MATH 215B: Differential Topology

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Introduction

An outline of topics we'll be covering, as well as other information about the course, can be found on Canvas and also at https://math.stanford.edu/~ionel/Math215B-w23.html. Homework will be assigned weekly (due on Thursdays on Gradescope) with lowest grade dropped, and there will be a midterm and final exam (dates announced soon). As is normally the case, we can work with others on homework but write up the solutions separately. Lecture notes will also be on Canvas and potentially have a bit more detail than what is covered in lecture. And office hours are posted on the website as well.

1 January 9, 2023

Before talking about differential manifolds, we'll make sure we're on the same page about topological manifolds. All of our spaces in this class will be Hausdorff, and all maps will be continuous, unless otherwise specified.

Roughly, a topological space M is a topological manifold if it looks locally like $\mathbb{R}^n$, with some additional assumptions of being Hausdorff and compact. Here's a more precise definition (there are various ways to make definitions):

Definition 1

An ***n*-dimensional topological manifold** is a Hausdorff, paracompact topological space M such that for any $x \in M$, there is an open neighborhood U of x in M such that U is homeomorphic to $\mathbb{R}^n$ (we could also say the ball in $\mathbb{R}^n$).

Recall that being **Hausdorff** means that any two distinct points can be separated by open sets, and being **paracompact** means that every open cover has an open refinement (of smaller sets) which is locally finite. Paracompactness implies the existence of a continuous partition of unity, which means that M is metrizable (that is, we have a metric on our manifold) – this is a very useful thing for us to have.

Example 2 (Non-examples of manifolds)

The **line with two origins** satisfies all of these assumptions for being a manifold except being Hausdorff. To construct this space, take two copies of $\mathbb{R}$ (call them $\mathbb{R} \times \{0\}$ and $\mathbb{R} \times \{1\}$), and then define $X = \mathbb{R} \times \{0, 1\}/\sim$, where points $(x, 0)$ and $(x, 1)$ are identified for all $x \neq 0$ (but the two origins stay distinct). We can check that all of the other assumptions here are satisfied. On the other hand, the **long line** satisfies all requirements except paracompactness – we can search up the construction ourselves, though it does require ordinals.

Example 3 (Examples of manifolds)
Any open set $U \subseteq \mathbb{R}^n$ is open, any discrete set (countable or not) is a 0-dimensional manifold, and the unit sphere in $\mathbb{R}^{n+1}$ is an n-dimensional manifold.

Often our manifolds will be either compact or **σ-compact** (meaning that they are a countable union of compact sets) – both of these imply paracompactness. So some authors imply σ-compactness in these definitions, which also implies **separability** for manifolds (which means that M contains a countable dense subset) These assumptions will all come into play later on, but we won't assume them in general here.

Definition 4
An **atlas** on a manifold M is a collection of charts $\{(U_\alpha, \phi_\alpha)\}$ which cover M. A **chart** is a pair (U, ϕ), where $U \subset M$ is an open set and $\phi : U \to V$ is a homeomorphism from U to an open set $V \subseteq \mathbb{R}^n$. On the overlap of any two charts (U_α, ϕ_α) and (U_β, ϕ_β) (that is, on $U_\alpha \cap U_\beta$), we thus get a **transition function** $\phi_{\beta\alpha} = \phi_\beta \circ \phi_\alpha^{-1}$ (appropriately restricted) mapping from $\phi_\alpha(U_\alpha \cap U_\beta)$ to $\phi_\beta(U_\alpha \cap U_\beta)$.

In other words, a single chart maps part of M to an open set of $\mathbb{R}^n$, and an atlas puts them together to cover all of the manifold. But if U_α and U_β are two open sets which overlap in M, then V_α and V_β both have a portion that correspond to $U_\alpha \cap U_\beta$, and the point of the transition function $\phi_{\alpha\beta}$ is to get us a map between subsets of $\mathbb{R}^n$ that relate those different charts. If we have an atlas, we can then think of **gluing charts together to form a manifold** via the transition functions, so M is homeomorphic to the space $(\bigsqcup_\alpha V_\alpha)/\sim$, where $\sim$ identifies $x \in V_\alpha$ with $y \in V_\beta$ if and only if $\phi_{\beta\alpha}(x) = y$.

We will focus on smooth manifolds in this class:

Definition 5
An atlas $\mathcal{A}$ on M is **of class C^k** if all transition functions are of class C^k (meaning that they are k times differentiable with all derivatives up to degree k continuous). An atlas is **smooth** if all transition functions are C^∞.

(We can always complete an atlas to a **maximal atlas** of class C^k, which will come up later.)

Definition 6
A **smooth manifold** M is a Hausdorff, paracompact space endowed with a C^∞ atlas.

In applications, it may seem like there is a difference betwen proving things for C^k manifolds and smooth manifolds. But it turns out that every C^1 manifold has a unique smooth structure by tossing out some charts, and thus C^1 diffeomorphisms becomes C^∞ diffeomorphisms and so on. So the only important distinction is between C^0 (topological) and C^∞ (smooth). And there are indeed topological manifolds which do not admit any smooth structure (see Donaldson's theorem in four dimensions) and also manifolds that admit multiple such structures (for example, Milnor proved that S^7 has several).

Example 7 (Graphs and level sets)
Let $f : U \to \mathbb{R}^m$ be a smooth function with $U \subseteq \mathbb{R}^n$ an open set. Then the **graph** of f is a smooth manifold of dimension n with a single global chart, and if c is a **regular value** of f (meaning that the differential df is surjective at all points in the inverse image), then the **level set** $f^{-1}(c)$ is a smooth manifold of dimension $n - m$ (by the implicit function theorem).

Example 8

The unit sphere $S^n \subseteq \mathbb{R}^{n+1}$ (of points satisfying $|x_0|^2 + \cdots + |x_n|^2 = 1$) is a smooth manifold because it is a level set by the previous example.

Example 9

We have the **real projective space** $\mathbb{RP}^n$ and **complex projective space** $\mathbb{CP}^n$ – recall for example that $\mathbb{CP}^n$ is the space of one-dimensional complex linear subspaces of $\mathbb{C}^{n+1}$, given the topology induced by the quotient

$$\mathbb{CP}^n = (\mathbb{C}^{n+1} \setminus 0)/\mathbb{C}^*$$

where the $\mathbb{C}^*$ (that is, nonzero complex numbers) action is given by $\lambda(z_0, \cdots, z_n) = (\lambda z_0, \cdots, \lambda z_n)$.

This space is Hausdorff (though it would not be Hausdorff if we didn't remove the origin from $\mathbb{C}^{n+1}$ – exercise). Indeed,

$$(\mathbb{C}^{n+1} \setminus 0)/\mathbb{C}^* = S^{2n+1}/S^1,$$

where $S^{2n+1} \subseteq \mathbb{C}^{n+1}$ is the unit sphere and S^1 is a subset of $\mathbb{C}^*$. And recall an important property of quotient spaces: if a group G is acting on a space X and X and G are compact (and Hausdorff), so is X/G. In particular, we get **homogeneous coordinates** on $\mathbb{CP}^n$, which we denote $[z_0, \cdots, z_n]$, and the way that we find charts is to observe that in these homogeneous coordinates we have $[z_0, \cdots, z_n] = [1, \frac{z_1}{z_0}, \cdots, \frac{z_n}{z_0}]$ whenever $z_0 \neq 0$, so we can use the last n coordinates as usual coordinates. (So we get $(n+1)$ different charts, and we just need to define the transition functions in terms of ratios between those coordinates.) But the point is that $\mathbb{RP}^n$ and $\mathbb{CP}^n$ are both compact smooth manifolds.

Example 10

Very similarly, the **Grassmannian** is the space of k-dimensional planes (linear subspaces) of a vector space V, and it is also a smooth manifold. The case $k = 1$ is basically the previous example, and we call it the **projectivization** of V and denote it $\mathbb{P}(V)$.

One question to keep in mind is to ask when the quotient of a manifold is itself a manifold – we'll address this later.

Example 11 (Example, but cautionary)

Working on $\mathbb{R}$, consider the atlas with one chart $\phi : \mathbb{R} \to \mathbb{R}$ sending $\phi(t) = t^3$. This yields a smooth manifold, but the chart is not C^1-compatible with the standard chart (given by the identity).

Definition 12

Let M be a smooth manifold. A map $f : M \to \mathbb{R}$ is C^k if for all charts ϕ_α, the map $f \circ \phi_\alpha^{-1}$ (from an open set of $\mathbb{R}^n$ to $\mathbb{R}$) is C^k. (In fact, we just need to check one chart at each point $x \in M$.)

We should check that that this definition is independent of our choice of charts (because change of charts are smooth).

Definition 13

Let M and N be two smooth manifolds. A map $f : M \to N$ is **smooth** if for **all** charts (not just for a single atlas) (U, ϕ) on M and (V, ϕ) on N such that $f(U) \subseteq V$, the composition $\phi^{-1} \circ f \circ \psi$ is smooth.

In other words, f is smooth at x if f is continuous and there **exists** a chart (U, ϕ) on M at x and a chart (V, ψ) on M at $f(x)$ with the above composition smooth at $\phi(x)$ (and this is again well-defined). From now on, things will be smooth, and note that smoothness implies continuity.

Definition 14

A (smooth) **diffeomorphism** between two smooth manifolds is a bijection $f : M \to N$ such that f and f^{-1} are smooth. we will write this as $M \simeq N$.

Example 15

We know that $\mathbb{RP}^1$ is the space of lines in $\mathbb{R}^2$. Then $\mathbb{RP}^1$ is diffeomorphic to S^1, and similarly $\mathbb{CP}^1$ is diffeomorphic to S^2.

In particular, when we said that manifolds may have several smooth structures earlier, we were looking at equivalence classes **up to diffeomorphism**. And it turns out that $\mathbb{R}^4$ has infinitely many smooth structures that are not diffeomorphic to each other (this is a result of Taubes).

Next, to define the differential of a smooth map $M \to N$, we need the notion of a tangent bundle. Since this is a graduate level course, we'll first discuss bundles, which will require more notation. There are two types of bundles – first of all, fiber bundles look "locally like a product but globally like a twist," and a good example to keep in mind is that the Mobius strip is a bundle over a circle.

Definition 16

A **fiber bundle** is a map $p : E \to B$ from a space E to a base space B with **fiber** F if for all $x \in B$, there is some neighborhood U of x in B such that there is a homeomorphism $\psi : p^{-1}(U) \to U \times F$ such that the following diagram commutes (where π is the projection from $U \times F$ onto the U coordinate):

$$\begin{tikzcd} p^{-1}(U) \arrow[r, "\psi"] \arrow[dr, "p"'] & U \times F \arrow[d, "\pi"] \\ & U \end{tikzcd}$$

We call ψ a **local trivialization** of the bundle, since locally we just have a product. But globally we may not have a product:

Example 17

We always have the trivial bundle $B \times F \to B$ given by production, and any covering map is a fiber bundle where the fiber is a discrete set. And finally, the Mobius band is given by $M = \mathbb{R} \times [0, 1]/\sim$ with identification given by $(x, 0) \sim (-x, 1)$; then the fiber is $\mathbb{R}$ and the base is $[0, 1]$ with endpoints identified, which is the same as S^1. More explicitly, our fiber bundle is the map $p : \mathbb{R} \times [0, 1]/\sim \to [0, 1]/_{0\sim 1} = S^1$.

2 January 12, 2023

Our first homework assignment is posted and is due next week, and lecture notes are also available on Canvas. In general, lecture notes will have slightly more detail than we discuss in class – for example, we didn't discuss **manifolds with boundary** last time. We will talk about manifolds without boundary in this class, but we should keep manifolds with boundaries in mind (and there's a homework problem on it) – the basic idea is that instead of looking locally like $\mathbb{R}^n$ everywhere, a manifold looks locally like $\mathbb{R}^n$ at **interior points** and like the half-plane $\mathbb{H}^n \subseteq \mathbb{R}^n$ at **boundary points**. We must show that this is well-defined (for example, we can't be an interior point in one chart and a boundary point in another because there's no homeomorphism between them).

Proposition 18

The boundary of a manifold with boundary is a manifold without boundary.

One tricky point is that the product of two smooth manifolds without boundary is another smooth manifold without boundary, but **the product of two smooth manifolds with nonempty boundary will be a topological manifold with boundary but not a smooth manifold with boundary** (the usual example is to take the product of an interval with itself, forming a square). What we end up with is a "manifold with boundary and corners," but we won't define that.

At the end of last lecture, we also started discussing bundles, moving towards discussing tangent bundles. Specifically, we defined a **fiber bundle** (also called **locally trivial fibration**) with fiber F to be a total space E mapping (under a map p) to a base space B, such that for each point in B there is a neighborhood U where the inverse image $p^{-1}(U)$ is homeomorphic to $U \times F$ in a way commuting with p and the projection $p_U : U \times F \to U$. (So we have a continuous family of fibers over the points in B.)

For any point $x \in B$, we call its preimage $p^{-1}(x)$ the **fiber at x** and denote it E_x – note that under the homeomorphism $p^{-1}(U) \xrightarrow{\psi} U \times F$, we have $E_x \simeq F$. (We call F the **abstract fiber**.) We explained the Mobius strip as an example last time, and we'll mention another example one now:

Definition 19

We have the **Hopf fibration** $p : S^3 \to S^2$ by thinking of $S^2 = S^3/S^1$ – this is an S^1 fiber bundle in which the fibers are circles. More generally, we can project S^{2n+1} to $S^{2n+1}/S^1 = \mathbb{CP}^n$; this is also an S^1 bundle.

This can be proved by hand using local coordinates and find the local trivialization ourselves, but later we'll talk about group actions on manifolds and see that such fibrations often arise.

Definition 20

An **morphism** of two fiber bundles $p_1 : E_1 \to B_1$ and $p_2 : E_2 \to B_2$ is a pair of (continuous) maps $(\phi, \tilde{\phi})$ where $\phi : B_1 \to B_2$ maps between bases and $\tilde{\phi} : E_1 \to E_2$ maps between the total space, such that "the square of maps commutes" ($\phi \circ p_1 = p_2 \circ \tilde{\phi}$). An **isomorphism** of two fiber bundles is defined similarly but with homeomorphisms as requirement.

In particular, the diagram commuting means that fibers are taken to fibers.

Example 21

Recall that $\mathbb{CP}^n$ is the space of one-dimensional complex linear subspaces of $\mathbb{C}^{n+1}$. The **tautological bundle** $E \to \mathbb{CP}^n$ is defined as follows: we define $E \subseteq \mathbb{CP}^n \times \mathbb{C}^{n+1}$ via

$$E = \{(\ell, v) : v \in \ell\},$$

with projection given by $(\ell, v) \mapsto \ell$. In other words, we have a line over every point (subspace) in $\mathbb{CP}^n$ which is the subspace itself (so the fibre at ℓ is ℓ, and the abstract fiber is $\mathbb{C}$).

We get such a tautological bundle for $\mathbb{RP}^n$ and the Grassmannian as well, since they are defined similarly as spaces of subspaces. Sometimes this bundle is also denoted τ.

Example 22

Consider the tautological bundle $\tau \to \mathbb{RP}^1 = S^1$ – this is a line bundle over the circle, and in fact it is isomorphic to the Mobius band bundle over S^1. On the other hand, it is not isomorphic to the trivial bundle (we cannot make it isomorphic to a product, because it twists).

We'll focus now on a special case:

Definition 23

A **vector bundle** $p : E \to B$ is defined the same way as a fiber bundle, but now requiring that the fibers E_x are vector spaces and that the local trivializations are fiberwise linear.

Here, B can still be any topological space, but we're saying that if we take a map $p^{-1}(U) \to U \times V$ and restrict to the particular fiber at x, we get a linear map between vector spaces. And we can then define similar notions for vector bundles being morphisms or isomorphisms – the definition are the same but we now also require fiberwise linearity for the maps.

Just like we can think about manifolds as gluing together subsets of $\mathbb{R}^n$, we can think of a bundle as **gluing together local trivializations via transition functions**. In other words, suppose we have a family (U_α, ψ_α) of local trivializations $\psi_\alpha : p^{-1}(U_\alpha) \to U_\alpha \times F$, we have **transition functions** (also called **clutching functions**) $\psi_{\beta\alpha} : \psi_\alpha(p^{-1}(U_\alpha \cap U_\beta)) \to \psi_\beta(p^{-1}(U_\alpha \cap U_\beta))$ defined by $\boxed{\psi_{\beta\alpha} = \psi_\beta \circ \psi_\alpha^{-1}}$. In other words, we have two "copies" of $(U_\alpha \cap U_\beta) \times F$ (one from α and one from β) and we want a way of describing how to relate the two.

The transition functions $\psi_{\alpha\beta}$ satisfy a few properties: (1) they are continuous on their domain and in fact homeomorphisms, (2) $\psi_{\alpha\alpha}$ is the identity for any α, and (3) on any triple overlap $U_\alpha \cap U_\beta \cap U_\gamma$ (or more precisely its preimage under p), we have $\psi_{\alpha\beta} \circ \psi_{\beta\gamma} = \psi_{\alpha\gamma}$. But as mentioned above, since these transition functions take fibers to fibers homeomorphically, we can equivalently use the maps $\psi_\alpha : p^{-1}(U_\alpha) \to U_\alpha \times F$ to think of transition functions as maps $\psi_{\beta\alpha} : (U_\alpha \cap U_\beta) \times F \to (U_\alpha \cap U_\beta) \times F$:

$$\begin{tikzcd}
(U_\alpha \cap U_\beta) \times F \arrow[r, "\psi_{\beta\alpha}"] \arrow[dr] & (U_\alpha \cap U_\beta) \times F \arrow[d] \\
& U_\alpha \cap U_\beta
\end{tikzcd}$$

If we start with a point $(x, v) \in (U_\alpha \cap U_\beta) \times F$ in the top left, then under $\psi_{\beta\alpha}$, we must send x to x because fibers are taken to fibers, and for any fixed x we get a homeomorphism $F \to F$ depending on x and depending on which local trivializations we're on. Thus we can write

$$\psi_{\beta\alpha}(x, v) = (x, g_{\beta\alpha}(x)(v)).$$

In other words, $g_{\beta\alpha}$ is a continuous map from $U_\alpha \cap U_\beta$ to Homeo(F) because $\psi_{\beta\alpha}$ is continuous. Then these $g_{\beta\alpha}$s also satisfy the same properties (1), (2), and (3) as before, and now we're thinking about them as homeomorphisms of F. And in fact given a collection of these clutching functions, we get a bundle:

Proposition 24

Let U_α be a covering of a base B, and suppose we have clutching functions $g_{\beta\alpha}$ satisfying properites (1), (2), (3) above. Then we get a bundle $E \to B$ by defining

$$E = \left(\bigsqcup_\alpha U_\alpha \times F\right) / \sim,$$

with $\sim$ identifying $(x, v) \in U_\alpha \times F$ with $(y, u) \in U_\beta \times F$ if $x = y$ and $g_{\beta\alpha}(x)(v)$, and then having the map $p : E \to B$ induced by the projection $U_\alpha \times F \to U_\alpha$.

The point is that the conditions $g_{\alpha\alpha} = \text{id}$ and $g_{\alpha\beta} \circ g_{\beta\gamma} = g_{\alpha\gamma}$ show us that this $\sim$ is in fact an equivalence relation. And objects in this bundle are then equivalence classes, which we can denote $[x, v]$. (Notice in particular that for a vector bundle, the fiber is something like $\mathbb{R}^n$, and then the transition functions are elements of $\text{GL}(\mathbb{R}^n)$.)

With this preparation, we're ready to specialize to the objects that we really care about. The goal is to take a smooth manifold M and define its **tangent bundle** $TM \to M$, which is a vector bundle over M, such that if we have two smooth manifolds M and N and a smooth map $f : M \to N$, we get a differential map $df : TM \to TN$ which is a vector bundle morphism.

Example 25

In the case where $U \subseteq \mathbb{R}^n$ is an open set, we should have $TU = U \times \mathbb{R}^n$, where a point (x, v) projects to x under the bundle map. Intuitively, the idea is that the tangent space consists of a point in the set together with a vector pointing in some direction, which we should think of as a tangent vector at that point.

We'll now basically take charts on our manifold and clutch these together to get the tangent bundle to the manifold. The idea is that a map $f : \mathbb{R}^n \to \mathbb{R}^m$ (or similarly a map between open subsets of those spaces) has a differential df represented by the matrix of partial derivatives.

Definition 26

Let $\mathcal{A}$ be an atlas on a smooth manifold M with charts (U_α, ϕ_α). Define the **tangent bundle** TM to be the quotient

$$TM = \left(\bigsqcup_\alpha U_\alpha \times \mathbb{R}^n\right) / \sim,$$

where $\sim$ identifies $(x, v) \in U_\alpha \times \mathbb{R}^n$ with $(y, u) \in U_\beta \times \mathbb{R}^n$ if $x = y$ and $u = (d\phi_{\beta\alpha})_x(v)$.

(Here $\phi_{\beta\alpha}$ are the transition functions on the manifold – they are smooth maps between open subsets of $\mathbb{R}^n$ – so in coordinates $(d\phi_{\beta\alpha})_x$ is basically the linear transformation given by the Jacobian matrix at x.) The idea is that each copy of $\mathbb{R}^n$ is the tangent space at a point in U_α, and we're doing the same gluing procedure as we've previously been doing. There are various other interpretations of tangent vectors / bundles that we can use too (instantaneous velocities along a path, derivations, germs, or coordinates), and it's important to be able to switch between them for different purposes – we'll discuss this next time.

3 January 17, 2023

Last lecture, we discussed bundles (in which we glue together local trivializations) and in particular tangent bundles. Tangent bundles $TM \to M$ to a smooth manifold M can be defined in several equivalent ways, which all boil down to working locally in charts and patching them together. One way is to take an atlas (U_α, ϕ_α), look at the various copies $U_\alpha \times \mathbb{R}^n$, and then glue them together via the transition functions $\phi_{\beta\alpha}$, gluing $(x, v) \in U_\alpha \times \mathbb{R}^n$ to $(y, w) \in U_\beta \times \mathbb{R}^n$ if and only if $x = y$ (on the manifold) and $w = (D\phi_{\beta\alpha})v$ (where $D\phi_{\beta\alpha}$ is the derivative of the change of charts $\phi_\beta \circ \phi_\alpha^{-1}$, which is basically the Jacobian matrix in coordinates). So we're thinking of v as the "tangent vector" in a particular direction at x, and a tangent vector should be thought of as an equivalence class $[(x, v)]$. We then get the projection map $\pi : TM \to M$ sending $[x, v]$ to x.

As an exercise for us, there is a more natural way to do this: define the tangent space of an open set in $\mathbb{R}^n$ and glue those together using the differential of the transition functions (we're trying to make a distinction between derivatives and differentials for now, since differentials map from tangent spaces to tangent spaces):

Proposition 27

Using this alternate definition of the tangent space, $TM \to M$ is a vector bundle over M, and if M is a C^k manifold (meaning transition functions are of class C^k) then TM is a C^{k-1} manifold of dimension $2\dim(M)$.

(Basically the $U_\alpha \times \mathbb{R}^n$s should be thought of as the local trivializations, and the fiber at $x \in M$, denoted T_xM, is $\mathbb{R}^n$.) Talking more now about the **tangent vectors** $[(x, v)]$, there are some intrinsic definitions that can be made that make the objects less abstract:

- Geometrically, we can imagine drawing a path on M and letting v be the **velocity vector along that path**. (We're assuming M is a smooth manifold here, and that's the standing assumption.)

Definition 28

Let $\gamma : I \to M$ be a path, where $I = (-\varepsilon, \varepsilon)$ is an interval containing 0. Then T_xM is the collection of smooth paths $\gamma : (-\varepsilon, \varepsilon) \to M$ with $\gamma(0) = x$, modulo the equivalence relation

$$\gamma_1 \sim_x \gamma_2 \text{ if } (f \circ \gamma_1)'(0) = (f \circ \gamma_2)'(0)\ \forall f : M \to \mathbb{R} \text{ smooth.}$$

The idea is that we haven't defined what it means to take a derivative of a map $I \to M$, but if we compose it with a map $M \to \mathbb{R}$ we get a map $(-\varepsilon, \varepsilon) \to \mathbb{R}$. And this is also equivalent to having an equivalence relation

$$\gamma_1 \sim_x \gamma_2 \text{ if } D(\phi_\alpha \circ \gamma_1)'(0) = (\phi_\alpha \circ \gamma_2)'(0)$$

for all charts (or just for one chart) ϕ_α on a neighborhood of x in M. The point is that $\phi_\alpha \circ \gamma$ is now a map from $(-\varepsilon, \varepsilon) \to \mathbb{R}^n$. (Right now, T_xM doesn't look like $\mathbb{R}^n$ just yet, but we'll see soon how that comes up.)

Definition 29

Let $f : M \to N$ be a smooth map between smooth manifolds. Then $df : TM \to TN$ is a smooth map which is also a bundle map (meaning it's fiberwise linear), defined in the following way: $(df)_x$ maps T_xM to $T_{f(x)}N$ linearly by setting

$$(df)_x[\gamma] = [f \circ \gamma].$$

(Here we think of T_xM with the "equivalence class of paths" definition.)

- Alternatively, tangent vectors tell us how to **differentiate smooth functions in a particular direction**. So we're thinking about "derivations," which turn out to be directional derivatives of smooth functions $f : M \to \mathbb{R}$ in this case.

Definition 30

A **derivation** on M at the point $x \in M$ is a linear map $X : C^\infty(M) \to \mathbb{R}$ satisfying Leibniz rule $X(fg) = (Xf) \cdot g(x) + f(x) \cdot (Xg)$. The space of derivations at $x \in M$ is denoted $\text{Der}_x(M)$.

Clearly $\text{Der}_x(M)$ is a vector space (since we can add two such derivations and multiply by constants) – the point is that we can define this to be our tangent space T_xM as well, and this vector space is $\mathbb{R}^n$.

