark 96. *The matrix $M^{-1}K$ is symmetric in this problem, and in particular it is* **positive definite***, so we will find nonnegative eigenvalues. And one note from 8.03 is that after finding all but one normal mode, we can find the last one by examining the* **trace** *of the matrix (which is also the sum of the eigenvalues).*

22.2 Extending the problem

Now let's say we have N masses instead of 4. Our Lagrangian is nicely symmetric as before, and we'll actually still have the same equations of motion:

$$m\ddot{x}_n = -2kx_n + kx_{n+1} + kx_{n-1}.$$

Again, we define $x_0 = x_N$ and $x_1 = x_{N+1}$. We could write out our equations in matrix form and try to find each of the normal modes as we did above, but that's annoying and more complicated in general.

Instead, we can think about the motion of the masses as a wave, since "bumping" one mass will make the other masses propagate. Inspired by that, we can use the ansatz

$$x_n \propto e^{i(\kappa n - \omega t)}.$$

This is actually the same normal mode form as before, but we encode the amplitude in the $e^{i\kappa n}$ part. This means we still have a normal mode, and it tells us the formula of a propagating wave! This will help us avoid a lot of matrix calculations. Since we want $x_0 = x_N$ (periodicity), we have

$$e^{-i\omega t} = e^{\kappa N - \omega t} \implies e^{i\kappa N} = 1 \implies \kappa = \frac{2m\pi}{N},$$

where m is an integer between 0 and $N-1$ inclusive. This means we basically have a periodic boundary condition –

plugging in the definition of x_n into our equations of motion, we have

$$x_n = e^{i\left(\frac{2\pi m}{N}n - \omega t\right)} \implies -m\omega^2 x_n = -2kx_n + kx_{n-1} + kx_{n+1}.$$

Notice that the ratio between x_{n+1} and x_n is just $e^{i\kappa}$, which means

$$-m\omega^2 x_n = -2kx_n + ke^{-2\pi im/N}x_n + ke^{2\pi im/N}x_n.$$

Dividing through by x_n, we see that

$$-m\omega^2 = -2k + ke^{-2\pi im/N} + ke^{2\pi im/N} \implies \omega^2 = \omega_0^2(2 - e^{-2\pi im/N} + e^{2\pi im/N}).$$

But the exponential terms simplify to a single cosine term, since the imaginary parts cancel! This gives us the final expression

$$\boxed{\omega^2 = \omega_0^2\left(2 - 2\cos\left(\frac{2\pi a}{N}\right)\right)},$$

where $0 \leq a \leq N-1$. To confirm that this is consistent with our above example, note that when $N = 4$,

$$\omega^2 = \omega_0^2\left(2 - 2\cos\left(\frac{\pi a}{2}\right)\right).$$

Plugging in $a = 1$ or $a = 3$ yields $\omega^2 = 2\omega_0^2$, plugging in $a = 0$ gives $\omega^2 = 0$, and plugging in $a = 2$ gives $\omega^2 = 4\omega_0^2$. Math works, and the takeaway is that there is actually some wave propagation going on.

Fact 97

The matrix here is actually the negative of the Laplacian matrix $L = D - A$, where D is the diagonal matrix encoding the degrees of a graph's vertices, and A is its adjacency matrix.

23 January 23, 2019 (Lecture)

There is a problem set due on Friday, but we have already covered everything needed to do all of those problems. As always, we should go to office hours if we have questions.

23.1 Review

We started studying Hamiltonian dynamics yesterday, defining the **Hamiltonian** as a function of position and momentum:

$$H(q_i, p_i) = \sum p_i\dot{q}_i - L,$$

where the generalized momenta p_i are defined via

$$p_i \equiv \frac{\partial L}{\partial \dot{q}_i}.$$

Plugging this into the Lagrangian, we found that the action took the form

$$S = \int \left(\sum_i p_i\dot{q}_i - H(q_i, p_i)\right) dt,$$

and to minimize the action, we need to satisfy **Hamilton's equations**

$$\boxed{\frac{\partial H}{\partial p_i} = \dot{q}_i, \frac{\partial H}{\partial q_i} = -\dot{p}_i}.$$

Again, as a sidenote, we also have

$$\frac{\partial H}{\partial t} = -\frac{\partial L}{\partial t}.$$

These are called the **canonical equations of motion** because they treat position and momentum in a symmetric way. In contrast to the single Lagrangian second-order equation, we now have two first-order differential equations.

By the way, it may seem like p and x are actually related, because we ordinarily define $p = m\dot{x}$. But the idea is that when we do our variational calculus, there's no reason p and x need to be dependent on each other, because we're doing those calculations **off of the actual physical path**.

23.2 Canonical transformations

Suppose we have a coordinate q and momentum p, such that

$$\frac{\partial H(q,p,t)}{\partial p} = \dot{q}, \frac{\partial H(q,p,t)}{\partial q} = -\dot{p},$$

that is, the Hamilton equations are valid. Our goal is to find some other variables Q, P (by applying a transformation to q and p) such that

$$\frac{\partial H(Q,P,T)}{\partial P} = \dot{Q}, \frac{\partial H(Q,P,T)}{\partial Q} = -\dot{P}.$$

The idea is to try to rewrite the Hamiltonian into something easier to solve.

Remark 98. *$H(q,p,t)$ and $H(Q,P,t)$ aren't the same function in terms of the arguments! Instead, they're ways of representing the* **same quantity** *(the Hamiltonian of the system), using* **different variables**. *So if we want, we can replace H with H' whenever it appears with P and Q.*

Writing out the action, we must have

$$\delta S = \delta \int (p\dot{q} - H(q,p,t))\, dt = \delta \int \left(P\dot{Q} - H(Q,P,T)\right) dt.$$

Basically, the action should always be the same in both cases - otherwise, we haven't actually defined a reasonable transformation from q, p to Q, P. One way to make this happen is to just set the bracketed terms to be essentially equal at any given time t:

$$\boxed{p\dot{q} - H(q,p,t) = P\dot{Q} - H(Q,P,t) + \frac{dG}{dt}},$$

where $\frac{dG}{dt}$ is just some total time derivative (which we already know doesn't do anything to the overall action). In principle, G can be a function of any of q, p, Q, P, and t, but we usually choose it just in terms of t and any two of the other coordinates.

23.3 A special kind of transformation

We often use $\boxed{G = -QP + F(q,P,t)}$ – we'll soon why we make this choice. Then the chain rule tells us that

$$\frac{dG}{dt} = \frac{\partial G}{\partial t} + \frac{\partial G}{\partial Q}\dot{Q} + \frac{\partial G}{\partial P}\dot{P} + \frac{\partial G}{\partial q}\dot{q}$$

(the lowercase p doesn't show up). Plugging in the form of G into this expression, we end up with

$$\frac{dG}{dt} = \frac{\partial F}{\partial t} - P\dot{Q} - Q\dot{P} + \frac{\partial F}{\partial P}\dot{P} + \frac{\partial F}{\partial q}\dot{q}.$$

Then plugging this in to our boxed equation above,

$$p\dot{q} - H(q,p,t) = P\dot{Q} - H(Q,P,t) + \left(\frac{\partial F}{\partial t} - P\dot{Q} - Q\dot{P} + \frac{\partial F}{\partial P}\dot{P} + \frac{\partial F}{\partial q}\dot{q}\right).$$

The $P\dot{Q}$ terms cancel. One way to make the remaining equation true is to set "corresponding" terms equal: for example, the only terms with $\dot{q}$ on either side are $p\dot{q}$ and $\frac{\partial F}{\partial q}\dot{q}$, so we can just set those equal: $\boxed{p = \frac{\partial F}{\partial q}}$. Similarly, we can set $\boxed{Q = \frac{\partial F}{\partial P}}$, and now we're just left with

$$\boxed{H(Q,P,T) = H(q,p,t) + \frac{\partial F}{\partial t}}.$$

Here F is called a **generating function**, because it "generates" the evolution of H and therefore of the dynamics.

Fact 99

We can turn a damped oscillator into an undamped oscillator by using a transformation like this – then, the coordinates will be changing with time, but they will oscillate sinusoidally. And there's other more sophisticated uses of these transformations as well.

23.4 An example of a canonical transformation

Example 100

Suppose the Hamiltonian for a system is

$$H(q,p,t) = \frac{p^2}{2m} - aq\sin(\omega t).$$

This is the Hamiltonian of a charged particle that is interacting with an oscillating electric field.

By Hamilton's equations,

$$\dot{p} = -\frac{\partial H}{\partial q} = a\sin(\omega t), \dot{q} = \frac{\partial H}{\partial p} = \frac{p}{m}.$$

However, this is annoying to solve, so we can instead introduce the generating function

$$F = qP - \frac{a}{\omega}q\cos(\omega t).$$

Applying the steps described above, we have that

$$p = \frac{\partial F}{\partial q} = P - \frac{a}{\omega}\cos(\omega t) \implies P = p + \frac{a}{\omega}\cos(\omega t), Q = \frac{\partial F}{\partial P} = q.$$

In other words, we redefine the momentum $p \to P$ to get rid of the annoying part of the Hamiltonian, while keeping the position unchanged. So now the Hamiltonian for our new coordinates is

$$H(Q,P,t) = \frac{p^2}{2m} - aq\sin(\omega t) + aq\sin(\omega t) = \frac{\left(P - \frac{a}{\omega}\cos(\omega t)\right)^2}{2m}.$$

Now $\dot{P} = -\frac{\partial H}{\partial Q} = 0$, so P is actually a conserved quantity (constant). The other Hamilton equation tells us that

$$\dot{Q} = \frac{\partial H}{\partial P} = \frac{1}{m}\left(P - \frac{a}{\omega}\cos(\omega t)\right).$$

But since P is a constant, we can just integrate Q, and we have

$$\boxed{q(t) = Q(t) = Q(0) + \frac{P}{m}t - \frac{a}{m\omega^2}\sin(\omega t)}.$$

Fact 101

If a Hamiltonian doesn't depend on p or q, then the other one will be constant! In 8.09, we sometimes do a transformation that makes the Hamiltonian completely constant, so all quantities end up being conserved.

23.5 More examples of generating functions

1. Let's say $F = qP + (\delta t)H$: this will basically make the Hamiltonian the generating function. The idea with the qP part is to first set $p = P$ and then modify the coordinates with the second part of the generating function. This gives us

   $$p = \frac{\partial F}{\partial q} = P + \delta t\frac{\partial H}{\partial q} = P - \dot{p}\,\delta t \implies \boxed{P = p + \frac{dP}{dt}\delta t = p(t+\delta t)}.$$

   Similarly,

   $$Q = \frac{\partial F}{\partial P} = q + \delta t\frac{\partial H}{\partial P} = q + \delta t\frac{\partial H}{\partial p} = q + \dot{q}\,\delta t \implies \boxed{Q = q(t+\delta t)}.$$

   since P and p are basically equivalent (up to leading order). So the Hamiltonian "generates time-translation:" it **moves systems forward in time.**

2. Next, consider a generating function $F = (q + \delta q)P$. Then

   $$p = \frac{\partial F}{\partial q} = P$$

   (leaving the momentum unchanged), and

   $$Q = \frac{\partial F}{\partial P} = q + \delta q.$$

   So if our generating function contains a p_x term, for example, that will translate us in the x-direction. In other words, **momentum translates the system in space.**

3. For the next example, consider a three-dimensional system with a position vector $\vec{r}$ and momentum vector $\vec{p}$, and let's say we have

   $$F = \vec{r}\cdot\vec{P} + (\vec{r}\times\vec{P})_z\delta\theta.$$

   Here the first term is the equivalent of the qP term in multiple dimensions, meaning it is

   $$xP_x + yP_y + zP_z + \delta\theta(yP_x - xP_y).$$

   We know that

   $$p_z = \frac{\partial F}{\partial z} = P_z,$$

so the momentum in the z-direction is not changed. Similarly,

$$Z = \frac{\partial F}{\partial P_z} = z,$$

so this transformation does not alter the z-component at all. But the other directions are more interesting:

$$p_x = \frac{\partial F}{\partial x} = P_x - P_y\,\delta\theta, p_y = \frac{\partial F}{\partial y} = P_y + P_x\,\delta\theta.$$

Combining these, $P_x = p_x + \delta\theta\, p_y, p_y = P_y - \delta\theta\, p_x$. Similarly, looking at the position,

$$X = \frac{\partial F}{\partial P_x} = x + y\,\delta\theta, Y = \frac{\partial F}{\partial P_y} = y - x\,\delta\theta.$$

Since θ is small, this gives

$$P_x \approx p_x \cos\delta\theta + p_y \sin\delta\theta, P_y = p_y \cos\delta\theta - p_x \sin\delta\theta, P_z = p_z.$$

$$X \approx x\cos\delta\theta + y\sin\delta\theta, Y \approx y\cos\delta\theta - x\sin\delta\theta, Z = z.$$

So **the angular momentum in the z-direction generates rotations around the z-axis**.

4. These all follow a general pattern: if $F = qP + \varepsilon G$, we'll have

$$\delta q = \varepsilon\frac{\partial G}{\partial p}, \delta p = -\varepsilon\frac{\partial G}{\partial q}.$$

23.6 Quantum mechanical states

We'll finish by showing how this plays out in quantum mechanics. Suppose we have a quantum mechanical state $|\psi\rangle$, such that we know the wavefunction $|\psi\rangle$ at time t. Our goal is to get it at some later time by finding $|\psi(t+\delta t)\rangle$.

The physical interpretation of the wavefunction is that $|\psi|^2$ gives us the probability that we'll find a particle somewhere. The total probability has to be 1, so

$$|\psi(t+\delta t)\rangle = U(\delta t)|\psi(t)\rangle$$

for some **unitary operator** U. The easiest way to do this is to write our U as $e^{-iH\,\delta t}$, since H translates things in time. The problem is that the argument of an exponential has to be unitless. H has units of energy and δt has units of time, so we have to divide by something to fix that: $\hbar$ happens to work! If we take $\delta t \to 0$, we find that

$$|\psi(t+\delta t)\rangle = e^{-i\frac{H\delta t}{\hbar}}|\psi(t)\rangle \approx \left(1 - i\frac{\delta t}{\hbar}H\right)|\psi(t)\rangle.$$

Rearranging terms,

$$|\psi(t+\delta t)\rangle - |\psi(t)\rangle = \delta|\psi\rangle \approx -\frac{i\delta t}{\hbar}H|\psi(t) \implies \boxed{H|\psi(t)\rangle = i\hbar\frac{\partial|\psi\rangle}{\partial t}},$$

which is the famous Schrodinger equation! The main conceptual jump here is the definition of a wavefunction, and the fact that this is related to the physical probability.

