1201 Algbra 1 Notes

Based on the 2016 autumn lectures by Prof F E A Johnson

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Algebra MATH1201

Prof. Francis Johnson

CHAPTER 1. § Elementary Linear Algebra § (Equations)

1.1 A note about linear equations and notation

$x+y=1$ (✓) $\qquad$ $\underset{\uparrow 1}{x^2+y^2=1}$ (X) $\qquad$ $x+y+z=1$ plane (✓)

Recommended Textbooks:
- H. Anton
  Linear Algebra & Geometry
- Schaum Outline Series
  Linear Algebra
- Lang's Algebra V. good

$x^2+y^2-z^2=1$ surface (X)

, we need 6 variables to illustrate the relationship

um of one point.

of letters. So we subscript variables.

$=1$

$X = (x_1, \ldots$ $\qquad$ OR $\qquad$ $X = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_6 \end{pmatrix}$ Column Vector

Row Vector

It can also be $x_1+x_2-x_3-x_4+2x_5-3x_6=1$

- The general equation in n-dimensions looks like

$\boxed{a_1x_1+a_2x_2+\cdots+a_nx_n=b}$ where $x_1, \cdots x_n$ are variables

Think as follows:

$\underset{\sim}{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$ represents "variables"

$\underset{\sim}{a} = (a_1, a_2, \cdots a_n)$ represents "constants"

- How about 2 equations in n unknowns?

$a_1x_1+a_2x_2+a_3x_3+\cdots+a_nx_n=b$

$c_1x_1+c_2x_2+c_3x_3+\cdots+c_nx_n=d$

Eventually, we will run out of letters.

Answer: Use double indices for coefficients.

$$S = \begin{cases} a_{11}x_1+a_{12}x_2+a_{13}x_3+\cdots+a_{1n}x_n=b_1 \\ a_{21}x_1+a_{22}x_2+a_{23}x_3+\cdots+a_{2n}x_n=b_2 \\ \vdots \qquad\qquad\qquad\qquad \vdots \\ a_{m1}x_1+a_{m2}x_2+a_{m3}x_3+\cdots+a_{mn}x_n=b_m \end{cases}$$

General System of m linear equations in n unknowns.

Cayley C. 1830 :

- Go back to a single equation.

$a_1x_1 + a_2x_2 + \cdots + a_nx_n = b$

$\underset{\sim}{a} = (a_1, a_2, \cdots, a_n)$ Row Vector

$\underset{\sim}{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$ Column Vector

- Define $\boxed{\begin{aligned} a.x &= a_1x_1 + a_2x_2 + \cdots + a_nx_n \\ &= \sum_{r=1}^{n} a_r x_r \end{aligned}}$

So a single equation looks like $\boxed{a.x = b}$

- In a general system S,

these coefficients are arranged as an $m \times n$ matrix.

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} = (a_{ij})_{\substack{1 \le i \le m \\ 1 \le j \le n}} \quad \underline{\underline{OR}} \quad (a_{kl})_{\substack{1 \le k \le m \\ 1 \le l \le n}}$$

Note: The 1st index represents the row number.

The 2nd index represents the column number.

- EXAMPLE:

$$\begin{cases} x_1 - x_2 + 2x_3 - x_4 = 1 \\ 2x_1 + x_2 - x_3 - x_4 = 2 \\ -x_1 - x_2 \quad\quad + x_4 = 0 \end{cases}$$ (3 equations in 4 variables)

Coefficient matrix $= \begin{pmatrix} 1 & -1 & 2 & -1 \\ 2 & 1 & -1 & -1 \\ -1 & -1 & 0 & 1 \end{pmatrix}$

1·2 Matrix Product

Cayley's Great Idea

$A = (a_{ij})$ $\substack{1 \le i \le m \\ 1 \le j \le n}$ $m \times n$ matrix

$B = (b_{jk})$ $\substack{1 \le j \le n \\ 1 \le k \le p}$ $n \times p$ matrix

AB is then the $m \times p$ matrix defined as follows

$$\begin{aligned} (AB)_{ik} &= (i^{th} \text{ row of } A)(k^{th} \text{ column of } B) \\ &= (a_{i1}, a_{i2}, a_{i3}, a_{i4}, \cdots a_{in}) \begin{pmatrix} b_{1k} \\ b_{2k} \\ \vdots \\ b_{nk} \end{pmatrix} \\ &= a_{i1}b_{1k} + a_{i2}b_{2k} + \cdots + a_{in}b_{nk} \\ &\boxed{= \sum_{j=1}^{n} a_{ij}b_{jk}} \end{aligned}$$

EXAMPLE:

$$A = \begin{pmatrix} 1 & 0 & 5 \\ -1 & 7 & 2 \end{pmatrix} \qquad B = \begin{pmatrix} 1 & -1 \\ 0 & 7 \\ 5 & 2 \end{pmatrix}$$

$[2\times 3]$ $\qquad$ $[3\times 2]$

$$AB = \begin{pmatrix} 26 & 9 \\ 9 & 54 \end{pmatrix} \qquad BA = \begin{pmatrix} 2 & -7 & 3 \\ -7 & 49 & 14 \\ 3 & 14 & 29 \end{pmatrix}$$

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Recap:

$$S = \begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m \end{cases}$$

$\underset{\sim}{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$ variable factor $n\times 1$ matrix

$\underset{\sim}{b} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix}$ $m\times 1$ matrix

$A = (a_{ij})$ $\begin{matrix} 1\le i\le m \\ 1\le j\le n \end{matrix}$ coefficient matrix $\qquad \boxed{S = A\underset{\sim}{x} = \underset{\sim}{b}}$

Matrix Multiplication

$A = (a_{ij})$ $\begin{matrix} 1\le i\le m \\ 1\le j\le n \end{matrix}$ $\qquad B = (b_{jk})$ $\begin{matrix} 1\le j\le n \\ 1\le k\le p \end{matrix}$

no. of columns A = no. of rolls B

So product AB is defined.

$$(AB)_{ik} = \sum_{j=1}^{n} A_{ij}B_{jk}$$

EXAMPLE: $\quad A = \begin{pmatrix} 1 & -1 & 0 \\ 2 & 0 & -1 \end{pmatrix} \qquad B = \begin{pmatrix} 1 & 1 \\ -1 & 2 \\ 2 & -1 \end{pmatrix}$

$$AB = \begin{pmatrix} 2 & -1 \\ 0 & 3 \end{pmatrix} \quad 2\times 2 \text{ matrix}$$

$$BA = \begin{pmatrix} 3 & -1 & -1 \\ 3 & 1 & -2 \\ 0 & -2 & 1 \end{pmatrix} \quad 3\times 3 \text{ matrix}$$

1·3 Basic Facts about Matrix Multiplication

(0) If A is $m\times n$, B is $n'\times p$, $\qquad$ "iff" ≡ if and only if

then AB is defined iff $n = n'$

(1) Suppose A is $m\times n$, B is $n\times p$,

then AB is defined, but BA is defined iff $p = m$

(2) If AB, BA both defined, then in general they have different sizes.

If AB, BA have the same size, then A and B are both square ($n\times n$ say)

(3) If A is $n\times n$, B is $n\times n$, both products are defined, but in general $AB \neq BA$

EXAMPLE:

$A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ $B = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

$\Rightarrow AB = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ $BA = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$

In this case, $BA = -AB$. BUT this is very unusual.

Matrix Addition

$A = (A_{ij})$ $\begin{matrix} 1 \le i \le m \\ 1 \le j \le n \end{matrix}$ $B = (B_{ij})$ $\begin{matrix} 1 \le i \le m \\ 1 \le j \le n \end{matrix}$

$\boxed{(A+B)_{ij} = A_{ij} + B_{ij}}$ (Adding corresponding indices)

EXAMPLE: $A = \begin{pmatrix} 1 & -1 & 2 \\ 0 & 3 & 4 \end{pmatrix}$ $B = \begin{pmatrix} 0 & 1 & -1 \\ 1 & -3 & -2 \end{pmatrix}$

$A + B = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 0 & 2 \end{pmatrix}$

(4) A, B are $m \times n$, C is $n \times p$

$(A+B).C = AC + BC$ Right Distributivity

Also, A, B are $m \times n$, D is $p \times m$

$D(A+B) = DA + DB$ Left Distributivity

(5) Associativity of Multiplication

A is $m \times n$, B is $n \times p$, C is $p \times q$

$(AB).C = A.(BC)$ Associativity

(6) Zero Matrix

For each size (any m, n), there is a zero matrix $0_{ij} = 0$

2×3 $0 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ 3×2 $0 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}$

$0.A = 0$

$A.0 = 0$

Beware: Let A be $m \times n$,

so $A^2 = A.A$ is defined Nilpotent Matrix

There are many cases where $A^2 = 0$ but $A \neq 0$.

EXAMPLE: $A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ $A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$

$A^2 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ but $A \neq 0$

(7) Identity Matrix (I_n)

- For each $n \geq 2$, there was a special $n \times n$ matrix I_n.

$I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ $I_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ $I_4 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

- Formal Def : $(I_n)_{ij} = \begin{cases} 1 & i=j \\ 0 & i \neq j \end{cases}$

- Traditional Notation:

$$\boxed{\delta_{ij} = \begin{cases} 1 & i=j \\ 0 & i \neq j \end{cases}}$$ Kronecker Delta

- <u>Properties of I_n</u> :

Let A be $m \times n$,

then AI_n is defined and $AI_n = A$

Likewise, if B is $n \times p$, $I_n B = B$

- Prove that $AI_n = A$ if A is $m \times n$.

Proof : $A = (A_{ij})$ $\begin{matrix} 1 \le i \le m \\ 1 \le j \le n \end{matrix}$

$I_n = (\delta_{jk})$ $\begin{matrix} 1 \le j \le n \\ 1 \le k \le p \end{matrix}$

Use definition.

$$(AI_n)_{ik} = \sum_{j=1}^{n} A_{ij} (I_n)_{jk}$$

$$= \sum_{j=1}^{n} A_{ij} \delta_{jk}$$

k is fixed while j varies from 1 to n so picked out the fixed terms (1)

$$= A_{ik} \delta_{\boxed{kk}} + \sum_{\boxed{j \neq k}}^{n} A_{ij} \delta_{jk}$$

when $j \neq k$, $\delta_{jk} = 0$

$$= A_{ik} + 0$$

$$= A_{ik}$$

$\therefore (AI_n)_{ik} = A_{ik}$

$\Rightarrow \boxed{AI_n = A}$

- EXAMPLE : $\begin{cases} 2x_1 + 3x_2 = 1 \\ x_1 + 2x_2 = 2 \end{cases}$

Three Basic Operations :

(I) Add $\lambda Eq^n(i)$ to $Eq^n(j)$ where λ is some number

(II) Multiply $Eq^n(i)$ by $\lambda \neq 0$

(III) Permute the order in which the equations are written.

Add $(-2)Eq^n(2)$ to $Eq^n(1)$: $\begin{matrix} -x_2 = -3 \\ x_1 + 2x_2 = 2 \end{matrix}$

Add $2Eq^n(1)$ to $Eq^n(2)$: $\begin{cases} -x_2 = -3 \\ x_1 = -4 \end{cases}$

Swap $Eq^n(1)$ and $Eq^n(2)$: $\begin{cases} x_1 = -4 \\ -x_2 = -3 \end{cases}$

Multiply $Eq^n(2)$ by -1: $\begin{cases} x_1 = -4 \\ x_2 = 3 \end{cases}$

1.4 Elementary Row Operations

- Consider a system of linear equations:

$$x_1 + x_2 + x_3 = 1$$
$$x_1 - x_2 + x_3 = 3$$
$$x_1 + x_2 - x_3 = -1$$

- What can we do? Three things:

(I) $\mathcal{E}(i,j;\lambda)$ Adds $\lambda \text{Row}(j)$ to $\text{Row}(i)$

i.e. new $\text{Row}(i)$ = old $\text{Row}(i) + \lambda \text{Row}(j)$

All other rows stay the same.

(II) $\mathcal{D}(i;\lambda)$ Multiplies $\text{Row}(i)$ by λ, provided $\lambda \neq 0$.

(III) $\mathcal{P}(i,j)$ Swaps $\text{Row}(i)$ and $\text{Row}(j)$

- We are going to show that we can perform these three operations by multiplying on LEFT by a suitable matrix.

$$A = \begin{pmatrix} a & b \\ c & d \\ e & f \end{pmatrix} \qquad \cancel{j=1 \quad i=2}$$

$\mathcal{E}(2,1;\lambda)$ Adds $\lambda \text{Row}(1)$ to $\text{Row}(2)$

$$\begin{pmatrix} 1 & 0 & 0 \\ \lambda & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} a & b \\ c & d \\ e & f \end{pmatrix} = \begin{pmatrix} a & b \\ \lambda a + c & \lambda b + d \\ e & f \end{pmatrix}$$

$\mathcal{E}(1,3;\mu)$

$$\begin{pmatrix} 1 & 0 & \mu \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} a & b \\ c & d \\ e & f \end{pmatrix} = \begin{pmatrix} a+\mu e & b+\mu f \\ c & d \\ e & f \end{pmatrix}$$

QUESTION. Which matrix perform $\mathcal{E}(i,j;\lambda)$?

Elementary Matrices

- 2×2: $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ $\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ $\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$

Greek letter "epsilon" → $\epsilon(1,1)$ $\epsilon(1,2)$ $\epsilon(2,1)$ $\epsilon(2,2)$

3×3: $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ $\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ $\begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ $\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

$\epsilon(1,1)$ $\epsilon(1,3)$ $\epsilon(2,1)$ $\epsilon(3,3)$

- Informal Def:

$\epsilon(i,j)$ is $n\times n$ matrix which has 1 in position (i,j) and 0s everywhere else.

- Formal Def :

$$\epsilon(i,j) = \left(\epsilon(i,j)_{rs}\right) \quad \begin{matrix} 1 \le r \le n \\ 1 \le s \le r \end{matrix}$$

$$\boxed{\epsilon(i,j)_{rs} = \delta_{ir}\delta_{js}}$$

Check:

$$\epsilon(i,j)_{rs} = \begin{cases} 1 & r=i \text{ and } s=j \\ 0 & \text{otherwise} \end{cases}$$

QUESTION: Let A be $m \times n$

Let $\epsilon(i,j)$ be elementary $m \times m$.

What is $\epsilon(i,j)A$?

EXAMPLE: $$\underset{\epsilon(1,3)}{\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}} \underset{A}{\begin{pmatrix} a & b \\ c & d \\ e & f \end{pmatrix}} = \begin{pmatrix} e & f \\ 0 & 0 \\ 0 & 0 \end{pmatrix}$$

" takes out the j^{th} row & put it into the i^{th} row & kills everything else "

- Proposition: Let $A = (a_{st})$ $\begin{matrix} 1 \le s \le m \\ 1 \le t \le n \end{matrix}$

Let $\epsilon(i,j)$ be elementary $m \times m$ matrix.

$\epsilon(i,j)A$ is the matrix where

i^{th} row $= j^{th}$ row of A

All other rows $\equiv 0$

Proof: $$[\epsilon(i,j)A]_{rt} = \sum_{s=1}^{m} \epsilon(i,j)_{rs} A_{st}$$ $(AB)_{rt} = A_{rs} B_{st}$

$$= \sum_{s=1}^{m} \delta_{ir}\delta_{js} A_{st}$$

sum over s separate cases $s=j$ & $s \ne j$

the case where $s=j$.

$$= \delta_{ir}\delta_{jj}A_{jt} + \sum_{s \ne j}^{m} \delta_{ir}\delta_{js}A_{st}$$

$\delta_{jj} = 1$

$$= \delta_{ir}A_{jt} + 0$$

when $s \ne j$ $\delta_{js} = 0$ by def

$$= \begin{cases} A_{jt} & r = i \\ 0 & r \ne i \end{cases}$$

Q.E.D.

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- Recap: Elementary Row Operations

Basic Matrices

- Corollary: If λ is a "number",

then $\lambda\epsilon(i,j)A$ is a matrix whose i^{th} row $= \lambda$ (j^{th} row of A)

All other rows = 0

- Def: $E(i,j;\lambda) = I_m + \lambda \epsilon(i,j)$

  e.g. $m=3$, then:
  $E(2,3;\lambda) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & \lambda \\ 0 & 0 & 1 \end{pmatrix}$

- Th: Let A be $m \times n$,
  then the matrix $E(i,j;\lambda)A$ is the matrix obtained from A by performing the operation $\mathcal{E}(i,j;\lambda)$

  $\mathcal{E} \equiv$ operation $E \equiv$ matrix
  (Curly letters) (Straight letters)

Proof: $[I_m + \lambda\epsilon(i,j)]A = A + \lambda\epsilon(i,j)A$

so k^{th} row of $[I_m + \lambda\epsilon(i,j)]A = k^{th}$ row of A, where $k \neq i$ and k is a numbe

(This is because k^{th} row of $\lambda\epsilon(i,j)A = 0$)

and i^{th} row of $[I_m + \lambda\epsilon(i,j)]A = i^{th}$ row of A + i^{th} row of $\lambda\epsilon(i,j)A$

$= i^{th}$ row of $A + \lambda . j^{th}$ row of A QED

EXAMPLE: $E(2,3;\lambda) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & \lambda \\ 0 & 0 & 1 \end{bmatrix}$ $A = \begin{pmatrix} a & b \\ c & d \\ e & f \end{pmatrix}$

$\Rightarrow E(2,3;\lambda)A = \begin{pmatrix} a & b \\ c+\lambda e & d+\lambda f \\ e & f \end{pmatrix}$

- Def: Let A be an $m \times n$ matrix,
  we say that A is invertible when there exists a matrix B $(m \times m)$ s.t. $AB = I_m$ and $BA = I_m$.
  When this happens, write $\boxed{B = A^{-1}}$

EXAMPLE: $A = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$ $B = \begin{pmatrix} 1 & -2 \\ 0 & 1 \end{pmatrix}$

$\Rightarrow AB = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ $BA = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

In this case, A is invertible and $A^{-1} = \begin{pmatrix} 1 & -2 \\ 0 & 1 \end{pmatrix}$

$A = E(1,2;2)$

$B = E(1,2;-2)$

- Prop: $E(i,j;\lambda)$ is invertible
  and $E(i,j;\lambda)^{-1} = E(i,j;-\lambda)$

Proof: coming soon ◡̈

Operation		Matrix $(i \neq j)$
① $\mathcal{E}(i,j;\lambda)$	$\sim$	$E(i,j;\lambda) = I_m + \lambda\epsilon(i,j)$
② $\mathcal{D}(i;\lambda)$ $(\lambda \neq 0)$	$\sim$	$\Delta(i;\lambda) = I_m + (\lambda - 1)\epsilon(i,i)$

EXAMPLE: $A = \begin{pmatrix} a & b \\ c & d \\ e & f \\ g & h \end{pmatrix}$ perform $\mathcal{D}(2;\lambda)$

Get $\begin{pmatrix} a & b \\ c & d \\ e & f \\ g & h \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ (\lambda-1)c & (\lambda-1)d \\ 0 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} a & b \\ \lambda c & \lambda d \\ e & f \\ g & h \end{pmatrix}$

$\boxed{A + (\lambda-1)\epsilon(i,i)A}$

- k^{th} row of $\epsilon(i,i)A = 0$ $(k \neq i)$

  i^{th} row of $\epsilon(i,i)A = i^{th}$ row of A

- k^{th} row of $(\lambda-1)\epsilon(i,i)A = 0$ $(k \neq i)$

  i^{th} row of $(\lambda-1)\epsilon(i,i)A = (\lambda-1)(i^{th}$ row of $A)$

This can be transformed into

$[I_m + (\lambda-1)\epsilon(i,i)]A$

- k^{th} row of $[I_m + (\lambda-1)\epsilon(i,i)]A = k^{th}$ row of A $(k \neq i)$

  i^{th} row of $[I_m + (\lambda-1)\epsilon(i,i)]A = i^{th}$ row of $A + (\lambda-1)(i^{th}$ row of $A)$

  $= \lambda.(i^{th}$ row of $A)$

- $\mathcal{D}(i;\lambda)$ multiplies i^{th} row of A by $\lambda \neq 0$

  $\mathcal{D}(i;\frac{1}{\lambda})$ multiplies i^{th} row of A by $\frac{1}{\lambda} \neq 0$

  $\Delta(i;\lambda)$ invertible and

  $\boxed{\Delta(i;\lambda)^{-1} = \Delta(i,\frac{1}{\lambda})}$

③ $\mathcal{P}(i,j)$ interchanges i^{th} & j^{th} row

- We expect that $P(i,j)$ = matrix obtained from I_m by swapping i^{th} & j^{th} rows.

EXAMPLE: $m=4$, then $P(2,3) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

- Does it work?

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \\ e & f \\ g & h \end{pmatrix} = \begin{pmatrix} a & b \\ e & f \\ c & d \\ g & h \end{pmatrix}$$

$I_m - \epsilon(i,i) - \epsilon(j,j) + \epsilon(i,j) + \epsilon(j,i)$

$P(i,j)^{-1} = P(i,j)$

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Operation		Matrix
$\mathcal{E}(i,j;\lambda)$	$\sim$	$E(i,j;\lambda) = I_m + \lambda\epsilon(i,j)$

$\mathcal{D}(i;\lambda) \quad (\lambda\neq 0) \qquad \sim \qquad \Delta(i;\lambda) = I_m + (\lambda-1)\in(i,i)$

$\mathcal{P}(i,j) \qquad \sim \qquad P(i,j) = I_m - \in(i,i) - \in(j,j) + \in(i,j) + \in(j,$

$(i\neq j)$

- On Exercise 2,

we need to show $\boxed{\in(i,j)\in(k,l) = \begin{cases} \in(i,l) & j=k \\ 0 & j\neq k \end{cases}}$

Assume this is true, we have

corollary $\quad \in(i,j)\in(i,j) = 0 \quad i\neq j$

$\in(i,i)\in(i,i) = \in(i,i)$

- Prop. If $i\neq j$, then

$E(i,j;\lambda)$ is invertible

and $E(i,j;\lambda)^{-1} = E(i,j;-\lambda)$

Proof: $E(i,j;\lambda)E(i,j;-\lambda)$

$= [I_m + \lambda\in(i,j)][I_m - \lambda\in(i,j)]$

$= I_m^2 + \lambda I_m\in(i,j) - \lambda I_m\in(i,j) - \lambda^2\in(i,j)\in(i,j)$

$= I_m - \lambda^2\in(i,j)\in(i,j) \qquad \in(i,j)\in(i,j)=0 \quad i\neq j$

$= I_m$

$\Rightarrow E(i,j;\lambda)E(i,j;-\lambda) = I_m = E(i,j;-\lambda)E(i,j;\lambda)$ QED

- Prop. If $\lambda\neq 0$, then

$\Delta(i;\lambda)$ is invertible

and $\Delta(i;\lambda)^{-1} = \Delta(i;\frac{1}{\lambda})$

Proof: $\Delta(i;\lambda) = I_m + (\lambda-1)\in(i,i)$

$\Delta(i;\frac{1}{\lambda}) = I_m + (\frac{1}{\lambda}-1)\in(i,i) = I_m + \left(\frac{1-\lambda}{\lambda}\right)\in(i,i)$

$\Rightarrow \Delta(i;\lambda)\Delta(i,\frac{1}{\lambda})$

$= [I_m + (\lambda-1)\in(i,i)][I_m + \left(\frac{1-\lambda}{\lambda}\right)\in(i,i)]$

$= I_m^2 + (\lambda-1)I_m\in(i,i) + (\frac{1-\lambda}{\lambda})I_m\in(i,i) + (\lambda-1)\left(\frac{1-\lambda}{\lambda}\right)\in(i,i)\in(i,i)$

$= I_m + (\lambda-1+\frac{1-\lambda}{\lambda})I_m\in(i,i) + (\lambda-1)(\frac{1-\lambda}{\lambda})\in(i,i)$

$= I_m + \frac{(\lambda-1)^2}{\lambda}I_m\in(i,i) - \frac{(\lambda-1)^2}{\lambda}\in(i,i)$

$= I_m$ QED

- Prop. If $i\neq j$, then

$P(i,j)$ is invertible

and $P(i,j)^{-1} = P(i,j)$ self-inverse

Proof: $P(i,j)P(i,j)$

$= [I_m - \in(i,i) - \in(j,j) + \in(i,j) + \in(j,i)][I_m - \in(i,i) - \in(j,j) + \in(i,j) + \in(j,i)]$

$$= I_m^2 + 2[\epsilon(i,j) + \epsilon(j,i) - \epsilon(i,i) - \epsilon(j,j)] I_m + \epsilon(i,i) + 0 - \epsilon(j,j) - 0 + \epsilon(j,j) - 0$$
$$- \epsilon(j,i) - 0 - \epsilon(i,j) + 0 + \epsilon(i,i) - \epsilon(j,i) - 0 + \epsilon(j,j) + 0$$
$$= I_m$$

QED

1.4 How to solve a system of linear equations?

$$S = \begin{cases} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \cdots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \cdots + a_{2n}x_n = b_2 \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + a_{m3}x_3 + \cdots + a_{mn}x_n = b_m \end{cases} \qquad S = \{A\underset{\sim}{x} = \underset{\sim}{b}\}$$

- Prop.- Suppose that I operate on the left by an invertible matrix P, then the solutions don't change.

  $S = \{A\underset{\sim}{x} = \underset{\sim}{b}\}$ $\qquad S' = \{PA\underset{\sim}{x} = P\underset{\sim}{b}\}$

  - If $\underset{\sim}{x}$ is a solution to S, then $\underset{\sim}{x}$ is also a solution to S'.
  - Conversely, if $\underset{\sim}{x}$ is a solution to S',

  $$PA\underset{\sim}{x} = P\underset{\sim}{b}$$

  - Multiply on the left by P^{-1} to get

  $$P^{-1}PA\underset{\sim}{x} = P^{-1}P\underset{\sim}{b} \qquad \text{where } P^{-1}P = I_m$$

  So, $\qquad A\underset{\sim}{x} = \underset{\sim}{b}$

  - So $\underset{\sim}{x}$ is a solution to S. QED

- (0) For some systems, solutions are obvious.

  EXAMPLE: $\left\{ \underset{\sim}{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \text{ satisfying } x_1 + x_2 + x_3 = 1 \right\}$

  $$x_1 = 1 - x_2 - x_3$$

  $$\underset{\sim}{x} = \begin{pmatrix} 1 - x_2 - x_3 \\ x_2 \\ x_3 \end{pmatrix} \qquad \text{(has infinitely many solutions)}$$

  As x_2 and x_3 vary, we can get all possible solutions. (2-dimension)

(I) $\begin{cases} x_1 \quad + x_3 + x_4 + x_5 + x_6 - x_7 = 1 \\ x_2 - x_3 - x_4 + x_5 - x_6 + x_7 = 2 \end{cases}$

$$\underset{\sim}{x} = \begin{bmatrix} 1 - x_3 - x_4 - x_5 - x_6 + x_7 \\ 2 + x_3 + x_4 - x_5 + x_6 - x_7 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \end{bmatrix} \qquad \text{(infinitely many solutions)} \quad \text{(5-dimension)}$$

1·4·1 How do you recognise in advance which systems have obvious solutions?

Reduced Row Echelon Form

$$\begin{pmatrix} 1 & ? & 0 & ?? & 0 \\ 0 & 0 & 1 & ??? & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

EXAMPLE : $\begin{pmatrix} 1 & 2 & 0 & 5 & 3 & 0 & 1 \\ 0 & 0 & 1 & -2 & 7 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 & 1 & 3 \end{pmatrix}$ ← Reduced Row Echelon

$\begin{pmatrix} 1 & 2 & 1 & 5 & 3 & 0 & 1 \\ 0 & 0 & 1 & -2 & 7 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 & 1 & 3 \end{pmatrix}$ ← NOT reduced row echelon

- A reduced row echelon matrix satisfies the following criteria:

① In any non-zero row, the 1st non-zero entry is 1 (leading 1)

② In the column of leading 1, all other entries must be 0.