Example 31

Suppose $M = \mathbb{R}^n$. Then a derivation is a map $X : C^\infty(\mathbb{R}^n) \to \mathbb{R}$, and for any $f : \mathbb{R}^n \to \mathbb{R}$ and for any fixed $v \in \mathbb{R}^n$ (this takes the role of our "direction") we have the formula

$$Xf = \left.\frac{d}{dt}\right|_{t=0} f(x + tv).$$

So basically we can think of $f(x + tv)$ as a map $\mathbb{R} \to \mathbb{R}$, which is a path traveling in direction v, and X is literally the directional derivative of $f : \mathbb{R}^n \to \mathbb{R}$ at x in the direction v. We do need to check that this is a derivation, but those properties follow from ordinary calculus. And what we want to prove is that all derivations at x are obtained this way (that is, all derivations on $\mathbb{R}^n$ at x are directional derivatives in some direction $v \in \mathbb{R}^n$), and then we can move things to general manifolds by using our charts.

The best way to prove this is to use coordinates, then pass to local coordinates: let $(x_1, \cdots, x_n)$ be coordinates on $\mathbb{R}^n$, and let $(e_1, \cdots, e_n)$ be a standard basis of $\mathbb{R}^n$ corresponding to that choice of coordinates – define $\left.\frac{\partial}{\partial x_i}\right|_x \in T_x\mathbb{R}^n$ to be the corresponding tangent vector to $\mathbb{R}^n$ in the direction $e_i \in \mathbb{R}^n$, either as a derivation or as a tangent space to a path (whichever way we want to define it). Now if M is a smooth manfiold, suppose $(x_1, \cdots, x_n)$ are **local** coordinates on M, meaning that we're identifying U_α with some open set $V_\alpha \in \mathbb{R}^n$ and $x_1, \cdots, x_n$ are the coordinate functions for V_α. In other words, $(x_1, \cdots, x_n)$ **is the map** $\phi_\alpha : U \to \mathbb{R}^n$. Then we can define $\left(\frac{\partial}{\partial x_i}\right)_x \in T_xM$ to be the vector corresponding to $(x, e_i) \in T_x\mathbb{R}^n$ under the local trivialization of the tangent bundle TM. (So we can take a path class in $\mathbb{R}^n$ and look at the inverse image of the chart map.) Then a basis of T_xM is given by

$$\left(\left.\frac{\partial}{\partial x_1}\right|_x, \cdots \left.\frac{\partial}{\partial x_n}\right|_x \right).$$

(We can think of this with whichever definition of tangent vectors we'd like, and various references pick just one of these definitions, but the point is that we need it to behave the same way under change of charts – for any smooth map $f : M \to N$, we get a bundle map $df : TM \to TN$ which is fiberwise linear, meaning $df_x : T_xM \to T_xN$ is a linear transformation $\mathbb{R}^n \to \mathbb{R}^n$. And what's important is that we have a chain rule: if $M \xrightarrow{f} N \xrightarrow{g} P$, then

$$d(g \circ f) = dg \circ df$$

(we can check this in coordinates). We thus get a functor from the category **Man** (in which objects are smooth manifolds and morphisms are smooth maps) to bundles **Bun** (in which objects are topological bundles and morphisms are bundle maps, meaning they are continuous and fiberwise linear).

Remark 32. *One motivating question we may ask is whether TM is trivial (meaning that our manifold is* **parallelizable***). For example, we can have TS^1 be trivial by letting $S^1 = \mathbb{R}/\mathbb{Z}$, using the single coordinate t and having $t \to \frac{\partial}{\partial t}$ be the global fiberwise basis. But we may also ask similar questions about TS^n, $T\mathbb{RP}^n, T\mathbb{CP}^n$, or for a Riemann surface.*

Definition 33

A **section** of a bundle $\pi : E \to B$ is a (continuous) map $s : B \to E$ such that $\pi \circ s = \mathrm{id}_B$.

We can think of taking a point $x \in B$ and getting a point $s_x \in E_x$ in the fiber above x, such that $x \to s_x$ is continuous (meaning that the map $x \mapsto (x, s_x)$ continuous in the local trivialization as a function from U to $U \times F$, where U contains x and is open in B). Not all fiber bundles have sections, but vector bundles do always have a special one:

Example 34

Let $E \to B$ be a vector bundle. Then because each E_x is a vector space, we get the **zero section** mapping each $x \in B$ to $0 \in E_x$.

Definition 35

A **vector field** on a smooth manifold M is a section of the tangent bundle TM. In other words, for each point $x \in M$, we associate a tangent vector $v_x \in T_xM$. This vector field is **smooth** if the association is a smooth map in charts (as a map from a neighborhood of $\mathbb{R}^n$ to $\mathbb{R}^{2n}$).

Lemma 36

Suppose $E \to B$ is a trivial vector bundle. Then there is at least one nowhere vanishing section.

Indeed, since $E \to B$ is trivial, it is isomorphic to $B \times \mathbb{R}^n$, and we have n linearly independent vectors so we have n linearly independent sections. So for any $v \in \mathbb{R}^n$ we get a constant section and we can use global isomorphism $E \simeq B \times \mathbb{R}^n$ to get a section on E. And this, along with the hairy ball theorem (which says that any smooth vector field on S^2 always has a zero), proves that the tangent space TS^2 is nontrivial.

(Later on in the class, we'll prove the Poincaré-Hopf theorem, which says that if M has a nowhere vanishing vector field, then the Euler characteristic of M must be zero. So that's something we should keep in mind – it's a generalization of the hairy ball theorem.)

We'll finish today's class by mentioning some operations on vector bundles – the point is that operations on vector spaces will extend to operations on the whole bundle (which is a continuous family of vector spaces). Since bundles are locally products, given two vector bundles E_1, E_2 over the same base B, we can get $E_1 \oplus E_2$ and $E_1 \otimes E_2$, and given a single bundle we can get $\det(E)$ or $\Lambda^k(E)$. We can do this by either (1) first doing these operations on local trivializations and showing they patch together through transition functions, or (2) using the universal properties of the direct sum, tensor product, determinant, and so on.

To see an example of (1) in action, if we have a bundle $\pi : E \to B$ and a map $f : B' \to B$, we can define the **pullback** $f^*E \to B'$ to be a bundle over B', where we just pull the fibers back to the base B'. So a more explicit definition is that

$$f^*E = \{(y, v) \in B' \times E : f(y) = \pi(v)\},$$

but the point is that the fiber of f^*E at x is just $E_{f(x)}$. (And the projection $f^*E \to B'$ maps (y, v) to y.) So one way to define the direct sum is to take the product space $E_1 \times E_2$ and have

$$E_1 \oplus E_2 = \Delta^*(E_1 \times E_2),$$

where $\Delta : B \to B \times B$ is the diagonal map $\Delta(x) = (x, x)$.

4 January 19, 2023

(This class will no longer have a midterm, but we will still have a final – the weekly problem sets are basically enough effort for us throughout the term.)

Our topic for today is **immersions, submersions, and submanifolds** – we'll give some precise definitions, examples, and counterexamples, as well as things to be aware of for differences between references. For all of this lecture, we'll have $f : M \to N$ denote a smooth map between smooth manifolds (without boundary – if we want, we can look through the material again and see how much of the theory goes through with boundary), where M has dimension m and N has dimension n. We thus get a differential $df : TM \to TN$, where the differential $df_x : T_xM \to T_{f(x)}N$ at a point $x \in M$ is a linear map between vector spaces, so that df is a vector bundle map.

The rank of a linear map is independent of our choice of basis or of charts, so the rank of the map df_x is well-defined. That allows us to make the following definition:

Definition 37

We say that f is an **immersion** if df_x is injective for all x, and it is a **submersion** if df_x is surjective for all x. Also, f is an **embedding** if it is an immersion and a homeomorphism onto its image, and N is a **submanifold** of M if the inclusion $\iota : N \hookrightarrow M$ is an embedding.

We should be careful that different authors mean different things when they say "embedding" or "submanifold:" for example, others may consider "embedded submanifold" versus "immersed submanifold." The point has to do with the topology: being a homeomorphism onto its image says that we have the induced topology from N. And sometimes double points (self-intersections) are allowed, so injectivity is removed as an assumption for the map f.

Example 38

Take a circle and gluing two diametrically opposite points in the plane to form a figure-8, or taking the real line and "curving back" so that the image of the positive real line approaches 0 in the limit both yield immersions. However, we get the wrong topology – in the first case the map is not injective, and in the second a neighborhood of that intersection point becomes two open neighborhoods. So both of these examples are not embeddings.

Remark 39. *As a sidenote, there's lots of linear algebra that may be useful – we can discuss subbundles, metrics, orthogonal complements, and so on – and we should read the reference posted in lecture notes for this. The point is always to work with charts / local trivializiations and patch those together with change of charts. Many of our arguments can be made locally, but not all – immersion and submersion are local statements, but embedding is a global statement.*

Example 40

Fix some $\alpha \in \mathbb{R}$ and consider the map $f : \mathbb{R} \to S^1 \times S^1$ sending $t \mapsto (\exp(it), \exp(i\alpha t))$. This map is in fact an immersion (local coordinates in terms of t), but if α is irrational then $f(t)$ is an injective immersion but not an embedding – in fact its image is dense in $S^1 \times S^1$, so it would have to have the trivial topology. But f is an embedding locally (if we restrict to a small neighborhood of a point); this will turn out to be a more general fact that we will prove.

Example 41

We can check that graphs of smooth functions are submanifolds, and open subsets of M are submanifolds.

Notice that either being an immersion or being a submersion means that df_x has maximal rank. This is an "open condition", so if df has maximal rank at a point, then it will have maximal rank in a neighborhood of that point, and a sufficiently small C^1 perturbation of f will also have maximal rank. But there are cases where we care about a map having constant rank which is not maximal (and that is no longer an open condition – **rank can go up under arbitarily small deformations**).

Example 42

A **Lie group** G is both a group and a smooth manifold, such that the group structure and manifold structure are compatible – in other words, the group multiplication $G \times G \to G$ and inverse map $G \to G$ are smooth maps. Some examples include $O(n)$ and other matrix groups. A **Lie group homomorphism** $\phi : G_1 \to G_2$ is then a smooth map which is also a group homomorphism. Then Lie group homomorphisms always have constant rank (and this is an exercise for us).

Theorem 43 (Inverse function theorem for $\mathbb{R}^n$)

Suppose $f : \mathbb{R}^n \to \mathbb{R}^n$ is a smooth map (just for simplicity – it doesn't have to be defined on all of $\mathbb{R}^n$) with $f(0) = 0$ and $(df)_0 : T_0\mathbb{R}^n \to T_0\mathbb{R}^n$ (the Jacobian matrix) an isomorphism. Then we change coordinates on the domain so that f is the identity in those coordinates (in other words, we can find a local inverse of f); more precisely, there are $\delta, \varepsilon > 0$ and a locally defined smooth function $g : B_\delta(0) \to B_\varepsilon(0)$ such that $f \circ g = \text{id}$ on $B_\delta(0)$. More generally, if $f \in C^k$, then $g \in C^k$.

We won't prove this result in class, but it holds pretty generally – we can use the fixed point theorem, so it extends to Banach spaces using the contraction principle. But the point is that we are just using g to change coordinates in the domain.

Theorem 44 (Constant rank theorem)
Let $f : M \to N$ be a smooth map between manifolds, and suppose the rank of $(df)_x$ is some constant k for $x \in M$ (this is all a local statement, so we can just say this is true on a subset). Then for any $p \in M$ there are local coordinates $\{x_1, \cdots, x_m\}$ on M at p and $\{y_1, \cdots, y_n\}$ on N at $f(p)$, such that $f(x_1, \cdots, x_m) = (x_1, \cdots, x_k, 0, \cdots, 0)$.

In particular, if f is an immersion, then $f(x_1, \cdots, x_m) = (x_1, \cdots, x_m, 0, \cdots, 0)$, and if it is a submersion, then $f(x_1, \cdots, x_m) = (x_1, \cdots, x_n)$. So locally, immersions are basically inclusions and submersions are basically projections. We can read the proof on our own – the point is to prove in local charts, so that we can reduce M to an open set of $\mathbb{R}^m$ and N to an open set of $\mathbb{R}^n$. So we can first prove a linear algebra statement that a single rank-k matrix can be made of the form mentioned, and then the continuous deformation can be adjusted with more change of coordinates. With this, the fact we were alluding to in Example 40 immediately follows:

Corollary 45
Let $f : M \looparrowright N$ be an immersion. Then f is a local embedding, meaning that for any $p \in M$ there is some neighborhood U of p in M such that $f|_U$ is an embedding.

(In particular, if we have self-intersections in f, we can avoid having them by taking a small enough neighborhood.)

Definition 46
Let $f : M \to N$ be a smooth map. A point $x \in M$ is a **regular point** if df_x is surjective, and a point $p \in N$ is a **regular value** if the inverse image $f^{-1}(p)$ consists of regular points. Points that are not regular points are called **critical points**.

Corollary 47 (Regular value theorem / implicit function theorem)
Suppose $f : M \to N$ is smooth, and c is a regular value of f. Then the level set $P = f^{-1}(c)$ is a submanifold of M with tangent space $T_x P = \ker(df_x)$.

Proof. Being a regular value implies that f is a submersion in some neighborhood of the level set $f^{-1}(c)$, so by the constant rank theorem we can change coordinates so that f looks like a projection. Looking in coordinates, we can then get an explicit description of the kernel of df_x which is exactly what we claim it to be. □

Example 48
Since the unit sphere $S^n \subseteq \mathbb{R}^{n+1}$ is a level set of $f(x_0, \cdots, x_n) = x_0^2 + \cdots + x_n^2$, it is indeed a submanifold.

Example 49
Consider the height function of a torus (so imagine a donut standing up on a table and measuring the height above the table at each point) – there are then four critical values and the remaining ones are regular values. We'll come back to this when we think about Morse's theorem

Definition 50

Suppose $N \subseteq M$ is a submanifold (with N having dimension n and M having dimension m). The **codimension** $\text{codim}\, N$ is $\dim(M) - \dim(N)$.

In particular, at any regular value c, the codimension $\text{codim}\, f^{-1}(c)$ is $\dim M - \dim N$.

We'll skip the easy version of the Whitney embedding theorem for now (and return to the harder version later) since we're a bit short on time. Instead, we'll discuss **normal bundles** – suppose P is a submanifold of M, so in particular $i : P \to M$ is an embedding. Differentiating this yields a bundle map $d_i : TP \to TM$, so that $T_xP \subseteq T_xM$ is a linear subspace for any $x \in P$ – in particular, $(di)_x$ is always an injection. Thus as we let x vary, we get a tangent space TP sitting inside $TM|_P = \iota^* TM$ (that is, we have a bnudle over the whole manifold, and we restrict it to P via pullback by inclusion). We then claim that TP is actually a **subbundle** (so that we have a linear subspace, and in a local trivializion of $TM|_P$ we have $\mathbb{R}^n$ at each point, and TP basically is a rotating family of planes) – again, this is something for us to read on our own.

Definition 51

Let P be a submanifold of M. The **algebraic normal bundle** to P in M is the quotient **bundle** (not manifold) $N = N_{P/M}$ of $TP \subseteq TM|_P$, in which the fiber of N at a point x is $N_x = T_xM/T_xP$ for all $x \in P$. A **geometric normal bundle**, also denoted N, is any complement to TP inside $TM|_P$, meaning that it is a subbundle N of $TM|_P$ so that $TM|_P = TP \oplus N$ – in other words, $T_xM = T_xP \oplus N_x$ for all x.

It turns out such a complement always exists – for example, we can pick a metric on TM (coming from an inner product which can be defined continuously, and then patching with a partition of unity), and then given $F \subseteq E$ a subbundle, we can put a metric on E and define $F^\perp$ to be the orthogonal complement with respect to that metric, so that $E = F \oplus F^\perp$. And the algebraic and geometric normal bundles are isomorphic because we're quotienting in both caess.

Lots of these constructions can be made functorial, but we won't lean too much into that perspective – one thing we can say is that taking the differential is an assignment (which we can make into a continuous map)

$$\text{Imm}(M, N) \xrightarrow{f \to df} \text{Mono}(TM, TN)$$

(here the monomorphisms are bundle maps that are injective). So if we have two manifolds, we may ask whether an immersion of manifolds exists (whether there are obstructions), and that's what we're moving towards in this class by studying things at the linear level for bundle maps. And we may also ask whether we can homotope from one immersion to another through immersions, or whether any immersion can be approximated by an embedding (though we need a topology to ask this question). The answer to this last question is no – we cannot approximate the figure-8 with embeddings.

We'll advertise two results that are beyond the scope of this class to prove, but we'll be able to start moving towards answering some of these questions stated. It turns out that if M is compact and strictly smaller in dimension than N, then the map $f \to df$ is a weak homotopy equivalence, so information about topology of immersions can be gathered from studying bundle maps. And Smale's theorem says that there is a homotopy between the usual embedding $\iota : S^2 \subseteq \mathbb{R}^3$ and $-\iota$ through immersions, which basically tells us that we can actually turn a sphere inside out using only immersions in a concrete way – we can watch videos online if we want to see this in action!

5 January 24, 2023

Some of the lecture content has been rearranged – next time we'll discuss transversality and Sard's theorem, and today we'll discuss the easy Whitney embedding theorem (the harder one coming later in the class), the tubular neighborhood theorem, Ehresmann's theorem, and group actions. All of this is basically discussing proper immersions and submersions, so we'll start by reviewing the definition there. Recall that an injective immersion is not necessarily an embedding, since we want a homeomorphism onto the image. If we add the requirement that the immersion is also **proper**, meaning that inverse images of compact sets are compact, then it becomes an embedding. (In particular, if we have a continuous map $f : M \to N$ and M is compact, then f is proper. And if $f : M \to N$ is proper and continuous, then it is in fact a closed map. So if it is a bijection – for example if f is an injective immersion, we have a bijection between M and the image in N – it will be a homeomorphism since the inverse is also continuous.)

Example 52

Embedding an open interval into a plane in the usual way is not a proper map, because the inverse image of some box containing one of the endpoints is a half-open interval. On the other hand, embedding the interval into the diameter of an open disk would be proper. Finally, the "figure-6 embedding" that we mentioned previously is not proper, because we can look at some closed box around the almost-intersection-point.

Theorem 53 (Easy Whitney embedding theorem)

Assume M is a smooth compact manifold without boundary (the proof does work with boundary too). Then M can be smoothly embedded into $\mathbb{R}^n$ for some n.

So even though we talked about abstract manifolds, compactness allows us to think of a manifold as a submanifold of $\mathbb{R}^n$. (And then we can worry about how large or small n can be later.)

Proof sketch. Let (U_i, ϕ_i) be coordinate charts on M. Since M is compact, we can cover it with finitely many open balls B_i such that $\overline{B}_i \subseteq U_i$ for all i (by picking balls around each point in M small enough to be contained in some chart, then using compactness). Then each B_i is a subset of M and is diffeomorphic (homeomorphic if we don't assume smooth) to a ball in $\mathbb{R}^m$.

Now for each i, choose a bump function β_i whose support is contained in U_i, such that β_i is identically 1 when restricted to the ball B_i. (So basically β_i is 0 outside of U_i and 1 in B_i and smoothly varies in between.) Specifically, we can define these on $\mathbb{R}^m$ and then map via the charts to the manifold. If we have k charts (U_i, ϕ_i) on M, then we can define a map $\phi : M \to \mathbb{R}^{m\times k}$ (really $\mathbb{R}^m$ times itself k times) given by

$$\phi(x) = (\beta_i(x)\phi_i(x))_{i=1}^k.$$

So basically we're using the charts but multiplying them down with the bump function. We can then check that ϕ is a proper injective immersion, hence an embedding. (Proper is automatic because M is compact, and for injectivity we use that β_i is 1 on the ball, so being the same means that ϕ_i must agree on the ball for both embeddings. And it suffices to check that we have an immersion on some B_i.) $\square$

We can also drop the assumption that M is compact:

Corollary 54 (Extension of easy Whitney embedding)

Let M be a second countable (smooth) manifold (with or without boundary). Then M can be properly (smoothly) embedded in $\mathbb{R}^n$ for some n.

The idea is to use a **smooth exhaustion function** $f : M \to \mathbb{R}$, meaning that the map is proper and bounded below (equivalently, the inverse image of $(-\infty, c]$, which is basically a "sub-level-set," is compact for all $c \in \mathbb{R}$). Such a function always exists when M is second countable. Then we can find a locally finite countable cover by charts with open balls U_i; if ρ_k is a partition of unity subordinate to that cover (which we can get from the bump functions via $\frac{\beta_k(x)}{\sum \beta_k(x)}$), then we can look at

$$f(x) = \sum_{k=1}^{\infty} k\rho_k(x).$$

Then again we can write down some map and show that it is a proper embedding. The point is that a connected paracompact manifold is actually second countable, so the only difference is that paracompact manifolds can actually have uncountably many connected components.

Theorem 55 (Tubular neighborhood theorem)

Let S be a compact submanifold of M. Then we can find a neighborhood of S so that M looks like the normal bundle. In other words, there is some **tubular neighborhood** ν of S (that is, there exists a diffeomorphism $N \to \nu$, where N is the total space of the normal bundle of S).

Proof. One proof uses the constant rank theorem. First, we'll assume S is a submanifold of $\mathbb{R}^m$, not necessarily compact but properly embedded. Looking at the tangent space T_xS at a point $x \in S$, we can consider the map $x \mapsto T_xS \subseteq \mathbb{R}^m$. This is called the **Gauss map** – we map x to the corresponding tangent subspace in the embedding into $\mathbb{R}^m$. Then taking the geometric normal bundle of S in $\mathbb{R}^m$ (where at each x we have $N_x = (T_xS)^\perp$), we can define a map from the normal bundle $\phi : N \to \mathbb{R}^m$ by the formula

$$\phi : (x, v) \mapsto x + v.$$

(This makes sense because $x \in S \subseteq \mathbb{R}^m$ and $v \in N_x \subseteq \mathbb{R}^m$.) Take the differential $d\phi$ restricted to the zero section of N (in other words, differentiating ϕ, then restricting to the part of the map where $v = 0$); we claim this map is full rank. Indeed, the restriction of this map to TS is an injection because S was properly embedded in $\mathbb{R}^m$, and then restriction to the fiber of the normal bundle N_x is also injective. (Basically we should think of this as being "block digonal" in the x and v parts.) So we have an immersion, and since the tangent space is a direct sum of the tangent space at S and the normal fiber. We claim this fact will be true in a neighborhood U of S in the normal bundle – that is, if U is sufficiently small, then ϕ is injective. We can prove this by contradiction. If S is compact, then ϕ restricted to a compact neighborhood is proper, so we do get an embedding. (It still works if S is not compact, but we have to do a bit more work.) But this yields the tubular neighborhood by choosing a small enough neighborhood, and the projection is $(x, v) \mapsto x$.

So now for the general case, because S is compact, we can restrict to a small neighborhood and assume M is second countable. Embed M in $\mathbb{R}^m$, so that we have S embedded into M, embedded into $\mathbb{R}^n$. Then find a tubular neighborhood of M by our first step, defining a map $\phi : N \to M$ (where N is the normal bundle of S in M) via $(x, v) \mapsto \pi(x + v)$. ☐

Theorem 56 (Ehresmann's theorem)

Any proper submersion is a locally trivial fibration (that is, a fiber bundle $\pi : E \to B$).

Remark 57. *This is false if we don't have a proper map: for example, the map $\mathbb{R}^n - \{0\} \to \mathbb{R}^n$ given by $x \mapsto x$ is a submersion but not locally trivial because of the hole.*

Proof. The idea here basically comes from the implicit function theorem – naively, define $E_x = \pi^{-1}(x)$, though we don't yet know that the E_xs are diffeomorphic. But because π is a submersion, by the implicit function theorem we know that there are local coordinates at any point in E_x in which we have a product. The trouble is that it's not clear how to patch the local coordinates together to get a product overall of the form $F \times U$.

So what we're going to do is show that the normal bundle of each fiber is trivial – specifically, it's the pullback of the tangent bundle at a point in the base. Then we can use the tubular neighborhood theorem to get the local trivialization we want. In more detail, since π is a submersion, each fiber $E_x = \pi^{-1}(x)$ is a submanifold. (Level sets turn out to be proper submanifolds, since inverse images of a point are compact.) The implicit function theorem then says that $T_pE_x = \ker(d\pi)_p$ – the tangent space is modeled by the kernel of the differential $d\pi_p : T_pE \to T_{\pi(p)}(B)$, where $\pi(p) = x$. If we do this at every point, we thus get a subbundle of TE which we can call $\text{Vert} = \ker(d\pi)$ (it's called a **vertical subbundle**). (One thing we must check is that it patches correctly on the transition functions, but it does.)

Now choose a complement to this subbundle at each point – the point is that we can find a complementary subbundle, called the "horizontal bundle," in some way (though it may not be canonical) such that $TE = \text{Vert} \oplus \text{Horiz}$. (In other words, $T_pE = \ker(d\pi)_p \oplus H_p$.) For example, we may take H_p to be the orthogonal complement of Vert with respect to some choice of metric on TE. But now $d\pi$ is a linear map $(d\pi)_p : T_pE \to T_{\pi(p)}B$, and if we restrict it to the horizontal space (meaning that we take $(d\pi)_p$ and restrict it to H_p, giving us a map $H_p \to T_{\pi(p)}(B)$), we get an isomorphism because it takes the vertical space to zero and thus this map passes to the quotient; then since π is a submersion and we have the same rank in both cases, we do get a bijection. Thus the restriction of the horizontal space to the fiber E_x gives us an isomorphism to the fixed vector space T_xB. The horizontal space is normal to the fibers so it is trivial, and then we get the product by the tubular neighborhood theorem. $\square$

Finally, we'll talk a bit about group actions: suppose G is a Lie group acting on a manifold M. We say the action is **smooth** if the map $G \times M \to M$ is smooth. We'll use without proof the following result:

Fact 58 (Slice theorem)

Assume that G acts smoothly on M with the action being free and proper (here free meaning that $G_x = \{g \in G : gx = x\}$ is just the identity for all x, and proper meaning that the map $(g, x) \mapsto g \cdot x$ is proper). Then M/G is a smooth manifold with $\pi : M \to M/G$ a submersion.

This is basically just about proving a bunch of intermediate results, using flows and the implicit function theorem. But we should be careful – without a proper action the quotient may not be Hausdorff.

Example 59

$\mathbb{CP}^n$ is the image of S^{2n+1} under a group action by S^1, and similarly the Grassmannian is also a similar quotient.