Fact 102

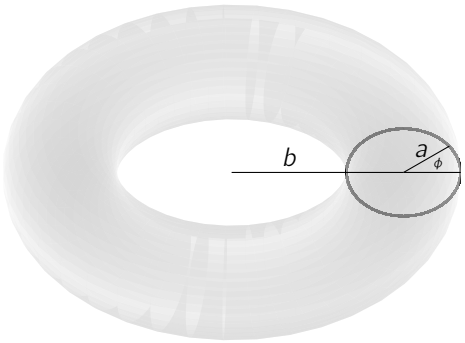
This equation is non-relativistic, because it's hard to "conserve probability" in the relativistic case. Dirac came along and tried to fix this with relativistic energy, but his model predicted that the probability of finding a particle was not 1 (it's possible to make an antiparticle). He thought this was nonsense, but it was actually the first sign that **antimatter** exists.

24 January 23, 2019 (Recitation)

Example 103

Consider a particle constrained to move on a torus. There are no forces on the particle, so the potential is zero.

First, we want to choose a sensible coordinate system. Let the distance from the center to the edge of the torus be b, and let the radius of the torus be a. Choose coordinates so that the torus sits in the xy-plane.



Then if we define ϕ to be the angle of elevation along the vertical cross-section (as shown), we have $z = a\sin\phi$. Setting θ be the angle of the particle in its xy-plane projection, the particle is a distance of $r = b + a\cos\phi$ away from the origin, and we have

$$x = r\cos\theta, y = r\sin\theta.$$

Taking a few derivatives,

$$\dot{x} = \dot{r}\cos\theta - r\dot{\theta}\sin\theta, \dot{y} = \dot{r}\sin\theta + r\dot{\theta}\cos\theta, z = a\dot{\phi}\cos\phi.$$

So we can write our Lagrangian out as usual:

$$L = K = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) = \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\theta}^2 + a^2\dot{\phi}^2\cos^2\phi\right),$$

since the cross-terms cancel and $\sin^2\theta + \cos^2\theta = 1$. Now since $r = a + b\cos\phi$, $\dot{r} = -a\dot{\phi}\sin\phi$, and

$$L = \frac{1}{2}m(a^2\dot{\phi}^2 + r^2\dot{\theta}^2) = \boxed{\frac{1}{2}m(a^2\dot{\phi}^2 + (b + a\cos\phi)^2\dot{\theta}^2)}.$$

We can think of this intuitively as the motion of the particle being split into two perpendicular coordinates. From here, we can start constructing a Hamiltonian: the generalized momenta are

$$p_\theta = \frac{\partial L}{\partial\dot{\theta}} = mr^2\dot{\theta} = m(b + a\cos\phi)^2\dot{\theta}$$

and

$$p_\phi = \frac{\partial L}{\partial \dot{\phi}} = ma^2\dot{\phi}.$$

The Hamiltonian is then

$$p_\theta\dot{\theta} + p_\phi\dot{\phi} - L = mr^2\dot{\theta}^2 + ma^2\dot{\phi}^2 - \frac{1}{2}m(a^2\dot{\phi}^2 + r^2\dot{\theta}^2) = \frac{1}{2}m(r^2\dot{\theta}^2 + a^2\dot{\phi}^2),$$

which is just the kinetic energy. (This happens whenever the potential is zero.) However, we want our Hamiltonian to be in terms of our position and momentum, so we really want to rewrite as

$$H = \frac{1}{2}m\left(r^2\left(\frac{p_\theta}{mr^2}\right)^2 + a^2\left(\frac{p_\phi}{ma^2}\right)^2\right) = \boxed{\frac{p_\theta^2}{2m(a\cos\phi + b)^2} + \frac{p_\phi^2}{2ma^2}}.$$

But we have a trick here: the Euler-Lagrange equations reveal that

$$\frac{\partial L}{\partial \theta} = \frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} \implies \frac{dp_\theta}{dt} = \frac{\partial L}{\partial \theta} = 0$$

in this case! So p_θ is actually a constant here, and this helps us a lot.

For example, if ϕ is constrained to be $\frac{\pi}{2}$ at the beginning and end of a trip, we can guess that $\phi = \frac{\pi}{2}$ for all time. This means $mb^2\dot{\theta}$ is a constant, so θ is linear with time. Also, if θ is 0 at the beginning and end of its trip, and $m(b + a\cos\phi)^2\dot{\theta}$ is a constant, we can just have $\dot{\theta} = 0$ for all time. We can also see this via the Hamilton equation

$$\frac{\partial H}{\partial p_\theta} = \dot{\theta} = \frac{p_\theta}{m(a\cos\phi + b)^2}.$$

The whole point is not to find specific paths: if we're given initial conditions, **we can make a physical guess and verify the equations from there**.

25 January 24, 2019 (Lecture)

Our third problem set is due tomorrow. By the way, we can use any software for the optimization that we want – Python or Mathematica work fine.

25.1 Review

Remember that after defining the generalized momenta $p_i \equiv \frac{\partial L}{\partial \dot{q}_i}$, we defined the Hamiltonian to be $H = \sum p_i\dot{q}_i - L$. After doing some optimization with the action, we found Hamilton's equations

$$\frac{\partial H}{\partial p_i} = \dot{q}_i, \frac{\partial H}{\partial q_i} = -\dot{p}_i.$$

These are the canonical equations of motion because they treat position and momentum symmetrically; the similarly named canonical transformations preserve these equations. We found that if we write a generating function in the form

$$F = qP + \varepsilon G(q, p),$$

we can create a change of coordinates $\delta q = \varepsilon\frac{\partial G}{\partial p}, \delta p = -\varepsilon\frac{\partial G}{\partial q}$. For example, if we set G to be the Hamiltonian and ε to be a small time interval, all positions and momenta are just translated forward by that small time interval. So the Hamiltonian is the generator of translations in time.

25.2 Ensemble systems: the Liouville equation

We know that if we have a bunch of particles and initial conditions, we can use the Hamiltonian to evolve the system. Classical mechanics is deterministic – we can just use the initial state and that will tell us the state for all future time. However, with a system like a room with many particles of gas, we can't know the initial state (there's too much information). Instead, we **measure and extract general properties** (like temperature and pressure), and we can use those to understand the general behavior of the system.

Basically, given some set of initial conditions like temperature and pressure, we can use probabilities to figure out properties. We can say that we have some idea of what the system will look like, and we can track the **states** instead of the particles themselves.

Example 104

Consider a small volume element (a hypercube) in phase space with a bunch of coordinates dp_i and dq_i. We want to know how many particles are in this box - that is, what's the general proximity of molecules?

We know how to calculate quantities in a volume: we just integrate

$$N = \int \rho \, dV,$$

where our volume element in phase space looks like

$$dV = \prod_i dq_i dp_i.$$

Here, ρ is just the general idea of "density" – we're making the assumption now that there are so many particles that we can treat this as a continuous system. Then the new question we care about is how the number of states in this volume element changes over time? That is, we want to find

$$\frac{dN}{dt} = \boxed{\iiint \frac{\partial \rho}{\partial t} dV} = -\oiint \vec{J} \cdot d\vec{s} = -\boxed{\iiint \vec{\nabla} \cdot \vec{J} \, dV}$$

by the divergence theorem. Here, $\vec{J} = \frac{\Delta N}{\Delta t \Delta A}$ is the "current density," $\vec{s}$ is the normal vector (different for each face of the hypercube, pointing out of the box), and we want to see how many things are coming and going in and out of the box. Now we have dVs in two different expressions, so if this is true for any arbitrary volume, we must have

$$\boxed{\frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot \vec{J} = 0}.$$

This is called the **continuity equation**, and it is basically a mathematical fact for all conserved quantities, which is why we see it in all fields (electromagnetism, fluid mechanics, and so on)! So now we (actually) define the current density: if we have a fixed surface with some thickness $dq_i = \dot{q}_i dt$ (since distance is velocity times time) and area dA, then the number of particles that move through the surface is

$$dN = \rho \dot{q}_i \, dt \, dA.$$

So this allows us to define the **current** in the q_i direction to be

$$J_{q_i} = \frac{d^2 N}{dt dA} = \rho \dot{q}_i.$$

This is like $\vec{J} = \rho \vec{v}$ in electromagnetism: the current density is the charge density times the velocity. But in our case,

$\dot{p}$ is also a velocity, since momentum is being treated as an ordinary coordinate in phase space.

So now by the continuity equation,

$$\frac{\partial \rho}{\partial t} + \sum_i \left(\frac{\partial(\rho \dot{q}_i)}{\partial q_i} + \frac{\partial(\rho \dot{p}_i)}{\partial p_i} \right) = 0,$$

and by the chain rule, the term in parentheses becomes

$$\frac{\partial \rho}{\partial q_i}\dot{q}_i + \rho \frac{\partial \dot{q}_i}{\partial q_i} + \frac{\partial \rho}{\partial p_i}\dot{p}_i + \rho \frac{\partial \dot{p}_i}{\partial p_i},$$

but now we can finally use Hamilton's equations! Since

$$\frac{\partial \dot{q}_i}{\partial q_i} = \frac{\partial^2 H}{\partial q_i \partial p_i} = \frac{\partial^2 H}{\partial p_i \partial q_i} = -\frac{\partial \dot{p}_i}{\partial p_i},$$

two of our terms cancel. This leaves us with

$$\frac{\partial \rho}{\partial t} + \sum_i \left[\frac{\partial \rho}{\partial q_i}\dot{q}_i + \frac{\partial \rho}{\partial p_i}\dot{p}_i \right] = 0$$

But by the chain rule, this is just $\frac{d\rho}{dt}$, so the point is that the total time derivative of density must be zero!

Proposition 105

The density of states in phase space is constant, regardless of the system.

This can be stated as "The total volume occupied in phase space is constant over time," and this is the basis of statistical mechanics! The whole point is that we're starting to get into uncertainty and probabilistic calculations.

Rewriting the expression we had above, looking at just one dimension, we know that

$$\frac{\partial \rho}{\partial t} = -\left(\frac{\partial \rho}{\partial q}\dot{q} + \frac{\partial \rho}{\partial p}\dot{p} \right).$$

Let's assume we have a Hamiltonian $H(q, p)$ that does not depend on time. We guess that

$$\rho \propto e^{-\beta H(p,q)}$$

for some constant β, noticing that there is no time dependence for the density ρ. Plugging this in, we're left with

$$0 = -\left(\dot{q}\left(-\beta \frac{\partial H}{\partial q} \right)\rho + \dot{p}\left(-\beta \frac{\partial H}{\partial p} \right)\rho \right),$$

and this indeed simplifies (by Hamilton's equations) to

$$-\dot{q}(-\dot{p}) - \dot{p}\dot{q} = 0.$$

The only thing we need to worry about is that β has to have inverse units of energy so that the exponential is well-defined. Conveniently, thermodynamics provides us with a useful constant, and we can write

$$\rho \propto e^{-(H(p,q))/(k_B T)},$$

where k_B is the Boltzmann constant. This probability distribution is related to the concept of a **partition function**, which we'll learn in 8.044.

Fact 106

Partition functions tell us the probability we'll find the particle in any given state.

Again, all of this makes more sense if we have "Avogadro's number of particles," so that the assumption of continuous density is reasonable.

25.3 Poisson brackets

We can go back and plug in Hamilton's equations into the expression we had for $\frac{d\rho}{dt}$ to find that

$$0 = \frac{\partial \rho}{\partial t} + \sum_i \left(\frac{\partial \rho}{\partial q_i} \frac{\partial H}{\partial p_i} - \frac{\partial \rho}{\partial p_i} \frac{\partial H}{\partial q_i} \right).$$

Definition 107

Given two arbitrary functions F, G, their **Poisson bracket** is

$$[F, G] = \sum_i \left(\frac{\partial F}{\partial q_i} \frac{\partial G}{\partial p_i} - \frac{\partial F}{\partial p_i} \frac{\partial G}{\partial q_i} \right).$$

This definition allows us to write the above equation as

$$\frac{\partial \rho}{\partial t} = -[\rho, H] = [H, \rho].$$

Fact 108

The Poisson bracket has the important property that

$$[F, G]_{q,p} = [F, G]_{Q,P};$$

that is, **the brackets are invariant under canonical transformations**.

In general, for any function F, we will have (by the chain rule and Hamilton's equations)

$$\boxed{\frac{dF}{dt} = \frac{\partial F}{\partial t} + [F, H]}.$$

We'll combine this with some previous ideas: let G be a generating function where $F = qP + \varepsilon G(q, p)$. Then plugging G in above, using that $\frac{\partial G}{\partial t} = 0$ (no explicit time-dependence), we learn that

$$\boxed{[G, H] = 0 \implies \frac{dG}{dt} = 0}.$$

This means we have a conserved quantity whenever the bracket (with the Hamiltonian) is zero!

Example 109

Setting $G = H$, clearly $[H, H] = 0$ (by plugging into the definition), so the Hamiltonian is conserved as long as it does not have explicit time-dependence.

A good way to tie this concept into what we've learned is to discuss **symmetry** in a similar way as we did for Lagrangian mechanics. Namely, how does the Hamiltonian change if we generate a transformation with $F = qP + \varepsilon G(q,p)$? By the chain rule,

$$\delta H = \frac{\partial H}{\partial q}\delta q + \frac{\partial H}{\partial p}\delta p = \frac{\partial H}{\partial q}\left(\varepsilon\frac{\partial G}{\partial p}\right) = \frac{\partial H}{\partial p}\left(-\varepsilon\frac{\partial G}{\partial q}\right) = \varepsilon[H,G].$$

So this tells us that

$$\boxed{[H,G]=0 \implies \delta H = 0}.$$

Rephrased, if the bracket of H and G is zero, meaning that "G generates a continuous symmetry of the system," the bracket of G and H is also zero, so "G is a conserved quantity."

Proposition 110

If $[G_i, H] = 0$ and $[G_j, H] = 0$, then

$$[[G_i, G_j], H] = 0.$$

This statement is usually not helpful on its own – it will usually tell us trivial resulst. But this tells us that

$$[G_i, G_j] = c_{ijk}G_k$$

where G_k is some other generator and c_{ijk} is some constant.

Fact 111

The set of generators being closed under the Poisson bracket gives us a **Lie algebra**! For example, angular momenta generate a Lie algebra, and we'll learn more about this in 8.05.

And as a final note, commutators show up in quantum mechanics as well: lots of this discussion here follows if we make the transformation

$$[F,G] \implies i\hbar[F,G].$$

26 January 24, 2019 (Recitation)

Example 112

Consider a function $z = f(r)$ in cylindrical coordinates, which defines a wire that a particle is constrained to move on. (For example, we could have $f(r) = -\frac{1}{2}r^2 + \frac{1}{4}r^4$.) This wire rotates around the z-axis with angular frequency ω, and gravity also acts on the particle.

