1301 Applied Mathematics 1 Notes

Based on the 2009 autumn lectures by Dr A Wynn

05/10/09

Part I : Probability

1. set theory

1.1. sets and subsets

<u>Defⁿ</u> : A set is a collection of objects thought of as a whole

<u>Defⁿ</u> : the objects, of which a set is a collection, are called elements or members of the set.

e.g. if A is a set and x is an element of A we write $x \in A$

if x does not belong to A we write $x \notin A$

e.g. $A = \{x_1, x_2, \ldots, x_n\}$

$A = \{x : x \text{ is an integer}\} = \{0, \pm 1, \pm 2, \ldots\}$

: Such that

Subset

$A \subset B$, $B \supset A$: means "the set A is a subset of B"

$A \subseteq B$: means " —— " ——— "

$A \subsetneq B$, $B \supsetneq A$: means "A is a subset of B, but $A \neq B$"

Often we consider a large set S which contains all the sets we are interested in for a particular problem. (handout 2 Sets : Ex.. Venn diagram)

1.2. set operations

· Intersection

$A \cap B = \{x : x \in A \ \& \ x \in B\}$

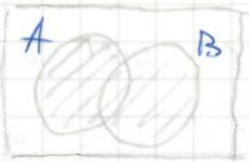
Rules of sets: $A \cup B = B \cup A$ $\quad AB = BA$ Commutative law

$(E \cup F) \cup G = E \cup (F \cup G)$ $\quad (EF)G = E(FG)$ Associative law

$(E \cup F)G = EG \cup FG$ $\quad EF \cup G = (E \cup G)(F \cup G)$ Distribution l.

· Union

$A \cup B = \{x : x \in A \text{ or } x \in B\}$



· Complement –

the ~~co~~ complement of a set A in S is the set of elements (in S) which are not in A.

we write A' ("A prime") for the complement of A

$A' = \{x : x \notin A\}$

· Disjoint or Mutually exclusive

A & B ~~can &~~ are disjoint if $A \cap B = \phi$

e.g. $A = (A')'$

$S' = \phi$

$\phi' = (S')' = S$

· Relative Complement :

$A \setminus B$ is the set of elements in A which are not in B

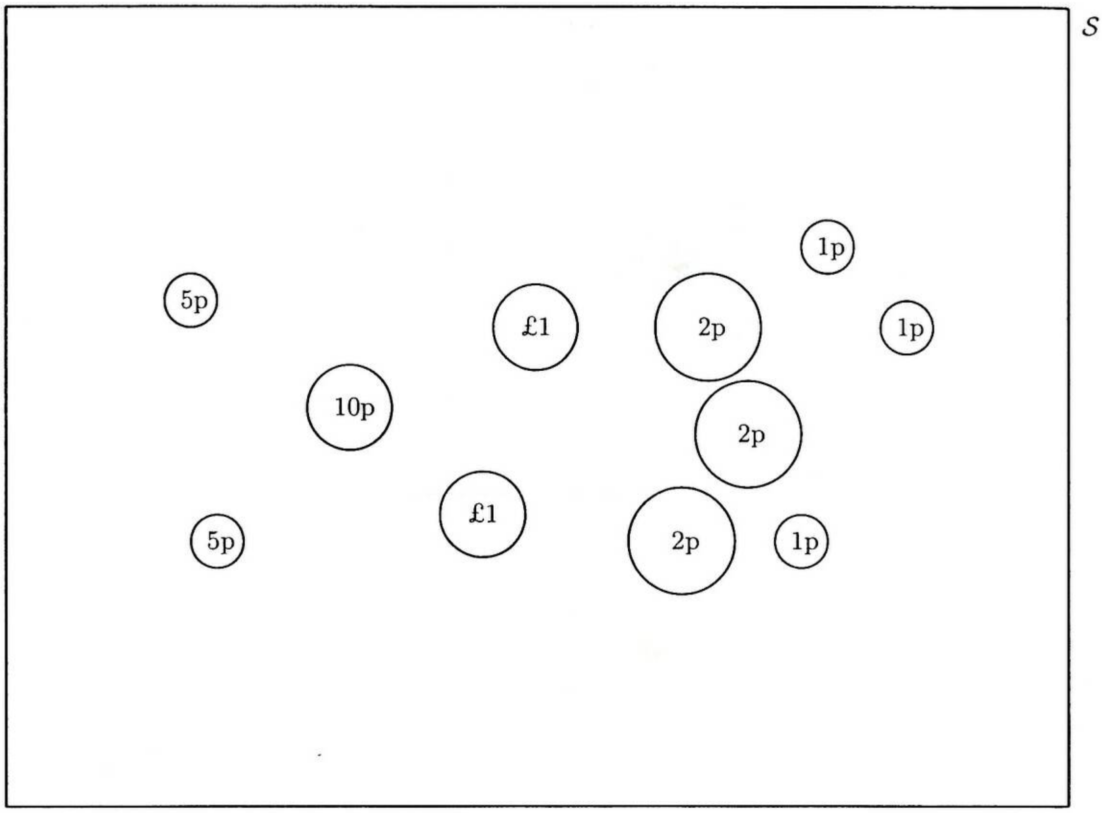
$A \setminus B = \{x : x \in A \text{ and } x \notin B\}$

e.g. let A and B be two sets

then $(A \setminus B) \cup (A \cap B) = A$

$(A \setminus B) \cup B = A \cup B$

Handout 2 Sets: Example Venn diagram



Here S is the set of all coins in my pocket the day I wrote this handout.