We can actually upgrade this conclusion to saying that $\pi : M \to M/G$ is a **principal G-bundle** – the important part here is the description in terms of local trivializations.:

Definition 60

Let G be a topological Lie group. A **principal G-bundle** is a fiber bundle $\pi : P \to B$ with extra properties: **(1)** the fiber is the group G, **(2)** the total space P has a free, fiberwise G-action $G \times P \to P$ sending $(g, x) \mapsto gx$, **(3)** the induced action on the fiber $\pi^{-1}(x)$ is transitive (meaning that it has just a single orbit and is really a copy of G), **(4)** there are local trivializations $\psi : \pi^{-1}(U) \to U \times G$ which are equivariant, meaning that $\psi(g \cdot x) = g(\psi(x))$ for all $g \in G$ and $x \in \pi^{-1}(U)$.

So the local trivialization respects the action of G.

Fact 61

If we look at the transition functions for such a principal G-bundle, usually we have $g_{\alpha\beta} : U_\alpha \cap U_\beta \mapsto \text{Homeo}(F)$, and in our case that maps factors through G by the last condition (fiberwise we are just multiplying by some $g \in G$). So $g_{\alpha\beta}$ maps from $U_\alpha \cap U_\beta$ to G and sends (g, y) to gy – in other words, the action of G on M is a group homomorphism from G to $\text{Homeo}(M)$.

We'll come back to use this in the future, so it's good to read up on it if we're not too familiar with these descriptions.

6 January 26, 2023

Today's topics are **transversality and Sard's theorem**, both of which we may have already seen in $\mathbb{R}^n$. The idea with transversality is to reinterpret the constant rank and regular value theorems in another way:

Definition 62

Two (smooth) submanifolds S_1, S_2 of M are **transverse** at an intersection point x, denoted $S_1 \pitchfork_x S_2$, if the tangent planes span the whole tangent space, meaning that $T_x S_1 + T_x S_2 = T_x M$. We say S_1 and S_2 are transverse if this holds at all intersection points.

(We don't need to have a direct sum in the sum above.)

Definition 63

Let $f : P \to M$ be a map between (smooth) manifolds, and let $S \subseteq M$ be a smooth manifold. We say that f is **transverse** to S, denoted $f \pitchfork S$, if $df(T_x P) + T_{f(x)} M = T_{f(x)} M$ for all $x \in P$ with $f(x) \in S$.

In other words, we again look at the two tangent planes, but one of them is coming from the image of the map f.

Example 64

A map f is transverse to a point if and only if p is a regular value of f (since the tangent space of a point is trivial). Also, S_1 is transverse to S_2 if and only if the inclusion incl_{S^1} is transverse to S_2, and so on, so the definitions are consistent. Additionally, $f \pitchfork S$ if and only if the graph of f is transverse to $P \times S$.

The regular value theorem says that the inverse image of a regular point is a submanifold, and now transversality allows us to generalize that result:

Proposition 65

Let M be a manifold and $f : P \to S$ be a map. If $f \pitchfork S$, then $f^{-1}(S)$ is a submanifold of the domain P. We already know the tangent bundle by the constant rank theorem, and the normal bundle is the pullback of the normal bundle of S, with $\operatorname{codim} f^{-1}(S) = \operatorname{codim}(S)$. So in particular for any submanifolds $S_1 \pitchfork S_2$, the intersection $S_1 \cap S_2$ is a submanifold with $T(S_1 \cap S_2) = TS_1 \cap TS_2$ (with $\dim(S_1 \cap S_2) = \dim(S_1) + \dim(S_2) - \dim(M)$).

The proof here is a formal manipulation – we're basically always using tricks of the sort in the example above to get to the implicit function theorem.

Proof sketch. The first statement is the most important, and then the statements about the normal bundle implies everything else. Checking that $f^{-1}(S)$ is a submanifold is a local statement (until we check that it's actually a homeomorphism onto the image). Recall that the inverse image of a point is a properly embedded submanifold, and we'll reduce to that case.

Without loss of generality (by choosing local coordinates) we can assume $M = S \times N$ is a product of S with its normal bundle (basically some $\mathbb{R}^n$). Then $S \hookrightarrow S \times N$ via the zero section is an embedding, and we can consider the projection $\pi : S \times N \to N$. We have $S = \pi^{-1}(0)$, and we must check that the composite map $\pi \circ f : P \to N$ has 0 as a regular value since $f^{-1}(s) = (\pi \circ f)^{-1}(0)$. But we have the assumption that $f \pitchfork S$, so $(\pi \circ f) \pitchfork 0$ and $d(\pi \circ f)$ is onto (because the tangent directions to f go to zero under the projection so we don't need to span those directions). Thus, at least locally (leading us to a local trivialization), the normal direction to $f^{-1}(s)$ is exactly to pullback of N. (This is what we discussed last time – the normal direction to the level sets under a submersion are exactly the pullback of some tangent vector in the image.) The same is true for transition functions (first pull back a local trivialization, and perform a change of trivializations). Since everything pulls back, we do have a submanifold. $\square$

Definition 66

Two maps f_1, f_2 into a manifold M are **transverse** denoted $f_1 \pitchfork f_2$, if and only if $f_1 \times f_2 \pitchfork \Delta$ (where Δ denotes the diagonal in $M \times M$). The **fiber product** of P_1 and P_2 along f_1, f_2 is $(f_1 \times f_2)^{-1}\Delta$.

In other words, we're considering the points $\{(x_1, x_2) \in P_1 \times P_2 : f_1(x_1) = f_2(x_2)\}$. And as a direct corollary, f_1 being transverse to f_2 means that the fiber product is a submanifold. This is a similar definition as for pullback bundles, and we can now calculate things like the normal bundle, the dimension of the fiber product, and so on. And we can read the lecture notes for more detail.

We can now turn to Sard's theorem – for Sard's theorem, Whitney embedding, and some other theorems, we need our manifolds to be **second countable** instead of just paracompact. We'll start with the version for smooth maps $f : \mathbb{R}^n \to \mathbb{R}^m$ and take it to maps on manifolds by working in charts. '

Theorem 67 (Sard)

Suppose we have a map $f : \mathbb{R}^n \to \mathbb{R}^m$ of class C^k, where $k > \max(0, n - m)$. Then the set of critical values of f is Borel measure zero in $\mathbb{R}^m$.

As stated, the result extends to finite-dimensional manifolds by using that second countability yields a countable family of charts, and the union of countably many sets of measure zero is also measure zero. But it also extends to Fredholm maps between separable infinite-dimensional **Banach manifolds** (defined just like for finite-dimensional manifolds, but instead of locally being $\mathbb{R}^n$ we have a separable Banach space) – then $n - m$ is the index of df, defined to be $\dim\ker(df) - \dim\operatorname{coker}(df)$. This goes under the name of **Sard-Smale**.

Remark 68. *The inequality in Sard's theorem is indeed sharp – Whitney showed an example of a C^1 map $\mathbb{R}^2 \to \mathbb{R}^1$ in which the critical locus (the set of critical points) is one-dimensional with image containing an arc γ. So the image contains an interval in $\mathbb{R}$. The idea is that the tangent plane is always horizontal, but the image manages to "climb."*

Definition 69

In the result above, a set $A \subseteq \mathbb{R}^n$ has **Borel measure zero** if for all $\varepsilon > 0$, there is some cover of A by countably many balls of total volume at most ε (under the Lebesgue measure).

We now start to see why secound countability was crucial – the countable union of Borel measure zero sets is itself measure zero. In particular, if we have a C^1 map $f : \mathbb{R}^n \to \mathbb{R}^n$, then the function takes measure-zero sets to measure-zero (since we have a bound on the "expansion size" on compact sets), and then for any $f : \mathbb{R}^n \to \mathbb{R}^m$ a C^1 map where $m > n$, the image of f is indeed measure zero because we can cover $\mathbb{R}^n$ by a very thin slab inside $\mathbb{R}^m$. (C^1 is crucial here – otherwise we could use the space-filling curve.)

Definition 70

A set $A \subseteq M$ is measure zero if A intersected with any chart is measure zero in $\mathbb{R}^n$ (or equivalently, for just a single chart intersecting an arbitrary point).

(This is indeed a well-defined notion by the previous proposition and because M is second countable.)

Example 71

A closed measure-zero set is **nowhere dense** (since the complement is open and dense).d

Dense sets behave well set theoretically except under countable intersections, and they in fact behave nicely under intersections too:

Theorem 72 (Baire's second category theorem)

A countable intersection of open, dense sets in M is dense in M if either M is a complete metric space or locally compact Hausdroff (this is where Sard-Smale comes into play). In particular, this holds in a finite-dimensional manifold M.

We say that a set A is **comeagre** (sometimes also a **Baire set**) if it contains a countable intersection of open and dense sets. We can then say that a point is **generic** in a manifold if it belongs to such a set. And taking countable intersections of such keeps us within this class of comeagre sets.

Theorem 73 (Sard's theorem, smooth version)

Let $f : M \to N$ be a smooth map between second countable manifolds. Then the locus of critical values of f are measure zero, and in particular (from the proof) the set of regular values are comeagre.

The proof will be skipped here (it's in the lecture notes), but it's nice – the idea to first look in the domain for the critical locus of f. Then cover that locus into pieces that are submanifolds, and use the fact that **maps into a larger dimension have image measure zero**. For that, to have a critical point of a map $M \to \mathbb{R}$ requires the differential to be onto, but that really means that it vanishes. So if the equation cuts tranversally, we will have a submanifold, and otherwise we have a locus where the first derivatives vanish. Then we can cover by level sets to reduce the derivative order by 1 and repeat. Finally, for high enough order

Fact 74

In particular, the results from before on manifolds still hold for smooth maps: for example, if we have a map $f : M \to N$ with $\dim(N) > \dim(M)$, then the image of f has measure zero.

Example 75

From algebra, we know that the generic polynomial of degree $k \geq 1$ in one variable has only simple zeros – we can now prove this using Sard's theorem.

Let M be the space of degree-k polynomials (which is a manifold by reading off coefficients). Consider pairs (x, p) where $x \in \mathbb{R}^n$ and p is a polynomial. The bad set is the points x that are a zero of p that are not simple, meaning that we want $P(x) = 0$ but $dP(x) = 0$. We just need to show that we can indeed hit any values, so this map is transverse to 0 and thus the bad locus is a submanifold of P; look at the projection to P and apply Sard's theorem to get the result.

Recall that Whitney's embedding theorem tells us that we can always embed in $\mathbb{R}^n$ for some n, and there's a strengthening of it which allows us to say some things about how big n can be:

Proposition 76

Let $M \hookrightarrow \mathbb{R}^k$ be a (smooth) embedding. Then the generic projection of M onto a hyperplane $\mathbb{R}^{k-1} \subseteq \mathbb{R}^n$ is immersed if $k > 2\dim(M)$ and embedded if $k > 2\dim(M) + 1$.

Proof sketch. The point is to prove that the "bad equations cut transversally" – this is always the idea with how to solve these questions. To define the projection, we must choose a hyperplane in $\mathbb{R}^k$ along with a normal vector v to it. (In fact, we can fix the plane and move the vector v.) The bad locus includes the points where the projection is not injective (meaning that $x_1 \neq x_2$ and with the line along v containing both), as well as those where the projection is not an immersion (meaning that the line at x is tangent to M at x). We just need to phrase this into equations and show that they cut transversally, and we'll talk a bit more about this next time. □

7 January 31, 2023

We'll discuss Whitney's embedding and approximation theorems today – last time, we considered an embedding $M \hookrightarrow \mathbb{R}^n$ and fixed some hyperplane $\mathbb{R}^{n-1} \subseteq \mathbb{R}^n$. Then the generic projection of M onto $\mathbb{R}^{n-1}$ is an immersion when $n > 2\dim(M)$ and injective if $n > 2\dim(M) + 1$. Combining these (plus some properness) we get an embedding if M is compact.

Proof. We may parameterize projections onto a fixed hyperplane by choosing some direction v in $\mathbb{R}^n$ not in the hyperplane $\mathbb{R}^{n-1}$. In other words, we are choosing some element of $\mathbb{RP}^{n-1} \setminus \mathbb{RP}^{n-2}$, but we're going to work with the upper half-sphere of $\mathbb{R}^n$ instead (that is, making sure our vector v always has last coordinate positive). Then we have $\mathbb{R}^n = \mathbb{R}^{n-1} \oplus \mathbb{R}v$, and for any v we get a projection $\pi_v : \mathbb{R}^n \to \mathbb{R}^{n-1}$ by only preserving the first part of the direct sum. We want to describe the "bad locus" of unit vectors v.

First of all, $\pi_V(M)$ is not an immersion if the line $\mathbb{R}v$ is tangent to M. To write that as a zero locus, we consider the map $(TM \setminus 0_M) \to S^{n-1}$ (since $TM \setminus 0_M$ all give us a well-defined direction of tangency) sending $w \mapsto \frac{w}{|w|}$ (so we normalize it to some unit vector). We can check that this is a smooth map (we have to do it in coordinates, but clearly it is), and now if $\dim(TM) > n-1$ then Sard's theorem tells us the image of this map is measure zero, so that the generic unit vector is not in the image. But this isn't quite good enough, since $\dim(TM) = 2\dim(M)$ and thus we need to cut out one more dimension. The last step is to restrict to unit vectors in TM as well (this is sometimes called the **sphere bundle** $S(TM) = \{(x, v) : v \in T_xM, ||v|| = 1\}$), and now we get a map $S(TM) \to S^{n-1}$ wish sends w to w. And we check that that $S(TM)$ is a manifold because the equation cuts transversally to TM, and $\dim S(TM) = 2\dim(M) - 1$. So indeed the condition for Sard's theorem is satisfied if $n > 2\dim(M)$.

The second part is very similar: we'll think about the condition of being "not injective" as not having two distinct points $x_1, x_2 \in M$ such that the line spanned by x_1, x_2 is in the direction of v. So the bad locus is the set of lines spanned by two points in M: that is, it is the image of the map $(M \times M) \setminus \Delta \to S^{n-1}$ (where Δ denotes the diagonal, since the points need to be distinct) sending $(x_1, x_2) \mapsto \frac{x_1 - x_2}{|x_1 - x_2|}$. I($M \times M \setminus \Delta$ is indeed a manifold because it's an open subset of a manifold.) Again we must show that this is a smooth map, and then a generic point will not be in the image by Sard's theorem (that is, the image has measure zero) if the dimension of the domain $2\dim(M)$ is strictly less than $n - 1$. This is exactly what we wanted to show. □

The point is then that we get a strengthening of the Whitney embedding theorem:

Theorem 77 (Medium Whitney embedding theorem)
Any smooth compact n-dimensional manifold M can be embedded in $\mathbb{R}^{2n+1}$ and immersed in $\mathbb{R}^{2n}$.

Proof. This follows from the previous result and the existence of **some** embedding $M \hookrightarrow \mathbb{R}^N$ for finite N (by easy Whitney). Specifically, by induction we can keep projecting to a hyperplane until the inequalities stop holding, and we use the fact that the map is **proper** and an injective immersion. □

Without compactness, this result also extends like the easy version:

Theorem 78 (Extension of medium Whitney embedding)
Any second countable smooth n-dimensional manifold M (with or without boundary) can be properly embedded in $\mathbb{R}^{2n+1}$ and immersed in $\mathbb{R}^{2n}$.

Theorem 79 (Whitney approximation theorem)
Let M be an n-dimensional compact manifold. If $k \geq 2n + 1$, then any smooth map $f : M \to \mathbb{R}^k$ can be approximated by an embedding in the C^0 measure – that is, for all $\varepsilon > 0$, there is an embedding $g : M \to \mathbb{R}^k$ such that $||f(m) - g(m)||_{C^0} = \sup_{x \in M} |f(x) - g(x)| < \varepsilon$.

Proof. Let $h : M \to \mathbb{R}^N$ be an embedding of M for some N, and now consider the map $(f, h) : M \to \mathbb{R}^{k+N}$ mapping $m \mapsto (f(m), h(m))$. This is an embedding, because it's an embedding in one of the factors. Again by induction, the generic projection onto a hyperplane $M \to \mathbb{R}^{k+N-1}$ is an embedding if $2\dim(M) < k + N - 1$, and then we can repeat until we get down up until $k \geq 2n + 1$. (More explicitly, by induction, if $M \hookrightarrow \mathbb{R}^{k+s}$ is an embedding, then the generic linear projection into $\mathbb{R}^{k+s-1}$ is an embedding if $k + s > 2\dim(M) + 1$, and we can do this until $s = 2\dim(M) - k + 2$. We then get a map $x \mapsto (f(x), \text{stuff})$ where the generic projection is an embedding, and in particular for any $\varepsilon > 0$ we can find such an embedding so that the projection of $(f(x), \text{stuff})$ down to the first coordinate is within the right distance from f. □

Again, this result can be upgraded:

Theorem 80 (Extension of Whitney approximation theorem)
Let M be a compact n-dimensional manifold. Then for any $r \geq 1$, the set of embeddings $\text{Emb}(M, \mathbb{R}^k)$ is dense in $C^r(M, \mathbb{R}^k)$ if $k \geq 2n + 1$, and the set of immersions $\text{Imm}(M, \mathbb{R}^k)$ is dense in $C^r(M, \mathbb{R}^k)$ if $k \geq 2n$.

Notice that nowhere in the previous proof did we actually use smoothness – the version of Sard's theorem that we actually needed was that a C^1 map has image measure zero. And denseness in C^r means that we can approximate any map within ε C^r distance (which is the sup of the distances across all derivatives up to order r).

Fact 81
There is in fact a strong Whitney's embedding theorem, which is stated as follows: if M is a compact n-dimensional manifold for $n > 0$, then M can be embedded in $\mathbb{R}^{2n}$ and immersed in $\mathbb{R}^{2n-1}$.

This proof requires a **surgery** and cannot be converted to an approximation theorem – basically the easy fact we can prove is that we can immerse M in $\mathbb{R}^{2n}$ with at most transverse double points (so we can avoid having three points on the same line, and we can make sure that when two points **are** on the same line then the projection of their tangent spaces span everything), and then we need to do a trick to remove intersections.

On the other hand, this bound is indeed optimal:

Example 82
The 2-dimensional manifold $\mathbb{RP}^2$ does not embed in $\mathbb{R}^3$ (if it were to embed, look at the unit normal vectors and show that $\mathbb{RP}^2$ would need to be oriented, which is not true); alternatively, any connected hypersurface in S^3 ($\mathbb{R}^3$ plus a point) must separate it into two pieces. And similarly, $\mathbb{RP}^2$ does not immerse into $\mathbb{R}^2$ (our homework). But $\mathbb{RP}^2$ can in fact be immersed in $\mathbb{R}^3$ with at most double and a single triple point (Boy's surface).

We'll now mention a few important facts about topologies on function spaces – Hirsch's book is a good reference for this, and a lot of this involves using Sard-Smale on function spaces.

Definition 83
Let $C^k(M, N)$ be the class of functions of class C^k (with all partial derivatives up to degree k continuous). We can place various topologies on $C^k(M, N)$, such as the topology of uniform convergence in C^k on all compact sets (since we're not assuming M is compact).

This is motivated by the "compact open topology" of continuous functions $C^0(M, N)$, in which the basis is indexed by compact sets $K \in M$ and open $U \in N$, where the corresponding open set indexed by K and U is the set $\{f \in C(M, N) : f(K) \subset U\}$. So if N is a metric space, a sequence of functions converging in this topology uniformly converges on compact sets.

Example 84
For any compact M, the space $C^k(M, \mathbb{R}^p)$ is a Banach space under the C^k norm (it is a vector space because we can add functions) $||f - g||_{C^k} = \max_{\alpha, |\alpha| \leq k} \sup_{x \in M} |\partial^\alpha f - \partial^\alpha g|$.

We're next going to discuss **smooth bundles and smooth sections**, switching directions to vector fields and differential forms from a top-down perspective.

Definition 85

A bundle $\pi : E \to B$ is **smooth** if E, B are smooth manifolds and the projection is a smooth map.

For example, if $\pi : E \to B$ is a smooth bundle, then π is actually a submersion – the converse is true by Ehresmann's theorem if we have a proper map.

Definition 86

Suppose $\pi : E \to M$ is a smooth **vector** bundle. A **smooth section** of $\pi : E \to M$ is a smooth map $S : M \to E$ such that $\pi \circ s = \text{id}$. Then (abusing notation) let $\Gamma(E) = \Gamma(M; E)$ be the set of smooth sections of E over M.

(The idea is that $s_x \in E_x$ smoothly varies in x.) Then $\Gamma(E)$ is a vector space and in fact a $C^\infty(M)$-module, since we can add two smooth sections fiberwise (the map $(\Gamma(E) \times \Gamma(E) \to \Gamma(E)$ sends (s_1, s_2) to $s_1 + s_2$, sending $(s_1 + s_2)(x) = s_1(x) + s_2(x))$, and we can multiply by smooth functions (the map $C^\infty(M) \times \Gamma(E) \to \Gamma(E)$ sends (f, s) to fs, with $(f \cdot s)(x) = f(x)s(x) \in E_x$).

Example 87

If $E = \varepsilon$ is the trivial line bundle, then $\Gamma(E) = C^\infty(M)$ is just the ordinary set of smooth functions $f : M \to \mathbb{R}$. We can again give $\Gamma(E)$ various topologies – we often avoid $\mathbb{C}^\infty$ because it's a Frechet space instead of a Banach space, we don't have the inverse function theorem, and so on – but working with C^k often works well.

Vector fields are then sections of TM, and differential forms are sections of $\Lambda^k(TM)$:

Definition 88

Let M be a smooth manifold. A (smooth) **vector field** on M is a smooth section V of TM, meaning that for each $x \in M$ we have some $V_x \in T_xM$ in a way that varies smoothly. We will let $\chi(M) = \Gamma(TM)$ denote the space of smooth vector fields on M.

We can add them and multiply by smooth functions, but what is most important is to look at the "flow" of a vector field, in which we look at integral curves (which are curves which are everywhere tangent to our vector field). In other words, if V is a smooth vector field, we want to find γ such that $\frac{d\gamma}{dt} = V(\gamma(t))$ for all $t \in \mathbb{R}$, and we give some initial condition $\gamma(0) = x$ for some fixed point $x \in M$. This is essentially an ODE because we can write it in coordinates, and by the theorem of existence and uniqueness of solutions (and dependence on initial conditions) there is some **maximal** interval $(-a_x, b_x)$ on which we have a solution $\gamma : (-a_x, b_x) \to M$. We'll assume that M is compact, so that a solution will exist for all t by a covering argument (alternatively, we can show the maximal interval is both open and closed). We then define the **flow**

$$\gamma_t(x) = \gamma(t) \text{ for the unique solution of } \left\{\frac{d\gamma}{dt} = V(\gamma(t)), \gamma(0) = x\right\}.$$

We'll show next time that γ_t is in fact a diffeomorphism for all $t \in \mathbb{R}$, basically because existence and uniqueness implies that $\phi_t \circ \phi_s = \phi_{t+s}$ for all t, s (finding a solution for time t, then going for another time s, is the same as just running the original flow for time $t + s$) and ϕ_0 is the identity map. So therefore a flow on M can be corresponded to a path of diffeomorphisms starting at the identity, and thus we have $\chi(M)$ "=" $T_{\text{id}}\text{Diff}(M)$. (Indeed, the derivative of ϕ_t at $t = 0$ is the same as differentiating γ at time 0, but this is exactly $V(x)$.) And finally we can view this as a map $\Phi : R \times M \to M$ sending $(t, x) \mapsto \Phi_t(x)$, and this is in fact smooth by ODE theory.

8 February 2, 2023

Today will cover **vector fields and differential forms**, discussing flows, the Lie bracket and derivative, differential forms, and de Rham cohomology. Recall from last time that if we have a (smooth) vector field V (meaning that at each point we have a vector in the tangent space, varying smoothly), then we get a flow by finding integral curves of the vector field, meaning that we look for paths γ solving $\frac{d\gamma}{dt}(t) = V(\gamma(t))$ (that is, the tangent vector points in the direction and magnitude of V) with some initial condition $\gamma(0) = x$. Existence and uniqueness were discussed last time, but sometimes we can only solve the equation on a small interval $\gamma : (-\varepsilon, \varepsilon) \to M$, motivating the following definition:

Definition 89

A vector field is **complete** (really, its flow is complete) if the differential equation above has a solution defined on all of $\mathbb{R}$ for any initial conditions.

Example 90

If V is compactly supported, then it is complete (because it is zero away from a compact set, so we can always extend it further beyond any finite interval).

From now assume V is complete, and let $\gamma_x : \mathbb{R} \to M$ be the unique solution of our ODE with $\gamma(0) = x$. We'll now adjust our notation so that we can think about these flows in various ways: write $\phi_t(x) = \gamma_x(t) = \Phi(t, x)$, and notice that $\Phi : \mathbb{R} \times M \to M$ is a smooth function (it's smooth in t, and it's also smooth in x by "smooth dependence on initial conditions"). Thus $\phi_t : M \to M$ is a smooth map, and it has the property that $\phi_t \circ \phi_s = \phi_{t+s}$ for all $t, s \in \mathbb{R}$ (by uniqueness). So because $\phi_0 = \text{id}$ and ϕ_{-t} is the inverse of ϕ_t, we see that we have a smoothly varying family of diffeomorphisms $M \to M$. (We **can** think of this as a smooth map between manifolds if we give $\text{Diff}(M)$ a manifold structure, but here we're just thinking of smoothness in the sense mentioned before). All of these different notations are all called "flow," but this last one $\phi : \mathbb{R} \to \text{Diff}(M)$ is both a smooth path and group homomorphism.

We can thus look at the "derivatives" of the flow (the only issue being that we haven't given a smooth topology on $\text{Diff}(M)$ yet). We claim that if we look at the infinitesimal derivative of ϕ_t when at $\phi_0 = \text{id}$ at the point x, then that's just going to be

$$\boxed{V(x) = \left.\frac{d}{dt}\right|_{t=0} \gamma_x(t) = \left.\frac{d}{dt}\right|_{t=0} \phi_t(x) = (d\Phi)_{(0,x)}\left(\frac{\partial}{\partial t}\right)}$$

(this is just changing the notation and using the definition of our ODE – the last one is saying that we're taking the partial derivative in the t-direction at our initial condition).

Theorem 91

Let M be a smooth manifold. Then there is a one-to-one correspondence between complete smooth vector fields on M and smooth group homomorphisms $\phi : \mathbb{R} \to \text{Diff}(M)$ (also known as one-parameter subgroups of $\text{Diff}(M)$).

(Recall that if M is compact, then we can drop the completeness requirement.)