③ Rows must be "stepped" (Echelon)

④ Zero rows finish last

- EXAMPLE 1:

Consider the system $\begin{cases} x_1 + 2x_2 \quad + 5x_4 + 3x_5 = 1 \\ x_3 - 2x_4 + 7x_5 = 2 \\ x_6 = 3 \end{cases}$

$A\underset{\sim}{x} = \underset{\sim}{b}$

$$A = \begin{pmatrix} 1 & 2 & 0 & 5 & 3 & 0 \\ 0 & 0 & 1 & -2 & 7 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \qquad \underset{\sim}{b} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$$

$x_1 \; x_2 \; x_3 \; x_4 \; x_5 \; x_6$

- Write the variables underneath
  & circle the variables under leading 1s.
- Eliminate circled variables

$x_1 = 1 - 2x_2 - 5x_4 - 3x_5$

$x_2 = x_2$

$x_3 = 2 + 2x_4 - 7x_5$

$x_4 = x_4$

$x_5 = x_5$

$x_6 = 3$

Canonical Solution

$$\begin{pmatrix} 1 - 2x_2 - 5x_4 - 3x_5 \\ x_2 \\ 2 + 2x_4 - 7x_5 \\ x_4 \\ x_5 \\ 3 \end{pmatrix}$$

- EXAMPLE 2: (More typical)

$$S = \begin{cases} x_1 + x_2 + x_3 - x_4 - x_5 = 1 \\ x_1 + x_2 - x_3 - x_4 + x_5 = 1 \\ x_1 + x_2 + 3x_3 - x_4 - 3x_5 = 1 \end{cases}$$

$S = \{A\underset{\sim}{x} = \underset{\sim}{b}\}$ $\quad A = \begin{pmatrix} 1 & 1 & 1 & -1 & -1 \\ 1 & 1 & -1 & -1 & 1 \\ 1 & 1 & 3 & -1 & -3 \end{pmatrix} \quad \underset{\sim}{b} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$

<u>Step 1.</u> Form augmented matrix $(A|b)$

$$(A|b) = \left(\begin{array}{ccccc|c} 1 & 1 & 1 & -1 & -1 & 1 \\ 1 & 1 & -1 & -1 & 1 & 1 \\ 1 & 1 & 3 & -1 & 3 & 1 \end{array}\right)$$

<u>Step 2.</u> Perform row operations until I get to reduced row Echelon form.

$$\xrightarrow{\mathcal{E}(2,1,-1),\ \mathcal{E}(3,1;-1)} \left(\begin{array}{ccccc|c} 1 & 1 & 1 & -1 & -1 & 1 \\ 0 & 0 & -2 & 0 & 2 & 0 \\ 0 & 0 & 2 & 0 & -2 & 0 \end{array}\right) \xrightarrow{\mathcal{E}(3,2;1)} \left(\begin{array}{ccccc|c} 1 & 1 & 1 & -1 & -1 & 1 \\ 0 & 0 & -2 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

$$\xrightarrow{\mathcal{D}(2;-\frac{1}{2})} \left(\begin{array}{ccccc|c} 1 & 1 & 1 & -1 & -1 & 1 \\ 0 & 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right) \xrightarrow{\mathcal{E}(1,2;-1)} \left(\begin{array}{ccccc|c} 1 & 1 & 0 & -1 & 0 & 1 \\ 0 & 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

$x_1 \ x_2 \ x_3 \ x_4 \ x_5$

<u>Step 3.</u> Write variables underneath & circle variables under leading 1

<u>Step 4.</u> Eliminate circled variables & write out canonical solution.

$$\begin{cases} x_1 + x_2 - x_4 = 1 \\ x_3 - x_5 = 0 \end{cases}$$

Canonical solution: $\begin{pmatrix} 1 - x_2 + x_4 \\ x_2 \\ x_5 \\ x_4 \\ x_5 \end{pmatrix}$

- EXAMPLE 3:

$$S = \begin{cases} x_1 + x_2 + x_3 + x_4 + x_5 = 1 \\ x_1 + 2x_2 + 3x_4 + 4x_4 + 5x_5 = 2 \\ x_1 - x_3 - 2x_4 - 3x_5 = 0 \end{cases}$$

Augmented matrix $(A|b) = \left(\begin{array}{ccccc|c} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 & 5 & 2 \\ 1 & 0 & -1 & -2 & -3 & 0 \end{array}\right) \xrightarrow[\mathcal{E}(3,1;-1)]{\mathcal{E}(2,1;-1)} \left(\begin{array}{ccccc|c} 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 & 4 & 1 \\ 0 & -1 & -2 & -3 & -4 & -1 \end{array}\right)$

$$\xrightarrow{\mathcal{E}(3,2;1)} \left(\begin{array}{ccccc|c} 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 & 4 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right) \xrightarrow{\mathcal{E}(1,2;-1)} \left(\begin{array}{ccccc|c} 1 & 0 & -1 & -2 & -3 & 0 \\ 0 & 1 & 2 & 3 & 4 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

$x_1 \ x_2 \ x_3 \ x_4 \ x_5$

$$\begin{cases} x_1 - x_3 - 2x_4 - 3x_5 = 0 \\ x_2 + 2x_3 + 3x_4 + 4x_5 = 1 \end{cases}$$

Canonical solution: $\begin{pmatrix} x_3+2x_4+3x_5 \\ 1-2x_3-3x_4-4x_5 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix}$

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EXAMPLE 4:

$$S=\begin{cases} x_1+x_2+x_3+x_4=1 \\ x_1+x_2-x_3+x_4=1 \\ x_1+x_2-x_3-x_4=1 \end{cases}$$

$A\underset{\sim}{x}=\underset{\sim}{b}$ $\qquad A=\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & 1 \\ 1 & 1 & -1 & -1 \end{pmatrix}$ $\qquad \underset{\sim}{b}=\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$

augmented matrix $\left(\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & 1 & 1 \\ 1 & 1 & -1 & -1 & 1 \end{array}\right)$

$$\xrightarrow[\mathcal{E}(3,1;-1)]{\mathcal{E}(2,1;-1)} \left(\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & -2 & 0 & 0 \\ 0 & 0 & -2 & -2 & 0 \end{array}\right)$$

$$\xrightarrow[\mathcal{D}(3;-\frac{1}{2})]{\mathcal{D}(2;\frac{1}{2})} \left(\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \end{array}\right)$$

$$\xrightarrow[\mathcal{E}(3,2;1)]{\mathcal{E}(1,2;1)} \left(\begin{array}{cccc|c} 1 & 1 & 0 & 1 & 1 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right)$$

$$\xrightarrow{\mathcal{D}(2;-1)} \left(\begin{array}{cccc|c} 1 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right)$$

$$\xrightarrow{\mathcal{E}(1,3;-1)} \left(\begin{array}{cccc|c} 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right) \text{ is in reduced form}$$

$\quad x_1 \; x_2 \; x_3 \; x_4$

- Write variables underneath & circle variables under leading 1^s.
- Eliminate

$$S'=\begin{cases} x_1+x_2 = 1 \\ x_3 = 0 \\ x_4 = 0 \end{cases}$$

※ If you start with 4 variables, you should end up with 4.

Canonical solution is $\begin{pmatrix} 1-x_2 \\ x_2 \\ 0 \\ 0 \end{pmatrix}$

All solutions obtained by letting x_2 vary.

1.5 Inverting a matrix

How to find the inverse of an invertible matrix?

- EXAMPLE: $A=\begin{pmatrix}0&0&1\\0&1&1\\-1&1&0\end{pmatrix}$ is a 3×3 matrix

Form "big augmented matrix" $(A|I_3)$ (3×6 matrix)

$$(A|I_3)=\left(\begin{array}{ccc|ccc}0&0&1&1&0&0\\0&1&1&0&1&0\\-1&1&0&0&0&1\end{array}\right)$$

Reduce: $\xrightarrow{\mathcal{P}(1,3)}\left(\begin{array}{ccc|ccc}-1&1&0&0&0&1\\0&1&1&0&1&0\\0&0&1&1&0&0\end{array}\right)$

$\xrightarrow{\mathcal{E}(1,2;-1)}\left(\begin{array}{ccc|ccc}-1&0&\boxed{-1}&0&-1&1\\0&1&\boxed{1}&0&1&0\\0&0&1&1&0&0\end{array}\right)$

$\xrightarrow[\mathcal{E}(1,3;1)]{\mathcal{E}(2,3;-1)}\left(\begin{array}{ccc|ccc}-1&0&0&1&-1&1\\0&1&0&-1&1&0\\0&0&1&1&0&0\end{array}\right)$

$\xrightarrow{\mathcal{D}(1;-1)}\left(\begin{array}{ccc|ccc}1&0&0&-1&1&-1\\0&1&0&-1&1&0\\0&0&1&1&0&0\end{array}\right)$

"$(A_3|I_3)\to(I_3|A^{-1})$"

Claim: $A^{-1}=\begin{pmatrix}-1&1&-1\\-1&1&0\\1&0&0\end{pmatrix}$

Check: $AA^{-1}=\begin{pmatrix}0&0&1\\0&1&1\\-1&1&0\end{pmatrix}\begin{pmatrix}-1&1&-1\\-1&1&0\\1&0&0\end{pmatrix}=\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}$

- Why does this method work?

 - Prop: If A,B are invertible $n\times n$ matrices, then the product AB is invertible and $(AB)^{-1}=B^{-1}A^{-1}$

 Proof: $(AB)(B^{-1}A^{-1})=A(BB^{-1})A^{-1}$
 $=AI_nA^{-1}$
 $=AA^{-1}$
 $=I_n$

 NEVER write $A^{-1}=\frac{1}{A}$

 $(B^{-1}A^{-1})(AB)=B^{-1}(A^{-1}A)B$
 $=B^{-1}I_nB$
 $=B^{-1}B$
 $=I_n$ QED

 Note: reversal in order

 - Generalisation:

 If $A_N,A_{N-1},\cdots A_2,A_1$ are all invertible $n\times n$ matrices, then the product $A_NA_{N-1}\cdots A_2A_1$ is invertible and its inverse $(A_NA_{N-1}\cdots A_2A_1)^{-1}=A_1^{-1}A_2^{-1}\cdots A_{N-1}^{-1}A_N^{-1}$

Again, note reversal of order.

- Here's why this method works

✓ Start with an $n\times n$ matrix A

Form "big augmented matrix" $(A|I_n)$ ($n\times 2n$ matrix)

Now reduce.

✓ Every time I do a row operation, I multiply on the LEFT by invertible matrix.

✓ Start with $(A|I_n)$

After 1st operation $\to (X_1A|X_1)$ where X_1 is invertible

After 2nd operation $\to (X_2X_1A|X_2X_1)$ where X_2 & X_1 are invertible

$\vdots$

After N^{th} operation $\to (X_NX_{N-1}\cdots X_2X_1A|X_NX_{N-1}\cdots X_2X_1)$ and all x_i are invertible

✓ Suppose after N operations, the left hand side of the "big augmented matrix" is I_n, i.e.

$(X_NX_{N-1}\cdots X_2X_1A|X_NX_{N-1}\cdots X_2X_1) = (I_n|B)$

then $B = X_NX_{N-1}\cdots X_2X_1$

$(A|I_n) \rightsquigarrow (BA|B) = (I_n|B)$

$BA = I_n \Rightarrow B = A^{-1}$

✓ I should really check that $AB=I_n$, but as we'll see, if A,B are $n\times n$ and $BA=I_n$, then $AB=I_n$ (preview)

- EXAMPLE: $A=\begin{pmatrix}2&3\\1&2\end{pmatrix}$

$$(A|I_2)=\left(\begin{array}{cc|cc}2&3&1&0\\1&2&0&1\end{array}\right)\xrightarrow{\mathcal{P}(1,2)}\left(\begin{array}{cc|cc}1&2&0&1\\2&3&1&0\end{array}\right)\xrightarrow{\mathcal{E}(2,1;-2)}\left(\begin{array}{cc|cc}1&2&0&1\\0&-1&1&-2\end{array}\right)$$

$$\xrightarrow{\mathcal{E}(1,2;2)}\left(\begin{array}{cc|cc}1&0&2&-3\\0&-1&1&-2\end{array}\right)\xrightarrow{\mathcal{D}(2;-1)}\left(\begin{array}{cc|cc}1&0&2&-3\\0&1&-1&2\end{array}\right)$$

$\therefore A^{-1}=\begin{pmatrix}2&-3\\-1&2\end{pmatrix}$

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- EXAMPLE ①:

$A=\begin{pmatrix}2&3\\1&2\end{pmatrix}$

We know that $A^{-1}=\begin{pmatrix}2&-3\\-1&2\end{pmatrix}$

Start with $(A|I_n)=\left(\begin{array}{cc|cc}2&3&1&0\\1&2&0&1\end{array}\right)$

$$\xrightarrow{\mathcal{P}(1,2)} \left(\begin{array}{cc|cc} 1 & 2 & 0 & 1 \\ 2 & 3 & 1 & 0 \end{array}\right)$$

Reduce & keep record

$$\xrightarrow{\mathcal{E}(2,1;-2)} \left(\begin{array}{cc|cc} 1 & 2 & 0 & 1 \\ 0 & -1 & 1 & -2 \end{array}\right)$$

$$\xrightarrow{\mathcal{E}(1,2;2)} \left(\begin{array}{cc|cc} 1 & 0 & 2 & -3 \\ 0 & -1 & 1 & -2 \end{array}\right)$$

$$\xrightarrow{\mathcal{D}(2;-1)} \left(\begin{array}{cc|cc} 1 & 0 & 2 & -3 \\ 0 & 1 & -1 & 2 \end{array}\right)$$

$\therefore \Delta(2;-1)E(1,2;2)E(2,1;-2)P(1,2)(A|I_2)$

$\downarrow$

$(I|\Delta(2;-1)E(1,2;2)E(2,1;-2)P(1,2)) = A^{-1}$

So I've written A^{-1} as a product of elementary invertible matrices.

So $A^{-1} = \Delta(2;-1)E(1,2;2)E(2,1;-2)P(1,2)$

So $A = P(1,2)^{-1}E(2,1;-2)^{-1}E(1,2;2)^{-1}\Delta(2;-1)^{-1}$

Note: reversal of order

$A = P(1,2)E(2,1;2)E(1,2;-2)\Delta(2;-1)$

Check: $A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} 1 & -2 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

$= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & -2 \\ 2 & -3 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

$= \begin{pmatrix} 2 & -3 \\ 1 & -2 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

$= \begin{pmatrix} 2 & 3 \\ 1 & 2 \end{pmatrix} = A$

- EXAMPLE ②: $A = \begin{pmatrix} 1 & 2 & 0 \\ 1 & 3 & 1 \\ 0 & -2 & -1 \end{pmatrix}$

1) Find A^{-1}

2.) Express A^{-1} as a product of elementary matrices.

3) Thereby, express A as a product.

1). $(A|I_n) = \left(\begin{array}{ccc|ccc} 1 & 2 & 0 & 1 & 0 & 0 \\ 1 & 3 & 1 & 0 & 1 & 0 \\ 0 & -2 & -1 & 0 & 0 & 1 \end{array}\right)$

$$\xrightarrow{\mathcal{E}(2,1;-1)} \left(\begin{array}{ccc|ccc} 1 & 2 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & -1 & 1 & 0 \\ 0 & -2 & -1 & 0 & 0 & 1 \end{array}\right)$$

$$\xrightarrow[\mathcal{E}(3,2;2)]{\mathcal{E}(1,2;-2)} \left(\begin{array}{ccc|ccc} 1 & 0 & -2 & 3 & -2 & 0 \\ 0 & 1 & 1 & -1 & 1 & 0 \\ 0 & 0 & 1 & -2 & 2 & 1 \end{array}\right)$$

$$\xrightarrow[\mathcal{E}(2,3;-1)]{\mathcal{E}(1,3;2)} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -1 & 2 & 2 \\ 0 & 1 & 0 & 1 & -1 & -1 \\ 0 & 0 & 1 & -2 & 2 & 1 \end{array}\right)$$

$A^{-1} = \begin{pmatrix} -1 & 2 & 2 \\ 1 & -1 & -1 \\ -2 & 2 & 1 \end{pmatrix}$

Check: $A^{-1}A = \begin{pmatrix} -1 & 2 & 2 \\ 1 & -1 & -1 \\ -2 & 2 & 1 \end{pmatrix}\begin{pmatrix} 1 & 2 & 0 \\ 1 & 3 & 1 \\ 0 & -2 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

2.) $A^{-1} = E(2,3;-1)E(1,3;2)E(3,2;2)E(1,2;-2)E(2,1;-1)$

3) $A = \underbrace{E(2,1;1)}_{} E(1,2;2) E(3,2;-2) \underbrace{E(1,3;-2)}_{} E(2,3;1)$

2^{nd} row, 1^{st} column is 1. 1^{st} row, 3^{rd} column is -2.

Check:

$$\underbrace{\begin{pmatrix}1&0&0\\1&1&0\\0&0&1\end{pmatrix}\begin{pmatrix}1&2&0\\0&1&0\\0&0&1\end{pmatrix}}_{X}\underbrace{\begin{pmatrix}1&0&0\\0&1&0\\0&-2&1\end{pmatrix}\begin{pmatrix}1&0&-2\\0&1&0\\0&0&1\end{pmatrix}}_{Y}\underbrace{\begin{pmatrix}1&0&0\\0&1&1\\0&0&1\end{pmatrix}}_{Z}$$

$$X = \begin{pmatrix}1&2&0\\1&3&0\\0&0&1\end{pmatrix} \quad Y = \begin{pmatrix}1&0&-2\\0&1&0\\0&-2&1\end{pmatrix} \quad Z = \begin{pmatrix}1&0&0\\0&1&1\\0&0&1\end{pmatrix}$$

$$\begin{pmatrix}1&2&-2\\1&3&-2\\0&-2&1\end{pmatrix}\begin{pmatrix}1&0&0\\0&1&1\\0&0&1\end{pmatrix} = \begin{pmatrix}1&2&0\\1&3&1\\0&-2&-1\end{pmatrix}$$

※ If a matrix is not invertible, then after row operations, we'll get

$$\left(\begin{array}{cccc|c}1&0&0&\cdots&\cdots\\0&1&0&\cdots&\cdots\\ &\vdots& & &\\0&0&0&\cdots&\cdots\end{array}\right)$$

0^s row at the end

i.e. we cannot get I_n

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Chapter 2. § Propositional Logic §

2·1 Truth Tables

① 'NOT': '$\neg$' (negation)

- p is a basic statement

e.g. p = 'It is raining.' then $\neg p$ = 'It is not raining.'

-

p	$\neg p$	$\neg\neg p$
T	F	T
F	T	F

called truth tables

- $\neg\neg p$ is the same as p $\boxed{\neg\neg p \equiv p}$

② 'AND': '$\wedge$' (conjunction)

- q = 'It is cold.' '$p \wedge q$' ≡ 'p and q'

Note: p & q are independent of each other

-

p	q	$p \wedge q$
T	T	T
	F	F
F	T	F
	F	F

③ 'OR':

- Exclusive 'or' / Latin 'aut'

-

p	q	$p \S q$
T	T	F
	F	T
F	T	T
	F	F

Johnson's notation ☺ ➡ We DON'T use it.

'either p or q, but not both'

- '$\vee$' (disjunction)

- Inclusive 'or' / Latin 'vel'

-

p	q	$p \vee q$
T	T	T
	F	T
F	T	T
	F	F

'either p or q, possibly both'

④ 'IMPLIES': '$\Rightarrow$' (implication)

- '$p \Rightarrow q$' ≡ 'if p, then q'
-

p	q	$p \Rightarrow q$
T	T	T
	F	F
F	T	T
	F	T

- Explanation:

p	q	$(\neg p) \vee q$	$\neg p$
T	T	T	F
	F	F	F
F	T	T	T
	F	T	T

So $\neg p \vee q$ has the same truth table as $p \Rightarrow q$

p	$\neg p$	$(\neg p) \vee p$
T	F	T
F	T	T

$p \Rightarrow q$ means raughly if p happens then q happens.

Since $(\neg p) \vee p$ is always true,

$p \Rightarrow q$ is true as long as q is true OR p is true.

- Say two statements are equivalent ('≡') when they have the same truth table.

2.2 Standard Identities

(I). $p \wedge q \equiv q \wedge p$
  . $p \vee q \equiv q \vee p$

} commutativity

(II). $p \wedge (q \wedge r) \equiv (p \wedge q) \wedge r$
  . $p \vee (q \vee r) \equiv (p \vee q) \vee r$

} associativity

(III). $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$
  equivalent to $p.(q+r) = pq + pr$
  . $p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$

} distributivity

(IV). $p \equiv p \wedge p$
  . $p \equiv p \vee p$

} idempotent

Proof of $p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$:

p	q	r	$p\vee(q\wedge r)$ $\equiv$	$(p\vee q)\wedge(p\vee r)$	$p\vee q$	$p\vee r$	$q\wedge r$
T	T	T	T	T	T	T	T
		F	F	T	T	T	F
	F	T	T	T	T	T	F
		F	F	T	T	T	F
F	T	T	T	T	T	T	T
		F	F	F	T	F	F
	F	T	F	F	F	T	F
		F	F	F	F	F	F

same!

De Morgan's Laws

- How does 'negation' behave ?

$\neg(p\wedge q)\equiv(\neg p)\vee(\neg q)$
$\neg(p\vee q)\equiv(\neg p)\wedge(\neg q)$ } duality

Proof.

p	q	$p\wedge q$	$\neg(p\wedge q)$ $\equiv$	$(\neg p)\vee(\neg q)$	$\neg p$	$\neg q$
T	T	T	F	F	F	F
	F	F	T	T	F	T
F	T	F	T	T	T	F
	F	F	T	T	T	T

same!

p	q	$\neg(p\vee q)$	$(\neg p)\wedge(\neg q)$	$\neg p$	$\neg q$
T	T	F	F	F	F
	F	F	F	F	T
F	T	F	F	T	F
	F	T	T	T	T

same!

2.3 • How many signs do we need ? Switching between logical symbols

(1) $p\Rightarrow q \equiv (\neg p)\vee q$

$\neg p\Rightarrow q \equiv (\neg\neg p)\vee q \equiv p\vee q$

So $\boxed{p\vee q \equiv \neg p \Rightarrow q}$

(2) $\neg(p\wedge q) \equiv (\neg p)\vee(\neg q)$

$\neg\neg(p\wedge q) \equiv \neg(\neg p\vee\neg q)$

$p\wedge q \equiv \neg(p\Rightarrow\neg q)$

So $\boxed{p\wedge q \equiv \neg(p\Rightarrow\neg q)}$

(3) $p \wedge \neg q \equiv \neg(p \Rightarrow \neg\neg q)$

$\equiv \neg(p \Rightarrow q)$

So $\boxed{p \Rightarrow q \equiv \neg(p \wedge \neg q)}$

Mon. 24/10/16

MATH1401: Algebra I

Prof. Francis Johnson

- Propositional logic is logic of constant propositions.
- Recap (switching between logical symbols):

We can get by with just 2 signs: $\neg, \Rightarrow$ (✓)

$\neg, \vee$ (✓)

$\neg, \wedge$ (✓)

For example, if I just take $\neg, \Rightarrow$, then

$p \vee q \equiv \neg\neg p \vee q$

$\boxed{p \vee q \equiv \neg p \Rightarrow q}$

$p \wedge q \equiv \neg\neg p \wedge q$

$\equiv \neg(\neg p \vee \neg q)$

$\boxed{p \wedge q \equiv \neg(p \Rightarrow \neg q)}$

also called 'nand' sometimes

- In fact, we just need one sign. Historically called <u>Sheffer's Stroke Function</u>.

p	q	$p \mid q$	$p \mid (p \mid q)$	$p \Rightarrow q$
T	T	F	T	T
	F	T	F	F
F	T	T	T	T
	F	T	T	T

$\boxed{p \mid q \equiv \neg(p \wedge q)}$

$\boxed{p \mid (p \mid q) \equiv p \Rightarrow q}$

p	$p \mid p$
T	F
F	T

$\boxed{p \mid p \equiv \neg p}$

2.4 Proof by Contradiction

- $p \Rightarrow q \equiv (\neg p) \vee q$

$\equiv (\neg p) \vee (\neg\neg q)$

$\equiv (\neg\neg q) \vee (\neg p)$

$\boxed{p \Rightarrow q \equiv (\neg q) \Rightarrow (\neg p)}$ ← principle of proof by contradiction

- ✓ $(\neg q) \Rightarrow (\neg p)$ is the <u>contrapositive</u> of $p \Rightarrow q$
  and they are equivalent (have the same truth table)
  ✓ $q \Rightarrow p$ is called <u>converse</u> of $p \Rightarrow q$
  $q \Rightarrow p \not\equiv p \Rightarrow q$
- <u>Def.</u> $\boxed{p \Leftrightarrow q \equiv (p \Rightarrow q) \wedge (q \Rightarrow p)}$ '$\Leftrightarrow$' $\equiv$ 'iff'
- $p, q, r, s \ldots$ are basic propositions that can be T or F.
  However, $p \wedge q \Rightarrow r$ is called a <u>composite proposition</u>.

<u>Def.</u> Composite propositions can be either
(I) always T tautology
(II) sometimes T, sometimes F contingent
(III) always F contradiction

eg. (I) $(p \Rightarrow q) \Rightarrow (\neg q \Rightarrow \neg p)$ tautology
(II) $p \wedge q \Rightarrow r$ contingent
since if p, q are F, r is T, true
if p, q are T, r is F. false
(III) $p \wedge (\neg p)$ contradiction

Note: <u>A contradiction is the same as the negation of a tautology.</u>
e.g. $p \vee (\neg p)$ is always T | $\neg(p \vee \neg p) \equiv \neg p \wedge \neg\neg p$
$\equiv p \wedge \neg p$ is always F

2.5 Valid Arguments

- e.g. a) If it is cold or raining, we shall stay inside.
  b) We are outside and it is not cold.
  c) Therefore, it is not raining.