Let C be the set of all copper coins $\{1p's, 2p's\}$

Let D be the set of silver coins $\{5p's, 10p's\}$

Let E —— " —— coins worth an even numb. of pence

F —— " —— coins worth more than 4p $\to \{5p, 10p, £1\}$

G —— " —— ~~m~~ which are not round (nothing)

$G = \emptyset$ - the empty set - is the set with no elements

1.2. set operation

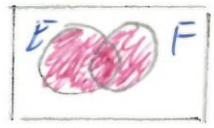
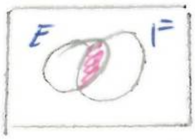
$D \cap E = \{10p\}$

$C \cup E$ is everything in S, except the ~~tho~~ two 5p coins

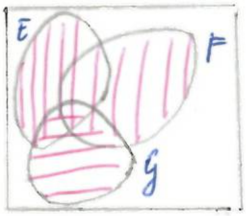
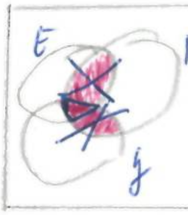
C and D are disjoint

$C \setminus E = \{1p \text{ coins}\}$

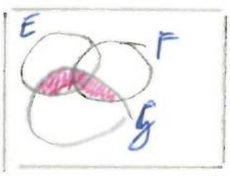
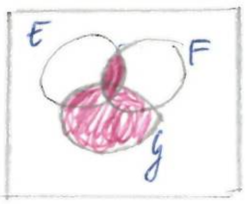
graphical proovements of set laws:

1)   Commutative l.

$E \cup F = F \cup E$ $\qquad EF = FE$

2)   Associative l.

$(E \cup F) \cup G = E \cup (F \cup G) \qquad (EF)G = E(FG)$

3)   Distributive l.

$(E \cup F)G = EG \cup FG \qquad EF \cup G = (E \cup G)(F \cup G)$

$$(E \cup G)(F \cup G) = EF \cup GF \cup EG \cup \cancel{GG}^{\,G} = EF \cup G \cup (GF \cup EG) =$$

$$= EF \cup G \cup \underbrace{(F \cup E)G}_{\text{part of the } G} = EF \cup (G \cup \overbrace{\cancel{\text{part of } G}}^{\text{subset } (G)}) = EF \cup G$$

<u>Определение</u> Вероятностью $\mathbb{P}(A)$ события в данном опыте называется отношение числа M исходов опыта, благоприятствующих событию A, к общему числу N возможных исходов опыта, образующих полную группу равновероятных попарно несовместных событий

$$\mathbb{P}(A) = \frac{M}{N}$$

<u>Определение</u> Событие B называется благоприятствующим событию A, если наступление события B влечёт за собой наступление события A. (Пример A = чётное число, $B = 4$)

01/10/09.

2.2. Sample spaces and Events

e.g. ~~To~~ two fair coins

The poss. outcomes:

(i) HH

(ii) HT

(iii) TT

Sample Space $S = \{HH, HT, TT\}$

An event is a subset of the sample space

e.g. Let A be the event if getting 2 heads $A = \{HH\}$

Let B ——— " ——— 1 head $B = \{HH, HT\}$

Since A has only one element it is called simple event.

B has 2 elements, $B = \{HH, HT\} = \{HH\} + \{HT\}$

So the event B happens if either HH or HT

$\emptyset$ null event - event that consists of outcomes

Пространство событий = sample space

2. Probability and sample spaces

2.1. Probability model.

Rolling a fair dice once . outcomes - 1, 2, 3, 4, 5, 6, each outcome has probability $\frac{1}{6}$

A probability has 2 components

(i) A set S → set of possible outcomes

(ii) Assign → probability to each outcome

what means?

assign

1) присваивать

Le Color

Assign Probabilities

$$\mathbb{P}(HH) = 1/4 \; ; \quad \mathbb{P}(HT) = 1/2 \; ; \quad \mathbb{P}(TT) = 1/4$$

I choose $S = \{ HH, HT, TT \}$

Might have been better to let

$$S = \{ (H,H), (H,T), (T,H), (T,T) \}$$

In this case $\mathbb{P}((H,H)) = \frac{1}{4}$; $\mathbb{P}((H,T)) = \frac{1}{4}$

$$\mathbb{P}((T,H)) = \frac{1}{4} \; ; \quad \mathbb{P}((T,T)) = \frac{1}{4}$$

$A = \{ H, H \}$ $\qquad A \cap B = \{ H, H \} = A$

$B = \{ HH, HT \}$ $\qquad A \cup B = \{ HH, HT \} = B$

$\cancel{= \{ (H,H), (HT), (T,H) \}}$

2.3. Probability axioms

Let $E \subset S$ be an event. A **probability** is a positive real number written $\mathbb{P}(E)$, which is called the "probability of E".