Proof sketch. We've already seen how to go from the vector field to the flow from our above definitions. And for the other direction, if we're given a path or group homomorphism $\phi : \mathbb{R} \to \text{Diff}$, we can define V from our boxed formula above – we then check that the flow of V is exactly ϕ by uniqueness. $\square$

Note that if we wanted to differentiate with respect to ϕ_t instead of ϕ_0, we get

$$V(\phi_t(x)) = \frac{d}{dt}\phi_t(x) = (d\phi)_{(t,x)}\frac{\partial}{\partial t}.$$

And now if M is compact, we can define the space $\chi(M)$ of smooth functions f on M (that is, smooth sections of TM), and we get a one-to one correspondence $X(M) \leftrightarrow T_{\text{id}}\text{Diff}(M)$, since the tangent space should be instantaneous velocity of any path. But if M is not compact and V is some vector field, there may not be any $\varepsilon > 0$ such that all flows are defined on $(-\varepsilon, \varepsilon)$ (meaning that we get no map $\phi : (-\varepsilon, \varepsilon) \times M \to M$. Indeed, if $M = \mathbb{R} \setminus 0$ and our vector field is just $X = \frac{\partial}{\partial t}$, then we must have $\phi_t(x) = x + t$ and thus no ε works because we will always have arbitily close elements to 0.

Example 92
The S^1 action on M (take a point and rotate it) can also be thought of as a 1-periodic flow, specifically given by $\phi(e^{it}, x) = e^{it}x$. (And alternatively we can be thinking of this as an action $S^1 \to \text{Diff}(M)$.)

Definition 93
Let V be a vector field on M, A point $x \in M$ is a **singular point** of V if $V(x) = 0$.

In particular, this means $\phi_t(x) = x$ for all t and we have a fixed point of the flow. (We can think of "flow" as coming from "flow of water," so having $V(x) = 0$ means we do not have any flow there.) And another way to say this is that we're in the zero locus of V, or that we're looking at the intersection of the "graph" of V in TM with the zero section. And the point is that if x is a nonsingular point and $V(x) \neq 0$, then we can change coordinates so that the flow is basically a product, meaning that we can find a **local slice** S (a codimension-1 manifold) which is transverse to V_x at x. We then get an embedding $\gamma_y : (-\varepsilon, \varepsilon) \to M$ for sufficiently small ε and y sufficiently "close to- V" (since transversality is an open condition), so we have local coordinates $(y \in S, t)$ and we in fact get an embedding in top dimension. But if we have a singular point, then the local models can be very wild.

We'll return to all of this when we do Poincaré-Hopf and intersection theory, but for now we'll move on to some different definitions.

Remark 94. *Recall that a vector field (which we will call X) on M can be thought of as a* **derivation** *over $C^\infty(M)$, meaning we get a map $C^\infty(M) \to C^\infty(M)$ sending f to Xf.*

Specifically, we have $Xf = df(X)$, meaning that for any $p \in M$, we map to $(df)_p X_p \in \mathbb{R}$ – that is, we consider the map $df_p : T_pM \to T_{f(p)}\mathbb{R} \cong \mathbb{R}$, so we're basically differentiating in the direction of the vector field. This is a $\mathbb{R}$-linear map, and it satisfies Liebniz's rule $X(fg) = (Xf)g = Xf(Xg)$ for all $f, g \in C^\infty(M)$ (this is just the product rule rephrased). Any vector field gives us such a function, and in fact any derivation satisfying these properties comes from a vector field. So once we believe this,

Definition 95
The **Lie bracket** of two vector fields X, Y is denoted $[X, Y] = XY - YX$ and defined by setting $[X, Y]f = X(Yf) - Y(Xf)$ for all $f \in C^\infty(M)$.

We must check that the right-hand side is indeed well-defined and a derivation ($\mathbb{R}$-linear in both X and Y and satisfying the Liebniz rule), but the Lie bracket of two vector fields is a vector field. The idea is that we can associate to any Lie group a Lie algebra with a Lie bracket, and that's what this is doing here – the point is to explain "how badly X and Y fail to commute." The main properties to keep in mind are that this is indeed $\mathbb{R}$-linear in both X and Y, skew-symmetric (so that $[X, Y] = -[Y, X]$), and that we satisfy the **Jacobi identity**

$$0 = [[X, Y], Z] + [[Y, Z], X] + [[Z, X], Y].$$

So the space of vector fields is now made into a Lie algebra, and we should keep it as a note in our mind for differential geometry.

Remark 96. *For any Lie group G, we get its* **Lie algebra** $\mathfrak{g} = T_e G$*, where we regard this as the space of* **left-invariant** *vector fields on G. (Since we have a group action, we can then move the tangent space to any other point in G.) We can thus show that any $v \in T_e G$ extends uniquely to some left-invariant vector field V satisfying $(dL_g)(V) = V$ (where $L_g : G \to G$ is the multiplication-by-g map), and we can also show that the Lie bracket of two left-invariant vector fields is itself left-invariant by the formula for the Lie bracket.*

We also have a notion of a **Lie derivative** in the direction of a vector field X, denoted $\mathcal{L}_X$ – we should think of this as the **infinitesimal change along the flow**. So if we know how to differentiate functions in some direction (df applied to X), then derivative of a vector field can be made clear using $\mathbb{R}^n$, but it doesn't change well under charts so it's not well-defined in general. What we actually want to say is that the Lie derivative of a vector field Y in the direction of the vector field X is itself a vector field: since the flow ϕ_t is a diffeomorphism, we can set

$$\mathcal{L}_X Y = \left.\frac{d}{dt}\right|_{t=0} (\phi_t^{-1})^*(Y)$$

(the -1 means we're putting the inverse, or going negative in time, and so we have the pullback of Y regarded as a section by the diffeomorphism ϕ_t^{-1}). So basically we let our vector at Y flow along X and we look at how it changes at time 0. But it turns out $\mathcal{L}_X Y = [X, Y]$, so we haven't actually done anything different here – in particular, we see that

$$\mathcal{L}_{[X,Y]} = [\mathcal{L}_X, \mathcal{L}_Y] = \mathcal{L}_X \mathcal{L}_Y - \mathcal{L}_Y \mathcal{L}_X,$$

and we have a Lie algebra homomorphism $X \mapsto \mathcal{L}_X$. And the key thing to keep in mind is that the Lie bracket is zero if and only if the flows of X and Y commute, which is the same as saying that Y is preserved by the flow of X.

That's all we'll say about vector fields for now, and we'll very briefly start with the linear algebra of differential forms. We'll let V be a vector space over $\mathbb{R}$ or $\mathbb{C}$ (so that we have a manifold) and we consider the **tensor algebra**

$$T(V) = \bigoplus_{k=0}^{\infty} V^{\otimes k},$$

in which an element of $T(V)$ is a linear combination of pure tensors $\sum x_1 \otimes \cdots \otimes x_k$. We then get an algebra with the product $(x, y) \mapsto x \times y$, and if we look only at the skew-symmetric part, we get

$$\Lambda^* V = TV/\sim, \quad x \otimes y \sim -y \otimes x,$$

where $\Lambda^k V = V^{\otimes k}/\sim$. We then get a wedge product on the quotient which we denote $x \wedge y$ (projecting from the tensor product). Then the dual of $((V^{\otimes})^n)$ is $\mathrm{Hom}(V \otimes \mathbb{R}, \mathbb{R})$, and its elements are of the form $\omega : V \times \cdots \times V \to \mathbb{R}$ (where the product has k terms) which is multilinear and skew-symmetric. We then call this the alteranting form, and we denote it $\boxed{A^k V = \Lambda^k(V^*) = (\Lambda^k V)^*}$. Next time, we'll put this on a manifold and have it smoothly vary with V the tangent space of the manifold.

9 February 7, 2023

We'll continue our discussion of differential forms and de Rham cohomology – we won't go through all the formulas, but we'll go through the main properties and talk about the de Rham isomorphism between de Rham cohomology and singular cohomology.

Recall that for a vector space V, an **alternating k-form** is a map $\alpha : V^k \to \mathbb{R}$ which is multilinear and skew-symmetric. In particular, the space of all such forms $A^k(V)$ can be thought of as either $\Lambda^k(V^*)$ or $(\Lambda^k V)^*$. So now if we let M be a smooth manifold, a **differential form** is a smoothly varying family $x \mapsto \alpha_x$ of alternating forms $\alpha_x : (T_x M)^k \to \mathbb{R}$. Here's the precise definition – the idea is as always to understand it for an open set of $\mathbb{R}^n$ first and then move it to manifolds using charts (though we can also think about TM as a bundle and wanting the map to vary smoothly in both x and $v \in T_x M$):

Definition 97

Let $U \subseteq \mathbb{R}^n$ be an open subset. A k-form is a smooth map $\alpha : U \to A^k(\mathbb{R}^n)$ (since $T_x U$ is canonically $\mathbb{R}^n$ at each point).

In particular, a 0-form means that at each x we map from the zero vector space to $\mathbb{R}$, so we just have a function $f : U \to \mathbb{R}$. Then df is a 1-form – normally we think of it as a map between the tangent bundles, but we can think of it instead as a map $x \mapsto df_x$, where $df_x : T_x U \to T_{f(x)}\mathbb{R}$ is regarded as a linear map $T_x U \to \mathbb{R}$. So if $(x_1, \cdots, x_n)$ are coordinates on $\mathbb{R}^n$, then we have the 1-forms $dx_1, \cdots, dx_n$. In these coordinates, a k-form can then be written as a sum over multi-indices

$$\alpha = \sum_I a_I dx_I \implies \alpha(x) = \sum_I a_I(x) dx_I,$$

where each $I = (i_1, \cdots, i_k)$ is a multi-index $1 \le i_1 < \cdots < i_k \le n$, each $a_I : U \to \mathbb{R}$ is a (smooth – still everything is smooth) function, and where $dx_I = dx_{i_1} \wedge \cdots \wedge dx_{i_n}$ (we often drop the wedges). So this extends to manifolds via charts, meaning that any k-form on M looks like $\sum_I a_I dx_I$ in local coordinates $(x_1, \cdots, x_n)$ on M. Then a 0-form on a manifold is still a smooth function $f : M \to \mathbb{R}$, and df is still a 1-form.

Definition 98

Let M be an n-dimensional manifold. A **volume form** on M is an n-form ω such that in local coordinates, we have $\omega(x) = a(x) dx_1 \cdots dx_n$ with the function $a(x) \neq 0$ for all x. Let $\Omega^k M$ denote the space of smooth k-forms on M.

The idea is that a volume form is a nowhere-vanishing top-dimensional form, since skew-symmetry means everything vanishes once we get past n-forms (in fact $\dim \Lambda^k V = \binom{n}{k}$ for a vector space V). We may think of $\Omega^k M$ as the space of sections of $\Lambda^k(TM)^*$ or as $\Lambda^k(T^*M)$ (here T^*M is also called the **cotangent bundle**).

Definition 99

The **differential** d is a map $d : \Omega^k(M) \to \Omega^{k+1}(M)$ defined in coordinates via

$$d\left(\sum_I a_I dx_I\right) = \sum_I da_I \wedge dx_I.$$

In particular, da_I is a 1-form and dx_I is a k-form, so this does yield a $(k+1)$-form. It's also well-defined independent of choices, and for a 0-form the map $d : \Omega^0(M) \to \Omega^1(M)$ takes in a function f and yields the usual df. By this definition, the map is $\mathbb{R}$-linear, and importantly $d^2 = 0$, since for any function $f : M \to \mathbb{R}$ we have $d^2 f = 0$ because second derivatives commute. Thus we get a de Rham complex

$$0 \to \Omega^0(M) \xrightarrow{d} \Omega^1(M) \to \cdots \xrightarrow{d} \Omega^k(M) \xrightarrow{d} \Omega^{k+1}(M) \xrightarrow{d} \cdots$$

which is a chain complex (meaning $d^2 = 0$), and thus we can take its homology, called its **de Rham cohomology** (because the index goes up)

$$H^k_{dR}(M) = \ker\left(\Omega^k(M) \xrightarrow{d} \Omega^{k+1}(M)\right) / \operatorname{im}\left(\Omega^{k-1}(M) \xrightarrow{d} \Omega^k(M)\right).$$

Here, we say that we are taking the set of closed k-forms, modding out by the exact k-forms:

Definition 100

A k-form α is **closed** if $d\alpha = 0$ and **exact** if $\alpha = d\beta$ for some β.

We'll now talk about the pullback (which will turn out to be a natural functor) – given a smooth manifold we get a space of differential forms, and now given a smooth map $f : M \to N$ between smooth manifolds we get a pullback map $f^* : \Omega^k(N) \to \Omega^k(M)$ given by

$$(f^*\alpha)_x(v_1, \cdots, v_k) = \alpha_{f(x)}(df_x(v_1), \cdots, df_x(v_k))$$

for any $x \in M$ and $v_1, \cdots, v_k \in T_x M$. We can then check that $d \circ f^* = f^* \circ d$, meaning that $d(f^*\alpha) = f^*(d\alpha)$ – in fact the formula for d was cooked up so that this holds – and thus f^* is actually a **chain map**, thus descending to the homology of the complex. In other words, f^* is also a map on kth de Rham cohomology. Additionally, if we have a composition of maps $M \xrightarrow{f} N \xrightarrow{g} P$, then $(g \circ f)^* = f^* \circ g^*$. In other words, the map $M \to \Omega^\bullet(M)$ or $M \to H^\bullet_{dR}(M)$ sending f to f^* is a contravariant functor from the category of manifolds **Man** with smooth maps to the category of chain complexes with chain maps, and also a functor to the category of vector spaces ($H^k_{dR}(M)$ is a vector space) with the corresponding morphisms.

We're now ready to discuss the de Rham isomorphism, which will be defined on the chain complex level. Suppose $\cdots \xrightarrow{\partial} C_k(M) \xrightarrow{\partial} C_{k-1}(M) \xrightarrow{\partial} \cdots$ be the singular chain complex, where $C_k(M)$ is the space of k-dimensional singular chains (linear combinations of simplices) on M and ∂ is the boundary map. The idea is that we will now define an integration map $I : \Omega^k(M) \to C^k(M; \mathbb{R}) = \operatorname{Hom}(C_k(M); \mathbb{R})$ by defining the image of the morphism for each (generator) simplex σ:

$$I(\alpha)(\sigma) = \int_\sigma \alpha$$

for all $\alpha \in \Omega^k(M)$ and each $\sigma : \Delta_k \to M$. (Here the integral $\int_\sigma \alpha$ is defined via pullback: we have $\int_\sigma \alpha = \int_{\Delta^k} \sigma^*\alpha$, where Δ^k is the standard simplex in $\mathbb{R}^{k+1}$, so integration does make sense.) Then by Stokes' theorem on $\mathbb{R}^n$, we see that I is a chain map – that is, replacing α with $d\alpha$ means we're just integrating on the boundary of the corresponding simplex, and notationally this means $I \circ \delta = d \circ I$. Thus integration induces a map $I_* : H^k_{dR}(M) \to H^k(M, \mathbb{R})$ in cohomology.

Theorem 101 (de Rham isomorphism)

The chain complex map $I : \Omega^\bullet(M) \to C^\bullet(M; \mathbb{R})$ (basically $\Omega^k(M) \to C^k(M; \mathbb{R})$ and then letting k vary) is a chain homotopy, and thus it induces a canonical (in fact natural in the sense of functors) isomorphism between de Rham cohomology $H^k_{dR}(M)$ and singular cohomology with $\mathbb{R}$-coefficients $H^k(M, \mathbb{R})$.

(It'll turn out to preserve the ring structure at the cohomology level, but that's for us to discuss next time.)

Main idea of proof. We'll focus on just the isomorphism part – we wish to show that both de Rham cohomology and singular cohomology have the same properties and that those properties uniquely determine them: **(1)** homotopy invariance, **(2)** the Poincaré lemma, a special case of homotopy invariance but also for special M in fact an explicit calculation for the isomorphism, and **(3)** the preservation of the Mayer-Vietoris sequence. Restating **(1)**, we wish to show that smoothly homotopic maps $f_0, f_1 : M \to N$ induce chain homotopic maps f_i^* in the de Rham complex $\Omega^\bullet$ and thus induce equal maps in the homology $H^\bullet_{dR}$. And one way to state **(2)** is as follows: let $U \subseteq \mathbb{R}^n$ be an open ball (or an open convex subset). Then any closed k-form (for $k > 0$) is exact, and specifically $H^k_{dR}(U) = 0$ for all $k > 0$ and $H^0_{dR}(U) = \mathbb{R}$.

For a sketch of the proof of homotopy invariance (which yields (2) as a special case), we claim that by naturality it suffices to prove the result for the sections $s_i : M \to M \times \mathbb{R}$ (for $i = 0, 1$) defined via $s_i(x) = (x, i)$ (which are clearly homotopic). Indeed, if f_i are homotopic, then we have a map $F : M \times I \to N$ and then $f_i = F \circ s_i$. But then if $s_0^* = s_1^*$, then by naturality we have $(F \circ s_0)^* = (F \circ s_1)^*$. And s_0^* and s_1^* are equal in de Rham, because both of them are inverses of the projection $\pi : M \times \mathbb{R} \to M$. To show that π^* and s^* are then inverses in de Rham, take $s = s_0$. Since s is a map $M \to M \times \mathbb{R}$, s^* is a map $H^k_{dR}(M \times \mathbb{R}) \to H^k(M)$ and π^* maps backwards between those groups. We know that $\pi \circ s = \text{id}$, so $s^* \circ \pi^*$ is exactly the identity by naturality. But for the other direction, $\pi^* \circ s^*$ is **chain homotopic** to the identity by construction: we will construct $K : \Omega^k(M \times \mathbb{R}) \to \Omega^{k-1}(M \times \mathbb{R})$ so that $\text{id} - \pi^* \circ s^* = \pm(K \circ d + d \circ K)$ with signs appropriately chosen. Specifically, we want to integrate dt: if we take $(x, t) \in M \times \mathbb{R}$, we will map

$$a_I dx_I \xrightarrow{K} 0, \quad a_I dt dx_J \xrightarrow{K} \int_0^t a_I(x, t) dt dx_I$$

where I is a $(k-1)$-multi-index in the second expression. Then we can check that the chain homotopy property does hold.

For Mayer-Vietoris, suppose U, V are open in M – we have a short exact sequence $0 \to \Omega^k(U \cup V) \to \Omega^k(U) \oplus \Omega^k(V) \to \Omega^k(U \cap V) \to 0$, where the first map sends α to $(\alpha|_U, \alpha|_V)$ (in other words, the pullback of the inclusion map) and the second map sends (α_1, α_2) to $(\alpha_1 - \alpha_2)|_{U \cap V}$, and thus that induces a long exact sequence in (de Rham co)homology $\cdots \to H^k(U \cup V) \to H^k(U) \oplus H^k(V) \to H^k(U \cap V) \xrightarrow{\delta} H^{k+1}(U \cup V) \to \cdots$. Then the map I will turn out to complete the corresponding diagram between the two Mayer-Vietoris sequences (it maps the corresponding terms and makes the diagram commute). The point now is the following:

Proposition 102

Assume we have a statement $P(U)$ about open sets U of manifolds M, such that $P(U)$ is true if U is diffeomorphic to some convex open set in $\mathbb{R}^n$ (Poincaré lemma). Furthermore, suppose that if $P(U), P(V), P(U \cap V)$ are true, then $P(U \cup V)$ is true (Mayer-Vietoris), and suppose that if $\{U_\alpha\}$ are disjoint and $P(U_\alpha)$ is true for each α, then $P(\bigsqcup_\alpha U_\alpha)$ is true. Then $P(M)$ will hold for all manifolds M (remember open sets of manifolds are themselves manifolds).

So in this case $P(U)$ is the statement of de Rham's theorem itself (that I gives us an isomorphim and a chain map). To understand why this proposition holds, note that if M is compact then we just need (a) and (b) (cover by finitely many open sets), and otherwise use an exhaustion function (a proper smooth map into $\mathbb{R}$) and break into bands based on preimages, then applying Mayer-Vietoris to the "odd" and "even" bands that slightly overlap. □

As a remark, there's also another step where we need to show, using Whitney approximation, that cohomology defined using continuous simplices is equivalent to cohomology defined using smooth simplices.

10 February 9, 2023

We'll discuss de Rham theory and Poincaré duality today – specifically, we'll need to discuss compactly supported de Rham cohomology and then see how Poincaré duality shows up. We'll then talk about the cup and wedge product and Poincaré duality for a submanifold, and perhaps talk about orientation, integrating forms on manifolds, and Stokes' theorem on manifolds (it's really just that change-of-charts is orientation-preserving, and we can integrate using a partition of unity and do it on $\mathbb{R}^n$).

Reviewing from MATH 215A, recall that if M is an oriented closed (without boundary) **topological** manifold, then M has a fundamental class $[M] \in H_n(M; \mathbb{Z})$ in the usual singular homology of M, such that the map $\text{PD} : H^k(M; \mathbb{Z}) \to H_{n-k}(M; \mathbb{Z})$ given by $\alpha \mapsto \alpha \cap [M]$ is an isomorphism. And we can pass this result to $\mathbb{R}$-coefficients and dualize, and in particular this gives us a **nondegenerate** pairing

$$\text{PD} : H^k(M; \mathbb{R}) \times H^{n-k}(M; \mathbb{R}) \to H^n(M; \mathbb{R}) \to \mathbb{R}$$

given by $(\alpha, \beta) \mapsto \alpha \cup \beta \mapsto \langle \alpha \cup \beta, [M] \rangle$ (which makes the cap product $\cap$ the dual of the cup product $\cup$). We now want to understand how this all looks under the de Rham isomorphism, since we've shown that the cohomology groups are isomorphic – we should expect that we then get a pairing $H^k_{\text{dR}}(M) \times H^{n-k}_{\text{dR}}(M) \to H^n_{\text{dR}}(M) \to \mathbb{R}$ which sends $(\alpha, \beta) \to \alpha \wedge \beta \to \int_M \alpha \wedge \beta$. (So cup goes to wedge product, and cap is essentially integrating against the manifold.)

The version we'll prove for de Rham cohomology allows us to work with not-necessarily-closed manifolds, but we'll need compactly supported de Rham cohomology for that.

Definition 103

Let M be a **smooth** manifold, not necessarily compact. The set of compactly supported differential k-forms is denoted $\Omega^k_c(M)$, and these also form a chain complex $\cdots \to \Omega^k_c(M) \xrightarrow{d} \Omega^{k+1}_c(M) \to \cdots$ (since the differential of zero is zero) with homology denoted $H^k_c(M)$.

(For a compact manifold, Ω^k_c is the same as the ordinary Ω^k.) It turns out that this homology theory may behave quite differently from the usual one, but it's useful because we can define it for open subsets (and thus work with charts). In particular, for an open subset $U \subseteq M$, extending forms by 0 defines a map $\iota : \Omega^k_c(U) \to \Omega^k_c(M)$ (which we can't do with usual de Rham). Obviously ι is a chain map, since we can extend by zero before or after differentiating, and thus we induce a map $\iota_U : H^k_c(U) \to H^k_c(M)$. (So we can kind of regard this as a **homology** on M "rel. ends," since we're ignoring the parts at infinity.) And Mayer-Vietoris thus goes in the opposite direction – we have an exact sequence

$$0 \to \Omega^k_c(U \cap V) \to \Omega^k_c(U) \oplus \Omega^k_c(V) \to \Omega^k_c(U \cup V) \to 0,$$

with the maps $\alpha \mapsto (\iota_U \alpha, -\iota_V \alpha)$ and $(\alpha, \beta) \to \iota_{U \cup V} \alpha + \iota_{U \cup V} \beta$, thus inducing a short exact sequence in homology. And now if we have a map $f : M \to N$ and we want a pullback map $f^* : \Omega^k_c(N) \to \Omega^k_c(M)$, we need f to be both smooth and **proper** (since we obviously want inverse images of compact sets to be compact), but then we get a map $f^* : H^k_c(N) \to H^k_c(M)$. In other words, we get functors $\Omega^\bullet_c(\cdot), H^\bullet_c(\cdot)$, where we go from a category of manifolds with morphisms given by proper maps to the category of chain complexes / vector spaces.

Example 104

We have $H^n_c(\mathbb{R}^n) = H^n(S^n) = \mathbb{R}$ (and $H^k_c(\mathbb{R}^n) = 0$ for all other k, including zero, since the only compactly supported constant function is zero). On the other hand, the usual de Rham cohomology is $H^n_{\text{dR}}(\mathbb{R}^n) = H^n_{\text{dR}}(\text{point}) = 0$ for all $n > 0$ (by the homotopy axiom).

(Really, what's happening here is that we're doing cohomology on the one-point compactification, except reduced homology for dimension zero.)

Theorem 105 (Poincaré duality for de Rham cohomology)

For all smooth oriented n-dimensional manifolds M (without boundary, but not necessarily compact), the pairing $\mathrm{PD} : H^k_{\mathrm{dR}}(M) \to H^{n-k}_c(M)^*$, where $\mathrm{PD}(\alpha)$ is the linear functional given by

$$(PD)(\alpha)(\beta) = \int_M \alpha \wedge \beta$$

is an isomorphism.

If $\alpha \in H^k_{\mathrm{dR}}(M)$ and $\beta \in H^{n-k}_c(M)$, then wedging their representatives gives us something compactly supported and thus we can integrate it on M. And if we assume that this map is well-defined, we can prove that we do have an isomorphism with the same proof as that of the de Rham isomorphism. Indeed consider the statement $P(U) = \{\mathrm{PD}$ is true for $U\}$ for any open set U in M. We know that Mayer-Vietoris holds, that H^n behaves correctly on points, and this behaves well under disjoint union as well, so we can apply Proposition 102. But to check well-definedness, we need to make sure $\int_M \alpha \wedge \beta$ makes sense and that this descends to cohomology by showing that we have a chain map. That'll use Stokes' theorem, so we'll defer all of this to the end of class if we have time.

