MATH0005 Algebra 1 Notes

Based on the 2018 autumn lectures by Prof F E A Johnson

ALGEBRA 1

Recommended texts:

① Howard Anton : Linear Algebra

② Schawn Outline series : Linear Algebra (S. Lipschutz)

③ Serge Lang : Algebra .

ALGEBRA 1

LINEAR EQUATIONS: $\underbrace{\text{variables}}$ have exponent $= 1$.
$$\left\{\begin{array}{l} x+y=1 \\ x+y+z=1 \\ w \equiv x+y+z=1 \end{array}\right.$$

If we run out of symbols we:

→ $x_1+x_2+x_3+x_4 \dots = 1$ → typical linear equation with n-variables.

$$a_1x_1+a_2x_2+\dots+a_nx_n=b.$$

- **SYSTEM OF EQUATIONS**:

1st equation: $a_1x_1+\dots+a_nx_n=b$
2nd equation: $c_1x_1+\dots+c_nx_n=d$ } If we run out of symbols:

↓

we double index de coeficients

TYPICAL LINEAR SYSTEM OF m EQUATIONS WITH N VARIABLES

$$\mathcal{S} \to \left\{\begin{array}{l} a_{11}x_1+\dots+a_{1n}x_n=b_1 \\ a_{21}x_1+\dots+a_{2n}x_n=b_2 \\ \vdots \\ a_{m1}x_1+\dots+a_{mn}x_n=b_m \end{array}\right.$$

MATRIX OF COEFFICIENTS.

$$\begin{pmatrix} a_{11} & \dots & a_n \\ \vdots & & \\ a_{m1} & \dots & a_{mn} \end{pmatrix}$$

Convention: (a_{ij}) $\left.\begin{array}{l} 1\le i\le m \\ 1\le j\le n \end{array}\right\}$ 1st index represents equation number. 2nd index indicates which variable it is coeficient of

$$\mathcal{S}\left\{\begin{array}{l} x_1+2x_2=-1 \\ 2x_1+3x_2=-1 \end{array}\right\}$$ has a unique solution $\left.\begin{array}{l} x_1=1 \\ x_2=-1 \end{array}\right\}$ written $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -1 \\ -1 \end{pmatrix}$.

$a_1x_1+\dots+a_nx_n=b$ → we symplify it by: $\underset{\sim}{a} = (a_1, \dots, a_n)$ Row vector

$\underset{\sim}{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$ Column vector

Define $\boxed{\underset{\sim}{a}\cdot\underset{\sim}{x} = a_1x_1+\dots+a_nx_n}$ product of vectors. $\boxed{\underset{\sim}{a}\cdot\underset{\sim}{x}=b}$ equation

MATRIX MULTIPLICATION

$A = (a_{ij})$ $\quad 1 \le i \le m$, $1 \le j \le n$. $\left\{ \begin{array}{l} m = \text{rows} \\ n = \text{columns} \end{array} \right.$.

$m \times n$ matrix

$B = (b_{jk})$ $\quad 1 \le j \le n$, $1 \le k \le p$

$n \times p$ matrix

Define AB to be the $m \times p$ matrix .

As follows $AB = (c_{ik})$ $\quad 1 \le i \le m$, $1 \le k \le p$.

$m \times p$ matrix .

Where $c_{ik} = (i^{th}\text{ row } A) \cdot (k^{th}\text{ column of } B)$.

example .

$$A = \begin{pmatrix} 1 & 2 & -1 \\ 0 & -2 & 3 \end{pmatrix} \qquad B = \begin{pmatrix} 1 & 0 \\ 2 & -3 \\ 4 & 5 \end{pmatrix}$$

2×3 $\qquad$ 3×2

$$A \times B = 2 \times 3 \text{ matrix} = \left(\begin{array}{c|c} C_{11} & C_{12} \\ \hline C_{21} & C_{22} \end{array} \right)$$

$C_{11} = (1^{st}\text{ row } A) \cdot (1^{st}\text{ column } B) = (1, 2, -1) \cdot \begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix} = 1 + \cancel{4} - \cancel{4} = 1$.

$C_{12} = (1^{st}\text{ row } A) \cdot (2^{nd}\text{ column } B) = (1, 2, -1) \cdot \begin{pmatrix} 0 \\ -3 \\ 5 \end{pmatrix} = -6 - 5 = -11$.

$C_{21} = (2^{nd}\text{ row } A) \cdot (1^{st}\text{ column } B) = (0, -2, 3) \cdot \begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix} = 8$.

$C_{22} = 0 + 6 + 15 = 21$.

$$AB = \begin{pmatrix} 1 & -11 \\ 8 & 21 \end{pmatrix} \qquad \underset{3 \times 3}{BA} = \begin{pmatrix} 1 & 0 \\ 2 & -3 \\ 4 & 5 \end{pmatrix} \cdot \begin{pmatrix} 1 & 2 & -1 \\ 0 & -2 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 2 & -1 \\ 2 & 10 & -11 \\ 4 & -2 & 11 \end{pmatrix}$$

$AB \ne BA$.

Formal definition :

$A = (a_{ij})$ $\quad 1 \le i \le m$, $1 \le j \le n$.

$B = (b_{jk})$ $\quad 1 \le j \le n$, $1 \le k \le p$.

$A_{m \times n}$ $\qquad$ $B_{n \times p}$.

In this case

$AB = AB_{ik}$ $\quad 1 \le i \le m$, $1 \le k \le p$.

$$\boxed{AB_{ik} = \sum^{n} a_{ij} b_{jk} = a_{11} b_{1k} + a_{12} b_{2k} + \ldots + a_{1n} b_{nk}}$$

Some obvious facts:

0) If A is $m\times n$.
B is $n\times p$.

AB defined $\Longleftrightarrow n = n'$
Is if and only if
nº of columns A = nº rows B.

1) If A is $m\times n$.
B is $n\times p$.

AB is defined but BA is defined only when $m=p$.

So if A is 2×3 B is 3×4, AB defined but BA not defined.

2) If A is $m\times n$.
B is $n\times m$.

then both products are defined

$$\left.\begin{array}{l} AB \text{ is } m\times m. \\ BA \text{ is } n\times n. \end{array}\right\} \text{They may not be the same size.}$$

SAME size when $m=n$ $\underbrace{\text{(square)}}_{\text{matriz cuadrada}}$.

3) Suppose A is $m\times m$.
B is $m\times m$.

both products defined but normally $AB \neq BA$

Example: $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$ $\qquad B = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$.

$$AB = \begin{pmatrix} -2 & 1 \\ -4 & 3 \end{pmatrix} \qquad BA = \begin{pmatrix} 3 & 4 \\ -1 & -2 \end{pmatrix}.$$

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$A_{m\times n} = (A_{ij})$ $\quad 1 \le i \le m$, $1 \le j \le n$

$B_{n\times p} = (B_{jk})$ $\quad 1 \le j \le n$, $1 \le k \le p$

The product AB is defined by $AB = ((AB)_{ik})$ $\quad 1 \le i \le m$, $1 \le k \le p$.

$$\boxed{AB_{ik} = \sum_{j=1}^{n} A_{ij}\, B_{jk}.}$$

$$A_{m\times \underline{n}}\; B_{\underline{n}\times p} = C_{m\times p}.$$

$$\mathcal{S} = \begin{cases} a_{11}x_1 + \dots + a_{1n}x_n = b_1 \\ a_{21}x_1 + \dots + a_{2n}x_n = b_2 \\ \vdots \qquad\qquad\qquad \vdots \\ a_{m1}x_1 + \dots + a_{mn}x_n = b_m \end{cases}$$

System of m equations and n variables: SINGLE MATRIX EQUATION.

$\underset{\sim}{Ax} = \underset{\sim}{b}$, where:

$A = (a_{ij})$ $\begin{matrix} 1 \leq i \leq m \\ 1 \leq j \leq n \end{matrix}$ matrix of coeficients

$\underset{\sim}{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$ $n\times 1$ matrix, column vector n.

$\underset{\sim}{b} = \begin{pmatrix} b_1 \\ \vdots \\ b_m \end{pmatrix}$ $m\times 1$ matrix.

<u>Example</u>:

$$\begin{cases} 2x_1 + 5x_2 = -1 \\ x_1 + 3x_2 = -1 \end{cases} \qquad \underset{A}{\begin{pmatrix} 2 & 5 \\ 1 & 3 \end{pmatrix}} \cdot \underset{\underset{\sim}{x}}{\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}} = \underset{\underset{\sim}{b}}{\begin{pmatrix} -1 \\ -1 \end{pmatrix}}$$

- Arithmetic of matrices

1) ADDITION: $A = (a_{ij})$ $\begin{matrix} 1 \leq i \leq m \\ 1 \leq j \leq n \end{matrix}$ $B = (b_{ij})$ $\begin{matrix} 1 \leq i \leq m \\ 1 \leq j \leq n \end{matrix}$.

$$A + B = (a_{ij} + b_{ij}) \quad \begin{matrix} 1 \leq i \leq m \\ 1 \leq j \leq n \end{matrix}$$

i.e: add corresponding entries.

$\boxed{eg}$

$$\begin{pmatrix} 1 & -1 & 1 \\ 0 & 2 & 3 \end{pmatrix} + \begin{pmatrix} 2 & 1 & 3 \\ 1 & 2 & 3 \end{pmatrix} = \begin{pmatrix} 8 & 0 & 4 \\ 1 & 4 & 6 \end{pmatrix}$$

They must be the same size

* <u>Properties</u>:

- $A + B = B + A$ (commutativity)
- $(A+B)+C = A+(B+C)$ (associativity)

* [For each size $m\times n$ there is a zero matrix

$\underset{\sim}{0}_{ij} = 0$ for all ij.

$\boxed{eg}$ 2×3 $\underset{\sim}{0} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$.]

- $A + \underset{\sim}{0} = \underset{\sim}{0} + A = A$ (additive identity)

- If A is $m\times n$
  B is $n\times p$.
  C is $p\times q$

Then AB is defined BC is defined, $(AB)C$ is defined and $A(BC)$ is defined (multiplicative associativity)

It is true that: $(AB)C = A(BC)$. Proof eventually.

- Distributive laws

A is $m \times n$ | $A \cdot (B+C) = AB + AC$

B, C $n \times p$ | $(B+C)D = BD + CD$.

D $p \times q$

- In general $AB \neq BA$

Identity matrix I_n (each $n \geq 1$).

$$I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} . \quad I_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad I_4 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Kronecker delta:

$$\delta_{ij} = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases} \qquad 1 \leq i \leq n \quad 1 \leq j \leq n$$

$$\delta_{ij} = \begin{pmatrix} 1 & \\ 0 & \end{pmatrix} \qquad 1_{11} \ i = j \qquad i = 2 \ \ j = 1 \ \ 0_{21} \ i \neq j$$

Definition: $I_n = (\delta_{ij})$ $1 \leq i \leq n$, $1 \leq j \leq n$ $\rightarrow I_n = \begin{pmatrix} 1_{11} & 0_{12} \\ 0_{21} & 1_{22} \end{pmatrix}$ $i = j$, $i \neq j$ los n^{os} con subíndices iguales $i = j$.

I_n is identity for matrix multiplication:

* Property: $A = (a_{ij})_{1 \leq i \leq m,\ 1 \leq j \leq n}$ $m \times n$

Then $I_m A = A$
and $A I_n = A$

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} A \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

1ª columna

2 columna

Proof: $(I_m A)_{ik} = \sum_{j=1}^{m} \delta_{ij} a_{jk} = \underbrace{\delta_{ii}}_{\text{by def}=1} \cdot a_{ik} + \sum_{j \neq i} \delta_{ij} a_{jk} =$ $j=1$ $j=i$

↳ Cuando los subíndices son iguales entonces va a ser un 1.

(most δ_{ij} are zero)

$= 1 \cdot a_{ik} + 0 \cdot \delta_{ij} = a_{ik}$. $(I_m A)_{ik} = a_{ik}$; $I_m A = A$ ....

Def^n. Let $A = (a_{ij})$ $1 \leq i \leq n$, $1 \leq j \leq n$ $n \times n$.

Say that A is invertible ($A = A^{-1}$) when there exists B (also $n \times n$) s.t

$$\boxed{\underset{A^{-1}}{AB} = I_n = \underset{A^{-1}}{BA}}$$

In ordinary arithmetic if $a \neq 0$ there exists $a^{-1} = \frac{1}{a}$; $aa^{-1} = 1 = a^{-1}a$. FALSE FOR MATRICES

⇓

No sirve solo con que $A \neq 0$, como vemos en el ejemplo, aunque $A \neq 0$ puede no existir inversa.

Example: 1st) $A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \neq \underset{\sim}{0}$.

le

2nd) → $A \cdot \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} c & d \\ 0 & 0 \end{pmatrix} \neq I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

A is not invertible

2x2 ADJUGATE

Adjugate

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} = \begin{pmatrix} ad-bc & 0 \\ 0 & ad-bc \end{pmatrix}$$

$$A^{-1} = \frac{1}{|A|} (\text{adj} A)^{-1}$$

$$\downarrow$$

$$\begin{pmatrix} M_C \cdot (-1)^{i+j} \end{pmatrix}$$

$$A^{-1} = \frac{1}{|A| \cdot (\text{adj} A)}$$

$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is invertible $\Longleftrightarrow$ $ad - bc \neq 0$

determinant

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \begin{pmatrix} \frac{d}{ad-bc} & \frac{-b}{ad-bc} \\ \frac{-c}{ad-bc} & \frac{a}{ad-bc} \end{pmatrix}$$

Dummy indices: el nombre, indices no importa

$$A = (a_{ij})_{\substack{1 \leq i \leq m \\ 1 \leq j \leq n}} = (a_{kj})_{\substack{1 \leq k \leq m \\ 1 \leq j \leq n}}$$

Indices are dummies.

$$\int_0^1 x\,dx = \frac{1}{2} = \int_0^1 y\,dy =$$

Example.

$$\begin{cases} 2x_1 + 5x_2 = -1 \\ x_1 + 3x_2 = -1 \end{cases}$$

→ NEVER SUBSTITUTE.

Subtract
restas

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1st) Add (−2) Eq(2) to Eq(1).

$$\begin{cases} 0 + (-x_2) = 1 \\ x_1 + 3x_2 = -1 \end{cases}$$

2nd) Add 3 Eq(1) to Eq(2).

$$\begin{cases} -x_2 = 1 \\ x_1 = 2 \end{cases}$$

3rd) swap Eq(1) and Eq(2).

$$\begin{cases} x_1 = 2 \\ -x_2 = 1 \end{cases}$$

4th) Multiply Eq(2) by (−1).

$$\begin{cases} x_1 = 2 \\ x_2 = -1 \end{cases}$$

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$$

- Three types of operations

① Add $\boxed{\lambda \text{ Eq}(j) \text{ to Eq}(i) = \mathcal{E}(i,j,\lambda)}$ ← NAME OF OPERATION

② Multiply equation(i) by $\lambda \neq 0$ = $\mathcal{M}(i,\lambda)$ ← NAME OF OPERATION

③ Interchange order of Eq(i) and Eq(j) = $\mathcal{P}(i,j)$ ← NAME OF OPERATION

In matrix terms: $\mathcal{E}(i,j;\lambda)$; Add λ Row(j) to Row(i).

$\mathcal{M}(i,\lambda)$; Multiply Row(i) by $\lambda \neq 0$

$\mathcal{P}(i,j)$: Interchange Row(i) under Row(j).

1st operation: $\mathcal{E}(i,j;\lambda)$

$$A = \begin{pmatrix} a & b \\ c & d \\ e & f \\ g & h \end{pmatrix}$$

Apply $\mathcal{E}(4,1;\lambda)$ to A = Add λ Row(1) to Row(4).

$$\Big\downarrow \ \mathcal{E}(4,1;\lambda)$$

$\boxed{A}$

$$\begin{pmatrix} a & b \\ c & d \\ e & f \\ g+\lambda a & h+\lambda b \end{pmatrix}$$

Now perform this operation by matrix multiplication

$\boxed{B}$

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \lambda & 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \\ e & f \\ g & h \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \\ e & f \\ g+\lambda a & h+\lambda b \end{pmatrix}$$

The matrix that performs the operation $\mathcal{E}(i,j;\lambda)$ is called $E(i,j;\lambda)$,

so $E(4,1,\lambda) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \lambda & 0 & 0 & 1 \end{pmatrix}$

We'll see that to perform $\mathcal{E}(i,j;\lambda)$ we form a matrix with 1's on diagonal 0's everywhere except (i,j) place, λ in position (i,j).

Basic matrices:

Fix $n \geq 2$

Let i,j be indices $1 \leq \begin{matrix} i \\ j \end{matrix} \leq n$.

Define $\epsilon(i,j)$ to be $n \times n$ matrix with $\begin{cases} 1 \text{ in position } (i,j) \\ 0 \text{ everywhere else.} \end{cases}$

↳ basic matrix

Example n=3

$$\underset{\epsilon(1,1)}{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}} \quad \underset{\epsilon(1,2)}{\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}} \quad \underset{\epsilon(1,3)}{\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}} \quad \underset{\epsilon(2,1)}{\begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}} \quad \underset{\epsilon(2,2)}{\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}}$$

$$\underset{\epsilon(2,3)}{\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}} \quad \underset{\epsilon(3,1)}{\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}} \quad \underset{\epsilon(3,2)}{\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}} \quad \underset{\epsilon(3,3)}{\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}}$$

$n=2$

$$\underset{E(1,1)}{\begin{pmatrix}1&0\\0&0\end{pmatrix}} \quad \underset{E(1,2)}{\begin{pmatrix}0&1\\0&0\end{pmatrix}} \quad \underset{E(2,1)}{\begin{pmatrix}0&0\\1&0\end{pmatrix}} \quad \underset{E(2,2)}{\begin{pmatrix}0&0\\0&1\end{pmatrix}}.$$

<u>Question</u>: What is the effect of multiplying A on left by $E(i,j)$?

$$\underset{E(2,3)}{\begin{pmatrix}0&0&0\\0&0&1\\0&0&0\end{pmatrix}}\underset{A}{\begin{pmatrix}a&b\\c&d\\e&f\end{pmatrix}}=\begin{pmatrix}0&0\\e&f\\0&0\end{pmatrix}$$

<u>Answer</u>: Take j^A Row (A), put into new i^A Row. Everything else $=0$.

To do this formally we need a definition:

Formal <u>defn</u>: $\underbrace{E(i,j)_{rs}}_{\text{indices}} = \underbrace{\delta_{ir}\,\delta_{js}}_{\text{Kronecker deltas}}$.

$E(2,3)$ should have one nonzero entry (actually 1) in position $(2,3)$. Everything else $=0$.

Check the definition: $E(2,3)_{rs} = \delta_{2r}\cdot\delta_{3s}$

$(r,s)=(1,1)$ $\quad \delta_{21}\delta_{31}=0$

$(r,s)=(1,2)$ $\quad \delta_{21}\delta_{32}=0$

$(r,s)=(1,3)$ $\quad \delta_{21}\delta_{33}=0\times1=0$.

$(r,s)=(2,1)$ $\quad \delta_{22}\delta_{31}=1\times0=0$.

$(r,s)=(2,2)$ $\quad \delta_{22}\delta_{32}=1\times0=0$.

$(r,s)=(2,3)$ $\quad \delta_{22}\delta_{33}=1\times1=1$.

but $E(2,3)_{23}=\delta_{22}\delta_{33}=1$, so I get 1 in position $(2,3)$.

$E(2,3)_{31}=\delta_{23}\delta_{31}=0$.

$E(2,3)_{32}=\delta_{23}\delta_{32}=0$.

$E(2,3)_{33}=\delta_{23}\delta_{33}=0\cdot1=0$.

$$\boxed{\text{Defn } E(i,j)_{rs}=\delta_{ir}\cdot\delta_{js}}$$

What is $E(i,j)A$? → Demostración multiplicación

Use defn of Matrix product:

$$[\underbrace{E(i,j)}_{E(i,j)_{rs}}\underbrace{A}_{A_{st}}]_{rt}=\sum_{s=1}^{n}E(i,j)_{rs}A_{st}=\sum_{s=1}^{\overset{\text{→ dimension}}{n}}\delta_{ir}\delta_{js}A_{st}=$$

↳ lo divido en $\left\langle\begin{matrix}s=j\\ s\neq j\end{matrix}\right.$

$$=\underbrace{\delta_{ir}\delta_{jj}}_{s=j}A_{jt}+\sum_{s\neq j}\delta_{ir}\overset{=0}{\delta_{js}}A_{st}.$$

But $\delta_{js} = 0$ when $s \neq j$; $\delta_{jj} = 1$.

$$[E(i,j)A]_{rt} = \delta_{ir} \cdot A_{jt} .$$

i fixed, j fixed, r can be anything between 1 and n.

$$[E(i,j)A]_{rt} = \delta_{ir} \cdot A_{jt} . = \begin{cases} A_{jt} & r = i \\ 0 & r \neq i . \end{cases}$$

when $r \neq i$ this is zero.

When $r = i$ this is A_{jt} .

That is

prop: The i^{th} Row of $E(i,j)A$ = j^{th} Row of A .

Everyother row of $E(i,j)A = 0$.

$$\boxed{E(i,j;\lambda) = I_n + \lambda E(i,j)} \to \begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix} + \begin{pmatrix}0&0&0\\0&0&0\\\lambda&0&0\end{pmatrix} = \begin{pmatrix}1&0&0\\0&1&0\\\lambda&0&1\end{pmatrix}$$

Suppose A is $n \times p$ matrix. We want to perform the operation $\mathcal{E}(i,j,\lambda)$ on A .

by left multiplying by a suitable $n \times n$ matrix → E

Example: I gave was: $\mathcal{E}(4,1,\lambda) = \begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\\lambda&0&0&1\end{pmatrix}$

We need to show this generally; I've shown that $E(i,j)A$ takes j^{th} Row A and puts into i^{th} Row. Kills everything else.

So $\lambda E(i,j)A$ consists of λ. j^{th} Row (A) inserted into new i^{th} Row.

Everything else = 0 . → la operación que estamos explicando

So $\underbrace{A + \lambda E(i,j) A}$ = New   A = old matrix .

is the matrix unit .

New r^{th} row = old j^{th} row for $r \neq i$ $A + \lambda E(i,j)\overset{0}{A} = A$.

New i^{th} row = old i^{th} row + λ old j^{th} row .

So we've shown: Th: $[I_n + \lambda E(i,j)]A$. is matrix obtained from A by performing $\mathcal{E}(i,j;\lambda)$.

$\begin{pmatrix}1&0&0\\0&1&0\\\lambda&0&1\end{pmatrix}(A)$ = OPERATION

Def^n $E(i,j;\lambda) = I_n + \lambda E(i,j)$.

- $E(i,j;\lambda)A$ is matrix obtained from A by performing $\mathcal{E}(i,j;\lambda)$.

↳ multiplying way .

$\mathcal{E}(i,j;-\lambda)$ reverses $\mathcal{E}(i,j;\lambda)$

Example: $E(1,2;5)$ 2×2 ; $\begin{pmatrix}1&5\\0&1\end{pmatrix}\begin{pmatrix}1&-5\\0&1\end{pmatrix} = \begin{pmatrix}1&0\\0&1\end{pmatrix} E E^{-1} = I$

prop (Fix n, fix i, j, $1 \le \overset{i}{j} \le n$ $i \neq j$)

$$\boxed{E(i,j;\lambda) \cdot E(i,j;\mu) = E(i,j;\lambda+\mu)}$$

$E(1,2;\lambda) \cdot E(1,2;\mu) = E(i,j;\lambda+\mu)$

$$\begin{pmatrix}1&\lambda\\0&1\end{pmatrix}\begin{pmatrix}1&\mu\\0&1\end{pmatrix} = \begin{pmatrix}1&\lambda+\mu\\0&1\end{pmatrix}$$

prop $E(i,j;\lambda) = I_n + \lambda \mathcal{E}(i,j)$ $\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix} \cdot \begin{pmatrix}0&0&0\\0&0&0\\\lambda&0&0\end{pmatrix} = \begin{pmatrix}1&0&0\\0&1&0\\\lambda&0&1\end{pmatrix}$

$E(i,j;\mu) = I_n + \mu \cdot \mathcal{E}(i,j)$

$E(i,j;\lambda) \cdot E(i,j;\mu) = \left[I_n + \lambda \mathcal{E}(i,j)\right] \cdot \left(I_n + \mu \mathcal{E}(i,j)\right) =$

$= I_n + \lambda \mathcal{E}(i,j) + \mu \mathcal{E}(i,j) + \lambda\mu\, \mathcal{E}(i,j) \cdot \mathcal{E}(i,j)$

However if $i \neq j$ then $\boxed{\mathcal{E}(i,j) \cdot \mathcal{E}(i,j) = 0}$.

$$\left[\mathcal{E}(i,j) \cdot \mathcal{E}(i,j)\right]_{rt} = \sum_{s=1}^{n} \mathcal{E}(i,j)_{rs} \cdot \mathcal{E}(i,j)_{st} = \sum_{s=1}^{n} \delta_{ir} \delta_{js} \delta_{is} \delta_{jt} =$$

1º Sustituye la 1ª $\delta_{js} = \delta_{jj} = 1$

2º Sustituye $\delta_{is} = \delta_{ij} = 0$

$$= \delta_{ir} \cdot \delta_{jj} \cdot \delta_{is} \cdot \delta_{jt} = \delta_{ir} \cdot \delta_{is} \cdot \delta_{jt} = 0$$

When $s = j$, $s \neq j$ $\delta_{is} = 0$.

$E(i,j;\lambda) \cdot E(i,j;\mu) = I_n + \lambda \mathcal{E}(i,j) + \mu \mathcal{E}(i,j) = I_n + (\lambda + \mu) \cdot \mathcal{E}(i,j) =$

$= \Sigma(i,j;\lambda+\mu)$. $\in O$.

taking $\mu = -\lambda$.

$\boxed{E(i,j;\lambda) \cdot \mathcal{E}(i,j;-\lambda) = E(i,j;0) = I_n}$.

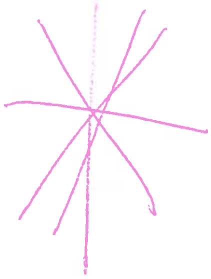
prop $E(i,j,\lambda)$ is invertible

$E(i,j;\lambda)^{-1} = E(i,j,-\lambda)$.

$\Sigma(i,j;\lambda) \Longleftrightarrow E(i,j;\lambda) = I_n + \lambda \mathcal{E}(i,j)$

$\mathcal{A}(i;\lambda) \Longleftrightarrow A(i,\lambda)$.

$\mathcal{P}(i,j) \Longleftrightarrow P(i,j)$.



