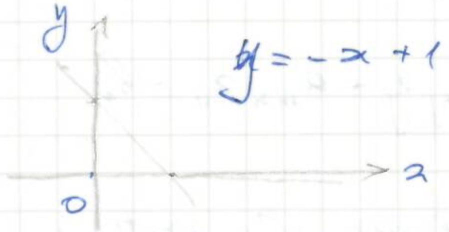
# 1201 Algebra 1 Notes

## Based on the 2009 autumn lectures by Prof F E A Johnson

# Linear Algebra

06.10.09.

$x + y = 1$ , but not $x^2 + \cancel{xy} + y^2 = 1$




$x + y + z = 1$ - plane

$p + q + r + x + y + z = 1$ - ?

## How to write equation with more variables.

$$2x_1 + 3x_2 - x_3 + x_4 + x_5 + x_6 - x_7 = 1$$

$$\underline{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \end{pmatrix}$$ ← column vector

$a_1 x_1 + a_2 x_2 + \ldots + a_n x_n = b$ - general (single) linear equation with $n$ unknowns

$c_1 x_1 + c_2 x_2 + \ldots + c_n x_n = d$

Note: still we will run out of variables!

$$\begin{cases} a_{11} x_1 + a_{12} x_2 + \ldots + a_{1n} x_n = b_1 \\ a_{21} x_1 + a_{22} x_2 + \ldots + a_{2n} x_n = b_2 \\ \vdots \\ a_{i1} x_1 + a_{i2} x_2 + \ldots + a_{in} x_n = b_i \quad \text{- } i^{th}\text{-equat.} \\ a_{n1} x_1 + a_{n2} x_2 + \ldots + a_{nn} x_n = b_n \end{cases}$$

$$S = \begin{cases} a_{11}x_1 + a_{12}x_2 + \dots + a_{1j}x_j + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \dots + a_{2j}x_j + a_{2n}x_n = b_2 \\ a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mj}x_j + a_{mn}x_n = b_m \end{cases}$$

General system of $m$ linear eq$^n$s with $n$ unknowns

$$x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \text{ - solution vector}$$

$$b = \begin{pmatrix} b_1 \\ \vdots \\ b_m \end{pmatrix} \text{ -}$$

$$A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ & & \vdots & \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix} \text{ - Coeficients from}$$

$m \times n$ matrix

numb. of equations/lines

numb. of unknowns

e.g. $\begin{cases} 2x_1 + x_2 - x_3 = 1 \\ x_1 - x_2 + 2x_3 = 2 \end{cases}$

$$x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \quad b = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \quad A = \begin{pmatrix} 2 & 1 & -1 \\ 1 & -1 & 2 \end{pmatrix}$$

# Matrix Multiplication

$a_1 x_1 + a_2 x_2 + \dots + a_n x_n = b$

$$(a_1, a_2, \dots, a_n)\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = b$$

Def$^n$ if $\underset{\sim}{a} = (a_1, \dots, a_n)$ - Row vector

$$\underset{\sim}{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \quad - \text{ Column vector}$$

Define $\underset{\sim}{a} \cdot \underset{\sim}{x} = a_1 x_1 + a_2 x_2 + \dots + a_n x_n = \sum_{r=1}^{n} a_r x_r$

In the general system:
the $i^{th}$ equation:

$a_{i1} x_1 + a_{i2} x_2 + \dots + a_{in} x_n = b_i$

$$(a_{i1} \dots a_{in})\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = b_i$$

Row vector is $1 \times n$ ~~vector~~ matrix.
Column vector is $n \times 1$ matrix.

$$A = \underset{m \times n}{\begin{pmatrix} a_{11} & \dots & a_{1n} \\ \\ a_{m1} & \dots & a_{mn} \end{pmatrix}} \qquad B = \underset{n \times p}{\begin{pmatrix} b_{11} & \dots & b_{1p} \\ \vdots & & \\ b_{n1} & \dots & b_{np} \end{pmatrix}}$$

$AB$ defined to be $m \times p$ matrix, where
$(i, k)^{th}$ is ($i^{th}$ Row $A$) ($k^{th}$ column of $B$)

$$A = \begin{pmatrix} 1 & -1 \\ 0 & 1 \\ 2 & 1 \end{pmatrix} \qquad B = \begin{pmatrix} 1 & -1 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

$3 \times 2$ $\qquad$ $2 \times 3$

$AB$ is $3 \times 3$ $\quad (2,2)^{th}$ entry $\begin{pmatrix} 0 & 1 \end{pmatrix}\begin{pmatrix} -1 \\ 1 \end{pmatrix} = 1$

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$$A = \begin{pmatrix} a_{11} & \dots & a_{1n} \\ a_{i1} & \dots & a_{in} \\ a_{m1} & & a_{mn} \end{pmatrix} \qquad B = \begin{pmatrix} b_{11} & b_{1k} & b_{1p} \\ \vdots & & \vdots \\ b_{n} & b_{nk} & b_{np} \end{pmatrix}$$

$m \times n$ $\qquad$ $n \times p$

'$AB$ is defined to be the $m \times p$ matrix. Such that entry in $(i, k)$ position:

$$(AB)_{ik} = (i^{th} \text{ Row } A) \cdot (k^{th} \text{ col. of } B)$$

$$(a_{i1}, \dots, a_{in}) \times \begin{pmatrix} b_{1k} \\ \vdots \\ b_{nk} \end{pmatrix} = \sum_{j=i}^{n} a_{ij} b_{jk} =$$

$$= a_{ij} b_{ik} + a_{i2} b_{2k} + \dots + a_{in} b_{nk}$$

$$\boxed{(AB)_{ik} = \sum_{j=1}^{n} a_{ij} b_{jk}}$$

$i$ = numb of row

$k$ = numb. of column

Example:

$$A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{pmatrix} \qquad B = \begin{pmatrix} 1 & -1 & 1 \\ -1 & 0 & -1 \end{pmatrix}$$

$3 \times 2$ $\qquad$ $2 \times 3$

$AB$ is $3 \times 3$

$$\begin{pmatrix} AB_{11} & AB_{12} & AB_{13} \\ AB_{21} & AB_{22} & AB_{33} \\ AB_{31} & AB_{32} & AB_{33} \end{pmatrix} = \begin{pmatrix} -1 & -1 & -1 \\ -1 & -3 & -1 \\ -1 & -5 & -1 \end{pmatrix}$$

$(AB)_{11} = (1, 2)\begin{pmatrix}1\\-1\end{pmatrix} = 1-2=-1$

$(AB)_{12} = (1, 2)\begin{pmatrix}-1\\0\end{pmatrix} = -1+0=-1$

$(AB)_{13} = (1, 2)\begin{pmatrix}1\\-1\end{pmatrix} = 1-2=-1$

$(AB)_{21} = (3, 4)\begin{pmatrix}1\\-1\end{pmatrix} = 3-4=-1$

$(AB)_{22} = (3, 4)\begin{pmatrix}-1\\0\end{pmatrix} = -3$

$(AB)_{23} = (3, 4)\begin{pmatrix}1\\-1\end{pmatrix} = -1$

$(AB)_{3\cancel{1}} = (5, 6)\begin{pmatrix}1\\-1\end{pmatrix} = -1$

$(AB)_{32} = -5$

$(AB)_{33} = -1$

2) $\underset{2\times 2}{BA} = \begin{pmatrix}3 & 4\\-6 & 8\end{pmatrix}$

$$\boxed{\begin{array}{ll} A = (a_{ij})_{\substack{1\le i\le m\\1\le j\le n}} & B = (b_{jk})_{\substack{1\le j\le N\\1\le k\le p}} \\ m\times n & N\times p\end{array}}$$

0) $AB$ is defined when $n=N$

1) ~~then~~ $AB$ is $m\times p$ matrix

$AB_{ik} = \sum_{j=1}^{n} a_{ij} b_{jk} = a_{i1} b_{1k} + a_{i2} b_{2k} + \dots a_{in} b_{nk}$

2) $BA$ is defined, when $m = p$

3) Even if both $AB$, $BA$ are defined they may be different sizes.

In particular $AB$, $BA$ are same size if $m = n = p$ i.e. both matrices are square.

4) Even if $A, B$ both square (say $n \times n$) so $AB$, $BA$ both defined and $n \times n$. However in general $AB \neq BA$.

$$\underset{A}{\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}} \quad \underset{B}{\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}} \qquad AB = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \quad BA = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

$\Rightarrow$ $AB = BA$ is rare occurrence

$$\mathcal{S} = \begin{cases} a_{11} x_1 + \dots + a_{1n} x_n = b_1 \\ \vdots \\ a_{m1} x_1 + \dots + a_{mn} x_n = b_m \end{cases}$$

$A\underline{x} = \underline{b}$ - single matrix equation

Example

1) $X = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \qquad X^2 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

2) Zero matrix

For any $m, n$ there is a $m \times n$ matrix $\underset{\sim}{0}$

$$\boxed{0_{ij} = 0}$$

3) Identity matrix

For each $n$ there is a special $n \times n$ matrix.

$$(I_n)_{ij} = \begin{cases} 1 & i=j \\ 0 & i \neq j \end{cases} = \delta_{ij}$$ - Kronecker Delta

$$A = \begin{pmatrix} a & b & c \\ d & e & f \end{pmatrix}$$

$$I_2 A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a & b & c \\ d & e & f \end{pmatrix} = \begin{pmatrix} a & b & c \\ d & e & f \end{pmatrix} = A$$

$$A I_3 = \begin{pmatrix} a & b & c \\ d & e & f \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} a & b & c \\ d & e & f \end{pmatrix} = A$$

Show of $A$ is $m \times n$

$$\boxed{\begin{aligned} I_m A &= A \\ A I_n &= A \end{aligned}}$$

Kronecker Delta - the function, denoted $\delta_{ij}$, of two variables $i$ & $j$, that takes the value 1 when $i=j$ and is zero otherwise

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$1201^*$ – exercise I

N2 $A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix}$

$P(n): A^n = \begin{pmatrix} 1 & n & f(n) \\ 0 & 1 & 2n \\ 0 & 0 & 1 \end{pmatrix}$

Prove $P(n)$ by induction

$P(1)$: $A = \begin{pmatrix} 1 & 1 & f(1) \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix}$

this is true ~~for~~ and $f(1) = 0$

Suppose $P(k)$ is true

$$A^k = \begin{pmatrix} 1 & n & f(k) \\ 0 & 1 & 2k \\ 0 & 0 & 1 \end{pmatrix}$$

then

$$A^{k+1} = \begin{pmatrix} 1 & k & f(k) \\ 0 & 1 & 2k \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & k+1 & f(k)+2k \\ 0 & 1 & 2k+2 \\ 0 & 0 & 1 \end{pmatrix}$$

is for req$^d$ form, where $f(k+1) = f(k) + 2k$

i.e. $P(k) \Rightarrow P(k+1)$

So by induction $P(n)$ holds $\forall\, n \geq 1$

$f(1) = 0$

$f(k+1) = f(k) + 2k \quad (k \geq 1)$

$f(2) = f(1) + 2 \times 1$

$f(3) = f(2) + 2 \times 2$

$f(4) = f(3) + 2 \times 3$

$2 \times 1 + \quad + 2 \times (n-1)$

$\Rightarrow f(n) = \cancel{f(3)} + \left( f(n-1) + f(n-2) + \dots f(3) + f(2) + f(1) + 2 \times (n-1) \right) =$

N3 $\quad A = \begin{pmatrix} 1 & \alpha & \gamma \\ 0 & 1 & \beta \\ 0 & 0 & 1 \end{pmatrix}$

## Exercise - 2

N1 $\quad E(i,j)_{rs} = \delta_{ir} \delta_{js}$

$\begin{array}{c c} & \begin{matrix} 1 & 2 & 3 \end{matrix} \\ \begin{matrix} 1 \\ 2 \\ 3 \end{matrix} & \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \end{array} \quad r, s = 1, 2, 3$

$a_{ij} = i + j = \begin{array}{c c} \begin{matrix} 1 & 2 \end{matrix} & \\ \begin{vmatrix} 2 & 3 \\ 3 & 4 \end{vmatrix} & \begin{matrix} 1 \\ 2 \end{matrix} \end{array}$

$E(1,2) E(2,3)$

$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

$i,j$ - entry of $E(1,2)\, E(2,3) = \sum_{k=1}^{3} E(1,2)_{ik} E(2,3)_{kj} =$

$= \sum_{k=1}^{3} \delta_{1i} \delta_{2k} \delta_{2k} \delta_{3j} = \delta_{1i} \delta_{3j} \sum_{k=1}^{3} \delta_{2k} \delta_{2k} = \delta_{1i} \delta_{3j} = (i,j)$ - entry of $E(1,3)$

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## Invertible matrices.

Let $A = a_{ij}$ $\begin{smallmatrix} 1\le i\le n \\ 1\le j\le n \end{smallmatrix}$ be $n \times n$ matrix

Say that $A$ is invertible
If there exists $n \times n$ matrix $B$

$$AB = I_n \quad \text{and} \quad BA = I_n$$

Beware: ~~A~~ $n\times m$ zero matrix may possibly ~~that~~ not be invertible

e.g. $A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $A \neq 0$ but $A$ is not invertible

$$\underset{A}{\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}} \underset{B}{\begin{pmatrix} a & b \\ c & d \end{pmatrix}} = \begin{pmatrix} c & d \\ 0 & 0 \end{pmatrix} \neq I_n \neq \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

утверждение

Proposition:
Let $A = (a_{ij})$ $\begin{smallmatrix} 1\le i\le m \\ 1\le j\le n \end{smallmatrix}$

Then $I_m A = A = A I_m$

Proof ~~eg~~ $X = (x_{rs})$ $Y = (y_{st})$

Recall $(XY)_{rt} = \sum_{s=1}^{max(s)} x_{rs} y_{st}$

Кронекер делта

$(I_n)_{jk} = (\delta_{jk})$; $(\delta_{jk}) = \begin{cases} 1 & j = k \\ 0 & j \neq k \end{cases}$

$$\underset{\begin{smallmatrix} 1\le i\le m \\ 1\le k\le n \end{smallmatrix}}{(AI)_{ik}} = \sum_{j=1}^{n} a_{ij} \delta_{jk}$$

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$$A = (a_{ij})$$

$$I_n = (\delta_{ij})$$

$$\delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

$$A\, I_n = \cancel{A(a_j)}\ A$$

$$(A I_n)_{ij} = \sum_{k=1}^{n} \underbrace{a_{ik}\, \delta_{kj}} = a_{ij}$$

unless $k = j$

& if $k = j$ then it's $a_{ij}$

$$\sum_{j=1}^{n} a_{ij}\,\delta_{jk} = a_{i1}\,\delta_{1k} + a_{i2}\,\delta_{2k} + \dots + a_{in}\,\delta_{nk}$$

all terms zero except $j = k$

$$= a_{ik}\,\delta_{kk}$$
$$= a_{ik}$$

$$(A I_n)_{ik} = a_{ik}$$

$A I_n = A$ Do $I_n A = A$ as exercise

Example

$$\begin{cases} x_1 + x_2 = 1 & [1] \\ x_1 - x_2 = 2 & [2] \end{cases}$$

1st operation: Allowed to add eq. $j$ to eq. $i$

$[1] + [2]$ :

$$\begin{cases} 2x_1 = 3 \\ x_1 - x_2 = 2 \end{cases}$$

2nd operation: $[1] \cdot \frac{1}{2}$ :

$$\begin{cases} x_1 = 1{,}5 & [1] \\ x_1 - x_2 = 2 & [2] \end{cases}$$

3rd operation : $-[1] + [2] = [1]$

$$\begin{cases} x_1 = 1{,}5 & [1] \\ -x_2 = 0{,}5 & [2] \end{cases}$$

4th op : $[2] \times -1$ :

$$\begin{cases} x_1 = 1{,}5 \\ x_2 = -0{,}5 \end{cases}$$

Ex. $I_n A = A$

$$(I_n A)_{ik} = \sum_{j=1}^{n} \delta_{ij} a_{jk} = a_{ik} \qquad \text{Q.E.D.}$$

We shall use 3 operation:

I $\mathcal{E}(i,j;\lambda)$ = operation which changes $i^{th}$ row,

$$\boxed{\text{New}(i^{th}\text{ Row}) = \text{old } i^{th}\text{ Row} + \lambda\, j^{th}\text{ Row}}$$

All other rows stay the same

II $\mathcal{D}(i,\lambda)$ multiplies $i^{th}$ Row by $\lambda$ $(\lambda \neq 0)$

$$\boxed{\text{New}(i^{th}\text{ Row}) = \text{old } i^{th}\text{ Row} \times \lambda}$$

All others rows stay the same.

III $\mathcal{P}(i,j)$ swaps $i^{th}$ Row, with $j^{th}$ Row

$$A = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & k \end{pmatrix}$$

$\mathcal{E}(3,2;\lambda)$ = adds $\lambda$ Row 2 to Row 3:

в одной пози-ции

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & \lambda & 1 \end{pmatrix}\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & k \end{pmatrix} = \begin{pmatrix} a & b & c \\ d & e & f \\ g+\lambda d & h+\lambda e & k+\lambda f \end{pmatrix}$$

$\parallel$

$E(3,2;\lambda)$

Le Color

## Basic Matrices

Pick n and fix

$\epsilon(i,j)$ is defined to be $n\times n$ matrix with 1 in position $(i,j)$, 0 ~~otherwise~~ elsewhere

the values at which should stop

~~column~~ row ~~column~~

Formally $\boxed{\epsilon(i,j)_{rs} = \delta_{ir}\,\delta_{js}} = \begin{cases} 1 & \overbrace{r=i}^{\text{row}} \ \& \ \overbrace{s=j}^{\text{column}} \\ 0 & \text{otherwise} \end{cases}$

running values

$n = 3$

$$\epsilon(1,1) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad \epsilon(2,3) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

$\epsilon(i,j)\ B \qquad B \ n\times p$

$$B = (b_{jk})_{\substack{1\le j\le n \\ 1\le k\le p}}$$

$$(\epsilon(i,j)B)_{rt} = \sum_{s=1}^{n} \epsilon(i,j)_{rs} B_{st} = \sum_{s=1}^{n} \delta_{ir}\,\delta_{js} B_{st} =$$

$$= \delta_{ir} B_{jt}$$

$\Rightarrow \epsilon(i,j)\ B$ is the matrix whose $i^{th}$ Row $= j^{th}$ of B all other rows $= 0$.

$[I_n + \lambda\,\epsilon(i,j)]\,B$

$I_n B + \lambda\,\epsilon(i,j)\,B$

$B + \lambda\,\epsilon(i,j)\,B$

$i^{th}$ Row = old $i^{th}$ Row + $\lambda\, j^{th}$ Row of B.

$I_n = \begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix}$ is $(I_n)_{ij} = \delta_{ij}$

matrix $\leftarrow$ $\boxed{E(i,j)}$

$E(i,j)_{rs} =$ s.t. if $i=r$ & $j=s$ then - it's one

$= \delta_{i,r}\,\delta_{sj}$

second filter.

How filters $\delta_{ir}\,\delta_{sj}$ works?

$E(i,j)_{rs} = \delta_{ir}\,\delta_{js} = \sum_{k=1}^{n} \delta_{ik}\,\delta_{kj}$

$$\begin{array}{cc} & \begin{array}{ccc} & 2 & \end{array} \\ 2 & \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \end{array} \cdot \begin{array}{c} \begin{array}{ccc} & 1 & \end{array} \\ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \end{array} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$\Rightarrow$ Due to $\delta$ - is not a matrix it is a function.

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Find general sol$^n$ of $n$:

$$\begin{aligned} x_1 + x_2 \qquad\quad - x_4 &= 2 \\ x_1 + x_2 + x_3 \qquad\quad &= 3 \\ 2x_1 + 2x_2 + x_3 - x_4 &= 5 \end{aligned}$$

① Writing in matrix form:

$$\begin{pmatrix} 1 & 1 & 0 & -1 \\ 1 & 1 & 1 & 0 \\ 2 & 2 & 1 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \\ 5 \end{pmatrix}$$

② Form "~~augmated~~ augmented" matrix:

augmented = расширенная

$$\left(\begin{array}{cccc|c} 1 & 1 & 0 & -1 & 2 \\ 1 & 1 & 1 & 0 & 3 \\ 2 & 2 & 1 & -1 & 5 \end{array}\right)$$

③ Now use 3 types of operations: $E(i, j; \lambda)$, $D(i, \lambda)$, $P(i, j)$:

$$\Rightarrow \left(\begin{array}{cccc|c} 1 & 1 & 0 & -1 & 2 \\ 1 & 1 & 1 & 0 & 3 \\ 2 & 2 & 1 & -1 & 5 \end{array}\right) \xrightarrow[E(3,1;-2)]{E(2,1;-1)} \left(\begin{array}{cccc|c} 1 & 1 & 0 & -1 & 2 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \end{array}\right) \rightarrow$$

$$\xrightarrow{E(3,2;-1)} \left(\begin{array}{cccc|c} 1 & 1 & 0 & -1 & 2 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

– reduced row Echelon form

$x_1 \quad x_2 \quad x_3 \quad x_4$

~~Wright~~ write out variables $x$, circle under leading 1s

④ Write ~~down~~ out reduced system

$$\begin{cases} x_1 + x_2 - \quad x_4 = 2 \\ \qquad\qquad x_3 + x_4 = 1 \end{cases}$$

$$\therefore x_1 = 2 - x_2 + x_4$$
$$x_2 = x_2$$
$$x_3 = 1 - x_4$$
$$x_4 = x_4$$

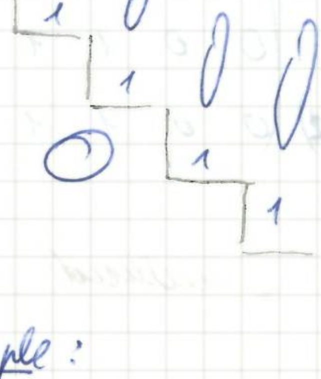
⑤ Eliminate circle variables:

$$\begin{pmatrix} 2 & -x_2 + x_4 \\ & x_2 \\ 1 & -x_4 \\ & x_4 \end{pmatrix}$$

Solution variables no restriction on $x_1$, $x_4$ - independent
Dependent variables - circled variables.