Remark 106. *Taking the dual of the statement of Poincaré duality for de Rham cohomology, we find that we also have an isomorphism $H^k_c(M) \cong H^{n-k}(M)^{**}$, where the double dual may not be the same as the original space in general because H^{n-k} may be infinite-dimensional (because the dual of an infinite product is the direct sum), for instance if M is an infinite union of points. But if M is compact or the total space of some vector bundle $E \to M$ over a compact manifold, then the de Rham cohomology will be finite-dimensional, and we have $H^{n-k}(M)^* \cong H^k_c(M)$ and $H^{n+r-k}(E)^* \cong H^k_c(E)$, where r is the rank of the bundle E. But now looking at ordinary de Rham cohomology, in which the homotopy axiom $H^*_{dR}(E) \cong H^*_{dR}(M)$, we can then use Poincaré duality to get $H^{*+r}_c(E) \cong H^*_c(M)$ and we don't get homotopy invariance.*

There's a few useful notions that we may want to use when working with these objects:

- A **good cover** of a manifold is a cover by charts so that all intersections are homeomorphic to $\mathbb{R}^n$. It turns out that smooth manifolds always have good covers (the proof uses a metric and considers the exponential map in differential geometry) and compact manifolds have finite good covers. So with this, we can get Kunneth for finitely supported de Rham.
- Smooth manifolds can be triangulated – that is, they are homeomorphic to a simplicial complex, so in particular we can calculate de Rham using simplicial calculations. But we should be careful that not all **topological** manifolds can be triangulated.

We'll now turn to cup and wedge and the Poincaré dual of a submanifold – we've seen that usual simplicial cohomology yields a functor from manifolds into chain complexes, and we can restrict that to de Rham. Specifically, if $f : M \to N$ is a smooth map between manifolds, then the pullback $f^* : \Omega^k(N) \to \Omega^k(M)$ yields a map $f^* : H^k_{\mathrm{dR}}(N) \to H^k_{\mathrm{dR}}(M)$ which commands with the natural transformation I given by the de Rham isomorphism. But now $\Omega^k(M)$ has a wedge product satisfying

$$d(\alpha \wedge \beta) = d\alpha \wedge \beta = (-1)^p \alpha \wedge d\beta$$

where α is a p-form, and thus it descends to some element Λ in de Rham cohomology. We then have

$$\Lambda : H^k_{\mathrm{dR}}(M) \times H^p_{\mathrm{dR}}(M) \to H^{k+p}_{\mathrm{dR}}(M),$$

and we want to know what this corresponds to under the de Rham isomorphism, which recall is a map $I : \Omega^k(M) \to C^k(M)$ given by $I(\omega)(\sigma) = \int_\sigma \omega$ for any k-form σ and any k-simplex σ.

Theorem 107

Let M be a closed manifold. Then the de Rham isomorphism I is a map $H^k_{\text{dR}}(M) \to H^k(M; \mathbb{R})$ taking $\wedge$ to $\cup$, and in cohomology we thus get a map $I(\alpha \wedge \beta) = I(\alpha) \cup I(\beta)$.

These facts are false on the chain level – it's not true that I takes the wedge product to the cup product there. And in fact the proof is quite involved, and we'll skip it here – there is a standard textbook reference (Bott-Tu) proof using spectral sequences and Cech cohomology. We must use **diagonal approximation**, showing that if ω, η, p, q are cochains and σ_p is a "front face" and σ_q is a "back face", then

$$\int_\Delta (\omega \wedge \eta) = \int_{\sigma_p} \omega \int_{\sigma_q} \eta.$$

We can now think about Poincaré duality on a submanifold – there are two different statements:

Proposition 108 (Poincaré duality of a submanifold)

Let M be a smooth oriented manifold, not necessaily compact, with $\dim M = n$, and let S be a properly embedded **oriented** submanifold with $\dim S = s$. Then we may look at integration $\mathcal{I}_S : \Omega^s_c(M) \to \mathbb{R}$ along S (integrating the top-dimensional form) given by $\mathcal{I}_s(\alpha) = \int_S \alpha$. By Poincaré duality, we may think of $\mathcal{I}_s \in H^s_c(M)^* \cong H^{n-s}(M)$ by Poincaré duality. Thus (under this isomorphism) there is a class $[\eta_s] \in H^{n-s}_{\text{dR}}(M)$ of codimension s characterized by $\int_S \omega = \int_M \eta_s \wedge \omega$ for all $\omega \in H^k_c(M)$.

We thus have a cohomology class η_s (dropping brackets) with the property that integrating on any compactly supported k-form on S, that's the same as integrating $\eta \wedge \omega$ on the whole manifold.

Alternatively there is another Poincaré dual class defined as follows: let $\iota : S \hookrightarrow M$ be a closed submanifold, compact without boundary. Then we can associate to such a submanifold S a **compactly supported** element $\tau_S \in H_c(M)$ characterized instead by

$$\int_S \omega = \int_M \tau_S \wedge \omega \quad \forall \omega \in H^k(M)$$

where ω now no longer needs to be compactly supported. So these constructions are very similar, but we're dual to different things in the two cases. The point is that because we're now working with a compactly supported submanifold, by the tubular neighborhood theorem we can take a neighborhood U of S in M. Then using Poincaré duality on U, we can choose τ_s to be compactly supported within U and extend it by 0 to τ_S on M. So what we basically have is a "delta function" on S, as close as possible to being supported on our submanifold.

This will be relevant for intersection theory later on – for example, we can compute the Poincaré dual of a point in $\mathbb{R}^n$, and any $\beta(x)dx$ (where β is a bump function and $dx = dx_1 \cdots dx_n$) will work there. And we can also compute the Poincaré dual of $\mathbb{R}^2 \setminus \{0\}$ and find something similar.

We'll finish by discussing **orientation** briefly:

Definition 109

A vector bundle $E \to B$ is **orientable** if its determinant line bundle $\det E$ is trivial.

(The determinant is the top-dimensional exterior product, with clutching function given by determinants.) Note that M is orientable if and only if its tangent bundle is orientable, and we need to check that this agrees with the algebraic topology notion of orientable so that we still have the same Poincaré duality. For example, if $L \to B$ is a line bundle, then L is orientable if and only if its determinant is trivial, which is the same as saying that L is trivial, which is equivalent to having a nowhere vanishing section. Indeed, use a metric and notice that this is equivalent to the unit sphere bundle $S(L)$ being trivial, but $S(L)$ is an S^0 bundle, meaning we have $\{\pm 1\} = \mathbb{Z}_2$ over each point in B. So trivial means trivial as a cover or as a bundle or in whichever other way we want, and for a bundle this is equivalent to finding trivializations such that transition / clutching functions all have positive determinant.

11 February 14, 2023

Today's class will talk some more about orientations, particularly integrating forms on manifolds, and then discuss **connections and curvature**. Recall from last time that a vector bundle $E \to M$ is orientable if the determinant line bundle is orientable, which is the same as being trivial. (This is also the same as saying that the sphere bundle, which is a two-fold cover, is disconnected, and also for a general vector bundle that there is a collection of local trivializations of E such that the clutching functions are orientation-preserving. And this is because the clutching functions of $\det E$ come from looking at the original clutching functions $U_\alpha \cap U_\beta \to GL(\mathbb{R}^k)$ and taking the determinant of those linear transformations.)

Definition 110
A manifold M is **orientable** if the tangent bundle TM is orientable.

Since $\det E$ is the same as $\Lambda^{\text{top}}(E)$, we are asking for $\Lambda^{\text{top}} TM$ to be orientable, which is the same as having a volume form ω on M (which is a top-dimensional form which is nowhere vanishing, so basically a nonzero section of $\Lambda^{\text{top}} T^*M$). The determinant is nice because it works well with sums (we get the tensor product bundle) and duals, so sums and duals of orientable bundles are also orientable. In MATH 215A, recall that an **orientation** n-dimensional topological manifold is a consistent choice of generators of the relative homology $H_n(M, M \setminus x; \mathbb{Z})$ as x varies, and similarly an orientation of a rank k bundle is a consistent choice of $H_k(E_x, E_x \setminus 0; \mathbb{Z})$. (By excision this is the same as just having H_n of the ball, but that's not a canonical relationship.)

Example 111
$\mathbb{R}^n$ is canonically oriented by the volume form $dx = dx_1 \wedge \cdots \wedge dx_n$ coming from the standard coordinates. Similarly, the hyperplane $\mathbb{H}^n$ where $x_1 \geq 0$ is also oriented by $dx_1 \wedge \cdots \wedge dx_n$, and at the boundary ∂H where $x_1 = 0$ we have coordinates $(x_2, \cdots, x_n)$ and thus there is a standard volume form $dx_2 \wedge \cdots \wedge dx_n$. We then have oriented ∂H so that the normal vector $\frac{\partial}{\partial x_1}$ is the first vector in our basis and points "inside" the manifold.

So this type of reasoning also extends to manifolds with boundary using the same convention (using charts): an orientation on M determines and orientation on ∂M such that the normal vector points inside. With that, we can define integration of forms on manifolds in a straightforward way: for any open set $U \subseteq \mathbb{R}^n$ and any compactly supported top-dimensional form $\omega(x) = f(x) dx_1 \cdots dx_n$, then $\int_U \omega = \int_U f(x) dx$ can be integrated in the ordinary way. So then we can use a partition of unity for a **general oriented manifold** M: let $\{U_\alpha\}$ be a cover by charts (ϕ_α, U_α) with positive change of charts, and suppose $1 = \sum_\alpha \rho_\alpha$ is a partition of unity where each ρ_α is supported on U_α. Then we want to define our integral on $\mathbb{R}^n$, so the formula would be

$$\int_M \omega = \sum_\alpha \int_M \rho_\alpha \cdot \omega = \sum_\alpha \int_{\phi_\alpha(U_\alpha)} (\rho_\alpha \circ \phi_\alpha^{-1})(\phi_\alpha^{-1})^* \omega = \sum_\alpha \int (\phi_\alpha^{-1})^* (\rho_\alpha \omega),$$

since we need to move each integral by pulling back the form and then integrating on the corresponding open set. We must check that this is well-defined independent of choices – in particular, we can only define this if $\omega \in \Omega_c^n(M)$ so that the sum can be made finite. But then it has all of the expected properties including change of variables, and it also satisfies Stokes' theorem:

Theorem 112 (Stokes' theorem on manifolds)
Suppose M is an n-dimensional oriented manifold with boundary ∂M, such that the boundary s oriented with the normal vector pointing inside. Then $\int_M d\omega = \int_{\partial M} \omega$ for all compactly supported n-forms ω on M.

(This follows directly from the usual Stokes' theorem by using partition of unity to move from $\mathbb{R}^n$ to M.)

Now if we want to integrate lower-dimensional forms, we have to integrate on submanifolds. Suppose S is a k-dimensional submanifold of M, such that we have an inclusion $\iota : S^k \hookrightarrow M$. Then for all $\omega \in \Omega_c^k(M)$ we basically integrate $\int_S \omega = \int_S \iota^* \omega$ (so this is really just an abuse of notation and we're restricting ω to the submanifold). Hopefully a lot of this is familiar to us, and it is mostly setup to help us with the missing parts of the de Rham theorem.

We'll now turn to differential geometry concepts and consider connections and curvature. The idea is to try to figure out a natural notion of "directional derivative" of a section of a vector bundle $E \to B$ (that is, how the section changes as we move in some direction v on M).

Definition 113
Let $E \to M$ be a vector bundle of rank k. A **differential form on M with values in E** are sections of $\mathrm{Hom}(\Lambda^k M, E) = E \otimes \Lambda^k(TM)^*$. The set of such forms is denoted $\Omega^k(M; E)$ or just $\Omega^k(E)$.

Writing down what this is in coordinates, remember that such a k-form on M with values in E is an assignment at each point $\omega_x : T_xM \times \cdots \times T_xM \to E_x$ (that is, we take in k tangent vectors to M and give some element of the fiber), and we do this in a way which varies smoothly as x varies in M. So then 0-forms with values in E are sections (so $\Omega^0(M, E) = \Gamma(E)$), and 1-forms with values in E are maps from $T_xM \to E_x$.

In differential geometry there are various definitions of connections that are useful to switch between, but we'll only think about two of them:

Definition 114
A **connection** (also **covariant derivative**) on E is an $\mathbb{R}$-linear transformation $\nabla : \Omega^0(M; E) \to \Omega^1(M; E)$ satisfying **Liebniz's rule** $\nabla(f \cdot s) = df \otimes s + f \nabla s$ for any $s \in \Omega^0(E)$ and any $f \in C^\infty(M)$.

In other words, we have a map $\Gamma(E) \to \Omega^1(M, E)$ which we can think of as assigning a map $T_xM \to E_x$ to each section. (Indeed, we can check that all of the terms in the Liebniz rule actually live in the right space.) For any $x \in M$, $v \in T_xM$, we can then define ∇_v to satisfy

$$(\nabla_v s)_x = (\nabla s)_x(v) \in E_x,$$

and we call this the "covariant derivative of s in the direction v." Then if we let v vary, we'll get a similar formula for vector fields, but we won't do that just yet.

Lemma 115

The difference of two connections is a 1-form on M with values in $\mathrm{End}(E)$ (which we can think of as $E \otimes E^*$). In other words, we have $\nabla^1 - \nabla^2 = A \in \Omega^1(M, \mathrm{End}(E))$. Conversely, if ∇ is a connection on E and $A \in \Omega^1(M, \mathrm{End}(E))$ is a 1-form, then $\nabla + A$ is a connection on E.

In other words, the space of connections is an affine space on the vector space $\Omega^1(M; \mathrm{End}(E))$.

Proof. Let $A = \nabla^1 - \nabla^2$; this is $\mathbb{R}$-linear because both terms on the right-hand side are, and

$$A(fs) = (df \otimes s + f\nabla_1 s) - (df \otimes s + f\nabla_2 s) = f(\nabla_1 s - \nabla_2 s) = f(As),$$

so A can be regarded as a 1-form in $\Omega^1(M; \mathrm{End}(E))$. Indeed, since A takes 0-forms on M with values in E to 1-forms of M with values in E, as before we may let $A_v s = (As)_x v$ for all $v \in T_x M$, $x \in M$, and $s \in \Gamma(E)$. Since A is linear in s and As is a 1-form, it is also linear in v. So A_v takes in a section in $\Gamma(E)$ and returns something in E_x, and we want to show that A_v depends only on the value s_x at s, not on any derivatives (this is called being **tensorial**). If we can prove this then we're done, because then the map $A_v : \Gamma(E) \to E_x$ descends to an $\mathbb{R}$-linear map $E_x \to E_x$, which is just an endomorphism in $\mathrm{End}(E_x)$. And now if we look at the map $v \mapsto A_v$ (that is, a map $T_x M \to \mathrm{End}(T_x M)$), it is $\mathbb{R}$-linear because it is $\mathbb{R}$-linear in v, and we just need to check that A_v depends on the value at a point. (This is sort of like thinking of a two-parameter function as a function in one of the two variables – the point is that we have s and v, and in this case we're freezing v and letting s vary, then letting v vary.)

So to prove that A_v is indeed only dependent on the value, we must use that $A_v(fs) = f(As)$. Since we're defining $A_v(fs) = f(x)A_v s$, by linearity it's enough to show that if $s(x) = 0$ then $A_v s = 0$. This is best checked in local trivializations $\pi^{-1}(U) \cong U \times \mathbb{R}^k$ – we can choose a basis $\{e_1, \cdots, e_k\}$ in $\mathbb{R}^k$ and bring it by trivialization to $\pi^{-1}(U)$, yielding a basis in each fiber E_x for $x \in U$ (here x is now varying). So then a section S over U can be written as $s(x) = \sum_{i=1}^k s^i(x) e_i$, where the s_is are functions on U. But now if we fix some point $x_0 \in U$, then $s(x_0) = 0$ implies that $s_i(x_0) = 0$ for all i, so if we calculate

$$A_v(s) = A_v\left(\sum_{i=1}^k s_i e^i\right) = \sum_{i=1}^k A_v(s_i e^i) = \sum_{i=1}^k s_i(x_0) A_v(e^i) = 0.$$

Note that this proof isn't quite correct – it's fine if s is supported on U, because then the coefficients s_i are supported on U by extending by zero. And otherwise we can use a partition of unity on M subordinated to a cover by local trivializations (which is always helpful when we have something linear): write $1 = \sum \rho_\alpha$, so that $s = \sum_\alpha \rho_\alpha s$. And now $A_v s = \sum_\alpha A_v(\rho_\alpha s)$, so if $s(x_0) = 0$ then $(\rho_\alpha s)(x_0) = 0$ as well so $A_v s = 0$. $\square$

Lemma 116

Connections exist on any vector bundle $E \to M$ and pull back under smooth maps (to $f^*\nabla$ on f^*E). In other words, a way of taking derivatives and a map f into E gives a way of taking derivatives on f^*E.

We can take directional derivatives of functions if things are not curved, but in general we need formulas in local trivializations.

Proof. Let $g_{\alpha\beta}$ be the clutching functions $g_{\alpha\beta} : U_\alpha \cap U_\beta \to \mathrm{GL}(\mathbb{R}^k)$. A section s of E can then be thought of as a bunch of functions $s : U_\alpha \to \mathbb{R}^k$, patched together via $s_\alpha = g_{\alpha\beta} s_\beta$. So a connection ∇ on E (restricted to U_α)

is $\nabla|_{U_\alpha} = d + A_\alpha$, where A_α is a 1-form $\Omega^1(U_\alpha; \text{End}(\mathbb{R}^k))$. These A_αs are called **connection 1-forms** – they don't actually patch together correctly in the usual way, and the formula is

$$A_\beta = g_{\alpha\beta}^{-1} dg_{\alpha\beta} + g_{\alpha\beta}^{-1} A_\alpha g_{\alpha\beta}$$

on the overlap $U_\alpha \cap U_\beta$. (Basically take ∇ of $s_\alpha = g_{\alpha\beta}s_\beta$ and use Liebniz's rule. Since these are endomorphisms, if they patched together we'd just have the second term, so the first one is a correction term.) And conversely, any such collection of 1-forms A_αs on U_α with values in $\mathbb{R}^k$ satisfying that condition will yield a connection on E, since we can do the usual writing of a partition of unity and letting $A_\alpha = \sum_\gamma \rho_\gamma g_{\gamma\alpha}^{-1} dg_{\gamma\alpha}$ – we just need to check that we do satisfy the formula above, but it's a direct calculation. $\square$

12 February 16, 2023

We'll talk today about two different topics, (1) connections and curvature (a differential geometry topic) and (2) classification of bundles (an algebraic topology one), both of which will both be useful for next lecture when we discuss characteristic classes. Recall from last lecture that in a smooth vector bundle $E \to M$, a connection is a map $\Omega^0(M; E) \to \Omega^1(M; E)$ which is $\mathbb{R}$-linear and satisfies Leibniz's rule (meaning that for any smooth function f and any section $s \in \Omega^0(M; E)$, we have $\nabla(fs) = (df) \otimes s + f\nabla s$. In particular, $\nabla = d + A_\alpha$ on a local trivialization U_α, where A_α is a connection 1-form; in other words, because our section now becomes a function $s : U_\alpha \to \mathbb{R}^k$, ∇s becomes $(\nabla s)_\alpha = ds_\alpha + A_\alpha s_\alpha$, where A_α is a linear trnasformation from the fiber to itself (since we want to take the point s_α in the fiber and map it to some other point in the fiber linearly|).

Example 117

Consider the unit sphere $S^n \subseteq \mathbb{R}^{n+1}$, and consider the vector bundle $E = TS^n$ over S^n. Then sections of E are vector fields, and for any such section $s \in \Gamma(TS^n) = \chi(S^n)$ and any direction $v \in T_xS^n$ we have

$$\nabla_v s = \text{proj}_{TS^n}(ds).$$

In other words, we want a way to take the derivative along a vector field, and we do this by taking the derivative in $\mathbb{R}^{n+1}$ and projecting. It's left for us to show that this actually satisfies Leibniz's rule.

Let ∇ be a connection on a vector bundle $E \to M$. We may extend such a map to ∇ (also denoted $d^\nabla : \Omega^k(M, E) \to \Omega^{k+1}(M, E)$) from k-forms with values in E to $(k+1)$-forms with values in E by R-linearity and Liebniz's rule: on any k-form η on M and section s, we have

$$\nabla(\eta \cdot s) = (d\eta)s + (-1)^k \eta \cdot \nabla s$$

(Notice that this is kind of like defining the differential of a form) – we did it first for functions and then did it on generators based on that using Leibniz's rule. So we have a way to differentiate k-forms.

Definition 118

The **curvature** of ∇ is $F_\nabla = d^\nabla \circ d^\nabla = \nabla \cdot \nabla$.

In other words, for any s we have $(F_\nabla s) = \nabla(\nabla s)$, and we may expect that this well involve two derivatives of s. But it turns out it doesn't: if we calculate what it looks like at a point x using Leibniz's rule, it turns out $(F_\nabla s)_x$ only depends on s_x and not derivatives of s. So in fact F_∇ is a section in a bundle, specifically a 2-form on M with values in

End(E) (this is kind of like the proof from last time where we subtracted to make Leibniz's rule). For a more explicit formula, we have (taking in two elements $X, Y \in TM$)

$$F_\nabla(x, y)s = \nabla_x \nabla_y s - \nabla_y \nabla_x s - \nabla_{[x,y]} s.$$

Note that in general F_∇ is nonzero, and in fact (from the equation we've just described) this encodes the failure of second derivatives of a section to commute. (If the bundles were trivial then everything is functions, but otherwise the second derivatives are very unlikely to commute.) If the curvature is zero, then we say that the connection ∇ is **flat**, and then $\cdots \to \Omega^k(M, E) \xrightarrow{d^\nabla} \Omega^{k+1}(M, E) \to \cdots$ becomes a chain complex because $d^\nabla \circ d^\nabla = 0$. So in such a flat bundle, we actually get the de Rham cohomology of M with values in E_c.

There's a few different geometric interpretations we can have for connections (we won't need these for what comes next, but it'll still be useful for intuition):

- Given a connection γ in M, we can **parallel transport** our vector by ensuring that the derivative $\nabla_{\frac{d\gamma}{dt}} s$ is zero (since we want the section to be "constant" or "parallel"). But because we have a formula $\nabla = d + A$ in local coordinates, we really have a first-order ODE but only along the path γ (so only in terms of t), with initial condition $I(S(\gamma(0))) = s_0 \in E_{\gamma(0)}$. So this is similar to the situation with flow but with fibers, and transport determines an isomorphism

  $$\mathrm{PT}_t : E_{\gamma(0)} \simeq E_{\gamma(t)}$$

  (the same thing happens as with flows, because running for time s and then running for an additional time t is the same as running for $s + t$ time). This then allows us to define an isomorphism of fibers of E along the path γ, and we can recover ∇_s via

  $$\nabla_v s = \frac{d}{dt}\bigg|_{t=0}$$

  (The idea is that without the PT_t^{-1}, we'd just have the ordinary derivative, but we need to first pull back the fiber at $\gamma(t)$ to the fiber at $\gamma(0)$ so that the spaces on which they exist are identified.) And finally, really the parallel transport depends on the path γ we take, so we should really be writing $\mathrm{PT}_t = \mathrm{PT}^\gamma_{0,t}$.

- For a more geometric viewpoint, notice that for a general smooth fiber bundle $pi : E \to B$, we can always define the **Ehresmann connection** (for fiber bundles the previous definition doesn't work but it does not here), which is a choice of horizontal spaces in a splitting of the following short exact sequence: we know that there is a well-defined **vertical space**, where the fibers are the kernel of the map $d\pi$ (by the implicit function theorem). Then if we write $0 \to \ker(d\pi)_x \to T_x E \xrightarrow{d\pi} T \to T_{\pi(x)} B \to 0$, a choice of horizontal space H_x is the map $T_{\pi(x)} B \to T$ which demonstrates the splitting. We then have $T_x E = V_x \oplus H_x$ at each x.

  This then extends to G-bundles, and our splitting must be G-equivariant (for vector fields we comparably want compatibility with the linear structure).

Remark 119. *The pullback of ∇ is the pullback of F_∇: in other words,*

$$f^* F_\nabla = F_{f^*\nabla}.$$

So if we have maps f between two manifolds, then we can pull back bundles and connections and the curvatures pull back as well.

We'll now talk about classification of bundles, and we would see it done in more detail in a class after 215A. Here, we'll just mention some useful statements but do very few proofs – principal G-bundles are the easiest for which to classify, so we'll briefly mention those first.

Theorem 120
There is a 1-to-1 correspondence between vector bundles up to isomorphism and principal G-bundles up to isomorphism.

In other words, for real vector bundles (though the same holds for complex ones), we have a one-to-one correspondence between (for example) real vector bundles $E \to X$ and $\text{Princ}_{\text{GL}(\mathbb{R}^k)}(X)$. We won't talk about the proof during lecture, but the idea is that given a vector bundle, we can send it to the **frame bundle** of E by taking $(x \in X, \text{basis of } \text{E}_x)$. And this is a principal bundle because change-of-basis indeed comes from $\text{GL}(\mathbb{R}^k)$. And for the other direction, if we have some action $G = \text{GL}(\mathbb{R}^k)$ on $V = \mathbb{R}^i$, we see that if $P \to X$ is a principal G-bundle and V a vector space with a G-action, we can consider $(P *_G V) = (P \times V)/G$. So we just need to check that these are well-defined and inverses.

If we now want to classify vector bundles up to isomorphism, which is equivalent to principal GL-bundles up to isomorphism, here is the motivation (coming from Whitney's embedding theorem): given $E \to X$ where X is compact (and where we can assume X is a manifold, though this is not needed), there is an embedding of $E \hookrightarrow \varepsilon^N$ for some sufficiently large N (we don't use charts, but we do use local trivializations). Then $E \to X$ is the pullback of the tautological bundle over the Grassimannian $T \to G_k$. Now if we view each fiber E_x as including into $\mathbb{R}^N$ (that is, as an inclusion into the corresponding trivial bundle for the embedding), the map $E \to X$ is the pullback of the tautological bundle $\tau \to G_k(\mathbb{R}^N)$. (We proved this on our homework: we have $f : X \to G_k(\mathbb{R}^N)$, so we indeed have $E = f^*\tau$.) Taking $N \to \infty$, we then get a map $f : X \to G_k\mathbb{R}^\infty$ – thus by fixing any embedding, we will get such a map f.

Theorem 121 (Steenrod classification theorem, special case)
The map f is a **classifying map**. In other words, the pullback of f is the tautological bundle of the Grassmannian, and if two pullbacks are equivalent then the two bundles are isomorphic:

$$\text{Vect}^k_{\mathbb{R}}(X) \xrightarrow{\cong} [X, G_k(\mathbb{R}^\infty)],$$

and we also have the analogous statement for $\mathbb{C}$. (Here recall that $[X, Y]$ denotes the set of maps $X \to Y$ up to homotopy.)

