1201 Algebra 1 Notes

Based on the 2010 autumn lectures by Prof F E A Johnson

5-10-2010

- $x^2 + xy + y^2 = 1$ Quadratic

- $x+y = 1$ / linear
- $x+y+z=1$

- Instead of using x, y, z ......
We pick single letter x_1
★ Variable indexed, x_1, x_2, x_3 ...... x_n.

★ Also index coefficient.

$$\begin{cases} a_1 x_1 + a_2 x_2 + a_3 x_3 + a_4 x_4 = b \\ \text{Single linear equation in 4 variables} \\ \\ c_1 x_1 + c_2 x_2 + c_3 x_3 + c_4 x_4 = d \end{cases}$$

Soon back in same position

$a_{11} x_1 + a_{12} x_2 + a_{13} x_3 + \dots\ a_{1n} x_n = b_1$
$a_{21} x_1 + a_{22} x_2 + a_{23} x_3 + \dots\ a_{2n} x_n = b_2$

$a_{i1} x_1 + a_{i2} x_2 + \dots\ a_{ij} x_j + a_{in} x_n = b_i$
i^{th} Equation ... j^{th} equation

$$S = \begin{cases} a_{11} x_1 + a_{12} x_2 + \dots\ a_{ij} x_j + a_{in} x_n = b_1 \\ \\ a_{i1} x_1 + \dots\ a_{ij} x_j + a_{in} x_n = b_i \\ \\ a_{m1} x_1 + \dots\ a_{mn} x_n = 1 \end{cases}$$

m linear equations
in n unknowns

$$a_1 x_1 + a_2 x_2 + \dots a_n x_n = b$$

$$\underbrace{(a_1, \dots a_n)}_{\text{Row Vector}} \underbrace{\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}}_{\text{Column vector}} = a_1 x_1 + a_2 x_2 + \dots a_n x_n$$

<u>Definition</u>

Column vector

<u>Row Vector</u>

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \underbrace{\begin{pmatrix} x_1 \\ \vdots \\ x_2 \\ \vdots \\ x_n \end{pmatrix}}_{x} = \begin{pmatrix} a_{11}x_1 + a_{12}x_2 + \dots a_{1n}x_n \\ a_{21}x_1 + \dots \dots a_{2n}x_n \end{pmatrix}$$

$\underbrace{m \text{ rows} \quad n \text{ columns}}_{A}$ <u>Def</u>^{n} m rows, 1 column

n rows
1 column

$$A x = b$$

where $x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$ $\quad b = \begin{pmatrix} b_1 \\ \vdots \\ b_m \end{pmatrix}$

By an $m \times n$ matrix

$$A = (a_{ij}) \quad \begin{matrix} 1 \le i \le m \\ 1 \le j \le n \end{matrix}$$

mean a collection of mn numbers arranged in m rows and n column.

eg. $\begin{pmatrix} 2 & 1 & 0 & 5 \\ -1 & 1 & 0 & -1 \\ 0 & 0 & 1 & 2 \end{pmatrix} = A$ i = row

j = column

- 3×4 matrix. What is $a_{34} = 2$
- $a_{23} = 0$

Special Case : row vector $m \times 1$ matrix

col. vector $1 \times n$ matrix

$$(a_1 \ldots\ldots a_n)\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = a_1x_1 + a_2x_2 + \ldots a_nx_n$$

- Basic Idea

A, B matrices

(i,j) entry of $AB = (i^{th}$ row of $A)\begin{pmatrix} j^{th} \text{ column} \\ \text{of } B \end{pmatrix}$

$\boxed{\text{Example}}$ $A = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & 2 \end{pmatrix}$ $B = \begin{pmatrix} 1 & 1 \\ -1 & 0 \\ 2 & 3 \end{pmatrix}$

2×3 3×2

$\rightsquigarrow 1^{st}$ row of $A = (1, -1, 0)$

$\rightsquigarrow 1^{st}$ column of $B = \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}$

eg. $(AB)_{11} = (1, -1, 0)\begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}$

$= 1.1 + (-1)(-1) + 0.2$

$= 2$

eg. $\left.\begin{array}{l} (AB)_{12} = 1 \\ (AB)_{21} = 3 \\ (AB)_{22} = 6 \end{array}\right\}$ $AB = \begin{pmatrix} 2 & 1 \\ 3 & 6 \end{pmatrix}$

$BA = \begin{pmatrix} 1 & 1 \\ -1 & 0 \\ 2 & 3 \end{pmatrix}\begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & 2 \end{pmatrix}$

$(BA)_{11} = 1$ $(BA)_{12} = 0$

$(BA)_{13} = 2$ $(BA)_{21} = -1$

$(BA)_{22} = 1$ $(BA)_{23} = 0$

$(BA)_{31} = 2$ $(BA)_{32} = 1$

$(BA)_{33} = 6$

$$BA = \begin{pmatrix} 1 & 0 & 2 \\ -1 & 1 & 0 \\ 2 & 1 & 6 \end{pmatrix}$$

RULE 0 If $A = (a_{ij})$ $\begin{matrix} 1 \le i \le m \\ 1 \le j \le n \end{matrix}$ $B = (b_{kl})$ $\begin{matrix} 1 \le k \le p \\ 1 \le l \le q \end{matrix}$

$m \times \underline{n}$ $p \times q$

- AB is defined <u>iff</u> $n = p$ ✓
  no. of col. A = no. of row of B. ✓

Define If $A = (a_{ij})$ $\begin{matrix} 1 \le i \le m \\ 1 \le j \le n \end{matrix}$

$B = (b_{jk})$ $\begin{matrix} 1 \le j \le n \\ 1 \le k \le p \end{matrix}$

Then AB is defined and

$(AB)_{\underline{ik}} = (i^{th} \text{ Row } A)(k^{th} \text{ col. } B)$

- If: i^{th} row of $A = (a_{i1}, a_{i2} \dots a_{in})$, l^{th} col $B = \begin{pmatrix} b_{1l} \\ b_{2l} \\ \vdots \\ b_{nl} \end{pmatrix}$

$$(AB)_{il} = a_{i1} b_{1l} + a_{i2} b_{2l} + \dots a_{in} b_{nl}$$

$$= \sum_{k=1}^{n} a_{ik} b_{kl} \text{ //}$$

$$(AB)_{il} = \sum_{k=1}^{n} a_{ik} b_{kl} = \left(\sum_{k=1}^{n} A_{ik} B_{kl} \right) \text{ //}$$

Rule 0 : AB defined only when $|$ cols of $A| = |$rows of $B|$ ✓

Rule 1 : $(AB)_{il} = \sum_{k=1}^{n} A_{ik} B_{kl}$ (i^{th} row of A $\times$ l^{th} col. of B)

n = cols of A
rows of B

Rule 2 if AB defined, then BA need not be defined.

eg. $\underset{(1\times2)}{A = (1, 2)}$, $\underset{(2\times3)}{B = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & 2 \end{pmatrix}}$

AB defined , BA not defined.

Rule 3 if AB, BA both defined
they need not be same size ✓
eg. A : 2×3 $B = 3\times2$
AB : 2×2 BA is 3×3

Rule 4 if AB are square $(n\times n)$ matrices then , usually but not always $AB \neq BA$
AB , BA both defined and same size

However they are normally different.

Example : $A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ $B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

$$\underbrace{AB = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad BA = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}}_{AB \neq BA\ !}$$

-) For numbers $x^2 = 0 \Rightarrow x = 0$
Not so for matrices

$$\underset{x}{\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}} . \underset{x^2}{\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

→ Zero matrix

$m \times n$. 0

$0_{ij} = 0$ for all i, j

(3×4) $\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = 0$

(4×3) $\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

$\boxed{0X = X0 = 0}$

$n \times n$ matrices

Special matrix I_n

$I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $I_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, $I_4 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

Formal Definition : ★ $(I_n)_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$

Kronecker delta (δ_{ij}) , $\delta_{ij} = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$

Characteristic Property

$I_n X = X$ $X = (n \times p)$

unchanged

$Y I_n = Y$ $Y (m \times n)$

Example

$$A = \begin{pmatrix} a & b & c \\ d & e & f \end{pmatrix} \quad (2\times 3)$$

$$AI_3 = \begin{pmatrix} a & b & c \\ d & e & f \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} a & b & c \\ d & e & f \end{pmatrix}$$

$$I_2 A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} a & b & c \\ d & e & f \end{pmatrix} = \begin{pmatrix} a & b & c \\ d & e & f \end{pmatrix}$$

Proposition if A is $m\times n$ then,

$$AI_n = A$$

Proof : $(AI_n)_{ik} = \sum_{j=1}^{n} A_{ij}(I_n)_{jk} = \sum_{j=1}^{n} A_{ij}\delta_{jk}$ ✓

$$\sum_{j=1}^{n} A_{ij}\delta_{jk} = A_{ik}\underline{\delta_{kk}} \quad , \quad \delta_{kk} = 1 \text{ refer to Def}^{n}$$

$$= A_{ik}$$

- $\delta_{ij} = \begin{cases} 1 & i=j \\ 0 & i\neq j \end{cases}$

$$(AI_n)_{ik} = A_{ik}$$

so. $AI_n = A$

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- Lets take $n=4$, $i=2$, $k=3$

$$(AI_n)_{23} = A_{21}\delta_{13} + A_{22}\delta_{23} + A_{23}\delta_{33} + A_{24}\delta_{43}$$

$\Rightarrow$ $j=1$ $\quad j=2$ $\quad j=3$ $\quad j=4$

$$0 + 0 + A_{23}\underbrace{\delta_{33}}_{=1} + 0 = A_{23}$$

since $(i\neq j) \therefore \Rightarrow 0$

- existential quantification
- there exists ; there is ; there are

With numbers

$x \neq 0 \Rightarrow \exists\, x^{-1} ; \; x x^{-1} = 1$ ✓

$= x^{-1} x$

not so with matrices. ✓

$x = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ has no inverse ✓

Definition An $n \times n$ matrix X is said to be invertible when $\exists$ $n \times n$ matrix Y such that $YX = XY = I_n$

eg. $Z = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ $\quad \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

$Z^{-1} = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$

Y is then called the inverse of X

$Y = X^{-1}$. not $\frac{1}{X}$

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Algebaric properties of matrix

Matrix sum : A, B are $m \times n$

$A + B$ is also $m \times n$

$(A+B)_{ij} = A_{ij} + B_{ij}$ (add corresponding entries)

$A + B = B + A$

- $A + (B + C) = (A + B) + C$ Assoc.
- $A + B = B + A$ Commutative.
- $0 + A = A$ (Additive identify)
- $A(BC) = (AB)C$
- $AI_n = A$
- $I_n A = A$ | A is $m \times n$

- $A(B+C) = AB + AC$ left distributive
- $(A+B)C = AC + BC$ Right distributive

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$$S = \begin{cases} a_{11}x_1 + \cdots\cdots a_{1n}x_n = b_1 \\ a_{21}x_1 + \cdots\cdots a_{2n}x_n = b_2 \\ \\ a_{m1}x_1 + \cdots\cdots a_{mn}x_n = b_m \end{cases} \longrightarrow \quad Ax = b$$

$$A = \begin{pmatrix} a_{11} & & a_{1n} \\ \vdots & & \vdots \\ a_{m1} & & a_{mn} \end{pmatrix}$$

We write in m matrix form

matrix of coefficients.

$$x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \qquad k = \begin{pmatrix} k_1 \\ \vdots \\ k_m \end{pmatrix}$$

How to recognize ~~the solutio~~ when you have a solution.

Example :

$$\begin{array}{c} \begin{pmatrix} 1 & 2 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \\ \begin{matrix} x_1 & x_2 & x_3 & x_4 & x_5 \end{matrix} \\ A \end{array} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$$

Comes from
$$\begin{aligned} x_1 + 2x_2 + x_4 &= 1 \\ x_3 + x_4 &= 2 \\ x_5 &= 3 \end{aligned}$$

Eliminate circled ~~coefficients~~ variables

$$\begin{aligned} x_1 &= 1 - 2x_2 - x_4 \\ x_2 &= x_2 \\ x_3 &= 2 - x_4 \\ x_4 &= x_4 \\ x_5 &= 3 \end{aligned}$$

General solution
$$\begin{pmatrix} 1 - 2x_2 - x_4 \\ x_2 \\ 2 - x_4 \\ x_4 \\ 3 \end{pmatrix}$$

- there are independent variables $\Rightarrow$ ($x_n = x_n$) no constraint
  (x_2, x_4)
- x_1, x_3, x_5 dependent variables

* For independent variable can substitute any numerical value you like !! ✓

Example: $\begin{pmatrix} 1 & 2 & 3 & 0 & 2 \\ 0 & 0 & 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} 4 \\ 3 \end{pmatrix}$

$x_1 \quad x_2 \quad x_3 \quad x_4 \quad x_5$

Circle variables under leading is

$$x_1 + 2x_2 + 3x_3 + 2x_5 = 4$$
$$x_4 + x_5 = 3$$

Eliminate
~~Elimate~~ circle variables

$$\begin{pmatrix} 4 - 2x_2 - 3x_3 - 2x_5 \\ x_2 \\ x_3 \\ 3 - x_5 \\ x_5 \end{pmatrix} \leftarrow \text{General solution}$$

* x_1 x_4 dependent
* x_2 x_3 x_5 independent

- (Reduced) Row Echelon matrices "

$$\begin{matrix} 1 & ? & 0 & ? & ? & 0 & ? & 0 \\ 0 & 0 & 1 & ? & ? & 0 & ? & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & ? & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{matrix}$$

* Observe the pattern
"Stepped when the column only contains one '1' and zero(s) "
~ similiar to δ_{ij}

a) First non-zero entry in any row must be 1 (leading 1)

b) The rest of the column in which leading 1 occ must be 0

c) Stepped

d) zero rows come after non zero rows

Example

$$\begin{pmatrix} 1 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \text{ is Row Echelon}$$

$$\begin{pmatrix} 1 & 2 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad \underline{\underline{\text{not}}}\,!$$

$$\begin{pmatrix} 1 & 2 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \quad \underline{\underline{\text{not}}}\,!$$

Example

$$\begin{cases} x_1 + x_2 - x_3 + x_4 = 1 \\ x_1 - x_2 + x_3 + x_4 = 1 \\ x_4 + 3x_2 - 3x_3 + x_4 = 1 \end{cases}$$

Write in matrix form

$Ax = b$

$$\underbrace{\begin{pmatrix} 1 & 1 & -1 & 1 \\ 1 & -1 & 1 & 1 \\ 1 & 3 & -3 & 1 \end{pmatrix}}_{A} \underbrace{\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}}_{\underline{x}} = \underbrace{\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}}_{\underline{b}}$$

Addition: $\begin{pmatrix} 2x_1 + 3x_2 = -1 \\ x_1 + 2x_2 = -1 \end{pmatrix}$

* 1) Add one equation to another

* 2) Multiply one " by $\lambda \neq 0$

* 3) Swap the order in which equations come

Form Augmented matrix $A \,\vdots\, b$

$$\left(\begin{array}{cccc|c} 1 & 1 & -1 & 1 & 1 \\ 1 & -1 & 1 & 1 & 1 \\ 1 & 3 & -3 & 1 & 1 \end{array}\right)$$

$$(A \,\vdots\, b)$$

* 1) Add Row j to Row i

* 2) Multiply Row i by $\lambda \neq 0$

* 3) Interchange order of rows

- kill 1^{st} column after Row 1
- Add (-1) Row 1 to Row 2
- Add (-1) Row 1 to Row 3

$$\Rightarrow \left(\begin{array}{cccc|c} 1 & 1 & -1 & 1 & 1 \\ 0 & -2 & 2 & 0 & 0 \\ 0 & 2 & -2 & 0 & 0 \end{array}\right) \xrightarrow{\text{Multiply Row 2 by } -1/2} \left(\begin{array}{cccc|c} 1 & 1 & -1 & 1 & 1 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & 2 & -2 & 0 & 0 \end{array}\right)$$

$$\left(\begin{array}{cccc|c} 1 & 1 & -1 & 1 & 1 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & 2 & -2 & 0 & 0 \end{array}\right)$$

– Add (-1) Row2 to Row 1

– (-2) Row 2 to Row 3

$\downarrow$

$$\left(\begin{array}{cccc|c} 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

This is row echelon

circle variables under leading 1s $\rightarrow$ x_1 x_2 x_3 x_4

① (-2) Row 3 to Row 3 [WRONG NOTES ?]

② $\left(\begin{array}{cccc|c} 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$?

Reduced System

$$\begin{cases} x_1 + x_4 = 1 \\ x_2 - x_3 = 0 \end{cases}$$

Eliminate circled variables $\begin{pmatrix} 1 - x_4 \\ x_3 \\ x_3 \\ x_4 \end{pmatrix}$ ← General solution

x_3 and x_4 are arbitary

Let $x_3 = 1$, $x_4 = 2$

$(-1) + 1 + (-1) + 2 = 1$

$(-1) - 1 + 1 + 2 = 1$

Example
$$\begin{cases} x_1 + x_2 + x_3 + x_4 = 1 \\ x_1 + x_2 - x_3 + x_4 = 2 \\ x_1 - x_2 + x_3 + x_4 = 3 \end{cases}$$

- Write system in Augmented (1st step)

$$\left(\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & 1 & 2 \\ 1 & -1 & 1 & 1 & 3 \end{array}\right) \xrightarrow[(-1)R_1 \text{ to } R_3]{\text{Add } (-1)R_1 \text{ to } R_2} \left(\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & -2 & 0 & 1 \\ 0 & -2 & 0 & 0 & 2 \end{array}\right)$$

Play the game!!
Get into Row Echelon Form

Swap R_2 and R_3

$$\left(\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 1 \\ 0 & -2 & 0 & 0 & 2 \\ 0 & 0 & -2 & 0 & 1 \end{array}\right)$$

Multiply R_2 by $-\frac{1}{2}$

$$\left(\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & -2 & 0 & 1 \end{array}\right)$$

- kill the rest of Col 2
- Add (-1) R2 to R_1

$$\left(\begin{array}{cccc|c} 1 & 0 & 1 & 1 & 2 \\ 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & -2 & 0 & 1 \end{array}\right)$$

Multiply R_3 by $-\frac{1}{2}$ →

$$\left(\begin{array}{cccc|c} 1 & 0 & 1 & 1 & 2 \\ 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & -\frac{1}{2} \end{array}\right)$$

• if leave this we have to substitute for x_3
NEVER SUBSTITUTE

Kill Rest of Col 3
Add (-1) R3 to R1

$$\begin{array}{ccccc} \ast & & & & \\ \ast & & & & \\ \ast & \left(\begin{array}{cccc|c} 1 & 0 & 0 & 1 & \frac{5}{2} \\ 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & -\frac{1}{2} \end{array}\right) \\ \ast & & & & \\ \ast & & & & \end{array}$$

x_1 x_2 x_3 x_4

dependent!

- Write out Reduced System

$$\begin{aligned} x_1 + \quad x_4 &= \tfrac{5}{2} \\ x_2 \quad &= -1 \\ x_3 \quad &= -\tfrac{1}{2} \\ x_4 &= x_4 \end{aligned}$$

★ General Sol.

$$\begin{pmatrix} -\frac{5}{2} & -x_4 \\ -1 & \\ -\frac{1}{2} & \\ & x_4 \end{pmatrix}$$

When we reduce we are allowed 3 operations.

① $\mathcal{E}(i, j; \lambda)$ adds λ Row (j) to Row(i)

(+) (Row(j) stays same, Row (i) changes)

where $(i \neq j)$

② $\mathcal{D}(i, \lambda)$ is : multiplies Row (i) by $\lambda \neq 0$

③ $P(i, j)$: Interchange Row(i) and Row (j)

- We are going to learn how to do these by Matrix Mult.

$$\begin{pmatrix} a & b & c & d \\ e & f & g & h \end{pmatrix}$$

$\downarrow$ $\mathcal{E}(1,2:3)$

Add (3) row(j) to row(i) $= \times \begin{pmatrix} 1 & 3 \\ 0 & 1 \end{pmatrix}$

$$\begin{pmatrix} a+3e & b+3f & c+3g & d+3h \\ e & f & g & h \end{pmatrix}$$

[$E(i,j:\lambda)$ • "1" along diagonal]
- "λ" in $(i,j)^{th}$ position
- "0" anywhere else

$n=4$

$$E(2,3:-5) \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -5 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \lambda \text{ in the position } \overset{i,j}{(2,3)}$$

$$\begin{pmatrix}1&0&0\\0&9&0\\0&0&1\end{pmatrix}\begin{pmatrix}a&b&c&d\\e&f&g&h\\k&l&m&n\end{pmatrix} \xrightarrow{D(2,9)} \begin{pmatrix}a&b&c&d\\9e&9f&9g&9h\\k&l&m&n\end{pmatrix}$$

$$\boxed{\Delta(i,\lambda)\begin{cases}\bullet\ \text{has "0" off diagonal}\\ \bullet\ (i,i)^{th}\text{ position is }\lambda \leftarrow\\ \bullet\ (k,k)^{th}\text{ position is } 1 \quad , k\neq 1\end{cases}}$$

- $\Delta(i,\lambda)_{rs} = \begin{cases}0 & r\neq s\\ \lambda & r=s=i\\ 1 & r=s\neq i\end{cases}$

- $\Delta(2,9) = \begin{pmatrix}1&0&0\\0&9&0\\0&0&1\end{pmatrix}$

- Swap row_3 with row_4

$$\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&0&1\\0&0&1&0\end{pmatrix} \times \begin{bmatrix}a&b&c&d\\e&f&g&h\\k&l&m&n\\w&x&y&z\end{bmatrix} \qquad \begin{bmatrix}a&b&c&d\\e&f&g&h\\w&x&y&z\\k&l&m&n\end{bmatrix}$$

How can I tell that $E(i,j;\lambda)$, $\Delta(i,\lambda)$, $P(i,j)$ do what they are supposed to do ?
(elementary matrices)

Basic Matrices

$$\epsilon(1,1) = \begin{pmatrix}1&0&0\\0&0&0\\0&0&0\end{pmatrix} \qquad \epsilon(3,2) = \begin{pmatrix}0&0&0\\0&0&0\\0&1&0\end{pmatrix}$$

- $\boxed{\text{Definition}}$ $\epsilon(i,j)_{rs} = \delta_{ir}\,\delta_{js}\begin{cases}1 & r=i \text{ and } s=j\\ 0 & \text{otherwise}\end{cases}$

In English: 1 in $(i,j)^{th}$ position
0 anywhere else.

(i,j) is fixed
(r,s) is variable

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$\mathcal{E}(i, j; \lambda)$

takes λ (jth row) and add it to i^{th} row.

All ~~must~~ rows apart from i^{th} stay the same.

We are going to find a matrix $E(i, j; \lambda) A$ is the matrix obtained from A via $\mathcal{E}(i, j : \lambda)$

<u>Example</u> : $\begin{pmatrix} 1 & 3 \\ 0 & 1 \end{pmatrix} \underbrace{\begin{pmatrix} a & b & e \\ c & d & f \end{pmatrix}}_{A} = \begin{pmatrix} a+3c & b+3d & e+3f \\ c & d & f \end{pmatrix}$

operation $\mathcal{E}(1, 2; 3)$

<u>Basic matrices</u>

- $\epsilon(i, j)$ is the $n \times n$ matrix with only one non-zero entry. It has 1 in $(i,j)^{th}$ place.

<u>Formal def^n</u> : $\epsilon(i,j)_{rs} = \delta_{ir}\,\delta_{js}$ $\left(= \begin{cases} 1 & r=i \text{ and } s=j \\ 0 & \text{otherwise} \end{cases} \right.$

This is definition

eg. $n=3$, $\epsilon(2,3) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$

If $n=4$

$\epsilon(4,3) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$ ← This is what it looks like.

