MATH0003 Analysis 1 Notes

Based on the 2018 autumn lectures by Prof L Parnovski

ANALYSİS 1

ANALYSIS 1 (1101)

< Leonid Parnovski
office: 607 (after Tuesday lecture)

Problem classes → 9-10 am on Tuesdays.

5% Hw → 10 pieces → 8 marked < ★ compulsory problems / NO ★ voluntees. } STUDENT NUMBER (Room 502).
5% midsessional exams
90% Final exam.

Analysis → Calculus
theoretical basis.

→ Definitions
→ Axioms : statements that do not need proofs } Theorems <— proof.

Ex. Consider the infinte series: $\left(1-\frac{1}{2}\right)+\left(\frac{1}{3}-\frac{1}{4}\right)+\left(\frac{1}{5}-\frac{1}{6}\right)+\ldots=\ln 2>0$

we $\times\frac{1}{2}$ $\Rightarrow \frac{1}{2}-\frac{1}{4}+\frac{1}{6}-\frac{1}{8}+\frac{1}{10}-\frac{1}{12}+\ldots=\frac{\ln 2}{2}$ } $=\frac{3\ln 2}{2}$

Now we have: $\left(1-\frac{1}{2}+\frac{1}{2}\right)+\left(\frac{1}{3}-\frac{1}{4}-\frac{1}{4}\right)+\left(\frac{1}{5}-\frac{1}{6}+\frac{1}{6}\right)+\left(\frac{1}{7}-\frac{1}{8}-\frac{1}{8}\right)\ldots=$

$=1+\frac{1}{3}-\frac{1}{2}+\frac{1}{5}+\frac{1}{7}-\frac{1}{4}\ldots=\frac{3\ln 2}{2}$.

START OF THE COURSE

$\mathbb{N}\subset\mathbb{Z}\subset\mathbb{Q}\subset\mathbb{R}\subset\mathbb{C}$ — $\subset$ → include (forma parte).

- $\mathbb{N}$ → natural numbers
- $\mathbb{Z}$ → enteros
- $\mathbb{Q}$ → rational numbers
- $\mathbb{R}$ → real numbers (infinitos)
- $\mathbb{C}$ → complex numbers

$\mathbb{N}=\{1,2,3,4,\ldots\}$ → ⚠ 0 IS NOT A NATURAL NUMBER $\boxed{0\notin\mathbb{N}}$

→ are elements of the set of $\mathbb{N}$ numbers.

* Addition: $n, m \in \mathbb{N}$ ⟶ (leads to) $n+m\in\mathbb{N}$ → belongs to the thing in the right.

* multiplication $n, m\in\mathbb{N}$ ⟶ $n\cdot m \in \mathbb{N}$

* Ordering: $\forall$ $n, m\in\mathbb{N}$ we have either $m>n$ or $m<n$ or $m=n$
$\forall$ → whichever (for all)

⚠ ANNOTATION: $\exists$ means "there exists"

$\neg m>n \iff m\le n$.
$\neg$ → not (true)
$\iff$ is equivalent to / if and only if

→ Properties of addition and multiplication

① $m+n=n+m$.
② $(m+n)+p=m+(n+p)$
③ $m\cdot n=n\cdot m$
④ $(m\cdot n)\cdot p=m\cdot(np)$
⑤ $(m+n)\cdot p=mp+np$.

→ Properties of ordering

① $m > n,\ n > p \underset{\text{implies}}{\Rightarrow} m > p$.

→ Problems with $\mathbb{N}$ numbers : solving equations .

$x + m = n$ solution : $x = n - m$.

no solution if $m \geq n$ (no hay solución natural porque no sabemos si el resultado va a ser un $\mathbb{N}$ number) .

$\mathbb{Z}$ (enteros) $= \{ \dots, -2, -1, 0, 1, 2, 3 \}$

$m \cdot x = n$ means $x = \frac{n}{m}$

$\mathbb{Q}$ (rational numbers) $= \left\{ \frac{p}{q} \underset{\text{such that}}{:} p, q \in \mathbb{Z},\ q \neq 0 \right\} =$

→ we are talking about a set .

(es solo para que se diferencien más) .

$= \left\{ \frac{p}{q} , p, q \in \mathbb{R} ,\ q \neq 0 \right\} =$

$\underbrace{\frac{p}{q} = \frac{p \cdot m}{q \cdot m}}_{\text{It's the same } \mathbb{Q} \text{ number.}}$ We can assume that p and q are coprime

no common prime factors .
(no se pueden simplificar)

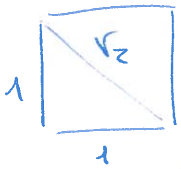
$= \left\{ \frac{p}{q} : , p \in \mathbb{Z} , q \in \mathbb{N} ; p \text{ and } q \text{ are co-prime} \right\}$

- ✱ Multiplication $\frac{p}{q} \cdot \frac{m}{n} = \frac{p \cdot m}{q \cdot n}$
- ✱ Addition : $\frac{p}{q} + \frac{m}{n} = \frac{p \cdot n + m \cdot q}{q \cdot n}$

→ Still problems (with rational numbers) :

① $x^2 = 2$ → irrational (there are not rational solution to this equation) .

② $x^2 = -1$ (next year) .



⚠ Th (theorem) .

The irrationality of $\sqrt{2}$

$\underset{\text{not existe}}{\nexists}\ x \underset{\text{pertenece}}{\in} \mathbb{Q} \underset{\text{such that}}{\text{s.t.}}\ x^2 = 2$.

Lemma (Less important theorem used to prove important Ths) .

If $n \in \mathbb{Z}$ is odd, then n^2 is also odd .

Proof: n is odd $\Rightarrow n = 2k+1, k \in \mathbb{Z} \Rightarrow n^2 = (2k+1)^2 = 4k^2 + 4k + 1 =$
$= \boxed{2 \cdot (2k^2 + 2k) + 1 \text{ is odd}}$. ▨ Q E D } demostrado.

(follows from lemma).

Corollary: (por lo tanto) (consecutivamente).
If n^2 is even, then n is also even

Proof of th-m

→ By contradiction. (estamos deduciendo que nunca va a haber un $\mathbb{Q}$ number que al cuadrado siga siendo co-prime).
assume that claim is false and deduce that our assumptions were not satisfied (or $0 = 1$).
Therefore the claim is true.
Assume $\exists\, x = \frac{p}{q} \in \mathbb{Q}$, s.t. $x^2 = 2$.

W L O G: we assume p and q are coprime.
(without loss of generally)
i.e. we can't achieve this to be true.

$x^2 = \frac{p^2}{q^2} = 2$, so, $p^2 = 2q^2 \Rightarrow p^2$ is even then corollary $\Rightarrow p$ is even,
$p = 2 \cdot k$, $k \in \mathbb{Z}$, so $\underset{(2k)^2 = 4k^2;}{\overset{\parallel}{p^2}} = 2q^2 \Rightarrow q^2 = 2k^2 \Rightarrow q^2$ is even $\Rightarrow$ corollary $\Downarrow$ q is even.

so $\underbrace{p \text{ and } q}_{}$ are not co-prime. (Como q y p al cuadrado dejan de ser co-prime, dejan de ser racionales, y el claim por lo tanto no existe).
Contradiction because they are both even
This, $\nexists\, x \in \mathbb{Q}$ s.t. $x^2 = 2$ ▨

4th October 2018

$\mathbb{R}$

(real numbers)

Ex: $\pi, \sqrt{2}$

- Addition: as they are infinite ¿how to we start the sum?
- Multiply: " ¿how do we do it?
- Every $\mathbb{R}$ no has modulus (absolute value).
  ⇓ (distance)

$x \in \mathbb{R}$

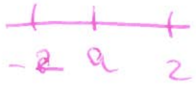
$$|x| = \begin{cases} x, & x \geq 0 \\ -x, & x < 0 \end{cases}$$

$x, y \in \mathbb{R}$ the distance between x and y is $|x - y|$.

$-|x| \leq x \leq |x|$

$-2 < a < 2$

$$|a| < 2 \begin{cases} a < 2 \\ a > -2 \end{cases}$$



Ex: Can we approximate $\sqrt{2}$ by rationals?

i.e. Can we find $\frac{p}{q}$ which is close to $\sqrt{2}$?. In other words, $\frac{p^2}{q^2}$ should be close to 2

$$\left|\frac{p^2}{q^2}-2\right| \text{ is small?}$$

Then, hopefully: $q^2\left|\frac{p^2}{q^2}-2\right| = |p^2-2q^2|$ is not very large.

But $|p^2-2q^2| \in \mathbb{N} \underset{\text{or}}{\cup} \{0\}$.

It can not be 0 as $\frac{p}{q} \neq \sqrt{2}$.

porque si $|p^2-2q^2| = 0$ entonces si que se cumple que $\frac{p}{q} = \sqrt{2}$, y eso no puede pasar

The smallest possibility is 1

Then $|p^2-2q^2| = 1$ and $\boxed{p^2-2q^2=\pm 1}$.

⇓

Pell's equation

Programme:

1st: Solve Pell's equation.

2nd: Use solutions to approximate $\sqrt{2}$.

Step 2: assuming we have Pell's equation solution:

$$p^2-2q^2=\pm1, \quad p,q>0, \quad p,q\in\mathbb{N}$$

How big $\left|\frac{p}{q}-\sqrt{2}\right|$ is?

$$\frac{p^2}{q^2}-2=\frac{\pm1}{q^2}, \text{ so } \left|\underbrace{\left(\frac{p}{q}-\sqrt{2}\right)\cdot\left(\frac{p}{q}+\sqrt{2}\right)}_{\text{Identidad notable}}\right| = \left|\pm\frac{1}{q^2}\right|$$

$$\text{so } \underbrace{\left|\frac{p}{q}-\sqrt{2}\right| \cdot \left|\frac{p}{q}+\sqrt{2}\right|}_{} = \frac{1}{q^2}, \text{ so } \left|\frac{p}{q}-\sqrt{2}\right| = \frac{1}{q^2\cdot\left(\frac{p}{q}+\sqrt{2}\right)}$$

this no is positive

en caso de producto $|AB| = |A|\cdot|B|$ ≠ en caso de suma $|A+B| \neq |A|+|B|$.

Use: if $0<a<b$, then $\frac{1}{a} > \frac{1}{b}$ (careful: wrong if $a<0$)

We have: $\frac{p}{q}+\sqrt{2} > \sqrt{2} > 1$. Therefore,

$$\boxed{\left|\frac{p}{q}-\sqrt{2}\right| = \frac{1}{q^2\left(\frac{p}{q}+\sqrt{2}\right)} < \frac{1}{q^2}}$$

Step 1: Solve Pell's equation: (very clever solution)

Take $p_n, q_n \in \mathbb{N}$ for $n=1,2,3\ldots$ defined by:

$p_1 = q_1 = 1$;

$p_{n+1} = p_n + 2q_n$

$q_{n+1} = p_n + q_n$.

Proposition $\forall n$ we have ① $p_n^2 - 2q_n^2 = \pm 1$, ② $\boxed{q_n \geq n}$.

se explica con las 1^{as} formulas, voy añadiendo cosas ⊕.

n	p_n	q_n	p_n/q_n
1	1	1	1
2	3	2	$3/2 = 1.5$
3	7	5	$7/5 = 1.4$
4	17	12	$17/12 = 1.4166$
5	41	29	$41/29 = 1.41379$

Proof of proposition (by induction).

Need to prove : $n=1$ is ok ?

① $p_1^2 - 2q_1^2 = \pm 1 = 1 - 2 = -1$.

② $q_1 = 1 \geq 1$

Assuming that $n=k$ is fine, i.e. $p_k^2 - 2q_k^2 = \pm 1$, $q_k \geq k$.

Then: $n = k+1$

① $q_{k+1} = q_k + p_k \geq k + p_k \geq k+1$ ✓

also,

② $p_{k+1}^2 - 2q_{k+1}^2 = (p_k + 2q_k)^2 - 2\cdot(p_k + q_k)^2 = p_k^2 + 4p_k \cdot q_k + 4q_k^2 - 2p_k^2 - 2q_k^2 - 4p_k q_k = -p_k^2 + 2q_k^2 = -\underbrace{(p_k^2 - 2q_k^2)}_{\pm 1} = \mp 1$ ✓ ▨

- $\underbrace{\text{Axioms}}_{\text{postulado}}$ **for $\mathbb{R}$**

Suppose, X is a set with 2 binary operations, $+$ and $\cdot$

(i.e. given arbitrary $x, y \in X$, we can define $x+y \in X$, $x \cdot y \in X$)

<u>definition</u> : we say that X is a field, if it satisfies the following 9 axioms (of arithmetic).

addition

A1 ⇒ $\forall a, b, c \in X$.
$(a+b)+c = a+(b+c)$ ⇒ associative laws.

A2 ⇒ $a+b = b+a$ ⇒ commutative law

A3 ⇒ $\exists$ (there exists) a special element $0 \in X$ s.t. $a+0=a$ $\forall a \in X$
(0 is called the additive identity).

A4 ⇒ $\forall a \in X$ $\exists$ $\underset{\text{another element}}{b} \in X$ s.t. $a+b=0$ (then $b=-a$ is called an additive inverse)

[<u>Remark</u> A1, A3, A4 mean that X is a group and $\underbrace{A1-4}_{A1, A2, A3, A4}$ mean that X is an $\underbrace{\text{abelian group}}_{\text{grupo que cumple las propiedades}}$.]

multiplication

A5 $\Rightarrow$ $a\cdot(b\cdot c)=(a\cdot b)\cdot c$ (associative laws)

A6 $\Rightarrow$ $a\cdot b=b\cdot a$.

A7 $\Rightarrow$ $\exists$ a special element, $1 \in X$ s.t $\forall a \in X$ we have

$a\cdot 1=a$. Moreover $1 \neq 0$.

(1 is called a multiplicative identity)

A8 $\Rightarrow$ $\forall a \in X$, (and $a\neq 0$) $\exists\, b \in X$ s.t $a\cdot b=1$.(b is called the multiplicative inverse) .

BOTH { A.9 $\Rightarrow$ $a\cdot(b+c)=ab+ac$ (distributive law) .

We say that X is an ordered field if, additionally we have a relation $\leq$, satisfying the axioms of order .

A.10 $\Rightarrow$ $\forall\, a, b \in X$ we have $a\leq b$ or $b\leq a$. (totality)

A.11 $\Rightarrow$ $a\leq b$ and $b\leq a \iff a=b$ (antisymmetry) .

A.12 $\Rightarrow$ $a\leq b$ and $b\leq c \Rightarrow a\leq c$ (transitivity)

A.13 $\Rightarrow$ $a\leq b$, $\underbrace{c \in X}_{\text{(arbitrary element of X)}} \Rightarrow a+c \leq b+c$.

A.14 $\Rightarrow$ $a\leq b,\ c\in X,\ c\geq 0.\ \Rightarrow ac\leq bc$.

definition: An ordered field X is satisfying an archimedian property, if $\forall a \in X$ $\exists\ n=1+1+1+1\dots+1\in \mathbb{N}$ s.t $n>a$.

def: $a<b$ means $a\leq b$ and $a\neq b$.

Th: $\forall\, x \in X$ we have $x\cdot 0=0$.

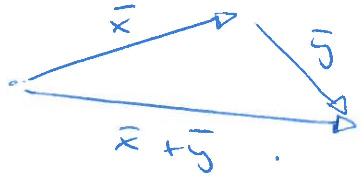
$\boxed{\text{proof}}$ $0+0=0 \Rightarrow$ A.3 Therefore, $$\begin{array}{c} x\cdot(0+0)=x\cdot 0\\ \parallel \text{ A.9}\\ x\cdot 0+x\cdot 0\end{array}$$

$\exists\ -(x\cdot 0)$ Therefore, $x\cdot 0+\underbrace{x\cdot 0+(-x\cdot 0)}_{0} = \underbrace{x\cdot 0+(-x\cdot 0)}_{0}$.

Therefore, $x\cdot 0=0$.

Th $|x+y| \le |x|+|y|$ · (THE TRIANGLE $\underbrace{\text{INEQUALITY}}_{\text{desigualdad}}$) $\forall\ x,y \in \mathbb{R}$

Remark



Proof: we know $a \le b \iff a^2 \le b^2$ if $a,b \ge 0$

Therefore, $|x+y| \le |x|+|y| \iff |x+y|^2 \le (|x|+|y|)^2$;

$(x+y)^2 = \cancel{x^2}+2yx+\cancel{y^2} \le \cancel{|x|^2}+2|x|\cdot|y|+\cancel{|y|^2}$; $\boxed{2yx \le 2|x|\cdot|y|}$,

$y^2=|y|^2$

This is true

which trivially holds ▨ . Since $2|x|\cdot|y| = |2xy|$.

Corollary

$$\left|\sum_{j=1}^{n} x_j\right| \le \sum_{j=1}^{n} |x_j| \Rightarrow |x_1+x_2+\dots+x_n| \le |x_1|+|x_2|+|x_3| \dots + |x_n| .$$

Proposition $|u-v| \ge |\bar{u}|-|\bar{v}|$. (INVERSE TRIANGLE INEQUALITY)

Proof

$$\Updownarrow$$

$|u| \le |u-v|+|v|$; $|\underbrace{(u-v)}_{x}+\underbrace{v}_{y}| \le |\underbrace{u-v}_{x}|+|\underbrace{v}_{y}|$;

; $|x+y| \le |x|+|y|$: $\begin{cases} x := u-v \\ y := v \end{cases}$.

↓ this is how I define sth.

↑

Then we have reduced to the triangle inequality

Corollary $|u-v| \ge \big||u|-|v|\big|$

Proof: we know that $|u-v| \ge |u|-|v|$; $|v-u| \ge |v|-|u|$.

$|u-v| = |v-u|$.

Now we notice that $\big||u|-|v|\big|$ is either $|u|-|v|$ or $|v|-|u|$.

In both cases $|u-v| \ge$ than both ▨

definition (i) A subset S of an ordered field X is bounded above, if $\exists\, H \in X$ st: $x \leq H,\ \forall x \in S$ (then H is called an $\underbrace{\text{upper bound}}_{\text{restricción por arriba}}$ for S).

(ii) S is bounded bellow, if $\exists\, h \in X$ (a lower bound for S) s.t $h \leq x,\ \forall x \in S$.

Examples:

① $S = \{1, 2, 17\}$, $X = \mathbb{Q}$

Bounded above. Upper bounds: 17, 18, 1000,000, ...
The smallest upper bound is 17. S is bounded below, Lower bounds: , 1, 0, -17..
The biggest lower bound is 1.

definition in addition: a set $S \subset X$ is bounded, if it is bounded above and bellow.

② $S = \{x \in \mathbb{Q},\ x > 0\}$ $X = \mathbb{Q}$

$X = \mathbb{Q}$: It is not bounded above, but it is bounded below, and lower bounds are any $h \leq 0$. The largest lower bound is 0.

③ $S = \left\{x \in \mathbb{Q} \underset{\text{s.t.}}{:}\ x > 0 \text{ and } x^2 < 2 \ (\text{i.e } x < \sqrt{2})\right\}$ $X = \mathbb{Q}$

S is bounded below, the largest lower bound is 0.
Upper bounds: 17, 1000, ...
any $H \in \mathbb{Q}$ s.t $H > \sqrt{2}$ is an upper bound of S.

Claim: There is no smallest upper bound of S.

proof: Suppose, H, a smallest upper bound exists, $H \in \mathbb{Q}$.
Then $H \neq \sqrt{2}$, so $H > \sqrt{2} \Rightarrow H - \sqrt{2} > 0 \Rightarrow \dfrac{1}{H - \sqrt{2}} > 0$.
By archimedean property (to be proved) $\exists\, n \in \mathbb{N}$ st
$n > \dfrac{1}{H - \sqrt{2}}$; $H - \sqrt{2} > \dfrac{1}{n}$ and $H - \dfrac{1}{n} > \sqrt{2}$. Denote
$H_1 = H - \frac{1}{n} \in \mathbb{Q}$. Then $\sqrt{2} < H_1 = H - \frac{1}{n} < H$:

$H_1 < H$: So H is not the smallest upper bound, but it is an upper bound. ▨

October 11th 2018

Example (4th):

$X = \mathbb{R}$, $S = \{x \in \mathbb{R}, 0 < x < \sqrt{2}\}$

There is a smallest upper bound, namely $\sqrt{2}$.

especificamente

Definition

X is a complete ordered field. In addition to A1-A14 it satisfies:

A.15 (the axiom of completeness).

Suppose $S \in X$, $\underbrace{S \neq \emptyset}$

Subconjunto de un conjunto que no tiene ningún elemento en él.

(i) If S is bounded above; it has a smallest upper bound of S denoted $\sup S$ and called the supremum of S (no pertenece a S pero es upper bound).

(ii) If S is bounded below, it has a largest lower bound of S denoted $\inf S$ called the infimum of S

As we have seen, $\mathbb{Q}$ does not satisfy the axiom ✱ PREGUNTAR TUTOR

Theorem: There is essentially 1 and only 1 example of complete ordered field, which we call $\mathbb{R}$ ✱ PREGUNTAR TUTOR.

Theorem (The archimedian property): The set of $\mathbb{N}$ numbers, $\mathbb{N} \subset \mathbb{R}$ is not bounded above.

Proof: Assume $\mathbb{N}$ is bounded above then, by the completeness axiom $\exists H = \sup \mathbb{N}$.

Then $(H-1) < H$ is not an upper bound.

Therefore, $\exists n \in \mathbb{N}$ st. $n > H-1$.

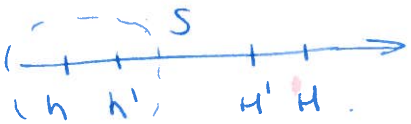
Then, $\underbrace{n+1}_{\mathbb{N}} > H$, so it is not an upper bound of $\mathbb{N}$. CONTRADICTION ▨

Definition: Suppose $S \subset \mathbb{R}$

(i) $H = \sup S$ $\underbrace{\text{IFF}}$

If and only if.

① $x \in S \Rightarrow x \leq H$.



② $H' < H \Rightarrow H'$ is not an upper bound of S. (ie. $\exists x \in S$, st $x > H'$)

(ii) $h = \inf S$ IFF.

① $x \in S \Rightarrow x \geq h$.