The way to think about this is that any $E \to X$ gives rise to some map $f : X \to G_k(\mathbb{R}^n)$, so any bundle over X is the pullback of the tautological bundle by f, unique up to homotopy.

Remark 122. *In general the Steenrod classification theorem holds for principal G-bundles where the base X is compact. So if we want to classify G-bundles up to isomorphism, we have a classifying space BG, over which lies a universal bundle EG, and any principal G-bundle $P \to X$ we get a unique map $X \to BG$.*

We'll now look at some of the consequences of these facts for classifying line bundles: recall that the space of lines in S^n is $\mathbb{RP}^n$, so $G_1(\mathbb{R}^\infty) = \mathbb{RP}^\infty$, and similarly $G_1(C^\infty) = \mathbb{CP}^\infty$ with the tautological bundle.

Theorem 123
There is a one-to-one correspondence between line bundles $\text{Vect}^1_{\mathbb{R}}$ over X and homotopy classes of maps $[X; \mathbb{RP}^\infty] = H^1(X; \mathbb{Z}_2)$. In parallel, we have $\text{Vect}^1_{\mathbb{C}^1} = [X, \mathbb{CP}^\infty] = H^2(X; \mathbb{Z})$ (note the difference in coefficients).

The first equality comes from a natural map: over $\mathbb{R}$, we want to show that a bundle $E \to X$ corresponds to a homotopy class of maps $f : X \to \mathbb{RP}^\infty$ such that $f^*\tau = E$. But such a map corresponds to $f^*\omega_1$, where

$\omega_1 \in H^1(\mathbb{RP}^\infty; \mathbb{Z}_2) \cong \mathbb{Z}_2$ is the generator. And over $\mathbb{C}$, we instead have correspondence to f^*c_1, where $c_1 \in H^2(\mathbb{CP}^\infty \mathbb{Z})$ (These two corresponding objects are the first Stiefel-Whitney class and Chern class), respectively, where recall that we have $H^*(\mathbb{RP}^k; \mathbb{Z}_2) = \mathbb{Z}_2\langle \mu \rangle / \mu^{k+1} = 0$ (a class in H^1 but Poincaré dual (in $\mathbb{Z}$-2 coefficients) to the line $\mathbb{RP}^1$ sitting inside $\mathbb{RP}^k$.

Corollary 124

Over $\mathbb{RP}^n$, we only have two line bundles (the trivial one and the tautological one). And when $X = S^n$ and $n \geq 2$, we only have the trivial bundle because the line bundles are classified by $H^1(S^n; \mathbb{Z}_2) = 0$.

We do need homotopy invariance to show that everything descends to cohomology, so the following result was used implicitly to show that things were well-defined:

Theorem 125 (Homotopy invariance)

Let $E \to B$ be any fiber bundle (in particular a vector bundle), and let $f_0, f_1 : X \to B$ be two homotopic maps. Then $f_0^* E \cong f_1^* E$.

In particular, we see that if B is just a single point then the bundle is trivial.

Corollary 126

Up to isomorphism, principal G-bundles (in particular vector bundles) over S^n are classified by $\pi_{n-1}(G)$.

The idea is that we can write $S^n = D^n_+ \cup_{S^{n-1}} D^n_-$ (two balls glued over the equator), and now if we have any map $E \to S^n$ restricting to the upper or lower hemisphere gives us a trivial bundle, so we have two local trivializations and just need to clutch them together. Then for a principal G-bundle, the clutching functions take values in G (for a vector bundle we'd do linear transformations instead), and now (modding out by homotopy) that gives us a map $g_{12} : S^{n-1} \to G$, which means we're getting an element of $\pi_{n-1}(G)$. So this classifying theorem is very powerful and we'll use its consequences!

13 February 21, 2023

We'll discuss **characteristic classes** today, which will provide a criterion to tell us when two bundles are not isomorphic. We stated a result that vector bundles up to isomorphism are in one-to-one correspondence with homotopy classes of maps from the base X to an infinite Grassmannian, and in particular for line bundles that's $\mathbb{RP}^\infty$ or $\mathbb{CP}^\infty$. But it's not realistic to go into that Grassmannian. So we'll associate bundles up to isomorphism with cohomology classes (and different classes means we don't have isomorphic bundles). Specifically, we'll consider Chern classes for complex vector bundles, Stiefel-Whitney classes for real vector bundles, and Euler classes for real (sometimes oriented) bundle, which come from the Thom class. We'll mostly state properties as black boxes (since we would see the full proofs in a further algebraic topology class):

Definition 127

The **total Chern class** of a complex vector bundle $E \to X$ is a mixed cohomology class

$$c(E) = 1 + c_1(E) + c_2(E) + \cdots ,$$

where $c_i(E) \in H^{2i}(X, \mathbb{Z})$ is called the **ith Chern class** of E and is uniquely characterized by the following axioms: **(1)** isomorphic bundles have the same total Chern class, **(2) [naturality]** $c(f^*E) = f^*c(E)$, **(3) [multiplicativity]** $c(E \oplus F) = c(E) \cup c(F)$, **(4) [rank]** $c_0(E) = 1$ and $c_k(E) = 0$ for all $k > \text{rank}(E)$, and **(5) [normalization]** the tautological bundle $\tau \to \mathbb{CP}^n$ has $c_1(\tau) = -h$, where $h \in h^2(\mathbb{CP}^n; \mathbb{Z})$ is the **hyperplane class**, which is the positive generator of $H^2(\mathbb{CP}^n; \mathbb{Z}) \cong \mathbb{Z})$ Poincare dual to a complex line in $H_2(\mathbb{CP}^n)$.

For the tautological bundle we can only have c_0 and c_1, and the reason we have it is that otherwise just having 1 would satisfy all of the properties. And the idea is that in a complex vector bundle viewed as a real bundle, we have a canonical orientation (because when we do change of complex trivializations, the real determinant of a complex-valued matrix is always positive – the form would be $dx \wedge dy$ in coordinates if $z = x + iy$).

We get a parallel story for real vector bundles as well, leading us to **Stiefel-Whitney classes** $w(E) = 1 + w_1(E) + \cdots$, but now with $w_i(E) \in H^i(X, \mathbb{Z}_2)$ (so H^i instead of H^{2i}, and $\mathbb{Z}_2$ instead of $\mathbb{Z}$). The axioms are the same, except that in $\mathbb{Z}_2$ we only have one generator so we don't have to worry about sign, and we then have $w_1(\tau) = h$, where h is again Poincaré dual to a line.

We won't prove existence or uniqueness of these classes, but we can read a book by Milnor and Stasheff ("Characteristic Classes") if we want to see a construction. For now, we can take it as a black box and do some examples:

Example 128

We know that $TS^n \oplus \varepsilon \cong \varepsilon^{2n+1}$ (where ε is the normal bundle), and thus we can show $w_i(TS^n) = 0$ for all $i > 0$.

Indeed, w_i of the trivial bundle is zero for all $i > 0$ by naturality (since the trivial bundle is a pullback by a constant map, and the cohomology of a point is nothing in higher degrees), and then we can use multiplicativity to see that the total Chern class is $w(TS^n) = 1$.

Example 129

In a previous homework, we computed the tangent space of $\mathbb{RP}^n$ and showed that $T\mathbb{RP}^n \oplus \varepsilon \cong \tau^* \otimes \varepsilon^{n+1}$ (on the right-hand side this is the same thing as saying $(n+1)$ copies of the dual of the tautological bundle). So again because $w(\varepsilon) = 1$, this means $w(T\mathbb{RP}^n) = (1+h)^{n+1} \in H^2(\mathbb{RP}^n; \mathbb{Z}_2)$, and we get a similar answer for $\mathbb{CP}^n$ – here we're using a fact about duals which is stated later in this lecture.

In particular, this proves that $T\mathbb{CP}^n$ is nontrivial because we have nontrivial. And this also shows that $T\mathbb{RP}^n$ is nontrivial unless $n = 2^k - 1$ for some integer k, since (by some properties of the binomial coefficients, for example $(1+h)^{2^\ell} = 1 + h^{2^\ell} \mod 2$) one of those coefficients will be odd. (Here remember that $H^2(\mathbb{RP}^n; \mathbb{Z}_2) \cong \mathbb{Z}_2[h]/h^{n+1}$.) A further result by Adams actually shows that $\mathbb{RP}^1, \mathbb{RP}^3$, and $\mathbb{RP}^7$ are the only ones that are parallelizable.

The idea is that the top Stiefel-Whitney / Chern classes are **obstructions** to the existence of a nowhere-zero-section:

Proposition 130

Let E be a complex (resp. real) bundle. If $c_{\text{top}}(E) \neq 0$ (resp. $w_{\text{top}}(E) \neq 0$), where top is the rank of the bundle, then E does not have a nowhere vanishing section.

Proof. If s were a nowhere vanishing section, it would span a line subbundle that splits off, and we can write $E = \ell \oplus F$ (where $F = \ell^\perp$ is of rank one less than the original bundle). Thus $c(E) = 1 \cdot c(F)$ and $c(F)$ vanishes at top rank of E, so $c_{\text{top}}(E)$ must be zero (or w in the corresponding spots for a real bundle). $\square$

Proposition 131

If $n = 2^k$, then $\mathbb{RP}^n$ cannot be immersed into $\mathbb{R}^{2n-2}$. (In particular, the strong Whitney theorem is sharp.)

Proof. If we could have such an immersion, then $T\mathbb{RP}^n$ could be embedded into the trivial bundle ε^{2n-2}, and the complement Q would have rank $n-2$. Then $w(T\mathbb{RP}^n)w(Q) = 1$, and looking at mod 2 binomial coefficients again gives us a contradiction (specifically we find we must have $w_{n-1}(Q)$, which is impossible). $\square$

It turns out that line bundles are in fact classified by their classes w_1 or c_1, and we in fact have a stronger general result:

Theorem 132

Let L be a line bundle. The assignment $L \to w_1(L)$ or $L \to c_1(L)$ give one-to-one correspondences $\text{Pic}_{\mathbb{R}}(X)$ (line bundles up to isomorphism) with $H^1(X; \mathbb{Z}_2)$ or $\text{Pic}_{\mathbb{C}}(X)$ with $H^2(X; \mathbb{Z})$. Furthermore these correspondences are **group homomorphisms**; in other words, $w_1(L_1 \otimes L_2) = w_1(L_1) + w_1(L_2)$ (or the equivalent statement for c_1).

This result does not hold for higher-rank bundles – there are rank-2 bundles with the same characteristic classes but are not isomorphic. (Though to prove that, we need to show that some maps into higher-dimensional Grassmannians are not homotopic, which we don't have the tools for right now.) And there are other useful properties as well: for example, we can show that

$$c_k(E^*) = (-1)^k c_k(E), \quad c_1(\det E) = c_1(E).$$

The best way to prove these is using the **splitting principle**, which explains that if E were a direct sum of line bundles these results follow immediately. (Since the inverse of a line bundle is exactly the dual, thus this is true for direct sums of line bundles. Not every bundle splits as a direct sum of line bundles, but we can pull back to a space in which it splits – this uses a more recent fact on our homework that pulling back over the projectivization splits off a line bundle, so continuing that repeatedly will do the job. So we get a splitting manifold $f : X' \to X$ with $f^*E = L_1 \oplus \cdots \oplus L_k$, and since the map $H^*(X) \to H^*(X')$ is injective in cohomology, any result that holds for $H^*(X')$ also holds for $H^*(X)$.)

The focus of our lecture here is really the **Thom class and Thom isomorphism**. Throughout this section, we'll let $E \to X$ be a real rank n vector bundle.

Definition 133

Let $E^+ = E \cup \infty$ be the **one-point compactification** of E (in which the topology is that the open neighborhoods of ∞ are $Y \setminus K$ for compact sets K). For example, the one-point compactification of $\mathbb{RP}^n$ is S^n. The **Thom space** of E is $T(E) = E^+$.

One way to construct this is to let $T(E) = D(E)/S(E)$, where $D(E)$ is the unit disk bundle and $S(E)$ is the sphere bundle. Then the boundary indeed becomes a single point (the point at infinity) and the interior of $D(E)$ is homeomorphic to E.

We'll assume E is oriented and work with $\mathbb{Z}$ coefficients – since the fiber at E at x is $E_x \cong \mathbb{R}^n$, we have the generator $u_x \in H^n(E_x, E_x \setminus 0; \mathbb{Z})$. Then recall that being orientable means that we have a consistent choice of u_x.

Theorem 134

Suppose $E \to X$ is an oriented rank-n vector bundle. Then there is a unique class $u \in H^n(T(E); \mathbb{Z})$, called the **Thom class**, which restricts to each fiber via $\iota_x^* u = u_x$ for all $x \in X$, where $\iota_x : E_x^+ \to E^+ = T(E)$ is the inclusion map. Furthermore (this is a version of Poincaré duality) the cup product with u

$$\cup\omega : H^k(X; \mathbb{Z}) \to H^{k+n}(T(E), \mathbb{Z})$$

is an isomorphism.

We get a similar result with $\mathbb{Z}_2$ coefficents, in which E no longer needs to be orientable. The idea is that first we can show this result for a trivial bundle (meaning it's true in local trivializations) and that we can extend it by Mayer-Vietoris to glue them together and using compactness.

There are a few other properties: the Thom class is multiplicative, meaning that

$$T(E \times F) = T(E) \times T(F)/\sim,$$

where $T(E)$ has fiber S^n and $T(F)$ has fiber S^m so we need to identify the points at infinity and quotient to get the compactification of S^{m+n}. And we check uniqueness by seeing that it restricts to each fiber. Indeed, E is a vector bundle over X so deformation retracts to X, so $H^*(E) \cong H^*(X)$ (in particular, a section $s : X \to E$ and the projection map $\pi : E \to X$ induce inverse maps in cohomology).

Definition 135

Let $E \to X$ be an oriented real bundle of rank n. Then the **Euler class** of $E \to X$ is $\chi(E) = s^* u(E)$, the pullback of the Thom class by a section s of E (such as the zero section).

More explicitly, we have $\chi(E) \in H^n(X; \mathbb{Z})$, though there is also a version living in $\mathbb{Z}_2$ coefficients where we do not need orientability. Since E includes into the pair $(E, E \setminus 0)$, we get a reverse map in cohomology $H^n(E, E \setminus 0) \to H^n(E) \xrightarrow{s^*} H^n(X)$, and then $u \in H^n(E, E \setminus 0)$ maps into $s^* u$ under this composite map – that's the Euler class.

Fact 136

Both the Thom class and Euler class are isomorphism invariant and natural under pullbacks, the Euler class of a sum satisfies $\chi(E \oplus F) = \chi(E) \cup \chi(F)$ (this follows from the property for the Thom class) – more generally we actually have $\chi(E \times F) = \chi(E) \otimes \chi(F)$. Finally, if E has a nowhere vanishing section then $\chi(E) = 0$.

We'll see some more connections between all of these objects next time!

14 February 23, 2023

We'll start **intersection theory** today, but first we'll mention a few important concepts related to last lecture which give some geometric interpretations. We'll expand more on some of this in a few lectures if time permits:

- The top Chern / Stiefel-Whitney class is the Euler class of the bundle, which is Poincaré dual to the zero locus of a generic section. We'll move towards proving this with today's material.
- The image of the total Chern class as a class in de Rham $H^*_{\text{dR}}(M)$ (ignoring the $\mathbb{Z}$ coefficient, working with $\mathbb{R}$ instead) is represented by a universal polynomial in the curvature F_∇ of a connection on E (this is called Chern-Weil theory). Recall that $dF_\nabla = 0$ (the **Bianchi identity**). This polynomial is kind of like the characteristic polynomial: we have
$$c(E) = \left[\det\left(I - \frac{1}{2\pi i}F_\nabla\right)\right] \in H^*_{\text{dR}}(M),$$
so in particular $c_1(E) = \frac{i}{2\pi}[F_\nabla] \in H^2_{\text{dR}}(M)$. And this is useful in differential geometry or later on in differential topology.

Last time, we discussed the Thom isomorphism: for an oriented vector bundle $E \to X$ of rank r, we can define the Thom space $T(E) = E^+$ to be the one-point compactification of E (or when X is compact, take the disk bundle and collapse the boundary to a point). The Thom class is then an element of $H^r(T(E); \mathbb{Z})$ such that it restricts to the generator U_x of $H^r(E_x^+, E_x^+ \setminus 0, \mathbb{Z})$ for any fiber E_x^+. Furthermore, taking the cup product with u yields an isomorphism $H^k(X) \cong H^{r+k}(T(E))$.

Remark 137. *We'll elaborate a bit more about how we actually get such a cup product: for example, we can look at the relative cup product $\cup : H^k(Y) \times H^m(Y, A) \to H^{m+k}(Y, A)$. (In terms of de Rham, $H^m(Y, A)$ is forms on Y which vanish on A, so cupping with any other form will make something else that vanishes on A. But really we should just think of it as a quotient of chain complexes.) So if we now apply this to the space where $Y = D(E)$ is the disk bundle and $A = S(E)$ is the sphere bundle, we do get a cup product $H^k(D(E)) \times H^r(D(E), S(E)) \to H^{r+k}(D(E), S(E))$. But now by the homotopy axiom the first term is $H^k(X)$, and since $r > 0$ by the long exact sequence of the pair $H^r(D(E), S(E))$ is the same as $H^r(D(E)/S(E)) = H^r(T(E))$, and same for the other term.*

We'll use this isomorphism now to get a version of intersection theory. The plan is as follows: we'll first define an intersection product in homology and show that it is Poincaré dual to the cup product in cohomology. Then we will show that the Euler class of E is indeed dual to the zero locus of a generic section of E (that is, $s^{-1}(0)$ for a section s transverse to the zero section as a submanifold).

Everything will be restricted to smooth, closed (compact without boundary) manifolds here, and they'll be oriented as well (otherwise we must use $\mathbb{Z}_2$ coefficients instead of $\mathbb{Z}$). For any such manifold M of dimension m, suppose P, Q are submanifolds of M of dimension p, q, also of that same form. If P and Q are transverse, the $P \cap Q$ is a submanifold of codimension $\text{codim}(P \cap Q) = \text{codim}(P) + \text{codim}(Q)$, so $P \cap Q$ has dimension $p + q - m$. We wish to define the intersection product such that the fundamental classes satisfy $[P] \cdot [Q] = [P \cap Q]$ in $H^*(M, \mathbb{Z})$.

Theorem 138 (Informal statement)
Let $S \subset M$ be an embedding of smooth oriented closed manifolds. Then the Poincaré dual of $[S]$ is the Thom class u_S of the normal bundle of S in M.

Given this, if we know that $PD[S] = u_S$ under Poincaré duality, then the statement $[P] \cdot [Q] = [P \cap Q]$ (we want to show) corresponds to $PD[P] \cup PD[Q] = PD[P \cap Q]$, which is the same as saying that the Thom classes of $P, Q, P \cap Q$ satisfy $u_P \cap u_Q = u_{P \cap Q}$. And this makes sense since $N_{P \cap Q}$ is basically $N_P \oplus N_Q$ (except the restriction maps), and the Thom class is multiplicative. The only problem is that we need to figure out how all of these equalities make sense. (It's not true that every homology class can be represented as the fundamental class of a smooth oriented submanifold, but it is true for $\mathbb{Q}$-coefficients. So up to multiplication by an integer this is true.)

So we'll now get started making all of this more rigorous: let $S \subset M$ again be an embedding of smooth oriented closed manifolds, and denote the normal bundle by N_s so that $TM|_S = TS \oplus N_S$. Since tangent bundles are oriented, this induces an orientation on N_S. By the tubular neighborhood theorem, we have a neighborhood ν_s of S in M, which is diffeomorphic to some neighborhood of the zero section of S.

Theorem 139 (Slightly less informal statement)
Take the same notation as above. Since Poincaré duality is given by taking the cap product with the fundamental class $[M]$, we are saying (essentially, see the proof below for more specifics) that $u_S \cap [M] = [S]$ (where this is holding in the homology $H_*(M;\mathbb{Z})$, though there's also a version where we get this to hold in $H_*(S;\mathbb{Z})$). Furthermore, the Poincaré dual $\eta_S \in H_{\text{dR}}(M)$ of S is represented by the Thom class u_S, where this holds in $H^*_{\text{dR}}(M)$.

Proof. To work towards a proof of this result, we first want to collapse the complement of the tubular neighborhood to a point via the **Thom collapse map** $\tau : M \to \nu_s^+ = N_s^+ = T(N_s)$. (To be more specific there is a diffeomorphism that identifies it, which we won't write down.) Since the Thom class lives in $T(N_s)$, we can look at the pullback map $\tau^* : H^*(N_s)) \to H^*(M)$ (where the normal bundle has rank r), and this map sends $u_S \in H^r(TN_s)$ to $\tau^* u_S$ in M. Since the Thom isomorphism (for the bundle N_S over S) goes from $H^*(S) \xrightarrow{\cup u_S} H^{*+r}(T(N_S))$, and then composing this with the τ^* pullback map yields a map to $H^{*+r}(M)$. So now $\tau^* u_S$ lives in cohomology of M, and now to turn $[S]$ into something in M we push using the inclusion map, and the real statement we want to prove is that

$$\tau^* u_S \cap [M] = \iota_*[S].$$

(For intuition, in terms of de Rham the Thom class is compactly supported in an ε-neighborhood of S and then extended by zero, and that's what τ^* is doing.) But now we have two Poincaré dual maps, one for M and one for S: we have

$$\cap[M] : H^k(M, \mathbb{Z}) \xrightarrow{\cong} H_{m-k}(M, \mathbb{Z}),$$

where the codimension of S is the rank of the normal bundle N_s, which is $r = m - s$ (this is the dimension in which the Thom class lives), and we also have a similar isomorphism

$$\cap[S] : H^k(S, \mathbb{Z}) \xrightarrow{\cong} H_{s-k}(S, \mathbb{Z}).$$

We'll now write down a commutative diagram first at a level of groups – everything is with $\mathbb{Z}$-coefficients, and recall that $m = r + s$:

$$\begin{tikzcd} H^0(S) \arrow[r, "\cup \tau^* u_S"] \arrow[d, "\cap[S]"] & H^{m-s}(M) \arrow[d, "\cap[M]"] \\ H_s(S) \arrow[r, "\iota"] & H_s(M) \end{tikzcd}$$

The idea is to show that this diagram commutes, since if we start with 1 in the top-left side: going along the top path, we get $1 \mapsto \tau^* u_S \mapsto \tau^* u_S \cap [M]$, and along the bottom path we get $1 \mapsto [S] \mapsto [S]$. The idea is to factor through the Thom collapse map, so that we instead have the following diagram:

$$\begin{tikzcd} H^0(S) \arrow[r, "\cup \tau^* u_S"] \arrow[d, "\cap[S]"] & H^{m-s}(\nu_S^+) \arrow[r, "\tau^*"] \arrow[d, dashed] & H^{m-s}(M) \arrow[d, "\cap[M]"] \\ H_s(S) \arrow[r] & H_s(\nu_S) \arrow[r] & H_s(M) \end{tikzcd}$$

The point is to restrict to the square on the left here – by the properties of the Thom collapse map, the right square does commute, and the dashed map we'll describe later. But the maps from $H^0(S)$ in the top left corner are isomorphism by the Thom isomorphism and Poincaré duality, and the bottom map $H_s(S) \to H_s(\nu_S)$ is an isomorphism by the homotopy axiom. So we also get an isomorphism induced for the dashed line, but then we would have to look at the commutativity on the right square more carefully, so we won't do that.

Instead, we'll claim that the naturality of this construction, the Thom class, and Thom isomorphism allow us to reduce the problem to the case where M is a fiberwise one-point compactification of the normal bundle N_S (so each fiber is now a sphere instead of a real space). Let's first look at that case – then let $E = N_S$ and let S_∞, S be the infinity and zero section (the infinity section is the fiberwise point that the sphere bundle collapsed to). The Thom space of E, $T(E)$, is then M except with the entire S_∞ collapsed. So now we can look at our diagram again (we'll just write u instead of u_S):

$$\begin{tikzcd} H^0(S) \arrow[r, "\cup\tau^* u_S"] \arrow[d, "\cap[S]"] & H^{m-s}(M/S_\infty) \arrow[r, "\tau^*"] \arrow[d, dashed, "\cap[M]"] & H^{m-s}(M) \arrow[d, "\cap[M]"] \\ H_s(S) \arrow[r] & H_s(M \setminus S_\infty) \arrow[r] & H_s(M) \end{tikzcd}$$

The point is that this special case of M is a smooth, closed, oriented manifold– we have local trivializations, and instead of $\mathbb{R}^n$s we have products of spheres. Then changes of charts are diffeomorphisms of spheres, which can be thought of just coming from linear transformations $\mathbb{R}^n \to \mathbb{R}^n$ (since ∞ will be taken to ∞ under any such isomorphism). So we have a smooth manifold with boundary, and orientation follows because the tangent space comes from the tangent space plus the normal space, each of which is oriented. So the map on the right $\cap[M]$ still works – it's the usual Poincaré duality – and we want to show that the middle dashed map is now the cap product with M, upon which the square on the right will commute. In other words, we must show that we have an isomorphism $\cap[M] : H^{m-s}(M/S_\infty) \to H_s(M \setminus S_\infty)$. For this we must use **Poincaré-Lefschetz duality** (which we should read in Hatcher or Bredon, along with Alexander duality), since $M \setminus S_\infty$ is an open manifold. The result is basically that for locally compact spaces (and in particular our manifolds are locally compact because everything is finite-dimensional), we have an isomorphism

$$\cap[M \setminus S_\infty] : H^*_{\text{compact}}(M \setminus S_\infty) \to H_*(M \setminus S_\infty),$$

where the **compactly supported cohomology** $H^*_{\text{cpt}}(X)$ is the direct limit $\varinjlim_{K \subseteq X} H^*(X, X \setminus K)$ – this corresponds to the compactly supported de Rham we discussed earlier on, since vanishing on $X \setminus K$ means the form is supported on K. So the red dashed map does come from Poincaré-Lefschetz and then naturality (of Poincaré-Lefschetz on the right, and from uniqueness of the Thom class on the left) shows us that both diagrams commute, hence the whole thing commutes and we get the desired boxed equality. □

15 February 28, 2023

We'll continue intersection theory and Poincaré duality today, but there's one important convention to be aware of for those of us who took 215A. **There are various conventions for the order of the cap product** $\cap$: for example, in Hatcher we take $[M] \cap \cdot$ (so homology capped with cohomology), while in Bredon we take $\cdot \cap [M]$ (cohomology capped with homology). We'll use the latter convention, and the point is that for $\alpha, \beta \in H^*$ and $a \in H_*$, we have the convention

$$\langle \alpha \cup \beta, a \rangle = \langle \alpha, \beta \cap a \rangle$$

while in Hatcher's version we'd have the α move over instead. But the point is that this implies $\boxed{(\alpha \cup \beta) \cap a = \alpha \cap (\beta \cap a)}$ (just by manipulating the identity above, since evaluating this cohomology class on a homology class yields the same answer in both cases), and we will make use of this formula.