Question = A is $m \times n$

$\epsilon(i,j)$ is the $m \times m$ basic matrix

What is $\epsilon(i,j)A$?

$$\underset{\epsilon(1,3)}{\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}} \underset{A}{\begin{pmatrix} a & b \\ c & d \\ e & f \end{pmatrix}} = \begin{pmatrix} e & f \\ 0 & 0 \\ 0 & 0 \end{pmatrix}$$

Answer $\epsilon(i,j)A$, takes i^{th} row of A : put it into i^{th} row and kills everything else.

Proof : $$[\epsilon(i,j)A]_{rs} = \sum_{i=1}^{m} \epsilon(i,j)_{rt} A_{ts}$$

$$= \sum_{i=1}^{m} \delta_{ir} \delta_{jt} A_{ts}$$

$$= \delta_{ir} \delta_{jj} A_{js}$$

$$= \delta_{ir} A_{js} = \begin{cases} A_{js} & r = i \\ 0 & r \neq i \end{cases}$$

$[\epsilon(i,j)A]_{is} = A_{js}$ | i^{th} row of LHS $= j^{th}$ row of A

$[\epsilon(i,j)A]_{ks} = 0 \quad k \neq i$ [QED] so $\lambda \epsilon(i,j) A$ is obtained by multiply j^{th} row A of λ and putting it i^{th} row. killing everything else.

so Define: $\boxed{E(i,j:\lambda) = I_n + \lambda \epsilon(i,j)}$

<u>Proposition</u>: $E(i,j;\lambda) A$ is the matrix obtained from A by operation $\mathcal{E}(i,j:\lambda)$

<u>Prop</u>: $E(i,j;\lambda) A = (I_n + \lambda \epsilon(i,j)) A = A + \lambda \epsilon(i,j) A$.

so i^{th} row of $A + \lambda \epsilon(i,j) A$ is i^{th} row $A + \lambda\, j^{th}$ row A.

$I_j : k \neq i$

k^{th} row $A + \lambda \epsilon(i,j) A$
$= k^{th}$ row $(+0)$ QED.

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \lambda & 0 & 1 \end{pmatrix}\begin{pmatrix} r & s \\ t & w \\ x & y \end{pmatrix} = \begin{pmatrix} r & s \\ t & w \\ \lambda r + x & \lambda s + y \end{pmatrix}$$

Added λ Row 1 to Row 3.

Claim $E(i,j;\lambda)$ is invertible $(i \neq j)$

$$\begin{pmatrix} 1 & \lambda \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & \mu \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & \lambda+\mu \\ 0 & 1 \end{pmatrix}$$

$$E(1,2;\lambda) E(1,2;\mu) = E(1,2;\lambda+\mu)$$

→ eg: $\begin{pmatrix} 1 & 99 \\ 0 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & -99 \\ 0 & 1 \end{pmatrix}$

$$E(i,j;\lambda)^{-1} = E(i,j;-\lambda)$$

Rule for multiplying $\epsilon(i,j)$

$$\epsilon(i,j)\,\epsilon(k,l) = \begin{cases} \epsilon(i,l) & j=k \\ 0 & j \neq k \end{cases}$$

For example:

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$\epsilon(1,2)\quad \epsilon(1,2) \qquad 2 \neq 1$

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

$\epsilon(1,2)\ \epsilon(2,1) = \epsilon(1,1)$

Prop: $\quad E(i,j;\lambda)^{-1} = E(i,j;-\lambda)$

Prop: $\quad$ Assume $\epsilon(i,j)\,\epsilon(i,j) = 0$

$$\left[I + \lambda\epsilon(i,j)\right]\left[I + \mu\epsilon(i,j)\right]$$

$$= \left[I + \lambda\epsilon(i,j) + \mu\epsilon(i,j) + \lambda\mu\,\underbrace{\epsilon(i,j)\,\epsilon(i,j)}_{=0}\right]$$

$$= I + (\lambda+\mu)\,\epsilon(i,j)$$

so I've shown

$E(i,j;\lambda)\,E(i,j;\mu) = E(i,j;\lambda+\mu)$

Put $\mu = -\lambda$

$E(i,j;\lambda)\,E(i,j;-\lambda) = E(i,j;0) = I_n$

so $E(i,j;-\lambda) = E(i,j;\lambda)^{-1}$ QED

$D(i,\lambda) \quad \lambda \neq 0$

Multiplies i^{th} row by λ leaves everything else the same.

We want a matrix $\Delta(i,\lambda)$
such that $\Delta(i,j)A$ is matrix obtained by performing $D(i,\lambda)$

<u>Guess</u> : $n=3$

$$\Delta(2,\lambda) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & \lambda-1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\Delta(2,\lambda) = I_2 + (\lambda-1)E(2,2)$$

<u>Definition</u>

$$\boxed{\Delta(i,\lambda) = I + (\lambda-1)E(i,i)}$$

<u>Prop</u> : $\Delta(i,j)A$ is the matrix obtained from A by the operation $D(i,\lambda)$.

i.e. multiply i^{th} row by λ and leave everything else the same.

<u>Proof</u> : $[I + (\lambda-1)E(i,i)]A = A + (\lambda-1)E(i,i)A$
$= i^{th}$ row $+ (\lambda-1)\, i^{th}$ row $A = \lambda(i^{th}$ row $A)$

We have seen $E(i,i)A$ kills everything apart from i^{th} row and takes i^{th} row and puts it in i^{th} row.

$(\lambda-1)E(i,i)A$ has all rows zero apart from i^{th} but

$$i^{th} \text{ row} = (\lambda-1)\, i^{th} \text{ row of } A$$

$$A + (\lambda - 1)\, E(i,i)\, A$$

k^{th} row : k^{th} row $A + 0$

$k \neq i$ QED.

→ $\Delta(i,\lambda)^{-1} = ?$

Prop : $\Delta(i,\lambda)\,\Delta(i,\mu) = \Delta(i,\lambda\mu)$

Proof : $[I + (\lambda - 1)E(i,i)][I + (\mu-1)E(i,i)]$

$= I + [(\lambda-1)+(\mu-1)]\,E(i,i) + (\lambda-1)(\mu-1)\,E(i,i)\,E(i,i)$

$= I + [(\lambda+\mu-2) + (\lambda\mu - \lambda - \mu + 1)]\,E(i,i)$

$= I + (\lambda\mu - 1)\,E(i,i)$ QED

Corollary : $\Delta(i,\lambda)^{-1} = \Delta(i,\frac{1}{\lambda}) = \Delta(i,\lambda^{-1})$ $\lambda \neq 0$

Proof : $\Delta(i,1) = I$

- $E(i,j;\lambda)A$ performs $\mathcal{E}(i,j;\lambda)$
- $\Delta(i,\lambda)A$ " $D(i,\lambda)$

$P(i,j)$ swaps i^{th} and j^{th} rows.

$P(i,j) \cdot I = P(i,j)$

so expect $P(i,j)$ to be I_n with I^{th} and j^{th} rows swapped

$n=4$

$$P(2,3) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$\Uparrow + \epsilon(2,3) + \epsilon(3,2)$

$$\Rightarrow \quad I = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \xRightarrow{\text{eg. } I - \epsilon(2,2) - \epsilon(3,3)} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\boxed{\underline{\text{Definition}}: \quad P(i,j) = I - \epsilon(i,i) - \epsilon(j,j) + \epsilon(i,j) + \epsilon(j,i)}$$

Prop $P(i,j)A$ is the matrix obtained from A by swapping i^{th} and j^{th} rows

Proof: $P(i,j)A = [I - \epsilon(i,i) - \epsilon(j,j) + \epsilon(i,j) + \epsilon(j,i)]A$

$= [I - \epsilon(i,i) - \epsilon(j,j)]A + [\epsilon(i,j) + \epsilon(j,i)]A$

$= (A - (i^{th} \text{ row } A) - j^{th} \text{ row } A) + \begin{pmatrix} j^{th} \text{ row of } \\ A \text{ in } i^{th} \text{ row} \end{pmatrix} + \begin{pmatrix} i^{th} \text{ row of } A \\ \text{in } j^{th} \text{ row} \end{pmatrix}$

i^{th} row of $\epsilon(i,j)A = j^{th}$ row of A

j^{th} row of $\epsilon(j,i)A = i^{th}$ row of A

And other rows of $[E(i,j) + \epsilon(j,i)]A$ are zero

QED

Expect that $P(i,j)^{-1} = P(i,j)$

Prop. $P(i,j)^{-1} = P(i,j)$

Proof : Directly from definition.

Assume
$$\epsilon(i,i)\,\epsilon(i,i) = \epsilon(i,i)$$
$$\epsilon(i,j)\,\epsilon(i,i) = 0 \qquad j \neq i$$
$$\epsilon(i,i)\,\epsilon(i,j) = \epsilon(i,j)$$

$$[I - \epsilon(i,i) - \epsilon(j,j) + \epsilon(i,j) + \epsilon(j,i)][I - \epsilon(i,i) - \epsilon(j,j) + \epsilon(i,j) + \epsilon(j,i)]$$

I	$\epsilon(i,i)$	$\epsilon(j,j)$	$\epsilon(i,j)$	$\epsilon(j,i)$
$\Rightarrow$ 1	-1	-1	1	1
	-1	-1	-1	-1
	+1	+1	+1	+1
	+1	+1	-1	-1
I	+0	+0	+0	+0

. QED

★ $E(i,j;\lambda)$, $\Delta(i,\lambda)$, $P(i,j)$ are called $\boxed{\text{Elementary matrices}}$

→ <u>Invertible matrices</u> :

How to find A^{-1} if it exists.

$$A = \begin{pmatrix} 2 & 3 \\ 1 & 2 \end{pmatrix} \quad \left(A^{-1} = \begin{bmatrix} 2 & -3 \\ -1 & 2 \end{bmatrix} \right)$$

$$|A|I| \longleftrightarrow [I|A^{-1}]$$

Matrix illustrated in easiest case

$$\left[\begin{array}{c|c} A & I_2 \end{array}\right]$$ psychological significance only

$$\left[\begin{array}{cc|cc} 2 & 3 & 1 & 0 \\ 1 & 2 & 0 & 1 \end{array}\right]$$

Perform Row operations (and keep track)

$$\left[\begin{array}{cc|cc} 2 & 3 & 1 & 0 \\ 1 & 2 & 0 & 1 \end{array}\right] \xrightarrow{P(1,2)} \left[\begin{array}{cc|cc} 1 & 2 & 0 & 1 \\ 2 & 3 & 1 & 0 \end{array}\right]$$

$$\downarrow E(2,1;-2)$$

$$\left[\begin{array}{cc|cc} 1 & 2 & 0 & 1 \\ 0 & -1 & 1 & -2 \end{array}\right]$$

$$\downarrow E(1,2;2)$$

$$\left[\begin{array}{cc|cc} 1 & 0 & 2 & -3 \\ 0 & -1 & 1 & -2 \end{array}\right]$$

$$\downarrow \Delta(2,-1)$$

$$\left[\begin{array}{cc|cc} 1 & 0 & 2 & -3 \\ 0 & 1 & -1 & 2 \end{array}\right]$$

$$I \mid A^{-1}$$

$$A \to P(1,2)A \to E(2,1;-2)P(1,2)A$$

$$\downarrow$$

$$E(1,2;2)E(2,1;-2)P(1,2)A$$

$$\downarrow$$

$$I = [\Delta(2,-1)E(1,2;2)E(2,1;-2)P(1,2)]A$$

$$\boxed{(xy)^{-1} = y^{-1}x^{-1}}$$

This tells us two extra things

$$A^{-1} = \Delta(2,-1)E(1,2;2)E(2,1;-2)P(1,2)$$

$$\begin{aligned} A &= \Delta(2,-1)E(1,2;2)E(2,1;-2)P(1,2) \\ &= P(1,2)^{-1}E(2,1;-2)^{-1}E(1,2;2)^{-1}\Delta(2,-1)^{-1} \\ &= P(1,2)E(2,1;2)E(1,2;-2)\Delta(2,-1), \end{aligned}$$

Check:

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\begin{pmatrix} 2 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 2 & 3 \\ 1 & 2 \end{pmatrix}$$

14-10-2010

$E(i,j;\lambda)$ $(i \neq j)$ $\quad E(i,j;\lambda)^{-1} = E(i,j,-\lambda)$

$\Delta(i,\lambda)$ $(\lambda \neq 0)$ $\quad \Delta(i,\lambda)^{-1} = \Delta(i,\lambda^{-1})$

$P(i,j)$ $(i \neq j)$ $\quad P(i,j)^{-1} = P(i,j)$

<u>Idea</u>: Want to inverse A $(n \times n)$

Form ~~2x2~~ $n \times 2n$ matrix

$$[A \mid I_n]$$

Now reduce, suppose I perform m Row operations to get from

$$[A \mid I_n] \rightsquigarrow [I_n \mid ?]$$

Then $? = A^{-1}$ [If you can't get I_n in LHS then A is not invertible]

Suppose matrices used are $Q_1 \ldots Q_m$

$$[A \mid I_n] \to Q_1[A \mid I_n] \to Q_2 Q_1 [A \mid I_n]$$
$$[Q_1 A \mid Q_1] \to [Q_2 Q_1 A \mid Q_2 Q_1]$$

Eventually get: $Q_m \ldots Q_2 Q_1 [A \mid I_n] = [Q_m \ldots Q_2 Q_1 A \mid Q_m \ldots Q_1]$

So if $(Q_m \ldots Q_1) A = I_n$

then* $Q_m \ldots Q_1 = A^{-1}$ $\quad$ * $[(XY)^{-1} = Y^{-1}X^{-1}]$

Also $A = (Q_m \ldots Q_1)^{-1}$, so you also get $A = Q_1^{-1} \ldots Q_m^{-1}$

* Note Reverse of Order

Example : $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 0 & 0 \end{bmatrix}$ First form : $\left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & -1 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 \end{array}\right]$

$[A \mid I_3]$

Now Reduce using Row Operations and keep track !!

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & -1 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & -2 & 0 & -1 & 1 & 0 \\ 0 & -1 & -1 & -1 & 0 & 1 \end{array}\right]$$

$E(2,1;-1)$
followed by $E(3,1;-1)$

$\Delta(2,-\tfrac{1}{2})$

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1/2 & -1/2 & 0 \\ 0 & -1 & -1 & -1 & 0 & 1 \end{array}\right]$$

$\downarrow$ $E(1,2;-1)$ followed by $E(3,2;1)$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1/2 & 1/2 & 0 \\ 0 & 1 & 0 & 1/2 & -1/2 & 0 \\ 0 & 0 & -1 & -1/2 & -1/2 & 1 \end{array}\right]$$

$\downarrow$ $E(1,3;1)$ followed by $\Delta(3,-1)$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1/2 & -1/2 & 0 \\ 0 & 0 & 1 & 1/2 & 1/2 & -1 \end{array}\right]$$

Claim : $\begin{bmatrix} 0 & 0 & 0 \\ 1/2 & -1/2 & 0 \\ 1/2 & 1/2 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 0 & 0 \end{bmatrix}^{-1}$

Check : $\begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 & 1 \\ 1/2 & -1/2 & 0 \\ 1/2 & 1/2 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$*\ A^{-1} = \Delta(3,-1)\, E(1,3;1)\, E(3,2;1)\, E(1,2;-1)\, \Delta(2,-\tfrac{1}{2})\, E(3,1;-1)\, E(2,1;-1)$

So $A = E(2,1;-1)^{-1} E(3,1;-1)^{-1} \Delta(2,-\tfrac{1}{2})^{-1} E(1,2;-1)^{-1} E(3,2;1)^{-1} E(1,3;-1)^{-1} \Delta(3,-1)^{-1}$

$A = E(2,1;1)\, E(3,1;1)\, \Delta(2,-2)\, E(1,2;1)\, E(3,2;-1)\, E(1,3;-1)\, \Delta(3,-1)$

Here is an example where A is <u>not</u> invertible

$$\underset{A}{\begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 3 & 1 \end{bmatrix}} \xrightarrow{\text{in form}} \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & -1 & 1 & 0 & 1 & 0 \\ 1 & 3 & 1 & 0 & 0 & 1 \end{array}\right] \rightarrow \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & -2 & 0 & -1 & 1 & 0 \\ 0 & 2 & 0 & -1 & 0 & 1 \end{array}\right]$$

$$\downarrow$$

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & -2 & 0 & -1 & 1 & 0 \\ \underbrace{0 \quad 0 \quad 0}_{\text{zero row}} & & & -2 & -1 & 1 \end{array}\right]$$

$A \rightarrow Q_3 Q_2 Q_1 A = B$ not invertible

If A^{-1} exists $A^{-1} Q_1^{-1} Q_2^{-1} Q_3^{-1} = B^{-1}$

But B is not invertible, so A is not invertible

PROPOSITIONAL LOGIC

p = 'It is raining'
q = 'It is cold'

These statements are both capable of being True or False and are independent.

We use four ~~sins~~ signs
① $\wedge$ (= and)
② $\vee$ (= or)
③ $\Rightarrow$ (= implies)
④ $\neg$ (= not)

· $p \wedge q$ = (It is raining and cold)

p	q	$p \wedge q$
T	T	T
	F	F
F	T	F
	F	F

Inclusive (in Latin 'vel')

p	q	$p \vee q$
T	T	T
	F	T
F	T	T
	F	F

and

$\vee, \underline{\wedge}, \neg, \Rightarrow$

19-10-2010

①

p	q	$p \wedge q$
T	T	T
	F	F
F	T	F
	F	F

~~disjunction~~
$\wedge$ conjunction

inclusive or $\vee$ (= vel)

②

p	q	$p \vee q$
T	T	T
	F	T
F	T	T
	F	F

$\vee$ disconjuction

~~Exclusive~~

③ 'not' · ($\neg$)

p	$\neg p$	$\neg\neg p$
T	F	T
F	T	F

$\neg\neg p \equiv p$

$\neg$ negation

Implication

p	q	$p = q$	
T	T	T	
	F	F	
F	T	T	
	F	T	

Example

(I) $(c \Rightarrow m) \wedge (m \Rightarrow s) \Rightarrow (c \Rightarrow \neg s)$

c = It is cold

m = It is cloudy

s = It is snowing

(If it cold it will be cloudy) $\wedge$ (If it is not cloudy it will snow)

Conclusion : Therefore, it is cold it will not snow

(II) $(c \Rightarrow m) \wedge (m \Rightarrow \neg s) \Rightarrow (s \Rightarrow \neg c)$

(if It is cold it will be cloudy) ∧ (If it is cloudy it will not snow)

Therefore if it snows, it is not ~~cold~~ cold.

(I)

c	m	s	A: $c \Rightarrow m$	B: $\neg m \Rightarrow s$	$A \wedge B$ (~~$c \Rightarrow \neg s$~~)	$c \Rightarrow \neg s$ (~~$A \wedge B \Rightarrow$~~)	$A \wedge B \Rightarrow (c \Rightarrow \neg s)$
T	T	T	T	T	T	F	F
		F	T	T	T	T	T
	F	T	F	T	F	F	T
		F	F	F	F	T	T
F	T	T	T	T	T	T	T
		F	T	T	T	T	T
	F	T	T	T	T	T	T
		F	T	F	F	T	T

← This is True 7/8 of the time but not all the time

(II)

c	m	s	A: $c \Rightarrow m$	B: $m \Rightarrow \neg s$	$A \wedge B$	$s \Rightarrow \neg c$	$A \wedge B \Rightarrow (s \Rightarrow \neg c)$
T	T	T	T	F	F	F	T
		F	T	T	T	T	T
	F	T	F	T	F	F	T
		F	F	T	F	T	T
F	T	T	T	F	F	T	T
		F	T	T	T	T	T
	F	T	T	T	T	T	T
		F	T	T	T	T	T

A compound proposition is universally valid (or tautology) when the final column is always True

A compound proposition is a contradiction when last Column is always False

Most propositions are neither tautologies nor contradiction They are called CONTINGENT.

Basic examples

① $p \vee \neg p$ (tautology)

p	$\neg p$	$p \vee \neg p$
T	F	T
F	T	T

② $p \wedge \neg p$ (contradiction)

p	$\neg p$	$p \wedge \neg p$
T	F	F
F	T	F

③

p	q	$p \Rightarrow q$ *	$\neg q$	$\neg p$	$\neg q \Rightarrow \neg p$ *
T	T	T	F	F	T
	F	F	T	F	F
F	T	T	F	T	T
	F	T	T	T	T

Two formulae are equivalent '$\equiv$' when they take truth values at same place

$$\underbrace{`p \Rightarrow q' \equiv `\neg q \Rightarrow \neg p'}_{\text{Proof by contradiction}}$$

'$\neg q \Rightarrow \neg p$' is contrapositive of '$p \Rightarrow q$'

- Not to be confused with CONVERSE

  $q \Rightarrow p$ is the converse of $p \Rightarrow q$

Exercise:

$(p \Rightarrow q) \Rightarrow (\neg q \Rightarrow \neg p)$ is a tautology

$(p \Rightarrow q) \Rightarrow (q \Rightarrow p)$ is CONTINGENT.

p	q	$p\wedge q$	$\neg(p\wedge q)$	$\neg p$	$\neg q$	$\neg p\wedge\neg q$
T	T	T	F	F	F	F
	F	F	T	F	T	T
F	T	F	T	T	F	T
	F	F	T	T	T	T

0) $\neg\neg p = p$

1) $\neg(p\wedge q) \equiv (\neg p)\vee(\neg q)$

1)' $\neg(p\vee q) \equiv (\neg p)\wedge(\neg q)$ ← CHECK IT

2) $p\vee(q\vee r) \equiv (p\vee q)\vee r$ | Assoc.

2)' $p\wedge(q\wedge r) \equiv (p\wedge q)\wedge r$

3) $p\vee q \equiv q\vee p$

3)' $p\wedge q \equiv q\wedge p$

4) $p\wedge(q\vee r) \equiv (p\wedge q)\vee(p\wedge r)$ | DISTRIBUTIVE

4)' $p\vee(q\wedge r) \equiv (p\vee q)\wedge(p\vee r)$

$\boxed{\text{'}\vee\text{' behaves like '+'} \quad \text{'}\wedge\text{' behaves like } \cdot}$ *

In distributive you get $a.(b+c) = a.b + ac$ *

but not $\cancel{a+(b.c) = (a+b)(a+c)}$

$\neg, \vee, \wedge, \Rightarrow$

How many signs do we need?

I can eliminate $\wedge$

$$p \wedge q \equiv \neg((\neg p) \vee (\neg q))$$

Can I eliminate $\Rightarrow$?

p	q	$p \Rightarrow q$	$\neg p$	$\neg p \vee q$
T	T	T	F	T
	F	F	F	F
F	T	T	T	T
	F	T	T	T

$$p \Rightarrow q \equiv (\neg p) \vee q$$

So I only need two signs $\neg, \vee$.

Or I could use $\neg, \Rightarrow$

$p \wedge q$ in terms of $\neg, \Rightarrow$?

$$p \wedge q \equiv \neg(p \Rightarrow \neg q)$$

$$p \vee q \equiv \neg p \Rightarrow q$$

You can also get away not $\neg, \wedge$. Exercise

Sheffers Stroke functions

p	q	$p \mid q$	$\phi(p \mid q)$
T	T	F	T
	F	T	F
F	T	T	T
	F	T	T

$$\neg p \equiv p \mid p.$$

$$p \Rightarrow q \equiv p \mid (p \mid q)$$

In Propositional Calculus dealing with constant statements.

$p =$ 'It is raining'
$\vdots$ etc.

