1102 Analysis 2 Notes

Based on the 2016 spring lectures by Prof D Vassiliev

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L1 Analysis 2

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Chapter 1 - Revision

Def 1.1

We say that $L = \lim_{n\to\infty} x_n$ if $\forall \varepsilon > 0\ \exists N \in \mathbb{N}$ s.t. $n > N \Rightarrow |x_n - L| < \varepsilon$

Important: N depends on ε, so people often write $N(\varepsilon)$.

Basic facts:

Limit of a sequence, if it exists, is unique.

Must also know Algebra of limits.

Now, recall the def-n of a limit of a function.

Given an $x_0 \in \mathbb{R}$ and a $\delta_0 > 0$, we call the set $(x_0 - \delta_0, x_0) \cup (x_0, x_0 + \delta_0) = (x_0 - \delta_0, x_0 + \delta_0) \setminus \{x_0\}$ a punctured neighbourhood of the point x_0.

I am looking at a function $f: D \to \mathbb{R}$ and I assume that x_0 is such that $\exists \delta_0 > 0$ s.t. $(x_0 - \delta_0, x_0) \cup (x_0, x_0 + \delta_0) \subset D$.

I want my function to be defined in some punctured neighbourhood of the point x_0, but not necessarily at the point x_0 itself.

Def 1.2 (ε-δ definition)

We say that $L = \lim_{x\to x_0} f(x)$ if $\forall \varepsilon > 0\ \exists \delta > 0$ s.t. $0 < |x - x_0| < \delta$ & $x \in D \Rightarrow |f(x) - L| < \varepsilon$.

Important: δ depends on x_0 & ε, so we can write $\delta(x_0, \varepsilon)$

Def 1.3 (Sequential definition)

We say that $L = \lim_{x\to x_0} f(x)$ if for any sequence $\{x_n\} \subset D \setminus \{x_0\}$ s.t. $\lim_{n\to\infty} x_n = x_0$ we have $\lim_{n\to\infty} f(x_n) = L$.

Def 1.2 & 1.3 are equivalent.

One sided limits:

- Right limit: $L = \lim_{x \to x_0^+} f(x)$ (write $x_0 < x < x_0 + \delta$ in def 1.2, or $x_0 < x_n$ in def 1.3).
- Left limit: $L = \lim_{x \to x_0^-} f(x)$ (write $x_0 - \delta < x < x_0$ in def 1.2, or $x_n < x_0$ in def 1.3).

Fact: $\lim_{x \to x_0} f(x)$ exists iff both one-sided limits exist and are equal, so $\lim_{x \to x_0} f(x) = \lim_{x \to x_0^+} f(x) = \lim_{x \to x^-} f(x)$ and all three limits are equal.

Notion of a limit at infinity.
(will be needed when dealing with improper integrals, note that in this course the limit itself will always be a finite $L \in \mathbb{R}$).

Def 1.4
We say that $L = \lim_{x \to \infty} f(x)$ if $\forall \varepsilon > 0 \ \exists X > 0$ s.t.
$|x| > X \ \& \ x \in D \Rightarrow |f(x) - L| < \varepsilon$.

Def 1.5
We say that $L = \lim_{x \to +\infty} f(x)$ if $\forall \varepsilon > 0 \ \exists X > 0$ s.t.
$x > X \ \& \ x \in D \Rightarrow |f(x) - L| < \varepsilon$.

Def 1.6
We say that $L = \lim_{x \to -\infty} f(x)$ if $\forall \varepsilon > 0 \ \exists X > 0$ s.t.
$x < -X \ \& \ x \in D \Rightarrow |f(x) - L| < \varepsilon$

Fact: $\lim_{x \to \infty} f(x)$ exists iff both one-sided limits exist and are equal, so $\lim_{x \to \infty} f(x) = \lim_{x \to +\infty} f(x) = \lim_{x \to -\infty} f(x)$ and all three limits are equal.

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Concept of continuity is very similar to the concept of a limit, but not the same.
When dealing with continuity we assume that $x_0 \in D$, i.e. function f is defined at the point x_0 itself.

Def 1.7 (ε, δ definition of continuity)
We say that the function $f: D \to \mathbb{R}$ is continuous at the point $x_0 \in D$ if $\forall \varepsilon > 0\ \exists \delta > 0$ s.t. $|x - x_0| < \delta$ & $x \in D$ $\Rightarrow |f(x) - f(x_0)| < \varepsilon$.

Def 1.8 (sequential definition of continuity)
We say that the function $f: D \to \mathbb{R}$ is continuous at the point $x_0 \in D$ if for any sequence $\{x_n\} \subset D$ such that $\lim_{n\to\infty} x_n = x_0$ we have $\lim_{n\to\infty} f(x_n) = f(x_0)$.

Note: Def 1.7 $\Leftrightarrow$ Def 1.8.

"Continuity at x_0" $\Leftrightarrow$ "Limit of $f(x)$ at x_0 exists and equals $f(x_0)$"

Def 1.9
We say that the function $f: D \to \mathbb{R}$ is continuous if it is continuous at every point $x_0 \in D$.

Must know:
Basic local properties of continuous functions:
- continuity of sum, product, quotient, inverse, composition and the inertia principle.

Lemma 1.1 (Inertia Principle).
Suppose $f: D \to \mathbb{R}$ is continuous at the point $x_0 \in D$. Then the following statements are true:
- If $f(x_0) > 0$ then $\exists \delta > 0$ s.t. $|x - x_0| < \delta$ & $x \in D \Rightarrow f(x) > 0$;
- If $f(x_0) < 0$ then $\exists \delta > 0$ s.t. $|x - x_0| < \delta$ & $x \in D \Rightarrow f(x) < 0$.

Should also know:

Global properties of continuous functions:

- Intermediate value theorem, attainment of bounds theorem (extreme value theorem).

Thm 1.1 (Bolzano-Weierstrass theorem)

Any bounded sequence of real numbers contains a convergent subsequence.

Chapter 2 - Cauchy sequences and uniform continuity

Def 2.1

A sequence of real numbers $\{x_n\}$ is said to be a Cauchy sequence if $\forall \varepsilon > 0 \ \exists N \in \mathbb{N}$ s.t. $m, n > N \Rightarrow |x_m - x_n| < \varepsilon$.

Compare with Def 1.1.

Thm 2.1 (Cauchy's general principle of convergence)

A sequence of real numbers $\{x_n\}$ converges iff it is a Cauchy sequence.

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<u>Thm 2.1</u> (Cauchy's general principle of convergence)
A sequence of real numbers $\{x_n\}$ converges iff it is a Cauchy sequence

<u>Proof</u>
<u>Part 1</u>
Suppose that the sequence $\{x_n\}$ converges. Need to prove that $\{x_n\}$ is a Cauchy sequence.
"Converges" means there is a limit, L.
Let ε be an arbitrary positive number.
Then by Def 1.1 $\exists N \in \mathbb{N}$ s.t. $n > N \Rightarrow |x_n - L| < \varepsilon/2$
Now take arbitrary m, n s.t. $m, n > N$.
Then we have

$$|x_m - x_n| = |(x_m - L) + (L - x_n)|$$
$$\leq |x_m - L| + |L - x_n| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

<u>Part 2</u>
Suppose that the sequence $\{x_n\}$ is a Cauchy sequence.
Need to prove that the sequence $\{x_n\}$ converges.
Set $\varepsilon = 1$ and apply Def 2.1.
We have: $\exists N \in \mathbb{N}$ s.t. $m, n > N \Rightarrow |x_m - x_n| < 1$.
In particular,

$$n > N \Rightarrow |x_n - x_{N+1}| < 1 \qquad (2.1)$$

But $|x_n| = |(x_n - x_{N+1}) + x_{N+1}|$

$$\leq |x_n - x_{N+1}| + |x_{N+1}| \qquad (2.2)$$

Combining (2.1) and (2.2) we get

$$|x_n| < |x_{N+1}| + 1$$

Hence $|x_n| \leq \max\{|x_1|, |x_2|, \dots, |x_N|, (|x_{N+1}|+1)\} \quad \forall n \in \mathbb{N}$
We see that our Cauchy sequence is bounded.
According to Bolzano-Weierstrass (Thm 1.1) the sequence $\{x_n\}$ contains a subsequence $\{x_{n_k}\}$ which converges to a limit L.
It only remains to prove that L is the limit of the whole sequence $\{x_n\}$.

Let ε be an arbitrary positive number.
Then by Def 1.1 $\exists K \in \mathbb{N}$ s.t. $k > K \Rightarrow |x_{n_k} - L| < \varepsilon/2$.
Also, by Def 2.1 $\exists M \in \mathbb{N}$ s.t. $m, n > M \Rightarrow |x_m - x_n| < \varepsilon/2$
Let us now <u>fix</u> a number $k \in \mathbb{N}$ s.t. $k > K$ & $n_k > M$
Then for $n > M$ we have

$$\begin{aligned}|x_n - L| &= |(x_n - x_{n_k}) + (x_{n_k} - L)| \\ &\leq |x_n - x_{n_k}| + |x_{n_k} - L| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \quad \square\end{aligned}$$

"Metric Space" = set on which you have a concept of distance.
[Abstract way of looking at things]
Say, in $\mathbb{R}$, the distance between & y is | y|.
A metric space is said be complete if Thm 2 holds.
We showed that $\mathbb{R}$ is a complete metric space.
Example on an incomplete metric space: $\mathbb{Q}$

<u>Def 2.2</u>
We say that the function $f: D \to \mathbb{R}$ is <u>uniformly continuous</u> if $\forall \varepsilon > 0 \ \exists \delta > 0$ s.t. $|x - y| < \delta$ and $x, y \in D \Rightarrow |f(x) - f(y)| < \varepsilon$.

Compare Def 2.2 with Defs 1.7 and 1.9.
Uniform continuity $\Rightarrow$ continuity,
but continuity $\not\Rightarrow$ uniform continuity.

<u>Thm 2.2</u> (Uniform continuity theorem)
If $f: [a, b] \to \mathbb{R}$ is continuous then it is uniformly continuous.
Important: here the interval $[a, b]$ is bounded and closed.

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Proof (by contradiction)

Suppose f is not uniformly continuous. Then $\exists \varepsilon > 0$ s.t. $\forall \delta > 0$ $\exists x, y \in [a,b]$ s.t. $|x-y| < \delta$ and $|f(x) - f(y)| \geq \varepsilon$.

Put $\delta = \frac{1}{n}$, $n = 1, 2, \ldots$

Then for each $n = 1, 2, \ldots$ we get $x_n, y_n \in [a,b]$ s.t.

$|x_n - y_n| < \frac{1}{n}$ (2.3)

and $|f(x_n) - f(y_n)| \geq \varepsilon$ (2.4)

Since $x_n \in [a,b]$, the sequence x_n is bounded.

By Bolzano-Weierstrass (Thm 1.1), the sequence $\{x_n\}$ has a convergent subsequence, $\{x_{n_k}\}$, which converges to some c. This c has to be in $[a,b]$, as if were not, $|x_{n_k} - c|$ would be bounded below by $a-c$ or $b-c$ (depending on whether $c < a$ or $c > b$), contradicting the fact that x_{n_k} converges to c.

Formula (2.3) implies

$$|y_{n_k} - c| = |(y_{n_k} - x_{n_k}) + (x_{n_k} - c)|$$
$$\leq |y_{n_k} - x_{n_k}| + |x_{n_k} - c|$$
$$< \frac{1}{n_k} + |x_{n_k} - c| \to 0 \quad (\text{as } x_{n_k} \to c)$$

as $k \to \infty$. Hence, the subsequence $\{y_{n_k}\}$ also converges to c.

The function f is continuous at the point c, so by the sequential def. of continuity (Def 1.8),

$$\lim_{k\to\infty} f(x_{n_k}) = f(c) \quad , \quad \lim_{k\to\infty} f(y_{n_k}) = f(c).$$

By the algebra of limits (for sequences) we have

$$\lim_{k\to\infty} (f(x_{n_k}) - f(y_{n_k})) = 0.$$

By the definition of a limit of a sequence (Def 1.1),

$$\exists K \in \mathbb{N} \text{ s.t. } k > K \Rightarrow |f(x_{n_k}) - f(y_{n_k})| < \varepsilon.$$

In particular,

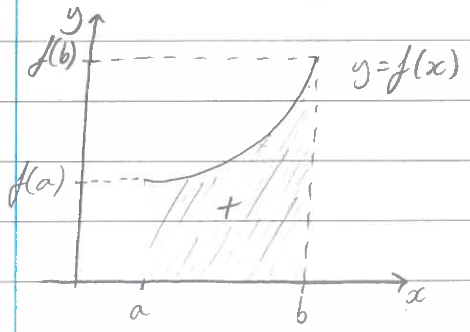
$$|f(x_{n_{K+1}}) - f(y_{n_{K+1}})| < \varepsilon,$$

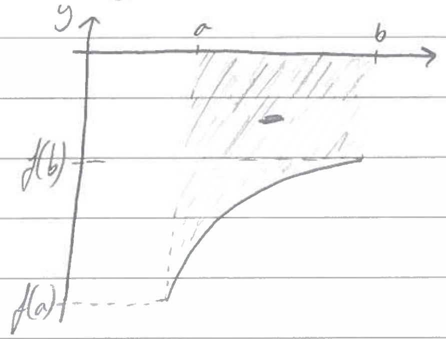
which contradicts (2.4). $\square$

Chapter 3 - Integration

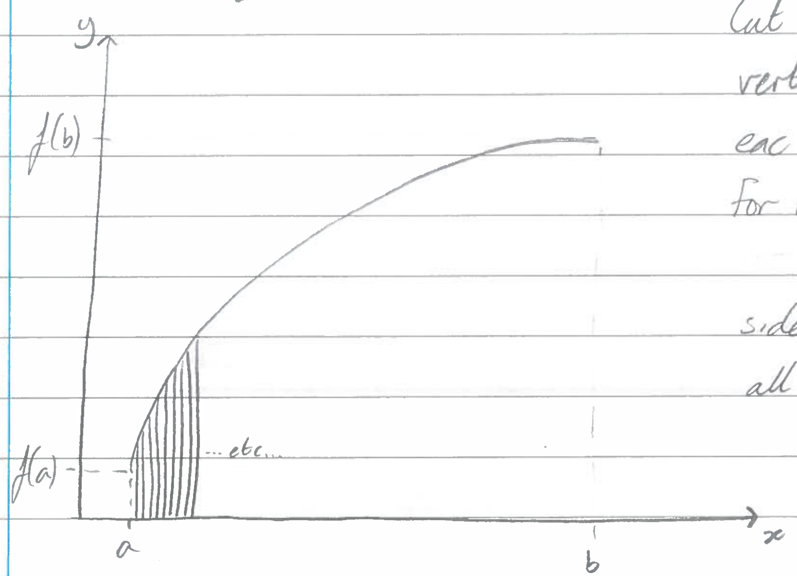
Our aim is to define $\int_a^b f(x)\,dx$.