✓ Is this argument valid?
✓ Let p be 'it is cold'
q be 'it is raining'
r be 'we are inside', then
✓ a) is $p \vee q \Rightarrow r$
b) is $(\neg r) \wedge (\neg p)$
The whole argument is then $\underbrace{(p \vee q \Rightarrow r) \wedge (\neg r) \wedge (\neg p)}_{LHS} \Rightarrow \neg q$

✓ Calculate truth table.

p	q	r	$p \vee q \Rightarrow r$	$(\neg r)\wedge(\neg p)$	LHS	$\neg q$	LHS $\Rightarrow \neg q$
T	T	T	T	F	F	F	T
		F	F	F	F	F	T
	F	T	T	F	F	T	T
		F	F	F	F	T	T
F	T	T	T	F	F	F	T
		F	F	T	F	F	T
	F	T	T	F	F	T	T
		F	T	T	T	T	T

- e.g.(2).

  a) $p \wedge q \Rightarrow r$

  b) $\neg r \wedge \neg p$

  The whole statement is $(p\wedge q \Rightarrow r) \wedge (\neg r) \wedge (\neg p) \Rightarrow \neg q$

p	q	r	$p\wedge q \Rightarrow r$	$(\neg r)\wedge(\neg p)$	LHS	$\neg q$	LHS $\Rightarrow \neg q$
T	T	T	T	F	F	F	T
		F	F	F	F	F	T
	F	T	T	F	F	T	T
		F	T	F	F	T	T
F	T	T	T	F	F	F	T
		F	T	T	T	F	F
	F	T	T	F	F	T	T
		F	T	T	T	T	T

This argument is not valid (because at least one line gives a false)

Fri. 28/10/16

MATH1401: Algebra I

Prof. Johnson

- Textbook: Notes on Logics by P.T. Johnstone

We have done logic of constant propositions. $\neg, \wedge, \vee, \Rightarrow$

2.6 Logic of Variable Propositions (Predicate Logic)

- $P(x)$ where x varies.
  - ✓ Suppose $x \in \{0,1\}$, $P(x) = 'x^2>0'$.
    In this case, $P(0)$ is false, but $P(1)$ is true.
  - ✓ Suppose $x \in \{0,1,2\}$, $P(x) = 'x^2>0'$
    $P(0)$ F $\quad$ $P(1)$ T $\quad$ $P(2)$ T
  - ✓ Suppose $x \in \{1,2\}$, $P(x) = 'x^2>0'$
    $P(1)$ T $\quad$ $P(2)$ T
- The values over which x is allowed to vary is called the domain (of discussion). Denoted by $\mathcal{D}$.
  - Then $P(x)$ is a statement about objects in $\mathcal{D}$.
  - We are allowed to use $\neg, \wedge, \vee, \Rightarrow$.
    Plus 2 new signs.
  - ✓ '$\forall$' Universal Quantifier ($\equiv$ 'for all')
    $(\forall x)P(x)$ means that for any value of x in $\mathcal{D}$, $P(x)$ is true.
  - ✓ '$\exists$' Existential Quantifier ($\equiv$ 'these exists')
    $(\exists x)P(x)$ means that for at least one x in $\mathcal{D}$, $P(x)$ is true.
- EXAMPLE:
  - ✓ $\mathcal{D} = \{0,1\}$
    Clearly, I can form $P(0)$ and I can form $P(1)$.
    $(\forall x)P(x)$, in this case, means $P(0) \wedge P(1)$.
    $(\exists x)P(x)$, in this case, means $P(0) \vee P(1)$.
  - ✓ $\mathcal{D} = \{0,1,2\}$
    $(\forall x)P(x)$, in this case, means $P(0) \wedge P(1) \wedge P(2)$
    $(\exists x)P(x)$, in this case, means $P(0) \vee P(1) \vee P(2)$

The Convention (for algebra)
$\mathbb{N} = \{0, 1, 2, \cdots\}$

  - ✓ Suppose $\mathcal{D} = \mathbb{N}$,
    $(\forall x)P(x) \sim P(0) \wedge P(1) \wedge P(2) \wedge P(3) \wedge \cdots \wedge P(n) \wedge P(n+1) \wedge \cdots$ formulae of ∞ length (infinite)
    $(\exists x)P(x) \sim P(0) \vee P(1) \vee P(2) \vee P(3) \vee \cdots \vee P(n) \vee P(n+1) \vee \cdots$
    We introduce '$\forall$', '$\exists$' predicator to avoid formulae of ∞ length.

2.6.1 <u>How to negate a quantifier?</u>

- Start with $\mathcal{D} = \{0,1\}$,
  $(\forall x)P(x) = P(0) \wedge P(1)$
  $\neg(\forall x)P(x) = \neg(P(0) \wedge P(1))$

$= \neg P(0) \vee \neg P(1)$

$= (\exists x) \neg P(x)$

✓ $\mathcal{D} = \{0,1,2\}$

$(\forall x) P(x) = P(0) \wedge P(1) \wedge P(2)$

$\neg(\forall x) P(x) = \neg P(0) \vee \neg P(1) \vee \neg P(2)$

$= (\exists x) \neg P(x)$

- First rule of negation

$$\boxed{\neg(\forall x) P(x) = (\exists x) \neg P(x)}$$

- Second rule of negation

$$\boxed{\neg(\exists x) P(x) = (\forall x) \neg P(x)}$$

e.g. $(\exists x) P(x) = P(0) \vee P(1)$

$\neg(\exists x) P(x) = \neg(P(0) \vee P(1))$

$= \neg P(0) \wedge \neg P(1)$

$\Rightarrow \neg(\exists x) P(x) = (\forall x) \neg P(x)$

2·6·2 Interchange of Order of Quantifiers

- The order in which quantifiers come is extremely important.

EXAMPLE: Suppose $\mathcal{D} = \{0,1\}$, $P(x,y) =$ '$x \neq y$'

Compare $(\forall x)(\exists y) P(x,y) \leftarrow$ TRUE

and $(\exists y)(\forall x) P(x,y) \leftarrow$ FALSE

✓ $(\exists y) P(x,y) = P(x,0) \vee P(x,1)$

$\therefore (\forall x)(\exists y) P(x,y) = (\forall x) [P(x,0) \vee P(x,1)]$

$= [P(0,0) \vee P(0,1)] \wedge [P(1,0) \vee P(1,1)]$

$= (F \vee T) \wedge (T \vee F)$

$= T \wedge T$

$= T$

✓ $(\forall x) P(x,y) = P(0,y) \wedge P(1,y)$

$\therefore (\exists y)(\forall x) P(x,y) = [P(0,0) \wedge P(1,0)] \vee [P(0,1) \wedge P(1,1)]$

$= [F \wedge T] \vee [T \wedge F]$

$= F \vee F$

$= F$

$$\boxed{\text{Axiom: } (\exists y)(\forall x) P(x,y) \Rightarrow (\forall x)(\exists y) P(x,y)}$$

- However, two quantifiers of same type commute.

i.e. $\boxed{\begin{array}{l}(\forall x)(\forall y)P(x,y) = (\forall y)(\forall x)P(x,y) \\ (\exists x)(\exists y)P(x,y) = (\exists y)(\exists x)P(x,y)\end{array}}$

MATH1201 Propositional Logic

Basic Propositions

1. Commutativity: $p \wedge q \equiv q \wedge p$

   $p \vee q \equiv q \vee p$

2. Associativity: $p \wedge (q \wedge r) \equiv (p \wedge q) \wedge r$

   $p \vee (q \vee r) \equiv (p \vee q) \vee r$

3. Distributivity: $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$

   $p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$

4. Idempotent: $p \equiv p \wedge p$

   $p \equiv \quad p \vee p$

De Morgan Laws

1. duality: $\neg (p \wedge q) \equiv (\neg p) \vee (\neg q)$

   $\neg (p \vee q) \equiv (\neg p) \wedge (\neg q)$

2. $p \vee q \equiv \neg p \Rightarrow q$

   $p \wedge q \equiv \neg (p \Rightarrow \neg q)$

   $p \Rightarrow q \equiv \neg (p \wedge \neg q)$

Sheffers Stroke Function

$p | q \equiv \neg (p \wedge q)$

$p | (p | q) \equiv p \Rightarrow q$

$p | p \equiv \neg p$

Proof by Contradiction / iff

Contrapositive: $p \Rightarrow q \equiv (\neg q) \Rightarrow (\neg p)$

$p \Leftrightarrow q \equiv (p \Rightarrow q) \wedge (q \Rightarrow p)$

Logic of Variable Propositions

$\neg (\forall x) P(x) = (\exists x) \neg P(x)$

$\neg (\exists x) P(x) = (\forall x) \neg P(x)$

$(\exists y)(\forall x) P(x, y) \Rightarrow (\forall x)(\exists y) P(x, y)$

$(\forall x)(\forall y) P(x, y) \equiv (\forall y)(\forall x) P(x, y)$

$(\exists x)(\exists y) P(x, y) \equiv (\exists y)(\exists x) P(x, y)$

MATH1201 Propositional Logic

Basic Propositions

1. commutativity: $p \wedge q \equiv q \wedge p$
   $p \vee q \equiv q \vee p$

2. associativity: $p \wedge (q \wedge r) \equiv (p \wedge q) \wedge r$
   $p \vee (q \vee r) \equiv (p \vee q) \vee r$

3. distributivity: $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$
   $p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$

4. idempotent: $p \equiv p \wedge p$
   $p \equiv p \vee p$

De Morgan Laws

1. $\neg (p \wedge q) \equiv (\neg p) \vee (\neg q)$ duality
   $\neg (p \vee q) \equiv (\neg p) \wedge (\neg q)$

2. $p \vee q \equiv \neg p \Rightarrow q$
   $p \wedge q \equiv \neg (p \Rightarrow \neg q)$
   $p \Rightarrow q \equiv \neg (p \wedge \neg q)$

Sheffer's Stroke Function

$p | q \equiv \neg (p \wedge q)$

$p | (p | q) \equiv p \Rightarrow q$

$p | p \equiv \neg p$

Proof by Contradiction / iff

$p \Rightarrow q \equiv (\neg q) \Rightarrow (\neg p)$ contrapositive

$p \Leftrightarrow q \equiv (p \Rightarrow q) \wedge (q \Rightarrow p)$

Logic of Variable Propositions

$\neg (\forall x) P(x) = (\exists x) \neg P(x)$

$\neg (\exists x) P(x) = (\forall x) \neg P(x)$

$(\exists y)(\forall x) P \Rightarrow (\forall x)(\exists y) P$

$(\forall x)(\forall y) P \equiv (\forall y)(\forall x) P$

$(\exists x)(\exists y) P \equiv (\exists y)(\exists x) P$

Fri. 28/10/16 (continued)

MATH1401: Algebra I

Prof. Johnson

Chapter 3 § Set Theory §

3.1 Sets

- '$\in$' : '$x \in A$' $\equiv$ 'x is a member of A' 'belongs to' / 'is a member of'

  ✓ Every set starts & ends with a curly bracket.

  e.g. $\{0,1,2,3,5,7\} = \{7,5,2,1,3,0\}$

  ① In a set, the order in which elements are written is NOT significant.

  Note: To stress order, use round brackets.

  e.g. $(0,1,2) \neq (2,1,0)$

  ② If A and B are sets, then $A=B$ iff $x \in A \Leftrightarrow x \in B$

  i.e. a set is determined by its members and nothing else.

  Two sets are the same if all of their elements are the same.

- '$\subset$' : ※ Do not confuse '$\in$' with '$\subset$'.

  ✓ Def. $B \subset A$ iff $x \in B \Rightarrow x \in A$, 'is a subset of'

  so $A=B$ iff $(A \subset B) \wedge (B \subset A)$

- EXAMPLE: $A = \{0, 1, \{0,1\}, \{0,2\}, \{2\}\}$

Question.	$0 \in A$	T	
	$\{0\} \in A$	F	$\leftarrow$ $\{0\}$ is not an element of A
	$2 \in A$	F	
	$\{2\} \in A$	T	$\leftarrow$ $\{2\}$ is an element of A
	$\{2\} \subset A$	F	$\leftarrow$ because $\{2\} \subset A$ iff $2 \in A$ but $2 \notin A \Rightarrow \{2\} \not\subset A$
	$\{\{2\}\} \subset A$	T	$\leftarrow$ because $\{2\} \in A \Rightarrow \{\{2\}\} \subset A$

Note. $\{0,1,1\} = \{0,1\}$

- To list elements of a set is quite naive.

More usually, we define sets by means of properties possessed by elements.

eg. $\{x \mid x \text{ is a TFL bus}\}$

✓ In general, $\left\{ \begin{array}{c|c} \text{typical} & \text{defined} \\ \text{element} & \text{property} \end{array} \right\}$

e.g. $\{x \in \mathbb{R} \mid x^2 > 3\}$

✓ $A = \{x \mid P_A(x)\}$ where P_A is the defining property of A.

Note. $\boxed{\{x \in \mathbb{R} \mid x^2 < 0\}}$ is an example of an <u>empty set</u>.
Notation: $\emptyset$

So if you're defining sets by properties, you will inevitably get an empty set.

Note: $\emptyset \subset A \quad \forall A$

✓ $|A|$ = number of elements in A

Note: $X = \{\emptyset\}$ is <u>non-empty</u>, since $\emptyset \in X$

so $0 = |\emptyset| \Rightarrow$ <u>0 is natural</u>

3.2 Predicates of Sets

- $A = \{x \mid P_A(x)\}$

  $B = \{x \mid P_B(x)\}$

✓ Union: $\boxed{A \cup B = \{x \mid P_A(x) \vee P_B(x)\}}$ i.e. $A \cup B$ consists of those elements that are in A or in B.

$P_{A \cup B}(x) = P_A(x) \vee P_B(x)$

✓ Intersection: $\boxed{A \cap B = \{x \mid P_A(x) \wedge P_B(x)\}}$

✓ Difference: $\boxed{A - B = \{x \mid P_A(x) \wedge \neg P_B(x)\}}$ i.e. $A - B$ consists of those elements which are in A but not in B.

$P_{A-B} = P_A(x) \wedge \neg P_B(x)$

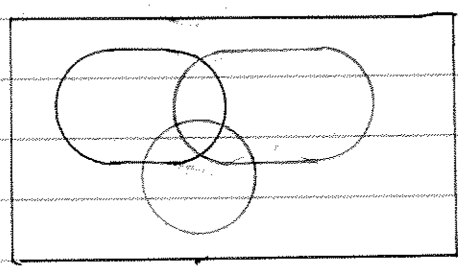
Note: Union / Intersection / Difference of sets ≡ another <u>set</u>.

- <u>Venn Diagram</u>

  A (black) B (red) C (green)

   $\equiv A \cap B$  $\equiv A \cap B \cap C$

   $\equiv C - A \cup B$

  Note: disadvantage = cannot represent ≥ 4 sets because 4 sets requires 3D.

- <u>Direct Product of Sets</u>

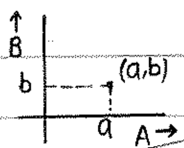
  ✓ Formally, $(a,b) = (a',b')$ iff $a = a'$ <u>AND</u> $b = b'$ called 'ordered pair'.

  so $(a,b) = (b,a)$ iff $a = b$

  ✓ $A = \{x \mid P_A(x)\}$

  $B = \{x \mid P_B(x)\}$

  $\boxed{A \times B = \{(x,y) \mid P_A(x) \wedge P_B(y)\}}$

  1st element from 1st set (A)

  2nd element from 2nd set (B)

  In general, $\boxed{B \times A \neq A \times B}$

e.g. $A = \{0,1,2\}$, $B = \{0,1,2,3\}$

$$\begin{array}{ccc}
(0,3) & (1,3) & (2,3) \\
(0,2) & (1,2) & (2,2) \\
(0,1) & (1,1) & (2,1) \\
(0,0) & (1,0) & (2,0)
\end{array}$$

✓ $|A|$ = card of A = number of elements in A

$$\boxed{|A \times B| = |A| . |B|}$$

- Which of the following is a function?

$$\left.\begin{array}{l} f(x) = x+1 \\ g(x) = \dfrac{1}{x+1} \\ h(x) = \sqrt{x+1} \end{array}\right\} \text{These are formulae.}$$

Mon. 31/10/16

MATH1201: Algebra I

Prof. Johnson

3.3 Mappings (Functions)

3.3.1 <u>What are mappings?</u>

- Informal Def.

  Let A, B be sets.

  By a mapping / function

$$f : \underset{\text{domain of } f}{A} \rightarrow \underset{\text{codomain of } f}{B}$$

  we mean a rule which assigns to each $a \in A$ a single element $f(a) \in B$.

EXAMPLE:

$A = B = \mathbb{Q}$ (rationals)

✓ could take as my rule $f(a) = a+1$

✓ whereas for $g(a) = \frac{1}{a+1}$, this isn't going to work

✓ However, if I take $A = \mathbb{Q} - \{1\}$, ← means A contains all rationals but 1

then $g : \mathbb{Q} - \{1\} \rightarrow \mathbb{Q}$

$g(a) = \frac{1}{a+1}$ is a mapping.

✓ Take $A = \{x \in \mathbb{R} \mid x \geq -1\}$, then

$h : A \rightarrow \mathbb{R}$

$h(x) = +\sqrt{x+1}$

Note. $\cancel{h: \mathbb{C} \to \mathbb{C}}$ ← This example is related to Riemann Surface.

$\cancel{h(x) = \sqrt{x+1}}$

- <u>Composition of mappings</u>:

Let A, B, C be sets.

$f: A \to B$ $\quad\quad$ $g: B \to C$

Therefore, $g \circ f: A \to C$ is the mapping

$\boxed{(g \circ f)(a) = g(f(a))}$ Composition

circle

✓ EXAMPLE:

$f: \mathbb{R} \longrightarrow \mathbb{R}$ $\quad\quad$ $g: \mathbb{R} \longrightarrow \mathbb{R}$

$f(x) = x^2+1$ $\quad\quad$ $g(x) = \cos x$

$(g \circ f)(x) = \cos(x^2+1)$

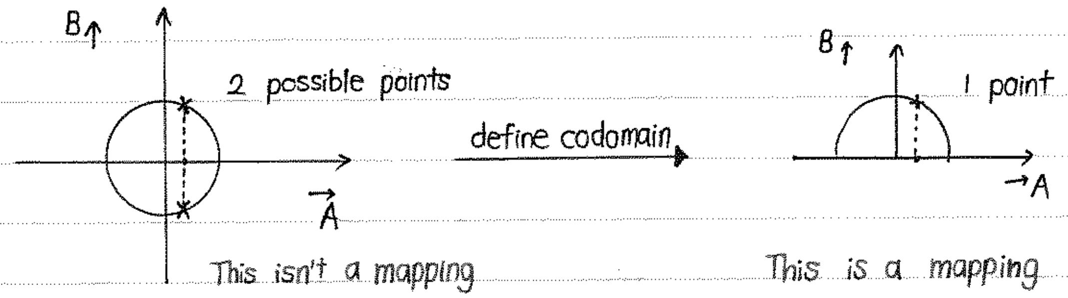
where $(f \circ g)(x) = \cos^2 x + 1$

Are they the same ?

$(g \circ f)(0) = \cos 1 < 1$

$(f \circ g)(0) = 2$

$f \circ g \neq g \circ f$



This isn't a mapping $\quad\quad$ This is a mapping

✓ Composition is associative.

$A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} D$

Then $h \circ g: B \to D$ $\quad\quad$ $g \circ f: A \to C$

$(h \circ g) \circ f: A \to D$ $\quad\quad$ $h \circ (g \circ f): A \to D$

Preposition: $\boxed{h \circ (g \circ f) = (h \circ g) \circ f}$

Proof: Let $a \in A$

$(h \circ (g \circ f))(a) = h((g \circ f)(a)) = h(g(f(a)))$

$((h \circ g) \circ f)(a) = (h \circ g)(f(a)) = h(g(f(a)))$ $\quad\quad$ QED

- <u>Identity mapping</u>

A set

$$\boxed{\begin{array}{l} Id_A : A \to A \\ Id_A(a) = a \quad \forall\, a \in A \end{array}}$$

'Id_A' ≡ 'I_A' ≡ 'Id'

- <u>Invertible mapping</u>

  Suppose A, B are sets and $f: A \to B$ is the mapping,

  We say that f is invertible where there exists mapping $g: B \to A$

  $$A \xrightarrow{f} B \xrightarrow{g} A \xrightarrow{f} B$$

  such that $\boxed{\begin{array}{l} g \circ f = Id_A \\ f \circ g = Id_B \end{array}}$ and

  If this is the case, we say that g is inverse mapping to f.

  Notation: $g = f^{-1}$ Note: $f^{-1} \neq \frac{1}{f}$

✓ EXAMPLE ①.

$f: \mathbb{R} \to \mathbb{R}$

$f(x) = 2x+1$

Q. Is f invertible? If so, what is f^{-1}?

$g(x) = \frac{x-1}{2}$

$$(f \circ g)(x) = f\left(\frac{x-1}{2}\right) = 2\left(\frac{x-1}{2}\right) + 1 = x$$

$$(g \circ f)(x) = \frac{(2x+1)-1}{2} = \frac{2x}{2} = x$$

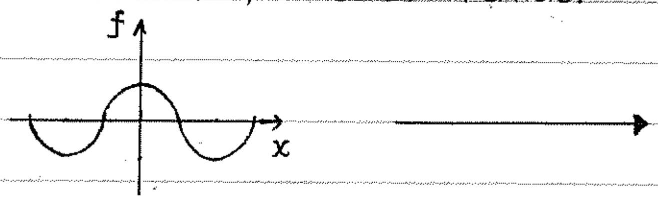
$\Rightarrow f^{-1}(x) = \frac{x-1}{2}$

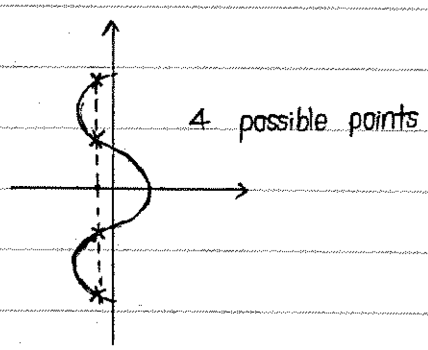
✓ EXAMPLE ②:

$f: \mathbb{R} \to \mathbb{R}$

$f(x) = \cos x$

In this case, f is not invertible.





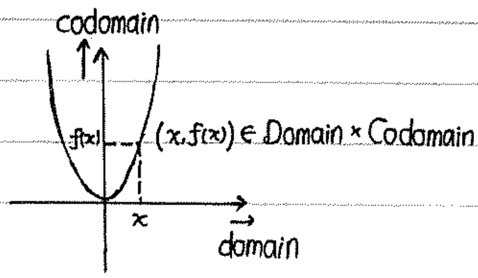
We'll deal with this formally very soon.

- <u>Formal Def.</u>

  ✓ Graph of a mapping.

  $f: \mathbb{R} \to \mathbb{R}$

  $f(x) = x^2$



✓ Formal Def.

Let A, B be sets.

By a mapping $f: A \to B$, I mean a subset

$$f \subset A \times B$$

with the following properties:

1) $\forall a \in A, \exists b \in B$ s.t. $(a, b) \in f$

(I then write $b = f(a)$) $\longrightarrow$ It is defined.

2) $\forall a \in A$, $(a, b) \in f$ and $(a, b') \in f$

$\Rightarrow b = b'$ $\longrightarrow$ There is only one point. (one-to-one)

✓ $f: A \to B$

Suppose I have an inverse $f^{-1}: B \to A$.

$f^{-1} \subset B \times A$

I)' $\forall b \in B, \exists a \in A$ s.t. $ba \in f^{-1}$

II)' $\forall b \in B, (b, a) \in f^{-1} \mid (b, a') \in f^{-1} \Rightarrow a = a'$

Question. Given a mapping $f: A \to B$.

What do I need to check before I can conclude that f is invertible?

3.3.2 Injectivity, Surjectivity and Bijectivity

① Injectivity: '单射函数'

Say that $f: A \to B$ is injective when given $a, a' \in A$,

$\boxed{f(a) = f(a') \Rightarrow a = a'}$ 'one-to-one'

✓ EXAMPLE ①:

$f: \mathbb{Q} \to \mathbb{Q}$

$f(x) = 2x + 1$

Then f is injective because

$f(x) = f(x') \Rightarrow 2x + 1 = 2x' + 1$

$x = x'$

✓ EXAMPLE ②:

$g: \mathbb{Q} \to \mathbb{Q}$

$g(x) = x^2 + 1$

Is g injective?

$g(1) = 2 = g(-1) = 2$

BUT $1 \neq -1 \Rightarrow$ not injective

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② Surjectivity: 「满射函数」

We say that f is surjective when

$$\boxed{\forall b \in B,\ \exists a \in A \text{ s.t. } b = f(a)}$$

✓ Informally, surjectivity means every element of codomain is hit by something in the domain.

✓ EXAMPLE: $f: \mathbb{R} \to \mathbb{R}$

$f(x) = 2x+1$ is surjective.

Let $y \in \mathbb{R}$.

Then, we need to write $y = 2x+1$

$x = \frac{y-1}{2}$ — x in terms of y

$\therefore \forall y \in \mathbb{R},\ \exists x = \frac{y-1}{2}$ s.t. $f(x) = y$

• Recall: $f: A \to B$ is invertible when $\exists$ mapping $g: B \to A$ s.t. $g \circ f = Id_A$ and $f \circ g = Id_B$

<u>Prop.</u> Let $f: A \to B$ be invertible, then

1) f is injective and

2) f is surjective

$$A \underset{g}{\overset{f}{\rightleftarrows}} B$$

Proof:

1.) Let $g: B \to A$ s.t. $g \circ f = Id_A$ and $f \circ g = Id_B$

Suppose $f(a) = f(a')$,

apply g to both sides: $g(f(a)) = g(f(a'))$

$(g \circ f)(a) = (g \circ f)(a')$

$Id_A(a) = Id_A(a')$

$a = a'$ QED

2) Let $b \in B$, put $a = g(b)$

$f(a) = f(g(b)) = (f \circ g)(b) = Id_B(b) = b$

$\Rightarrow f$ is surjective QED

③ Bijectivity

A mapping f is said to be bijective when it is both injective and surjective.

- Def. So by the previous example, I've shown that

  An invertible mapping is bijective.

- In fact, the converse is also true.

$p \Rightarrow q$

'converse'

$\equiv$

$q \Rightarrow p$

✓ Recall: $\boxed{\begin{array}{l} f: A \to B \text{ is a mapping} \\ \Leftrightarrow f \subset A \times B \text{ s.t.} \\ \text{I) } \forall a \in A, \exists b \in B, (a,b) \in f \\ \text{II) If } (a,b) \in f \text{ and } (a,b') \in f \\ \quad \Rightarrow b = b' \end{array}}$

✓ So if $f \subset A \times B$,

we define $f^{-1} \subset B \times A$ ← Note: this definition does not yet assume f is mapping.

$\Leftrightarrow f^{-1} = \{(b,a) \mid (a,b) \in f\}$

$f \subset A \times B$	$f^{-1} \subset B \times A$
I) $\forall a \in A, \exists b \in B$ s.t $(a,b) \in f$	I)' $\forall a \in A, \exists b \in B$ s.t $(b,a) \in f^{-1}$ ← implies f^{-1} is surjective
II) $(a,b) \in f$ and $(a,b') \in f \Rightarrow b = b'$	II)' $(b,a) \in f^{-1}$ and $(b',a) \in f^{-1} \Rightarrow b = b'$ ← implies f^{-1} is injective
III) $(a,b) \in f$ and $(a',b) \in f \Rightarrow a = a'$	III)' $(b,a) \in f^{-1}$ and $(b,a') \in f^{-1} \Rightarrow a = a'$
IV) $\forall b \in B, \exists a \in A$ s.t. $(a,b) \in f$	(IV)' $\forall b \in B, \exists a \in A$ s.t. $(b,a) \in f^{-1}$

Note: I), II) are conditions for f to be a mapping.

III) is the condition for f to be injective.

IV) is the condition for f to be surjective.

WHILE

III)', IV)' are conditions for f^{-1} to be a mapping.

✓ So if f satisfies I), II), III) & IV), then

f^{-1} satisfies I)', II)', III)' & IV)'

$\Rightarrow f^{-1}$ is a mapping and also f^{-1} is bijective

- Th. I) Let $f: A \to B$ be a mapping, then

  f is invertible $\Leftrightarrow$ f is bijective

II) If that is the case, then

f^{-1} is also a bijective mapping.

- <u>Prop.</u> Let $\left.\begin{array}{l} f: A \to B \\ g: B \to C \end{array}\right\}$ mappings

If f, g are invertible / bijective, then

$g \circ f$ is also invertible / bijective.

Proof:

If f, g are invertible with inverses f^{-1}, g^{-1},

$$A \underset{f^{-1}}{\overset{f}{\rightleftarrows}} B \underset{g^{-1}}{\overset{g}{\rightleftarrows}} C$$

$$\begin{aligned}
(f^{-1} \circ g^{-1}) \circ (g \circ f) &= f^{-1} \circ (g^{-1} \circ g) \circ f \\
&= f^{-1} \circ Id_B \circ f \\
&= f^{-1} \circ f \\
&= Id_A
\end{aligned}$$

$$\begin{aligned}
(g \circ f) \circ (f^{-1} \circ g^{-1}) &= g \circ (f \circ f^{-1}) \circ g^{-1} \\
&= g \circ Id_B \circ g^{-1} \\
&= g \circ g^{-1} \\
&= Id_C
\end{aligned}$$

QED

Notice that

$\boxed{(g \circ f)^{-1} = f^{-1} \circ g^{-1}}$ reversal of order !