In Mathematics we deal with variable statements
$P(x) =$ '$x \geq 4$' If x is an integer this represents ∞ many constant Statements

$Q(x) =$ '$x \leq 6$'
$P(x) \wedge Q(x) =$ '$4 \leq x \leq 6$'
Need two more ideas
Universal Quantifies
$\sim (\forall x) P(x)$ - for every x under discussion $P(x)$ is true.

Existential Quantifies

$(\exists x) P(x)$

$\sim$ for at least one x under discussion $P(x)$ is true.

We need to learn how to manipulate $\forall, \exists$.
- $\mathscr{D}$ = domain of discussion
might be $\mathbb{R}, \mathbb{Z}, \mathbb{Q}$
Easiest case $\{0, 1\}$

$P(x)$: $\mathscr{D} = (0,1)$
two possible statements $P(0), P(1)$
eg. $P(x) =$ '$x = 1$'

What does $(\forall x) P(x)$ look like ? $\rightarrow P(0) \wedge P(1)$
$(\forall x) P(x) \equiv P(0) \wedge P(1)$

Example: $\mathcal{D} = \{0, 1, 2\}$

$$(\forall x) P(x) \equiv P(0) \wedge P(1) \wedge P(2)$$

SNAG $\mathcal{D} = \mathbb{N} = \{0, 1, 2, \cdots n, n+1 \ldots\}$

$$(\forall x) P(x) = P(0) \wedge P(1) \wedge P(2) \cdots \wedge P(n) \wedge P(n+1) \ldots$$

$(\forall x) P(x)$ looks finite but is infinite in this case.

Example $\mathcal{D} = \{0, 1\}$

$$(\exists x) P(x) = P(0) \vee P(1)$$

$\mathcal{D} = \{0, 1, 2\}$

$$(\exists x) P(x) = P(0) \vee P(1) \vee P(2)$$

$\mathcal{D} = \mathbb{N}$, $(\exists x) P(x) = P(0) \vee P(1) \vee P(2) \ldots \vee P(n) \vee P(n+1) \ldots$

Example: $\neg (\forall x) P(x)$ Take $\mathcal{D} = \{0, 1\}$

$$(\forall x) P(x) \equiv P(0) \wedge P(1)$$
$$\neg (\forall x) P(x) \equiv \neg (P(0) \wedge P(1))$$
$$\equiv \neg P(0) \vee \neg P(1)$$
$$\equiv (\exists x) \neg P(x)$$

Example $\mathcal{D} = \{0, 1, 2\}$

$$(\forall x) P(x) = P(0) \wedge P(1) \wedge P(2)$$
$$\neg (\forall x) P(x) \equiv \neg (P(0) \wedge P(1) \wedge P(2))$$
$$\equiv \neg P(0) \vee \neg P(1) \vee \neg P(2)$$
$$\equiv (\exists x) \neg P(x)$$

- $\neg (\forall x) P(x) \equiv (\exists x) \neg P(x)$ → We take $\neg (\forall x) P(x) \equiv (\exists x) \neg P(x)$ to be true for all possible $\mathcal{D}$.

Difficulty: $\mathcal{D}$ is infinite, eg. $\mathbb{N}$

$$\boxed{\bullet \quad \neg(\exists x)P(x) \equiv (\forall x)\neg P(x)}$$

eg: $\mathcal{D} = \{0, 1\}$

$$(\exists x)P(x) = P(0) \vee P(1)$$

$$\neg(\exists x)P(x) = \neg(P(0) \vee P(1))$$

$$= \neg P(0) \wedge \neg P(1)$$

$$= (\forall x)\neg P(x)$$

21-10-2010

<u>Order</u>: $(0,1) \neq (1,0)$

(a,b) ordered ~~place~~ pair

Rules of inequality for ordered pairs

$(a,b) = (c,d)$ iff $a=c$ and $b=d$

In set theory order is not fundermental

We use curly brackets $\{0,1\} = \{1,0\}$

Set theory has one primitive notion belonging to $\in$

$0 \in \{0,1\}$

$1 \in \{0,1\}$

$2 \notin \{0,1\}$

Two (essentially different) ways of describing a set.

<u>Baby Way</u> : List the elements

<u>Example</u> : $X = \{1, 2, 3, 4, 5, 6\}$

How many elements? <u>SIX</u>

$Y = \{1, \{2,3\}, 4, \{5,6\}\}$

This has FOUR ELEMENTS, 1, $\{2,3\}$, 4, $\{5,6\}$

$\{2,3\} \in Y$

<u>SUBSETS</u>

Suppose A, B sets, Write $B \subset A$, when

'$x \in B \Rightarrow x \in A$' is <u>true</u>

If x belongs to B, then x belongs to A

- DON'T CONFUSE WITH '$\subset$' and '$\in$'

Example : $A = \{0, 1, \{2,3\}, \{4,5\}, 2, 4\}$

$\{2,3\} \in A$ ✓ $\qquad \{2,4\} \in A$ ✗

$\{2,3\} \subset A$ ✗ $\qquad \{2,4\} \subset A$ ✓

$x \in A$ iff $\{x\} \subset A$

For infinite sets listing isn't effective procedure

SOPHISTICATED WAY

Idea. define a set by means of a property that its elements have "Defining property".

$A = \{x \mid P_A(x)\}$

$B = \{x \mid x$ is a green London bus$\}$

typical element , Defining property

So have $P_B(x)$ = 'x is a green London bus'

eg. $X = \{x \in \mathbb{Z}, \ 3 \le x \le 100\}$

-) UNION

$A = \{x \mid P_A(x)\}$

$B = \{x \mid \cancel{P_A(x) \vee} \ P_B(x)\}$

$A \cup B = \{x \mid P_A(x) \vee P_B(x)\}$

-) Intersection :

$A \cap B = \{x \mid P_A(x) \wedge P_B(x)\}$

subset.

$B \subset A$ when $P_B(x) \Rightarrow P_A(x)$

→ VENN DIAGRAM

- You can represent all possible relations between 3 sets A, B, C in two dimensions.

For Four sets A, B, C, D . you need three dimensions

For five sets AAGHH! You need logic.

→ <u>Mappings ≡ Functions</u>

$f(x) = x^2$. is so far only a FORMULA

<u>Rough Definition</u> :

→ A, B sets $f : A \to B$

A mapping $f : A \to B$ is a "rule" which associates to each $a \in A$, a single element $f(a) \in B$.

A is <u>domain</u> of f

B is <u>codomain</u> of f

$\mathbb{R}$ = real nos

$f : \mathbb{R} \to \mathbb{R}$, $f(x) = x^2$ is <u>now</u> a mapping

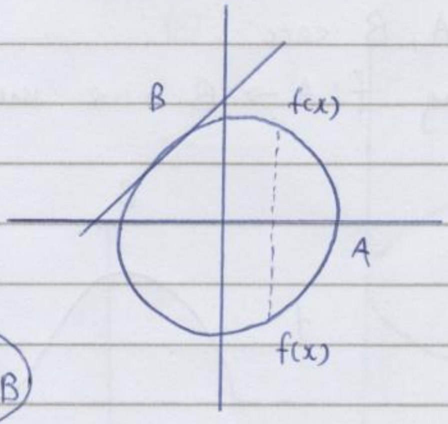
~ $g(x) = \dfrac{x}{x-1}$

$g : \mathbb{R} - \{1\} \to \mathbb{R}$ is a mapping

26-10-2010

$x^2+y^2=1$

$y=f(x)$



Curve $\subset A \times B$

A, B sets

By a mapping

$$f: A \to B$$

I mean a 'rule' which given $a \in A$ assigns to it a single element $f(a) \in B$

(a,b) denotes the ordered

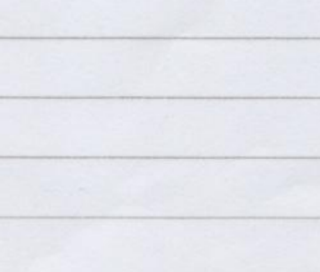
$(a,b) = (a',b')$ iff $a=a'$, $b=b'$

If A, B are sets $A \times B = \{ (a,b) \mid a \in A \text{ and } b \in B \}$

<u>Product Set</u>

$P_{A \times B}(a,b) = P_A(a) \wedge P_B(b)$ Cartesian Product (Descrete)

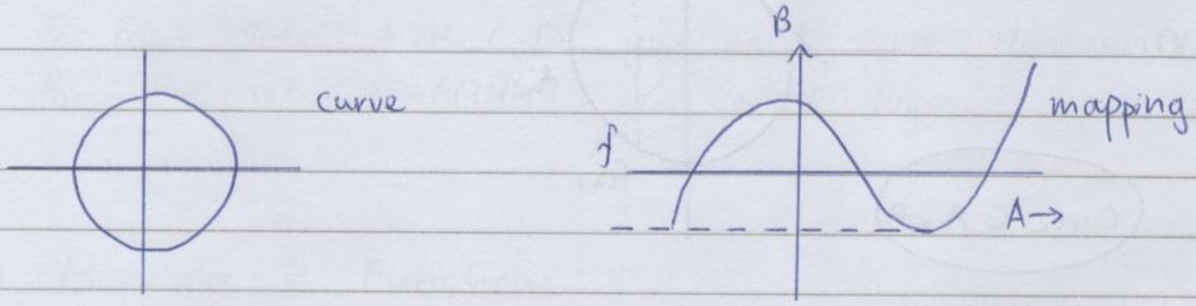
Idea $A \times B$



accurate if
$A = B = \mathbb{R}$

Formal definition : A, B sets

By a mapping $f: A \to B$, we mean a subset $(f \subset A \times B)$



such that i) $\forall a \in A \ \exists b \in B$ such that $(a,b) \in f$

(Write $b = f(x)$ $\quad (a,b) \in f$

In English, for every $a \in A$, there is an $f(a) \in B$

ii) If $(a,b) \in f$ and $(a,b') \in f$, then $b = b'$

In English, "$f(a)$ is a single element of B"

Example : i) $A = B = \mathbb{R}$

$f: \mathbb{R} \longrightarrow \mathbb{R}$ $\quad f(x) = x^2$ is a perfectly good mapping

ii) $g: \mathbb{R} \longrightarrow \mathbb{R}$

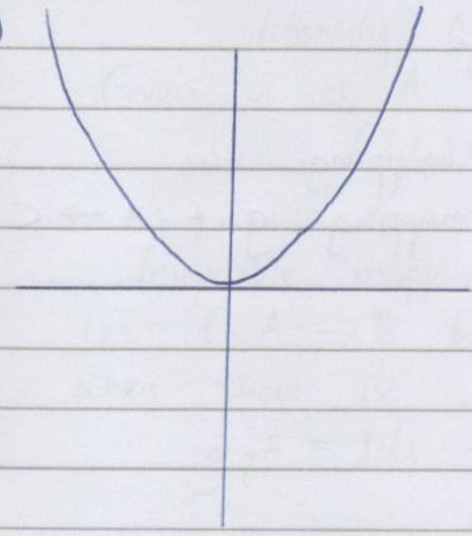
$g(x) = \sqrt{x}$

(a) is not a mapping, (b) $g(x)$ is not defined for $x < 0$

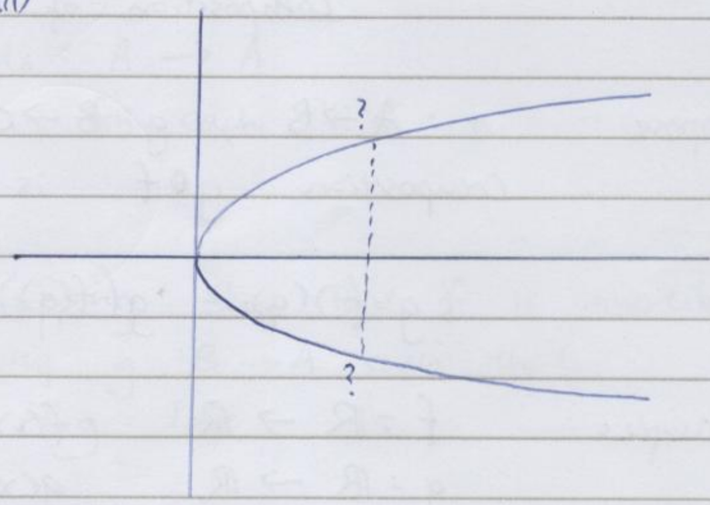
(iii) $h: \mathbb{R}_+ \to \mathbb{R}_+$

$h(x) = \sqrt{x}$ (still not a mapping because haven't specified which square root)

(i)



(ii)



The casual thing to do here is to restrict the codomain

$$h : \mathbb{R}_+ \to \mathbb{R}_+$$
$$h(x) = \sqrt{x}$$

<u>Snag</u>: $g : \mathbb{C} \to \mathbb{C}$

$g(z) = \sqrt{z}$

This is NEVER A FUNCTION

To summarise:

A mapping $f : A \to B$ must satisfy

i) $\forall a \in A \ \exists b \in B : (a,b) \in f$
$\boxed{b = f(a)}$

ii) If $(a,b) \in f$ and $(a,b') \in f$
then $b = b'$

Composition of mappings

Suppose : $f : A \to B$, $g : B \to C$ are mappings the composition $g \circ f$ is the mapping $g \circ f : A \to C$

$$(g \circ f)(a) = g(f(a))$$

Examples : $f : \mathbb{R} \to \mathbb{R}$ $\quad f(x) = \sin x$

$g : \mathbb{R} \to \mathbb{R}$ $\quad g(x) = x^2 + 1$

Then $\to$ $g \circ f(x) = \sin^2(x) + 1 = (|\sin x|^2 + 1)$

$\to$ $f \circ g(x) = \sin(x^2 + 1)$

$\to$ $f \circ g \neq g \circ f$

$f \circ g \leq 1$

while $g \circ f \leq 2$

- Composition is usually non-commutative
- Composition is always associative

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[rr, bend right, "g \circ f"'] & B \arrow[r, "g"] \arrow[rr, bend left, "h \circ g"] & C \arrow[r, "h"] & D \end{tikzcd}$$

Prop : $h \circ (g \circ f) = (h \circ g) \circ f$

Proof : $[h \circ (g \circ f)](a) = h((g \circ f)(a))$

$= h(g(f(a)))$

$[(h \circ g) \circ f](a) = (h \circ g)(f(a))$

$= h(g(f(a)))$

QED

Identity Mapping

Given a set A , $Id_A : A \to A$

$$(Id_A)(x) = x \text{ , for each } x \in A$$

Invertible mappings

Let $f : A \to B$ be a mapping. Say that f is invertible when there is a mapping $g : B \to A$ such that

$$g \circ f = Id_A \quad \text{and} \quad f \circ g = Id_B$$

$$\log : \mathbb{R}_+ \longrightarrow \mathbb{R}$$

$$\log(x) = \int_1^x \frac{dt}{t}$$

Napier

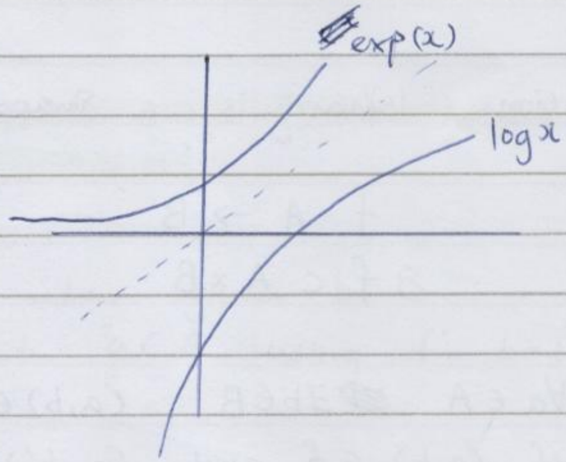
Newton

$$\exp : \mathbb{R} \longrightarrow \mathbb{R}_+$$

$$\exp(x) = \sum_{r=0}^{\infty} \frac{x^r}{r!}$$

$$\exp(\log(x)) = x$$

$$\log(\exp(x)) = x$$



Not every mapping is invertible, MOST MAPPINGS ARE NOT INVERTIBLE !

eg: $f : \mathbb{R} \to \mathbb{R}_+ : \{x : x \geq 0$

$f(x) = x^2$ not invertible

Notation: Write inverse of f as f^{-1}

not as $\frac{1}{f}$

X | A, B subsets of X

$$X-(A\cup B) = (X-A)\cap(X-B)$$

Model Proof : $X-(A\cup B) = \{x : P_X(x) \wedge \neg P_{A\cup B}(x)\}$

Do it by manipulating defining predicts

$$P_{A\cup B}(x) = P_A(x) \vee P_B(x)$$

$$P_X(x) \wedge \neg P_{A\cup B}(x) \equiv P_X(x) \wedge \neg(P_A(x) \vee P_B(x))$$

$$= (P_X(x) \wedge \neg P_A(x)) \wedge P_X(x) \wedge \neg P_B(x)$$

$$= P_{X-A}(x) \wedge P_{X-B}(x)$$

$$X - A\cup B = \{x : P_{X-A}(x) \wedge P_{X-B}(x)\}$$

$$= (X-A)\cap(X-B)$$

Question : When is a mapping invertible?

$f : A \to B$ $\qquad$ $f^{-1} : B \to A$

$f \subset A \times B$ $\qquad$ Want $f^{-1} \subset B \times A$

i) $\forall a \in A$ ~~$\exists$~~ $\exists b \in B$: $(a,b) \in f$

ii) If $(a,b) \in f$ and $(a,b') \in f$, then $b=b'$ $\qquad$ f

iii) $\forall b \in B$ $\exists a \in A$, $(a,b) \in f$

iv) If $(a,b) \in f$ and $(a',b) \in f$ then $a=a'$

i)' $\forall b \in B$ $\exists a \in A$ $(b,a) \in f^{-1}$

ii)' If $(b,a) \in f^{-1}$ and $(b,a') \in f^{-1}$ then $a=a'$ $\qquad$ f^{-1}

iii)' $\forall a \in A$ $\exists b \in B$: $(b,a) \in f^{-1}$

• Definition : $f^{-1} = \{(b,a) \in B \times A \text{ s.t. } (a,b) \in f\}$

iv)' If $(b,a) \in f^{-1}$ $(b',a) \in f^{-1}$, then $b=b'$

minor image

If $f: A \to B$ is a mapping f must satisfy i) ~~ii~~ ii)

If f has an inverse mapping f^{-1} then f must ~~satif~~ satisfy iii) and iv)

(because f^{-1} must satisfy i)' and ii)'

i) and ii) defines the conditions for mapping

iii) is called SURJECTIVITY

Definition : A mapping $f: A \to B$ is surjectivity when $\forall b \in B, \exists a \in A ; b = f(a)$

iv) is called INJECTIVITY

Definition : A mapping $f: A \to B$ is injective when $f(a) = f(a') \Rightarrow a = a'$

In English : f is surjective when given $b \in B$. I can hit it with $a \in A$ using f $b = f(a)$.

$f: \mathbb{R} \to \mathbb{R}$, $f(x) = 2x + 1$ is surjective

Given $y \in \mathbb{R}$. I can find $x = \frac{y-1}{2}$ s.t. $f(x) = y$

eg: $f: \mathbb{Z} \to \mathbb{Z}$

$f(x) = 2x + 1$

f is <u>not surjective</u> because $f(x)$ is odd. Can't hit 2 using an <u>integer</u>.

eg: $f : \mathbb{R} \to \mathbb{R}$
$f(x) = 3x - 2$
Easy way without thought!!
suppose $f(x) = f(x')$
$3x - 2 = 3x' - 2$
$x = x' \Rightarrow x = x'$
$\therefore$ It is injective

eg: $f : \mathbb{R} \Rightarrow \mathbb{R}$ $f(x) = x^2$
$f(1) = f(-1)$ and $1 \neq -1$
$\therefore$ Not injective

X * STANDARD MISTAKE ALL MAPPINGS ARE INJECTIVE
confusing: $f(a) = b$ and $f(a) = b'$
$\Rightarrow b = b'$ X

with $f(a) = b$ $f(a') = b$
$\Rightarrow a = a'$ ✓

I've proved

Prop: If $f : A \to B$ is an invertible mapping then f is both injective and surjective.
and the same proof shows

Prop: If $f : A \to B$ is a mapping which is both injective and surjective
then $f^{-1} : B \to A$ is a mapping and is also injective and surjective

Proof $(i) \equiv (iii)'$

$(ii) \equiv (iv)'$

$(iii) \equiv (i)'$

$(iv) \equiv (ii)'$ QED

A mapping which is both injective and surjective is called <u>BIJECTIVE</u>

so, <u>Then</u> $f : A \to B$ is invertible iff f is <u>BIJECTIVE</u>.

-> Two sets A, B are equivalent when $\exists$ bijective $\beta : A \to B$

28-10-2010

$f : A \to B$ mapping

f is invertible when

$\exists g : B \to A$ s.t.

$g \circ f = Id_A$ and $f \circ g = Id_B$

$f : A \to B$ is bijective when

1) $f(a) = f(a') \Rightarrow a = a'$

[INJECTIVE] and

2) $\forall b \in B \ \exists a \in A$

$f(a) = b$ [SURJECTIVE]

I proved f, Then f is injective $\Leftrightarrow$ f is bijective

Permutations

Take the set : $\{1, 2, \dots n\}$

A permutation on n "letters" is a bijective mapping

$f = \{1, \dots, n\} \longrightarrow \{1, \dots n\}$

$\boxed{n=2}$ $\{1, 2\}$

$1 \to 1$

$2 \to 2$

$1 \to 1$

$2 \to 1$

is not injective

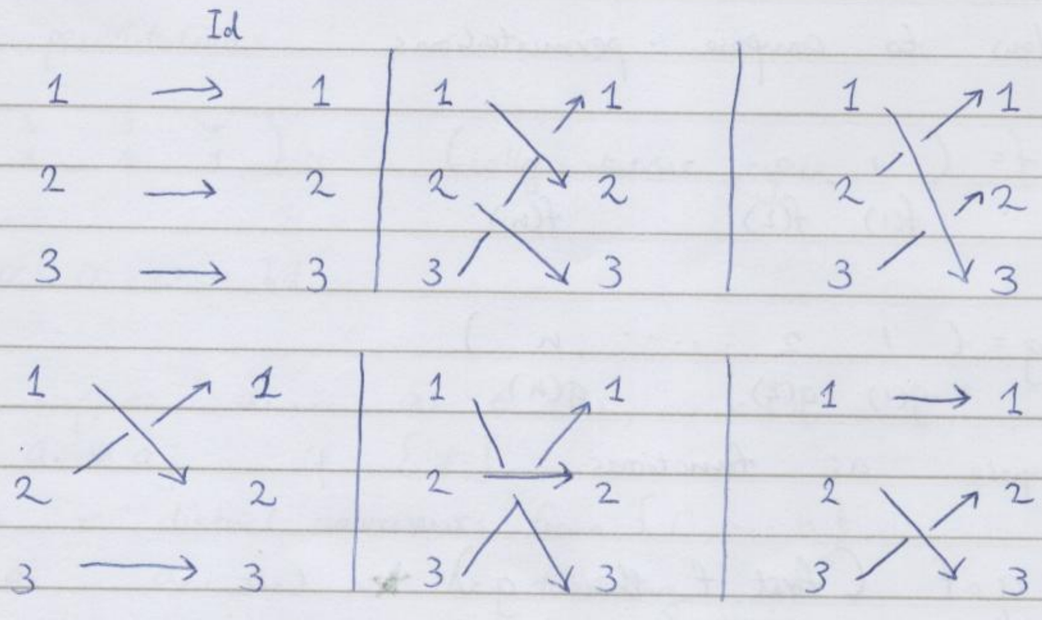
$1 \to 2$

$2 \to 1$

Twisted

$\to$ So there are $\underline{2}$ bijections $\{1,2\} \to \{1,2\}$

$\boxed{n=3}$



∴ There are six bijections $\{1,2,3\} \longrightarrow \{1,2,3\}$

How many injections $\{1, \ldots n\} \to \{1, \ldots n\}$?