Basic idea: $\int_a^b f(x)\,dx$ is "signed area under the curve."





Riemann integral (Idea)



Cut the area into narrow vertical strips and approximate eac strip y a rectangle. For a rectangle we now how esbmat [illegible] tiply sides nen add up areas of all rectangles

Until further notice, we will be working on a closed bounded interval $[a,b]$.

Def 3.1

A function $f:[a,b]\to\mathbb{R}$ is said to be bounded if $\exists m, M \in \mathbb{R}$ s.t. $m \le f(x) \le M$, $\forall x \in [a,b]$.

The set of all bounded functions $f:[a,b]\to\mathbb{R}$ will be denoted by $\mathcal{B}[a,b]$.

Until further notice, we will be working with bounded functions only.

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Def 3.2

(i) A partition P of the interval $[a,b]$ is a finite sequence of real numbers $P = \{x_0, x_1, \dots, x_n\}$ such that $a = x_0 < x_1 < \dots < x_{n-1} < x_n = b$.
The set of all partitions of a given interval $[a,b]$ will be denoted by $\mathcal{P}[a,b]$.

(ii) The mesh (norm / width) of the partition $P = \{x_0, x_1, \dots, x_n\}$ is the number $\|P\| := \max_{i=1,\dots,n} (x_i - x_{i-1})$

(iii) Let $P, Q \in \mathcal{P}[a,b]$. We say that Q is a refinement of P if $P \subset Q$, ie. every point of P is a point of Q.

Example 3.1

$P = \{0, \frac{1}{2}, 1\}$, $Q = \{0, \frac{1}{2}, \frac{3}{4}, 1\}$ are partitions of the interval $[0,1]$, with $\|P\| = \frac{1}{2}$ and $\|Q\| = \frac{1}{2}$.
Here, Q is a refinement of P.

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Example 3.2

$P = \{0, \frac{1}{2}, 1\}$ and $Q = \{0, \frac{1}{3}, 1\}$ are partitions of $[0,1]$ we have $\|P\| = \frac{1}{2}$ and $\|Q\| = \frac{2}{3}$.
Neither is a refinement of the other.

Example 3.3

Take P and Q as in the previous example and consider the partition
$R := P \cup Q = \{0, \frac{1}{3}, \frac{1}{2}, 1\}$.
This is called common refinement of P and Q.

Example 3.4

Let $n \in \mathbb{N}$, then $P = \{0, \frac{1}{n}, \frac{2}{n}, \dots, \frac{n-1}{n}, 1\}$ is a partition of the interval $[0,1]$ into n subintervals of equal length. Generalisation to an arbitrary interval $[a,b]$.
$x_i := a + \frac{b-a}{n} i$, $i = 0, 1, \dots, n$

When Q is a refinement of P we will write $P \subset Q$.

Lemma 3.1

If $P, Q \subset \mathcal{P}[a,b]$ and $P \subset Q$ then $\|P\| > \|Q\|$
(ie. mesh decreases with refinement)
Proof - obvious! $\square$

often in exams $\rightarrow$ Def 3.3

Let $f \in \mathcal{B}[a,b]$, $P \in \mathcal{P}[a,b]$, $P = \{x_0, x_1, \dots, x_n\}$.
The lower Darboux sum of f with respect to the partition P is defined as
$$L(f, P) := \sum_{i=1}^{n} m_i (x_i - x_{i-1})$$
where $m_i := \inf_{x \in [x_{i-1}, x_i]} (f(x))$.

The <u>upper Darboux sum</u> of f with respect to the partition P is defined as

$$U(f, P) := \sum_{i=1}^{n} M_i (x_i - x_{i-1})$$

where $M_i := \sup_{x \in [x_{i-1}, x_i]} (f(x))$

$\inf_{x \in [x_{i-1}, x_i]} (f(x))$ (3.1) means $\inf\{y \in R \mid y = f(x) \text{ for some } x \in [x_{i-1}, x_i]$ (3.2)

I am taking the infimum of the values of the fuction $f(x)$, not x itself.

<u>Lemma 3.2</u>

If the partition P' is a refinement of the partition P with one extra point, then

$$L(f, P) \leq L(f, P'), \quad U(f, P') \leq U(f, P)$$

<u>Proof</u>

Write P as $a = x_0 < x_1 < x_2 < \dots < x_{n-1} < x_n = b$

Write P' as $a = x_0 < x_1 < \dots < x_{k-1} < x' < x_k < \dots < x_{n-1} < x_n = b$

Then

$$L(f, P') - L(f, P) = \left(\inf_{x \in [x_{k-1}, x']} f(x)\right)(x' - x_{k-1}) + \left(\inf_{x \in [x', x_k]} f(x)\right)(x_k - x')$$
$$- \left(\inf_{x \in [x_{k-1}, x_k]} f(x)\right)(x_k - x_{k-1})$$

But $\inf_{x \in [x_{k-1}, x']} f(x) \geq \inf_{x [x_{k-1}, x_k]} f(x)$

and $\inf_{x \in [x', x_k]} f(x) \geq \inf_{x \in [x_{k-1}, x_k]} f(x)$

So

$$L(f, P') - L(f, P) \geq \left(\inf_{x \in [x_{k-1}, x_k]} f(x)\right)(x' - x_{k-1}) + \left(\inf_{x \in [x_{k-1}, x_k]} f(x)\right)(x_k - x')$$
$$- \left(\inf_{x \in [x_{k-1}, x_k]} f(x) \; f(x)\right)(x_k - x_{k-1})$$

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So

$$L(f, P') - L(f, P) \geq \left(\inf_{x \in [x_{k-1}, x_k]} f(x)\right)\left[(x' - x_{k-1}) + (x_k - x') - (x_k - x_{k-1})\right] \geq 0$$

Upper sums handled similar.

[Note $A \subset B \Rightarrow \inf B \leq \inf A$ and $\sup B \geq \sup A$]

Lemma 3.3

If the partition P' is a refinement of the partition P then

$L(f, P) \leq L(f, P')$, $U(f, P') \leq U(f, P)$.

Proof

Repeated application of lemma 3.2 $\square$.

Theorem 3.1

For any partitions $P, Q \in \mathcal{P}[a,b]$

$L(f, P) \leq U(f, Q)$.

Proof

Let $R := P \cup Q$ be a common refinement of P and Q.

Then by lemma 3.3,

$L(f, P) \leq L(f, R)$, $U(f, R) \leq U(f, Q)$

But, obviously, $L(f, R) \leq U(f, R)$, so

$L(f, P) \leq L(f, R) \leq U(f, R) \leq U(f, Q)$

So $L(f, P) \leq U(f, Q)$. $\square$

Def 3.4

Let $f \in \mathcal{B}[a,b]$.

The <u>lower Riemann integral</u> of f over $[a,b]$ is defined as

$$\underline{\int_a^b} f(x)\,dx := \sup_{P \in \mathcal{P}[a,b]} (L(f,P))$$

The <u>upper Rieman integral</u> of f over $[a,b]$ is defined as

$$\overline{\int_a^b} f(x)\,dx := \inf_{P \in \mathcal{P}[a,b]} U(f,P)$$

Corollary 3.1

Let $f \in \mathcal{B}[a,b]$. Then

$$\underline{\int_a^b} f(x)\,dx \le \overline{\int_a^b} f(x)\,dx \qquad (3.3)$$

Proof

By Thm 3.1, $L(f,P) \le U(f, \quad)$ for y p f partitions P and Q. Taking sup emem over $\mathcal{P}$,] we get

$$\underline{\int_a^b} f(x)\,dx \le U(f,Q) \qquad (3.4)$$

Note that formula (3.4) holds for any partition Q. Taking infimum over $Q \in \mathcal{P}[a,b]$ we get (3.3). $\square$

Corollary 3.1 justifies the follo ng defin

Def 3.5

The function $f \in \mathcal{B}[a,b]$ is said to be <u>Riemann integrable</u> on $[a,b]$ if

$$\underline{\int_a^b} f(x)\,dx = \overline{\int_a^b} f(x)\,dx.$$

In this case the common value of the lower and upper Riemann integrals is denoted by

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$$\int_a^b f(x)\,dx$$

and is called the <u>Riemann integral</u> of f over $[a,b]$. The set of all Riemann integrable functions $f:[a,b]\to\mathbb{R}$ will be denoted by $\mathcal{R}[a,b]$

<u>Observation</u>

$\mathcal{R}[a,b] \subset \mathcal{B}[a,b]$ but $\mathcal{R}[a,b] \neq \mathcal{B}[a,b]$.

So not every bounded function is Riemann integrable.

<u>Theorem 3.2</u> (Riemann's Criterium for Integrability)

Let $f \in \mathcal{B}[a,b]$. Then $f \in \mathcal{R}[a,b]$ iff $\forall \varepsilon > 0$ $\exists P \in \mathcal{P}[a,b]$ such that

$$U(f,P) - L(f,P) < \varepsilon \qquad (3.5).$$

<u>Proof</u>

<u>Part 1</u>

Suppose $\forall \varepsilon>0$ $\exists P \in \mathcal{P}[a,b]$ s.t. (3.5) holds.
Need to prove that $f \in \mathcal{R}[a,b]$.

Let ε be an arbitrary positive number. Choose a $P \in \mathcal{P}[a,b]$ s.t. (3.5) holds. Then

$$U(f,P) - \varepsilon \underset{\substack{\uparrow \\ (3.4)}}{<} L(f,P) \underset{\substack{\uparrow \\ \text{Def } 3.4}}{\le} \underline{\int}_a^b f(x)\,dx \underset{\substack{\uparrow \\ \text{Cor } 3.1}}{\le} \overline{\int}_a^b f(x)\,dx \underset{\substack{\uparrow \\ \text{Def } 3.4}}{\le} U(f,P)$$

So $0 \le \overline{\int}_a^b f(x)\,dx - \underline{\int}_a^b f(x)\,dx < \varepsilon$

As $\varepsilon > 0$ is arbitrary, this implies

$$\overline{\int}_a^b f(x)\,dx - \underline{\int}_a^b f(x)\,dx = 0$$

which means that $f \in \mathcal{R}[a,b]$.

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<u>Thm 3.2</u> (Riemann's Criterion for Integrability)
Let $f \in \mathcal{B}[a,b]$. Then $f \in \mathcal{R}[a,b]$ iff $\forall \varepsilon > 0 \ \exists P \in \mathcal{P}[a,b]$ s.t. $U(f,P) - L(f,P) < \varepsilon$ (3.5)

<u>Proof</u>
<u>Part 2</u>
Suppose $f \in \mathcal{R}[a,b]$.
Need to prove that $\forall \varepsilon > 0 \ \exists P \in \mathcal{P}[a,b]$ s.t. (3.5) holds.
Let ε be an arbitrary positive number.
Choose $P', P'' \in \mathcal{P}[a,b]$ s.t.

$$\int_a^b f(x)\,dx - \frac{\varepsilon}{2} < L(f, P') \quad (3.6)$$

$$U(f, P'') < \int_a^b f(x)\,dx + \frac{\varepsilon}{2}. \quad (3.7)$$

Let $P := P' \cup P''$ be a common refinement of P' and P''.
Then by Lemma 3.3, $L(f,P') \le L(f,P)$, $U(f,P) \le U(f,P'')$.
Also, obviously, $L(f,P) \le U(f,P)$, so

$$L(f,P') \le L(f,P) \le U(f,P) \le U(f,P''). \quad (3.8)$$

Formulae (3.6) - (3.8) imply

$$\int_a^b f(x)\,dx - \frac{\varepsilon}{2} < L(f,P) \le U(f,P) < \int_a^b f(x)\,dx + \frac{\varepsilon}{2}$$

which gives (3.5). $\square$

I will prove Riemann integrability of certain classes of functions.
I will denote the set of all continuous functions $f: [a,b] \to \mathbb{R}$ by $C[a,b]$ (C is a vector space).

__Thm 3.3__ (Attainment of Bounds Thm - Extreme Value Thm)
If $f:[a,b] \to \mathbb{R}$ is continuous then it achieves its global maximum at some $c \in [a,b]$ and its global minimum at some $d \in [a,b]$

We see that $C[a,b] \subset \mathcal{B}[a,b]$.

