Imperial College London

[MP2 2014]

B.Sc. and M.Sci. EXAMINATIONS 2014

SECOND YEAR STUDENTS OF PHYSICS

MATHEMATICAL METHODS

Date Monday 2nd June 2014 2.00 pm - 4.00 pm

DO NOT OPEN THIS PAPER UNTIL THE INVIGILATOR TELLS YOU TO.

Answer ALL of Question 1 and TWO questions out of Questions 2, 3, 4 and 5.

A mathematical formulae sheet is provided

[*Before starting, please make sure that the paper is complete; there should be 6 pages, with a total of FIVE questions. Ask the invigilator for a replacement if your copy is faulty.*]

1. (i) Consider the complex exponential function $f(z) = e^z$. Show that f is differentiable for all $z \in \mathbb{C}$.

(ii) Determine the residue at $z = 0$ of

$$f(z) = \frac{e^z - 1}{z^4}$$

(iii) Determine the Fourier transform $\hat{f}(k)$ of the function

$$f(x) = e^{-\alpha x^2} \qquad \text{for all } x \in \mathbb{R} \text{ and } \alpha > 0.$$

(You may use without proof the fact that $\int_{-\infty}^{\infty} \exp(-u^2)\, du = \sqrt{\pi}$)

(iv) Use contour integration to compute the integral

$$I = \int_{-\infty}^{\infty} \frac{dx}{(1+x^2)(4+x^2)}.$$

(v) Consider the sequence

$$a_n = \frac{1+n^2}{n^2} \in \mathbb{Q}. \tag{1}$$

Use the standard absolute value norm and show that the sequence is a Cauchy sequence.

Consider the sequence

$$b_n = \left(\frac{1+n}{n}\right)^n \in \mathbb{Q}. \tag{2}$$

From the fact that $e^x = \lim_{n\to\infty} \left(1 + \frac{x}{n}\right)^n$ explain why b_n must be a Cauchy sequence and also explain why b_n cannot be strongly convergent in $\mathbb{Q}$ given that $e \notin \mathbb{Q}$.

(vi) Consider the two dimensional plane. Use the calculus of variations to show that the shortest path between two points in the plane is a straight line.

(vii) Show that for a 3×3 matrix $\mathbf{A}$

$$\det \mathbf{A} = \epsilon_{ijk}\, a_{1i}\, a_{2j}\, a_{3k}.$$

(viii) Explain the Newton-Raphson algorithm and find the real positive root to two decimal places of

$$\cos(x) = x^2.$$

2. (i) Consider the two functions

$$f_1(z) = \frac{z^2 + \frac{1}{4}}{z\left(z - \frac{1}{2}\right)^2}$$

$$f_2(z) = \exp\left(\frac{1}{z-1}\right).$$

Identify the singularities, and determine their nature, for both functions.

(ii) Compute the residues of the poles of f_1 and f_2.

(iii) Let $\mathcal{C}$ denote a circle of radius 2 with centre at the origin and positive orientation. Compute the integral

$$\int_{\mathcal{C}} [f_1(z) + f_2(z)]\, dz.$$

(iv) Suppose the Fourier transform of a function $g(x)$ is given by

$$\hat{g}(k) = \frac{1}{k-i}.$$

Find the function $g(x)$ for all values of $x \in \mathbb{R}$ including $x = 0$.

(v) The double Fourier transform

$$\hat{f}(k,\omega) = \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dt\, f(x,t) e^{-i(kx+\omega t)}$$

of a function $f(x,t)$ satisfies the equation

$$i\,\omega \hat{f} + k^2 \hat{f} - 1 = 0.$$

Find the corresponding differential equation satisfied by $f(x,t)$.