$\mathcal{E}(i,j) \to$ matrix (with 0's and λ) $\begin{pmatrix}0&0&0\\0&0&0\\\lambda&0&0\end{pmatrix}$

$E(i,j;\lambda) \to$ matrix (with 0's, 1's and λ) $\begin{pmatrix}1&0&0\\0&1&0\\\lambda&0&1\end{pmatrix}$

$\Sigma(i,j;\lambda) \to$ operation

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Matrix **Operation**

$$\boxed{E(i,j,\lambda)} \xleftrightarrow{\text{correspond}} \mathcal{E}(i,j;\lambda)$$ Add → λ row (j) to row (i).

$E(i,j;\lambda)A$ is the matrix obtained from A by $\mathcal{E}(i,j;\lambda)$.

2nd operation: $\mathcal{D}(i,j)$: Multiply row(i) by $\lambda \neq 0$.

The matrix that we use is: $\Delta(i,\lambda)$

Matrix **Operation**

$$\boxed{\Delta(i,\lambda)} \longleftrightarrow \mathcal{D}(i,j).$$

$\Delta(i,j)\cdot A$ is the matrix obtained from A by $\mathcal{D}(i,\lambda)$.

- How to define $\Delta(i,\lambda)$:

  Lets start with: $E(i,i,\mu)$.

  $E(i,i,\mu)A$ adds μ row(i) to row(i). Everything else stays the same.

  So i^{th} row of $E(i,i,\mu)A = (i^A \text{ Row } A) + \mu\,(i^A \text{ Row } A) =$

  $= (\mu+1)\cdot i^A$ Row (A). Everything else stays the same.

  Take $\lambda = \mu + 1$ or $\mu = \lambda - 1$.

  Then definition of $\Delta(i,\lambda)$.

$$\boxed{\Delta(i,\lambda) = I_n + (\lambda-1)\,E(i,i) = E(i,i;\lambda-1)}$$

Example $n=3$, $i=2$:

$$\boxed{\Delta(2,\lambda) = I_3 + (\lambda-1)\,E(2,2)} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ 0 & \lambda-1 & 0 \\ 0 & 0 & 0 \end{pmatrix} =$$

$$= \boxed{\begin{pmatrix} 1 & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & 1 \end{pmatrix}}$$

Therefore:

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} a & b \\ c & d \\ e & f \end{pmatrix} = \begin{pmatrix} a & b \\ \lambda c & \lambda d \\ e & f \end{pmatrix}$$

If $\lambda \neq 0$, then $\Delta(i,\lambda)$ is invertible

and $\Delta(i,\lambda)^{-1} = \Delta(i,\lambda^{-1}) = \Delta\left(i,\frac{1}{\lambda}\right)$

Example $m=4$.

$$\underset{\Delta(3,4)}{\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&4&0\\0&0&0&1\end{pmatrix}}\cdot\underset{\Delta(3,\frac{1}{4})}{\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&\frac{1}{4}&0\\0&0&0&1\end{pmatrix}}=\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix}$$

$i\neq j$.

$$E(i,j\cdot\lambda)^{-1}=E(i,j,-\lambda).$$

$$\Delta(i,\lambda)^{-1}=\Delta\left(i,\frac{1}{\lambda}\right)\quad\lambda\neq 0$$ ⇒ Para que sea inversa $\lambda\neq 0$.

3rd operation $P(i,j)$.

We want

$P(i,j)\longleftrightarrow P(i,j)$. Interchanges i^{th} and j^{th} rows.

To be such that:

$P(i,j)A$ by swapping i^{th} and j^{th} rows.

We need to show $P(i,j)$ exists. If it exists, it must be true that

$$P(i,j)^{-1}=P(i,j).$$

If $P(i,j)$ exists then $P(i,j)=P(i,j)I_m$.

$\|$
which would
be obtained by
switching i^{th} and j^{th}
rows of I_m.

Example: $m=4$.

$P(1,3)I_4$ ⇒ Switch 1^{st} and 3^{rd} rows of I_4:

$$\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix}\longrightarrow\begin{pmatrix}0&0&1&0\\0&1&0&0\\1&0&0&0\\0&0&0&1\end{pmatrix}$$ This is our candidate for $P(1,3)$.

Does it work?

$$\begin{pmatrix}0&0&1&0\\0&1&0&0\\1&0&0&0\\0&0&0&1\end{pmatrix}\cdot\begin{pmatrix}a&b\\c&d\\e&f\\g&h\end{pmatrix}=\begin{pmatrix}e&f\\c&d\\a&b\\g&h\end{pmatrix}$$

formal def^n of $P(i,j)$.

$$\boxed{P(i,j) = I_m - E(i,i) - E(j,j) + E(i,j) + E(j,i)}$$

SYSTEMS OF OPERATIONS:

$$\mathcal{S} = \begin{cases} a_{11}x_1 + \dots + a_{1n}x_n = b_1 \\ \vdots \\ a_{m1}x_1 + \dots + a_{mn}x_n = b_m \end{cases}$$

$$A\underline{x} = \underline{b} \qquad A = (a_{ij}) \quad \begin{matrix} 1 \leq i \leq n \\ 1 \leq j \leq m \end{matrix}$$

$$A = (a_{ij}) \qquad \underline{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \qquad \underline{b} = \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix}$$

Certain systems of equations have obvious solutions:

<u>Example</u>:
$$\begin{cases} \textcircled{x_1} + x_2 - x_4 + 2x_6 = 1 \\ \textcircled{x_3} + x_4 + x_6 = 2 \\ \textcircled{x_5} - x_6 = 3 \end{cases}$$

In a matrix form:

$$\begin{array}{cccccc} \textcircled{1} & 1 & 0 & -1 & 0 & 2 \\ 0 & 0 & \textcircled{1} & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & \textcircled{1} & -1 \\ x_1 & x_2 & x_3 & x_4 & x_5 & x_6 \end{array} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$$

ROWS ARE STEPPED: in all of the rows the first no is a 1.

1st Circle variables underneath leading 1st.

2nd Eliminate the circle variables

$$\begin{aligned} x_1 &= 1 - x_2 + x_4 - 2x_6 \\ x_2 &= x_2 \\ x_3 &= 2 - x_4 - x_6 \\ x_4 &= x_4 \\ x_5 &= 3 + x_6 \\ x_6 &= x_6 \end{aligned}$$

$$\begin{pmatrix} \textcircled{x_1} \\ x_2 \\ \textcircled{x_3} \\ x_4 \\ \textcircled{x_5} \\ x_6 \end{pmatrix} = \begin{pmatrix} 1 - x_2 + x_4 - 2x_6 \\ x_2 \\ 2 \quad x_4 - x_6 \\ x_4 \\ 3 \quad + x_6 \\ x_6 \end{pmatrix}$$

Canonical solution

(Reduced) Row $\boxed{\text{Echelon}}$ Matrix:

$\xrightarrow{}$ rows are stepped.

1) In any nonzero row 1st element you meet must be 1 (leading 1).
2) The rest of the column of a leading 1 must be 0.
3) Rows are stepped in <u>Echelon form</u>.
4) Any zero row comes after any non zero row.

Example:

$$\begin{pmatrix} 1 & 5 & 0 & 4 & 0 & -1 \\ 0 & 0 & 1 & 3 & 0 & 2 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{pmatrix}$$

WRONG!!!

$$\begin{pmatrix} 1 & 5 & \textcircled{2} & 4 & 0 & -1 \\ 0 & 0 & 1 & 3 & 0 & 2 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{pmatrix}$$ → How do I change it? take 2row2 and add it to 1.

$$\begin{pmatrix} 1 & 5 & 0 & 4 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 3 & 0 & 2 \end{pmatrix}$$ → It is not stepped!!!

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$$\mathcal{S} = \begin{cases} a_{11}x_1 + \cdots + a_{1n}x_n = b_1 \\ \vdots \qquad\qquad \vdots \qquad \vdots \\ a_{m1}x_1 + \cdots + a_{mn}x_n = b_m \end{cases}.$$

$$A = (a_{ij}) \quad \begin{matrix} 1 \le i \le m \\ 1 \le j \le n \end{matrix}$$

$$A\underset{\sim}{x} = \underset{\sim}{b}$$

$$\underset{\sim}{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \qquad b = \begin{pmatrix} b_1 \\ \vdots \\ b_m \end{pmatrix}$$

$$A\underset{\sim}{x} = \underset{\sim}{b}$$

$$\Big\downarrow \text{Simplify}$$

$$E_1 A \underset{\sim}{x} = E_1 b.$$

$$\downarrow$$

$$E_2E_1A\underset{\sim}{x} = E_2E_1\underset{\sim}{b}$$. Simpler still. End up here with row reduced.

$$E_pE_{p+1}\cdots E_1Ax = E_pE_{p+1}\cdots + E_1\underset{\sim}{B}$$

$$E_1 = E(i,j,\lambda) \text{ or } \Delta(i,\lambda) \text{ or } P(i,j).$$

Form aumengted matrix:

A is $m \times n$ b is $m \times 1$.

So $(A|b)$ is $m \times (n+1)$.

Example :

①

$$f = \begin{cases} x_1 - x_2 + x_3 - x_4 + x_5 - x_6 = 1 \\ x_1 - x_2 + 2x_3 \quad - x_5 - x_6 = 2 \\ x_1 - x_2 \quad 2x_4 + 3x_5 - x_6 = 0 \end{cases}$$

1st From augmented matrix $(A|b)$.

$$(A|b) = \left(\begin{array}{cccccc|c} 1 & -1 & 1 & -1 & 1 & -1 & 1 \\ 1 & -1 & 2 & 0 & -1 & -1 & 2 \\ 1 & -1 & 0 & -2 & 3 & -1 & 0 \end{array}\right)$$

2nd Perform operations to augmented matrix in reduced Row Echelon form:

$$\left(\begin{array}{cccccc|c} \textcircled{1} & -1 & \boxed{1} & -1 & 1 & -1 & 1 \\ 0 & 0 & 1 & 1 & -2 & 0 & 1 \\ 0 & 0 & \boxed{-1} & -1 & 2 & 0 & -1 \end{array}\right)$$

I want the rest to be 0.

$\boxed{1}$ $\Sigma(2,1;-1)$.

Add (-1) row(1) to row(2).

$\boxed{2}$ $\Sigma(3,1;-1)$.

$$\left(\begin{array}{cccccc|c} 1 & -1 & 1 & -1 & 1 & -1 & 1 \\ 0 & 0 & 1 & 1 & -2 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

$\boxed{3}$ $\Sigma(3,2;1)$.

$$\begin{array}{c} \left(\begin{array}{cccccc|c} 1 & -1 & 0 & -2 & 3 & -1 & 0 \\ 0 & 0 & 1 & 1 & -2 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right) \\ \begin{array}{cccccc} \textcircled{x_1} & x_2 & \textcircled{x_3} & x_4 & x_5 & x_6 \end{array} \end{array}$$

$\boxed{4}$ $\Sigma(1,2,-1)$.

This is matrix reduced to Echelon form.

3rd Write variables underneath.

Circle these under leading 1's.

Write out reduced system

Eliminate circled variables

$$\begin{cases} \textcircled{x_1} - x_2 \quad - 2x_4 + 3x_5 - x_6 = 0 \\ \qquad \textcircled{x_3} + x_4 - 2x_5 \quad = 1 \end{cases}$$

Canonical solution

$x_1 = x_2 + 2x_4 - 3x_5 + x_6$.

$x_2 = x_2$

$x_3 = 1 - x_4 + 2x_5$

$x_4 = x_4$

$x_5 = x_5$.

$x_6 = x_6$

General solution

$$\begin{pmatrix} x_2 + 2x_4 - 3x_5 + x_6 \\ x_2 \\ 1 - x_4 + 2x_5 \\ x_4 \\ x_5 \\ x_6 \end{pmatrix}$$

②

$F_2 \Leftarrow\!\!> F_2 + 3F_3$.

$$\mathcal{S} : \begin{cases} x_1 + x_2 + x_3 - x_4 + x_5 - x_6 = 4 \\ x_1 - x_2 + x_3 - x_4 + x_5 - x_6 = 5 . \\ x_1 + 5x_2 + x_3 - x_4 + x_5 - x_6 = 2 . \end{cases}$$

Write down the augmented matrix:

$$(A|b) = \left(\begin{array}{cccccc|c} 1 & 1 & 1 & -1 & 1 & -1 & 4 \\ 1 & -1 & 1 & -1 & 1 & -1 & 5 \\ 1 & 5 & 1 & -1 & 1 & -1 & 2 \end{array}\right)$$

Perform operations to augmented matrix m to reduced Row Echelon Forr

$$\left(\begin{array}{cccccc|c} 1 & 1 & 1 & -1 & 1 & -1 & 4 \\ 0 & -2 & 0 & 0 & 0 & 0 & \wedge \\ 0 & 4 & 0 & 0 & 0 & 0 & -2 \end{array}\right)$$

$\boxed{1}$ $\Sigma(2,1,-1)$.

$\boxed{2}$ $\Sigma(3,1,-1)$.

$\boxed{3}$ $\Sigma(3,2,2)$.

$$\left(\begin{array}{cccccc|c} 1 & 1 & 1 & -1 & 1 & -1 & 4 \\ 0 & -2 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

$\boxed{4}$ $\alpha(2,-1/2)$

$$\left(\begin{array}{cccccc|c} 1 & 1 & 1 & -1 & 1 & -1 & 4 \\ 0 & 1 & 0 & 0 & 0 & 0 & -1/2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

$\boxed{5}$ $\Sigma(1,2,-1)$.

$\boxed{6}$ Reduced system is then

$$\left(\begin{array}{cccccc|c} 1 & 0 & 1 & -1 & 1 & -1 & 9/2 \\ 0 & 1 & 0 & 0 & 0 & 0 & -1/2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \textcircled{x_1} & \textcircled{x_2} & x_3 & x_4 & x_5 & x_6 & \end{array}\right)$$

$$\begin{cases} x_1 + x_3 - x_4 + x_5 - x_6 = 9/ \\ x_2 = -1/ \end{cases}$$

Canonical solution = General solution

$$\begin{pmatrix} 9/2 - x_3 + x_4 - x_5 + x_6 \\ -1/2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{pmatrix}$$

so many solutions x_3, x_4, x_5, x_6 arbitrary .

③

$$S\begin{cases} x_1 + x_2 = 1 \\ x_1 + x_2 + x_3 = 5 \\ x_2 + x_3 = 4 \end{cases}$$

$$\left(\begin{array}{ccc|c} 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 5 \\ 0 & 1 & 1 & 4 \end{array}\right) \xrightarrow{\Sigma(2,1,-1)} \left(\begin{array}{ccc|c} 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 4 \\ 0 & 1 & 1 & 4 \end{array}\right) \xrightarrow{\Sigma(3,2,-1)} \left(\begin{array}{ccc|c} 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 4 \\ 0 & 1 & 0 & 0 \end{array}\right)$$

$$\xrightarrow{\Sigma(1,3,-1)} \left(\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 4 \\ 0 & 1 & 0 & 0 \end{array}\right) \xrightarrow{P(2,3)} \begin{array}{c} \left(\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 4 \end{array}\right) \\ \textcircled{x_1}\ \textcircled{x_2}\ \textcircled{x_3} \end{array} \longrightarrow \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 4 \end{pmatrix}$$

INVERSE MATRICES

$A = (a_{ij})$ $\quad 1 \leq i \leq n$, $1 \leq j \leq n$

$n \times n$ matrix

How do we find:

a) whether A is invertible?
b) If so what is A^{-1}

STEP 1 : form $n \times 2n$ matrix:

n rows . $(A \mid I_n)$. $\longrightarrow (E_1 A \mid E_1 \quad) \rightarrow (E_2 E_1 A \mid E_2 E_1 \quad)$

STEP 2 $\quad$ 2n columns .

Suppose after m steps: $(E_m \ldots E_1 A \mid E_m \ldots E_1)$

$\Downarrow$ I_n $\qquad$ $\Downarrow$ Then A^{-1} .

Example :

$$A = \begin{pmatrix} 3 & 2 & 1 \\ 2 & 2 & 1 \\ 1 & 1 & 1 \end{pmatrix}.$$

Form $(A \mid I_3) = \left(\begin{array}{ccc|ccc} 3 & 2 & 1 & 1 & 0 & 0 \\ 2 & 2 & 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 & 1 \end{array}\right)$

Perform operations to get 'I_n'

$\boxed{1}$ $P(1,3)$.
$\boxed{2}$ $E(2,\textcircled{1},-2)$
$\boxed{3}$ $\Sigma(3,\textcircled{1},-3)$
$\boxed{4}$ $P(2,3)$
$\boxed{5}$ $\Sigma(1,\textcircled{2},1)$
$\boxed{6}$ $\Delta(2,-1)$

es la que se queda = ✱

$$\left(\begin{array}{ccc|ccc} 1 & 1 & 1 & \bigcirc & 0 & 1 \\ 2 & 2 & 1 & 0 & 1 & 0 \\ 3 & 2 & 1 & 1 & 0 & 0 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 & 1 & -2 \\ 0 & -1 & -2 & 1 & 0 & -3 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 0 & 0 & 1 \\ 0 & -1 & -2 & 1 & 0 & -3 \\ 0 & 0 & -1 & 0 & 1 & -2 \end{array}\right)$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & -1 & -1 & 0 & -2 \\ 0 & -1 & -2 & 1 & 0 & -3 \\ 0 & 0 & -1 & 0 & 1 & -2 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|ccc} 1 & 0 & -1 & 1 & 0 & 2 \\ 0 & 1 & 2 & -1 & 0 & 3 \\ 0 & 0 & -1 & 0 & 1 & -2 \end{array}\right)$$

$\boxed{7}$ $E(1,3;-1)$
$\boxed{8}$ $E(2,3;2)$
$\boxed{9}$ $\Delta(3,-1)$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -1 & 0 \\ 0 & 1 & 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 0 & 1 & -2 \end{array}\right) \to \left(\underbrace{\begin{array}{ccc} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}}_{I_3} \middle| \underbrace{\begin{array}{ccc} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{array}}_{A^{-1}} \right)$$

claim this i

Check

$$\begin{pmatrix} 3 & 2 & 1 \\ 2 & 2 & 1 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\boxed{A^{-1} = \Delta(3,-1), E(2,3;2), E(1,3;-1), \Delta(2,-1), E(2,1;1), P(2,3), E(3,1,-3), E(2,1,-2), P(1,3).}$$

SUMMARY

$A \longrightarrow (A \mid I_n)$ $\quad A^{-1} =$ $\quad \overset{2^\circ}{\Delta}(\ )\,\overset{1^\circ}{E}(2,1;-1) I_n$

Perform a sequence of row operations controlled by matrices $X_1 \ldots X_m$.

$$(A \mid I_n) \to (X_1 A \mid X_1) \to (X_2 X_1 A \mid X_2 X_1) \to \cdots (X_m X_{m-1} \cdots X_2 X_1 A \mid X_m X_{m-1} \cdots X_1)$$

We finish when the LHS → left hand side

$X_m \ldots X_1 A = I_n$.

Put $B = X_m \ldots X_1 \to (I_n \mid B)$ So $BA = I_n$, so $B = A^{-1}$ $(I_n \mid A^{-1})$.

In the above example:

$$A^{-1} = \underset{X_1}{\Delta(3,-1)}, \underset{X_2}{E(2,3;2)}, \underset{X_3}{E(1,3;-1)}, \underset{X_4}{\Delta(2,-1)}, \underset{X_5}{E(2,1;1)}, \underset{X_6}{P(2,3)}, \underset{X_7}{E(3,1;-3)},$$
$$\underset{X_8}{E(2,1,-2)}, \underset{X_9}{P(1,3)}. \longrightarrow \boxed{9 \text{ Operations}}$$

i.e. $A^{-1} = X_9 X_8 X_7 X_6 X_5 X_4 X_3 X_2 X_1$.

$A = X_1^{-1} X_2^{-1} X_3^{-1} X_4^{-1} X_5^{-1} X_6^{-1} X_7^{-1} X_8^{-1} X_9^{-1}$

Suppose Y, Z invertible $n \times n$.

So I have $YY^{-1} = Y^{-1}Y = I_n$.

$ZZ^{-1} = Z^{-1}Z = I_n$.

Proposition: Let Y, Z, be invertible $n \times n$. Then YZ is also invertible and $(YZ)^{-1} = \underbrace{Z^{-1}Y^{-1}}$.

*REVERSE ORDER

$$(YZ) \cdot (Z^{-1}Y^{-1}) = Y(ZZ^{-1})Y^{-1} = Y I_n Y^{-1} = YY^{-1} = I_n.$$

$$(Z^{-1}Y^{-1}) \cdot (YZ) = Z^{-1}(Y^{-1}Y)Z = Z^{-1} I_n Z = Z^{-1}Z = I_n \quad \text{QED}.$$

$A^{-1} = X_9 X_8 \dots$

$A = X_1^{-1} X_2^{-1} \dots$

$A = P(1,3), E(2,1;2), E(3,1;3), P(2,3); E(2,1;-1), \Delta(2;1), E(1,3;1),$

$E(2,3;-2), \Delta(3,-1).$

So if A is invertible provided we keep track of operations and their order, we can not only find A^{-1}, but we can also express A as product of elementary matrices.

Example:

$$A = \begin{pmatrix} 0 & 0 & 1 \\ 0 & -1 & -1 \\ 1 & 1 & 2 \end{pmatrix}$$

$$(A|I_3) = \left(\begin{array}{ccc|ccc} 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & -1 & -1 & 0 & 1 & 0 \\ 1 & 1 & 2 & 0 & 0 & 1 \end{array}\right) \xrightarrow{P(1,3)} \left(\begin{array}{ccc|ccc} 1 & 1 & 2 & 0 & 0 & 1 \\ 0 & -1 & -1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \end{array}\right) \xrightarrow{E(1,2;1)}$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & 1 & 1 \\ 0 & -1 & -1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \end{array}\right) \xrightarrow[E(2,3;1)]{E(1,3;-1)} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -1 & 1 & 1 \\ 0 & -1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \end{array}\right) \xrightarrow{\Delta(2,-1)}$$

$$\underbrace{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}}_{I_3} \underbrace{\begin{pmatrix} -1 & 1 & 1 \\ -1 & -1 & 0 \\ 1 & 0 & 0 \end{pmatrix}}_{}$$

$$A^{-1} = \underset{X_9}{\Delta(2,-1)} \underset{X_8}{E(2,3;1)} \underset{X_7}{E(1,3;-1)} \underset{X_6}{E(1,2;-1)} \underset{X_5}{P(1,3)}.$$

Check:

$$A \cdot A^{-1} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & -1 & -1 \\ 1 & 1 & 2 \end{pmatrix} \cdot \begin{pmatrix} -1 & 1 & 1 \\ -1 & -1 & 0 \\ 1 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \checkmark$$

$A^{-1} = X_9 X_8 X_7 X_6 X_5$, $A = X_5^{-1} X_6^{-1} X_7^{-1} X_8^{-1} X_9^{-1}$

$A = P(1,3) E(1,2;-1) E(1,3;-1) E(2,3;1) \Delta(2,-1).$

October 20th 2018

Example:

$$A = \begin{pmatrix} 1 & -1 & 1 \\ 1 & 1 & 1 \\ 1 & -3 & 1 \end{pmatrix} \xrightarrow[E(1,3;-1)E(1,2;-1)]{\text{Suppose I start performing operations}} \begin{pmatrix} 1 & -1 & 1 \\ 0 & 2 & 0 \\ 0 & -2 & 0 \end{pmatrix} \xrightarrow{E(2,3;1)} \begin{pmatrix} 1 & -1 & 1 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

zero row $\Rightarrow |A| = 0$.

Not invertible, ¿why?

UNA MATRIZ NO ES INVERTIBLE SI → zero row

① Prop: Suppose i^{th} Row $B = 0$, then i^{th} Row $BC = 0$ for any C.

Proof: $BC_{ik} = \sum \underset{\overset{\|}{0}}{B_{ij}} C_{jk} = 0$.

② Proposition: If B is a square matrix and i^{th} Row $B = 0$, then B is not invertible.

Proof: Suppose B is invertible, then $BB^{-1} = I_n$. This implies i^{th} Row of $I_n = 0$ $\boxed{\text{False}}$. So B is not invertible QED.

③ Proposition: let A be a square matrix and X invertible matrix.
If XA has a zero row then A is not invertible.

Proof: Suppose A is invertible $AA^{-1} = I_n$; $(XA) \cdot A^{-1} = X$ (which is invertible). But XA has a zero row, so X has a zero row: CONTRADICTION

In the example:

$$A = \begin{pmatrix} 1 & -1 & 1 \\ 1 & 1 & 1 \\ 1 & -3 & 1 \end{pmatrix} \quad \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \quad \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix} \quad \begin{pmatrix} 0 & 0 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Take $X = E(3,2;1)\,E(3,1;-1)\,E(2,1,-1)$

XA has a zero Row. X invertible so A not invertible QED.

"START COURSE":

PROPOSITIONAL LOGIC:

and, or, not, implies

① Start with statements which can be either True (T) or False (F).

eg: p = it is raining, q = it is cold.

ATOMIC STATEMENTS.

TRUTH TABLES ①

p	q	p and q
T	T	T
	F	F
F	T	F
	F	F

→ $p \wedge q$.

AND = $\wedge$

Conjunction → only is True when both are true.

② OR

Two possible senses:

$\boxed{2.1}$ Inclusive or: Latin = Vel

$\boxed{2.2}$ Exclusive or: Latin = aut

2.1 INCLUSIVE OR = $\vee$ = DISJUNCTION

p	q	$p \vee q$
T	T	T
	F	T
F	T	T
	F	F

only True when either p or q is true, if not is false

2.2 EXCLUSIVE OR = $\{$ = JOHNSON'S SIGN

p	q	$p \{ q$
T	T	F
	F	T
F	T	T
	F	F

It is true when either p or q is true. It is false when both are true or both are false

③ NOT = $\neg$ = Negation

p : it is raining

q : it is not raining

p	$\neg p$	$\neg\neg p$
T	F	T
F	T	F

② COMPOUND STATEMENTS

A compound statement is one which involves at least 2 atomic statements.

eg: $p \vee q$, $(p \vee q) \wedge r$.

p	q	r	$(p \vee q) \wedge r$	$p \vee q$	$(p\wedge r) \vee (q \wedge r)$	$p \wedge r$	$q \wedge r$
T	T	T	T	T	T	T	T
		F	F	T	F	F	F
	F	T	T	T	T	T	F
		F	F	T	F	F	F
F	T	T	T	T	T	F	T
		F	F	T	F	F	F
	F	T	F	F	F	F	F
		F	F	F	F	F	F

Observe: $(p \vee q) \wedge r$, $(p \wedge r) \vee (q \wedge r)$

have same truth tables: TRUE / FALSE under exactly the same conditions

when 2 compound statements have same Truth Table we regard them as equivalent : $\equiv$

$$\boxed{(p \vee q) \wedge r \equiv (p \wedge r) \vee (q \wedge r)}$$

eg $p \equiv \neg\neg p$

p	$\neg p$	$\neg\neg p$
T	F	T
F	T	F

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Summary:

① and → $\wedge$
② or → $\vee$
③ not → $\neg$
④ implies → $\Rightarrow$

} Atomic statements.