__Thm 3.4__
$C[a,b] \subset \mathcal{R}[a,b]$
So any continuous function is Riemann integrable.

__Proof__
Consider an $f \in C[a,b]$.
Let ε be an arbitrary positive number.
Then by Thm 2.2 and Def 2.2, $\exists \delta$ s.t.
$|x-y| < \delta$ & $x,y \in [a,b] \Rightarrow |f(x) - f(y)| < \varepsilon/2(b-a)$
Choose a partition $P = \{x_0, x_1, \dots, x_n\}$ with $\|P\| < \delta$.
Then $M_i - m_i \leq \varepsilon / 2(b-a)$;
here $M_i = \sup_{x \in [x_{i-1}, x_i]} f(x)$, $m_i = \inf_{x \in [x_{i-1}, x_i]} f(x)$

$$U(f,P) - L(f,P) = \sum_{i=1}^{n} (M_i - m_i)(x_i - x_{i-1}) \leq \frac{\varepsilon}{2(b-a)} \underbrace{\sum_{i=1}^{n} (x_i - x_{i-1})}_{b-a}$$

$$= \varepsilon/2 < \varepsilon$$

and Riemann's Criterion (Thm 3.2) tells us that $f \in \mathcal{R}[a,b]$. ☐

__Def 3.6__
A function $f: D \to \mathbb{R}$ is said to be __increasing__ on D if $x_1 < x_2$ & $x_1, x_2 \in D \Rightarrow f(x_1) \leq f(x_2)$.

__Def 3.7__
A function $f: D \to \mathbb{R}$ is said to be __decreasing__ on D if $x_1 < x_2$ & $x_1, x_2 \in D \Rightarrow f(x_1) \geq f(x_2)$.

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Def 3.8

A function $f : D \to \mathbb{R}$ is said to be monotonic on D if it is either increasing on D or decreasing on D.

Note: if $f : [a,b] \to \mathbb{R}$ is monotonic then it is bounded. Indeed, suppose f is increasing, then

$$f(a) \le f(x) \le f(b).$$

Thm 3.5

If $f : [a,b] \to \mathbb{R}$ is monotonic, then it is Riemann integrable on $[a,b]$.

Proof

For definiteness, assume f to be increasing.
Consider the partition P, divided into n subintervals of equal length,

$$x_i = a + i\frac{b-a}{n}, \quad i = 0, 1, 2, \dots, n$$

Then $$L(f,P) = \sum_{i=1}^{n} \underset{\uparrow \atop f(x_{i-1})}{m_i} \overbrace{(x_i - x_{i-1})}^{\frac{b-a}{n}} = \frac{b-a}{n} \sum_{i=1}^{n} f(x_{i-1})$$

$$= \frac{b-a}{n} \sum_{i=0}^{n-1} f(x_i)$$

Similarly $$U(f,P) = \sum_{i=1}^{n} \underset{\uparrow \atop f(x_i)}{M_i} \overbrace{(x_i - x_{i-1})}^{\frac{b-a}{n}} = \frac{b-a}{n} \sum_{i=1}^{n} f(x_i)$$

Now, let ε be an arbitrary positive number.
Choose n so large that $\frac{b-a}{n}(f(b) - f(a)) < \varepsilon$.
Then $$U(f,P) - L(f,P) = \frac{b-a}{n}(f(x_n) - f(x_0))$$

$$= \frac{b-a}{n}(f(b) - f(a)) < \varepsilon$$

and Riemann's criterion tells us that $f \in \mathcal{R}[a,b]$.
The case of decreasing functions is handled similarly. $\square$

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Several lemmata needed.

Lemma 3.4

Let the partition P' be a refinement of the partition P with one extra point, and let M be an upper bound for $|f|$ on $[a, b]$. Then

$L(f, P') - 2M\|P\| \le L(f, P) \le L(f, P')$,

$U(f, P') \le U(f, P) \le U(f, P') + 2M\|P\|$.

Proof

For definiteness, give for lower sums. In view of Lemma 3.2 we need only to prove the left inequality

$L(f, P') - 2M\|P\| \le L(f, P)$

or, equivalently,

$|L(f, P') - L(f, P)| \le 2M\|P\|$

Arguing as in proof of lemma 3.2

$$|L(f, P') - L(f, P)| = \left| \left(\inf_{x \in [x_{k-1}, x']} f(x)\right)(x' - x_{k-1}) + \left(\inf_{x \in [x', x_k]} f(x)\right)(x_k - x') - \left(\inf_{x \in [x_{k-1}, x_k]} f(x)\right)(x_k - x_{k-1}) \right|$$

$$\le \left|\inf_{x \in [x_{k-1}, x']} f(x)\right|(x' - x_{k-1}) + \left|\inf_{x \in [x', x_k]} f(x)\right|(x_k - x') + \left|\inf_{x \in [x_{k-1}, x_k]} f(x)\right|(x_k - x_{k-1})$$

$$\le M(x' - x_{k-1}) + M(x_k - x') + M(x_k - x_{k-1})$$

$$= 2M(x_k - x_{k-1}) \le 2M\|P\| \quad \square$$

$$\left[\begin{array}{c} -M \le f(x) \le M \\ -M \le \inf_{x \in \dots} f(x) \le M \\ \left|\inf_{x \in \dots} f(x)\right| \le M \end{array}\right]$$

Lemma 3.5

Let the partition P' be a refinement of the partition P with k extra points and let M be an upper bound for $|f|$ on $[a, b]$. Then

$L(f, P') - 2kM\|P\| \le L(f, P) \le L(f, P')$,

$U(f, P') \le U(f, P) \le U(f, P') + 2kM\|P\|$

Proof - Repeated application of lemma 3.4. $\square$

Thm 3.6 (Darboux's Theorem)

Let $f \in \mathcal{B}[a,b]$.

Then $\forall \varepsilon > 0 \ \exists \delta > 0$ s.t. $\|P\| < \delta$

$$\Rightarrow \underline{\int_a^b} f(x)\,dx - \varepsilon < L(f,P) \le \underline{\int_a^b} f(x)\,dx, \qquad (3.9)$$

$$\overline{\int_a^b} f(x)\,dx \le U(f,P) < \overline{\int_a^b} f(x)\,dx + \varepsilon. \qquad (3.10)$$

Proof

Let ε be an arbitrary positive number.

By the definition of supremum

$\exists Q \in \mathcal{P}[a,b]$ s.t.

$$\underline{\int_a^b} f(x)\,dx - \frac{\varepsilon}{2} < L(f,Q) \qquad (3.11)$$

Let Q have k points and let M be a pper bound for $|f|$ on $[\,,b]$.

Choose a $\underline{\delta} > 0$ s.t. $2kM\underline{\delta} < \frac{\varepsilon}{2}$. (3.12)

Let $P \in \mathcal{P}[a,b]$ with $\|P\| < \underline{\delta}$. (3.13)

Set $R := P \cup Q$. Then R has no more than k extra points compared to P, so by lemma 3.5,

$L(f,R) - 2kM\|P\| \le L(f,P)$. (3.14)

Also, by lemma 3.3,

$L(f,Q) \le L(f,R)$. (3.15)

Combining formulae (3.11)-(3.15) we get the left inequality (3.9).

The right inequality (3.9) is obvious - follows from def of lower Riemann integral.

A similar argument for upper sums gives a $\bar{\delta} > 0$.

$\|P\| < \delta \Rightarrow (3.10)$.

It only remains to set $\delta := \min\{\underline{\delta}, \bar{\delta}\}$. $\square$

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Def 3.9

A sequence of partitions $P_n \in \mathcal{P}[a,b]$ with $\lim_{n\to\infty} \|P_n\| = 0$ is said to be a limiting sequence.

Thm 3.7

Let $f \in \mathcal{B}[a,b]$ and $\{P_n\}$ be a limiting sequence of partitions. Then $\lim_{n\to\infty} L(f,P_n) = \underline{\int_a^b} f(x)\,dx$, $\lim_{n\to\infty} U(f,P_n) = \overline{\int_a^b} f(x)\,dx$, and, hence, $f \in \mathcal{R}[a,b]$

iff $\lim_{n\to\infty} L(f,P_n) = \lim_{n\to\infty} U(f,P_n)$.

Proof

Exercise sheet 3.

Thm 3.8

Let $f,g \in \mathcal{B}[a,b]$ and suppose that $f(x) \neq g(x)$ only at a finite number of points. Then
$\underline{\int_a^b} f(x)\,dx = \underline{\int_a^b} g(x)\,dx$, $\overline{\int_a^b} f(x)\,dx = \overline{\int_a^b} g(x)\,dx$.

In particular, $f \in \mathcal{R}[a,b]$ iff $g \in \mathcal{R}[a,b]$.

Proof

Exercise sheet 4, hint: use Thm 3.7 $\square$.

Thm 3.9 (Properties of the Riemann Integral)

Linear properties

(i) Let $f,g \in \mathcal{R}[a,b]$. Then $f+g \in \mathcal{R}[a,b]$ and
$\int_a^b (f(x)+g(x))\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$.

(ii) Let $f \in \mathcal{R}[a,b]$ and $\alpha \in \mathbb{R}$. Then $\alpha f \in \mathcal{R}[a,b]$ and
$\int_a^b \alpha f(x)\,dx = \alpha \int_a^b f(x)\,dx$.

Order property
Let $f, g \in \mathcal{R}[a,b]$ and $f \geq g$,
then $\int_a^b f(x)\,dx \geq \int_a^b g(x)\,dx$.

Domain splitting property
Let $f \in B[a,b]$ and let $c \in (a,b)$. Then $f \in \mathcal{R}[a,b]$
iff $f \in \mathcal{R}[a,c]$ and $f \in \mathcal{R}[c,b]$ then
$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$.

Triangle inequality
Let $f \in \mathcal{R}[a,b]$. Then $|f| \in \mathcal{R}[a,b]$ and
$\left|\int_a^b f(x)\,dx\right| \leq \int_a^b |f(x)|\,dx$. (3.16)

Cauchy-Schwarz inequality
Let $f, g \in \mathcal{R}[a,b]$. Then $fg \in \mathcal{R}[a,b]$ and
$\left|\int_a^b f(x)g(x)\,dx\right| \leq \sqrt{\int_a^b (f(x))^2\,dx}\sqrt{\int_a^b (g(x))^2\,dx}$ (3.17)

Finite dimensional versions (3.16) & (3.17)
$$\left|\sum_{k=1}^n f_k\right| \leq \sum_{k=1}^n |f_k|$$

$$\left|\sum_{k=1}^n f_k g_k\right| \leq \sqrt{\sum_{k=1}^n f_k^2}\sqrt{\sum_{k=1}^n g_k^2}$$

Cauchy-Schwarz in 3D:
$\underset{\sim}{u} \cdot \underset{\sim}{v} = u_1 v_1 + u_2 v_2 + u_3 v_3$ (dot or inner product)
$\|\underset{\sim}{u}\| = \sqrt{\underset{\sim}{u} \cdot \underset{\sim}{u}} = \sqrt{(u_1)^2 + (u_2)^2 + (u_3)^2}$
$|\underset{\sim}{u} \cdot \underset{\sim}{v}| \leq \|\underset{\sim}{u}\| \cdot \|\underset{\sim}{v}\|$
$\underset{\sim}{u} \cdot \underset{\sim}{v} = \|\underset{\sim}{u}\| \cdot \|\underset{\sim}{v}\| \cos\theta$

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<u>Extended notation</u>: now a may be bigger than b.
Defining Riemann integral in this more general setting:

- $\int_a^a f(x)\,dx := 0$

- if $a > b$, we say that $\int_a^b f(x)\,dx$ exists if $\int_b^a f(x)\,dx$ exists and put $\int_a^b f(x)\,dx := -\int_b^a f(x)\,dx$

Further on I will denote by I an interval, bounded or unbounded, open or closed at either end. I will sometimes write $I = \langle c, d \rangle$ where angular brackets indicate that end points may be included (square brackets) or excluded (round brackets). The end points c, d may be real numbers or symbols $\pm\infty$.

<u>Types of intervals</u>

(c, d), $(c, d]$, $[c, d)$, $[c, d]$

$(-\infty, d)$, $(-\infty, d]$, $[c, +\infty)$, $(c, +\infty)$, $(-\infty, +\infty)$

I is an interval
$I = \langle c, d \rangle$

Def 3.10
A function $f : I \to \mathbb{R}$ is said to be locally Riemann integrable over I if $\forall a, b \in I$ we have $f \in \mathcal{R}[a,b]$. The set of all Riemann integrable functions over the interval I will be denoted $\mathcal{R}_{loc}(I)$

Example 3.5
$f : (-1, 1) \to \mathbb{R}$
$f(x) = \frac{1}{1-x^2}$. Then $\underbrace{f \in \mathcal{R}_{loc}(-1,1)}_{\text{should have written}}$ [Here $I = (-1,1)$]
$[a,b] \subset (-1,1)$
$\mathcal{R}_{loc}(\underbrace{(-1,1)}_{I})$

Example 3.6
$f : \underbrace{\mathbb{R}}_{I} \to \mathbb{R}$, $f(x) = x^2$
$f \in \mathcal{R}_{loc}(\mathbb{R})$

General facts
- $C(I) \subset \mathcal{R}_{loc}(I)$
- if $f : I \to \mathbb{R}$ is monotonic, then $f \in \mathcal{R}_{loc}(I)$
- if $I = [c,d]$ is a close bounded interval, then
  $\mathcal{R}_{loc}(I) = \mathcal{R}(I) \equiv \mathcal{R}_{loc}[c,d] = \mathcal{R}[c,d]$

Def 3.11
Let $f : I \to \mathbb{R}$. A primitive of f is a function $F : I \to \mathbb{R}$ which is continuous on I, differentiable on the interior of I and satisfies $F'(x) = f(x)$ on the interior of I.