## Reduced Row Echelon

General Shape



Example:

$$\begin{matrix} 1 & 2 & 0 & 4 & -1 & 0 \\ 0 & 0 & 1 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{matrix}$$

## Examples of not echelon

$$\cancel{\begin{matrix}1&2&0&4&-1&0\\0&0&1&3\end{matrix}} \quad \begin{pmatrix}1&2&0\\0&0&1&\dots\\0&0&0\end{pmatrix}$$

$$\begin{pmatrix}1&2&0&4&-1&0\\0&0&0&0&0&1\\0&0&1&3&0&0\end{pmatrix}$$

e.g. $\begin{cases}x_1 - x_2 + x_3 + 2x_5 = -1\\ x_1 + x_2 + 3x_3 = 3\\ x_1 + x_2 + 3x_3 + x_4 + x_5 = 6\end{cases}$

$$\Rightarrow \left(\begin{array}{ccccc|c}1&-1&1&0&2&-1\\1&1&3&0&0&3\\1&1&3&1&1&6\end{array}\right) \xrightarrow[E(3,1;-1)]{E(2,1;-1)} \left(\begin{array}{ccccc|c}1&-1&1&0&2&-1\\0&2&2&0&-2&4\\0&2&2&1&-1&7\end{array}\right) \longrightarrow$$

$$\xrightarrow[D(2,\frac{1}{2})]{E(3,2;-1)} \left(\begin{array}{ccccc|c}1&-1&1&0&2&-1\\0&1&1&0&-1&2\\0&0&0&1&1&3\end{array}\right) \xrightarrow{E(1,2;1)} \left(\begin{array}{ccccc|c}1&0&2&0&1&1\\0&1&1&0&-1&2\\0&0&0&1&1&3\end{array}\right)$$

Reduced system:

$$x_1 + 2x_3 + x_5 = 1$$
$$x_2 + x_3 - x_5 = 2$$
$$x_4 + x_5 = 3$$

General sol$^n$: $\qquad x_3, x_5$ – ordinary

$$\begin{matrix}1 & -2x_3 - x_5\\ 2 & -x_3 + x_5\\ & x_3\\ 3 & -x_5\\ & x_5\end{matrix}$$

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$$\begin{cases} x_1 - x_2 + x_3 + \quad + x_5 + 4x_6 = 2 \\ x_1 - x_2 + x_3 \quad\quad + 5x_6 = 0 \\ x_1 - x_2 + \quad + x_4 \quad + 2x_6 = 1 \end{cases}$$

Write in augmented matrix form:

$$\left(\begin{array}{cccccc|c} 1 & -1 & 1 & 0 & 1 & 4 & 2 \\ 1 & -1 & 1 & 0 & 0 & 5 & 0 \\ 1 & -1 & 0 & 1 & 0 & 2 & 1 \end{array}\right) \xrightarrow[\mathcal{E}(3,1;-1)]{\mathcal{E}(2,1;-1)}$$

Play the game! Reduce to Row Echelon form.

$$\left(\begin{array}{cccccc|c} 1 & -1 & 1 & 0 & 1 & 4 & 2 \\ 0 & 0 & 0 & 0 & -1 & 1 & -2 \\ 0 & 0 & -1 & 1 & -1 & -2 & -1 \end{array}\right) \xrightarrow{P(2,3)}$$

$$\left(\begin{array}{cccccc|c} 1 & -1 & 1 & 0 & 1 & 4 & 2 \\ 0 & 0 & -1 & 1 & -1 & -2 & -1 \\ 0 & 0 & 0 & 0 & -1 & 1 & -2 \end{array}\right) \xrightarrow[\mathcal{E}(1,2;-1)]{\mathcal{D}(2,-1)}$$

$$\left(\begin{array}{cccccc|c} 1 & -1 & 0 & 1 & 0 & 2 & 1 \\ 0 & 0 & 1 & -1 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 & -1 & 1 & -2 \end{array}\right) \xrightarrow[\mathcal{D}(3,-1)]{\mathcal{E}(2,3;1)}$$

$$\left(\begin{array}{cccccc|c} 1 & -1 & 0 & 1 & 0 & 2 & 1 \\ 0 & 0 & 1 & -1 & 0 & 3 & -1 \\ 0 & 0 & 0 & 0 & 1 & -1 & 2 \end{array}\right)$$

$x_1$ $x_2$ $x_3$ $x_4$ $x_5$ $x_6$ - circle variables under leading 1s

Now write reduced system & eliminate circled variables:

$$\begin{cases} x_1 - x_2 \quad + x_4 \quad + 2x_6 = 1 \\ \quad x_3 - x_4 \quad \cancel{3}x_5 = -1 \\ \quad\quad x_5 - x_6 = 2 \end{cases}$$

Write down in a solution vector.

$$\begin{pmatrix} 1 + x_2 - x_4 - 2x_6 \\ x_2 \\ -1 + x_4 - 5x_6 \\ x_4 \\ 2 + x_6 \\ x_6 \end{pmatrix}$$

here $x_2, x_4, x_6$ independent
No restrictions on them.
$\infty$ many solutions

# Basic Matrices

Fix $n \geq 2$

Kronecker $\quad \epsilon(i,j)$ informally is the $n \times n$ matrix having 1 in $(i,j)^{th}$ position and 0 elsewhere

$\delta_{ab} = \begin{cases} 1 & a=b \\ 0 & a \neq b \end{cases}$

$$\boxed{\epsilon(i,j)_{rs} = \delta_{ir}\delta_{js}}$$

$\delta_{ir}\delta_{js} = \begin{cases} 1 & r=i \;\&\; s=j \\ 0 & \text{otherwise} \end{cases}$

e.g. $\epsilon(2,1) = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$, $\epsilon(1,2) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$

$$\boxed{\epsilon(i,j)\cdot\epsilon(k,l) = \begin{cases} \epsilon(i,l) & j=k \\ 0 & j \neq k \end{cases}}$$

e.g. $n=3$

$$\epsilon(1,2)\,\epsilon(2,3) = \epsilon(1,3)$$

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$[\epsilon(1,2)\,\epsilon(2,3)]_{rt} = \sum_{s=1}^{3} \epsilon(1,2)_{rs}\,\epsilon(2,3)_{st} = \sum_{s=1}^{3} \delta_{1r}\delta_{2s}\delta_{2s}\delta_{3t}$$

$\delta_{22} = 1$

$\delta_{2s} = 0 \quad s \neq 2$

$$= \delta_{1r}\, 1 \cdot 1 \cdot \delta_{3t} = \epsilon(1,3)$$

what does it mean if matrix has fixed row/column?

$\exists\ \delta_{is}$ if $i$ is not changing

$\Rightarrow$ row which changes from $s=1$ to $s=n$

по определению (стр. 8') $\delta_{is}$ - функция, а не матрица

---

Ex.

Let $A = \cancel{\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & g \end{pmatrix}}$ $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{pmatrix} \cancel{\begin{matrix} a_{13} & a_{14} \\ a_{23} & a_{24} \\ a_{33} & a_{34} \end{matrix}}$

$3\times 2$

$\underset{3\times3}{E}(2,3) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} =$

$(E(2,3)A)_{rt} = \sum_{s=1}^{3} E(2,3)_{rs} a_{st} = \sum_{s=1}^{3} \delta_{2r}\delta_{3s} a_{st} =$

$1 \le r \le 3,\ 1 \le t \le 2$

$= \qquad = \delta_{2r} a_{3t} = \begin{cases} a_{3t} & 2=r \\ 0 & 2\ne r \end{cases}$

m.e. $\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ a_{31} & a_{32} \\ 0 & 0 \end{pmatrix}$

16/10/09

$\epsilon(i,j)_{rs} = \delta_{ir}\delta_{js}$ $\quad m \times m$

Let $A$ be an $m \times n$ matrix

$$A = (a_{st})_{\substack{1 \le s \le m \\ 1 \le t \le n}}$$

What is $\epsilon(i,j)A$? $\quad m \times n$

Calculate $\displaystyle [\epsilon(i,j)A]_{rt} = \sum_{s=1}^{m} \epsilon(i,j)_{rs}\, a_{st}$

$$\sum_{s=1}^{m} \delta_{ir}\delta_{js}\, a_{st} = \delta_{ir}\delta_{jj}\, a_{jt} \ (+\text{ zeros})$$

$$\cancel{= \delta_{ir}\delta_{jt}\, a}$$

$$\therefore \quad \boxed{[\epsilon(i,j)A]_{rt} = \delta_{ir}\, a_{jt} = \begin{cases} a_{jt} & r = i \\ 0 & r \ne i \end{cases}}$$

$i^{th}$ Row of $\epsilon(i,j)A = j^{th}$ Row of $A$
all other rows of $\epsilon(i,j)A$ are $0$

e.g. $n = 4 \quad m = 3 \quad \epsilon(2,3)A$

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} a & b & c & d \\ e & f & g & h \\ k & l & m & n \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ k & l & m & n \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

**Corollary**

$i^{th}$ Row of $\lambda\,\epsilon(i,j)A = \lambda\, j^{th}$ Row of $A$, all other rows ~~are fixed~~ of $\lambda\,\epsilon(i,j)A = 0$

~~Coll~~ Corollary

$(A + \lambda \epsilon(i,j) A)$ is precise ~~(precisely)~~ the matrix obtained by adding $\lambda$ Row $j$ to Row $i$ leaving all other rows fixed.

Corollary

$[I_m + \lambda \epsilon(i,j)] A$ is matrix obtained by applying $\mathcal{E}(i,j;\lambda)$ to $A$

Def$^n$ $\boxed{E(i,j;\lambda) = I_m + \lambda\epsilon(i,j)}$ , where $i \neq j$

So $m=3$ $E(2,3;\lambda) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & \lambda \\ 0 & 0 & 1 \end{pmatrix}$

So eg $m=3$ $E(3;1;\mu) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \mu & 0 & 1 \end{pmatrix}$

Example $E(2,3;\lambda)A$ $A = \begin{pmatrix} a & b & c & d \\ e & f & g & h \\ k & l & m & n \end{pmatrix}$

$[I + \lambda\epsilon(2,3)]A = A + \lambda\epsilon(2,3)A$

$$\begin{pmatrix} a & b & c & d \\ e & f & g & h \\ k & l & m & n \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 & 0 \\ \lambda k & \lambda l & \lambda m & \lambda n \\ 0 & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} a & b & c & d \\ e+\lambda k & f+\lambda l & g+\lambda m & h+\lambda n \\ k & l & m & n \end{pmatrix}$$

[$E(i,j;\lambda)$ is invertible]

$\boxed{E(i,j,\lambda)^{-1} = E(i,j,-\lambda)}$ $i \neq j$

$(I - \lambda\epsilon(i,j))(I + \lambda\epsilon(i,j)) = \cancel{\lambda\epsilon(i,j)} - \lambda^2\epsilon(i,j)\epsilon(i,j) = I - \lambda^2\epsilon(i,j)\epsilon(i,j)$

$(I + \lambda\epsilon(i,j))$

Q Does $\epsilon(i,j)\,\epsilon(i,j) = 0$?

$\epsilon(i,j)\,\epsilon(i,j) =$

e.g. $\begin{pmatrix}1&0&0\\0&0&0\\0&0&0\end{pmatrix}\begin{pmatrix}1&0&0\\0&0&0\\0&0&0\end{pmatrix}=\begin{pmatrix}1&0&0\\0&0&0\\0&0&0\end{pmatrix}$ $\quad i=j$

$\begin{pmatrix}0&1&0\\0&0&0\\0&0&0\end{pmatrix}\begin{pmatrix}0&1&0\\0&0&0\\0&0&0\end{pmatrix}=\begin{pmatrix}0&0&0\\0&0&0\\0&0&0\end{pmatrix}$ $\quad i\neq j$

But $\epsilon(i,j)\,\epsilon(i,j) = 0$ since $i \neq j$

So $\big(I + x\,\epsilon(i,j)\big)\big(I - x\,\epsilon(i,j)\big) = I$

give the same result

$E(i,j,\lambda) \sim E(i,j,\lambda) = I + x\,\epsilon(i,j)$ ← matrix /expression

Dewendue

$\Delta(i,x) \sim$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} a & b & c & d \\ e & f & g & h \\ k & l & m & n \end{pmatrix} = \begin{pmatrix} a & b & c & d \\ \lambda e & \lambda f & \lambda g & \lambda h \\ k & l & m & n \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ 0 & (\lambda-1) & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Def$^n$ $\boxed{\Delta(i,\lambda) = I + (\lambda-1)\,\epsilon(i,j)}$ , here $i = j$

Prop. $\Delta(i,\lambda)A$ is the matrix obtained by multiplying ($i^{th}$ Row of $A$) by $\lambda$, all other rows stay the same.

Proof $\{I + (\lambda-1)\,\epsilon(i,i)\}A = A + (\lambda-1)\,\epsilon(i,i)A$

$\epsilon(i,j)A$ is matrix where $i^{th}$ Row $= j^{th}$ Row of $A$ all other rows are zero.

For $j \neq i$ $j^{th}$ Row of $[I + (\lambda-1)\epsilon(i,i)]A = j^{th}$ Row of $A$

$i^{th}$ Row of $[I + (\lambda-1)\epsilon(i,i)]A =$

$= i^{th}$ Row of $A + (\lambda-1)\, i^{th}$ Row of $A$

$= \lambda\; i^{th}$ Row of $A$

Q.E.D

**Prop**

If $\lambda \neq 0$ $\Delta(i,\lambda)$ is invertible and $\boxed{\Delta(i,\lambda)^{-1} = \Delta(i,\frac{1}{\lambda})}$

**Proof**

$$\Delta(i,\lambda)\Delta(i,\tfrac{1}{\lambda})$$

$$= \left[I + (\lambda - 1)\,\epsilon(i,i)\right]\left[I + \left(\tfrac{1}{\lambda} - 1\right)\epsilon(i,i)\right] =$$

$$= I + (\lambda-1)\,\epsilon(i,i) + \frac{1-\lambda}{\lambda}\,\epsilon(i,i) + \frac{(\lambda-1)(1-\lambda)}{\lambda}\,\epsilon(i,i)^2$$

**But** $\epsilon(i,i)^2 = \epsilon(i,i)$

$(1-\lambda)^2 =$
$= 1 - 2\lambda + \lambda^2$

$$= I + \left[(\lambda - 1) + \frac{(1-\lambda)}{\lambda} + - \frac{(1-\lambda)^2}{\lambda}\right]\epsilon(i,i) =$$

$-\frac{(1-\lambda)^2}{\lambda} =$

$$= I + \Big[\underbrace{(\cancel{\lambda} - \cancel{1}) + (\cancel{\tfrac{1}{\lambda}} - \cancel{1}) + (\cancel{2} - \cancel{\tfrac{1}{\lambda}} - \cancel{\lambda})}_{=0}\Big]\,\epsilon(i,i) = I$$

$= 2 - \frac{1}{\lambda} - \lambda$

$P(i,j)$ swaps Row $i$ & $j$
if $P(i,j)$ realizes $\mathcal{P}(i,j)$, then $P(i,j)$ is obtained from $I$ by applying $\mathcal{P}(i,j)$

$n = 4$

$\mathcal{P}(2,3)$ should be $\begin{pmatrix} 1&0&0&0\\ 0&0&1&0\\ 0&1&0&0\\ 0&0&0&1 \end{pmatrix}$

$$\begin{pmatrix} 1&0&0&0\\ 0&0&1&0\\ 0&1&0&0\\ 0&0&0&1 \end{pmatrix}\begin{pmatrix} a&b\\ c&d\\ e&f\\ g&h \end{pmatrix} = \begin{pmatrix} a&b\\ e&f\\ c&d\\ g&h \end{pmatrix}$$

**H/W** ~~Explore~~ How to express $P(i,j)$ in term of $\epsilon(i,j)$?

Express $\mathcal{P}(i,j)$ in terms of $\epsilon(i,j)$

{ Exploring $\mathcal{P}(i,j)$ }

$$\mathcal{P}(1,2) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} + \begin{pmatrix} -1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = 1$$

$$\mathcal{P}(i,j) = I - \epsilon(i,i) - \epsilon(j,j) + \epsilon(i,j) + \epsilon(j,i)$$

$$\boxed{\mathcal{P}(i,j) = I - \epsilon(i,i) - \epsilon(j,j) + \epsilon(i,j) + \epsilon(j,i)}$$

20.10.09.

$P(i,j)$ $m \times m$ matrix s.t. $P(i,j)A$ is obtained from $A$ by $P(i,j)$

$P(i,j)$ has to be the matrix obtained by swapping $i^{th}$ and $j^{th}$ rows of $I_m$

$P(i,j)\, I_m = P(i,j)$

$m=4$ $P(1,3)$ $I_m$ operating in $I_4$:

$$\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{bmatrix} \times P(1,3) = \begin{bmatrix}0&0&1&0\\0&1&0&0\\1&0&0&0\\0&0&0&1\end{bmatrix}$$

Prop $\boxed{P(i,j) = I_m - \epsilon(i,i) - \epsilon(j,j) + \epsilon(i,j) + \epsilon(j,i)}$

$$\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{bmatrix} - \begin{bmatrix}1&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0\end{bmatrix} - \begin{bmatrix}0&0&0&0\\0&0&0&0\\0&0&1&0\\0&0&0&0\end{bmatrix} + \begin{bmatrix}0&0&1&0\\0&0&0&0\\0&0&0&0\\0&0&0&0\end{bmatrix} + \begin{bmatrix}0&0&0&0\\0&0&0&0\\1&0&0&0\\0&0&0&0\end{bmatrix}$$

Prop $\boxed{P(i,j)^{-1} = P(i,j)}$

Proof Need to show $P(i,j)\,P(i,j) = I$

NB! $\epsilon(i,i)^2 = \epsilon(i,i)$

$$\left[I - \epsilon(i,i) - \epsilon(j,j) + \epsilon(i,j) + \epsilon(j,i)\right]^2 = I$$

$\epsilon(i,j)\,\epsilon(k,l) = \epsilon(i,l)$ if $k=j$

$$\begin{aligned}
&I - \epsilon(i,i) - \epsilon(j,j) + \epsilon(i,j) + \epsilon(j,i)\\
&- \epsilon(i,i) + \epsilon(j,j) + \epsilon(i,j) - \epsilon(j,i)\\
&+ \epsilon(i,i) - \epsilon(j,j) - \epsilon(i,j) + \epsilon(j,i)\\
&+ \epsilon(i,i) + \epsilon(j,j) - \epsilon(i,j) - \epsilon(j,i)\\
\hline
&I \qquad \cancel{\epsilon(i,j)}
\end{aligned}$$

Paradox of $\epsilon(i,j)\cdot\epsilon(k,l)$

To theory: 1. $\epsilon(i,j)\cdot\epsilon(k,l) = \begin{cases}\epsilon(i,l) & \text{if } j=k \\ 0 & \text{if } j\neq k\end{cases}$

$$2.\ \epsilon(i,i)\cdot\epsilon(i,i) = \epsilon(i,i)$$

NB! $\epsilon(i,j)$ - is a matrix not a function

## Q How to find $A^{-1}$ in practise?

Suppose 1. $A$ is $m\times m$, $B$ is also $m\times m$

2. $B\cdot A$ ; $BI_m = B$

So $B(A|I_m) = (BA|B)$

So if $BA = I_m$ then $B = A^{-1}$

$(A|I_m) \longrightarrow (I_m|A^{-1})$, $A^{-1} = B$

т.к. мы говорили что A - квадратная матрица ⇒ можно утверждать if $BA=I$ then $AB=I$

**Example** $A = \begin{bmatrix} 0 & 0 & 1 \\ 0 & -1 & 2 \\ 1 & 1 & 0 \end{bmatrix}$ to find $A^{-1}$

**step I** From the $m\times 2m$ matrix $(A|I_m)$

here $\left(\begin{array}{ccc|ccc} 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & -1 & 2 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 \end{array}\right) \longrightarrow$

**step II** Reduce to Row Echelon Form

$\xrightarrow{P(1,3)} \left(\begin{array}{ccc|ccc} 1 & 1 & 0 & 0 & 0 & 1 \\ 0 & -1 & 2 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \end{array}\right) \xrightarrow{E(2,3;-2)} \left(\begin{array}{ccc|ccc} 1 & 1 & 0 & 0 & 0 & 1 \\ 0 & -1 & 0 & -2 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \end{array}\right) \xrightarrow[D(2,-1)]{E(1,2;1)} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -2 & 1 & 1 \\ 0 & 1 & 0 & 2 & -1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \end{array}\right)$ $(I_3|B)$, $A^{-1}$

**[step III]** check

$\begin{pmatrix} 0 & 0 & 1 \\ 0 & -1 & 2 \\ 1 & 1 & 0 \end{pmatrix}\begin{pmatrix} -2 & 1 & 1 \\ 2 & -1 & 0 \\ 1 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \Rightarrow$ correct

Could be more than one variant of B-?

Again $A = \begin{pmatrix} 0 & 0 & 1 \\ 0 & -1 & 2 \\ 1 & 1 & 0 \end{pmatrix}$ then left multiplied

NB! $\cancel{(xy)^{-1} = x^{-1}y^{-1}}$

by $D(2,-1)E(1,2,1)E(2,3,-2)P(1,3) = A^{-1}$

**Prop:** If $x, y$ are invertible $m\times m$ matrix, then $\boxed{(xy)^{-1} = y^{-1}\cdot x^{-1}}$

**Proof:** $(xy)y^{-1}x^{-1} = x(yy^{-1})x^{-1} = xI_nx^{-1} = xx^{-1} = I$

$(y^{-1}x^{-1})xy = y^{-1}Iy = y^{-1}y = I$ Q.E.D.

22. Mär

So $A = P(1,3)^{-1}\, E(2,3,-2)^{-1}\, E(1,2,1)^{-1}\, \Delta(2,-1)^{-1}$

$\Rightarrow A = P(1,3)\, E(2,3,2)\, E(1,2,-1)\, \Delta(2,-1)$

check

$$\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 2 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & -1 & 2 \\ 1 & 1 & 0 \end{pmatrix}$$

$P(3,1) \quad E(2,3,2) \quad E(1,2,-1) \quad \Delta(2,-1)$

22.10.08.

# FORMAL LOGIC

## Proposition* Logic.

Proposition* = утверждение

not, and, or, implies

[Пропозиционная логика = оперируем высказываниями]

Simple proposition
"It is cold"
"It is raining"

} capable of being only true – T or false – F.

**not** negation $\neg$

$p$ = 'It is raining' ; $\Rightarrow$ $\neg p$ = 'It's not raining'

| $p$ | $\neg p$ | $\neg\neg p$ |
|---|---|---|
| T | F | T |
| F | T | F |

$\Rightarrow p = \neg(\neg p)$

**and** conjunction $\wedge$

$p$ = 'It is cold'
$q$ = 'It is raining'

| $p$ | $q$ | $p \wedge q$ = '$p$ and $q$' |
|---|---|---|
| T | T | T |
| | F | F |
| F | T | F |
| | F | F |

inclusive or
in Latin, 'vel'

**or** disjunction $\vee$

'$p \vee q$' means either $p$ or $q$, possibly both

| $p$ | $q$ | $p \vee q$ |
|---|---|---|
| T | T | T |
| | F | T |
| F | T | T |
| | F | F |

statement = утверждение

utterance = высказывание

**Def$^n$** Proposition - it is a statement* which can be either true or false.

exclusive or — In latin means 'aut'

xor no name no symbol $\oplus$

| $p$ | $q$ | $p \veebar q$ |
|---|---|---|
| T | T | F |
| | F | T |
| F | T | T |
| | F | F |

Implies $\Rightarrow$

| $p$ | $q$ | $p \Rightarrow q$ |
|---|---|---|
| T | T | T |
| | F | F |
| F | T | T |
| | F | T |

I would say it is not exactly true, but 'not contradicted'.

Q Why is true the truth table of $\Rightarrow$?
Go back to $\Gamma$

| $p$ | $\neg p$ | $p \vee (\neg p)$ |
|---|---|---|
| T | F | T |
| F | T | T |

$\Rightarrow$ $p \vee (\neg p)$ is always true

'$p \Rightarrow q$' is meant to symbolyse 'if $p$ then $q$'
start by noting $(\neg p) \vee p$ is true
~~$(\neg p) \vee p$~~ but if $p$ then $q$

if p happens then q happens

So ~~Now~~ replace $p$ by $q$

$(\neg p) \vee q \equiv$ '$p \Rightarrow q$'

| $p$ | $q$ | $\neg p \vee q$ | $\neg p$ |
|---|---|---|---|
| T | T | T | F |
| | F | F | F |
| F | T | T | T |
| | F | T | T |

So take the def$^n$ of '$\Rightarrow$' to be

Def$^n$ $\boxed{\text{'}p \Rightarrow q\text{'} \equiv \text{'}(\neg p) \vee q\text{'}}$ $\rightarrow$ ?

25 Example 1) $p \wedge q \Rightarrow q$

| $p$ | $q$ | $p \wedge q \Rightarrow q$ | $p \wedge q$ |
|---|---|---|---|
| T | T | T | T |
| T | F | T | F |
| F | T | T | F |
| F | F | T | F |

Def^n A compound proposition* which only takes value T is called tautology

Compound proposition* = составное (сложное) утверждение.

e.g. $p \vee (\neg p)$ is tautology
$p \wedge q \Rightarrow p$ — " — ...

2) $p \vee q \Rightarrow q$ is not a tautology

| $p$ | $q$ | $p \vee q \Rightarrow q$ | $p \vee q$ |
|---|---|---|---|
| T | T | T | T |
| T | F | F | T |
| F | T | T | T |
| F | F | T | F |

contingent* = зависимый от обстоятельств.

Contingent proposition - prop. which takes values T or F.

3) Two compound statements are equivalent $\equiv$, when they take same truth values

e.g. 1) $p \wedge q \equiv q \wedge p$
2) $p \vee q \equiv q \vee p$

4) "Proof by Contradiction"

| $p$ | $q$ | $\neg q \Rightarrow \neg p$ | $\neg q$ | $\neg p$ | $p \Rightarrow q$ |
|---|---|---|---|---|---|
| T | T | T | F | F | T |
| | F | F | T | F | F |
| F | T | T | F | T | T |
| | F | T | T | T | T |

So '$\neg q \Rightarrow \neg p$' $\equiv$ '$p \Rightarrow q$' :
i.e. to show $p \Rightarrow q$ sometimes easier
to show $\neg q \Rightarrow \neg p$

(28) 23.10.09

Example {If [it rains] or [is cold] we shall not go to town}
[We are in town] : ⇒ therefore [it's not raining]

2) If [it rains] and [is cold] we — " —
We are in town : therefore it is not raining

Let: ~~Let~~

$p$: it rains
$q$: it is cold
$r$: we are in town

⇒ first example:
$$(p \vee q \Rightarrow \neg r) \wedge r \Rightarrow \neg p$$

⇒ second ex.:
$$(p \wedge q \Rightarrow \neg r) \wedge r \Longrightarrow \neg p$$

Truth T. (1)

| $p$ | $q$ | $r$ | $(p \vee q \Rightarrow \neg r) \wedge r \Rightarrow \neg p$ | $p \vee q$ | $p \vee q \Rightarrow \neg r$ | $(p \vee q \Rightarrow \neg r) \wedge r$ |
|---|---|---|---|---|---|---|
| T | T | T | T | T | F | F |
|  |  | F | T | T | T | F |
|  | F | T | T | T | F | F |
|  |  | F | T | T | T | F |
| F | T | T | T | T | F | F |
|  |  | F | T | T | T | F |
|  | F | T | T | F | T | T |
|  |  | F | T | F | T | F |

Le Color

(29) (ii)

| $p$ | $q$ | $r$ | $(p\wedge q \Rightarrow \neg r)\wedge r \Rightarrow \neg p$ | $p\wedge q$ | $\neg r$ | $p\wedge q \Rightarrow \neg r$ | $(p\wedge q \Rightarrow \neg r)\wedge r$ | $\neg p$ |
|---|---|---|---|---|---|---|---|---|
| T | T | T | T | T | F | F | F | F |
| | | F | T | T | T | T | F | |
| | F | T | F | F | F | T | T | |
| | | F | T | F | T | T | F | |
| F | T | T | T | F | F | T | T | T |
| | | F | T | F | T | T | F | |
| | F | T | T | F | F | T | T | |
| | | F | T | F | T | T | F | |

Equivalent statements

$(a+b)c = ac+bc$

<u>don't get:</u>

$(a\cdot b\cdot k)+c =$
$= (a+c)\cdot(b+c)\cdot(k+c)$

$$\begin{cases} p\wedge q \equiv q\wedge p \\ p\vee q \equiv q\vee p \end{cases}$$ commutative

$$\begin{cases} p\wedge(q\wedge r) \equiv (p\wedge q)\wedge r \\ p\vee(q\vee r) \equiv (p\vee q)\vee r \end{cases}$$ Associative

$$\begin{cases} (p\wedge q)\vee r \equiv (p\vee r)\wedge(q\vee r) \\ (p\vee q)\wedge r \equiv (p\wedge r)\vee(q\wedge r) \end{cases}$$ Distributive

| $p$ | $q$ | $\neg(p\wedge q)$ | $p\wedge q$ | $\neg p$ | $\neg q$ | $\neg p\vee\neg q$ |
|---|---|---|---|---|---|---|
| T | T | F | T | F | F | F |
| | F | T | F | F | T | T |
| F | T | T | F | T | F | T |
| | F | T | F | T | T | T |

$$\boxed{\begin{array}{l} \neg(p\wedge q) \equiv (\neg p)\vee(\neg q) \\ \neg(p\vee q) \equiv (\neg p)\wedge(\neg q) \end{array}}$$ Duality

$$\boxed{\neg q \Rightarrow \neg p \equiv p\Rightarrow q)}$$ – 1.