$= n!$

Represent permutations(f) like this

$$\begin{pmatrix} 1 & 2 & 3 & \ldots\ldots & n-1 & n \\ f(1) & f(2) & f(3) & & f(n-1) & f(n) \end{pmatrix}$$

Prop : If $\left.\begin{array}{l} f : A \to B \\ g : B \to C \end{array}\right\}$ are bijective

then composite is also bijective

Proof : $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$

So $g \circ f$ is invertible so bijective

<u>QED</u>

How to compose permutations

$$f = \begin{pmatrix} 1, & 2, & \cdots, & n \\ f(1) & f(2) & & f(n) \end{pmatrix}$$

$$g = \begin{pmatrix} 1, & 2 & \cdots, & n \\ g(1) & g(2) & & g(n) \end{pmatrix}$$

Compose as functions

$g \circ f$ (first f then g) ★

$$\begin{array}{ccccc} 1 & 2 & 3 & (n-1) & n \\ \cancel{f(1)} & \cancel{f(2)} & \cancel{f(3)} & \cancel{f(n-1)} & \cancel{f(n)} \\ gf(1) & gf(2) & gf(3) & gf(n-1) & gf(n) \end{array} \begin{array}{l} \downarrow f \\ \downarrow g \end{array}$$

and cross out the middle line

Example: $n=4$

$$f = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 3 & 1 \end{pmatrix}$$

$$g = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 2 & 1 \end{pmatrix}$$

$$\underline{g \circ f} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ \cancel{2} & \cancel{4} & \cancel{3} & \cancel{1} \\ 3 & 1 & 2 & 4 \end{pmatrix} \begin{array}{l} \downarrow f \\ \downarrow g \end{array} \quad \Bigg| \quad \underline{f \circ g} : \begin{pmatrix} 1 & 2 & 3 & 4 \\ \cancel{4} & \cancel{3} & \cancel{2} & \cancel{1} \\ 1 & 3 & 4 & 2 \end{pmatrix} \begin{array}{l} \downarrow g \\ \downarrow f \end{array}$$

$$g \circ f = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 1 & 2 & 4 \end{pmatrix}, \qquad f \circ g : \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 3 & 4 & 2 \end{pmatrix}$$

$*\ \ g \circ f \neq f \circ g$

Cyclic permutations

$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \end{pmatrix}$ is a really obvious cycle

$$\sigma \circ \sigma \circ \sigma \circ \sigma = Id$$

In general suppose $a_1, \ldots, a_r \in \{1, \ldots, n\}$

$a_i \neq a_j$ if $\{\neq\}$

ie. take r district elements from $\{1, \ldots n\}$

Then $(a_1, a_2, \ldots a_{r-1}, a_r)$ is the following

$$(a_1 \ldots a_r)(a_i) = \begin{cases} a_{i+1} & \text{if } i < r \\ a_1 & \text{if } i = r \end{cases} \qquad \text{[FORMAL DEFINITION]}$$

$(a_1 \ldots a_r)(x) = x$ if $x \notin \{a_1 \ldots a_r\}$

eg①: $\underline{\underline{n=7}}$ $(2,5,6,3)$

$(2,5,6,3)(2) = 5$

$(2,5,6,3)(5) = 6$

$(2,5,6,3)(6) = 3$

$(2,5,6,3)(3) = 2$

$$(2,5,6,3) = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 1 & 5 & 2 & 4 & 6 & 3 & 7 \end{pmatrix}$$

$(2,5,6,3)(x) = x$

$x \notin \{2,5,6,3\}$

goes to next

eg②: $\underline{\underline{n=13}}$ $(1,7,9,6,11,13,12,4)$

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 \\ 7 & 2 & 3 & 1 & 5 & 11 & 9 & 8 & 6 & 10 & 13 & 4 & 12 \end{pmatrix}$$

$(a_1, \ldots a_r)$ is called a cycle of length r

$(a_1 \ldots a_r)^r = Id$

compose an r cycle with itself r times get back to Id.

Prop : If $a_1, \dots a_r \in \{1, \dots n\}$
$b_1, \dots b_s \in \{1, \dots n\}$

Suppose $\{a_1, \dots a_r\} \cap \{b_1, \dots b_r\} = \phi$
(sets have no common element)
then $(a_1, \dots a_r) \circ (b_1, \dots b_s)$
$= (b_1, \dots bs) \circ (a_1 \dots a_r)$

In English : " Disjoint Cycle Commute "

Example : $n = 9$
$\sigma = (1, 5, 7, 4)$ $\qquad \rho = (2, 9, 8, 3)$

$$\sigma \circ \rho = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ \cancel{1} & \cancel{9} & \cancel{2} & \cancel{4} & \cancel{5} & \cancel{6} & \cancel{7} & \cancel{3} & \cancel{8} \\ 5 & 9 & 2 & 1 & 7 & 6 & 4 & 3 & 8 \end{pmatrix} \begin{matrix} \\ \downarrow \rho \\ \downarrow \sigma \end{matrix}$$

$$\rho \circ \sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ \cancel{5} & \cancel{2} & \cancel{3} & \cancel{1} & 7 & 6 & 4 & 8 & 9 \\ 5 & 9 & 2 & 1 & 7 & 6 & 4 & 3 & 8 \end{pmatrix} \begin{matrix} \\ \downarrow \sigma \\ \downarrow \rho \end{matrix}$$

$\rho \circ \sigma = \sigma \circ \rho$
(because cycles are disjoint.)

Any permutation is a product of disjoint cycles
~~Example~~ : Example : $n = 10$

$$f = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\ 4 & 2 & 8 & 1 & 10 & 5 & 9 & 7 & 3 & 6 \end{pmatrix}$$

$f = (1,4)(2)(3,8,7,9,3)(5,10,6)$

same $\quad (2) = Id$ Degenerate cycle length 1

$f = (3, 8, 7, 9)(1,4)(5,10,6)$

2-11-2010

$f : \{1, \ldots n\} \to \{1 \ldots n\}$

permutation (ie. bijective)

1) $f = C_1 C_2 \ldots C_K$ where each C_i is a cyclic permutation

$C_i = a_1^i, \ldots a^i$

and C_i is a disjoint from C if $i \neq j$

2) Disjoint cycles <u>commute</u>.

Example : $f = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 \\ 11 & 6 & 4 & 1 & 7 & 9 & 5 & 10 & 2 & 8 & 3 \end{pmatrix}$

$$f = \underset{C_1}{(1, 11, 3, 4)} \underset{C_2}{(2, 6, 9)} \underset{C_3}{(5, 7)} \underset{C_4}{(8, 10)}$$

$$C_i C_j = C_j C_i \quad \text{for all } i, j$$

Definition : If σ is a permutation on $\{1, \ldots n\}$

then $\text{order}(\sigma) = \min\{ k \geq 1 \text{ such that } \sigma^k = Id \}$

$\text{ord}(C_1) = 4$, $\text{ord}(C_2) = 3$, $\text{ord}(C_3) = 2 = \text{ord}(C_4)$

$$\begin{aligned}
(C_1 C_2 C_3 C_4)^4 &= C_1 C_2 C_3 C_4 \, C_1 C_2 C_3 C_4 \, C_1 C_2 C_3 C_4 \, C_1 C_2 C_3 C_4 \\
&= C_1^4 C_2^4 C_3^4 C_4^4 \\
&= C_2^4 \qquad , * \; C_1^4 = C_3^4 = C_4^4 = Id \\
&= C_2
\end{aligned}$$

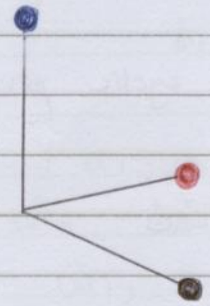
$$(C_1 C_2 C_3 C_4)^{12} = C_2^3 = Id.$$

Proposition : If $\sigma = C_1 C_2 \ldots C_K$

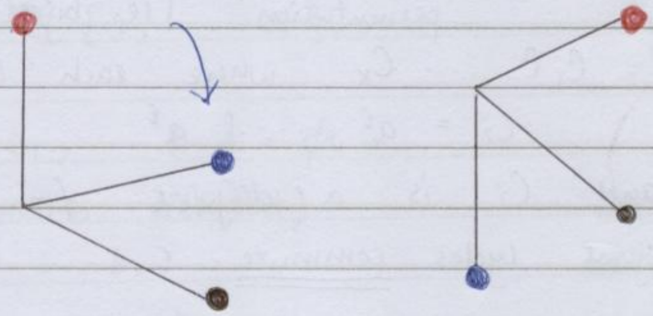
where $C_1 \ldots C_K$ disjoints cycles.

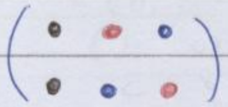
$\text{ord}(r) = \text{LCM}[\text{length}(C_1), \ldots \text{length}(C_K)]$

(I)



(II)



 $\begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}$

$$\text{Sign} = + \left[\begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix} \right.$$

These take $(I) \to (I)$, $(II) \to (II)$]

$$\text{sign} = -1 \left[\begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix} \right]$$

These all swap (I) and (II)

REVERSE ORIENTATION

Definition: A transposition is a cycle of length 2 (i,j)

Example $n = 5$

$$(2,4) = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 4 & 3 & 2 & 5 \end{pmatrix}$$

We first show

Proposition: A cycle of length n is a product of $(n-1)$ transpositions

~~Proof~~: $\Longrightarrow$

Proof : $(1\ 3\ 2\ 4\ 7) = (1,7)(1,4)(1,2)(1,3)$

$$\begin{array}{l}\left[\begin{array}{ccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 3 & 2 & 1 & 4 & 5 & 6 & 7 \\ 3 & 1 & 2 & 4 & 5 & 6 & 7 \\ 3 & 4 & 2 & 1 & 5 & 6 & 7 \\ 3 & 4 & 2 & 7 & 5 & 6 & 1 \end{array}\right] \begin{array}{l} (1,3) \\ (1,2) \\ (1,4) \\ (1,7) \end{array}\end{array}$$

$(a_1\ a_2 \cdots a_n) = (a_1, a_n) \ldots\ldots ($

$(n-1)$ transpositions.

"Contrary" Rule : A cycle of cross length is a product of an odd number of transpositions.

A cycle of odd length is a product of an even number of transpositions.

A transposition is called adjacent when it has the form $(k, k+1)$

eg- $(2,3) = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 3 & 2 & 4 & 5 \end{pmatrix}$ is adjacent

Proposition: Any transposition is a product of an odd number of adjacent transpositions.

Proof : $(2,6) = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 3 & 2 & 4 & 5 & 6 \\ 1 & 6 & 2 & 4 & 5 & 3 \\ 1 & 6 & 3 & 4 & 5 & 2 \end{pmatrix} \begin{array}{l} (2,3) \\ (3,6) \\ (2,3) \end{array}$

gap $= 6-2 = 4$

If $T = (i,j)$, define the gap$(T) = |j-i|$ gap$(3,6) = 3$

Put $T = (i,j)$, if gap$(T) = 1$, then already adjacent

So doesn't have anything to do.

⇒) suppose proved gap < k that transposition of gap < k is a product of adjacents.

suppose $k = |j - i|$

Write $(i,j) = (i, i+1)(i+1, j)(i, i+1)$

$\text{gap } (i+1, j) = k - 1$

So $(i+1, j) = \sigma_1 \dots \sigma_{2N+1}$ where each σ_i adjacent

$(i,j) = (i, i+1)\sigma_1 \dots \sigma_{2N+1}(i, i+1)$

product of $2N+3$ is adjacent QED

<u>In fact</u>: A transposition of gap k is a product of $(2k-1)$ adjacent transposition

$(2,6) = (2,3)(3,6)(2,3)$

$(3,6) = (3,4)(4,6)(3,4)$

$(4,6) = (4,5)(5,6)(4,5)$ ok.

$(2,6) = (2,3)(3,4)(4,5)(5,6)(4,5)(3,4)(2,3)$

gap $= 4$, no of adjacents $= 7 = (2 \times 4) - 1$

To summerise

(i) Every permutation is a product of cycles.

(ii) Any cycle is a product of transpositions.

(iii) Any transposition is a product of adjacent transposition

So. <u>Prop</u>: Any permutation of $\{1 \dots n\}$ is a product of adjacent transpositions.

<u>Rough Definition</u>: Take $\sigma : \{1, \dots n\} \circlearrowleft$ permutation

Write as product of adjacent transpositions

$$\sigma = T_1 \dots T_N, \quad \text{sign}(\sigma) = \begin{cases} +1 & N \text{ even} \\ -1 & N \text{ odd} \end{cases}$$

Laplaces' formula for sign (σ)

$\sigma : \{1, \ldots n\} \circlearrowleft$ permutation

$$L(\sigma) = \prod_{1 \le i < j \le n} \left(\frac{\sigma(j) - \sigma(i)}{j - i} \right)$$

We'll show that $L(\sigma) = +1$ when σ is a product of an even number of adjacent transpositions.

$L(\sigma) = -1$ . . . . . . ODD.

Write $L(\sigma) = \prod_{1 \le i < j \le n} \sigma(j) - \sigma(i)$

Note : $L(\sigma) = \dfrac{L(\sigma)}{L(Id)}$

Prop : $T = (k, (k+1))$ [adjacent transpositions]

Then $L(\sigma \tau) = -L(\sigma)$

~~Going to split $L(\sigma)$ into 7 factors~~

Fixing $\tau = (k, k+1)$

For an arbitary permutation p.

Write $L_1(p) = \prod_{i<j<k} p(j) - p(i)$

$L_2(p) = \prod_{i<k=j} p(j) - p(i)$

$L_3(p) = \prod_{i<k<k+1=j} p(j) - p(i)$

$L_4(p) = p(k+1) - p(k) = \prod_{i=k<k+1=j}$

$L_5(p) = \prod_{i=k<k+1<j} p(j) - p(i)$

$L_6(p) = \prod_{i=k+1<j} p(j) - p(i)$

$L_7(p) = \prod_{k+1<i<j} p(j) - p(i)$

$L(p) = L_1(p) L_2(p) L_3(p) L_4(p) L_5(p) L_6(p) L_7(p)$

Now compute $L_i(\sigma\tau)$ $\tau = (k, k+1)$

$L_1(\sigma\tau) = L_1(\sigma)$

$L_2(\sigma\tau) = L_3(\sigma)$

$L_3(\sigma\tau) = L_2(\sigma)$

$L_4(\sigma\tau) = -L_4(\sigma)$

$L_5(\sigma\tau) = L_6(\sigma)$

$L_6(\sigma\tau) = L_5(\sigma)$

$L_7(\sigma\tau) = L_7(\sigma)$

$L(\sigma\tau) = -L(\sigma)$

$\Rightarrow$

Corollary : If $\sigma = \tau_1 \cdots \tau_N$
and each τ_i adjacent transpositions
Then $L(\sigma) = (-1)^N L(Id)$

$$L(\tau_1 \cdots \tau_{N-1}, \tau_N) = -L(\tau_1 \cdots \tau_{N-1})$$
$$= +L(\tau_1 \cdots \tau_{N-2})$$
$$= -L(\tau_1 \cdots \tau_{N-3})$$
$$= (-1)^N L(Id)$$

QED

This shows Corollary : Impossible to write a permutation both as a product of an EVEN of adjacent transpositions
ODD of adjacent transpositions

Proof : $\sigma = \tau_1 \cdots \tau_N$
If N even $L(\sigma) > 0$ If N ODD $L(\sigma) < 0$

Corollary : If I write $\sigma = \tau_1 \cdots \tau_N$ (τ_i adjacent transpositions)
$L(\sigma) = (-1)^N L(Id)$

$$\frac{L(\sigma)}{L(Id)} = (-1)^N$$

$$\Rightarrow L(\sigma) = (-1)^N = \begin{cases} +1 & N \text{ even} \\ -1 & N \text{ odd} \end{cases}$$

QED.

4-11-2010

Example : $\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 \\ 8 & 11 & 5 & 7 & 3 & 9 & 10 & 6 & 1 & 12 & 13 & 4 & 2 \end{pmatrix}$

i) Decompose σ as product of disjoint cycles.

ii) Compute ord (σ)

iii) compute sign (σ)

Ans(i) $\sigma = (1,8,6,9)(2,11,13)(3,5)(4,7,10,12)$

(ii) ord(σ) = 12 = LCM (4, 3, 2, 4)

cycle of ODD Length is a product an EVEN no. of transpositions.
EVEN ...... ODD....
Every transposition is a product of an odd no. of adj. trans.
so every cycle of ODD length is a product of an EVEN no. of adj trans.
EVEN .... ODD...

sign (cycle of length N) = $(-1)^{N-1}$

(iii) $(1,8,6,9)(2,11,13)(3,5)(4,7,10,12)$
$= (-1) \quad (+1) \quad (-1) \quad (-1)$ [MULTIPLY]

sign(σ) = -1

FIELD

Typical example of a FIELD is $\mathbb{Q}$

- Add, subtract, multiply and divide by $\neq 0$

Formal definition: By a field $\mathbb{F}$ we mean the following

$$\mathbb{F} = (\mathbb{F}, +, 0, \cdot, 1)$$

where i) $\mathbb{F}$ is a set, $0 \in \mathbb{F}$, $1 \in \mathbb{F}$, $0 \neq 1$

ii) $+ : \mathbb{F} \times \mathbb{F} \to \mathbb{F}$ is a mapping

We write $a+b$ rather than $+(a,b)$

such that (1) $a+(b+c) = (a+b)+c$ Assoc.

(2) $a+b = b+a$ Commut.

(3) $a+0 = 0+a = a$ IDENTITY

for all $a, b, c \in \mathbb{F}$

(iii) For each $a \in \mathbb{F}$ there exists $-a \in \mathbb{F}$, $a+(-a) = 0$ (Additive inverse)

(iv) $\cdot : \mathbb{F} \times \mathbb{F} \to \mathbb{F}$ mapping, written $a.b$ instead of $\cdot(ab)$

(1) $a \cdot (b \cdot c) = (a \cdot b) . c$ Assoc.

(2) $ab = ba$ Commut.

(3) $a . 1 = 1 . a = a$ Identity

(v) For each $a \in \mathbb{F} - \{0\}$, $\exists a^{-1} \in \mathbb{F}$

$a . a^{-1} =$, Mult. inverse

(1) $a \cdot (b+c) = ab + ac$ Left

(2) $(b+c)a = ba + ca$ Right

} Distributive

Examples : i) $\mathbb{Q}$ is a Field.

ii) $\mathbb{R}$ is a field.

iii) $\mathbb{C}$ is a field.

$\mathbb{Z}$ is not a field. **NO** multiplicative inverse for 2, eg. or 3.

Finite example : Take $\mathbb{F}_2$ is a field with 2 elements $\{0, 1\}$

(Thinks : 0 ~ Even integers)

1 ~ Odd integers

+	0	1
0	0	1
1	1	0

$\cdot$	0	1
0	0	0
1	0	1

$1^{-1} = 1$

$\mathbb{F}_3 = \{0, 1, 2\}$ field with 3 elements.

Thinks $0 = \{$ integers exactly divisible by 3 $\}$

$1 = \{$ integers with remainder $= 1 \bmod 3 \}$

$2 = \{$ integers with remainder $= 2 \bmod 3 \}$

+	0	1	2
0	0	1	2
1	1	2	0
2	2	0	1

$\cdot$	0	1	2
0	0	0	0
1	0	1	2
2	0	2	1

$2^{-1} = 2$. $1^{-1} = 1$

New example :

$\{0, 1, 2, 3\}$ considered as remainders mod 4

$+$	0	1	2	3
0	0	1	2	3
1	1	2	3	0
2	2	3	0	1
3	3	0	1	2

$\cdot$	0	1	2	3
0	0	0	0	0
1	0	1	2	3
2	0	2	0	$\underline{\underline{2}}$
3	0	3	2	1

[NOT A FIELD]

~ 2 has no multiply inverse

$2 \neq 0$, $2^2 = 0$

Now take $\{0, 1, 2, 3, 4\}$ as remainders mod 5

Just look at mult.

$\cdot$	0	1	2	3	4
0	0	0	0	0	0
1	0	1	2	3	4
2	0	2	4	1	3
3	0	3	1	4	2
4	0	4	3	2	1

$= \mathbb{F}_5$

n	remainders mod. n
2	Field
3	Field
4	Not a Field
5	Field
6	Not a Field
7	Field

{ Remainders mod n } is a field $\iff$ n is prime.

Final example

$\mathbb{Q}(\sqrt{2})$

elements look like $a + b\sqrt{2}$. $a, b \in \mathbb{Q}$

$$a + b\sqrt{2} = c + d\sqrt{2} \iff a = c \text{ and } b = d$$

Addition : $(a + b\sqrt{2}) + (c + d\sqrt{2}) = (a+c) + (b+d)\sqrt{2}$

$$0 = 0 + 0\sqrt{2}$$

Mult. $(a + b\sqrt{2})(c + d\sqrt{2})$ $\left(= ac + 2bd + (ad + bc)\sqrt{2} \right)$

$$\parallel$$

$$ac + 2bd + (ad + bc)\sqrt{2}$$

$$1 = 1 + 0\sqrt{2}$$

$$(a + b\sqrt{2})^{-1} = \left(\frac{a - b\sqrt{2}}{a^2 - 2b^2} \right)$$

$(a + b\sqrt{2})(a - b\sqrt{2})$

$= a^2 - 2b^2 + 0\sqrt{2}$

$a^2 - 2b^2 \neq 0$ of $ab \neq 0$

$\sqrt{2}$ is not in $\mathbb{Q}$

16-11-2010

< Linear Algebra over a general field $\mathbb{F}$ >

Let $\mathbb{F}$ be a field, eg. $\mathbb{F} = \mathbb{Q}$

$$\mathbb{F}^n = \left\{ \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}, x_i \in \mathbb{F} \right\}$$ column vector

$n \times 1$ ~~matrix~~ matrices

operations connected with $\mathbb{F}^n$

<u>Addition</u>: $+ : \mathbb{F}^n \times \mathbb{F}^n \to \mathbb{F}^n$

$$\underbrace{\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}}_{\underline{x}} + \underbrace{\begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix}}_{\underline{y}} = \underbrace{\begin{pmatrix} x_1 + y_1 \\ x_2 + y_2 \\ \vdots \\ x_n + y_n \end{pmatrix}}_{\underline{x}+\underline{y}}$$

Scalar multiplication

$$\cdot, \ \mathbb{F} \times \mathbb{F}^n \to \mathbb{F}^n$$

$$\lambda . \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} \lambda x_1 \\ \lambda x_2 \\ \vdots \\ \lambda x_n \end{pmatrix}$$

Multipying vector $\underline{x} \in \mathbb{F}^n$, scalar $\lambda \in \mathbb{F}$

zero vector $\underline{0} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$

-) <u>Various Properties</u>

<u>Additive</u>

$$\underline{x} + (\underline{y} + \underline{z}) = (\underline{x} + \underline{y}) + \underline{z}$$

$$\underline{x} + \underline{y} = \underline{y} + \underline{x}$$

$$\underline{x} + \underline{0} = \underline{0} + \underline{x} = \underline{x}$$

$$\underline{x} + (-\underline{x}) = 0$$

where $-\underline{x} = (-1)x$

Multiplicative

$$\lambda . (\mu . \underline{x}) = (\lambda \mu) . \underline{x}$$
$$1 . x \doteq \underline{x}$$

Distributive
$$(\lambda + \mu) . \underline{x} = \lambda \underline{x} + \mu \underline{x}$$
$$\lambda (\underline{x} + \underline{y}) = \lambda \underline{x} + \lambda \underline{y}$$

Definition : Let $\mathbb{F}$ be a field
By a vector space V over a $\mathbb{F}$
we mean the following data.
$$V = (V, 0, +, \cdot) \text{ where}$$

V is a set and $0 \in V$ ("zero vector")

$$+ : V \times V \to V \quad \text{mappings}$$
$$\cdot : \mathbb{F} \times V \to V$$

such that

Additive properties

- $x + (y + z) = (x + y) + z$
- $x + y = y + z$
- $x + 0 = 0 + x = x$
- $x + \underset{(-x)}{(-1)x} = 0$

Multiplicative properties :

$$\lambda . (\mu . x) = (\lambda \mu) x$$
$$1 . x = x$$

Distributive :
$$(\lambda + \mu) . x = \lambda x + \mu x$$
$$\lambda (x + y) = \lambda x + \lambda y$$

Example 1: $\mathbb{F}^n$ is a vector space $\mathbb{F}$
(standard examples)

Example 2: $V = \left\{ \begin{pmatrix} x \\ -x \end{pmatrix} \quad x \in \mathbb{F} \right\}$ observe that $V \subset \mathbb{F}^2$
Is $V = \mathbb{F}^2$?