A probability must satisfy the following 3 axioms:

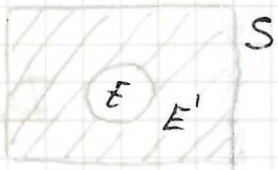
$$\boxed{\begin{array}{l} 1.\ \cancel{\mathbb{P}(E) \geq 0} \quad 0 \leq \mathbb{P}(E) \leq 1 \\ 2.\ \mathbb{P}(S) = 1 \qquad E \cap F = \emptyset \\ \text{proposition } 3.\ \text{For two disjoint events } E, F \subset S: \\ \qquad \mathbb{P}(E \cup F) = \mathbb{P}(E) + \mathbb{P}(F) \end{array}}$$

Axioms: For any sequence of mutually exclusive events $E_1, E_2, \ldots$ (that is, events for which $E_i E_j = \emptyset$, $i \neq j$)

$$\boxed{\mathbb{P}\left(\bigcup_{i=1}^{\infty} E_i\right) = \sum_{i=1}^{\infty} \mathbb{P}(E_i)}$$

, where $\mathbb{P}(E)$ is the prob. of E.

Remember $E' = \{x : x \notin E\}$



So $E \cap E' = \phi$

So E and E' are disjoint events

$$\Rightarrow \mathbb{P}(E) + \mathbb{P}(E') \overset{\text{3 Axiom}}{=} \mathbb{P}(E \cup E') = \mathbb{P}(S) \overset{\text{2 Axiom}}{=} 1$$

$$\boxed{\mathbb{P}(E) = 1 - \mathbb{P}(E')} \qquad (1)$$

1) Similarly $A \setminus B$ and $A \cap B$ are disjoint events and $(A \setminus B) \cup (A \cap B) = A$ (lect. 1)

By axioms 3 $\mathbb{P}(A) = \mathbb{P}((A \setminus B) \cup (A \cap B)) =$

$= \mathbb{P}(A \setminus B) + P(A \cap B)$ [i]

2) Also $A \setminus B$ and B are disjoint, and

$(A \setminus B) \cup B = A \cup B$

By axiom 3 $\mathbb{P}(A \cup B) = \mathbb{P}(A \setminus B) + \mathbb{P}(B)$ [ii]

By [i] and [ii]

$$\mathbb{P}(A \cup B) = \mathbb{P}(A \setminus B) + \mathbb{P}(B)$$
$$= \mathbb{P}(A) - \mathbb{P}(A \cap B) + \mathbb{P}(B)$$

$$\Rightarrow \boxed{\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cap B)} \qquad (2),$$

where If A & B are mutually exclusive

$\Rightarrow \mathbb{P}(A \cap B) = 0$

$\Rightarrow$ formula 2 $\rightarrow$ axiom of Prob. 3.

Proposition 4.2. if $E \subset F \Rightarrow \mathbb{P}(E) \leq \mathbb{P}(F)$

2.4. Probability in discrete sample spaces

Discrete means that there are only finitely many outcomes <u>or</u> there are ~~many~~ infinitely many outcomes and they can be written in a sequence $e_1, e_2, e_3, \ldots$

Sample Space $S = \{e_1, e_2, e_3, e_4, \ldots\}$

Event $E_1 = \{e_1\}$, $E_2 = \{e_2\}$ ← simple events

For each event $A \subset S$, A is a union of simple events E_i. Since each pair E_i, E_j is disjoint $(E_i \cap E_j)$. This means $\mathbb{P}(A)$ is just a sum of $\mathbb{P}(E_i)$'s

Note: if $i \neq j$ $E_i \cap E_j = \{e_i\} \cap \{e_j\} = \phi$

Example: DIE

$S = \{1, 2, 3, 4, 5, 6\}$

$E_1 = \{1\}$ = event that I roll a 1

$E_2 = \{2\}$ = ——— " ——— 2

$\vdots$

$E_6 = \{6\}$

For fair die let $\mathbb{P}(E_1) = \frac{1}{6}$, $\mathbb{P}(E_2) = \frac{1}{6}$, … $\mathbb{P}(E_6) = \frac{1}{6}$

$E_V = \{2, 4, 6\}$ = Event that I get an even numb.

$E_{V5} = \{2, 4, 6, 5\}$ = Die rolls an even numb. <u>or</u> a 5

$E_{V5} = E_V \cup E_5$, $E_V = E_2 \cup E_4 \cup E_6$

discrete sample space - it is a sample space which is containing a finite number of possibilities or an unending sequence with as many elements as there are whole numbers.

continuous sample space - it is a sample space contains an infinite number of possibilities equal to the number of points on a line segment.

e.g. $\mathbb{P}(E_V) = \mathbb{P}(E_2) + \mathbb{P}(E_4) + \mathbb{P}(E_6) = \frac{1}{6} + \frac{1}{6} + \frac{1}{6} = \frac{1}{2}$

e.g. $\mathbb{P}(E_{V^c}) = P(E_1) + \mathbb{P}(E_5)$

2.5: Equally likely outcomes.