$(p \wedge q) \vee r \equiv (p \vee r) \wedge (q \vee r)$

① $p \vee (q \vee r) \equiv (p \vee q) \vee r$
$p \wedge (q \wedge r) \equiv (p \wedge q) \wedge r$.
} ASSOCIATIVITY

② $p \vee q = q \vee p$
$p \wedge q = q \wedge p$
} COMMUTATIVITY.

③ $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$
$p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$
} DISTRIBUTIVITY

④ $p \equiv p \vee p$
$p \equiv p \wedge p$
} IDEMPOTENCE.

} DE MORGAN'S LAW.
↓
Actually due to George Boole.

p	p	$p \vee p$	$p \wedge p$
T	T	T	T
F	F	F	F .

Analogy between + and $\vee$.
$\cdot$ and $\wedge$

In arithmetic just got:

$a \cdot (b+c) = ab + ac$.

$\equiv$

p	q	$p \wedge q$	$\neg(p \wedge q)$	$\neg p$	$\neg q$	$(\neg p) \vee (\neg q)$
T	T	T	F	F	F	F
	F	F	T	F	T	T
F	T	F	T	T	F	T
	F	F	T	T	T	T

⑤ $\left\{\begin{array}{l}\neg(p \wedge q) \equiv (\neg p) \vee (\neg q) \\ \neg(p \vee q) \equiv (\neg p) \wedge (\neg q)\end{array}\right\}$ DUALITY

→ saco fuera y cambio signo

④ IMPLIES = $p \Rightarrow q \equiv (\neg p) \vee q$

p	q	$p \Rightarrow q$
T	T	T
	F	F
F	T	T
	F	T

MOTIVATION:

- $\neg p \vee p$ is always T.

p	$\neg p$	$\neg p \vee p$
T	F	T
F	T	T

- $p \Rightarrow q$ means if p happens then q happen

p	$\neg p$	q	$\neg p \vee q \equiv p \Rightarrow q$
T	F	T	T
		F	F
F	T	T	T
		F	T

✱

"p implies q" = If p then q.
If p is true q should be true.
If q is true, p doesn't matter.

- $p \Rightarrow q \equiv (\neg p) \vee q$
- $\neg p \Rightarrow q \equiv (\neg\neg p) \vee q$

But $\neg\neg p \equiv p$.

$\neg p \Rightarrow q \equiv p \vee q$.

- $\boxed{p \vee q \equiv} \boxed{\neg p \Rightarrow q}$
- $\boxed{p \wedge q \equiv} (\neg\neg p) \wedge (\neg\neg q) \equiv \neg(\neg p \vee \neg q) \equiv \boxed{\neg(p \Rightarrow \neg q)}$.

So we have expressed 'or' and 'and' in terms of 'implies', 'not'.

Can obviously express:

$\Rightarrow$ in terms of $\vee$, $\neg$: $p \Rightarrow q \equiv (\neg p) \vee q$

$p \Rightarrow q \equiv \neg p \vee q \equiv \neg p \vee \neg\neg q \equiv \neg(p \wedge \neg q)$

So $p \Rightarrow \equiv \neg(p \wedge \neg q)$. So it looks like we only need 2 symbols: $\neg$, $\Rightarrow$

In fact we only need one sign:

"SHEFFERS' STROKE FUNCTION"

p	q	p|q
T	T	~~F~~
	F	T
F	T	T
	F	T

→ $\neg p \equiv p|p$.

(Exercise express $\Rightarrow$ in terms of '|')

Propositional logic ~ Constant propositions

Predicate logic ~ Variable propositions

ex: $P(x) = x \geq 2$.

If x is a natural No, there are ∞ many such.

Ex: Consider propositions $P(x)$ where x is either 0 or 1.

so have two constant propositions $P(0)$, $P(1)$.

$\{0, 1\}$ = domain of discussion.

1) Is $P(x)$ true for every occurance of x?.

2) Is $P(x)$ true for at least one occurence of x.

UNIVERSAL QUANTIFIER

$(\forall x)\ P(x)$ means $P(x)$ is true for every x in Domain

EXISTENTIAL QUANTIFIER

$(\exists x)\ P(x)$ means $P(x)$ is true for at least one x in Domain.

$\mathcal{D}$ = Domain.

$\mathcal{D} = \{0, 1\}$

$(\forall x).\ P(x) \equiv P(0) \wedge P(1)$

$\mathcal{D} = \{0, 1, 2\}$

$(\forall x)\ P(x) \equiv P(0) \wedge P(1) \wedge P(2)$.

$\mathbb{N} = \{0, 1, 2, 3 \dots n, n+1 \dots\}$.

$(\forall x)\ P(x) \equiv P(0) \wedge P(1) \wedge P(2) \wedge \dots P(n) \wedge P(n+1) \wedge \dots$.

We use $(\forall x)\ P(x)$ to avoid ∞ long formulae.

$(\exists x)\ P(x) = P(0) \vee P(1)$.

$\mathcal{D} = \{0, 1\}$

$\mathcal{D} = \{0, 1, 2\}$

$(\exists x)\ P(x) = P(0) \vee P(1) \vee P(2)$.

$\mathcal{D} = \mathbb{N}$

$(\exists x)\ P(x) \equiv P(0) \vee P(1) \vee P(2) \vee P(n) \vee P(n+1) \vee$

How does not interact with $\forall$?

$\mathcal{D} = \{0,1\}$

Duality

at least 1 x no belong to P(x)

$(\forall x)\, P(x) = P(0) \wedge P(1)$.

$\neg (\forall x)\, P(x) = \neg\ P(0) \wedge P(1) = (\neg P(0)) \vee (\neg P(1)) \equiv (\exists x)\, (\neg P(x))$.

$\mathcal{D} = \{0,1,2\}$

$\neg\ (\forall x)\ P(x) = \neg\ (P(0) \wedge P(1) \wedge P(2) \equiv \neg\ P(0) \vee \neg P(1) \vee \neg P(2) \equiv (\exists x)(\neg P(x))$

We take as an axiom

no todos los valores.

① $\neg (\forall x)\ P(x) \equiv (\exists x)\ (\neg P(x))$ -

② $\neg (E x)\ P(x) \equiv (\forall x)\ \neg(P(x))$.

$\mathcal{D} = \{0,1\}$.

$(\exists x)\ P(x) \equiv P(0) \vee P(1)$.

$\neg (\exists x)\ P(x) \equiv \neg (P(0) \vee P(1)) \equiv \neg P(0) \wedge P(1) \equiv (\forall x) \cdot (\neg P(x))$.

$\mathcal{D} = \{0,1,2\}$

③ $(\exists x) \cdot P(x) \equiv P(0) \vee P(1) \vee P(2)$

④ $\neg (\exists x)\ P(x) \equiv \neg\ (P(0) \vee P(1) \vee P(2)) \equiv \neg\ P(0) \wedge \neg(P(1)) \wedge \neg P(2) \equiv$

$\equiv (\forall x) \neg P(x)$

ORDER OF QUANTIFIERS

$\mathcal{D} = \{0,1\}$ $P(x,y)$ x, y in $\mathcal{D}$

1ST $(\forall x)\ (\exists y)\ P(x,y)$ 2nd $\exists (y)\ (\forall x)\ P(x,y)$.

Take $P(x,y) = "x = y"$.

x, y in $\{0,1\}$.

$(\exists y)\ P(x,y) = P(x,0) \vee P(x,1)$.

1ST $(\forall x)(\exists y)\ P(x,y) = [P(0,0) \vee P(0,1)] \wedge [P(1,0) \vee P(1,1)]$

$P(0,0) = "0=0"$ T.

$P(1,0) = '1=0'$ F .

$P(0,1) = '0=1'$ F .

$P(1,1) = '1=1'$ T.

$P(0,0) \vee P(0,1)$ is T .

$P(1,0) \vee P(1,1)$ is T .

So in this model : $(\forall x)\ (\exists y)\ P(x,y)$ is T .

2nd $(\forall x)\ P(x,y) = P(0,y) \wedge P(1,y)$.

$(\exists y)\ (\forall x)\ P(x,y) = [P(0,0) \wedge P(1,0)] \vee [P(0,1) \wedge P(1,1)] \equiv F$.

$(T \wedge F) \vee (F \wedge T)$

so $(\forall x)(\exists y)\ P(x,y)$ is not equivalent to $(\exists y)(\forall x)\ P(x,y)$

ORDER OF QUANTIFIERS IS IMPORTANT.

In fact $\boxed{(\exists y)(\forall x)\ P(x,y) \Rightarrow (\forall x)(\exists y)\ P(x,y)}$ ✗

/ but converse is $\boxed{\text{false}}$.

However,

$$\boxed{\begin{array}{l}(\forall x)(\forall y) \equiv (\forall y)(\forall x)\\ (\exists x)(\exists y) \equiv (\exists y)(\exists x)\end{array}}$$ ✓

SET THEORY

$\{0,1,2\} = \{1,2,0\} = \{1,0,2\}$

* In set theory order unimportant.

- To indicate order we use round brackets ( )

  Ex. $(0,1) \neq (1,0)$

  $\{0,1\} = \{1,2\}$.

- SET THEORY HAS ONE PRIMITIVE SIGN '$\in$' = belonging = is a member of.

  $X = \{0,1,3\}$.

  $0 \in X,\ 1 \in X,\ 3 \in X,\ 2 \notin X$.

- A set is determined by its members.
- A, B sets $A = B \iff (x \in A \Rightarrow x \in B) \wedge (x \in B \Rightarrow x \in A)$.

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- Sets begin with { and end with }.
- Two ways of describing a set:

  ① NAIVE WAY: LIST the members.

  ② Use "defining property"

  ① → $\{0,1,2\}$

  ② → $\{x \in \mathbb{Z} ; 0 \le x \le 2\}$.

  $\{x \mid x \in \mathbb{Z} \wedge (0 \le x \le 2)\}$.

INCLUSION.

Suppose A, B are sets. Write '$A \subset B$' when for each $x \in A$, $x \in B$

$(\forall x)(x \in A \Rightarrow x \in B)$.

MISTAKES ⚠

→ Don't confuse membership with inclusion.

$$X = \{0, 1, \{0,1\}, \{1,2\}, \{1\}, \{0,1,2\}\}$$

$0 \in X$, ✓

$1 \in X$ ✓

$\{1\} \in X$ ✓

$\{1\} \subset X$ ✓

~~$\{0\} \in X$~~ ~~✗~~

~~$2 \in X$~~ ~~✗~~

$\{0\} \subset X$

2 no está solo en el set

~~$\{1,2\} \subset X$~~ ✗ → it is a subset included in the set = los elementos dentro del subset tienen que estar solos en el set.

$\{1,2\} \in X$ ✓ → it is a member.

↑ el propio elemento está

The usual way to define a set is by a defining property:

$$B = \{x \mid x \text{ is a LONDON BUS}\}.$$

↑ TFL

↑ Typical element ↑ Defining predicate.

$$X = \{x \mid (x \in \mathbb{R}) \wedge (x^2 = 2)\}.$$

$$Y = \{x \mid (x \in \mathbb{Z}) \wedge (x^2 = 2)\}.$$

$Y = \emptyset$ The set with no elements. EMPTY SET.

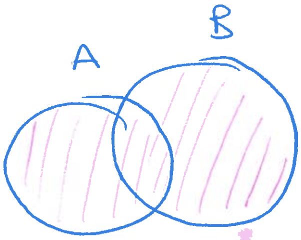
$$A = \{x \mid P_A(x)\}.$$

$P_A(x)$ is the predicate which defines A.

UNION

$\boxed{A \cup B} = \{ x \mid P_A(x) \vee P_B(x) \}$.

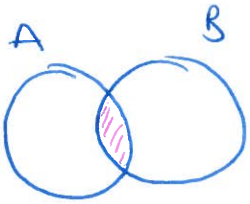
$x \in A \cup B \iff \boxed{(x \in A) \vee (x \in B)}$.



INTERSECTION

$\boxed{A \cap B} = \{ x \mid P_A(x) \wedge P_B(x) \}$.

$x \in A \cap B \iff \boxed{(x \in A) \wedge (x \in B)}$.



PROPERTIES OF ∪, ∩

1) $A \cup (B \cup C) = (A \cup B) \cup C$.

1) $A \cap (B \cap C) = (A \cap B) \cap C$

ASSOCIATIVITY

$P_{A \cup B}(x) = P_A(x) \vee P_B(x)$.

Proof: ①

$P_{A \cup (B \cup C)} = P_A(x) \vee (P_{B \cup C}(x) \equiv P_A(x) \vee \left(P_B(x) \vee P_C(x)\right) \equiv$

$\equiv \left(P_A(x) \vee P_B(x)\right) \vee P_C(x) \equiv P_{(A \cup B)}(x) \vee P_C(x) = P_{(A \cup B) \cup C}$

2) $A \cup B = B \cup A$

2) $A \cap B = B \cap A$

COMMUTATIVITY

3) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$.

3) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

DISTRIBUTIVE LAW.

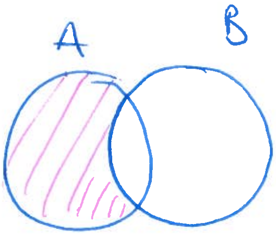
4) $A \cup A = A$

4) $A \cap A = A$

IDEMPOTENT.

DIFFERENCES

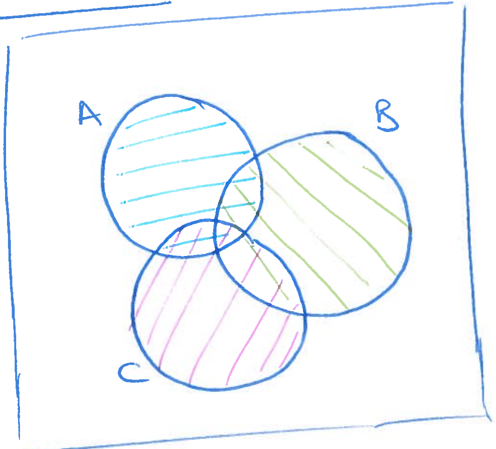
$A - B = \{ x \mid (x \in A) \wedge (x \notin B) \}$.



$P_{A-B}(x) = P_A(x) \wedge \neg P_B(x)$.

$A - A = \emptyset$

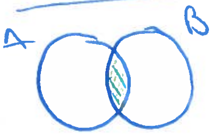
VENN DIAGRAM

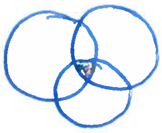


A = ▨

B = ▨

C = ▨

$A \cap B =$ 

$A \cap B \cap C =$ 

PRODUCT SETS:

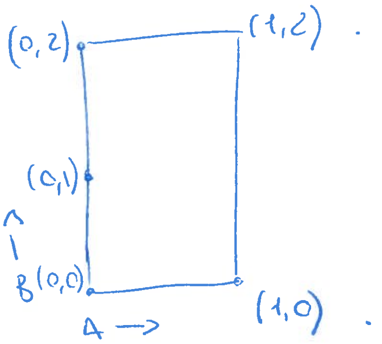
Ordered pairs (a,b).

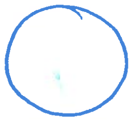
RULE OF EQUALITY: $(a,b) = (a',b') \iff (a=a') \wedge (b=b')$.

$$A \times B = \{ (a,b) \mid (a \in A) \wedge (b \in B) \}.$$

Ex: $A = [0,1] = \{ x \in \mathbb{R} : 0 \leq x \leq 1 \}$.

$B = [0,2] = \{ x \in \mathbb{R} ; 0 \leq x \leq 2 \}$.

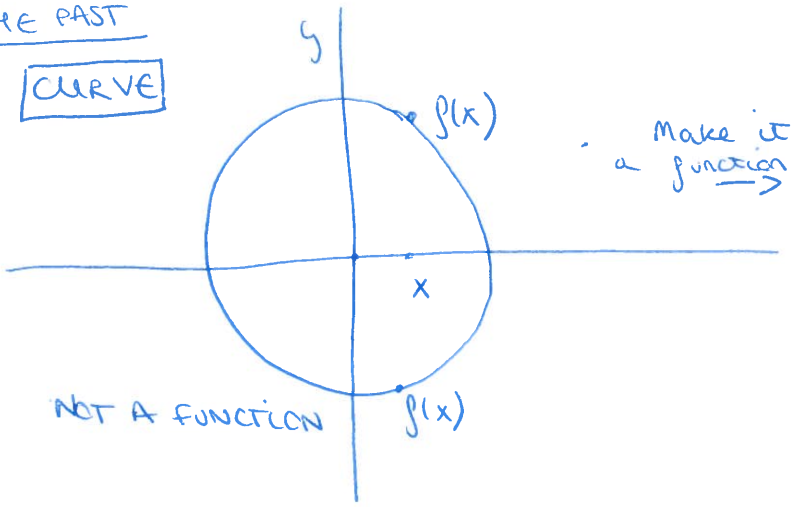


S^1 

$S^1 \times S^1$  TORUS (DUNKIN DONUT).

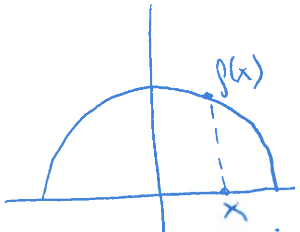
IN THE PAST

CURVE



NOT A FUNCTION

Make it a function →



A FUNCTION

Informal def^n of function

A, B sets

By a function $f: A \to B$.

we mean a rule which assigns to each $a \in A$ a single well defined element $f(a) \in B$.

A is domain of f.
B is codomain of f.

$f(x) = 2x+1$.

$$g(x) = \frac{1}{x+1}$$

$h(x) = x^2$.

$k(x) = \sqrt{x}$.

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FUNCTIONS / MAPPINGS

A, B sets.

By a function / mapping $f: A \to B$.

we mean $A \xrightarrow{f} B$.

1) F is a rule which assigns to each $a \in A$ a single well defined element $f(a) \in B$.

A is called the DOMAIN of f.

B is called the CODOMAIN of f

each $f(a)$ must be a single element of B.

Example: $\boxed{f: \mathbb{R} \to \mathbb{R}}$

$f(x) = x+2$. → Por cada $\mathbb{R}$ que pongo obtengo otro nº $\mathbb{R}$.

$$g(x) = \frac{1}{x+2}.$$

$g(-2)$ not defined.

so $g: R - \{-2\} \to \mathbb{R}$ is now a mapping.

$h(x) = \sqrt{x}$

Put $\mathbb{R}_{\geq 0} = \{ x \in \mathbb{R} \; : \; x \geq 0 \}$.

$h : \mathbb{R}_{\geq 0} \longrightarrow \mathbb{R} \quad h(x) = \sqrt{x}$.

If we make a convention that $\sqrt{x} \geq 0$ then we have a mapping.

• COMPOSITION OF MAPPINGS

Suppose I have mappings

$$A \xrightarrow{f} B \xrightarrow{g} C.$$

I get a mapping $g \circ f$.

$$g \circ f : A \longrightarrow C.$$

$$\boxed{(g \circ f)(a) = g(f(a))}.$$

COMPOSITION IS ALWAYS ASSOCIATIVE

$$C \xrightarrow{h} D.$$

$$A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} D.$$

$$A \xrightarrow{g \circ f.} C \xrightarrow{h} D$$

$$A \xrightarrow{f} B \xrightarrow{h \circ g.}$$

Proposition Composition is ASSOCIATIVE.

Proof: With the above

$$[h \circ (g \circ f)](a) = h(g \circ f(a)) = h(g(f(a))) =$$

$$[(h \circ g) \circ f](a) = (h \circ g)(f(a)) = h(g(f(a))) =$$

So $h \circ (g \circ f) = (h \circ g) \circ f$. QED.

How about commutativity?

$f : \mathbb{R} \longrightarrow \mathbb{R} \quad f(x) = x^2 + 1$.

$g : \mathbb{R} \longrightarrow \mathbb{R} \quad g(x) = \cos(x)$.

$$\left.\begin{aligned} (g \circ f)(x) &= \cos(x^2+1) \\ (f \circ g)(x) &= (\cos(x))^2 + 1 \end{aligned}\right\} \neq$$

They are not the same.

$$\left.\begin{aligned}(g \circ f)(0) &= \cos 1 < 1 \\ (f \circ g)(0) &= 2\end{aligned}\right] \text{ NOT THE SAME}$$

$\boxed{\text{Proposition}}$ COMPOSITION is NOT COMMUTATIVE USUALLY.

- Identity mapping

If A is a set define $Id_A : A \longrightarrow A$.

$$\boxed{Id_A(a) = a} \qquad \text{ex}: f(x) = x$$

- Invertible mapping

Let $f ; A \longrightarrow B$ be mapping. Say that f is invertible when there exists a mapping

$$\boxed{g : B \longrightarrow A}$$

such that

$$\begin{tikzcd} A \arrow[r, "f"] & B \arrow[r, "g"] & A \end{tikzcd}$$

$$\begin{cases} g \circ f = Id_A \; ; A \longrightarrow A. \\ f \circ g = Id_B \; ; B \longrightarrow B. \end{cases}$$

Example. $\mathbb{R}_{>0} = \{x \in \mathbb{R} \; ; x > 0\}$.

$\exp. : \mathbb{R} \longrightarrow \mathbb{R}_{>0}$. $\exp(x) = \sum_{r=0}^{\infty} \frac{x^r}{r!}$

$\log : \mathbb{R}_{>0} \longrightarrow \mathbb{R}$. $\log(x) = \int_1^x \frac{dt}{t}$.

$$\left\{\begin{aligned} \exp \circ \log (x) &= x \\ \log \circ \exp (x) &= x \end{aligned}\right\} \left(\begin{aligned} e^{\log x} &= x \\ \log(e^x) &= x \end{aligned} \right).$$

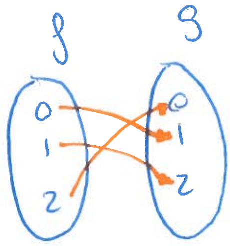
$f : \mathbb{R} \longrightarrow \mathbb{R}$ $f(x) = 2x + 1$. $g(x) = \frac{x-1}{2}$.

want $g : \mathbb{R} \longrightarrow \mathbb{R}$ $(f \circ g)(x) = x$

$(g \circ f)(x) = x$.

$$\{0, 1, 2\} \xrightarrow{f} \{0, 1, 2\}.$$

$f(0) = 1$	$g(0) = 2$
$f(1) = 2$	$g(1) = 0$
$f(2) = 0$	$g(2) = 1$.



$g \circ f = Id$ ⇾ $0 \longrightarrow 1 \longrightarrow 0$.

$f \circ g = Id$.

Prop: If $f: A \to B$ has an inverse mapping then the inverse mapping is unique.

Proof: Suppose $g: B \to A$, $h: B \to A$ } he supposes that there are more than 1 inverse.

Satisfy:

$$g \circ f = Id_A$$

$$f \circ g = Id_B$$

$$h \circ f = Id_A$$

$$f \circ h = Id_B$$

$h \circ (f \circ g) = h \circ Id_B = h$.

$(h \circ f) \circ g = Id_A \circ g = g$. → there is only 1 inverse.

But $h \circ (f \circ g) = (h \circ f) \circ g$, so $h = g$ QED.

If $f: A \to B$ is invertible wrote its inverse as $f^{-1}: B \to A$ NUNCA $f^{-1} = \frac{1}{f}$

- **INJECTIVITY**

Say that $f: A \to B$ is injective when $\boxed{f(a) = f(d) \Rightarrow a = d}$

example:

$f: \mathbb{Z} \to \mathbb{Z}$

$f(x) = 2x+1$. → Para cada y solo puede haber 1 x.

$\boxed{f \text{ is injective}}$ why?

Suppose $f(x) = f(x')$; $2x+1 = 2x'+1$; $2x = 2x'$; $x = x'$.

example:

$g: \mathbb{Z} \to \mathbb{Z}$ $g(x) = x^2$.

$g(1) = g(-1) = 1$.

but $1 \neq -1$. $\boxed{g \text{ is not injective}}$

It is injective → I can only put 1 nb that gives me the answer

Para cada y solo 1 x

- SURJECTIVITY

$f: A \to B$ is subjective $\forall b \in B \; \exists a \in A \mid f(a) = b$

In English, everything in B can be hit by something in A.

Example:

$f: \mathbb{R} \to \mathbb{R}$

$f(x) = 2x+1$.

$\boxed{f \text{ is surjective}}$ why?

Take $y \in \mathbb{R}$. Got to find $x \in \mathbb{R}$ $f(x) = y$. Take $x = \left(\frac{y-1}{2}\right)$

Example

$g: \mathbb{R} \to \mathbb{R}$

$g(x) = x^2$.

$\boxed{g \text{ is not surjective}}$

there is no $x \in \mathbb{R}$ $g(x) = -1$.

However $h: \mathbb{C} \to \mathbb{C}$

$h(z) = z^2$ then h is surjective (porque $\sqrt{h(z)}$ si que existe al ser $\mathbb{C}$)

Exercise: $f: \mathbb{R} \to \mathbb{R}$ $f(x) = 2x^3 - x = \begin{cases} 0{,}7 \\ -0{,}7 \\ 0 \end{cases}$

$g: \mathbb{Z} \to \mathbb{Z}$ $g(x) = 2x^3 - x$ → $0 = 2x^3 - x$ → no puedo meter ningún número entero que de 0

f is surjective but not injective → porque tengo muchas soluciones $\mathbb{R}$
→ para TODAS tiene que haber 1 entero

g is injective but not surjective
→ solo tengo 1 solución $\mathbb{Z}$

A mapping $f: A \to B$ is BIJECTIVE when it is both INJECTIVE AND SURJECTIVE.

Proposition An invertible mapping is bijective

Proof: Suppose $f: A \to B$ is invertible with:

$f^{-1} = g;\; B \to A$

$g \circ f = Id_A$.

$f \circ g = Id_B$.

$\boxed{f \text{ is injective}}$

Suppose $f(a) = f(a')$.

Apply g. $g(f(a)) = g(f(a'))$
$$\underset{a}{\overset{\|}{g(f(a))}} = \underset{a}{\overset{\|}{g(f(a'))}}$$

So $a = a'$.

$\boxed{f \text{ is surjective}}$

let $b \in B$ put $a = g(b)$.

Apply f, $f(a) = \underbrace{f(g(b))}_{f \circ g = Id} = b$

$f(a') = b$.

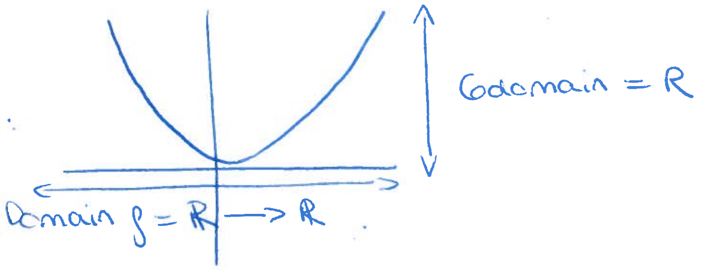
QED .