Throughout today, saying that $S \subseteq M$ is a submanifold means that both S and M are closed, oriented, and smooth. Recall that last time, we proved that whenever we have $S \subseteq M$, the Thom class u_S of the normal bundle of S is Poincaré dual to the fundamental class $[S]$ (in other words, $u_S \cap [M] = [S]$). And in fact, we get something more general if we use the boxed identity, namely that

$$(\alpha \cup u_S) \cap [M] = \alpha \cap [S] \quad \forall \alpha \in H^*(M).$$

So now we can define intersection theory by hand and show it yields an invariant, or (our approach) we'll define it in homology to be the Poincaré dual of $\cup$, and then we'll find its geometric interpretation in terms of actual intersections of manifolds. An intersection product in homology is then of the form

$$H_*(M) \times H_*(M) \xrightarrow{\cdot} H_*(M)$$

sending (a, b) to $a \cdot b$ (Latin letters will denote homology, and greek letters will denote cohomology), where we define the dashed arrow in the following way (and note that the vertical arrows are all isomorphisms, so we can in fact traverse them in reverse):

$$\begin{tikzcd}
H_*(M) & \times & H_*(M) \arrow[r, dashed, "\cdot"] & H_*(M) \\
H^*(M) \arrow[u, "\cap[M]"] & \times & H^*(M) \arrow[u, "\cap[M]"] \arrow[r, "\cup"] & H^*(M) \arrow[u, "\cap[M]"]
\end{tikzcd}$$

In other words, we will define (so that we are Poincaré dual to the cup product)

$$a \cdot b = D^{-1}(Da \cup Db) = Da \cap b,$$

where $D : H_*(M) \to H^*(M)$ is the **inverse** of the Poincaré dual map. Furthermore, since $\alpha \cup \beta = (-1)^{\dim(\alpha)\dim(\beta)} \beta \cup \alpha$ (dimension k meaning that we live in $H^k(M)$), and Poincaré duality takes dimension to codimension, we see that in homology we have

$$a \cdot b = (-1)^{\operatorname{codim}(a)\operatorname{codim}(b)} b \cdot a \quad \forall a, b \in H_*(M).$$

Remembering that when we said $u_S \cap [M] = [S]$ last time, we really meant that we push forward $[S]$ into homology and use the Thom collapse map τ^* to modify the Thom class, so the real statement is that $\boxed{\tau^* u_S \cap [M] = \iota_*[S]}$.

Theorem 140

Suppose P, Q are submanifolds of M (again, all closed, oriented, and smooth), and assume that P is transverse to Q. Then $P \cap Q$ is a smooth manifold (also closed) with a natural orientation, and $[P \cap Q] = [P] \cdot [Q]$.

In particular, if we think about all of these fundamental classes going to the Thom classes of normal bundles under Poincaré duality, we must have

$$u_{P \cap Q} = u_P \cup u_Q;$$

really we need to restrict these classes to $P \cap Q$ and collapse, but the morally correct fact is that we get Thom classes via cup product. And furthermore, even if P and Q are not transverse, we can perturb them slightly (via flows) so that they are.

Proof. The main idea is as follows: we proved that the normal bundle of $P \cap Q$ is the sum of the normal bundles of P and Q (after restriction to the intersection). And we'll prove that the Thom class of a direct sum is the cup product of the Thom classes on our current homework, so that all fits together.

First, we should note that P being transverse to Q implies (by the implicit function theorem) that $P \cap Q$ is a closed smooth manifold, and we have a description of its normal bundle. Looking at the orientations, we know that P, Q, M are oriented, so TP, TQ, TM are oriented bundles, and in particular $TP \oplus N_P = TM|_P$, so N_P is also oriented canonically (and so is N_Q) in a way so that the direct sum is orientation-preserving. But then restricting $TQ \oplus N_Q = TM|_Q$ to the intersection $P \cap Q$, which is a submanifold of Q, the normal bundle in Q will be the restriction of N_P to $P \cap Q$. We then have

$$T_x Q = T_x(P \cap Q) \oplus (N_P)_x$$

fiberwise at any point $x \in P \cap Q$, meaning that

$$T_x M = T_x(P \cap Q) \oplus N_{P,x} \oplus N_{Q,x}.$$

And this is how we induce the orientation on $P \cap Q$: every term here has a canonical orientation except the one for $P \cap Q$, so we also induce an orientation on the intersection. And as we said before, the normal bundle $N_{P \cap Q, M}$ is the restriction (so pullback by inclusion) $\iota^* N_{P,M}$ direct summed with the $\iota^* N_{Q,M}$, where the two ιs are the inclusions $P \cap Q \to P$ and $P \cap Q \to Q$. So now by **multiplicativity** of the Thom class we see that $u_{P \cap Q, M} = \iota^* u_{P,M} \cup \iota^* u_{Q,M}$. Then by Poincaré duality this then gives us the result (since another way to write $u_S \cap [M] = [S]$ is that the Poincaré dual in M, $D[S] = u_S$, is the Thom class of the normal bundle of S in M), because we now have by definition of the intersection product

$$[P] \cdot [Q] = D^{-1}(D[P] \cup D[Q]) = D^{-1}(u_{P,M} \cup u_{Q,M}) = D^{-1} u_{P \cap Q} = [P \cap Q],$$

as desired. □

And notice also that we can rewrite this last relation as $[P] \cdot [Q] = u_P \cap [Q]$.

Remark 141. *If manifolds are not necessarily oriented, then we have a $\mathbb{Z}_2$ version of this statement – the problem is that de Rham doesn't see torsion, so we do want orientations, but if just care about homology we can do $\mathbb{Z}_2$.*

The idea now is that if P and Q are of complementary dimension and intersect transversally, then we get a **intersection number** (also denoted $\cdot$)

$$P \cdot Q = \varepsilon_*([P] \cdot [Q]) \in H_0(M) \cong \mathbb{Z},$$

basically summing up the orientations at the individual points, where ε is the **augmentation map** sending $H_0(M) \to \mathbb{Z}$ (since 0-dimensional homology can be thought of as constants, and we can just take the value; equivalently think of this as the induced map in homology sending M to a single point).

Thus if M is m-dimensional, we get a nondegenerate pairing $H_k(M) \times H_{m-k}(M) \to \mathbb{Z}$ sending (a, b) to $\varepsilon(a \cdot b)$, which is Poincaré dual to the corresponding pairing $H^{m-k}(M) \times H^k(M) \to \mathbb{Z}$ sending (α, β) to $\langle \alpha \cup \beta, [M] \rangle$ (so here the augmentation map corresponds to pairing with $[M]$). More explicitly, we can think of this as a two-step process $H^{m-k}(M) \times H^k(M) \to H^m(M) \to \mathbb{Z}$ sending (α, β) to $\alpha \cup \beta$ to $\langle \alpha \cup \beta, [M] \rangle$.

Remembering that as a set (but also as oriented manifolds, up to a sign) $P \cap Q$ is the same as intersecting $P \times Q$ with the diagonal Δ (and this relates cross product to cup product). So the statement is then that

$$[P] \cdot [Q] = (-1)^m [P \times Q] \cdot [\Delta_M],$$

where $m = \dim(M)$. More concretely, we can think in terms of de Rham: let u_P be the Thom class in $H^k(\nu_P^+)$ (where ν_p is the tubular neighborhood we've been talking about, k is the codimension of P in M, and then we one-point-compactify), and for locally compact spaces we have whenever $k > 0$ that

$$H^k(\nu_P^+) = H^k_{\text{compact}}(\nu_P)$$

(and we get the reduced homology for $k = 0$). So in our case, the image of the Thom class of the normal bundle N_P in M under de Rham is the class $\eta_P \in H^*_{compact}(M)$ (the Poincaré dual of P) which we defined a few lectures ago (the Thom collapse map of u_P in $H^k_{\text{compact}}(\nu_P, \mathbb{R}) = H^k(\nu_P^+, \mathbb{R})$, which pulls back to $H^k(M, \mathbb{R})$). Remember that η_P can be thought of as compactly supported almost as a delta function in a small neighborhood of P in M. And then we find that

$$[P][Q] = [P \cap Q] = \int_M \eta_P \wedge \eta_Q$$

(since in de Rham, the cap becomes a wedge), which is the same as $\int_P \eta_Q = u_Q \cap [P]$.

We can now talk about **inverse images and degree**: for a continuous map $f : M \to N$ (we can work in topology or in smooth topology) between closed, oriented topological manifolds, we can define (using Poincaré duality) "wrong way maps" (also called **shriek maps**) where homology goes backward and cohomology goes forward. The idea is to pull back in homology by pulling back in cohomology and using Poincaré duality:

$$\begin{tikzcd} H_k(N) \arrow[r, dashed, "f_!"] \arrow[d, "D"] & H_{m-n+k}(M) \arrow[d, "D"] \\ H^{n-k}(N) \arrow[r, "f^*"] & H^{n-k}(M) \end{tikzcd}$$

Following the diagram, we then get a map

$$f_! : H_k(N) \to H_{m-n+k}(M), \quad f_!(a) = D^{-1} f^*(Da).$$

Geometrically, if $f : M \to N$ is a smooth map between smooth (oriented, closed) manifolds, then by Sard's theorem a generic point $p \in N$ is regular, so $f^{-1}(p)$ is smooth. The pushforward in homology is then

$$f_![\text{pt}] = [f^{-1}(\text{pt})],$$

and more generally if f is transverse to some manifold Q, then $f_![Q] = [f^{-1}Q]$. Since we have Poincaré duality on both M and N, in cohomology we then get

$$f^!\alpha = \int_{f^{-1}(\text{pt})} \alpha,$$

where we are integrating α on a generic fiber.

Definition 142

If M, N are two manifolds of the same dimension, and N is connected, then the **degree** of the map $f : M \to N$, an element of $\mathbb{Z}$, is well-defined as

$$f_*[M] = (\deg f)[N].$$

We can interpret this geometrically as follows:

Corollary 143

If $f : M \to N$ is a map between smooth manifolds, we then have

$$\deg f = [M][f^{-1}(p)] = \sum_{x \in f^{-1}(p)} \mathrm{sign}(x).$$

The picture is that if we pick a generic point $p \in N$, the inverse image in M is a 0-dimensional manifold with a canonical orientation, meaning we have a bunch of points each with a $+$ or $-$ sign. Then the degree is then the sum of the ± 1s.

We can then think about the geometric interpretation of the Euler class and self-intersection as follows: if $S \hookrightarrow M$ is a smooth, closed, oriented submanifold, we can define the **self-intersection** of S as follows: we can push S off of itself (by homework) to some perturbation S_t, which will be transverse to S generically, and define

$$S \cdot S = [S \cap S_t] \in H_{2s-m}(S),$$

and thus we can include it into $H_{2s-m}(M)$ if we want. And in particular if $m = 2s$ this then gives us something we can identify as a number:

Theorem 144

Let $S \subseteq M$ be a submanifold (we assume smooth, closed, oriented, but we don't need smoothness, just enough to have a normal bundle). Then the self intersection of S is Poincaré dual to the Euler class of its normal bundle, meaning that $S \cdot S = \chi(N_S)$ in $H_*(S; \mathbb{Z})$ (which we can include into $H_*(M; \mathbb{Z})$). In particular, the Euler class of an oriented vector bundle $E \to S$ is Poincaré dual to the zero locus $s^{-1}(0)$ of a generic section s of E (that is, a section transverse to zero).

We'll prove this next time!

16 March 2, 2023

Today will focus on the Euler class, Poincaré duality, and applications – remember that all manifolds today are closed, oriented, and smooth (and all bundles are oriented, or else we need $\mathbb{Z}_2$ coefficients), and we have (when $S \subset M$ is a submanifold) that $[S]$ is Poincaré dual to the Thom class u_S of the normal bundle, meaning that $u_S \cap [M] = [S]$ or equivalently $(\alpha \cup u_S) \cap [M] = \alpha \cap [S]$ for any α. Here we should think of the cohomology class $\alpha \in H^*(M)$ as a form in de Rham, and then the right-hand side is like $\int_S \alpha$. Then $\alpha \cup u_S$ corresponds to the intersection product of the Poincaré dual of α and the Poincaré dual of u_S (which is just S). So in words, the intersection of S with a cohomology class's Poincaré dual is the same as integrating α on that surface, and that's why we think of the Poincaré dual of S as a delta function supported normal to S.

We thus get a geometric interpretation of the Euler class of a vector bundle via self-intersections, which is Theorem 144 – basically if we take a generic section s of E (which will be transverse to the zero section) and look at the zero locus, then $\chi(E)$ (which is a cohomology class) satisfies $\chi(E) \cap [S] = [s^{-1}(0)]$ (these are homology classes in $H_*(S)$, which is the same as the homology of $H_*(E)$). Simultaneously, we can prove the following corollary: if $S \subset M$ is a submanifold, then the self-intersection $S \cdot S = \chi(N_S) \cap [S]$ is also this same homology class. (And this is the same as the previous statement by the tubular neighborhood theorem, since self-intersection can be made to happen in just a neighborhood of S.)

Proof of Theorem 144. Taking a tubular neighborhood of S, we can reduce to proving this result where M is the fiberwise one-point compactification of E (this is the same strategy as when we proved that $u_S \cap [M] = [S]$ – it yields a sphere bundle over S). So we have the same diagram, where $S \subset E$ is the zero section: pushing s off of itself to get s_t transverse to s, or equivalently using a section s to the normal bundle of S, which is E,

$$S \cdot S = [S \cap S_t] \in H_*(S)$$

and we want to prove that $[S \cap S_t] = \chi(E) \cap [S]$. We know $S \cdot S$ is Poincaré dual to $u_S \cup u_S$, which comes from the cross product:

$$\begin{tikzcd} u \times u \arrow[r, "\cup"] \arrow[d, "\mathrm{PD}(M)\cong"] & u \cup u \arrow[d, "\mathrm{PD}(M)\cong"] & u \arrow[l, "\cup u"'] \arrow[d, "\mathrm{PD}(S)\cong"] \\ S \times S \arrow[r, "\cdot"] & S \cdot S & u \cap [S] \arrow[l] \end{tikzcd}$$

So in other words, the following diagram commutes, where ν_S is the tubular neighborhood of S and ν_S^+ is where we identify the entire infinity section to a point):

$$\begin{tikzcd} H^r(\nu_s^+) \times H^r(\nu_s^+) \arrow[r, "\cup"] \arrow[d, "\cap[M]"] & H^{2r}(\nu_S^+) \arrow[d, "\cap[M]"] & H^r(S) \arrow[l, "\cup u"'] \arrow[d, "\cap[S]"] \\ H_{m-r}(S) \times H_{m-r}(S) \arrow[r, "\cdot"] & H_{m-2r}(\nu_S) & H_{m-2r}(S) \arrow[l, "\cong"] \end{tikzcd}$$

The blue arrow here comes from the Thom isomorphism theorem, the red one from the homotopy axiom (specifically, this is coming from the pushforward of the section), and here recall that $m - r = \dim(S)$. So the right square commutes (this is just the statement $(\alpha \cup u_S) \cap [M] = \alpha \cap [S]$), and the left square commutes by the definition of the intersection product. So going back to the first diagram, we see that self-intersection $S \cdot S = [S_t \cdot S] = [s^{-1}(0)]$ by definition, but then looking at the diagram this is the same as $s^* u \cap [S]$, and pulling back the Thom class yields the Euler class $\chi(E) \cap [S]$. $\square$

Example 145

Consider $M = \mathbb{CP}^2$ and let S be a complex line L (which we can think of as $\mathbb{CP}^1$ sitting inside $\mathbb{CP}^2$), which is really a sphere. In homogeneous coordinates $[z_0, z_1, z_2]$, we can let L be the set of points $[z_0, z_1, 0]$. (Recall that $\mathbb{CP}^2$ is like compactifying a point at infinity in each direction for $\mathbb{C}^2$.)

We then claim that $L \cdot L$ is a single point (since we can just solve for a nearby equation like $z_2 = \varepsilon z_1$); since these are complex lines we have a canonical orientation corresponding to taking v, iv, and intersecting two complex things gives us a **positive** orientation. Furthermore, because complex things are even-dimensional, we don't have to worry about signs of intersections.

Corollary 146

For the tautological line bundle τ over real or complex projective spaces $\mathbb{RP}^n, \mathbb{CP}^n$, its Euler class is $-h$, where h is the hyperplane class Poincaré dual to a complex line L.

So for $\tau \to \mathbb{CP}^n$, we have $c_1(\tau) = \chi(\tau)$ for a line bundle, which is Poincaré dual to the zero locus of a generic section, which is $-h \in H^2(\mathbb{CP}^n, \mathbb{Z})$.

Remark 147. *The negative sign here comes from the orientation of the tangent bundle –* $\chi(\tau) \in H^2(\mathbb{CP}^n;\mathbb{Z}) \cong \mathbb{Z}$*, and* h *generates the cohomology because intersection product with a line gives* 1*. So* $\chi(\tau) = -h$*, and testing it against a line by integrating on a two-dimensional class yields* $\chi(\tau) \cap L = \chi(\tau|_L)$ *by naturality, and thus this reduces to calculating the Euler class of the tautological bundle over a sphere* $\mathbb{CP}^1$*. Then we can use the section* $z \mapsto \overline{z} = \frac{|z|^2}{z}$*, and that has degree* -1*.*

And for $\tau \to \mathbb{RP}^n$, we have $w_1(\tau) = \chi(\tau) = h \in H^1(\mathbb{RP}^n, \mathbb{Z}_2)$ (where the Euler class is now in $\mathbb{Z}_2$). Here we're using the following fact:

Proposition 148

The top Chern class of a complex vector bundle E is the Euler class, meaning that $c_{\text{top}}(E) = \chi(E)$ is Poincaré dual to the zero locus of a transverse (generic) section. And the top Stiefel-Whitney class of a real vector bundle is similarly $w_{\text{top}}(E) = \chi_2(E)$, Poincaré dual mod 2 to the zero locus of a transverse section.

So we have an interpretation of not just the Euler class but also the top classes of a bundle. We won't go into the proof in detail during lecture, but the point is to take an axiomatic approach, showing that that $\chi(E)$ also satisfies the same axioms (normalization, naturality, and so on).

Theorem 149

The Euler characteristic of a (smooth, closed, oriented) manifold is the same as the Euler class of its tangent bundle. In other words (we're evaluating the Euler class on the fundamental class so that we actually get a number)

$$\chi(TM)[M] = \chi(M) = \sum_{i=0}^{m} (-1)^i \text{rank}(H_i(M)).$$

Proof outline. This is mostly a straightforward application of the previous theorem, but we do intersection with the diagonal $P \times Q \cap \Delta_M$ and then split the diagonal in homology (that is, split the Thom class of the diagonal in cohomology). More specifically, let Δ_M be the diagonal of M in $M \times M$. We then claim that the self-intersection $\Delta_M \cdot \Delta_M$ is both the left and right-hand side of what we want. Indeed, the tangent space to the diagonal is points of the form (v, v) with $v \in TM$, so the normal to Δ_M consists of vectors $(v, -v)$ with $v \in TM$, and this is isomorphic to the tangent bundle because we just added a negative sign (though the changing orientation may give us an extra factor of $(-1)^m$). Thus the self-intersection

$$\Delta_M \cdot \Delta_M = \chi(N_\Delta)[\Delta] = \chi(TM)[M],$$

since there is a map $\iota : M \to M \times M$ sending x to (x, x) which yields a diffeomorphism $M \to \Delta_M$. But we can also calculate self-intersection differently by the Kunneth decomposition.

If we choose a basis $\{h_i\}$ of $H_*(M)$, where all $h_i \in H_{\dim(h_i)}(M)$, and since the intersection is a nondegenerate pairing we have a dual basis $\{h^j\}$, meaning that $h_i \cdot h^j = \delta_{ij}$. Remember then that we pick up a sign if we change the order, and thus $h^j \cdot h_i = (-1)^{\text{codim}(h_i)\text{codim}(h^j)}\delta_{ij}$ (and if h_i, h^j do not have complementary dimension the pairing doesn't exist so we just declare it to be zero).

Lemma 150

Under Kunneth, the diagonal Δ_M of M splits as

$$[\Delta_M] = \sum (-1)^{\dim(h_i)} h_i \times h^i.$$

Proof of sketch. This is a calculation – we have

$$[\Delta_M] = \sum_{i,j} c_{ij} h_i \times h^j,$$

so if we want to find the coefficients c_{ij}, we should intersect with $h^k \times h_\ell$, upon which we get

$$\Delta \cdot (h^k \times h_\ell) = \sum_{i,j} c_{ij}(h_i \times h^j)(h^k \times h_\ell).$$

We can now use that $(a \times b) \cdot (c \times d) = (-1)^{\text{codim}(c)\text{codim}(b)}(a \cdot c) \times (b \cdot d)$ (this comes from how cup and cross product relate, since cup is cross product restricted to the diagonal) and then we use Poincaré duality), and similarly $a \cdot b = \Delta_M \cdot (a \times b)$. But then plugging everything back in yields

$$[\Delta_M](h^k \times h_\ell) = h^k \cdot h_\ell = (-1)^{\text{codim}(k)\text{codim}(\ell)} \delta_{\ell k}.$$

□

This finishes the proof that we're after, since we indeed get $(-1)^{\dim(h_i)}$ for each i and adding them together yields the Euler characteristic. □

Corollary 151

Let $S \subseteq M$ be a (smooth, closed, oriented) submanifold. If the Euler class of the normal bundle of S is nonzero, then S cannot be pushed off of itself (or else the Euler class will be nonzero). And in particular, if the Euler characteristic $\chi(M)$ is nonzero, then any vector field on M has at least one zero, and we've deduced a generalization of the hairy ball theorem (since for example $\chi(S^2) = 2$).

We'll look at more consequences next time, proving the Poincaré-Hopf and Lefschetz fixed point theorems!

17 March 9, 2023

Today, we'll discuss zeros of vector fields and fixed points of flows or diffeomorphisms (as an application of what we've done so far). The idea is to use intersection theory to show that (under certain conditions) fixed points must exist and can be counted up to sign.

Recall that (in this part of the class, if we want to work with $\mathbb{Z}$ coefficients then all manifolds are smooth – though C^1 is enough – closed, and oriented) if we have a manifold M with a submanifold S, then the Poincaré dual of S is the Thom class u_S of the normal bundle. We've also defined an intersection product in homology which is Poincaré dual to the cup product in cohomology, so that $[P] \cdot [Q]$ is Poincaré dual to $u_P \cup u_Q$. And if P and Q are transverse, then these are equal to $[P \cap Q] \xrightarrow{\text{PD}} u_{P \cap Q}$. We then found that if $E \to S$ is an oriented vector bundle, then the Euler class $\chi(E)$ (a cohomology class) evaluated at the homology $[S]$ is the self-intersection $S \cdot S$ of the zero section, which is $[s^{-1}(0)]$, the zero locus of a a section transverse to the zero section. So in short, the Poincaré dual to $\chi(E)$ (in S) is the (homology class) of the zero locus of a section of E transverse to the zero section (in particular, the generic section is transverse).

We'll now apply all of this to vector fields. If we let v be a vector field on M (that is, a section of the tangent bundle TM), recall that a zero of the vector field is also called a **singular point**.

Definition 152

A singular point x of the vector field v is **nondegenerate** if v is transverse to the zero section at x.

Since the set of singular points is also the intersection of v with the zero section, and because we showed (by "splitting the diagonal" in homology) that the Euler class satisfies

$$\chi(M) = \chi(TM)[M] = 0_M \cdot 0_M,$$

and this right-hand side is the sum of $\operatorname{sign}(x)$ over all singular points x in v if v has only nondegenerate singular points. (Indeed, 0_M and v are both m-dimensional, and we're intersecting inside the $(2m)$-dimensional space TM, so the intersection is a compact oriented 0-dimensional manifold, hence finitely many points because M is compact and oriented.)

More generally, if x is any **isolated zero** of a vector field on M (which doesn't mean the intersection has to be transverse – imagine something like $y = x^2$ intersecting the x-axis), we can generalize the notion of a sign and associate to x the **index** $\operatorname{ind}_x(v) \in \mathbb{Z}$ given by the local intersection number. In other words, **(1)** we deform v in a neighborhood of x so that it is transverse to the zero section, and then count (with sign) the number of intersection points. (We do need to check that this is well-defined independent of the (sufficiently small) perturbation we want, though.) Or we can use the following alternative method: **(2)** restrict v to a small ball $B_\varepsilon(x)$ in M, such that x is the only zero of v. Work in a local trivialization and choose a metric, and let $S_\varepsilon(x)$ be the sphere of radius ε around x in M. Then we can let $\operatorname{ind}_x(v)$ be the degree of the map $y \mapsto \frac{v(y)}{|v(y)|}$ from $S_\varepsilon(x)$ to S^{m-1} (here we're looking at the value in the fiber T_yM), and we again have to check that this is well-defined independent of the choices (of metric, local trivializations, and our sufficiently small ε). And we may show that these definitions are the same.

Example 153

If we consider a vector field in $\mathbb{R}^2$ which is pointing radially inward towards the origin, then the vector field has index 1.

Theorem 154 (Poincaré-Hopf)

Let M be a smooth, closed, oriented manifold, and let v be a vector field on M with only isolated zeros. Then the Euler characteristic of M is

$$\chi(M) = \sum_{x \in v^{-1}(0)} \operatorname{ind}_x(v).$$

Proof sketch. If v is transverse to the zero section, then the index $\operatorname{ind}_x(v)$ is the sign of x (with our second method, the idea is that locally the vector field looks linear so we just need to determine the sign); otherwise perturb to make it transverse. □

(In particular, any vector field on the sphere must have zeros because $\chi(S^2) \neq 0$.)