If a sample space S has $N(S)$ simple events $E_1, E_2, \ldots, E_{N(S)}$. Since E_i's are simple they are disjoint

$$\underbrace{\mathbb{P}(E_1) + \mathbb{P}(E_2) + \ldots + \mathbb{P}(E_{N(S)})}_{N(S)} \overset{Ax\,3}{=} \mathbb{P}(S) \overset{Ax\,2}{=} 1$$

$$\mathbb{P}(E_1) = \mathbb{P}(E_2) = \ldots = \mathbb{P}(E_{N(S)}) = \frac{1}{N(S)}$$

Example

Consider a 4-sided fair die. A trial consists two throws of the die.

I can ask: what is the probability of

(a) The sum of the two numbers <u>~~doesn't~~ not exceed 5</u>

(b) The sum of the two number is ~~odd~~

(c) The difference is not bigger than 1

(d) The numbers are the same.

<u>Answer</u>: the Sample space S

$(1,1)$ $(1,2)$ $(1,3)$ $(1,4)$

$(2,1)$ $(2,2)$ $(2,3)$ $(2,4)$

$(3,1)$ $(3,2)$ $(3,3)$ $(3,4)$

$(4,1) \quad (4,2) \quad (4,3) \quad (4,4)$

So S consist of 16 simple, equally likely events.

$$\mathbb{P}(\text{each event}) = \frac{1}{N(S)} = \frac{1}{16}$$

$$\mathbb{P}((a)) = \frac{10}{16}$$

$$\mathbb{P}((b)) = \frac{8}{16}$$

$$\mathbb{P}((c)) = \frac{10}{16}$$

$$\mathbb{P}((d)) = \frac{\cancel{10}\,4}{16}$$

2.6. Conditional probability.

"what is the probability of the event B, given we know an event A has already occured"

Example: Throwing a 6-sided die.

$S = \{1, 2, 3, 4, 5, 6\}$, each outcome has probability $\frac{1}{6}$

let $A = \{1, 2, 3\}$ and let $B = \{2, 4, 6\}$

what is the probability of B given I know A has ~~occur~~ occured?

We know we rolled a 1, 2, 3 each is equally likely, so each has prob. $\frac{1}{3}$. So the prob. we rolled an even numb. $= \frac{1}{3}$

So probability of B given A is $\frac{1}{3}$"

Note: $\mathbb{P}(A \cap B) = \mathbb{P}(\{2\}) = \frac{1}{6}$

$\mathbb{P}(A) = \frac{1}{2}$

<u>Def^n</u> (Conditional probability)

можно

The conditional probability of an event B given an even has occured:

вероятность части множества A. Работа с A как с S

$$\boxed{\mathbb{P}(B \mid A) = \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(A)}}$$

вероятность что выпадут эти события <u>глобально</u>

вероятность что случится какой-то из элементов этого множества A. <u>глобально</u>.

In the above example $\mathbb{P}(A \cap B) = \mathbb{P}(\{2\}) = \frac{1}{6}$,

$\mathbb{P}(A) = \mathbb{P}(\{1, 2, 3\}) = \frac{1}{2}$

$$\Rightarrow \mathbb{P}(B \mid A) = \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(A)} = \frac{\frac{1}{6}}{\frac{1}{2}} = \frac{1}{3}$$

So the def^n agrees with our intuition

Note: if A had not already occured, $\mathbb{P}(B)$

$\mathbb{P}(B) = \mathbb{P}(\{2, 4, 6\}) = \frac{1}{2}$

So in general $\mathbb{P}(B \mid A) \neq \mathbb{P}(B)$

If I replace A by S in the def^n of cond. prob., then

$$\mathbb{P}(B \mid S) = \frac{\mathbb{P}(S \cap B)}{\mathbb{P}(S)} = \frac{\mathbb{P}(B)}{1} = \mathbb{P}B$$

Le Color

18.10.09. Q

Какая связь между $\mathbb{P}(A)$ и $\mathbb{P}(A \cap B)$?

$$\left.\begin{aligned} \mathbb{P}(A \cap B) &= \frac{a}{N} \\ \mathbb{P}(A) &= \frac{n}{N} \end{aligned}\right\} \to \frac{a}{N} \leq \frac{n}{N}$$