3. (i) Use suffix notation and the permutation symbol to show that for any two vectors $\mathbf{a}$ and $\mathbf{b} \in \mathbb{R}^3$

$$\mathbf{a} \times \mathbf{b} \cdot \mathbf{a} = 0\,.$$

(ii) Explain if the following sets, A, B and C, of vectors constitutes a basis for $\mathbb{R}^3$.

$$A: \qquad \mathbf{a}_1 = (1,0,0),\ \mathbf{a}_2 = (1,1,1),\ \mathbf{a}_3 = (0,0,1),\ \mathbf{a}_4 = (-1,0,1)$$

$$B: \qquad \mathbf{b}_1 = (\frac{1}{2},1,-1),\ \mathbf{b}_2 = (0,-1,2),\ \mathbf{b}_3 = (-\frac{1}{2},-\frac{1}{2},0),$$

$$C: \qquad \mathbf{c}_1 = (1,1,1),\ \mathbf{c}_2 = (0,1,1),\ \mathbf{c}_3 = (-\frac{1}{2},0,0)$$

(iii) Consider $\mathbb{R}^\infty$ with the l_1 norm and show, using ϵ formalism, that the sequence $\mathbf{a}_n \in \mathbb{R}^\infty$ given by

$$\mathbf{a}_n = \left(1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, ..., \frac{1}{2^n}, 0, 0, ...\right)$$

is a Cauchy sequence.

(iv) Show that a strongly convergent series is always a Cauchy sequence.

(v) Use the Schwarz inequality

$$|\langle x, y\rangle| \le \langle x, x\rangle^{1/2} \langle y, y\rangle^{1/2}$$

to show that

$$\sum_{n=0}^{\infty} \frac{1}{\sqrt{2^n n!}} \le \sqrt{2e}.$$

(Hint: You may want to consider sequences constructed from the power series representations of e^x and $(1-x)^{-1}$.)

4. (i) Derive the Euler-Lagrange equation for the functional

$$J[y] = \int_{x_0}^{x_1} f(y, y', x)\, dx$$

where $y \in C^2([x_0, x_1], \mathbb{R})$ and is subject to the boundary conditions $y(x_0) = y_0$ and $y(x_1) = y_1$.

(ii) Show that if the kernel f does not explicitly depend on x, then the Euler-Lagrange equation implies

$$f - y' \frac{\partial f}{\partial y'} = \text{constant.}$$

(iii) Use the calculus of variations to derive Newton's second law for a particle moving along the x axis in a potential $U(x)$.

(iv) Let

$$J[y] = \int_0^2 \frac{\sqrt{1 + (y')^2}}{y}\, dx.$$

Consider curves starting at (0,1) and ending at (2,3) . Determine the curve that minimises the functional $J[y]$.

(v) Consider functions $y(x)$ for which $y(0) = 0$, $y(1) = 1/2$ and $\int_0^1 2y\, dx = 1/6$. Amongst this set of functions determine $y(x)$ such that it makes the functional

$$J[y] = \int_0^1 \{(y')^2 + 2y\}\, dx$$

stationary.

5. (i) Define the Kronecker delta symbol δ_{ij} .

(ii) Define the permutation symbol ϵ_{ijk} .

(iii) Use suffix notation to show that

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\mathbf{c}.$$

You may use without proof that $\epsilon_{ijk}\,\epsilon_{pqk} = \delta_{ip}\delta_{jq} - \delta_{iq}\delta_{jp}$.

(iv) Consider the transformation between two orthonormal systems S and S' . Show that the square length of a vector transforms as a tensor of rank zero.

(v) Consider the differential equation

$$\frac{dy}{dx} = f(x, y(x))$$

and derive the Runge-Kutta scheme

$$y_{n+1} = y_n + \frac{h}{2}\left[f(x_n, y_n) + f(x_{n+1}, y_n + f(x_n, y_n)h)\right].$$

(vi) Consider the equation

$$\frac{dy}{dx} = x\tanh(xy).$$

Use the Runge-Kutta scheme to estimate, to one significant decimal, the value of $y(x)$ at $x = 1.1,$ 1.2 and 1.3 given that $y(1) = 1$.

END OF PAPER