3.4 Permutations

- $f: \{1, 2, \cdots, n\} \to \{1, 2, \cdots, n\}$ mappings

If $n = 3 = m$, Identity permutation

$Id \begin{pmatrix} 1 \to 1 \\ 2 \to 2 \\ 3 \to 3 \end{pmatrix}$ can be written as $\begin{pmatrix} 1 & 2 & 3 \\ \downarrow & \downarrow & \downarrow \\ 1 & 2 & 3 \end{pmatrix}$

$\begin{pmatrix} 1 \to 1 \\ 2 \searrow\nearrow 2 \\ 3 \quad\; 3 \end{pmatrix}$ can be written as $\begin{pmatrix} 1 & 2 & 3 \\ \downarrow & \downarrow & \downarrow \\ 1 & 3 & 2 \end{pmatrix}$

- <u>Def.</u> A permutation on n letters is a bijective mapping

$\boxed{f: \{1, \cdots, n\} \to \{1, \cdots, n\}}$

Convention: $f = \begin{pmatrix} 1 & 2 & 3 & \cdots & n \\ f(1) & f(2) & f(3) & \cdots & f(n) \end{pmatrix}$

e.g. $f = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 5 & 3 & 4 & 1 & 2 \end{pmatrix}$

means $f(1) = 5$

$f(2) = 3$

$f(3) = 4$

$f(4) = 1$

$f(5) = 2$

- Composition of permutation:

$f = \begin{pmatrix} 1 & 2 & \cdots & n \\ f(1) & f(2) & \cdots & f(n) \end{pmatrix}$ $\qquad g = \begin{pmatrix} 1 & 2 & \cdots & n \\ g(1) & g(2) & \cdots & g(n) \end{pmatrix}$

e.g. $f = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 5 & 3 & 4 & 1 & 2 \end{pmatrix}$

$g = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 4 & 1 & 5 & 2 \end{pmatrix}$

How to calculate $g \circ f$?

$\overleftarrow{g \circ f} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ \cancel{5} & \cancel{3} & \cancel{4} & \cancel{1} & \cancel{2} \\ 2 & 1 & 5 & 3 & 4 \end{pmatrix} \begin{matrix} \downarrow f \\ \downarrow g \end{matrix}$ $\qquad f$ come 1^{st}, g comes 2^{nd}

Then cross out the middle line (row)

i.e. $g \circ f = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 1 & 5 & 3 & 4 \end{pmatrix}$

How to calculate f^{-1} ?

$f^{-1} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 4 & 5 & 2 & 3 & 1 \end{pmatrix}$ $\qquad$ 'upside-down'

sigma

- $\sigma_n = \{ f : \{1, \cdots, n\} \to \{1, \cdots, n\} \}$, f is bijective

σ_n is the self of permutations on n letters

$|\sigma_n| = n!$

This is because $f(n)$ has n possibilities

$f(n-1)$ $\cdots$ $(n-1)$ $\cdots$

$\vdots$ $\qquad$ $\vdots$

$f(1)$ $\cdots$ 1 $\cdots$

3.4.1 Cyclic Permutations

e.g. $f(1)=3$, $f(3)=5$, $f(5)=4$, $f(4)=1$ can be written as $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 2 & 5 & 1 & 4 \end{pmatrix}$

Shorthand: $(1,3,5,4)$ goes to

leave out 2 since $f(2)=2$

- Formal Def.

Let $a_1, a_2, \cdots, a_n \in \{1, \cdots, n\}$,

By the cycle $(a_1, a_2, \cdots, a_m)$,

$$(a_1, a_2, \cdots, a_m)(a_1) = a_2$$

$$(a_1, a_2, \cdots, a_m)(a_2) = a_3$$

$$\vdots \qquad \vdots$$

$$(a_1, a_2, \cdots, a_m)(a_{m-1}) = a_m$$

$$(a_1, a_2, \cdots, a_m)(a_m) = a_1$$

$$(a_1, a_2, \cdots, a_m)(x) = x \quad \text{if} \quad x \notin \{a_1, a_2, \cdots, a_m\}$$

✓ $\{a_1, a_2, \cdots, a_m\}$ is called support of the cycle

$C = (a_1, a_2, \cdots, a_n)$ is called a cycle of length n

✓ Def.

If $s \in \sigma_n$, then the order of s is the smallest integer $n \geq 1$ s.t.

$$\underbrace{s \circ s \circ \cdots \circ s}_{n} = Id$$

$$s^n = Id$$

* EXAMPLE: $s = (1,3,7)$ ← a shorthand

by def.

Then $\boxed{s(1)=3, \ s(3)=7, \ s(7)=1}$, $s(x)=x$ otherwise

$s^2(1)=7$, $s^2(3)=1$, $s^2(7)=3$, $s^2(x)=x$ otherwise

$s^3(1)=1$, $s^3(3)=3$, $s^3(7)=7$, $s^3(x)=x$ otherwise

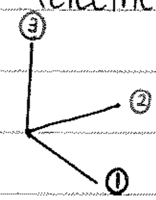
In this case, the order of s is 3.

*The order of cyclic permutation $(a_1, a_2, \cdots a_m)$ is simply m.

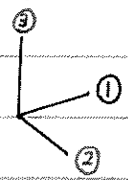
i.e. The order of a cyclic permutation is the length of the cycle.

✓ Applications:

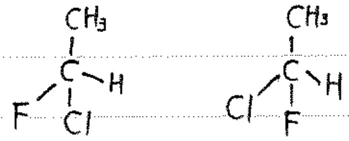
* Physics: Fleming's Right-hand Rule: (electric generators)



Fleming's Left-hand Rule: (electric motors)



$\begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}$

* Chemistry:  stereoisomer '立体异构'

3·4·1 Transposition

- Def. A transposition is a cycle of length 2.

e.g. $\tau = (3,5)$ ⟵ shorthand

$$(3,5)(x) = \begin{cases} 5 & x = 3 \\ 3 & x = 5 \\ x & x \notin \{3,5\} \end{cases}$$

$\tau^2 = Id$

✓ Prop. Any cycle is a product of transpositions.

Proof:

$C = (a_1, a_2, \cdots, a_m)$

$C = (a_1, a_m) \cdots (a_1, a_3)(a_1, a_2)$ QED

$$\left(\begin{array}{cccccc} a_1 & a_2 & a_3 & \cdots & a_{m-1} & a_m \\ a_2 & a_1 & a_3 & \cdots & a_{m-1} & a_m \\ a_2 & a_3 & a_1 & \cdots & a_{m-1} & a_m \\ & & & \vdots & & \\ a_2 & a_3 & a_4 & \cdots & a_1 & a_m \\ a_2 & a_3 & a_4 & \cdots & a_m & a_1 \end{array}\right) \begin{array}{l} \\ (a_1, a_2) \\ (a_1, a_3) \\ \\ (a_1, a_{m-1}) \\ (a_1, a_m) \end{array}$$

- In fact,

A cycle of length m is a product of $(m-1)$ transpositions.

✓ sign (of) transposition $= -1$

sign (of) cycle of length $m = (-1)^{m-1}$

i.e. a cycle of even length has sign -1

a cycle of odd length has sign 1

3·4·2 Disjoint Cycles

- Def.

We say two cycles

$C_1 = (a_1, a_2, a_3, \cdots, a_m)$
$C_2 = (b_1, b_2, b_3, \cdots, b_k)$ are disjoint

when $\{a_1, a_2, \cdots, a_m\} \cap \{b_1, b_2, \cdots, b_k\} = \emptyset$

e.g. $(1,3,7)$ & $(2,5,9,4)$ are disjoint.

- <u>Prop.</u> If C_1 and C_2 are disjoint cycles, then

$$C_1C_2 = C_2C_1$$

- <u>Th.</u> Any permutation can be written (uniquely up to order) as a product of disjoint cycles.

EXAMPLE ①:

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 \\ 4 & 5 & 9 & 7 & 3 & 10 & 8 & 1 & 2 & 12 & 6 & 11 \end{pmatrix}$$

$\sigma = \underbrace{(1,4,7,8)}_{4}(2,5,3,9)\underbrace{(6,10,12,11)}_{4}$ <u>shorthand</u>

sign = $(-1)(-1)(-1) = (-1)^3 = -1$

order = 4

EXAMPLE ②:

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 \\ 4 & 5 & 9 & 7 & 3 & 10 & 8 & 1 & 2 & 13 & 6 & 11 & 14 & 13 \end{pmatrix}$$

$\sigma = \underbrace{(1,4,7,8)}_{4}(2,5,3,9)\underbrace{(6,10,13,14,12,11)}_{6}$

sign = $(-1)^3 = -1$

'最小公倍数'

order = 12 ← lowest common multiples of the order of disjoint cycles.

In this case, 4 and 6 ⇒ lowest common multiple = 12

Note: { Laplace's Theorem

Any transposition is a product of an add no. of adjacent transpositions.

see "Chapter 7. Basis Theorem"

Mon. 14/11/16

MATH1201 Algebra I

Prof. Johnson

Chapter 4. § Fields and Vector Spaces §

4.1 Fields

- $\mathbb{N} = \{0, 1, 2, \cdots, n, n+1, \cdots\}$

  ✓ cardinals of finite sets 'cardinal' ≡ '基数'

  ✓ can $\begin{cases} \text{add} \\ \text{multiply} \end{cases}$, but cannot always $\begin{cases} \text{subtract} \\ \text{divide} \end{cases}$

$\mathbb{Z} = \{0, \pm 1, \pm 2, \cdots, \pm n, \pm(n+1), \cdots\}$

✓ can add, subtract, multiply, but cannot always divide (by non-zero)

$\mathbb{Q} = \{\frac{p}{q}, \ p, q \in \mathbb{Z}, q \neq 0\}$ rationals

✓ rule of equality: $\frac{p}{q} = \frac{p'}{q'} \Rightarrow pq' = p'q$

✓ $\mathbb{Q}$ is the first example of a field.

- Def.

域 By a field $\mathbb{F}$, I mean a 5-tuple $\mathbb{F} = (\mathbb{F}, +, \cdot, 0, 1)$ where

(i) $\mathbb{F}$ is a set

(ii) $0, 1 \in \mathbb{F}$, $0 \neq 1$

(iii) $+ : \mathbb{F} \times \mathbb{F} \to \mathbb{F}$ (means + is a mapping)

Write $a + b \equiv +(a, b)$

(iv) $\cdot : \mathbb{F} \times \mathbb{F} \to \mathbb{F}$

Write $a \cdot b = \cdot(a, b)$

An n-tuple is a (ordered) sequence of n elements

- These must satisfy the following rules.

✓ Additive Axioms

I) $a + (b + c) = (a + b) + c \quad \forall a, b, c \in \mathbb{F}$ associativity

II) $a + 0 = a = 0 + a \quad \forall a \in \mathbb{F}$ 0 is an additive identity

III) $\forall a \in \mathbb{F}, \exists (-a) \in \mathbb{F}$ s.t. additive inverse

$a + (-a) = 0 = (-a) + a$

IV) $\forall a, b \in \mathbb{F}, \ a + b = b + a$ commutativity

✓ Multiplicative Axioms

I)' $a \cdot (b \cdot c) = (a \cdot b) \cdot c \quad \forall a, b, c \in \mathbb{F}$ associativity

II)' $a \cdot 1 = 1 \cdot a = a \quad \forall a \in \mathbb{F}$ 1 is a multiplicative identity

III)' $a \cdot a^{-1} = 1 = a^{-1} \cdot a \quad \forall a \in \mathbb{F},\ a \neq 0$ multiplicative inverses

IV)' $\forall a, b \in \mathbb{F},\ a \cdot b = b \cdot a$ commutativity

✓ Distributive Axioms

$a \cdot (b+c) = a \cdot b + a \cdot c \quad \forall a, b, c \in \mathbb{F}$

$(b+c) \cdot a = b \cdot a + c \cdot a \quad \forall a, b, c \in \mathbb{F}$

e.g. 1) $\mathbb{Q}$ is a field

2) $\mathbb{R}$ is also a field

3) $\mathbb{C}$ is a field

4.1.1 Finite fields

• ✓ $\mathbb{F}_2$ arithmetic mod 2 ← modulus arithmetic 同余

- $\mathbb{F}_2 = \{\text{even}, \text{odd}\}$

$+ : \mathbb{F}_2 \times \mathbb{F}_2 \to \mathbb{F}_2$

+	even	odd
even	even	odd
odd	odd	even

$\cdot : \mathbb{F}_2 \times \mathbb{F}_2 \to \mathbb{F}_2$

·	even	odd
even	even	even
odd	even	odd

- In future, we write

$0 \equiv$ even

$1 \equiv$ odd

i.e. $\mathbb{F}_2 = \{0, 1\}$ a field with 2 elements

Thus,

+	0	1
0	0	1
1	1	0

·	0	1
0	0	0
1	0	1

(Addition mod 2)

✓ $\mathbb{F}_3$ arithmetic mod 3

$\mathbb{F}_3 = \{0, 1, 2\}$ ← possible remainders mod 3

+	0	1	2
0	0	1	2
1	1	2	0
2	2	0	1

·	0	1	2
0	0	0	0
1	0	1	2
2	0	2	1

e.g. $7 + 5 = 12 \equiv 0$

$7 \div 3$: remainder 1; $5 \div 3$: remainder 2; $12 \div 3$: no remainder

e.g. $5 \times 8 = 40 \equiv 1$

$5 \div 3$: remainder 2; $8 \div 3$: remainder 2; $40 \div 3$: remainder 1

✓ $\mathbb{F}_5$ arithmetic mod 5

$\mathbb{F}_5 = \{0, 1, 2, 3, 4\}$

+	0	1	2	3	4
0	0	1	2	3	4
1	1	2	3	4	0
2	2	3	4	0	1
3	3	4	0	1	2
4	4	0	1	2	3

·	0	1	2	3	4
0	0	0	0	0	0
1	0	1	2	3	4
2	0	2	4	1	3
3	0	3	1	4	2
4	0	4	3	2	1

$2^{-1} = 3 \mod 5$

$4^{-1} = 4 \mod 5$

e.g. $4+4 = 8 \equiv 3$

e.g. $3 \cdot 4 = 12 \equiv 2$

$\mathbb{F}_5$ is a field

✓ $\mathbb{Z}_4$ arithmetic mod 4

$\mathbb{Z}_4 = \{0, 1, 2, 3\}$

·	0	1	2	3
0	0	0	0	0
1	0	1	2	3
2	0	2	0	2
3	0	3	2	1

NOT a field

because 2 has no multiplicative inverse

✓ When is arithmetic mod a field ?

$\Leftrightarrow$ n is prime (Gauss)

Fri. 18/11/16

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4·1·2 Quadratic Fields

- Let $\mathbb{F}$ be a field which you understand, e.g. take $\mathbb{F} = \mathbb{Q}$ / $\mathbb{F} = \mathbb{R}$,

  If $\alpha \in \mathbb{F}$, ask:

  does the eqn $x^2 = \alpha$ have a solution in $\mathbb{F}$?

  e.g. $\mathbb{F} = \mathbb{R}$, $x^2 = -1$ has no solution in $\mathbb{R}$.

✓ Informal Idea:

- Construct a new element $\sqrt{\alpha}$.

- Consider expressions $x_1 + x_2\sqrt{\alpha}$: $x_i \in \mathbb{F}$
- Add & multiply in usual way.

① $(x_1 + x_2\sqrt{\alpha}) + (y_1 + y_2\sqrt{\alpha}) = (x_1 + y_1) + (x_2 + y_2)\sqrt{\alpha}$

② $(x_1 + x_2\sqrt{\alpha}) \cdot (y_1 + y_2\sqrt{\alpha}) = (x_1 y_1 + x_2 y_2 \alpha) + (x_1 y_2 + x_2 y_1)\sqrt{\alpha}$

③ zero: $0 = 0 + 0\sqrt{\alpha}$

$1 = 1 + 0\sqrt{\alpha}$

$(1 + 0\sqrt{\alpha})(x_1 + x_2\sqrt{\alpha}) = x_1 + x_2\sqrt{\alpha} \longrightarrow$ This implies 1 is an identity

✓ Do inverses exist?

- Trick: $(x_1 + x_2\sqrt{\alpha})(x_1 - x_2\sqrt{\alpha}) = x_1^2 - x_2^2\alpha + 0$

[Need to know that $x_1^2 - x_2^2\alpha \neq 0$.
Otherwise, $x_1^2 = x_2^2\alpha \Rightarrow \left(\frac{x_1}{x_2}\right)^2 = \alpha$
Contradiction.]

$$(x_1 + x_2\sqrt{\alpha})\left(\frac{x_1}{x_1^2 - x_2^2\alpha} + \frac{x_2}{x_1^2 - x_2^2\alpha}\sqrt{\alpha}\right) = 1$$

$x_1 - x_2\sqrt{\alpha}$ is called the <u>conjugate</u> to $x_1 + x_2\sqrt{\alpha}$.

- If you guarantee that $x_1^2 - x_2^2\alpha \neq 0$, when $(x_1, x_2) \neq (0,0)$, then you get a field called $\boxed{\mathbb{F}(\sqrt{\alpha})}$

Construct $\sqrt{\alpha}$ as follows.

<u>Formal Def.</u>

$F(\sqrt{\alpha}) = \mathbb{F} \times \mathbb{F}$

① Addition $(x_1, x_2) + (y_1, y_2) = (x_1 + y_1, x_2 + y_2)$

zero $0 = (0, 0)$

Think $\boxed{\begin{array}{l} x_1 + x_2\sqrt{\alpha} = (x_1, x_2) \\ \text{so } 1 = (1, 0) \\ \sqrt{\alpha} = (0, 1) \end{array}}$

② Multiplication $\boxed{(x_1, x_2) \cdot (y_1, y_2) = (x_1 y_1 + x_2 y_2 \alpha,\ x_1 y_2 + x_2 y_1)}$

$(0,1) \cdot (0,1) = (\alpha, 0) = \alpha(1, 0) = \alpha \cdot 1$

✓ EXAMPLE ①:

$\mathbb{F} = \mathbb{R}$, $\alpha = -1$

'$x^2 = \alpha$ has no solution in $\mathbb{F}$'
$\Downarrow$
'$\mathbb{F}(\sqrt{\alpha})$ is a field'

$x^2 = -1$ has no solution in $\mathbb{R}$.

So $\mathbb{R}(\sqrt{-1})$ is a field.
$\quad\ \ \ \|$
$\quad\ \ \mathbb{C}$

✓ EXAMPLE ②:

$$\mathbb{F} = \mathbb{F}_5 = \{0, 1, 2, 3, 4\}$$

$0^2 = 0$

$1^2 = 1$

$2^2 = \boxed{4 = -1}$ → This is because $4+1=0$, $4=-1$

$3^2 = 4 = -1$

$4^2 = 1$

$x^2 = 2$ has <u>no solution</u> in $\mathbb{F}_5$.

So $\mathbb{F}_5(\sqrt{2})$ <u>is a field</u> with <u>25 elements</u>

since $\mathbb{F}_5 = \underbrace{\mathbb{F}_5}_{5 \text{ elements}} \times \underbrace{\mathbb{F}_5}_{5 \text{ elements}}$

4.2 Vector Spaces

• Def.

Fix a field. (eg. $\mathbb{F} = \mathbb{Q}$)

By a vector space V over $\mathbb{F}$, I mean a 4-tuple $V = (V, +, \underline{0}, \cdot)$ where

1) V is a set

2) $\underline{0} \in V$

3) $+ : V \times V \to V$ mapping

$\underline{v} + \underline{w} = +(\underline{v}, \underline{w})$ addition

4) $\cdot : \mathbb{F} \times V \to V$ mapping

$\lambda \cdot \underline{v} = \cdot(\lambda, \underline{v})$ scalar multiplication

s.t. the following rules are held:

I) $\underline{u} + (\underline{v} + \underline{w}) = (\underline{u} + \underline{v}) + \underline{w} \quad \forall \underline{u}, \underline{v}, \underline{w} \in V$ associative

II) $\underline{u} + \underline{0} = \underline{u} = \underline{0} + \underline{u} \quad \forall \underline{u} \in V$ additive identity

III) $\forall \underline{u} \in V$, $\exists (-\underline{u})$ s.t.

$\underline{u} + (-\underline{u}) = \underline{0} = (-\underline{u}) + \underline{u}$ additive inverse

IV) $\underline{u} + \underline{v} = \underline{v} + \underline{u} \quad \forall \underline{u}, \underline{v} \in V$ commutative

I') $\lambda \cdot (\mu \cdot \underline{v}) = (\lambda\mu) \cdot \underline{v} \quad \forall \lambda, \mu \in \mathbb{F}, \forall \underline{v} \in V$

II') $1 \cdot \underline{u} = \underline{u} \quad \forall \underline{u} \in V$

III') $-\underline{u} = (-1) \cdot \underline{u}$

Plus two distributive laws.

D1) $\lambda \cdot (\underline{u} + \underline{v}) = \lambda \underline{u} + \lambda \underline{v} \quad \forall \lambda \in \mathbb{F}, \forall \underline{u}, \underline{v} \in V$

i.e. multiplication of λ distributes over addition

D2) $(\lambda+\mu).\underline{u} = \lambda\underline{u} + \mu\underline{u}$ $\quad\quad \forall \lambda,\mu \in \mathbb{F},\ \forall \underline{u} \in V$

✓ <u>Basic Example:</u>

($\mathbb{F}$ is fixed in advance)

$$\mathbb{F}^n = \left\{ \underline{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} : x_i \in \mathbb{F} \right\}$$

means vector space

$\mathbb{F}^n \neq \mathbb{F}_n$

means fields

① $\underline{x}+\underline{y} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} + \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix} = \begin{pmatrix} x_1+y_1 \\ x_2+y_2 \\ \vdots \\ x_n+y_n \end{pmatrix}$ addition

② $\lambda.\underline{x} = \lambda \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} \lambda x_1 \\ \lambda x_2 \\ \vdots \\ \lambda x_n \end{pmatrix}$ scalar multiplication

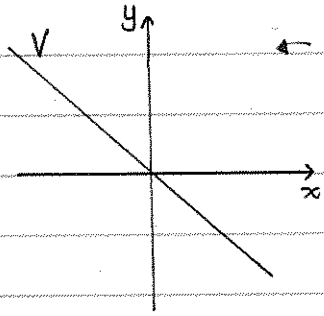
③ $\underline{0} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$

$\Rightarrow \mathbb{F}^n$ is a vector space over $\mathbb{F}$

✓ <u>EXAMPLE:</u>

$\mathbb{F} = \mathbb{R}$

$= \{\underline{x} \in \mathbb{R}^2 : x_1 + x_2 = 0\}$



← an 1-direction vector space sits in $\mathbb{F}^2$

Note: '$V \subset \mathbb{F}^2$ BUT $V \neq \mathbb{F}^2$' in this example.

means 'Every $\mathbb{F}^n$ is a vector space.

But not every vector space is an $\mathbb{F}^n$.'

V is a vector space.

Any $\underline{x} \in V$ can be written as $\underline{x} = \begin{pmatrix} x_1 \\ -x_1 \end{pmatrix}$.

If $\underline{y} = \begin{pmatrix} y_1 \\ -y_1 \end{pmatrix} \in V$, then

$$\underline{x}+\underline{y} = \begin{pmatrix} x_1+y_1 \\ -(x_1+y_1) \end{pmatrix}$$

$$\underline{0} = \begin{pmatrix} 0 \\ -0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \in V$$

$$\lambda\underline{x} = \lambda \begin{pmatrix} x_1 \\ -x_1 \end{pmatrix} = \begin{pmatrix} \lambda x_1 \\ -\lambda x_1 \end{pmatrix}$$

$\Rightarrow V \subset \mathbb{F}^2$, but $V \neq \mathbb{F}^2$ (counterexample: $\begin{pmatrix} 1 \\ 0 \end{pmatrix} \notin V$)

- $\mathbb{F}$ is a fixed field.

Let $A = \left(a_{ij} \,\middle|\, \begin{smallmatrix} 1 \le i \le m \\ 1 \le j \le n \end{smallmatrix}\right)$ $\quad m \times n$ matrix

$a_{ij} \in \mathbb{F}$

Consider the homogenous system S

$S = (A\underline{x} = \underline{0})$

$$S = \begin{cases} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = 0 \\ a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n = 0 \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n = 0 \end{cases}$$

✓ Temporary Notation:

$K_A = \{\underline{x} \in \mathbb{F}^n : A\underline{x} = \underline{0}\}$ solution set

$K_A \subset \mathbb{F}^n$

If $A \neq 0$, then $K_A \neq \mathbb{F}^n$.

✓ Prop

K_A is a vector space over $\mathbb{F}$.

Proof: - If $\underline{x} \in K_A$ and $\underline{y} \in K_A$, then $\underline{x} + \underline{y} \in K_A$

$$\left.\begin{array}{l} A\underline{x} = \underline{0} \text{ and } A\underline{y} = \underline{0} \Rightarrow A\underline{x} + A\underline{y} = \underline{0} + \underline{0} = \underline{0} \\ \qquad\qquad\qquad \Leftrightarrow A(\underline{x} + \underline{y}) = \underline{0} \end{array}\right\} \text{addition } (\checkmark)$$

- If $\underline{x} \in K_A$ and $\lambda \in \mathbb{F}$, then

$A \cdot (\lambda \underline{x}) = \lambda A \cdot \underline{x} = \lambda \cdot \underline{0} = \underline{0}$

So $\lambda \underline{x} \in K_A$ scalar multiplication (✓)

- Zero: $A \cdot \underline{0} = \underline{0}$

So $\underline{0} \in K_A$ zero (✓)

- The remaining axioms are automatically satisfied.

(satisfied already in $\mathbb{F}^n$).

Hence, K_A is a vector space over $\mathbb{F}$ ▨

4.2.1 Linear Independence

- Def.

Let V be a vector space over $\mathbb{F}$

Suppose $\underline{v}_1, \underline{v}_2, \underline{v}_3, \dots, \underline{v}_n \in V$

An expression of the form

$\boxed{\underline{v} = \lambda_1 \underline{v}_1 + \lambda_2 \underline{v}_2 + \lambda_3 \underline{v}_3 + \dots + \lambda_n \underline{v}_n}$ $\lambda_i \in \mathbb{F}$

is called a linear combination in $\underline{v}_1, \dots, \underline{v}_n$.

✓ Notice that I can always get $\underline{0}$ as a linear combination.

$\underline{0} = 0\underline{v}_1 + 0\underline{v}_2 + 0\underline{v}_3 + \dots + 0\underline{v}_n$

✓ Informal Def.

$\{\underline{v}_1, \dots, \underline{v}_n\}$ is said to be linearly independent (L.I.) when the ONLY way to get

$\underline{0}$ is with all coefficients $= 0$.

- Formal Def.

  $\{\underline{v_1}, \dots, \underline{v_n}\}$ is linearly independent (LI) when

  $$\boxed{\begin{array}{l} \lambda_1 \underline{v_1} + \lambda_2 \underline{v_2} + \dots + \lambda_n \underline{v_n} = \underline{0} \\ \Rightarrow \lambda_1 = \lambda_2 = \dots = \lambda_n = 0 \end{array}}$$

✓ EXAMPLE:

$$V = \mathbb{F}^4$$

$$\underline{e_1} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad \underline{e_2} = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \quad \underline{e_3} = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \quad \underline{e_4} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

Then $\{\underline{e_1}, \underline{e_2}, \underline{e_3}, \underline{e_4}\}$ are LI.

Proof: $\lambda_1 \underline{e_1} + \lambda_2 \underline{e_2} + \lambda_3 \underline{e_3} + \lambda_4 \underline{e_4}$ $(*)$

$$= \begin{pmatrix} \lambda_1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ \lambda_2 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ \lambda_3 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 0 \\ \lambda_4 \end{pmatrix}$$

$$= \begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \\ \lambda_4 \end{pmatrix}$$

So, if $(*) = 0$, then $\begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \\ \lambda_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$

$\Rightarrow \lambda_1 = \lambda_2 = \lambda_3 = \lambda_4 = 0$

$\Rightarrow \{\underline{e_1}, \underline{e_2}, \underline{e_3}, \underline{e_4}\}$ is LI. $\blacksquare$

- Generalisation:

$$V = \mathbb{F}^n$$

$$\underline{e_1} = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \quad \underline{e_2} = \begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \quad \dots \quad \underline{e_n} = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}$$

Then $\{\underline{e_1}, \underline{e_2}, \dots, \underline{e_n}\}$ is LI.