$\begin{pmatrix} 1 \\ 0 \end{pmatrix} \notin V \qquad V \neq \mathbb{F}^2$

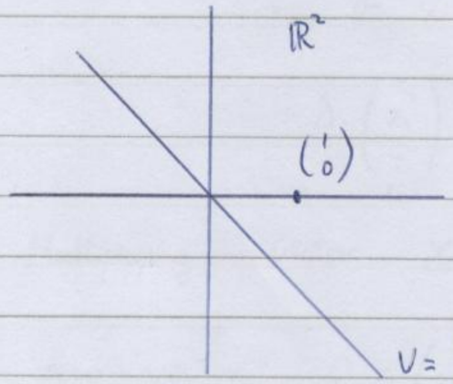
$\to$ This V is a vector space

$$\begin{pmatrix} x \\ -x \end{pmatrix} + \begin{pmatrix} y \\ -y \end{pmatrix} = \begin{pmatrix} x+y \\ -(x+y) \end{pmatrix} \quad , \quad \lambda \begin{pmatrix} x \\ -x \end{pmatrix} = \begin{pmatrix} \lambda x \\ -\lambda x \end{pmatrix}$$

$\Downarrow \quad \Downarrow \quad \Downarrow$

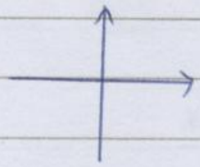
$V \quad V \quad V$

$0 = \begin{pmatrix} 0 \\ -0 \end{pmatrix}$

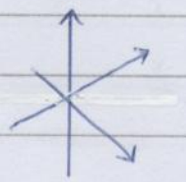


$V = \left\{ \begin{pmatrix} x \\ -x \end{pmatrix}, \ x \in \mathbb{R} \right\}$

$\mathbb{F}^1 = \mathbb{F}$ (1 dimension)

$\mathbb{F}^2 =$  (2 dimensions)

We will see V is 1 dimension
($\dim V = 1$)

$\mathbb{F}^3 =$  (3 dimensions)

Linear Independence

Let V be a vector space $/ \mathbb{F}$

$\underline{v_1}, \underline{v_2} \ldots \underline{v_n} \in V$

say that $\underline{v_1} \ldots \underline{v_n}$ are linearly independent over $\mathbb{F}$ (L.I)

when $\lambda_1 \underline{v_1} + \lambda_2 \underline{v_2} + \ldots \lambda_n \underline{v_n} = 0$

$\Rightarrow \lambda_1 = \lambda_2 = \ldots = \lambda_n = 0$

when $\lambda_1, \ldots \lambda_n \in \mathbb{F}$

An expression $\lambda_1 v_1 + \lambda_2 v_2 + \ldots \lambda_n v_n$ is called a linear combination of $v_1 \ldots v_n$.

In English the only way is get 0 as a linear combination is by having all coefficients $= 0$

Example : $\underline{e_1} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ $\underline{e_2} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ $\underline{e_3} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

$e_2 \in \mathbb{F}^3$

$\{\underline{e_1}, \underline{e_2}, \underline{e_3}\}$ are L.I over $\mathbb{F}$

Proof suppose: $\lambda_1 \underline{e_1} + \lambda_2 \underline{e_2} + \lambda_3 \underline{e_3} = 0$

$\lambda_1 e_1 = \begin{pmatrix} \lambda_1 \\ 0 \\ 0 \end{pmatrix}$, $\lambda_2 \underline{e_2} = \begin{pmatrix} 0 \\ \lambda_2 \\ 0 \end{pmatrix}$ $\lambda_3 \underline{e_3} = \begin{pmatrix} 0 \\ 0 \\ \lambda_3 \end{pmatrix}$

$$\lambda_1 \underline{e_1} + \lambda_2 \underline{e_2} + \lambda_3 \underline{e_3} = \begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \end{pmatrix}$$

If $\lambda_1 e_1 + \lambda_2 e_2 + \lambda_3 e_3 = 0 = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

$$\begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \text{ so } \begin{pmatrix} \lambda_1 = 0 \\ \lambda_2 = 0 \\ \lambda_3 = 0 \end{pmatrix}$$

Example 2: $q_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \quad q_2 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \quad q_3 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$

Then $\{q_1, q_2, q_3\}$ over L.I.

$$\lambda_1 q_1 + \lambda_2 q_2 + \lambda_3 q_3 = \begin{pmatrix} \lambda_1 \\ \lambda_1 \\ \lambda_1 \end{pmatrix} + \begin{pmatrix} \lambda_2 \\ \lambda_2 \\ 0 \end{pmatrix} + \begin{pmatrix} \lambda_3 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} \lambda_1 + \lambda_2 + \lambda_3 \\ \lambda_1 + \lambda_2 \\ \lambda_1 \end{pmatrix}$$

so if $\lambda_1 q_1 + \lambda_2 q_2 + \lambda_3 q_3 = \underline{0}$

$$\begin{pmatrix} \lambda_1 + \lambda_2 + \lambda_3 \\ \lambda_1 + \lambda_2 \\ \lambda_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad \begin{array}{l} \lambda_1 = 0 \\ \lambda_1 + \lambda_2 = 0, \quad \lambda_2 = 0 \\ \lambda_1 + \lambda_2 + \lambda_3 = 0, \quad \lambda_3 = 0 \end{array}$$

Eg. $\psi_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \quad \psi_2 = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} \quad \psi_3 = \begin{pmatrix} 2 \\ -1 \\ 2 \end{pmatrix}$

Are $\{\psi_1, \psi_2, \psi_3\}$ L.I over $\mathbb{Q}$?

$\psi_3 = 2\psi_1 - \psi_2$ Here none of coeff $= 0$

$$\boxed{2\psi_1 - \psi_2 - \psi_3 = 0}$$

Example: $\underline{v_1} = \begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \end{pmatrix} \quad \underline{v_2} = \begin{pmatrix} 1 \\ 1 \\ -1 \\ -1 \end{pmatrix} \quad \underline{v_3} = \begin{pmatrix} 1 \\ 1 \\ 1 \\ -1 \end{pmatrix} \quad \underline{v_4} = \begin{pmatrix} 1 \\ -1 \\ -1 \\ -1 \end{pmatrix}$

$$\left.\begin{array}{l} \underline{v_4} = \underline{v_1} + \underline{v_2} - \underline{v_3} \\ \text{so. } \underline{v_1} + \underline{v_2} - \underline{v_3} - \underline{v_4} = 0 \end{array}\right\} \text{ none of coeff} = 0$$

Dependence Relation

→ $\{\underline{v_1} \dots \underline{v_n}\}$ are linearly dependent (L.D) when $\exists$ linear combination

$$\lambda_1 v_1 + \dots \lambda_n v_n = 0$$

where at least one $\boxed{\lambda_i \neq 0}$

V vector space / $\mathbb{F}$

$\underline{v_1} \dots \underline{v_n} \in V$

$\underline{v_1} \dots \underline{v_n}$ are L.I over $\mathbb{F}$ when

$$\lambda_1 v_1 + \dots \lambda_n v_n = 0 \Rightarrow \lambda_1 = \lambda_2 = \dots \lambda_n = 0$$

$v_1, \dots v_n$ are L.D. over $\mathbb{F}$ when there exists a lower combination $\lambda_1 v_1 + \dots \lambda_n v_n = 0$ where at least one $\lambda_i \neq 0$.

<u>Standard example</u>

$$V = \mathbb{F}^n = \left\{ \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}, \; x_i \in \mathbb{F} \right\}$$

$$e_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \quad e_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \quad \dots \quad e_n = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix}$$

~~Exactly~~ Formally : $(e_i)_k = \begin{cases} 0 & k \neq i \\ 1 & k = 1 \end{cases}$

Prop : $\{e_1, \dots e_n\}$ is. LI over $\mathbb{F}$

Proof : $\lambda_1 e_1 + \dots \lambda_n e_n = \begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \vdots \\ \lambda_n \end{pmatrix}$ so if $\lambda_1 e_1 + \dots \lambda_n e_n = 0 = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$

then $\lambda_1 = 0, \lambda_2 = 0, \dots \lambda_n = 0$

<u>Standard Mistake</u>.

"Every set is. L.I"

Take $\lambda_1 = 0, \lambda_2 = 0, \dots \lambda_n = 0$ X

$\lambda_1 \cancel{v_1} + \dots \lambda_n v_n = 0$

and $\lambda_1 = 0 \dots \lambda_n = 0$

<u>True statement</u>

You can always get 0 by having all coeff = 0

But L.I means "the only way to get 0 is by" **having** all coeffs = 0"

Spannings : V vector space / $\mathbb{F}$.

$v_1, v_2 \dots v_n \in V$

say that $v_1, v_2 \dots v_n$ spans V

when given ~~away~~ any vector $w \in V$

we can write

$$\underline{w} = \lambda_1 \underline{v_1} + \lambda_2 \underline{v_2} + \dots \lambda_n \underline{v_n}$$

For some $\lambda_i \in \mathbb{F}$

In English $\{v_1 \dots v_n\}$ spans V when every vectors $w \in V$ is a linear combination in $\{v_1 \dots v_n\}$

Standard example : $V = \mathbb{F}^n$

$$\underline{e}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \quad \underline{e}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \quad \underline{e}_n = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix}$$

Prop : $\{\underline{e}_1, \underline{e}_2, \dots \underline{e}_n\}$ spans $\mathbb{F}^n$

Proof : Given $\underline{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \in \mathbb{F}^n \quad x_i \in \mathbb{F}$

$$\underline{x} = x_1 \underline{e}_1 + x_2 \underline{e}_2 + \dots x_n \underline{e}_n$$

$$\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} x_1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ x_2 \\ 0 \\ 0 \\ 0 \end{pmatrix} + \dots \begin{pmatrix} 0 \\ 0 \\ \vdots \\ x_n \end{pmatrix}$$

Example : $q_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \quad q_2 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \quad q_3 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$

Then $\{q_1, q_2, q_3\}$ spans $\mathbb{F}^3$

Given $\underline{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \in \mathbb{F}^3$

I have to write $\underline{x} = \lambda_1 q_1 + \lambda_2 q_2 + \lambda_3 q_3$

for some $\lambda_1, \lambda_2, \lambda_3 \in \mathbb{F}$ $\Rightarrow$

$$\lambda_1 q_1 + \lambda_2 q_2 + \lambda_3 q_3 = \begin{pmatrix} \lambda_1 + \lambda_2 + \lambda_3 \\ \lambda_1 + \lambda_2 \\ \lambda_1 \end{pmatrix}$$

Given $\underline{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$

$$\underline{x} = \lambda_1 x_1 + \lambda_2 x_2 + \lambda_3 x_3$$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} \lambda_1 + \lambda_2 + \lambda_3 \\ \lambda_1 + \lambda_2 \\ \lambda_1 \end{pmatrix}$$

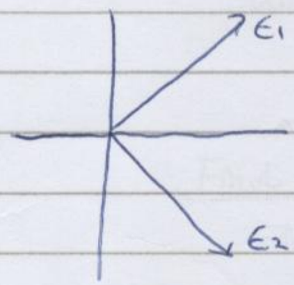
Take $\lambda_1 = x_3$, $\lambda_2 = x_2 - x_3$

$$\lambda_3 = x_1 - \lambda_2 - \lambda_3$$
$$= x_1 - x_2 + x_3 - x_3$$
$$= x_1 - x_2$$

$$\underline{x} = x_3 q_1 + (x_2 - x_3) q_2 + (x_1 - x_2) q_3$$

<u>Example</u>: $\mathbb{F} = \mathbb{Q}$

$\epsilon_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ $\qquad \epsilon_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$



$\{ \epsilon_1, \epsilon_2 \}$ spans $\mathbb{Q}^2$

Given $x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ need to write

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \lambda_1 \epsilon_1 + \lambda_2 \epsilon_2$$

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} \lambda_1 + \lambda_2 \\ \lambda_1 - \lambda_2 \end{pmatrix}$$

$$\lambda_1 = \frac{x_1 + x_2}{2}, \quad \lambda_2 = \frac{x_1 - x_2}{2}$$

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \left(\frac{x_1 + x_2}{2} \right) \underline{\epsilon_1} + \left(\frac{x_1 - x_2}{2} \right) \underline{\epsilon_2}$$

$\langle \underline{\epsilon_1}, \underline{\epsilon_2} \rangle$ also LI

$$\lambda_1 \epsilon_1 + \lambda_2 \epsilon_2 = 0$$

$$\begin{pmatrix} \lambda_1 + \lambda_2 \\ \lambda_1 - \lambda_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\left.\begin{aligned} 2\lambda_1 &= 0 \\ 2\lambda_2 &= 0 \end{aligned}\right| \begin{aligned} \lambda_1 &= 0 \\ \lambda_2 &= 0 \end{aligned}$$

Definition : V vector space / $\mathbb{F}$

$\{ \epsilon_1 \dots\dots \epsilon_n \} \in V$

$\{ \epsilon_1, \dots\dots \epsilon_n \}$ is a basis for V

when i) $\{ \epsilon_1, \dots\dots \epsilon_n \}$ is L.I and

ii) $\{ \epsilon_1, \dots\dots \epsilon_n \}$ spans V

Standard example : $\underline{e_1} = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$ $\underline{e_2} = \begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$ $e_n = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix}$

$\{ e_1 \dots e_n \}$ is a Basis for $\mathbb{F}^n$

This is the Standard Basis.

Example : $\epsilon_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ $\epsilon_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

Shown $\{ \epsilon_1 \, \epsilon_2 \}$ is a basis for $\mathbb{Q}^2$

$e_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ $e_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

is a basis for $\mathbb{Q}^2$

The Fundermental Theorem of Linear Algebra.

Basis Theorem

If V is a non zero vector space $\mathbb{F}$

~~then~~ then 1) V has at least one basis and

2) Any 2 ~~bis~~ basises for V have the same number of elements.

The number of elements in a basis of V is called

"the Dimension of V, $\dim(V)$".

example : $\dim(\mathbb{F}^2) = 2$

$\dim(\mathbb{F}^3) = 3$

$\dim(\mathbb{F}^n) = n$

Examples : $P = \left\{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \in \mathbb{F}^3 : x_1 + x_2 + x_3 = 0 \right\}$

P is a vector space

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} + \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} x_1+y_1 \\ x_2+y_2 \\ x_3+y_3 \end{pmatrix}$$

$$\begin{array}{ccc} \downarrow & \downarrow & \Downarrow \\ P & P & P \end{array}$$

$$x_1 + x_2 + x_3 = 0$$
$$y_1 + y_2 + y_3 = 0$$
$$(x_1+y_1) + (x_2+y_2) + (x_3+y_3) = 0$$

Mult : $\lambda \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} \lambda x_1 \\ \lambda x_2 \\ \lambda x_3 \end{pmatrix}$

$$\underset{\substack{\parallel \\ 0}}{\lambda(x_1+x_2+x_3)} = \underset{\substack{\parallel \\ 0}}{\lambda x_1 + \lambda x_2 + \lambda x_3}$$

$\dim P = 2$ ✓ (dimension depends on how many L.I variables you can find no coordinates!)

Find a basis for P

$$\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \quad \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$$
$$\epsilon_1 \qquad \epsilon_2$$

Example : $\{\epsilon_1, \epsilon_2\}$ is a basis for P

$$\lambda_1 \epsilon_1 + \lambda_2 \epsilon_2 = \begin{pmatrix} \lambda_1 \\ 0 \\ -\lambda_1 \end{pmatrix} + \begin{pmatrix} \lambda_2 \\ -\lambda_2 \\ 0 \end{pmatrix} = \begin{pmatrix} \lambda_1+\lambda_2 \\ -\lambda_2 \\ -\lambda_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\lambda_2 = 0, \lambda_1 = 0$$

$\{\epsilon_1, \epsilon_2\}$ is L.I

eg. $\{\underline{\epsilon}_1, \underline{\epsilon}_2\}$ spans P

Given $x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$ $\quad x_1 + x_2 + x_3 = 0$

$x_1 = -x_2 - x_3$

$$x = \begin{pmatrix} -x_2 - x_3 \\ x_2 \\ x_3 \end{pmatrix}$$

$$= (-x_3)\,\underline{\epsilon}_1 + (-x_2)\,\underline{\epsilon}_2$$

$\dim P = 2$

eg : $H = \left\{ \underline{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \in \mathbb{F}^n \right.$ such that $x_1 + x_2 + \dots x_n = 0$

$\dim H = n - 1$

18-11-2010

V vector space / $\mathbb{F}$

eg. $\epsilon_1, \ldots\ldots, \epsilon_n \in V$

$\{\epsilon_1, \ldots\ldots, \epsilon_n\}$ is a BASIS for V when

1) $\{\epsilon_1, \ldots\ldots, \epsilon_n\}$ is L.I.

2) $\{\epsilon_1, \ldots\ldots, \epsilon_n\}$ spans V

1) $\{\epsilon_1, \ldots \epsilon_n\}$ is L.I when

$$\lambda_1\epsilon_1, \ldots\ldots + \lambda_n\epsilon_n = 0 \Rightarrow \lambda_1 = \lambda_2 = \ldots \lambda_n = 0$$

2) $\{\epsilon_1, \ldots\ldots \epsilon_n\}$ spans V when any $v \in V$ can be expressed

$$V = \lambda_1\epsilon_1 + \ldots\ldots \lambda_n\epsilon_n, \quad \lambda_i \in \mathbb{F}$$

Basis theorem says.

1) Any (nonzero) vector space has a Basis

2) Any two bases for the same space have the same number of elements.

dim(V) = no. of elements in a BASIS for V.

Example : 1 $\mathbb{F}^n$ has basis $(e_1, \ldots\ldots e_n)$ so. $\dim(\mathbb{F}^n) = n$

$$e_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \qquad e_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \qquad e_n = \begin{pmatrix} 0 \\ \vdots \\ 1 \end{pmatrix}$$

Example 2 : $H = \{ \underline{x} \in \mathbb{F}^n : x_1 + x_2 + \ldots x_n = 0 \}$

$$\underline{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \qquad \dim H = n - 1$$

<u>Eg: n=4</u> $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}$ i) $x_1 + x_2 + x_3 + x_4 = 0$

$$\underline{\epsilon_1} = \begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \end{pmatrix} \qquad \underline{\epsilon_2} : \begin{pmatrix} -1 \\ 0 \\ 1 \\ 0 \end{pmatrix} \qquad \underline{\epsilon_3} : \begin{pmatrix} -1 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

(H) (H) (H)

Claim $\{\epsilon_1, \epsilon_2, \epsilon_3\}$ is a basis for H (n=4)

$\Rightarrow$

L.I. $\lambda_1 \epsilon_1 + \lambda_2 \epsilon_2 + \lambda_3 \epsilon_3 = 0$

$$\begin{pmatrix} -\lambda_1 \\ \lambda_1 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} -\lambda_2 \\ 0 \\ \lambda_2 \\ 0 \end{pmatrix} + \begin{pmatrix} -\lambda_3 \\ 0 \\ 0 \\ \lambda_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} -\lambda_1 & -\lambda_2 & -\lambda_3 \\ \lambda_1 & & \\ & \lambda_2 & \\ & & \lambda_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$\Rightarrow \lambda_1 = 0, \quad \lambda_2 = 0, \quad \lambda_3 = 0$

$\{\epsilon_1, \epsilon_2, \epsilon_3\}$ spans H

Let $\underline{x} \in H$ so $x_1 + x_2 + x_3 + x_4 = 0$

so $x_1 = -(x_2 + x_3 + x_4)$

$$x_2 \epsilon_1 + x_3 \epsilon_2 + x_4 \epsilon_3 = \begin{pmatrix} -x_2 \\ x_2 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} -x_3 \\ 0 \\ x_3 \\ 0 \end{pmatrix} + \begin{pmatrix} -x_4 \\ 0 \\ 0 \\ x_4 \end{pmatrix} = \begin{pmatrix} -(x_2+x_3+x_4) \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}$$

- <u>More general example</u>: Let A be an $m \times n$ matrix over field $\mathbb{F}$

$A = (a_{ij}) \quad 1 \le i \le m \quad a_{ij} \in \mathbb{F}$
$\quad 1 \le j \le m$

Let $T_A : \mathbb{F}^n \to \mathbb{F}^m$ given by
$T_A(\underline{x}) = A\underline{x}$

$$\begin{pmatrix} a_{1n} & a_{1i} \\ \vdots & \vdots \\ a_{j1} & a_{jn} \end{pmatrix} \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$$

Provisional notation:
Define $K_A = \{\underline{x} \in \mathbb{F}^n ; A\underline{x} = 0\}$

Prop: K_A is a vector space ; $K_A \subset \mathbb{F}^n$

Proof: <u>Addition</u> Suppose $x, y \in K_A$ [need $x + y \in K_A$]

$A\underline{x} = 0, \quad Ay = 0$

$A\underline{x} + Ay = 0$

$A(\underline{x} + \underline{y}) = 0 \Rightarrow$ so $\underline{x} + \underline{y} \in K_A$

Multiply by scalar : Let $\underline{x} \in K_A$, $\lambda \in \mathbb{F}$

[Need $\lambda \underline{x} = K_A$]

$A\underline{x} = 0$, $\lambda A \underline{x} = \lambda \underline{0} = 0$

$\lambda a_{ij} = a_{ij} \lambda$ for i,j

so $A(\lambda \underline{x}) = 0$ .a $\lambda \underline{x} \in K_A$

zero $A\underline{0} = 0$ so $0 \in K_A$

All like other axioms automatically hold because we are inside $\mathbb{F}^n$ where they already hold.

QED

Question: How do we compute $\dim(K_A)$?

Example : $A = \begin{pmatrix} 1 & 1 & -1 & 0 & -1 & 0 \\ 1 & 0 & 1 & 0 & 1 & 1 \\ 1 & -1 & -1 & -1 & 1 & 1 \\ 1 & 2 & 1 & 1 & -1 & 0 \end{pmatrix}$

what is K_A ?