Theorem 3.10 (Fundamental Theorem of Calculus)

Let $f \in \mathcal{R}[a,b]$ and let F be a primitive of f. Then

$$\int_a^b f(x)\,dx = F(b) - F(a) = \underbrace{F(x)\Big|_a^b}_{\text{short way of writing.}}$$

Proof

Let $P = \{x_0, x_1, \dots, x_n\}$ be a partition of $[a,b]$. Applying the Mean Value theor to the function F on the subinterval $[x_{i-1}, x_i]$ we get

$F(x_i) - F(x_{i-1}) = f(\xi_i)(x_i - x_{i-1})$

$\boxed{\text{MVT: Let } f:[a,b] \to \mathbb{R} \text{ be cont on } [a,b] \text{ and diff on } (a,b). \text{ Then } \exists\, \xi \in (a,b) \text{ s.t. } \frac{f(b)-f(a)}{b-a} = f'(\xi)}$

for some $\xi_i \in (x_{i-1}, x_i)$. But

$m_i \le f(\xi_i) \le M_i$

Hence $m_i(x_i - x_{i-1}) \le F(x_i) - F(x_{i-1}) \le M_i(x_i - x_{i-1})$

Summing up over i from 1 to n we get

$$L(f,P) \le \underbrace{\sum_{i=1}^{n} \big(F(x_i) - F(x_{i-1})\big)} \le U(f,P)$$

$$F(x_1) - F(x_0) + F(x_2) - F(x_1) + \dots + F(x_n) - F(x_{n-1})$$

$$= F(x_n) - F(x_0) = F(b) - F(a)$$

So $L(f,P) \le F(b) - F(a) \le U(f,P)$. (3.18)

(3.18) is true for any partition.

Taking a limiting sequence of partitions $\{P_k\}$ and letting $k \to \infty$ we get

$$\lim_{k\to\infty} L(f, P_k) \le F(b) - F(a) \le \lim_{k\to\infty} U(f, P_k)$$

Def 3.12

Let $f \in \mathcal{R}_{loc}(I)$. An indefinite integral of f is a function $F: I \to \mathbb{R}$ defined by $F(x) = \int_a^x f(t)\,dt$ for some $a \in I$.

Fact: two indefinite integrals differ by a constant.

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Mean Value Theorem

$$\ldots\ldots \frac{f(b)-f(a)}{b-a} = f'(\xi)$$

Proof of the Fundamental Theorem of Calculus

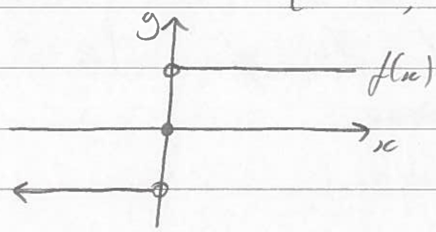
$$F(x_i) - F(x_{i-1}) = \underbrace{f(\xi)}_{\substack{\shortparallel \\ F'(\xi)}}(x_i - x_{i-1})$$

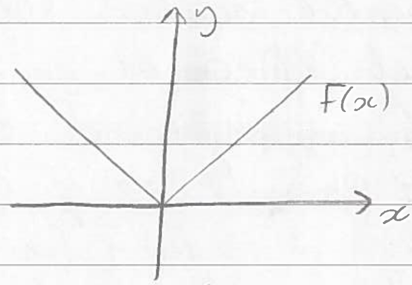
$F'(\xi)$ by definition of a primitive.

$a \to x_{i-1}$
$b \to x_i$
$f \to F$
$f' \to f$

I claim that taking an indefinite integral is a "smoothing" operator.

$$f(x) = \mathrm{sgn}(x) = \begin{cases} 1, & x > 0 \\ 0, & x = 0 \\ -1, & x < 0 \end{cases} \quad , \quad F(x) := \int_0^x \mathrm{sgn}(t)\,dt = |x|$$



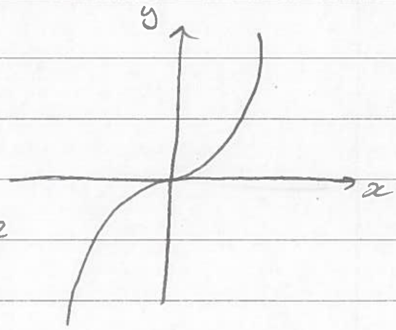


f not continuous.

f continuous, but still not perfect : not differentiable.

Take indefinite integral again:

$$\tilde{F}(x) : \int_0^x F(t)\,dt = \int_0^x |t|\,dt = \frac{x|x|}{2}$$



$\tilde{F}$ is differentiable but second derivative doesn't exist at 0.

Thm 3.11 (Properties of the Indefinite Integral)
Let $f \in \mathcal{R}_{loc}(I)$ and let $F: I \to \mathbb{R}$ be an indefinite interal of f. Then
(i) F is continuous on I.
(ii) F is differentiable at each interior point $x_0 \in I$ at which f is continuous, and at such a point $F'(x_0) = f(x_0)$.
(iii) if f is continuous on I, then F is a primitive of f.

Proof
(i) Take $\forall x_0 \in I$. Must prove F is continuous at x_0.
Case 1:
x_0 is an interior point of I. Let ε be an arbitrary positive number. Choose a $\delta_0 > 0$ s.t. $[x_0 - \delta_0, x_0 + \delta_0] \subset I$.
Let $M > 0$ be an upper bound for $|f|$ on $[x_0 - \delta_0, x_0 + \delta_0]$.
This upper bound exists because
$$f \in \mathcal{R}_{loc}(I) \Rightarrow f \in \mathcal{R}[x_0 - \delta_0, x_0 + \delta_0]$$
$$\Rightarrow f \in \mathcal{B}[x_0 - \delta_0, x_0 + \delta_0].$$
Now choose a $\delta > 0$ st. $\delta < \delta_0$ and $M\delta < \varepsilon$.
Then for any $x \in (x_0 - \delta_0, x_0 + \delta_0)$ we have
$$|F(x) - F(x_0)| = \left| \int_a^x f(t)\,dt - \int_a^{x_0} f(t)\,dt \right|$$
$$= \left| \int_{x_0}^x f(t)\,dt \right| \le \left| \int_{x_0}^x |f(t)|\,dt \right| \le \left| \int_{x_0}^x M\,dt \right|$$
$$= M|x - x_0|$$
$$< M\delta < \varepsilon.$$

Case 2:
x_0 is a left hand endpoint of I, prove similarly. Use one sided neighbourhoods of x_0.
Case 3:
x_0 is a right endpoint of I.

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(ii) Let x_0 be an interior point of I at which f is continuous. Let ε be an arbitrary positive number. Then $\exists\, \delta > 0$ s.t. $(x_0 - \delta_0, x_0 + \delta_0) \subset I$ and $|x - x_0| < \delta \Rightarrow |f(x) - f(x_0)| < \frac{\varepsilon}{2}$.
(Here I used continuity.)
Hence, for any $x \in (x_0 - \delta_0, x_0 + \delta_0)$, $x \neq x_0$, we have

$$\left|\frac{F(x) - F(x_0)}{x - x_0} - f(x_0)\right| = \left|\frac{\int_a^x f(t)\,dt - \int_a^{x_0} f(t)\,dt}{x - x_0} - \frac{\int_{x_0}^x f(x_0)\,dt}{x - x_0}\right|$$

[note: $f(x_0) = \text{const}$]

$$= \left|\frac{\int_{x_0}^x (f(t) - f(x_0))\,dt}{x - x_0}\right|$$

$$\leq \frac{\left|\int_{x_0}^x |f(t) - f(x_0)|\,dt\right|}{|x - x_0|}$$

$$\leq \frac{\left|\int_{x_0}^x \frac{\varepsilon}{2}\,dt\right|}{|x - x_0|} = \frac{\frac{\varepsilon}{2}|x - x_0|}{|x - x_0|}$$

$$= \frac{\varepsilon}{2} < \varepsilon.$$

By the definition of a limit of a function, this means

$$\lim_{x \to x_0}\left(\frac{F(x) - F(x_0)}{x - x_0} - f(x_0)\right) = 0 \Leftrightarrow \lim_{x \to x_0}\left(\frac{F(x) - F(x_0)}{x - x_0}\right) = f(x_0)$$

$\Leftrightarrow F'(x_0) = f(x_0)$ by definition of a derivative.

(iii) follows from (i), (ii) and definition of a primitive. $\square$

Remark 3.1

If we know that the interval I is open and the function f is continuous, then the proof of 3.11 becomes shorter. There is no need to prove part (i) because it follows from (ii).

Example

Find derivative of function $G(x) = \int_{e^{-x}}^{e^{x}} \sin(t^2)\, dt$.

Put $F(y) := \int_0^y \sin(t^2) dt$. Then $G(x) = F(e^x) - F(e^{-x})$

Hence, by Chain Rule and Thm 3.11

$$G'(x) = F'(e^x)e^x + F'(e^{-x})e^{-x} = \sin(e^{2x})\, e^x + \sin(e^{-2x})\, e^{-x}$$

Corollary 3.2 (Integration by parts)

Let $f, g \in C[a,b]$ and let F, G be primitives of f, g respectively.

Then

$$\int_a^b f(x) G(x)\, dx + \int_a^b F(x) g(x)\, dx = \underbrace{\Big[F(x) G(x)\Big]_a^b}_{F(b)G(b) - F(a)G(a)} \qquad (3.19)$$

Proof

Set $H(x) := F(x) G(x)$ Then H is continuous on $[a,b]$ and differentiable on (a,b).

Moreover by the product rule

$$H'(x) = f(x) G(x) + F(x) g(x), \quad \forall x \in (a,b)$$

Thus H is a primative of $fG + Fg$, so by the Fundamental theorem of Calculus,

$$\int_a^b f(x) G(x) + F(x) g(x)\, dx = H(x)\Big|_a^b = \Big[F(x) G(x)\Big]_a^b$$

$\square$

Formula (3.19) can be rewritten as

$$\int_a^b F'(x) G(x)\, dx + \int_a^b F(x) G'(x)\, dx = \Big[F(x) G(x)\Big]_a^b \qquad (3.20)$$

Only problem with (3.20) is that it requires the use of one-sided derivatives: I tried to avoid defining $F'(a)$ or $F'(b)$.

The most common form of the integration by parts formula is

$$\int_a^b F'(x)\,G(x)\,dx = -\int_a^b F(x)\,G'(x)\,dx + \Big[F(x)G(x)\Big]_a^b \qquad (3.21)$$

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Corollary 3.3 (Change of Variable - Integration by Substitution)
Let I be an open interval and let $\varphi: I \to \mathbb{R}$ be continuously differentiable. Let J be an interval of positive length such that $\varphi(I) \subset J$ and let $f: J \to \mathbb{R}$ be continuous. Then for any $a, b \in I$

$$\int_a^b f(\underbrace{\varphi(t)}_{x}) \underbrace{\varphi'(t)\,dt}_{dx} = \int_{\varphi(a)}^{\varphi(b)} f(x)\,dx$$

Formal argument

$$x = \varphi(t) \quad , \quad \frac{dx}{dt} = \varphi'(t) \quad , \quad dx = \varphi'(t)\,dt$$

Proof
Introduce function $F(s) := \int_c^s f(x)\,dx$, then

consider $(F \circ \varphi)(t) = F(\varphi(t))$,
differentiate it and apply the Fundamental theorem of Calculus. $\square$

Thm 3.12 (Integral test for the convergence of series)
Let r be a natural number and let $f: [r, +\infty) \to [0, +\infty)$ be a decreasing function.
Then the series $\sum_{k=r}^{\infty} f(k)$ converges iff $\lim_{n\to\infty} \int_r^n f(x)\,dx$ exists.

[f is Riemann integrable on $[r, n]$ as it is monotone.]

Proof
Put $T_n := \sum_{k=1}^{n} f(k) - \int_r^n f(x)\,dx \qquad (3.22)$
for $n \in \mathbb{N}$, $n \geq r$.
We have

$$T_n - T_{n+1} = \int_n^{n+1} f(x)\,dx - f(n+1)$$

$$= \int_n^{n+1} f(x)\,dx - \int_n^{n+1} f(n+1)\,dx$$

$$= \int_n^{n+1} \underbrace{(f(x) - f(n+1))}_{\geq 0 \text{ when } x \in [n, n+1]} dx \geq 0$$

So, sequence $\{T_n\}$ is decreasing, as $T_n - T_{n+1} \geq 0$.
Now I estimate T_n from below:

$$T_n = \sum_{k=r}^{n-1} \int_k^{k+1} f(k)\, dx + f(n) - \sum_{k=r}^{n-1} \int_k^{k+1} f(x)\, dx$$

$$= \sum_{k=r}^{n-1} \int_k^{k+1} \underbrace{(f(k) - f(x))}_{\geq 0 \text{ when } x \in [k, k+1]} dx + f(n) \geq f(n)$$

But f is nonnegative $\Rightarrow f(n) \geq 0 \Rightarrow T_n \geq 0$.

So $\{T_n\}$ is decreasing and bounded below by zero.
Hence, $\{T_n\}$ converges.
Suppose that the series $\sum_{k=r}^{\infty} f(k)$ converges. Then formula (3-22), the fact that the sequence $\{T_n\}$ converges, and the algebra of limits tells us that $\lim_{n \to \infty} \int_r^n f(x)\, dx$ exists.

Indeed: $$\int_r^n f(x)\, dx = \underbrace{\sum_{k=r}^{n} f(k)}_{\substack{\text{converges} \\ \text{as } n \to \infty.}} - \underbrace{T_n}_{\substack{\text{converges} \\ \text{as } n \to \infty.}}$$

Conversely, suppose that $\lim_{n \to \infty} \int_r^n f(x)\, dx$ exists.
Then (3-22), the fact that the sequence $\{T_n\}$ converges, and the algebra of limits tells us that the series $\sum_{k=r}^{\infty} f(k)$ exists. $\square$

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Does $\sum_{k=1}^{\infty} \frac{1}{k}$ converge or not?

Consider $f : [1, +\infty) \to [0, +\infty)$, $f(x) = \frac{1}{x}$.
This f is decreasing on $[0, +\infty)$ and f takes nonnegative values. Can apply Thm 3.12: $\sum_{k=1}^{\infty} \frac{1}{k}$ converges iff $\lim_{n\to\infty} \int_1^n \frac{1}{x}\,dx$ exists.