We'll show the converse. Every bijective mapping is invertible

FORMAL DEF^n OF MAPPING : he's using the formal def to prove this

$f: \mathbb{R} \longrightarrow \mathbb{R}$ $f(x) = x^2$

I can draw its graph:



The graph is a subset of domain × codomain. It consists of the set.

$\{(x, f(x)) ; x \in \text{Domain}\}$.

Def^n: let A, B be sets.

By a mapping $f: A \longrightarrow B$.

I mean a subset $f \subset A \times B$.

which satisfies the following 2 conditions .

(If $(a,b) \in f$ we think $b = f(a)$)

definicion de funcion

(I) $\forall a \in A \; \exists b \in B \; (a,b) \in f$ ⟶ ($b = f(a)$ is defined)

(II) If $(a,b) \in f$ and $(a,b') \in f$ then $b = b'$.

($f(a)$ is a single element)

f is injective when: ⟶ para cada y solo puede haber 1 x

(III) $(a,b) \in f$ and $(a',b) \in f \Rightarrow a = a'$.

f is surjective when :

(IV) $\forall b \in B \quad \exists a \in A \quad (a,b) \in f$ → Todas las y se pueden obtener con una x que pertenezc al set de x.

Suppose $f < A \times B$.

Define $f^{-1} < B \times A$.

by : $(b,a) \in f^{-1} \Longleftrightarrow (a,b) \in f$.

CONDITIONS FOR F TO BE A BIJECTIVE MAPPING :

es funcion

(I) : $\forall a \in A \quad \exists b \in B \quad (a,b) \in f$.

(II) $[(a,b) \in f] \wedge [(a,b') \in f] \Rightarrow b = b'$.

es inyect y subj

(III) $[(a,b) \in f] \wedge [(a',b) \in f] \Rightarrow a = a'$.

(IV) $\forall b \in B \quad \exists a \in A \quad (a,b) \in f$. $f(x) = y = 2x + 1$

CONDITIONS FOR F^{-1} TO BE A BIJECTIVE MAPPING

$B \longrightarrow A$

$f^{-1}(x) = x = \frac{y-1}{2}$

funcion

(I)' $\forall b \in B \quad \exists a \in A \quad (b,a) \in f^{-1}$.

(II)' $[(b,a) \in f^{-1}] \wedge [(b,a') \in f^{-1}] \Rightarrow a = a'$.

Note (I)' ≡ (IV). (III)' ≡ II.

(II)' ≡ (III). (IV)' ≡ I.

f is bijective then f is invertible, which means f^{-1} exits

So if f is bijective, then it satisfies (III) and (IV). So f^{-1} satisfies (I)' and (II)' (F^{-1} is a mapping).

bijective

(III)' $[(b,a) \in f^{-1}] \wedge [(b',a) \in f^{-1}] \Rightarrow b = b'$.

(IV)' $\forall a \in A \quad \exists b \in B \quad (b,a) \in f^{-1}$

To summarise :

$f : A \to B$ is a bijective mapping

iff
$\Longleftrightarrow f$ satisfies (I), (II), (III), (IV).
$\Longleftrightarrow f^{-1}$ satisfies (I)', (II)', (III)', (IV)'.
$\Longleftrightarrow f^{-1}$ is also a bijective mapping.

Two sets A, B have the same **CARDINALITY** when $\exists$ bijective mapping.

$$f : A \longrightarrow \boxed{B} \to |B| =$$

- **Permutations**

Let $n \geq 1$ be an integer and consider the set of all bijective mappings

$$f : \{1, \dots, n\} \longrightarrow \{1, \dots n\}.$$

$n=2$. Lets look all mappings: $\{1,2\} \longrightarrow \{1,2\}$.

$$\begin{tikzcd} 1 \arrow[r] & 1 \\ 2 \arrow[ru] & 2 \end{tikzcd}$$

NOT IN OR SUR.

Identity
$$\begin{tikzcd} 1 \arrow[r] & 1 \\ 2 \arrow[r] & 2 \end{tikzcd}$$

$\boxed{T}$ Special name
$$\begin{tikzcd} 1 \arrow[rd] & 1 \\ 2 \arrow[ru] & 2 \end{tikzcd}$$

$$\begin{tikzcd} 1 \arrow[rd] & 1 \\ 2 \arrow[r] & 2 \end{tikzcd}$$

$$O_n = \{ f ; \{1, \dots n\} \longrightarrow \{1, \dots n\}$$

f is bijective mapping ?..

$\sigma_2 = \{ Id, T \}$ where $T(1) = 2$ $T(2) = 1$. $1 \xrightarrow{T} 2 \xrightarrow{T} 1$

Note that $T \circ T = Id$.

Proposition Let $A \xrightarrow{f} B \xrightarrow{g} C$.

be invertible mappings. Then $g \circ f ; A \to C$ is also invertible

$$\boxed{\underbrace{(g \circ f)^{-1}}_{A \to C} = \underbrace{f^{-1} \circ g^{-1}}_{A \to B \to C}}.$$

Proof
$$g \circ f \circ f^{-1} \circ g^{-1} = g \circ Id \circ g^{-1}$$
$$= g \circ g^{-1} = Id$$

$$f^{-1} \circ g^{-1} \circ g \circ f = f^{-1} \circ Id \circ f.$$
$$= f^{-1} \circ f.$$
$$= Id$$

QED.

σ_3 has 6 elements $= 3! = 3 \cdot 2 \cdot 1 = 6$.

$$Id \begin{cases} 1 \longrightarrow 1 \\ 2 \longrightarrow 2 \\ 3 \longrightarrow 3 \end{cases}$$

$3! = 3 \cdot 2 \cdot 1$

$$\sigma = \begin{cases} 1 \longrightarrow 2 \\ 2 \longrightarrow 3 \\ 3 \longrightarrow 1 \end{cases} \rightarrow \sigma^{-1} = \begin{cases} 2 \longrightarrow 1 \\ 3 \longrightarrow 2 \\ 1 \longrightarrow 3 \end{cases}$$

$$\sigma^2 = \begin{cases} 1 \longrightarrow 3 \\ 2 \longrightarrow 1 \\ 3 \longrightarrow 2 \end{cases} \quad 1 \to 2 \to 3.$$

$$T = \begin{cases} 1 \longrightarrow 2 \\ 2 \longrightarrow 1 \\ 3 \longrightarrow 3 \end{cases}$$

$$\sigma \circ T = \begin{cases} 1 \longrightarrow 3 \\ 2 \longrightarrow 2 \\ 3 \longrightarrow 1 \end{cases} \qquad T \longrightarrow \sigma$$

$$\sigma^2 \circ T = \begin{cases} 1 \longrightarrow 1 \\ 2 \longrightarrow 3 \\ 3 \longrightarrow 2 \end{cases} \qquad T \longrightarrow \sigma^2.$$

$T^2 = Id$

$\sigma^2 = \sigma^{-1} \longrightarrow B \to A$.

so $\sigma^3 = Id$

$T \circ \sigma(1) = 1$

$T \circ \sigma(2) = 3$.

$T \circ \sigma(3) = 2$.

$T \circ \sigma = \sigma^2 \circ T$.

$\boxed{\neq \sigma \circ T}$

$\sigma_3 = \{\enclose{circle}{1}, \sigma, \sigma^2, \tau, \sigma\tau, \sigma^2\tau.\}$.

$\sigma^3 = 1 \qquad \tau^2 = 1$.

$\tau\sigma = \sigma^2\tau$.

σ_n has $n!$ elements .

1st november 2018 .

Def^n A set X is finite when either i) $X = \emptyset$ or

ii) $\exists$ bijective mapping $g : \{1, \dots, n\} \to X$.

(so $\exists$ also $g^{-1} : X \to \{1, \dots, n\}$: Bijective .

If X is finite $\varphi : \{1 \dots n\} \to X$ bijective we write

$|X| = n$

A → B

$|X| =$ cardinal of X .

$|\emptyset| = 0$

Proposition . Let X, Y be finite sets with $|X| = |Y| = \underbrace{n}_{\{1, \dots, n\}} \geq 1$.

Let $f : X \to Y$ be a mapping

Then, i) f injective $\overset{\text{implies}}{\Longrightarrow}$ f surjective .

ii) f surjective $\Longrightarrow$ f injective .

Proof i) By induction on n .

obviously true on $n=1$. → $n = 1 \geq 1$ ✓

Suppose for $n-1$ . .

$\boxed{f : X \to Y \text{ is injective}}$.

Cont $X = \{x_1, \dots, x_n)$.

$f(x_n) \in Y$.

Put $X' = X - \{x_n\}$ $\quad f(x_n) \in Y$

$f ; X' \to Y - \{f(x_n)\}$ still injective .

Now $|X'| = n-1 = |Y - \{f(x_n)\}|$ → proposition

By inductive hypothesis → creo que es: si es injective y tienen la misma cardinality será surjective

$f : X' \to Y - \{f(x_n)\}$ is surjective

so $f : X \to Y$ also surjective .

ii) Supposse $f: X \to Y$ is surjective.

Cont $Y = \{y_1, \dots, y_n\}$.

f is surjective, so choose $x' \in X$; $f'(x') = y_n$.

Now $f: X - \{x'\} \to Y - \{y_n\}$. still surjection.

By inductive hypothesis → Si es surjective y = cardinality será injective

$f: X - \{x'\} \to Y - \{y_n\}$ is injective.

But $f(x') \neq f(x'')$ for any $x'' \in X - \{x'\}$

because $f(x') = y_n$ $f(x'') \in \{y_1, \dots, y_{n-1}\}$ → quito y_n.

So $f: X \to Y$ also injective QED.

Corollary: Let X, Y be finite sets with $|X| = |Y|$ and let

let $f: X \to Y$ be a mapping Then:

f is injective $\iff$ f is surjective.
$\iff$ f is bijective.

False for infinite sets.

eg $\mathbb{N} = \{0, 1, \dots, n, n+1 \dots$

$f: \mathbb{N} \to \mathbb{N}$ $f(n) = n+1$,

is injective but not surjective → porque ex el 0 no se puede obtener metiendo un nº N, tendría que ser (-1)

also, $g: \mathbb{N} \to \mathbb{N}$

$g(0) = 0$.

$g(n) = n-1$ $1 \leq n$.

g is surjective but not injective.

Consider $f: X \to X$ $|X| = n$.

Proposition If $|X| = n$ then there are precisely $n!$ bijective mappings

$f: X \to X$.

Proof By induction on n.

True for $n = 1$ (obviously)

Suppose proved for $n-1$.

Cont $X = \{x_1 \dots x_n\}$.

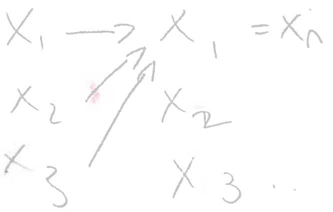
Consider $f: X \to X$.

How many ways can I choose $f(x_n)$?

n ways of choosing $f(x_n)$ → puedo meter en $f(x)$ ↑ tantas X como $n!$

Make a specific choice; para demostrarlo con $n-1$.

$f(x_n) = x'$.



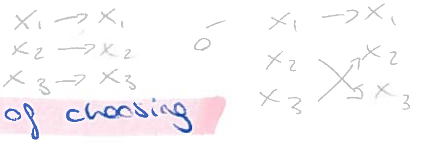
3 ways of getting X_n.

Now have a bijective mapping

$f : X - \{x_n\} \to X - \{x'\}$.

Both sides have cardinal $n-1$.

So try induction, having chosen $f(x_n)$ there $(n-1)!$ ways of completing mapping →



ó ó . . . $n!$ ways.

So ∃ $n \cdot (n-1)! = n!$ ways of choosing

$f : X \to X$ bijective QED.

Def^n : $\sigma_n = \{ f : \{1, \dots, n\} \to \{1, \dots, n\}$ bijective mapping.

so $|\sigma_n| = n!$

Proposition : If $f, g \in \sigma_n$ then $f \circ g \in \sigma_n$, $f^{-1} \in \sigma_n$.

$f : \{1 \dots n\} \to \{1, \dots n\}$.

Write f down as follows:

$$f = \begin{pmatrix} 1 & 2 & \dots & n \\ f(1) & f(2) & \dots & f(n) \end{pmatrix}$$

eg :

$$f = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 \\ 3 & 8 & 7 & 11 & 1 & 10 & 5 & 6 & 4 & 2 & 9 \end{pmatrix}.$$

$f(1) = 3 \quad f(2) = 8 \quad \dots \quad f(11) = 9$.

Composition of permutations

$n = 5 \quad f = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 4 & 5 & 1 & 3 & 2 \end{pmatrix} \quad g = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 2 & 4 & 1 & 5 \end{pmatrix}$

$$g \circ f = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ \cancel{4} & \cancel{5} & \cancel{1} & \cancel{3} & \cancel{2} \\ 1 & 5 & 3 & 4 & 2 \end{pmatrix} \begin{matrix} \} f \\ \} g \end{matrix}.$$

first f then g.

$$g \circ f = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 5 & 3 & 4 & 2 \end{pmatrix}.$$

$$f^{-1} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 5 & 4 & 1 & 2 \end{pmatrix}.$$

$$\rho = \begin{pmatrix} 1 & 2 & \cdots & n \\ \rho(1) & \rho(2) & \cdots & \rho(n) \end{pmatrix}$$. FULL FORM of ρ.

There is an abbreviated form : cycle notation :

CYCLIC PERMUTATIONS

Ex : $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 4 & 2 & 6 & 3 & 5 & 1 \end{pmatrix}$

$= (1, 4, 3, 6)$.

Ex : $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 5 & 6 & 1 & 2 & 3 & 7 & 4 \end{pmatrix}$

$= \big(\quad (2, 6, 7, 4)(1, 5, 3) \big) = (1, 5, 3)(2, 6, 7, 4)$.

Def^n : Let $\{a_1, a_2, \ldots, a_k\} \subset \{1, 2, \ldots n\}$.

Define the cyclic permutation

$(a_1, a_2, \ldots, a_k)$ as follows .

$$(a_1, a_2, \ldots a_k)(a_i) = \begin{cases} a_{i+1} & \text{if } 1 \leq i \leq k-i \\ a_1 & \text{if } i = k \end{cases} .$$

$(a_1, a_2 \ldots a_k)(x) = x$ if $x \notin \{a_1 \ldots a_k\}$.

$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & x \\ 2 & 3 & 4 & 5 & 1 & x \end{pmatrix}(1, 2, 3, 4, 5)(x)$

Disjoint cycles :

Let $\{a_1, \ldots, a_k\} \subset \{1, \ldots n\}$

$\{b_1, \ldots b_m\} \subset \{1, \ldots n\}$.

Say that $(a_1 \ldots a_k)$, $(b_1 \ldots b_m)$

are disjoint cycles when $\{a_1 \ldots a_k) \cap \{b_1 \ldots b_m\} = \emptyset$

Theorem

Let $\rho \in \sigma_n$ → bijective mapping then either :

i) $\rho = Id$ or

ii) ρ is a cycle or :

iii) ρ is a product of disjoint cycles .

iv) Disjoint cycles commute .

ex :

$$\rho = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 \\ 3 & 8 & 7 & 11 & 1 & 10 & 5 & 6 & 4 & 2 & 9 \end{pmatrix}$$

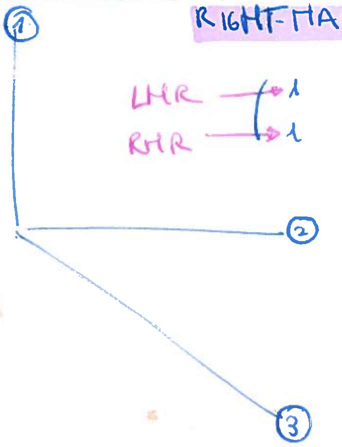
$(4, 11, 9)$.

$= \big(\swarrow(2, 8, 6, 10)$. $(1,3,7,5)$

$= (4,11,9)(1,3,7,5)(2,8,6,10) = (1,3,7,5)(2,8,6,10)(4,11,9)$.

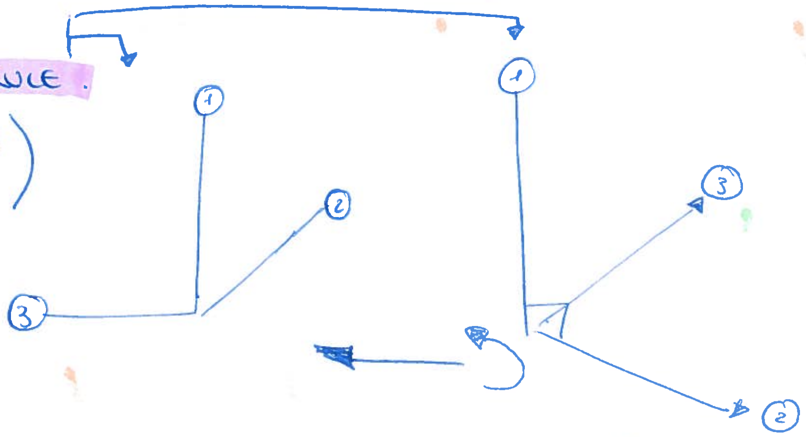
november 2nd 2018

LEFT-HAND RULE



RIGHT-HAND RULE.

$$\begin{array}{c} \text{LHR} \to \\ \text{RHR} \to \end{array}\begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}$$



To each permutation:

$$f = \begin{pmatrix} 1 & 2 & \dots & n \\ f(1) & f(2) & \dots & f(n) \end{pmatrix}$$

we associate a sign $+1$ or -1.

$+1$ → preserves orientation

-1 → reverses orientation

✱ IMPORTANT:

Example: $\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 3 & 4 & 5 & 1 \end{pmatrix}$ separado $\boxed{(1,2,3,4,5) = (1,5)(1,4)(1,3)(1,2)}$.

↳ Bijective mapping

Generalisation ✱

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 1 & 3 & 4 & 5 \\ 2 & 3 & 1 & 4 & 5 \\ 2 & 3 & 4 & 1 & 5 \\ 2 & 3 & 4 & 5 & 1 \end{pmatrix}$$

→ Switch 1 and 2. $(1,2)$

→ Switch 2 and 3 $(2,3)$

→ Switch 1 and 4 $(1,4)$

→ Switch 1 and 5 $(1,5)$.

✱, GENERALISATION

$$(a_1, a_2, a_3, \dots, a_n) = (a_1, a_n)(a_1, a_{n-1}) \dots (a_1, a_3)(a_1, a_2)$$

Cycle of length n $=$ $\underbrace{(n-1)}$

there are $(n-1)$ multiplication terms.

A transposition is a cycle of length 2.

ex: $(2,7)$ ó $(7,100)$.

Proposition A cycle of length n is a product of $(n-1)$ transpositions so:

I) A cycle of ODD length is a product of any EVEN nb of transpositions

II) A cycle of Even length is a product of any odd nb of transpositions

Last time we showed that:

Theorem Any permutation is a product of disjoint cycles.

Corollary: Any permutation is a product of transpositions

Defn: let $\sigma \in S_n$. Write $\text{sign}(\sigma) = +1$ when σ is product of EVEN nb of transpositions

Sign $\sigma = -1$ when σ is product of ODD nb of transpositions

Need to show at some point that: it is not possible to write same permutations as both product of even nb of terms and also as product of odd nb of terms.

How to calculate $\text{sign}(\sigma)$:?

Ex 1 $\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 \\ 5 & 6 & 7 & 11 & 3 & 8 & 1 & 2 & 10 & 4 & 9 \end{pmatrix}$.

1st Write σ as a product of disjoint cycles: 2nd and count length of cycles.

$$\sigma = \overbrace{(1,5,3,7)}^{3 \text{ transpositions}} \; (2,6,8) \; (4,11,9,10)$$

$$\quad (-1) \qquad\qquad (+1) \qquad\qquad (-1)$$

$\text{sign}(\sigma) = +1$.

Ex 2

$p = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 \\ 6 & 9 & 4 & 3 & 12 & 11 & 10 & 13 & 5 & 8 & 1 & 2 & 7 \end{pmatrix}$.

$$p = \overbrace{(1,6,11)}^{2 \text{ transposition}} \underbrace{(2,9,5,12)}_{3 \text{ transpositions}} (3,4) \; (7,10,8,13).$$

$\text{Sign}(\sigma) = (+1) \cdot (-1)(-1)(-1) = \boxed{-1}$

$\text{order}(p) = 12$.

ORDER OF PERMUTATION

$\text{ord}(\sigma) = \min \{ N \geq 1 \; ; \; \sigma^N = 1 \}$

example: $(1,5,3,4)$ has order 4.

Order of $(a_1 a_2, \dots a_n) = n$. Iterate a cycle of length n, n times and get the identity

$(1,5,3,4)$.

volver a conseguir el = nº del principio

example: $\sigma = \underbrace{(1,5,3,4)}_{X_1}^{\text{order } 4} \underbrace{(2,7,6)}_{X_2}^{\text{order } 3}$.

$\underbrace{X_1 X_2}_{\text{order } 12} = \underbrace{X_2 X_1}_{\text{order } 12}$ (Disjoint cycles) ciclo → porque el mcm de 4 y 3 es 12.

Proposition: If X_1, X_2 are disjoint cycles.

$$(X_1 X_2)^N = X_1^N X_2^N .$$

(This is only true because)

$$X_2 X_1 = X_1 X_2$$

Proposition: If $X_1, \dots, X_m$ are disjoint cycles

$\text{Order}(X_1 \dots X_m) = \text{LCM}(\text{ord}(X_1), \dots, \text{ord}(X_m))$ least community multiple
$= \text{LCM}(\text{length}(X_1), \dots \text{length}(X_m))$.

$(X_1, \dots, X_m)^N = X_1^N X_2^N \dots X_m^N$. Because they commute.

Ex:

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 \\ 5 & 7 & 2 & 10 & 8 & 4 & 3 & 11 & 1 & 12 & 9 & 6 \end{pmatrix}.$$

$\sigma = (1,5,8,11,9)(2,7,3)(4,10,12,6)$ $\quad \text{sign}\,\sigma = -1$.

$\text{ord}(\sigma) = \text{LCM}(5,3,4) = 60$.

Say that a transposition (i,j) is ADJACENT when $|j-i| = 1$

ex: $(3,4)$ $(6,7)$.

but not $(1,3)$.

Proposition: Any transposition is a product of an ODD nb of adjacent transpositions

Proof: Define $\text{gap}(i,j) = |j-i|$

If $\text{gap}(i,j) = k$ Then (i,j) is a product of $(2k-1)$ adjacent transpositions

If gap $=1$ nothing to prove !, it is already adjacent.

Suppose proved for gap $= k-1$.

Suppose gap $(i,j)=k$.

$$\left(\begin{array}{ccc} i & i+1 & j \\ \cancel{i+1} & \cancel{i} & j \\ j & \cancel{i} & i+1 \\ \cancel{j} & \cancel{i+1} & i \\ j & i+1 & i \end{array}\right) \begin{array}{l} \rightarrow (i,i+1) \\ \rightarrow (i+1,j) \\ \rightarrow (j,i+1) \end{array}$$

gaps

$$(i,j) = \underset{1}{(i,\ i+1)}\ \underset{k-1}{\overbrace{(i+1,\ j)}^{(j-i)-1=k-1}}\ \underset{1}{(i,\ i+1)}.$$

product of odd nb of adjacent transpositions $2k-1$.

$2k-3 = 2(k-1)-1$.

By induction $(i+1,j) = T_1\,T_2 \cdots T_{2k-3}$

$$(i,j) = (i,\ i+1)\,T_1 \cdots T_{2k-3}\,(i,(i+1))$$

Adjacent transpositions

get $\underbrace{1+(2k-3)+1 = 2k-1}_{\text{odd nb of transpositions}}$

QED.

✱ IMPORTANT

$\mathbb{N} = \{0,1,2 \ldots n, n+1, \ldots\}$ natural nos.

Can add, multiply, but usually can't substract and almost never divide

$\mathbb{Z} = \{0, \pm1, \pm2 \ldots \pm n, \pm(n+1) \ldots\}$ integers.

Can add, substract, multiply, but DIVISION STILL A PROBLEM.

$$\mathbb{Q} = \left\{ \frac{m}{n} \ ;\ \begin{array}{l} m,n \in \mathbb{Z} \\ n \neq 0 \end{array} \right\}$$

Rule of Equality $\dfrac{m}{n} = \dfrac{m'}{n'} \iff mn' = m'\cdot n$

CAN add, substract, multiply and divide

✱ write them as product of adjacen transpositions.

$(2,4) = (2,3)(3,4)(2,3)$

$(1,4) = (1,2)(2,3)(3,4)(2,3)(1,2)$

$(2,6) = (2,3)(3,4)(4,5)(5,6)(4,5)(3,4)$ (2,

Defn By a field I mean

$\mathbb{F} = (\mathbb{F}, +, 0, \cdot, 1)$.

where $\mathbb{F}$ is a set.

$0,1 \in \mathbb{F}$ $\quad 0 \neq 1$

sume - 2 n^os de $\mathbb{F}$

$+ ;\ \mathbb{F} \times \mathbb{F} \longrightarrow \mathbb{F}$ is a mapping

$(x,y) \longrightarrow x+y$.

write $x+y$ rather than $+(x,y)$.

$+$ is assumed to have following properties.

$x+(y+z) = (x+y)+z$ ASSOCIATIVE

$x+y = y+x$ COMMUTATIVE

$x+0 = 0+x = x$ (0 is ADDITIVE IDENTITY)

$\forall x \in \mathbb{F} \quad \exists (-x) \in \mathbb{F} \quad x + (-x) = (-x) + x = 0$ ADDITIVE INVERSE

MULTIPLICATIVE AXIOMS:

$\cdot,\ \mathbb{F} \times \mathbb{F} \longrightarrow \mathbb{F}$

$(x, y) \longrightarrow x \cdot y$ (not $\cdot(x,y)$) .

Such that $(x \cdot y) \cdot z = x \cdot (y \cdot z)$ ASSOCIATIVE

$x \cdot y = y \cdot x$ COMMUTATIVE

$x \cdot 1 = 1 \cdot x = x$ $\quad 1$ is MULTIPLICATIVE IDENITY

$\forall x \in \mathbb{F} - \{0\}\ \exists\ x^{-1}; \quad x \cdot x^{-1} = x^{-1} \cdot x = 1$. MULTIPLICATIVE INVERSE

(Non zero elements have multiplicative inverse)

DISTRIBUTIVE AXIOM

$(x+y) \cdot z = x \cdot z + y \cdot z$.

$z \cdot (x+y) = z \cdot x + z \cdot y$.

Anything with the above properties is called a field

es: $\mathbb{Q} = (\mathbb{Q}, +, 0, \cdot, 1)$ is a field .