Still do not get the example

стр. 8 7->21

$P(R_1 R_2) = P(R_1) P(R_2 | R_1)$

$\left(\frac{2}{3}\right)\left(\frac{7}{11}\right)$

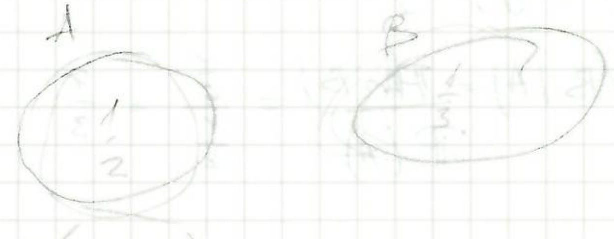
a – кол-во общих элементов между A и B

n – кол-во элементов в A

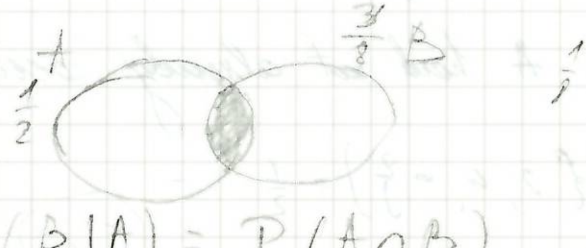
N – общее количество всех элементов.

$$\frac{P(A \cap B)}{\mathbb{P}(A)} = \frac{a}{N} : \frac{n}{N} = \frac{a}{n}$$ – вероятность того что случилась B зная что выпала A.

St.



$P(B|A) = 0$

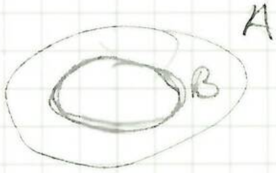


$$P(B|A) = \frac{P(A \cap B)}{\mathbb{P}(A)} =$$

$$\mathbb{P}(B|A) = \frac{\mathbb{P}(A\cap B)}{\mathbb{P}(A)}$$

$$\Rightarrow \boxed{\mathbb{P}(A)\,\mathbb{P}(B|A) = \mathbb{P}(A\cap B)}$$

if not $\mathbb{P}(B|A)$



Def

Events A and B are said to be *independent* if

$\Rightarrow \mathbb{P}(B|A) = \mathbb{P}(B)$

$$\boxed{\mathbb{P}(A\cap B) = \mathbb{P}(A)\cdot\mathbb{P}(B)} \qquad [1]$$

If [1] does not hold, A and B are called *dependent events*

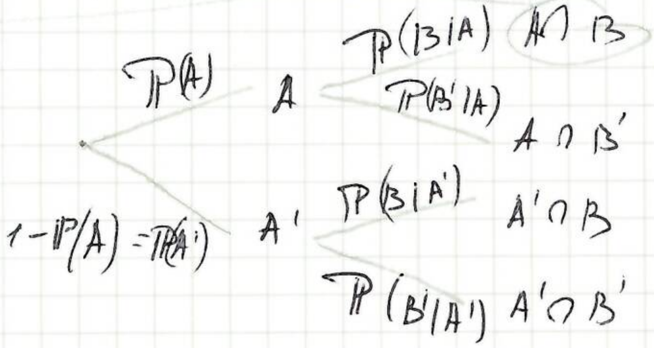
if A and B are independent:

$$\mathbb{P}(B|A) = \frac{\mathbb{P}(A\cap B)}{\mathbb{P}(A)} = \frac{\mathbb{P}(A)\cdot\mathbb{P}(B)}{\mathbb{P}(A)} = \mathbb{P}(B)$$

2.7. Trees

Suppose I have 2 events A and B

why?



For example, $\mathbb{P}(A\cap B) = \mathbb{P}(A)\,\mathbb{P}(B|A)$

I could have replace A and B in the above diagram to get $\mathbb{P}(A\cap B) = \mathbb{P}(B)\cdot\mathbb{P}(A|B)$

[illegible]: События которые not equally likely.
N: в конце их указываем.

Handout 3 Trees

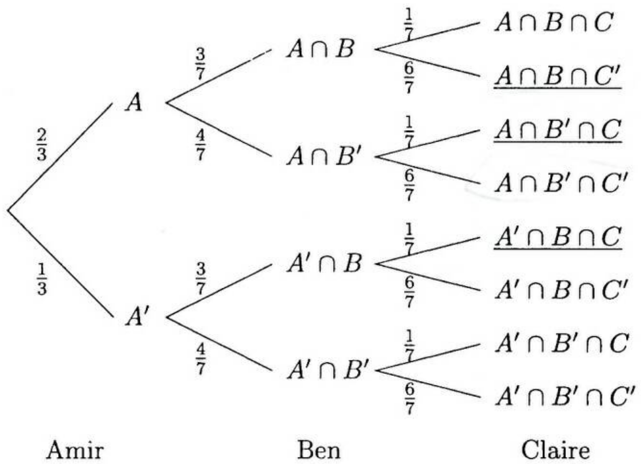
и можем мы учитывать разные вероятности.

Example:

Suppose that Amir is in the Union at 2pm on two days out of every three, whereas Ben is there about three times a week and Claire only about once a week. How likely is it that they will meet?