$\lim_{n\to\infty} \ln n$ doesn't exist. Hence, $\sum_{k=1}^{\infty} \frac{1}{k}$ diverges.

- It is important to check that f is nonnegative and decreasing before using Thm (3.12).

- Be careful not to mix up the discrete variable k and continuous variable x.

Short way of stating Thm 3.12:
Under certain conditions,
$$\sum_{k=r}^{\infty} f(k) \text{ converges iff } \underbrace{\int_r^{\infty} f(x)\,dx}_{\substack{\text{Improper integral,}\\ \text{to be defined in the}\\ \text{last chapter of the}\\ \text{course.}}} \text{ exists.}$$

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L9

Chapter 4 - Power Series

$$\sum_{n=0}^{\infty} a_n x^n$$

Here $a_0, a_1, \dots$ are given real numbers and $x \in \mathbb{R}$ is an independent variable.
Convention: $x^0 = 1 \ \forall x \in \mathbb{R}$

Any power series converges for $x = 0$, $\sum_{n=0}^{\infty} a_n 0^n = a_0$.
But it is not clear for which other x (if any) the series converges.

Recall from Math 1101:
- For a given $x \in \mathbb{R}$, power series converges $\Rightarrow \lim_{n\to\infty} a_n x^n = 0$, but $\lim_{n\to\infty} a_n x^n = 0 \not\Rightarrow$ power series converges.
- Concept of absolute convergence.

Given an $x \in \mathbb{R}$, we say that our power series <u>converges absolutely</u> for this x if the series $\sum_{n=0}^{\infty} |a_n x^n|$ converges.
Note absolute convergence $\Rightarrow$ convergence, but convergence $\not\Rightarrow$ absolute convergence.
Short way of saying "our power series converges absolutely" is $\sum_{n=0}^{\infty} |a_n x^n| < \infty$.

Introduce the set $S := \{ r \geq 0 \mid \sum_{n=0}^{\infty} a_n x^n \text{ converges for } x = r \text{ or } x = -r \}$. (4.1)

Obviously, $\{0\} \subset S \subset [0, +\infty)$

<u>Def 4.1</u>
The radius of convergence of a power series is the extended real number $R := \sup S$ where S is the set (4.1).

"Extended real number" means we allow R to take value ∞.
"$R = \infty$" is a short way of saying "the set S is unbounded (from above)."

Any power series has a rad of [illegible], $0 \le R \le \infty$.

Example 4.1

$\sum_{n=0}^{\infty} x^n$

To find R, fix $x \neq 0$ and apply ratio test.

$$\left|\frac{x^{n+1}}{x^n}\right| = |x| \to |x|$$

Ratio test tells us that the series converges absolute for $|x| < 1$ and diverges for $|x| > 1$. We have divergence for $x = 1$ or $x = -1$ because the necessary condit for convergence of a series is not satisfied.

So $S = [0, 1)$, $R = 1$.

For $|x| < 1$ we have $\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$

Example 4.2

$\sum_{n=0}^{\infty} \frac{x^n}{n!}$ To find R, fix $x \neq 0$ and apply the ratio test.

$$\left|\frac{x^{n+1}}{(n+1)!} \div \frac{x^n}{n!}\right| = \left|\frac{x^{n+1}}{(n+1)!} \times \frac{n!}{x^n}\right| \quad | \quad \to 0 \text{ as } n \to \infty$$

so the series converges absolutely $\forall x \in R$, so [,
So $R = \infty$

Example 4.3

$\sum_{n=0}^{\infty} n! x^n$. Fix $x \neq 0$. Observe that $n! x^n$ is unbounded.

If you wish to apply the ratio test:

$$\left|\frac{(n+1)!\,x^{n+1}}{n!\,x^n}\right| = (n+1)|x| \text{ diverges to } +\infty \text{ as } n \to +\infty.$$

So series diverges $\forall x \neq 0$.
So $S = \{0\}$, $R = 0$

Thm 4.1

If $|x| < R$ then the power series converges absolutely and if $|x| > R$ then the power series diverges.

Note: Thm 4.1 is nontrivial. It tells us that the set S defined by (4.1) is an interval, it doesn't have holes in it.
In particular, we cannot have the situation where the series converges for $x = 1$, diverges for $x = 2$ and converges for $x = 3$.

Proof

The fact that series diverges for $|x| > R$ follows from the definition of R: indeed, the fact that $|x| > R$ and Def 4.1 imply that $|x| \notin S$, S being the set (4.1).
So only need to prove absolute convergence for $|x| < R$.
Suppose $|x| < R$. Then, by def. of R, there exists a $y \in \mathbb{R}$ s.t. $|x| < |y| \leq R$ and $\sum_{n=0}^{\infty} a_n y^n$ converges.
Convergence of $\sum_{n=0}^{\infty} a_n y^n$ implies that $\lim_{n\to\infty} a_n y^n = 0$, which, in turn, implies that the sequence $\{a_n y^n\}$ is bounded. Thus, $\exists M \geq 0$ s.t. $|a_n y^n| \leq M$, $\forall n = 0, 1, 2, \ldots$
Hence, $|a_n x^n| = |a_n y^n| \left|\frac{x}{y}\right|^n \leq M \left|\frac{x}{y}\right|^n$

But $\sum_{n=0}^{\infty} M \left|\frac{x}{y}\right|^n < \infty$ (converges as $|x| < |y|$ so $\left|\frac{x}{y}\right| < 1$)
Hence, $\sum_{n=0}^{\infty} |a_n x^n|$ converges by the Comparison Test. $\square$

Thm 4.1 doesn't tell us what happs at R and $x=-R$.

Suppose $R \neq 0$. Then the interval $(-R, R)$ is called the <u>interval of convergence</u>.

A power series defines a function
$$f:(-R,R) \to \mathbb{R}, \quad f(x) = \sum_{n=0}^{\infty} a_n x^n.$$

Consider the power s e
$$\sum_{n=0}^{\infty} a_n x^n \quad (4.2) \quad \text{and} \quad \sum_{n=1}^{\infty} n a_n x^n \quad (4.3)$$

<u>Lemma 4.1</u>

The two power series (4.2) ( ), hav the same radii of convergence.

<u>Proof</u>

Denote radii of convergence of (4.2) (3) by R and R' respectively.

Let us prove first that $R' \leq R$ (4.4).

Suppose (4.4) is false. Then $R' > R$ and we can choose an x s.t. $R < x < R'$. By Thm 4.1, series (4.3) converges absolutely, whereas series (4.2) <u>diverges</u>. But $|a_n x^n| \leq |n a_n x^n|$, $n = 1, 2, \ldots$ so series (4.) <u>converges absolutely</u> by the Comparison Test. #

Let us now prove ( )

Suppose (4.5) is false. Then $R > R'$ and we can choose $y, z \in \mathbb{R}$ s.t. $R' < y < z < R$.

By Thm 4.1, series $\sum_{n=0}^{\infty} a_n z^n$) converges a. o. ly whereas $\sum_{n=1}^{\infty} n a_n y^n$ (4.7) <u>diverges</u>.

But $|n a_n y^n| = |a_n z^n| \left(n \left|\frac{y}{z}\right|^n\right)$ (4.8).

It is known from Math 1101 that $\lim_{n \to \infty} \left(n \overbrace{\left|\frac{y}{z}\right|}^{<1}{}^n\right) = 0$

so $n \left|\frac{y}{z}\right|^n \leq M \quad \forall n = 1, 2, \ldots$

Formula (4.8) implies that $|n a_n y^n| \leq M |a_n z^n|$.

Hence, (4.7) converges absolutely by the Comparison Test. * This proves (4.5). $\square$

Introduce power series

$$\sum_{n=1}^{\infty} n a_n x^{n-1} \quad (4.9)$$

Lemma 4.2

The two power series, (4.2) and (4.9) have the same radii of convergence.

Proof

Both power series converge for $x=0$. For $x \neq 0$ we have $\sum_{n=1}^{\infty} n a_n x^{n-1} = \frac{1}{x} \sum_{n=1}^{\infty} n a_n x^n$ so LHS and RHS converge or diverge for exactly the same x.
The result follows from lemma 4.1. $\square$

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$$\sum_{n=0}^{\infty} a_n x^n \qquad (4.2)$$

$$\sum_{n=1}^{\infty} n a_n x^n \qquad (4.3)$$

$$\sum_{n=1}^{\infty} n a_n x^{n-1} \qquad (4.9)$$

$$\sum_{n=k}^{\infty} \frac{n!}{(n-k)!} a_n x^{n-k} \qquad (4.10)$$ [formal k^{th} derivative of (4.2)]

where k is a nonnegative integer.

Lemma 4.3

The two power series (4.2) & (4.10) have the same radii of convergence

Proof

Repeated application of lemma 4.2 □.

Lemma 4.4

$$\left| \frac{(x_0+h)^n - x_0^n}{h} - n x_0^{n-1} \right| \leq \frac{n(n-1)|h|}{2} (|x_0| + |h|)^{n-2}$$

$\forall x_0, h \in \mathbb{R}, \; h \neq 0, \; \forall n = 2, 3, 4, \ldots$

$$a^2 - b^2 = (a-b)(a+b)$$
$$a^3 - b^3 = (a-b)(a^2+ab+b^2)$$
$$\vdots$$
$$a^n - b^n = (a-b)(\ldots) \quad , \quad a = x_0 + h, \; b = x_0$$

$$\frac{(x_0+h)^n - x_0^n}{h} - n x_0^{n-1} = \sum_{j=0}^{n-1} (x_0+h)^j x_0^{n-1-j} - n x_0^{n-1}$$

$$= \sum_{j=1}^{n-1} \left((x_0+h)^j x_0^{n-1-j} - x_0^{n-1} \right) \qquad (j=0 \text{ term}) = 0$$

$$= \sum_{j=1}^{n-1} x_0^{n-1-j} \left((x_0+h)^j - x_0^j \right)$$

$$= h \sum_{j=1}^{n-1} x_0^{n-1-j} \sum_{i=0}^{j-1} (x_0+h)^{j-1-i} x_0^i$$

so by the triangle inequality

$$\left|\frac{(x_0+h)^n - x_0^n}{h} - n x_0^{n-1}\right| \le |h| \sum_{j=1}^{n-1} |x_0|^{n-1-j} \sum_{i=0}^{j-1} (|x_0|+|h|)^{j-1-i} |x_0|^i$$

$$\le |h| \sum_{j=1}^{n-1} (|x_0|+|h|)^{n-1-j} \sum_{i=0}^{j-1} (|x_0|+|h|)^{j-1-i} (|x_0|+|h|)^i$$

$$\le |h| \sum_{j=1}^{n-1} (|x_0|+|h|)^{n-1-j} \sum_{i=0}^{j-1} (|x_0|+|h|)^{j-1} (|x_0|+|h|)^i$$

$$= |h| \sum_{j=1}^{n-1} (|x_0|+|h|)^{n-2} \sum_{i=0}^{j-1} 1$$

$$= |h| (|x_0|+|h|)^{n-2} \sum_{j=1}^{n-1} j$$

$$= |h| (|x_0|+|h|)^{n-2} \frac{n(n-1)}{2} \quad \square.$$

Thm 4.2

Let $\sum_{n=0}^{\infty} a_n x^n$ be a power series with radius of convergence $R>0$ and let $f:(-R,R) \to \mathbb{R}$ be the sum of this series. Then f is differentiable on the interval $(-R,R)$ and $f'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1}$

Proof

Fix an x_0 and a h_0 s.t. $0 < h_0 < R - |x_0|$ and let h be such that $0<|h|<h_0$. Then by Lemmata and 4.3 and 4.4

$$\left|\frac{\sum_{n=0}^{\infty} a_n (x_0+h)^n - \sum_{n=0}^{\infty} a_n x_0^n}{h} - \sum_{n=1}^{\infty} n a_n x_0^{n-1}\right|$$

$$= \left| \sum_{n=2}^{\infty} a_n \left(\frac{(x_0+h)^n - x_0^2}{h} - n x_0^{n-1}\right)\right|$$

$$\le \sum_{n=2}^{\infty} \left| a_n \left(\frac{(x_0+h)^n - x_0^2}{h} - n x_0^{n-1}\right)\right|$$

$$\le \frac{|h|}{2} \sum_{n=2}^{\infty} n(n-1)|a_n|(|x_0|+|h|)^{n-2} \le \frac{|h|}{2} \sum_{n=2}^{\infty} \underbrace{n(n-1)|a_n|(|x_0|+h_0)^{n-2}}_{\text{doesn't depend on } h\text{, just a number.}}$$

$$\longrightarrow 0 \text{ as } h \to 0$$

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Note: Lemma 4.3 plays an important role in our proof, it guarantees that all series in the above argument converge absolutely

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Corollary 4.1

Let $\sum_{n=0}^{\infty} a_n x^n$ be a power series with radius of convergence $R>0$, and let $f:(-R,R) \to \mathbb{R}$ be the sum of this series. then f is infinitely differentiable on the interval $(-R, R)$ and

$$f^{(k)}(x) = \sum_{n=k}^{\infty} \frac{a_n n!}{(n-k)!} x^{n-k}$$

Proof

Repeated application of Th 4.2 and Lemma 4.3 $\square$.