$\mathbb{Z} = (\mathbb{Z}, +, 0, \cdot, 1)$ is $\boxed{\text{NOT}}$ a field, no multiplicative inverse .

Example 2: $\mathbb{R}$ is a field .

$\mathbb{F}_2$ field with 2 elements .

$\mathbb{F}_2 = \{\text{EVEN}, \text{ODD}\}$.

EVEN = 0 → si el resultado es un nº par pongo 0 .

ODD = 1 . → si el resultado = odd pongo −1 .

+	EVEN	ODD
EVEN	EVEN	ODD
ODD	ODD	EVEN

→

+	0	1
0	0	1
1	1	0

·	0	1
0	0	0
1	0	1

$\mathbb{F}_3$ field with 3 elements $\{0, 1, 2\}$.

$\boxed{2^{-1} = 2 \quad 1^{-1} = 1}$.

+	0	1	2
0	0	1	2
1	1	2	0
2	2	0	1

·	0	1	2
0	0	0	0
1	0	1	2
2	0	2	1

$\boxed{2^{-1} = 2 \quad 1^{-1} = 1}$

Let's try some idea with $\{0, 1, 2, 3\}$ remainders mod 4.

·	0	1	2	3
0	0	0	0	0
1	0	1	2	3
2	0	2	0	2
3	0	3	2	1

$\mathbb{Z}/4$ not a field.
There is a field $\mathbb{F}_4$
with 4 elements.

$\{0, 1, 2, 3, 4\}$ remainders mod 5.

·	1	2	3	4
0	0	0	0	0
1	0	2	3	4
2	0	4	1	3
3	0	1	4	2
4	0	3	2	1

This is a field $\mathbb{F}_5$

$2^{-1} = 3 \quad 3^{-1} = 2 \quad 4^{-1} = 4$.

Take remainders mod p.

$\{0, 1, \dots, p-1\}$ is a field $\mathbb{F}_n \iff p$ is prime

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$\mathbb{F}$ field

$$\mathbb{F}^n = \left\{ x = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} ; x_n \in \mathbb{F} \right.$$

- Addition

$+ : \mathbb{F}^n \times \mathbb{F}^n \to \mathbb{F}^n$

$$\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}, \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix} \longrightarrow \begin{pmatrix} x_1 + y_1 \\ x_2 + y_2 \\ \vdots \\ x_n + y_n \end{pmatrix}$$

- Scalar multiplication
  - ; $\mathbb{F} \times \mathbb{F}^n \longrightarrow \mathbb{F}^n$

$$\Big(\lambda, \underset{\underset{\sim}{x}}{\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}}\Big) \longrightarrow \underset{\lambda \underset{\sim}{x}}{\begin{pmatrix} \lambda x_1 \\ \lambda x_2 \\ \vdots \\ \lambda x_n \end{pmatrix}}$$

- Zero: $\underset{\sim}{0} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$

Thy have the following properties:

$\boxed{1}$ ADDITIVE PROPERTIES:

- $\underset{\sim}{x} + (\underset{\sim}{y} + \underset{\sim}{z}) = (\underset{\sim}{x} + \underset{\sim}{y}) + \underset{\sim}{z}$: Associative
- $\underset{\wedge}{x} + \underset{\sim}{y} = \underset{\wedge}{y} + \underset{\wedge}{x}$: commutative
- $\underset{\sim}{0} + \underset{\wedge}{x} = \underset{\wedge}{x} + \underset{\wedge}{0} = \underset{\wedge}{x}$: identity
- $\forall \underset{\wedge}{x} \in \mathbb{F}^n, \exists -x \in \mathbb{F}^n$; $x + (-\underset{\wedge}{x}) = \underset{\wedge}{0}$: inverses

$\boxed{2}$ SCALAR MULTIPLICATION PROPERTIES

- $\lambda \cdot (\underset{\sim}{x} + \underset{\sim}{y}) = \lambda \cdot \underset{\wedge}{x} + \lambda \underset{\wedge}{y}$: distributive
- $(\lambda + \mu) \cdot \underset{\wedge}{x} = \lambda \underset{\wedge}{x} + \mu \underset{\wedge}{x}$: distributive
- $1 \cdot \underset{\wedge}{x} = \underset{\wedge}{x}$: identity
- $0 \cdot \underset{\wedge}{x} = \underset{\wedge}{0}$: inverse
- $\lambda (\mu \underset{\wedge}{x}) = (\lambda \mu) \cdot \underset{\wedge}{x}$: associative

Def^n Let $\mathbb{F}$ be a field. By a vector space, V over $\mathbb{F}$, I mean:

$V = (V, +, \underset{\wedge}{0}, \cdot)$ where:

i) V is a set.

ii) $\underset{\wedge}{0} \in V$.

iii) $+ ; V \times V \longrightarrow V$ is a mapping which satisfies the properties above.

$$(\underset{\wedge}{x}, \underset{\wedge}{y}) \longrightarrow \underset{\wedge}{x} + \underset{\wedge}{y}.$$

- $\underset{\wedge}{x} + (\underset{\wedge}{y} + z) = (\underset{\wedge}{x} + \underset{\wedge}{y}) + z$ Associative
- $\underset{\wedge}{x} + \underset{\wedge}{y} = \underset{\wedge}{y} + \underset{\wedge}{x}$: commutative.
- $\underset{\wedge}{0} + \underset{\wedge}{x} = \underset{\wedge}{x} + \underset{\wedge}{0} = \underset{\wedge}{x}$: identity.
- $\forall \underset{\wedge}{x} \in V, \exists (-x) \in V$: $x + (-x) = \underset{\wedge}{0}$: inverses.

iv) $\cdot \, ; \mathbb{F} \times V \longrightarrow V$ mapping

$(\lambda, x) \longrightarrow \lambda \cdot x$.

such that ⟶ they satisfy the scalar mult. properties .

Example

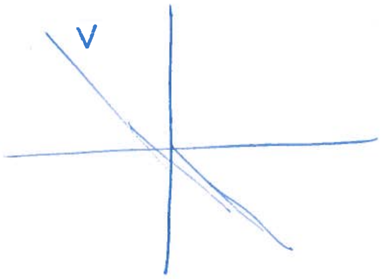
i) $\mathbb{F}^n$ is a vector space over $\mathbb{F}$ for all $n \geq 2$

ii) For $n=1$, $\mathbb{F}$ is a vector space over $\mathbb{F}$.

↳ un pto

Snag : Not every vector space is actually an $\mathbb{F}^n$. $\mathbb{F}^{-1}$ = no existe

Example .. $\mathbb{F} = \mathbb{Q}$



$$V = \left\{ \begin{pmatrix} \lambda \\ -\lambda \end{pmatrix} ; \lambda \in \mathbb{F} \right\} .$$

$$V = \{ V, +, \underline{0}, \cdot \} .$$

$+ : \; V \times V \longrightarrow V$.

$$\begin{pmatrix} \lambda \\ -\lambda \end{pmatrix} + \begin{pmatrix} \mu \\ -\mu \end{pmatrix} = \begin{pmatrix} \lambda + \mu \\ -(\lambda + \mu) \end{pmatrix} .$$

$\cdot \, : \; \mathbb{F} \times V \longrightarrow V$

$$\left(\mu , \begin{pmatrix} \lambda \\ -\lambda \end{pmatrix} \right) \longrightarrow \begin{pmatrix} \mu \lambda \\ -\mu \lambda \end{pmatrix}$$

$$\underline{0} = \begin{pmatrix} 0 \\ -0 \end{pmatrix} .$$

λ ,   no pertenece a $V = \begin{pmatrix} \lambda \\ -\lambda \end{pmatrix}$

$V \subset \mathbb{F}^2$.

$V \neq \mathbb{F}^2$ why not ? : $\begin{pmatrix} 1 \\ 0 \end{pmatrix} \notin V$ $-\lambda$

Let V be a vector space over $\mathbb{F}$

Let $\underline{v}_1, \dots, \underline{v}_n \in V$.

Say that $\{ \underline{v}_1 , \dots , \underline{v}_n \}$ is .. Linearly independent when

FORMAL DEFINITION

$$\boxed{\begin{aligned} &\lambda_1 \underline{v}_1 + \dots + \lambda_n \underline{v}_n = \underline{0} \quad \lambda_i \in \mathbb{F} \\ &\Rightarrow \lambda_1 = \lambda_2 = \dots = \lambda_n = 0 . \end{aligned}}$$

An expression of the form $\lambda_1 \underline{v}_1 + \lambda_2 \underline{v}_2 + \dots + \lambda_n \underline{v}_n$ $(\lambda_i \in \mathbb{F})$ is called a LINEAR COMBINATION in $\{ \underline{v}_1 , \underline{v}_2 , \dots , \underline{v}_n \}$

Note that taking $\lambda_1 = \lambda_2 = \dots = \lambda_n = 0$ always GIVES . $0 \underline{v}_1 + 0 \underline{v}_2 + \dots + 0 \underline{v}_n = \underline{0}$.

- What the def^n says is that $\{\underset{\sim}{v}_1, \dots, \underset{\sim}{v}_n\}$ is L.I iff the only way to get 0 is by having all coefficients $=0$.

Example: $V = \mathbb{F}^3$.

$$\underset{\sim}{e}_1 = \begin{pmatrix}1\\0\\0\end{pmatrix} \quad \underset{\sim}{e}_2 = \begin{pmatrix}0\\1\\0\end{pmatrix} \quad \underset{\sim}{e}_3 = \begin{pmatrix}0\\0\\1\end{pmatrix}.$$

Claim that $\{\underset{\sim}{e}_1, \underset{\sim}{e}_2, \underset{\sim}{e}_3\}$ is L.I

so suppose: $\lambda_1 \underset{\sim}{e}_1 + \lambda_2 \underset{\sim}{e}_2 + \lambda_3 \underset{\sim}{e}_3 = \underset{\sim}{0} = \begin{pmatrix}0\\0\\0\end{pmatrix}$

$$\lambda_1\begin{pmatrix}1\\0\\0\end{pmatrix} + \lambda_2\begin{pmatrix}0\\1\\0\end{pmatrix} + \lambda_3\begin{pmatrix}0\\0\\1\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix}.$$

$$\begin{pmatrix}\lambda_1\\0\\0\end{pmatrix} + \begin{pmatrix}0\\\lambda_2\\0\end{pmatrix} + \begin{pmatrix}0\\0\\\lambda_3\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix}.$$

$$\begin{pmatrix}\lambda_1\\\lambda_2\\\lambda_3\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix}$$ and read! $\lambda_1 = 0$, $\lambda_2 = 0$, $\lambda_3 = 0$.

Example 2 $V = \mathbb{F}^3$.

$$\underset{\sim}{\varphi}_1 = \begin{pmatrix}1\\0\\0\end{pmatrix} \quad \underset{\sim}{\varphi}_2 = \begin{pmatrix}1\\1\\0\end{pmatrix} \quad \underset{\sim}{\varphi}_3 = \begin{pmatrix}1\\1\\1\end{pmatrix}.$$

$\{\underset{\sim}{\varphi}_1, \varphi_2, \varphi_3\}$ also L.I.

Take

$$\lambda_1 \underset{\sim}{\varphi}_1 + \lambda_2 \underset{\sim}{\varphi}_2 + \lambda_3 \underset{\sim}{\varphi}_3 = \underset{\sim}{0}$$

$$\begin{pmatrix}\lambda_1\\0\\0\end{pmatrix} + \begin{pmatrix}\lambda_2\\\lambda_2\\0\end{pmatrix} + \begin{pmatrix}\lambda_3\\\lambda_3\\\lambda_3\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix}.$$

$$\begin{pmatrix}\lambda_1 + \lambda_2 + \lambda_3\\ \lambda_2 + \lambda_3\\ \lambda_3\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix}.$$

read: $\lambda_3 = 0$

$\lambda_2 + \lambda_3 = 0$; $\lambda_2 = 0$

$\lambda_1 + \lambda_2 + \lambda_3 = 0$; $\lambda_1 = 0$.

Example : $V = \mathbb{F}^3$

$$\underset{\sim}{t}_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \quad \underset{\sim}{t}_2 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \quad \underset{\sim}{t}_3 = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}.$$

is not L.I:

$2\underset{\sim}{t}_1 - \underset{\sim}{t}_2 = \underset{\sim}{t}_3$ so we have

so $2\underset{\sim}{t}_1 - \underset{\sim}{t}_2 - \underset{\sim}{t}_3 = 0$.

and at least one coefficient $\neq 0$

$\lambda_1 = 2, \quad \lambda_2 = -1, \quad \lambda_3 = -1$.

$$\boxed{\lambda_3 \neq 0}$$

Vector space $/ \mathbb{F}$ → todos los vectores del espacio vectorial se obtienen de los vectores indep.

Let $\underset{\sim}{v}_1, \dots, \underset{\sim}{v}_n \in V$.

Say that $\{\underset{\sim}{v}_1, \underset{\sim}{v}_2, \dots, \underset{\sim}{v}_n\}$ $\overset{\text{alcanza}}{\text{spans}}$ V when

$\forall \underset{\sim}{v} \in V, \exists \lambda_1, \dots, \lambda_2 \in \mathbb{F}$ s.t $\underset{\sim}{v} = \lambda_1 \underset{\sim}{v}_1 + \dots + \lambda_n \underset{\sim}{v}_n$

↳ un vector se obtiene de la combinación de 3 independientes

In english: every vector $\underset{\sim}{v}$ in V . c

can be expressed as a LINEAR COMBINATION in $\{\underset{\sim}{v}_1, \dots, \underset{\sim}{v}_n\}$.

Ex : $\underset{\sim}{e}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \quad \underset{\sim}{e}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \quad \underset{\sim}{e}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$.

SPAN $\mathbb{F}^3$: why?

$\underset{\sim}{x} \in \mathbb{F}^3 \qquad \underset{\sim}{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$

$\underset{\sim}{x} = x_1 \underset{\sim}{e}_1 + x_2 \underset{\sim}{e}_2 + x_3 \underset{\sim}{e}_3$.

Exercise: $\varphi_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \quad \varphi_2 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \quad \varphi_3 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$.

$\{\varphi_1, \varphi_2, \varphi_3\}$ also SPAN $\mathbb{F}^3$.

son base de $\mathbb{F}^3$

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V = vector space $/ \mathbb{F}$.

$\underset{\sim}{v}_1, \underset{\sim}{v}_2, \dots, \underset{\sim}{v}_n \in V$.

- $\{v_1, \dots, v_n\}$ are L.I when : $\boxed{\lambda_1 \underset{\sim}{v}_1 + \lambda_2 \underset{\sim}{v}_2 + \lambda_3 \underset{\sim}{v}_3 = 0}$; $(\lambda \in \mathbb{F})$.
  ; $\boxed{\lambda_1 = \lambda_2 = \lambda_3 = 0}$.
- $\{v_1, \dots, v_n\}$ $\boxed{\text{spans}}$ V when : $\forall \underset{\sim}{x} \in V \ \exists \lambda_1, \dots, \lambda_n \in \mathbb{F}$ s.t.

$$\boxed{\underset{\sim}{x} = \lambda_1 \underset{\wedge}{v}_1 + \dots + \lambda_n \underset{\wedge}{v}_n}$$

STANDARD EXAMPLES:

$$V = \mathbb{F}^n = \left\{ \underline{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} ; x_i \in \mathbb{F} \right\}$$

$$\underset{\wedge}{e}_1 = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \\ 0 \end{pmatrix} \quad \underset{\wedge}{e}_2 = \begin{pmatrix} 0 \\ 1 \\ \vdots \end{pmatrix} \quad \underset{\wedge}{e}_{n-1} = \begin{pmatrix} \vdots \\ 1 \\ 0 \end{pmatrix} \quad \underset{\wedge}{e}_n = \begin{pmatrix} \vdots \\ 1 \end{pmatrix}$$

el 1 va en la pósicion n-1

en la pos n

es : $n = 4$.

$$\underline{e}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad \underline{e}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \quad \underline{e}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \quad \underline{e}_4 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

Proposition $\{\underset{\wedge}{e}_1, \dots, \underset{\wedge}{e}_n\}$ is always L.I.

Proof Suppose $\lambda_1 \underset{\wedge}{e}_1 + \dots + \lambda_n \underline{e}_n = \underline{0} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$

$$\lambda_1 \underline{e}_1 = \begin{pmatrix} \lambda_1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \qquad \lambda_2 \underline{e}_2 = \begin{pmatrix} 0 \\ \lambda_2 \\ \vdots \\ 0 \end{pmatrix}$$

so $\lambda_1 \underset{\wedge}{e}_1 + \dots + \lambda_n \underset{\wedge}{e}_n = \begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \vdots \\ \lambda_4 \end{pmatrix}$

so if $\lambda_1 \underset{\wedge}{e}_1 + \lambda_2 \underline{e}_2 + \dots + \lambda_n \underline{e}_n = 0 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$

Then, reading $\lambda_1 = 0$, $\lambda_2 = 0$, ... $\lambda_n = 0$

so i.e. $\{\underset{\wedge}{e}_1, \dots, e_n\}$ is L.I.

Proposition $\{e_1, \ldots, e_n\}$ always spans $\mathbb{F}^n$.

Proof: $$x = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = x_1 e_1 + x_2 e_2 + x_3 e_3 \quad + x_n e_n$$

$n = 4$:
$$x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = x_1 \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + x_2 \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} + x_3 \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}.$$

Def^n Let V be a vector space $/\mathbb{F}$. Let $E_1, \ldots, E_n \in V$. Say that $\{E_1, E_2, \ldots, E_n\}$ is a **BASIS** for V when $E_1, \ldots, E_n$ is L.I and SPANS V.

Our previous calculations show that:

Th^m $\{e_1, \ldots, e_n\}$ form a basis for $\mathbb{F}^n$.

eg: $n = 3$.

$e_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ $e_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ $e_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$. bases for $\mathbb{F}^3$.

BASIS THEOREM

If V is a vector space $(V \neq 0)$, then:

i) V has at least 1 basis.

ii) Any two bases for V have same nb of elements.

(e_1, e_2, e_3) → Hay 3 e, es decir 3 elementos

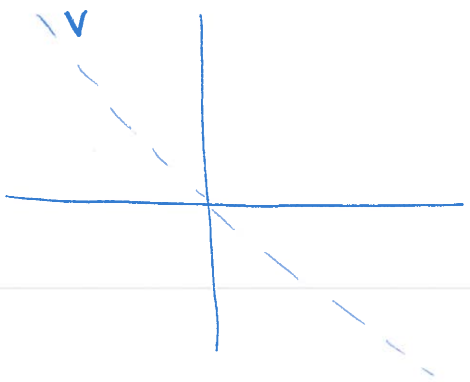
Def^n: If V is a vector space $(V \neq 0)$

$\dim(V)$ = nb of elements in any basis for V.

$\dim(V)$ = dimension of V.

Proposition: $\dim(\mathbb{F}^n) = n$.

Proof $\{e_1, \ldots e_n\}$ is L.I and spans.



$\mathbb{F} = \mathbb{Q}$

Example 1

$$V = \left\{ \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \mathbb{Q}^2 \; ; \; x_1 + x_2 = 0 \right\}.$$

V is a vector space $\left(\cancel{= \mathbb{F}^2}\right)$.

Addition

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} x_1 + y_1 \\ x_2 + y_2 \end{pmatrix} \in V.$$

$$x_1 + x_2 = 0.$$

$$y_1 + y_2 = 0.$$

$$\Rightarrow (x_1 + y_1) + (x_2 + y_2) = 0.$$

Scalar multiplication

$$\underset{\wedge}{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in V \; ; \; \lambda \in \mathbb{F}.$$

$$\lambda \underset{\wedge}{x} = \begin{pmatrix} \lambda x_1 \\ \lambda x_2 \end{pmatrix} \Rightarrow \lambda (x_1 + x_2) = 0$$

$$\Rightarrow \lambda x_1 + \lambda x_2 = 0.$$

multiplico el $v \in V$ por un nº de $\mathbb{F}$ y me da un vector que tb $\in V$.

Question: $\dim (V) = ?$.

$$\left(\underset{\sim}{0} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \in V. \right)$$

$\dim (V) = 1$.

Does it have a basis?

Yes: $\varphi = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ $\{ \varphi \}$ spans V.

$$\underset{\wedge}{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} ; \; x_1 + x_2 = 0 \; ; \quad \underset{\wedge}{x} = x_1 \underset{\wedge}{\varphi}.$$

$\{\underset{\wedge}{\varphi}\}$ is L.I

$$\lambda \underset{\wedge}{\varphi} = \underset{\wedge}{0}$$

$$\begin{pmatrix} \lambda \\ -\lambda \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \text{Read } 0 = \lambda.$$

Example 2:

$$V = \left\{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \in \mathbb{Q}^3 \; ; \; x_1 + x_2 + x_3 = 0 \right\}.$$

V is a vector space.

Addition:

$$\underset{\wedge}{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \in V. \quad \underset{\wedge}{y} = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} \in V \; ; \; \underset{\wedge}{x} + \underset{\wedge}{y} = \begin{pmatrix} x_1 + y_1 \\ x_2 + y_2 \\ x_3 + y_3 \end{pmatrix} \in V.$$

- **Scalar multiplication**

$\underset{\wedge}{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$; $x_1 + x_2 + x_3 = 0$. zero : $\underset{\wedge}{0} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \in V$.

$\underset{\wedge}{x} \in V$

$\lambda \underset{\wedge}{x} = \begin{pmatrix} \lambda x_1 \\ \lambda x_2 \\ \lambda x_3 \end{pmatrix} = \lambda (x_1 + x_2 + x_3) = 0$.

$\lambda \underset{\wedge}{x} \in V$.

All other axioms are satisfied as satisfied already in $\mathbb{Q}^3$.

Bases for V

$\varphi_1 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$ $\varphi_2 = \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}$.

$\varphi_1, \varphi_2 \in V$.

- $\lambda_1 \varphi_1 + \lambda_2 \varphi_2 = \begin{pmatrix} -\lambda_1 \\ -\lambda_2 \\ \lambda_2 + \lambda_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$.

Read: $\lambda_1 = 0$, $\lambda_2 = 0$.

so $\{\varphi_1, \varphi_2\}$ is L.I .

- Spans V:

$\lambda_1 \varphi_1 + \lambda_2 \varphi_2 = \begin{pmatrix} -\lambda_1 \\ -\lambda_2 \\ \lambda_2 + \lambda_1 \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$

So suppose $\underset{\wedge}{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$ s.t $x_1 + x_2 + x_3 = 0$.

$\boxed{\text{So } \dim(V) = 2}$.

nº de φ que hay

want to choose $\lambda_1, \lambda_2 \in \mathbb{F}$ s.t:

$\underset{\wedge}{x} = \lambda_1 \underset{\wedge}{\varphi_1} + \lambda_2 \underset{\wedge}{\varphi_2}$. Take $-\lambda_1 = x_1$, $-\lambda_2 = x_2$.

then $x_3 = \lambda_1 + \lambda_2$.

Let $A = (a_{ij})_{\substack{1 \le i \le m \\ 1 \le j \le n}}$.

$m \times n$ over $\mathbb{F}$ ($a_{ij} \in \mathbb{F}$) .

$$\left\{ \begin{array}{l} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = 0 \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n = 0 \end{array} \right.$$

Def: $K_A = \{ \underset{\wedge}{x} \in \mathbb{F}^n : A\underset{\wedge}{x} = 0 \}$

$A\underset{\wedge}{x} = 0$.

K_A is the solution set to system

(El set del espacio vectorial está formado por todos los vectores que cumplen $Ax = 0$)

Proposition: K_A is a vector space / $\mathbb{F}$.

Proof: observe: $K_A \subset \mathbb{F}^n$.

- **Addition** suppose $\underset{\sim}{x}, \underset{\sim}{y} \in K_A$.

$$\text{so} \quad \left.\begin{array}{l} A\underset{\sim}{x} = 0 \\ A\underset{\sim}{y} = 0 \end{array}\right\} \begin{array}{l} \Rightarrow A\underset{\sim}{x} + A\underset{\sim}{y} = 0 \\ \Rightarrow A(\underset{\sim}{x} + \underset{\sim}{y}) = 0. \end{array}$$

so if $\underset{\sim}{x}, \underset{\sim}{y} \in K_A$.

$\underset{\sim}{x} + \underset{\sim}{y} \in K_A$.

- **Scalar multiplication** $\underset{\sim}{x} \in K_A$; $\lambda \in \mathbb{F}$.

$A\underset{\sim}{x} = 0$

$\lambda(A\underset{\sim}{x}) = 0 \Rightarrow A(\lambda\underset{\sim}{x}) = 0$

so $\lambda\underset{\sim}{x} \in K_A$.

Zero : $A\underset{\sim}{0} = \underset{\sim}{0}$ so $\underset{\sim}{0} \in K_A$.

All other axioms satisfied automatically, already satisfy in $\mathbb{F}^n$.

Let's find out how to find : $<$ i) Basis for K_A ii) dim K_A.

First consider case where A is in reduced row echelon form

eg :

$$A = \begin{array}{c} \begin{pmatrix} 1 & 1 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & -1 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 & 1 & -1 & 1 \end{pmatrix} \\ \begin{matrix} \enclose{circle}{x_1} & x_2 & \enclose{circle}{x_3} & x_4 & \enclose{circle}{x_5} & x_6 & x_7 \end{matrix} \end{array}.$$

$\boxed{F = \mathbb{Q}}$

Solve $A\underset{\sim}{x} = 0$.

General solution :

$$X = \begin{pmatrix} -x_2 - x_4 - x_6 - x_7 \\ x_2 \\ x_4 + x_6 - x_7 \\ x_4 \\ + x_6 - x_7 \\ x_6 \\ x_7 \end{pmatrix}$$

obvious basis for K_A

$x_2 = 1,\ x_4 = x_6,\ x_7 = 0$.

$$E_1 = \begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

First obvious choice

Second obvious choice :

$x_2 = 0,\ x_4 = 1,\ x_6 = x_7 = 0$.

$$\underset{\sim}{E_2} = \begin{pmatrix} -1 \\ 0 \\ 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

4th obvious choice :

$x_2 = x_4 = x_6 = 0,\ x_7 = 1$

$$\underset{\sim}{E_4} = \begin{pmatrix} -1 \\ 0 \\ -1 \\ 0 \\ -1 \\ 0 \\ 1 \end{pmatrix}$$

3rd obvious choice :

$x_2 = x_4 = 0, \quad x_6 = 1, \quad x_7 = 0$

$$\underset{\sim}{E}_3 = \begin{pmatrix} -1 \\ 0 \\ 1 \\ 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}$$

$$\underset{\sim}{E}_1 = \begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad \underset{\sim}{E}_2 = \begin{pmatrix} -1 \\ 0 \\ 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad \underset{\sim}{E}_3 = \begin{pmatrix} -1 \\ 0 \\ 1 \\ 0 \\ 1 \\ 1 \\ 0 \end{pmatrix} \quad \underset{\sim}{E}_4 = \begin{pmatrix} -1 \\ 0 \\ -1 \\ 0 \\ -1 \\ 0 \\ 1 \end{pmatrix}$$

Claim this is a basis for K_A .