Our Lagrangian is

$$L = K - U = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) - mgz.$$

Since our particle is stuck on the wire, we must have $z = f(r), x = r\cos\theta, y = r\sin\theta$. Taking derivatives yields

$$\dot{x} = \dot{r}\cos\theta - r\dot{\theta}\sin\theta, \quad \dot{y} = \dot{r}\sin\theta + r\dot{\theta}\cos\theta, \quad \dot{z} = f'(r)\cdot\dot{r},$$

and if we plug everything in, we'll find that

$$L = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2 + f'(r)^2\dot{r}^2) - mgf(r).$$

Since $\dot{\theta} = \omega$ by the problem statement, this kills one of our degrees of freedom. This reduces the Lagrangian to

$$\boxed{L = \frac{1}{2}m\dot{r}^2(1 + f'(r)^2) + \frac{1}{2}m\omega^2 r^2 - mgf(r)}.$$

Now by the Euler-Lagrange equation for r, we have

$$\frac{\partial L}{\partial r} = \frac{d}{dt}\frac{\partial L}{\partial \dot{r}} \implies \boxed{m\dot{r}^2 f'(r)f''(r) + m\omega^2 r - mgf'(r) = \frac{d}{dt}\left(m\dot{r}(1 + f'(r)^2)\right)}.$$

Taking the time-derivative and using the product rule, this will all simplify to

$$0 = m\ddot{r}(1 + f'(r)^2) + m\dot{r}^2 f'(r)f''(r) - m\omega^2 r + mgf'(r).$$

We'll now ask a more concrete question: **is there a nonzero r_0 for which there is a steady state solution?** For this to happen, we'll want $r(t) = r_0, r'(t) = r''(t) = 0$, which means

$$0 = -m\omega^2 r_0 + mgf'(r_0) \implies \boxed{gf'(r_0) = \omega^2 r_0.}$$

Example 113

If $f(r) = \frac{1}{3}r^3$, then the equation that needs to be satisfied is

$$gr_0^2 = \omega^2 r_0 \implies r_0 = \frac{\omega^2}{g}.$$

Once we have an equilibrium r_0, we can think about **small oscillations** around that equilibrium. Let's say $r(t) = r_0 + \varepsilon(t)$ for some small perturbation – note that all terms that have ε^2 in them can be treated as effectively zero. Now

$$\dot{r}(t) = \dot{\varepsilon}(t) \implies f'(r)f''(r) \to 0.$$

We can further simplify by noting that

$$f'(r(t)) = f'(r_0 + \varepsilon(t)) = f'(r_0) + f''(r_0)\varepsilon(t) + \cdots$$

by the Taylor series expansion around r_0. This means $f'(r(t))^2$ is, to first order, $f'(r_0)^2 + 2f'(r_0)f''(r_0)\varepsilon$. Plugging this into the equation of motion,

$$0 = m\ddot{\varepsilon}(1 + f'(r_0)^2 + 2f'(r_0)f''(r_0)\varepsilon) - m\omega^2(r_0 + \varepsilon) + mg(f'(r_0) + f''(r_0)\varepsilon).$$

The term with $\ddot{\varepsilon} \cdot \varepsilon$ drops out, and we simplify this to

$$0 = m\ddot{\varepsilon}(1 + f'(r_0)^2) - m\omega^2\varepsilon + mgf''(r_0)\varepsilon + (-m\omega^2 r_0 + mgf'(r_0)).$$

And now the term in parentheses is zero, because r_0 is a fixed point! This means we're left with (dividing through by m)

$$0 = \ddot{\varepsilon}(1 + f'(r_0)^2) + (gf''(r_0) - \omega^2)\varepsilon.$$

This is a **simple harmonic motion**, and therefore our small displacement will oscillate with a frequency ω' of

$$\omega'^2 = \frac{gf''(r_0) - \omega^2}{1 + f'(r_0)^2}.$$

This is a stable equilibrium if and only if the restoring force is opposing the direction of displacement. That is, we need $\boxed{f''(r_0) > \frac{\omega^2}{g}}$ (since the denominator is always positive).

Example 114

For the example $f(r) = \frac{1}{3}r^3$, this condition becomes

$$2gr_0 > \omega^2.$$

Plugging in the value of r_0 from above, we find that our equilibrium point will be stable. The intuitive reason for this is that the centrifugal force is proportional to r^2, while the curve's height is proportional to r^3, so the bead doesn't want to move further up.

On the other hand, if $f(r) = r$, we have a fixed point of $r_0 = \frac{g}{\omega^2}$. However, no equilibrium point is stable, because $gf''(r_0) = 0$. Thus, there will not be nice oscillations in this case.

By the way, we can check that the force in the θ direction is $2m\omega\dot{r}$ (even though the bead is constrained to the wire, we've forced circular motion). The idea is that we have to do work on the bead to keep the bead spinning at frequency ω! This means **the energy is not equal to the Hamiltonian**: instead, we'll have $E = H + W$.

27 January 25, 2019

Today we're going to talk about **Lagrange points**, the first "selective topic" of this course.

27.1 Two particles (plus a third)

Suppose we have two particles m_1 and m_2 on the x-axis, separated by a distance R. If we define the reduced mass as $\mu = \frac{m_1 m_2}{m_1 + m_2}$, then m_2 is a distance $R\frac{\mu}{m_2}$ away from the center of mass, and m_1 is similarly a distance $R\frac{\mu}{m_1}$ away. Fix the center of mass to be the origin; then, the coordinates of the two masses are

$$M_1 = \left(-\frac{R\mu}{m_1}, 0\right), \quad M_2 = \left(\frac{R\mu}{m_2}, 0\right).$$

We work with a **co-rotating frame**, so that the frame is rotating at some frequency Ω, but the particles look like they are stationary from our frame. (In other words, the two particles are rotating at a frequency Ω in the rest frame as well.) For simplicity, let's say that the two masses are in a circular orbit. Then

$$\mu\frac{v^2}{R} = \frac{Gm_1m_2}{R^2} \implies (R\Omega)^2 = \frac{G(m_1 + m_2)}{R}.$$

This gives us something similar to Kepler's third law:

$$\Omega^2 = \frac{G(m_1 + m_2)}{R^3}.$$

Now, let's add a third mass m_3 that is much less massive than the other two (so it has no impact on the other two masses). Normally, the three-body problem is not analytically solvable, but we can assume the coordinate frame is stationary. Also, let's say that the third mass orbits in the same orbital plane as the other two bodies - essentially, we're just asking the following question:

Problem 115

Is there anywhere in this plane (x, y) that we can put this third mass in the rotating frame such that it will stay stationary?

The solutions to this problem are called **Lagrange points**, and they have practical applications! Without doing too much work, we can already find one: when $m_1 \gg m_2$, Ω^2 is approximately $\frac{Gm_1}{R^3}$, meaning that we can treat mass m_2 as rotating around an (almost-fixed) mass m_1. So we can put the third mass on the diametrically opposite side of m_1: this point is often called $\boxed{L_3}$, and it's a distance of $R + O(1)\frac{m_2}{m_1}$ away from m_1.

27.2 Using the Lagrangian

To solve for the other stationary points, we want to write down a Lagrangian. But the rotating frame is not inertial, and trying to solve in our co-rotating frame creates Coriolis and centrifugal terms, which are annoying to deal with. Instead, let's write everything in the inertial frame first.

We denote variables in the inertial frame with an i subscript. The kinetic energy of the third mass is

$$K = \frac{1}{2}m_3(\dot{r}_i^2 + r_i^2\dot{\theta}_i^2).$$

In the rotating frame, $r = r_i$ (we're just rotating around the center of mass, or origin), and $\theta = \theta_i - \Omega t$. Clearly, the derivatives are then $\dot{r} = \dot{r}_i$ and $\dot{\theta} = \dot{\theta}_i - \Omega$. So our actual kinetic energy becomes

$$K = \frac{1}{2}m_3(\dot{r}^2 + r^2(\dot{\theta} + \Omega)^2).$$

Unfortunately, the mass m_3 is moving about two particles, both of which are not centered at the origin, so we actually want to transform the problem back into Cartesian coordinates. But we should remember that we're not interested in the general solution – we want the case where $\dot{r} = \dot{\theta} = 0$. So this means we can simplify the problem to

$$K = \frac{m_3}{2}r^2\Omega^2 = \frac{m_3}{2}\Omega^2(x^2 + y^2).$$

The potential energy is also easy to write down conceptually: it is

$$U = -\frac{Gm_1m_3}{r_{13}} - \frac{Gm_2m_3}{r_{23}}.$$

By the distance formula, we also know that

$$r_{13}^2 = \left(x + R\frac{\mu}{m_1}\right)^2 + y^2, \quad r_{23}^2 = \left(x - R\frac{\mu}{m_2}\right)^2 + y^2.$$

Our Lagrangian is just $L = K - U$, and remember that we artificially removed all the dot terms by choosing a stationary point. Writing the expression for r_{13}^2 as differentials,

$$2r_{13}dr_{13} = 2\left(x + R\frac{\mu}{m_1}\right)dx + 2ydy$$

This tells us the partial derivatives

$$\frac{\partial r_{13}}{\partial x} = \frac{x + R\frac{\mu}{m_1}}{r_{13}}, \quad \frac{\partial r_{13}}{\partial y} = \frac{y}{r_{13}}.$$

Now let's plug into our EL equations. Luckily, the right-hand side of $\frac{\partial L}{\partial x} = \frac{d}{dt}\frac{\partial L}{\partial \dot{x}}$ is just 0, so we just have the equation

$\frac{\partial L}{\partial x} = 0$. Differentiating,

$$m_3\Omega^2 x - \frac{\partial U}{\partial x} = 0.$$

Writing this out,

$$m_3\Omega^2 x = \frac{Gm_1m_3}{r_{13}^2}\frac{\partial r_{13}}{\partial x} + \frac{Gm_2m_3}{r_{23}^2}\frac{\partial r_{23}}{\partial x}.$$

We can divide through by m_3 and replace for the expressions we found, and we end up with

$$\boxed{\Omega^2 x = \frac{Gm_1}{r_{13}^3}\left(x + R\frac{\mu}{m_1}\right) + \frac{Gm_2}{r_{23}^3}\left(x - R\frac{\mu}{m_2}\right)}.$$

The EL equation for y is slightly simpler:

$$\frac{\partial L}{\partial y} = \frac{d}{dt}\frac{\partial L}{\partial \dot{y}} \implies \boxed{\Omega^2 y = \frac{Gm_1}{r_{13}^3}y + \frac{Gm_2}{r_{23}^3}y}.$$

27.3 Solutions not on the line of m_1 and m_2

If $y \neq 0$, we can cancel the ys, and we need

$$\Omega^2 = \frac{Gm_1}{r_{13}^3} + \frac{Gm_2}{r_{23}^3}.$$

Plugging this into the first equation,

$$x\left(\frac{Gm_1}{r_{13}^3} + \frac{Gm_2}{r_{23}^3}\right) = \frac{Gm_1}{r_{13}^3}\left(x + R\frac{\mu}{m_1}\right) + \frac{Gm_2}{r_{23}^3}\left(x - R\frac{\mu}{m_2}\right).$$

The terms with x all cancel, and we're left with something simple:

$$0 = \frac{GR\mu}{r_{13}^3} - \frac{GR\mu}{r_{23}^3},$$

so we just need $r_{13} = r_{23}$. This means our stationary point not on the y-axis must satisfy

$$\Omega^2 = \frac{G(m_1+m_2)}{R^3} = \frac{G(m_1+m_2)}{r_{13}^3} \implies r_{13} = r_{23} = R.$$

So there are two solutions where y is positive or negative, but both lie where the **three masses form an equilateral triangle**! These are called $\boxed{L_4}$ and $\boxed{L_5}$.

27.4 Finding the last two solutions

The rest of our Lagrange points have $y = 0$. In this case, our distances are along the x-axis:

$$r_{13} = \left|x + \frac{\mu}{m_1}\right|, r_{23} = \left|x - R\frac{\mu}{m_2}\right|.$$

Plugging this in to the linear equation above for x, we have that

$$\Omega^2 x = \pm\frac{Gm_1}{\left(x + R\frac{\mu}{m_1}\right)^2} \pm \frac{Gm_2}{\left(x - R\frac{\mu}{m_2}\right)^2}.$$

Replacing Ω^2 with its known value and dividing through by G, this becomes

$$\frac{(m_1+m_2)}{R^3}x = \pm\frac{m_1}{\left(x+R\frac{\mu}{m_1}\right)^2} \pm \frac{m_2}{\left(x-R\frac{\mu}{m_2}\right)^2}.$$

It turns out that these solutions will be closer to the smaller mass: intuitively, we can't get too close to the big mass without getting pulled in. Let's define $d \equiv r_{23}$.

The Lagrange point $\boxed{L_1}$ between the two masses satisfies $d = R\frac{\mu}{m_2} - x \implies x = R\frac{\mu}{m_2} - d$. Then $r_{13} = R - d$, and correcting all of the $\pm$ signs in this case,

$$\frac{m_1}{(R-d)^2} - \frac{m_2}{d^2} = \frac{(m_1+m_2)}{R^3}\left(R\frac{\mu}{m_2} - d\right) = \frac{m_1}{R^2} - \frac{(m_1+m_2)}{R^3}d.$$

Now to take advantage of the fact that $m_1 \gg m_2 \implies d \ll R$, we can take leading order expansions. This leads to

$$\frac{m_1}{R^2}\left(1+2\frac{d}{R}\right) - \frac{m_2}{d^2} = \frac{m_1}{R^2} - \frac{(m_1+m_2)}{R^3}d.$$

Canceling common terms,

$$\frac{3m_1 d}{R^3} = \frac{m_2}{d^2} \implies d = R\left(\frac{m_2}{3m_1}\right)^{1/3}.$$

Example 116

For the sun, $\frac{m_2}{m_1}$ is about 10^{-6}. This means d is about 1 percent of the distance from the earth to the sun.

On the other hand, $\boxed{L_2}$ is on the other side of our second mass (farther away from the first mass). This time $d = x - R\frac{\mu}{m_2}$, and $r_{13} = R + d$, so plugging in and dividing by G,

$$\frac{m_1}{(R+d)^2} + \frac{m_2}{d^2} = \frac{(m_1+m_2)}{R^3}\left(d + R\frac{\mu}{m_2}\right) = \frac{d(m_1+m_2)}{R^3} + \frac{m_1}{R^2}.$$

Doing the same first-order approximation, we end up with

$$\frac{m_1}{R^2}\left(1-\frac{2d}{R}\right) + \frac{m_2}{d^2} = \frac{m_1 d}{R^3} + \frac{m_1}{R^2} \implies \frac{m_2}{d^2} = \frac{3m_1 d}{R^3},$$

so $d = R\left(\frac{m_2}{3m_1}\right)^{1/3}$, meaning we have the same distance (to leading order) just on the other side!