30 **Proof** $(\neg q \Rightarrow \neg p) \equiv p \Rightarrow q$

$q \Rightarrow p$ is the "converse" of $p \Rightarrow q$

$\neg q \Rightarrow \neg p$ is the "contrapositive" of $p \Rightarrow q$

So $(\neg q \Rightarrow \neg p) \equiv p \Rightarrow q$

$(q \Rightarrow p) \not\equiv p \Rightarrow q$

Used four signs: $\neg, \wedge, \vee, \Rightarrow$.

I can eliminate $\wedge$, by $\neg(p \wedge q) \equiv (\neg p) \vee (\neg q)$

I can eliminate $\vee$, by $p \vee q \equiv \neg p \Rightarrow q$

**Define**

| p | q | p\|q |
|---|---|---|
| T | T | F |
|   | F | T |
| F | T | T |
|   | F | T |

Just need to use one sign

Sheffer's Stroke Function 1900.

used and designed for use in computers.

$$\boxed{p|q \equiv \neg(p \wedge q)}$$

$$\neg(p \wedge q) \equiv (\neg p) \vee (\neg q)$$

$$p \vee q \equiv \neg p \Rightarrow q$$

$$\neg p \vee q \equiv p \Rightarrow q$$

$$p \wedge (q \wedge r) = (p \wedge q) \wedge p \wedge r$$

$$p \vee (q \vee r) = (p \vee r) \vee (p \vee q)$$

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$p \Rightarrow q \equiv (\neg p) \vee q$

$\neg p \Rightarrow q \equiv \neg\neg p \vee q \qquad \neg\neg p \equiv p$

So $[p \vee q \equiv (\neg p) \Rightarrow q]$

Likewise:

$\neg(p \wedge q) \equiv \neg p \vee \neg q$

$\neg\neg(p \wedge q) = \neg(\neg p \vee \neg q)$

So $[p \wedge q \equiv \neg(\neg p \vee \neg q) \equiv \neg(p \Rightarrow \neg q)]$

So can eliminate $\vee, \wedge$

$p \vee q \equiv (\neg p) \Rightarrow q$

$p \wedge q \equiv \neg(p \Rightarrow \neg q)$

| $p$ | $q$ | $p \mid q$ | $p \mid (p \mid q)$ |
|---|---|---|---|
| T | T | F | T |
|  | F | T | F |
| F | T | T | T |
|  | F | T | T |

'$\mid$' = sheffers stroke function.

$\boxed{p \mid q \equiv \neg(p \wedge q)}$

So $p \mid q \equiv \neg(p \wedge q)$

$p \mid (p \mid q) = \neg(p \wedge (p \mid q)) = \neg(p \wedge (\neg(p \wedge q))) =$

$= \neg p \wedge (p \wedge q) = \neg p \vee \neg(p \wedge q)$

## Logic of Variable Proposition (Predicate Calculus)

First attempt:

Suppose our Universe consists of two elements

$0, 1$.

~~consider~~ Consider statements about $0, 1$

e.g. $P(x) = {}'x = 0'$

So $P(0)$ is T, $P(1)$ is F

Still have $\neg, \wedge, \vee, \Rightarrow$

But in addition, get two extra ways of making statements.

Existential statement:

'There exists $x$ such that $P(x)$ is T'

Universal statement:

'For each $x$, $P(x)$ is T'

$P(0)$, $P(1)$

when talking about only two elements $0, 1$

Universal statement $P(0) \wedge P(1)$

Existential st. $P(0) \vee P(1)$

More ambitious:

Suppose $U$ now have three elements $0, 1, 2$

Universal st. $P(0) \wedge P(1) \wedge P(2)$

Exist. st. $P(0) \vee P(1) \vee P(2)$

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Mathematics!

Make statements about $\mathbb{N} = \{0, 1, 2, \dots, n, n+1, \dots\}$

snag*: get formula of $\infty$ length
$\left\{ \begin{array}{l} \text{Universal statement}: P(0) \wedge P(1) \wedge P(2) \wedge \dots P(n) \wedge P(n+1) \wedge \dots \\ \text{Exist. statement}: P(0) \vee P(1) \vee P(2) \vee \dots P(n) \vee P(n+1) \vee \dots \end{array} \right.$

snag: 1. zarbozdna 2. npenamabal

Two new signs:

$\forall$ Universal Quantifier*

Quantifier* = квантор

$\Rightarrow (\forall x)\ P(x) \sim$ '$P_x$ T for every $x$'

Thinks $\forall x\, P(x) \sim$ '$P(0) \wedge P(1) \wedge \dots \wedge P(n) \wedge P(n+1) \wedge \dots$'

$\exists$ Existential quantifier

$\Rightarrow (\exists x)\ P(x) \sim$ '$P(x)$ T for at least one $x$'

Thinks $(\exists x) P(x) \sim$ '$P(0) \vee P(1) \vee P(2) \vee \dots$'

task: Need to relate $\forall$, $\exists$ in terms of what we already know

Q How do they interact with $\neg$?

Back to universe $= \{0, 1\}$

$(\forall x) P(x) = P(0) \wedge P(1)$

$\neg (\forall x) P(x) \equiv \neg (P(0) \wedge P(1)) = \neg P(0) \vee \neg P(1) = \exists (x) \neg P(x)$

$\exists (x)\, Q(x) = Q(0) \vee Q(1) =$

$$\boxed{\begin{array}{l} \neg(\forall x) P \equiv (\exists x) \neg P \\ \neg(\exists x) P \equiv (\forall x) \neg P \end{array}}$$

Universe $= \{0, 1, 2\}$

$(\exists x) P(x) = P(0) \vee P(1) \vee P(2)$

$$\neg(\exists x)\, P(x) = \neg(P(0) \vee P(1) \vee P(2))$$
$$= \neg P(0) \vee \neg P(1) \vee \neg P(2)$$
$$= (\forall x)\, \neg P(x)$$

$\forall$ : For any

$\exists$ : There exists

$\rightarrow$ $\neg\forall$ is there exists

e.g. $\forall$ n integer

n is div by 2

$\exists$ n = 3 , which is not div by 3

$$\neg\{\forall n, P(n)\} = \{\exists n, \neg P(n)\}$$

квантор (кванторное слово от лат. quantum – сколько) – слово стоящее перед субъектом суждения (все, ни один, некоторые и т. д.) и указывающее, относится ли суждение ко всему объему понятия, выражающего субъект, или к его части.

Пример: „Ни один злой человек не бывает счастлив" – суждение

субъект(S) = „...злой человек..."

предикат(P) = понятие „счастлив"

связка = " не бывает

Кванторное слово = „ни один".

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ordered set

$a, b \quad (1,2) \neq (2,1)$

$a, b, c \quad (1,2,3) \neq (3,2,1)$

In a set (properly speaking) we disregard order and we signify this by using curly brackets.

$\{1,2\} = \{2,1\}$

$\{1,2,3\} = \{3,2,1\}$

Most primitive description of a set is to list elements

$A = \{1,3,5,7\} (= \{7,1,3,5\})$

$3 \in A$ is True

$\in$ belong to

$\subset$ ~~$\in$~~ subset

Let $A, B$ sets

Say that $B \subset A$ 'B is contained in A'

$\forall x \quad x \in B \Rightarrow x \in A$

$A = \{1,3,5,7\} \quad B = \{1,7\}$

then $B \subset A$. Don't confuse '$\in$' $\neq$ '$\subset$'

Example 1) $A = \{0, 1, \{0,2\}\}$

$\{0,1\} \subset A \Rightarrow$ true $\qquad 2 \in A \Rightarrow$ false

$\{0,2\} \subset A \Rightarrow$ false $\qquad \{0,2\} \in A \Rightarrow$ True

In order for $\{0,2\} \subset A$ to be true we have to have $2 \in A$

2) $A = \{0, 2, \{0, 1\}, \{0, 2\}, 3, \{3\}\}$

$\{0\} \subset A \Rightarrow$ true $\qquad \{0\} \in A \Rightarrow$ false

$\{0, 1\} \subset A \Rightarrow$ false $\qquad \{0, 1\} \in A \Rightarrow$ true

$\{0, 2\} \subset A \Rightarrow$ true $\qquad \{0, 2\} \in A \Rightarrow$ true

$\{0, 3\} \subset A \Rightarrow$ true $\qquad \{0, 3\} \in A \Rightarrow$ false

3) $\{0, 2, \{0, 1\}, \{0, 2\}, 3, \{3\}\} \neq$

$\neq \{0, 2, 0, 1, 0, 2, 3, 3\}$

$\|$

$\{0, 2, 1, 3\} = \{0, 1, 2, 3\}$

## Sets in Mathematics

$\mathbb{N} = \{0, 1, 2, \dots, n, n+1, \dots\}$

$\mathbb{Z} = \{0, 1, -1, 2, -2, \dots, n, -n, \dots\}$

$E = \{x \in \mathbb{N} : x = 2n \text{ for some } n\}$

// Here we described set by means of a property

// this is usual way to describe sets.

$A = \{x : x \text{ is left hand drive car}\}$

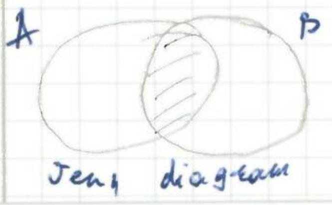
$\boxed{X = \{x : P(x)\}}$ - standard discription

typical element $\qquad$ defining property

e.g. $A = \{x : P_A(x)\}$ $\quad P_A$ defining a property of $A$

$B = \{x : P_B(x)\}$ $\quad P_B$ — " ——— $B$

$\boxed{Q \text{ when is } B \subset A? \text{ when } 'P_B(x) \Rightarrow P_A(x)'}$



$= A \cap B$ $\qquad$ Note: do not use Venn diagram!

Jenn diagram

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## Operations on Sets.

$A = \{x : P_A(x)\}$ $\quad$ $B = \{x : P_B\}$

$\boxed{A \cap B = \{ x : P_A(x) \wedge P_B\, x\}}$ - intersection of A & B

$\boxed{A \cup B = \{ x : P_A(x) \vee P_B(x) \}}$ - union

$\boxed{A - B = \{ x : P_A(x) \wedge (\neg P_B(x))\}}$ - complement

## Standard Rules for manipulating Sets (De Morgan's laws)

$$\begin{cases} A \cup B = B \cup A \\ A \cap B = B \cap A \end{cases}$$

$$\begin{cases} A \cup (B \cup C) = (A \cup B) \cup C \\ A \cap (B \cap C) = (A \cap B) \cap C \end{cases}$$

$$\begin{cases} A \cap (B \cup C) = (A \cap B) \cup (A \cap C) \\ A \cup (B \cap C) = (A \cup B) \cap (A \cup C) \end{cases}$$

$X - (A \cup B) = (X - A) \cap (X - B)$

$\{x : P_X(x) \wedge \neg (P_A(x) \vee P_B(x)) \}$

$P_X(x) \wedge (\neg P_A(x) \wedge \neg P_B(x))$

$[P_X(x) \wedge \neg P_A\, x] \wedge [P_X(x) \wedge \neg P_B\, x]$

$X - (A \cap B) = (X - A) \cup (X - B)$

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$A = \{a : P_A(a)\}$

set — braces only — typical element — Defining prop.

$A \cap B: \quad P_{A \cap B}(a) = P_A(a) \wedge P_B(a)$

$A \cup B: \quad P_{A \cup B}(a) = P_A(a) \vee P_B(a)$

$A - B: \quad P_{A-B}(a) = P_A(a) \wedge \neg P_B(a)$

Empty set

'$a \in \phi$' is always false

$A = \{1, 2\} \qquad B = \{3, 4\}$

$\phi = A \cap B$

## Product of Sets.

$(x, y)$ ordered pair

$(x,y) = (x', y')$ if $x = x'$ & $y = y'$

$\{x, y\}$ unordered pair

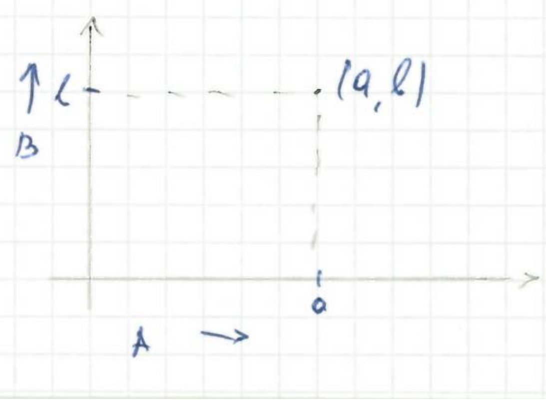
$(x,y) = \{x, \{x,y\}\}$ $\qquad \{\{x,y\}, \{x,y\}\}$

$A, B$ sets

$$\boxed{A \times B = \{(x,y) : x \in A \ \& \ y \in B\}}$$

formally

$$\boxed{P_{A \times B}(x,y) = P_A(x) \wedge P_B(y)}$$



e.g.

$A = \{x : x \in \mathbb{N}\}$

$B = \{x : x \in \mathbb{Z}\}$

$A \cap B = \{x : (x \in \mathbb{N}) \cap (x \in \mathbb{Z})\}$

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De Morgan's Laws:

$$\begin{cases} A \cap B = B \cap A \\ A \cup B = B \cup A \end{cases}$$ commutative

$$\begin{cases} (A \cap B) \cap C = A \cap (B \cap C) \\ (A \cup B) \cup C = A \cup (B \cup C) \end{cases}$$ associative

$$\begin{cases} A \cup (B \cap C) = (A \cup B) \cap (A \cup C) \\ A \cap (B \cup C) = (A \cap B) \cup (A \cap C) \end{cases}$$ Distributive

$$\begin{cases} A \cap A = A \\ A \cup A = A \end{cases}$$

$$\boxed{\begin{aligned} A - (B \cap C) &= (A - B) \cup (A - C) \\ A - (B \cup C) &= (A - B) \cap (A - C) \end{aligned}}$$ Complement

Proof: Complement law

Defining prop. of $A - (B \cup C)$

$$P_{A-(B \cup C)} = P_A(x) \wedge \neg (P_B(x) \vee P_C(x)) =$$

// $p \wedge \neg(q \vee r) = p \wedge (\neg q \wedge \neg r) = (p \wedge \neg q) \wedge (p \wedge \neg r) =$

$$= (P_A(x) \wedge \neg P_B(x)) \wedge (P_A(x) \wedge \neg P_C(x)) = P_{A-B}(x) \wedge P_{A-C}(x) =$$

$$= P_{(A-B) \cap (A-C)}(x)$$

But $P_{(A-B) \cap (A-C)} = P_{A-B}(x) \wedge P_{A-C}(x)$

So $P_{A-(B \cup C)}(x) \equiv P_{(A-B) \cap (A-C)}(x) =$

$$\therefore A - (B \cup C) = (A - B) \cap (A - C)$$

# Mappings / Functions

domain

codomain = values of function

**Def$^n$**: Mapping (or Function) $f$ set $A$ to set $B$ is a subset $f \subset A \times B$, that for any $a \in A$ there exists a single element $b \in B$ that $(a, b) \in f$.

$$\boxed{\begin{array}{l} f: A \to B \equiv f \subset A \times B \text{ that} \\ \text{(i) } \forall a \in A \ \exists b \in B \ (a,b) \in f \\ \text{(ii) } \forall (a,b) \in f \text{ and } \forall (a',b') \in f: \\ \qquad a = a' \Rightarrow b = b' \end{array}}$$

**Def$^n$** Injective mapping - is a mapping that for all elements in codomain there is no more then one element in domain that $(a, b) \in f$.

$$\boxed{\begin{array}{l} f: A \to B \text{ is injective} \Rightarrow \\ f(a) = f(a') \Rightarrow a = a' \end{array}}$$

**Def$^n$** Surjective mapping - is a mapping that for all elements in codomain there is at least one element in domain that $(a, b) \in f$.

$$\boxed{\begin{array}{l} f: A \to B \text{ is surjective} \Rightarrow \\ \forall b \in B \ \exists a \in A: f(a) = b \end{array}}$$

## Composition of Mapping

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[rr, bend right, "g\circ f(a)"'] \arrow[rrr, bend right=50, "h\circ g\circ f(a)"'] & B \arrow[r, "g"] & C \arrow[r, "h"] & D \end{tikzcd}$$

**Def$^n$** $\boxed{g\circ f(a) = g(f(a))}$

**Prop.** $\boxed{(h\circ g)\circ f = h\circ(g\circ f)}$ $\Rightarrow$ composition is associative

## Inverse Mapping

If $f \subset A\times B$ Define $f^{-1} \subset B\times A$
by $(b,a) \in f^{-1} \Leftrightarrow (a,b)\in f$

**Conclusion**: if $f^{-1}$ is mapping then $f$ is bijective
if $f$ is mapping then $f^{-1}$ is bijective

**Thm** Let $f: A \to B$ be a mapping
$f$ is bijective $\to$ $f^{-1}$ is bijective

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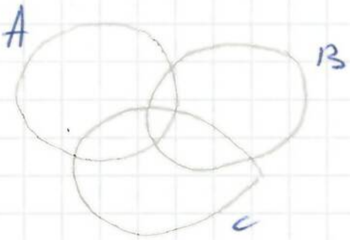
investigation

## Problem with Venn Diagrams




Conclusion $\Rightarrow$ Do not use it

Abel 1820 defined function

## Functions (or mappings*)

Let $A$, $B$ be sets.

Informal Def$^n$: A function $f$(~~$A \to B$~~) is a Rule which ~~using~~ assigns to each $a \in A$ a well defined element $f(a) \in B$ and writes $f: A \to B$, $b = f(a)$

mapping = function

mapping* = 1. соответствие (m) отображение

domain* = область

$A$ = domain* of function

$B$ = values of function (Codomain)

Example $A = B = \mathbb{Q}$

$f(a) = a + 2$

$f(x) = x + 2$

domain of function* область определения функции

Beware: A function is not a formula

i.e. $g(x) = \frac{x^2}{x-1}$ - is not a function.

Q How to make $g(x)$ into a function!

$g: \mathbb{R} - \{1\} \longrightarrow \mathbb{R}$

$g(x) = \frac{x^2}{x-1}$ is a function.

Domain of function - is the set of values of the independent variable for which a function is defined.

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$h(z) = \sqrt{z}$

$\cancel{k}: \mathbb{R}_+ \longrightarrow \mathbb{R}_+$

~~Show~~ Need to specify that by $\sqrt{z}$ mean $\sqrt{z} \geq 0$

$h : \mathbb{C} \to \mathbb{C}$ $\cancel{h}$

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Example Let $A$ be a set
Let $Id_A : A \to A$ Identity Mapping
be the mapping $Id_A(a) = a$

## Composition of Mapping

Example $A = B = C = \mathbb{R}$
Consider $f : \mathbb{R} \to \mathbb{R}$ $f(x) = x^2+1$
$g : \mathbb{R} \to \mathbb{R}$ $g(x) = \cos(x)$

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[rr, bend right, "g\circ f"'] & B \arrow[r, "g"] & C \end{tikzcd}$$

Def$^n$ $f : A \to B$ mapping
$g : B \to C$ mapping

$$\boxed{(g\circ f)(a) = g(f(a))}$$

$(g\circ f)(x) : \cos(x^2+1)$
$(f\circ g)(x) : \cos^2(x)+1 = |\cos(x)|^2+1$

Composition is always associative !

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[rr, bend right, "g\circ f"'] & B \arrow[r, "g"] \arrow[rr, bend left, "h\circ g"] & C \arrow[r, "h"] & D \end{tikzcd}$$

Prop. $$\boxed{(h\circ g)\circ f = h\circ(g\circ f)}$$

Proof $[(h\circ g)\circ f](a) = (h\circ g)(f(a)) = h(g(f(a)))$
$[h\circ(g\circ f)](a) = h\circ(g\circ f(a)) = h(g(f(a)))$

# Injective Mapping

**Def^n** $f: A \to B$ is injective when

$$\boxed{f(a) = f(a') \Rightarrow a = a'}$$

**Example** Consider

1) $f: \mathbb{R} \to \mathbb{R}$

$f(x) = 3x + 2$

$f$ is injective. why?

$f(x) = 3x + 2$

$f(x') = 3x' + 2$

if $f(x) = f(x')$

$\therefore 3x + 2 = 3x' + 2$

$3x = 3x'$

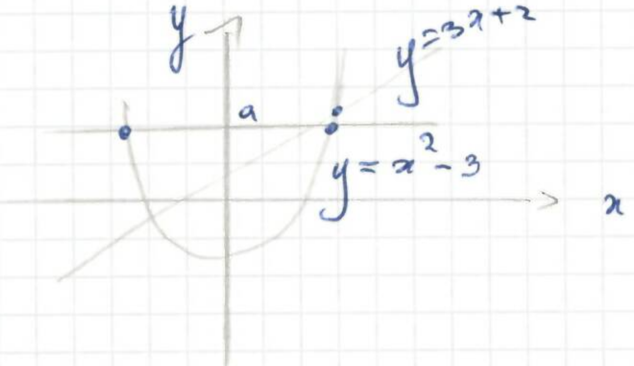
$x = x'$

2) $g: \mathbb{R} \to \mathbb{R}$

$g(x) = x^2$

$g$ is not injective. why?

$$\left.\begin{aligned} g(1) &= 1 \\ g(-1) &= 1 \end{aligned}\right\} \text{ so } g(1) = g(-1) \text{ (prop. of injection)} \text{ but } 1 \neq -1$$




$y = a$ crossing $y = x^2 - 3$ twice,

$y = 3x + 2$ once

$\Rightarrow y = 3x + 2$ is injective

**English** Function is injective when $\forall$ its value has one or none defined value in the domain.

## Surjective Mapping

**Def**$^n$ Let $f: A \to B$ be a mapping

$f$ is surjective, when

for each $b \in B$ there exists $a \in A$

such that $f(a) = b$

$$\boxed{\forall b \in B \;\; \exists a \in A : f(a) = b}$$

**Example:** 1) $f: \mathbb{R} \to \mathbb{R}$

$f(a) = 3a+2$

Is $f$ is surjective? Yes

Given $y \in \mathbb{R}$ I have to "hit" $y$ by $f$

I have to find $a$ ; $f(a) = y$

Take $a = \frac{y-2}{3}$ $\qquad 3a+2 = y$

2) $h: \mathbb{R} \to \mathbb{R}$

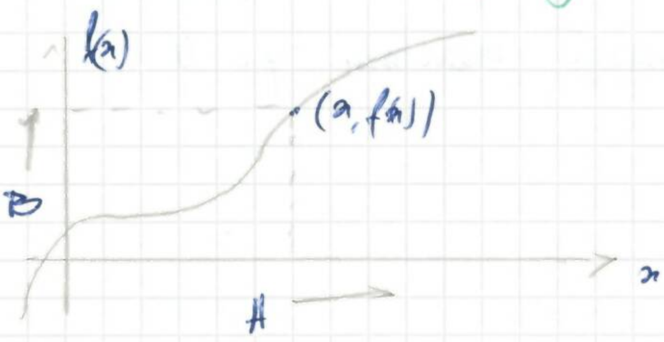
$h(a) = a^2+1$

Is $h$ surjective? No

How do I hit $0$?

**English** Function is surjective when $\forall$ its value has at least one defined value in domain.

# Definition of Mapping / Function




$(x, f(a)) \in A \times B$

**Def$^n$** Let $A, B$ be sets.
~~By a~~ mapping $f: A \to B$ is
I mean a subset $f \subset A \times B$, such that

(i) $\forall a \in A \ \exists b \in B \quad (a, b) \in f$ (write $b = f(a)$, when $(a, b) \in f$)
(ii) $\forall (a, b) \in f, \ \forall (a', b') \in f \qquad a = a' \Rightarrow b = b'$
i.e. $f(a)$ is a single element.