② $h' > h \Rightarrow h'$ is not a lower bound of S (i.e. $\exists x \in S$, st $x < h'$) ⑤

Definition Suppose $S \subset \mathbb{R}$

(i) S has a maximum, if $\exists\, x_m \in S$ st $x \in S \Rightarrow x \leq x_m$.

notation: $x_m = \max S$. → pertenece a S y es upper bound

(ii) S has a minimum, if $\exists\, x_m \in S$ st $x \in S \Rightarrow x \geq x_m$.

notation $x_m = \min S$. → pertenece a S y es lower bound

Theorem: NO SIEMPRE EXISTE MAXIMUM PERO SÍ SIEMPRE SUPRIMUM.

① If $\exists\, x_m = \max S$, then $\sup S = x_m$. → Si hay max, max=sup, pero Sup no siempre ≠ max

② If $\exists\, x_m = \min S$, then $\inf S = x_m$.

Proof of:

$x_m = \max S$.

1st: x_m is an upper bound.

2nd: Suppose, $H' < x_m$, then $\exists\, x = x_m \in S$ st $x_m > H'$ so H' is not an upper bound.

Therefore x_m is the smallest upper bound $= \max S$.

Examples.

① $[a,b] = \{x \in \mathbb{R},\ a \leq x \leq b\}$

$a, b \in \mathbb{R}$, $a < b$ $\max S = b$

$\min S = a$.

② $S = (a,b) = \{x \in \mathbb{R},\ a < x < b\}$

$\sup S = b$.

$\inf S = a$ There is no max or min

③ $S = [a,b) = \{x \in \mathbb{R},\ a \leq x < b\}$

$\min S = a$.

$\sup S = b$ no max.

④ $S = [a, +\infty) = \{x \in \mathbb{R},\ x \geq a\}$

bounded below, not above, $\min S = a$.

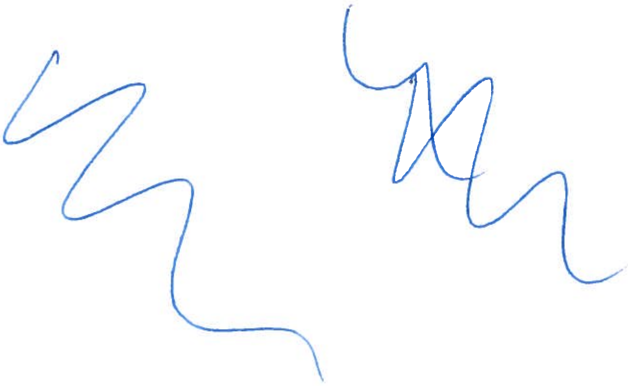
⑤ Finite sets $S = \{x_1, x_2, \dots x_n\}$.

always have $\max S$ and $\min S$.

⑥ $S = \{1 - \frac{1}{n},\ n \in \mathbb{N}\}$

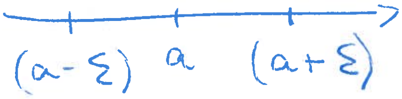
S is bounded, $\min S = 0$, $\sup S = 1$, no max.

- Axioms of an ordered field (need to know completeness axiom, but not arithmetic and order off the top of your head
- Conditionally convergent series can be rearranged to have any sum.
- Pell's equation
- e is irrational
- convexity of e^x and the arithmetic mean inequality (HW10),
- Functions that are continuous but nowhere differentiable (HW 10 #7)
- Radius of convergence of power series
- Term-by-term differentiation of power series



Definition: Suppose $a \in \mathbb{R}$, $\varepsilon > 0$

The ε - $\underbrace{\text{neighbourhood}}_{\text{entorno}}$ of a is $(a-\varepsilon, a+\varepsilon) = \{x \in \mathbb{R}, (x-a) < \varepsilon\}$.



PREGUNTAR TUTOR

SEQUENCES

Example: $X_n = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \dots + \frac{1}{2^n} = 1 - \frac{1}{2^n}$.

It looks like $X_n \to 1$ as $n \to \infty$

Definition: A sequence (of real numbers) is an assignment to any $n \in \mathbb{N}$ of a real number. $X_n \in \mathbb{R}$ (This is a mapping: $\mathbb{N} \to \mathbb{R}$).

Notation

$\langle X_n \rangle$ or $\langle X_n \rangle_{n=1}^{\infty}$ is a sequence and numbers X_n are called the terms (elements) of the sequence.

The range of $\langle X_n \rangle$
- Is boundend above if its $\overset{\text{dominio}}{\text{range}}$ is bounded above.
- Is bounded below if its range is bounded below.
- Is bounded if its range is bounded

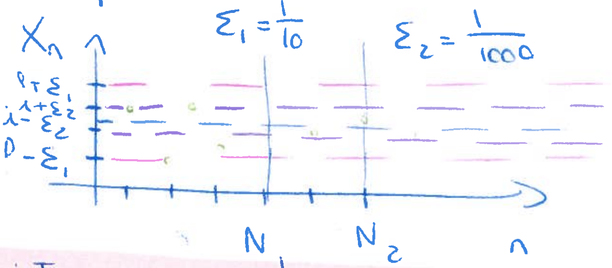
Examples

2 $\langle X_n \rangle$, given by $x_n = n$.

range $= \mathbb{N}$, bounded below.

3. $X_n = (-1)^n$ $-1, 1, -1, 1 \dots$

range $= \{1, -1\}$.



Definition: The most important definition in this course

A sequence $\langle X_n \rangle$ converges to a limit: $\ell \in \mathbb{R}$, if the following holds: $\forall \varepsilon > 0, \exists N \in \mathbb{R}$ s.t $n \in \mathbb{N}$, $n > N \Rightarrow |X_n - \ell| < \varepsilon$

Definition: $\langle X_n \rangle$ converges to ℓ if given any $\varepsilon > 0$, we can find a number $N \in \mathbb{R}$ st all terms X_n with $n > N$ are inside ε-neighborhood.

Notation $\lim_{n \to \infty} x_n = \ell$, or $\lim x_n = \ell$ or $x_n \to \ell$.

Remark 1. $\ell \in \mathbb{R}$, not $\pm \infty$.

2. Some people require $N \in \mathbb{N}$.

Definition: If n does not converge to any $\ell \in \mathbb{R}$, we say that it diverges.

Example: $X_n = \frac{1}{n}$ claim $X_n \to 0$

Proof given $\varepsilon > 0$, we need to find $N \in \mathbb{R}$ st $n > N \Rightarrow \underset{\frac{1}{n}}{\underset{\|}{|x_n - 0|}} < \varepsilon$.

$$\frac{1}{n} < \varepsilon \iff n > \frac{1}{\varepsilon}.$$

Take: $N = \frac{1}{\varepsilon}$.

Then $\forall n > N = \frac{1}{\varepsilon}$ we have $|X_n - 0|$ so $X_n \to 0$.

October 16th 2018

def^n: $x_n \to \ell$ as $n \to \infty \iff \lim_{n \to \infty} x_n = L$.

means that $\forall \varepsilon > 0$ $\exists N \in \mathbb{R}$ s.t for each $n > N$, $n \in \mathbb{N}$ we have $|x_n - \ell| < \varepsilon$.

Example:

① $X_n = \frac{1}{n}$, then $L = 0$.

$\uparrow$ $\lim_{n \to \infty} \frac{1}{n} = 0$.

$\frac{1}{n} \to 0$, take $N = \frac{1}{\varepsilon}$ (Last lecture).

② $\underbrace{\frac{1}{2^n} \to 0}$. Given $\varepsilon > 0$ we need to find N s.t for each $n > N$ we have $|X_n - L| < \varepsilon \Rightarrow \frac{1}{2^n} < \varepsilon$.

$\lim_{n \to \infty} \frac{1}{2^n} = 0$.

HW 1 question 4 $\downarrow$

$2^n > \frac{1}{\varepsilon}$. We know that $2^n = (1+1)^n \geq 1 + n > n$.

This means that if $n > \frac{1}{\varepsilon}$, this implies that $2^n > \frac{1}{\varepsilon}$.

Take $N = \frac{1}{\varepsilon}$. Then if $n > N \Rightarrow n > \frac{1}{\varepsilon} \Rightarrow 2^n > \frac{1}{\varepsilon} \Rightarrow \frac{1}{2^n} < \varepsilon \Rightarrow$

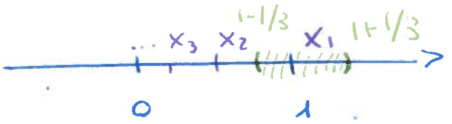
$\Rightarrow |x_n - 0| < \varepsilon$. $\underbrace{\text{Thus}}_{\text{Por lo tanto}}$, $x_n \to 0$.

③ $x_n = \frac{1}{n}$; $\ell = 1$.

Consider $\ell = 1$, cause we know $\ell = 0$.

Proof that $x_n \not\to \ell$ = $\boxed{\text{Claim}}$.

def. $x_n \not\to \ell$ means that . $\exists\ \varepsilon > 0$ s.t $\forall N \in \mathbb{R}$, $\exists\ n > N$, $n \in \mathbb{N}$ s.t $|x_n - \ell| \geq \varepsilon$.



Take $\varepsilon = \frac{1}{3}$.

absolute value

Then $|x_n - \ell| \geq \frac{1}{3} \Rightarrow \left|\frac{1}{n} - 1\right| \geq \frac{1}{3}$; $\frac{n-1}{n} \geq 1/3$. Then $\forall\ n \geq 2$ we have $\frac{n-1}{n} \geq \frac{1}{3}$.

This means that the inequality $|x_n - 1| < \varepsilon$ holds only for $n = 1$ (And this must work $\forall n$ in order to be true the claim.)

⚠ * Como n solo pueden ser ℕ va probando hasta tener $n \geq 2$

Example: $x_n = \frac{2n^2 - 1}{n^2 + 1} \to 2$.

Proof: Given $\varepsilon > 0$, we need to find $N \in \mathbb{R}$ st $n > N$ we have

$|x_n - \ell| < \varepsilon$; $|x_n - 2| < \varepsilon$; $\left|\frac{2n^2 - 1}{n^2 + 1} - 2\right| < \varepsilon$;

; $\left|\frac{\cancel{2n^2} - 1 - \cancel{2n^2} - 2}{n^2 + 1}\right| < \varepsilon$; $\left|\frac{-3}{n^2 + 1}\right| < \varepsilon$; $\frac{3}{n^2 + 1} < \varepsilon$;

$n^2 + 1 > \frac{3}{\varepsilon} \Longleftrightarrow n^2 > \frac{3}{\varepsilon} - 1$.

I consider 2 cases as I don't know if $\frac{3}{\varepsilon} - 1$ is > 0.

Case 1: $\frac{3}{\varepsilon} - 1 \geq 0$, then $n^2 > \frac{3}{\varepsilon} - 1 \Longleftrightarrow n > \sqrt{\frac{3}{\varepsilon} - 1}$.

Case 2: $\frac{3}{\varepsilon} - 1 < 0$, then $n^2 > \frac{3}{\varepsilon} - 1$ ALWAYS.

Thus, we take $N = \begin{cases} \sqrt{\frac{3}{\varepsilon} - 1}, & \frac{3}{\varepsilon} - 1 \geq 0 \\ \text{Cualquier n° porque siempre es } \in, & \frac{3}{\varepsilon} - 1 < 0 \end{cases}$

Remark Suppose we have found $N \in \mathbb{R}$ s.t $\forall n > N$ we have $|x_n - \ell| < \varepsilon$. Then any number $\tilde{N} > N$ would also work $\left(n > \tilde{N} \Rightarrow |x_n - \ell| < \varepsilon\right)$

In particular we can find $\tilde{N} \in \mathbb{N}$ which works.

Back to last example: $x_n = \frac{2n^2-1}{n^2+1} \Rightarrow \lim \frac{2n^2-1}{n^2+1} = \lim \frac{2n^2-1}{n^2+1} \cdot \frac{1/n^2}{1/n^2} =$

$$= \lim \frac{2 - \overset{\to 0}{\frac{1}{n}} \cdot \overset{\to 0}{\frac{1}{n}}}{1 + \underset{\to 0}{\frac{1}{n}} \cdot \underset{\to 0}{\frac{1}{n}}} = \boxed{\frac{2}{1}}$$

↑ Should be. ✓

We'll try to prove it:

Theorem (algebra of limits):

Suppose, $x_n \to x$, $y_n \to y$.
Then:

1) $x_n + y_n \to x + y$. (sum rule)

2) $x_n \cdot y_n \to xy$ (product rule).

3) $\frac{x_n}{y_n} \to \frac{x}{y}$, assuming $y \neq 0$, $y_n \neq 0$ $\forall n$ (quotient rule).

October 20th 2018

Theorem (algebra of limits)

a) Sum rule:

$$\left.\begin{array}{l} x_n \to x \\ y_n \to y \end{array}\right\} \Rightarrow x_n + y_n \to x + y$$

Proof (take one).

Need to show $\left|(x_n + y_n) - (x + y)\right|$ becomes small for large n.
(that the difference between $\rightleftarrows$ is small)

$\boxed{\text{know}}$. $x_n \to x$, so given any $\varepsilon > 0$ $\exists N \in \mathbb{R}$ s.t $\forall n > N$ we have $|x_n - x| < \varepsilon$.

$\boxed{\text{also}}$ $y_n \to y$, so $\exists \tilde{N}$ (different number) $\in \mathbb{R}$ s.t $\forall n > \tilde{N}$ we have $|y_n - y| < \varepsilon$.

Take $\hat{N} = \max\{N, \tilde{N}\}$, then $\forall n > \hat{N}$ we satisfy both $|x_n - x| < \varepsilon$ and $|y_n - y| < \varepsilon$.

Therefore, $\left|(x_n + y_n) - (x + y)\right| = \left|(x_n - x) + (y_n - y)\right| \overset{\text{triangle inequality}}{\leq} |\underset{\wedge}{x_n} - x| + |\underset{\wedge}{y_n} - y| < 2\varepsilon$

Idea: Take $\frac{\varepsilon}{2}$ instead of ε.

Proof! (official version)

Given $\varepsilon > 0$, we can find $N_1 \in \mathbb{R}$ s.t $\forall n > N_1$ we have $|x_n - x| < \frac{\varepsilon}{2}$.

Given $\varepsilon > 0$, we can also find $N_2 \in \mathbb{R}$ s.t $\forall n > N_2$ we have $|y_n - y| < \frac{\varepsilon}{2}$

Put $N = \max\{N_1, N_2\}$.

Then $\forall n > N$ we have both $|x_n - x| < \frac{\varepsilon}{2}$ and $|y_n - y| < \frac{\varepsilon}{2}$

and therefore $|(x_n + y_n) - (x + y)| = |(x_n - x) + (y_n - y)| \leq |x_n - x| + |y_n - y|$

$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$

So $\lim_{n \to \infty} (x_n + y_n) = x + y$ ▨

Th (SANDWICH THEOREM) = (TWO POLICEMEN THEOREM).

Suppose, $\langle x_n \rangle$, $\langle y_n \rangle$ and $\langle z_n \rangle$ are three sequences s.t:

$x_n \leq y_n \leq z_n$

Suppose, $x_n \to l$ and $z_n \to l$

then $y_n \to l$.

Proof: Given $\varepsilon > 0$ $\boxed{x_n \to l}$ $\exists N_1$ s.t $\forall n > N_1$ we have $|x_n - l| < \varepsilon$ $*_1$

$\boxed{z_n \to l}$. $\exists N_2$ s.t $\forall n > N_2$ $(z_n - l)$ we have $|z_n - l| < \varepsilon$ $*_2$

$*_1 \iff -\varepsilon < x_n - l < \varepsilon \iff l - \varepsilon < x_n < \varepsilon + l$.

$*_2 \iff l - \varepsilon < z_n < l + \varepsilon$.

Take $N = \max\{N_1, N_2\}$

Then $\forall n > N$ we have $l - \varepsilon \leq x_n \leq y_n \leq z_n \leq l + \varepsilon$, so $l - \varepsilon < y_n < l + \varepsilon$,

and $-\varepsilon < y_n - l < \varepsilon \iff |y_n - l| < \varepsilon$, so $y_n \to l$ ▨

Example

$x_n = \frac{(-1)^n}{n^2}$

$-1, \frac{1}{4}, \frac{-1}{9}, \frac{1}{16} \ldots$ we just notice $|x_n| = \frac{1}{n^2}$, so

$-\frac{1}{n^2} \leq x_n \leq \frac{1}{n^2}$ and $\frac{1}{n^2} = \underset{\substack{\downarrow \\ 0}}{\frac{1}{n}} \cdot \underset{\substack{\downarrow \\ 0}}{\frac{1}{n}} \to 0$ (by product rule)

Similarly, $-\frac{1}{n^2} \to 0$, so by sandwich, $\frac{(-1)^n}{n^2} \to 0$.

Recall: Suppose, $S \subset \mathbb{R}$ is bounded, if $\exists\, a, b \in \mathbb{R}$ s.t

$\forall x \in S$ we have $a \leq x \leq b$.

Lemma Suppose, $S \subset \mathbb{R}$, then S is bounded $\iff \exists M > 0$

s.t $\forall x \in S$ $|x| \leq M$.

$\Leftarrow$ **Proof**: Suppose, $\exists M > 0$ s.t $\forall x \in S$ we have $|x| \leq M$

$$\Updownarrow$$

$$-M \leq x \leq M.$$

Take $a = -M$, $b = M$.

$\Rightarrow$ Assume, S is bounded, so $\exists\, a, b \in \mathbb{R}$ s.t $\forall x \in S$ we have $a \leq x \leq b$.

Take $M = \max\{|a|, |b|\}$. Then $M \geq |b| \geq b$, also $M \geq |a| \geq -a$,

so $a \geq -M$; $-M \leq a$

Therefore $-M \leq a \leq x \leq b \leq M$, so $|x| \leq M$ ▨

A sequence $x_n = (-1)^n$ is bounded, but does not converge (to be proved later)

Th. Suppose, x_n converges, then it is bounded

Proof: Suppose, $x_n \to \ell$ then for $\varepsilon = 17$ we have $\exists N$ s.t $\forall n > N$ we have $|x_n - \ell| < \varepsilon$.

WLOG → without loss of generality

we assume $N \in \mathbb{N}$, then $|x_n| = |(x_n - \ell) + \ell| \overset{\text{triangle inequality}}{\leq} |x_n - \ell| + |\ell| < |\ell| + 17$

Take $M := \max\{|x_1|, |x_2|, \ldots |x_n|, |\ell| + 17\}$

Then $\forall n$ we have $\boxed{|x_n| \leq M} \to -M < x_n < M$

so $\langle x_n \rangle$ is bounded ▨ $\quad$ it is bounded.

b) Product rule

Proof: product rule

$\hookrightarrow$ $\left.\begin{array}{l} x_n \to x \\ y_n \to y \end{array}\right\} \Rightarrow x_n y_n \to xy$.

Proof: (take one)

Given $\varepsilon > 0$, $\exists N_1$ st $\forall n > N_1$ we have $|x_n - x| < \varepsilon$,

$\exists N_2$ s.t $\forall n > N_2$ we have $|x_n - y| < \varepsilon$

Take $N = \max\{N_1, N_2\}$.

Suppose, $n > N$. How can we estimate the difference between x_ny_n and x

$$|x_ny_n - xy| = |x_ny_n - x_ny + x_{\wedge}y - xy| = |x_n(y_n - y) + y\cdot(x_n - x)|$$

$$\le |x_n|\underset{\substack{\wedge\\ \varepsilon}}{|y_n - y|} + |y|\underset{\substack{\wedge\\ \varepsilon}}{|x_n - x|} < \varepsilon\cdot(|x_n| + |y|).$$

We know that x_n converges, so it is bounded, so $\exists M$ s.t

$|x_n| \le M$ $\forall n$. Therefore $\varepsilon\cdot(|x_n| + |y|) \le \varepsilon(M + |y|)$.

Idea: take $\frac{\varepsilon}{M+|y|}$ instead of ε.

Proof: (official version)

Since $\langle x_n \rangle$ converges, it is bounded, so $\exists M$ s.t $|x_n| < M$ $\forall n$.

Given $\varepsilon > 0$, $\exists N_1$ st $\forall n > N_1$ we have $|x_n - x| < \frac{\varepsilon}{M+|y|}$.

Similarly, $\exists N_2$, $\forall n > N_2$ we have $|y_n - y| < \frac{\varepsilon}{M+|y|}$.

Take $N = \max\{N_1, N_2\}$.

Then $\forall n > N$ we have $|x_ny_n - xy| = |x_n(y_n - y) + y\cdot(x_n - x)| \le$

$$|x_n|\cdot|y_n - y| + |y|\cdot|x_n - x| \le M\cdot\frac{\varepsilon}{M+|y|} + |y|\cdot\frac{\varepsilon}{M+|y|} = \varepsilon.$$

Thus, $x_ny_n \to xy$ ▨

3) Quotient rule

$$\left.\begin{array}{l} x_n \to x \\ y_n \to y \\ y \neq 0 \\ y_n \neq 0 \quad \forall n \end{array}\right| \Longrightarrow \frac{x_n}{y_n} \to \frac{x}{y}$$

$$\frac{x_n}{y_n} = \underset{\substack{\downarrow\\ x}}{x_n}\cdot\underset{\substack{\downarrow \text{ If I can prove it}\\ 1/y}}{\frac{1}{y_n}} \xrightarrow{\text{Then}} \frac{x}{y}.$$

It is enough to show $\frac{1}{y_n} \to \frac{1}{y}$

Since $y_n \to y$, $\forall \varepsilon > 0$ $\exists N$ s.t $\forall n > N$ we have $|y_n - y| < \varepsilon$.

How big can $\left|\frac{1}{y_n} - \frac{1}{y}\right|$ be?

$$\left|\frac{1}{y_n} - \frac{1}{y}\right| = \left|\frac{y - y_n}{y_ny}\right| = \frac{|y - y_n|}{|y_n|\cdot|y|} < \frac{\varepsilon}{|y_n|\cdot|y|}.$$

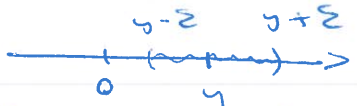
The sequence converges, so it is bounded.

Is it true that $\langle \frac{1}{|y_n|} \rangle$ is bounded?

Lemma Suppose, $y_n \to y$, $\forall n$ $y_n \neq 0$, $y \neq 0$ then $\exists c > 0$ s.t $|y_n| \geq c$ $\forall n$.

(This implies $\frac{1}{|y_n|} \leq \frac{1}{c}$).