✓ EXAMPLE ①:

$$V = \mathbb{F}^3$$

$$\underline{\varphi_1} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \qquad \underline{\varphi_2} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \qquad \underline{\varphi_3} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

Then, $\{\underline{\varphi_1}, \underline{\varphi_2}, \underline{\varphi_3}\}$ is LI.

Proof: $\lambda_1 \underline{\varphi_1} + \lambda_2 \underline{\varphi_2} + \lambda_3 \underline{\varphi_3} = \begin{pmatrix} \lambda_1 \\ \lambda_1 \\ \lambda_1 \end{pmatrix} + \begin{pmatrix} \lambda_2 \\ \lambda_2 \\ 0 \end{pmatrix} + \begin{pmatrix} \lambda_3 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} \lambda_1 + \lambda_2 + \lambda_3 \\ \lambda_1 + \lambda_2 \\ \lambda_1 \end{pmatrix}$

So if $\lambda_1 \underline{\varphi_1} + \lambda_2 \underline{\varphi_2} + \lambda_3 \underline{\varphi_3} = \underline{0}$, then $\begin{pmatrix} \lambda_1 + \lambda_2 + \lambda_3 \\ \lambda_1 + \lambda_2 \\ \lambda_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

So $\lambda_1 = 0$ | $\lambda_1 + \lambda_2 = 0$, But $\lambda_1 = 0$, So $\lambda_2 = 0$ | $\lambda_1 + \lambda_2 + \lambda_3 = 0$, But $\lambda_1 + \lambda_2 = 0$, So $\lambda_3 = 0$

$\Rightarrow \{\underline{\psi_1}, \underline{\psi_2}, \underline{\psi_3}\}$ is LI. ▨

✓ EXAMPLE ②:

$\underline{\psi_1} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \quad \underline{\psi_2} = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} \quad \underline{\psi_3} = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \quad \underline{\psi_4} = \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix}$

Then $\{\underline{\psi_1}, \underline{\psi_2}, \underline{\psi_3}, \underline{\psi_4}\}$ is not LI.

(It is linearly dependent)

Work over $\mathbb{Q}$

Proof: Solve $\lambda_1\underline{\psi_1} + \lambda_2\underline{\psi_2} + \lambda_3\underline{\psi_3} + \lambda_4\underline{\psi_4} = \underline{0}$

$$\begin{pmatrix} \lambda_1 + \lambda_2 \qquad + 2\lambda_4 \\ \lambda_1 - \lambda_2 + \lambda_3 - \lambda_4 \\ \lambda_1 + \lambda_2 + \lambda_3 + 3\lambda_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\left(\begin{array}{cccc:c} 1 & 1 & 0 & 2 & 0 \\ 1 & -1 & 1 & -1 & 0 \\ 1 & 1 & 1 & 3 & 0 \end{array}\right)$$

Reduce to row echelon form:

$$\xrightarrow[\mathcal{E}(3,1;-1)]{\mathcal{E}(2,1;-1)} \left(\begin{array}{cccc:c} 1 & 1 & 0 & 2 & 0 \\ 0 & -2 & 1 & -3 & 0 \\ 0 & 0 & 1 & 1 & 0 \end{array}\right)$$

$$\xrightarrow{\mathcal{D}(2;\frac{1}{2})} \left(\begin{array}{cccc:c} 1 & 1 & 0 & 2 & 0 \\ 0 & -1 & 1/2 & -3/2 & 0 \\ 0 & 0 & 1 & 1 & 0 \end{array}\right)$$

$$\xrightarrow{\mathcal{E}(1,2;1)} \left(\begin{array}{cccc:c} 1 & 0 & 1/2 & 1/2 & 0 \\ 0 & -1 & 1/2 & -3/2 & 0 \\ 0 & 0 & 1 & 1 & 0 \end{array}\right)$$

$$\xrightarrow[\mathcal{E}(2,3;-\frac{1}{2})]{\mathcal{E}(1,3;-\frac{1}{2})} \left(\begin{array}{cccc:c} 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & -2 & 0 \\ 0 & 0 & 1 & 1 & 0 \end{array}\right)$$

$$\xrightarrow{\mathcal{D}(2;-1)} \left(\begin{array}{cccc:c} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 2 & 0 \\ 0 & 0 & 1 & 1 & 0 \end{array}\right)$$

$\lambda_1 \quad \lambda_2 \quad \lambda_3 \quad \lambda_4$

$\lambda_1 = 0$	Take $\lambda_1 = 0$
$\lambda_2 = -2\lambda_4$	$\lambda_2 = -2$
$\lambda_3 = -\lambda_4$	$\lambda_3 = -1$
$\lambda_4 = \lambda_4$	$\lambda_4 = 1$

Then $\quad \underline{0} - 2\underline{\psi}_2 - \underline{\psi}_3 + \underline{\psi}_4 = \underline{0}$

Since $-2 \neq 0,\ -1 \neq 0,\ 1 \neq 0,$

We have a linear combination in which not all $\lambda_i = 0$. ▨

- Def.

✓ $\{\underline{v}_1, \underline{v}_2, \dots, \underline{v}_n\}$ are linearly dependent (LD) when $\exists$ linear combination

$$\lambda_1 \underline{v}_1 + \lambda_2 \underline{v}_2 + \lambda_3 \underline{v}_3 + \dots + \lambda_n \underline{v}_n = \underline{0}$$

in which at least one $\lambda_i \neq 0$.

✓ Such an expression in which some $\lambda_i \neq 0$ is called a dependence relation.

✓ In the previous example,

$0\underline{\psi}_1 - 2\underline{\psi}_2 - \underline{\psi}_3 + \underline{\psi}_4 = \underline{0}$ is a dependence relation.

Mon. 21/11/16

MATH1201 : Algebra I

Prof. Johnson

Recap:

Def 1. linearly independent

V is a vector space over $\mathbb{F}$.

$\{\underline{v}_1, \underline{v}_2, \dots, \underline{v}_n\} \subset V$.

$\{\underline{v}_1, \underline{v}_2, \dots, \underline{v}_n\}$ is LI when

$$\sum_{i=1}^{n} \lambda_i \underline{v}_i = \underline{0} \Rightarrow \lambda_1 = \lambda_2 = \dots = \lambda_n = 0$$

4.2.2 Spanning

- Def.

V is a vector space over $\mathbb{F}$.

$\{\underline{v}_1, \underline{v}_2, \dots, \underline{v}_n\} \subset V$.

We say $\{\underline{v}_1, \underline{v}_2, \dots \underline{v}_n\}$ spans V when

$\forall \underline{w} \in V,\ \exists \lambda_1, \lambda_2, \dots, \lambda_n \in \mathbb{F}$ s.t. $\underline{w} = \sum_{i=1}^{n} \lambda_i \underline{v}_i$

✓ In English, (informal)

Every vector $\underline{w} \in V$ can be expressed as a linear combination $\{\underline{v_1}, \cdots, \underline{v_n}\}$.

✓ EXAMPLE ①:

Let $\underline{e_1} = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$ $\quad \underline{e_2} = \begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$ $\quad \cdots \quad$ $\underline{e_n} = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}$

be the standard vectors in $\mathbb{F}^n$, then $\{\underline{e_1}, \underline{e_2}, \cdots, \underline{e_n}\}$ spans $\mathbb{F}^n$.

Proof:

Let $\underline{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \in \mathbb{F}^n$

$$\underline{x} = x_1 \underline{e_1} + x_2 \underline{e_2} + \dots + x_n \underline{e_n}$$

$$= \begin{pmatrix} x_1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ x_2 \\ \vdots \\ 0 \end{pmatrix} + \dots + \begin{pmatrix} 0 \\ \vdots \\ \vdots \\ x_n \end{pmatrix}$$ ▨

✓ EXAMPLE ②.

$$V = \left\{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \in \mathbb{F}^3 ; \ x_1 + x_2 + x_3 = 0 \right\}$$

$\underline{\varphi_1} = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$ $\qquad \underline{\varphi_2} = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$

Then $\{\underline{\varphi_1}, \underline{\varphi_2}\}$ spans V.

Proof: · $x_1 + x_2 + x_3 = 0 \longrightarrow$ could substitute either x_1, x_2 or x_3

$$\begin{cases} x_1 = -x_2 - x_3 \\ x_2 = \ \ x_2 \\ x_3 = \qquad x_3 \end{cases}$$

Then $\underline{x} = x_2 \underline{\varphi_1} + x_3 \underline{\varphi_2}$

$$= \begin{pmatrix} -x_2 \\ x_2 \\ 0 \end{pmatrix} + \begin{pmatrix} -x_3 \\ 0 \\ x_3 \end{pmatrix}$$

$$= \begin{pmatrix} -x_2 - x_3 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

$\Rightarrow \{\underline{\varphi_1}, \underline{\varphi_2}\}$ spans V.

take out the coefficients of $\underline{\varphi}$

$M(\mathrm{Id})^{\mathcal{E}}_{\Phi} = [M(\mathrm{Id})^{\Phi}_{\mathcal{E}}]$

standard basis

To show $\underline{\Phi}$ spans V.

① express $\underline{\mathcal{E}}$ in terms of $\underline{\Phi}$

e.g. $\underline{e_n} = \lambda_1 \underline{\varphi_1} + \dots + \lambda_n \underline{\varphi_n}$

② $\forall \underline{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \in V$, we have

$$\underline{x} = x_1 \underline{e_1} + \dots + x_n \underline{e_n}$$

$$= x_1 (\lambda_1 \underline{\varphi_1} + \dots + \lambda_n \underline{\varphi_n}) + \dots$$

$$= \underline{\varphi_1} (\qquad) + \dots + \underline{\varphi_n} (\qquad)$$ ▨

4·2·3 Basis and Dimensions

· Def. $\qquad$ '/$\mathbb{F}$' ≡ 'over $\mathbb{F}$'

Let V be a vector space over $\mathbb{F}$

Let $\{\underline{E_1}, \underline{E_2}, \cdots, \underline{E_n}\} \subset V$.

We say that $\{\underline{E_1}, \underline{E_2}, \cdots, \underline{E_n}\}$ is a basis for V when

1) $\{\underline{E}_1, \underline{E}_2, \ldots, \underline{E}_n\}$ is LI.

and 2) $\{\underline{E}_1, \underline{E}_2, \cdots, \underline{E}_n\}$ spans V.

- Prop.

Let $\underline{e}_1 = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$, $\underline{e}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$, $\cdots$, $\underline{e}_n = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}$ be the standard vectors in $\mathbb{F}^n$. Then $\{\underline{e}_1, \underline{e}_2, \cdots, \underline{e}_n\}$ is the basis for $\mathbb{F}^n$.

Proof:

$\{\underline{e}_1, \underline{e}_2, \ldots, \underline{e}_n\}$ is LI. (proved in last lecture)

$\{\underline{e}_1, \cdots, \underline{e}_n\}$ spans $\mathbb{F}^n$. (just proved) ▨

✓ EXAMPLE(②):

$$V = \left\{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \in \mathbb{F}^3 \,\wedge\, : x_1 + x_2 + x_3 = 0 \right\}$$

$$\underline{\varphi}_1 = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} \qquad \underline{\varphi}_2 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$$

Then $\{\underline{\varphi}_1, \underline{\varphi}_2\}$ is the basis for $\mathbb{F}^3$.

Proof:

$\{\underline{\varphi}_1, \underline{\varphi}_2\}$ spans V (shown in the previous pg)

$$\lambda_1 \underline{\varphi}_1 + \lambda_2 \underline{\varphi}_2 = \underline{0}$$

$$\begin{pmatrix} -\lambda_1 \\ \lambda_1 \\ 0 \end{pmatrix} + \begin{pmatrix} -\lambda_2 \\ 0 \\ \lambda_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$\Rightarrow \lambda_1 = \lambda_2 = 0$

$\Rightarrow \{\underline{\varphi}_1, \underline{\varphi}_2\}$ is LI. ▨

- Main Theorem. The Basis Theorem

Let V be a (non-zero) vector space over $\mathbb{F}$, then

(i) V has at least one basis.

(ii) Any two basis for V have the same number of elements.

- Def.

The number of elements in a basis for V is called the dimension of V, written

$\dim(V)$ (or $\dim_{\mathbb{F}}(V)$)

so $\dim(\mathbb{F}^n) = n$ because it has a basis $\underline{e}_1, \cdots, \underline{e}_n$ with n elements.

✓ EXAMPLE ①:

$$V = \left\{ \underline{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} : x_1 + x_2 + x_3 = 0 \right\}$$

$\underline{\varphi}_1 = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$ $\underline{\varphi}_2 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$ is a basis for V.

Then $\dim(V) = 2$.

✓EXAMPLE ②:

$$W = \left\{ \underline{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \mathbb{F}^2 : x_1 + x_2 = 0 \right\}$$

Then $\dim(W) = 1$, since

$$W = \left\{ x_2 \begin{pmatrix} -1 \\ 1 \end{pmatrix} : x_2 \in \mathbb{F} \right\}$$

$\Rightarrow \left\{ \begin{pmatrix} -1 \\ 1 \end{pmatrix} \right\}$ is a basis for W.

✓EXAMPLE ③:

$A = (a_{ij})_{\substack{1 \le i \le m \\ 1 \le j \le n}}$ $m \times n$ matrix over $\mathbb{F}$

$$K_A = \{ \underline{x} \in \mathbb{F}^n : A\underline{x} = \underline{0} \}$$

Compute $\dim(K_A)$, so first find a basis for K_A.

e.g. Easy Case.

Assume $A = \begin{pmatrix} 1 & -1 & 0 & 3 & 0 & -1 \\ 0 & 0 & 1 & 2 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{pmatrix}$ 3×6

Take $\mathbb{F} = \mathbb{Q}$.

$x_1 \quad x_2 \quad x_3 \quad x_4 \quad x_5 \quad x_6$

$A\underline{x} = \underline{0}$ Write out general solution.

$$\begin{aligned} x_1 &= x_2 - 3x_4 + x_6 \\ x_2 &= x_2 \\ x_3 &= -2x_4 - x_6 \\ x_4 &= x_4 \\ x_5 &= -x_6 \\ x_6 &= x_6 \end{aligned}$$

$$\underline{x} = \begin{pmatrix} x_2 - 3x_4 + x_6 \\ x_2 \\ -2x_4 - x_6 \\ x_4 \\ -x_6 \\ x_6 \end{pmatrix}$$

Therefore, the obvious basis for K_A:

1st choice:

$x_2 = 1 \quad x_4 = 0 \quad x_6 = 0$

$$\underline{\varphi}_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

2^{nd} choice:

$x_2 = 0 \qquad x_4 = 1 \qquad x_6 = 0$

$$\underline{\varphi_2} = \begin{pmatrix} -3 \\ 0 \\ -2 \\ 1 \\ 0 \\ 0 \end{pmatrix}$$

3^{rd} choice:

$x_2 = 0 \qquad x_4 = 0 \qquad x_6 = 1$

$$\underline{\varphi_3} = \begin{pmatrix} 1 \\ 0 \\ -1 \\ 0 \\ -1 \\ 1 \end{pmatrix}$$

$\therefore \underline{x} = x_2 \underline{\varphi_1} + x_4 \underline{\varphi_2} + x_6 \underline{\varphi_3}$

$\Rightarrow \{\underline{\varphi_1}, \underline{\varphi_2}, \underline{\varphi_3}\}$ spans K_A.

We know that

$$\underline{x} = x_2 \underline{\varphi_1} + x_4 \underline{\varphi_2} + x_6 \underline{\varphi_3}$$

$$= \begin{pmatrix} ? \\ x_2 \\ ? \\ x_4 \\ ? \\ x_6 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

So, $x_2 = x_4 = x_6 = 0$

Hence, $\{\underline{\varphi_1}, \underline{\varphi_2}, \underline{\varphi_3}\}$ is LI.

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MATH1201, Algebra I

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- Basis Theorem (Coming Soon)
  1) V has at least one basis for V
  2) Any two basis for V have the same number of elements. ($\dim V$)

✓ $\{\underline{E_1}, \dots, \underline{E_n}\}$ is a basis for V when

(i) $\{\underline{E_1}, \dots, \underline{E_n}\}$ is LI.

(ii) $\{\underline{E_1}, \dots, \underline{E_n}\}$ spans V.

If $\underline{x} \in V$, we can write

$$\underline{x} = x_1\underline{E_1} + x_2\underline{E_2} + x_3\underline{E_3} + \dots + x_n\underline{E_n} \quad \text{①} \quad \text{where } x_1, \dots, x_n \in \mathbb{F}.$$

Linear independence tells us that this expression is unique.

i.e. if $\underline{x} = y_1 \underline{E_1} + y_2 \underline{E_2} + y_3 \underline{E_3} + \dots + y_n \underline{E_n}$, then $\quad \cdots ②$

$x_1 = y_1, \ x_2 = y_2, \ \dots, \ x_n = y_n$

Proof. $① - ②: \ 0 = \sum_{i=1}^{n} (x_i - y_i) \underline{E_i}$

$\Rightarrow x_i - y_i = 0 \quad \forall i$

$\Rightarrow x_i = y_i \quad \forall i$ ▨

- Prop.

If $\{E_1, \dots, E_n\}$ is a basis for V, then for each $x_i \in V$, there is a unique expression

$$\boxed{\underline{x} = x_1 \underline{E_1} + \dots + x_n \underline{E_n}}$$

Fri. 25/11/16 (cont.)

MATH1201 : Algebra I

Prof. Johnson

Chapter 5. § Linear Mapping §

5·1 Linear Mappings and Connection to Matrices

5·1·1 Def.

Let V, W be vector spaces over $\mathbb{F}$.

$T: V \to W$ is a mapping.

We say that T is linear when

(i) $T(\underline{x}+\underline{y}) = T(\underline{x}) + T(\underline{y}) \quad \forall\, \underline{x}, \underline{y} \in V$

(ii) $T(\lambda \underline{x}) = \lambda T(\underline{x})$

✓ Standard Example:

$V = \mathbb{F}^n \qquad W = \mathbb{F}^m$

$A = (a_{ji})_{\substack{1\le j\le m \\ 1\le i\le n}} \qquad a_{ji} \in \mathbb{F} \qquad$ ~~$m \times n$ matrix~~

Define $T_A : \mathbb{F}^n \to \mathbb{F}^m$ by $T_A(\underline{x}) = \underset{\substack{\uparrow \\ \text{matrix provided}}}{A}\underline{x}$ where $\underline{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$

T_A is obviously linear.

Proof: $T_A(\underline{x}+\underline{y}) = A(\underline{x}+\underline{y})$

$= A\underline{x} + A\underline{y}$

$= T_A(\underline{x}) + T_A(\underline{y})$

$T_A(\lambda \underline{x}) = A(\lambda \underline{x})$

$= \lambda A \underline{x}$

$= \lambda T_A(\underline{x})$ ▨

· <u>To what extent does an arbitrary linear map look like a standard example?</u>

✓ EXAMPLE ②: (Differentiation)

Let $V = \{a_0.1 + a_1 x + a_2 x^2 + a_3 x^2 : a_0, a_1, a_2, a_3 \in \mathbb{F}\}$

Define $D: V \to V$ to be differentiation.

Denote $a(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3$, then

differentiate $a(x)$ $\longrightarrow D(a) = a_1 + 2a_2 x + 3a_3 x^2$

Then we can prove that $D(a+b) = D(a) + D(b)$

$D(\lambda a) = \lambda D(a)$ ▨

- Take V be a vector space with basis $\{\underline{E_1}, \dots, \underline{E_n}\}$,
  W be some other vector space.

What do I need to do to specify a linear map $T: V \to W$?

✓ Ans. It is enough to specify the values $T(\underline{E_1}), \dots, T(\underline{E_n})$.

✓ Proof:

Let $\underline{w_1}, \dots, \underline{w_n} \in W$.

Consider the following mapping

$$T: V \to W$$

$T(\underline{x}) = x_1 \underline{w_1} + x_2 \underline{w_2} + \dots + x_n \underline{w_n}$ where $\underline{x} = x_1 \underline{E_1} + x_2 \underline{E_2} + x_3 \underline{E_3} + \dots + x_n \underline{E_n}$ is the unique expression for $\underline{x}$.

Claim. (1) T is linear.

(2) $T(\underline{E_i}) = \underline{w_i}$

Proof of (1):

$$\underline{x} = \sum_{i=1}^{n} x_i \underline{E_i}, \quad \underline{y} = \sum_{i=1}^{n} y_i \underline{E_i}$$

Then,

$$\underline{x} + \underline{y} = \sum_{i=1}^{n} x_i \underline{E_i} + \sum_{i=1}^{n} y_i \underline{E_i} = \sum_{i=1}^{n} (x_i + y_i) \underline{E_i}$$

Since $T(\underline{x}) = \sum_{i=1}^{n} x_i \underline{w_i}$, $T(\underline{y}) = \sum_{i=1}^{n} y_i \underline{w_i}$,

$$T(\underline{x} + \underline{y}) = \sum_{i=1}^{n} (x_i + y_i) \underline{w_i} = T(\underline{x}) + T(\underline{y})$$

$\Rightarrow T$ is additive.

$$\lambda \underline{x} = \sum_{i=1}^{n} (\lambda x_i) \underline{E_i}$$

So, $T(\lambda \underline{x}) = \sum_{i=1}^{n} (\lambda x_i) \underline{w_i} = \lambda \sum_{i=1}^{n} x_i \underline{w_i} = \lambda T(\underline{x})$

Therefore, T is linear. ▨ (1)

Proof of (2): effectively replace "j" with "i"

$\underline{E_i} = \sum_{j=1}^{n} \delta_{ij} \underline{E_j}$ is the unique expression of $\underline{E_i}$ in terms of $\underline{E_1}, \underline{E_2}, \dots, \underline{E_n}$.

$$T(\underline{E_i}) = \sum_{j=1}^{n} \delta_{ij} \underline{w_j} = \underline{w_i}$$ ▨ (2)

So we have shown the following.

✓ Th.

A linear map T is determined (uniquely) by the values $T(\underline{E_1}), T(\underline{E_2}), \ldots, T(\underline{E_n})$, where $\{\underline{E_1}, \ldots, \underline{E_n}\}$ is a basis for domain.

- Property 1:

If $T: V \to W$ is linear, then $T(0) = 0$

Proof: $0\underline{v} = 0\underline{v} + 0\underline{v}$

$T(0\underline{v}) = T(0\underline{v} + 0\underline{v}) = T(0\underline{v}) + T(0\underline{v})$ by def.

Add $-T(0\underline{v})$ to both sides:

$0 = T(0\underline{v}) + T(0\underline{v}) - T(0\underline{v})$

$0\underline{w} = T(0\underline{v}) + 0$

So, $T(0\underline{v}) = 0\underline{w} \Rightarrow T(0) = 0$ ▨

- Property 2: Matrix Multiplication

Let $T: U \to V$; $S: V \to W$ be linear, then

$S \circ T: U \to W$ is also linear

Proof: Let $\underline{x}, \underline{y} \in U$

$$
\begin{aligned}
(S\circ T)(\underline{x}+\underline{y}) &= S(T(\underline{x}+\underline{y})) \\
&= S(T(\underline{x}) + T(\underline{y})) \quad \text{since } T \text{ is linear} \\
&= S(T(\underline{x})) + S(T(\underline{y})) \quad \text{since } S \text{ is linear} \\
&= (S\circ T)(\underline{x}) + (S\circ T)(\underline{y})
\end{aligned}
$$

$$
\begin{aligned}
(S\circ T)(\lambda\underline{x}) &= S(T(\lambda\underline{x})) \\
&= S(\lambda T(\underline{x})) \\
&= \lambda S(T(\underline{x})) \\
&= \lambda (S\circ T)(\underline{x})
\end{aligned}
$$

▨

5.1.2 Associating a Matrix with a Linear Map

- Let V be a vector space with basis $\mathscr{E} = \{\underline{E_1}, \underline{E_2}, \ldots, \underline{E_n}\}$,

  W be a vector space with basis $\Phi = \{\underline{\varphi_1}, \underline{\varphi_2}, \ldots, \underline{\varphi_m}\}$.

Then $\dim(V) = n$, $\dim(W) = m$.

Let $T: V \to W$ be linear.

Let $\underline{w_1}, \underline{w_2}, \ldots, \underline{w_n} \in W$ be the vectors s.t.

$$T(\underline{E_i}) = \underline{w_i}$$

Then $\underline{w_i}$ has a unique expression in terms of $\underline{\varphi_1}, \underline{\varphi_2}, \ldots, \underline{\varphi_m}$.

$$\underline{w_i} = \sum_{j=1}^{m} a_{ji} \underline{\varphi_j}$$

$$\text{So,} \quad T(\underline{E_i}) = \sum_{j=1}^{m} a_{ji}\,\underline{\varphi_j}$$

- Def.

$$M(T)^{\Phi}_{\mathcal{E}} = (a_{ji})_{\substack{1\le j\le m\\ 1\le i\le n}} \qquad \text{where } T: V\to W \text{ is linear}$$

called "matrix of T wrt $\mathcal{E}$ on the left and Φ on the right"

$\mathcal{E} = \{\underline{E_1}, \underline{E_2}, \dots, \underline{E_m}\}$ is a basis for V

$\Phi = \{\underline{\varphi_1}, \underline{\varphi_2}, \dots, \underline{\varphi_n}\}$ is a basis for W

determined by $\boxed{T(\underline{E_i}) = \sum_{j=1}^{m} a_{ji}\,\underline{\varphi_j}}$

✓ Convention chosen so as to coincide with standard example.

i.e. Let $A = (a_{ji})_{\substack{1\le j\le m\\ 1\le i\le n}}$ $\quad a_{ji}\in\mathbb{F}$

Take $\{\underline{E_1}, \underline{E_2}, \dots, \underline{E_n}\}$ to be standard basis for $\mathbb{F}^n$.

$$\underline{E_1} = \begin{pmatrix}1\\0\\ \vdots\\ 0\end{pmatrix}, \quad \underline{E_2} = \begin{pmatrix}0\\1\\ \vdots\\ 0\end{pmatrix}, \dots, \quad \underline{E_{n-1}} = \begin{pmatrix}0\\ \vdots\\ 1\\ 0\end{pmatrix}, \quad \underline{E_n} = \begin{pmatrix}0\\ \vdots\\ 0\\ 1\end{pmatrix}$$

Take $\{\underline{\varphi_1}, \underline{\varphi_2}, \dots, \underline{\varphi_n}\}$ to be standard basis for $\mathbb{F}^m$.

$$T_A : \mathbb{F}^n \to \mathbb{F}^m$$

$$T_A(\underline{x}) = A\underline{x}$$

so that $\boxed{M(T_A)^{\Phi}_{\mathcal{E}} = A}$

Proof: Calculate $T_A(\underline{E_i})$

$$T_A(\underline{E_i}) = \begin{pmatrix} a_{11} & \cdots & a_{1i} & \cdots & a_{1n}\\ \vdots & & \vdots & & \vdots\\ a_{j1} & \cdots & a_{ji} & \cdots & a_{jn}\\ \vdots & & \vdots & & \vdots\\ a_{m1} & \cdots & a_{mi} & \cdots & a_{mn}\end{pmatrix} \cdot \begin{pmatrix}0\\ \vdots\\ 1\\ \vdots\\ 0\end{pmatrix} \leftarrow i^{th} \text{ position (row)}$$

$$= \begin{pmatrix}a_{1i}\\ a_{2i}\\ \vdots\\ a_{mi}\end{pmatrix}$$

$$= \sum_{j=1}^{m} a_{ji}\,\underline{\varphi_j}$$

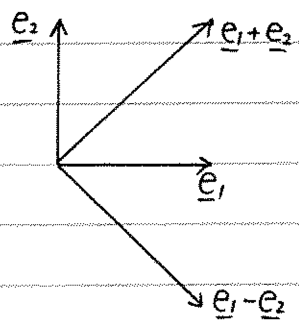
That is $M(T_A)^{\Phi}_{\mathcal{E}} = A$. ▨

✓ EXAMPLE:

Let $A = \begin{pmatrix}2 & 1\\ 1 & 2\end{pmatrix}$. $\quad 2\times 2$ matrix over $\mathbb{Q}$.