**Example** i) $f(x) = 3x + 2$
$f = \{(x, 3x+2), x \in \mathbb{R}\}$

(ii) $g(x) = x^2$
$g = \{(x, x^2), x \in \mathbb{R}\}$

$f$: injective $(a, b) \in f$ and $(a', b) \in f \Rightarrow a = a'$
$f$: surjective $\forall b \in B \ \exists a \in A \ (a, b) \in f$

~~Отображение~~ Функция F множества X в множество Y есть подмножество $F \subset X \times Y$, такое, что для любого $x \in X$ существует единственный элемент $y \in Y$, такой, что $(x, y) \in F$

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$f \subset A \times B$

Four conditions you need to be clear about

(i) $\forall a \in A \quad \exists b \in B \quad (a, b) \in f$

(ii) $(a, b) \in f$ and $(a, b') \in f \Rightarrow b = b'$

f is a mapping
sufficient 2 cond. both

(iii) $(a, b) \in f$ and $(a', b) \in f \Rightarrow a = a'$ - f is injective

(iv) $\forall b \in B \ \exists a \in A \ (a, b) \in f$ - f is surjective

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$A = B = \{1, 2\}$

$f : \{1, 2\} \to \{1, 2\}$

i. 1) $f(1) = 1$ $\quad 1 \to 1$ identity
2) $f(2) = 2$ $\quad 2 \to 2$ injective, surjective => Bijective

ii. 1) $f(1) = 1$ $\quad 1 \to 1$ $f(1) = f(2)$, $1 \neq 2$, not injective
2) $f(2) = 1$ $\quad 2 \to 1$ not surjective

iii 1) $f(1) = 2$ $\quad 1 \to 2$ $f(1) \neq f(2)$, $1 \neq 2$ and $f(1) \neq f(2)$
2) $f(2) = 1$ $\quad 2 \to 1$ Bijective

iv) 1) $f(1) = 2$ $\quad 1 \to 2$ not injective
2) $f(2) = 2$ $\quad 2 \to 2$ not surjective

**cardinal number**
n: 1) кардинальное число множества
2) мощность множества
a: главный, основной, количественный (числ.)

$A, B$ – two sets, have the same cardinal number*
when $\exists$ bijective.

$f : A \to B$

$g : \{1, 2, 3\} \to \{1, 2\}$

$1 \to 1$ $\quad\quad 1 \to 1$
$2 \to 2$ $\quad\quad 2 \to 2$
$3 \to 2$ $\quad\quad 3 \to 1$

etc. Here there are no injective mapping

Back to formulations:

$f \subset A \times B$

I picked out 4 conditions

(i) $\forall a \in A \;\; \exists b \in B : (a, b) \in f$
(ii) $(a, b) \in f$ and $(a, b') \in f \to b = b'$
(iii) $(a, b) \in f$ and $(a', b) \in f \Rightarrow a = a'$
(iv) $\forall b \in B \;\; \exists a \in A : (a, b) \in f$

cardinal number (cardinal) - the number of elements in a mathematical set.

(47)

Anything satisfying (i) & (ii) is called a mapping
(iii) is injective
(iv) is surjective

exponentials $\exp : \mathbb{R} \longrightarrow \mathbb{R}_+ = \{x \in \mathbb{R} \; x > 0\}$

$$\exp(x) = \sum_{r=0}^{\infty} \frac{x^r}{r!}$$

$\log x = \int^x \frac{dt}{t}$ $\quad \log : \mathbb{R}_+ \longrightarrow \mathbb{R}$

$\exp(\log(x)) = x \qquad x > 0$

$\log(\exp(x)) = x \qquad x \in \mathbb{R}$

$\log(x) = y$

if $\exp(y) = x$

$(x, y) \in \log$ iff. (if and only if) $(y, x) \in \exp$

Inverse mapping

if $f \subset A \times B$ Define $f^{-1} \subset B \times A$
by $(b, a) \in f^{-1} \Leftrightarrow (a, b) \in f$

(i)' $\forall\, b \in B \; \exists\, a \in A : (b, a) \in f^{-1}$
(ii)' $(b, a) \in f^{-1}$ and $(b, a') \in f^{-1} \Rightarrow a = a'$
(iii)' $(b, a) \in f^{-1}$ and $(b', a) \in f^{-1} \Rightarrow b = b'$
(iv)' $\forall\, a \in A \; \exists\, b \in B \; (b, a) \in f^{-1}$

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**conclusion**

(i)' for $f^{-1}$ $\Leftrightarrow$ (iv) for $f$
(ii)' for $f^{-1}$ $\Leftrightarrow$ (iii) for $f$
(iii)' for $f^{-1}$ $\Leftrightarrow$ (ii) for $f$
(iv)' for $f^{-1}$ $\Leftrightarrow$ (i) for $f$

**Prop.** If $f^{-1}$ is mapping then $f$ is bijective.
If $f$ is a mapping then $f^{-1}$ is bijective.

**theorem** Let $f : A \to B$ be a mapping

$f$ is bijective $\Leftrightarrow$ $f^{-1}$ is a bijective

**Observe** If $f$ is bijective

$$A \xrightarrow{f} B \qquad B \xrightarrow{f^{-1}} A$$

$$(f^{-1} \circ f)(a) = a \qquad f^{-1} \circ f = Id_A$$
$$(f \circ f^{-1})(b) = b \qquad f \circ f^{-1} = Id_B$$

**Example** $\sin : \mathbb{R} \to [-1;1]$ is surjective, not ~~subjec~~ injective

$$\sin\left(\frac{\pi}{2}\right) = \sin\left(\frac{5\pi}{2}\right)$$

$\sin : \left[\frac{\pi}{2}, \frac{3\pi}{2}\right] \to [-1;1]$ is bijective

$\sin^{-1} : [-1, 1] \to \left[\frac{\pi}{2}, \frac{3\pi}{2}\right]$

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General Statement

$f: A \to B$ mapping. Then $f$ has an universe mapping $f^{-1}: A \to B \iff f$ is bijective

---

$\Rightarrow$ $f$ bijective $\Rightarrow$ $f^{-1}$ exists

$f: A \to B$ mapping is said to be invertible iff. $\exists$ mapping $g: B \to A$
such that $g \circ f = Id_A$
and $f \circ g = Id_B$

---

$\Rightarrow$ If such a $g$ exists then $f$ is bijective

Fields set of numbers in which can do standard arithmetic:
$\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$, Finite fields.

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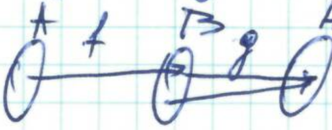
Suppose $fa = fa'$

Then $g(f(a)) = g(f(a'))$

i.e. $(g \circ f)(a) = (g \circ f)a'$

since $g \circ f$ inj, $a = a'$, i.e. $f$ is inj

$g \circ f$ inj $\Rightarrow$ $g$ inj. false




$A = \{a\}$ $B = \{c, d\}$

$f(a) = c$, $g(c) = g(d) = a$

$g \circ f(a) = a$.

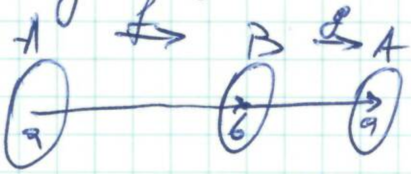
$g \circ f$ surj $\Rightarrow$ $g$ surj

Let $a \in A$ then since $g \circ f$ surjective,

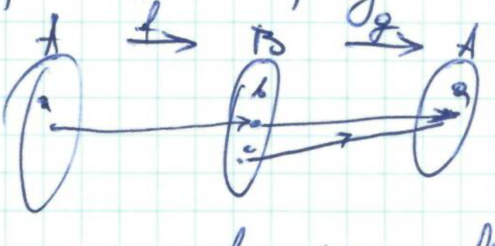
$\exists a' \in A$ s.t. $(g \circ f)(a') = a$.

Let $b = f(a')$. Then $g(b) = a$

i.e. $g$ surjective.




$g \circ f = 1_A \Rightarrow f \circ g = 1_B$ ?




$g \circ f = 1_A$ $f \circ g(c) = f(a) = b$ $\rightarrow f \circ g \neq 1_B$

<u>OR</u> $f : \mathbb{R} \to \mathbb{R}_{\geq 0}$

$f(a) = a^2$

$g : \mathbb{R}_{\geq 0} \to \mathbb{R}$, $g(a) = +\sqrt{a}$

$(A \cup B) \times (C \cup D) = (A \times C) \cup (B \times D)$

$A = \{a\}, \quad B = \emptyset \qquad A \cup B = a$

$C = \emptyset, \quad D = \{b\} \qquad C \cup D = \{b\}$

$A \times C \cup B \times D = \emptyset \neq A \cup B \times C \cup D = \{(a, b)\}$

N5.3.

$A = \{1, 2\}, \quad B = \{1, 2, 3\}$

a) f ll

(i) $f : \mathbb{R} \to \mathbb{R}$

$f(x) = x^3 + x$

inf $x = y \Rightarrow f x = f(y)$

$x < y \Rightarrow f(x) < f(y)$

need to solve f strictly inc.

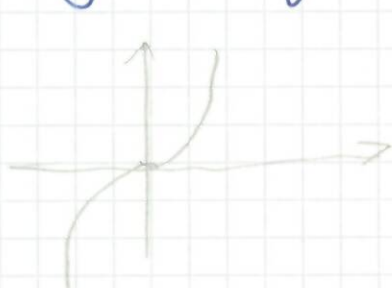
i) $f'(x) = 3x^2 + 1 > 0$ so $f$ strictly increasing

~~Proof~~ ii) Suppose $y > x$: $y^3 + y - x^3 + x = \underbrace{y^3 - x^3}_{>0} + (y - x)$

$= (y - x)(\underbrace{y^2 + xy + y^2 + 1}_{>0}) > 0$

// $y^n - x^n = (y - x)(y^{n-1} + \dots + x^{n-1})$

ii



$f(x) \to \infty \quad x \to \infty$

$f(x) \to -\infty$ & $x \to -\infty$

Given $y \in \mathbb{R}$. $\exists x_1, x_2 \in \mathbb{R}$ s.t

$f(x_1) < y, \ f(x_2) > y$

f continuous function

$\exists x \in \mathbb{R}$ s.t. $f(x) = 4$.

**Injective:** $f: A \to B$

$f(a) = f(a') \Rightarrow a = a'$

i.e. $a \neq a' \Rightarrow f(a) \neq f(a')$

**Surjective**

$\forall\, b \in B\ \exists\, a \in A$ s.t. $f(a) = b$

WS. 3

(i) $f: \mathbb{R} \to \mathbb{R}$

(ii) $f: \mathbb{Z} \to \mathbb{Z} \quad f(x) = x^3 + x$

Already seen $x < y \Rightarrow f(x) < f(y)$ :

$x \neq y \Rightarrow f(x) \neq f(y)$ i.e. $f$ injective

Not surjective e.g. $3 \neq f(x)$ for any $x \in \mathbb{Z}$

$[\, f(1) = 2 < 3,\ f(2) = 10 > 3 \,]$

(iii) $f: \mathbb{Z} \to \mathbb{Z} \quad f(x) = \left\lfloor \frac{5x+3}{5} \right\rfloor = \left\lfloor x + \frac{3}{5} \right\rfloor = x$

$\therefore f = id_{\mathbb{Z}}$ so inj. and surj.

(x) $f: \mathbb{R} \to \mathbb{R} \quad f(x) = \left\lfloor \frac{5x+3}{5} \right\rfloor = \left\lfloor x + \frac{3}{5} \right\rfloor$

4) $f: A \to B,\ g: B \to A$

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[rr, bend right, "g \circ f"'] & B \arrow[r, "g"] & A \end{tikzcd} \qquad \text{e.g.} \quad \begin{tikzcd} \mathbb{R} \arrow[r, "\sin"] \arrow[rr, bend right, "x \mapsto (\sin x)^2"'] & \mathbb{R} \arrow[r, "2"] & \mathbb{R} \end{tikzcd}$$

$x \to \sin x \quad x \to x^2$

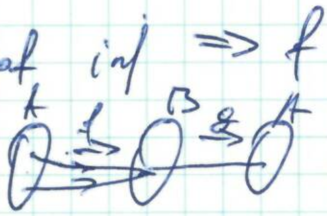
$g \circ f(a) = g(f(a))$

$f$ 1) inj $\Rightarrow g \circ f$ inj

— surj. $\Rightarrow$

---

$g \circ f$ inj $\Rightarrow f$ inj $\qquad \Rightarrow f \circ g(a) = f\sqrt{a} = (\sqrt{a})^2 = a, \quad f \circ g = id$




$g \circ f(-1) = g(1) = 1 \qquad g \circ f \neq id$

Def$^n$ vector space over $\mathbb{F}$ is a quantiple

$\mathbb{F}^n = (\mathbb{F}^n, +, 0, \cdot)$

1. $\mathbb{F}^n$ is a set ($\mathbb{F}^n = \{ \underset{\sim}{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} : x_i \in \mathbb{F} \}$), $\underset{\sim}{0} \in \mathbb{F}^n$
2. $+ : \mathbb{F}^n \times \mathbb{F}^n \to \mathbb{F}^n$ is a mapping satisfying:

   $(\underset{\sim}{x}, y) \to \underset{\sim}{x} + y$

   I $\underset{\sim}{x} + (y + \underset{\sim}{z}) = (\underset{\sim}{x} + y) + \underset{\sim}{z}$ Assoc.

   II $\underset{\sim}{x} + y = y + \underset{\sim}{x}$ Comm.

   III $\underset{\sim}{x} + \underset{\sim}{0} = \underset{\sim}{0} + \underset{\sim}{x} = \underset{\sim}{x}$ Ident.

   IV $\forall \underset{\sim}{x} \in \mathbb{F}^n \ \exists (-\underset{\sim}{x}) \in \mathbb{F}^n : \underset{\sim}{x} + (-\underset{\sim}{x}) = 0$ Inv.
3. $\cdot : \mathbb{F} \times \mathbb{F}^n \to \mathbb{F}^n$ is a mapping satisfying:

   $(\lambda, \underset{\sim}{x}) \to \lambda \underset{\sim}{x}$

   I $\lambda \cdot (\mu \underset{\sim}{x}) = (\lambda \mu) \underset{\sim}{x}$ Assoc.

   II $\lambda \underset{\sim}{x} = \underset{\sim}{x} \lambda$ Commut.

   III $1 \cdot \underset{\sim}{x} = \underset{\sim}{x} = \underset{\sim}{x} \cdot 1$ Ident.

   IV ——— " ——— ———
4. Final axiom:

   $\lambda (\underset{\sim}{x} + y) = \lambda \underset{\sim}{x} + \lambda y$

   $(\lambda + \mu) \underset{\sim}{x} = \lambda \underset{\sim}{x} + \mu \underset{\sim}{x}$ Distributive

Def$^n$ $\underset{\sim}{v}_1 \dots \underset{\sim}{v}_n$, vectors in a vector space over $\mathbb{F}$ are linearly independent iff.

$$\boxed{\lambda_1 \underset{\sim}{v}_1 + \lambda_2 \underset{\sim}{v}_2 + \dots \lambda_n \underset{\sim}{v}_n = 0 \Rightarrow \lambda_1 = \dots = \lambda_n = 0}$$

Def$^n$ $\underset{\sim}{v}_1 \dots \underset{\sim}{v}_m$, vectors in a vector space over $\mathbb{F}$ are spanning $V$ iff.

$$\boxed{\forall \underset{\sim}{x} \in V \ \exists x_1, \dots, x_m \in \mathbb{F} \text{ that } \underset{\sim}{x} = x_1 \underset{\sim}{v}_1 + \dots x_m \underset{\sim}{v}_m}$$

**Def$^n$** $\underset{\sim}{e}_1, \dots \underset{\sim}{e}_n$, vectors in a vector space $V \mid \mathbb{F}$ are **basis for $V$**, when

$$\boxed{\begin{array}{l} 1)\ \underset{\sim}{e}_1, \dots \underset{\sim}{e}_n \text{ are LI} \\[1em] 2)\ \underset{\sim}{e}_1, \dots, \underset{\sim}{e}_n \text{ span } V \end{array}}$$

**Def$^n$** $\boxed{\dim(V) = \text{number of elements in a basis for } V}$

**Def$^n$** $K_A$ = set of solutions of (homogenious) system of equations

$$\boxed{K_A = \left\{ \underset{\sim}{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \in \mathbb{F}^n : A x = 0 \right\}}$$

**Note** $K_A < \mathbb{F}^n$, i.e. it is a vector space over $\mathbb{F}$

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# FIELDS

$\mathbb{N} = \{0, 1, 2, \ldots, n, n+1 \ldots\}$ - how many things $\{$in set$\}$

$+ : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$

$(x, y) \to x+y$

$0 + x = x = x + 0$

$\mathbb{Z} = \{0, \pm 1, \pm 2, \ldots, \pm n\}$ -> integers

Now add & subtract

In $\mathbb{N}$ $\mathbb{Z}$ you can multiply
But can't divide.

$\mathbb{Q} = \{\frac{m}{n} : m, n \in \mathbb{Z}, n \neq 0\}$

can add subtract & divide
Two special symbols

| | | |
|---|---|---|
| $0$ | $x + 0 = x = 0 + x$ | Additive identity |
| $1$ | $x \cdot 1 = x = 1 \cdot x$ | Mult. Identity |

Anything that looks like ~~this~~ is called a <u>field</u>.

Formal Def

quintuple = a set of 5 similar things considered as a unit.

By a field $\mathbb{F}$, we mean a quintuple*

$\mathbb{F} = (F, +, 0, \cdot, 1)$ where ~~$\mathbb{F}$ is a set~~

1. $\mathbb{F}$ is a set, $0, 1 \in \mathbb{F}$ $0 \neq 1$

2. $+ : \mathbb{F} \times \mathbb{F} \to \mathbb{F}$ mapping (additive)

Instead of $+(a, b)$ we write $a + b$

$(a, b) \longmapsto a + b$ such that

$a + (b + c) = (a + b) + c$ Asoc.

$a + b = b + a$ Comm

$a + 0 = a = 0 + a$ Ident.

$\forall a \in \mathbb{F} \; \exists (-a) \in \mathbb{F}$ $a + (-a) = 0 = (-a) + a$ Inverse

(iii) $\cdot \; \mathbb{F} \times \mathbb{F} \mapsto \mathbb{F}$ mapping (mult.)

$(a, b) \to a \cdot b$

$$a \cdot (b \cdot c) = (a \cdot b) \cdot c \qquad \text{Assoc}$$

$$a \cdot b = b \cdot a \qquad \text{Comm}$$

$$a \cdot 1 = a = 1 \cdot a \qquad \text{Ident.}$$

for $a \in \mathbb{F} - \{0\}$ $\exists a^{-1} \in \mathbb{F}$ $a^{-1} \cdot a = 1 = a \cdot a^{-1}$ Inverse

Finally we need

(iv) $\left.\begin{array}{l} a \cdot (b + c) = a \cdot b + a \cdot c \\ (b + c) \cdot a = b \cdot a + c \cdot a \end{array}\right\}$ distributive laws,

Anything that satisfies these axioms is a <u>Field</u>

---

$\mathbb{Q}$ is a first example of a <u>field</u>

$\mathbb{R}$ also is a field.

$\mathbb{C} = \{a + ib : a, b \in \mathbb{R}\}$ - field
$i^2 = -1$

Some other examples.

Arithmetic mod 2

$\mathbb{F}_2 = \{0, 1\}$ with addition, multipl. defined by

| + | 0 | 1 |
|---|---|---|
| 0 | 0 | 1 |
| 1 | 1 | 0 |

| $\cdot$ | 0 | 1 |
|---|---|---|
| 0 | 0 | 0 |
| 1 | 0 | 1 |

Very useful

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Arithmetic mod 3

$\mathbb{F}_3 = \{0, 1, 2\}$

| + | 0 | 1 | 2 |
|---|---|---|---|
| 0 | 0 | 1 | 2 |
| 1 | 1 | 2 | 0 |
| 2 | 2 | 0 | 1 |

| $\cdot$ | 0 | 1 | 2 |
|---|---|---|---|
| 0 | 0 | 0 | 0 |
| 1 | 0 | 1 | 2 |
| 2 | 0 | 2 | 1 |

Arithmetic mod 4 $\quad \{0, 1, 2, 3\} = \mathbb{Z}/4$ - not a field

| + | 0 | 1 | 2 | 3 |
|---|---|---|---|---|
| 0 | 0 | 1 | 2 | 3 |
| 1 | 0 | 2 | 3 | 0 |
| 2 | 2 | 3 | 0 | 1 |
| 3 | 3 | 0 | 1 | 2 |

| $\cdot$ | 0 | 1 | 2 | 3 |
|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 0 |
| 1 | 0 | 1 | 2 | 3 |
| 2 | 0 | 2 | 0 | 2 |
| 3 | 0 | 3 | 2 | 1 |

as

~~no inverse~~

1) $2^{-1}$ does not exist

2) $2 \cdot 2 = 0$ and $2 \neq 0$.

## Another Example: $\mathbb{Q}(\sqrt{2})$

Elements are formed $\quad a + b\sqrt{2} \quad (\sqrt{2})^2 = 2$

$$a + b\sqrt{2} + c + d\sqrt{2} = a + c + (b+d)\sqrt{2}$$

$$(a + b\sqrt{2})(c + d\sqrt{2}) = ac + 2bd + (ad + bc)\sqrt{2}$$

$$a + b\sqrt{2} = c + d\sqrt{2} \quad \text{iff.} \quad a = c, \; b = d$$

$$1 = 1 + 0\sqrt{2}$$

$$a + b\sqrt{2} \neq 0$$

$$(a + b\sqrt{2})^{-1} = \frac{1}{(a + b\sqrt{2})} = \frac{a - b\sqrt{2}}{a^2 - 2b^2}$$

Have to know that

$$a^2 - 2b^2 \neq 0 \quad \text{if} \quad a, b \in \mathbb{Q}$$

$$2 \neq \left(\frac{a}{b}\right)^2$$

(55) entries

17.11.09. 1201

$\mathbb{F}$ field (eg. $\mathbb{F} = \mathbb{Q}$)

$\mathbb{F}^n = \{ \underset{\sim}{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} : x_i \in \mathbb{F} \}$

I In $\mathbb{F}^n$ we can add

$+ \quad \mathbb{F}^n \times \mathbb{F}^n \longrightarrow \mathbb{F}^n$

$(x, y) \longrightarrow x + y$

$x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \quad y = \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix} \quad \Rightarrow \quad \underset{\sim}{x} + y = \begin{pmatrix} x_1 + y_1 \\ x_2 + y_2 \\ \vdots \\ x_n + y_n \end{pmatrix}$

II We can also multiply

$x \in \mathbb{F}^n$ by $\lambda \in \mathbb{F}$

$\cdot : \quad \mathbb{F} \times \mathbb{F}^n \longrightarrow \mathbb{F}^n$

$(\lambda, \underset{\sim}{x}) \longrightarrow \lambda \cdot \underset{\sim}{x}$

$\lambda \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} \lambda x_1 \\ \vdots \\ \lambda x_n \end{pmatrix}$

e.g. $\underset{\sim}{x} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \quad \lambda = 5 \quad \lambda \underset{\sim}{x} = \begin{pmatrix} 5 \\ 10 \\ 15 \end{pmatrix}$

Scalar multication.

$F^n$ with $+, \cdot$ have diffuent properties

1) $\underset{\sim}{x} + (y + \underset{\sim}{z}) = (\underset{\sim}{x} + y) + \underset{\sim}{z}$ Assoc

2) $\underset{\sim}{x} + y = y + \underset{\sim}{x}$ Comm.

3) $\underset{\sim}{x} + \underset{\sim}{0} = \underset{\sim}{0} + \underset{\sim}{x} = \underset{\sim}{x}$ Identity

4) if $-\underset{\sim}{x} = (-1)\underset{\sim}{x}$ then

$\underset{\sim}{x} + (-\underset{\sim}{x}) = 0$

5) $\lambda \cdot (\mu \underset{\sim}{x}) = (\lambda \mu) \cdot x$ $\quad \lambda \mu \in \mathbb{F}, \underset{\sim}{x} \in \mathbb{F}^n$

6) $1 \cdot \underset{\sim}{x} = \underset{\sim}{x}$ $\quad 1 \in \mathbb{F} \quad x \in \mathbb{F}^n$

7) $\lambda \cdot (\underset{\sim}{x} + y) = \lambda \cdot \underset{\sim}{x} + \lambda \cdot y$ $\quad \lambda \in \mathbb{F}, \lambda, \mu \in \mathbb{F}, x \in \mathbb{F}^n$

(56)

8) $(\lambda+\mu)\cdot \underset{\sim}{x} = \lambda \underset{\sim}{x} + \mu \underset{\sim}{x} \quad \lambda \in \mathbb{F},\ \lambda\mu \in \mathbb{F},\ \underset{\sim}{x} \in \mathbb{F}^n$

rough Def$^n$ Anything with these properties in called a vector vector space over $\mathbb{F}$

Formally a vector space over $\mathbb{F}$ consists of following $(V, +, 0, \cdot)$ where $V$ is a set, $0 \in V$

$+ : V \times V \to V \quad \underset{\sim}{x} + y \quad (= +(x,y))$

is a mapping satisfying

1) $x + (y+z) = (x+y) + z$

2) $x+y = y+x$

3) $x + 0 = 0 + x = x$

4) $\forall\, x \in V \quad \exists -x \in V : x + (-x) = 0$

$\cdot : \mathbb{F} \times V \to V \quad \lambda \cdot \underset{\sim}{x} \quad (= \cdot(\lambda, x))$

is a mapping s.t.