Proof:



Given $\varepsilon > 0$ $\exists N$ s.t $\forall n > N$ we have $|y_n - y| < \varepsilon$.

take $\varepsilon = \frac{|y|}{2}$, then $n > N \Rightarrow |y_n| = |y - (y - y_n)| \underset{\uparrow}{\geq} |y| - |\underset{\overset{\wedge}{\varepsilon}}{y - y_n}| >$

$> |y| - \varepsilon =$

inverse triangle inequality

$= |y| - \frac{|y|}{2} = \frac{|y|}{2} > 0$. This holds for $n > N$.

Put $c := \min\left\{|y_1|, |y_2|, \dots, |y_n|, \frac{|y|}{2}\right\} > 0$.

Then $|y_n| > c$ $\forall c$.

October 23rd:

Know: $x_n \to x$, $y_n \to y$.

$\forall n$ $y_n \neq 0$, $y \neq 0$.

trying to prove: $\frac{x_n}{y_n} \to \frac{x}{y}$ QUOTIENT RULE.

It is enough to prove that $\frac{1}{y_n} \to \frac{1}{y}$

Notice: if $|y_n - y| < \varepsilon$, then $\left|\frac{1}{y_n} - \frac{1}{y}\right| = \frac{|y - y_n|}{|y_n \cdot y|} < \frac{\varepsilon}{|y_n||y|}$.

Lemma 1

Under the assumptions above $\exists c > 0$ s.t $\forall n$ $|y_n| > c$.

then: $\frac{1}{|y_n|} < \frac{1}{c}$.

Lemma 2

$\frac{1}{y_n} \to \frac{1}{y}$.

Proof of the quotient rule:

$$\frac{|y - y_n|}{|y_n||y|} < \frac{\varepsilon}{\underset{\hat{c}}{|y_n|}|y|} < \frac{\varepsilon}{c|y|}$$

By previous Lemma we can find c: $\exists c$ s.t $\frac{1}{|y_n|} < \frac{1}{c}$ $\forall n$.

Idea: take $\varepsilon = \varepsilon \cdot c|y|$.

Given $\varepsilon > 0$ st $\forall n > N$ we have $|y_n - y| < \varepsilon c|y|$.

Then $\left| \frac{1}{y_n} - \frac{1}{y} \right| = \frac{|y_n - y|}{|y_n|\,|y|} < \varepsilon \cdot \alpha \cancel{|y|} \cdot \frac{1}{\cancel{|y|}} \cdot \frac{1}{\cancel{\alpha}} = \varepsilon$.

Therefore, $\frac{1}{y_n} \to \frac{1}{y}$

Now by product rule we have $\frac{x_n}{y_n} = \underset{x}{\underset{\downarrow}{x_n}} \cdot \underset{\frac{1}{y}}{\underset{\downarrow}{\frac{1}{y_n}}} \to \frac{x}{y}$ ▨

def^n: A sequence $\langle x_n \rangle$ diverges to $+\infty$, ($x_n \to +\infty$, or $\lim x_n = +\infty$) if $\forall M \in \mathbb{R}\ \exists N \in \mathbb{R}$ s.t. $\forall n > N$ we have $x_n > M$.

Similarly: $\langle x_n \rangle$ diverges to $-\infty$, if. $\forall \underset{\Uparrow}{m} \in \mathbb{R}$, $\exists N \in \mathbb{R}$ s.t $\forall n > N$

large modulus, negative .

we have $x_n < m$.

Example: $x_n = n^2$.

Claim $x_n \to +\infty$

Proof: Given $m \in \mathbb{R}$, we need to find N s.t $\forall n > N$ we have $n^2 > M$.

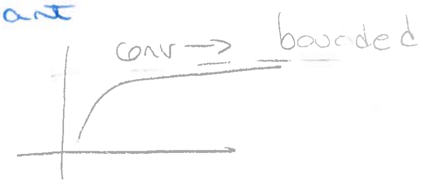
$\to n^2$ siempre > 0 .

① If $M > 0$, $n^2 > M \iff n > \sqrt{M}$

Take $N = \sqrt{M}$.

② If $m < 0$, take $N =$ which ever nb we want eg: 17 .

Overall $= \begin{cases} \sqrt{m} & m \geq 0 \\ 17 & m < 0 \end{cases}$



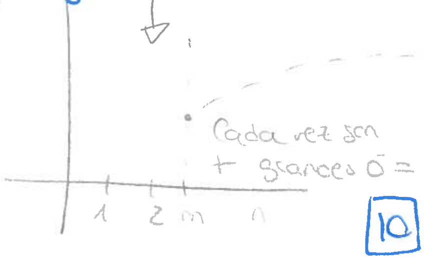
Recall: $\langle x_n \rangle$ convergent $\Rightarrow$ bounded .

$x_n = (-1)^n$ is bounded, but not convergent .(to be proved) .

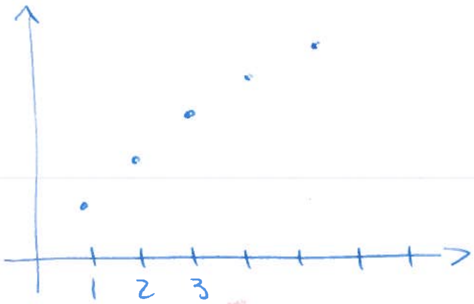
def: a sequence $\langle x_n \rangle$ is said to be • increasing, if $n > m \Rightarrow x_n \geq x_m$

→ no tiene escalera

- strictly increasing, if $n > m \Rightarrow x_n > x_m$. and
- decreasing if $n > m \Rightarrow x_n \leq x_m$, and
- strictly decreasing if $n > m \Rightarrow x_n < x_m$.



- $\langle x_n \rangle$ is **MONOTONE**, if it is $\underbrace{\text{increasing or decreasing}}_{\text{solo 1 de las dos}}$.



Th

(1) If $\langle x_n \rangle$ increases and is bounded (above), then $\lim x_n = \sup x_n =$
$= \sup \{ x_n , n \in \mathbb{N} \}$.

(2) If $\langle x_n \rangle$ decreases and is bounded (below), then
$\lim x_n = \inf x_n = \inf \{ x_n , n \in \mathbb{N} \}$.

Corollary:
If $\langle x_n \rangle$ is bounded and monotone, it converges.

October 25th 2018

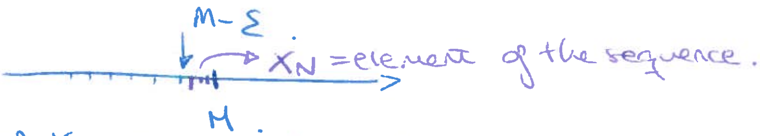
Th Let $\langle x_n \rangle$ be a sequence, put $S := \{ x_n , n \in \mathbb{N} \}$.

(1) If $\langle x_n \rangle$ increases and is bounded (above), then it converges:
$\lim x_n = \sup S$.

② If $\langle x_n \rangle$ decreases and is bounded below, then $\lim x_n = \inf S$.

Proof of 1.

Put $M := \sup S$.



Then M is an upper bound of x_n, so $\forall n$ $x_n \leq M$
Suppose, $\varepsilon > 0$, then $M-\varepsilon$ is not an upper bound of S, so $\exists x_N$ st $x_N \in x_n$ s.t $x_N > M-\varepsilon$ (i.e. $\exists N$ s.t $x_N > M - \varepsilon$) Then, since $\langle x_n \rangle$ increases, $\forall n > N$ we have $x_n \geq x_N > M-\varepsilon$. Thus, $\forall n > N$.

$M - \varepsilon < x_N \leq x_n \leq M < M + \varepsilon$; i.e. $|x_n - M| < \varepsilon$; so $x_n \to M$ ▨

__Example__: consider a sequence $\langle x_n \rangle$ given by $X_1 = 1$ and

$$X_{n+1} = \frac{1}{3} \cdot (x_n + 1) \quad \forall n \geq 1$$

$n=0 \quad X_1 = 1$

$n=1 \quad X_2 = \frac{2}{3} \longrightarrow \frac{1}{3}(X_1+1) = \frac{2}{3}$

$n=2 \quad x_3 = \frac{5}{9} = \frac{1}{3} \cdot (x_2 + 1) = \frac{1}{3} \cdot \left(\frac{2}{3}+1\right) = \frac{5}{9}$

__Claim 1__: $x_n \geq \frac{1}{2}$, $\forall n$ (it is bounded below).

__Proof__: Induction

$X_1 = 1 \geq \frac{1}{2}$. because it was convenient

If $x_n \geq \frac{1}{2}$, then $X_{n+1} = \frac{1}{3}(x_n+1) \geq \frac{1}{3}\left(\frac{1}{2}+1\right) = \frac{1}{2}$.

__Claim 2__: $\langle x_n \rangle$ decreases:

__Proof__:

$X_{n+1} \leq x_n$, or $x_n - x_{n+1} \geq 0$.

But $x_{n+1} = \frac{1}{3}(x_n+1)$ and $x_n \geq \frac{1}{2}$.

Therefore, $x_n - x_{n+1} = x_n - \frac{1}{3}(x_n+1) = \frac{2x_n}{3} - \frac{1}{3} \underset{\substack{\uparrow \\ x_n \geq \frac{1}{2}}}{\geq} \frac{2}{3} \cdot \frac{1}{2} - \frac{1}{3} = 0$

So $\langle x_n \rangle$ decreases.

Use the theorem.

$\langle x_n \rangle$ decreases, is bounded below $\Longrightarrow$ it converges.

Put $l = \lim x_n$.

Denote $y_n = X_{n+1} = \frac{1}{3}(x_n+1)$.

By HW 3, Question 4(a)

$\hookrightarrow x_n \to l$ then $x_{n+1} \to l$.

$y_n \to l$.

Therefore, $l = \lim y_n = \lim \frac{1}{3}(x_n+1) \overset{\text{algebra of limits}}{=} \frac{1}{3}(l+1)$; so

$3l = l+1$, $l = \frac{1}{2}$

Thus, $\lim x_n = \frac{1}{2}$.

Example 2. **FIBONACCI SEQUENCE**.

$x_1 = x_2 = 1$, $x_{n+2} = x_{n+1} + x_n$, $\forall n \in \mathbb{N}$.

$1, 1, 2, 3, 5, 8 \dots$ ←-------

Compute: $l = \cancel{\lim} x_n$. Not allowed !

$x_{n+2} = x_{n+1} + x_n$, so $\boxed{l = 2l}$ so $l = 0$.
↳HW3 exercise 4.

$\downarrow \qquad \downarrow \qquad \downarrow$

$l \qquad l \qquad l$

Be careful!! (si la sequence va creciendo entonces no tiene sentido que $l=0$).

$\langle x_n \rangle$ is a sequence:
- convergent $\Rightarrow$ bounded.
- bounded and monotone $\Rightarrow$ convergent.
- bounded $\not\Rightarrow$ convergent.
- bounded $\Rightarrow$?

Subsequences.

$x_n = \frac{1}{n}$.

new n, j_n of whom x_{j_n} depend and

$n = \cancel{1}, 2, \cancel{3}, \cancel{4}, 5, 6, \cancel{7}, 8$ I only take these because

$x_n = \cancel{1}, \frac{1}{2}, \cancel{1/3}, \cancel{1/4}, 1/5, 1/6, \cancel{1/7}, 1/8$.

$n = 1, 2, 3, 4$

$j_n = 2, 5, 6, 8 \dots$

$x_{j_n} = 1/2, 1/5, 1/6, 1/8, \dots$

→ a new sequence with the elements of the sequence n that I choose.

→ a new sequence with the elements of x_n when $n \in j_n$.

$j_1 = 2 \qquad x_{j_1} = 1/2$

$j_2 = 5 \qquad x_{j_2} = 1/5$

$j_3 = 6. \qquad x_{j_3} = 1/6$.

def^n: Suppose $\langle x_n \rangle$ is a sequence and $\langle j_n \rangle$ is a strictly increasing sequence of $\mathbb{N}$ nbs. We call a new sequence $\langle x_{j_n} \rangle$ a subsequence of $\langle x_n \rangle$.

Examples :

1) $X_n = (-1)^n$ $\quad -1, 1, -1, 1, \dots$

Suppose, $j_n = 2n$. Then $X_{2n} = 1$, subsequence $1, 1, 1$.

Suppose, $j_n = 2n+1$. Then $X_{2n+1} = -1$.

2) $X_n = \frac{1}{n} + \sin\left(\frac{\pi n}{2}\right)$. $\quad 1+1, \frac{1}{2}+0, \frac{1}{3}-1, \frac{1}{4}+0, \frac{1}{5}+1, \dots$

Suppose, $j_n = 1+4n$. Then $X_{4n+1} = \frac{1}{4n+1} + 1$

vueltas

$\sin\left(\frac{(4n+1)\pi}{2}\right) = \sin\left(\frac{4n\pi}{2} + \frac{\pi}{2}\right) = 1$ $\quad \to 90$

$\lim X_{4n+1} = 1$.

Suppose, $j_n = 2n$. Then $X_{2n} = \frac{1}{2n}$.

$\lim X_{2n} = 0$.

Similarly : $X_{4n+3} \to -1$.

Th: Suppose, $X_n \to \ell$, then any subsequence $\langle X_{j_n} \rangle$ satisfies $X_{j_n} \to \ell$.

where $\ell \in \mathbb{R}$ or $\ell = \pm\infty$.

Proof. (given only for $\ell \in \mathbb{R}$).

Claim : any strictly increasing sequence of natural nbs, j_n satisfies $j_n \geq n$.

Proof of claim : Induction :

$j_1 \in \mathbb{N}$, so $j_1 \geq 1$

Suppose, $j_n \geq n$. Then $j_{n+1} > j_n$. , so $j_{n+1} \geq j_n + 1 \geq n+1$.

Proof of theorem

We know that : given $\varepsilon > 0$, $\exists N$ s.t $\forall n > N$ we have $|X_n - \ell| < \varepsilon$.

We need to show that $|X_{j_n} - \ell| < \varepsilon$ for large n.

But if $n > N$ then $j_n \geq n > N$, so $|X_{j_n} - \ell| < \varepsilon$

Therefore, $X_{j_n} \to \ell$ $\quad$ ▨

Corollary 1: Suppose, $x_n \to l$. Then $x_{n+1} \to l$ and

$\lim_{n \to \infty} x_{n+k} = l \ , \forall k \in \mathbb{N}$.

Corollary 2: If a sequence $\langle x_n \rangle$ has 2 subsequences convergent to two different limits, then $\langle x_n \rangle$ is divergent.

Example: $x_n = (-1)^n$

$\lim x_{2n+1} = -1 \quad \lim x_{2n} = 1$.

Therefore, $\langle x_n \rangle$ diverges.

Th BOLZANO - WEIERSTRASS:

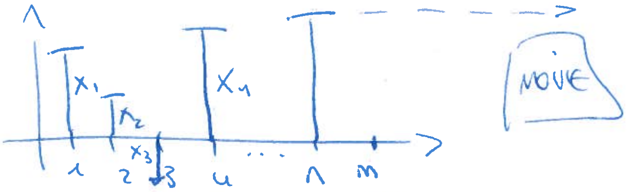
Every bounded sequence has a convergent subsequence.

Lemma: Every sequence has a monotone subsequence.

altura.

Proof:

We say that seat n is convenient if $m > n$ s.t $\boxed{x_m \leq x_n}$.



Then the seat n is not convenient, if $m > n$ s.t $x_m > x_n$.

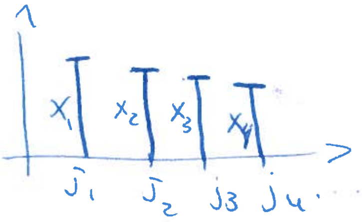
There are 2 cases of n is convenient:

① Case 1: There are infinitely many convinient seats.

② Case 2. There are finitely many convinient seats.

$\boxed{1} \to$ Suppose, seats $j_1, j_2, j_3, \ldots$ are convenient put them in increasing order.

$j_1 < j_2 < j_3 < \ldots$ are convenient.



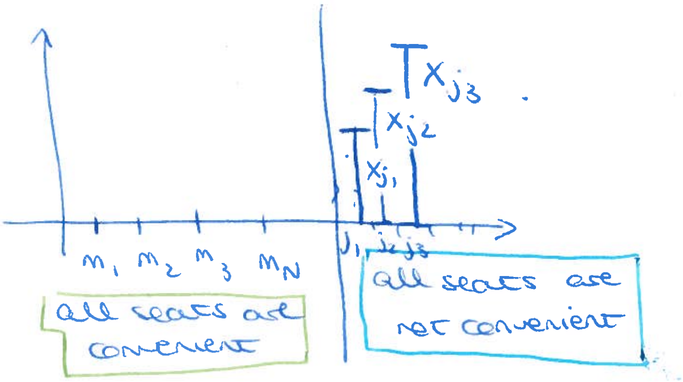
Then seat j_1 is convenient, so $x_{j_1} \geq x_{j_2}$.

Then seat j_2 is convenient, so $x_{j_2} \geq x_{j_3}$.

J_n is convenient $\Rightarrow x_{j_n} \geq x_{j_{n+1}}$

Therefore, $\langle x_{j_n} \rangle$ is decreasing.

$\boxed{2} \to$ Suppose, $m_1 < m_2 < m_3 < \dots < m_N$ are all convenient seats.



Denote $j_1 = m_N + 1$. (put $j_1 = 1$ if there are not convenient seats).

Seat j_1 and all seats to the right of j_1 are inconvenient.

The seat j_1 is inconvenient $\Rightarrow \exists\, j_2 > j_1$ s.t $x_{j_2} > x_{j_1}$.

j_2 is inconvenient $\Rightarrow \exists\, j_3 > j_2$ s.t $x_{j_3} > x_{j_2}$.

j_n is inconvenient $\Rightarrow \exists\, j_{n+1} > j_n$ s.t $x_{j_{n+1}} > x_{j_n}$.

Thus, the subsequence:

$\langle x_{j_n} \rangle$ is increasing

30th November.

Th (Bolzano-Weierstrass)

Every bounded sequence has a convergent subsequence.

Lemma:

Every sequence has a monotone subsequence. ▨

Proof of theorem:

Let $\langle x_n \rangle$ be bounded. By lemma it has a monotone subsequence $\langle x_{j_n} \rangle$. This subsequence $\langle x_{j_n} \rangle$ is bounded, so it converges to its supremum (if it increases) and to its infinum (if it decreases)

Example: Consider the following sequence: $x_n = 1, \frac{1}{2}, \frac{1}{3}, \frac{2}{3}, \frac{1}{4}, \frac{2}{4}, \frac{3}{4}, \dots$

x_n (all rationals in $(0,1]$)

Then $\forall a \in (0,1]$ $\exists$ a subsequence convergent to a.

(HW question).

INFINITE SERIES:

Sequences $\langle a_n \rangle : a_1, a_2, a_3, \dots$

Series: $\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \dots$

Examples:

① (good) $1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \dots = 2$.

② (horrible) $0 = \underset{\underset{1-1}{\shortparallel}}{0} + 0 + 0 + \dots = (1-1) + (1-1) + (1-1) + \dots = 1 - 1 + 1 - 1 + 1 \underset{\underset{-(1-1)}{\shortparallel}}{(-1+1)} - 1 \dots =$

$= 1 - (1-1) - (1-1) - (1-1) - \dots = 1 - 0 - 0 - 0 = 1$.

def^n:

Infinite summation:

Let $N \in \mathbb{N}$. The partial sums of infinite series $\sum_{n=1}^{\infty} a_n$ are numbers $S_N = \sum_{n=1}^{N} a_n = a_1 + a_2 + \dots + a_N$

These partial sums form a sequence $\langle S_N \rangle_{N=1}^{\infty}$. $\to = \langle S_N \rangle_{N=1}^{2} = (a_1, a_1 + a_2)$

definition: We say that infinite series $\sum_{n=1}^{\infty} a_n$ converges to $l \in \mathbb{R}$, if the sequence $\langle S_N \rangle_{N=1}^{\infty}$ converges to l.

Notation:

$$\sum_{n=1}^{\infty} a_n = l \iff \lim_{N \to \infty} S_N = l.$$

$$\shortparallel$$

$$\sum_{n} a_n = \sum a_n.$$

def: If the sequence $\langle S_N \rangle$ diverges, we say that $\sum_{n=1}^{\infty} a_n$ diverges

Proposition:

If $\sum a_n$ converges then:

1) The sequence $\langle a_n \rangle$ converges, and $\lim a_n = 0$.

2) $\sum_{n=k}^{\infty} a_n = a_k + a_{k+1} + a_{k+2} + \dots \xrightarrow{k \to \infty} 0$.

(The tails of convergent series go to 0).

3) If $c \in \mathbb{R}$, then $\sum_{n=1}^{\infty} (c \cdot a_n)$ converges and $\sum_{n=1}^{\infty} (c \cdot a_n) =$

$$= c \cdot \sum_{n=1}^{\infty} a_n$$

$$\sum_{n=1}^{\infty}(c\cdot a_n) \;\|\; ca_1 + ca_2 + ca_3 + \dots$$

$$c \cdot \sum_{n=1}^{\infty} a_n \;\|\; c(a_1 + a_2 + a_3 + \dots)$$

4) If $\sum_{n=1}^{\infty} b_n$ also converges, then $\sum_{n=1}^{\infty} (a_n + b_n)$ converges, and .

$$\sum_{n=1}^{\infty} (a_n + b_n) = \sum a_n + \sum b_n .$$

Proof

① $S_N = \sum_{n=1}^{N} a_n = a_1 + a_2 + a_3 + \dots + a_N$

$S_{N-1} = \sum_{n=1}^{N-1} a_n = a_1 + a_2 + \dots + a_{N-1}$.

Then : $a_N = S_N - S_{N-1} \xrightarrow{N \to \infty} l - l = 0$, so $\lim a_N = 0$.

$$S_N \xrightarrow{N \to \infty} l \qquad S_{N-1} \xrightarrow{N \to \infty} l .$$

$\lim a_n = 0$.

③ $\sum_{n=1}^{\infty} (c \cdot a_n) = \lim_{N \to \infty} \sum_{n=1}^{N} (c \cdot a_n) = \lim_{N \to \infty} (c \cdot a_1 + c\,a_2 + \dots + c \cdot a_N) =$

$= \lim_{N \to \infty} \left[c \cdot (a_1 + a_2 + \dots + a_N) \right] \underset{\uparrow}{=} c \cdot \lim_{N \to \infty} (a_1 + a_2 + \dots + a_N) = c \cdot \sum_{n=1}^{\infty} a_n$.

algebra of limits .

④ Similar . $\boxed{//}$

Examples: 2 (revisited) .

$$1 - 1 + 1 - 1 + 1 - 1 + \dots = \sum_{n=0}^{\infty} (-1)^n$$

$a_n = (-1)^n \not\to 0$, so $\sum (-1)^n$ diverges .