Corollary 4.2

Let $\sum_{n=0}^{\infty} a_n x^n$ be a power series with radius of convergence $R>0$, and let $f:(-R,R) \to \mathbb{R}$ be the sum of this series. Then

$$a_n = \frac{f^{(n)}(0)}{n!}$$

Proof

Apply Corollary 4.1 with $x=0$.

$$f^{(k)}(0) = \frac{a_k k!}{0!} = a_k k!$$

(The only term that survied is $n=k$ as $x=0$)

So $a_k = \frac{f^{(k)}(0)}{k!}$

Replacing k with n we get

$$a_n = \frac{f^{(n)}(0)}{k!}$$

Thus, for a power series with positive radius of convergence, we can write

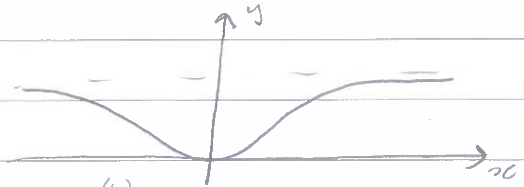
$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n \qquad (4.11)$$

true for $x \in (-R, R)$

Warning: Not every infinitely differentiable functn can be written as a power series.

Example: $f: \mathbb{R} \to \mathbb{R}$,

$$f(x) = \begin{cases} e^{-1/x^2} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$



$f^{(k)}(0) = 0$, $k = 0, 1, 2, \ldots$

If f were given by a power series we would have

$$f(x) = \underbrace{\sum_{n=0}^{\infty} \frac{0}{n!} x^n}_{\equiv 0} \text{ for } x \in (-R, R).$$

Theorem 4.3

Let $\sum_{n=0}^{\infty} a_n x^n$ be a power series with radius of convergence $R > 0$ and let $f: (-R, R) \to \mathbb{R}$ be the sum of this series. Then

$$\int_0^x \left(\sum_{n=0}^{\infty} a_n t^n \right) dt = \sum_{n=0}^{\infty} \frac{a_n x^{n+1}}{n+1}, \quad \forall x \in (-R, R) \qquad (4.12).$$

Note: series in the RHS of (4.12) has same radius of convergence as $\sum_{n=0}^{\infty} a_n x^n$.

Indeed, use Lemma 4.3 viewing $\sum_{n=0}^{\infty} a_n \frac{x^{n+1}}{n+1}$ as the original power series. Then $\sum_{n=0}^{\infty} a_n x^n$ is the formal derivative of $\sum_{n=0}^{\infty} a_n \frac{x^{n+1}}{n+1}$.

Example 4.4

The exponential function is defined as

$$\exp(x) = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \ldots = \sum_{n=0}^{\infty} \frac{x^n}{n!} \qquad (4.13)$$

We know (Example 4.2) that in this, $R = \infty$.

So $\exp(x)$ is defined on the whole real line, $\exp: \mathbb{R} \to \mathbb{R}$.

By Thm 4.2 exp is differentiable and

$$\exp'(x) = \sum_{n=1}^{\infty} \frac{x^{n-1}}{(n-1)!} \underset{\substack{\uparrow \\ \text{introduce } m = n-1}}{=} \sum_{m=0}^{\infty} \frac{x^m}{m!} \underset{\substack{\uparrow \\ \text{rename } m \\ \text{into } n}}{=} \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

Remark 4.1

The exponential function can also be defined as

$$\exp x := \lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n \qquad (4.14)$$

Turns out (4.13) and (4.14) are equivalent.

Euler's number

$$e := \exp 1 = \sum_{n=0}^{\infty}\frac{1}{n!} = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n \approx 2.718281828$$

[note Leo Tolstoy was born in 1828!]

It is known (?? Analysis 1) that

$$\exp(x) = e^x$$

Example 4.5

Trigonometric functions cos and sin:

$$\cos x := 1 - \frac{x^2}{2} + \frac{x^4}{24} - \dots = \sum_{k=0}^{\infty}(-1)^k\frac{x^{2k}}{(2k)!}, \qquad (4.15)$$

$$\sin x := x - \frac{x^3}{6} + \frac{x^5}{125} - \dots = \sum_{k=0}^{\infty}(-1)^k\frac{x^{2k+1}}{(2k+1)!}. \qquad (4.16)$$

$R = \infty$, so cos and sin are infinitely differentiable functions, $\mathbb{R}\to\mathbb{R}$. Moreover $\cos' x = -\sin x$, $\sin' x = \cos x$.

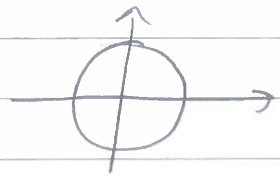
Consider power series

$$\sum_{n=0}^{\infty} a_n z^n \qquad (4.17)$$

where the coefficients a_n are real, but the independent variable (argument) z is complex, i.e. $z = x + iy \in \mathbb{C}$, $x, y \in \mathbb{R}$, $i^2 = -1$. Here I am looking at the issue of extending a real power series to the complex plane.

Fact: Thm 4.1 remains true and no need to change the proof. As a result, instead of an interval of convergence $(-R, R)$ we get a disc of convergence

$\{z \in \mathbb{C} \mid |z| < R\}$. Note $|z| = \sqrt{x^2+y^2}$ in the complex plane.



Thm 4.1 now reads: if $|z| < R$ then the series converges absolutely, if $|z| > R$ then the series diverges.

Why is it so useful to extend a power series into the complex plane?

Example 4.6

I claim that for any $z \in \mathbb{C}$ we have

$$\cos z = \frac{e^{iz} + e^{-iz}}{2}, \quad (4.18) \qquad \sin z = \frac{e^{iz} - e^{-iz}}{2i}. \quad (4.19)$$

In particular, (4.18) and (4.19) hold for real z, which is non trivial.

Example 4.7

Look at function $\frac{1}{1+x^2}$, $x \in \mathbb{R}$.

I know that $\frac{1}{1+x^2} = \sum_{k=0}^{\infty} (-1)^k x^{2k}$, for $|x| < 1$

Geometric progression with ratio $-x^2$.

Here $R = 1$.

Wondering: why $R=1$, not $R=\infty$.

Function $\frac{1}{1+x^2}$ is infinitely differentiable on $\mathbb{R}$.

What goes wrong when $|x| = 1$?

To understand, extend function $\frac{1}{1+x^2}$ into the complex plane: look at $\frac{1}{1+z^2}$ where $z \in \mathbb{C}$. Observe that

$$\frac{1}{1+z^2} = \frac{i}{2}\left(\frac{1}{z+i} - \frac{1}{z-i}\right) \quad (4.20)$$

Our function has singularities at $z = \pm i$

Example 4.8

$$f(z) := \begin{cases} e^{-\frac{1}{z^2}}, & z \neq 0 \\ 0, & z = 0 \end{cases} \qquad (4.21)$$

To understand what's wrong with this function, put $z = iy$, $y \neq 0$. Then

$f(z) = e^{1/y^2}$ horrible singularity as $y \to 0$.

We didn't see this horrible singularity when we were sitting on the real line.

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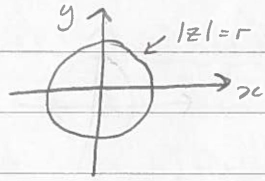
Even more general power series

$$\sum_{n=0}^{\infty} c_n z^n \quad (4.23)$$

where $\begin{cases} c_n = a_n + ib_n \in \mathbb{C}, & a_n, b_n \in \mathbb{R}, \\ z = x + iy \in \mathbb{C}, & x, y \in \mathbb{R}. \end{cases}$

$$S := \left\{ r \geq 0 \,\middle|\, \sum_{n=0}^{\infty} c_n z^n \text{ converges for some } z \in \mathbb{C} \text{ with } |z| = r \right\}$$



$R := \sup S$

Chapter 5 – L'Hôpital's Rule and Taylor's Theorem

Thm 5.1 (Rolle's theorem)

Let $f : [a,b] \to \mathbb{R}$ be continuous on $[a,b]$ and differentiable on (a,b) and let $f(a) = f(b)$. Then $\exists\, \xi \in (a,b)$ s.t. $f'(\xi) = 0$.

Thm 5.2 (Mean Value Theorem)

Let $f : [a,b] \to \mathbb{R}$ be continuous on $[a,b]$ and differentiable on (a,b). Then $\exists\, \xi \in (a,b)$ s.t.

$$f'(\xi) = \frac{f(b) - f(a)}{b-a} \quad (5.1)$$

Thm 5.3 Cauchy's Generalisation of the Mean Value theorem)

Let $f, g : [a,b] \to \mathbb{R}$ be continuous on $[a,b]$ and differentiable on (a,b). Then $\exists\, \xi \in (a,b)$ s.t.

$$(g(b) - g(a)) f'(\xi) = (f(b) - f(a)) g'(\xi) \quad (5.2)$$

Note: Thm 5.2 is a special case of Thm 5.3, with $g(x) = x$.

Proof of Thm 5.3

Introduce a new function

$$h(x) := (g(b) - g(a))f(x) - (f(b) - f(a))g(x)$$

We have $h(a) = f(a)g(b) - f(b)g(a) = h(b)$.
Application of Thm 5.1 to function $h(x)$ gives the required result. □

Thm 5.4 (L'Hopital's Rule)

Let $f, g : (a,b) \to \mathbb{R}$ be differentiable and let $c \in (a,b)$ be such that $f(c) = g(c) = 0$ and $g'(x) \neq 0$ for $x \neq c$.
Then

$$\lim_{x \to c} \frac{f(x)}{g(x)} = \lim_{x \to c} \frac{f'(x)}{g'(x)} \qquad (5.3)$$

provided the latter limit exists.

Remark

I claim that conditions of Thm 5.4 ensure that $g(x) \neq 0$ for $x \neq c$. Indeed, suppose that $g(x) = 0$ for some $x \in (a,b)$, $x \neq c$.

Case 1 : $x > c$

Apply Rolle's Thm to function g on the interval $[c, x]$: get a $\xi \in (c, x)$ st. $g'(\xi) = 0$. ※ contradiction!

Case 2 : $x < c$

Similar (interval is $[x, c]$).

Proof of Thm 5.4

Suppose $\lim_{x \to c} \frac{f'(x)}{g'(x)}$ exists and equals L. then

$$\lim_{x \to c^+} \frac{f'(x)}{g'(x)} = L \qquad (5.4)$$

and

$$\lim_{x \to c^-} \frac{f'(x)}{g'(x)} = L \qquad (5.5)$$

Then in order to prove (5.3), need to show that

$$\lim_{x \to c^+} \frac{f(x)}{g(x)} = L \qquad (5.6)$$

and

$$\lim_{x \to c^-} \frac{f(x)}{g(x)} = L \qquad (5.7).$$

Let us prove 5.6.

Take an arbitrary sequence $\{x_n\} \subset (c, b)$ s.t.

$$\lim_{n \to \infty} x_n = c \qquad (5.8).$$

Applying Thm 5.3 on the interval $[c, x_n]$ we get a sequence $\{\xi_n\}$ s.t. $c < \xi_n < x_n$ (5.9)

and $(g(x_n) - \underbrace{g(c)}_{0}) f'(\xi_n) = (f(x_n) - \underbrace{f(c)}_{0}) g'(\xi_n)$

which can be rewritten as

$$\frac{f(x_n)}{g(x_n)} = \frac{f'(\xi)}{g'(\xi)} \qquad (5.10)$$

Formula (5.8), (5.9) and the Sandwich Principle imply $\lim_{n \to \infty} \xi_n = c$ (5.11).

Formulae (5.4), (5.11) and sequential definition of a limit (Def 1.3, one sided version) imply

$$\lim_{n \to \infty} \frac{f'(\xi_n)}{g'(\xi_n)} = L \qquad (5.12)$$

Formulae (5.10) and (5.12) imply

$$\lim_{n \to \infty} \frac{f(x_n)}{g(x_n)} = L \qquad (5.13)$$

Formulae (5.13) and the sequential def. of a limit (Def 1.3, one sided version) imply (5.6).

[at last step it was important that sequence $\{x_n\}$ is arbitrary]. $\square$

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Example 5.1

$$\lim_{x\to 0} \frac{\sin x}{x} = \lim_{x\to 0} \frac{\cos x}{1} = \lim_{x \to 0} \cos x = \cos(0) = 1$$

Example 5.2

$$\lim_{x\to 0} \frac{1-\cos x}{x^2} = \lim_{x\to 0} \frac{\sin x}{2x} = \frac{1}{2} \lim_{x\to 0} \frac{\sin x}{x} = \frac{1}{2} \times 1 = 1$$

[$\lim_{x\to c} \frac{f(x)}{g(x)}$ Before applying L'Hopital's Rule, check $f(c) = g(c) = 0$]

Consider a function $f:(a,b) \to \mathbb{R}$ which is $n+1$ times differentiable (n is a nonnegative integer).
Fix an $x_0 \in (a,b)$ and put

$$P_{n,x_0}(x) := f(x_0) + f'(x_0)(x-x_0) + \frac{f''(x_0)}{2!}(x-x_0)^2 + \dots + \frac{f^{(n-1)}(x_0)}{(n-1)!}(x-x_0)^{n-1} + \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n. \quad (5.14)$$

This is Taylor's polynomial of degree n.

$$R_{n,x_0}(x) := f(x) - P_{n,x_0}(x). \quad (5.15)$$

This is the <u>remainder term</u>, or <u>error term</u>.
When $x_0 = 0$, the subscript x_0 is often dropped.

$$f(x) = P_{n,x_0} + R_{n,x_0}. \quad (5.16)$$

Question: How should we estimate the remainder term?

<u>Thm 5.5</u> (Taylor's Theorem, with Lagrange's remainder)
Let $f:(a,b) \to \mathbb{R}$ be $(n+1)$ times differentiable and let $x_0 \in (a,b)$. Then for any $x \in (a,b)$, $x \neq x_0$, there exists some ξ strictly between x_0 and x such that

$$R_{n,x_0}(x) = \frac{f^{(n+1)}(\xi)}{(n+1)!}(x-x_0)^{n+1}. \quad (5.17)$$

"Strictly between x_0 and x" means:
- if $x_0 < x$, then $\xi \in (x_0, x)$
- if $x < x_0$, then $\xi \in (x, x_0)$.