5) $\lambda \cdot (\mu x) = (\lambda \cdot \mu) x$

6) $1 \cdot x = x \cdot 1 \qquad 1 \in \mathbb{F} \quad x \in V$

7) $\lambda (x+y) = \lambda \cdot x + \lambda y$ ~~$(\lambda+\mu)\cdot x = \lambda x + \mu x$~~ } Distribution

8) $(\lambda + \mu) \cdot x = \lambda \cdot x + \mu x$

Example 1) $\mathbb{F}^n = (\mathbb{F}^n, +, 0, \cdot)$ is a vector space / $\mathbb{F}$ (= over $\mathbb{F}$)

$n=2 \quad \mathbb{F}^2 = \left\{ \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} : x_1, x_2 \in \mathbb{F} \right\}$

$n=3 \quad \mathbb{F}^3 = \left\{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}, \ x_i \in \mathbb{F} \right\}$

2) $V = \left\{ \begin{pmatrix} x \\ -x \end{pmatrix} : x \in \mathbb{F} \right\}$

$V \subset \mathbb{F}^2$

Addition $\begin{pmatrix} x \\ -x \end{pmatrix} + \begin{pmatrix} y \\ -y \end{pmatrix} = \begin{pmatrix} x+y \\ -(x+y) \end{pmatrix} \in V$

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S. Multiplication $\lambda \cdot \begin{pmatrix} x \\ -x \end{pmatrix} = \begin{pmatrix} \lambda x \\ -\lambda x \end{pmatrix} \in V$

## Linear independence

Suppose $V$ $(= (V, +, 0, \cdot))$
is a vector space / $\mathbb{F}$

Let $v_1, \dots v_n \in V$

Say that $\{\underset{\sim}{v_1}, \dots, v_n\}$ are linear independent when $\lambda_1 v_1 + \lambda_2 v_2 + \dots + \lambda_n v_n = 0$

$\Rightarrow \lambda_1 = \lambda_2 = \lambda_3 \dots = \lambda_n = 0$

Example 1) $V = \mathbb{F}^3$ $\quad e_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ $\quad e_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ $\quad e_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

Then $\{e_1, e_2, e_3\}$ are L.I.

Suppose

$$\lambda_1 e_1 + \lambda_2 e_2 + \lambda_3 e_3 = 0 = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} \lambda_1 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ \lambda_2 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ \lambda_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

2) $\underset{\sim}{\epsilon_1} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ $\quad \underset{\sim}{\epsilon_2} = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$ $\quad \underset{\sim}{\epsilon_3} = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$

Claim $\{\epsilon_1, \epsilon_2, \epsilon_3\}$ are LI

Suppose $\lambda_1 \epsilon_1 + \lambda_2 \epsilon_2 + \lambda_3 \epsilon_3 = 0$

$$\begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \end{pmatrix} + \begin{pmatrix} \lambda_1 \\ -\lambda_2 \\ \lambda_3 \end{pmatrix} + \begin{pmatrix} \lambda_1 \\ 0 \\ -\lambda_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

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$$\begin{pmatrix} \lambda_1 + \lambda_2 + \lambda_3 = 0 \\ \lambda_1 - \lambda_2 = 0 \\ \lambda_1 + \lambda_2 - \lambda_3 = 0 \end{pmatrix} \Rightarrow \lambda_1 = \lambda_2 \Rightarrow$$

$$\Rightarrow \left.\begin{matrix} 2\lambda_1 - \lambda_3 = 0 \\ 2\lambda_1 + \lambda_3 = 0 \end{matrix}\right\} \Rightarrow 4\lambda_1 = 0 \Rightarrow \lambda_1 = 0$$

$$\Rightarrow \lambda_2 = 0$$

$$\Rightarrow \lambda_3 = 0$$

3) $\varphi_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ $\varphi_2 = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$ $\varphi_3 = \begin{pmatrix} 3 \\ 1 \\ 3 \end{pmatrix}$

These are not LI (= they are linearly Dependent = LD)

$$2\varphi_1 + \varphi_2 = \varphi_3$$

$$2\varphi_1 + \varphi_2 - \varphi_3 = 0$$

$$\lambda_1 = 2 \quad \lambda_2 = 1 \quad \lambda_3 = -1$$

Here I have ~~ex~~ an expression for $\underset{\sim}{0}$ in terms of $\varphi_i$'s where coefficients are $\neq 0$

$$\lambda_1\varphi_1 + \lambda_2\varphi_2 + \lambda_3\varphi_3 = 0 \text{ but } \lambda_1 \neq 0$$

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$\underset{\sim}{v}_1 \dots \underset{\sim}{v}_n$ vectors in a vector space $V / \mathbb{F}$

---

$\{\underset{\sim}{v}_1 \dots v_n\}$ are LI iff.

$$\lambda_1 \underset{\sim}{v}_1 + \lambda_2 \underset{\sim}{v}_2 + \dots \lambda_n \underset{\sim}{v}_n = 0 \Rightarrow \lambda_1 = \lambda_2 = \dots = \lambda_n = 0$$

---

In English, the <u>only</u> way to get $\phi$ is the obvious way i.e. all coef. are zero

Beware! 1) <u>Standard Mistake</u>:

$$v_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \quad v_2 = \begin{matrix} 1 \\ -1 \\ 1 \end{matrix} \quad v_3 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

$\underset{\sim}{v}_1 + \underset{\sim}{v}_2 - 2v_2 = 0$, but also I have

$\cancel{0\underset{\sim}{v}_1 + 0\underset{\sim}{v}_2 + 0 v_3 = 0 \Rightarrow LI}$

In this example

$v_1 + v_2 - 2v_2 = 0$ is a dependence relation~~ship~~:

and $\{v_1, v_2, v_3\}$ are LD

2) Standard vector space $\mathbb{F}^n = \{\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} ; x_i \in \mathbb{F}\}$

Define $e_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$ $e_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \dots e_{n-1} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 1 \\ 0 \end{pmatrix}$ $e_n = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix}$

$\{e_1, \dots, e_n\}$ = are linearly independent

why?

$$\lambda_1 e_1 + \lambda_2 e_2 + \dots + \lambda_n e_n = 0$$

$$\begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \\ \vdots \\ \lambda_n \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

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3) $v_1 = \begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \end{pmatrix}, \quad v_2 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \end{pmatrix}, \quad v_3 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ -1 \end{pmatrix}, \quad v_4 = \begin{pmatrix} 0 \\ 1 \\ 1 \\ 1 \end{pmatrix}$

are they LI?

Suppose $\lambda_1 v_1 + \lambda_2 v_2 + \lambda_3 v_3 + \lambda_4 v_4 = 0$

$$\begin{pmatrix} \lambda_1 \\ -\lambda_1 \\ \lambda_1 \\ -\lambda_1 \end{pmatrix} + \begin{pmatrix} \lambda_2 \\ \lambda_2 \\ \lambda_2 \\ 0 \end{pmatrix} + \begin{pmatrix} \lambda_3 \\ 0 \\ 0 \\ -\lambda_3 \end{pmatrix} + \begin{pmatrix} 0 \\ \lambda_4 \\ \lambda_4 \\ \lambda_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{cases} \lambda_1 + \lambda_2 + \lambda_3 = 0 \\ -\lambda_1 + \lambda_2 + \lambda_4 = 0 \\ \lambda_1 + \lambda_2 + \lambda_4 = 0 \\ -\lambda_1 - \lambda_3 + \lambda_4 = 0 \end{cases}$$

$$\left(\begin{array}{cccc|c} 1 & 1 & 1 & 0 & 0 \\ -1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 & 0 \\ -1 & 0 & -1 & 1 & 0 \end{array}\right) \xrightarrow[\substack{\mathcal{E}(3,1;-1) \\ \mathcal{E}(4,1;1)}]{\mathcal{E}(2,1;1)} \left(\begin{array}{cccc|c} 1 & 1 & 1 & 0 & 0 \\ 0 & 2 & 1 & 1 & 0 \\ 0 & 0 & -1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 \end{array}\right) \xrightarrow[\cancel{\mathcal{E}}]{\mathcal{P}(2,4)}$$

$$\rightarrow \left(\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & -1 & 1 & 0 \\ 0 & 2 & \cancel{0} & 1 & 0 \end{array}\right) \xrightarrow[\mathcal{E}(4,2;-2)]{\mathcal{E}(1,2;-1)} \left(\begin{array}{cccc|c} 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & -1 & 0 \end{array}\right) \xrightarrow{\mathcal{E}}$$

$$\rightarrow \left(\begin{array}{cccc|c} 1 & 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \end{array}\right) \longrightarrow \begin{array}{c} \left(\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right) \\ \begin{array}{cccc} \lambda_1 & \lambda_2 & \lambda_3 & \lambda_4 \end{array} \end{array}$$

General solution:

$$\lambda_1 = 0 \qquad \begin{pmatrix} 0 \\ -\lambda_4 \\ \lambda_4 \\ \lambda_4 \end{pmatrix}$$
$$\lambda_2 = -\lambda_4$$
$$\lambda_3 = \lambda_4$$
$$\lambda_4 = \lambda_4$$

(61) $0\,\underset{\sim}{v}_1 - \lambda_4 v_2 + \lambda_4 v_3 + \lambda_4 v_4 = 0$ ( $\lambda_4$ - could be any num. )

$\lambda_4 = 1$ $\therefore$ $0\,\underset{\sim}{v}_1 - v_2 + v_3 + v_4 = 0$

$-v_2 + v_3 + v_4 = 0$ No, it is a LD relⁿ.

Spanning: Let $\underset{\sim}{v}_1, \dots, \underset{\sim}{v}_n \in V$ (vector Space)

Say that $v_1, \dots, v_n$ span $V$ when

$$\boxed{\begin{array}{l}\forall \underset{\sim}{x} \in V \cdot \exists x_1, \dots, x_n \in \mathbb{F} \text{ such that} \\ \underset{\sim}{x} = x_1 \underset{\sim}{v}_1 + x_2 \underset{\sim}{v}_2 + \dots + x_n \underset{\sim}{v}_n\end{array}}$$

An expression of the form

$\lambda_1 \underset{\sim}{v}_1 + \dots + \lambda_n v_n \qquad \lambda_i \in \mathbb{F}$

is called a linear combination in $v_1, \dots v_n$

<u>In English</u>: $\underset{\sim}{v}_1 \dots v_n$ span $V$ when any vector $\underset{\sim}{x} \in V$ can be expressed as a linear combination in $\{\underset{\sim}{v}_1 \dots \underset{\sim}{v}_n\}$

Example: 1) $e_1 = \begin{pmatrix}1\\0\end{pmatrix}$ $e_2 = \begin{pmatrix}0\\1\\0\end{pmatrix}$ ... $e_n = \begin{pmatrix}0\\1\end{pmatrix}$, with $e_1, \dots, e_n \in \mathbb{F}^n$

$e_1, \dots e_n$ span $\mathbb{F}^n$: Take an arbitrary vector $\underset{\sim}{x} = \begin{pmatrix}x_1\\x_2\\\vdots\\x_n\end{pmatrix} \in \mathbb{F}^n$

I can write:

$\underset{\sim}{x} = x_1 \underset{\sim}{e}_1 + x_2 \underset{\sim}{e}_2 + \dots + x_n \underset{\sim}{e}_n$

2) $\underline{\epsilon}_1 = \begin{pmatrix}1\\1\\1\end{pmatrix}$ $\underline{\epsilon}_2 = \begin{pmatrix}0\\1\\1\end{pmatrix}$ $\underline{\epsilon}_3 = \begin{pmatrix}0\\0\\1\end{pmatrix}$

Claim $\{\underline{\epsilon}_1, \underline{\epsilon}_2, \underline{\epsilon}_3\}$ span $\mathbb{F}^3$

why? Take $\underline{v} = \begin{pmatrix}\lambda_1\\\lambda_2\\\lambda_3\end{pmatrix} \in \mathbb{F}^3$

I have to find $x_1, x_2, x_3 \in \mathbb{F}$ such that

$x_1 \underline{\epsilon}_1 + x_2 \underline{\epsilon}_2 + x_3 \underline{\epsilon}_3 = \begin{pmatrix}\lambda_1\\\lambda_2\\\lambda_3\end{pmatrix} = \underline{v}$

Vector space (векторное пространство) = линейное пространство - множество элементов называемых векторами, для которых определены операции сложения и умножение на число

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$$\begin{pmatrix} x_1 \\ x_1 \\ x_1 \end{pmatrix} + \begin{pmatrix} 0 \\ x_2 \\ x_2 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ x_3 \end{pmatrix} = \begin{pmatrix} x_1 \\ x_1 + x_2 \\ x_1 + x_2 + x_3 \end{pmatrix}$$

I want $x_1 = \lambda_1$

$x_1 + x_2 = \lambda_2$

$x_1 + x_2 + x_3 = \lambda_3$

Put $x_1 = \lambda_1$

$x_2 = \lambda_2 - \lambda_1$

$x_3 = \lambda_3 - \lambda_2$

e.g. Take $\underline{v} = \begin{pmatrix} 1 \\ 2 \\ 5 \end{pmatrix} \therefore x_1 = 1, \ x_2 = 1 \ \ x_3 = 3$

$$\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} + 3\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 5 \end{pmatrix}$$

<u>Basis</u>: $\underline{e}_1, \dots \underline{e}_n$ vectors in $V$ vector space $/ \mathbb{F}$

Def$^n$ $\underline{e}_1, \dots, \underline{e}_n$ is a basis for $V$, when

$$\boxed{\begin{array}{l} \text{i) } \underline{e}_1, \dots, \underline{e}_n \text{ is LI} \\ \wedge \\ \text{ii) } \underline{e}_1, \dots, \underline{e}_n \text{ is span } V \end{array}}$$

Eventually we will prove

Any (non-zero) vector space $V$ has a basis

Any two bases for $V$ have same number of elements:

$$\boxed{\dim(V) = \text{number of elements in basis for } V}$$

$V = \left\{ \begin{pmatrix} x \\ -x \end{pmatrix} : x \in \mathbb{R} \right\} \quad e = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \quad \dim(V) = 1$

$\forall x \in V, \ v = \lambda \underline{e}$ for some $\lambda$

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## Linear Independence

$v_1, \dots v_n \in V$ are LI
when $\lambda_1 v_2 + \dots + \lambda_n v_n = 0$
$\Rightarrow \lambda_1 = \lambda_2 = \lambda_3 = \dots \lambda_n = 0$

## Spannity

$\underset{\sim}{v}_1, \cdots \underset{\sim}{v}_n$ span $V$ when $\forall \underset{\sim}{x} \in V \; \exists \; x_1, \dots, x_n \in \mathbb{F} \;\; \underset{\sim}{x} = x_1 \underset{\sim}{v}_1 + x_2 \underset{\sim}{v}_2 + \dots + x_n \underset{\sim}{v}_n$

## Basis

$\underset{\sim}{e}_1, \dots, \underset{\sim}{e}_n$ form a basis for $V$
when 1) $\underset{\sim}{e}_1, \dots, \underset{\sim}{e}_n$ is LI
and 2) $\underset{\sim}{e}_1, \dots, \underset{\sim}{e}_n$ span $V$

## Basis THM

If $V$ is a non-zero vector space
then 1) $V$ has a basis
2) ~~only~~ any two ~~elements~~ basis have same no of elements

The dimension of $V$ = no of elements in a basis for $V$

Example 1) $\dim(\mathbb{F}^2) = 2$

$e_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad e_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, $\{e_1, e_2\}$ is LI & spans $\mathbb{F}^2$

2) $\dim(\mathbb{F}^3) = 3$

$e_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \quad e_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \quad e_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \Rightarrow \{e_1, e_2, e_3\}$ is LI & spans $\mathbb{F}^3$

3) $\dim(\mathbb{F}^n) = n$

$e_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \dots e_n = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \Rightarrow \{e_1 \dots e_n\}$ is LI & spans $\mathbb{F}^n$

(64) 4) $\mathbb{F}$ field

$$V = \left\{ x \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \in \mathbb{F}^3, \ x_1 + x_2 + x_3 = 0 \right\}$$

V - vector space.

Addition on V

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} + \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} x_1 + y_1 \\ x_2 + y_2 \\ x_3 + y_3 \end{pmatrix} \qquad \begin{matrix} x_1 + x_2 + x_3 = 0 \\ y_1 + y_2 + y_3 = 0 \end{matrix}$$

$$(x_1 + y_1) + (x_2 + y_2) + (x_3 + y_3) = 0$$

Scalar multiplication:

$$\lambda \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} \lambda x_1 \\ \lambda x_2 \\ \lambda x_3 \end{pmatrix} \qquad \begin{matrix} x_1 & x_3 = 0 \\ \lambda x_1 & = 0 \end{matrix}$$

$$\text{zero } \underset{\sim}{0} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \qquad 0 + 0 + 0 = 0$$

$\dim(V) = 2$

$$(1, 1, 1) \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0$$

A

$(1, 1, 1)$ is already in R.E.F

$$\begin{matrix} (1, & 1, & 1) \\ x_1 & x_2 & x_3 \end{matrix} \qquad x_1 = -x_2 - x_3$$

$$\begin{pmatrix} -x_2 - x_3 \\ x_2 \\ x_3 \end{pmatrix} =$$

Take $x_2 = 1 \quad x_3 = 0$

$$e_1 = \begin{pmatrix} -1 \\ +1 \\ 0 \end{pmatrix} \quad e_2 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \qquad \{e_1, e_2\} = LI \ \& \ \text{span } V$$

(65)

## Generalization of Last Example.

Let $A = \begin{pmatrix} a_{11} & \dots & a_{1n} \\ \vdots & & \\ a_{m1} & \dots & a_{mn} \end{pmatrix}$ $m \times n$ matrix over $\mathbb{F}$

Put $K_A = \left\{ \underline{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \in \mathbb{F}^n : A\underline{x} = 0 \right\}$

$K_A$ - set of solutions of (homogenious) system of eq$^{ns}$

Note $K_A \subset \mathbb{F}^n$

$K_A$ is a vector space / $\mathbb{F}$

<u>Addition</u>:

If $\underline{x}, \underline{y} \in K_A$

$A\underline{x} = 0$ & $A\underline{y} = 0$

so $A\underline{x} + A\underline{y} = 0$

$A(\underline{x} + \underline{y}) = 0 \quad (\underline{x} + \underline{y}) \in K_A$

<u>Scalar Multipl.</u>

$x \in K_A$, $\lambda \in \mathbb{F} \Rightarrow \lambda\underline{x} \in K_A$

$A\underline{x} = 0$

$\sum_{j=1}^{n} a_{ji} x_j = 0$

$\lambda$

$\lambda A\underline{x} = 0$

$\lambda\left(\sum_{j=1}^{n} a_{ji} x_j = 0\right)$

$\sum_{j=1}^{n} a_{ji}(\lambda x_j) = 0$

$A(\lambda \underline{x}) = 0$

<u>Zero</u>

$A\underline{0} = 0$, so $\underline{0} \in K_A$

All other properties hold in $\mathbb{F}^n$ already.

Q How to calculate $\dim K_A$?

(66) Example 1) (Easy case, $A$ is RE form)

$$A = \begin{pmatrix} 1&2&0&1&0&1 \\ 0&0&1&0&0&0 \\ 0&0&0&0&1&1 \end{pmatrix} \quad 3\times 6$$

$$K_A = \{ \underline{x} \in \mathbb{F}^6 : A\underline{x} = 0 \}$$

General Solution:

$$\left.\begin{aligned} x_1 &= -2x_2 - x_4 - x_6 \\ x_2 &= x_2 \\ x_3 &= -x_6 \\ x_4 &= x_4 \\ x_5 &= -x_6 \\ x_6 &= x_6 \end{aligned}\right\} \begin{pmatrix} -2x_2 - x_4 - x_6 \\ x_2 \\ -x_6 \\ x_4 \\ -x_6 \\ x_6 \end{pmatrix}$$

There is an obvious choice of basis

$$\begin{aligned} x_2 &= 1 \\ x_4 &= 0 \\ x_6 &= 0 \end{aligned} \quad \epsilon_1 = \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \qquad \left.\begin{aligned} x_2 &= 0 \\ x_4 &= 1 \\ x_6 &= 0 \end{aligned}\right\} \epsilon_2 = \begin{pmatrix} -1 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \qquad \begin{aligned} x_2 &= 0 \\ x_4 &= 0 \\ x_6 &= 0 \end{aligned} \quad \epsilon_3 = \begin{pmatrix} -1 \\ 0 \\ -1 \\ 0 \\ -1 \\ 1 \end{pmatrix}$$

$\epsilon_1, \epsilon_2, \epsilon_3$ are LI

$$\{ \lambda_1 \epsilon_1 + \lambda_2 \epsilon_2 + \lambda_3 \epsilon_3 = \begin{pmatrix} -2\lambda_1 - \lambda_2 - \lambda_3 \\ \lambda_1 \\ -\lambda_3 \\ \lambda_2 \\ -\lambda_3 \\ \lambda_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \}$$

$$\underset{0}{\overset{\shortparallel}{}}$$

$\therefore \lambda_1 = \lambda_2 = \lambda_3 = 0$

They Spans $K_A$

if $\underline{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_6 \end{pmatrix} \in K_A$

$\underline{x} = x_2 \epsilon_1 + x_4 \epsilon_2 + x_6 \epsilon_3$

So $\dim K_A = 3$, with basis $\{\epsilon_1, \epsilon_2, \epsilon_3\}$

(64) 2) In general $A$ would not be in REF (= row echelon form)

$$A = \begin{pmatrix} 1 & 1 & -1 & 1 & -1 & 1 \\ 1 & -1 & 0 & 1 & 0 & 1 \\ 1 & 3 & -2 & 1 & -2 & 1 \end{pmatrix} \quad 3\times 6 \qquad \mathbb{F} = \mathbb{Q}$$

$K_A = \{ x \in \mathbb{Q}^6 ; \ A x = 0$

Reduce $A$ to REF and solve

$$\left(\begin{array}{cccccc|c} 1 & 1 & -1 & 1 & -1 & 1 & 0 \\ 1 & -1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 3 & -2 & 1 & -2 & 1 & 0 \end{array}\right) \longrightarrow \left(\begin{array}{cccccc|c} 1 & 1 & -1 & 1 & -1 & 1 & 0 \\ 0 & -2 & 1 & 0 & 1 & 0 & 0 \\ 0 & 2 & -1 & 0 & -1 & 0 & 0 \end{array}\right) \longrightarrow \left(\begin{array}{cccccc|c} 1 & 1 & -1 & 1 & -1 & 1 & 0 \\ 0 & -1 & 1/2 & 0 & 1/2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right) \longrightarrow$$

$$\longrightarrow \left(\begin{array}{cccccc|c} 1 & 0 & -1/2 & 1 & -1/2 & 1 & 0 \\ 0 & 1 & -1/2 & 0 & -1/2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

$x_1 \ x_2 \ x_3 \ x_4 \ x_5 \ x_6$

$\therefore$ General solution:

$$\begin{pmatrix} \frac{1}{2}x^3 - x_4 + \frac{1}{2}x_5 - x_6 \\ \frac{3}{2}x^4 + \frac{x^5}{2} \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{pmatrix} \Rightarrow \dim K_A = 4$$

Basis

$$e_1 = \begin{pmatrix} +\frac{1}{2} \\ 1/2 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad e_2 = \begin{pmatrix} -1 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \quad e_3 = \begin{pmatrix} 1/2 \\ 1/2 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \quad e_4 = \begin{pmatrix} -1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

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**Prop** $V$ vector space $/\mathbb{F}$

$\underline{e}_1, \dots, \underline{e}_n$ basis for $V$

For each $\underline{x} \in V$ $\underline{x}$ can be expressed as linear comb $\{e_1, \dots e_n\}$ in a unique way.

**Proof.** Suppose

$$\underline{x} = a_1\underline{e}_1 + \dots + a_n\underline{e}_n$$

$$\underline{x} = x_1'\underline{e}_1 + \dots + x_n'\underline{e}_n$$

$$0 = \underline{x} - \underline{x}' = (x_1 - x_1')\underline{e}_1 + \dots (x_n - x_n')\underline{e}_n$$

But $\underline{e}_1, \dots, \underline{e}_2$ are LI

So $x_1 - x_1' = 0$ $\quad x_2 - x_2' = 0$ $\quad \dots$ $\quad x_n - x_n' = 0$

$x_1 = x_1'$ $\quad x_2 = x_2'$ $\quad x_n = x_n'$ Q

linear mapping: линейное отображение

## Linear Mapping*

**Def^n** Let $V, W$ are vector space over field $\mathbb{F}$.

mapping: отображение

Let $T: V \longrightarrow W$ be a mapping

Say that $T$ is linear iff.

(i) $T(\underline{x} + \underline{y}) = T(\underline{x}) + T(\underline{y})$ says preserves addition

(ii) $T(\lambda \underline{x}) = \lambda T(\underline{x})$ says T preserves scal. mult.