We have used corollary from (1) . If $a_n \not\to 0$, then $\sum a_n$ diverges .

Examples : 3 (geometric series) .

$$\sum_{n=0}^{\infty} x^n , \quad x \in \mathbb{R}$$

$$\| \quad 1 + x + x^2 + \dots$$

Bernoulli

If $|x| \geq 1$, then $|x^n| = |x|^n = (1+h)^n \overset{\downarrow}{\geq} 1 + hn \geq 1$ como $|x^n| \geq 1$

So $|x^n| \not\to 0$ and $\sum x^n$ diverges

1st november 2018 .

Geometric series

$\sum_{n=0}^{\infty} x^n$ if $|x| \geq 1$.

We have $|x^n| \geq 1$.

Therefore, $x^n \not\to 0$

so the series diverges . (we know $\sum_{n=1}^{\infty} a_n$ converges $\Rightarrow a_n \xrightarrow{n\to\infty} 0$ so $a_n \not\to 0 \Rightarrow$ series diverges) .

Suppose $|x| < 1$

Then $x^n \to 0$.

Does de series converge? .

Partial sum :

$$S_N = 1 + x + x^2 + \dots + x^N .$$

$$x S_N = x + x^2 + x^3 \dots + x^N + x^{N+1}$$

$$(1-x) S_N = 1 - x^{N+1} \text{ , so}$$

$|x|<1$

$x^{N+1} \xrightarrow[N\to\infty]{} 0$.

$$S_N = \frac{1-x^{N+1}}{1-x} = \underset{\frac{1}{1-x}}{\underset{\downarrow N\to\infty}{\frac{1}{1-x}}} - \underset{0}{\underset{\downarrow N\to\infty}{\boxed{\frac{x^{N+1}}{1-x}}}} \to \frac{1}{1-x}$$

Therefore,

$$\sum_{n=0}^{\infty} x^n = \begin{cases} \frac{1}{1-x} , & |x| < 1 \\ \text{diverges} & |x| \geq 1 \end{cases} .$$

Example 4 : (telescoping series) .

$\sum_{n=1}^{\infty} \frac{1}{n \cdot (n+1)}$ The partial sums:

$$S_N = \sum_{n=1}^{N} \frac{1}{n \cdot (n+1)} = \frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \dots + \frac{1}{N(N+1)}$$

Notice : $\frac{1}{n \cdot (n+1)} = \frac{1}{n} - \frac{1}{n+1}$.

Therefore

$$S_N = \left(\frac{1}{1} - \cancel{\frac{1}{2}}\right) + \left(\cancel{\frac{1}{2}} - \cancel{\frac{1}{3}}\right) + \left(\cancel{\frac{1}{3}} - \frac{1}{4}\right) + \dots + \left(\cancel{\frac{1}{N-1}} - \cancel{\frac{1}{N}}\right) + \left(\cancel{\frac{1}{N}} - \frac{1}{N+1}\right) = 1 - \frac{1}{N+1} \to 1$$

Therefore,

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = 1 .$$

As the partial sums converge to 1 then the serie converges to 1.

Example

$$\sum_{n=1}^{\infty} \frac{2^n}{3^n+1}$$

We know:

we have showed that $\sum |x|^n \begin{cases} \frac{1}{1-x}, & |x|<1 \\ \text{diverges} & |x| \geq 1 \end{cases}$

$$\frac{2^n}{3^n+1} < \frac{2^n}{3^n} \quad , \text{ and } \quad \sum_{n=1}^{\infty} \frac{2^n}{3^n} = \sum \left(\frac{2}{3}\right)^n .$$

converges (geometric series with $\frac{2}{3} < 1$).

Then $\sum \frac{2^n}{3^n+1}$ should converge?

Th (the comparison test).

If $0 \leq a_n \leq b_n$ $\forall n$ and $\sum_{n=1}^{\infty} b_n$ converges, then $\sum_{n=1}^{\infty} a_n$ also converges and $\sum_{n=1}^{\infty} a_n \leq \sum_{n=1}^{\infty} b_n$.

Corollary: Under the above conditions, if $\sum a_n$ diverges then $\sum b_n$ diverges.

Recall: Suppose $\langle x_n \rangle$ is an increasing sequence. Then either it is bounded (above) and then it converges to its supremum or it is unbounded and then it diverges to $+\infty$.

Proof of the theorem:

Consider partial sums: $A_N = \sum_{n=1}^{N} a_n$ and $B_N = \sum_{n=1}^{N} b_n$.

Then we know that $0 \leq A_N \leq B_N$. We also know: $a_n > 0$ $(a_1, a_1+a_2, a_2+a_1+a_3 \ldots\ldots)$

Since $a_n, b_n \geq 0$ the sequences $\langle A_N \rangle$ and $\langle B_N \rangle$ are increasing. They converge $\iff$ they are bounded.

Since $\sum_{n=1}^{\infty} b_n$ converges, $B_N \rightarrow B_\infty = \lim_{N \to \infty} B_N = \sup B_N$.

In particular $A_N \leq B_N \leq B_\infty$ $\forall N$

So the sequence $\langle A_N \rangle$ is bounded, so it converges (since it increases) to $\sup A_N$.

$$\underbrace{\sup A_N}_{} \leq B_\infty$$

→ So A_N is bounded and therefore it converges

upper bound .

Smallest upper bound .

Therefore,

$\sum a_n$ converges and $\sum_{n=1}^{\infty} a_n \leq \sum_{n=1}^{\infty} b_n$.

Example: we will prove later that $\sum_{n=1}^{\infty} \frac{1}{n^3}$ converges. How about

$$\sum_{n=1}^{\infty} \frac{1}{n^3-2} \;?$$

Can we use comparison test?

Two problems:

① $\frac{1}{n^3-2}$ is sometimes negative

② $\frac{1}{n^3-2} \leq \frac{1}{n^3}$ is not TRUE .

Proposition 3 (lo de la C) .

IDEA : $\sum \frac{1}{n^3}$ converges $\Rightarrow \sum \frac{2}{n^3}$ also converges .

For which n we have $\frac{1}{n^3-2} \leq \frac{2}{n^3}$?

$$n^3 \leq 2 \cdot (n^3-2) \;;\; n^3 \leq 2n^3-4 \;;$$

$$n^3 \geq 4 \;.$$

$\frac{1}{n^3-2} \leq \frac{2}{n^3} \iff n^3 \geq 4$. OK if $n \geq 2$.

Therefore, since $\sum_{n=2}^{\infty} \frac{2}{n^3}$ converges, the series $\sum_{n=2}^{\infty} \frac{1}{n^3-2}$ also converges, by comparison test and

$$\sum_{n=1}^{\infty} \frac{1}{n^3-2} = -1 + \underset{\substack{\uparrow \\ \text{converges}}}{\sum_{n=2}^{\infty} \frac{1}{n^3-2}} \leq -1 + \sum_{n=2}^{\infty} \frac{2}{n^3}$$

Def^n: We say that $\sum_{n=1}^{\infty} a_n$ absolutely converges if $\sum_{n=1}^{\infty} |a_n|$ converges.

If $\sum_{n=1}^{\infty} a_n$ converges but $\sum_{n=1}^{\infty} |a_n|$ diverges, we say that

$\sum_{n=1}^{\infty} a_n$ is conditionally convergent.

some ⊕ and some ⊖

positive everywhere

Theorem: If $\sum_{n=1}^{\infty} |a_n|$ converges then $\sum_{n=1}^{\infty} a_n$ converges.

Proof Given a_n, we define:

$$\boxed{a_n^+} = \begin{cases} a_n \; (=|a_n|) & \text{if } a_n \geq 0 \\ 0 & \text{if } a_n < 0 \end{cases}$$ (the positive part).

they are both ⊕.

$$\boxed{a_n^-} = \begin{cases} 0 & \text{if } a_n \geq 0 \\ -a_n & \text{if } a_n < 0 \end{cases}$$ (the negative part).

$-a_n = |a_n| \rightarrow a_n = 3 \quad -a_n = 3$.

Example: If $a_n = (-1)^n \cdot \frac{1}{n}$.

$(a_n) = -1, \frac{1}{2}, -\frac{1}{3}, \frac{1}{4} \ldots$

$a_n^+ = 0, \frac{1}{2}, 0, \frac{1}{4}, \ldots$

$a_n^- = 1, 0, \frac{1}{3}, 0, \ldots$

Then $\boxed{a_n = a_n^+ - a_n^- \text{ and } |a_n| = a_n^+ + a_n^-}$

Also, $\boxed{0 \leq a_n^{\pm} \leq |a_n|}$.

By comparison test, if $\sum |a_n|$ converges $\Rightarrow$ $\sum a_n^+$ and $\sum a_n^-$ converge.

Therefore, $\sum a_n = \sum (a_n^+ - a_n^-)$ converges and $\sum a_n = \sum a_n^+ - \sum a_n^-$

Example:

$\sum_{n=1}^{\infty} x^n$ absolutely converges, if $|x| < 1$.

Also, $-\frac{1}{1\cdot 2}, -\frac{1}{2\cdot 3} + \frac{1}{3\cdot 4} - \frac{1}{4\cdot 5} + \ldots$ converges absolutely.

Th The infinite series $\sum_{n=1}^{\infty} \frac{1}{n^p} = 1 + \frac{1}{2^p} + \frac{1}{3^p} + \ldots$

converges $\Longleftrightarrow$ $p > 1$.

Remark If $p = \frac{a}{b} \in \mathbb{Q}$ then $n^p = \sqrt[b]{n^a}$. Otherwise we will define it later.

We need to prove:

① If $p>1$ then $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges.

② If $p \le 1$ then $\sum_{n=1}^{\infty} \frac{1}{n^p}$ diverges.

Proof: Put $S_N = \sum_{N=1}^{N} \frac{1}{n^p}$. then $\langle S_N \rangle_{N=1}^{\infty}$ increases

It converges $\iff$ it is bounded.

It diverges $\iff$ unbounded to $+\infty$.

Case ①: $p \le 1$. We need to prove that $\langle S_N \rangle$ is unbounded

Since $n^p \le n$, we know that $\frac{1}{n^p} \ge \frac{1}{n}$, we have . eso.

for $N = 2^m$ we have $S_N = S_{2^m} = \frac{1}{1^p} + \frac{1}{2^p} + \dots + \frac{1}{(2^m)^p} \ge$

$$\ge \frac{1}{1} + \frac{1}{2} + \underbrace{\left(\frac{1}{3} + \frac{1}{4}\right)}_{2 \text{ terms}} + \underbrace{\left(\frac{1}{5} + \dots + \frac{1}{8}\right)}_{4 \text{ terms}} + \underbrace{\left(\frac{1}{9} + \dots + \frac{1}{16}\right)} + \dots + \underbrace{\left(\frac{1}{2^{(m-1)}+1} + \dots + \frac{1}{2^m}\right)}_{m-1} \ge$$

$$\ge 1 + \frac{1}{2} + \underbrace{\left(\frac{1}{4} + \frac{1}{4}\right)}_{2} + \underbrace{\left(\frac{1}{8} + \frac{1}{8} + \dots + \frac{1}{8}\right)}_{4} + \underbrace{\left(\frac{1}{16} + \overset{8}{\dots} + \frac{1}{16}\right)}_{8} + \dots + \underbrace{\left(\frac{1}{2^m} + \dots + \frac{1}{2^m}\right)}_{2^{m-1}} =$$

$$= 1 + \frac{1}{2} + 2 \cdot \left(\frac{1}{4}\right) + 4 \cdot \left(\frac{1}{8}\right) + 8 \cdot \left(\frac{1}{16}\right) + \dots + 2^{m-1} \cdot \left(\frac{1}{2^m}\right) =$$

$$= 1 + \underbrace{\frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \dots + \frac{1}{2}}_{\frac{1}{2} \cdot \text{ⓜ} = \text{términos}} = 1 + \frac{m}{2} \to \infty$$ as $m \to \infty$. Therefore

$\langle S_N \rangle$ is unbounded, so $S_N \to +\infty$.

and $\sum \frac{1}{n^p}$ diverges.

Remark: If $1 \ge p > 0$ $\frac{1}{n^p} \to 0$ as $n \to \infty$, however $\sum \frac{1}{n^p}$ diverges

the series $\sum \frac{1}{n} = \frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \dots$ is called the harmonic series.

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November 13th 2018

Suppose $\sum_{n=1}^{\infty} a_n$ is an infinite series with $a_n \geq 0$, $\forall n$. Then, the sequence $\langle S_N \rangle_{N=1}^{\infty}$ of partial sums, $S_N = \sum_{n=1}^{N} a_n$, is increasing and either:

i) is bounded and converges, or

ii) is unbounded and diverges to $+\infty$.

Th $\sum_{n=1}^{\infty} \frac{1}{n^p}$, $p > 0$ converges $\overset{iff}{\iff}$ $p > 1$.

$\boxed{\text{Case 1}}$ $p \leq 1$ (proved).

$\boxed{\text{Case 2}}$ $p > 1$ Take

$N = 2^m - 1$ Then:

$$S_N = S_{2^m - 1} = \frac{1}{1^p} + \frac{1}{2^p} + \cdots + \frac{1}{(2^m-1)^p} = \frac{1}{1^p} + \overbrace{\left(\frac{1}{2^p} + \frac{1}{3^p}\right)}^{2 \text{ terms}} + \overbrace{\left(\frac{1}{4^p} + \cdots + \frac{1}{7^p}\right)}^{4 \text{ terms} \ldots 2^n} +$$

$$+ \underbrace{\left(\frac{1}{8^p} + \cdots + \frac{1}{15^p}\right)}_{8} + \underbrace{\left(\frac{1}{16^p} + \cdots\right)}_{16} + \cdots + \underbrace{\left(\frac{1}{(2^{m-1})^p} + \cdots + \frac{1}{(2^m-1)^p}\right)}_{2^{m-1}} \leq$$

$$\leq \frac{1}{1} + \overbrace{\left(\frac{1}{2^p} + \frac{1}{2^p}\right)}^{2} + \overbrace{\left(\frac{1}{4^p} + \frac{1}{4^p} + \frac{1}{4^p} + \frac{1}{4^p}\right)}^{4} + \underbrace{\left(\frac{1}{8^p} + \frac{1}{8^p} + \cdots + \frac{1}{8^p}\right)}_{8} + \cdots +$$

$$+ \underbrace{\left(\frac{1}{(2^{m-1})^p} + \cdots + \frac{1}{(2^{m-1})^p}\right)}_{2^{m-1}} = \frac{1}{1} + 2 \cdot \left(\frac{1}{2^p}\right) + 4 \cdot \left(\frac{1}{4^p}\right) + 8 \cdot \left(\frac{1}{8^p}\right) + (2^{m-1}) \cdot \left(\frac{1}{(2^{m-1})^p}\right) =$$

$$= 1 + \frac{1}{2^{p-1}} + \frac{1}{4^{p-1}} + \frac{1}{8^{p-1}} + \cdots + \frac{1}{(2^{m-1})^{p-1}} = 1 + \frac{1}{2^{p-1}} + \frac{1}{(2^{p-1})^2} + \frac{1}{(2^{p-1})^3} + \cdots +$$

$$\left[\frac{1}{\underbrace{(2^{m-1})^{-1}}_{a^{-1}} \cdot \underbrace{(2^{m-1})^p}_{a^p}} \quad a^{-1} \cdot a^p = a^{p-1} \right]$$

$$\cdots + \frac{1}{(2^{p-1})^{m-1}} = \sum_{n=0}^{m-1} \frac{1}{(2^{p-1})^n} \leq$$

$$\leq \sum_{n=0}^{\infty} \frac{1}{(2^{p-1})^n} = \sum_{n=0}^{\infty} \left(\frac{1}{2^{p-1}}\right)^n.$$

NOTE: $p > 1 \Rightarrow p - 1 > 0 \Rightarrow 2^{p-1} > 2^0 = 1 \Rightarrow \frac{1}{2^{p-1}} < 1$. Therefore,

$\sum_{n=0}^{\infty} \left(\frac{1}{2^{p-1}}\right)^n$ is a $\underbrace{\text{convergent}}$ geometric series.

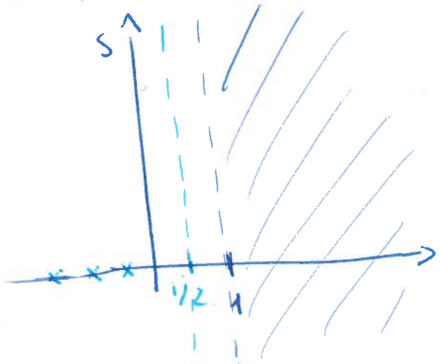
es convergente porque acabamos de demostrar que existe un upper bound que pertenece al Σ (porque es más pequeño que 1) y a la vez más grande que el resto.

Thus, we have found an upper bound $\left(\sum_{n=0}^{\infty} \left(\frac{1}{2^{R-1}} \right)^n \right)$ of $< S_{2^m-1} >$, so $< S_{2^m-1} >_{m=1}^{\infty}$ is bounded, so $S_{2^m-1} \not\to +\infty$, so $S_N \not\to +\infty$, so $< S_N >$ is bounded, so $< S_N >$ converges, so $\sum_{n=a}^{\infty} \frac{1}{n^p}$ converges if $p > 1$ ▨

REMARK: Rieman zeta series

$$\boxed{\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}}$$ is called the Rieman zeta - function

For $n=1$, the function converges.



[For which n in the complex plane $\zeta(s) = 0$]

$$\zeta(2) = \boxed{\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}}$$ Se sabe.

ζ (odd integer) = ?

$\zeta(3) \notin \mathbb{Q}$: this is the only thing they know.

We know that: $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \dots + \frac{1}{n^{p}}$ diverges (harmonic series diverge)
$\hookrightarrow p = 1 \equiv \frac{1}{n}$ (aunque tiende a 0, hemos demostrado que si $p \leq 1$ entonces diverge).

But the series $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \dots$ converges.

Why?

Th: (alternating series test)

Suppose $< a_n >_{n=0}^{+\infty}$ is a decreasing sequence, $a_n \geq 0$, $\lim_{n \to \infty} a_n = 0$. Then $\sum_{n=0}^{\infty} (-1)^n \cdot a_n = a_0 - a_1 + a_2 - a_3 + \dots$ converges.

$S_{2N+1} = 1 - \frac{1}{2}$

$S_{2(N+1)+1} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4}$

$\} S_{2(N+1)+1} - S_{2N+1} = \left(1 - \frac{1}{2}\right) \oplus \frac{1}{3} - \frac{1}{4} = \oplus$

Proof:

(split the sequence of partial sums in $\left\langle \begin{matrix} \text{even} \\ \text{odd} \end{matrix} \right)$.

Consider 2 sequences of partial sums :

$$S_{2N} = \sum_{n=0}^{2N} (-1)^n \cdot a_n = a_0 - a_1 + a_2 - \cdots - a_{2N-1} + a_{2N}$$

and $S_{2N-1} = \sum_{n=0}^{2N-1} (-1)^n a_n = a_0 - a_1 + \cdots + a_{2N-2} - a_{2N-1}$

del par: voy a restar el grande (el primero)

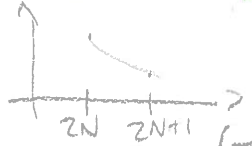
Claim 1: $\langle S_{2N} \rangle$ is decreasing

Proof: $S_{2(N+1)} - S_{2N} = \underbrace{\overset{\text{eg } \frac{1}{4}}{a_{2N+2}} - \overset{\text{eg } \frac{1}{3}}{a_{2N+1}}} . = \ominus$

*pag 16 atrás

$$S_{2(N+1)} = \sum_{n=0}^{2(N+1)} (-1)^n (a_n) = a_0 - a_1 + a_2 - \cdots - a_{2N-1} + a_{2N} - \underline{a_{2N+1} +}$$

$\underbrace{+ a_{2N+2}}_{\text{remain}}$. $\Big(\text{Since } \underbrace{a_{2N+2} \le a_{2N+1}}_{\text{no tenga en cuenta } (-1)^n \text{ porque sólo tengo en cuenta } \langle a_n \rangle} \Big)$.



Impar: resto el pequeño

Claim 2 $\langle S_{2N-1} \rangle$ is increasing

Proof: $S_{2(N+1)-1} - S_{2N-1} = S_{2N+1} - S_{2N-1} = \underline{a_{2N} - a_{2N+1}} = \oplus$

*pag 14 atrás

eg $\frac{1}{3} - \frac{1}{4}$

$$S_{2N+1} = \sum_{n=0}^{2N+1} (-1)^n \cdot a_n = a_0 - a_1 + \cdots + a_{2N-2} - a_{2N-1} + \underbrace{a_{2N} - a_{2N+1}}_{\text{remain}} \ge 0$$

(Since $a_{2N} \ge a_{2N+1}$)

Claim 3 $S_{2N} \ge S_{2N-1}$

Proof

$$S_{2N} - S_{2N-1} = a_{2N} \ge 0$$

Therefore, $\underbrace{S_1 \le S_3 \le \cdots}_{\text{impares}} \le S_{2N-1} \le S_{2N} \le \underbrace{S_{2N-2} \le \cdots S_2}_{\text{pares}}$.

In particular:

$S_{2N} \ge S_1$, so $\langle S_{2N} \rangle$ is decreasing bounded below sequence, so it converges .

We also know :

$S_{2N-1} \le S_2$, so $\langle S_{2N-1} \rangle$ is an increasing bounded above sequence, so it converges .

Claim 4. $\lim S_{2N} = \lim S_{2N-1}$.

Proof. We know .

$S_{2N} - S_{2N-1} = a_{2N}$, so $\lim S_{2N} - \lim S_{2N-1} = \lim a_{2N} = 0$

(our assumption)

Thus, $\lim S_{2N} = \lim S_{2N-1}$

Now problem 6(g), HW4 implies that $\langle S_N \rangle$ converges, so our series

$a_0 - a_1 + a_2 - a_3 + \dots$ converges ▨

si pongo $|a_n|$ diverge, pero a_n converge

Corollary, the series $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \dots$ converges conditionally.

Question: given $\langle a_n \rangle$, can we find out whether $\sum_{n=1}^{\infty} a_n$ converges?

1. If $a_n \not\to 0$, series diverges.
2. Suppose, $a_n \geq 0 \ \forall n$. Then we can use:
   - (A) Comparison test.
   - (A') limit comparison test. (HW5).
   - (A'') Improved comparison test (proved later).
   - (B) Ratio test. } to be discussed.
   - (C) Root test.
3. If not all a_n are positive, does the series $\sum |a_n|$ converge? If so, then $\sum a_n$ converges absolutely
4. If $\sum |a_n|$ diverges, then perhaps we can apply alternating series test.
5. Give up.