Suppose $f(x)$ is a polynomial of degree n.
What can we say about the remainder term $R_{n,x_0}(x)$?
$R_{n,x_0}(x) = 0$.

$$f(x) = f(x_0) + f'(x_0)(x-x_0) + \frac{f''(x_0)(x-x_0)^2}{2!} + \dots + \frac{f^{(n-1)}(x_0)(x-x_0)^{(n-1)}}{(n-1)!}$$
$$+ \frac{f^{(n)}(x_0)(x-x_0)^n}{n!} + \frac{f^{(n+1)}(\xi)(x-x_0)^{(n+1)}}{(n+1)!}. \quad (5.18)$$

For $n=0$, formula (5.18) becomes
$$f(x) = f(x_0) + f'(\xi)(x-x_0)$$
$$\Updownarrow$$
$$\frac{f(x) - f(x_0)}{x - x_0} = f'(\xi)$$

So Taylor's Thm can be viewed as a generalisation of the Mean Value Thm.

Proof of Thm 5.5
For definiteness consider the case $x > x_0$.
Fix x and consider the function
$$g(t) := R_{n,t}(x) = f(x) - P_{n,t}(x)$$
$$= f(x) - f(t) - f'(t)(x-t) - \frac{f''(t)(x-t)^2}{2!} - \dots - \frac{f^{(n-1)}(t)\,(x-t)^{n-1}}{(n-1)!}$$
$$- \frac{f^{(n)}(t)\,(x-t)^n}{n!}. \quad (5.19)$$

where $t \in (a,b)$ is the independant variable.
Since $f:(a,b) \to \mathbb{R}$ is $(n+1)$ times differentiable, $g:(a,b) \to \mathbb{R}$ is

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differentiable (once).
Differentiating (5.19), we get

$$g'(t) = 0 - \cancel{f'(t)} + \left[\cancel{f'(t)} - \cancel{f''(t)(x-t)}\right] + \left[\cancel{f''(t)(x-t)} - \cancel{\frac{f'''(t)}{2!}(x-t)^2}\right]$$

$$+ \dots + \left[\cancel{\frac{f^{(n-1)}(t)}{(n-2)!}(x-t)^{n-2}} - \cancel{\frac{f^{(n)}(t)}{(n-1)!}(x-t)^{n-1}}\right]$$

$$+ \left[\cancel{\frac{f^{(n)}(t)}{(n-1)!}(x-t)^{n-1}} - \frac{f^{(n+1)}(t)}{n!}(x-t)^n\right]$$

$$= -\frac{f^{(n+1)}(t)}{n!}(x-t)^n$$

Applying Cauchy's Generalisation of the MVT to the functions $g(t)$ and $h(t) := (x-t)^{n+1}$ on the interval $[x_0, x]$ we get

$$\frac{g(x) - g(x_0)}{h(x) - h(x_0)} = \frac{g'(\xi)}{h'(\xi)}$$

$$= \frac{-\frac{f^{(n+1)}(\xi)}{n!}(x-\xi)^n}{-(n+1)(x-\xi)^n}$$

$$= \frac{f^{(n+1)}(\xi)}{(n+1)!}$$

for some $\xi \in (x_0, x)$. But

$$g(x_0) = R_{n,x_0}(x), \quad g(x) = 0, \quad h(x_0) = (x-x_0)^{n+1}, \quad h(x) = 0,$$

so $$\frac{-R_{n,x_0}(x)}{-(x-x_0)^{n+1}} = \frac{f^{(n+1)}(\xi)}{(n+1)!} \Rightarrow (5.17). \quad \square$$

Thm 5.6 (Taylor's theorem with Cauchy's Remainder)

Let $f:(a,b)\to\mathbb{R}$ be $(n+1)$ times differentiable and let $x_0\in(a,b)$. Then for any $x\in(a,b)$, $x\neq x_0$, there exists some ξ strictly between x_0 and x such that

$$R_{n,x_0}(x) = \frac{f^{(n+1)}(\xi)}{n!}(x-\xi)^n(x-x_0) \qquad (5.20).$$

So
$$f(x) = f(x_0) + f'(x_0)(x-x_0) + \frac{f''(x_0)}{2!}(x-x_0)^2 + \dots + \frac{f^{(n-1)}(x_0)}{(n-1)!}(x-x_0)^{n-1}$$
$$+ \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n + \frac{f^{(n+1)}(\xi)}{n!}(x-\xi)^n(x-x_0) \qquad (5.21)$$

Proof of Thm 5.6

Starts the same as for Thm 5.5.
Applying the MVT (NOT Cauchy's generalisation) to the function $g(t)$ on the interval $[x_0, x]$, we get

$$\frac{g(x)-g(x_0)}{x-x_0} = g'(\xi) = -\frac{f^{(n+1)}(\xi)}{n!}(x-\xi)^n$$

for some $\xi\in(x_0,x)$.

$$-\frac{R_{n,x_0}(x)}{x-x_0} = -\frac{f^{(n+1)}(\xi)}{n!}(x-\xi)^n \Rightarrow (5.20)$$

Suppose $a<0<b$. If we choose $x_0=0$, then Taylor's formula is called Maclaurin's formula.

- Maclaurin's formula with Lagrange's remainder (5.17):
  for any $x\in(a,b)$, $x\neq 0$, there exists some ξ strictly between 0 and x such that
  $$f(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \dots + \frac{f^{(n-1)}(0)}{(n-1)!}x^{n-1} + \frac{f^{(n)}(0)}{n!}x^n + \frac{f^{(n+1)}(\xi)}{(n+1)!}x^{n+1}.$$

- Maclaurin's formula with Cauchy's remainder (5.20):
  $$\dots\ f(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \dots + \frac{f^{(n-1)}(0)}{(n-1)!} + \frac{f^{(n)}(0)}{n!}x^n + \frac{f^{(n+1)}(\xi)}{n!}(x-\xi)^n x.$$

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Compare with

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n \qquad (4.11)$$

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Overall, Lagrange's remainder (5.17) is easier to remember and easier to use than Cauchy's remainder (5.20). When dealing with specific mathematical problems, my recommendation is to try using Lagrange's remainder first, and, if this doesn't give the required result, try using Cauchy's remainder.

When $x_0 = 0$ Taylor's formula is called <u>Maclaurin's formula</u>. Maclaurin's formula is probably the most important special case of Taylor's formula.
The function f below is assumed to be defined on an interval (a, b) such that $a < 0 < b$.

- Maclaurin's formula with remainder in Lagrange's form (5.17):
  $\forall x \in (a,b),\ x \neq 0,\ \exists\ \xi$ strictly between 0 and x s.t.
  $$f(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \dots + \frac{f^{(n)}(0)}{n!}x^n + \frac{f^{(n+1)}(\xi)}{(n+1)!}x^{n+1}.$$

- Maclaurin's formula with remainder in Cauchy's form (5.20):
  $\forall x \in (a,b),\ x \neq 0,\ \exists\ \xi$ strictly between 0 and x s.t.
  $$f(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \dots + \frac{f^{(n)}(0)}{n!}x^n + \frac{f^{(n+1)}(\xi)}{n!}(x-\xi)^n x.$$

We see that Maclaurin's formula is very similar to a power series. The difference is that a power series has infinitely many terms and no remainder term. Compare with (4.11).

<u>Example 5.3</u>

Let us write down Maclaurin's formula with Lagrange's remainder for $\exp : \mathbb{R} \to \mathbb{R}$. We have
$$\exp^{(k)} x = \exp x, \quad k = 0, 1, 2, \dots$$
so $\exp^{(k)}(0) = 1, \quad k = 0, 1, 2, \dots$
and so we write

$$\exp x = 1 + x + \frac{x^2}{2!} + \dots + \frac{x^n}{n!} + \frac{\exp(\xi)}{(n+1)!} x^{n+1} \quad (5.22)$$

Here x is an arbitrary nonzero real number and ξ is some real number strictlly between 0 and x. The latter means that $\xi \in (0, x)$ if $x > 0$, and $\xi \in (x, 0)$ if $x < 0$.

The number n in (5.22) is an arbitrary non negative integer (chosen by the user of the formula). Of course, ξ depends on x and on n.

Can one obtain the (infinite) power series for exp (see (4.14)) from Maclaurin's formula (5.22)? Let us fix an x and examine what happens to the remainder term

$$R_n(x) = \frac{\exp(\xi)}{(n+1)!} x^{n+1}$$

when $n \to \infty$.

Case 1: $x > 0$

We have

$$|R_n(x)| = \frac{\exp(\xi)}{(n+1)!} x^{n+1} < \exp(x) \frac{x^{n+1}}{(n+1)!}$$

It is known that for any fixed $x \in \mathbb{R}$

$$\lim_{n \to \infty} \frac{x^n}{n!} = 0$$

so $\lim_{n \to \infty} R_n(x) = 0$.

Case 2: $x < 0$

We have

$$|R_n(x)| = \frac{\exp \xi}{(n+1)!} |x|^{n+1} < \frac{|x|^{n+1}}{(n+1)!}$$

so in this case we also get $\lim_{n \to \infty} R_n(x) = 0$.

We see that the power series (4.14) can be obtained from Maclaurin's formula (5.22).

Example 5.4

Let us write down Maclaurin's formula with Lagrange's remainder for $f:(-1,+\infty)\to\mathbb{R}$, $f(x)=\ln(1+x)$.

We have

$$f^{(k)}(x) = (-1)^{k-1}\frac{(k-1)!}{(1+x)^k}, \quad k=1,2,\ldots$$

so $f^{(k)}(0) = (-1)^{k-1}(k-1)!$, $\quad k=1,2,\ldots$

Also $f(0)=0$.

Hence Maclaurin's formula reads

$$f(x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \ldots + (-1)^{n-1}\frac{x^n}{n} + \frac{(-1)^n x^{n+1}}{(n+1)(1+\xi)^{n+1}} \quad (5.23).$$

Here x is an arbitrary real number satisfying the inequalities $x>-1$ and $x\neq 0$, and ξ is some real number strictly between 0 and x.

Looking at (5.23) we conjecture the following power series for $\ln(1+x)$:

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \ldots = \sum_{n=1}^{\infty}(-1)^{n+1}\frac{x^n}{n} \quad (5.24).$$

The power series (5.24) obviously converges for $x\in(-1,1]$ (for $x=1$ use Leibniz's test, ie, convergence test for an alternating series, the moduli of whose terms tend to zero) but we still need to prove that the sum of this series does indeed give $\ln(1+x)$.

In order to do this, we fix an $x\neq 0$ and examine what happens to the remainder term when $n\to\infty$.

We consider two cases.

Case 1: $0<x\le 1$

Our remainder term in Lagrange's form is

$$R_n(x) = \frac{(-1)^n x^{n+1}}{(n+1)(1+\xi)^{n+1}} \quad (5.25)$$

We have

$$|R_n(x)| = \frac{x^{n+1}}{(n+1)(1+\xi)^{n+1}} < \frac{x^{n+1}}{n+1} \leq \frac{1}{n+1} \to 0$$

as $n \to \infty$.

Case 2 : $-1 < x < 0$

Unfortunately, we encounter a problem: looking at (5.25) it is difficult to see why $R_n(x) \to 0$ as $n \to \infty$.
In fact, careful examination of formula (5.25) shows that one can only use it to prove that $R_n(x) \to 0$ as $n \to \infty$ for $-\frac{1}{2} < x < 0$.
The way around this difficulty is to rewrite the remainder term in Cauchy's form (5.20).
This allows one to prove that $R_n(x) \to 0$ as $n \to \infty$ for all $-1 < x < 0$. To be done in Exercise Sheet 7.

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Example 5.5

$f : (-1, +\infty) \to \mathbb{R}$, $f(x) = (1+x)^\alpha$, $\alpha \in \mathbb{R}$

Maclaurin's formula?

Then derive power series expansion

$$(1+x)^\alpha = \sum_{n=0}^{\infty} \underbrace{\binom{\alpha}{n}}_{\text{"}\alpha \text{ choose } n\text{"}} x^n, \quad \forall x \in (-1, 1)$$

$$\binom{\alpha}{0} := 1, \quad \binom{\alpha}{n} := \frac{\alpha(\alpha-1)\dots(\alpha-n+1)}{n!} \quad \text{for } n = 1, 2, \dots$$

Chapter 6 - Improper Integrals

Riemann integral: assumption that interval is bounded and closed and function is bounded.

Now I will be any interval, $I = \langle c, d \rangle$.

Function f will be assumed to be from $\mathcal{R}_{loc}(I)$

Note: f is not necessarily bounded.

Aim: to define (if possible) $\int_c^d f(x)\,dx$

Def 6.1

Let $f \in \mathcal{R}_{loc}(I)$, $I = \langle c, d \rangle$

(i) We say that f is integrable at c in the improper sence if $\exists p \in I$ s.t. $\lim_{a \to c^+} \int_a^p f(x)\,dx$.

We will denote this limit by $\int_c^p f(x)\,dx$.

(ii) We say that f is integrable at d in the improper sence if $\exists p \in I$ s.t. $\lim_{b \to d^-} \int_p^b f(x)\,dx$.

We will denote this limit by $\int_p^d f(x)\,dx$.

(iii) We say that f is integrable over I in the improper sense if both (i) & (ii) hold for the same ρ. We will then write

$$\int_c^d f(x)\,dx := \int_c^\rho f(x)\,dx + \int_\rho^d f(x)\,dx$$

<u>Note:</u>

- The pluses and minuses in Def 6.1 refer to one-sided limits.

  If $c=-\infty$, "$a \to c^+$" means $a \to -\infty$

  If $d=+\infty$, "$b \to d^-$" means $d \to +\infty$.
- Choice of ρ is really irrelevent. If limits exist for the particular ρ, they exist for any other. $\int_c^d f(x)\,dx$ does not depend on the choice of ρ.

If $I=[c,d]$ (closed bounded interval) then the improper integral = usual Riemann integral.