**Example** Let $V$ be the set of polynomial over $\mathbb{Q}$

$$f(t) = a_0 + a_1 t + a_2 t^2 + \dots a_n t^n$$

$V$ is a vector space

Addition in $V$

$$g(t) = b_0 + b_1 t + \dots + b_n t^n$$

$$(f+g)(t) = (a_0 + b_0) + (a_1 + b_1)t + (a_2 + b_2)t^2 + \dots (a_n + b_n)t^n$$

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Scalar mult.:

$\lambda f(t) = \lambda a_0 + \lambda a_1 t + \lambda a_2 t^2 + \dots \lambda a_n t^n$

zero:

$0(t) = 0 + 0t + 0t^2 + \dots + 0t^n$

Consider differentiation:

$D: V \to V \qquad D(f) = \frac{df}{dt}$

So $D(1) = ?$ $\quad D(t) = 1 \quad D(t^2) = 2t$ etc.

$D$ is linear:

$D(f+g) = D(g) + D(f)$

$D(\lambda f) = \lambda D(f) \qquad \lambda$ - const

Example $V = \mathbb{F}^n \qquad W = \mathbb{F}^n$

and let $A = (a_{ij})_{\substack{1 \le j \le n \\ 1 \le i \le n}}$ $\quad m \times n$ matrix / $\mathbb{F}$

Define $T_A : \mathbb{F}^n \to \mathbb{F}^m$ by

$T_A(\underline{x}) = A\underline{x}$ (matrix product)

$$\left.\begin{aligned} T_A(x+y) &= A(x+y) = Ax + Ay = T_A(x) + T_A(y) \\ T_A(\lambda x) &= A(\lambda \underline{x}) = \lambda A\underline{x} = \lambda T_A(\underline{x}) \end{aligned}\right\} \Rightarrow T_A \text{ - linear}$$

Example: We'll show that the standard example is "typical"
observation The standard basis for $\mathbb{F}^n$

$e_1 = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \quad e_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \dots e_n = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix}$

Let $A = \begin{pmatrix} a_{11} & \dots & a_{1n} \\ a_{s1} & \dots & a_{sn} \\ \vdots & & \\ a_{n1} & \dots & a_{nn} \end{pmatrix}$

Mult. each basis vector $e_1, \dots e_n$ on left

$A\underline{e}_1 = \begin{pmatrix} a_{11} & \dots & a_{1n} \\ \vdots & & \vdots \\ a_{s1} & \dots & a_{sn} \\ \vdots & & \vdots \\ a_{n1} & \dots & a_{nn} \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} a_{11} \\ \vdots \\ a_{s1} \\ \vdots \\ a_{n1} \end{pmatrix}$ - 1st C. $\quad A\underline{e}_2 = \begin{pmatrix} a_{12} \\ \vdots \\ a_{s2} \\ \vdots \\ a_{n2} \end{pmatrix}$ ← 2nd C. $\quad A\underline{e}_3$

$= a_{11}\underline{e}_1 + a_{21}\underline{e}_2 + \dots + a_{n1}\underline{e}_n \qquad = a_{12}\underline{e}_1 + a_{22}\underline{e}_2 + \dots + a_{m}e_n$

$A e_i = \sum_{j=1}^{n} a_{ji}\, \underline{e}_j = T_A(e_i)$

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$n = 3$

$e_1 = \begin{pmatrix}1\\0\\0\end{pmatrix}$ $e_2 = \begin{pmatrix}0\\1\\0\end{pmatrix}$ $e_3 = \begin{pmatrix}0\\0\\1\end{pmatrix}$ ; $\begin{pmatrix}a\\b\\c\end{pmatrix} = a\underline{e}_1 + b\underline{e}_2 + ce_3$

In general: $T : V \to W$ linear

Let $\underline{\epsilon}_1, \dots \underline{\epsilon}_n$ be basis for $V$

$\underline{\varphi}_1, \dots \underline{\varphi}_m$ ... $W$

Apply $T$ in terms of to each of $\underline{\epsilon}_1, \dots \underline{\epsilon}_n$

$T(\underline{\epsilon}_1) \in W$ express it as lin combination in $\varphi_1, \dots \varphi_m$

$T(\underline{\epsilon}_1) = ?\varphi_1 + ??\varphi_2 + \dots ???\varphi_m$

$$\begin{cases} T(\underline{\epsilon}_1) = a_{11}\varphi_1 + a_{21}\varphi_2 + \dots a_{m1}\varphi_m \\ T(\underline{\epsilon}_2) = a_{12}\varphi_1 + a_{22}\varphi_2 + \dots a_{m2}\varphi_m \\ \vdots \\ T(\underline{\epsilon}_n) = a_{1n}\varphi_1 + a_{2n}\varphi_2 + \dots a_{mn}\varphi_m \end{cases}$$

General Convention:

$\left.\begin{matrix}\epsilon_1 \dots \epsilon_n \text{ basis for } V \\ \varphi_1 \dots \varphi_m \dots W\end{matrix}\right\}$ $T : V \to W$ linear

$$\boxed{T(\underline{\epsilon}_i) = \sum_j^m a_{ji}\varphi_j} = a_{1i}\varphi_1 + a_{2i}\varphi_2 + a_{3i}\varphi_3 + a_{4i}\varphi_3 + \dots a_{mi}\varphi_m =$$

$=$

$$T(\underline{\epsilon}_i) = \sum_{j=1}^n a_{ji}e_j = \begin{pmatrix}a_{1i}\\ \vdots \\ a_{ni}\end{pmatrix} = a_{1i}\underline{\epsilon}_1 + a_{2i}\underline{\epsilon}_2 + \dots a_{ni}\underline{\epsilon}_n$$

$$T(\underline{\epsilon}_i) = \begin{pmatrix}a_{1i}\\ \vdots \\ a_{mi}\end{pmatrix} = a_{1i}\varphi_1 + a_{2i}\varphi_2 + \dots a_{mi}\varphi_m$$

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Example $V = \mathbb{Q}^2 = \{ \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} ; x_i \in \mathbb{Q} \}$

$A = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$ let

$T_A : \mathbb{Q}^2 \to \mathbb{Q}^2$ linear mapping

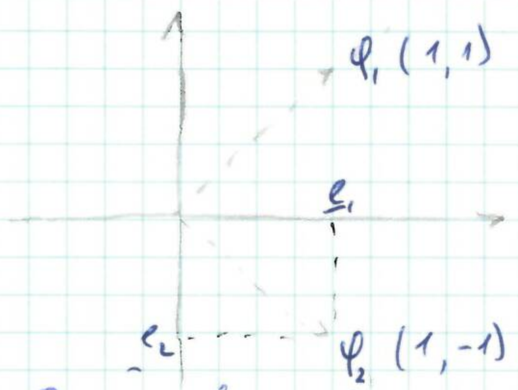
$T_A(\underset{\sim}{x}) = A\underset{\sim}{x}$

$e_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad e_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$T_A(\underset{\sim}{e_1}) = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \; (= e_1 + 2e_2)$

$T_A(\underset{\sim}{e_2}) = \begin{pmatrix} 2 \\ 1 \end{pmatrix} \; (= e_2 + 2e_1)$

$\{\varphi_1, \varphi_2\}$ is a basis for $\mathbb{Q}^2$




and consider effect of $T_A$ on $\varphi_1, \varphi_2$

$$T_A(\varphi_1) = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ 3 \end{pmatrix} = 3\varphi_1 = 3\varphi_1 + 0\varphi_2$$

$$T_A(\varphi_2) = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \end{pmatrix} = -\varphi_2 = 0\varphi_1 + (-1)\varphi_2$$

what is matrix of $T_A$ wrt (with respect to) $\varphi_1, \varphi_2$

$\begin{pmatrix} 3 & 0 \\ 0 & -1 \end{pmatrix}$ In this basis $T_A$ is easier to understand.

Composition

Suppose

$S: W \to U$ is also linear

Already have basis $\Phi = \{\varphi_1 \dots \varphi_m\}$ for $W$

Suppose have basis $\Psi = \{\psi_1 \dots \psi_p\}$ for $U$

I can consider $\left[ M(S)_{\Phi}^{\Psi} = (b_{kj})_{\substack{1 \le k \le p \\ 1 \le j \le m}} \quad p \times m \right]$

$$\begin{tikzcd} \overset{\varepsilon}{V} \arrow[r, "T"] \arrow[rr, bend right, "S \circ T"'] & \overset{\Phi}{W} \arrow[r, "S"] & \overset{\Psi}{U} \end{tikzcd}$$

$S \circ T$ also linear

$M(S \circ T)_{\varepsilon}^{\Psi}$, $M(S)_{\Phi}^{\Psi}$, $M(T)_{\varepsilon}^{\Phi}$

$p \times n \qquad p \times n \qquad m \times n$

Theorem: w.r.t. above

$$\boxed{M(S \circ T)_{\varepsilon}^{\Psi} = M(S)_{\Phi}^{\Psi} \, M(T)_{\varepsilon}^{\Phi}}$$

Proof

$M(T)_{\varepsilon}^{\Phi} = (a_{ji})_{\substack{1 \le j \le m \\ 1 \le i \le n}}$, $M(S)_{\Phi}^{\Psi} = (b_{kj})_{\substack{1 \le k \le p \\ 1 \le j \le m}}$

I suppose $M(S \circ T)_{\varepsilon}^{\Psi} = (c_{ki})_{\substack{1 \le k \le p \\ 1 \le i \le n}}$

i.e. $(S \circ T)(e_i) = \sum_{k=1}^{p} c_{ki} \psi_k$

$T(e_i) = \sum_{j=1}^{m} a_{ji} \varphi_j$

$S \circ T(e_i) = S\left(\sum_{j=1}^{m} a_{ji} \varphi_j\right) = \sum_{j=1}^{m} a_{ji} S(\varphi_j) = \sum_{j=1}^{m} a_{ji} \left\{ \sum_{k=1}^{p} b_{kj} \psi_k \right\} =$

$= \sum_{j=1}^{m} \sum_{k=1}^{p} a_{ji} b_{kj} \psi_k = \sum_{j=1}^{m} \sum_{k=1}^{p} b_{kj} \cdot a_{ji} \psi_k = \sum_{k=1}^{p} \left\{ \sum_{j=1}^{m} b_{kj} a_{ji} \right\} \psi_k$

$\| a_{ji}, b_{kj} \in \mathbb{F}$ so $a_{ji} b_{kj} = b_{kj} a_{ji}$

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## Technical Observation (changing order of summation)

$$\gamma = \begin{pmatrix} \gamma_{11} & \gamma_{12} \\ \gamma_{21} & \gamma_{22} \end{pmatrix}$$

$$\sum_{j=1}^{2} \sum_{i=1}^{2} \gamma_{ij} = \sum_{j=1}^{2} \{\gamma_{1j} + \gamma_{2j}\} = \{\gamma_{11} + \gamma_{21}\} + \{\gamma_{12} + \gamma_{22}\}$$

$$\sum_{i=1}^{2} \sum_{j=1}^{2} \gamma_{ij} = \sum_{i=1}^{2} \{\gamma_{i1} + \gamma_{i2}\} = \{\gamma_{11} + \gamma_{12}\} + \{\gamma_{21} + \gamma_{22}\}$$

because addition is commutative they are (=)

In general : Interchange of order of summation

$$\boxed{\sum_{i=1}^{m} \sum_{j=1}^{n} \gamma_{ij} = \sum_{j=1}^{n} \sum_{i=1}^{m} \gamma_{ij}}$$

## Matrix associated to a linear mapping:

$T : V \to W$ linear

Let $\epsilon_1, \dots \epsilon_n$ be a basis for $V$

$\varphi_1, \dots, \varphi_m$ — " — $W$

Express each $T(\epsilon_i)$ in terms of $(\varphi_1 \dots \varphi_m)$

$$\boxed{T(\epsilon_i) = \sum_{j=1}^{m} a_{ji} \varphi_j \qquad m \times n}$$

Write $\left[ M(T)_{\varepsilon}^{\Phi} = (a_{ji})_{\substack{1 \le j \le m \\ 1 \le i \le n}} \right]$

This is the matrix of $T$ wrt $\varepsilon$ on left $\Phi$ on right.

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But I also have

$$(S \circ T)(e_i) = \sum_{k=1}^{p} c_{ki} \psi_k$$

So $c_{ki} = \sum_{j=1}^{m} b_{kj} a_{ji}$ This is precisely definition of matrix product

QED.

In terms that Physicist, whom understand
Matrix product $\leftrightarrow$ composition.

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Recall: $T : V \to W$ $S : W \to U$

linear

Basis $\mathcal{E} = \{\epsilon_1, \ldots, \epsilon_n\}$ for $V$

$\Phi = \{\varphi_1 \ldots \varphi_m\}$ for $W$

$\Psi = \{\psi_1 \ldots \psi_p\}$ for $U$

$M(T)_{\mathcal{E}}^{\Phi} = (a_{ji})_{\substack{1 \le j \le m \\ 1 \le i \le n}}$ where $T(\epsilon_i) = \sum_{j=1}^{m} a_{ji} \varphi_j$

matrix of $T$ wrt $\mathcal{E}$ on left, $\Phi$ on right

$M(S)_{\Phi}^{\Psi} = (b_{kj})_{\substack{1 \le k \le p \\ 1 \le j \le n}}$ when $S(\varphi_j) = \sum_{k=1}^{p} b_{kj} \psi_k$

$(S \circ T)(\epsilon_i) = \sum_{k=1}^{p} c_{ki} \psi_k$ So $M(S \circ T)_{\mathcal{E}}^{\Psi} = c_{ki}\,{}_{\substack{1 \le k \le p \\ 1 \le i \le n}}$

I showed

$M(S \circ T)_{\mathcal{E}}^{\Psi} = M(S)_{\Phi}^{\Psi} \, M(T)_{\mathcal{E}}^{\Phi}$ $\quad \frac{\Psi}{\mathcal{E}} \quad \frac{\Psi}{\Phi} \frac{\Phi}{\mathcal{E}}$

(Composition of linear maps) $\Leftrightarrow$ (product of matrices)

Important special case:

$V = W$ & $T = id$

and suppose $\mathcal{E} = \{\epsilon_1, \ldots \epsilon_n\}$

$\Phi = \{\varphi_1, \ldots, \varphi_n\}$

are both basis for $V$

I can express each $\varphi_1$ as a linear combination in $\epsilon_1 \ldots \epsilon_n$

$$\varphi_j = \sum_{i=1}^{n} a_{ij} \epsilon_i$$

I can express and each $\epsilon_i$ as lin comb in $\varphi_1 \ldots \varphi_n$

$$\epsilon_i = \sum_{j=1}^{n} b_{ji} \varphi_j$$

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$$M(Id)^{\mathcal{E}}_{\Phi} = (a_{ij})_{\substack{1 \le i \le n \\ 1 \le j \le n}} \qquad (1)$$

Similarly

$$M(Id)^{\Phi}_{\mathcal{E}} = (b_{ji})_{\substack{1 \le j \le n \\ 1 \le i \le n}} \qquad (2)$$

what is relation between (1) & (2) ?

$$\boxed{M(Id)^{\mathcal{E}}_{\mathcal{E}} = M(Id)^{\mathcal{E}}_{\Phi}\, M(Id)^{\Phi}_{\mathcal{E}} = (a_{ij})(b_{ji})}$$

$Id$

Prop. $\boxed{\text{For any basis } \mathcal{E} \text{ of } V \; M(Id)^{\mathcal{E}}_{\mathcal{E}} = Id}$

Proof $Id(\epsilon_i) = \epsilon_i = \sum_{j=1}^{n} \delta_{ji} \epsilon_j$

$Id(\epsilon_1) = \epsilon_1 + 0\epsilon_2 + \dots + 0\epsilon_n$

$Id(\epsilon_2) = 0\epsilon_1 + \epsilon_2 + \dots + 0\epsilon_n$

So we proved prop.

Prop $\boxed{M(Id)^{\Phi}_{\mathcal{E}} = \left[M(Id)^{\mathcal{E}}_{\Phi}\right]^{-1}}$

example $V = \mathbb{F}^3$

$\mathcal{E} = \{e_1, e_2, e_3\}$ = standard basis

$\Phi = \{\varphi_1, \varphi_2, \varphi_3\}$

$\varphi_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = e_1 \quad \varphi_2 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = e_1 + e_2 \quad \varphi_2 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = e_1 + e_2 + e_3$

Write down $M(Id)^{\mathcal{E}}_{\Phi}$

i.e. express $\{\varphi_j\}$ ~~a~~ in terms of $\{e_i\}$

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$$M(Id)^{\mathcal{E}}_{\Phi} = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

Now write down $M(Id)^{\Phi}_{\mathcal{E}}$ i.e. express $\{e_i\}$ in terms of $\{\varphi_j\}$

$e_1 = 1\varphi_1 + 0\varphi_2 + 0\varphi_3$

$e_2 = -\varphi_1 + \varphi_2 + 0\varphi_3$

$e_3 = 0\varphi_1 - \varphi_2 + \varphi_3$

$\Rightarrow \begin{array}{c} \\ \varphi_1\varphi_2\varphi_3 \\ \\ \end{array}\begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix}$

From 1st principles

$e_1 = \varphi_1$ $\quad e_2 = ?$ $\quad e_1 + e_2 = \varphi_2$, $e_2 = -e_1 + \varphi_2 = -\varphi_1 + \varphi_2$

$e_2 = -\varphi_1 + \varphi_2$ $\quad e_3 = -e_1 + e_2 + \varphi_3 = \cancel{-\varphi_1} + \varphi_1 - \varphi_2 + \varphi_3 = -\varphi_2 + \varphi_3$

$M(Id)^{\Phi}_{\mathcal{E}} = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix}$ we claim $\begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}^{-1}$

$$\begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Back to $A = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$

$T_{\mathcal{E}} : \mathbb{Q}^2 \to \mathbb{Q}^2$

$T(\underset{\sim}{x}) = Ax$ $\quad \left\{ e_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad e_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\} = \mathcal{E}$

$\left\{ \varphi_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad \varphi_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \right\} = \Phi$

We know $M(T)^{\Phi}_{\Phi} = M(Id)^{\Phi}_{\mathcal{E}} \cdot M(T)^{\mathcal{E}}_{\mathcal{E}} \cdot M(Id)^{\mathcal{E}}_{\Phi}$

$T = Id \cdot T \cdot Id$

First compute $M(T)^{\mathcal{E}}_{\mathcal{E}} = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$

$Ae_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ $\quad Ae_2 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$

we showed $M(T)^{\Phi}_{\Phi} = \begin{pmatrix} 3 & 0 \\ 0 & -1 \end{pmatrix}$

We also know $M(Id)^{\mathcal{E}}_{\Phi}$

i.e. express $\varphi_j$ in terms of ~~$\varphi_1$ $\varphi_2$~~ $e_i$

$M(Id)^{\mathcal{E}}_{\Phi} = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$ $\quad \varphi_1 = e_1 + e_2$

$\varphi_2 = e_1 - e_2$

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So I should be able to compute $M(Id)^{\Phi}_{\mathcal{E}} = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}^{-1}$

$$\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}^{-1} = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} = \begin{pmatrix} 1/2 & 1/2 \\ 1/2 & -1/2 \end{pmatrix}$$

So now

$$M(T)^{\Phi}_{\Phi} = \begin{pmatrix} 1/2 & 1/2 \\ 1/2 & -1/2 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} = \begin{pmatrix} 3/2 & 3/2 \\ -1/2 & 1/2 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} = \begin{pmatrix} 3 & 0 \\ 0 & -1 \end{pmatrix}$$

$T: V \to V$ linear

Two basis $\mathcal{E}: \{e_1, \dots e_n\}$ $\Phi = \{\varphi_1 \dots \varphi_n\}$ for $V$

$M(T)^{\mathcal{E}}_{\mathcal{E}}$ = matrix of $T$ wrt $\mathcal{E}$ on both sides

$M(T)^{\Phi}_{\Phi}$ ... ... $\Phi$

$$M(T)^{\Phi}_{\Phi} = M(Id)^{\Phi}_{\mathcal{E}}\, M(T)^{\mathcal{E}}_{\mathcal{E}}\, M(Id)^{\mathcal{E}}_{\Phi}$$

matrix of basis change

$A' = PAP^{-1}$

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Recall: $T: V \to W$ linear

$\mathcal{E} = \{e_1, \dots e_n\}$ basis for $V$

$\mathcal{F} = \{\varphi_1, \varphi_2 \dots \varphi_m\}$ basis for $W$

We get an $m \times n$ matrix $M(T)^{\mathcal{F}}_{\mathcal{E}}$

Define by $M(T)^{\mathcal{F}}_{\mathcal{E}} = (\tau_{ji})_{\substack{1 \le j \le m \\ 1 \le i \le n}}$

$$T(e_i) = \sum_{j=1}^{m} \tau_{ji} \varphi_j$$

If $W = V$ usual (though not necessary)

we take $\mathcal{F} = \mathcal{E}$, i.e. same basis on each side

$$M(T)^{\mathcal{E}}_{\mathcal{E}} = A$$

---

Examples 1) $P_n(\mathbb{Q}) = \{$polynomials of degree $\le n$ with coefficients in $\mathbb{Q}\}$

i.e. $P_n(\mathbb{Q}) = \{a_0 + a_1 t + \dots + a_n t^n$, where $a_1, \dots, a_n \in \mathbb{Q}\}$

Theorem Rule of equality:

$$a_0 + a_1 t + \dots + a_n t^n = b_0 + b_1 t + \dots + b_n t^n$$

iff. $\forall i \; a_i = b_i$

—

So $1, t, \dots t^n$ span $P_n(\mathbb{Q})$ and they are LI by

So $\{1, t, \dots, t^n\}$ is a basis for $P_n(\mathbb{Q})$

$\dim P_n(\mathbb{Q}) = n+1$

$a(t) = a_0 1 + a_1 t + \dots a_n t^n$

Take $D: P_n(\mathbb{Q}) \to P_n(\mathbb{Q})$

$$D(a(t)) = \frac{d}{dt} a(t)$$

$D$ is linear $D(a+b) = D(a) + D(b)$

$D(\lambda a) = \lambda D(a)$ $(\lambda \in \mathbb{Q})$

(80) Take $n=3$ $\mathcal{E}=\{1, t, t^2, t^3\}$

Find $M(D)^{\mathcal{E}}_{\mathcal{E}}$

$D(1) = 0$ $\qquad D(e_1) = 0e_1 + 0e_2 + 0e_3 + 0e_4$

$D(t) = 1$ $\qquad D(e_2) = 1e_1 + 0e_2 + 0e_3 + 0e_4$

$D(t^2) = 2t$ $\qquad D(e_3) = 2e_2 + 0e_1 + 0e_3 + 0e_4$

$D(t^3) = 3t$ $\qquad D(e_4) = 0e_1 + 0e_2 + 3e_3 + 0e_4$

$$M(D)^{\mathcal{E}}_{\mathcal{E}} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & \boxed{2} & 0 \\ 0 & 0 & 0 & \boxed{3} \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

---

$$f(e_i) = \sum_{j=1}^{m} a_{ji} \varphi_j \qquad A = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ a_{m1} & \cdots & a_{mn} \end{pmatrix}$$

2) $V = \{ f : \mathbb{R} \to \mathbb{R} \}$ of the form $f(x) = \lambda_1 \exp(x) + \lambda_2 x \exp(x) + \lambda_3 x^2 \exp(x) \}$ where $\lambda_1, \lambda_2, \lambda_3 \in \mathbb{R}$

$\varphi_1 = \exp(x)$ $\quad \varphi_2 = x\exp(x)$ $\quad \varphi_3 = x^2 \exp(x)$

$\{\varphi_1, \varphi_2, \varphi_3\}$ span $V$. They are, in fact, LI

Again have <u>linear</u> map

$D : V \to V$ $\qquad D(f) = \frac{df}{dx}$

Now compute $M(D)^{\mathcal{F}}_{\mathcal{F}}$

$\mathcal{F} = \{\varphi_1, \varphi_2, \varphi_3\}$

$$D(\varphi_1) = \frac{d}{dx}\exp(x) = \exp(x) = \varphi_1 + 0\varphi_2 + 0\varphi_3$$

$$D(\varphi_2) = \frac{d}{dx}(x\exp(x)) = \exp(x) + x\exp(x) = \varphi_1 + \varphi_2 + 0\varphi_3$$

$$D(\varphi_3) = \frac{d}{dx}(x^2\exp(x)) = 2x\exp(x) + x^2\exp(x) = 2\varphi_2 + \varphi_3 + 0\varphi_1$$

$$M(D)^{\mathcal{F}}_{\mathcal{F}} = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix}$$

$$D^2 = \frac{d^2}{dx^2}$$

$$M(D^2)^{\mathcal{F}}_{\mathcal{F}} = M(D)^{\mathcal{F}}_{\mathcal{F}}\, M(D)^{\mathcal{F}}_{\mathcal{F}} = \begin{pmatrix}1&1&0\\0&1&2\\0&0&1\end{pmatrix}\begin{pmatrix}1&1&0\\0&1&2\\0&0&1\end{pmatrix} = \begin{pmatrix}1&2&2\\0&1&4\\0&0&1\end{pmatrix}$$