$K_A = \{ \underline{x} \in \mathbb{F}^6 . A\underline{x} = 0 \}$

$$\begin{pmatrix} 1 & 1 & -1 & 0 & -1 & 0 \\ 1 & 0 & 1 & 0 & 1 & 1 \\ 1 & -1 & -1 & -1 & 1 & 1 \\ 1 & 2 & 1 & 1 & -1 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{cases} x_1 + x_2 - x_3 - x_5 = 0 \\ x_1 + \quad + x_3 + x_5 + x_6 = 0 \\ x_1 - x_2 - x_3 - x_4 + x_5 + x_6 = 0 \\ x_1 + 2x_2 + x_3 + x_4 - x_5 = 0 \end{cases}$$

K_A = solution set to $A\underline{x} = 0$

Solve by row reduction

$$\begin{pmatrix} 1 & 1 & -1 & 0 & -1 & 0 \\ 0 & -1 & 2 & 0 & 2 & 1 \\ 0 & 2 & 0 & 1 & -2 & -1 \\ 0 & 1 & 2 & 1 & 0 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 1 & 0 & 1 & 1 \\ 0 & -1 & 2 & 0 & 2 & 1 \\ 0 & 0 & 4 & 1 & 2 & 1 \\ 0 & 0 & 4 & 1 & 2 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 1 & 0 & 1 & 1 \\ 0 & 1 & -2 & 0 & -2 & -1 \\ 0 & 0 & 1 & 1/4 & 1/2 & 1/4 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\downarrow$$

Reduced matrix $\begin{pmatrix} 1 & 0 & 0 & -1/4 & 1/2 & 3/4 \\ 0 & 1 & 0 & 1/2 & -1 & -1/2 \\ 0 & 0 & 1 & 1/4 & 1/2 & 1/4 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \leftarrow \begin{pmatrix} 1 & 0 & 0 & -1/4 & 1/2 & 3/4 \\ 0 & 1 & 0 & 1/2 & -1 & -1/2 \\ 0 & 0 & 1 & 1/4 & 1/2 & 1/4 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$

$x_1 \; x_2 \; x_3 \; x_4 \; x_5 \; x_6$

General element of K_A looks like.

$$\begin{pmatrix} \frac{x_4}{4} - \frac{x_5}{2} - \frac{3x_6}{4} \\ -\frac{x_4}{2} + x_5 + \frac{x_6}{2} \\ -\frac{x_4}{4} - \frac{x_5}{2} - \frac{x_6}{4} \\ x_4 \\ x_5 \\ x_6 \end{pmatrix}$$

First choice: $x_4 = 1 \quad x_5 = 0 \quad x_6 = 0$

$$\epsilon_1 = \begin{pmatrix} 1/4 \\ -1/2 \\ -1/4 \\ 1 \\ 0 \\ 0 \end{pmatrix}$$

Obvious second choice: $\epsilon_2 = \begin{pmatrix} -1/2 \\ 1 \\ -1/2 \\ 0 \\ 1 \\ 0 \end{pmatrix} \quad x_4 = 0,\ x_5 = 1,\ x_6 = 0$

Third obvious choice: $\epsilon_3 = \begin{pmatrix} -3/4 \\ 1/2 \\ -1/4 \\ 0 \\ 0 \\ 1 \end{pmatrix} \quad x_4 = 0, x_5 = 0, x_6 = 1$

$\{\epsilon_1, \epsilon_2, \epsilon_3\}$ spans K_A Because $\underline{x} = x_4 \epsilon_1 + x_5 \epsilon_2 + x_6 \epsilon_3$

$\{\epsilon_1, \epsilon_2, \epsilon_3\}$ are L.I.

$$\lambda_1 \epsilon_1 + \lambda_2 \epsilon_2 + \lambda_3 \epsilon_3 = \begin{pmatrix} \text{Mess} \\ \vdots \\ \lambda_1 \\ \lambda_2 \\ \lambda_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad \dim K_A = 3$$

$$\lambda_1 = \lambda_2 = \lambda_3 = 0$$

23-11-2010

< Linear Maps >

V, W vector spaces / $\mathbb{F}$ (field)

$T: V \to W$ is a <u>mapping</u>

Say that T is linear when $T(\underline{x}+\underline{y}) = T(\underline{x}) + T(\underline{y})$ (

$T(\lambda \underline{x}) = \lambda T(\underline{x})$ (Pre· scalar mult.)

Eg. d/dx

$d/dx(f+g) = df/dx + dg/dx$

$d/dx(\lambda f) = \lambda\, df/dx$ (λ is constant)

Standard example :

Fix field $\mathbb{F}$, $A : (A_{ji})_{\substack{1\le j\le m \\ 1\le i\le n}}$ $m\times n$ matrix over $\mathbb{F}$

$T_A : \mathbb{F}^n \to \mathbb{F}^m$ $\quad$ $T_A(\underline{x}+\underline{y}) = T_A(\underline{x}) + T_A(\underline{y})$

$T_A(\underline{x}) = A\underline{x}$ $\quad$ $T_A(\lambda\underline{x}) = \lambda T_A(\underline{x})$

(matrix prod.)

LINEAR !

- We are going to show that every linear matrices "Like the standard example"

Example: $A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{21} & a_{23} \end{pmatrix}$, $\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{21} & a_{23} \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{matrix} x_1 a_{11} + x_2 a_{12} + x_3 a_{13} + \\ x_1 a_{21} + x_2 a_{22} + x_3 a_{23} \end{matrix}$

2×3 $\quad$ $T_A(\underline{x}) = A\underline{x}$

$T_A : \mathbb{F}^3 \longrightarrow \mathbb{F}^2$, Apply to $\underline{e}_1, \underline{e}_2\ \underline{e}_3$

$T_A(\underline{x}) = A\underline{x}$ $\quad$ $T_A(e_1) = \begin{pmatrix} a_{11} \\ a_{21} \end{pmatrix}$, $T_A(e_2) = \begin{pmatrix} a_{12} \\ a_{22} \end{pmatrix}$, $T_A(e_3) = \begin{pmatrix} a_{13} \\ a_{23} \end{pmatrix}$

$e_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ $\quad e_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ $\quad e_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

Standard Example

$e_r \mapsto r^{th}$ column of A.

$E_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ $\quad E_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$\Rightarrow$

$$T(\epsilon_i) = \sum_{j=1}^{m} a_{ji}\,\varphi_j \qquad i = 1 \dots n.$$

Coeff. of $T(\epsilon_1) = 1^{st}$ column
$T(\epsilon_2) = 2^{nd}$ column [NOT ROW!]
$T(\epsilon_n) = n^{th}$ column

<u>Example</u>: $V = \{ae^x + bxe^x + cx^2e^x + dx^3e^x : a, b, c, d \in \mathbb{R}\}$
V is a vector space of dim $= 4$

V has basis:
$\epsilon_1 = e^x$, $\epsilon_2 = xe^x$, $\epsilon_3 = x^2e^x$, $\epsilon_4 = x^3e^x$
Take $D: V \to V$ $D = d/dx$

$D(\epsilon_1) = 1.\epsilon_1 + 0\,\epsilon_2 + 0\,\epsilon_3 + 0.\epsilon_4$

$D(\epsilon_2) = 1\epsilon_1 + 1.\epsilon_2 + 0.\epsilon_3 + 0.\epsilon_4$ $\qquad \left[\frac{d}{dx}(xe^x) = \overset{\epsilon_1}{e^x} + \overset{\epsilon_2}{xe^x}\right]$

$D(\epsilon_3) = 0\epsilon_1 + 2\epsilon_2 + 1.\epsilon_3 + 0.\epsilon_4$ $\qquad \left[\frac{d}{dx}(x^2e^x) = \underset{2\epsilon_2}{2xe^x} + \underset{\epsilon_3}{x^2e^x}\right]$

$D(\epsilon_4) = 0.\epsilon_1 + 0.\epsilon_2 + 3\epsilon_3 + 1.\epsilon_4$

In this example, matrix of D is 4×4

$$D = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$D^2 = \begin{pmatrix} 1 & 2 & 2 & 0 \\ 0 & 1 & 4 & 6 \\ 0 & 0 & 1 & 6 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

$\frac{d^2}{dx^2}(3xe^x - x^3e^x) =$
$3xe^x - x^3e^x = 3\epsilon_2 - \epsilon_4 \sim \begin{pmatrix} 0 \\ 3 \\ 0 \\ -1 \end{pmatrix}$

$$D^2\begin{pmatrix} 0 \\ 3 \\ 0 \\ -1 \end{pmatrix} = \begin{pmatrix} 6 \\ -3 \\ -6 \\ -1 \end{pmatrix}$$

$\Rightarrow \frac{d^2}{dx^2}(3xe^x - x^3e^x) = 6e^x - 3xe^x - 6x^2e^x - x^3e^x$

Example: $T : V \to W$

$(\epsilon_1 \dots\dots \epsilon_n)$ basis for V $\quad \dim V = n$

$\{\varphi_1 \dots\dots \varphi_m\}$ basis for W $\quad \dim W = m$

Associate a $m \times n$ matrix to T as follows

$$T(\epsilon_i) = \sum_{j=1}^{m} a_{ji} \varphi_j$$

The matrix of T W.R.T $\mathcal{E} = \{\epsilon_1 \dots\dots \epsilon_n\}$ on left.

$\Phi = \{\varphi_1 \dots\dots \varphi_m\}$ right.

$$M(T)_{\mathcal{E}}^{\Phi} = (a_{ji}) \quad \begin{matrix} 1 \le j \le m \\ 1 \le i \le n \end{matrix}$$

The example I had before

$W = V = \{ae^x + bxe^x + cx^2e^x + dx^3e^x\}$

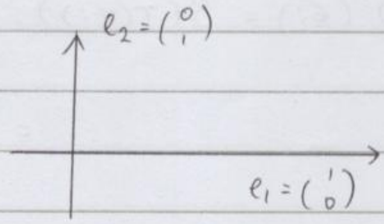
$\mathcal{E} = \{e^x, xe^x, x^2e^x, x^3e^x\} = \Phi$

$$M(D)_{\mathcal{E}}^{\mathcal{E}} = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Example: $T : \mathbb{R}^2 \to \mathbb{R}^2$

$$T\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_1 + 2x_2 \\ 2x_1 + x_2 \end{pmatrix}$$

$\mathcal{E} = \{e_1, e_2\}$



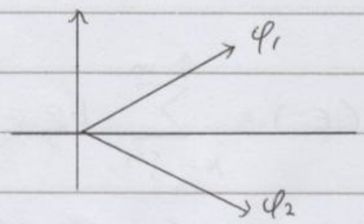
$T(e_1) = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$

$$M(T)_{\mathcal{E}}^{\mathcal{E}} = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$$

$T(e_2) = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$

Take: eg: $\varphi_1 = e_1 + e_2 = \begin{pmatrix} 3 \\ 3 \end{pmatrix}$

$\varphi_2 = e_1 - e_2$



$T(\varphi_1) = \begin{pmatrix} 3 \\ 3 \end{pmatrix} = 3\varphi_1 + 0\varphi_2$

$T(\varphi_2) = \begin{pmatrix} -1 \\ 1 \end{pmatrix} = 0\varphi_1 - 1\varphi_2$

$$M(T)_{\Phi}^{\Phi} = \begin{pmatrix} 3 & 0 \\ 0 & -1 \end{pmatrix}$$

- $M(T)_{\mathcal{E}}^{\mathcal{E}} = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$
- $M(T)_{\Phi}^{\Phi} = \begin{pmatrix} 3 & 0 \\ 0 & -1 \end{pmatrix}$
- $M(T)_{\Phi}^{\mathcal{E}} = \begin{pmatrix} 3 & -1 \\ 3 & 1 \end{pmatrix}$

$T(\varphi_1) = \begin{pmatrix} 3 \\ 3 \end{pmatrix} = 3e_1 + 3e_2$

$T(\varphi_2) = \begin{pmatrix} -1 \\ 1 \end{pmatrix} = -e_1 + e_2$

$\langle$ Composition Formula $\rangle$

$$U \xrightarrow{T} V \xrightarrow{S} W$$

U, V, W vector spaces

S, T linear $\left(\begin{array}{c} S \circ T \text{ is} \\ \text{also} \end{array} \right)$

$\mathcal{E} = \{ \epsilon_1 \ldots\ldots \epsilon_n \}$ basis for U

$\Phi = \{ \varphi_1 \ldots\ldots \varphi_m \}$ basis for V

$\Psi = \{ \psi_1 \ldots\ldots \psi_p \}$ basis for W

so got $M(T)^{\Phi}_{\mathcal{E}}$ $\quad M(S)^{\Psi}_{\Phi}$ $\quad M(S \circ T)^{\Psi}_{\mathcal{E}}$

Composition Formula.

$$\boxed{M(S \circ T)^{\Psi}_{\mathcal{E}} = M(S)^{\Psi}_{\Phi} \; M(T)^{\Phi}_{\mathcal{E}} \quad \text{(matrix product)}}$$

Proof: $M(T)^{\Phi}_{\mathcal{E}} = (a_{ji})$ $\qquad T(\epsilon_i) = \sum_{j=1}^{m} a_{ji} \varphi_j$

$M(S)^{\Psi}_{\Phi} = (b_{kj})$ $\qquad S(\varphi_j) = \sum_{k=1}^{p} b_{kj} \psi_k$

$$(S \circ T)(\epsilon_i) = S(T(\epsilon_i)) = S\left(\sum_{j=1}^{m} a_{ji} \varphi_j \right)$$

$$= \sum_{j=1}^{m} a_{ji} S(\varphi_j)$$

$$= \sum_{j=1,\, k=1}^{m\ \ p} a_{ji} b_{kj} \psi_k \quad , \; a_{ji} b_{kj} = b_{kj} a_{ji}$$

$$(S \circ T)(\epsilon_i) = \sum_{k=1}^{p} \sum_{j=1}^{m} b_{kj} a_{ji} \psi_k$$

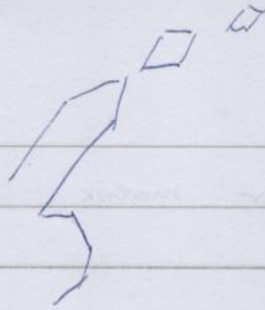
$\underset{M(S)^{\Psi}_{\Phi}}{B = (b_{kj})}$ $\quad \underset{M(T)^{\Phi}_{\mathcal{E}}}{A = (a_{ji})}$ $\quad (BA)_{ki} = \sum_{j=1}^{m} b_{kj} a_{ji}$

$$(S \circ T)(\epsilon_i) = \sum_{k=1}^{p} (BA)_{ki} \psi_k$$

$\rightarrow M(S \circ T)^{\Psi}_{\mathcal{E}} = B \cdot A = M(S)^{\Psi}_{\Phi} \; M(T)^{\Phi}_{\mathcal{E}}$

QED

Formula for matrix product is chosen so this is True!



25-11-2010

$U \xrightarrow{T} V \xrightarrow{S} W$

U, V, W vector spaces

$\mathcal{E} = \{e_1, \ldots e_n\}$ for U
$\Phi = \{\varphi_1, \ldots \varphi_k\}$ for V
$\Psi = \{\psi_1, \ldots \psi_p\}$ for W

$\boxed{M(S)^{\Psi}_{\Phi} = \text{matrix of } S \ \Phi \text{ left } \Psi \text{ right}}$

$\boxed{M(T)^{\Phi}_{\mathcal{E}} = \text{matrix of } T \ \mathcal{E} \text{ left } \Phi \text{ right}}$

<u>Composition formula</u>

$$M(S \circ T)^{\Psi}_{\mathcal{E}} = M(S)^{\Psi}_{\Phi} \ M(T)^{\Phi}_{\mathcal{E}}$$

$$\begin{array}{ccc} \Psi & & \Phi \\ & \nearrow & \\ \Phi & & \mathcal{E} \end{array}$$

very special case

$U = V = W$ and $\Psi = \mathcal{E}$

and $S = T = Id$

so I've got two bases $\mathcal{E} = \{e_1, \ldots e_n\}$
$\Phi = \{\varphi_1, \ldots \varphi_n\}$

for some vector space V

Express each φ_i in terms of $\mathcal{E}$

$$\underline{\varphi_i} = \sum_{j=1}^{n} a_{ji} \underline{e_j}$$

What is $A = (a_{ji})$

Apply Id to Q_i

$$Id(Q_i) = \sum_{j=1}^{n} a_{ji} e_j$$

so $A = M(Id)^{\mathcal{E}}_{\Phi}$

$\Rightarrow$

Now express each e_i in terms of Φ

$$e_k = \sum_{l=1}^{n} b_{lk} \varphi_l$$

$$B = b_{lk}$$
$$B = M(Id)_{\varepsilon}^{\Phi}$$

$$B = A^{-1} \text{ and } A = B^{-1}$$

**
Prop: If ε, Φ bases for V
then $M(Id)_{\varepsilon}^{\Phi} = \left(M(Id)_{\Phi}^{\varepsilon} \right)^{-1}$

Proof: first calculate

$$M(Id)_{\varepsilon}^{\varepsilon} = Id(e_i) = e_i$$

$$e_1 = 1.e_1 + 0.e_2 + \dots\dots 0 e_n$$
$$e_2 = 0.e_1 + 1e_2 + 0.e_3 + \dots 0e_n$$
$$e_n = 0.e_1 + 0e_2 + \dots\dots 1e_n$$

$M(Id)_{\varepsilon}^{\varepsilon} = I_n$ $\qquad$ $M(Id)_{\Phi}^{\Phi} = I_n$

use composition formula

$$M(Id)_{\varepsilon}^{\varepsilon} = M(Id)_{\Phi}^{\varepsilon} M(Id)_{\varepsilon}^{\Phi} \quad , Id = Id \cdot Id$$
$$I_n = A.B$$
$$M(Id)_{\Phi}^{\Phi} = M(Id)_{\varepsilon}^{\Phi} M(Id)_{\Phi}^{\varepsilon}$$
$$I_n = B.A$$

Not quite so special case

$U = V = W$ $\qquad$ $T: V \to V$ linear

just two bases $\varepsilon = \{e_1 \dots e_n\}$

$\Phi = \{\varphi_1 \dots \varphi_n\}$

were interested in relation between $M(T)_{\varepsilon}^{\varepsilon}$, $M(T)_{\Phi}^{\Phi}$

Change of Basis Formula

$$\boxed{M(T)_{\Phi}^{\Phi} = M(Id)_{\varepsilon}^{\Phi} \; M(T)_{\varepsilon}^{\varepsilon} \; M(Id)_{\Phi}^{\varepsilon}}$$

Proof: $T = Id \cdot T \cdot Id$

use composition formula

$$M(T)_{\Phi}^{\Phi} = M(Id)_{\varepsilon}^{\Phi} \; M(T \cdot Id)_{\Phi}^{\varepsilon}$$

$$= M(Id)_{\varepsilon}^{\Phi} \; M(T)_{\varepsilon}^{\varepsilon} \; M(Id)_{\Phi}^{\varepsilon} \qquad \text{QED.}$$

Example: $V = \mathbb{Q}^2$

$T: \mathbb{Q}^2 \to \mathbb{Q}^2 \quad T\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} \frac{5x_1}{2} - \frac{x_2}{2} \\ -\frac{x_1}{2} + \frac{5x_2}{2} \end{pmatrix}$

$\varepsilon = \{e_1, e_2\}$, $\Phi = \left\{ \underset{\varphi_1}{\begin{pmatrix} 1 \\ 1 \end{pmatrix}} \underset{\varphi_2}{\begin{pmatrix} -1 \\ 1 \end{pmatrix}} \right\}$

First calculate: $M(T)_{\varepsilon}^{\varepsilon}$

$$T\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 5/2 \\ -1/2 \end{pmatrix} T$$
$$= 5/2\, e_1 - \frac{e_2}{2}$$

$$T\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -1/2 \\ 5/2 \end{pmatrix} = -\frac{e_1}{2} + \frac{5}{2} e_2$$

$$M(T)_{\varepsilon}^{\varepsilon} = \begin{pmatrix} 5/2 & -1/2 \\ -1/2 & 5/2 \end{pmatrix}$$

Question: what is $M(T)_{\Phi}^{\Phi}$

$M(T)_{\Phi}^{\Phi} = M(Id)_{\varepsilon}^{\Phi} \; M(T)_{\varepsilon}^{\varepsilon} \; M(Id)_{\Phi}^{\varepsilon}$, what is the matrix in T with respect to Φ left and Φ right

$M(Id)_{\Phi}^{\varepsilon}$? $\varphi_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} = e_1 + e_2$: $\varphi_2 = \begin{pmatrix} -1 \\ 1 \end{pmatrix} = -e_1 + e_2$

$$M(Id)_{\Phi}^{\varepsilon} = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$$

$$M(Id)_{\varepsilon}^{\Phi} = \left(M(Id)_{\Phi}^{\varepsilon}\right)^{-1} = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}^{-1} = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$$

So $M(T)_{\Phi}^{\Phi} = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} 5/2 & -1/2 \\ -1/2 & 5/2 \end{pmatrix}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$

$= \frac{1}{2}\begin{pmatrix} 2 & 2 \\ -3 & 3 \end{pmatrix}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$

$= 1/2 \begin{pmatrix} 4 & 0 \\ 0 & 6 \end{pmatrix}$

$= \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}$

From $1^{st.}$ principles

$$\left.\begin{aligned} T(\varphi_1) &= T\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \end{pmatrix} = 2\varphi_1 + 0\varphi_2 \\ T(\varphi_2) &= T\begin{pmatrix} -1 \\ 1 \end{pmatrix} = \begin{pmatrix} -3 \\ 3 \end{pmatrix} = 3\varphi_2 + 0\varphi_1 \end{aligned}\right\} M(T)_{\Phi}^{\Phi} = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}$$

$\rightarrow$ $V = \mathbb{Q}_3$ $\quad T\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -x_1 + 2x_2 \\ x_2 + 2x_3 \\ 2x_1 - 2x_2 + 3x_3 \end{pmatrix}$

$\mathcal{E} = \{e_1, e_2, e_3\}$ $\qquad \Phi = \left\{\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}\right\}$

$\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ $\qquad\qquad \varphi_1 \quad \varphi_2 \quad \varphi_3$

Eassy to write down

$M(T)_{\mathcal{E}}^{\mathcal{E}} = \begin{pmatrix} -1 & 2 & 0 \\ 0 & 1 & 2 \\ 2 & -2 & 3 \end{pmatrix}$

$M(Id)_{\Phi}^{\mathcal{E}} = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}$

Find $M(T)_{\Phi}^{\Phi}$

Change of Basis

$$M(T)_{\Phi}^{\Phi} = M(Id)_{\varepsilon}^{\Phi} \; M(T)_{\varepsilon}^{\varepsilon} \; M(Id)_{\Phi}^{\varepsilon}$$

$$M(Id)_{\varepsilon}^{\Phi} = \left(M(Id)_{\Phi}^{\varepsilon} \right)^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}^{-1}$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ -1 & -1 & 0 \\ 1 & -1 & 1 \end{pmatrix}$$

$$M(T)_{\Phi}^{\Phi} = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 1 & -1 & 1 \end{pmatrix} \begin{pmatrix} -1 & 2 & 0 \\ 0 & 1 & 2 \\ 2 & -2 & 3 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} -1 & 2 & 0 \\ 1 & -1 & 2 \\ 1 & -1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 2 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix} \,//$$

<u>Linear maps (again!)</u>

V, W vector spaces

$T: V \to W$

$$\begin{cases} T(\underline{x}+\underline{y}) \\ T(\lambda \underline{x}) = \lambda T(\underline{x}) \end{cases}$$

Prop: If T is linear

$T(\underline{0}) = \underline{0}$

Proof: $T(\underline{0}) = T(\underline{0}+\underline{0}) = T(\underline{0}) + T(\underline{0})$

Add $-T(\underline{0})$ to each side

$0 = -T(0) + T(0)$

$= -T(0) + T(0) + T(0)$

$= 0 + T(0)$

$0 = T(0)$ QED.