Fact 117

It turns out that L_4 and L_5 are actually stable equilibrium points. In contrast, L_1, L_2, L_3 are not stable, but the potentials are pretty flat. For example, if we put something stationary at L_1, it will stay there for a few months in real life.

27.5 Applications

There are Trojan and Greek asteroid systems that sit in the L_4 and L_5 Lagrange points of the Jupiter-Sun system! (We can then have "big battles" at L_1 and L_2, because those are the easiest points of passage between L_4 and L_5.) L_1 and L_2 are interesting in their own right as well. We've put SOHO, the Solar and Heliospheric Observatory, at L_1 for 20 years or so. SOHO actually undergoes a complicated orbit, called a **halo orbit**, so that the Coriolis effect

makes the point very stable. Similarly, WMAP and other cosmic background radiation satellites are at L_2 to protect themselves from the sun's radiation.

28 January 25, 2019 (Recitation)

Fact 118

Problem set 3 solutions were discussed during recitation – this has been omitted.

29 January 28, 2019 (Lecture)

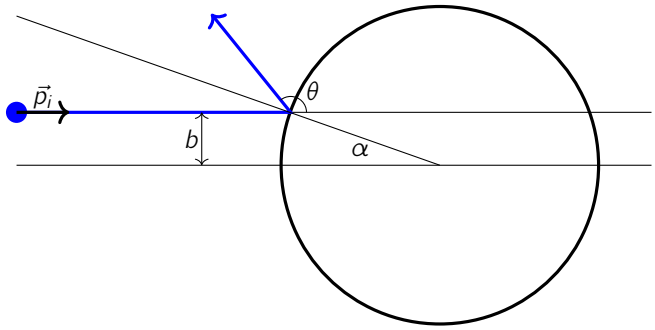
Today, we're going to begin our discussion of **scattering**. The basic premise is as follows: let's say we have two particles a and b. Then conservation of energy and momentum tell us that

$$\sum_j E_i^j = \sum_j E_f^j, \quad \sum_j \vec{p}_i^j = \sum_j \vec{p}_f^j.$$

If we work in the center of mass frame with two particles that are approaching each other, then we know that $|\vec{p}_a^i| = |\vec{p}_b^i|$, and if we're given the angle at which the particles scatter off, we can figure out the final properties of momentum and energy as well. But we want to deal with this scattering problem without being explicitly told the angle of scattering, and this will lead us to some interesting physics in the next few days.

29.1 An example

Consider a massive sphere of radius R (which doesn't move), and consider a particle with momentum $\vec{p}_i$ that approaches it at a distance b above its equator, as shown below.



Notice that the angle of scattering is $\theta = \pi - 2\alpha$, since the angle of incidence and reflection are the same (and both equal to α). Then $\vec{p}_f$ has the same magnitude as $\vec{p}_i$, but the component **parallel to the normal** is flipped.

Also, $b = R\sin\alpha$, meaning there is a one-to-one relationship between b and α. Since there is also a one-to-one relationship between θ and α, there is a one-to-one mapping between b and θ.

This is a simple case, but the more interesting question is **what happens if we have a beam of particles**. Let's assume all of them have momentum p_i, and our beam has a radius $R_b \gg R$.

Problem 119

Given a solid angle $d\Omega$ (relative to the massive sphere), what's the probability that a particle scatters into that solid angle?

To answer this question, we define a **differential cross-section** as

$$\frac{d\sigma}{d\Omega} = \frac{\left(\frac{\text{beam particles scattered into } d\Omega}{\Delta t}\right)}{\text{luminosity } L}$$

where L is defined to be the number of beam particles divided by $\Delta t \pi R_b^2$. This is a normalization factor: it helps divide out the properties of the beam, so that particles that miss the sphere don't count.

We can integrate to get a total **cross-section** (which we'll soon see that we can think of as cross-sectional area)

$$\sigma \equiv \int \left(\frac{d\sigma}{d\Omega}\right) d\Omega.$$

Now to get the total number of beam scatters, plug in the definition of $\frac{d\sigma}{d\Omega}$ to find that

$$\frac{\text{beam scatters}}{\delta t} = \sigma L = \sigma \frac{\text{beam particles}}{\delta t \pi R_b^2},$$

so that the total probability of scattering is

$$p = \frac{\sigma}{\pi R_b^2}.$$

29.2 Doing some calculations

Consider a ring of the beam with radius (offset) between b and $b + db$. Then by the definition of luminosity, since the area of the ring is $2\pi b\, db$,

$$\frac{(\text{beam particles in radius } b \text{ to } b + db)}{\delta t} = L \cdot 2\pi b\, db.$$

Because of the one-to-one mapping described above, we can also look at the particles that get scattered into the corresponding solid angle: the same expression then evaluates to

$$L \frac{d\sigma}{d\Omega} 2\pi \sin\theta d\theta.$$

(The $\sin\theta d\theta$ comes from spherical coordinates.) Setting these equal, we find that

$$\boxed{\frac{d\sigma}{d\Omega} = \frac{b}{\sin\theta}\left|\frac{db}{d\theta}\right|},$$

and this equation holds true as long as we have azimuthal symmetry in the problem! Plugging in the properties of the sphere, and using $\theta = \pi - 2\alpha$, we end up with

$$\frac{d\sigma}{d\Omega} = \frac{R\sin\alpha}{\sin(\pi - 2\alpha)}\left|\frac{db}{d\alpha}\frac{d\alpha}{d\theta}\right| = \frac{R\sin\alpha}{\sin 2\alpha}\left|R\cos\alpha\left(-\frac{1}{2}\right)\right|,$$

and this further simplifies to

$$\frac{R^2 \sin\alpha\cos\alpha}{2 \cdot 2\sin\alpha\cos\alpha} = \frac{R^2}{4}.$$

So for this case, the differential cross-section is $\frac{R^2}{4}$, which is (importantly) isotropic: its value doesn't depend on θ or ϕ. That leads us to the following result:

Proposition 120

Scattering a beam of particles off of a sphere **uniformly populates the solid angle**.

Since the total solid angle of a sphere is 4π,

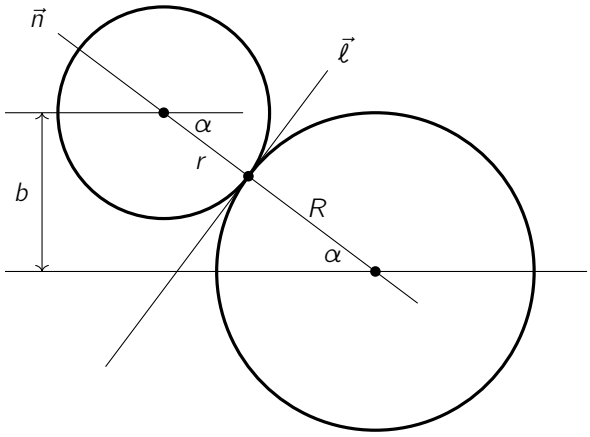
$$\sigma = \int \left(\frac{d\sigma}{d\Omega}\right) d\Omega = \frac{R^2}{4} 4\pi = \pi R^2,$$

and the cross-section turns out to just be the cross-sectional area exposed to the beam in this case.

In general, we often only have the scattering data and not the shape of the massive particle – our goal is to infer something about the target that is being hit. And the idea is that we can learn something based on the value of the differential cross-section $\frac{d\sigma}{d\Omega}$.

29.3 Extending the problem

Instead of having a infinitesimally small point particle bouncing off an infinitely massive sphere, now let's say we have two spheres that are hitting each other. Again, we work in the center-of-mass frame: this means the two spheres start off with equal and opposite momenta.



After the two particles rebound, they will still have equal and opposite momenta in this frame, and we want to find the new scattering angle. We have all forces only along the normal axis n, so we just flip the momentum in the n-direction while keeping the momentum in the ℓ-direction constant.

So that means the dynamics look similar as before: the scattering angle will be $\pi - 2\alpha$, and now we just have $b = (R+r)\sin\alpha$ instead! Everything will be the same, except now the total cross-section is $\sigma = \pi(R+r)^2$, and we're essentially pretending we have a single big particle at the center.

Fact 121

In particle physics, two particles will interact via the **Yukawa potential**

$$V(r) \propto \frac{e^{-mrc/\hbar}}{r}.$$

When we study weak gravity (with a graviton) and electromagnetism (with a photon), the corresponding particle has mass $m = 0$, so this reduces to the Coulomb potential proportional to $\frac{1}{r}$. But it turns out the particles behind

the weak force have large masses, so they can only interact meaningfully at very short distances. Strong nuclear interactions have a problem too: the Yukawa potential is only accurate at long distances.

Basically, the interactions between quarks at close distances don't allow them to get too far away from each other! So in a nucleus, neutrons and protons (nucleons) like to be within 1 femtometer of each other (also called 1 Fermi), and there is a force of about 2 tons between two nucleons at that range. This large force means that hitting a nucleon is actually a lot like hitting a wall and bouncing off.

29.4 Neutrons

Let's scatter a neutron off of Uranium-235. This isotope makes up less than 1 percent of natural uranium, and it occasionally undergoes spontaneous fission, which allows neutrons to start racing around.

Typically, all of these neutrons have the same energy:

$$K_n \approx 2 \text{ MeV},$$

which corresponds to a speed of about $v_n \approx 0.06c$, meaning we can basically ignore relativity. Naively, we should say that the scattering cross-section is

$$\pi(8 \text{ fm} + 1 \text{ fm})^2 \approx 250 \text{ fm}^2 = 2.5 \text{ b}.$$

The "b" here stands for "barn," and that unit name comes from scientists wanting to use a name that no one could understand. (The idea is from the idiom "can't hit the broad side of a barn," since Uranium is pretty large.) It turns out that the particle interactions make Uranium "look" a bit bigger, so the actual cross-section ends up being about 7 barns.

We'll think of this scattering as having a **neutron at rest** and a beam of uranium coming at it at speed v_n (since uranium is not moving in the lab frame). Then

$$\frac{(\text{number of } n\text{-}U \text{ scatters})}{\Delta t} = \sigma L = \sigma \rho_u v_n,$$

where ρ_u is the number density of uranium and we're using a similar equation to $\vec{J} = \rho \vec{v}$ in the last equality. Then the mean time between scatters is the inverse of the rate of scatters, meaning

$$\boxed{\tau = \frac{1}{\sigma \rho_u v_n}}.$$

We're interested in the **mean free path**, which is how far the neutron travels between scatters: this distance (in the lab frame) is

$$\overline{d} = v_n \tau = \frac{1}{\sigma \rho_u}.$$

(This is true for particles that aren't spherical, too.) Plugging in our numbers, we find that $\overline{d}$ is about 3 centimeters, which is pretty long. (One reason for this is that there are no long-term electromagnetic interactions for neutrons!) So neutrons are very dangerous sources of radiation – they go through rock and whatever else we try to use as shielding.

But there's another case – neutrons don't need to bounce off Uranium-235. They can instead turn it into Uranium-236, which is unstable and (through fission) decays into smaller nuclei plus neutrons (sometimes Krypton-92, Barium-141, and 3 neutrons). The probability this happens is about $p_f = 20\%$, so the probability of inducing fission on the ith interation is

$$p_f^i = p_f(1 - p_f)^{i-1},$$

with the $(1-p_f)$ factors because we can't do fission on the first $i-1$ interactions. So the expected value is

$$\overline{n}_f = \sum i p_f (1-p_f)^{i-1}.$$

This is not a difficult series to evaluate – we can use a trick to write it as

$$= -p_f \frac{d}{dp_f}\left(\sum (1-p_f)^i\right) = -p_f \frac{d}{dp_f}\left[\frac{1}{1-(1-p_f)}\right] = \frac{1}{p_f}.$$

So the mean number will be about 5 in our case, which means that by the central limit theorem (or hand-waving), the distance traveled before fission is about

$$\overline{d}_f = \sqrt{n_f} \cdot \overline{d} \approx 7 \text{ cm},$$

and the average time of decay is (notice the n_f instead of $\sqrt{n_f}$, because this measurement depends on the path and not the displacement)

$$\tau_f = \frac{\overline{d} n_f}{v_n} \approx 10^{-8} \text{ s}.$$

Every time we do fission, we create 2.5 new neutrons. If our block of uranium has diameter less than 7 cm, it's pretty likely we won't have a chain reaction (because the neutron will be out of the block quickly). But if P is the probability that the neutron n induces fission, and $\nu = 2.5$ is the number of neutrons created, then the number of neutrons is

$$n(t+\delta t) = n(t) + \frac{\delta t}{\tau_f} n(t)(\nu P - 1).$$

In words, the new electrons come from the rate of fission, times the number of neutrons we currently have, and then νP tells us the number of new neutrons per fission, while -1 tracks the ones that disappear from fission. Taking $\delta t \to 0$, we can solve the differential equation

$$\frac{dn}{dt} = \frac{\nu P - 1}{\tau_f} n \implies n(t) = n(0) e^{\left(\frac{\nu P - 1}{\tau_f}\right) t}.$$

This means that the chain reaction is exponential, and behavior depends on whether $\nu P > 1$ or $\nu P < 1$. So for Uranium-235, we need $P > \frac{1}{\nu} \approx 40\%$. This is achieved by having a sphere of about 9 centimeters: taking the density of uranium, this is about 50 kilograms of total uranium.

Fact 122

This is how weapons used to be made: we start with pieces of "sub-critical" pieces of uranium. When these pieces are pushed together, we start getting a chain reaction, and within a microsecond, lots of energy is released. The modern strategy, though, is to compress a plutonium core for a similar effect.

Tomorrow, we'll study how to do something more sensible with scattering.

30 January 28, 2019 (Recitation)

We'll do another problem similar to the bead on a wire from last class, where we find oscillations around the equilibrium position.

Example 123

Consider a vortex in a plane (this is the shape created if we swirl water in a water bottle). This creates a strong swirling motion near the center, and the velocity fields look like

$$\vec{u} = \frac{\gamma}{r}\hat{\theta},$$

where $\hat{\theta}$ is the unit vector that constitutes the angle of rotation and γ is the strength of the vortex.