How Def 6.1 works for particular integrals.

If $I=[1,+\infty)$, then existance of $\int_1^{+\infty} f(x)\,dx$ means existance of $\lim_{b\to+\infty}\int_1^b f(x)\,dx$.

If $I=(0,1]$, then existance of $\int_0^1 f(x)\,dx$ means existance of $\lim_{a\to 0^+}\int_a^1 f(x)\,dx$

<u>Example 6.1</u>

Examine existance and evaluate, if possible,

$\int_0^1 x^\alpha\,dx$ (6.1) where $\alpha\in\mathbb{R}$. Here $I=(0,1]$

$f:(0,1]\to\mathbb{R}$, $f(x)=x^\alpha$

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Case 1: $\alpha \geq 0$

In this case function f admits a continuous extension to $x=0$: for $\alpha > 0$ put $f(0)=0$, for $\alpha = 0$ put $f(0)=1$.

Hence, integral (6.1) exists in the normal Riemann sense:

$$\int_0^1 x^\alpha \, dx = \frac{x^{\alpha+1}}{\alpha+1}\Big|_0^1 = \frac{1}{\alpha+1}$$

Case 2: $\alpha < 0$

Cannot define $f(0)$ by continuity.

Have to use Def 6.1. Start by choosing $p \in [0, 1]$.

Choose $p=1$. Need to look at

$$\lim_{a \to 0^+} \int_0^1 x^\alpha \, dx. \quad (6.2)$$

We have
$$\int_a^1 x^\alpha \, dx = \begin{cases} \frac{x^{\alpha+1}}{\alpha+1}\Big|_a^1 & \text{if } \alpha \neq -1 \\ \log x \Big|_a^1 & \text{if } \alpha = -1 \end{cases}$$

$$= \begin{cases} \frac{1-a^{\alpha+1}}{\alpha+1} & \text{if } \alpha \neq -1 \\ -\log a & \text{if } \alpha = -1 \end{cases}$$

note: limit (6.2) exists only when $-1 < \alpha < 0$, in which case the limit is $\frac{1}{\alpha+1}$.

Combing both cases: integral (6.1) exists iff $\alpha > -1$, in which case $\int_0^1 x^\alpha \, dx = \frac{1}{\alpha+1}$

Example 6.2

Examine the existance and evaluate, if possible,

$$\int_1^{+\infty} x^\alpha \, dx \qquad (6.3).$$

Here $f: \underbrace{[1, +\infty)}_{I} \to \mathbb{R}$, $f(x) = x^\alpha$, $\alpha \in \mathbb{R}$.

Use Def 6.1 with $p=1$. We are looking at

$$\lim_{b \to +\infty} \int_1^b x^\alpha \, dx \qquad (6.4)$$

$$\int_1^b x^\alpha \, dx = \begin{cases} \left. \dfrac{x^{\alpha+1}}{\alpha+1} \right|_1^b & \text{if } \alpha \neq -1 \\ \left. \ln x \right|_1^b & \text{if } \alpha = -1 \end{cases}$$

$$= \begin{cases} \dfrac{b^{\alpha+1} - 1}{\alpha+1} & \text{if } \alpha \neq -1 \\ \ln b & \text{if } \alpha = -1 \end{cases}$$

We see that the limit (6.4) exists only if $\alpha < -1$, in which case the limit $-\frac{1}{\alpha+1}$.

Final answer: integral (6.4) exists iff $\alpha < -1$, in which case $\int_1^{+\infty} x^\alpha \, dx = \frac{-1}{\alpha+1}$

Thm 6.1

Let $f: I \to [0 + \infty)$ be a locally Riemann integrable function. Then f is integrable over I iff there exists a constant K s.t. for any $a, b \in I$, $a < b$, we have,

$$\int_a^b f(x) \, dx \leq K \qquad (6.5)$$

Proof

Denote $I = \langle c, d\rangle$ (as usual).

Part 1

Suppose that $\int_c^d f(x)dx$ exists. Need to prove $\exists K$ s.t. $\forall a, b \in I^c$, $a < b$, we have (6.5).

Put $K = \int_c^d f(x)\,dx$.

Suppose $\exists a, b \in I$, $a < b$, s.t. (6.5) fails, i.e. $\int_a^b f(x)dx > K = \int_c^d f(x)\,dx$

Choose a $p \in (a, b)$ and, using domain splitting property of Riemann integral, rewrite inequality as

$$\int_a^p f(x)\,dx + \int_p^b f(x)dx > \int_c^d f(x)dx \qquad (6.6)$$

Let $a \to c^+$, after which let $b \to d^-$.
The integrals on LHS of (6.6) can only get bigger, and we get

$$\int_c^p f(x)dx + \int_p^d f(x)\,dx > \int_c^d f(x)dx$$

$$\Rightarrow \int_c^d f(x)\,dx > \int_c^d f(x)\,dx. \quad \#$$

Part 2

Suppose $\exists K$ s.t. $\forall a, b \in I$, $a < b$, we have (6.5).
Need to prove $\int_c^d f(x)dx$ exists.

Fix a $p \in I$, $p \neq c$, $p \neq d$, and consider set

$$\left\{ \int_a^p f(x)dx \;\middle|\; a \in I,\ c < a < p \right\} \qquad (6.7)$$

The set (6.7) is bounded above by K, hence it has a supremum which we will denote by F.
Given an arbitrary $\varepsilon > 0$, choose an $a = a(\varepsilon)$ so that $F - \varepsilon < \int_a^p f(x)dx \leq F$.

Since F is nonnegative, for any $a' < a$, $a' \in I$, we have

$$F - \varepsilon < \int_{a'}^{p} f(x)\,dx \leq F$$

$$\Downarrow$$

$$\left| \int_{a'}^{p} f(x)\,dx - F \right| < \varepsilon$$

which means that

$$\lim_{a' \to c^+} \int_{a'}^{p} f(x)\,dx = F.$$

Thus we proved integrability at the left endpoint of the interval.

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Thm 6.2 (Comparison Theorem for Improper Integrals)
Let $f, g \in \mathcal{R}_{loc}(I)$, $I = \langle c, d \rangle$, and suppose that
- $0 \le f(x) \le g(x) \quad \forall\ x \in I.$
- $\int_c^d g(x)\,dx$ exists

Then $\int_c^d f(x)\,dx$ exists

Proof
Given arbitrary $a, b \in I$, $a < b$, we have
$$\int_a^b f(x)\,dx \le \int_a^b g(x)\,dx \qquad (6.8)$$

But according to Thm 6.1 $\exists K$ s.t.
$$\int_a^b g(x)\,dx \le K \qquad (6.9).$$

irrespective of the choice of a and b.

Combining (6.8) and (6.9) we get $\int_a^b f(x)\,dx \le K$
irrespective of the choice of a and b.
But according to Thm 6.1 this implies that
$\int_c^d f(x)\,dx$ exists. $\square$

Thm 6.3
Suppose that $f \in \mathcal{R}_{loc}(I)$ and $|f|$ is integrable over I. Then f is integrable over I.
(Integrable in the improper sense!)

Proof
Introduce functions
$$f^+(x) := \frac{|f(x)| + f(x)}{2}, \quad f^-(x) := \frac{|f(x)| - f(x)}{2}$$

Easy to see
$$\begin{cases} 0 \le f^+(x) \le |f(x)| & (6.10) \\ 0 \le f^-(x) \le |f(x)| & (6.11) \end{cases}$$

(consider cases $f(x) \geq 0$ and $f(x) < 0$)
By def 3.10 and Thm 3.9 we have
$f^+, f^- \in \mathcal{R}_{loc}(I)$.

Formula (6.10) and Thm 6.2 imply that
f^+ is integrable over I.
Similarly, formula (6.11) and Thm 6.2 imply that
f^- is integrable over I.
But $f(x) = f^+(x) - f^-(x)$, so integrability of
f follows by linearity. $\square$

Example 6.3

Let us prove the existance of the improper
integral $\displaystyle\int_0^{+\infty} \frac{\cos x}{1+x^3}\,dx$ (6.12).

As there is no problem at $x=0$, in order to
prove the existance of the integral (6.12)
it is sufficient to prove existance of integral
$$\int_1^{\infty} \frac{\cos x}{1+x^3}\,dx \quad (6.13).$$

By Thm 6.3, in order to prove existence of
integral (6.13) is sufficient to prove existence
of integral
$$\int_1^{+\infty} \frac{|\cos x|}{1+x^3}\,dx$$

Compare functions $\frac{|\cos x|}{1+x^3}$ and $\frac{1}{x^3}$ on $[1,+\infty)$

We have
$0 \leq \frac{|\cos x|}{1+x^3} \leq \frac{1}{x^3}$, so by Thm 6.2, in order
to prove the existance of integral (6.14), it is
sufficient to prove the existance of

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$$\int_1^{+\infty} \frac{1}{x^3}\, dx \qquad (6.15)$$

But existence of (6.15) is a standard fact, see Example 6.2.

Example 6.4

Let us prove the existence of the improper integral $\displaystyle\int_0^{+\infty} \frac{\sin x}{x}\, dx$ (6.16)

$I = (0, +\infty) \leftarrow$ the Dirichlet Interval.

Do we have a problem at $x=0$?
Does $\displaystyle\lim_{x\to 0^+} \frac{\sin x}{x}$ exist? Yes, it equals 1 (Example 5.1).

So integrand emits a continuous extension to $x=0$. We are effictively integrating a continuous function ove $[0, +\infty)$. So no problem at $x=0$.
Sufficient to prove existance of

$$\int_1^{+\infty} \frac{\sin x}{x}\, dx \qquad (6.17).$$

Argument from previous example won't work.

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$$\int_0^{+\infty} \frac{\sin x}{x}\,dx \qquad (6.16)$$

$\vdots$

$$\int_1^{+\infty} \frac{\sin x}{x}\,dx \qquad (6.17)$$

To prove existence of (6.17), look at

$$\int_1^b \frac{\sin x}{x}\,dx \underset{\substack{\uparrow \\ \text{integration by parts}}}{=} -\frac{\cos x}{x}\Big|_1^b - \int_1^b \frac{\cos x}{x^2}\,dx$$

$$= \underbrace{-\frac{\cos b}{b}}_{\substack{\to 0 \\ \text{as } b \to +\infty}} + \cos 1 - \int_1^b \frac{\cos x}{x^2}\,dx$$

To prove the existence of (6.17) it is sufficient to prove existence of

$$\lim_{b\to+\infty} \int_1^b \frac{\cos x}{x^2}\,dx.$$

So problem reduced to proving existence of the integral

$$\int_1^{+\infty} \frac{\cos x}{x^2}\,dx \qquad (6.18)$$

By Thm 6.3, to prove existence of (6.18), sufficient to prove existence of

$$\int_1^{+\infty} \frac{|\cos x|}{x^2}\,dx \qquad (*)$$

We have

$$0 \le \frac{|\cos x|}{x^2} \le \frac{1}{x^2} \qquad \forall x \ge 1$$

By Thm 6.2, to prove existence of (*) it is sufficient to prove existence of

$$\int_1^{+\infty} \frac{1}{x^2}\,dx.$$

But existence of the latter is a standard fact (Example 6.2).

Turns out, that
$$\int_0^{+\infty} \frac{\sin x}{x}\, dx = \frac{\pi}{2} \qquad (6.19)$$

(6.19) can be proven without complex analysis. Consider the function
$$f(a) := \int_0^{+\infty} e^{-ax} \frac{\sin x}{x}\, dx \quad \text{where } a > 0.$$

Note $\int_0^{+\infty} e^{-x}\, dx = 1$

$$\int_0^b e^{-x}\, dx = 1 - e^{-b} \to 1 \text{ as } b \to +\infty$$

Differentiate with respect to a, then solve the resulting differential equation to get, in the end, $f(0)$.
Here one needs properties of improper integrals depending on a parameter.

Another important integral: Gaussian integral.
$$\int_{-\infty}^{+\infty} e^{-x^2}\, dx = \sqrt{\pi} \qquad (6.20)$$

Proving existence of (6.20) is easy.
Sufficient to prove existence of $\int_0^{+\infty} e^{-x^2} dx$.

Sufficient to prove existence of $\int_1^{+\infty} e^{-x^2} dx$.

Compare with $\int_1^{+\infty} e^{-x}\, dx$, $e^{-x^2} \leq e^{x}$ for $x \geq 1$

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$$\left(\int_{-\infty}^{+\infty} e^{-x^2}\,dx\right)\left(\int_{-\infty}^{\infty} e^{-y^2}\,dy\right)$$

$$= \iint e^{-x^2-y^2}\,dx\,dy$$

Method to find $\int_{-\infty}^{+\infty} e^{-x^2}\,dx$

then switch to polar coordinates.

Rephrasing Thm 3.12 in more compact form.

Thm 6.4

Let r be a natural number and let $f:[r,+\infty) \to [0,+\infty)$ be a decreasing function. Then the series $\sum_{k=r}^{+\infty} f(k)$ converges iff the improper integral $\int_r^{+\infty} f(x)\,dx$ exists.

Proof

Thm 3.12 tells us $\sum_{k=r}^{+\infty} f(k)$ converges iff $\lim_{n\to\infty} \int_r^n f(x)\,dx$ exists.

Observe now that for any $b \in \mathbb{R}$, $b \geq r$, we have

$$\int_r^{[b]} f(x)\,dx \leq \int_r^b f(x)\,dx \leq \int_r^{[b]+1} f(x)\,dx$$

Hence, $\lim_{n\to\infty} \int_r^n f(x)\,dx$ exists iff $\lim_{b\to\infty} \int_r^b f(x)\,dx$.

In other words $\lim_{n\to\infty} \int_r^n f(x)\,dx$ exists iff the improper integral $\int_r^{+\infty} f(x)\,dx$ exists (converges) $\square$