$\{\underset{\sim}{E}_1, \underset{\sim}{E}_2, \underset{\sim}{E}_3, \underset{\sim}{E}_4\}$ spans K_A

Take x :

$$\underset{\sim}{x} = x_{\textcircled{2}} \underset{\sim}{E}_1 + x_{\textcircled{4}} \underset{\sim}{E}_2 + x_6 \underset{\sim}{E}_3 + x_7 \underset{\sim}{E}_4 .$$

so if $\underset{\sim}{x} = \underset{\sim}{0}$; $\underset{\sim}{x} = x_2 \underset{\sim}{E}_1 + x_4 \underset{\sim}{E}_2 + x_6 \underset{\sim}{E}_3 + x_7 \underset{\sim}{E}_4$.

<u>Read</u> (de la solución grande) : $x_2 = 0$; $x_4 = 0$; $x_6 = 0$; $x_7 = 0$.

$$\underset{\sim}{x} = \begin{pmatrix} -x_2 - x_4 - x_6 - x_7 \\ x_2 \\ x_4 + x_6 - x_7 \\ x_4 \\ x_6 - x_7 \\ x_7 \end{pmatrix} = \begin{pmatrix} ? \\ 0 \\ ? \\ 0 \\ ? \\ 0 \end{pmatrix}$$

$\dim(K_A) = 4$

A more general example :

$$A = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 & 1 & -1 & 1 \\ 1 & 1 & 1 & -1 & -1 & -1 & 1 \end{pmatrix}$$

$K_A = \{ \underset{\sim}{x} ; A \underset{\sim}{x} = 0 \}$

To solve $A \underset{\sim}{x} = 0$ we first reduce A to row echelon form : $\mathbb{F} = \mathbb{Q}$.

$$\to \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & -2 & -2 & 0 & -2 & 0 \\ 0 & 0 & 0 & -2 & -2 & -2 & 0 \end{pmatrix} \xrightarrow[\text{2 and 3 by } 1/2]{\text{Multiply row}} \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & -1 & -1 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 & -1 & -1 & 0 \end{pmatrix}$$

$$\to \begin{pmatrix} 1 & 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & -1 & -1 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 & -1 & -1 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 & -1 & -1 & 0 \end{pmatrix} \longrightarrow$$

$$A' = \left(\begin{array}{ccccccc} 1 & 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 \\ \enclose{circle}{x_1} & x_2 & \enclose{circle}{x_3} & x_4 & \enclose{circle}{x_5} & x_6 & x_7 \end{array}\right)$$

General solution to $A\underset{\sim}{x} = 0$.

$$\underset{\sim}{x} = \begin{pmatrix} -x_2 - x_5 - x_7 \\ x_2 \\ +x_5 \\ -x_5 - x_6 \\ x_5 \\ x_6 \\ x_7 \end{pmatrix}$$

make the obvious choices:

$x_2 = 1$, $x_5 = x_6 = x_7 = 0$.

$$\underset{\sim}{E}_1 = \begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$x_5 = 1$ $x_2 = x_6 = x_7 = 0$

$$\underset{\sim}{E}_2 = \begin{pmatrix} -1 \\ 0 \\ 1 \\ -1 \\ 1 \\ 0 \\ 0 \end{pmatrix}$$

$x_6 = 1$ $x_5 = x_3 = x_2 = 0$

$$\underset{\sim}{E}_3 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ -1 \\ 0 \\ 1 \\ 0 \end{pmatrix}$$

$$\underset{\sim}{E}_4 = \begin{pmatrix} -1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

If $\underset{\sim}{x} \in K_A$.

$$\underset{\sim}{x} = x_{\enclose{circle}{2}} \underset{\sim}{E}_1 + x_5 \underset{\sim}{E}_2 + x_6 \underset{\sim}{E}_3 + x_7 \underset{\sim}{E}_4 .$$

$$V = \left\{ \underset{\sim}{x} \in \mathbb{Q}^{10} : x_1 + x_2 + x_3 + x_4 + x_5 + x_6 + x_7 + x_8 + x_9 + x_{10} = 0 \right\} \quad \mathbb{F} = \mathbb{Q} .$$

$\dim(V) = 9$.

$$A\underset{\sim}{x} = 0 \quad \rightarrow \quad A = \left(\begin{array}{cccccccccc} 1, & 1, & 1, & 1, & 1, & 1, & 1, & 1, & 1, & 1 \\ \enclose{circle}{x_1} & x_2 & x_3 & x_4 & x_5 & x_6 & x_7 & x_8 & x_9 & x_{10} \end{array}\right).$$

General solution: $A\underset{\sim}{x} = 0$.

$$\begin{pmatrix} -x_2 - x_3 - \cdots - x_{10} \\ x_2 \\ x_3 \\ \\ \end{pmatrix} \qquad E_1 = \begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \\ \vdots \end{pmatrix} \qquad E_2 = \begin{pmatrix} -1 \\ 0 \\ 1 \\ 0 \\ \vdots \end{pmatrix} \text{ etc.}$$

November 22nd

Basis Th: Let V be a vector space $/ \mathbb{F}$.

$(V \neq 0)$

1) There exists at least 1 basis for V.

2) Any 2 bases for V have the same no of elements.

$\dim_{\mathbb{F}}(V)$ = nb of such elements.

Proof. early December.

Proposition Let $\{E_1, \dots, E_n\}$ be a basis for V.

If $\underset{\wedge}{x} \in V$, then $\underset{\wedge}{x}$ has a unique expression as a linear combination

$\underset{\wedge}{x} = x_1 E_1 + \dots + x_n E_n$.

Proof: As $\{E_1, \dots, E_n\}$ spans V, then if $\underset{\wedge}{x} \in V$, I can write

$\underset{\wedge}{x} = x_1 E_1 + \dots + x_n E_n$ for some x_i.

Claim this expression is unique.

Suppose $\underset{\wedge}{x} = x_1' E_1 + \dots + x_n' E_n$.

$$\underset{\substack{\| \\ 0}}{\underline{x} - \underline{x}} = \underbrace{(x_1' - x_1)}_{\lambda_1 = 0} E_1 + \dots + \underbrace{(x_n' - x_n)}_{\lambda_2 = 0} E_n.$$

So now

$(x_1' - x_1) E_1 + \dots + (x_n' - x_n) E_n = \underline{0}$.

But $E_1, \dots, E_n$ are LI so

$x_1' - x_1 = 0 \quad \dots \quad x_n' - x_n = 0$.

i.e $x'_1 = x_1 \quad \dots \quad x_n' = x_n$ QED

Example:

$\underline{e}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \quad \underline{e}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \quad \underline{e}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ ← Standard basis

$E_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \quad E_2 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \quad E_3 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ → also a bases

Both bases for $\mathbb{F}^3$.

$$\underline{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \boxed{x_1 \underline{e}_1 + x_2 \underline{e}_2 + x_3 \underline{e}_3} .$$

Also :

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ x_3 \end{pmatrix} = \begin{pmatrix} x_1 - x_3 \\ x_2 - x_3 \\ 0 \end{pmatrix} + \begin{pmatrix} x_3 \\ x_3 \\ x_3 \end{pmatrix}$$

$$\begin{pmatrix} x_1 - x_3 \\ x_2 - x_3 \\ 0 \end{pmatrix} = \begin{pmatrix} x_1 - x_2 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} x_2 - x_3 \\ x_2 - x_3 \\ 0 \end{pmatrix}$$

$$\boxed{\underline{x} = (x_1 - x_2)\underline{E}_1 + (x_2 - x_3)\underline{E}_2 + x_3 \underline{E}_3}$$

two expressions different because of the basis but in each basis the vector has a uniquely expression.

- Linear mappings :

Let V, W be vector spaces over $\mathbb{F}$.

Let $T : V \longrightarrow W$ be a mapping .

T is said to be Linear when :

1) $T(\underline{x} + \underline{y}) = T(\underline{x}) + T(\underline{y}) \quad \forall\ \underline{x}, \underline{y} \in V$.

2) $T(\lambda \underline{x}) = \lambda T(\underline{x}) \quad \forall\ \underline{x} \in V$ and $\forall\ \lambda \in \mathbb{F}$.

Prop: If $T ; V \longrightarrow W$ is linear then $T(\underline{0}) = \underline{0}$

$f(x) = y$

$T(0_V) = 0_W$.

$\underline{0}_V = \underline{0}_V + \underline{0}_V$.

Apply T :

$T(\underline{0}_V) = T(\underline{0}_V + \underline{0}_V)$;

$T(\underline{0}_V) = T(\underline{0}_V) + T(\underline{0}_V)$.

Add $-T(0_V)$ to each side

$$T(0_V) - T(0_V) = T(0_V) + T(0_V) - T(0_V) .$$

$$\parallel$$

$$0_W = T(0_V) + 0_W .$$

$$0_W = T(0_V) .$$

QED .

Standard example

let $A = (a_{ij})_{\substack{1 \le i \le m \\ 1 \le j \le n}}$ be $m \times n$ matrix $/ \mathbb{F}$.

columnas

Consider $T_A : \mathbb{F}^n \longrightarrow \mathbb{F}^m$

defined by $T_A(\underline{x}) = A\underline{x}$.

m filas n filas

$\overbrace{}^{Ax \in \mathbb{F}^M}$ (tiene m filas)

i.e.:

$$T_A(x) = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ & & \\ a_{m1} & \cdots & a_{mn} \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n \end{pmatrix}$$

Then T_A is a linear

Why?:

$$T_A(\underline{x} + \underline{y}) = A(\underline{x} + \underline{y}) = A\underline{x} + A\underline{y} = T_A(\underline{x}) + T_A(\underline{y}).$$

$$T_A(\lambda x) = A(\lambda x) = \lambda Ax = \lambda T_A(x).$$

Differentiation as a linear map:

$\underline{a}$ ⊗ → basis $\quad \underline{a} = \begin{pmatrix} a_1 \\ a_2 \\ a \\ \vdots \\ a_n \end{pmatrix}$

Let $\mathbb{F}$ be a field.

Put $P_n(\mathbb{F}) = \{ a_0 \cdot 1 + a_1 \cdot x + a_2 \cdot x^2 + \cdots + a_n x^n \}$ where $a_i \in \mathbb{F}$.

$\underline{v}(3,2,1)$ $\quad \underline{v}\ x = 3\underline{i} + 2\underline{j} + \underline{k}$ $\quad \downarrow \underline{i} \quad \downarrow \underline{j} \quad \downarrow \underline{k}$ ... (por eso x es base)

$P_n(\mathbb{F})$ = polynomial of degree $\le n$. Regarded as formal expressions.

Rule of equality:

$a_0 \cdot 1 + a_1 x + \cdots + a_n x^n = b_0 \cdot 1 + b_1 \cdot x + \cdots + b_n x^n$ iff $a_i = b_i \ \forall i$.

$P_n(\mathbb{F})$ is a vector space over $\mathbb{F}$.

Add polynomials in obvious way: multiply by $\lambda \in \mathbb{F}$.

$$P_2(\mathbb{F}) = \{ a_0 \cdot 1 + a_1 x + a_2 x^2 \mid a_i \in \mathbb{F} \}.$$

$$P_3(\mathbb{F}) = \{ a_0 \underset{x^0}{\boxed{1}} + a_1 \underset{x^1}{\boxed{x}} + a_2 \underset{x^2}{\boxed{x^2}} + a_3 \underset{x^3}{\boxed{x^3}} \mid a_i \in \mathbb{F} \}$$

: forman la base $\{x^0, x^1, x^2, x^3\}$

$\dim P_n(\mathbb{F}) = n+1.$

↓ para polinomios

Define D : $P_n(\mathbb{F}) \longrightarrow P_n(\mathbb{F})$.

$$D(\underline{a}(x)) = \left(\frac{da}{dx}\right).$$

ex:

$D(1) = 0$.

$D(x) = 1$

$D(x^2) = 2 \cdot x$.

Then

$$D(a+b) = \frac{d}{dx}(a+b) = \frac{da}{dx} + \frac{db}{dx} = D(a) + D(b)$$

$$D(\lambda a) = \lambda D(a) \quad (\lambda \in \mathbb{F}).$$

The matrix of a linear map

$T: V \longrightarrow W$ linear.

Basis $\Sigma = \{E_1, \dots, E_n\}$ basis for V.

$\Phi = \{\varphi_1, \dots, \varphi_m\}$ basis for W.

Consider $T(E_1) \in W$.

$\{\varphi_1, \dots, \varphi_m\}$ basis for W.

so $T(E_1)$ has a unique expression as a linear combination in $\{\varphi_1, \varphi_m\}$

$$T(E_1) = ?\,\varphi_1 + ??\,\varphi_2 + \dots + ????\dots?\,\varphi_m.$$

How do I name coefficients?

Need 2 indices.

$$\boxed{\begin{aligned} T(E_1) &= a_{11}\varphi_1 + a_{21}\varphi_2 + \dots + a_{m1}\varphi_m. \\ T(E_2) &= a_{12}\varphi_1 + a_{22}\varphi_2 + \dots + a_{m2}\varphi_m. \\ T(E_n) &= a_{1n}\varphi_1 + a_{2n}\varphi_2 + \dots + a_{mn}\varphi_m. \end{aligned}}$$

$(\varphi_1 a_{11} + \varphi_2 a_{21} + \dots + \varphi_m a_{m1}) \rightarrow$ es una contecúa

$$\boxed{T(E_j) = \sum_{i=1}^{m} a_{ij}\,\varphi_i}$$: convention

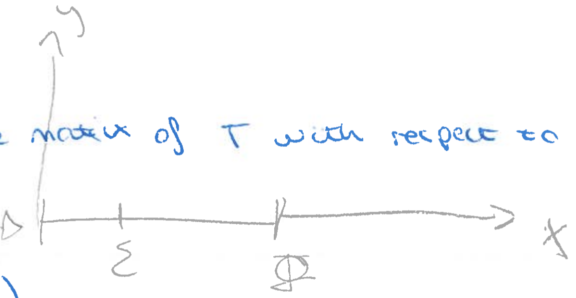
el número de vectores de la base

Def^n : $\mathcal{E} = \{ \epsilon_1, \dots \epsilon_n \}$ basis for V.

$\Phi = \{ \varphi_1, \dots, \varphi_m)$ basis for W.

$T : V \longrightarrow W$

$$T(E_j) = \sum_{i=1}^{m} a_{ij} \varphi_i .$$



$\mathcal{M}(a_{ij})_{\substack{1 \le i \le m \\ 1 \le j \le n}}$ the matrix of T with respect to $\mathcal{E}$ on left Φ on rigth.

$$\mathcal{M}(T)^{\Phi}_{\mathcal{E}} = (a_{ij})$$

$D : P_3(\mathbb{F}) \longrightarrow P_3(\mathbb{F})$.

Basis take:

$* = \{ 1, x, x^2, x^3 \}$ → basis of $W = \mathcal{E} = \Phi$

$D(1) = 0 + 0x + 0x^2 + 0x^3$

$D(x) = 1 + 0x + 0x^2 + 0x^3$

$D(x^2) = 2x + 0 + 0x^2 + 0x^3$

$D(x^3) = 3x^2 + 0 + 0x + 0x^2$

$$\mathcal{M}(T)^{*}_{x} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

November 23rd 2018

- Interchange of order of summation

$\Gamma(i,j)$ things that can be added together.

Assume $\oplus$ is associative and commutative.

→ propiedad de la suma.

$T(i,j)_{\substack{1 \le i \le m \\ 1 \le j \le n}}$.

$$\boxed{\sum_{i=1}^{m} \sum_{j=1}^{n} \Gamma(i,j) = \sum_{j=1}^{n} \sum_{i=1}^{m} \Gamma(i,j)}$$

$m = 2$ $1 \le i \le 2$

$n = 3$ $1 \le j \le 3$.

eg: $\Gamma(1,1)$, $\Gamma(1,2)$, $\Gamma(1,3)$

$\Gamma(2,1)$, $\Gamma(2,2)$, $\Gamma(2,3)$

$$\sum_{j=1}^{3} \Gamma(i,j) = \left(\Gamma(i,1) + \Gamma(i,2) + \Gamma(i,3)\right)$$

$$\sum_{i=1}^{2} \sum_{j=1}^{3} \Gamma(i,j) = \left(\Gamma(1,1) + \Gamma(1,2) + \Gamma(1,3)\right) + \left(\Gamma(2,1) + \Gamma(2,2) + \Gamma(2,3)\right)$$

- **Determination of a linear map on a basis**

$T : V \longrightarrow W$ linear.

Let $\{E_1, \dots, E_n\}$ be a basis for V.

Proposition T is completely determined by values $T(E_1), \dots, T(E_n)$.

Proof: Let $\underline{x} = x_1E_1 + \dots + x_nE_n$, be the unique expression of $\underline{x}$ in terms of $E_1, \dots, E_n$.

$T(x) = T(x_1E_1 + \dots + x_nE_n) = T(x_1E_1) + \dots + T(x_nE_n) =$ (T preserves +).

$= x_1 \cdot T(E_1) + \dots + x_n T(E_n)$.

So if I know $T(E_1), \dots, T(E_n)$, I know $T(\underline{x})$ for any $\underline{x} \in V$.

QED/

Even better:

$E_1, \dots, E_n$ basis for V.

Choose $\underline{w}_1, \underline{w}_2, \dots, \underline{w}_n \in W$.

Then there exists a unique linear map $T : V \longrightarrow W$ s.t.

$T(\underline{E}_1) = \underline{w}_1$, $T(\underline{E}_2) = \underline{w}_2$ .... $T(\underline{E}_n) = \underline{w}_n$

Proof: Write $\underline{x} = x_1E_1 + \dots + x_nE_n$.

Define $T(\underline{x}) = x_1\underline{w}_1 + \dots + x_n\underline{w}_n$.

T is linear $T(E_i) = w_i$

- The matrix associated to a linear map.

$T: V \longrightarrow W$ linear.

$\{E_1, \dots E_n\}$ basis for U $\quad$ $\{\varphi_1, \dots, \varphi_m\}$ basis for W.

T is completely determined by : (Express each $T(E_j)$ in terms of $\varphi_1, \dots, \varphi_m$

$$T(E_1) = a_{11}\varphi_1 + a_{21}\varphi_2 + \dots + a_{m1}\varphi_m.$$

$$T(E_{2}) = a_{12}\varphi_1 + a_{22}\varphi_2 + \dots + a_{m2}\varphi_m.$$

son la j

$$\vdots$$

$$T(E_n) = a_{1n}\varphi_1 + a_{2n}\varphi_2 + \dots + a_{mn}\varphi_n$$

i.e:
$$T(E_j) = \sum_{i=1}^{m} a_{ij}\varphi_i \quad (j = 1, \dots, n)$$

fija j en cada linea y hace la suma sobre i= hasta m

Proposition:

Let $T: U \longrightarrow V$; $S: V \longrightarrow W$ be linear.

Then $S \circ T : U \longrightarrow W$ is linear

Proof: Let $x, y \in U$.

$$(S \circ T)(x + y) = S(T(x + y)) =$$

$$(T \text{ is linear}) = S(T(x) + T(y)) \underset{\substack{\uparrow \\ S \text{ is linear}}}{=} S(T(x)) + S(T(y)) =$$

$$= (S \circ T)(x) + (S \circ T)(y).$$

$$(S \circ T)(\lambda x) = S(T(\lambda x)) \underset{\substack{\uparrow \\ T \text{ is linear}}}{=} S(\lambda T(x) = \lambda . S(T(x)) = \lambda(S \circ T)(x)$$

So $S \circ T$ is linear QED.

$$T(E_j) = \sum_{i=1}^{m} a_{ij}\,\varphi_i$$

Def^n $$m(T)^{\Phi}_{\Sigma} = (a_{ij})_{\substack{1\le i\le m\\ 1\le j\le m}}$$

$\downarrow$

Matrix of T with respect to Σ on the left and Φ in the right.

$m(T)^{\Phi}_{\Sigma}$ = matrix of T.
wrt Σ on left
Φ on right

$$\begin{tikzcd} T: V \arrow[r] \arrow[d] & W \arrow[d] \\ \text{basis } \Sigma & \text{basis } \Phi \end{tikzcd}$$

- **Composition formula:**

Suppose we have linear maps

$$\begin{tikzcd} U \arrow[r, "T"] \arrow[d, phantom, "\parallel"] & V \arrow[r, "S"] \arrow[d, phantom, "\parallel"] & W \arrow[d, phantom, "\parallel"] \\ \{E_1, \dots, E_p\} \arrow[d, phantom, "\parallel"] & \{\varphi_1, \dots \varphi_n\} \arrow[d, phantom, "\parallel"] & \{\psi_1, \dots \psi_m\} \arrow[d, phantom, "\parallel"] \\ \Sigma & \Phi & \Psi \end{tikzcd}$$

We now have a matrix

$M(T)^{\Phi}_{\Sigma}$ and also $M(S\circ T)^{\Psi}_{\Sigma}$

$M(S)^{\Psi}_{\Phi}$

Proposition: $$\underset{\text{composition}}{M(S\circ T)}{}^{\Psi}_{\Sigma} = M(S)^{\Psi}_{\Phi} \underset{\text{matrix product}}{\cdot} M(T)^{\Phi}_{\Sigma}$$

Proof: write $T(E_k) = \sum_{j=1}^{n} a_{jk}\,\varphi_j$ $\quad M(T)^{\Phi}_{\Sigma} = (a_{jk})_{\substack{1\le j\le n\\ 1\le k\le p}}$.

$$S(\varphi_j) = \sum_{i=1}^{m} b_{ij}\,\psi_i \qquad M(S)^{\Psi}_{\Phi} = (b_{ij})_{\substack{1\le i\le m\\ 1\le j\le n}}.$$

$$(S\circ T)(E_k) = \sum_{i=1}^{m} c_{ik}\,\psi_i \quad (1\le k\le p)$$

Want to express c_{ik} in terms of (a_{jk}) (b_{ij}).

$$(S\circ T)(E_k) = S(T(E_k)) = S\left(\sum_{j=1}^{n} a_{jk}\varphi_j\right) =$$

$$= \sum_{j=1}^{n} a_{jk} S(\varphi_j) = \sum_{j=1}^{n} a_{jk}\left\{\sum_{i=1}^{m} b_{ij}\psi_i\right\} =$$

$$= \sum_{j=1}^{n}\sum_{i=1}^{m} a_{jk} b_{ij}\psi_i =$$

($a_{jk} b_{ij} = b_{ij} a_{jk}$ elements of the field).

$$= \sum_{j=1}^{n}\sum_{i=1}^{m} b_{ij} a_{jk}\psi_i \underset{\uparrow}{=} \sum_{i=1}^{m}\left\{\sum_{j=1}^{n} b_{ij} a_{jk}\right\}\psi_i$$

interchange order of summation

$$(S\circ T)(E_k) = \sum_{i=1}^{m}\left\{\sum_{j=1}^{n} b_{ij} a_{jk}\right\}\psi_i .$$

So $C_{ik} = \sum_{j=1}^{n} b_{ij} a_{jk}$. $B = (b_{ij})$ $A = (a_{jk})$.

i.e. $M(S\circ T)^{\Psi}_{\Sigma} = BA = M(S)^{\Psi}_{\Phi}\cdot M(T)^{\Phi}_{\Sigma}$

SUMMARY

$$\begin{tikzcd}
\mathcal{U} \arrow[r, "T"] \arrow[d, equal] & V \arrow[r, "S"] \arrow[d, equal] & W \arrow[d, equal] \\
\Sigma \text{ basis} & \Phi \text{ basis} & \Psi \text{ basis}
\end{tikzcd}$$

$$M(S\circ T)^{\Psi}_{\Sigma} = M(S)^{\Psi}_{\Phi}\, M(T)^{\Phi}_{\Sigma}.$$

- **CHANGE of basis**

$$V \xrightarrow{T} V .$$

Suppose I have 2 basis for V.

$\Sigma = \{E_1, \dots, E_n\}$ $\Phi = \{\varphi_1, \dots, \varphi_n\}$.

$M(T)^{\Sigma}_{\Sigma} = (a_{ij})$ $1 \le i \le n$, $1 \le j \le n$

$T(E_j) = \sum^{n} a_{ij} E_i$.

$$M(T)^{\Phi}_{\Phi} = (b_{ij}) \quad \begin{matrix} 1 \le i \le n \\ 1 \le j \le n \end{matrix}$$

$$T(\varphi_j) = \sum_{i=1}^{n} b_{ij}\, \varphi_i$$

The change of basis formula allows us to calculate (b_{ij}) in terms of (a_{ij})

Simplest case $T = Id$.

$Id_V : V \longrightarrow V$.

$\Sigma = \{E_1, \dots, E_n\}$ $\quad \Phi = \{\varphi_1, \dots, \varphi_n\}$.

Both are bases for V.

Express φ_j in terms of $E_1, \dots, E_n$.

$$\varphi_j = \sum_{i=j}^{n} a_{ij} \cdot E_i \qquad \boxed{A = (a_{ij})}$$

Express E_i in terms of $\varphi_1, \dots, \varphi_n$

$$E_i = \sum_{j=1}^{n} b_{ji}\, \varphi_j \qquad \boxed{B = (b_{ji})}$$

Question: What is relation between A, B?

We'll see that both A, B are invertible $\boxed{B = A^{-1}}$

Formally, writing

$$\varphi_j = \sum_{i=1}^{n} a_{ij} E_i$$

Then, $A = (a_{ij}) = M(Id)^{\Sigma}_{\Phi}$

Writing $E_i = \sum_{j=1}^{n} b_{ji}\, \varphi_j$

$$B = (b_{ij}) = M(Id)^{\Phi}_{\Sigma}$$

Prop. Let $\Sigma = \{E_1, \dots, E_n\}$ be a basis for V.

$M(Id)^{\Sigma}_{\Sigma} = I_n$.

$$\left.\begin{aligned} E_1 &= 1E_1 + 0E_2 + \dots + 0E_n \\ E_2 &= 0E_1 + 1E_2 + \dots + 0E_n \\ &\vdots \\ E_n &= 0E_1 + 0E_2 + \dots + 1E_n \end{aligned}\right\} \boxed{M(Id)^{\Sigma}_{\Sigma} = I_n}$$

Proposition : Let $\Sigma = \{E_1, \ldots, E_n\}$ basis for V.