Switching to Cartesian coordinates, we can find that

$$\hat{\theta} = \frac{-y\hat{x} + x\hat{y}}{r},$$

so we have a velocity field of the form

$$\vec{u} = \frac{\gamma}{r^2}(-y\hat{x} + x\hat{y}).$$

But the problem becomes more interesting if we have multiple vortices – they will then swirl around each other, since the centers of the vortices aren't fixed. We can account for that by defining $\vec{u}_{ij}$ to be the velocity field at $\vec{r}_i$ (where there is a vortex of strength γ_i) from $\vec{r}_j$ (where there is a vortex of strength γ_j). If we set $\vec{r}_j = 0$ for convenience, then

$$\vec{u}_{ij} = \frac{\gamma_j}{r_i^2}(-y_i\hat{x} + x_i\hat{y}).$$

But in general, if $\vec{r}_j$ is at (x_j, y_j), then the velocity contribution is

$$\vec{u}_{ij} = \gamma_j \frac{y_j - y_i}{|\vec{r}_i - \vec{r}_j|^2}\hat{x} + \gamma_j \frac{x_i - x_j}{|\vec{r}_i - \vec{r}_j|^2}\hat{y}.$$

Remember this is the velocity vector at $\vec{r}_i$ from a vortex at $\vec{r}_j$, so now we want to look at the velocity field from all N vortices to get

$$\vec{u}_i = \sum_{j \neq i} \left(\gamma_j \frac{y_j - y_i}{|\vec{r}_i - \vec{r}_j|^2}\hat{x} + \gamma_j \frac{x_i - x_j}{|\vec{r}_i - \vec{r}_j|^2}\hat{y} \right).$$

To get dynamics out of this expression, remember that since the vortices are the objects moving in the fluid, we just have $\vec{u}_i = (\dot{x}_i, \dot{y}_i)$. So now we have (breaking up the components)

$$\dot{x}_i = \sum_{j \neq i} \gamma_j \frac{y_j - y_i}{|\vec{r}_i - \vec{r}_j|^2}, \quad \dot{y}_i = \sum_{j \neq i} \gamma_j \frac{x_i - x_j}{|\vec{r}_i - \vec{r}_j|^2}.$$

We can multiply to get some symmetry:

$$\boxed{\gamma_i \dot{x}_i = \sum_{j \neq i} \gamma_i \gamma_j \frac{y_j - y_i}{|\vec{r}_i - \vec{r}_j|^2}, \quad \gamma_i \dot{y}_i = \sum_{j \neq i} \gamma_i \gamma_j \frac{x_i - x_j}{|\vec{r}_i - \vec{r}_j|^2}}.$$

Now, we have a surprising claim:

Proposition 124

The Hamiltonian for the vortex system is

$$H = -\frac{1}{2} \sum_{k \neq j} \gamma_k \gamma_j \ln |\vec{r}_k - \vec{r}_j|,$$

where, our parameters are $\vec{r}_i = (x_i, y_i)$.

Proof. Usually, we write the Hamiltonian in terms of position and momentum, and to get our Hamiltonian in that form, we will assert that

$$p_i = \gamma_i y_i, x_i = x_i.$$

In other words, we've turned the y-coordinate into a kind of momentum in phase space! Now the Hamilton's equations

$$\dot{x}_i = \frac{\partial H}{\partial p_i}, \dot{p}_i = -\frac{\partial H}{\partial x_i} \quad \text{for } 1 \leq i \leq N$$

need to be satisfied. Indeed,

$$\dot{x}_i = \frac{\partial H}{\partial(\gamma_i y_i)} \implies \gamma_i \dot{x}_i = \frac{\partial H}{\partial y_i} = -\frac{1}{2}\sum_{k\neq j} \gamma_k \gamma_j \frac{1}{|\vec{r}_k - \vec{r}_j|}\frac{\partial |\vec{r}_k - \vec{r}_j|}{\partial y_i},$$

and since $|\vec{r}_k - \vec{r}_j| = \sqrt{(x_k - x_j)^2 + (y_k - y_j)^2}$, we can use the chain rule to find that the last partial derivative in the above expression is

$$\frac{\partial |\vec{r}_k - \vec{r}_j|}{\partial y_i} = \frac{y_k - y_j}{|\vec{r}_k - \vec{r}_j|}(\delta_{ik} - \delta_{jk})$$

(where δ is the Kronecker delta, which is 1 if the two variables are the same and 0 otherwise). When substituting in, we find that the $\frac{1}{2}$ in the Hamiltonian goes away, because either k or j can be equal to i, which gives an additional factor of 2. Finally, we find that

$$\gamma_i \dot{x}_i = -\frac{1}{2}\sum_{k\neq j} \gamma_k \gamma_j \frac{1}{|\vec{r}_k - \vec{r}_j|}\frac{y_k - y_j}{|\vec{r}_k - \vec{r}_j|}(\delta_{ik} - \delta_{jk}) = -\sum_{j\neq i} \gamma_i \gamma_j \frac{y_i - y_j}{|\vec{r}_i - \vec{r}_j|^2},$$

which means Hamilton's equations are indeed satisfied. We can similarly check this for the other Hamilton's equations, showing that indeed our Hamiltonian is correct. □

For comparison, recall that in electromagnetism, the energy between two particles is

$$E = \frac{q_1 q_2}{4\pi\varepsilon_0 r}.$$

This energy goes to 0 in three dimensions as $r \to \infty$. However, in two dimensions, the force is proportional to $\frac{1}{r}$, and integration tells us that

$$E = \frac{q_1 q_2}{4\pi\varepsilon_0} \ln r.$$

So this Hamiltonian is analogous to that of a 2-dimensional charge, where charges play the roles of the γs! This has to do with superfluids (which have no viscosity): we can set up these vortices in 2 dimensions, and the interactions look a lot like charged particles.

The key takeaway here is that we can use the Hamiltonian formalism in interesting ways (like making one of the two spatial dimensions into a kind of "momentum").

31 January 29, 2019 (Lecture)

Our project is due tomorrow – we should remember to email our submission, which includes both the code and the pdf of the short writeup to Professor Williams.

31.1 Review

Recall that last time, we started talking about the scattering properties of a particle. To analyze this, we had a beam of particles hit a central particle, and to divide out the properties of the beam, we define a differential cross-section $\frac{d\sigma}{d\Omega}$, which is just the number of beam particles into the solid angle $d\Omega$ per unit time, divided by the luminosity L (which measures beam particles per unit area per unit time).

We also defined a cross-section, which is the result of integrating the differential cross-section across all angles. When we plugged that into our definition, we found that the number of beam particles that scatter per unit time is equal to σL, so the probability of scattering is just σ per area of the beam.

The quantity b is called the **impact parameter**, and we found the general form of the differential cross-section (assuming azimuthal symmetry and a one-to-one between b and θ) to be

$$\frac{d\sigma}{d\Omega} = \frac{b}{\sin\theta}\left|\frac{db}{d\theta}\right|.$$

The first thing we did last time was to bounce a small particle off a sphere: if the impact point is at an angle α from the normal, the scattering angle is $\theta = \pi - 2\alpha$. This led us to

$$\frac{d\sigma}{d\Omega} = \frac{R^2}{4} \implies \sigma = \pi R^2.$$

This meant that the cross-section was just the "cross-sectional area that can get hit" (from the point of view of the beam). Similarly, we found that two spheres bouncing off each other gave a cross-section of $\pi(R+r)^2$.

One application of this came out of bouncing a neutron off of Uranium-235. We found that the mean free path of a neutron between interactions was $\overline{d} = 3$ cm, and since there is a 20 percent chance of fission each time, there is an average of 5 interactions between fissions and a distance of about $3\sqrt{5} \approx 7$ centimeters traveled between fissions in about 10^{-8} seconds.

The next thing to consider was how often a chain reaction is created. We found that if ν is the number of neutrons created per fission and P is the probability of interacting, we have the exponential function

$$n(t) = n(0)e^{\left(\frac{\nu P - 1}{\tau_f}\right)t}.$$

The critical point of whether this exponentially blows up or decays comes at $\nu P = 1$, and there are applications to weaponry and other fields.

31.2 Extending to other examples

We've been working in our lab frame, so that the mass M that the neutron collides with is stationary. But we'll now switch to the center of mass frame, so that the momenta of the colliding particles are equal and opposite. Conveniently, in elastic scattering, due to conservation of energy and momentum, the momenta must be equal and opposite afterwards as well.

The center-of-mass frame is offset by the lab frame by a velocity V such that

$$m_n(v_n - V) = MV \implies V = \frac{m_n}{M + m_n}v_n \equiv \gamma v_n.$$

This means the velocity of the neutron in the center-of-mass frame after the collision is

$$\vec{v}_{cm}^{\,n} = (v_n - V)\cos\theta\,\hat{x} + (v_n - V)\sin\theta\,\hat{y} = v_n\left((1-\gamma)\cos\theta\,\hat{x} + (1-\gamma)\sin\theta\,\hat{y}\right).$$

To see how much energy the neutron lost (on average) while scattering, we go back to the lab frame: the velocity of the neutron then looks like

$$\vec{v}_{lab}^{\,n} = v_n\left(((1-\gamma)\cos\theta+\gamma)\hat{x} + (1-\gamma)\sin\theta\hat{y}\right).$$

Expanding out the kinetic energy, we find that

$$E_n^f = \frac{1}{2}m_n v_n^2\left((1-\gamma)^2\cos^2\theta+\gamma^2+2\gamma(1-\gamma)\cos\theta+(1-\gamma^2)\sin^2\theta\right).$$

We want to find the average value over all scattering angles, but we know that our distribution is uniform over solid angles, so $\cos\theta$ is uniform! This means that (integrating $\cos\theta$ from -1 to 1)

$$\langle E_n^f\rangle = \frac{1}{2}mv^2\frac{\int_{-1}^{1}\left((1-\gamma)^2+\gamma^2+2\gamma(1-\gamma)\cos\theta\right)d(\cos\theta)}{\int_{-1}^{1}d(\cos\theta)}.$$

Normally, when we integrate spherically, we have a density function $\rho(\theta)$ in both the numerator and denominator, but this is just 1 here because the sphere's properties are isotropic. So the calculation becomes very easy: the average of $\cos\theta$ is 0, and our final answer is

$$\boxed{\langle E_n^f\rangle} = E_n^i\left(\left(1-\frac{m_n}{m_n+M}\right)^2+\frac{m_n^2}{(m_n+M)^2}\right) = \boxed{E_n^i\left(\frac{M^2+m_n^2}{(M+m_n)^2}\right)}.$$

We can confirm that this makes sense in the limits $M \gg m_n$ and $m_n \gg M$. Also, if $M = m_n$, which means we're scattering spheres off other equally-sized spheres, we lose half our energy on average.

31.3 A nuclear reactor

Usually uranium is stored in a **fuel rod**. (We want those rods to be pretty small, so that we don't get a critical mass.) To reduce the energy, this uranium is surrounded by another substance to scatter off of: we can't use neutrons, since they have a lifetime of 8 minutes, and we can't use hydrogen, because it is explosive. Instead, we use **heavy water**, which is water but with hydrogen replaced with deuterium (to prevent the neutrons from being absorbed into the hydrogens).

We then introduce another material called a **control rod**, which likes to absorb neutrons. Then the way to start a nuclear reactor is to have the control rod completely in, and then slowly pull the rod out so that νP increases (since the neutrons aren't getting absorbed anymore).

Fact 125

Radioactive decay can produce particles that move faster than the speed of light **in a specific substance** with high index of refraction! This is called **Cherenkov radiation**.

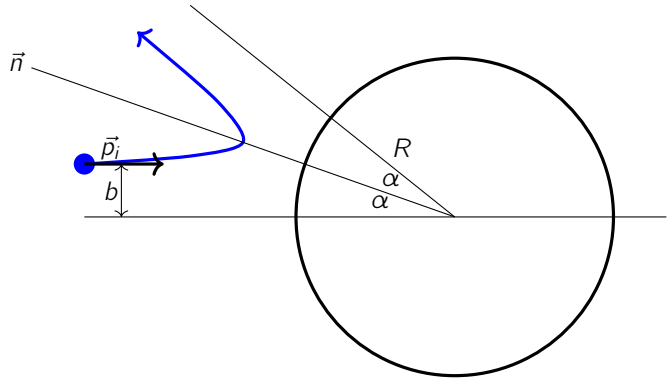
31.4 The plum pudding model

At one time, the plum pudding model was the most commonly accepted model of the atom. Let's model a gold atom as a cube: the volume is this cube would then be

$$V = L^3 = \frac{M_{Au}}{\rho_{Au}N_A} = \frac{0.2\text{kg/mol}}{19300\text{kg/m}^3\cdot 6\times 10^{23}} \implies L \approx 2.6\times 10^{-10}\text{m},$$

where M_{Au} is the molar mass of gold, ρ_{Au} is the density, and N_A is Avogadro's number. Naively, we'd then think that the plum pudding part of the atom would have radius about 10^{-10} meters.