Th (Improved comparison test).

Suppose, $0 \leq a_n \leq b_n$. Then:

(i) $\sum_{n=0}^{\infty} b_n$ converges $\Rightarrow \sum_{n=0}^{\infty} a_n$ converges.

(ii) $\sum a_n$ diverges $\Rightarrow \sum b_n$ diverges.

Proof of (i) (sketch of proof).

$$\sum_{n=0}^{\infty} a_n = \sum_{n=0}^{N} a_n + \sum_{n=N+1}^{\infty} a_n$$

finite sum

converges, by comparison test.

November 15th 2018

Improved comparison test:

$0 \leq a_n \leq b_n$ $\quad \sum b_n$ converges $\Rightarrow \sum a_n$ converges.

Recall: In geometric series, we have $a_n = x^n$, $x \in \mathbb{R}$.

$\sum a_n = \sum x^n$ converges iff $|x| < 1$.

Idea 1:

$$|x| = \underbrace{\frac{|a_{n+1}|}{|a_n|} = \frac{|x^{n+1}|}{|x^n|}}$$

If this expresion is < 1 then it will be convergent, if not, divergent.

Idea 2:

$$|x| = \sqrt[n]{|x|^n} = \sqrt[n]{|a_n|}$$

Th: The ratio test: Suppose, $\sum_{n=1}^{\infty} a_n$ is an infinite series and $\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = \ell \in \mathbb{R}$. Then:

(i) If $\ell < 1$, then $\sum a_n$ converges.

(ii) If $\ell > 1$, then $\sum a_n$ diverges.

(iii) If $\ell = 1$, no conclusion.

Proof

(i) I know $\lim \frac{|a_{n+1}|}{|a_n|} = \ell < 1$

choose $\varepsilon > 0$ s.t $\boxed{\ell + \varepsilon < 1}$ (take $\varepsilon = \frac{1-\ell}{2}$).

para demostrarlo con $\ell < 1$ de manera que $\frac{|a_{n+1}|}{|a_n|} < \boxed{\ell + \varepsilon} < 1$. entonces se que $|X| < 1$, y por tanto $\sum a_n$ converge

Then $\exists N \in \mathbb{N}$ s.t $\left|\frac{|a_{n+1}|}{|a_n|} - \ell\right| < \varepsilon$.

$\forall n > N$

$$-\varepsilon < \frac{|a_{n+1}|}{|a_n|} - \ell < \varepsilon \iff \ell - \varepsilon < \frac{|a_{n+1}|}{|a_n|} < \ell + \varepsilon \Rightarrow$$

$$\Rightarrow |a_{n+1}| < (\ell + \varepsilon) \cdot |a_n| \quad \forall n > N.$$

Suppose $k \in \mathbb{N}$.

↳ para encontrar una sequence más grande que se que converge

termino más grande que (aunque lo multiplique por un nº pequeño, va a seguir siendo más grande que si otro th multiplicado

Then

$$|a_{N+k}| < (\ell+\varepsilon)\,|a_{N+k-1}| < (\ell+\varepsilon)^2\,|a_{N+k-2}| < (\ell+\varepsilon)^3|a_{N+k-3}|$$

$$< \cdots < (\ell+\varepsilon)^{k-1}\,|a_{N+1}| = (\ell+\varepsilon)^k \frac{|a_{N+1}|}{(\ell+\varepsilon)}$$

Since $\sum_{k=1}^{\infty} (l+\varepsilon)^k \cdot \frac{|a_{N+1}|}{(l+\varepsilon)} = \frac{|a_{N+1}|}{(l+\varepsilon)} \cdot \sum_{k=1}^{\infty} (l+\varepsilon)^k$ converges.

(a geometric series with $l+\varepsilon < 1$).

The improved comparison test implies that $\sum |a_n|$ converges and, thus $\sum a_n$ absolutely converges.

$\to 1 < l - \varepsilon < \frac{|a_{n+1}|}{|a_n|}$. entonces $\Rightarrow 1 < |X|$, asique diverge.

(ii) Choose $\varepsilon > 0$ s.t $l - \varepsilon > 1$.

Then $\exists N \in \mathbb{N}$ s.t $\forall n > N$ we have $\left| \frac{|a_{n+1}|}{|a_n|} - l \right| < \varepsilon$

$$l - \varepsilon < \frac{|a_{n+1}|}{|a_n|} < \varepsilon + l$$

$$|a_{n+1}| > (l-\varepsilon) \cdot |a_n| \quad \forall n > N.$$

Therefore, if $K \in \mathbb{N}$ we have $|a_{N+k}| > (l-\varepsilon)|a_{N+k-1}| >$
$> (l-\varepsilon)^2 |a_{N+k-2}| > \dots > (l-\varepsilon)^{k-1} |a_{N+1}|$

However, since (Bernoulli's inequality)
$(l-\varepsilon) > 1$, we have $(l-\varepsilon)^k \to +\infty$ as $k \to \infty$.

So $|a_{N+k}| \to +\infty$ as $k \to \infty$, so $a_n \to 0$ as $n \to \infty$,
so $\sum a_n$ diverges. ▨

Th (root test)

Suppose $\lim_{n \to \infty} \sqrt[n]{|a_n|} = l \in \mathbb{R}$ then:

(i) If $l < 1$, $\sum_{n=1}^{\infty} a_n$ converges absolutely.

(ii) If $l > 1$, then $\sum_{n=1}^{\infty} a_n$ diverges

(iii) If $l = 1$, no conclusion

Applications → to power series of the form: $\sum_{n=0}^{\infty} b_n x^n$.

Example: definition. For $x \in \mathbb{R}$ we define $\exp(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!} =$
$= 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$

Proposition. $\forall x \in \mathbb{R}$ this series converges.

Proof: Use the ratio test. $a_n = \frac{x^n}{n!}$, so $\frac{|a_{n+1}|}{|a_n|} = \frac{\frac{x^{n+1}}{(n+1)!}}{\frac{x^n}{n!}} =$

$= \frac{x^{n+1}}{x^n} \cdot \frac{n!}{(n+1)!} = \frac{x}{n+1} \to 0$ as $n \to \infty$ so $l = 0$.

and $\sum_{n=0}^{\infty} \frac{x^n}{n!}$ converges by the ratio test.

We want to prove that $\exp(x+y) = \exp(x) \cdot \exp(y)$.

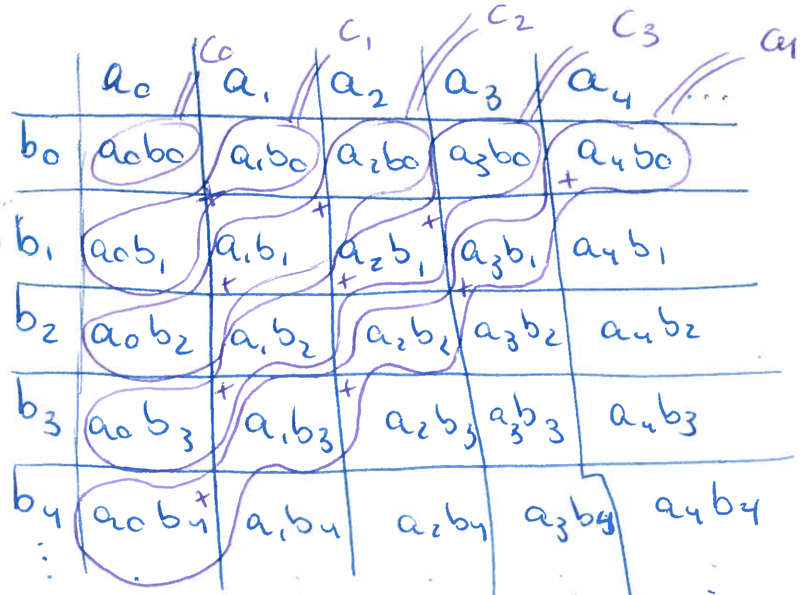
(we define exp. as infinite series).

Multiplying series:

$(a_0 + a_1) \cdot (b_0 + b_1) = a_0b_0 + a_0b_1 + \dots$

$(a_0 + a_1 + a_2 + \dots)(b_0 + b_1 + b_2 + \dots) = a_0b_0 + a_0b_1 + \dots$

we don't know which multiplication goes next, because order is important



	a_0	a_1	a_2	a_3	a_4
b_0	a_0b_0	a_1b_0	a_2b_0	a_3b_0	a_4b_0
b_1	a_0b_1	a_1b_1	a_2b_1	a_3b_1	a_4b_1
b_2	a_0b_2	a_1b_2	a_2b_2	a_3b_2	a_4b_2
b_3	a_0b_3	a_1b_3	a_2b_3	a_3b_3	a_4b_3
b_4	a_0b_4	a_1b_4	a_2b_4	a_3b_4	a_4b_4

Given 2 infinite series $\sum_{n=0}^{\infty} a_n$ and $\sum_{n=0}^{\infty} b_n$, we define

$c_0 = a_0b_0$

$c_1 = a_0b_1 + a_1b_0$

$c_n = a_0b_n + a_1b_{n-1} + \dots + a_{n-1}b_1 + a_nb_0$.

$$c_n = \sum_{k=0}^{n} a_k \cdot b_{n-k}$$

Is it true that $\left(\sum_{n=0}^{\infty} a_n\right) \cdot \left(\sum_{n=0}^{\infty} b_n\right) \overset{?}{=} \sum_{n=0}^{\infty} c_n = \boxed{\sum_{n=0}^{\infty} \sum_{k=0}^{\infty} a_k b_{n-k}}$

CAUCHY PRODUCT

Th (Cauchy products)

If $\sum_{n=0}^{\infty} a_n$ and $\sum_{n=0}^{\infty} b_n$ converge absoluttely, then

$\sum c_n$ also converges absoluttely, and (*) holds:

$$\left(\sum a_n\right)\left(\sum b_n\right)=\sum c_n .$$

Proof: Later (may be proved).

Corollary, $\exp(x+y) = \exp(x)\cdot\exp(y)$.

Binomial expansion

$$\exp(x+y) = \sum_{n=0}^{\infty}\frac{(x+y)^n}{n!} = \sum_{n=0}^{\infty}\frac{1}{n!}\left(\overbrace{\sum_{k=0}^{n}\binom{n}{k}x^k\cdot y^{n-k}}^{=(x+y)^n}\right) =$$

$$= \sum_{n=0}^{\infty}\frac{1}{\cancel{n!}}\cdot\sum_{k=0}^{n}\frac{\cancel{n!}}{k!(n-k)!}x^k\cdot y^{n-k} = \sum_{n=0}^{\infty}\sum_{k=0}^{n}\frac{x^k}{k!}\cdot\frac{y^{n-k}}{(n-k)!}$$

Cauchy product:

$$\exp(x) = \sum_{n=0}^{\infty}\frac{x^n}{n!}, \quad a_n = \frac{x^n}{n!}$$

$$\exp(y) = \sum_{n=0}^{\infty}\frac{y^n}{n!}, \quad b_n = \frac{y^n}{n!}$$

$$c_n = \underbrace{\sum_{k=0}^{n} a_k\cdot b_{n-k}}_{\text{lo obtengo de antes que he multiplicado } a_n\cdot b_n} = \sum_{k=0}^{n}\frac{x^k}{k!}\cdot\frac{y^{n-k}}{(n-k)!}, \text{ so}$$

$$\exp(x)\cdot\exp(y) = \sum c_n = \sum_{n=0}^{\infty}\sum_{k=0}^{n}\frac{x^k}{k!}\cdot\frac{y^{n-k}}{(n-k)!} = \exp(y+x)$$

Limits of functions and continuity

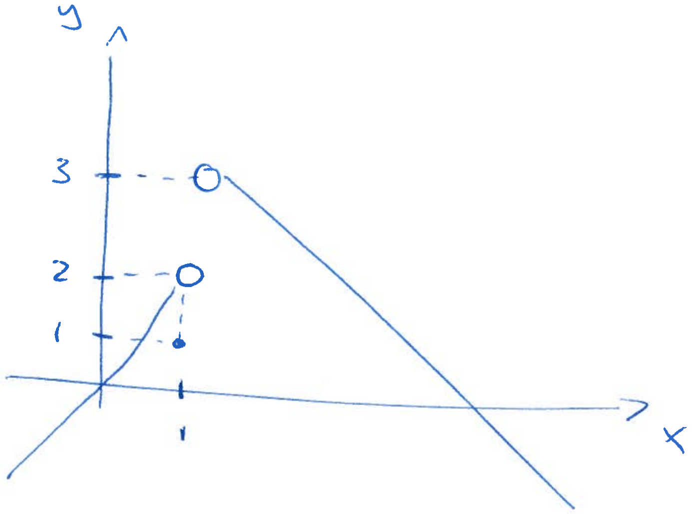
$$f : \underset{\substack{\uparrow \\ \text{Domain of } f.}}{\mathbb{R}} \to \underset{\substack{\uparrow \\ \text{Codomain}}}{\mathbb{R}}$$

Example ①

$f : \mathbb{R} \to \mathbb{R}$.

Defined by:

$$f(x) = \begin{cases} 4-x, & \text{if } x > 1. \\ 2x, & \text{if } x < 1. \\ 1 & \text{if } x = 1. \end{cases}$$



Intuition • as $x \to 1^+$ (x approaches 1 from above, or from the right,

$f(x) \longrightarrow 3$

• as $x \to 1^-$ (from below or from the left),

$f(x) \to 2$.

f is discontinuous as $x=1$, but continuous at $x \neq 1$.

def 1) $f: (a,b) \longrightarrow \mathbb{R}$

We say that $\lim_{x \to a^+} f(x) = l \in \mathbb{R}$.

|| NOTATION

$\left(\lim_{x \searrow a} f(x) = l \right)$ if $\forall \varepsilon > 0 \ \exists \underset{\substack{\uparrow \\ \text{delta}}}{\delta} > 0$ s.t $\overset{\text{whenever}}{\downarrow} x \in (a, a+\delta)$

this implies $|f(x) - l| < \varepsilon$

$$\Updownarrow$$

$\left(f(x) \in (l-\varepsilon, l+\varepsilon) \right)$.

20th November 2018 :

$f:(a,b) \longrightarrow \mathbb{R}$

means $f(x)$ is defined for $x \in (a,b)$ and $f(x) \in \mathbb{R}$ $\forall x \in (a,b)$.

def:

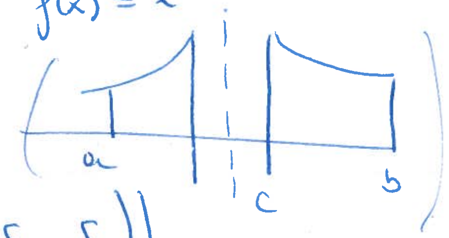
a) Suppose, $f: (a,b) \longrightarrow \mathbb{R}$

we say $\lim_{x \to a^+} f(x) = \lim_{x \searrow a} f(x) = l$, if $\forall \varepsilon > 0 \ \exists \delta > 0$

s.t if $x \in (a, a+\delta)$, then we have $|f(x) - l| < \varepsilon$.

b) $\lim\limits_{x \to b^-} f(x) = \lim\limits_{x \nearrow b} f(x) = \ell$ if $\boxed{\forall \varepsilon > 0 \ \exists \delta > 0 \text{ s.t if } x \in (b-\delta, b) \text{ then we have } |f(x) - \ell| < \varepsilon}$

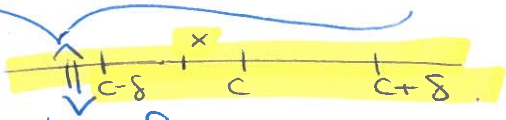
c) Suppose, $c \in (a,b)$ and f is defined on (a,b), except possibly c (i.e. f is defined on $(a,c) \cup (c,b)$). We say that $\lim\limits_{x \to c} f(x) = \ell$, if $\lim\limits_{x \to c^+} f(x) = \lim\limits_{x \to c^-} f(x) = \ell$



Equivalent definition (take $\delta = \min(\delta_1, \delta_2)$)

$\forall \varepsilon > 0 \ \exists \delta > 0$ s.t $\underbrace{x \in (c, c+\delta) \text{ or } x \in (c-\delta, c)}$

$\Rightarrow |f(x) - \ell| < \varepsilon$.



$\Updownarrow$

$|x-c| < \delta$ and $x \neq c$.

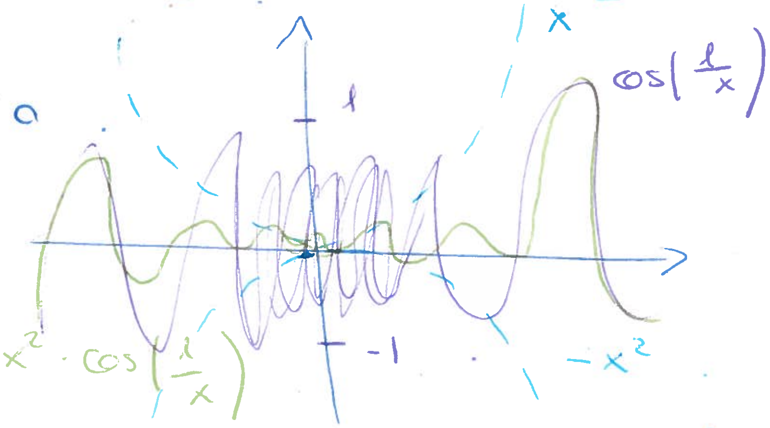
$0 < |x-c| < \delta$.

def' $\lim\limits_{x \to c} f(x) = \ell$, if $\forall \varepsilon > 0 \ \exists \delta > 0$ s.t

$0 < |x-c| < \delta \Rightarrow |f(x) - \ell| < \varepsilon$

Example 2 :

$f(x) = 1 + x^2 \cdot \cos\left(\frac{1}{z}\right)$ for $x \neq 0$



Claim $\lim\limits_{x \to 0} f(x) = 1$

Proof. Given $\varepsilon > 0$ we need to find $\delta > 0$ s.t $0 < \underset{|x|}{\underset{\parallel}{|x-0|}} < \delta$

$\Rightarrow \underset{\left|x^2 \cdot \cos\left(\frac{1}{x}\right)\right| < \varepsilon}{\underset{\parallel}{|f(x) - 1|}} < \varepsilon$.

We notice that $\left|x^2 \cdot \cos\left(\frac{1}{x}\right)\right| \leq |x^2| = x^2$

Therefore, if $x^2 < \varepsilon$, then $\left|x^2 \cdot \cos\left(\frac{1}{x}\right)\right| < \varepsilon$.

However, $x^2 < \varepsilon \iff |x| < \sqrt{\varepsilon}$. Take $\delta = \sqrt{\varepsilon}$. Then

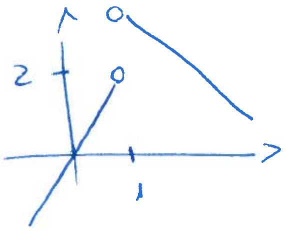
$0 < |x| < \delta = \sqrt{\varepsilon} \Rightarrow x^2 < \varepsilon \Rightarrow \left|x^2 \cos\left(\frac{1}{x}\right)\right| \leq x^2 < \varepsilon$.

Thus, $\lim_{x \to 0} f(x) = 1$.

Remark: f is not defined at 0, but we do not care.

Example 1 : (re-visited)

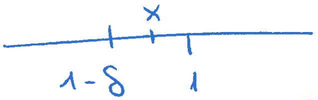
$$f(x) = \begin{cases} 4-x, & x > 1. \\ 2x, & x < 1. \\ 1, & x = 1. \end{cases}$$



$\overbrace{1-\delta < x < 1}^{0 < |x-1| < \delta}$

Claim $\lim_{x \to 1^-} f(x) = 2$.

Proof: Given $\varepsilon > 0$ we need to find $\delta > 0$ s.t $1-\delta < x < 1$ $\Rightarrow |f(x)-2| < \varepsilon$, $x < 1 \Rightarrow f(x) = 2x$, so $|f(x)-2| = |2x-2| < \varepsilon$ $\iff 2|x-1| < \varepsilon \iff |x-1| < \frac{\varepsilon}{2}$ Take $\delta = \frac{\varepsilon}{2}$. Then, this implies $|x-1| < \delta = \frac{\varepsilon}{2}$, if $1-\delta < x < 1$



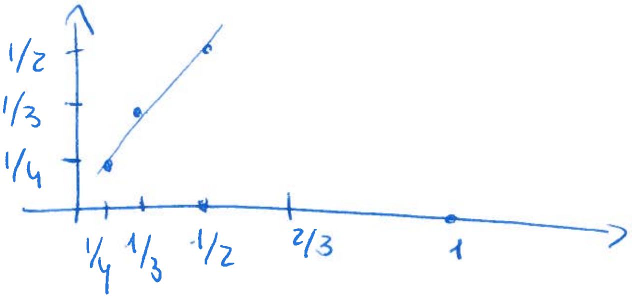
so $|f(x)-2| = |2x-2| = 2|x-1| < \varepsilon$.

Thus $\lim_{x \to 1^-} f(x) = 2$.

Example 3 : (more advanced).

$f : (0,1) \longrightarrow \mathbb{Q}$

$$f(x) = \begin{cases} 0, & x \notin \mathbb{Q} \\ 1/q & x \in \mathbb{Q},\ x = \frac{p}{q}, \end{cases}$$ p and q are coprime.



Claim $\lim_{x \to 1^-} f(x) = 2$.

Claim $\forall c \in (0,1)$.

$\lim_{x \to c} f(x) = 0$.

Proof Given $\varepsilon > 0$, we need to find $\delta > 0$ s.t $0 < |x-c| < \delta$ $\Longrightarrow |f(x)| < \varepsilon$. Take $N \in \mathbb{N}$ s.t $\frac{1}{N} < \varepsilon$. Then

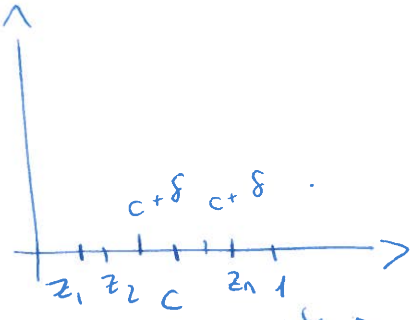
There are finitely many points, $z_1, z_2, \ldots, z_n$ s.t

s.t $f(z_j) \in \left\{ \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots, \frac{1}{N} \right\}$

in other words, $\{ z_1, \ldots, z_n \} = \left\{ \frac{1}{2}, \frac{1}{3}, \frac{2}{3}, \frac{1}{4}, \frac{2}{4}, \frac{3}{4}, \frac{1}{5}, \ldots, \frac{N-1}{N} \right\}$

Then $x \in (0,1)$, $x \neq z$,

we have $|f(x)| < \varepsilon$



Take $\delta = \min\{ |c - z_1|, |c - z_2|, \underset{\substack{\uparrow \\ \text{throw away}}}{\ldots}, |c - z_n| \} > 0$

$|c - z_j|$, if $c \neq z$.