Prop: if T is linear

$$T\left(\sum_{i=1}^{N} \lambda_i \varphi_i\right) = \sum_{i=1}^{N} \lambda_i T(\varphi_i)$$

30-11-2010

<Linear Maps>

$T. V \to W$, V, W vector spaces/$\mathbb{F}$

$T(\underline{x}+\underline{y}) = T(\underline{x}) + T(\underline{y})$

$T(\lambda \underline{x}) = \lambda T(\underline{x})$

We study linear maps because they are EASY

linear maps are completely determined by what happened on a BASIS

$\{\epsilon_1, \ldots \epsilon_n\}$ is a basis for V

Make a choice of $w_1, w_2, \ldots w_n \in W$

Decide that $T(\underline{\epsilon_1}) = \underline{w_1}$, $T(\underline{\epsilon_2}) = \underline{w_2}$ .... $T(\underline{\epsilon_n}) = \underline{w_n}$

Prop: Given a basis $\epsilon_1 \ldots \epsilon_n$ for V

and vectors $w_1 \ldots w_n$

There exists a unique linear map

$T. V \to W$ such that

$T(\epsilon_i) = w_i$ for all i

Proof: let $\underline{x} \in V$: know that can express $\underline{x} = x_1\epsilon_1 + \ldots x_n\epsilon_n$ ($x_i \in \mathbb{F}$)

Define $T(\underline{x}) = x_1w_1 + \ldots x_nw_n$

ie. replace ϵ_i by w_i

1) T is a mapping.

To each $x \in V$, I've assigned $T(\underline{x}) \in W$

$T(\underline{x})$ is uniquely ~~defined~~ specified by x.

If $x = x_1'\epsilon_1 + \ldots x_n'\epsilon_n$

$x = x_1\epsilon_1 + \ldots x_n\epsilon_n$

$0 = x - x' = (x_1 - x_1')\epsilon_1 + \ldots (x_n - x_n')\epsilon_n$

$\epsilon_1 \ldots\ldots \epsilon_n$, LI

so $x_1 - x_1' = x_2 - x_2' = \cdots x_n - x_n' = 0$

$x_1' = x_1$, $x_2' = x_2$, $x_n' = x_n$

$$T(\underline{x}) = x_1'\epsilon_1 + \ldots\ldots x_n'\epsilon_n$$
$$= x_1\epsilon_1 + \ldots\ldots x_n\epsilon_n$$

Unique expansion for $T(x)$

T is linear

$$y = y_1\epsilon_1 + \ldots\ldots y_n\epsilon_n$$
$$x+y = (x_1+y_1)\epsilon_1 + \ldots\ldots (x_n+y_n)\epsilon_n$$
$$= x_1\epsilon_1 + \ldots x_n\epsilon_n + \ldots y_1\epsilon_1 + y_n\epsilon_n$$

$$T(x+y) = (x_1+y_1)w_1 + \ldots\ldots (x_n+y_n)w_n$$
$$= x_1 w_1 + \ldots x_n w_n + y_1 w_1 + \ldots y_n w_n$$
$$= T(x) + T(y)$$

Finally: $T(\epsilon_1) = w_1$, $T(\epsilon_r) = w_r$

$$\epsilon_1 = 1.\epsilon_1 + 0.\epsilon_2 + \ldots 0\epsilon_n$$
$$T(\epsilon_1) = 1w_1 + 0 = w_1$$

Likewise $\epsilon_r = 0 + 0 + \ldots + 1.\epsilon_r + 0 \ldots\ldots$

$$T(\epsilon_r) = 0 + 1.w_r + 0 = w_r$$

QED//

Matrix representation of a linear map.

$T: V \to W$, $\mathcal{E} = \{ \epsilon_1, \dots \epsilon_n \}$ basis for V

$\Phi = \{ \varphi_1, \dots \varphi_m \}$ basis for W

To specify T need to specify

$\underline{w}_1, \dots \underline{w}_n \in W$ $\quad T(\underline{\epsilon}_i) = \underline{w}_r$

Express w_r in terms of $\varphi_1 \dots \varphi_n$

$$w_r = \sum_{s=1}^{m} a_{sr} \varphi_s$$

$$M(T)_{\mathcal{E}}^{\Phi} = (a_{sr}) \quad \begin{matrix} 1 \leq s \leq m \\ 1 \leq r \leq n \end{matrix}$$

Prop: If $\left. \begin{matrix} T: U \to V \\ S: V \to W \end{matrix} \right\}$ linear

then $S \circ T : U \to W$ is linear

Proof: $(S \circ T)(\underline{x} + \underline{y}) = S(T(\underline{x} + \underline{y}))$ T linear

$= S(T(x) + T(y))$ S linear

$= S(T(x)) + S(T(y))$

$(S \circ T)(x+y) = (S \circ T)(x) + (S \circ T)(y)$

Similarly: $(S \circ T)(\lambda \underline{x}) = \lambda (S \circ T)(\underline{x})$ QED.

We saw
$$M(S\circ T)_{\mathcal{E}}^{\Psi} = M(S)_{\Phi}^{\Psi} \cdot M(T)_{\mathcal{E}}^{\Phi}$$

Any matrix over $\mathbb{F}$ is the matrix of some linear map.

Proof: if $A = (a_{sr})_{\substack{1 \le s \le m \\ 1 \le r \le n}}$ $\quad a_{sr} \in \mathbb{F}$

let $\mathcal{E} = \{e_1 \ldots\ldots e_n\}$ standard basis of $\mathbb{F}^n$
$\mathcal{E}' = \{E_1, \ldots\ldots E_n\}$ " $\mathbb{F}^n$
Define $T(e_r) = \sum_{s=1}^{m} a_{sr} E_s \in W$
Then T is uniquely specified and
Furthermore $M(T)_{\mathcal{E}}^{\mathcal{E}'} = A = (a_{sr})$ QED.

Example: Differentiation is a linear map.

$$V = \{ \lambda_1 \sin(x) + \lambda_2 \cos(x) + \lambda_3 x \sin(x) + \lambda_4 x \cos(x) \}$$
$$\lambda_i \in \mathbb{R}$$

$\{\sin(x), \cos(x), x\sin(x), x\cos(x)\}$ are L.I

~ Think about of take $x = 0, \pi/2, \pi/4 \ldots$

$\{\sin(x), \cos(x), x\sin(x), x\cos(x)\}$ span V

by definition of V $\dim(V) = 4$

so $\Phi = \{\sin(x), \cos(x), x\sin(x), x\cos(x)\}$ is a basis for V

Take $D: V \to V$ to be $D(f) = df/dx$

D is linear.

Compute $M(D)_{\Phi}^{\Phi}$

$D(\sin(x)) = \cos(x)$

$D(\cos(x)) = -\sin(x)$

$D(x\sin(x)) = \sin(x) + x\cos(x)$

$D(x\cos(x)) = \cos(x) - x\sin(x)$

⇨

$D(\varphi_1) = 0.\varphi_1 + 1.\varphi_2 + 0.\varphi_3 + 0.\varphi_4$

$D(\varphi_2) = (-1)\varphi_1 + 0.\varphi_2 + 0.\varphi_3 + 0.\varphi_4$

$D(\varphi_3) = 1.\varphi_1 + 0.\varphi_2 + 0.\varphi_3 + 1.\varphi_4$

$D(\varphi_4) = 0.\varphi_1 + 1.\varphi_2 + (-1)\varphi_3 + 0.\varphi_4$

$$M(D)_{\Phi}^{\Phi} = \begin{pmatrix} 0 & -1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

$D.D = d^2/dx^2$

$$\boxed{M.(D.D)_{\Phi}^{\Phi} = M(D)_{\Phi}^{\Phi} \times M(D)_{\Phi}^{\Phi}} \; *$$

$$= \begin{pmatrix} 0 & -1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & -1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} -1 & 0 & 0 & -2 \\ 0 & -1 & 2 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

$d^2/dx^2 : (\sin(x) + \cos(x) - x\cos(x))$

express it.

$$\sin(x) + \cos(x) - x\cos(x) \approx \begin{pmatrix} 1 \\ 1 \\ 0 \\ -1 \end{pmatrix}$$

$d^2/dx^2 \, (\sin(x) + \cos(x) - x\cos(x))$

$$\sim \begin{pmatrix} -1 & 0 & 0 & -2 \\ 0 & -1 & -2 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 0 \\ -1 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 0 \\ 1 \end{pmatrix}$$

$d/dx\,(d/dx\,(\sin(x) + \cos(x) - x\cos(x)))$

$d/dx\,(\cos(x) - \sin(x) - \cos(x) + x\sin(x))$

$d/dx\,(-\sin(x) + x\sin(x))$

$= -\cos(x) + \sin x + x\cos(x)$ //

$V = \{ \lambda_1 \sin x_1 + \lambda_2 \cos(x) + \lambda_3 x \sin(x) + \lambda_4 x \cos(x) \}$

$D : V \to V \qquad D = d/dx$

$$M(D) = \begin{pmatrix} 0 & -1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

Notic that D is invertible!

$$\left(\begin{array}{cccc|cccc} 0 & -1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \end{array}\right)$$

$$\left(\begin{array}{cccc|cccc} 0 & -1 & 0 & 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \end{array}\right)$$

$$\left(\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & -1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & -1 & 0 \end{array}\right) D^{-1}$$

(refer to above)

Example: $\int \sin(x) + 2x\cos(x) = \int D^{-1}$

$\sin(x) + 2x\cos(x) \sim \begin{pmatrix} 1 \\ 0 \\ 0 \\ 2 \end{pmatrix}$

Multiply from left to right!

Apply D^{-1} to $\begin{pmatrix} 1 \\ 0 \\ 0 \\ 2 \end{pmatrix}$

$$\begin{pmatrix} 0 & 1 & 1 & 0 \\ -1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 2 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 2 \\ 0 \end{pmatrix} \sim \cos(x) + 2x\sin(x)$$

$$\int \sin(x) + 2x\cos(x)\, dx = \cos(x) + 2x\sin(x)$$

Ignore constant of integration!

Vector subspaces:

V vector space / $\mathbb{F}$

let $U \subset V$

Say that U is a vector subspace of V when

$\boxed{\text{i)}}$ if $\underline{x}, \underline{y} \in U$ then $\underline{x}+\underline{y} \in U$ (we already know $\underline{x}+\underline{y} \in V$)

$\boxed{\text{ii)}}$ if $x \in U$, $\lambda \in \mathbb{F}$, then $\lambda x \in U$

$\boxed{\text{iii)}}$ $0 \in U$

Prop: If U is a vector subspace of V, then U is a vector space in its own right.

Proof: All axioms are satisfied (Check!!)

$T: V \to W$ linear

Define $\mathrm{Ker}(T) = \{ x \in V \,;\, T(x) = 0 \}$

$\mathrm{Ker}(T)$ is called the KERNEL of T.

Prop: If $T: V \to W$ linear, then $\mathrm{Ker}(T)$ is a vector subspace of V

Proof: (i) If $\underline{x}, \underline{y} \in \mathrm{Ker}(T)$

then $T(\underline{x}) = 0$, $T(\underline{y}) = 0$

so $T(\underline{x}+\underline{y}) = T(\underline{x}) + T(\underline{y})$

$= 0 + 0$

$= 0$

so, $\underline{x}+\underline{y} \in \mathrm{Ker}(T)$

(ii) if $x \in \mathrm{Ker}(T)$, $\lambda \in \mathbb{F}$

$T(\lambda \underline{x}) = \lambda T(\underline{x}) = \lambda 0 = 0$

so $\lambda \underline{x} \in \mathrm{Ker}(T)$

(iii) Also, $T(0) = 0$, so $0 \in \mathrm{Ker}(T)$ QED

$T : V \to W$ linear

$$\boxed{\text{Define} \quad Im(T) = \left\{ \underline{w} \in W : \text{for some } x \in V \atop T(\underline{x}) = \underline{w} \right\}}$$

In English, $Im(T)$ is the set of all $w \in W$ which you can hit using T.

Prop: If $T : V \to W$ is linear then $Im(T)$ is a vector subspace of W

Proof: Suppose $w, W \in Im(T)$

Write $w = T(x)$, $W = T(x')$

$w + w' = T(x) + T(x') = T(x + x')$

so $w + w' \in Im(T)$

If $w \in Im(T)$ $\lambda \in \mathbb{F}$

$w = T(\underline{x})$

$\lambda w = \lambda T(\underline{x}) = T(\lambda \underline{x})$

$\lambda w \in Im(T)$

$0 \in Im(T)$; $T(0) = 0$ QED

Elementary Linear Algebra only has two theorems.

1) BASIS THEOREM

2) Kernel Rank Theorem :

↳ if $T: V \to W$ linear then

$$\dim \operatorname{Ker}(T) + \dim \operatorname{Im}(T) = \dim(V)$$

Example: A is $m \times n$ matrix

$K_A = \{ x \in \mathbb{F}^n : Ax = 0 \}$

(solution set)

Put $T_A : \mathbb{F}^n \to \mathbb{F}^m$

$T_A(\underline{x}) = A\underline{x}$

Then observe that $K_A = \operatorname{Ker}(T_A)$

2-11-2010

Basis Theorem

If V is a (non-zero) vector space
then i) V has at least one basis
ii) any two bases for V have same number of elements
(= dim V)

Kernel Rank

If $T: V \to W$ is linear

$$\dim(\ker(T)) + \dim(\mathrm{Im}(T)) = \dim V$$

Special Case: If V is $\{0\}$ then V has no basis
but we define $\dim\{0\} = 0$

Proof $T: V \to W$
(General case) $\ker(T) \neq \{0\}$
and $\mathrm{Im}(T) \neq \{0\}$

Let $\{\epsilon_1, \dots \epsilon_k\}$ be a basis for $\ker(T)$
$\{\varphi_1, \dots \varphi_m\}$ be a basis for $\mathrm{Im}(T)$

$\varphi_1 \in W$
For each r $1 \leq r \leq m$
Choose a vector
$\epsilon_{k+R} \in V$ such that
$T(\epsilon_{k+R}) = \varphi_r$
So now I have a set
$\{\epsilon_1, \dots \epsilon_k, \epsilon_{k+1}, \dots \epsilon_{k+m}\}$

Need to show

a) $\{ \epsilon_1 , \ldots \epsilon_{k+m} \}$ is linearly independent and

b) $\{ \epsilon_1 , \ldots \epsilon_{k+m} \}$ spans V

Proof of a suppose $\sum_{i=1}^{k+m} \lambda_i \epsilon_i = 0$ $*$

I have to show $\lambda_i = 0$ for all i

~ Apply T : $T\left(\sum_{i=1}^{k+m} \lambda_i \epsilon_i \right) = \sum_{i=1}^{k+m} \lambda_i T(\epsilon_i)$

But $\{ \epsilon_1 , \ldots \epsilon_k \} \in Ker(T)$ so $= \sum_{i=k+1}^{k+m} \lambda_i T(\epsilon_i)$

$T(\epsilon_1) = T(\epsilon_2) = T(\epsilon_3) = \ldots T(\epsilon_k) = 0 = \sum_{r=1}^{m} \lambda_{k+r} T(\epsilon_{k+r})$

$= \sum_{r=1}^{m} \lambda_{k+r} \varphi_r$

But $\sum_{i=1}^{k+m} \lambda_i \epsilon_i = 0$

so $T\left(\sum_{i=1}^{k+m} \lambda_i \epsilon_i \right) = T(0) = 0$

so $\sum_{r=1}^{m} \lambda_{k+r} \varphi_r = 0$

But $\{ \varphi_1 \ldots \varphi_m \}$ so. L.I

so $\lambda_{k+1} = \lambda_{k+2} = \ldots \lambda_{k+m} = 0$

Substitute back in $(*)$ get.

$\sum_{i=1}^{k} \lambda_i \epsilon_i = 0$ But $\{ \epsilon_1 , \ldots \epsilon_i \}$

is a basis for $Ker(T)$

$\lambda_1 = \lambda_2 = \ldots = \lambda_k = 0$

So from the association $\sum_{i=1}^{k+m} \lambda_i \epsilon_i = 0$

deduce $\lambda_1 = \lambda_2 = \lambda_k = \lambda_{k+1} \ldots = \lambda_{k+m} = 0$ QED (a)

Proof of (b)

Given $x \in V$ Need to show that

$$\underline{x} = \sum_{i=1}^{k+m} \underline{x}_i \underline{\epsilon}_i \quad \text{for some } x_i \in \mathbb{F}$$

Let $x \in V$, $T(x) \in \mathrm{Im}(T)$

so write $T(\underline{x}) = \sum_{i=1}^{m} \lambda_i \varphi_i \qquad T(\epsilon_{k+i}) = \varphi_i$

Define $$x' = \sum_{i=1}^{m} \lambda_i T(\epsilon_{k+i})$$

$$= \sum_{i=1}^{m} \lambda_i \varphi_i = T(x)$$

so $T(x - x') = T(x) - T(x') = 0$

So $x - x' \in \mathrm{Ker}(T)$

so write $$x - x' = \sum_{i=1}^{k} x_i \epsilon_i$$

$$x = \sum_{x=i}^{k} x_i \epsilon_i + \sum_{i=1}^{m} \lambda_i \epsilon_{k+i}$$

Define $x_{k+i} = \lambda_i$ so.

$$\underline{x} = \sum_{i=1}^{k+m} x_i \epsilon_i \qquad \text{QED (b)}$$

So general case $Ker(T) \neq 0$, $Im(T) \neq 0$

$$\dim(V) = k+m = \dim Ker(T) + \dim Im(T)$$

Two special case

(I) : $Im(T) = `0'$ $\dim Im(T)$

Then $T \equiv 0$, $Ker(T) = V$

so $\dim(V) = \dim Ker(T) + 0 = \dim Ker(T) + \dim Im(T)$

(II) $Ker(T) = 0$ $\dim Ker(T) = 0$

Let $\{\varphi_1 \ldots\ldots \varphi_m\}$ be the basis for $Im(T)$

Define $\epsilon_i = \varphi_i$ proceed as before

$x - x' \in Ker(T)$ $x - x' = 0$

$x = \sum_{i=1}^{m} x_i \epsilon_i$ so $\{\epsilon_1, \ldots \epsilon_m\}$ spans

proof of L.I same as
QED as before

Example : $A = \begin{pmatrix} 1 & 1 & -1 & -1 & 1 \\ 1 & -1 & 1 & -1 & 1 \\ 1 & 3 & -3 & -1 & 1 \end{pmatrix}$ $\mathbb{F} = \mathbb{Q}$

Consider the standard linear map

$T_A : \mathbb{Q}^5 \to \mathbb{Q}^3$

$T_A(\underline{x}) = A\underline{x}$

First find basis for $Ker(T_A)$

$Ker(T_A) = \{x : A\underline{x} = 0\}$

$$\begin{pmatrix} 1 & 1 & -1 & -1 & 1 \\ 1 & -1 & 1 & -1 & 1 \\ 1 & 3 & -3 & -1 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & -1 & -1 & 1 \\ 0 & -2 & 2 & 0 & 0 \\ 0 & 2 & -2 & 0 & 0 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} 1 & 1 & -1 & -1 & 1 \\ 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 0 & -1 & 1 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$x_1 \; x_2 \; x_3 \; x_4 \; x_5$

$\Rightarrow$

$$\begin{pmatrix} x_4 - x_5 \\ x_3 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} \qquad \epsilon_1 = \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}$$

$$\epsilon_2 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \qquad \epsilon_3 = \begin{pmatrix} -1 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

is a basis for Ker T

$T_A : \mathbb{Q}^5 \to \mathbb{Q}^3$

$\dim \operatorname{Ker} T_A + \dim \operatorname{Im} T_A = 5$

$\dim \operatorname{Ker} T_A = 3$

$\dim \operatorname{Im} T_A = 2 \quad (= 5-3)$

$$\begin{pmatrix} 1 & 1 & -1 & -1 & 1 \\ 1 & -1 & 1 & -1 & 1 \\ 1 & 3 & -3 & -1 & 1 \end{pmatrix} \to$$

$$\begin{array}{c} \begin{pmatrix} 1 & 0 & 0 & -1 & 1 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \\ \begin{matrix} x_1 & x_2 & x_3 & x_4 & x_5 \end{matrix} \end{array}$$

<u>Prop</u> : The columns in original matrix A : which <u>lie above the circle variables</u> form a basis for $\operatorname{Im} T_A$

<u>Proof</u> : This is clear if A is <u>row reduced</u>

So suppose A is not row reduced.

Row reduce A to A'

$A' = PA$, P invertible

or $A = QA'$, $Q = P^{-1}$ invertible

when A row reduced

$\to$ $\operatorname{Im} A$ spanned by standard basis vector

let $\{e_1 \dots e_n\}$ be a standard basis vectors

$Ae_i = i^{th}$ column of A

BASIS THEOREM

7-12-2010

V vector space / $\mathbb{F}$
(non-zero)

(i) V has at least one basis

(ii) Any two bases have the same number of elements ($\mathcal{I}$ $\dim(V)$)

• Prop : If $\{v_1, \dots v_k\}$ is L.I in vector space V then $\boxed{v_i \neq 0}$ for each i

Proof : suppose $v_r = 0$

Then $0 \cdot v_1 + \dots 0 \cdot v_{r-1} + \underbrace{1 \cdot v_r}_{(=0)} + 0 \cdot v_{r+1} + \dots 0 \cdot v_k = 0$

is a dependence relation QED

• Prop (Little Exchange Lemma)

Let $\{w_1 \dots w_m\}$ be a spanning set for V and let $\underline{v} \neq 0$

suppose $\underline{v} = \lambda_1 v_1 + \dots + \lambda_m v_m$, then $\lambda_r \neq 0$

Then $\{w_1, \dots w_{r-1} | v | w_{r+1}, \dots w_m\}$ also spans V

(In English) I've exchanged $\underline{w_r}$ for $\underline{v}$

Proof : $\underline{v} = \lambda_r \underline{w_r} + \sum_{i \neq r} \lambda_i w_i$

$$\lambda_r w_r = v - \sum_{i \neq r} \lambda_i w_i \quad \text{and } \lambda_r \neq 0$$

$$\boxed{w_r = \left(\frac{1}{\lambda_r}\right) v + \sum_{i \neq r} \left(\frac{-\lambda_i}{\lambda_r}\right) w_i \quad *}$$

We know $\{w_1, \dots w_{r-1} | w_r | w_{r+1}, \dots w_m\}$ spans V

Want to show that

$\{w_1, \dots w_{r-1} | v | \underline{w_{r+1}}, \dots \underline{w_m}\}$ also spans V

Let $\underline{x} \in V$ and write $\underline{x} = \sum_{i=1}^{m} \mu_i w_i$

$$\boxed{\underline{x} = \mu_r \cdot \underline{w_r} + \sum_{i \neq r} \mu_i w_i}$$

⇨

Substitute for $\underline{w}_r$

$$x = \mu_r \left\{ \left(\frac{1}{\lambda_r}\right) \underline{v} + \sum_{i \neq r} \left(\frac{-\lambda_i}{\lambda_r}\right) w_i \right\} + \sum_{i \neq r} \mu_i w_i$$

$\boxed{\text{Collect terms}}$

$$\underline{x} = \left(\frac{\mu_r}{\lambda_r}\right) \underline{v} + \sum_{i \neq r} \left(\mu_i - \frac{\mu_r \lambda_i}{\lambda_r} \right) \underline{w_i}$$

so I've expressed an arbiltary vector $\underline{x} \in V$ as a linear combination $\{ w_1, \dots w_{r-1}, v', w_{r+1}, \dots w_m \}$ QED

Exchange Lemma (Full Version)

• Prop : Let $\{ \underline{w}_1, \dots w_m \}$ be a spanning set for V.
Let $\{ \underline{v}_1, \dots v_k \}$ be a L.I set in V
Then (i) $k \leq m$ and (ii) there exists a spanning set $\{ u_1, \dots u_m \}$ for V such that
$u_i = v_i$ for $1 \leq i \leq k$ and $u_i \in \{ w_1, \dots w_m \}$ for $k < i$

Proof : By induction on k Induction Base $k = 1$ has already been done (Little Exchange Lemma)

Assume proved for $k-1$, so
(i) $k-1 \leq m$
(ii) there exists a spanning set $\{ u_1', \dots u_m' \}$ which
$u_i' = v_i$, $1 \leq i \leq k-1$ and $u_i' \in \{ w_1, \dots w_m \}$ $k-1 < i$

Express v_k as a linear combination

$$v_k = \sum_{i=1}^{m} \lambda_i u_i'$$

$$v_k = \sum_{i=1}^{k-1} \lambda_i v_i + \sum_{i=k}^{m} \lambda_i u_i'$$

$v_k \neq 0$ because $\{ \underline{v}_1, \dots \underline{v}_k \}$ is L.I

so Some $\lambda_i \neq 0$

⇨

If $\lambda_i = 0$ for $k \leq i \leq m$

I get a dependence relation $v_k = \sum_{i=1}^{k-1} \lambda_i \underline{v}_i$ [Contradiction!]