But now, let's imagine bouncing an alpha particle off of the gold atom. Our impact parameter b and momentum $\vec{p_i}$ are the same, but our picture looks slightly different since the two particles are both positively charged and will repel each other:



By symmetry, this process should be symmetrical around $\vec{n}$, which explains why the two labeled angles are both α. Then the scattering angle is again $\theta = 180 - 2\alpha$, but we know that the electrical potential is of the form $U(r) = \frac{k}{r}$, where $k = \frac{2e \cdot 79e}{4\pi\varepsilon_0}$ (from the electrical interaction between the alpha particle and gold). This means that we can characterize the motion of the alpha particle more explicitly! Kepler's law for gravitational orbits gives us

$$\frac{\beta}{r} = 1 + \varepsilon \cos\theta,$$

where θ is tracking the trajectory of the particle (not the scattering angle) and we have parameters

$$\beta = \frac{\ell^2}{m(GMm)}, \varepsilon = \sqrt{1 + \frac{2E\ell^2}{m(GmM)^2}}.$$

And in this situation, all we've really changed is that GmM becomes k, and the signs of the interactions are flipped:

$$\beta = -\frac{\ell^2}{mk}, \varepsilon = \sqrt{1 + \frac{2E\ell^2}{mk^2}}.$$

By convention, we normally choose θ to be 0 when r achieves its minimum value. We don't want that here – we want to make the minimum occur at $\theta = \pi - \alpha$ as in our diagram above. So the correct equation becomes

$$\frac{\beta}{r} = 1 + \varepsilon \cos(\theta - (\pi - \alpha)) = 1 - \varepsilon \cos(\theta + \alpha).$$

Remember that we're still trying to find the relationship between the impact parameter b and the scattering angle θ. So we want to solve for α in terms of the other parameters: the easiest way is to look at the part of the motion where $\theta \to \pi$, because then $R \to \infty$ and the left hand side is zero. This tells us that

$$0 = 1 - \varepsilon \cos\alpha \implies \cos(\alpha) = \frac{1}{\varepsilon}.$$

Define $\gamma \equiv \frac{k}{2E}$. Notice that our electrical potential is zero at the point when we're infinitely far away, so the total energy (potential plus kinetic) for the alpha particle must be $E = \frac{1}{2}mv_0^2$. The angular momentum will be $\ell = bmv_0$ (since $\ell = r \times p$), and what's important is that **both E and ℓ are conserved throughout the motion**! So now because $\frac{\ell^2}{m} = b^2 \cdot 2E = \frac{b^2 k}{\gamma}$,

$$\cos\alpha = \frac{1}{\sqrt{1 + \frac{2E}{k}\frac{\ell^2}{m}\frac{1}{k}}} = \frac{1}{\sqrt{1 + \frac{1}{\gamma}\frac{b^2 k}{\gamma}\frac{1}{k}}} = \frac{\gamma}{\gamma^2 + b^2}.$$

So now we have a relationship between b and α that we can use! The rest is really just algebra. The easiest way is

to now make the observation that

$$\tan\alpha = \frac{b}{\gamma} \implies b = \gamma \tan\left(\frac{\pi-\theta}{2}\right) = \gamma\cot\frac{\theta}{2}.$$

So we find that

$$\frac{db}{d\theta} = -\frac{\gamma}{2\sin^2\left(\frac{\theta}{2}\right)},$$

and therefore

$$\frac{d\sigma}{d\Omega} = \frac{b}{\sin\theta}\left|\frac{db}{d\theta}\right| = \frac{\gamma\cot\frac{\theta}{2}}{\sin\theta}\cdot\frac{\gamma}{2\sin^2\frac{\theta}{2}} = \frac{\gamma^2}{4\sin^4\frac{\theta}{2}} \implies \boxed{\frac{d\sigma}{d\Omega} = \frac{k^2}{(4E)^2\sin^4\frac{\theta}{2}}}.$$

And this means we've found the differential cross-section for the plum pudding model, if we neglect the recoil of the gold.

Fact 126

In Rutherford's famous gold foil experiment, he fired these alpha particles at a sheet of gold, and the recoil was as if "firing a 15-inch shell at a piece of tissue paper and it came back and hit you." The only logical explanation for this type of behavior is what we know today is a nucleus!

Basically, we can estimate the radius of the central scattering sphere via

$$\frac{1}{2}mv^2 = \frac{k}{r_{min}},$$

and it's experimentally clear that the size of the central sphere is many orders of magnitude smaller than the plum pudding model indicates.

We'll finish the lecture with a cautionary tale "so we don't do something stupid:" there were once a bunch of sub-critical plutonium rods, and some engineers put them next to each other. The supervisor came in and told them to move them apart. But water (and therefore people) are a good **neutron moderator**, so this decision was dumb and could have resulted in a chain reaction! Luckily, nothing bad ever happened: the lesson is essentially "don't try to be a hero."

32 January 29, 2019 (Recitation)

32.1 A system that varies with time

Example 127

Suppose we have a time-dependent Lagrangian

$$L = f(t)\left(\frac{m}{2}\dot{x}^2 - V(x)\right),$$

where, for example, $f(t) = e^{\gamma t}$ and $V(x) = \frac{1}{2}m\omega^2x^2$. This is the Lagrangian that corresponds to a damped harmonic oscillator.

Let's first write out the equation of motion. By the Euler-Lagrange equation for x,

$$\frac{\partial L}{\partial x} = \frac{d}{dt}\frac{\partial L}{\partial \dot{x}} \implies -f(t)V'(x) = \frac{d}{dt}\left(mf(t)\dot{x}\right),$$

which simplifies to

$$mf(t)\ddot{x} = -mf'(t)\dot{x} - f(t)V'(x) = \implies \boxed{m\ddot{x} = -m\frac{f'(t)}{f(t)}\dot{x} - V'(x)}.$$

This is an equation of forces, which we can qualitatively write as

$$F = F_{\text{damping}} + F_{\text{potential}},$$

where

$$F_{\text{damping}} = -m\frac{f'(t)}{f(t)}\dot{x}$$

is a term proportional to the velocity. This is why we pick $f(t) = e^{\gamma t}$: we can make the constant $-m\frac{f'(t)}{f(t)}$ constant by making f an exponential function, or we could make it decay with respect to time: for instance, if $f(t) = t$, then we would have a damping term proportional to $\frac{1}{t}$.

Notice that we're extending our mathematical formalism now – the Lagrangian is now dependent on time, and it is **not simply the kinetic energy minus the potential energy**.

Switching to Hamiltonian formalism, we can write the generalized momentum as

$$p = \frac{\partial L}{\partial \dot{x}} = mf(t)\dot{x}.$$

Our Hamiltonian is therefore

$$H = \dot{x}p - L = mf(t)\dot{x}^2 - L = \boxed{\frac{m}{2}f(t)\dot{x}^2 + f(t)V(x)}.$$

This has a term corresponding to the kinetic energy and a term corresponding to the potential energy, but we can rewrite it in terms of x and p (instead of $\dot{x}$):

$$H = \frac{p^2}{2mf(t)} + f(t)V(x).$$

Notice that H is no longer conserved, since it explicitly depends on time.

32.2 A canonical transformation

We'll switch gears now. Let's say we have a Hamiltonian of the form

$$H(p, q) = \frac{p^2}{2} + \frac{1}{2q^2}.$$

We've dropped the extra constants, like m, which we shouldn't worry about right now – essentially, we have a potential that is proportional to q^{-2}, so if we put a particle somewhere and send it towards the origin, it will bounce off.

The equations of motion are

$$\dot{p} = -\frac{\partial H}{\partial q} = \frac{1}{q^3}, \quad \dot{q} = \frac{\partial H}{\partial p} = p,$$

so we have $\ddot{q} = \frac{1}{q^3}$, which is an ugly equation. But the underlying motion is very simple – the motion of the particle continues until its kinetic energy is zero and then bounces off – so there should be an easier equation of motion we can get out of this with a change of coordinates.

Let's try a generating function $F(q, P) = P \ln q$. The purpose of a function like F is to automatically provide a way to do a canonical transformation, since not all functions $Q(q, p)$ and $P(q, p)$ give transformations that preserve

Hamilton's equations. Recall that the rules of F force

$$\frac{\partial F}{\partial q} = p, \quad \frac{\partial F}{\partial P} = Q,$$

and the new form of the Hamiltonian is $K(Q,P) = H(p,q) + \frac{\partial F}{\partial t}$. In our case, $\frac{\partial F}{\partial t}$ is zero, and the equations give us

$$\frac{\partial F}{\partial q} = p \implies p = \frac{P}{q}$$

and

$$\frac{\partial F}{\partial p} = Q \implies Q = \ln q \implies q = e^Q.$$

Because H and K are the same function, our Hamiltonian in P and Q coordinates will be

$$\boxed{K(Q,P)} = \frac{p^2}{2} + \frac{1}{2q^2} = \frac{(Pe^{-Q})^2}{2} + \frac{1}{2(e^Q)^2} = \boxed{2e^{-2Q}(P^2+1)}.$$

The tradeoff we're making here is that the Hamiltonian becomes uglier, but our equation of motion may become nicer. Applying the new Hamilton's equations,

$$\dot{P} = -\frac{\partial K}{\partial Q} = (P^2+1)e^{-2Q}, \quad \dot{Q} = \frac{\partial K}{\partial P} = Pe^{-2Q}.$$

But notice that $\dot{P} = 2K$, and K isn't dependent on time. This means K is a conserved quantity, and therefore $\dot{P}$ is conserved as well! So we'll set $K = E$ to be constant, which yields $\dot{P} = 2E \implies P = 2Et + C$. Looking again at the Hamiltonian,

$$E = \left(\frac{P^2+1}{2}\right) e^{-2Q}$$

is constant, so we can isolate Q to get

$$e^{2Q} = \frac{p^2+1}{2E} \implies e^Q = \sqrt{\frac{P^2+1}{2E}}.$$

But we picked our new coordinates so that $q = e^Q$! So the actual equation of motion is

$$q(t) = \sqrt{\frac{p(t)^2+1}{2E}}.$$

Since $P = pq$, and we wrote down P in terms of E above, we can plug that in and simplify to find

$$q(t) = \sqrt{2Et^2 + 2Ct + \frac{C^2+1}{2}}.$$

Since $q(t)p(t) = P(t) = 2Et + C$, we must have an initial condition of the form $q(0)p(0) = C$.

Example 128

Consider the case where $q(0) = 1$ and $p(0) = -1$, so the particle starts at a position of 1 and moves towards the origin.

Then our solution becomes

$$q(t) = \sqrt{2Et^2 - 2t + 1},$$

which indeed satisfies the qualitative description we're expecting. Asymptotically, $q(t) = \sqrt{2E} \cdot t$ as $t \to \infty$.

We'll finish by explaining how could we have potentially found our generating function F. First, we can rewrite the

original Hamiltonian as

$$H(p,q) = \frac{1}{2q^2}((pq)^2 + 1).$$

If we try defining $P = pq$, we get $H(p,q) = \frac{1}{2q^2}(P^2+1)$. Then

$$\frac{\partial F}{\partial q} = p = \frac{P}{q} \implies F(P,q) = P \ln q,$$

and this is nice because the $\ln q$ creates an exponential term in the new Hamiltonian – since the derivative of an exponential is a constant times itself, this helps us relate P and Q to the (constant) energy in the arguments above. In general, finding the right F is really about playing around with the coordinates and doing a lot of trial and error.

33 January 30, 2019 (Lecture)

33.1 Review

We've been talking about scattering for the past two days, and we derived the generic formula

$$\frac{d\sigma}{d\Omega} = \frac{b}{\sin\theta}\left|\frac{db}{d\theta}\right| \implies \sigma = \int \left(\frac{d\sigma}{d\Omega}\right) d\Omega.$$

Yesterday, we went through an important example where a particle is fired at a target with a repulsive potential $U(r) = \frac{k}{r}$. The particle will be deflected by the repulsive force: just like in the case when the particles actually collide, we have $\theta = \pi - 2\alpha$, but we found the relationship between θ and b (the impact parameter) here to be

$$\frac{2Eb}{k} = \cot\frac{\theta}{2}.$$

This yields the very famous formula

$$\frac{d\sigma}{d\Omega} = \frac{k^2}{(4E)^2 \sin^4\frac{\theta}{2}},$$

which is also called the **Rutherford scattering cross-section**. The discrepancy between this calculation and experimental results proved that the plum pudding model must be incorrect – instead, atoms must have a central positively-charged nucleus! (This was one of the most important discoveries in physics.)

33.2 Scattering off electrons

Consider a charged particle with mass m_x and high initial momentum $p = m_x v$ in the x-direction, fired (from infinitely far away) at another charged particle with mass M. Let's say that the scattering angle is small, and the path is only deflected by an angle $\theta \ll 1$. (The model here is of charged particles going through materials, and the particle of mass M is an electron.)

This deflection adds a transverse momentum p_T to the particle's motion. It's a good enough approximation to say that the x-direction's momentum is almost the same for the particle, especially since the two particles have very different masses. Therefore,

$$p_T \approx m_x v \sin\theta \approx m_x v\theta$$

is the magnitude of the transverse momentum for both masses (by conservation of momentum). We're dealing with

charged particles here, so we can use the Rutherford equation to approximate

$$\frac{2Eb}{k} = \cot\frac{\theta}{2} \approx \frac{2}{\theta} \implies \theta \approx \frac{k}{Eb}.$$

Since the particle is approaching from infinitely far away, its energy is $E = \frac{1}{2}mv^2$, which tells us that

$$\theta = \frac{2k}{m_x v^2 b} \implies p_t \approx m_x v \frac{2k}{m_x v^2 b} = \frac{2k}{bv}.$$

To find how much energy was lost by the m_x particle, we can just see how much energy the M particle gained. We are neglecting the momentum gained in the x-direction for this second particle, so the m_x particle has a change in energy of

$$\delta E = -\frac{p_T^2}{2M} = -\frac{2k^2}{Mb^2v^2}.$$

This means that (for example) we lose more energy when we scatter off of an electron than a proton for the same impact parameter! But an atom has Z times more electrons than nuclei, so we can essentially just treat our mass M to be the mass of the electron m_e, since that's where most of the scattering is going to happen.

33.3 Back to the beam

We now shift to the rest frame of our charged particle m_x, which means we will see a large beam of electrons coming at it. We need to integrate to see how many electrons we encounter as a function of the impact parameter b. Let's say we have a cylinder of depth dx that we pass through, and we are considering a cross-section of a ring with radius between b and $b + db$. Then the energy lost as the particle passes through the cylinder can be approximated as

$$\Delta E = \left(-\frac{2k^2}{m_e b^2 v^2}\right)\left(\frac{Z\rho}{m_A}\right)(2\pi b\, db\, dx),$$

where Z is the atomic number, ρ is the density, and m_A is the molar mass, meaning the second term (in parentheses) gives us the number of electrons per unit volume. Moving the differentials around, we have that

$$\frac{d^2E}{dx\,db} = -\frac{4\pi k^2 Z\rho}{m_e b v^2 m_A}.$$

Here Z, ρ, m_A are properties of the material, v is a property of the projectile, m_e is a constant, and k depends on both the material and the projectile. Since the value of b is effectively random, and we care about $\frac{dE}{dx}$, we can integrate over b to find

$$\frac{dE}{dx} = -\frac{4\pi k^2 Z\rho}{m_e v^2 m_A} \log\left(\frac{b_{max}}{b_{\min}}\right).$$

In practical situations, nature will give us some values of $b_{\min}$ and $b_{\max}$, and we can approximate m_A as A atomic mass units, where A is the atomic number. Collecting all the constants, we find that

$$\frac{dE}{dx} = -0.3\,\frac{\text{MeV}\cdot\text{cm}^2}{\text{g}}\left(\frac{Z\rho}{A}\right)\left(\frac{q_x}{v/c}\right)^2 \log\left(\frac{b_{max}}{b_{\min}}\right),$$

where q_x is the charge (as a multiple of the charge of a proton). All that's left for us to do is estimate the smallest and largest possible values of b. To find $b_{\min}$, we can use the quantum uncertainty principle, which tells us that

$$\Delta p \Delta x \geq \frac{\hbar}{2}.$$

This means that the smallest length scale (in the spatial dimension) is on the order of

$$b_{\min} \approx \frac{\hbar}{2p} = \frac{\hbar}{2m_e v}.$$

We also have a cutoff for $b_{\max}$: we can only interact with electrons that are at various (discrete) energy levels, and the maximum impact parameter turns out to be

$$b_{\max} = O(1)\frac{\hbar v}{\langle I \rangle},$$

where $\langle I \rangle$ is the ionization energy. In this case, the $O(1)$ number happens to just be 1, and our final result is that (plugging everything back in)

$$\frac{b_{\max}}{b_{\min}} \approx \frac{2m_e v^2}{\langle I \rangle} \implies \boxed{\frac{dE}{dx} = -0.3\frac{\text{MeV} \cdot \text{cm}^2}{\text{g}} \left(\frac{Z\rho}{A}\right)\left(\frac{q_x}{v/c}\right)^2 \log\left(\frac{2m_e v^2}{\langle I \rangle}\right)}.$$

This is called the **Bethe equation**, and it is accurate as long as $\frac{v}{c}$ is small (nonrelativistic) but large enough to be much larger than the speed of an electron! From here, we can find the energy lost per unit length by simply Googling a few quantities.