Solution:

Let A, B, C stand for the events of each being there on a particular day. $I\!P(A) = 2/3$, $I\!P(B) = 3/7$, $I\!P(C) = 1/7$. If all three choose their days independently the tree diagram is as below:



We see that $I\!P(A \cap B \cap C) = (2/3)(3/7)(1/7) = 2/49$ so they are all there together roughly once every $3\frac{1}{2}$ weeks. Also the underlined terms give the occasions on which exactly two of them are there. Thus

$$\begin{aligned}
I\!P(\text{exactly two there}) &= I\!P((A \cap B \cap C') \cup (A \cap B' \cap C) \cup (A' \cap B \cap C)) \\
&= I\!P(A \cap B \cap C') + I\!P(A \cap B' \cap C) + I\!P(A' \cap B \cap C) \\
&= (2/3)(3/7)(6/7) + (2/3)(4/7)(1/7) + (1/3)(3/7)(1/7) = 47/147
\end{aligned}$$

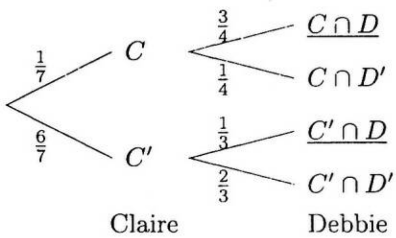
(since the three possibilities are disjoint the simple addition law holds).

Example:

Suppose now there is a fourth member of the group, Debbie. The probability that Debbie goes to the Union is 3/4 if Claire collects her, but only 1/3 if she has to go on her own. What is the probability that Debbie goes to the Union on any particular day?

Solution:

The tree now looks like (conditional probability: do Claire first)



What we want is $I\!P(D)$. This is $I\!P(C \cap D) + I\!P(C' \cap D)$ since these two events are disjoint. From the tree we see that this is $(1/7)(3/4) + (6/7)(1/3) = 11/28$.

In general for two events A and B we have $I\!P(B) = I\!P(B \cap A) + I\!P(B \cap A')$ which we can write as $I\!P(B) = I\!P(B|A)I\!P(A) + I\!P(B|A')I\!P(A')$ using the multiplication law.

1) A = event Annie goes to the union $\mathbb{P}(A) = 2/3$

B = event Ben goes to the union $\mathbb{P}(B) = 3/4$

C = event Claire goes to the union $\mathbb{P}(C) = 1/4$

2) $\mathbb{P}(D|C) = \frac{3}{4}$

$\mathbb{P}(D|C') = \frac{1}{3}$

Two events A, B, $\mathbb{P}(B) = \mathbb{P}(B \cap A) + \mathbb{P}(B \cap A')$

$$\mathbb{P}(B) = \mathbb{P}(B|A) \cdot \mathbb{P}(A) + \mathbb{P}(B \cap A') \cdot \mathbb{P}(A')$$

3. Sample space with many events 12/10/09.

So we need help with counting:

permutation
перестановка

$$\boxed{\text{Combinatorics} = \underline{\text{Permutions}} + \text{combinations}}$$

Recall that if each event in our sample space has equal probability. Then if A is an event:

кол-во
элементов в A

$$\boxed{P(A) = \frac{N(A)}{N(S)}}$$

Example:

If there are lots of events, this approach is not useful.

A pack of cards $S_1 = 4$ suits {clubs, Diamonds, hearts, spades}

$S_2 = 13$ face values $\{2, 3, 4, \ldots, J, Q, K, A\}$

Q How many pairs (x, y) are there if $x \in S_1, y \in S_2$

A $4 \times 13 = 52$ pairs

This is an example of a multiplication rule

> if we have r operation and the i^{th} operation can be performed n_i ways
> then the r operations can be performed
> $n_1 \times n_2 \times n_3 \times \ldots \times n_r$ ways.

Example: In the card example $n_1 = 4$

$n_2 = 13$

Example:

Suppose I take samples (x, y) from the same set $S = \{a_1, a_2, \dots a_n\}$

$$\left.\begin{matrix} a_1a_1 & a_1a_2 \dots a_1a_n \\ a_2a_1 & a_2a_2 \dots a_2a_n \\ \vdots & \\ a_na_1 & a_na_2 \dots a_na_n \end{matrix}\right\}$$ there are n^2 samples of the form (x, y)

Note:

(i) Repetition is allowed
e.g. a_3a_3 is accepted.

(ii) Order is important
e.g. a_1a_2 & a_2a_1 are different pairs.

In this example:

if we don't allow repetitions
then there are $n^2 - n$ pairs (x, y) [removing diagonal entries]

Another way:

first element of the pairs picked n ways.
Second element can be chosen in $(n-1)$ ways
$\therefore$ there are $n(n-1) = n^2 - n$ pairs

do not allow repetitions

if ~~order~~ ~~does~~ is not important. Then we view
e.g. a_1a_2 as the same as a_2a_1.