$\Phi = \{\varphi_1, \ldots, \varphi_n\}$.

$$M(Id)^{\Sigma}_{\Phi} = a_{ij} \qquad M(Id)^{\Phi}_{\Sigma} = (b_{ji}) = B.$$
$$= A$$

Then A, B are invertible and $B = A^{-1}$

Proof : $$M(Id)^{\Sigma}_{\Sigma} = M(Id \underset{\substack{\uparrow \\ \text{composition}}}{\circ} Id)^{\Sigma}_{\Sigma}$$

$$= M(Id)^{\Sigma}_{\Phi} \; M(Id)^{\Phi}_{\Sigma}$$

So $Id = AB$

$$M(Id)^{\Phi}_{\Phi} = M(Id)^{\Phi}_{\Sigma} \; M(Id)^{\Sigma}_{\Phi}$$

$$I_n = BA.$$

So $AB = BA = I$.

$$V = \mathbb{Q}^3 = \left\{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} : x_i \in \mathbb{Q} \right\}.$$

$$\Sigma = \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}. \text{ basis}$$

$$\Phi = \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \right\} \text{ basis}$$

Take $T : \mathbb{Q}^3 \longrightarrow \mathbb{Q}^3$ to be. Calculate $M(T)^{\Sigma}_{\Sigma}$

$$T\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 2x_1 + x_2 - x_3 \\ 3x_2 + x_3 \\ 4x_3 \end{pmatrix}$$

T is linear (he knows it).

$$T\begin{pmatrix}1\\0\\0\end{pmatrix}=\begin{pmatrix}2\\0\\0\end{pmatrix}$$

$$T\begin{pmatrix}0\\1\\0\end{pmatrix}=\begin{pmatrix}1\\3\\0\end{pmatrix} \qquad m(T)^{\Sigma}_{\Sigma}=\begin{pmatrix}2&1&-1\\0&3&1\\0&0&4\end{pmatrix} \qquad \text{Find } m(T)^{\Phi}_{\Phi}$$

$$T\begin{pmatrix}0\\0\\1\end{pmatrix}=\begin{pmatrix}-1\\1\\4\end{pmatrix}$$

Use composition formula.

$$T = Id \circ T \circ Id .$$

$$m(T)^{\Phi}_{\Phi} = m(Id)^{\Phi}_{\Sigma}\, m(T\circ Id)^{\Sigma}_{\Phi} =$$

$$= m(Id)^{\Phi}_{\Sigma}\; m(T)^{\Sigma}_{\Sigma}\; m(Id)^{\Sigma}_{\Phi}$$

Know $m(T)^{\Sigma}_{\Sigma}$

also know $m(Id)^{\Sigma}_{\Phi}=\begin{pmatrix}1&1&0\\0&1&1\\0&0&1\end{pmatrix}$

Check inverse of $=\begin{pmatrix}1&-1&1\\0&1&-1\\0&0&1\end{pmatrix}=m(Id)^{\Phi}_{\Sigma}$

So now:

$$m(T)^{\Phi}_{\Phi}=\begin{pmatrix}1&-1&1\\0&1&-1\\0&0&1\end{pmatrix}\begin{pmatrix}2&1&-1\\0&3&1\\0&0&4\end{pmatrix}\begin{pmatrix}1&1&0\\0&1&1\\0&0&1\end{pmatrix}=$$

$$=\begin{pmatrix}2&-2&-2\\0&3&-3\\0&0&4\end{pmatrix}\begin{pmatrix}1&1&0\\0&1&1\\0&0&1\end{pmatrix}=\begin{pmatrix}2&0&0\\0&3&0\\0&0&4\end{pmatrix}$$

November 29th

$T: V \longrightarrow W$ linear

Basis $\Sigma = [E_1, \dots, E_n]$ for V.

$\Phi = [\varphi_1, \dots, \varphi_m]$ for W.

$$T(E_j) = \sum_{i=1}^{m} a_{ij}\,\varphi_i$$

Define $m(T)^{\Phi}_{\mathcal{E}} = (a_{ij})_{\substack{1\le i\le m\\ 1\le j\le n}}$.

→ Matiz $= \begin{pmatrix}\varphi_1 & \varphi_1 & \varphi_1\\ \varphi_2 & \varphi_2 & \varphi_2\end{pmatrix}$

Linear maps $\longrightarrow$ matrices

$T \longmapsto m(T)^{\Phi}_{\mathcal{E}}$

Conversely, if $(a_{ij})_{\substack{1\le i\le m\\ 1\le j\le n}}$.

determine linear map: $T: V \to W$.

by $T(E_j) = \sum_{i=1}^{m} a_{ij}\,\varphi_i$

This gives a unique linear map. In this case: $m(T)^{\Phi}_{\mathcal{E}} = A$.

Under this correspondence

$\left\{\text{Composition of linear maps}\right\} \longleftrightarrow \left\{\text{matrix product}\right\}$.

$$m(S\circ T)^{\Psi}_{\mathcal{E}} = m(S)^{\Psi}_{\Phi}\;\; m(T)^{\Phi}_{\mathcal{E}}$$

As composition is Associative then matrix product is also associative.

Standard example! (Again).

let $A = (a_{ij})_{\substack{1\le i\le m\\ 1\le j\le n}}$.

$m\times n$ matrix over a field $\mathbb{F}$

let $\mathcal{E} = \left[\underset{E_1}{\begin{pmatrix}1\\0\\\vdots\\\vdots\end{pmatrix}}, \underset{E_2}{\begin{pmatrix}0\\1\\0\\0\\0\end{pmatrix}}, \ldots\ldots, \underset{E_n}{\begin{pmatrix}0\\1\end{pmatrix}}\right]$ standard basis for $\mathbb{F}^n$.

$\Phi = \left[\underset{\varphi_1}{\begin{pmatrix}1\\0\end{pmatrix}}, \underset{\varphi_2}{\begin{pmatrix}0\\1\\0\end{pmatrix}}, \ldots\ldots, \underset{\varphi_m}{\begin{pmatrix}0\\1\end{pmatrix}}\right]$ standard basis for $\mathbb{F}^m$.

In this case $M(T)^{\Phi}_{\mathcal{E}} = A$

$T_A : \mathbb{F}^n \longrightarrow \mathbb{F}^m$.

$T_A(\underline{x}) = A\underline{x}$. matrix product .

Prop $M(T_A)^{\Phi}_{\mathcal{E}} = A$.

Check :

$$T_A(E_1) = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} = \begin{pmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{pmatrix} = \sum_{i=1}^{m} a_{i1} \varphi_i .$$

$$T_A(E_2) = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{pmatrix} = \begin{pmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{pmatrix} = \sum_{i=1}^{m} a_{i2} \varphi_i$$

ie $T_A(E_1) = 1^{st}$ column

$T_A(E_2) = 2^{nd}$ column

$\vdots$

$T_A(E_n) = n^{th}$ column

⚠ IMPORTANTE .

Define $V = \{ a_1 \exp(x) + a_2 x \exp(x) + a_3 x^2 \exp(x) \}$.

||

$a(x)$

$a_1, a_2, a_3 \in \mathbb{Q}$.

V is a vector space / $\mathbb{Q}$.

$\underbrace{\exp(x),\ x\exp(x),\ x^2\exp(x)}_{\text{basis .}}$ are linearly independent (Exercise)

$\dim(V) = 3$.

Basis $\Phi = \{ \exp(x), x\exp(x), x^2\exp(x) \}$.

Define $D : V \longrightarrow V$ by $D\, a(x) = \frac{da}{dx}$

Calculate $M(D)^{\Phi}_{\Phi}$

$D(\exp(x)) = \exp(x) + 0 x \cdot \exp(x) + 0 \cdot x^2 \cdot \exp(x)$.

$D(x \cdot \exp(x)) = \exp(x) + x \exp(x)$

$D(x^2 \exp(x)) = 2x \cdot \exp(x) + x^2 \exp(x)$

$$M(D)^{\Phi}_{\Phi} = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix}$$

Suppose I take $D^2 = D \circ D$.

$$M(D^2)^{\Phi}_{\Phi} = M(D)^{\Phi}_{\Phi} \; M(D)^{\Phi}_{\Phi} = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 2 \\ 0 & 1 & 4 \\ 0 & 0 & 1 \end{pmatrix}$$

Now take $D^3 = D \circ D \circ D$.

$$M(D^3)^{\Phi}_{\Phi} = M(D)^{\Phi}_{\Phi} \; M(D^2)^{\Phi}_{\Phi} = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 & 2 \\ 0 & 1 & 4 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 3 & 6 \\ 0 & 1 & 6 \\ 0 & 0 & 1 \end{pmatrix}.$$

Solve: $\left(\frac{d^3}{dx^3} - \frac{d^2}{dx^2} + \frac{d}{dx}\right) a(x) = x \cdot \exp(x) - 2 \cdot x^2 \cdot \exp(x)$.

$$M(D^3 - D^2 + 1)^{\Phi}_{\Phi} = \underbrace{\begin{pmatrix} 0 & 1 & 4 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix}}_{D^3 - D^2} + \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 4 \\ 0 & 1 & 4 \\ 0 & 0 & 1 \end{pmatrix}$$

Find its inverse:

$$\left(\begin{array}{ccc|ccc} 1 & 2 & 4 & 1 & 0 & 0 \\ 0 & 1 & 4 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|ccc} 1 & 2 & 0 & 1 & 0 & -4 \\ 0 & 1 & 0 & 0 & 1 & -4 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right) \longrightarrow \begin{pmatrix} 1 & -2 & 4 \\ 0 & 1 & -4 \\ 0 & 0 & 1 \end{pmatrix}.$$

Matrix of $(D^3 - D^2 + D)^{-1} = \begin{pmatrix} 1 & -2 & 4 \\ 0 & 1 & -4 \\ 0 & 0 & 1 \end{pmatrix}$

$(D^3 - D^2 + D) \; a(x) = x \cdot \exp(x) - 2x^2 \exp(x)$.

$a(x) = (D^3 - D^2 + D)^{-1} \; (x \exp(x) - 2x^2 \exp(x))$

$$\begin{pmatrix} 1 & -2 & 4 \\ 0 & 1 & -4 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ -2 \end{pmatrix} = \begin{pmatrix} -10 \\ 9 \\ -2 \end{pmatrix} \sim -10 \exp(x) + 9 x \exp(x) - 2x^2 \exp(x)$$

$$(D|I) \sim \left(\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix} \middle| \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right)$$

$$D^{-1} = \begin{pmatrix} 1 & -1 & 2 \\ 0 & c & -2 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\int \left(2 \cdot \exp(x) - 5x \exp(x) + 1000\, x^2 \exp(x) \right) . dx$$

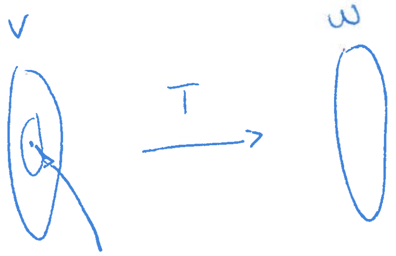
$$D^{-1} \begin{pmatrix} 2 \\ -5 \\ 1000 \end{pmatrix} = \begin{pmatrix} 1 & -1 & 2 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 2 \\ -5 \\ 1000 \end{pmatrix} = \begin{pmatrix} 2007 \\ -2005 \\ 1000 \end{pmatrix}$$

$$= 2007 \exp(x) - 2005\, x \exp(x) + 1000x^2 \exp(x) .$$

$T : V \longrightarrow W$ linear.

Define $\underbrace{\ker(T)}_{\uparrow} = \{ \underline{x} \in V : T(x) = 0 \}$.

kernel of T .



$\ker(T)$.

Proposition $\ker(T) \subset V$ $\ker(T)$ is a vector space.

Proof : $0 \in \ker(T)$ $T(\underline{0}) = 0$.

- **Addition** $x \in \ker(T)$ $y \in \ker(T) \Rightarrow x + y \in \ker(T)$.

$$T(x+y) = T(x) + T(y) = 0 + 0 = 0 .$$

- **Scalar multiplication** $x \in \ker(T)$, $\lambda \in \mathbb{F}$.

$$T(\lambda x) = \lambda\, T(x) = \lambda \cdot 0 = 0 .$$

$$\Rightarrow \lambda \cdot x \in \ker(T) .$$

The remaining axioms are automatically satisfied because they are true already in V .

November 30th 2018

Let V be a vector space / $\mathbb{F}$. and let $\mathcal{U} \subset V$

Say $\mathcal{U}$ is a vector subspace of V when :

i) $0 \in \mathcal{U}$.

ii) If $\underline{x}, \underline{y} \in \mathcal{U}$ then $\underline{x} + \underline{y} \in \mathcal{U}$.

iii) If $x \in \mathcal{U}$. $\lambda \in \mathbb{F}$, $\lambda \underline{x} \in \mathcal{U}$.

Prop If $\mathcal{U} \subset V$ is a vector subspace then :

$\mathcal{U}$ is itself a vector space .

Proof we have all necessary elements of structure and all axioms satisfied because already satisfied in V .

Definition : let $T : V \longrightarrow W$ be linear , define :

$$\ker(T) = \{ x \in V : T(\underline{x}) = 0 \}$$

↑ kernel of T .

Proposition : $\ker(T)$ is a vector subspace of V (= domain of T) .

Proof : i) $0 \in \ker(T)$

$T(\underline{0}) = \underline{0}$

ii) If $x, y \in \ker(T)$.

$T(x) = 0$ $T(y) = 0$.

so $T(x+y) = T(x) + T(y) = 0 + 0 = 0$.

iii) If $x \in \ker(T)$, $\lambda \in \mathbb{F}$.

$T(x\lambda) = \lambda T(x) = \lambda \cdot 0 = 0$.

$\lambda x \in \ker(T)$ QED .

Def^n let $T : V \longrightarrow W$ be linear .

Define $\mathrm{Im}(T) = \{ w \in W ; \exists v \in V \;\; T(v) = w \}$.

i.e. all elements of co-domain hit by T .

Image of T .

Proposition If $T: V \longrightarrow W$ linear Then $Im(T)$ is a vector subspace of W.

Proof i) $0_W \in Im(T)$ because $T(0_V) = 0_W$.

ii) Let $w_1, w_2 \in Im(T)$. Choose $v_1, v_2 \in V$. s.t. $T(v_1) = w_1$ $T(v_2) = w_2$

$$\left.\begin{array}{l} T(v_1+v_2) = T(v_1) + T(v_2) = w_1 + w_2 \end{array}\right\} \text{so } w_1+w_2 \in Im(T)$$

iii) If $w \in Im(T)$ $\lambda \in \mathbb{F}$. Choose $v \in V : T(v) = w$.

$T(\lambda v) = \lambda T(v) = \lambda w$ so $\lambda w \in Im(T)$

QED.

Theorem: Kernel - Rank Th-m

If $T: V \longrightarrow W$ linear

then $\dim \ker(T) + \dim Im(T) = \dim V$.

We'll assume Basis Theorem is TRUE.

Basis Theorem. If V is a non-zero vector space then:

i) V has at least one basis.

ii) Any two basis have same no of elements (= dim).

Convention!

$\dim(0) = 0$

i.e. zero-vector space has dimension $= 0$.

Proof: of Kernel - Rank Theorem.

$T: V \longrightarrow W$ linear

First, consider "Generic case", where $\ker(T) \neq 0$ and $Im(T) \neq 0$.

Put $k = \dim \ker(T)$.
$m = \dim Im(T)$.

We have to show that $k + m = \dim(V)$

Choose a basis $\{E_1, \dots, E_k\}$ for $\ker(T)$.

Choose a basis $\{\varphi_1, \dots, \varphi_m\}$ for $Im(T)$

Now choose $E_{k+1}, \dots, E_{k+m}$ in V, so that $\boxed{T(E_{k+i}) = \varphi_i.}$

E_{k+1} hits φ_1
E_{k+10} hist φ_{10}

Los primeros al ser $\ker(T)$ $T(E_x) = 0$, el resto serán $= \varphi$.

V tiene base $E_1, \dots E_{k+m}$

Claim $\{E_1, \dots, E_{k+m}\}$ is a basis for V.

$\{E_1, \dots, E_{k+m}\}$ is L.I.

Suppose $\boxed{\lambda_1 E_1 + \lambda_2 E_2 + \dots + \lambda_k E_k + \dots + \lambda_{k+m} E_{k+m} = 0}$

$\lambda_i \in \mathbb{F}$

I have to show each $\lambda_i = 0$.

Apply T to

- $T(\lambda_1 E_1 + \dots + \lambda_k E_k + \lambda_{k+1} E_{k+1} + \dots + \lambda_{k+m} E_{k+m}) = 0$

because $T(0) = 0 \rightarrow$ esto se cumple por ser lineal

so $\lambda_1 T(E_1) + \dots + \lambda_k T(E_k) + \lambda_{k+1} T(E_{k+1}) + \dots + \lambda_{k+m} T(E_{k+m}) = 0$

① • But $E_1, \dots, E_k \in$ ker(T)

so $T(E_1) = \dots = T(E_k) = 0$.

② So $\lambda_{k+1} T(E_{k+1}) + \dots + \lambda_{k+m} T(E_{k+m}) = 0$

i.e. $\lambda_{k+1} \varphi_1 + \dots + \lambda_{k+m} \varphi_m = 0$.

- But $\{\varphi_1, \dots, \varphi_m\}$ is L.I

BASIS FOR Im(T).

so $\lambda_{k+1} = \lambda_{k+2} = \dots = \lambda_{k+m} = 0$

- Substitute in $\underbrace{\lambda_1 E_1 + \dots + \lambda_k E_k = 0}_{\text{me queda eso}}$. But $\{E_1, \dots, E_k\}$ is L.I,

BASIS for kernel(T).

Hence, $\lambda_1 = \lambda_2 = \dots = \lambda_k = 0$, so final $\lambda_i = 0$ for all i

$1 \le i \le k+m$.

i.e. $\{E_1, \dots, E_{k+m}\}$ is L.I

We also claim that $\{E_1, \dots, E_{k+m}\}$ spans V.

- Let $\underset{\wedge}{x} \in V$ (I've got to show $\underset{\wedge}{x} = x_1 E_1 + \dots + x_{k+m} E_{k+m}$ for some $x_1, \dots, x_{k+m} \in \mathbb{F}$)

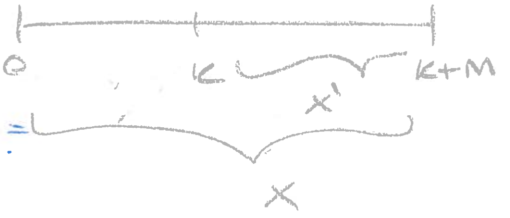
- Consider $T(\underline{x}) =$ Im(T).

and express $T(\underline{x})$ in terms of φ apply T.

$\{\varphi_1, \dots, \varphi_m\}$

$T(\underline{x}) = \xi_1 \underline{\varphi_1} + \dots + \xi_m \underline{\varphi_m}$ → Porque de $T(E_1)$ hasta $T(E_k) = 0$ entonces te queda $\underline{\varphi_1}, \underline{\varphi_m}$

$(\xi_i \in \mathbb{F})$.



- Put $\underline{x}' = \xi_1 \underline{E_{k+1}} + \dots + \xi_m E_{k+m}$.

$T(\underline{x}') = \xi_1 T(E_{k+1}) + \dots + \xi_m T(E_{k+m}) =$

$= \xi_1 \underline{\varphi_1} + \dots + \xi_m \underline{\varphi_m} = T(\underline{x})$.

- Consider, $\underline{x} - \underline{x}'$

$x - x' =$ desde 0-K.

$T(\underline{x} - \underline{x}') = T(x) - T(x') = 0$

so $x - x' \in \ker(T)$ so I can write $x - x' = x_1 \underline{E_1} + \dots + x_k \underline{E_k}$

So $\underline{x} = x_1 E_1 + \dots + x_k E_k + x' =$

$= x_1 E_1 + \dots + x_k E_k + \xi_1 E_{k+1} + \dots + \xi_m E_{k+m}$

- To make this formally correct, put $x_{k+i} = \xi_i$.

$$\boxed{x = x_1 E_1 + x_2 E_2 + \dots + x_k E_k + x_{k+1} E_{k+1} + \dots + x_{k+m} E_{k+m}}$$

So $E_1, \dots, E_{k+m}$ spans V.

Hence, $\{E_1, \dots, E_{k+m}\}$ is basis for V.

so $\dim(V) = k + m = \dim \ker(T) + \dim \operatorname{Im}(T)$

QED (Generic case).

$\dim \operatorname{Im}(T)$ is the rank of T.

$T: V \longrightarrow W$ linear

$\dim(V) = \dim \ker(T) + \dim \operatorname{Im}(T)$.

I've proved Generic case where $\ker(T) \neq 0$ $\operatorname{Im}(T) \neq 0$.

so both have basis.

Special case I $\operatorname{Im}(T) = 0$ so $\ker(T) = V$.

because $T(x) = 0$ $\forall x \in V$.

$\dim V = \dim \ker(T) = \dim \ker(T) + \dim \operatorname{Im}(T)$.

Special case II $\ker(T) = 0$

Prop: Let $T: V \longrightarrow W$ be linear.

Then T is injective $\Longleftrightarrow \ker(T) = 0$

↳ solo 1 vector da 0 $(T(0) = 0)$

Proof: $\Rightarrow$ Suppose T is injective and $\underline{x} \in \ker(T)$ so $T(\underline{x}) = 0$ and $\underbrace{T(\underline{0}) = \underline{0}}_{\text{es lineal}}$.

Since T is injective $\underline{x} = \underline{0} \Rightarrow \ker(T) = \{0\}$

$\Leftarrow$ Suppose $\ker(T) = \{0\}$ and that $T(x) = T(x')$.

so $T(x - x') = T(x) - T(x') = 0$ so $x - x' = 0$ so $x = x'$ and T is injective QED.

Special case II

Suppose $\ker(T) = \{0\}$

Let $E_1, \dots, E_n$ be a basis for V.

Claim $T(E_1), \dots, T(E_n)$ is a basis for $\text{Im}(T)$.

Let $\underline{w} \in \text{Im}(T)$

- Choose $\underline{x} \in V : T(\underline{x}) = w$

  Write $\underline{x} = x_1 E_1 + \dots + x_n E_n$.

  $\underline{w} = T(\underline{x}) = x_1 T(E_1) + \dots + x_n T(E_n)$

  so $T(E_1), \dots, T(E_n)$ spans $\text{Im}(T)$.

- For $\lambda_1 T(E_1) + \dots + \lambda_n T(E_n) = 0$.

  $T(\lambda_1 E_1 + \dots + \lambda_n E_n) = 0$.

  But $T(0) = 0$ and T is injective so

  $\lambda_1 E_1 + \dots + \lambda_n E_n = 0$

  so $\lambda_1 = \dots = \lambda_n = 0$

- $\{E_1, \dots, E_n\}$ is L.I QED

Back to standard example:

$A = (a_{ij})_{\substack{1 \le i \le m \\ 1 \le j \le n}}$ $m \times n$ matrix $/\mathbb{F}$.

$T_A : \mathbb{F}^n \longrightarrow \mathbb{F}^m$ $T_A(\underline{x}) = A\underline{x}$

Recall I defined $K_A = \{ x \in \mathbb{F}^n : A\underline{x} = 0 \}$

And so $K_A = \ker(T_A)$.

So now abolish K_A notation.

example : $A = \begin{pmatrix} 1 & 1 & -1 & 1 & 1 & -1 \\ 1 & 1 & 1 & -1 & 1 & -1 \\ 1 & 1 & -3 & 3 & 1 & -1 \end{pmatrix}$

over $\mathbb{Q}$.

$T_A : \mathbb{Q}^6 \longrightarrow \mathbb{Q}^3$

$T_A(\underline{x}) = A \cdot \underline{X}$.

So $\ker(T_A)$ = solution set of homogeneous system $A\underline{X}$.

Reduce A to echelon form

$$\begin{pmatrix} 1 & 1 & -1 & 1 & 1 & -1 \\ 0 & 0 & 2 & -2 & 0 & 0 \\ 0 & 0 & -2 & 2 & 0 & 0 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 1 & -1 & 1 & 1 & -1 \\ 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\longrightarrow \begin{array}{cccccc} 1 & 1 & 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ \boxed{x_1} & x_2 & \boxed{x_3} & x_4 & x_5 & x_6 \end{array}$$

son la sol. del sistema cuando $AX=0$.

General solution :

$$\begin{pmatrix} -x_2 - x_5 + x_6 \\ x_2 \\ x_4 \\ x_4 \\ x_5 \\ x_6 \end{pmatrix}$$

$\dim \ker(T_A) = \boxed{4}$.

so $\dim \operatorname{Im}(T_A) = 2 = 6-4$.

↳ Nb of circle elements

las que no son 0.

COMMON MISTAKES

Take the columns above circled variables in REDUCED matrix

In this case $\cancel{\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}}$.

CORRECT METHOD

Take columns above circled variables in

ORIGINAL MATRIX

In this case $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \begin{pmatrix} -1 \\ 1 \\ -3 \end{pmatrix}$: basis for Im.

Example 2:

$$A = \begin{pmatrix} 2 & 1 & 1 & 1 & 1 \\ 3 & 2 & -1 & 1 & 1 \\ 1 & 1 & -2 & 0 & 0 \end{pmatrix}$$

and reduce:
$$\begin{pmatrix} 1 & 1 & -2 & 0 & 0 \\ 0 & -1 & 5 & 1 & 1 \\ 0 & -1 & 5 & 1 & 1 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 0 & 3 & 1 & 1 \\ 0 & 1 & -5 & -1 & -1 \\ 0 & 0 & 0 & 0 & 0 \\ \boxed{x_1} & \boxed{x_2} & & & \end{pmatrix}$$

So $\begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$ is a basis for $Im(T_A)$

Suppose

$$\left\{ \begin{array}{l} a_{11} x_1 + \dots + a_{1n} x_n = b_1 \\ \vdots \\ a_{m1} x_1 + \dots + a_{mn} x_n = b_m \end{array} \right.$$

First write as matrix eq^n:

$A\underline{x} = \underline{b}$

$T_A(\underline{x}) = \underline{b}$

So suppose $\underline{y}$ satisfies $A\underline{y} = \underline{b}$.

Suppose $A\underline{z} = \underline{b}$ is some other solution

$A\underline{y} - A\underline{z} = \underline{b} - \underline{b} = 0$.

$A(\underline{y} - \underline{z}) = 0$

$\underline{y} - \underline{z} \in ker(T_A)$.

$\underline{z} - \underline{y} \in ker(T_A)$.

Put $\underline{x} = \underline{z} - \underline{y}$; $\underline{x} + \underline{y} = \underline{z}$

so typical solution $\underline{z}$ has form $\underline{z} = \underline{y} + \underline{x}$.

$\underline{y}$ particular solution $\underline{x} \in ker(T_A)$. $A\underline{x} = 0$.

- **BASIS THEOREM:**

Let V be a non-zero vector space over field $\mathbb{F}$.

1) There exists at least one basis for V. (EXISTENCE).

2) Any two basis for V have same nb of elements $(= \dim_{\mathbb{F}}(V))$.
(UNIQUENESS)

Proposition: Let $\{w_1, \dots, w_n\} \subset V$.

V is a vector space $/\mathbb{F}$.

If $0 \in \{w_1, \dots, w_n\}$ then

$\{w_1, \dots, w_n\}$ is not LI.