So calculate $\frac{d^2}{dx^2}(2\exp(x) + 11x\exp(x) - 5x^2\exp(x))$

$$\exp(x) \sim \begin{pmatrix}1\\0\\0\end{pmatrix} \quad x\exp(x) \sim \begin{pmatrix}0\\1\\0\end{pmatrix} \quad x^2\exp x \sim \begin{pmatrix}0\\0\\1\end{pmatrix}$$

$$f(x) \sim \begin{pmatrix}2\\11\\-5\end{pmatrix}$$

$$D^2(f) \sim \begin{pmatrix}1&2&2\\0&1&4\\0&0&1\end{pmatrix}\begin{pmatrix}2\\11\\-5\end{pmatrix} = \begin{pmatrix}14\\-9\\-5\end{pmatrix} \sim 14\exp(x) - 9x\exp(x) - 5x^2\exp(x)$$

Q Is $\begin{pmatrix}1&1&0\\0&1&2\\0&0&1\end{pmatrix}$ invertible? Yes

$$\left(\begin{array}{ccc|ccc}1&1&0&1&0&0\\0&1&2&0&1&0\\0&0&1&0&0&1\end{array}\right) \longrightarrow \left(\begin{array}{ccc|ccc}1&0&0&1&-1&2\\0&1&0&0&1&-2\\0&0&1&0&0&1\end{array}\right)$$

$$M(D^{-1})^{\mathcal{F}}_{\mathcal{F}} = \begin{pmatrix}1&-1&2\\0&1&-2\\0&0&1\end{pmatrix} \qquad D^{-1} = \int \text{ so we can integrate}$$

Example $\int x^2\exp(x)\,dx$

corresponds to
$$\overbrace{\begin{pmatrix}1&-1&2\\0&1&-2\\0&0&1\end{pmatrix}}^{D^{-1}}\begin{pmatrix}0\\0\\1\end{pmatrix} = \begin{pmatrix}2\\-2\\1\end{pmatrix}$$

Claim

$$\int x^2\exp(x)\,dx = 2\exp(x) - 2x\exp(x) + x^2\exp(x)$$

$$D = \cancel{2\exp(x)} - \cancel{2x\exp(x)} - \cancel{2\exp(x)} + x^2\exp(x) + \cancel{2x\exp(x)}$$
$$= x^2\exp(x).$$

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1) $\begin{cases} a_{11}x_1 + \dots a_{1n}x_n = b_1 \\ a_{21}x_1 + \dots a_{2n}x_n = b_2 \\ \vdots \\ a_{m1}x_1 + \dots a_{mn}x_n = b_m \end{cases}$ System of linear eq.$^n$s

OLD STYLE

$A\underline{x} = b$, $A = \begin{pmatrix} a_{11} \dots a_{1n} \\ \vdots \\ a_{m1} \dots a_{mn} \end{pmatrix} = (a_{ij})_{\substack{1 \le i \le m \\ 1 \le j \le n}}$

< 1840

NEW STYLE => linear mapping < 1970

$T : \mathbb{F}^n \to \mathbb{F}^m$

standard basis $e_1, \dots, e_n$

$E_1, \dots, E_m$

$T(e_i) = \sum_{j=1}^{m} a_{ji} E_j \qquad i = 1 \dots n$

**Prop** Prop. If $T : U \to V$, $S : V \to W$ linear over $\mathbb{F}$

then $S \circ T : U \to W$ is also linear over $\mathbb{F}$

**Proof** 'T linear' means $T(\underline{x} + y) = T(\underline{x}) + T(y)$

$T(\lambda \underline{x}) = \lambda T(\underline{x})$

'S linear' means $S(\underline{\xi} + y) = S(\xi) + S(y)$

$S(\lambda \xi) = \lambda S(\xi)$

Compare:

$S \circ T(x+y) = S(T(x+y)) = S(T(x) + T(y)) = S(T(x)) + S(T(y)) =$

$= (S \circ T)(x) + (S \circ T)(y)$

$S \circ T(\lambda x) = S(T(\lambda x)) = S(\lambda T(x)) = \lambda S(T(x)) = \cancel{\lambda}$

$= \lambda (S \circ T)(x)$ QED.

OLD MAtrix $\longleftrightarrow$ NEW Linear map.

**Prop** Let $\{e_1 \dots e_n\}$ be basis for $V$
$\{\epsilon_1 \dots \epsilon_m\}$ — " — for $W$
and let $A = (a_{ij})_{\substack{1 \le i \le m \\ 1 \le j \le n}}$ be $m \times n$ matrix over $\mathbb{F}$

Define $T: V \to W$ by

$$\underline{x} \in V \quad \underline{x} = \sum_{i=1}^{n} x_i e_i$$

$$T(\underline{x}) = \sum_{i=1, j=1}^{n\ m} x_i a_{ji} \epsilon_j$$

then $T$ is linear (map) and $M(T)_{\varepsilon}^{\epsilon} = A$

**Proof** $\underline{x} = \sum x_i e_i \qquad y = \sum y_i e_i$

$$\underline{x} + y = \sum_i (x_i + y_i) e_i$$

$$T(\underline{x} + y) = \sum_{i=1\, j=1}^{n\ m} (x_i + y_i) a_{ji} \epsilon_j = \sum_{i=1\, j=1}^{n\ m} x_i a_{ji} \epsilon_j + \sum_{ij} y_i a_{ji} \epsilon_j =$$

$$= T(x) + T(y)$$

$$T(\lambda \underline{x}) = T\left(\sum \lambda x_i e_i\right) = \sum_{i,j} \lambda x_i a_{ji} \epsilon_j = \lambda \left(\sum_{i,j} x_i a_{ji} \epsilon_j\right) = \lambda T(x)$$

so $T$ is linear $\quad e_i = 0 \dots + 1 \cdot e_i + \dots 0$

Also $T(e_i) = \sum_{j=1}^{n} a_{ji} \epsilon_j$

by def$^n$ $M(T)_{\varepsilon}^{\epsilon} = A = (a_{ji})_{\substack{1 \le j \le m \\ 1 \le i \le n}}$ Q.E.D.

84 $M(S\circ T)_{\varepsilon}^{\Psi} = M(S)_{\Phi}^{\Psi} \, M(T)_{\varepsilon}^{\Phi}$

composition matrix product

composition is always Assoc. ~~Addition~~ & matrix prod. is also assoc.

**Prop** Let $\underset{m\times n}{A}$ $\underset{n\times p}{B}$ $\underset{p\times q}{C}$ : matrices over $\mathbb{F}$

then $(A(B)C = A(BC)$

**Proof** Let $T_A : \mathbb{F}^n \to \mathbb{F}^m$ $T_A(x) = Ax$

$T_B : \mathbb{F}^p \to \mathbb{F}^n$ $T_B(y) = By$

$T_C : \mathbb{F}^q \to \mathbb{F}^p$ $T_C(z) = Cz$

$$\begin{tikzcd} \mathbb{F}^q \arrow[r, "T_C"] \arrow[rr, bend right, "T_B\circ T_C"'] & \mathbb{F}^p \arrow[r, "T_B"] \arrow[rr, bend left, "T_A\circ T_B"] & \mathbb{F}^n \arrow[r, "T_A"] & \mathbb{F}^m \end{tikzcd}$$

comp. is assoc.

$T_A\circ(T_B\circ T_C) = (T_A\circ T_B)\circ T_C$

$T_A\circ(T_B\circ T_C)(z) = A(BC)z$

$(T_A T_B)T_C = (AB)Cz$

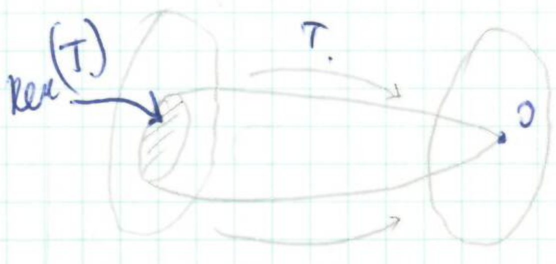
so $A(BC)z = (AB)Cz$

true for all $z$ so $A(BC)$, $(AB)C$ have identical columns,

$j = 1 \dots q$ QED.

**Def$^n$ 85** Let $T: V \to W$ be a linear map ($V, W$ vector space over $\mathbb{F}$)

$\ker(T) := \{ \underline{x} \in V : T(\underline{x}) = 0 \}$
(kernel of $T$)




**Def$^n$** Let $V$ be a vector space $/\mathbb{F}$. Let $U \subset V$
Say that $U$ is a vector space of $V$
iff. (1) $\forall \underline{x}, y \in U$, $\underline{x} + y \in U$
(2) $\forall x \in U\ \forall \lambda \in \mathbb{F}$ $\lambda \underline{x} \in U$
(3) $0 \in U$

**Prop** If $U$ is a vector space of $V$ then $U$ is a vector space in its own right

**Proof** $U$ has addition, scalar mult., zero and all other axioms automatically satisfied, because satisfied in $V$ Q.E.D.

**Prop** Let $T: V \to W$ be lin., Then $\ker(T)$ is a ~~vector sp~~ subspace of $V$

**Proof** Let $x, y \in \ker(T)$ $T(x) = 0$ $T(y) = 0$, $T(x+y) = Tx + Ty = 0 + 0 = 0$
So $x + y \in \ker(T)$

$x \in \ker(T)$, $\lambda \in \mathbb{F}$ - $T(x) - 0$

$T(\lambda x) = \lambda T(x) = \lambda 0 = 0$

So $\lambda x \in \ker(T)$

 Finally $T(\underline{0}) = \underline{0}$

Q why?

$0 = 0 + 0$

$T(0) = T(0) + T(0)$

Add $-T(0)$ to each side

$T(0) - T(0) = T(0) + (T(0) - T(0))$

$0 = T(0) = 0$

$0 = T0$ QED.

Defn Let $T: V \to W$ be linear

$Im(T) := \{ \underline{w} \in W : \exists \underline{a} \in V : T(\underline{a}) = \underline{w} \}$

Image of $T$ / i.e. the set of vectors you can hit with $T$

Prop If $T: V \to W$ linear
then the $Im\ T)$ is a vector subspace of $W$

Proof Let $w_1, w_2' \in Im(T)$

$T(\underline{a}) = \underline{w}$ ~~$T a_1$~~ $_2$ $T(a') = w'$

$T(\underline{a}_1 + \underline{a}') = T(\underline{a}_1) + T(a_2') = w_1 + v_2'$

So $w_1 + w_2 \in Im(T)$

$w \in Im(T)$ $\lambda \in \mathbb{F}$

$T(a) = w$ $T(\lambda a) = \lambda T(a) = \lambda w$

so $\lambda w \in Im(T)$

$T(0) = 0$ so $0 \in Im(T)$

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OLD | NEW
--- | ---

| OLD | | NEW |
|---|---|---|
| Matrix | $\Longleftrightarrow$ | Linear map |
| matrix prod. | $\Longleftrightarrow$ | Comp. of lin. map. |
| Solution set $A\underline{x} = 0$ | $\Longleftrightarrow$ | $\{x : Tx = 0\} = \ker(T)$ |

kernel of T

$A\underline{x} = \underline{b}$ for which $\underline{b}$ $\Longleftrightarrow$ $\operatorname{Im}(T)$

$Ax = 0$ does $\exists$ solution?

$$T : V \to W$$
$$\ker(T) \subseteq V \qquad \operatorname{Im}(T) \subseteq W$$

~~rank~~ kernel ~~them~~

$$\dim \ker(T) + \dim \operatorname{Im}(T) = \dim V$$

| OLD | NeW |
| --- | --- |
| $A\underline{x} = b$ A is $m \times n$ <br> $\begin{cases} a_1 x_1 + \dots + a_{1n} x_n = b_1 \\ a_2 x_1 + \dots + a_{2n} x_n = b_2 \\ \vdots \end{cases}$ | $T: V \to W$ <br> linear map <br> $T(\underline{x}) = \underline{b}$ |
| $A\underline{x} = 0$ <br> homogeneous system | $\ker(T) = \{\underline{x} \in V, T(x) = 0\}$ |
| Ask for which B <br> $Ay = b$ has a solution | $\operatorname{Im}(T) = \{\underline{b} \in W : \exists \underline{x} \in V \; T(x) = b\}$ |

## Kernel Rank Theorem

Let $T: V \to W$ be a linear map. Then

$$\boxed{\dim \ker(T) + \dim \operatorname{Im}(T) = \dim V}$$

I will use Basis Theore (shall to be proved)

[ Basis Theorem: Any non zero vector space $V$ has a basis. Any two bases for $V$ have same no of elements

**Proof**: Suppose $\ker(T) \neq \phi$ and $\operatorname{Im}(T) \neq \phi$

Let $e_1 \dots e_k$ be a basis for $\ker(T)$

Let $f_1 \dots f_m$ be a basis for $\operatorname{Im}(T)$

so $\dim \ker(T) = k$, $\dim \operatorname{Im}(T) = m$

For each $i$ 1

8(b)

For each $i$ $1 \le i \le m$ choose vector $e_{k+i} \in V$ such that $T(e_{k+i}) = \varphi_i$ ($\varphi_i \in Im(T)$)

So now I have vectors

$e_1, \dots e_{k+m} \in V$

I claim these form a basis for $V$

Need to show $e_1, \dots e_{k+m}$ is LI

and $e_1, \dots e_{k+m}$ is ~~basis for~~ span $V$

Suppose $\lambda_1 e_1 + \dots + \lambda_k e_k + \lambda_{k+1} e_{k+1} + \dots + \lambda_{k+m} e_{k+m} = 0$

(go to show $\lambda_1 = \lambda_2 = \dots = \lambda_{k+m} = 0$) — ποσεις Τ σε όλους ρητώς

$$T\left(\sum_{i=1}^{k+m} \lambda_i e_i\right) = T\left(\sum_{i=1}^{k} \lambda_i e_i + \sum_{j} \lambda_j e_j\right) = T(0) = 0$$

$$\sum_{j=1}^{k} \lambda_i T(e_i) + \sum_{j=1}^{m} \lambda_{k+j} T(e_{k+j}) = 0$$

$T(e_i) = 0 \quad 1 \le i \le k$

$T(e_{k+j}) = \varphi_j \quad 1 \le j \le m$ (by choice)

So $\sum_{j=1}^{m} \lambda_{k+j} \varphi_j = 0$ But $\varphi$ are LI, as they form basis for Image

So $[\lambda_{k+1} = \lambda_{k+2} = \dots = \lambda_{k+m} = 0]$

subst. bac.

$\lambda_1 e_1 + \dots + \lambda_k e_k = 0$

But $e_1, \dots e_k$ are LI (basis for $\ker(T)$)

So $[\lambda_1 = \lambda_2 = \dots = \lambda_k = 0]$

$\Rightarrow \forall i \; 1 \le i \le k+m \quad \lambda_i = 0$ QED (LI.

9) Now have to show $e_{k+1}, \dots, e_{k+m}$ span $V$

Let $\underline{x} \in V$, $T(\underline{x}) \in Im(T)$

Write $T(\underline{x}) = \mu_1 \varphi_1 + \dots + \mu_m \varphi_m$

($\varphi_1, \dots \varphi_m$ span $Im(T)$)

Define $y = \mu_1 e_{k+1} + \dots + \mu_m e_{k+m} \in V$

so $T(y) = \mu_1 T(e_{k+1}) + \dots + \mu_m T(e_{k+m}) =$

$= \mu_1 \varphi_1 + \dots\dots + \mu_m \varphi_m = T(\underline{x})$

So $T(x) - T(y) = 0$

$T(x - y) = 0$

So $\underline{x} - y \in \ker(T)$

So $\underline{x} - y = \lambda_1 e_1 + \dots + \lambda_k e_k$ for such $\lambda_i$

$\underline{x} = \lambda_1 e_1 + \dots + \lambda_k e_k + \mu_1 e_{k+1} + \dots + \mu_m e_{k+m}$

This linear combination in $e_1 \dots e_{k+m}$

so $e_1 \dots e_{k+m}$ span $V$

$\Rightarrow$ $e_1 \dots e_{k+m}$ is a basis for $V$

$$\dim(V) = k+m = \dim(\ker T) + \dim(Im\, T)$$

Special cases:

1) $\underline{\ker(T) = \phi}$ (here $k=0$). Take basis $\varphi_1 \dots \varphi_m$ for $Im\, T$

$e_1 \dots e_m \in V$

$T(e_i) = \varphi_i$

Same proof shows $e_1, \dots e_m$ is ~~b~~ a basis for $V$

$\dim(V) = 0 + \dim(Im(T))$, $(k=0)$

2) $\underset{(m=0)}{\underline{Im(T) = \phi}}$ then $\forall x \in V$ $T(x) = 0$

i.e. $V = \ker(T)$

$$\lambda \begin{pmatrix}1\\1\\3\end{pmatrix} + \lambda_2 \begin{pmatrix}1\\-1\\1\end{pmatrix} = 0$$

$$\lambda_1 + \lambda_2 = b_1$$

$$\lambda_1 - \lambda_2 = b_2 \qquad \lambda_1 = 0 \quad \lambda_2 = 0$$

$$3\lambda_1 + \lambda_2 = b_3$$

$$2\lambda_1 = b_1 + b_2 \quad \Rightarrow \lambda_1 = \frac{b_1 + b_2}{2}$$

$$\lambda_2 = b_1 - \frac{b_1 + b_2}{2} = \frac{b_1}{2} - \frac{b_2}{2}$$

$$\lambda_1 = b_1$$

$$\lambda_2 = b_2$$

$$2\lambda_1 + \lambda_1 = b_3$$

**Example** 81: $A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ 3 & 1 & 3 \end{pmatrix}$

Look at $T_A : \mathbb{Q}^3 \to \mathbb{Q}^3$

$$T_A(x) = A\underset{\sim}{x}$$

To find $\ker(T_A)$ we solve $A(x) = 0$

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 1 & -1 & 1 & 0 \\ 3 & 1 & 3 & 0 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & -2 & 0 & 0 \\ 0 & -2 & 0 & 0 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$$\begin{array}{ccc} x_1 & x_2 & x_3 \end{array}$$

$\therefore\ x_1 = -x_3$

$x_2 = 0$

$x_3 = x_3$

So $\dim \ker(T) = 1$ $\quad \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$ is a basis for $\ker(T)$

Now consider for which $b \in \mathbb{Q}^3$ there exists a solution

$$A x = \underline{b} \qquad b = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$$

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & b_1 \\ 1 & -1 & 1 & b_2 \\ 3 & 1 & 3 & b_3 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 1 & 1 & b_1 \\ 0 & -2 & 0 & -b_1 + b_2 \\ 0 & -2 & 0 & -3b_1 + b_3 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 1 & 1 & b_1 \\ 0 & -2 & 0 & -b_1 + b_2 \\ 0 & 0 & 0 & -2b_1 - b_2 + b_3 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 0 & 1 & \frac{b_1}{2} + \frac{b_2}{2} \\ 0 & -1 & 0 & \frac{b_1}{2} - \frac{b_2}{2} \\ 0 & 0 & 0 & -2b_1 - b_2 + b_3 \end{array}\right)$$

Necessary condition is

$$-2b_1 - b_2 + b_3 = 0$$

$$b_3 = 2b_1 + b_2$$

$\begin{pmatrix} 1 \\ 1 \\ 3 \end{pmatrix} \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} = 0$

$$\operatorname{Im}(T) = \left\{ \begin{pmatrix} b_1 \\ b_2 \\ 2b_1 + b_2 \end{pmatrix} \ b_1, b_2 \in \mathbb{Q} \right. !$$

$\begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$

So $\dim \operatorname{Im}(T) = 2 \ \cap\ \dim \mathbb{Q}^3 = 3$

$\dim \ker(T) = 1 \qquad 3 = 1 + 2$

Theorem **Basis Theorem.**

Let $V$ be a (non zero) vector space over a field $\mathbb{F}$.
then 1) $V$ has at least one basis
2) any two basis have the same number of elements.

Proof I I'll prove (2) first

Lemma 1 **Exchange Lemma** (Steinitz - 1910)
Let $\{v_1, \dots, v_k\}$ be LI subset of vector space $V$
and $\{w_1, \dots, w_n\}$ be a spanning set for $V$

Then
i) $k \le n$ and
ii) there exists a spanning set $\{w_1', \dots, w_n'\}$
such that $w_r' = v_r$ for $1 \le r \le k$ and
$w_r' \in \{w_1', \dots, w_n'\}$ for $k < r$

First we prove:
Lemma 2 **Exchange Lemma** (Baby version)
Let $v \in V$ $v \neq 0$ and let $\{w_1 \dots w_m\}$ be a spanning set.
Express $v = \lambda_1 w_1 + \dots + \lambda_i w_i + \dots + \lambda_m w_m$
If $\lambda_i \neq 0$ then $\{w_1, \dots, w_{i-1}; v; w_{i+1}, \dots, w_m\}$
is a spanning set.
(In English I've swapped $v$ for $w_i$)

proof 2 $v = \lambda_1 w_1 + \dots + \lambda_i w_i + \dots + \lambda_m w_m$
(I can do this because $w_1 \dots w_m$ span $V$)

Suppose $\lambda_i \neq 0 \rightarrow v = \lambda_i w_i + \sum_{r \neq i} \lambda_r w_r$
$v \neq 0$ So since $\lambda_i \neq 0$

9a) $w_i = \frac{1}{\lambda_i} v + \sum_{r \neq i} \left(\frac{-\lambda_r}{\lambda_i}\right) w_r \quad \left(\lambda_i \neq 0, \text{ so } \frac{1}{\lambda_i} \in F\right)$

I claim that $\{w_1, \dots, w_{i-1}; v; w_{i+1}, \dots w_m\}$ still span $V$

Let $\underline{x} \in V$ (I have to write $\underline{x}$ as lin. comb. of m)

Write $\underline{x} = \sum_{r=1}^{m} \mu_r w_r = \mu_i w_i + \sum_{r \neq i} \mu_r w_r$

Substitute $w_i$ using

$$x = \mu_i \left(\frac{v}{\lambda_i}\right) + \mu_i \left(\sum_{r \neq 0} \left(\frac{-\lambda_r}{\lambda_i}\right) w_r\right) + \sum_{r \neq i} \mu_r w_r$$

$$x = \frac{\mu_i}{\lambda_i} v + \sum_{r \neq i} \left(\mu_r - \frac{\mu_i \lambda_r}{\lambda_i}\right) w_r$$

Proof 1 Proof by induction on $k$

Let $P(k)$ be the statement for $k \geq 1$

i.e. $P(k)$ $\begin{cases} \text{If } \{v_1 \dots v_k\} \text{ is LI and } \{w_1 \dots w_m\} \text{ span} \\ \text{then i) } k \leq m \text{ and} \\ \text{ii) } \exists \text{ spanning set } \{w_1' \dots w_m'\} \\ \quad w_r' = v_r \ \ r \leq k,\ w_r' \in \{w_1 \dots w_m\} \ \ k < r \end{cases}$ S

$P(1)$ is true by Baby Version ($v_1 = v$)

suppose $P(k-1)$ is proved.

$\{v_1 \dots v_k\}$ original LI set

$\{w_1 \dots w_m\}$ " spanning set

By hypothesis $P(k-1)$ I've got a new spanning set.

$\{u_1, \dots u_{k-1}, u_k, \dots u_m\}$ in which $u_i = v_i \quad i \leq k-1$

9ii $v_k = \lambda_1 u_1 + \dots + \lambda_{k-1} u_{k-1} + \lambda_k u_k + \dots + \lambda_m u_m$

Now $v_k \neq 0$ because $v_1, \dots v_k$ is LI

"If $v_k = 0$ $\quad 0v_1 + 0v_2 + \dots + 0v_{k-1} + 1v_k = 0$

Dependence relation"

If $\lambda_k = \dots = \lambda_m = 0$ I would have

$v_k = \lambda_i u_i + \dots + \lambda_k u_{k-1} = \lambda_i v_1 + \dots + \lambda_{k-1} v_{k-1}$

(Dependence rel$^n$. contradiction)

Therefore $\lambda_r \neq 0$ for some $r$ $\quad k-1 < r$

Swap $v_k$ and $u_r$ by baby lemma.

Now I have a spanning set

$\{u_1, \dots u_{k-1} \mid u_k \dots u_{r-1} \mid v_k \mid u_{r+1} \dots u_m\}$

Reorder Put $w'_s = w_s = v_s$ for $s < k-1$

$w'_k = v_k$; $w'_s = u_{s-1}$ for $k+1 \leq s \leq r$

$w'_s = u_s$ $\quad r+1 \leq s$

So now $\{ \underset{v_1}{\overset{\shortparallel}{w'_1}} \dots \underset{v_k}{\overset{\shortparallel}{w'_k}} \mid w'_{k+1} \dots w'_m \}$ - spanning set

and $w'_{k+1} \dots w'_m \in \{w_1 \dots w_m\}$

So $P(k-1) \to P(k)$ $\qquad$ Q.E.D.

II Now I'll prove (1):

Let $\{\epsilon_1 \dots, \epsilon_m\}$ $\{\varphi_1 \dots \varphi_n\}$ be bases for $V$

$\{\epsilon_1 \dots \epsilon_n\}$ is LI $\{\varphi_1 \dots \varphi_n\}$ spans so

$\{\varphi_1 \dots \varphi_n\}$ is LI $\{\epsilon_1 \dots \epsilon_m\}$ spans, so $n \leq m$ so $m = n$ QED

Basis Theorem

Let $V$ be a non-zero vector space $/ \mathbb{F}$, then

1) $V$ has a basis
2) Any two bases have same number of elements.