33.4 Further questions

Given this derivation, it's now valuable to think about the **range of the particle** before it runs out of energy. (The real-life model to keep in mind now is of radiation going into the human body.) We could integrate the expression

$$\text{Range} = \int_E^0 \frac{dE}{dE/dx},$$

but this is difficult to do analytically, and we'll simplify this by pretending that the log term stays constant (so we don't need to change v as a function of x). We can regroup a few terms and write

$$\frac{dE}{dx} = -\left(0.3\frac{Z\rho}{A}c^2 \log\left(\frac{2m_e v^2}{\langle I \rangle}\right)\right)\frac{q_x^2}{v^2} \cdot \frac{m_x/2}{m_x/2},$$

where we define the term in parentheses (which is constant) to be α. Plugging back into the integral, we find that

$$\text{Range} = \frac{2}{\alpha q_x^2 m_x}\int_0^E dE = \frac{E^2}{\alpha q_x^2 m_x}.$$

Therefore, the range of the particle scales as the energy squared, is inversely proportional to the charge squared, and is inversely proportional to the mass.

This result isn't exact, for a variety of reasons. Experimentally, at a certain point, we start having **radiative losses** because of radiating photons, and the approximations we've made also break down when the electron speed is faster than the speed of our particle.

Also, because energy loss is proportional to $\frac{1}{E}$, the energy loss has a pronounced increase as the particle gets deeper into the material, which is called a **Bragg peak**. But this doesn't happen if we have a photon beam, so if we have a tumor in our body, we can figure out how deep it is and tune our proton beam so that it only gives a large dose at a **specific depth** in the skin. Once we overlap a bunch of different such beams, we have an effective way to do chemotherapy – this is what people mean by the phrase "**proton therapy**."

33.5 Final tidbits about scale

Neutrino scattering does occur, and (experimentally) it looks like the effective radius is 10^{-5} times the radius of a proton, which is a small cross-section. We've seen smaller cross-sections as well: protons in the LHC often travel 11,000 times around a 27 km circumference tube, meaning that the probability of collision is very small.

The ATLAS group computed a bunch of cross-sections: proton-proton scatterings are so high-energy that the Coulomb interactions don't matter, so we're within 20 percent to just take $\pi(R+r)^2$. On the other hand, Higgs bosons have a cross-section of 10^{-6} proton-radii, and we've measured to a precision of about 10^{-9}. Even at this small scale, the standard model still holds up, which is "pretty frustrating."

34 January 30, 2019 (Recitation)

Fact 129

Today's lecture covered the planned material for today's recitation, so this hour ended up being an "office hour" of sorts. Basically, the calculations from the bead-on-a-wire problem were repeated – since they are described above, notes from this recitation have been omitted.

35 January 31, 2019 (Lecture)

35.1 Introduction

There have been no snow days this IAP, so we're further into the "selected topics" than anyone has ever been in this class.

Today, we're discussing a topic from general motion called "principal axes." To begin, imagine a pendulum free to swing in three dimensions. At any point, if the vector from the fixed point to the mass is $\vec{r}$ and the direction of the momentum is $\vec{p}$, then the angular momentum vector is $\vec{\ell} = \vec{r} \times \vec{p}$.

It's good to keep in mind that the velocity vector is (generically) $\vec{v} = \vec{\omega} \times \vec{r}$. Notice then that $\vec{\omega} = \omega \hat{z}$, so it is **not true** that $\vec{\ell} = I\vec{\omega}$ (even though the analogous scalar statement is true)!

35.2 Looking more carefully at the moment of inertia

Consider a solid (three-dimensional) body of any shape, and let $\hat{z}$ be an axis through its center of mass. Suppose the body rotates around $\hat{z}$, with an angular velocity denoted by ω.

If we look at a differential element $\vec{r}_j$ of the body, then the angular momentum of that element is

$$\vec{\ell} = \sum_j \vec{r}_j \times \vec{p}_j = \sum_j = \vec{r}_j m_j \times (\vec{\omega} \times \vec{r}_j),$$

where we can notice that ω is the same for the whole body. Using a vector identity, this simplifies to

$$\sum_j m_j \left((\vec{r}_j \cdot \vec{r}_j)\vec{\omega} - (\vec{r}_j \cdot \vec{\omega})\vec{r}_j \right).$$

These terms simplify further: $\vec{r}_j \cdot \vec{r}_j$ is r_j^2, $\vec{\omega}$ is $\omega\hat{z}$, and $\vec{r}_j \cdot \vec{\omega} = z_j\omega$, where z_j is the z-coordinate of this element. Substituting these values in and factoring out the ω, the angular momentum becomes

$$\vec{\ell} = \omega \sum_j m_j \left(\rho_j^2 \hat{z} - z_j x_j \hat{x} - z_j y_j \hat{y}\right),$$

where $\rho_j^2 \equiv x_j^2 + y_j^2$ is defined to be the squared distance to the axis of rotation. (Remember that everything here is **rotation around the z-axis.** Then the z-component of the moment of inertia is

$$I_{zz} \equiv \sum m_j \rho_j^2,$$

but the other two terms in the $\hat{x}$ and $\hat{y}$ direction may look less familiar:

$$I_{xz} = -\sum m_j x_j z_j = I_{zx},$$

and similar expressions show up whenever the two indices for I are different. These expressions are called **products of inertia** – the most general expression is that

$$\begin{bmatrix} \ell_x \\ \ell_y \\ \ell_z \end{bmatrix} = \begin{bmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{yx} & I_{yy} & I_{yz} \\ I_{zx} & I_{zy} & I_{zz} \end{bmatrix} \begin{bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{bmatrix}.$$

This is effectively something called a "second-rank tensor."

Remark 130 (Aside: why doesn't this come up in 8.01?)**.** *In mechanics, we often look at rotations about a fixed axis, and in general we define that axis to be the z-axis. Then we find generically that*

$$\vec{\ell} = \begin{bmatrix} I_{xz}\omega \\ I_{yz}\omega \\ I_{zz}\omega \end{bmatrix}.$$

But out of these three terms, we often only extract the last one, and we call it I_ω. Additionally, we often consider objects that inherently have a lot of **symmetry***: consider a sphere with differential elements A and B on opposite sides of the y-axis. Then the contribution of A to the moment of inertia is $I_{yz} = -m_j z_j y_j$, while the contribution of B is $I_{yz} = -m_j z_j (-y_j)$, and these will add up to 0. So for a sphere, all the contributions for products of inertia are actually zero, and this kind of situation comes up often in 8.01.*

35.3 Principal axes

In the early to mid 18th century, people noticed that we could define three axes for rigid objects such that

$$\ell_i = I_i \omega_i, \quad i = 1, 2, 3.$$

For a sphere, any three axes will work, and for a cylinder, one of them must be along the long axis. These are called **principal axes of rotation**, and the idea is that they are attached to the object and move as the object rotates. Determining them turns out to be an eigenvalue problem:

Proposition 131

If $I\vec{\omega}' = \lambda\vec{\omega}'$ along some axis, then $\ell = \lambda\vec{\omega}'$ along the axis of the eigenvector.

So we need to find the values λ such that

$$\det \begin{bmatrix} I_{xx} - \lambda & I_{xy} & I_{xz} \\ I_{yx} & I_{yy} - \lambda & I_{yz} \\ I_{zx} & I_{zy} & I_{zz} - \lambda \end{bmatrix} = 0.$$

This is almost never reasonable to do except in specific problems and shapes, which is why we rarely see it in practice. But the concept is useful for knowing (for example) if our axes are **stable**, which doesn't require the full calculation of knowing where the axes are.

Remark 132. *Note that the Spectral Theorem tells us that because the* **inertia tensor** *is symmetric, the principal axes will be orthogonal.*

35.4 Free motion

In general rigid-body motion, it's not that easy to see what will happen to an object that can rotate freely. First of all, we can write down our Lagrangian in terms of the principal axes: there is no stated potential energy, so everything comes from the rotational kinetic energy

$$L = \frac{1}{2} \sum_{i=1}^{3} I_i \omega_i^2.$$

But we need to be careful; this Lagrangian is written in a non-inertial frame, because the principal axes are changing. So we can't get Lagrange equations of motion directly – we need to do some transformation into an ordinary reference frame. Luckily, we can use **Euler angles** get from any orientation to any other orientation by doing three rotations. In other words, there exist three Euler angles that convert our standard axes into our principal axes, and once we do this for our three angles ψ, θ, ϕ, we can write down our Euler-Lagrange equations. This leads to the following three equations (calculations omitted):

$$I_1 \dot{\omega}_1 + (I_3 - I_2)\omega_3 \omega_2 = 0,$$

$$I_2 \dot{\omega}_2 + (I_1 - I_3)\omega_1 \omega_3 = 0,$$

$$I_3 \dot{\omega}_3 + (I_2 - I_1)\omega_2 \omega_1 = 0.$$

Here, the equations measure the net torque on the object: if we don't have free motion (that is, there are external torques), the 0s in the right-hand side of the equation need to be replaced with torques along the corresponding principal axes.

Suppose, for simplicity of notation, that the principal axes ℓ_1, ℓ_2, ℓ_3 are in the positive x, y, z directions. For a conceptual understanding of the above equations, we ask ourselves **how we can change angular momentum along an axis**. One method is to speed up the rotation along that axis, which is the first term $I_j \dot{\omega}_j$ in each equation. To motivate the other term, suppose there are two angular momenta $I_1\omega_1$ and $I_2\omega_2$ in the xy-plane. Say that the particle is rotating with an angular velocity ω_3 about the z-axis, and we take an infinitesimal step $\Delta\theta_3 \ll 1$.

When that step is taken, what's the change in angular momentum for the other two axes? Because we've rotated our axes, the existing angular momentum in each direction needs to be projected to the adjusted axes. The change to the angular momentum $I_1\omega_1$ in the x-direction, or ℓ_1, is effectively nothing (since it changes by a $\cos \Delta\theta_3$ factor, which is 1 to first order). But along the ℓ_2 direction, we pick up an $I_1\omega_1 \sin \Delta\theta_3 \approx I_1\omega_1\Delta\theta_3$. Similarly, $\Delta\ell_1 = I_2\omega_2\Delta\theta_3$, and clearly $\Delta\ell_3 = 0$ under this rotation. Dividing through by Δt and then taking to zero, we have

$$\dot{\ell}_1 = -I_2\omega_2\omega_3, \dot{\ell}_2 = I_1\omega_1\omega_3, \dot{\ell}_3 = 0.$$

Repeating this argument for each of the three axes indeed accounts for all of the terms in the above equation.

35.5 Stability of motion

As with many ideas we've discussed in this class, suppose we start with some rotating angular velocity ω, and we perturb the motion by a tiny amount. We're interested in learning whether the motion stays close to the original motion (whether we have a stable equilibrium) – this is important for objects like rotating satellites, because our positioning is never exactly correct in real life.

The easiest case to study is where we start rotating about a principal axis (and have only small contributions along the other axes). Suppose this axis is ℓ_1, which means $\omega_1 \gg \omega_2, \omega_3$ with the perturbation. This means that to first order,

$$I_1\dot{\omega}_1 + (I_3 - I_2)\omega_2\omega_3 = 0 \implies \dot{\omega}_1 = 0.$$

The behavior along the other axes is more complicated. First of all, differentiate the second equation with respect to time and use $\dot{\omega}_1 = 0$ to find

$$I_2\ddot{\omega}_2 + (I_1 - I_3)\omega_1\dot{\omega}_3 = 0,$$

since $\dot{\omega}_1$ was just found to be zer o. Then we can substitute in the value of $\dot{\omega}_3$ from the third equation to get

$$I_2\ddot{\omega}_2 + (I_1 - I_3)\omega_1\left(-\frac{(I_2 - I_1)\omega_2\omega_1}{I_3}\right) = 0.$$

This is simple harmonic motion! Stability occurs if and only if

$$(I_1 - I_3)(I_1 - I_2) > 0,$$

since we need the constant in $\ddot{x} + \omega^2 x = 0$ to be positive. In other words, a perturbation sometimes gives us a simple harmonic oscillator, but at other times, we'll have exponentially increasing instability. The important constant here is

$$\Omega_2 = \omega_1\sqrt{\frac{(I_1 - I_3)(I_1 - I_2)}{I_3 I_2}} = \Omega_3.$$

This kind of stable behavior occurs (that is, Ω is real-valued) if I_1 is either the **largest** moment of inertia or the **smallest**. But if it's in the middle, this isn't true! Restated, this tells us the following result:

Theorem 133 (Intermediate Axis Theorem)
Rotating motion is stable about a particle's largest and smallest principal axis, but not the middle.

Example 134
Supersonic fighter jets will undergo coupled motion under rotation when the moment of inertia is too high (this is called **inerrtia coupling**). Chuck Yeager, the first man to go faster than the speed of sound in the X-1A, did a roll in the aircraft, which induced rolls around the other axes. Unfortunately, in the X-2, this happened again, and the pilot couldn't recover. Now, we have computers that automatically calculate the induced rolls and correct for them.

Example 135

Explorer 1 was designed to rotate around its principal axis. It was a very long cylinder, but it wouldn't rotate properly in practice (even though this wasn't the intermediate axis). It turns out that the issue was objects **not being perfectly rigid**: non-rigidity means the object wants to get into a minimum energy state, and it is advantageous to rotate about the axis with maximum I to get a smaller ω for minimal energy. (This is a general property of non-rigid objects.)

Example 136

Gravity Probe B was designed to test general relativity, and the way it avoids issues with wobbling is by making the gyroscope a perfect sphere! For this project, four of the most perfect spheres approximately the size of a ping pong ball were created – for reference, to scale, the largest mountain on Earth would be about seven feet tall.