Take $\underline{e_1} = \begin{pmatrix}1\\0\end{pmatrix}$, $\underline{e_2} = \begin{pmatrix}0\\1\end{pmatrix}$

$\underline{\varphi_1} = \underline{e_1} + \underline{e_2}$, $\underline{\varphi_2} = \underline{e_1} - \underline{e_2}$



Then

$$T_A(\underline{e_1}) = \begin{pmatrix}2 & 1\\ 1 & 2\end{pmatrix}\cdot\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}2\\1\end{pmatrix} = 2\underline{e_1} + \underline{e_2}$$

$$T_A(\underline{e_2}) = \underline{e_1} + 2\underline{e_2}$$

Since T is linear,

$$T_A(\underline{\varphi_1}) = T_A(\underline{e_1}+\underline{e_2})$$
$$= 3\underline{e_1} + 3\underline{e_2}$$
$$= 3\underline{\varphi_1}$$
$$T_A(\underline{\varphi_2}) = T_A(\underline{e_1}-\underline{e_2})$$
$$= \underline{e_1} - \underline{e_2}$$
$$= \underline{\varphi_2}$$

$$M(T_A)_{\mathcal{E}}^{\mathcal{E}} = A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$$

However, $M(T_A)_{\Phi}^{\Phi} = \begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix}$ called diagonalisation

choose a new basis!

5.2 Composition Formula

- Take U to be a vector space with basis $\{\underline{E_1}, \dots, \underline{E_n}\} = \mathcal{E}$

  V $\{\underline{\varphi_1}, \dots, \underline{\varphi_m}\} = \Phi$

  W $\{\underline{\psi_1}, \dots, \underline{\psi_p}\} = \Psi$

  $T: U \to V$ is linear and $S: V \to W$ is also linear.

  Let $M(T)_{\mathcal{E}}^{\Phi} = (a_{ji})_{\substack{1 \le j \le m \\ 1 \le i \le n}}$

  $M(S)_{\Phi}^{\Psi} = (b_{kj})_{\substack{1 \le k \le p \\ 1 \le j \le m}}$, then we also have

  $M(S \circ T) = (c_{ki})_{\substack{1 \le k \le p \\ 1 \le i \le n}}$

  (because $S \circ T$ is also linear)

- Ques.

  What's the relation between

  $M(S \circ T)_{\mathcal{E}}^{\Psi}$, $M(S)_{\Phi}^{\Psi}$ and $M(T)_{\mathcal{E}}^{\Phi}$?

Ans. $M(S \circ T)_{\mathcal{E}}^{\Psi} = M(S)_{\Phi}^{\Psi} \cdot M(T)_{\mathcal{E}}^{\Phi}$

Proof: [Compute $M(S \circ T)_{\mathcal{E}}^{\Psi}$]

$$(S \circ T)(\underline{E_i}) = S(T(\underline{E_i}))$$
$$= S\Big(\sum_{j=1}^{m} a_{ji}\underline{\varphi_j}\Big) \qquad T: U \to V$$
$$= \sum_{j=1}^{m} a_{ji} S(\underline{\varphi_j}) \qquad \text{since } S \text{ is linear}$$
$$= \sum_{j=1}^{m} a_{ji}\Big(\sum_{k=1}^{p} b_{kj}\underline{\psi_k}\Big) \qquad S: V \to W$$
$$= \sum_{j=1}^{m}\sum_{k=1}^{p} a_{ji} b_{kj}\underline{\psi_k}$$

Put $\boxed{a_{ji} b_{kj} = b_{kj} a_{ji} \text{ in } \mathbb{F}}$ ← commutativity

So $(S \circ T)(\underline{E_i}) = \sum_{j=1}^{m} \sum_{k=1}^{p} b_{kj} a_{ji} \underline{\psi_k}$

change the order of summation

$$= \sum_{k=1}^{p} \sum_{j=1}^{m} b_{kj} a_{ji} \underline{\psi_k} \qquad (\#)$$

If $B = (b_{kj}) = M(S)_{\Phi}^{\Psi}$

$A = (a_{ji}) = M(T)_{\mathcal{E}}^{\Phi}$, then $(BA) = (BA)_{ki}$ when

$$(BA)_{ki} = \sum_{j=1}^{m} b_{kj} a_{ji} \qquad (*)$$

sub $(*)$ into $(\#)$:

$$(S \circ T)(\underline{E_i}) = \sum_{k=1}^{p} (BA)_{ki} \underline{\psi_k}$$

So, $BA = M(S \circ T)_{\mathcal{E}}^{\Psi}$ since $M(S \circ T)_{\mathcal{E}}^{\Psi} = (BA)_{ki}$ $\quad 1 \leq k \leq p$, $1 \leq i \leq n$

$$\Rightarrow M(S)_{\Phi}^{\Psi} \cdot M(T)_{\mathcal{E}}^{\Phi} = M(S \circ T)_{\mathcal{E}}^{\Psi} \qquad ▨$$

- Interchanging the order of summation

✓ Let $A = \begin{pmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \end{pmatrix}$

$\sum_{j=1}^{3} c_{ij}$: For $i=1$, $= c_{11} + c_{12} + c_{13}$

For $i=2$, $= c_{21} + c_{22} + c_{23}$

So, $\sum_{i=1}^{2} \sum_{j=1}^{3} c_{ij} = (c_{11} + c_{12} + c_{13}) + (c_{21} + c_{22} + c_{23})$

$\sum_{i=1}^{2} c_{ij}$: For $j=1$, $= c_{11} + c_{21}$

For $j=2$, $= c_{12} + c_{22}$

For $j=3$, $= c_{13} + c_{23}$

So, $\sum_{j=1}^{3} \sum_{i=1}^{2} c_{ij} = (c_{11} + c_{21}) + (c_{12} + c_{22}) + (c_{13} + c_{23})$

✓ Claim:

$$\sum_{i=1}^{2} \sum_{j=1}^{3} c_{ij} = \sum_{j=1}^{3} \sum_{i=1}^{2} c_{ij}$$

Proof:

$$(c_{11} + c_{12} + c_{13}) + (c_{21} + c_{22} + c_{23}) = (c_{11} + c_{21}) + (c_{12} + c_{22}) + (c_{13} + c_{23})$$

✓ If addition is associative & commutative, you can interchange order of summation.

Mon. 28/11/16

MATH1201: Algebra I

Prof. Johnson

Recap:

- $T: V \to W$ is linear.

$\mathcal{E} = \{E_1, \dots, E_n\}$ basis for V

$\Phi = \{\varphi_1, \dots, \varphi_m\}$ basis for W

$M(T)_{\mathcal{E}}^{\Phi} = (a_{ji})_{\substack{1 \le j \le m \\ 1 \le i \le n}}$ $m \times n$ matrix $\rightarrow$ "transformation matrix"

$T(\underline{E_i}) = \sum_{j=1}^{m} \underbrace{a_{ji}}_{\text{"transformation matrix"}} \underline{\varphi_j}$

EXAMPLE 1:

- $\mathbb{F}$ is a field.

$P_n = \{a_0 + a_1 x + a_2 x^2 + \dots + a_n x^n : a_i \in \mathbb{F}\}$ This means P_n is the set of polynomials of degree $\le n$ with coefficients in $\mathbb{F}$.

Then,

basis: $\{1, x, x^2, \dots, x^n\}$

$\dim(P_n) = n+1$

$D: P_3 \to P_3$ Note: derivative of sum = sum of derivative

$D \equiv$ differentiation $\equiv \frac{d}{dx}$

✓We want $\underline{M(D)_{\mathcal{E}}^{\mathcal{E}}}$ $\longleftarrow$ a matrix D with $\mathcal{E}$ on the left and $\mathcal{E}$ on the right

$D(1) = 0$	$\underline{E_1} = 1$
$D(x) = 1$	$\underline{E_2} = x$
derivative of x^2 $\rightarrow D(x^2) = 2x$	$\underline{E_3} = x^2$
$D(x^3) = 3x^2$	$\underline{E_4} = x^3$

$$D(\underline{E_i}) = \sum_{j=1}^{4} a_{ji} \underline{E_j}$$

Since $D(1) = 0\underline{E_1} + 0\underline{E_2} + 0\underline{E_3} + 0\underline{E_4}$

$(a_{11}\underline{E_1} + a_{21}\underline{E_2} + a_{31}\underline{E_3} + a_{41}\underline{E_4}) \longleftarrow 1^{st}$ column

$$\Rightarrow M(D)_{\mathcal{E}}^{\mathcal{E}} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

put the coefficients in the 1^{st} column (NOT the 1^{st} row)

$D(\underline{E_2}) = D(x) = 1\underline{E_1} + 0\underline{E_2} + 0\underline{E_3} + 0\underline{E_4} \longleftarrow 2^{nd}$ column

$D(\underline{E_3}) = D(x^2) = 0\underline{E_1} + 2\underline{E_2} + 0\underline{E_3} + 0\underline{E_4} \longleftarrow 3^{rd}$ column

$D(\underline{E_4}) = D(x^3) = 0\underline{E_1} + 0\underline{E_2} + 3\underline{E_3} + 0\underline{E_4} \longleftarrow 4^{th}$ column

✓ How about $D^2 = D.D$ $\frac{d^2}{dx^2}$ 2^{nd} derivative ?

$$M(D^2)^{\mathcal{E}}_{\mathcal{E}} = M(D.D)^{\mathcal{E}}_{\mathcal{E}}$$

$$= M(D)^{\mathcal{E}}_{\mathcal{E}} . M(D)^{\mathcal{E}}_{\mathcal{E}}$$

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix} . \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 6 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Check using the same method.

$D^2(1) = 0$

$D^2(x) = 0$

$D^2(x^2) = 2$

$D^2(x^3) = 6x$

$$\Rightarrow M(D^2)^{\mathcal{E}}_{\mathcal{E}} = \begin{pmatrix} 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 6 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

✓ EXAMPLE 2:

$$V = \{(a_1 + a_2 x + a_3 x^2)\exp(x) : a_1, a_2, a_3 \in \mathbb{Q}\}$$

Then $\begin{cases} \text{basis: } \{\underset{(\underline{E}_1)}{\exp(x)}, \underset{(\underline{E}_2)}{x\exp(x)}, \underset{(\underline{E}_3)}{x^2\exp(x)}\} \\ \dim_{\mathbb{Q}}(V) = 3 \end{cases}$

Again, take $D: V \to V$ differentiation is always linear

Then,

$D(\underline{E}_1) = \exp(x) = \underline{E}_1$

$D(\underline{E}_2) = \exp(x)(1+x) = \underline{E}_1 + \underline{E}_2$

$D(\underline{E}_3) = \exp(x)(x^2+2x) = 2\underline{E}_2 + \underline{E}_3$

$$\Rightarrow M(D)^{\mathcal{E}}_{\mathcal{E}} = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix}$$

So,

$$\boxed{M(D^2)^{\mathcal{E}}_{\mathcal{E}}} = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix} . \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix}$$

means

2nd derivative

$$= \begin{pmatrix} 1 & 2 & 2 \\ 0 & 1 & 4 \\ 0 & 0 & 1 \end{pmatrix}$$

Similarly,

$$M(D^3)^{\mathcal{E}}_{\mathcal{E}} \left(= M(D^2)^{\mathcal{E}}_{\mathcal{E}} . M(D)^{\mathcal{E}}_{\mathcal{E}}\right)$$

$$= M(D)^{\mathcal{E}}_{\mathcal{E}} . M(D^2)^{\mathcal{E}}_{\mathcal{E}}$$

$$= \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix} . \begin{pmatrix} 1 & 2 & 2 \\ 0 & 1 & 4 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 3 & 6 \\ 0 & 1 & 6 \\ 0 & 0 & 1 \end{pmatrix}$$

✓ EXAMPLE 3:

Calculate $\frac{d^3}{dx^3}\big(\underbrace{2\exp(x) - 3x\exp(x) + 5x^2\exp(x)}_{a(x)}\big)$

Soln: Represent $a(x) = \begin{pmatrix} 2 \\ -3 \\ 5 \end{pmatrix}$

Then, $D^3(a(x)) = \begin{pmatrix} 1 & 3 & 6 \\ 0 & 1 & 6 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ -3 \\ 5 \end{pmatrix}$

$= \begin{pmatrix} 23 \\ 27 \\ 5 \end{pmatrix}$

This means $23\exp(x) + 27x\exp(x) + 5x^2\exp(x)$.

- <u>Integration</u>

$D \sim \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix}$ differentiation

Find D^{-1}.

$$\left(\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right) \xrightarrow{\mathcal{E}(2,3;-2)} \left(\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & -2 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right)$$

$$\xrightarrow{\mathcal{E}(1,2;-1)} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -1 & 2 \\ 0 & 1 & 0 & 0 & 1 & -2 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right)$$

Therefore, $D^{-1} = \begin{pmatrix} 1 & -1 & 2 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{pmatrix}$ integration

✓ EXAMPLE:

$$\int \{2\exp(x) - 3x\exp(x) + 5x^2\exp(x)\}\,dx$$

Soln: $\begin{pmatrix} 1 & -1 & 2 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ -3 \\ 5 \end{pmatrix} = \begin{pmatrix} 15 \\ -13 \\ 5 \end{pmatrix}$

$$\Rightarrow \int \{2\exp(x) - 3x\exp(x) + 5x^2\exp(x)\}\,dx$$

$$\sim 15\exp(x) - 13x\exp(x) + 5x^2\exp(x) + C$$

5.3 Change of Basis Formula

- Let V be a vector space with basis

$\mathcal{E} = \{\underline{E}_1, \underline{E}_2, \ldots, \underline{E}_n\}$
$\Phi = \{\underline{\varphi}_1, \underline{\varphi}_2, \ldots, \underline{\varphi}_n\}$ } both are basis for V.

✓ <u>1^{st} step</u>:

Express $\underline{\varphi_1}$ as a linear combination in $\{\underline{E_1}, \dots, \underline{E_n}\}$.

$$\underline{\varphi_1} = \sum_{j=1}^{n} a_{j1} \underline{E_j}$$

Likewise,

$$\underline{\varphi_2} = \sum_{j=1}^{n} a_{j2} \underline{E_j}$$

$$\vdots$$

In general, $\underline{\varphi_i} = \sum_{j=1}^{n} a_{ji} \underline{E_j}$

So we get a matrix

$$(a_{ji})_{\substack{1 \le j \le n \\ 1 \le i \le n}}$$

(a_{ji}) is simply $(a_{ji}) = M(U)_{\underline{\Phi}}^{\underline{\mathcal{E}}}$

$$\underline{\varphi_i} = Id(\underline{\varphi_i}) = \sum_{j=1}^{n} a_{ji} \underline{E_j}$$

Now, express $\underline{E_i}$ in terms of $\{\underline{\varphi_1}, \dots, \underline{\varphi_n}\}$

$$\underline{E_1} = \sum_{j=1}^{n} b_{j1} \underline{E_j}$$

Then, $\underline{E_i} = \sum_{j=1}^{n} b_{ji} \underline{E_j}$

Then, $b_{ji} = M(Id)_{\underline{\mathcal{E}}}^{\underline{\Phi}}$.

- Prop.

$$M(Id)_{\underline{\mathcal{E}}}^{\underline{\Phi}} = \left[M(Id)_{\underline{\Phi}}^{\underline{\mathcal{E}}}\right]^{-1}$$

Proof: (use the composition formula)

$M(Id)_{\underline{\mathcal{E}}}^{\underline{\mathcal{E}}} = M(Id)_{\underline{\Phi}}^{\underline{\mathcal{E}}} \cdot M(Id)_{\underline{\mathcal{E}}}^{\underline{\Phi}}$ by composition formula.

Likewise,

$$M(Id)_{\underline{\Phi}}^{\underline{\Phi}} = M(Id)_{\underline{\mathcal{E}}}^{\underline{\Phi}} \cdot M(Id)_{\underline{\Phi}}^{\underline{\mathcal{E}}}$$

$$\Rightarrow M(Id)_{\underline{\Phi}}^{\underline{\mathcal{E}}} = I_n / M(Id)_{\underline{\mathcal{E}}}^{\underline{\Phi}}$$

Fri. 02/12/16

MATH1201: Algebra 1

Prof. Johnson

Recap:

✓ V is a vector space.

$\dim(V) = n$

$\underline{\mathcal{E}} = \{\underline{E_1}, \dots, \underline{E_n}\}$ ‖ both basis for V

$\underline{\Phi} = \{\underline{\varphi_1}, \dots, \underline{\varphi_n}\}$

$T: V \to V$ linear

✓ Suppose we have a linear mapping represented by matrix $M(T)_{\mathcal{E}}^{\mathcal{E}}$ of T wrt $\mathcal{E}$.

i.e. $T(\underline{E_i}) = \sum_{j=1}^{n} a_{ji} \underline{E_j}$

$M(T)_{\mathcal{E}}^{\mathcal{E}} = (a_{ji})$ is known.

✓ T has a matrix $M(T)_{\Phi}^{\Phi}$ wrt Φ.

$$T(\underline{\varphi_i}) = \sum_{j=1}^{n} b_{ji} \underline{\varphi_j}$$

$M(T)_{\Phi}^{\Phi} = (b_{ji})$

5.3 Change of Basis

- Q. What is the relationship between $M(T)_{\Phi}^{\Phi}$ and $M(T)_{\mathcal{E}}^{\mathcal{E}}$?

✓ We first consider

$$Id: V \to V$$

$$Id(\underline{E_i}) = \underline{E_i} = \sum_{j=1}^{n} \delta_{ji} \underline{E_j}$$

It effectively replaces j with i.

So,

$$\boxed{M(Id)_{\mathcal{E}}^{\mathcal{E}} = I_n} = (\delta_{ji})$$

Similarly,

$$M(Id)_{\Phi}^{\Phi} = I_n$$

✓ However, we can express Φ in terms of $\mathcal{E}$.

$$\underline{\varphi_i} = \sum_{j=1}^{n} c_{ji} \underline{E_j}$$

i.e. $\boxed{M(Id)_{\Phi}^{\mathcal{E}} = (c_{ji})}$ $1 \le j \le n$, $1 \le i \le n$ ← transformation matrix ($\mathcal{E} \to \Phi$)

Likewise, we can express $\mathcal{E}$ in terms of Φ.

$$\underline{E_i} = \sum_{j=1}^{n} f_{ji} \underline{\varphi_j}$$

i.e. $M(Id)_{\mathcal{E}}^{\Phi} = (f_{ji})$ $1 \le j \le n$, $1 \le i \le n$

- Prop.

$$M(Id)_{\Phi}^{\mathcal{E}} = \left[M(Id)_{\mathcal{E}}^{\Phi} \right]^{-1}$$

✓ Proof: $M(Id)_{\Phi}^{\Phi} = M(Id \circ Id)_{\Phi}^{\Phi}$

$= M(Id)_{\mathcal{E}}^{\Phi} \cdot M(Id)_{\Phi}^{\mathcal{E}}$ by composition formula

$I_n = M(Id)_{\mathcal{E}}^{\mathcal{E}}$

$= M(Id \circ Id)_{\mathcal{E}}^{\mathcal{E}}$

$= M(Id)_{\Phi}^{\mathcal{E}} \cdot M(Id)_{\mathcal{E}}^{\Phi}$ by composition formula

Therefore,

$M(Id)_{\Phi}^{\mathcal{E}} = \left[M(Id)_{\mathcal{E}}^{\Phi} \right]^{-1}$ by def. ∎

✓ Carallary: Change of Basis Formula

$T: V \to V$ linear

$\left.\begin{array}{l}\mathcal{E} = \{E_1, \dots, E_n\} \\ \Phi = \{\varphi_1, \dots, \varphi_n\}\end{array}\right\|$ both basis for V

$$\boxed{M(T)^{\Phi}_{\Phi} = M(Id)^{\Phi}_{\mathcal{E}} . M(T)^{\mathcal{E}}_{\mathcal{E}} . M(Id)^{\mathcal{E}}_{\Phi}}$$

Proof: Use composition formula.

$$\begin{aligned} M(T)^{\Phi}_{\Phi} &= M(Id \circ T \circ Id)^{\Phi}_{\Phi} \\ &= M(Id)^{\Phi}_{\mathcal{E}} . M(T \circ Id)^{\mathcal{E}}_{\Phi} \\ &= M(Id)^{\Phi}_{\mathcal{E}} . M(T)^{\mathcal{E}}_{\mathcal{E}} . M(Id)^{\mathcal{E}}_{\Phi} \end{aligned}$$ ▨

- √ EXAMPLE ①.

Consider $T: \mathbb{Q}^3 \to \mathbb{Q}^3$.

$$T\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 4x_1 - 2x_2 + 2x_3 \\ 3x_2 - 2x_3 \\ -x_1 + 2x_2 - x_3 \end{pmatrix}$$

T is linear

- Take standard basis,

$$\mathcal{E} = \left\{ \underset{E_1}{\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}}, \underset{E_2}{\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}}, \underset{E_3}{\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}} \right\}$$

If this is the standard basis, just take the coefficients out.

Then,

$$M(T)^{\mathcal{E}}_{\mathcal{E}} = \begin{pmatrix} 4 & -2 & 2 \\ 0 & 3 & -2 \\ -1 & +2 & -1 \end{pmatrix}$$

- Can we find a basis Φ s.t. $M(T)^{\Phi}_{\Phi}$ is 'Nice' ?

Let's try
$$\Phi = \left\{ \underset{\varphi_1}{\begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}}, \underset{\varphi_2}{\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}}, \underset{\varphi_3}{\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}} \right\}$$

$M(Id)^{\varepsilon}_{\Phi}$
From standard basis to Φ:
just take the coefficients in Φ.

So, $$M(Id)^{\varepsilon}_{\Phi} = \begin{pmatrix} 2 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 1 \end{pmatrix}$$

Therefore, $$M(Id)^{\Phi}_{\mathcal{E}} = \left[M(Id)^{\mathcal{E}}_{\Phi}\right]^{-1} = \begin{pmatrix} 1 & -1 & 1 \\ -1 & 2 & -2 \\ 1 & -2 & 3 \end{pmatrix}$$

By change of basis formula,

$$M(T)^{\Phi}_{\Phi} = M(Id)^{\Phi}_{\mathcal{E}} M(T)^{\mathcal{E}}_{\mathcal{E}} M(Id)^{\mathcal{E}}_{\Phi}$$

$$= \begin{pmatrix} 1 & -1 & 1 \\ -1 & 2 & -2 \\ 1 & -2 & 3 \end{pmatrix} . \begin{pmatrix} 4 & -2 & 2 \\ 0 & 3 & -2 \\ -1 & 2 & -1 \end{pmatrix} . \begin{pmatrix} 2 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 3 & -3 & 3 \\ -2 & 4 & -4 \\ 1 & -2 & 3 \end{pmatrix} . \begin{pmatrix} 2 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

✓ EXAMPLE ②:

Consider $T : \mathbb{Q}^3 \to \mathbb{Q}^3$

$$T = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 4x_1 - 2x_2 + x_3 \\ 2x_2 \\ 3x_3 \end{pmatrix}$$

T is linear.

- Take standard basis.

$$\mathcal{E} = \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}$$

Then, $M(T)^{\mathcal{E}}_{\mathcal{E}} = \begin{pmatrix} 4 & -2 & 1 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}$

- Take another basis

$$\Phi = \left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \right\}$$

So, $M(Id)^{\mathcal{E}}_{\Phi} = \begin{pmatrix} 1 & -1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$

Therefore, $M(Id)^{\Phi}_{\mathcal{E}} = \begin{pmatrix} 1 & -1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}^{-1} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & -1 & 1 \end{pmatrix}$

So, $M(T)^{\Phi}_{\Phi} = M(Id)^{\Phi}_{\mathcal{E}} M(T)^{\mathcal{E}}_{\mathcal{E}} M(Id)^{\mathcal{E}}_{\Phi}$

$$= \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & -1 & 1 \end{pmatrix} . \begin{pmatrix} 4 & -2 & 1 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix} . \begin{pmatrix} 1 & -1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 2 & 0 \\ 0 & 0 & 3 \\ 4 & -4 & 4 \end{pmatrix} . \begin{pmatrix} 1 & -1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 4 \end{pmatrix}$$

- Check:

$$T(\underline{\varphi_1}) = T\begin{pmatrix}1\\1\\0\end{pmatrix} = \begin{pmatrix}2\\2\\0\end{pmatrix} = 2\underline{\varphi_1}$$

$$T(\underline{\varphi_2}) = T\begin{pmatrix}-1\\0\\1\end{pmatrix} = \begin{pmatrix}-3\\0\\3\end{pmatrix} = 3\underline{\varphi_2}$$

$$T(\underline{\varphi_3}) = T\begin{pmatrix}1\\0\\0\end{pmatrix} = \begin{pmatrix}4\\0\\0\end{pmatrix} = 4\underline{\varphi_3}$$

$$\Rightarrow M(T)_{\underline{\Phi}}^{\underline{\Phi}} = \begin{pmatrix}2 & 0 & 0\\0 & 3 & 0\\0 & 0 & 4\end{pmatrix}$$

Fri. 02/12/16 (cont)

MATH1201: Algebra I

Prof. Johnson

Chapter 6. § Kernel-Rank Theorem §

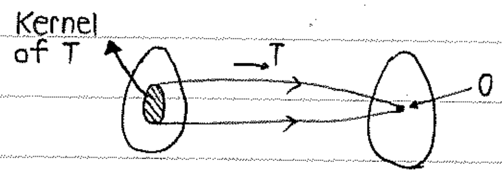
• Def.

$T: V \to W$ linear

$Ker(T) = \{\underline{v} \in V ; T(\underline{v}) = \underline{0}\}$ kernel of T

$Im(T) = \{\underline{w} \in W ; \exists \underline{v} \in V, T(\underline{v}) = \underline{w}\}$ image of T

(i.e. the set of all elements in W by T when applied to V)



• Th. Kernel-Rank Theorem

$\dim[Ker(T)] + \dim[Im(T)] = \dim(V) = \dim(\text{domain})$

• Def.

V is a vector space over $\mathbb{F}$.

$U \subset V$. We say that U is a vector subspace of V when

1) $\underline{0} \in U$

2) $\forall \underline{x}, \underline{y} \in U$, $\underline{x} + \underline{y} \in U$

3) $\forall \underline{x} \in U$, $\forall \lambda \in \mathbb{F}$, $\lambda \underline{x} \in U$

• Prop.

If $U \subset V$ is a vector subspace, then U is a vector space on its own right.

✓ Proof: – U has addition.

$+: U \times U \to U$

$(\underline{x}, \underline{y}) \to \underline{x} + \underline{y}$ by 2)

– U has scalar multiplication.

$\cdot : \mathbb{F} \times U \to U$

$(\lambda, \underline{x}) \to \lambda \underline{x}$ by 3)

– $\underline{0} \in U$ by 1)

– U has additive inverses.

$\underline{x} \in U$ then $-\underline{x} = (-1).\underline{x} \in U$ by 3)

taking $\lambda = -1$

- All remaining axioms are already satisfied because it is satisfied in V. $\boxtimes$

6·1 The Kernel is a Vector Space

• Prop.

$T: V \to W$ linear.

$Ker(T) = \{\underline{v} \in V ; T(\underline{v}) = \underline{0}\}$ is a vector subspace of V.

Proof: - $\underline{0}_V \in Ker(T)$ Ker(T) has a zero.

($since\ T(\underline{0}_V) = \underline{0}_V$)

- If $\underline{x}, \underline{y} \in Ker(T)$, then

$$T(\underline{x}+\underline{y}) = T(\underline{x}) + T(\underline{y}) \quad \text{since } T \text{ is linear}$$
$$= \underline{0} + \underline{0}$$
$$= \underline{0}$$

So, $\underline{x}+\underline{y} \in Ker(T)$ Ker(T) has addition.