Then $0 < |x - c| < \delta$ implies $x \neq z$ and thus $f(x) < \frac{1}{N} < \varepsilon$.

November 22nd 2018

Suppose, $S \subset \mathbb{R}$ is any subset of $\mathbb{R}$.
↳ subset of $\mathbb{R}$ nbs.

def: Suppose, $f : S \longrightarrow \mathbb{R}$ and $c \in S$. We say the function is continuous at c, if $\forall \varepsilon > 0$ $\exists \delta > 0$ s.t $|x - c| < \delta$ and $x \in S$ implies $|f(x) - f(c)| < \varepsilon$.

df f is continuous on S $(f \in C(S))$ if f is continuous at every point $c \in S$.

The class of all f continuous on S is denoted as $C(S)$.

Special cases.

(1) $S = (a,b)$, $c \in (a,b)$ Then f is continuous at c iff

$$\lim_{x \to c} f(x) = f(c)$$

(2) $S = [a,b]$ f is continuous on S means:

(i) for any $c \in (a,b)$

$$\lim_{x \to c} f(x) = f(c).$$

(ii) $\lim_{x \to a^+} f(x) = f(a)$ and $\lim_{x \to b^-} f(x) = f(b)$

Example 3:

$f: (0,1) \longrightarrow \mathbb{Q}$.

$$f(x) = \begin{cases} 0, & x \notin \mathbb{Q} \\ 1/q, & x = \frac{p}{q} \end{cases}.$$

Since $\lim_{x \to c} f(x) = 0$ $\forall c$, this means that f is continuous at all irrational points and discontinuous (not continuous) at all rationals.

Th (limits of functions and sequences)

(1) f is defined on (a,b) except possibly at $c \in (a,b)$. Then $\lim_{x \to c} f(x) = \ell$ iff the following statement holds

Suppose, $\langle x_n \rangle$ is a sequence with $x_n \in (a,b)$, $x_n \neq c$, $\lim x_n = c$. Then $\lim_{n \to \infty} f(x_n) = \ell$.

(**Remark**: $\langle f(x_n) \rangle_{n=1}^{\infty}$ is a sequence)

(2) $\lim_{x \to b^-} f(x) = \ell \iff$ for any sequence $\langle x_n \rangle$ with $x_n \in (a,b)$, $x_n < b$, $x_n \to b$, we have $\lim f(x_n) = \ell$.

Similarly $\lim_{x \to a^+} f(x) = \ell \iff \forall$ sequence $\langle x_n \rangle$ with $x_n \in (a,b)$, $x_n > a$, $x_n \to a$, we have $\lim f(x_n) = \ell$.

(3) $f : S \longrightarrow \mathbb{R}$, $c \in S$. f is continuous at c iff whenever $\langle x_n \rangle$ is a sequence with $x_n \to c$, $x_n \in S$ we have $\lim_{n\to\infty} f(x_n) = f(c)$

$\lim_{x\to c} f(x) = f(c) \to \lim_{n\to\infty} f(x_n) = f(c)$

Proof. of (3) other cases are similar
f is continuous at c, $\langle x_n \rangle$, $x_n \to c$ iff $\lim_{n\to\infty} f(x_n) = f(c)$

$\Rightarrow$ Suppose, f is continuous at c, $x_n \in S$, $x_n \to c$.

Need to prove : $\lim_{n\to\infty} f(x_n) = f(c)$.

Suppose, $\varepsilon > 0$
Since f is continuous at c, $\exists\, \delta > 0$ s.t $|x - c| < \delta$ $x \in S$
implies $\Rightarrow$ $|f(x) - f(c)| < \varepsilon$.

Since, $\lim_{n\to\infty} x_n = c$, given $\delta > 0$ $\exists\, N \in \mathbb{R}$ s.t $\forall n > N$ $|x_n - c| < \delta$.

This means that for $n > N$ we have : $|x_n - c| < \delta \Rightarrow |f(x_n) - f(c)| < \varepsilon$

This means that : $\lim_{n\to\infty} f(x_n) = f(c)$

$f(x_n) \to f(c)$ Lo niego
entonces x_n debería no $\to c$, por tanto, entonces Lo hay contradicción

$\Leftarrow$ we know that $\forall$ sequence $\langle x_n \rangle$ with $x_n \in S$, $x_n \to c$ we have $f(x_n) \to f(c)$ as $n \to \infty$.

We need to prove that f is continuous at c.
(i.e. $\forall\, \varepsilon > 0$ $\exists\, \delta > 0$ s.t $\forall x \in S$ with $|x - c| < \delta$ we have, $|f(x) - f(c)| < \varepsilon$).

Suppose, f is not continuous i.e. $\exists\, \varepsilon > 0$ $\forall\, \delta > 0$ $\exists\, x \in S$ s.t $|x - c| < \delta$ but $|f(x) - f(c)| > \varepsilon$.

Contradigo $f(x_n) \not\to f(c)$

Niego $f(x_n) \to f(c)$, entonces x_n debería $\not\to c$

In particular (take $\delta = \frac{1}{n}$). $\exists\, \varepsilon > 0$ s.t $\forall n \in \mathbb{N}$ (and there's put $\delta = \frac{1}{n}$). $\exists\, x = x_n \in S$ s.t $|x_n - c| < \frac{1}{n}$ but $|f(x_n) - f(c)| > \varepsilon$.

Then $|x_n - c| < \frac{1}{n} \Leftrightarrow -\frac{1}{n} < x_n - c < \frac{1}{n} \Leftrightarrow \overset{c}{\overset{\uparrow}{c - \overset{0}{\cancel{\frac{1}{n}}}}} < x_n < \overset{c}{\overset{\uparrow}{c + \overset{0}{\cancel{\frac{1}{n}}}}}$

So $x_n \to c$ (by the sandwich theorem).

However, since $|f(x_n)-f(c)| > \varepsilon$, this implies $f(x_n) \not\to f(c)$.

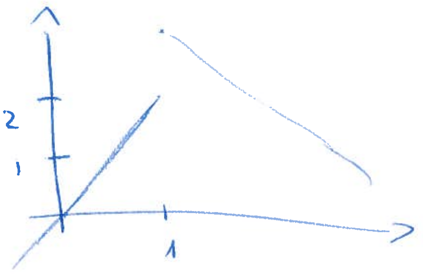
Thus, we have considerated a sequence $\langle x_n \rangle$ with $x_n \to c$,
but $f(x_n) \not\to f(c)$

Contradiction shows that f is continuous

Si $x_n \not\to c$ entonces f no es continua

Example 1 (yet again)

$$f(x) = \begin{cases} 2x, & x < 1. \\ 4-x, & x > 1 \\ 1, & x = 1. \end{cases}$$



$$\lim_{x \to 1^-} f(x) = ?$$

Suppose, $x_n < 1$ and $x_n \to 1$. Then $f(x_n) = 2x_n \to 2$

as $n \to \infty$ (by algebra of limits for sequences).

Then the theorem of limits of functions and sequences

implies $\lim_{x \to 1^-} f(x) = 2$

Similarly, if $x_n > 1$ and $x_n \to 1$, then $f(x_n) = 4 - x_n \to 3$ so

$\lim_{x \to 1^+} f(x) = 3$.

Thus, $\lim_{x \to 1} f(x)$ does not exist and is discontinuous at 1.

<u>Example</u>: Consider $f(x) = x^2$ and suppose, $c \in \mathbb{R}$ and $x_n \to c$.
Then $f(x_n) = x_n^2$, so $\lim_{n \to \infty} f(x_n) = \lim_{n \to \infty} x_n^2 = c^2 = f(c)$.
So by the Th. of functions and sequences $f(x) = x^2$ is continuous
on $\mathbb{R}$.

Th (algebra of limits for functions)

Suppose, $\lim_{x \to c} f(x) = A$, $\lim_{x \to c} g(x) = B$. Then:

1) $\lim_{x \to c^-} [f(x) + g(x)] = A + B$.

2) $\lim_{x\to c} [f(x)\cdot g(x)] = A\cdot B$.

3) If $B \neq 0$ and $g(x) \neq 0$. for $0<|x-c|<\delta$ then:

$$\lim_{x\to c} \frac{f(x)}{g(x)} = \frac{A}{B} .$$

Proof of 1) Suppose, $x_n \to c$.

Then $f(x_n) \to A$ and $g(x_n) \to B$ (by the Th-m of functions and sequences .

Therefore, $(f+g)\cdot(x_n) = f(x_n) + g(x_n) \to A+B$.

(sum rule for sequences) Applying the same Th-m again, we deduce that $\lim_{x\to c}[f(x)+g(x)] = A+B$.

(2) (3) Similar **proof**.

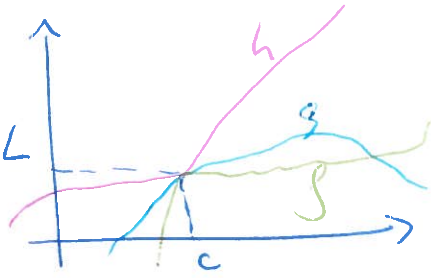
Th : (sandwich th-m for functions) .

Suppose, $\lim_{x\to c} f(x) = \lim_{x\to c} h(x) = L$. and

$f(x) \le g(x) \le h(x)$.

$\forall x$ with $|x-c| < \delta$ for some $\delta > 0$

Then $\lim_{x\to c} g(x) = L$.



Proof Suppose, $x_n \to c$

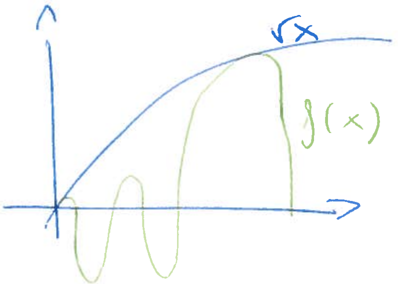
$$\text{Then } \underset{\substack{\downarrow \\ L}}{f(x_n)} \le g(x_n) \le \underset{\substack{\downarrow \\ L}}{h(x_n)} .$$

So $g(x_n) \to L$ and thus .

thus $\lim_{x\to c} g(x) = L$.

(**Remark**: all these Th-m work for one-sided limits as well)

Example :



$$f(x) = \sqrt{x} \cdot \cos\left(\frac{1}{x}\right) \qquad x > 0 .$$

Claim $\lim_{x \to 0^+} f(x) = 0$

Proof: $-\sqrt{x} \le f(x) = \sqrt{x} \cdot \cos\left(\frac{1}{x}\right) \le \sqrt{x}$.

Since $\lim_{x \to 0^+} \sqrt{x} = 0$ (HW 7),

the sandwich th-m implies $\lim_{x \to 0^+} f(x) = 0$.

Which functions are continuous?

1) $f(x) =$ constant (take $\delta = 17$).

2) $f(x) = x$ (take $\delta = \varepsilon$).

Th Suppose, $f, g : S \longrightarrow \mathbb{R}$ are continuous at $c \in S$. Then $f+g$, fg are continuous at c as well as $\frac{f}{g}$, assuming $g(c) \neq c$.

Proof: Suppose, $x_n \in S$, $x_n \to c$. Then $f(x_n) \to f(c)$, $g(x_n) \to g(c)$ therefore, $f(x_n) \cdot g(x_n) \to f(c) \cdot g(c)$, so $f \cdot g$ is continuous at c.

$f+g$ and $\frac{f}{g}$ are proved similarly.

Corollary: all polynomials are continuous functions on $\mathbb{R}$. All functions P/q where P and Q are polynomials are continuous at all points $c \in \mathbb{R}$ s.t $Q(c) \neq 0$

Th Suppose g is continuous at c and f is continuous at $g(c)$. Then $(f \circ g)(x) = f(g(x))$ is continuous at c.

Proof Suppose, $\lim_{n \to \infty} x_n = c$. Then $\lim_{n \to \infty} g(x_n) = g(c)$.

Since g is continuous at c).

Since f is continuous at $g(c)$, we have $\lim_{n \to \infty} f(g(x_n) = f(g(c))$

Thus, $(f \circ g)(x_n) \to (f \circ g) \cdot (c)$.

Therefore, $f \circ g$ is continuous at c.

November 24th 2018

f is continuous if ; $x_n \to x \Rightarrow \lim\limits_{n \to \infty} f(x_n) = f(x)$.

i.e : $f(\lim\limits_{n \to \infty} x_n) = \lim\limits_{n \to \infty} f(x_n)$.

f "respects" limits.

We have defined $\exp(x) = \sum\limits_{n=0}^{\infty} \frac{x^n}{n!}$.

Two results :

$\forall x, \exp(x) \geq 1+x$ (HW6).

also $\exp(x+y) = \exp(x) \cdot \exp(y)$.

Th exp is continuous on $\mathbb{R}$.

Proof

Step 1 exp is continuous at 0.

Proof Suppose $-1 < x < 1$.

then : $\exp(x) \geq 1+x$.

thus, $\exp(-x) \geq 1-x$, and $\exp(x)\exp(-x) = \exp(0) = 1$,

so $\exp(x) = \frac{1}{\exp(-x)} \leq \frac{1}{1-x}$.

Thus. for $-1 < x < 1$.

$$\begin{tikzcd} 1+x \arrow[d, "x \to 0"] & \leq \exp(x) \leq & \frac{1}{1-x} \arrow[d, "x \to 0"'] \\ 1 & & 1 \end{tikzcd}$$

By sandwich theorem, $\lim\limits_{x \to 0} \exp(x) = 1 = \exp(0)$.

So exp is continuous at 0.

Step 2 Suppose, $c \in \mathbb{R}$. Suppose, $\langle x_n \rangle$ is a sequence, $x_n \to c$. Thus, $x_n - c \to 0$.

Therefore,

$\exp(x_n) = \exp((x_n - c) + c) = \exp(x_n - c) \cdot \exp(c)$

$\exp(x_n - c) \cdot \exp(c) \longrightarrow \exp(c)$ as $n \to \infty$.

$\Big\downarrow n \to \infty$

$(\exp(\textcircled{c} - c)$ ↳ porque $x_n \to c$.

$\exp(0) = 1$

Since exp is continuous at 0 .

Thus, exp is continuous at c .

def

$$e = \exp(1) = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \dots = 2.7\,1828\,18\,284 .$$

Suppose, $f : [a,b] \longrightarrow \mathbb{R}$

What can we say about

$$f([a,b]) := \{ f(x) : x \in [a,b] \} .$$

Th 1

Every continuous function $f : [a,b] \longrightarrow \mathbb{R}$ is bounded .

(i.e. $f([a,b])$ is a bounded set)

So, it has inf and sup :

$$\boxed{m = \inf_{x \in [a,b]} f(x) = \inf \{ f(x) : x \in [a,b] \}} .$$

and

$$\boxed{M = \sup_{x \in [a,b]} f(x) = \sup f([a,b])}$$

Th 2

Every continuous function :

$f : [a,b] \longrightarrow \mathbb{R}$ attains its maximum and minimum :

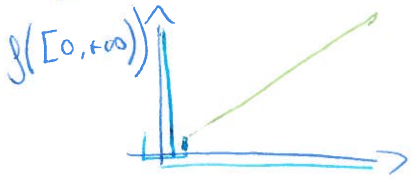
$\exists\, x_{min}, x_{max} \in [a,b]$ s.t $\forall x \in [a,b]$ we have

$$\boxed{m = f(x_{min}) \le f(x) \le f(x_{max}) = M}$$

Examples : (all assumptions are important)

1) $f : [0, +\infty) \longrightarrow \mathbb{R}$

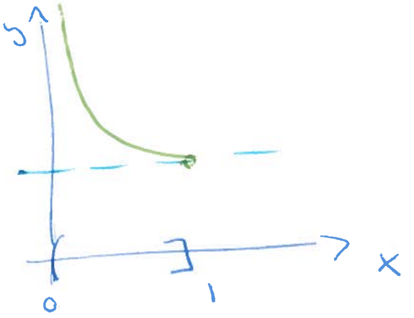
$f(x) = x$.



$f([0,+\infty)) = [0,+\infty)$ is bounded (f is not defined on a finite interval).

<u>Example 2</u>: $f:(0,1] \to \mathbb{R}$.

$f(x) = \frac{1}{x}$.



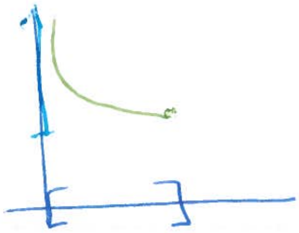
$f((0,1]) = [1,+\infty)$

f is unbounded ($(0,1]$ is not a closed interval).

<u>Example 3</u>

$f:[0,1] \to \mathbb{R}$.

$$f(x) = \begin{cases} 1/x, & x>0 \\ 1, & x=0 \end{cases}.$$



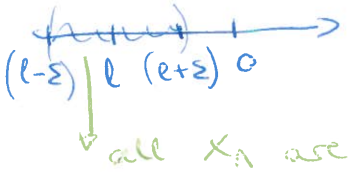
$f([0,1]) = [1,+\infty)$

is unbounded (f is not continuous)

Lemma

Suppose, $\langle x_n \rangle$ is a sequence, $x_n \geq 0$. $\forall n$, and $\lim x_n = l$. Then $l \geq 0$.

Proof Suppose, $l < 0$



all x_n are here for large n.

Choose ε s.t $l + \varepsilon < 0$

REMARK

Careful: $x_n > 0$ does not imply $\lim x_n > 0$
↳(puede ser 0).

<u>Corollary</u> Suppose $\langle x_n \rangle$ is a sequence, $x_n \in [a,b]$, and $x_n \to l$.

Then $l \in [a,b]$.

Reformulation of Bolzano-Weierstrass

Suppose, $\langle x_n \rangle$ is a sequence and $x_n \in [a,b]$ $\forall n$. Then there is a subsequence, x_{j_n}, s.t $x_{j_n} \to l \in [a,b]$ as $n \to \infty$.

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def: a set $S \subset \mathbb{R}$ is called sequentially compact, if $\forall$ sequence $\langle x_n \rangle$, $x_n \in S$, $\exists$ a subsequence $\langle x_{j_n} \rangle$, s.t $x_{j_n} \to x \in S$

We have proved that $[a,b]$ is sequentially compact.

Theorem 1: If $f:[a,b] \to \mathbb{R}$ is continuous, then $f([a,b])$ is bounded.

Theorem 2: $\exists$ $x_{min}, x_{max} \in [a,b]$ s.t $f(x_{min}) \leq f(x) \leq f(x_{max})$ $\forall x \in [a,b]$

Proof of Th:1.

Suppose, $f([a,b])$ is unbounded above, say then $\to$ claim.

our claim

(1st) $\forall H \in \mathbb{R}$ $\exists$ $x \in [a,b]$ s.t $f(x) > H$. In particular, $H = n \in \mathbb{N}$ is not an upper bound, so $\exists$ $\underbrace{x = x_n}_{\text{a point}} \in [a,b]$ s.t $f(x_n) \geq n$.

Therefore, $f(x_n) \longrightarrow +\infty$ as $n \to \infty$

our conditions

(2nd) By Bolzano-Weierstrass, $\exists$ a subsequence, $\langle x_{j_n} \rangle$ s.t $\lim\limits_{n \to \infty} x_{j_n} = x_\infty \in [a,b]$

But f is continuous at x_∞, so $f(x_{j_n}) \to f(x_\infty)$.

On the other hand, $f(x_{j_n}) \to +\infty$; our claim doesn't satisfy the conditions

Contradiction!

Therefore, $f([a,b])$ is bounded ▨

Proof of Th:2.

By th:1, $f([a,b])$ is bounded, so $\exists$ $M = \sup f([a,b]) = \sup\{f(x) \mid x \in [a,b]\}$

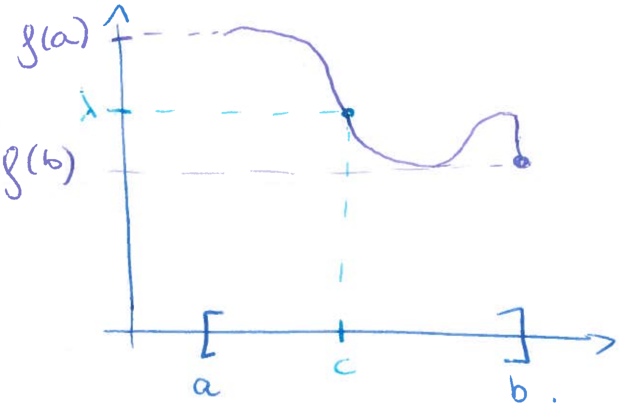
by HW3, Que 7, $\exists$ a sequence $\langle y_n \rangle$, $y_n \in f([a,b])$ s.t $y_n \to M$ as $n \to \infty$. Then $\exists x_n$ s.t $y_n = f(x_n)$, $x_n \in [a,b]$. By Bolzano-Weierstrass $\exists$ a subsequence $x_{j_n} \to x_{max} \in [a,b]$.

Then
$$M = \lim_{n \to \infty} y_n = \lim_{n \to \infty} f(x_n) = \lim_{n \to \infty} f(x_{j_n}) = f(\lim_{n \to \infty} x_{j_n}) = f(x_{max})$$

The proof of x_{min} is similar ▨

Theorem 3 : Intermediate value theorem :

Suppose, $f : [a,b] \to \mathbb{R}$ is continuous (on $[a,b]$) and $\lambda \in \mathbb{R}$ is between $f(a)$ and $f(b)$ (i.e. $f(a) \leq \lambda \leq f(b)$, or $f(b) \leq \lambda \leq f(a)$). Then $\exists\, c \in [a,b]$ s.t $f(c) = \lambda$!



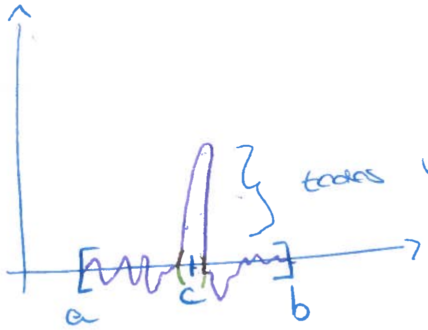
REMARK

These theorems imply that the "image" $f([a,b])$ of a closed interval is a closed interval $f([a,b]) = [f(x_{min}), f(x_{max})]$, if f is continuous.