if no repetition and order not important
then there are $\frac{n^2 - n}{2} = \frac{n(n-1)}{2}$ pairs

do not allow 1. repetitions & 2. order not important

1. $S = \{ O \quad O \quad O \quad O \quad O \}$ $n := size(S)$

$O \quad O \quad O \quad O \quad O$ r

n^r

2. $S = \{ O \; O \; O \; O \; O \}$ r

$n(n-1)(n-2)\dots(n-r+1) = \frac{n!}{(n-r)!}$

3. $r = n$ $\boxed{n!}$ - permutation of size n

3) $N_r = \{ 1 \; \cancel{4} \; 3 \; 4 \; 5 \}$ $size N_r = r$
$= \{ 2 \; 3 \; 4 \; 5 \; 1 \}$

$$N_r \cdot r! = \frac{n!}{(n-r)!}$$

$$N_r = \frac{n!}{r!(n-r)!}$$

$S = \{ O \quad O \; O \; O \; O \}$

$\cancel{O} \; O \; O \; O \; O$

$\cancel{O} \; \cancel{O} \; O \; O \; O$

order is important, repetition ~~not~~ allowed

General this: Given a set $S = \{a_1, a_2, \dots, a_n\}$

I ~~wt~~ want to pick a sample of size r from S?

$(a_1, a_2) \neq (a_2, a_1)$ $\quad (a_1, a_1)$

3.1. Order is important and repetition is allowed then these are n^r samples of size r.

3.2. Order is important, repetition not allowed

(a_2, a_1) different from (a_1, a_2)

n choices for first element
$n-1$ choices for 2^{nd} element
⋮
$(n-r+1)$ choices for r^{th} element

$\Rightarrow$

$\Rightarrow$ these are $n(n-1)(n-2)\dots(n-r+1)$ sample of size r

$$= \frac{n!}{(n-r)!} = n(n-1)(n-2)\dots$$

This is called the ~~of~~ arrang permutation of size r that can be taken from n elements.

If we take $r = n$, this tells us the number of ways:

we can arrange n elements $= \frac{n!}{(n-n)!} = \frac{n!}{0!} = n!$ permutations

3.3. If order is not important, repetition not allowed

Suppose there are N_r sets satisfying the assumption. Each such set contains r elements that can be ordered in $r!$ ~~the~~ ways.

permutation = перестановка

So the number of permutation

$$r!\,N_r = \frac{n!}{(n-r)!} \quad (\text{by } 3.2)$$

$$\Rightarrow \boxed{N_r = \frac{n!}{r!\,(n-r)!}}$$ – this is called the number of combinations of r elements

we write $\binom{n}{r}$ or ${}^nC_r$ for $\frac{n!}{r!(n-r)!}$

e.g. if $S = \{a_1, a_2, a_3, a_4\}$ $\quad a_1 a_4$

Permutations of size 2: $\quad a_1a_2,\ a_1a_3,\ a_1a_4$
$= 12$ $\quad \cancel{a_2a_1},\ \cancel{a_2a_3},\ \cancel{a_2a_4}$

Combinations $= 6$. $\quad \cancel{a_3a_1},\ a_3a_2,\ \cancel{a_3a_4}$

$\begin{pmatrix} 2! = 2 \\ r = 2 \end{pmatrix}$ $\quad a_4a_1,\ a_4a_2,\ a_4a_3$

Binomial

$$\boxed{\frac{4!}{2!\,2!} = \frac{24}{4} = 6}$$

Binomial expansion.

$\binom{n}{r}$ is the coefficient that arises from the power x^r in the expansion of $(1+x)^n$

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \dots + \frac{n(n-1)(n-2)\dots(n-r+1)x^r}{r!} + \dots + nx^{n-1} + x^n$$

$$\Rightarrow (1+x)^n = \sum_{r=0}^{n} \binom{n}{r} x^r$$

also $(a+b)^n = \left(a\left(1+\frac{b}{a}\right)\right)^n = a^n\left(1+\frac{b}{a}\right)^n = a^n \sum_{r=0}^{n} \binom{n}{r}\left(\frac{b}{a}\right)^r =$

$$= \sum_{r=0}^{n} \binom{n}{r} b^r a^{n-r}$$

Example :

How many subsets are there of a set of size n?

A subset of size 0 : $\binom{n}{0} = \frac{n!}{n!\,0!} = 1$ ($\emptyset$ empty set)

size 1 : $\binom{n}{1}$

size 2 : $\binom{n}{2}$

size r : $\binom{n}{r}$

size n : $\binom{n}{n}$

So, number of subset is

$$\binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \dots + \binom{n}{n} = \sum_{r=0}^{n} \binom{n}{r} = \sum_{r=0}^{n} \binom{n}{r} \cdot 1^r$$

by ⊕ $= (1+1)^n = 2^n$

14/10/09

Example: Birthdays

You have up to 365 people. what is ~~birthday~~ the probability that at least two have a birthday on the same day? (Assume there are 365 days)

Approach:

Work out probability that no two people share a birthday. then take 1 - this probability to get the answer

The sample space consists of ordered samples where repetition is allowed. The sample space S consists of points $(a_1, a_2, a_3, \dots a_n)$, where a_i represents the birthday of the i^{th} person.

S contains 365^n sample points, because each a_i can take 365 values.

Let A be the event "no two share a birthday"

Q How many sample points are in A?