Proof: Suppose $w_i = 0$ (uno de esos vectores es 0)

Consider $\sum_{j=1}^{n} \lambda_j w_j$ where $\lambda_j = \begin{cases} 1 & j = i \\ 0 & j \neq i \end{cases}$ $\rightarrow \lambda_i w_i = \lambda_i 0 \Rightarrow \lambda_i \neq 0$

then $\underbrace{\sum_{j=1}^{n} \lambda_j w_j = 0}_{d_i \cdot w_i + \underbrace{\lambda\_w\_ + \lambda\_w\_}_{\lambda = 0\ (i \neq j)} = 0}$ but $\lambda_i \neq 0$ so $\{w_1, \dots, w_n\}$ is not LI //

Proposition: Let V be a non zero vector space spanned by vectors $\{w_1, \dots, w_n\}$.

Then V has a basis with at most n elements.

Proof: By induction on n.

Let $P(n)$ be statement of the proposition.

- First prove $P(1)$:

→ $V \neq 0$ is spanned by a single element $\overbrace{\underline{w} = (\underline{w}_1)}^{n=1}$

④ → I claim that $\{w\}$ is L.I.

→ Suppose $\lambda \underline{w} = 0$

Got to show $\lambda = 0$.

→ Suppose not. Then multiply across by λ^{-1}.

$\lambda^{-1} \lambda \underline{w} = 0$ so $\underline{w} = 0$.

→ But $\{\underline{w}\}$ spans V.

so $V = 0$. Contradiction, so $P(1)$ is true.

→ Suppose $\overset{P(k)}{P(n-1)}$ is true.

Got to show $\overset{P(k+1)}{P(n)}$ is true.

V is spanned by $\{\underline{w}_1, \dots, \underline{w}_n\}$.

If $\{w_1, w_2, \dots, w_n\}$ is L.I, then $\{w_1, \dots, w_n\}$ is a basis for V.

Ⓑ → Suppose $\{w_1, \dots, w_n\}$ is not L.I.

so $\exists\ \lambda_1, \dots, \lambda_n \in \mathbb{F}$

$\lambda_1 w_1 + \dots + \lambda_n w_n = 0$.

$$\sum_{j=1}^{n} \lambda_j w_j = 0 \ ; \ \sum_{j \neq i} \lambda_j w_j = -\lambda_i$$

and $\lambda_i \neq 0$ for some i, so $\lambda_i w_i = \sum_{j \neq i} (-\lambda_j) w_j$.

→ Multiply across by λ_i^{-1}: $w_i = \sum_{j \neq i} \left(-\frac{\lambda_j}{\lambda_i}\right) w_j$

→ Take out $\underline{w}_i$.

Claim that $\{w_1, \dots, w_{i-1}, w_{i+1}, \dots, w_n\}$.

↳ al quitas w_i, no sigue span porque w_i en

Still spans V.

pasa nada se puede def función de w_j

→ let $\underline{x} \in V$. $\{w_1, \dots, w_n\}$ span V, so write

$$\underline{x} = \sum_{j=1}^{n} \mu_j w_j .$$

So $\underline{x} = \mu_i w_i + \sum_{j \neq i} \mu_j w_j$.

so $$\boxed{\underline{x} = \sum_{j \neq i} \left(\mu_j - \mu_i \frac{\lambda_j}{\lambda_i}\right) \cdot w_j}$$

→ so V is spanned by $\{w_1, W_{i-1}, w_{i+1}, \dots, w_n\}$.

which has $(n-1)$ elements.

→ porque al quitar w_i, ya el resto es LI y como tb spans, es ba

By induction, V has a basis with $\leq n-1$ elements in this case.

$(n-1) \leq n$, then in either case V has a basis with $\leq n$ elements

QED.

→ This proves EXISTENCE if V is assumed to be spanned by a FINITE nb of elements.

In all case, we'll meet ...., this shows existence.

If you believe AXIOM OF CHOICE, then every non-zero vector space has a basis regardless of whether it is finitely generated or not.

If not then FINE.

UNIQUENESS

Proof by E. Steinitz $\boxed{c\ 1908}$

Exchange lemma (Baby version)

Exchange Lemma (Baby version) → Suppose $\{w_1, \dots, w_n\}$ span V, and let $\underline{v} \in V$ $\boxed{\underline{v} \neq 0}$

Write $\boxed{\underline{v} = \sum_{j=1} \lambda_j w_j}$

para w_i se puede expresar como V, en función de w_j.

If $\lambda_i \neq 0$ then $\{w_1, \dots, w_{i-1}, \underline{v}, w_{i+1}, \dots, w_n\}$ spans V.

(i.e. swap $\underline{w}_1$ and $\underline{v}$)

Proof: write $\underline{v} = \lambda_i \underline{w}_i + \sum_{j \neq i} \lambda_j \underline{w}_j$ and $\lambda_i \neq 0$.

Multiply across by λ_i^{-1}

$$\boxed{\left(\frac{1}{\lambda_i}\right) \underline{v} = \underline{w}_i + \sum_{j \neq i} \left(\frac{\lambda_j}{\lambda_i}\right) \cdot \underline{w}_j}$$

$$\underline{w}_i = \left(\frac{1}{\lambda_i}\right) \underline{v} + \sum_{j \neq i} \left(-\frac{\lambda_j}{\lambda_i}\right) \cdot w_j$$

→ I claim that $\{w_1, \dots, w_{i-1}, \underline{v}, \underline{w}_{i+1}, \dots, \underline{w}_n\}$ spans V.

The hypothesis is that $\{w_1, \dots, w_n\}$ spans V

$\underline{x} \in V$ and write $\boxed{\underline{x} = \sum_{j=1} \mu_j \underline{w}_j}$

$$\underline{x} = \mu_i \underline{w}_i + \sum_{j \neq i} \mu_j \underline{w}_j.$$

$$\underbrace{\underline{x} = \left(\frac{\mu_1}{\lambda_1}\right) \underline{v} + \sum_{j \neq i} \left(\mu_j - \frac{\mu_i \lambda_j}{\lambda_i}\right) \cdot w_j}_{\text{el vector x se expresa en función de v y } w_j \text{ por eso estos span}}$$

So $\{\underline{w}_1, w_2, \dots, w_{i-1}, v, w \dots w_n\}$ spans V. QED ✓.

Full exchange lemma

Let V be a non zero vector space spanned by $\{\underline{w}_1, \dots, \underline{w}_n\}$

Suppose $\{\underline{v}_1, \dots, v_k\}$ is L.I ($\underline{v}_i \in V$).

Then:

i) $k \leq n$.

ii) There exists a spanning set $\{\underline{w}_1', \dots, \underline{w}_n'\}$ for V.

s.t $w_i' = v_i$ $(1 \leq i \leq k)$ $w_j' \in \{w_1, \dots, w_n\}$ for $k+1 \leq j$.

December 7th 2018

Exchange lemma (Steinitz, 1908):

Let $\{w_1, \dots, w_n\}$ spans vector space V.

and let $\{\underline{v}_1, \dots, \underline{v}_k\} \subset V$ be L.I. Then:

i) $k \leq n$.

dash

ii) $\exists$ a spanning set $\{\underline{w}_1', \dots, \underline{w}_n'\}$ for V s.t

$\underline{w}_1' = \underline{v}_1, \dots, \underline{w}_k' = \underline{v}_k$

$w_i' = v_i$ $(1 \leq i \leq k)$.

and s.t for $j > k$ .. $w_j' \in \{w_1, \dots, w_n\}$.

Proof By induction on k:

- The induction base is $k=1$.

We've already done this, to repeat, express v_1:

$$v_1 = \lambda_1 w_1 + \dots + \lambda_n w_n$$

- As $v_1 \in \{v_1, \dots, v_k\}$ which is L.I.

we know $v_1 \neq 0$

So chose i. $\lambda_i \neq 0$.

$$\underline{v}_1 = \lambda_i w_i + \sum_{j \neq i} \lambda_j w_j .$$

$$w_i = \left(\frac{1}{\lambda_i}\right) v + \sum_{j \neq i} \left(-\frac{\lambda_j}{\lambda_i}\right) w_j$$

- Claim: $\{w_1, \dots, w_{i-1} ; v ; w_{i+1}, \dots, w_n\}$.

spans V.

$$\text{If } \underline{x} = \sum \mu_j w_j = \mu_i \underline{w_i} + \sum_{j \neq i} \mu_j w_j$$

$$\underline{x} = \left(\frac{\mu_i}{\lambda_i}\right) \underline{v} + \sum_{j \neq i} \left(\mu_j - \mu_i \frac{\lambda_j}{\lambda_i}\right) w_j$$

So we've proved induction base.

- Induction step: Assume true for k-1
- Induction hypothesis then gives a spanning set of form

$$\{\underbrace{v_1, \dots, v_{k-1}}_{w_1' \text{ hasta } w_{k-1}'} ; w_k', \dots, w_n'\}$$

porque es true para k-1

where $k-1 \leq n$. $w_j' \in \{w_1, \dots, w_n\}$

$\{w_1, \dots, w_k\}$.

In this case $\boxed{k-1 < n}$

Why?

Otherwise if $n = k-1$ then $\{v_1, \dots, v_{k-1}\}$ spans V.

- This gives a contradiction by expressing

$$v_k = \sum_{i=1}^{k-1} \eta_i v_i$$

$\eta_k = -1$

$\eta_k \neq 0$

$$\sum_{i=1}^{k-1} \eta_i v_i + (-1) v_k = 0 .$$

contradicts L.I of $\{v_1, \dots, v_k\}$.

hemos dicho que eran L.I al principio

- As $k-1 < n$ then $k \leq n$, which is first part of conclusion.

$4-1 < 4$

To finish, take spanning set $\{v_1, \dots, v_{k-1} ; w_k', \dots, w_n'\}$

- Express v_k in terms of ⟶

Write

$$\underline{v_k} = \lambda_1 v_1 + \dots + \lambda_{k-1} v_{k-1} + \sum_{j=k}^{n} \lambda_i w_i' .$$

Observe $\underline{v_k} \neq 0$ as $\{v_1, \dots, v_k\}$ is L.I

- Claim that $\lambda_j \neq 0$ for some j.

  $k \le j \le n$ .. $\sum_{j=k}$

- Otherwise, if $\lambda_j = 0$ for $k \le j \le n$.

  I have a dependence relation.

$$v_k = \lambda_1 v_1 + \dots + \lambda_2 v_2$$

porque debería ser $0 = \lambda_1 v_1 + \dots + \lambda_2 v_2$

Contradicting linear independence of $\{v_1, \dots, v_k\}$.

So $\lambda_i \neq 0$ for some j. $k \le i \le n$.

$$V_k = \lambda_1 v_1 + \dots + \lambda_{k-1} V_{k-1} + \boxed{\lambda_i w_i'} + \sum_{\substack{j=k \\ j \neq i}}^{n} \lambda_j \cdot w_j'$$

$\lambda_i \neq 0$

Baby lemma (cambia w_i' por v_i

- So use Baby version and swap V_k and w_i'

  Now we have a spanning set of form $\{V_1, .., V_{k-1}, V_k\} \cup \{w_j' ; k \le j \le n, j \neq i\}$

  cardinal = $(n-k)$

  nb of elements

- Formally to complete proof:

  Put $w_i'' = v_i$ $1 \le i \le k$ and let $\{w_j'' ; k+1 \le j \le n\}$

  be some indexing of the set $\{w_j' : 1 \le j \le k, j \neq i\}$.

Q.ED

Corollary BASIS THEOREM.

Let V be a non-zero vector space, then:

i) V has at least one basis.

ii) Any two basis have same nb of elements.

Proof: i) already done.

ii) Suppose $\{E_1, \dots, E_m\}$ is a basis for V. and $\{\varphi_1, \dots, \varphi_n\}$ is also a basis /V

$\{E_1, \dots, E_m\}$ is L.I and $\{\varphi_1, \dots, \varphi_n\}$ spans. So $m \le n$.

Now $\{\varphi_1, \dots, \varphi_n\}$ is. LI. $\{E_1, \dots, E_m\}$ spans, so $n \le m$.

so $m \le n \le m$ so $m = n$. QED

Corollary : (of Exchange Lemma)

Suppose : V, W are vector spaces with $V \subset W$.

then, $V = W \iff \dim(V) = \dim(W)$.

Proof $\Rightarrow$ Trivial

$\Leftarrow$ Suppose, $\dim(V) = \dim(W)$.

- If $\dim(V) = \dim(W) = 0$ then $V = \{0\} = W$. So nothing to prove.
- So suppose $V \neq 0$ (so $W \neq 0$)

Let $\{E_1, \dots, E_n\}$ be a basis for V.

Let $\{\varphi_1, \dots, \varphi_n\}$ be a basis for W.

$\{\varphi_1, \dots, \varphi_n\}$ spans W. $\{E_1, \dots, E_n\}$ is L.I

By exchange lemma : $\{E_1, \dots, E_n\}$ spans W.

If $\underline{w} \in W$. write :

$$\underline{w} = \lambda_1 E_1 + \dots + \lambda_n E_n \in V.$$

because each $E_i \in V$. So $W \subset V \subset W$ so $V = W$

QED.

Back to Kernel - Rank Theorem:

$T : V \to W$ be linear.

$$\boxed{\dim(V) = \dim \ker(T) + \dim \operatorname{Im}(T)}$$

I showed T Injective $\iff \dim \ker(T) = 0$.

Proposition : T is surjective $\iff \dim \operatorname{Im}(T) = \dim(W)$.

Proof :

$\Rightarrow$ trivial

$\Leftarrow$ As $\operatorname{Im}(T) \subset W$ then $\dim \operatorname{Im}(T) = \dim(W)$

Implies $\operatorname{Im}(T) = W$ I.e. T surjective

• Isomorphisms .

Let V, W be vector spaces

Say that V, W are isomorphic, written $V \cong W$

when there exists a bijective linear map

$$T : V \longrightarrow W .$$

1) $V \cong V$ (Reflexivity)

$Id_V : V \longrightarrow V$ linear and bijective

2) If $V \cong W$ then $W \cong V$ (Symmetry)

Proof let $T : V \longrightarrow W$ be linear and bijective.

I claim $T^{-1} : W \to V$ is linear (and it is certainly bijective)

Let $w_1, w_2 \in W$. Consider

$$T^{-1}(w_1+w_2) - T^{-1}(w_1) - T^{-1}(w_2)$$

Apply T.

$$T\left(T^{-1}(w_1+w_2) - T^{-1}(w_1) - T^{-1}(w_2)\right) \overset{T \text{ is linear}}{=} TT^{-1}(w_1+w_2) - TT^{-1}(w_1) - TT^{-1}(w_2) =$$

$$= w_1 + w_2 - w_1 - w_2 = 0 .$$

But T is injective, so:

$$T^{-1}(w_1+w_2) - T^{-1}(w) - T^{-1}(w_2) = 0 .$$

i.e. $T^{-1}(w_1+w_2) = T^{-1}(w_1) + T^{-1}(w_2)$.

Likewise :

$$T\left(T^{-1}(\lambda w) - \lambda T^{-1}(w)\right) = 0$$

T is injective, so : $T^{-1}(\lambda w) - \lambda T^{-1}(w) = 0$, so,

$T^{-1}(\lambda w) = \lambda T^{-1}(w)$ and T^{-1} is linear.

QED// (Symmetry).

3) $U \cong V$ and $V \cong W$, then $U \cong W$. Transitivity

Proof :

If $T : U \to V$ is linear bijective,

$S : V \to W$ is linear and bijective

Then $S \circ T : U \to W$ is linear bijective

Proposition

If V, W are vector spaces / $\mathbb{F}$, then:

$V \cong W \iff \dim(V) = \dim(W)$.

$\Rightarrow$ Suppose $T : V \to W$ linear and bijective

Let $\{E_1, \dots, E_n\}$ be a basis for V.

Claim that $\{T(E_1), \dots, T(E_n)\}$ is a basis for W.

$\{T(E_1), \dots, T(E_n)\}$ is L.I.

Why?

$$\sum_{i=1}^{n} \lambda_i T(E_i) = 0$$

$$||$$

$$T\left(\sum_{i=1}^{n} \lambda_i E_i\right) = 0.$$

But T is injective so $\ker(T) = 0$, so

$$\sum_{i=1}^{n} \lambda_i E_i = 0$$ and $\{E_1, \dots, E_n\}$ is L.I, so $\lambda_1 = \dots = \lambda_n = 0$.

Claim $T(E_1), \dots, T(E_n)$ spans W.

- Let $\underline{w} \in W$. T surjective

Choose $\underline{v} \in V : T(\underline{v}) = \underline{w}$

Express $\underline{v} = \sum_{i=1}^{n} \lambda_i E_i$

$$\underline{w} = T(\underline{v}) = T\left(\sum_{i=1}^{n} \lambda_i E_i\right)$$

$$\underline{w} = \sum_{i=1}^{n} \lambda_i T(E_i)$$

$\{T(E_i) \dots T(E_n)\}$ spans V.

$T(E_1) \dots T(E_n)\}$ basis for W.

$\dim(W) = n = \dim(V)$ QED $\Rightarrow$

$\Leftarrow$

Conversely, if $\dim(V) = \dim(W)$

then $V \cong W$.

- Take basis $\{\epsilon_1, \dots, \epsilon_n\}$ for V.

$\{\varphi_1, \dots, \varphi_n\}$ for W.

Let $T : V \to W$ be the unique linear mapping determined by

$T(\epsilon_i) = \varphi_i$.

Let $S : W \to V$ be unique linear s.t $S(\varphi_i) = \epsilon_i$.

Then, $S \circ T(\epsilon_i) = \epsilon_i$ so $S \circ T = Id_V$.

Likewise, $(T \circ S)(\varphi_i) = \varphi_i$ so $T \circ S = Id_W$.

so $T : V \to W$ is a linear bijection.

$T^{-1} = S$.

so $V \cong W$.

QED//

December 13th:

PERMUTATIONS

$\sigma_n = \{\sigma : \{1, \dots, n\} \longrightarrow \{1, \dots, n\}$.

σ is bijective.

||

σ_n = group of permutations on n. letters.

$|\sigma_n| = n!$

If $\sigma; \tau \in \sigma_n \Rightarrow \sigma \circ \tau \in \sigma_n$.

I) If $\sigma \in \sigma_n$, we can write $\sigma = C_1 \cdots C_m$ where $C_1, \dots, C_m$ are disjoint cycles

II) Disjoint cycles commute

$\sigma^n = C_1^n \cdots C_m^n$

$C_i C_j = C_j C_i$

III) A transposition is a cycle of length 2. $t = (i, j)$.

Any cycle can be written as a product of transpositions

$$\underbrace{(a_1, \dots, a_n)}_{\text{length } n} = \underbrace{(a_1, a_n)(a_1, a_{n-1}) \dots (a_1, a_2)}_{(n-1) \text{ transpositions}}.$$

IV) Any transposition is a product of an ODD nb of adjacent transpositions.

$(i, i+1)$ is adjacent → Proof by induction on <u>gap</u>.

$\text{gap}(i,j) = |j-i| (= k)$ $\quad |j-(i+1)| = |j-i-1| = |k-1|$

$$(i,j) = \underbrace{(i, i+1)}_{\text{gap } 1} \underbrace{(i+1, j)}_{\text{gap}(k-1)} \underbrace{(i, i+1)}_{\text{gap } 1}.$$

se puede escribir como producto de $2k-1$ adyacent transp. → el gap en este caso es

A transposition of gap k, can be written as a product of $2k-1$ adjacent transpositions

$\underbrace{k-1}$

$2(k-1)-1$

$$2k-1 = 1 + [2(k-1) - 1] + 1.$$

<u>Corollary</u>: Any permutation can be written as a product of adjacent transpositions.

<u>Def</u>: Let $\sigma \in \mathfrak{S}_n$

Define: $\text{Sign}(\sigma) = \prod_{1 \le r < s \le n} \left(\frac{\sigma(s) - \sigma(r)}{s - r} \right)$ Laplace's formula

↑ product symbol

<u>Prop</u>: $\text{Sign}(\sigma \cdot \tau) = -\text{sign}(\sigma)$. if τ is an adjacent transposition.

Obvious consequence.

0) $\text{Sign}(Id) = 1$.

1) $\text{Sign}(\tau_1, \dots \tau_n) = (-1)^n$ if $\tau_1, \dots, \tau_n$ are adjacent transpositions

So if σ is a product of an EVEN nb of adjacent trans

$\text{sign}(\sigma) = +1$

if σ is a product of an ODD nb of adj trans

$\text{sign}(\sigma) = -1$

If $\sigma \in \sigma_n$ define $\mathcal{L}(\sigma) = \prod_{1 \leq r < s \leq n} \sigma(s) - \sigma(r)$.

so $\mathcal{L}(Id) = \prod_{1 \leq r < s \leq n} (s - r)$.

Proposition If $\tau = (i, i+1)$ is an adjacent transposition, then,

$$\mathcal{L}(\sigma t) = -\mathcal{L}(\sigma).$$

Proof $(i, i+1)$ are fixed.

(r, s) vary $r < s$.

Consider $\{r, s\} \cap \{i, i+1\}$.

Possibilities

(0) $\{r, s\} \cap \{i, i+1\} = \emptyset$.

(1) $\{r, s\} \cap \{i, i+1\} = i$.

splits into two $\begin{cases} r < i & s = i \quad 1(a) \\ r = i & i+1 < s \quad 1(b) \end{cases}$.

(2) $\{r, s\} \cap \{i, i+1\} = \{i+1\}$.

2(a) $r < i$ $\quad s = i + 1$.

2(b) $r = i + 1$ $\quad i + 2 \leq s$.

---

(3) $\{r, s\} \cap \{i, i+1\} = i, i+1$

i.e. $r = 1$, $s = i + 1$.

Here, $\tau = (i, i+1)$.

For any, $\sigma \in \sigma_n$ define $\mathcal{L}_0(\sigma), \mathcal{L}_1(\sigma), \mathcal{L}_2(\sigma) \mathcal{L}_3(\sigma), \mathcal{L}_4(\sigma), \mathcal{L}_5(\sigma)$

as follows.

$$\mathcal{L}_0(\sigma) = \prod_{(r,s) \text{ in case } 0} \sigma(s) - \sigma(r).$$

$$\mathcal{L}_1(\sigma) = \prod_{(r,s). \text{ case } 1(a)} \sigma(s) - \sigma(r).$$

$$\mathcal{L}_2(\sigma) = \prod_{(r,s) \text{ case } 1(b)} \sigma(s) - \sigma(r).$$

$$\mathcal{L}_3(\sigma) = \prod_{(r,s)\ \text{case 2a}} \sigma(s) - \sigma(r) .$$

$$\mathcal{L}_4(\alpha) = \prod_{(r,s)\ \text{case 2b}} \sigma(s) - \sigma(r) .$$

$$\mathcal{L}_5(\sigma) = \prod_{(r,S)\ \text{case 3}} \sigma(s) - \sigma(r) \quad \leftarrow \text{single factor .}$$

$$\mathcal{L}(\sigma) = \mathcal{L}_0(\sigma)\mathcal{L}_1(\sigma)\mathcal{L}_2(\sigma)\mathcal{L}_3(\sigma)\mathcal{L}_4(\sigma)\mathcal{L}_5(\sigma)$$

But (r,s) in case 0, then $\tau(r), \tau(s)$ also in case 0.

$\mathcal{L}_0(\sigma t) = \mathcal{L}_0(\sigma)$ If (r,s) in case 1(a), then $(\tau(r), t(s))$ in case 2a and viceversa.

$\mathcal{L}_1(\sigma t) = \mathcal{L}_3(\sigma)$.

$\mathcal{L}_3(\sigma t) = \mathcal{L}_1(\sigma)$.

(r,s) in case 1(b) $\Leftarrow\Rightarrow$ $(\tau(r), t(s))$ case 2(b).

$\mathcal{L}_2(\sigma t) = \mathcal{L}_4(\sigma)$.

$\mathcal{L}_4(\sigma t) = \mathcal{L}_2(\sigma)$

But

$\mathcal{L}_5(\sigma t) = -\mathcal{L}_5(\sigma)$.

$$\mathcal{L}(\sigma\tau) = \mathcal{L}_0(\sigma t)\mathcal{L}_1(\sigma\tau)\mathcal{L}_2(\sigma t)\mathcal{L}_3(\sigma t)\mathcal{L}_4(\sigma t)\mathcal{L}_5(\sigma t).$$

$\mathcal{L}_0(\sigma\tau) = \mathcal{L}_0(\sigma)$ $\qquad$ $\mathcal{L}_3(\sigma t) = \mathcal{L}_1(\sigma)$

$\mathcal{L}_1(\sigma\tau) = \mathcal{L}_3(\sigma)$ $\qquad$ $\mathcal{L}_4(\sigma t) = \mathcal{L}_2(\sigma)$.

$\mathcal{L}_2(\sigma t) = \mathcal{L}_4(\sigma)$ $\qquad$ $\mathcal{L}_5(\sigma t) = -\mathcal{L}_5(\sigma)$

Hence, we see that $\mathcal{L}(\sigma\tau) = -\mathcal{L}(\sigma)$ provided t is an adjacent transposition.

Corollary, If $\sigma = \tau_1, \dots, \tau_n$, τ_i adjacent transposition

$$\mathcal{L}(\sigma) = (-1)^n \mathcal{L}(\mathrm{Id}) .$$

Proof. By induction on n.

$n=1$ Take $\sigma = Id$, $\mathcal{L}(t) = -\mathcal{L}(Id)$ by above.

So suppose proved for $n-1$

$$\mathcal{L}(t_1, \dots, t_{n-1}, t_n) = -\mathcal{L}(T_1, \dots, T_{n-1}) = (-1)\cdot(-1)^{n-1} = (-1)^n$$

By induction.

Corollary

$\mathcal{L}(\sigma) = (-1)^n$ if σ is a

$\mathcal{L}(Id)$ product of n adjacent transposition

$\quad ||$

sign (σ)

December 14th

Definition of reduced row echelon form

$A = (a_{ij})$ $\begin{matrix} 1 \le i \le m \\ 1 \le j \le n \end{matrix}$

$A_{i,*} = (a_{ij})$ $1 \le j \le n$. ith row

Condition 1

If $A_{i,*} \neq 0$ and $A_{k,*} = 0$. then $i < k$.

In English, Zero rows come last.

Let $A_{1,*}, \dots, A_{r,*}$

be the non zero rows.

For $1 \le i \le r$.

$c(i) = \min\{ j ; a_{i,j} \neq 0 \}$.

i.e. $c(i)$ is the 1st column in which you get a non zero entry in

$A_{i,*}$.

$a_{i,c(i)} \neq 0$ but $a_{i,j} = 0$ if $j < c(i)$.

Condition 2 $a_{i,c(i)} = 1$.

Condition 3 $a_{k,c(k)} = 0$ if $k \neq i$.

Condition 4 $c(1) < c(2) < \ldots < c(r)$.

In English, the rows are stepped.