$f[a,b]$ {
- is bounded (th. 1).
- contains the end points (th: 2).
- contains everything in between (th: 3).

Lemma : (The inertia principle)

Suppose, $f : (a,b) \to \mathbb{R}$ is continuous at $c \in (a,b)$ and $f(c) > 0$ Then $\exists\, \delta > 0$ s.t $f(x) > 0 \;\; \forall\, x \in (c-\delta, c+\delta)$.



todos los valores entre $c-\delta$ y $c+\delta$ tienen $f(x) > 0$.

Proof : Take $\varepsilon = \frac{f(c)}{2} > 0$

Then $\exists\, \delta > 0$ s.t. $x \in (c-\delta, c+\delta) \Rightarrow \left| f(x) - f(c) \right| < \varepsilon = \frac{f(c)}{2}$

$$\Leftrightarrow \; -\frac{f(c)}{2} < f(x) - f(c) < \frac{f(c)}{2} \;\Leftrightarrow\; 0 < \frac{f(c)}{2} < f(x) < \frac{3f(c)}{2},$$

so $f(x) > 0$.

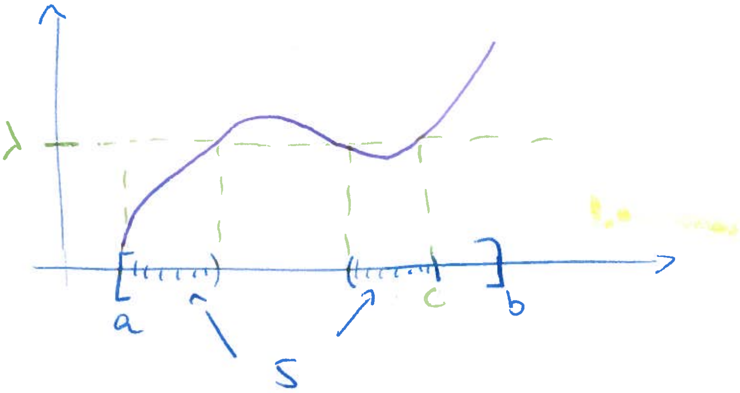
Corollary,

Suppose, f is continuous at $c \in (a,b)$ and $f(c) \begin{Bmatrix} > \\ < \end{Bmatrix} \lambda$. Then $\exists\ \delta > 0$ s.t $x \in (c-\delta, c+\delta) \Rightarrow f(x) \begin{Bmatrix} > \\ < \end{Bmatrix} \lambda$

Proof of Th-3

If $\lambda = f(a)$, or $\lambda = f(b)$ there is nothing to prove (take $c=a$ or $c=b$). Assume $f(a) < \lambda < f(b)$.

Put $S := \{x \in [a,b],\ f(x) < \lambda\}$.

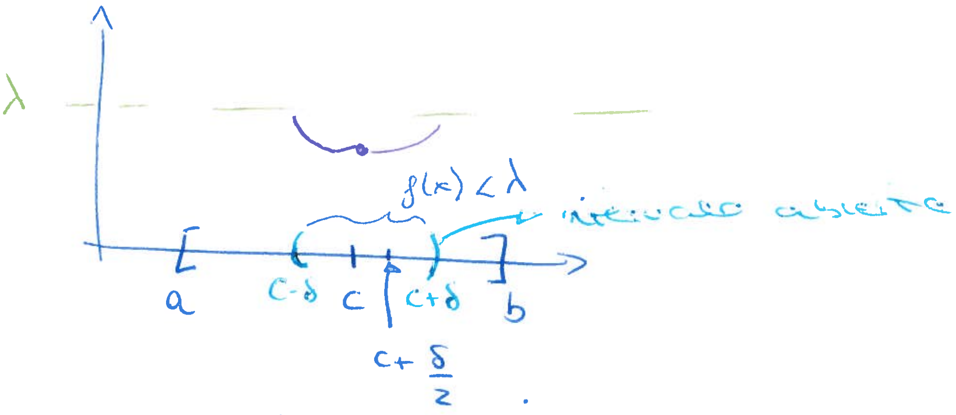


Then S is bounded, also, $a \in S$, so $S \neq \emptyset$ Put $c := \sup S$.

Claim $f(c) = \lambda$.

Proof:

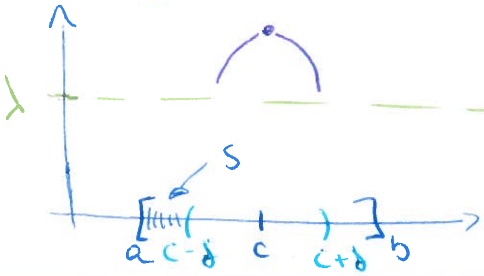
1) Assume $f(c) < \lambda$. Then by the corollary, $\exists\ \delta > 0$ s.t $x \in (c-\delta, c+\delta)$ $\Rightarrow f(x) < \lambda$.



In particular, $f\left(c+\frac{\delta}{2}\right) < \lambda$ Thus, $c+\frac{\delta}{2} \in S$, so c is not an upper bound of S

Contradiction

2) Assume $f(c) > \lambda$ Then $\exists\ \delta > 0$ s.t $x \in (c-\delta, c+\delta) \Rightarrow f(x) > \lambda$



This means $(c-\delta, c+\delta) \cap S = \emptyset$ and since c is an upper bound of S, $c-\delta$ is also an upper bound of S, so C is not the smallest upper bound.

Contradiction.

Thus, $f(c) = \lambda$ ▨

Suppose, $a > 0$. Then $a^{p/q} = \sqrt[q]{a^p} = \left(\sqrt[q]{a}\right)^p$

How do we define a^x, if $x \notin \mathbb{Q}$?.

def. Suppose, $A, B \subset \mathbb{R}$ and $f: A \to B$ is a bijection, i.e.

$\forall y \in B \ \exists \, ! \, x \in A$ s.t $f(x) = y$.

unique

injection

surjection

The inverse mapping (function) $f^{-1}: B \to A$ is defined by $f(x) = y \iff f^{-1}(y) = x$

i.e: $f^{-1} \circ f(x) = x$, $f \circ f^{-1}(y) = y$.

Example 1: $n \in \mathbb{N}$

$f(x) = x^n$, $f: [0, +\infty) \to [0, +\infty)$ is bijection

and $f^{-1}(y) = \sqrt[n]{y}$

Example 2: $\exp: \mathbb{R} \to (0, +\infty)$

$$\exp(x) = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

Claim exp is a bijection

Proof:

$\exp(x) \geq 1$ if $x \geq 0$

$\exp(x) > 1$ if $x > 0$.

If $x<0$, we have $\exp(x) = \frac{1}{\exp(\underbrace{-x})} \in (0,1)$.

$-x$ va a ser algo positivo, porque x es <0

Thus, $\exp(x)>0 \quad \forall x \in \mathbb{R}$.

Suppose, $x < x'$ then $x' = x+h$, $h>0$ and $\frac{\exp(x')}{\exp(x)} = \frac{\exp(x+h)}{\exp(x)} =$

$= \frac{\cancel{\exp(x)} \cdot \exp(h)}{\cancel{\exp(x)}} = \exp(h) > 1$, so $\exp(x') > \exp(x)$

Therefore, exp is strictly increasing and, thus, is an injection (i.e. $\forall y$ equation $f(x)=y$ has not more than 1 solution)

<u>Claim</u>: f is a surjection on to $(0,+\infty)$, i.e $\forall y \in (0,+\infty)\ \exists x \in \mathbb{R}$ s.t $\exp(x)=y$.

Since $\exp(x) \geq 1+x$ (HW 6) we have $\lim\limits_{x\to+\infty} \exp(x) = +\infty$

Also, if $\underset{(-x\to+\infty)}{x\to-\infty}$, $\lim\limits_{x\to-\infty} \exp(x) = \frac{1}{\exp(-x)} = 0$.

Thus, $\forall y \in (0,+\infty)\ \exists h_+$ s.t $\forall x > h_+$, we have $\exp(x) > y$.

Similarly, $\exists h_-$ s.t $\forall x < h_-$, we have $\exp(x) < y$.

$\underset{x\to-\infty}{\exp(x)} \to 0$

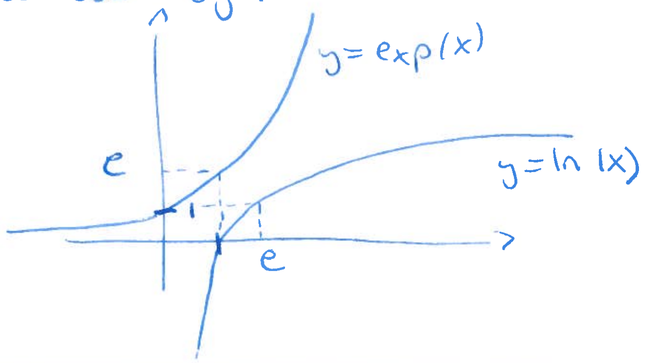
Therefore, $\exp(h_- - 1) < y < \exp(h_+ + 1)$ By intermediate value Theorem, $\exists x \in \mathbb{R}$ s.t $\exp(x)=y$.

<u>December 4th 2018</u>

$\exp : \mathbb{R} \to (0,+\infty)$ is a bijection

<u>def</u>: The inverse to exp is $\ln : (0,+\infty) \to \mathbb{R}$.

Sometimes use log.



Proposition for $x \in \mathbb{Q}$, we have $\exp(x) = e^x$

Proof (sketch)

Assume $x = n \in \mathbb{N}$

Then $\exp(y+z) = \exp(y) \cdot \exp(z)$.

So $\exp(n) = \exp(\underbrace{1+1+\dots+1}_{n}) = \exp(1) \cdot \exp(1) \dots \exp(1) =$

$= [\exp(1)]^n = e^n$.

The rest is similar

def. for any $a>0$ we have $a^x := \exp(x \cdot \ln a)$

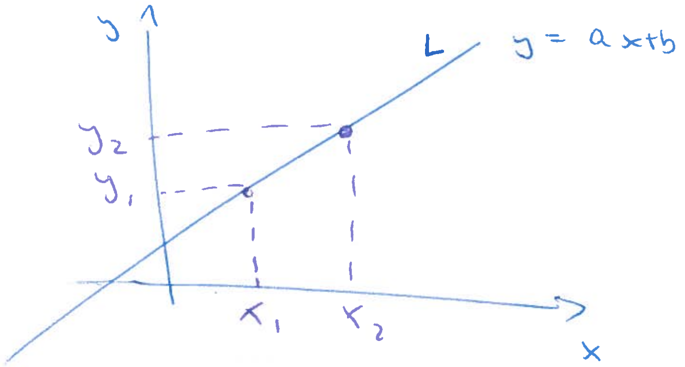
For $a \neq 1$, this is a bijection from $\mathbb{R}$ onto $(0,+\infty)$.

Differentiation

Slope of a curve

Toy case: slope of a straight line.

Suppose, $y = ax+b$ is a straight line.



We define the slope of L is a.

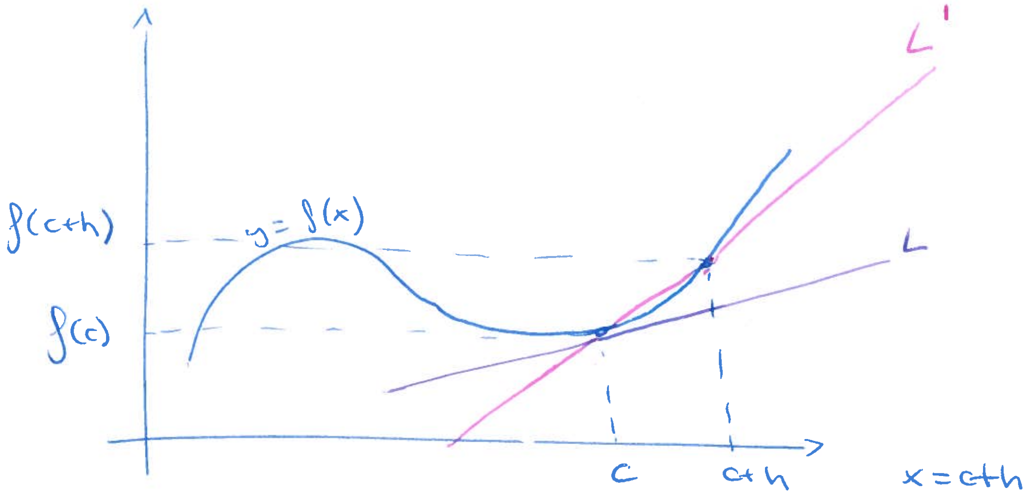
Suppose, (x_1, y_1) and (x_2, y_2) are on L.

Then, $\ominus\ y_1 = ax_1 + b$

$\oplus\ y_2 = ax_2 + b$

$y_2 - y_1 = a(x_2 - x_1)$, so the slope of $L = a = \dfrac{y_2 - y_1}{x_2 - x_1} = \dfrac{\text{the change in } y}{\text{the change in } x}$.

Step 2 The slope of $y = f(x)$ at $x = c$.



The slope of $L' = \dfrac{f(c+h) - f(c)}{(c+h) - c} = \dfrac{f(c+h) - f(c)}{h} \overset{x=c+h}{=} \dfrac{f(x) - f(c)}{x - c}$

Hope: $\exists \lim\limits_{h \to 0} (\text{slope of } L') = \text{slope of } L$.

def: Suppose, $f : S \to \mathbb{R}$, $S \subset \mathbb{R}$, S contains an open interval around c. We say that f is differentiable at c, if

$$\lim_{h \to 0} \frac{f(c+h) - f(c)}{h} = \lim_{x \to c} \frac{f(x) - f(c)}{x - c} \quad \text{exists}.$$

This limit is called the derivative of f at c, and is denoted by:

$$f'(c) = \frac{df}{dx}(c) = \frac{df}{dx}\Big|_{x=c}.$$

f is differentiable on S if it is differentiable at any point $c \in S$.

Examples

(1) $f(x) = 17$

$$f'(c) = \lim_{h \to 0} \frac{f(c+h) - f(c)}{h} = \lim_{h \to 0} \frac{17 - 17}{h} = \lim 0 = 0.$$

(2) $f(x) = x$

$$f'(c) = \lim_{h \to 0} \frac{\cancel{c} + K - \cancel{c}}{\cancel{K}} = \lim_{h \to 0} 1 = 1, \quad f' = 1.$$

(3) $f(x) = x^2$

$$\frac{f(c+h) - f(c)}{h} = \frac{(c+h)^2 - c^2}{h} = \frac{\cancel{c^2} + 2c\cancel{h} + h^{\cancel{2}} - \cancel{c^2}}{\cancel{h}} = 2c + h \xrightarrow{h \to 0} 2c$$

$f'(c) = 2c$, so $(x^2)' = 2x$.

(4) $f(x) = x^{17}$.

$$\frac{f(c+h)-f(c)}{h} = \frac{(c+h)^{17}-c^{17}}{h} = \frac{\cancel{c^{17}}+17c^{16}h+h^2\cdot\text{something}-\cancel{c^{17}}}{h} =$$

$$= 17c^{16} + h\cdot\text{something} \xrightarrow{h\to 0} \boxed{17c^{16}}.$$

In general, $(x^n)' = n\cdot x^{n-1}$ at least for $n\in\mathbb{N}$.

(5) $\Rightarrow$ $f(x) = \frac{1}{x}$, $x \neq 0$.

$$\frac{f(c+h)-f(c)}{h} = \frac{\frac{1}{c+h}-\frac{1}{c}}{h} = \frac{\frac{c-(c+h)}{(c+h)c}}{h} = \frac{-h}{(c+h)\cdot c\cdot h} =$$

$$= -\frac{1}{(c+h)\cdot c} \xrightarrow{h\to 0} \frac{-1}{c^2}$$

Thus, $\left.\frac{d\left(\frac{1}{x}\right)}{dx}\right|_{x=c} = -\frac{1}{c^2}$, so $\left(\frac{1}{x}\right)' = -\frac{1}{x^2}$.

(6) $\to$ $f(x) = \sqrt{x}$

$$\frac{f(c+h)-f(c)}{h} = \frac{\sqrt{c+h}-\sqrt{c}}{h} = \frac{\sqrt{c+h}-\sqrt{c}}{h}\cdot\frac{\sqrt{c+h}+\sqrt{c}}{\sqrt{c+h}+\sqrt{c}} =$$

$$= \frac{c+h-c}{h(\sqrt{c+h}+\sqrt{c})} = \frac{1}{\sqrt{c+h}+\sqrt{c}} \xrightarrow{h\to 0} \frac{1}{2\sqrt{c}}, \text{ so}$$

$$(\sqrt{x})' = \frac{1}{2\sqrt{x}}.$$

(7) $f(x) = |x|$, $c = 0$.

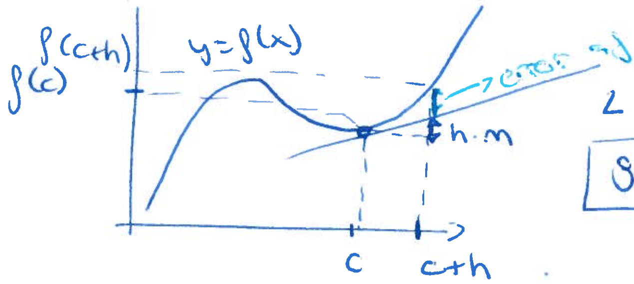
$$\frac{f(0+h)-f(0)}{h} = \frac{|h|}{h} =$$

$$= \begin{cases} 1, & h>0 \\ -1, & h<0 \end{cases}.$$

December 6th 2018

f is differentiable at c, if $\boxed{\lim_{h\to 0}\frac{f(c+h)-f(c)}{h} \underset{\substack{\uparrow \\ x=c+h}}{=} \lim_{x\to c}\frac{f(x)-f(c)}{x-c} = f'(c)}$ exist.

I want to show that this means that f is well approximated by a linear function → graph with straight line



error of approximation → (has to go to 0 as $h\to 0$ faster than h)

L.

$$\boxed{\begin{gathered} g(c+h) = f(c) + m\cdot h \\ m = f'(c) \end{gathered}}$$

↓

Linear approximation

Th (theorem of linear-approximation).

f is differentiable at c iff $f(c+h) = \underbrace{f(c) + m\cdot h}_{\text{Linear approximation}} + \overbrace{R(h)\cdot h}^{\text{error (remainder)}}$ ⊛ and $\lim_{h\to 0} R(h) = 0$.

↑

($\exists\, m \in \mathbb{R}$ and a function $R(h)$ s.t)

Then $m = f'(c)$

(The remainder $R(h)\cdot h$ goes to 0 faster than h)

Proof:

$\Rightarrow$ Suppose

$\exists\ f'(c) = \lim_{h\to 0}\frac{f(c+h)-f(c)}{h} = m$

We define $R(h) = \begin{cases} \frac{f(c+h)-f(c)}{h} - m, & h\neq 0 \\ 0, & h = 0 \end{cases}$.

Then $h\cdot R(h) = f(c+h) - f(c) - h\cdot m$, also, $\lim_{h\to 0} R(h) = \overset{f'(c)=m}{m} - m = 0$, so ⊛ holds.

$\Leftarrow$ Suppose ⊛ holds,

then: $\lim_{h\to 0}\frac{f(c+h)-f(c)}{h} = \lim_{h\to 0}\frac{m\cdot h + R(h)\cdot h}{h} = \lim_{h\to 0}\left[m + R(h)\right] = m$, so

f is differentiable and $f'(c) = m$ ▨

Th: If f is differentiable at c, then f is continuous at c.

Proof: Suppose, f is differentiable at c.

Then $\lim\limits_{x\to c} f(x) \underset{\substack{\uparrow \\ x=h+c}}{=} \lim\limits_{h\to 0} f(c+h) = \lim\limits_{h\to 0} \left(f(c) + mh + R(h)\cdot h \right) = f(c)$,

so f is continuous at c. ▨

Th Suppose, f and g are differentiable at c.

Then:

a) so is $f+g$, and $[f+g]'(c) = f'(c) + g'(c)$ (sum rule).

b) so is, $f\cdot g$ and $(f\cdot g)' = f'g + f\cdot g'$ (product rule).

Proof:

a) $$\lim_{h\to 0} \frac{(f+g)\cdot(c+h) - (f+g)\cdot(c)}{h} = \lim_{h\to 0} \frac{f(c+h)-f(c)}{h} + \lim_{h\to 0} \frac{g(c+h)-g(c)}{h} =$$

$= f'(c) + g'(c)$. ▨

b) Th-m about linear approximations implies:

$$f(c+h) = f(c) + f'(c)\cdot h + R_f(h)\cdot h.$$

and

$$g(c+h) = g(c) + g'(c)\, h + R_g(h)\cdot h.$$

and $\lim\limits_{h\to 0} R_f(h) = \lim\limits_{h\to 0} R_g(h) = 0$

Then $(f\cdot g)\cdot(c+h) = f(c+h)\cdot g(c+h) = \overbrace{f(c)\cdot g(c) + h\cdot\left[f(c)\, g'(c) + g(c)\cdot f'(c)\right]}^{\text{linear function}}$

$$+\, h\cdot\underbrace{\left[f(c)\cdot R_g(h) + g(c)\, R_f(h) + f'(c)\cdot g'(c)\cdot h + f'(c)\cdot R_g(h)\, h + g'(c)\cdot R_f(h)\cdot h + R_f(h)\cdot R_g(h)\cdot h \right]}_{R_{fg}(h)}.$$

and $R_{f\cdot g}(h) \to 0$ as $h \to 0$ (lo compruebo).

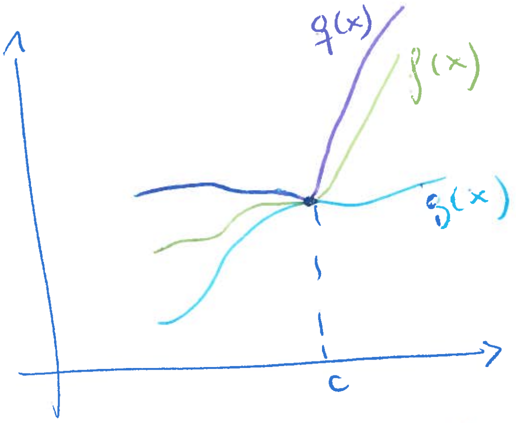
By the th-m about linear approximation $(f\cdot g)$ is differentiable at c,

and $(f\cdot g)' = f\cdot g' + f'\cdot g$.