- If $\underline{x} \in Ker(T)$, $\lambda \in \mathbb{F}$, then

$$T(\lambda\underline{x}) = \lambda T(\underline{x}) \quad \text{since } T \text{ is linear}$$
$$= \lambda \underline{0}$$
$$= \underline{0}$$

So, $\lambda\underline{x} \in Ker(T)$ Ker(T) has scalar multiplication. $\boxtimes$

6·2 The Image is a Vector Space

• Prop.

$T: V \to W$ linear

$Im(T) = \{\underline{w} \in W ; \exists \underline{v} \in V, T(\underline{v}) = \underline{w}\}$ is a vector subspace of W.

Proof: - $\underline{0}_W \in Im(T)$ because $T(\underline{0}_V) = \underline{0}_W$ Im(T) has a zero.

- If $\underline{w}_1, \underline{w}_2 \in Im(T)$, choose $\underline{v}_1, \underline{v}_2 \in V$ s.t. $T(\underline{v}_1) = \underline{w}_1$, $T(\underline{v}_2) = \underline{w}_2$

So,

$$T(\underline{v}_1 + \underline{v}_2) = T(\underline{v}_1) + T(\underline{v}_2)$$
$$= \underline{w}_1 + \underline{w}_2$$

So, $\underline{w}_1 + \underline{w}_2 \in Im(T)$ Im(T) has addition.

- If $\underline{w} \in Im(T)$, $\lambda \in \mathbb{F}$, choose $\underline{v} \in V$ s.t. $T(\underline{v}) = \underline{w}$.

Then,

$$T(\lambda\underline{v}) = \lambda T(\underline{v}) = \lambda \underline{w}$$

So, $\lambda \underline{w} \in Im(T)$. $Im(T)$ has scalar multiplication. ▨

6·3 The Kernel-Rank Theorem

· Standard Example:

✓ $A = (a_{ji})_{\substack{1 \le j \le m \\ 1 \le i \le n}}$ $m \times n$ matrix

$(a_{ji}) \in \mathbb{F}$

$T_A : \mathbb{F}^n \to \mathbb{F}^m$ linear

$\Leftrightarrow A\underline{x} = \underline{b}$ $\quad \underline{x} \in \mathbb{F}^n$, $\underline{b} \in \mathbb{F}^m$

$\Leftrightarrow T_A(\underline{x}) = \underline{b}$

- Suppose we have two solns, $\underline{x}$ and $\underline{y}$, then

$$\left.\begin{array}{l} A\underline{x} = \underline{b} \\ A\underline{y} = \underline{b} \end{array}\right\| \Rightarrow A(\underline{x} - \underline{y}) = \underline{b} - \underline{b} = \underline{0} \quad \text{i.e. } A\underline{z} = \underline{0}$$

- $Ker(T_A) = \{\underline{x} \in \mathbb{F}^n ; A\underline{x} = \underline{0}\}$

This is what we have previously called K_A.

i.e. In this case,

$Ker(T_A)$ = soln set to homogeneous (system) equation $A\underline{x} = \underline{0}$

- How about $Im(T_A)$?

General eqn: $A\underline{x} = \underline{b}$ $\quad \underline{x} \in \mathbb{F}^n$, $\underline{b} \in \mathbb{F}^m$

$Im(T_A) = \{\underline{b} \in \mathbb{F}^m$ s.t. $A\underline{x} = \underline{b}$ has a soln$\}$

· Th. Kernel-Rank Theorem

$T : V \to W$ linear, then

$$\boxed{\dim[Ker(T)] + \dim[Im(T)] = \dim(V)}$$

Proof: The typical case is where

$Ker(T) \neq \{\underline{0}\}$ and $Im(T) \neq \{\underline{0}\}$

Plus two special cases

(a) $Ker(T) = \{\underline{0}\}$

(b) $Im(T) = \{\underline{0}\}$

In these cases, we define dim (zero vector space) = 0.

$\{\underline{0}\}$ is the 0-dimensional vector space over $\mathbb{F}$, basis: $\emptyset$

① Typical Case:

$Ker(T) \neq \{\underline{0}\}$ so $Ker(T)$ has a basis

and $Im(T) \neq \{\underline{0}\}$ so $Im(T)$ also has a basis.

- Let $\{\underline{E}_1, \dots, \underline{E}_k\}$ be a basis for $Ker(T)$.

Let $\{\underline{\varphi}_1, \dots, \underline{\varphi}_m\}$ be a basis for $Im(T)$.

Choose $\{\underline{E}_{k+1}, \dots, \underline{E}_{k+m}\} \subset V$ so that $T(\underline{E}_{k+1}) = \underline{\varphi}_1$, …, $T(\underline{E}_{k+m}) = \underline{\varphi}_m$ $\quad \| \quad T(\underline{E}_{k+j}) = \underline{\varphi}_j$

- Claim: $\{\underline{E}_1, \underline{E}_2, \dots, \underline{E}_k, \underline{E}_{k+1}, \dots, \underline{E}_{k+m}\}$ is a basis for V.

So we need to show that

a) $\{\underline{E}_1, \dots, \underline{E}_{k+m}\}$ is LI.

and b) $\{\underline{E}_1, \dots, \underline{E}_{k+m}\}$ spans V.

- Proof of a):

Suppose $\sum_{i=1}^{k+m} \lambda_i \underline{E}_i = \underline{0}$ (*)

$\boxed{\text{Apply } T,}$ we get $\longrightarrow T\left((\lambda_1\underline{E}_1)\cup(\lambda_2\underline{E}_2)\cup\dots\cup(\lambda_{k+m}\underline{E}_{k+m})\right) = T(\underline{0})$

$\sum_{i=1}^{k+m} \lambda_i T(\underline{E}_i) = \underline{0}$ Since T is linear,

$\lambda_1 T(\underline{E}_1) + \lambda_2 T(\underline{E}_2) + \dots + \lambda_{k+m} T(\underline{E}_{k+m}) = \underline{0}$

as it is a basis for Ker(T) Since $\{\underline{E}_1, \dots, \underline{E}_k\} \subset \mathrm{Ker}(T)$,

$T(\underline{E}_1) = T(\underline{E}_2) = \dots = T(\underline{E}_k) = \underline{0}$ ⟵ by def. of Ker(T)

So $\sum_{i=1}^{m} \lambda_{k+i} T(\underline{E}_{k+i}) = \underline{0}$ (tail) ⟵ since $\sum_{i=1}^{k+m} \lambda_i T(\underline{E}_i) - \sum_{i=1}^{k} \lambda_i T(\underline{E}_i) = \underline{0} - \underline{0} = \underline{0}$

Since $T(\underline{E}_{k+i}) = \underline{\varphi}_i$,

$\sum_{i=1}^{m} \lambda_{k+i} \underline{\varphi}_i = \underline{0}$

Since $\{\underline{\varphi}_1, \underline{\varphi}_2, \dots, \underline{\varphi}_m\}$ is a basis for Im(T),

$\boxed{\{\underline{\varphi}_1, \underline{\varphi}_2, \dots, \underline{\varphi}_m\} \text{ is LI.}}$

So $\lambda_{k+1} = \lambda_{k+2} = \dots = \lambda_{k+m} = 0$.

Substitute back into (*),

$\sum_{i=1}^{k} \lambda_i \underline{E}_i = \underline{0}$

But $\boxed{\{\underline{E}_1, \underline{E}_2, \dots, \underline{E}_k\} \text{ is LI}}$ since it is a basis for Ker(T).

So $\lambda_1 = \lambda_2 = \dots = \lambda_k = 0$

Therefore,

$\sum_{i=1}^{k+m} \lambda_i \underline{E}_i = \underline{0} \Rightarrow \forall i \quad \lambda_i = 0$ ▨ LI

- Proof of b):

Let $\underline{v} \in V$.

We need to show that $\underline{v} = \sum_{i=1}^{k+m} \lambda_i \underline{E}_i$

$\boxed{\text{Apply } T}$, we get

$\underline{w} = T(\underline{v}) \in \mathrm{Im}(T)$ by def. of Im(T)

Then, since

$\boxed{\{\underline{\varphi}_1, \dots, \underline{\varphi}_m\} \text{ is a basis}}$ for Im(T),

we have

$T(\underline{v}) = \sum_{i=1}^{m} \mu_i \underline{\varphi}_i$, $\mu_i \in \mathbb{F}$

Put $\underline{v}' = \sum_{i=1}^{m} \mu_i \underline{E}_{k+i}$

Then,

$$T(\underline{v}') = \sum_{i=1}^{m} \mu_i \underline{\varphi}_i = T(\underline{v})$$

$T(\underline{v}') = \sum_{i=1}^{m} \mu_i T(\underline{E}_{k+i})$

But we have chosen $\{\underline{E}_{k+1}, \dots, \underline{E}_{k+m}\}$ so that $T(\underline{E}_{k+j}) = \underline{\varphi}_j$

So,

$$T(\underline{v} - \underline{v}') = T(\underline{v}) - T(\underline{v}') \quad \text{since } T \text{ is linear}$$
$$= \underline{0}$$

Therefore,

$$\boxed{\underline{v} - \underline{v}' \in \mathrm{Ker}(T)} \qquad \text{by def. of } \mathrm{Ker}(T)$$

So, write

$$\underline{v} - \underline{v}' = \sum_{i=1}^{k} \lambda_i \underline{E}_i$$
$$\Leftrightarrow \quad \underline{v} = \sum_{i=1}^{k} \lambda_i \underline{E}_i + \underline{v}'$$
$$= \sum_{i=1}^{k} \lambda_i \underline{E}_i + \sum_{j=1}^{m} \mu_j \underline{E}_{k+j}$$

So, $\underline{v}$ is a linear combination in $\{\underline{E}_1, \dots, \underline{E}_{k+m}\}$.

Finally, put

$$\lambda_{k+j} = \mu_j$$

Then,

$$\underline{v} = \sum_{i=1}^{k+m} \lambda_i \underline{E}_i$$

So, $\{\underline{E}_1, \dots, \underline{E}_{k+m}\}$ spans V

- Hence, $\dim(V) = k+m = \dim[\mathrm{Ker}(T)] + \dim[\mathrm{Im}(T)]$. ▨

Mon. 05/12/16

MATH1201: Algebra I

Prof. Johnson

• Th. Kernel-Rank Theorem

$T: V \to W$ linear

$$\boxed{\dim[\mathrm{Ker}(T)] + \dim[\mathrm{Im}(T)] = \dim(V)}$$

Note: $\mathrm{Rank}(T) = \dim[\mathrm{Im}(T)]$

✓ proved when both $\mathrm{Ker}(T) \neq \{\underline{0}\}$ and $\mathrm{Im}(T) \neq \{\underline{0}\}$.

✓ Two special cases:

1). $\mathrm{Ker}(T) = \{\underline{0}\}$ i.e. $\dim[\mathrm{Ker}(T)] = 0$

So it is sufficient to show that $\dim(V) = \dim[\mathrm{Im}(T)]$.

Proof:

Take basis $\{\varphi_1, \dots, \varphi_m\}$ for $Im(T)$.

Choose $\{\underline{E}_1, \dots, \underline{E}_m\} \subset V$ s.t. $T(\underline{E}_i) = \underline{\varphi}_i$

Claim: $\{\underline{E}_1, \dots, \underline{E}_m\}$ is a basis for V.

① $\{\underline{E}_1, \dots, \underline{E}_m\}$ is LI

Suppose we have a linear combination

$$\sum_{i=1}^{m} \lambda_i \underline{E}_i = \underline{0}$$

Apply T. Then we get

$$\sum_{i=1}^{m} \lambda_i T(\underline{E}_i) = \underline{0}.$$

$$T\left(\sum_{i=1}^{m} \lambda_i \underline{E}_i\right) = T(\lambda_1 \underline{E}_1) + \dots + T(\lambda_m \underline{E}_m)$$
$$= \lambda_1 T(\underline{E}_1) + \dots + \lambda_m T(\underline{E}_m)$$
$$= \sum_{i=1}^{m} \lambda_i T(\underline{E}_i)$$

Since $T(\underline{E}_i) = \underline{\varphi}_i$,

$$\sum_{i=1}^{m} \lambda_i \underline{\varphi}_i = \underline{0}$$

But because $\{\underline{\varphi}_1, \dots \underline{\varphi}_m\}$ is a basis for $Im(T)$,

$\{\underline{\varphi}_1, \dots, \underline{\varphi}_m\}$ is LI.

So, $\lambda_1 = \lambda_2 = \dots = \lambda_m = 0$

Therefore,

$$\sum_{i=1}^{m} \lambda_i \underline{E}_i = \underline{0} \Rightarrow \lambda_1 = \lambda_2 = \dots = \lambda_m = 0$$

▨ LI.

② Spanning.

We need to find $\lambda_1, \dots, \lambda_m \in \mathbb{F}$ s.t. $\underline{v} = \sum_{i=1}^{m} \lambda_i \underline{E}_i$

Apply T to $\underline{v}$:

$$T(\underline{v}) = Im(T)$$

Write $T(\underline{v}) = \sum_{i=1}^{m} \lambda_i \underline{\varphi}_i$

Put $\underline{v}' = \sum_{i=1}^{m} \lambda_i \underline{E}_i$, then

$$T(\underline{v}') = \sum_{i=1}^{m} \lambda_i T(\underline{E}_i) = \sum_{i=1}^{m} \lambda_i \underline{\varphi}_i = T(\underline{v})$$

So, we have

$$T(\underline{v} - \underline{v}') = T(\underline{v}) - T(\underline{v}') \quad \text{since } T \text{ is linear}$$
$$= \underline{0}$$

So, $\underline{v} - \underline{v}' = Ker(T) = \{\underline{0}\}$

$\Rightarrow \underline{v} - \underline{v}' = 0$

$\Leftrightarrow \underline{v} = \underline{v}'$

i.e. $\underline{v} = \sum_{i=1}^{m} \lambda_i \underline{E}_i$

▨ spanning

Therefore, $\dim(V) = \dim[Im(T)]$

▨

Special Case 2) $Im(T) = \underline{0}$

def. of $Im(T) = \{\underline{w} \in W, \exists \underline{v} \in V \text{ s.t. } T(\underline{v}) = \underline{w}\}$

So $\forall \underline{v} \in V$, $T(\underline{v}) = \underline{0}$

So $V = Ker(T)$. by def. of $Ker(T)$

Then, $\dim(V) = \dim[Ker(T)]$

i.e. $\dim(V) = \dim[Ker(T)] + \underbrace{\dim[Im(T)]}_{}$ with $\overset{0}{\shortparallel}$ ■

6.4 Connections with Injectivity, Surjectivity & Bijectivity

6.4.1 Injectivity

- Prop.

Let $T: V \to W$ be linear.

Then $Ker(T) = \{\underline{0}\} \Leftrightarrow T$ is injective.

i.e. $\dim[Ker(T)] = 0 \Leftrightarrow T$ is injective.

Injective:
$f(a_1) = f(a_2)$ $\Downarrow$ $a_1 = a_2$

✓ Proof:

($\Rightarrow$) Assume $Ker(T) = \{\underline{0}\}$

Then

$T(\underline{v}) = \underline{0} \Rightarrow \underline{v} = \underline{0}$ by def. of $Ker(T)$

Suppose $\underline{v}'$ & $\underline{v}'' \in V$ satisfy $T(\underline{v}') = T(\underline{v}'')$

Then

$T(\underline{v}' - \underline{v}'') = T(\underline{v}') - T(\underline{v}'') = \underline{0}$

So $\underline{v}' - \underline{v}'' = 0$

$\Leftrightarrow$ $\underline{v}' = \underline{v}''$

Therefore, T is injective. ■ ($\Rightarrow$)

($\Leftarrow$) Suppose $T(\underline{v}) = \underline{0}$ and $T(\underline{0}) = \underline{0}$, then

$\underline{v} = \underline{0}$

$\Rightarrow Ker(T) = \{\underline{0}\}$ ■ ($\Leftarrow$) ■

6.4.2 Surjectivity

- Prop.

Let $T: V \to W$ be linear.

Then T is surjective $\Leftrightarrow Im(T) = W$

i.e. T is surjective $\Leftrightarrow \dim[Im(T)] = \dim(W)$.

Surjective:
$\forall b \in B, \exists a \in A$
s.t. $f(a) = b$

✓ Proof: trivial

Look at def. of $Im(T)$.

✓ Corollary ①:

$T: V \to W$ linear

Then T is bijective iff $Ker(T) = \{\underline{0}\}$ & $Im(T) = W$

✓ Corollary ②:

$T: V \to W$ linear.

Then T is invertible $\Leftrightarrow$ $Ker(T) = \{\underline{0}\}$ & $Im(T) = W$

- Prop.

Let $T: V \to W$ be linear.

If T is bijective, then T^{-1} exists and $T^{-1}: W \to V$ is also linear.

✓ Proof: Let $\underline{w}_1, \underline{w}_2 \in W$, then

$$T\left(T^{-1}(\underline{w}_1 + \underline{w}_2) - T^{-1}(\underline{w}_1) - T^{-1}(\underline{w}_2)\right)$$
$$= TT^{-1}(\underline{w}_1 + \underline{w}_2) - TT^{-1}(\underline{w}_1) - TT^{-1}(\underline{w}_2) \quad \text{since } T \text{ is linear}$$
$$= \underline{w}_1 + \underline{w}_2 - \underline{w}_1 - \underline{w}_2$$
$$= \underline{0}$$

But T is injective.

$$\Rightarrow T^{-1}(\underline{w}_1 + \underline{w}_2) - T^{-1}(\underline{w}_1) - T^{-1}(\underline{w}_2) = T(\underline{0}) = \underline{0}$$
$$\Rightarrow T^{-1}(\underline{w}_1 + \underline{w}_2) = T^{-1}(\underline{w}_1) + T^{-1}(\underline{w}_2)$$

i.e. T^{-1} is additive.

Similarly,

$$T\left(T^{-1}(\lambda \underline{w}) - \lambda T^{-1}(\underline{w})\right)$$
$$= TT^{-1}(\lambda \underline{w}) - \lambda TT^{-1}(\underline{w})$$
$$= \lambda \underline{w} - \lambda \underline{w}$$
$$= \underline{0}$$

But T is injective.

$$\Rightarrow T^{-1}(\lambda \underline{w}) - \lambda T^{-1}(\underline{w}) = T(\underline{0}) = \underline{0}$$
$$\Rightarrow T^{-1}(\lambda \underline{w}) = \lambda T^{-1}(\underline{w})$$

Therefore, T^{-1} is also linear ▨

- Th.

Let A, B be $n \times n$ matrices over $\mathbb{F}$.

If $AB = I_n$, then $BA = I_n$.

✓ Proof: Let $T_A: \mathbb{F}^n \to \mathbb{F}^n$

$T_B: \mathbb{F}^n \to \mathbb{F}^n$ be linear map $T_A(\underline{x}) = A\underline{x}$

$T_B(\underline{x}) = B\underline{x}$

Therefore,

$T_{AB}(\underline{x}) = AB\underline{x}$

$= T_A T_B(\underline{x})$

Since $AB = I_n$,

$T_A T_B = Id_{\mathbb{F}^n} \leftarrow$ Identity matrix over $\mathbb{F}^n$

So T_A surjective.

If $\underline{w} \in \mathbb{F}^n$, then $T_A(T_B(\underline{w})) = \underline{w}$

Since $\dim[Ker(T_A)] + \dim[Im(T_A)] = n = \dim(\mathbb{F}^n)$,

and T_A surjective $\Rightarrow \dim[Im(T_A)] = \dim(\mathbb{F}^n)$,

$\Rightarrow \dim[Ker(T_A)] = 0$

So,

T_A is injective.

$\Rightarrow T_A$ is invertible.

$\Rightarrow T_A^{-1}$ is linear.

So we can write

$T_A^{-1} = T_C$ where $C = M(T_A^{-1})_{\mathcal{E}}^{\mathcal{E}}$ $\quad \mathcal{E}$ is the standard basis.

So $T_C T_A = Id_{\mathbb{F}^n}$

$\Rightarrow T_{CA} = Id_{\mathbb{F}^n}$

$\Rightarrow CA = Id_{\mathbb{F}^n}$

But since we have $AB = I_n$

$CA = I_n$,

$$\begin{cases} (CA)B = I_n . B = B \\ \quad \parallel \\ C(AB) = C . I_n = C \end{cases}$$

$\Rightarrow B = C$

i.e. $AB = I_n$

$BA = I_n$ $\blacksquare$

(von Neumann Property for $n \times n$ matrices)

Fri. 09/12/16

MATH1201: Algebra I

Prof. Johnson

Chapter 7. § The Basis Theorem §

- Th. Basis Theorem

  Let V be a non-zero vector space over $\mathbb{F}$, then

  (I) V has at least one basis. $\longrightarrow$ existence

  (II) Any two basis for V have the same number of elements $= \dim_{\mathbb{F}}(V)$

  $\hookrightarrow$ uniqueness

- Exchange Lemma

  V is a non-zero vector space over $\mathbb{F}$.

  $\{\underline{v}_1, \dots, \underline{v}_k\} \subset V$ is LI.

  $\{\underline{w}_1, \dots, \underline{w}_m\} \subset V$ spans.

  Then (i) $k \leq m$ and

  (ii) $\exists$ spanning set $\{\underline{w}_1', \dots, \underline{w}_m'\}$ for V s.t.

  $\underline{w}_i' = \underline{v}_i$ for $1 \leq i \leq k$

  AND $\underline{w}_i' \in \{\underline{w}_1, \dots, \underline{w}_m\}$ for $k \leq i$

7·1 Exchange Lemma

7·1·1 Baby Exchange Lemma

- Let $\underline{v} \in V$, $\underline{v} \neq \underline{0}$, and let $\{\underline{w}_1, \underline{w}_2, \dots, \underline{w}_m\}$ be a spanning set for V.

  Write $\underline{v} = \lambda_1 \underline{w}_1 + \lambda_2 \underline{w}_2 + \dots + \lambda_m \underline{w}_m$, $\lambda_j \in \mathbb{F}$. dummy variable, so subscript does not matter

  If $\lambda_i \neq 0$, then

  $\{\underline{w}_1, \dots, \underline{w}_{i-1}, \underline{v}, \underline{w}_{i+1}, \dots \underline{w}_m\}$ also spans V.

  here we swapped $\underline{w}_i$ for $\underline{v}$ as long as $\lambda_i \neq 0$

Proof: Since $\underline{v} = \sum_{j=1}^{m} \lambda_j \underline{w}_j$,

write $\underline{v} = \lambda_i \underline{w}_i + \sum_{j \neq i} \lambda_j \underline{w}_j$ where $\lambda_i \neq 0$ take out the "special" term

Then $\lambda_i \underline{w}_i = \underline{v} - \sum_{j \neq i} \lambda_j \underline{w}_j$ ($\lambda_i \neq 0$ so $\frac{1}{\lambda_i} \in \mathbb{F}$)

divide through by λ_i

$$\underline{w}_i = \left(\frac{1}{\lambda_i}\right)\underline{v} + \sum_{j \neq i} \left(-\frac{\lambda_j}{\lambda_i}\right)\underline{w}_j \quad (*)$$

We claim that $\{\underline{w}_1, \underline{w}_2, \dots, \underline{w}_{i-1}, \underline{v}, \underline{w}_{i+1}, \dots, \underline{w}_m\}$ spans V.

Let $\underline{x} \in V$, then we can write

$\underline{x} = \sum_{j=1}^{m} \xi_j \underline{w}_j$ since $\{\underline{w}_1, \dots, \underline{w}_m\}$ spans V

Then $\quad \underline{x} = \xi_i \underline{w_i} + \sum_{j\neq i} \xi_j \underline{w_j}$

substitute $\quad (*)$:

$$\underline{x} = \left(\frac{\xi_i}{\lambda_i}\right)\underline{v} + \xi_i \sum_{j\neq i}\left(-\frac{\lambda_j}{\lambda_i}\right)\underline{w_j} \quad + \sum_{j\neq i}\xi_j \underline{w_j}$$

$$= \left(\frac{\xi_i}{\lambda_i}\right)\underline{v} + \sum_{j\neq i}\left(\xi_j - \xi_i \cdot \frac{\lambda_j}{\lambda_i}\right)\underline{w_j}$$

Therefore, $\underline{x}$ is a linear combination in $\{\underline{w_1}, \underline{w_2}, \dots, \underline{w_{i-1}}, \underline{v}, \underline{w_{i+1}}, \dots, \underline{w_m}\}$.

i.e. $\{\underline{w_1}, \dots, \underline{w_{i-1}}, \underline{v}, \underline{w_{i+1}}, \dots, \underline{w_m}\}$ spans V. ▨

- We observe that

  any set which contains $\underline{0}$ cannot be LI.

  ✓ proof: Suppose $\{\underline{0}, \underline{u_2}, \dots, \underline{u_n}\}$, then we have

  $1.\underline{0} + 0.\underline{u_2} + 0.\underline{u_3} + \dots + 0\underline{u_n} = \underline{0}$ with at least one coefficient $\neq 0$.

7·1·2 Full Exchange Lemma:

- V is a non-zero vector space over $\mathbb{F}$.

  $\{\underline{v_1}, \dots, \underline{v_k}\} \subset V$ is LI,

  $\{\underline{w_1}, \dots, \underline{w_m}\} \subset V$ spans V

  Then 1) $k \leq m$ and

  2) $\exists$ spanning set $\{\underline{w_1'}, \dots, \underline{w_m'}\}$ for V s.t.

  $\underline{w_i'} = \underline{v_i}$ for $1 \leq i \leq k$

  AND $\underline{w_i'} \in \{\underline{w_1}, \dots, \underline{w_m}\}$ for $k < i$.

✓ Proof: (By induction on k)

- The case $k=1$ is precisely the Baby Exchange Lemma.

  We put $\underline{v} = \underline{v_1}$. Induction Base ✓.

  Then we get a spanning set $\{\underset{\underline{u_2}}{\underset{\downarrow}{\underline{w_1}}}, \dots, \underset{\underline{u_i}}{\underset{\downarrow}{\underline{w_{i-1}}}}, \underset{\underline{u_1}}{\underset{\downarrow}{\underline{v_1}}}, \underset{\underline{u_{i+1}}}{\underset{\downarrow}{\underline{w_{i+1}}}}, \dots, \underset{\underline{u_m}}{\underset{\downarrow}{\underline{w_m}}}\}$

  We can re-index the terms s.t.

  $\underline{u_1} = \underline{v_1}$, $\underline{u_j} = \underline{w_{j-1}}$ for $2 \leq j \leq i$ This means

  AND $\underline{u_j} = \underline{w_j}$ for $j \geq r+1$

  Then, we have

  $\{\underline{u_1}, \underline{u_2}, \dots, \underline{u_m}\}$ spans V.

  $\underline{u_1} = \underline{v_1}$
  $\underline{u_2} = \underline{w_1}$
  $\underline{u_3} = \underline{w_2}$
  $\vdots$
  $\underline{u_{i-1}} = \underline{w_{i-2}}$
  $\underline{u_i} = \underline{w_{i-1}}$

  $\underline{u_{i+1}} = \underline{w_{i+1}}$
  $\underline{u_{i+2}} = \underline{w_{i+2}}$
  $\vdots$
  $\underline{u_{m-1}} = \underline{w_{m-1}}$
  $\underline{u_m} = \underline{w_m}$

- Induction Step:

  1) Suppose true for $k-1$, so $k-1 \leq m$.

  And we have a spanning set $\{\underline{u_1}, \underline{u_2}, \dots, \underline{u_m}\}$ for V in which

  $\underline{u_i} = \underline{v_i}$ for $1 \leq i \leq k-1$ & $\underline{u_i} \in \{\underline{w_1}, \dots, \underline{w_m}\}$ for $i > k-1$.

If $k-1=m$, we would have

$\{\underline{v}_1, \dots, \underline{v}_{k-1}\}$ spans V.