Proof (2) Suppose $\{\epsilon_1 \ldots \epsilon_m\}$, $\{\varphi_1 \ldots \varphi_n\}$ are both basis

$\{\epsilon_1 \ldots \epsilon_m\}$ is LI, $\{\varphi_1 \ldots \varphi_n\}$ spans so $m \leq n$

$\{\varphi_1 \ldots \varphi_n\}$ is LI and $\{\epsilon_1 \ldots \epsilon_m\}$ spans so $n \leq m$

(proved in previous lecture)

so $m \leq n \leq m$ so $m = n$ Q.E.D.

Proof 1 Let $S(m)$ be the ~~flow~~ following statement

"If $V \neq 0$ and $V$ is spanned by a set $\{\varphi_1 \ldots \varphi_m\}$ with $m$ elements then $\{\varphi_1 \ldots \varphi_m\}$ contains a basis for $V$"

I'll prove each $S(m)$ is true $(m \geq 1)$

1) $S(1) = 1$ If $\varphi_1$ spans $V \neq 0$, then $\varphi_1$ is a basis for $V$

claim $\varphi_1$ is LI

otherwise if $\lambda_1 \varphi_1 = 0$ and $\lambda_1 \neq 0$

$\varphi_1 = \lambda_1^{-1} \lambda_1 \varphi_1 = \lambda_1^{-1} 0 = 0$

So $\varphi_1 = 0$ so everything in $V$ is zero

i.e. $V = 0$ contradiction

So $S(1)$ is true

2) $S(m-1) \Rightarrow S(m)$ $m \geq 2$

So suppose $\varphi_1 \ldots \varphi_m$ is a spanning set for $V$

If $\varphi_1, \ldots, \varphi_m$ is LI then $\{\varphi_1, \ldots, \varphi_m\}$ is a basis and I'm finished.

Q6 I have a rel$^n$ $\lambda_1 \varphi_1 + \dots \lambda_m \varphi_m = 0$
And at least one coefficient, say $\lambda_r$, is non-zero
$\lambda_r \neq 0$ :

$$\lambda_r \varphi_r = \sum_{j \neq r} (-\lambda_j) \varphi_j$$

so $$\varphi_r = \sum_{j \neq r} \frac{-\lambda_j}{\lambda_r} \varphi_j$$

I claim that $\{\varphi_1 \dots \varphi_{r-1} | \varphi_{r+1} \dots \varphi_m\}$ is a spanning set (I've excluded $\varphi_r$).
To say this, take $\underline{x} \in V$ and write

$$\underline{x} = \sum_{j=1}^{m} x_j \varphi_j = x_r \varphi_r + \sum_{j \neq r} x_j \varphi_i$$

$$\underline{x} = \sum_{j \neq r} \left( \frac{-x_r \lambda_j}{\lambda_r} \right) \varphi_j + \sum_{j \neq r} x_j \varphi_j$$

$$\underline{x} = \sum_{j \neq r} \left( x_j - \frac{x_r \lambda_j}{\lambda_r} \right) \varphi_j$$

i.e. $\{\varphi_1, \dots \varphi_{r-1} | \varphi_{r+1} \dots \varphi_n\}$ still a basis

So now $\{\varphi_1 \dots \varphi_{r-1} | \varphi_{r+1} \dots \varphi_m\}$ has $(m-1)$ elements

By induction hypothesis $S(m-1)$
$\{\varphi_1 \dots \varphi_{r+1} | \varphi_{r+1} \dots \varphi_m\}$ containts a basis.

So $\{\varphi_1 \dots \varphi_m\}$ containts a basis. QED.

# Isomorphism.

Two sets are essentially the same (have the same cardinal), when there is a bijective mapping between them.

**Def^n** Let $V, W$ be vector spaces over $\mathbb{F}$. $V, W$ are **isomorphic**

$$V \cong W$$

when there is a bijective linear map $T: V \to W$.

**Theorem** Let $V, W$ be vector spaces over $\mathbb{F}$. Then

$$\boxed{V \cong W \Leftrightarrow \dim V = \dim W}$$

**Proof** Let $\dim V = n$

I'll first show that $\mathbb{F}^n \cong V$. Take basis $\{v_1, \dots v_n\}$ for $V$

Let $\{e_1, \dots e_n\}$ be standard basis for $\mathbb{F}^n$

Construct linear mapping $T: \mathbb{F}^n \to V$.

$$\underline{x} = \begin{pmatrix} a_1 \\ \vdots \\ a_m \end{pmatrix} = a_1 e_1 + \dots + a_n e_n \in \mathbb{F}^n$$

Define

$$T(a) = a_1 v_1 + \dots + x_n v_n$$

$$y = y_1 v_1 + \dots + y_n v_n$$

$$T^{-1}(y) = y_1 e_1 + \dots + y_n e_n$$

$$\begin{array}{cccccc} e_1 & e_2 & \dots & e_n & \\ \downarrow & \downarrow & & \downarrow & T \\ v_1 & v_2 & \dots & v_n & \\ \downarrow & \downarrow & & \downarrow & T^{-1} \\ e_1 & e_2 & \dots & e_n & \end{array}$$

$T: F^n \to V$ is bijective lin. mapping (bijective as it has inverse $T^{-1}$)

Shown if $\dim V = n$ then $\mathbb{F}^n \cong V$

Suppose $\dim W = n$. Choose isomorphism

$$S: F^n \to W$$

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$S \circ T^{-1} : V \to W$ : $S \circ T^{-1}$ is linear & bijective

$$\begin{tikzcd} V \arrow[rr] \arrow[dr, "T^{-1}"'] & & W \\ & \mathbb{F}^n \arrow[ur, "S"'] & \end{tikzcd}$$

So span

$\dim V = \dim W \Rightarrow V \cong W$

Conversally suppose $V \cong W$

Let $T : V \to W$ be a bijective linear map.

Need to show $\dim V = \dim W$

Let $e_1, \ldots, e_n$ be a basis for $V$

Claim $T(e_1) \ldots, T(e_n)$ is a basis for $W$

LI. suppose:

$\lambda_1 T(e_1) + \ldots + \lambda_n T(e_n) = 0$

every linear map takes zero : $T(0) = 0$

so $T(\lambda_1 e_1 + \ldots + \lambda_n e_n) = 0$

but $T(0) = 0$

& $T$ is injective so

$\lambda_1 e_1 + \ldots + \lambda_n e_n = 0$

But $e_1, \ldots e_n$ are LI

so $\lambda_1 = \lambda_2 = \ldots = \lambda_n = 0$

i.e. $T(e_1) \ldots T(e_n)$ is LI

$T(e_1) \ldots T(e_n)$ span $W$

Choose $w \in W$. Choose $v \in V$ : $T(v) = w$ (surjective)

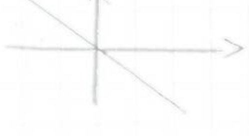
Write $v = \lambda_1 e_1 + \ldots + \lambda_n e_n$

$w = T(v) = \lambda_1 T(e_1) + \ldots + \lambda_n T(e_n)$

i.e. $Te_1 \ldots Te_n$ span $W$

QED

i.e. $V = \{ \binom{x}{x} : x \in \mathbb{R} \}$ 1-dimensional



$\cong \mathbb{R}$

99) 11.12.09. 1201

Recall 1) $S = \begin{cases} a_{11} x_1 + \dots + a_{1n} x_n = b_1 \\ a_{21} x_1 + \dots + a_{2n} x_n = b_2 \\ \vdots \\ a_{m1} x_1 + \dots + a_{mn} x_n = b_m \end{cases}$ 1830

2) $A \underline{x} = \underline{b}$ 1840

$A = \begin{pmatrix} a_{11} & \dots & a_{1n} \\ \vdots & & \\ a_{m1} & \dots & a_{mn} \end{pmatrix} \qquad \underline{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \qquad b = \begin{pmatrix} b_1 \\ \vdots \\ b_m \end{pmatrix}$

We solve this

3)

100 In the non-homogeneous case

$$A \underline{x} = \underline{b}$$

If I know at least one solution. (Particular solution, $z$)

say $z \Rightarrow A \underline{\underline{z}} = b$

Then any other solution to $A \underline{x} = \underline{b}$ has the form

$$x = \underline{x}' + \underline{\underline{z}}$$

where $A x' = 0$ solution of homogeneous system

i.e. $x' \in K_A$

$A x = b$

$A \underline{\underline{z}} = b$

$A(x - z) = \underline{b} - \underline{b} = 0$

So $x - z \in K_A$

Write $x - z = x' \in K_A \Rightarrow x = x' + z$

Remaining Question:

Q when does $A \underline{x} = \underline{b}$ have a solution?

Write $T(A) x - A x$

when does $\underline{b} \in Im(T_A)$?

| Original system | Reduced |
|---|---|
| $A x = b$ | $E A \underline{x} = E \underline{b}$ |

$$K_A = K_{EA}$$

Beware: $Im(T_A) \neq Im(T_{EA})$

However $Im(T_A) \cong Im(T_{EA})$

**Prop.** 101 Let $A = (a_{ji})_{\substack{1 \le j \le m \\ 1 \le i \le n}}$

$$A x_1 = \begin{pmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{pmatrix} \qquad A x_2 = \begin{pmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{pmatrix} \qquad \dots \quad A x_n = \begin{pmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{pmatrix}$$

$$\mathrm{Im}(T_A) \Rightarrow \mathrm{span}\{A x_1, \dots, A x_n\} = \{\lambda_1 A x_1 + \dots + \lambda_n A x_n\}$$

**Proof** Write down what are you doing?

$$A \underline{x} = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ \vdots & & & \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix} \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = x_1 A x_1 + x_2 A x_2 + \dots x_n A x_n$$

QED

To find a basis for $\mathrm{Im}(T_A)$ it is enough to find a maximal linearly independent set of columns. When system is reduced it is obvious.

**Example**

$$A = \begin{pmatrix} 1 & 2 & 0 & 1 & 0 & -1 & 4 \\ 0 & 0 & 1 & -1 & 0 & 2 & 3 \\ 0 & 0 & 0 & 0 & 1 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$x_1 \quad x_2 \quad x_3 \quad x_4 \quad x_5 \quad x_6 \quad x_7$$

You get $K_A$ from uncircled variables. To get Image, note that circled columns are LI and any other columns can be expressed in terms of them.

col 6

$$\begin{pmatrix} -1 \\ 2 \\ 1 \\ 0 \end{pmatrix} = -1\,\mathrm{col}(1) + 2\,\mathrm{col}(3) + \mathrm{col}(5)$$

So in a reduced system circled columns form a basis for Image.

102 $Ax = b \longrightarrow EAx = Eb$

Pick a basis $Col(i_1) \ldots Col(i_k)$ of $EA$

basis for $Im(T_{EA})$

$E^{-1} Col(i_1) \ldots E^{-1}(Col(i_k))$ gives a basis for $Im(T)$

Example $A = \begin{pmatrix} 1 & 1 & -1 & 1 & -1 \\ 1 & -1 & 1 & -1 & 1 \\ 1 & 3 & -3 & 3 & -3 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & -1 & 1 & -1 \\ 0 & -2 & 2 & -2 & 2 \\ 0 & 2 & -2 & 2 & -2 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & -1 & 1 & -1 \\ 0 & -1 & +1 & -1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$

$x_1 \; x_2 \; x_3 \; x_4 \; x_5$

Find a basis for $Im(T_A)$

Reduce

$\Rightarrow \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 3 \end{pmatrix}$ Form a basis for $Im\, T_A$

N 9.1.

Subspaces

$U \subseteq V$ vector space $\quad (u+v)+w = u+(v+w)$

$U \neq \emptyset$

Conditions are:

$\underline{u}_1, \underline{u}_2 \in U \Rightarrow \underline{u}_1 + \underline{u}_2 \in U$

$\underline{u} \in U, \lambda \in \mathbb{F} \Rightarrow \lambda \underline{u} \in U$

15. 12. 09. 1201

## Permutations

**Def$^n$** A ~~permutations~~ permutations on $n$ letters is bijective mapping
$f: \{1, \dots, n\} \to \{1, \dots, n\}$
Conventual to write of

$$f = \begin{pmatrix} 1 & 2 & 3 & \dots & n \\ f(1) & f(2) & f(3) & \dots & f(n) \end{pmatrix}$$

**Example** 1) $n = 2$
There are two permutations

$Id: \{1, 2\} \to \{1, 2\}$
$x \to x$

$$Id = \begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix}$$

$\tau : \{1, 2\} \to \{1, 2\}$
$\tau(1) = 2$
$\tau(2) = 1$

2) $n = 3$
There are six permutations

$$Id = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix} \quad \sigma = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix} \quad \tau = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}$$

A mapping $f: \{1 \dots n\} \to \{1 \dots n\}$ is bijective iff. $f$ is invertable.
If $g : \{1 \dots n\} \to \{1 \dots n\}$ is bijective
$(g \circ f)^{-1} = f^{-1} \circ g$
$\cancel{{}^{-1} \circ f) =}$ $(g \circ f)$

$\sigma = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}$ $\quad \sigma^2 = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}$

$\sigma \circ \sigma$

$\begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \\ 3 & 1 & 2 \end{pmatrix} \begin{matrix} \sigma \\ \sigma \end{matrix}$

$\tau = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}$

$\sigma\tau = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}$ $\quad \sigma^2\tau = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}$

$\begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \\ 3 & 2 & 1 \end{pmatrix} \begin{matrix} \tau \\ \sigma \end{matrix}$

$\begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 1 \\ 1 & 3 & 2 \end{pmatrix} \begin{matrix} \sigma \\ \tau \end{matrix}$ $\quad \sigma^3 = Id \quad \tau = 1$

$\tau\sigma = \sigma^2\tau$

**Beware** Composition of permutation is highly non commutative
However, as we'll see, there are permutation that commute

## Cyclic permutations.

**Example** $n = 5$

$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 3 & 4 & 5 & 1 \end{pmatrix}$

$n = 7$

$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 3 & 2 & 6 & 7 & 5 & 4 & 1 \end{pmatrix}$

Def$^n$ Let...
I mean)

(1) what does it mean?
(2)

$\{a_1, \dots, a_m\} \subset \{1, \dots n\}$, By the cycle $(a_1, a_2 \dots a_m)$
in $(a_1, \dots, a_m) : \{1 \dots n\} \to \{1 \dots n\}$

$(a_1, a_2 \ldots a_m)(a_1) = a_2$

$(a_1, a_2 \ldots a_m)(a_2) = a_3$ and $(a_1, \ldots a_m)(x) = x$

$\vdots$ if $x \notin \{a_1, \ldots a_m\}$

$(a_1, a_2 \ldots a_m)(a_{m-1}) = a_m$

$(a_1, a_2 \ldots a_m)(a_m) = a_1$

$(a_1, a_2 \ldots a_m)$ is a cycle of length $m$.

## Disjoint cycles

**Def$^n$** Let $(a_1, \ldots a_m), (b_1, \ldots b_k)$ be cycles. they said to be disjoint when

$$\boxed{\{a_1, \ldots a_m\} \cap \{b_1, \ldots b_k\} = \phi}$$

**Example** $n = 10$

$(1, 3, 5, 8, 7)$ $(2, 4, 6, 10)$ these are disjoint cycles ?

**Proposition** Let $(a_1, \ldots, a_m), (b_1, \ldots b_k)$ be disjoint

then $(a_1 \ldots a_m) \cdot (b_1, \ldots b_k) = (b_1 \ldots b_k) \cdot (a_1, \ldots, a_m)$

"Disjoint cycles commute"

Proof

of $\overset{x}{(1, 3, 5, 8, 7)} \cdot \overset{y}{(2, 4, 6, 10)}$

ie $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\ 1 & 4 & 3 & 6 & 5 & 10 & 7 & 8 & 9 & 2 \\ 3 & 4 & 5 & 6 & 8 & 10 & 1 & 7 & 9 & 2 \end{pmatrix} \begin{matrix} \} y \\ x \end{matrix} = x \cdot y$ $\Bigg|$ $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\ 3 & 2 & 5 & 4 & 8 & 6 & 1 & 7 & 9 & 10 \\ 3 & 4 & 5 & 6 & 8 & 10 & 1 & 7 & 9 & 2 \end{pmatrix} \begin{matrix} \} x \\ y \end{matrix} = y \cdot x$

$x \cdot y = y \cdot x$

Exercise

**Proposition** Any permutation is a product of disjoint cycles.

**Proof** Do it!

**Exercise** $n = 14$

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 \\ 5 & 8 & 9 & 14 & 7 & 10 & 11 & 12 & 3 & 4 & 1 & 2 & 6 & 13 \end{pmatrix}$$

$$(1,5,7,11)(2,8,12)(3,9)(4,14,13,6,10)$$

4 3 2 5

Q why we are doing this?

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$$

$$\begin{cases} a_{11}x_1 + a_{12}x_2 = b_1 & \cdot\, a_{22} \\ a_{21}x_1 + a_{22}x_2 = b_2 & \cdot\, a_{12} \end{cases}$$

$$-\begin{cases} a_{11}a_{22}x_1 + a_{12}a_{22}x_2 = a_{22}b_1 \\ a_{12}a_{21}x_1 + a_{12}a_{22}x_2 = a_{12}b_2 \end{cases}$$

$$a_{11}a_{22} - a_{12}a_{21}\ x_1 = ?$$

$$(a_{11}a_{22} - a_{12}a_{21})\,x_1 = ?$$

$$\det\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = a_{11}a_{22} - a_{12}a_{21} = a_{1\,\mathrm{id}(1)}a_{2\,\mathrm{id}(2)} - a_{1\tau(1)}a_{2\tau(2)}$$

$$(\tau(1) = 2,\ \tau(2) = 1)$$

$$\sum \qquad \tau = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$$

1 2

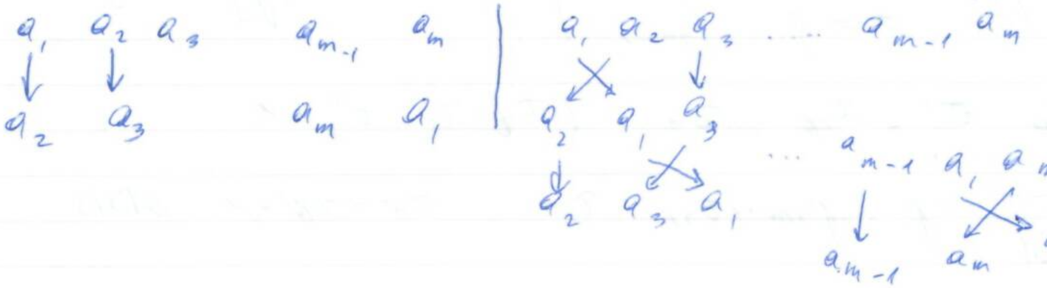
1

times with a −1 sign

# Permutations

I) Any permutations is a product of disjoint cycles.
II) Disjoint cycles commute.
III) Any cycle can be written as a product of transposition
[A transposition is a cycle of length 2 eg (2,5)]
IV) A cycle of length m is a product of (m−1) transpositions
V) Any transposition is a product of an odd number of adjecent transpositions [~ adjecent transposition ~ (i, i+1)]
VI) Any permutation is product of adjecent transp.
VII) The identity cannot be written a product of an ODD number of adjecent transpositions

Proof of III, IV: $(a_1, a_2, \dots, a_m) = \underbrace{(a_1, a_m)(a_1, a_{m-1})\dots(a_1, a_2)}_{(m-1)}$

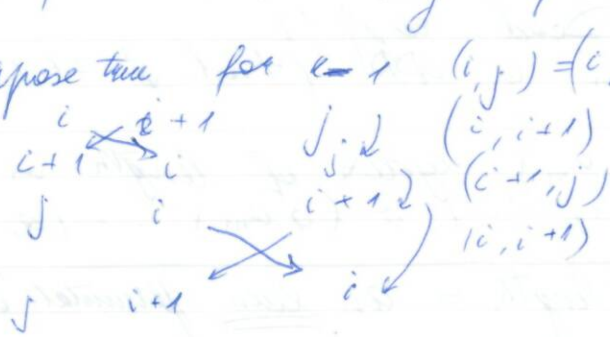


Define gap $(i,j) = |j-i|$

eg. gap $(2,5) = 3$

adjecent transpositions has gap=1
I'll prove if gap$(i,j)=k$
then $(i,j)$ is a prod. of $2k-1$ adjecent transpositions
OK for $k=1$. Nothing to prove.

Suppose true for $k-1$ $(i,j) = (i,i+1)(i+1,j)(i,i+1)$



By induction $(i+1,j)$ is a product of $2k-3$ adj. trans.
so $(i,j)$ is a prod. of $1+(2k+3)+1$ adj. trans
$\qquad\qquad\qquad\qquad\qquad\; \| $
$\qquad\qquad\qquad\qquad\; 2k+1$ QED. V

VI Any permutations is product of cycles

Any cycles is a product of transpositions

Any transposition is ... adj. trans

So any permutation is a product of adj. transpositions

Assuming VII

VII A permutation $\sigma$ is either a product

$\sigma = \tau_1 \dots \tau_{2N}$ (Even), $\tau_1 \dots \tau_2$ adjacent

or $\sigma = \tau_1 \dots \tau_{2N}$ (ODD) But not Both.

Proof of VII Suppose $\sigma = \tau_1 \dots \tau_{2N}$ | where $\tau_i, \rho_j$ adj. trans.

$\sigma = \rho_1 \cdots \rho_{2m+1}$

So $\sigma^{-1} = \tau_{2N} \dots \tau_1$ ($\tau_i^{-1} = \tau_1$ $\tau_2^2 = 1$

$\underset{Id}{\sigma\sigma^{-1}} = \rho_1 \dots \rho_{2m+1} \tau_{2N} \dots \tau_1$ $2m + 2N + 1$ ODD

Def$^n$

$$\operatorname{sign} \sigma \begin{cases} +1 & \text{when } \sigma \text{ is a product of an \underline{even} numb. of adj. trans} \\ -1 & \text{— " ——— \underline{odd} — " —} \end{cases}$$

Computing sign is Dead Easy!!

1) Any transposition is ODD (prod. of $2k-1$ adj. trans)

2) If. $\sigma = (a_1 \dots a_m)$, cycles of length $m$

$\operatorname{sign}(\sigma) = (-1)^{m-1}$ ($\sigma = (a_1 a_m) \cdots (a_1 a_2)$)

A cycle of odd length is an even permutation

A cycle of even length is an odd permutation

Example:

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 \\ 12 & 8 & 11 & 7 & 1 & 2 & 10 & 9 & 6 & 4 & 3 & 5 \end{pmatrix}$$

$$\sigma = (1,12,5)(2,8,9,6)(3,11)(7,4,10) \qquad \text{sign}(\sigma) = +1$$

$$+1 \qquad -1 \qquad -1 \qquad +1$$

Def$^n$ $\text{ord}\,\sigma = \min\{N \geq 1 : \sigma^N = 1\}$

so a cycle of length $m$ has ord $= m$

For any ~~de~~ arbitrary perm. $\sigma = c_1 \dots c_k$ $c_i$ disjoint cycles.

$$\text{ord}(\sigma) = LCM(\text{ord}(c_1) \dots \text{ord}(c_k))$$

$\sigma$ permutation of $\{1 \dots n\}$ Laplace's formula

Define $L(\sigma) = \prod_{i<j} (\sigma(j) - \sigma(i))$

So $L(Id) > 0$

Prop. if $\tau$ is adj. transp.
say $\tau = (r, r+1)$
then $L(\sigma\tau) = -L(\sigma)$

Proof: $L_1(\sigma) = \prod_{1 \leq i < j < r} \sigma(j) - \sigma(i)$

$L_2(\sigma) = \prod_{i<j=r} \sigma(j) - \sigma(i)$

$L_3(\sigma) = \prod_{i<r<r+1 \leq j}$

$L_4(\sigma) = \sigma(r+1) - \sigma(r)$ // $i=r$ $j=r+1$

$L_5(\sigma) = \prod_{i=r<r+1<j} \sigma_j - \sigma(i)$

$L_6(\sigma) = \prod_{i=r+1<j} \sigma(j) - \sigma(i)$

$L_7(\sigma) = \prod_{r+1<i<j} \sigma(j) - \sigma(i)$

$L_1(\sigma\tau) = L_1(\sigma)$
$L_2(\sigma\tau) = L_3(\sigma)$
$L_3(\sigma\tau) = L_2(\sigma)$
$L_4(\sigma\tau) = -L_4(\sigma)$
$L_7(\sigma\tau) = L_7(\sigma)$
$L_6(\sigma\tau) = L_5(\sigma)$
$L_5(\sigma\tau) = L_6(\sigma)$

so if $L(\tau_1 \dots \tau_N) = (-1)^N L(Id)$

if $\tau_1 \dots \tau_N$ are adj trans

if $N$ odd $\tau_1 \dots \tau_N \neq id$

(otherwise ( $L(Id) = -L(Id)$ $\quad L(Id) > 0$ ) Q.E.D.