Corollary

$$(f_1 \cdot f_2 \cdots f_n)' = f_1' \cdot f_2 \cdots f_n + f_1 f_2' f_3 \cdots f_n + \cdots + f_1 f_2 \cdots f_{n-1} f_n'$$

Th: (sandwich Th-m).



Suppose, $g(x) \le f(x) \le q(x)$ ∀ x in a neighbourhood of c.

and

$g(c) = q(c) = L$.

$g'(c) = q'(c) = m$.

Then f is differentiable at c, and $f'(c) = m$.

Proof:

we have

$L = g(c) \le f(c) \le q(c) = L$.

so $f(c) = L$. Therefore, $g(c+h) - g(c) \le f(c+h) - f(c) \le q(c+h) - q(c)$

Case ①: $h > 0$. Then:

$$\underset{\substack{\downarrow h\to 0 \\ g'(c)=m}}{\frac{g(c+h)-g(c)}{h}} \le \frac{f(c+h)-f(c)}{h} \le \underset{\substack{\downarrow h\to 0 \\ q'(c)=m}}{\frac{q(c+h)-q(c)}{h}}.$$

Thus, by the sandwich Theorem for the limits of functions,

$$\lim_{h\to 0^+} \frac{f(c+h)-f(c)}{h} = m.$$

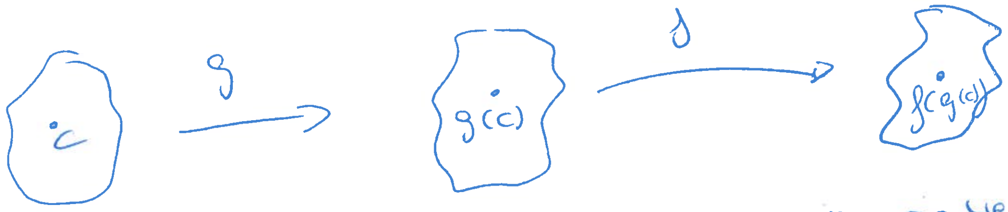
Case 2: $h<0$ then:

$$\underbrace{\frac{g(c+h)-g(c)}{h}}_{\xrightarrow{h\to 0^-}\ m} \le \frac{f(c+h)-f(c)}{h} \le \underbrace{\frac{g(c+h)-g(c)}{h}}_{\xrightarrow{h\to 0^-}\ m}$$

Thus, $\lim\limits_{h\to 0^-} \dfrac{f(c+h)-f(c)}{h} = m$.

Overall, $f'(c) = \lim\limits_{h\to 0} \dfrac{f(c+h)-f(c)}{h} = m$ ▨

Th-m (The chain rule) $(f\circ g)(x) = f(g(x))$.



Suppose, g is differentiable at c and f is differentiable at $g(c)$.
Then, $f\circ g$ is differentiable at c and $(f\circ g)'(c) = f'(g(c))\cdot g'(c)$.

Proof: We know (from assumptions)

$$g(c+h) = g(c) + g'(c)h + R_g(h)\cdot h$$

$$R_g(h) \xrightarrow[h\to 0]{} 0$$

also, $f(g(c)+k) = f(g(c)) + f'(g(c))\,k + R_f(k)\,k$.

$$R_f(k) \xrightarrow[k\to 0]{} 0$$

Therefore, $(f\circ g)(c+h) = f(g(c+h)) = f\big[g(c) + \underbrace{g'(c)h + R_g(h)\cdot h}_{k}\big] =$

→ linear approximation

$$= \boxed{f(g(c)) + f'(g(c))\cdot g'(c)\cdot h} +$$

$$h\cdot\Big[f'(g(c))\,\underset{\downarrow h\to 0}{R_g(h)} + \big(g'(c)+R_g(h)\big)\,\underset{\downarrow k\to 0,\ \text{as } h\to 0}{R_f(k)}\Big] =$$

Note: $k\to 0$ as $h\to 0$

$\lim\limits_{h\to 0} k = 0$.

$$= (f\circ g)(c) + \underset{\substack{\shortparallel \\ f'(g(c))\cdot g'(c)}}{m}\cdot h + h\cdot \overset{0}{R_{f\circ g}(h)}$$ and $R_{f\circ g}(h) \to 0$ as $h\to 0$.

Therefore, $f \circ g$ is differentiable at c and $(f \circ g)'(c) =$

$$= f'(g(c)) \cdot g'(c) .$$

Examples:

1) $(x^n)' = (x \cdot x \cdots x)' \underset{\uparrow \text{product rule}}{=}$ $n \in \mathbb{N}$.

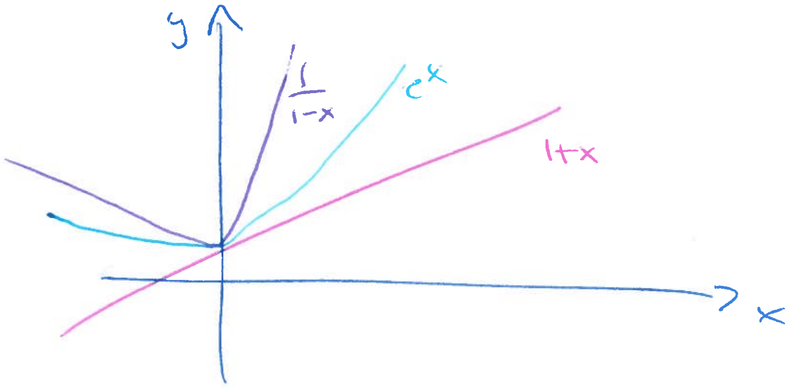
$$= \underbrace{1 \cdot x \cdot x \cdot x \cdot x}_{(n-1)} + \underbrace{1 \cdot x \cdots x}_{n-1} + \dots + \underbrace{1 \cdot x \cdots x}_{n-1} = n \cdot x^{n-1}$$

2) $f(x) = \exp(x) = e^x$

Step 1 $c = 0$

for $|x| < 1$

we have proved $1 + x \leq e^x \leq \frac{1}{1-x}$



$$1+x \Big|_{x=0} = \frac{1}{1-x} \Big|_{x=0} = 1 .$$

also,

$$(1+x)' \Big|_{x=0} = 1 \Big|_{x=0} = 1 .$$

$$\left(\frac{1}{1-x}\right)' \Big|_{x=0} = \frac{+1}{(1-x)^2} \Big|_{x=0} = 1 .$$

By the sandwich theorem for derivatives,

$$\frac{d(e^x)}{dx} \Big|_{x=0} = 1 .$$

Step 2:

$$\frac{d}{dx}e^x\Big|_{x=c} = \lim_{h\to 0}\frac{e^{c+h}-e^c}{h} = \lim_{h\to 0}\frac{e^c e^h - e^c}{h} =$$

$$= e^c\cdot \underbrace{\lim_{h\to 0}\frac{e^h - e^0}{h}}_{\substack{\| \\ (e^x)'\big|_{x=0} = 1}} = \boxed{e^c}\cdot\left(\frac{d}{dx}e^x\right)\Big|_{x=0} = e^c\cdot 1 = \boxed{e^c}$$

Therefore $\frac{d}{dx}e^x = e^x$.

December 11th

Corollary: $a>0$

$$\frac{d}{dx}(a^x) = \left[e^{x\ln a}\right]' \underset{\text{chain rule}}{=} e^{x\ln a}\cdot[x\ln a]' = \ln a\cdot a^x$$

Quotient rule:

$$\left(\frac{f}{g}\right)' = \left(f\cdot\frac{1}{g}\right)' \overset{\text{product rule}}{=} f'\cdot\frac{1}{g} + f\cdot\left(\frac{1}{g}\right)' \underset{\substack{\uparrow \\ \text{chain rule}}}{=} f'\cdot\frac{1}{g} + \left(-\frac{1}{g^2}\right)g'\cdot f =$$

$$= \frac{f'g - g'f}{g^2}$$

Inverse function:

$f: A \to B$ is a bijection
$\quad A \subset \mathbb{R},\ B \subset \mathbb{R}$

$\exists\, f^{-1}: B \to A \quad (f^{-1}\circ f)(x) = x$.

Suppose, f is differentiable at $a \in A$.
Suppose also that f is differentiable at $b = f(a)$. Then

$$\frac{d}{dx}x\Big|_{x=a} = \frac{d}{dx}\left((f^{-1}\circ f)(x)\right)\Big|_{x=a} = 1$$

$$\underset{\substack{\uparrow \\ \text{chain rule}}}{=} (f^{-1})'(b)\cdot f'(a) \quad \text{so} \quad (f^{-1})'(b) = \frac{1}{f'(a)}$$

Example. $y = \ln x \iff x = e^y$ $\quad f\ \ x \xrightarrow[\ln]{} y$, $\quad a = x$
$b = \ln x$.

Suppose $e^{a} = b$, so $a = \ln b$.

Then $\frac{d}{dx}[\ln x]\Big|_{x=b} = \frac{1}{[e^{y}]'}\Big|_{y=a} = \frac{1}{b}$.

Therefore, $[\ln x] = \frac{1}{x}$

Prop: $f(x) = x^{n}$, $\forall n \in \mathbb{R}$ is differentiable on $\{ x \in \mathbb{R}, x > 0$

and $(x^{n})' = n \cdot x^{n-1}$.

Proof for $x > 0$. write $x = e^{\ln x}$, so $x^{n} = e^{n \ln x}$, so $(x^{n})' = [e^{n \ln x}]' =$

$\underset{\substack{\uparrow \\ \text{chain rule}}}{=} e^{n \ln x} \cdot [n \ln x]' = x^{n} \cdot n \cdot \frac{1}{x} = n \cdot x^{n-1}$ ▨

Careful: $\sqrt{x}$ is defined for $x \geq 0$.
but is differentiable for $x > 0$.

def We say that a function $f: \underset{\mathbb{R}}{\underset{\cap}{S}} \to \mathbb{R}$ has a global maximum (global minimum) at $c \in S$, if $f(c) \geq f(x)$ $\forall x \in S$.
$(f(c) \leq f(x))$.

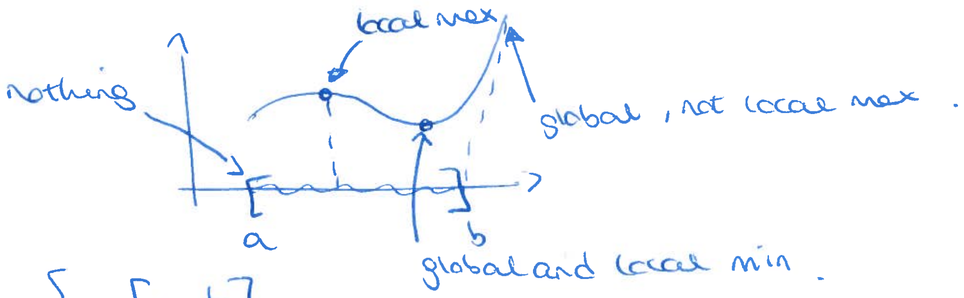
def $f: \underset{\mathbb{R}}{\underset{\cap}{S}} \to \mathbb{R}$ has a local maximum (local minimum) at $c \in S$, if $\exists \delta > 0$

s.t.

(i) $(c - \delta, c + \delta) \subset S$.

(ii) $\forall x \in (c - \delta, c + \delta)$ we have $f(x) \leq f(c)$ $\left(f(x) \geq f(c) \right)$

Extremum = maximum or minimum



$S = [a, b]$.

Theorem Suppose, c is a local extremum and f is differentiable at c, then $f'(c)=0$.

Proof: Suppose, c is a local minimum, i.e. $f(x) \geq f(c)$ $\forall x \in (c-\delta, c+\delta)$.

Then: $$f'(c) = \underbrace{\lim_{h \to 0^+} \frac{f(c+h)-f(c)}{h}}_{I} = \underbrace{\lim_{h \to 0^-} \frac{f(c+h)-f(c)}{h}}_{II}$$

Note: $f(c+h) - f(c) \geq 0$ for $\underbrace{|h| < \delta}_{-\delta < h < \delta}$.

for $h \in (0, \delta)$, we have $\frac{f(c+h)-f(c)}{h} \geq 0$.

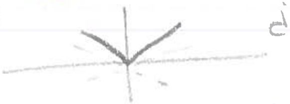
$$\Downarrow$$
$$I \geq 0.$$

Similarly, for $h \in (-\delta, 0)$ we have $\frac{f(c+h)-f(c)}{h} \leq 0$

$$\Downarrow$$
$$II \leq 0.$$

Thus, $0 \leq I = II \leq 0$, so $I = II = 0$, and $f'(c)=0$ ▨

Remark: $f(x)=|x|$ has a local min at 0, but f is not differentiable at 0.



¿cuál elijo?

def c, is a critical point of f, if $f'(c)=0$.

To find global extremum of f on $[a,b]$, find all critical points c, and compare all $f(c)$ with $f(a)$ and $f(b)$.

December 13th

Example:

$f: [0,2] \longrightarrow \mathbb{R}$

$f(x) = 1 + 5x - x^5$

Find max $f(x)$ and $x \in [0,2]$

min $f(x)$ $x \in [0,2]$.

Critical points: $f'(x) = 5 - 5x^4 = 0 \iff x^4 = 1$.

Real critical points: $x = \pm 1$

Only $x=1$ is on $[0,2]$. Therefore, $\max_{x \in [0,2]} f(x) \overset{\max}{=} \{f(0), f(1), f(2)\} =$

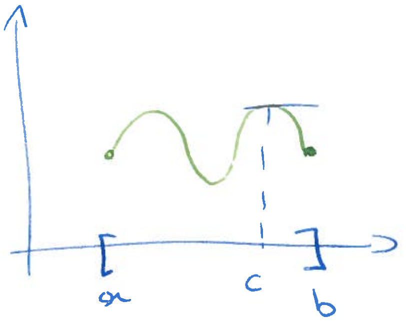
$= \max\{1, 5, -21\} = 5$

and $\min_{x \in [0,2]} f(x) = \min \{ f(0), f(1), f(2) \} = \min \{1, 5, -21\} = -21$.

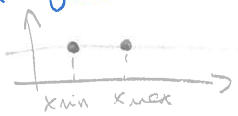
Th (Rolle's theorem).

Suppose, $a < b$, $f : [a,b] \longrightarrow \mathbb{R}$, f is continuous on $[a,b]$ and differentiable on (a,b) .

Suppose, $f(a) = f(b)$, then $\exists c \in (a,b)$ s.t $f'(c) = 0$.



Proof f is continuous on $[a,b]$, so f attains alcanza maximal and minimal values, i.e $\exists$ X_{min} and X_{max} s.t X_{min} is global min, and X_{max} global max .



Case 1 $f(X_{min}) = f(X_{max})$

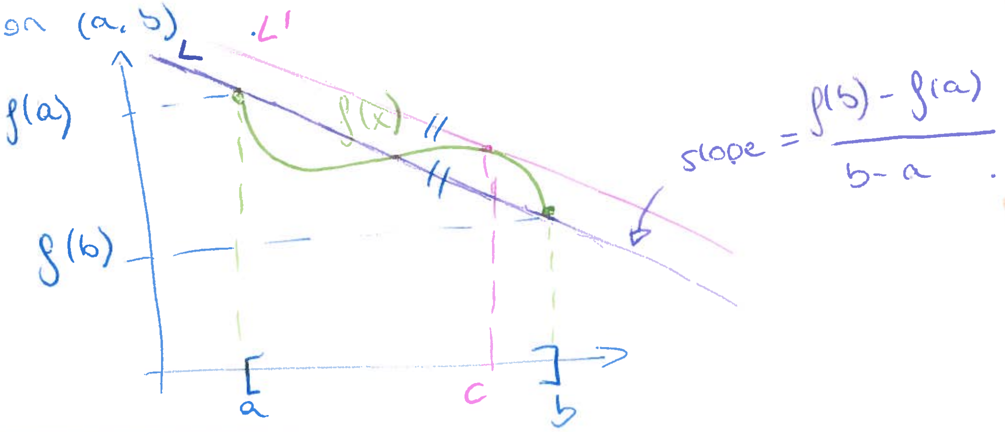
Then $f(x) = \text{const} = f(X_{min})$, so $f'(x) = 0$ $\forall x$, so take any $c \in (a,b)$ $f'(c) = 0$.

Case 2 $f(x_{min}) < f(x_{max})$ Then at least one of $f(X_{min})$ or $f(X_{max})$ is different from $f(a) = f(b)$, say $f(x_{min}) \neq f(a) = f(b)$

The $X_{min} \in (a,b)$. If we put $c = X_{min}$, X_{min} is a local minimum, so $f'(c) = 0$.

Th (mean value theorem)

Suppose, $f : [a,b] \longrightarrow \mathbb{R}$, is continuous on $[a,b]$ and differentiable on (a,b)



$\text{slope} = \dfrac{f(b) - f(a)}{b - a}$.

Then, $\exists c \in (a,b)$ s.t $f'(c) = \frac{f(b)-f(a)}{b-a}$.

slope of L' = slope of L.

Proof Denote:

$$m := \frac{f(b)-f(a)}{b-a} \quad \text{and} \quad g(x) = f(x) - mx .$$

Then, $g'(x) = f'(x) - m$.

Then, $g(a) = f(a) - \left(\frac{f(b)-f(a)}{b-a}\right)\cdot a =$

$$= \frac{f(a)\cdot(b-a) - af(b) + af(a)}{b-a} = \frac{bf(a) - \cancel{f(a)\cdot a} - af(b) + \cancel{af(a)}}{b-a} =$$

$$= \boxed{\frac{f(a)\cdot b - af(b)}{b-a}}$$

$$g(b) = f(b) - \left(\frac{f(b)-f(a)}{b-a}\cdot b\right) = \frac{\cancel{f(b)\cdot b} - f(b)\cdot a - bf(a) + \cancel{bf(b)}}{b-a} =$$

(=)

$$= \boxed{\frac{bf(a) - af(b)}{b-a}},$$

Thus, $g(a) = g(b)$ so by the Rolle's theorem $\exists c \in (a,b)$ s.t $g'(c) = 0$, and

$$\begin{aligned} &g'(c) = 0 \\ &\;\; \| \\ &f'(c) - m \\ &f'(c) = m \end{aligned}$$

▨

Corollary,

Suppose $f: [a,b] \to \mathbb{R}$, f is continuous on $[a,b]$ and differentiable on (a,b). Then:

(i) if $f'(x) > 0$ $\forall x \in (a,b)$ then f is strictly increasing on $[a,b]$.

i.e: $x_2 > x_1 \Rightarrow f(x_2) > f(x_1)$

(ii) if $f'(x) < 0$ $\forall x \in (a,b)$ then f is strictly decreasing

i.e. $x_2 > x_1 \Rightarrow f(x_2) < f(x_1)$.

(iii) $f'(x) = 0 \quad \forall x \in (a,b)$

$\Rightarrow f(x) = \text{const}$

Proof Suppose $a \le x_1 < x_2 \le b$.

Then, the mean value theorem implies that $\exists c \in (x_1, x_2)$ s.t

$$f'(c) = \frac{f(x_2)-f(x_1)}{x_2-x_1} = m$$

Then:

(i) $m>0 \Rightarrow f(x_2)-f(x_1) > 0$.

(ii) $m<0 \Rightarrow f(x_2)-f(x_1) < 0$,

(iii) $m=0 \Rightarrow f(x_2)-f(x_1) = 0$

Th Suppose, f is continuous on $[a,b]$, $c \in (a,b)$

(i) $\exists\, \delta > 0$ s.t $f'(x) > 0 \quad \forall x \in (c-\delta, c)$, and $f'(x) < 0 \quad \forall x \in (c, c+\delta)$, then c is a local max.

(ii) $\exists\, \delta > 0$ s.t $f'(x) < 0 \quad \forall x \in (c-\delta, c)$, $f'(x) > 0 \quad \forall x \in (c, c+\delta)$. Then c is a local minimum

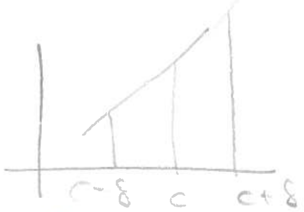
Proof

(i) $f'(x) > 0$ on $(c-\delta, c) \Rightarrow f$ is increasing on $[c-\delta, c]$,

i.e. $f(c) \ge f(x) \quad \forall x \in [c-\delta, c]$.

$f'(x) < 0$ on $(c, c+\delta) \Rightarrow f$ is decreasing on $[c, c+\delta]$.

i.e $f(c) \ge f(x) \quad \forall x \in [c, c+\delta]$



Thus, c is a local max ∎

Th: Suppose, $f : [a,b] \to \mathbb{R}$ is twice continuously differentiable on (a,b),

i.e $f''(x) = (f')'(x)$ exists and is continuous, and

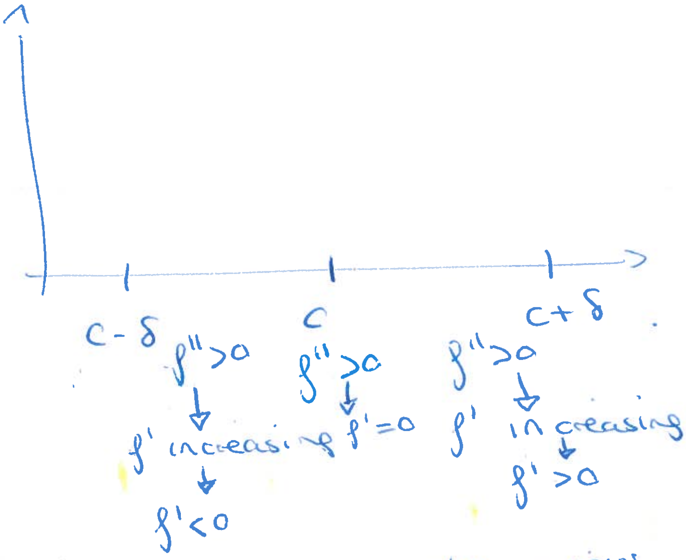
$c \in (a,b)$ is a critical point $(f'(c) = 0)$.

(i) Suppose, $f''(c) > 0$, then c is a local minimum.

(ii) $f''(c) < 0$, then c is a local maximum.

Proof

(i) $f''(c) > 0$, f'' is continuous $\Rightarrow \exists\, \delta > 0$ s.t $f''(x) > 0$ $\forall\, x \in (c-\delta, c+\delta)$



This reduces the statement to the previous theorem ▨