Then, we could write $\underline{v}_k = \sum_{j=1}^{k-1} \mu_j \underline{v}_j$ (as a linear combination of the new spanning set)

So, $\sum_{j=1}^{k-1} \mu_j \underline{v}_j + (-1).\underline{v}_k = \underline{0}$

But this is a dependence relation in $\{\underline{v}_1, \underline{v}_2, \dots, \underline{v}_k\}$ since $-1 \neq 0$.

Contradiction, as $\{\underline{v}_1, \dots, \underline{v}_k\}$ is LI.

So we must have $k \leq m$. ▨ 1)

2) Now we can write

$$\underline{v}_k = \sum_{j=1}^{k-1} \eta_j \underline{v}_j + \sum_{j=k}^{m} \eta_j \underline{u}_j \quad (\underline{v}_k \neq \underline{0} \text{ as } \{\underline{v}_1, \dots, \underline{v}_k\} \text{ is LI})$$

We claim that

$\eta_j \neq 0$ for some j $(k \leq j)$ since $\{\underline{v}_1, \dots, \underline{v}_k\}$ is LI

Otherwise, we get

$\underline{v}_k = \sum_{j=1}^{k-1} \eta_j \underline{v}_j$ which is a dependence rel^n "relation" in $\{\underline{v}_1, \dots, \underline{v}_k\}$.

Contradiction.

So $\eta_j \neq 0$ for some j $(k \leq j \leq m)$

WLOG, we can re-index s.t. $\eta_k \neq 0$.

$$\underline{v}_k = \sum_{j=1}^{k-1} \eta_j \underline{v}_j + \eta_k \underline{u}_k + \sum_{k<j} \eta_j \underline{u}_j$$

$$(\eta_k \neq 0)$$

Use Baby Exchange Lemma again.

Swap $\underline{u}_k$ for $\underline{v}_k$.

Now we have a spanning set

$\{\underline{v}_1, \dots, \underline{v}_k, \underline{u}_{k+1}, \dots, \underline{u}_m\}$ where $\underline{u}_{k+1}, \dots, \underline{u}_m \in \{\underline{w}_1, \dots, \underline{w}_m\}$ ▨

7.2 Basis Theorem

- Corollary: Uniqueness Part of Basis Theorem

  Let V be a non-zero vector space.

  Suppose $\{\underline{E}_1, \dots, \underline{E}_m\}$ and $\{\underline{\varphi}_1, \dots, \underline{\varphi}_n\}$ are basis for V.

  Then $m = n$.

Proof: Since $\{\underline{E}_1, \dots, \underline{E}_m\}$ is LI and $\{\underline{\varphi}_1, \dots, \underline{\varphi}_n\}$ spans V,

by the exchange lemma, $m \leq n$.

Since $\{\underline{\varphi}_1, \dots, \underline{\varphi}_n\}$ is LI and $\{\underline{E}_1, \dots, \underline{E}_m\}$ spans V, by the exchange lemma, $n \le m$.

Then, $m \le n \le m$

So we have $m = n$. ▨

• Carollary: Existence of Basis

V is a non-zero vector space

Then V has at least one basis.

Proof:

– V has at least one spanning set, namely V itself. (i.e. V spans V)

$V = \{\underline{v}_1, \dots, \underline{v}_n\}$ is the maximal LI set of V.

If $\{\underline{v}_1, \dots, \underline{v}_n\} \subset \{\underline{w}_1, \dots, \underline{w}_N\}$ and $n < N$, then $\{\underline{w}_1, \dots, \underline{w}_N\}$ is not LI

– We need to search through all possible spanning sets and choose one with the smallest # of elements.

Write your chosen minimal spanning set $\{\underline{\varphi}_1, \dots, \underline{\varphi}_n\}$.

– Claim: $\{\underline{\varphi}_1, \dots, \underline{\varphi}_n\}$ is LI.

If not, choose a dependence relⁿ.

$\lambda_1 \underline{\varphi}_1 + \lambda_2 \underline{\varphi}_2 + \dots + \lambda_i \underline{\varphi}_i + \dots + \lambda_n \underline{\varphi}_n = \underline{0}$ where $\lambda_i \neq 0$ for some i

We claim that $\{\underline{\varphi}_1, \dots, \underline{\varphi}_{i-1}, \quad \underline{\varphi}_{i+1}, \dots, \underline{\varphi}_m\}$ still spans.

empty space

We can write

$$\underline{\varphi}_i = \sum_{j \neq i} \left(-\frac{\lambda_i}{\lambda_j}\right) \underline{\varphi}_j + 0 \qquad (\#)$$

So if $\underline{x} \in V$, we have

$$\underline{x} = \xi_i \underline{\varphi}_i + \sum_{j \neq i} \xi_j \underline{\varphi}_j$$

Substitute (#):

$$\underline{x} = \xi_i \sum_{j \neq i} \left(-\frac{\lambda_i}{\lambda_j}\right) \underline{\varphi}_j + \sum_{j \neq i} \xi_j \underline{\varphi}_j$$

$$= \sum_{j \neq i} \left(\xi_j - \frac{\xi_i \lambda_i}{\lambda_j}\right) \underline{\varphi}_j \qquad \text{(a linear combination)}$$

So, $\{\underline{\varphi}_1, \dots, \underline{\varphi}_{i-1}, \quad \underline{\varphi}_{i+1}, \dots, \underline{\varphi}_m\}$ still spans, and is smaller than any supposedly smallest spanning set.

Contradiction.

Hence, $\{\underline{\varphi}_1, \dots, \underline{\varphi}_n\}$ both spans and is LI.

Hence, it is a basis for V. ▨

7.3 Isomorphism 同构

- Def.

We say that vector spaces V, W over $\mathbb{F}$ are isomorphic when there exists an invertible linear map $T: V \to W$, i.e. a bijective linear map.

Then write $V \cong W$, when $\exists$ invertible linear map $T: V \to W$.

Note:

① $V \cong W \Rightarrow W \cong V$

This is because $T^{-1}: W \to V$ is also an invertible map.

② $\boxed{U \cong V \text{ and } V \cong W \Rightarrow U \cong W}$

This is because $S: U \to V$ is invertible & linear

$T: V \to W$ is invertible & linear

Then $T \circ S: U \to W$ is invertible & linear

$(T \circ S)^{-1} = S^{-1} \circ T^{-1}$ reversal of order

③ $V \cong V$ $Id: V \to V$

'$\cong$' is a 'equivalence relation'

✓ We will show

$$V \cong W \Leftrightarrow \dim(V) = \dim(W)$$

✓ Prop.

Let $\{E_1, E_2, \dots, E_n\}$ be a basis for V.

Let $T: V \to W$ be an invertible linear map, then

(i) $\{T(E_1), T(E_2), \dots, T(E_n)\}$ is a basis for W.

so (ii) $\dim(W) = \dim(V) = n$.

'ETP' $\equiv$ enough to prove'

Proof: ETP (i)

Suppose $\sum_{i=1}^{n} \lambda_i T(E_i) = \underline{0}$, then

$T(\sum_{i=1}^{n} \lambda_i E_i) = \underline{0}$ since T is linear

But since T is injective,

$\sum_{i=1}^{n} \lambda_i E_i = \underline{0}$ since $T(\underline{0}) = \underline{0}$

So, $\{E_1, E_2, \dots, E_n\}$ is LI.

$\Rightarrow \lambda_1 = \lambda_2 = \dots = \lambda_n = 0$

So $\sum_{i=1}^{n} \lambda_i T(E_i) = \underline{0} \Rightarrow \lambda_1 = \lambda_2 = \dots = \lambda_n = 0$

i.e. $\{T(E_1), T(E_2), \dots, T(E_n)\}$ is LI.

Let $\underline{w} \in W$.

Put $\underline{v} = T^{-1}(\underline{w}) \in V$.

Since $\{\underline{E}_1, \underline{E}_2, \dots, \underline{E}_n\}$ spans V,

$$\underline{v} = \sum_{i=1}^{n} \lambda_i \underline{E}_i$$

So we have $\underline{w} = T(T^{-1}(\underline{w})) = T(\underline{v}) = T\left(\sum_{i=1}^{n} \lambda_i \underline{E}_i\right)$

Therefore, $\{T(\underline{E}_1), T(\underline{E}_2), \dots, T(\underline{E}_n)\}$ spans.

So, $\{T(\underline{E}_1), T(\underline{E}_2), \dots, T(\underline{E}_n)\}$ is a basis for V. ▨

✓ Carollary:

$V \cong W \Rightarrow \dim(V) = \dim(W)$

Note: converse is also true

✓ Prop.

If $\dim(V) = \dim(W)$, then $V \cong W$.

Proof: If $\dim(V) = \dim(W) = 0$, then

$V = \{\underline{0}\}$, $W = \{\underline{0}\}$

So $0 \to 0$ is iso. "iso" ≡ "isomorphic"

Suppose $\dim(V) = \dim(W) = n \geq 1$.

Let $\mathbb{F}^n$ be a standard vector space with standard basis $\{\underline{e}_1, \dots, \underline{e}_n\}$.

Let $\{\underline{E}_1, \underline{E}_2, \dots, \underline{E}_n\}$ be a basis for V.

· $\{\underline{\varphi}_1, \underline{\varphi}_2, \dots, \underline{\varphi}_n\}$ be a basis for W, then

$n = \dim(V) = \dim(W)$

Let $S: \mathbb{F}^n \to V$ be a linear map, then

$$S\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \sum_{i=1}^{n} x_i \underline{E}_i$$

So, S is linear & bijective. (S invertible)

Note, $S(\underline{e}_i) = \underline{E}_i$

Let $T: \mathbb{F}^n \to W$ be a linear map, then

$$T\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \sum_{i=1}^{n} x_i \underline{\varphi}_i$$

So, T is linear & bijective. (T inverible)

$\mathbb{F}^n \cong V$ and so $V \cong \mathbb{F}^n$.

Since $V \cong \mathbb{F}^n$ & $\mathbb{F}^n \cong W$,

$$V \cong W$$

$T \circ S^{-1} : V \to W$ is isomorphic. ▨

Mon. 12/12/16

MATH1201 : Algebra I

Prof. Johnson

- Compute basis for $Ker(T_A)$ and $Im(T_A)$

Suppose A : $m \times n$ matrix / $\mathbb{F}$, so $T_A : \mathbb{F}^n \to \mathbb{F}^m$

$$\boxed{T_A(\underline{x}) = A\underline{x}}$$

We can compute $Ker(T_A)$ by reducing A to row echelon form.

Then $\dim[Ker(T_A)]$ = no. of uncircled variables.

And we also know that

$$\dim[Ker(T_A)] + \dim[Im(T_A)] = \dim \mathbb{F}^n \qquad \text{Kernal-Rank Theorem}$$

$$\Leftrightarrow \dim[Im(T_A)] = \dim \mathbb{F}^n - \dim[Ker(T_A)]$$

"the rank of A" $\qquad = n -$ no. of uncircled variables

$$\Leftrightarrow \dim[Im(T_A)] = \text{no. of circled variables}$$

<u>How do we compute a basis for $Im(T_A)$?</u>

$$A\underline{x} = \begin{pmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ \vdots & & & & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$$

the n^{th} column of A

$$T_A(\underline{x}) = x_1 Col_1(A) + x_2 Col_2(A) + \dots + x_n \boxed{Col_n(A)}$$

where $Col_j(A) = j^{th}$ column of A $= \begin{pmatrix} a_{1j} \\ a_{2j} \\ \vdots \\ a_{mj} \end{pmatrix}$

So we get

✓ <u>Prop</u>

$\{Col_j(A)\}$ $1 \le j \le n$ is a spanning set for $Im(T_A)$

✓ <u>How to find a maximal LI subset of $\{Col_j(A)\}$, $1 \le j \le n$?</u>

- To see how to do this, consider the special case where A is in reduced row echelon form.

$$\begin{array}{cccccc} \multicolumn{6}{c}{\begin{pmatrix} 1 & -1 & 0 & 3 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}} \\ x_1 & x_2 & x_3 & x_4 & x_5 & x_6 \end{array}$$

In this case, a basis is given the columns which lie above the circled variables.

i.e. a basis for $Im(T_A)$ is

$$\{\varphi_1, \varphi_2, \varphi_3, \varphi_4\} = \left\{ \begin{matrix} 1^{st} \text{ col.} \\ \text{of } A \end{matrix}, \ Col_3(A), \ Col_5(A), \ Col_6(A) \right\}$$

✓ In general, if A is $m \times n$ and reduced to A' (also $m \times n$), we do this by left multiplication by an invertible matrix P.

$$A' = PA$$

$$T_{A'} = T_P \circ T_A$$

Since T_P is invertible, we get

✓ Prop.

$T_P : Im(T_A) \xrightarrow{\cong} Im(T_{A'})$ is an isomorphism.

$T_P^{-1} : Im(T_{A'}) \xrightarrow{\cong} Im(T_A)$

✓ Corollary:

To obtain a basis for $Im(T_A)$,

1) reduce A to A'

2) take the columns in A which lie above circled variables in A'.

✓ EXAMPLE:

Work over $\mathbb{Q}$.

$$A = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 & 1 & -1 & 1 \\ 1 & 1 & 3 & 3 & 1 & 3 & 1 \end{pmatrix}$$

Take $T_A : \mathbb{Q}^7 \to \mathbb{Q}^3$.

Find 1) basis for $Ker(T_A)$.

2) basis for $Im(T_A)$.

Soln: $A \to \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & -2 & -2 & 0 & -2 & 0 \\ 0 & 0 & 2 & 2 & 0 & 2 & 0 \end{pmatrix}$

$$\to \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & -1 & -1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\to \begin{pmatrix} 1 & 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$x_1 \quad x_2 \quad x_3 \quad x_4 \quad x_5 \quad x_6 \quad x_7$$

So $\left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 3 \end{pmatrix} \right\}$ form a basis for $Im(T_A)$.

Rank $A = \dim[Im(T_A)] = 2$

General soln to $A\underline{x} = \underline{0}$:

$$\begin{pmatrix} -x_2 - x_5 - x_7 \\ x_2 \\ -x_4 - x_6 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \end{pmatrix}$$

Make obvious choices to get basis for $Ker(T_A)$.

$$\left\{ \begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ -1 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ -1 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\}$$

Chapter 3

7·4 Permutation (contd. from previous lectures)

7·4·1 Laplace's Formula for sign(σ)

- $\sigma : \{1, 2, \dots, n\} \to \{1, 2, \dots, n\}$ is a permutation. (bijective mapping)

✓ Proof: Define $\mathcal{L}(\sigma) = \underbrace{\prod_{1 \le i < j \le n}}_{\text{"product"}} (\sigma(j) - \sigma(i))$

This has $\binom{n}{2} = \frac{n(n-1)}{2}$ terms → quite big & not practical.

greek letter "tau"

- Let τ be an adjacent transposition.

  $\tau = (k, k+1)$

  Swap k & $k+1$ for everything else.

- Prop

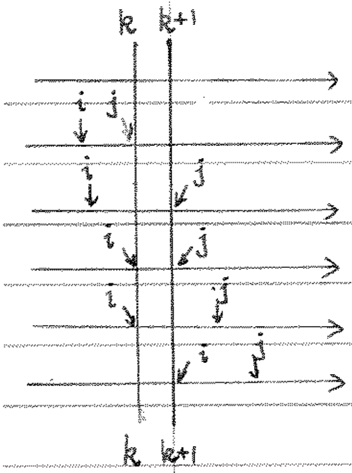
  $\mathcal{L}(\sigma\tau) = -\mathcal{L}(\sigma)$ if $\tau = (k, k+1)$ is adj trans

  adjacent transposition

- Fix τ in advance.

  $S = \{(i, j) : 1 \le i < j \le n\}$ can be decomposed into subsets.

adjacent transposition: $(i, i+1)$



$S_0 = \{(i,j) \in S : \{i,j\} \cap \{k, k+1\} = \emptyset\} \leftarrow i \neq k,\ i \neq k+1,\ j \neq k,\ j \neq k+1$

$S_1 = \{(i,k) : i < k\} \leftarrow j = k \Rightarrow (i,j) = (i,k)$

$S_2 = \{(i,k+1) : i < k\} \leftarrow j = k+1 \Rightarrow (i,j) = (i,k+1)$

$S_3 = \{(k, k+1)\} \leftarrow i = k,\ j = k+1 \Rightarrow (i,j) = (k, k+1)$

$S_4 = \{(k,j) : k+1 < j\} \leftarrow i = k \Rightarrow (i,j) = (k,j)$

$S_5 = \{(k+1, j) : k+1 < j\} \leftarrow i = k+1 \Rightarrow \ldots$

"rau"

Therefore, for any permutation ρ, we clearly have

(fixed in advance) $\boxed{\mathcal{L}(\rho) = \mathcal{L}_0(\rho)\mathcal{L}_1(\rho)\mathcal{L}_2(\rho)\mathcal{L}_3(\rho)\mathcal{L}_4(\rho)\mathcal{L}_5(\rho)}$

where $\mathcal{L}_i(\rho) = \prod_{(i,j) \in S_t} \left(\rho(j) - \rho(i)\right)$

- With $\tau = (k, k+1)$, we have

swap k & $k+1$

$\mathcal{L}_0(\sigma\tau) = \mathcal{L}_0(\sigma)$ stay the same e.g. $\mathcal{L}_1(\sigma\tau) = \{(i,k+1), i<k\}$

$= \prod_{(i,j)\in S_2} \sigma(j) - \sigma(i)$

$\mathcal{L}_1(\sigma\tau) = \mathcal{L}_2(\sigma)$ $\Big\rangle$ swap around

$\mathcal{L}_2(\sigma\tau) = \mathcal{L}_1(\sigma)$ $= \mathcal{L}_2(\sigma)$

$\mathcal{L}_3(\sigma\tau) = -\mathcal{L}_3(\sigma)$ change the sign $\mathcal{L}_3(\sigma\tau) = \{(k+1, k)\}$

$\mathcal{L}_4(\sigma\tau) = \mathcal{L}_5(\sigma)$ $\Big\rangle$ swap around $= -\{(k, k+1)\}$

$\mathcal{L}_5(\sigma\tau) = \mathcal{L}_4(\sigma)$ $= -\mathcal{L}_3(\sigma)$

✓ Corollary:

$\mathcal{L}(\sigma\tau) = -\mathcal{L}(\sigma)$ when τ is an adj trans

✓ Corollary:

If $\tau_1, \tau_2, \ldots, \tau_m$ are all adj. trans,

$\boxed{\mathcal{L}(\sigma\tau_1\tau_2\ldots\tau_m) = (-1)^m \mathcal{L}(\sigma)}$

special case: $\sigma = Id$

We cannot write Id as a product of an odd no. of adj trans.

This will give .... $\mathcal{L}(Id) = -1$

However, $\mathcal{L}(Id) = 1$.

Contradiction.

✓ Corollary. Laplace's Theorem

If $\tau_1, \ldots, \tau_m$ are all adj trans, then

$\mathcal{L}(\tau_1\tau_2\tau_3\ldots\tau_m) = (-1)^m \mathcal{L}(Id)$

- Laplace's Def. of sign(σ).

$\mathrm{sign}(\sigma) = \dfrac{\mathcal{L}(\sigma)}{\mathcal{L}(Id)}$

$$\text{sign}(\sigma) = \frac{\prod_{i<j} (\sigma(j) - \sigma(i))}{\prod_{i<j} (j-i)}$$

Fri 16/12/16

MATH1201 Algebra I

Prof. Johnson

$\sigma : \{1, \dots, n\}$ bijective

$$\text{sign}(\sigma) = \frac{\prod_{1\le i<j\le n} \sigma(j) - \sigma(i)}{\prod_{1\le i<j\le n} (j-i)} = \frac{\mathcal{L}(\sigma)}{\mathcal{L}(Id)}$$ where $\mathcal{L}(\sigma) = \prod_{1\le i<j\le n}$.

If $\sigma = \tau_1 \tau_2 \dots \tau_m$ where each τ_i is adj trans,

$$\text{sign}(\sigma) = (-1)^m.$$

7.4.2. Prop.

Any transposition is a product of an odd number of adj trans.

Proof: - (i,j) is any transposition.

Define $gap(i,j) = |j-i|$

We'll show that a transposition with $gap = k$ is a product of $(2k-1)$ adjacent transposition.

- Proof (by Induction on k).

$k=1$: nothing to prove since an odd adj trans has $gap(i,j) = 1$

This means $(i+1, j)$ with $gap = k-1$ is a product of an odd number of adj trans.

Assume true for $k-1$, and suppose $gap(i,j) = k$.

Then
$$\begin{pmatrix} i & i+1 & \dots & j \\ i+1 & i & \dots & j \\ j & i & \dots & i+1 \\ j & i+1 & \dots & i \end{pmatrix}$$
$(i, i+1)$ [adj trans]

$(i+1, j)$ according to assumption $|j-(i+1)| = |(j-i)-1| = |k-1|$ is true has $gap = k-1$

$(i, i+1)$ [adj trans]

i.e. $(i,j) = (i,i+1)(i+1,j)(i,i+1)$

By induction, since $gap(i,j) = k$, $gap(i+1,j) = k-1$

$(i+1, j)$ is a product of $2(k-1)-1 = 2k-3$ adj trans.
s.t. it's odd

So, (i,j) is a product of $\underbrace{1}_{(i,i+1)} + (2k-3) + \underbrace{1}_{(i,i+1)} = 2k-1$ adj trans. ▨

from above

✓ Corollary:

If ρ is a transposition, then

$$\text{sign}(\rho) = (-1)$$

Proof: $\rho = \tau_1\tau_2\cdots\tau_{2k-1}$ where τ_i is adjacent.

Therefore,

$$\text{sign}(\rho) = (-1)^{2k-1} = (-1).$$

$\blacksquare$

• Prop.

Let $C = (a_1, a_2, \cdots, a_n)$ be a cycle of length n.

Then C is a product of $(n-1)$ transpositions.

Proof:
$$\begin{pmatrix} a_1 & a_2 & \cdots & a_{n-1} & a_n \\ \vdots & \vdots & & \vdots & \vdots \\ a_2 & a_3 & \cdots & a_n & a_1 \end{pmatrix}$$

$$(a_1, \cdots, a_n) = \underbrace{(a_1, a_n)(a_1, a_{n-1})\cdots(a_1, a_3), (a_1, a_2)}_{(n-1)}$$

$\blacksquare$

✓ Corollary.

A cycle of length $(n-1)$ is a product of adjacent transpositions.

But every permutation is a product of (adjacent) transpositions, so

• Th.

Any permutation is a product of adjacent transpositions.

- Suppose $\sigma : \{1, 2, \cdots, n\}$ is bijective.

Then $\sigma = \tau_1\tau_2\cdots\tau_N$ where each τ_i is adj trans.

So $\text{sign}(\sigma) = (-1)^N$. ①

- Suppose $\rho : \{1, \cdots, n\}$ is also bijective.

Then $\rho = \theta_1\theta_2\cdots\theta_M$ where each θ_i is adj trans.

So $\text{sign}(\rho) = (-1)^M$ ②

- If we compose σ & ρ, we get

$$\sigma.\rho = \tau_1\tau_2\cdots\tau_N\theta_1\theta_2\cdots\theta_M \quad \text{where } \tau_i \,\&\, \theta_j \text{ adjacent}$$

Then
$$\begin{aligned} \text{sign}(\sigma\rho) &= (-1)^{N+M} \\ &= (-1)^N \cdot (-1)^M \end{aligned}$$

By ① and ②, we get

$$\text{sign}(\sigma\rho) = \text{sign}(\sigma)\text{sign}(\rho)$$

Therefore, we have proved

✓ Prop.

If $\sigma, \rho : \{1, \cdots, n\}$ bijective, then

$$\text{sign}(\sigma\rho) = \text{sign}(\sigma)\,\text{sign}(\rho)$$

✓ Corollary:

If $C = \{a_1, \dots, a_n\}$ is a cycle of length n, then

$$\text{sign}(C) = (-1)^{n-1}$$

Proof: $C = \underbrace{(a_1, a_n) \dots (a_1, a_3)(a_1, a_2)}_{(n-1)}$

and $\text{sign}(a_i, a_j) = (-1)$

$\Rightarrow \text{sign}(C) = \underbrace{(-1)(-1) \dots (-1)(-1)}_{(n-1)} = (-1)^{n-1}$ ▨

· Def.

We say a permutation σ is even when $\text{sign}(\sigma) = (+1)$

odd $\text{sign}(\sigma) = (-1)$

✓ So, a cycle of even length is odd.

a cycle of odd length is even.

✓ In general, if σ is a permutation, write

$\sigma = C_1 C_2 \dots C_M$ where C_i are disjoint cycles.

$$\boxed{\text{sign}(\sigma) = \text{sign}(C_1) \dots \text{sign}(C_M)}$$

✓ EXAMPLE:

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 \\ 5 & 6 & 9 & 13 & 8 & 12 & 14 & 11 & 2 & 15 & 1 & 3 & 4 & 10 & 7 \end{pmatrix}$$

$$= \underbrace{(1,5,8,11)}_{4}\underbrace{(2,6,12,3,9)}_{5}\underbrace{(4,13)}_{2}\underbrace{(7,14,10,15)}_{4}$$

sign $= (-1) \cdot (+1) \cdot (-1) \cdot (-1) = (-1)$

order $= 20$ (lowest common multiple of 4, 5 & 2)

✓ $\sigma_n = \{ f : \{1, \dots, n\} \to \{1, \dots, n\} \}$ where f is bijective.

Then $|\sigma_n| = n!$

$\sigma_n^{\text{even}} = \{ f \in \sigma_n \,;\, \text{sign}(f) = +1 \}$

$\sigma_n^{\text{odd}} = \{ f \in \sigma_n \,;\, \text{sign}(f) = -1 \}$

✓ Prop.

$$|\sigma_n^{\text{even}}| = |\sigma_n^{\text{odd}}| = \frac{n!}{2}$$

Proof: Let τ be your favourite transposition.

Consider $\tau_* : \sigma_n \to \sigma_n$.

$\tau_*(f) = \tau f$

τ_* is bijective $\Rightarrow \tau_*^{-1} = \tau_*$

$\tau_*\tau_*(f) = \tau_*(\tau f)$

$= \tau\tau f$

$= f$

$\Rightarrow \tau^2 = 1$

Now $\left.\begin{array}{l} \tau_* : \sigma_n^{even} \longrightarrow \sigma_n^{odd} \\ \tau_* : \sigma_n^{odd} \longrightarrow \sigma_n^{even} \end{array}\right\}$ self inverse

So $\tau_* : \sigma_n^{even} \longrightarrow \sigma_n^{odd}$ is bijective

$\text{sign}(\tau_*(f)) = \text{sign}(\tau)\text{sign}(f) = -\text{sign}(f)$

$|\sigma_n^{even}| = |\sigma_n^{odd}|$

& $|\sigma_n| = |\sigma_n^{even}| + |\sigma_n^{odd}|$

$= 2|\sigma_n^{even}|$

$|\sigma_n^{even}| = \frac{n!}{2}$

Likewise, $|\sigma_n^{odd}| = \frac{n!}{2}$ ▨

Exercise 9 (HW)

$P_5 = \{a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5,\ a_i \in \mathbb{Q}\}$

Trick: $\mathcal{D}^6 \equiv 0$ on P_5

Factorise: $\mathcal{D}^6 - 1 = (\mathcal{D}^2 - 1)(\mathcal{D}^4 + \mathcal{D}^2 + 1)$

Therefore, on P_5: $1 - \mathcal{D}^6 = (1 - \mathcal{D}^2)(\mathcal{D}^4 + \mathcal{D}^2 + 1)$

$1 = (1 - \mathcal{D}^2)(\mathcal{D}^4 + \mathcal{D}^2 + 1)$

So, $\mathcal{D}^4 + \mathcal{D}^2 + 1$ is invertible on P_5,

and its inverse is $1 - \mathcal{D}^2$

$(\mathcal{D}^4 + \mathcal{D}^2 + 1)(f) = x^3 + x + 2$

$\Rightarrow f = (1 - \mathcal{D}^2)(x^3 + x + 2)$

$= x^3 + x + 2 - 6x$

$= x^3 - 5x + 2$