Tomorrow will be a wrap-up lecture; we'll talk a bit about some other symmetries that exist in nature.

36 January 31, 2019 (Recitation)

Here are some recommended readings about geometric pictures of the Hamiltonian formalism:

- Nonlinear Dynamics and Chaos (Strogatz),
- Lectures on Classical Dynamics (Tong),
- Mathematical Methods of Classical Mechanics (Arnold).

Recall that our Hamiltonian for a simple harmonic oscillator looks like

$$H(p,q) = \frac{p^2}{2m} + \frac{1}{2}m\omega^2 q^2.$$

Because $\frac{\partial H}{\partial t} = 0$, the volume is preserved in phase space. More importantly, energy E is constant, so the orbits in phase space trace out ellipses of the form

$$E(p,q) = \frac{p^2}{2m} + \frac{1}{2}m\omega^2 q^2.$$

The p-intercepts of this intercept are at $\pm\sqrt{2mE}$, and the q-intercepts are at $\pm\sqrt{\frac{2E}{m\omega^2}}$. But to find the dynamics, we need to actually use Hamilton's equations $\dot{p} = -\frac{\partial H}{\partial q}$ and $\dot{q} = \frac{\partial H}{\partial p}$. Plugging into our Hamiltonian, this yields

$$\dot{p} = -m\omega^2 q, \quad \dot{q} = \frac{p}{m} \implies \ddot{q} = \frac{\dot{p}}{m} = -\omega^2 q.$$

So now we have a dynamical picture, and we can see what role time plays in our solution. If we set $q(t) = A\sin(\omega t)$, then we know that $p(t) = m\dot{q} = Am\omega\cos(\omega t)$, meaning that

$$E = \frac{p(t)^2}{2m} + \frac{1}{2}m\omega^2 q(t)^2 = \frac{m^2A^2\omega^2\cos^2(\omega t)}{2m} + \frac{1}{2}m\omega^2A^2\sin^2(\omega t) = \frac{1}{2}m\omega^2A^2$$

is independent of time, as expected. This tells us that the maximum amplitude A controls the energy of the system.

Example 137

Suppose the harmonic oscillator is initially stretched at an amplitude of $x = A$, and we release it from rest.

In phase space, the motion of the oscillator will trace out the ellipse clockwise. But let's say that we don't know exactly where the particle is in our phase space, since our measurement devices have errors. Then we can estimate the phase space location by drawing a circle around the point $(A, 0)$.

And this is why volume preservation in phase space is useful! We know that the trajectory of the circle is a translation of the circle, and the evolution of dynamics does not change its size. So if we know the initial position to within 10 percent accuracy, we'll always know it within 10 percent accuracy. This isn't so true in complex systems like weather, where we no longer have phase space volume preservation.

Remark 138. *Also, it's good to remember that Heisenberg's uncertainty principle becomes more noticeable at smaller scales, so each possible state essentially takes up a finite positive volume in state space. At macroscopic scales, we can basically think of them as points, though.*

For our next topic, consider the damped harmonic oscillator

$$\ddot{x} + \gamma\dot{x} + \omega^2 x = 0.$$

No matter what amplitude we start with, as time goes to infinity, we know exactly what the system will look like: it will be at rest at equilibrium. In phase space, this means that **all trajectories spiral in (exponentially) towards the origin**. The relevant piece of theory is that $\frac{\partial H}{\partial t}$ is not zero, so Liouville's theorem does not hold.

The origin in this case is called an **attractor** – let's describe a more interesting example of one.

Example 139

Consider the equation of motion

$$\ddot{x} + \varepsilon\dot{x}(x^2 - 1) + \omega^2 x = 0.$$

We can think of this system as having a harmonic force $-m\omega^2 x$, as well as a damping force $-m\varepsilon\dot{x}(x^2 - 1)$. But the interesting feature is the $(x^2 - 1)$ term, meaning that the **damping is position-dependent**.

If our particle starts at $x = 0$, the harmonic force will be 0, while the damping force is $m\varepsilon\dot{x}$. If we make $\dot{x}$ a small positive number, the damping force will push us in the positive direction too. But if $x > 1$, the damping force will start pushing us backwards, meaning the damping force will be restoring in this case. So the harmonic force pushes us towards 0, while the damping force is a "driving" force that pushes us towards 1 and -1.

For simplicity, let's use $\dot{x}$ instead of p for our phase space. Then there exists an "ideal" trajectory, which is a simple harmonic oscillator that passes through $x = 1$ and $x = -1$. And if we start anywhere outside that ideal trajectory, the path will approach that simple harmonic oscillator. This is called the **Van der Pol oscillator**, and it's an example of a system with a nontrivial attractor!

Finally, we'll discuss the idea of **chaos**. Let's set up a 3-D coordinate system, where there are some dynamical variables (not just x and p). We could have several kinds of attractors: we could have a loop like the Van der Pol oscillator, which causes the particle to move in a helix motion in phase space, or we could have more interesting attractors like a plane, where the system pulls particles onto the plane and then more complicated motion happens.

But describing this situation mathematically can be difficult. If a certain surface in phase space is an attractor, then mathematically, at large times, there's some complicated equation in terms of the parameters that we need to satisfy. But if we restrict our motion to fewer dimensions, we can have larger uncertainty in the restricted domains.

For example, initial condition information in the **Lorenz attractor** is practically useless! This kind of idea occurs for Hamiltonian systems too, but we may need four parameters to get this kind of chaotic behavior. (This is analogous to saying that "puff pastry imperfections are amplified a lot through repeated folding")

37 February 1, 2019 (Lecture)

37.1 Review

Throughout the course of this class, we've started from first principles and obtained Newton's laws, conservation of energy, momentum, and Noether's theorem in general. If we add on the speed of light being constant, we get relativity, and if we add the concepts wavefunctions, we can get quantum mechanics. The idea is that we can work out the laws of physics **once we know all of our symmetries**.

But we still want to find more symmetries of the universe – a more concrete question we are asking is "Are there any experiments we can do, for which the outcome will tell us something about nature?" If the answer is no, then we've found a symmetry. The setup we have here is that we can build any measuring apparatus inside of an isolated opaque box, and we're not allowed to bring objects outside (in a way that allows us to define relative quantities). In the first week of this class, we noted that postulates 1 and 2 tell us that there's no experiment we can do to learn absolute time, the location of the opaque box, or how it's oriented. This implies that we have conservation of energy, momentum, and time. Today, we're going to discuss some other symmetries.

37.2 CPT

There are certain "reversals" in physics that we can perform:

- **C** stands for charge conjugation, which involves replacing particles (matter) with anti-particles (basically anti-matter).
- **P** refers to parity, which requires performing a mirror reflection in an odd-dimensional space.
- **T** is time-reversal, which means running time backwards.

Notice that with CPT, we're taking $q \to -q$ for charge, $\vec{x} \to -\vec{x}$ for reflections, or $t \to -t$ for time. The question we're interested in is **whether these are symmetries of nature**. In other words, can we build an experiment that tells us if we have antimatter or matter, which direction time is moving in, or which hand is our left or right?

As a related note, there were four fundamental forces known to physicists in the 1950s:

- Gravity: we had GR by this time.
- Electromagnetism: we even had quantum electrodynamics developed. (Most of chemistry is E&M.)
- Strong nuclear force: it keeps the nucleus together.
- Weak nuclear force: its main purpose was making some particles (like neutrons) decay.

37.3 Checking laws of nature under CPT

Let's look at some different quantities in mechanics to see if the laws will change. First of all, the velocity of a particle

$$\vec{v} = \frac{d\vec{x}}{dt}$$

is flipped under P (parity) and T (time), but not C (charge). This also means that momentum changes under P and T. Curiously,

$$\vec{\ell} = \vec{x} \times \vec{p}$$

also doesn't change under C, does change under T, but does not change under P (because $\vec{x}$ and $\vec{p}$ both get flipped). This is why angular momentum is often called a "pseudo-vector."

Next, looking at gravity, the gravitational force

$$F = -\frac{GM_1M_2}{r^2}\hat{r}$$

doesn't depend on time, charge, or orientation, just the distance between the particles, so it's not affected by any of CPT. But we can look more closely at Newton's laws: Newton's first law has to do with inertial frames, while the third law is conservation of momentum, and both of those either get a minus sign in front of them or nothing. Therefore, we only need to check the second law

$$\vec{F} = \frac{d\vec{p}}{dt}.$$

This doesn't change under charge (if we're doing a Coulomb interaction,for example, we pick up a negative sign from both particles). Under a mirror symmetry, the second law also still holds, since both $\vec{F}$ and $\vec{p}$ have their directions flipped. Finally, under time, $\vec{F} = \frac{d(-\vec{p})}{d(-t)}$ still gives us the same equation.

Next, let's look at torque, since the weirdness with angular momentum may make us potentially concerned: the relevant equation is

$$\tau = \vec{r} \times \vec{F} = \frac{d\vec{\ell}}{dt}.$$

Both $\vec{r}$ and $\vec{F}$ change signs under P, both ℓ and t change under T, and nothing changes under C, so everything here is still valid. The takeaway is that **mechanics (and we can also check electromagnetism) behaves properly under CPT**. (Note that the right-hand rule of physics is **just a convention**.)

Time-reversal may seem like a confusing concept, but the reasons it is normally said to forbidden are thermodynamic (and therefore macroscopic in nature). So to physicists in 1955, as far as we know, charge conjugation, parity, and time reversal are all symmetries of nature.

37.4 Complications of modern physics

Ernst Mach was a physicist and a philosopher, and he mentioned that "even instinctive knowledge of so great a logical force as the principle of symmetry may lead us astray." He was the first person to go back and look critically at Newton's laws.

In 1956, Lee and Yang were talking to Feynman at a conference, and they decided to do an exhaustive study about parity violation. They found incredibly strong evidence that parity was conserved in strong and E&M forces, but they didn't for the weak nuclear force.

Fact 140

So over Christmas break, the Wu experiment happened. In this experiment, radioactive Cobalt-60 (which has 27 protons) goes through beta decay to an excited state of Nickel-60, which decays to ground-state Nickel.

Angular momentum has to be conserved throughout the process, and Cobalt's nucleus has a spin of $5\hbar$. So in this case, since we can't have orbital angular momentum along the electron-neutrino axis (as momentum is $\vec{r} \times \vec{p}$), we need to make Nickel (with 4 units of spin), the electron (with half a unit of spin), and an electron neutrino (with half

a unit of spin).

We make the Cobalt very cold and set up a magnetic field. Applying parity to a magnetic field doesn't do anything (since we have a cross-product in the definition). When decay happens, the Nickel basically doesn't move, but the electron and electron neutrino that are emitted can exit in many directions. It's interesting to consider this relative to the **direction of the magnetic field**, so if the magnetic field points up, we can measure whether the electron goes up and neutrino goes down or vice versa.

Under a parity transformation, the magnetic field doesn't change, and neither does angular momentum. Cobalt's spin doesn't change, and neither does Nickel, but the momenta of the electron and the neutrino do change. So the question of whether parity is conserved is whether the rate at which an electron goes up is the same as that of which a neutrino goes up. Experimentally, we can make this simpler to measure by flipping the magnetic field (by taking the whole experiment and flipping it 180 degrees), and putting an electron detector on one side to count the rate of emission.

It turns out weak interaction is actually **as weakly conserved as possible**! Every equation we write down in electromagnetism or mechanics involves only vectors or only pseudovectors. But the weak force is a vector minus a pseudovector (with equal weight), which is a quantity really bad at conserving things.

Fact 141

The Wu experiment proved that parity conservation could be violated, and Lee and Yang immediately got the Nobel Prize the next year for this.

To add to the complication, if we make a mirror image of the Wu experiment, we get a different result, but if we make a mirror image out of antimatter, we get identical results to the original experiment! So "CP" symmetry seems like it's what must be preserved here – left-handed antimatter is the same as right-handed matter.

Fact 142

This leads to Feynman's famous joke: we can tell another lifeform in the universe what right and left are unambiguously. But if we show up and meet the aliens, they might be made of antimatter.

For about six years, there weren't any issues. But in 1964, Cronin and Fitch observed a certain particle decay that violated CP. Unfortunately, it seemed that for relativity to be a good theory, CPT had to be conserved. So if CP could be violated, so must T, which implied that time-reversal wasn't a good symmetry of nature either!

At the time, physicisists knew about the existence of up, down, charm, and strange quarks, but had not yet observed the charm quark. In 1973, Kobayashi and Maskawa found that if top and bottom quark existed, then CP violation could be allowed. They won the Nobel Prize in 2008 – it took a long time because we needed experiments to verify some interactions between quarks. But there's no way to reverse a decay to test effects of time reversal, so we were forced for a long time to assume that it was just possible to violate it. Finally, in 2012, SLAC observed that weak decays do define an arrow of time – however, this has nothing to do with the arrow of time that we perceive!

In conclusion, we've discovered in the last century that CPT are not symmetries. We learned that with enough fundamental particles, we can break symmetries, but we don't know why there **are** so many different types of particles.

37.5 Current fundamental understanding

All of fundamental physics is based on Lagrangians and Hamiltonians, since forces in a Newtonian sense are not fundamental. We now think of "forces" as particles interacting by exchanging energy and momentum in a quantum mechanical sense.

On a higher level, we can get the form of every type of force using **gauge symmetries**. We can postulate the type of symmetry that derives Maxwell's equations, and we can get complete forms of the weak and strong force too with more complicated symmetries. It was originally thought that the weak force was different from electromagnetism - it turns out they can be unified into an **electroweak force**. But those two forces look different in our life because the Higgs field breaks the electroweak symmetry:

Example 143

Imagine a perfect cylinder, and suppose that we apply a force directly down its axis. This cylinder has to flex somehow, and this behavior can be probabilistic in quantum mechanics. This is called **spontaneous symmetry breaking**, and it somehow explains why photons have no mass but other particles have very high mass.

We don't actually know the source of electroweak symmetry breaking, though, and most interactions are still symmetries.

There are still many other unanswered questions: the symmetry of QCD should allow CP violation to occur in the strong force, but for some reason, this has not been observed. And there are other open questions too: for example, we don't know what most of the stuff (dark energy, dark matter) in the universe is. Also, if matter and antimatter are almost equivalent, why is everything around us made of matter? It sounds like there must be unknown sources of C violation as well, but we've never seen them.

Fact 144

Fermi once said that "Before I came here I was confused about this subject. Having listened to your lecture, I am still confused, but on a higher level."

38 February 1, 2019 (Recitation)

No one else came, so the recitation instructor decided not to go over the problem set. I suppose this is an interesting end to the class...