Hence $\lambda_r \neq 0$ for some r with $k \leq r \leq m$

By Little Exchange Lemma,

$\{ v_1, \ldots v_{k-1}, u'_k, \ldots\ldots u'_{r-1}, v_k, u'_{r+1}, \ldots u'_m \}$

spans (so $k \leq m$)

Re-index

$\{ v_1, \ldots\ldots v_{k-1}, v_k \mid u_{k+1}, \ldots\ldots u_m \}$ where

$\{ u_{k+1}, \ldots\ldots u_m \} = \{ u'_k, \ldots\ldots u'_{r-1} \mid u'_{r+1}, \ldots\ldots u'_m \}$

and $u_r \in \{ w_1, \ldots w_m \}$ for $k < j$

QED

Basis Theorem (Uniqueness part)

Let $\{ e_1, \ldots e_m \}$ $\{ \varphi_1 \ldots \varphi_n \}$ be bases for a vector space V

then $m = n$ (In English, any two bases have same number of elements)

Proof: $\{ e_1, \ldots e_m \}$ is L.I and $\{ \varphi_1, \ldots \varphi_n \}$ spans V, so by Exchange Lemma $m \leq n$

Also, $\{ \varphi_1, \ldots \varphi_n \}$ is L.I and $\{ e_1, \ldots e_m \}$ spans V so now

$n \leq m$ ~ so $m \leq n \leq m$

so $m = n$ QED

BASIS THEOREM

Let V be a non-zero vector space / $\mathbb{F}$

Then i) V has at least one basis and

ii) any two bases have same number of elements

ii) has been proved, still have to prove (i)

We'll prove

Prop: Any spanning set contains basis.

Proof: V non-zero vector space and suppose $\{\varphi_1, \dots \varphi_m\}$ spans V

~~Prove~~ Prove by induction on m that some subset of $\{\varphi_1, \dots \varphi_m\}$ is a basis

$m=1$: V spanned by $\{\varphi_1\}$

$V \neq 0$ so $\varphi_1 \neq 0$

Then $\{\varphi_1\}$ is L.I, $\lambda \varphi_1 = 0$

$\varphi_1 \neq 0$ so $\lambda = 0$

(If $\lambda \neq 0$, $\varphi_1 = \lambda^{-1} . \lambda \varphi_1 = \lambda^{-1} . 0 = 0$ contradiction)

so Induction Base is OK suppose proved for $m-1$.

Let $\varphi_1, \dots \varphi_m$ be a spanning set.

If $\varphi_1, \dots \varphi_m$ is also L.I, then $\{\varphi_1, \dots \varphi_m\}$ is a basis and we are finished.

If $\varphi_1, \dots \varphi_m$ is L.D, choose a dependence relation

$\lambda_1 \varphi_1 + \dots \lambda_r \varphi_r + \dots \lambda_m \varphi_m = 0$ in which $\lambda_r \neq 0$

so. $\varphi_r = \sum_{i \neq r} \left(\frac{-\lambda_i}{\lambda_r} \right) \varphi_i$

Claim that, $\{\varphi_1, \dots \varphi_{r-1} \mid \varphi_{r+1}, \dots \varphi_m\}$ spans

Let $\underline{x} \in V$ and express as linear combination.

$$\underline{x} = \sum_{i=1}^{m} x_i \varphi_i = x_r \varphi_r + \sum_{i \neq r} x_i \varphi_i$$

$$\underline{x} = \sum_{i \neq r} \left(- \frac{x_r \lambda_i}{\lambda_r} \right) \varphi_i + \sum_{i \neq r} x_i \varphi_i$$

$$\underline{x} = \sum_{i \neq r} \left(x_i - \frac{x_r \lambda_i}{\lambda_r} \right) \varphi_i$$

$\Rightarrow$

ie. $\{ e_1, \dots e_{r-1} \mid e_{r+1} \dots e_m \}$ spans V

By induction $\{ e_1, \dots e_{r-1} \mid e_{r+1}, \dots e_m \}$ contains a basis
Hence $\{ e_1, \dots e_m \}$ contains a basis QED

Corollary: Any non-zero vector space V contains a basis

Proof: Let $\{ e_1, \dots e_m \}$ be a spanning set for V
Then $\{ e_1, \dots e_m \}$ contains a basis QED

The assumption here is that V can be spanned by
• finite subset.
Is it TRUE without that assumption ??

ISOMORPHISM

For Sets :

$$X \cong Y \quad \text{when} \quad \exists\, f : X \to Y, \quad g : Y \to X \qquad \text{(Bijective correspondence)}$$

$$g \circ f = Id_X \quad \text{and} \quad f \circ g = Id_Y$$

For vector spaces :

you need your correspondence to preserve addition and scalar multiplication

ie. need linear mappings

$$f : V \to W \qquad g : W \to V$$

$$f \circ g = Id_W \,, \quad g \circ f = Id_V$$

Definition : Let V, W be vector spaces over $\mathbb{F}$. By an isomorphism we mean a bijective linear map

$$T : V \to W$$

Notice T has an inverse mapping $T^{-1} : W \to V$ because T bijective

Prop : If $T : V \to W$ is a linear bijective then $T^{-1} : W \to V$ is also linear

Proof : Let $w_1, w_2 \in W$

Need to show

$$T^{-1}(w_1 + w_2) = T^{-1}(w_1) + T^{-1}(w_2)$$

Consider $T(T^{-1}(w_1 + w_2) - T^{-1}(w_1) - T^{-1}(w_2))$

$$= T.T^{-1}(w_1 + w_2) - T.T^{-1}(w_1) - T.T^{-1}(w_2)$$

(because T is linear)

$$= w_1 + w_2 - w_1 - w_2$$

$$= 0 \qquad (= T(0))$$

But T is injective and $T(0) = 0$

so $T^{-1}(w_1 + w_2) - T^{-1}(w_1) - T^{-1}(w_2) = 0$ and

$$T^{-1}(w_1 + w_2) = T^{-1}(w_1) + T^{-1}(w_2)$$

similarly $T^{-1}(\lambda w) = \lambda T^{-1}(w)$ ⇨

$(T(T^{-1}(\lambda w) - \lambda T^{-1}(w)) = 0 = T(0)$

so $\quad T^{-1}(\lambda w) - \lambda T^{-1}(w) = 0$

QED

When $\exists$ linear bijection

$T: V \to W \quad$ write $\quad V \cong W$

Standard examples (dim)

$\mathbb{F}$	1
$\mathbb{F}^2$	2
$\mathbb{F}^3$	3
$\mathbb{F}^4$	4
$\vdots$	$\vdots$
$\mathbb{F}^n$	n

Proposition: Let V be a vector space / $\mathbb{F}$ and suppose that $\dim(V) = n$ (ie V has a basis with n elements)

then $V \cong \mathbb{F}^n$

Proof: Let $\{\varphi_1, \dots \varphi_m\}$ be a basis for V

Define $\gamma: \mathbb{F}^n \to V$

by $\gamma\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \sum_{i=1}^{n} x_i \varphi_i$, easy to see γ is linear (check it.)

γ is surjective:

let $\underline{w} \in V$, $\{\varphi_1, \dots \varphi_n\}$ spans V

so express $\underline{w} = \lambda_1 \varphi_1 + \dots \lambda_n \varphi_n$

for some $\lambda_1, \dots \lambda_n \in \mathbb{F}$

$\gamma\begin{pmatrix} \lambda_1 \\ \vdots \\ \lambda_n \end{pmatrix} = \underline{w}$ ie. γ is surjective

$\Rightarrow$

ν is injective :

Suppose $\nu\begin{pmatrix}x_1\\ \vdots \\ x_n\end{pmatrix} = \nu\begin{pmatrix}y_1\\ \vdots \\ y_n\end{pmatrix}$

$$\sum_{i=1}^{n} x_i \varphi_i = \sum_{i=1}^{n} y_i \varphi_i$$

so $\displaystyle\sum_{i=1}^{n} (x_i - y_i)\varphi_i = 0$

But $\{\varphi_1 \ldots \varphi_n\}$ is L.I

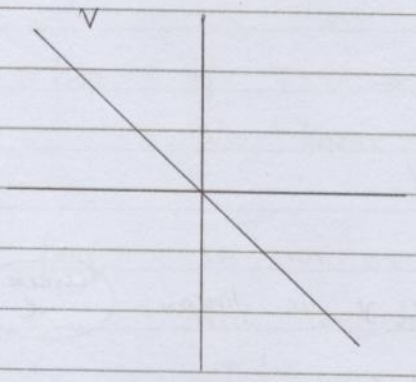
so $x_i - y_i = 0$ for all i

ie. $x_i = y_i$ for all i

ie) $\begin{pmatrix}x_1\\ \vdots \\ x_n\end{pmatrix} = \begin{pmatrix}y_1\\ \vdots \\ y_n\end{pmatrix}$ QED

Example

$V = \left\{ \begin{pmatrix}x_1\\ x_2\end{pmatrix} \in \mathbb{Q}^2 : x_1 + x_2 = 0 \right\}$



$V \cong \mathbb{R}$

$\boxed{V \neq \mathbb{R}}$ Beware

9-12-2010

V vector space of $\dim = n$ $\quad V \cong \mathbb{F}^n$

Proposition

Let $T: V \to W$ be a linear isomorphic (ie. bijective)
If $\{e_1, \dots e_m\}$ is a basis for V
then $\{T(e_1), \dots T(e_m)\}$ is a basis for W.

Proof: need to show
i) $T(e_1), \dots T(e_m)$ is L.I
ii) $T(e_1), \dots T(e_m)$ spans W

i) Suppose $\lambda_1 T(e_1) + \dots \lambda_m T(e_m) = 0$

This linear and $T(\lambda_1 e_1 + \dots \lambda_m e_m) = 0$
But T is injective and $T(0) = 0$

so $\lambda_1 q_1 + \dots \lambda_m e_m = 0$
But $\{e_1, \dots, e_m\}$ is L.I so

$\lambda_1 = \lambda_2 = \dots \lambda_m = 0$
So $\{T(e_1), \dots T(e_m)\}$ is L.I

Q.E.D

ii) Given $w \in W$ and we have to express
$w = \mu_1 T(e_1) + \dots \mu_m T(e_m)$

But T is surjective so denote

$w = T(\underline{x})$ for some $\underline{x} \in V$

$\{e_1, \ldots, e_m\}$ spans V so write

$\underline{x} = \mu_1 e_1 + \ldots \mu_m e_m$

$y = T(\underline{x}) = \mu_1 T(e_1) + \ldots \mu_m T(e_m)$

ie. $\{T(e_1) \ldots T(e_m)\}$ spans W QED.

Corollary: If $V \cong W$ then
$\dim(V) = \dim(W)$

Proof: Let $\{e_1, \ldots e_m\}$ be basis
for V (so $\dim(V) = m$)
Let $T: V \to W$ be an ismorphic
Then $\{T(e_1), \ldots T(e_m)\}$ is a basis for W ie.
$\dim(W) = m = \dim(V)$ QED

Put everything together

I) $V \cong W \Leftrightarrow \dim(V) = \dim(W)$

II) $V \cong \mathbb{F}^n \Leftrightarrow \dim(V) = n$

III) $\mathbb{F}^n \cong \mathbb{F}^m \Leftrightarrow n = m$

$\Rightarrow$

Proof of (I) : ($\Rightarrow$ already done)

suppose $\dim(V) = \dim(W)$ ($= m$, say)

Then (a) $\exists$ isomorphism

$T : \mathbb{F}^m \to V$

and (b) $\exists$ isomorphism $S : \mathbb{F}^m \to W$

Proof of (II) : ($\Leftarrow$) done last lecture

($\Rightarrow$) suppose $T : \mathbb{F}^n \to V$ is isomorphism

Let $\{e_1, \dots e_n\}$ be standard basis for $\mathbb{F}^n$

Then $\{T(e_1), \dots T(e_n)\}$ basis for V

so $\dim(V) = n$ QED (II)

dimension (= dim)

is the single numerical invarient of vector spaces.

What about linear maps

$T : V \Rightarrow W$ linear

$\mathrm{Ker}(T)$ is a vector subspace of V

$\mathrm{Im}(T)$ is a vector subspace of W

<Kernel Rank Theorem>

$$\dim \mathrm{Ker}(T) + \dim \mathrm{Im}(T) = \dim V$$

T linear

$$\begin{tikzcd} V \arrow[r, "T"] \arrow[d, "\cong"] & W \arrow[d, "\cong"] \\ \mathbb{F}^n \arrow[r, "T_A"] & \mathbb{F}^m \end{tikzcd}$$

$T_A(\underline{x}) = A\underline{x}$

A is $m \times n$ matrix over $\mathbb{F}$

" T is essentrally matrix multiplication "

Prop: Let $\mathcal{U}$ be a vector subspace of V ($U \subset V$)
Then $U = V \iff \dim(U) = \dim(V)$

Proof: $(\Rightarrow)$ Trivial
$(\Leftarrow)$ suppose
$\{e_1, \dots e_m\}$ is a basis for U and that $\dim(V) = m$
suppose that $\{e_1, \dots e_m\}$ does not span V.

Then $\exists\, \underline{v} \in V$ such that $\underline{v}$ cannot be expressed as a linear combination in.
in $\{e_1, \dots e_m\}$
so $\{e_1, \dots e_m\}$ must be L.I

$\lambda_1 e_1 + \dots \lambda_m e_m + \lambda_{m+1} = 0$

If $\lambda_{m+1} \neq 0$ then
$$\underline{v} = \sum_{i=1}^{m} \left(\frac{-\lambda e}{\lambda_{m+1}} \right) \quad \underline{\underline{\text{Contradiction}}}$$

so $\lambda_{m+1} = 0$
so $\lambda_1 e_1 + \dots + \lambda_m e_m = 0$
but $(e_1, \dots e_m)$ is L.I (basis for U)

so $\lambda_1 = \lambda_2 = \dots \lambda_{m+1} = 0$
so $m+1 \leq \dim(V)$ by Exchange Lemma
~~so~~ Contradiction $\dim(V) = m$

so $e_1, \dots e_m$ also spans V
so $V \subset U \subset V$
so $V = U$ $\underline{\underline{\text{QED}}}$

Prop : when is a linear map $T: V \to W$ injective ?

Answers : If and only if $\dim \mathrm{Ker}(T) = 0$

If $T: V \to W$ injective then $\mathrm{Ker}(T) = 0$ why !
suppose $T(x) = 0$ we know $T(0) = 0$
so $x = 0$ since T injective and $\dim(\mathrm{Ker}(T)) = 0$
$\Updownarrow$
~~dim~~ $\mathrm{Ker}(T) = 0$

Conversely suppose $\mathrm{Ker}(T) = 0$ suppose $T(x) = T(y)$
$T(x) - T(y) = 0$
$T(x-y) = 0 \Rightarrow x - y = 0$ so $x = y$ T is injective.

~ When is a linear map $T: V \to W$ surjective
Ans : iff $\dim(\mathrm{Im}(T)) = \dim W$
$\mathrm{Im}(T) = W$ then $\dim(\mathrm{Im}(T)) = \dim W$
Conversely if $\dim(\mathrm{Im}(T)) = \dim(W)$

by above $\mathrm{Im}(T) = W$ QED

14-12-2010

1) Every ~~expand~~ spanning contains a basis Dual statement

1)' Every L.I set is contained in a basis

Proof : Let $\{v_1, \dots v_n\}$ be L.I.
Let $\{\varphi_1, \dots \varphi_m\}$ be a basis

Exchange Lemma, says that $\exists$ basis $\varphi_1', \dots \varphi_m'$
in which $\varphi_1' = v_1 \quad 1 \leq i \leq k$

These statements can be expressed differently.

- Prop : Let V be a vector space $\dim V = m$
let $v_1, \dots v_m$ be polarised distinct elements of V
$(v_i \neq v_j \ , i \neq j)$

Then i) if $\{v_1, \dots v_m\}$ spans V, then $\{v_1, \dots v_m\}$ is a basis for V
ii) if $\{v_1, \dots v_m\}$ is L.I then $\{v_1, \dots v_m\}$ is a basis for V

Proof : Need to show $\{v_1, \dots v_m\}$ is also L.I.
Suppose not $\exists$ dependence relation

$$\lambda_1 v_1 + \dots \lambda_r v_r + \dots \lambda_m v_m = 0 \quad \text{with } \lambda_r \neq 0 \quad \text{so}$$

$$v_r = \sum_{i \neq r} \left(\frac{-\lambda_i}{\lambda_r} \right) v_i$$

so $\{v_1, \dots v_{r-1} \mid v_{r+1}, \dots v_m\}$

so $\dim(V) \leq m-1$ Contradiction since $(\dim V = m)$

ii) If $\{v_1, \dots v_m\}$ is L.I, then it is contained in a basis, say $\{v_1, \dots v_m, \dots v_n\}$ $m \leq n$

if $m < n$ then $\dim(V) = m < n = \dim(V)$,

Contradiction, so $m = n$ □

What Exchange Lemma says about linear matrices

Let $r \leq \min\{m, n\}$

Define $J_{mn}(r) = \left(\begin{array}{c|c} I_r & 0 \\ \hline 0 & 0 \end{array}\right)$

eg: $J_{3,5}(2) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$

$r = m < n$

$\left[\begin{array}{c|c} I_m & 0 \end{array}\right]$

$\left[\begin{array}{c} I_m \\ \hline 0 \end{array}\right]$ $r = n < m$

very special case

$r = m = n$ $\left[I_n \right]$ $n = m = r$

Theorem (First normal form)

If A is an $m \times n$ matrix $/ \mathbb{F}$ then exists

$A = P J_{\min}(r) Q$ where i) $r \le \min\{m,n\}$

ii) P is invertible $m \times m$

iii) Q is invertible $n \times n$

Proof: Let $T_A : \mathbb{F}^n \to \mathbb{F}^m$ be linear map $T_A(\underline{x}) = A\underline{x}$

Let $\mathcal{E} = \{e_1, \dots e_m\}$ be standard-basis of $\mathbb{F}^m$

and $\epsilon = \{\epsilon_1, \dots \epsilon_n\}$ be standard-basis of $\mathbb{F}^n$

$M(T_A)^{\mathcal{E}}_{\epsilon} = A$

Let $\psi_1 \dots \psi_r$ be a basis for $\mathrm{Im}(T_A)$ $\quad (r = \dim \mathrm{Im}(T_A))$

Extend $\psi_1, \dots \psi_r$ to be a basis $\{\psi_1, \dots \psi_m\}$ for $\mathbb{F}^m$

Let $\varphi_1, \dots \varphi_r \in \mathbb{F}^n$ be s.t. $T_A(\varphi_i) = \psi_i$ $\quad (h = \dim \mathrm{Ker}(T_A))$

Let $\varphi_{r+1} \dots \varphi_{r=k}$ $\quad (r \neq k = n)$ be a basis for $\mathrm{Ker}(T_A)$

Now $\{\varphi_1, \dots \varphi_r, \varphi_{r+1}, \dots \varphi_n\}$ spans $\mathbb{F}^n$

$T_A(\varphi_i) = \psi_i \qquad 1 \le i \le r$

$T_A(\varphi_j) = 0 \qquad r+1 \le j$

so $M(T_A)^{\Psi}_{\Phi} = J_{\min}(r)$

write $T_A = Id \circ T_A \circ Id$

$$M(T_A)^{\mathcal{E}}_{\epsilon} = M(Id)^{\mathcal{E}}_{\Psi} \cdot M(T_A)^{\Psi}_{\Phi} \cdot M(Id)^{\Phi}_{\epsilon}$$

$$\left.\begin{array}{l} \text{Put } P = M(Id)^{\mathcal{E}}_{\Psi} \\[2ex] \quad Q = M(Id)^{\Phi}_{\epsilon} \end{array}\right\} \text{both invertible}$$

so $A = P J_{\min}(r) Q$ $\qquad$ QED

r is called the rank of A

$r = \dim \mathrm{Im}(T_A)$

Revision Topics

1) System of linear equations : $A\underline{x} = b$

2) Multiply by invertible P (Row operations)

$$PA\underline{x} = \underline{b}$$

PA is reduced row echelon

$\Rightarrow$ $P^{-1}A = J_{mn}(r)\,Q$

to find Q, Q^{-1} are column operations

Transpose of a matrix (don't need to know this term)

~ A is $m \times n$ matrix

A^T is the transpose of A, is $n \times m$

$$(A^T)_{ji} = A_{ij} \qquad \begin{matrix} 1 \le i \le m \\ 1 \le j \le n \end{matrix}$$

$1 \le j \le n$
$1 \le i \le m$

Prop : $(AB)^T = B^T A^T$

Proof: Write $A^T = \alpha$ and $B^T = \beta$

$$\alpha_{ji} = A_{ij} \quad , \quad \beta_{kj} = B_{jk}$$

$$(\beta\alpha)_{ki} = \sum_j \beta_{kj}\,\alpha_{ji}$$

$$= \sum_j \alpha_{ji}\,\beta_{kj}$$

$$= \sum_j A_{ij} B_{jk} = (AB)_{ik}$$

$$(B^T A^T)_{ki} = (AB)_{ik}$$

$$= (AB)^T{}_{ki}$$

So $B^T A^T = (AB)^T$ <u>QED</u>

So to do column operations, we can in principle transpose do row operation, transpose back.

$$E(i,j;\lambda)^T = E(j,i;\lambda)$$
$$\Delta(i,\lambda)^T = \Delta(i,\lambda)$$
$$P(i,j)^T = P(i,j)$$

Row operations

$E(i,j;\lambda)A$ - adds λ row j to row i

$\Delta(i,\lambda)A$ - multiplies row i by λ

$P(i,j)A$ - swaps row i and row j

Column operations

- $AE(j,i;\lambda)$, adds λ ( ) to column j
- $A\Delta(i,\lambda)$, multiplies column i by λ
- $AP(i,j)$, swaps column i and column j.

$$\underset{A}{\begin{pmatrix} a & b \\ c & d \\ e & f \end{pmatrix}} \underset{E(1,2;\lambda)}{\begin{pmatrix} 1 & \lambda \\ 0 & 1 \end{pmatrix}} = \begin{pmatrix} a & \lambda a + b \\ c & \lambda c + d \\ e & \lambda e + f \end{pmatrix}$$

$$\begin{pmatrix} a & b \\ c & d \\ e & f \end{pmatrix} \underset{\Delta(2;\lambda)}{\begin{pmatrix} 1 & 0 \\ 0 & \lambda \end{pmatrix}} = \begin{pmatrix} a & b\lambda \\ c & d\lambda \\ e & f\lambda \end{pmatrix}$$