We'll now see another similar setup: let $f : M \to M$ be a diffeomorphism (such as a flow of a vector field), and let x be a fixed point of f (meaning that $f(x) = x$, or equivalently that we have an intersection point of Γ_f with the diagonal Δ_M, which is also the graph of the identity map). Then x is **nondegenerate** if Γ_f is transverse to Δ at x. Then the sign of x is given by

$$\operatorname{sign}(x) = \operatorname{sign} \det(I - df).$$

And if x is an isolated fixed point of f, then we can define an index $\text{ind}_x(f)$ similarly to how we did above by taking the local intersection number at x of Γ_f with Δ (here we're using that the normal bundle to the diagonal is isomorphic to the tangent bundle), or also as the appropriate map between spheres as before.

Theorem 155 (Lefschetz fixed point theorem)

Let M be a smooth, closed, oriented manifold, and let f be a diffeomorphism of M. If f has only isolated fixed points, then the local index is equal to the **Lefschetz number**

$$\sum_{x \in \text{Fix}(f)} \text{ind}_x(f) = L(f) = \sum_{k=0}^{m} (-1)^k \text{trace}(f_{*,k}),$$

where $f_{*,k}$ is the induced map of f on homology $H_k(M)$ (with real coefficients, but it doesn't matter), and we are taking its trace as a linear transformation.

Proof. The left-hand side is a sum of indices, which (as before) is the intersection is the intersection $\Gamma_f \cdot \Delta$. Since we have a splitting of (homology class of) the diagonal (in terms of a basis in homology and a dual basis), which we discussed last lecture, we can use the same approach to get a splitting of the graph Γ_f in homology:

$$[\Gamma_f] = \sum_i (-1)^{\dim(h_i)} h_i \times f_*(h^i)$$

where h_i is a basis of $H_*(M)$ and h^i is its dual basis with respect to the intersection product. (So in the case of the identity the map f_* is just the identity.) And we can show this by demonstrating that the two sides are equal if we pair against $h^i \times h_j$, which form a basis of $H_*(M \times M) = H_*(M) \otimes H_*(M)$ (here we're working with $\mathbb{R}$-coefficients). So because $\Delta = \Gamma_{\text{id}}$ we can plug things in and use properties of the cross product to get the result. $\square$

Corollary 156 (Topological Lefschetz fixed point theorem)

Let $f : X \to X$ be a homeomorphism (in fact sufficient to have X a CW complex and f continuous) with nonzero Lefschetz number $L(f)$. Then f has a fixed point.

This is basically a generalization of the Brouwer fixed point theorem, and in fact this tells us a way to count the sum of the indices of our fixed points (if they are isolated). In particular, if f is homotopic to the identity, f_* is just the identity map and the trace of id_V is just $\dim(V)$. Thus in this case we will have $L(f) = \sum (-1)^k \dim H_k = \chi(M)$ equal to the Euler characteristic.

Example 157

If v is a vector field and ϕ_t is a series of diffeomorphisms coming from the flow, then all of these maps are diffeomorphic to the identity, so $L(\phi_t) = \chi(M)$.

Example 158

If x is a zero of a vector field v, that yields a fixed point of ϕ_t. Thus a nondegenerate zero leads us to a nondegenerate fixed point of the flow, and this connects back to Poincaré-Hopf as well. More generally we can look at time-t periodic orbits, which also correspond to fixed points of ϕ_t, and find that $L(\phi_t) = \sum \text{ind}_x \phi_t$.

Remark 159. *In dynamical systems, we may start with a diffeomorphism $f : M \to M$ and consider the fixed points of its iterates f^i. We can then assemble the* **Lefschetz zeta function***, which is the formal power series*

$$\zeta_f(t) = \exp\left(\sum_{n=0}^{\infty} \frac{L(f^n)}{n} t^n\right).$$

We can in fact do some calculations and see that

$$\zeta_f(t) = \frac{\prod_{k \text{ odd}} \det(I - tf_{*k})}{\prod_{k \text{ even}} \det(I - tf_{*k})}$$

For example, the zeta function for any map homotopic to the identity is $\zeta = (1-t)^{-\chi(M)}$.

The last thing we'll do today is to discuss transversality a bit more – there are a few needed results about transversality that we needed when defining (for example) the intersection number. All of these follow from the following fact (which we proved), which is an application of Sard's theorem:

Theorem 160 (Thom transversality)
Assume that $F : P \times S \to M$ is transverse to a map $g : Q \to M$. Then for a generic $s \in S$, the map $F_s = F|_{P \times \{s\}}$ is generic to g.

Theorem 161
Let $E \to M$ be a vector bundle over a compact manifold M, and let $g : P \to E$ be a smooth map. Then we can deform the zero section of E to make it transverse to g, or we can deform g to make it transverse to the zero section.

In particular, this yields all of the transversality that we wanted in the earlier results (for example showing that we can make the zero section transverse to itself), since we can always perform the correct deformations.

Proof. We'll just prove the first statement. The idea here is that M is compact, so E can be embedded into some large trivial bundle ε^N of large rank. Writing $\varepsilon^N = E \oplus F$ (with $F = E^\perp$ with respect to some fixed metric), then we get a projection $\pi : \varepsilon^N \to E$. Then for each $y \in \mathbb{R}^N$ we have a constant section s_y, and for generic y the projection $\pi(s_y)$ is a section of E transverse to the zero section, since putting these maps all together yields a map $F : \mathbb{R}^n \times M \to \varepsilon^N \to E$ sending (y, x) to $s_y(x)$ to $\pi(s_y(x))$, and (since F is transverse to the zero section) the Thom transversality theorem applies. □

Next time, we'll discuss Morse theory, and as an application we'll be able to get estimates of the number of critical points of a (Morse) function.

18 March 14, 2023

Our topic for these last two lectures will be **Morse theory** – we'll present two perspectives, both the classical one and the "more modern" one which has more applications to current problems. We'll let M denote a closed (smooth) manifold, oriented when needed, and $f : M \to \mathbb{R}$ a smooth function. Recall that x is a **critical point** of f if df_x is not onto, and since we have a map onto $\mathbb{R}$ that's equivalent to having $df_x = 0$. (The typical example to keep in mind is the **height function** of the torus, where we stand it up and let f denote its height.) So the linearization of f vanishes at any critical point x, meaning we can instead look at the "quadratic" approximation. One definition uses a metric, but here is a more intrinsic one:

Definition 162

Let x be a critical point of f. The **Hessian** at x, denoted $\text{Hess}_x(f)$, is the symmetric bilinear form $T_xM \times T_xM \to \mathbb{R}$ defined by

$$\text{Hess}_x(f)(v, w) = V(Wf)_x,$$

where V, W are vector fields on M that extend v and w, and where we recall that $Vf = (df)v$ is the directional derivative of f in the direction of v.

Lemma 163

The definition above is well-defined (meaning that it is independent of the extension of vector field) and symmetric.

Proof. We'll first show that this is symmetric – indeed, $V(Wf)_x - W(Vf)_x = ([V, W]f)_x$, where the commutator $[V, W]$ is also a vector field and thus this becomes $df[V, W]$ evaluated at x, which is zero because x is a critical point. (So notably the Hessian is only symmetric at critical points – partial derivatives in different directions may not commute in general.) Then $h = Wf$ is a function on M, and $(Vh)_x = dh_x(V) = dh_x(v)$ only depends on the value of the vector field at the point x and is independent of extension. So then by symmetry the same argument shows that this only depends on teh value of W at the point x. $\square$

Example 164

When $M = \mathbb{R}^n$, the Hessian $\text{Hess}_x(f)$ is the usual matrix of second derivatives $\left(\frac{\partial^2 f}{\partial x_i x_j}(x)\right)$ in coordinates. So this also extends to charts and local coordinates on a general manifold.

Definition 165

A critical point x of f is **nondegenerate** if the Hessian $\text{Hess}_x f$ is nondegenerate as a bilinear form (that is, $\text{Hess}_x f(v, v) = 0$ if and only if $v = 0$).

Remember that if we have a map $f : M \to \mathbb{R}$, then we can think of df as a map $TM \to T\mathbb{R}$ but also as a map $x \mapsto df_x$. Then since $df_x : T_xM \to T\mathbb{R} = \mathbb{R}$ is a 1-form, we see that df is a section of the cotangent bundle T^*M and thus traces out a "graph" on M (where at x we have $(x, df_x) \in T_x^*M$). The critical points of f then form the zero locus of this section, and those points being nondegenerate corresponds to transversality to the zero section. (This is because intrinsically the Hessian at x is like the derivative of the derivative, so we basically have $(\nabla s)_x$ for some **connection** ∇ in E so that we still end up with a section. And then a calculation shows that at the zero section s, we have $(\nabla s)_x$ independent of the choice of connection.)

Definition 166

Let x be a nondegenerate critical point of f. The **index** $\text{ind}_x f$ is the number of negative eigenvalues of $\text{Hess}_x f$, counted with multiplicity (this is also related to the signature of the Hessian).

Here note that because the Hessian is a symmetric, nondegenerate bilinear form, it has only positive and negative real eigenvalues. And then (as we discussed last time) $\text{ind}_x(f)$ will be the local intersection number of S (that is, the intersection of df with the zero section).

Lemma 167 (Morse lemma)

Let p be an index k nondegenerate critical point of f. Then there are local coordinates $(x_1, \cdots, x_n)$ around p, such that

$$f(x) = f(p) - \sum_{i=1}^{k} x_i^2 + \sum_{i=k+1}^{n} x_i^2.$$

In other words, we can find coordinates so that we just have a quadratic function.

Proof. Assume without loss of generality that $f(p) = 0$ (just by shifting) and remember that in local coordinates we're working around 0. We may arrange for $f(x)$ to look like $\sum_{i,j=1}^n x_i x_j h_{ij}(x)$ near p, where the h_{ij}s are smooth functions with the values of $h_{ij}(0)$ forming the Hessian matrix. (The idea is to write

$$f(x) = \int_0^1 \frac{d}{dt} f(tx) dx,$$

and then we can use the chain rule and get an expression of the form $\sum x_i(\text{stuff})$, then repeat one more time.) From there, we can do the proof by induction: change coordinates so that $f(x) = \pm x_1^2 \pm x_2^2 \cdots \pm x_\ell^2 + \sum_{i,j=\ell+1}^n x_i x_j h_{ij}(x)$, and then we can change variables because the Hessian is nondegenerate – the formula is to keep $u_m = x_m$ for all $m \le \ell$ and then define

$$u_{\ell+1} = \sqrt{h_{\ell+1,\ell+1}(x)} \left(x_{\ell+1} + \sum_{m=\ell+2}^{n} x_m \frac{h_{m,\ell+1}(x)}{h_{\ell+1,\ell+1}(x)} \right)$$

if $h_{\ell+1,\ell+1}(x)$ is positive and otherwise we put a negative inside the square root. Then plugging it in either yields $x_{\ell+1}^2$ or $-x_{\ell+1}^2$; in particular, the number of negative signs will always be the same because it is the dimension of the negative eigenspaces of $\text{Hess}_x(f)$, which is independent of coordinates and equal to the index. $\square$

Corollary 168

If f has only nondegenerate critical points, then the critical points are isolated.

Indeed, we can see this from the local model, since $\sum \pm x_i^2$ only has a critical point at zero. But alternatively, nondegenerate critical points correspond to transverse intersection, meaning the intersection is a 0-dimensional manifold (and is thus a collection of isolated points).

Definition 169

A (smooth) function $f : M \to \mathbb{R}$ is **Morse** if all of its critical points are nondegenerate.

Example 170

Thinking back to the height function on a torus (which is Morse), there are four critical points – the bottom one looks like $x_1^2 + x_2^2$ (index 0), the top one looks like $-x_1^2 - x_2^2$ (index 2), and the middle two are saddle points (index 1).

Example 171

The **monkey saddle** $f(x, y) = x^3 + 3xy^2$ is not Morse, because 0 is a degenerate critical point. But adding a small deformation (such as $ax + by$ for small a, b) will make this a Morse function. And for a "worse example," something like $f(x) = e^{-1/x^2} \sin\left(\frac{1}{x}\right)$ is smooth but not Morse, since it vanishes at infinite order at 0.

We'll show that adding linear terms does work for making functions Morse in general. Notice that if M is a closed manifold and f is Morse, then the set of critical points is a 0-dimensional **compact** manifold, hence finite.

Lemma 172

Let M be a closed manifold. Then the generic function $f : M \to \mathbb{R}$ is Morse.

The idea is to do something similar to how we showed the generic projection works in the Whitney embedding theorem, producing a bunch of perturbations from a finite-dimensional space.

Proof. Embed M in $\mathbb{R}^N$ for some N, and look at linear perturbations of the form $F(a, x) = f_a(x) = f(x) + a \cdot x$, where $a \in \mathbb{R}^N$. So F is a map $\mathbb{R}^N \times M \to \mathbb{R}$ (where we use the embedding of M in $\mathbb{R}^N$), and we wish to show that dF is transverse to zero (so that by Thom transversality we see that df_a is transverse to zero for generic a). But this is true because regardless of the value of x, dF will be onto by choosing a appropriately. □

Unfortunately, the space of Morse functions is **not path-connected** in general – that is, if we look at $C^\infty(M)$ (with the compact open topology) and we take a generic path in $C^\infty(M)$, we will have finitely many instances where f_t still has an isolated critical point. (For example, if we start with a kink in the graph and straighten it out, there will be a point where the critical point is degenerate – thus the bad locus corresponds to birth or death of a pair of critical points and to df_t being **tangent** to the zero section.)

Definition 173

The **sublevel set** of f is the inverse image $M^a = f^{-1}((-\infty, a])$.

By transversality, but extended to manifolds with boundary (we just need to ask for the boundary to be transverse as well), we find that for all regular values a of f, M^a is a manifold with boundary (with boundary given by the level set $f^{-1}(a)$).

Proposition 174

Suppose $a < b$ and there are no critical values in $[a, b]$. Then $f^{-1}([a, b])$ is diffeomorphic to $f^{-1}(a) \times [a, b]$. Thus, M^a is diffeomorphic to M^b and we have $M^b = M^a \cup_{f^{-1}(a)} f^{-1}(a) \times [a, b]$ (where we glue along the boundary).

We'll just prove the first one. The idea is to "run the gradient flow" – we can imagine basically pouring liquid into our manifold and letting the height rise. The gradient does depend on the metric, though:

Proof. Choose a metric g on TM, which induces an isomorphism $TM \cong T^*M$ (since $g : T_xM \times T_xM \to \mathbb{R}$ allows us to map $v \mapsto g(v, \cdot)$). We can then take the section df of T^*M, and this maps under this isomorphism to a vector field ∇f (this is called the **gradient**, and it has nothing to do with the connection) given by

$$df(\cdot) = g(\nabla f, \cdot).$$

But the gradient is perpendicular to level sets, so we just need to scale it so that it varies appropriately: define $\rho : M \to \mathbb{R}$ to be

$$\rho(x) = \frac{1}{||(\nabla f)_x||^2}$$

on $f^{-1}([a, b])$, with norm measured with respect to g. (This is well-defined, since having no critical points means $\nabla f \neq 0$ everywhere on $f^{-1}([a, b])$. If we then bump this function down to zero outside $f^{-1}([a - 2\varepsilon, b + 2\varepsilon])$, and we look at the flow of the rescaled vector field

$$v = \rho(x)\nabla f.$$

We can then check that when we take its flow ϕ_t, we do get $\phi_{b-a} : M^b \to M^a$ a diffeomorphism, with

$$\frac{d}{dt}(\phi_t) = df(v) = \langle \nabla f, v\rangle = \langle \nabla f, \rho(x)\nabla f\rangle = \rho(x)||\nabla f||^2,$$

which is 1 within $[a, b]$ and 0 outside $[a - 2\varepsilon, b + 2\varepsilon]$. This yields the diffeomorphism. □

So in other words, the topology of sublevel sets only changes whenever we cross a critical value.

Definition 175

Let M be an m-dimensional compact manifold with boundary ∂M, and let $f : S^{k-1} \times D^{n-k} \to \partial M$ be a (not necessarily smooth) map. Then the space $M \cup_f (D^k \times D^{n-k})$ is said to be "obtained from M by attaching a k-dimensional handle."

This construction is topological, but the Morse lemma really implies the following:

Theorem 176

If x is a critical point of index k of a Morse function $f : M \to \mathbb{R}$, and $c = f(x)$ is a critical value with x the **only** critical point in $(c - \varepsilon, c + \varepsilon)$, then $M^{c+\varepsilon}$ is obtained from $M^{c-\varepsilon}$ by attaching a k-handle.

In other words, the D^k part is the negative eigenspace, and D^{n-k} is the positive eigenspace – this will relate to the direction in which the gradient flows to or away from x.

19 March 16, 2023

We'll continue Morse theory today, mostly discussing the classical case (with finite-dimensional manifolds) and then briefly mentioning the modern approach. Recall that the key example to keep in mind is the height function $f : M \to \mathbb{R}$ of a manifold, where the critical points are all nondegenerate (for example by seeing that the Hessian is nondegenerate or that df is transverse to the zero section). Then each critical point has an index, which is the number of negative eigenvalues of the Hessian, and at each such point p the function looks like $f(x_1, \cdots, x_n) = f(p) + \sum_i \pm x_i^2$ in local coordinates (with $\mathrm{ind}_p(f)$ equal to the number of negative signs).

The idea is that up to diffeomorphism, level sets and sublevel sets only change when we get to a critical point. Then if we suppose f has only a single critical point x of index k for each critical value c (imagine having a manifold with two "humps" at the same height – we can just slightly deform the function so that the points appear at different values of f), then we get the sublevel set $M^{c+\varepsilon} = f^{-1}(-\infty, c + \varepsilon]$ from the sublevel set $M^{c-\varepsilon} = f^{-1}(-\infty, c - \varepsilon]$ by "attaching a k-handle" – that is, taking the union with $D^k \times D^{n-k}$ and attaching via some map $S^{k-1} \times D^{n-k}$.

Example 177

If we think about the height function of a torus and consider the second lowest critical point (with index 1) at value c, notice that the level set at $c + \varepsilon$ consists of two circles (for the two sides of the torus). Then as we go down to $c - \varepsilon$, those two circles will "flow down." Pictorally, we can imagine a purse with a handle; the two circles of the torus will flow down to form the boundary of the top of the purse on the two sides of the handle. Then D^k corresponds to the "core" of the handle and S^{k-1} corresponds to the attaching point.

(And the proof that this attaching process works does in fact use some kind of gradient flow line idea.) In particular, this means that **any smooth closed** (really C^2 is enough) **manifold has the topology of a CW complex**, since we're attaching a k-cell D^k when we attach a k-handle:

Theorem 178

Let M be a smooth closed manifold, and let $f : M \to \mathbb{R}$ be a Morse function. Then M has the homotopy type of a CW complex, with one cell in dimension k for each index-k critical point of f.

For example, the height function on a torus has points of index $0, 1, 1, 2$, and indeed the usual construction where we identify opposite edges of a square is a CW complex with those cells. However, note that this result does **not** hold for topological manifolds.

Corollary 179

There is a chain complex, called the **Morse-Smale complex**

$$\cdots \xrightarrow{\partial} CM_k \xrightarrow{\partial} CM_{k-1} \xrightarrow{\partial} \cdots ,$$

such that CM_k is the free $\mathbb{Z}$-module generated by critical points of index k, and such that the homology is (naturally) isomorphic to the usual (singular, cellular, etc.) homology $H_*(M, \mathbb{Z})$.

In particular, we get bounds for the minimum number of critical points. We know that there will always be a minimum and a maximum for any function $f : M \to \mathbb{R}$ on a smooth closed manifold, but we can now say more:

Theorem 180 (Weak Morse inequalities)

Let $f : M \to \mathbb{R}$ be a Morse function on a smooth closed manifold M. Then the number of critical points of index k is at least $\text{rank}(H_k(M))$.

Example 181

Any Morse function on $M = \mathbb{CP}^n$ must have at least one critical point of index $2k$ for all $k \in \{0, 1, \cdots, n\}$, so in particular there are at least $(n + 1)$ critical points.

On the other hand, the sphere can have just two critical points (the minimum and the maximum). We in fact have a converse:

Theorem 182 (Reeb)

Let M be a smooth closed manifold of dimension n with a Morse function with exactly two critical points. Then M is **homeomorphic** to S^n.

Indeed, the first nontrivial sublevel set must be a ball (because we start off with a minumum), and then we get an attaching map with the maximum, meaning that M is $B^n \cup B^n$ with some attaching map along the boundary. This means (because the attaching map is a diffeomorphism, hence a homeomorphism, and then using some facts about $\text{Homeo}(S^n)$) M is homeomorphic to $B^n \cup_{\text{id}} B^n = S^n$.

On the other hand, M is not always diffeomorphic to S^n – Milnor found an exotic 7-sphere, which is a manifold homeomorphic but not diffeomorphic to the standard S^7, and he did so by using this attaching construction. He then classified the exotic smoooth structures on S^7 and found that there are 28 of them (coming from denominator of the Bernoulli numbers). In general there will be finitely many such smooth structures – the **Smale h-cobordism theorem** implies that in dimensions at least 5, the number of exotic smooth structures on S^n is $\pi_0(\text{Diff}(S^{n-1}))$, where the diffeomorphisms on S^{n-1} are exactly our attaching maps along the equator and we're considering them up to homotopy. And then a result of Milnor and Kervaire implies that (for $n \geq 5$) there are finitely many different smooth structures on S^n which form a finite-dimensional group under connected sum.

Morse theory is in fact a good tool for classifying one- and two-dimensional closed manifolds because we can only attach so many things – in one dimension we just have the circle, and in two dimensions we just have the sphere, torus, and connected sums of tori. That type of argument shows that there are no exotic spheres in dimension at most 3, but in fact the question for $n = 4$ is still open. In contrast, Taubes proved around 1980 using gauge theory has proved that there are infinitely many smooth structures on $\mathbb{R}^4$ – the way to tell them apart is by writing down a geometric PDE and extracting an invariant to count the number of solutions. And then Donaldson proved around 1990 that there are 4-dimensional topological manifolds that do not admit a smooth structure.

Remark 183. *Since then, there has been various progress constructing smooth structures on various closed* 4-*dimensional manifolds (which are distinguished by some larger homology $H_*(M)$). Even something like $\mathbb{CP}^2$ doesn't have enough homology for the question to be answered – the invariants aren't good enough – but it is known that the connected sum of three copies of $\mathbb{CP}^2$ does have infinitely many smooth structures.*

The same techniques that prove the weak Morse inequalities in fact tell us more:

Corollary 184 (Strong Morse inequalities)
Let $f : M \to \mathbb{R}$ be a Morse function on a smooth closed manifold M, and let $c_k(f)$ be the number of critical points of index k for f and $b_k(M)$ be the Betti numbers $\text{rank}(H_k(M))$. Then for all k,

$$\sum_{i=0}^{k} (-1)^{k-i} c_i(f) \geq \sum_{i=0}^{k} (-1)^{k-i} b_i(M).$$

In particular, adding the results for k and $k+1$ yields $c_{k+1}(f) \geq b_{k+1}(M)$, which is the weak inequality.

To conclude, we can also formulate the **Morse-Smale-Witten** version of Morse theory. Again let M be a closed manifold and $f : M \to \mathbb{R}$ be a Morse function. Our goal is to construct a complex like the Morse-Smale complex, generated by the critical points, and now specifying the differential (boundary) map to **count the number of gradient flow lines** of f. The idea is to pick a metric g on M – the metric gives a canonical isomorphism $TM \cong T^*M$, and thus $df \in T^*M$ corresponds to the vector field $\nabla f = \text{grad}(f)$ which is perpendicular to level sets of f. So we can consider the negative gradient flow of f, meaning that we consider integral curves – we can imagine pouring syrup on the top of the manifold and seeing how it flows down. (This is a closed manifold, so flow exists for all time.) We then want a solution $\gamma : \mathbb{R} \to M$ such that

$$\frac{d\gamma}{dt} + (\nabla f)(\gamma(t)) = 0,$$

which we then think of as a flow ϕ_t.

Definition 185

Let x be a critical point of f of index k. We can then define the **stable manifold**

$$W^s(x) = \{y \in M : \phi_t(y) \to x \text{ as } t \to \infty\}$$

and the **unstable manifold**

$$W^u(x) = \{y \in M : \phi_t(y) \to x \text{ as } t \to -\infty\}.$$

For example, on the torus, the "inner circle" forms the stable manifold on the bottom index-1 critical point, and then the circle sitting below that point forms the unstable manifold. This gives us a saddle picture. In general, we will find using the Morse lemma that W^s, W^u are homeomorphic to balls of dimension $n-k, k$ respectively, so $\dim(W^s(X)) = n - \text{ind}_x(f), \dim(W^u(x)) = \text{ind}_x(f)$.

Definition 186

A metric g is **Morse-Smale** with respect to f if $W^s(x)$ is transverse to $W^u(y)$ for all critical points x, y.

In such a situation, we can always consider the set of points with gradient flow lines from x to y (or alternatively go to x as $t \to -\infty$ and y as $t \to \infty$); this can be characterized by the set $W^u(x) \cap W^s(y)$, and it is a smooth manifold of dimension $\text{ind}_x(f) - \text{ind}_y(f)$. Since we have an $\mathbb{R}$-action which translates by time, we can mod this set out by our action (so if two points are along the same flow lines, they are identified), and that will give us the set of gradient flow lines. (Notice that if $x \neq y$, then this action is free since the flow cannot stay put.) Letting $\mathcal{M}(x, y)$ be the resulting quotient, we will get a smooth manifold (Hausdorff takes a bit of work, but we can basically take a slice near x) of dimension $\text{ind}_x(f) - \text{ind}_y(f) - 1$. And now we can define our complex: let $CM_k(f, g)$ be the chain complex again generated by critical points of index k in degree k, and now define

$$\partial x = \sum_{y \text{ of index } k-1} (\text{number of gradient flow lines (counting with sign) } x \to y) \cdot y.$$

Indeed, if the index of x is one more than the index of y, then $\mathcal{M}(x, y)$ is a zero-dimensional manifold, so it is a bunch of points (and we can check that it is oriented). We then just need to check that $\partial^2 = 0$ and that the homology agrees with the usual one. (Here note that the gradient flow lines do depend on the metric g, but it will turn out that the homology will still be naturally isomorphic to $H_*(M)$.)