365 choices for a_1

364 choices for a_2

363 — " — a_3

$\vdots$

$(365-n+1)$ choices for a_n

So numb. of sample points in A is

$$N(A) = 365 \times 364 \times 363 \times \dots \times (365-n+1)$$

$$\Rightarrow \mathbb{P}(A) = \frac{N(A)}{N(S)} = \frac{365 \times 364 \times \dots \times (365-n+1)}{(365)^n}$$

$\frac{365!}{(365-n)!(365)^n} = \cancel{365\times}$

Prob. (that at least two share a birthday) =

$= 1 - \mathbb{P}(A) = 1 - \frac{365!}{(365-n)!\,365^n}$

e.g. $n = 22$, $\mathbb{P}(A') = 0{,}476$

$n = 23$, $\mathbb{P}(A') = 0{,}507$

$n = 92$, $\mathbb{P}(A') = 0{,}9999965$

3-4 Permutations with identical objects

How many ways can you arrange the letters in the word ENGINEERING?

$3\times E$, $3\times N$, $2\times G$, $2\times I$, $1\times R$

If all the letters were different, there would be $11!$ arrangements.

$\Rightarrow$ the $3\times E$ can be arranged in $3!$ ways,

the $3\times N$ — " — in $3!$ ways

the $2\times G$ — " — in $2!$ ways,

the $2\times I$ — " — in $2!$ ways,

So the answer $= \frac{11!}{3!\,3!\,2!\,2!}$

In general, if I have n objects and

n_1 of type 1

n_2 of type 2 ...

Then the total number of different arrangements is $\frac{n!}{n_1!\,n_2!\,n_3!\dots}$

Example: ~~B~~irthdays:

$n = 365$ - number of people

$\mathbb{P}$(at least two have a birthday on a same day) =

$= \frac{M}{N}$ ~~......~~ ,

M = number of outcome where there at least 2 have a birthday on same day.

N = number of total different outcomes.

1) $N = 365^{\cancel{365}\, n}$

2) $\mathbb{P}$ (at least two ...) = 1 - $\mathbb{P}$(no two people share one birthday) $= \frac{365!}{(365-n)!\, 365^n}$

4. The Binomial and Poisson Distributions

4.1. Bernoulli trials

- These are repeated independent trials with only two outcomes per trial.
- The probabilities of these two events is fixed throughout the trials

These probabilities are usually called p and q

- p is the probability of a success
- q is the probability of a failure
- $p+q=1$ (e.g. tossing a fair coin $p=q=\frac{1}{2}$)
- the sample space for each trial is $\{s, f\}$, where s = success, f = failure.
- The sample space for n-trials contains events of the form:

$$\underbrace{ss\,fff\,ssfsf\,sssss}_{n \text{ trials}}$$

- There ~~prob.~~ are 2^n sample points for a sample space consisting of n trials.
- The prob. of the above event is

$$p\,p\,q\,q\,q\,p\,p\,q\,p\,q\,p\,p\,p\,p\,p$$

the prob. multiply because the trials were independent.

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4.2. Binominal distribution

Suppose we have n trials and we want to know the prob. of r success in our n trials.

- if the first r trials are success and then we have (n-r) failures the prob. of this is

$$\boxed{p^r q^{n-r}} \qquad (1)$$

- However, these r success don't have to come from the first r trials. They can come from any r (out of n) trials

- So we need to mult. the prob. (1) by the numb. of ways of choosing r places for there to be a success.

i.e. $\binom{n}{r}$

The prob. of exactly r successes in n independent trials each with probability p of success and q of failure is (r ∈ positive integer)

$$\boxed{b(r; n, p) = \binom{n}{r} p^r q^{n-r}}$$

The probabilities are called the binominal distribution

Need to check that $\mathbb{P}(S) = 1$

where $S = \{0 \text{ successes}, 1 \text{ success}, 2 \text{ successes}, \ldots, n \text{ success}\}$

$$\mathbb{P}(S) = \sum_{r=0}^{n} b(r; n; p) = \sum_{r=0}^{n} \binom{n}{r} p^r q^{n-r}$$

$$(\text{last lecture}) = (p+q)^n = 1^n = 1$$

Example:

if the probability of hitting a target is $\frac{2}{5}$ and 5 shots are fired, what is the probability of at least 2 hits?

Here I'm written $b(r) = b(r;n,p) = b(r;5,\frac{2}{5})$

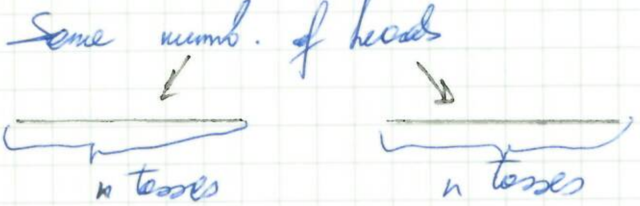
Use binomial distribution: $n=5$, $p=\frac{2}{5}$, $q=1-p=\frac{3}{5}$

we need $b(2)+b(3)+b(4)+b(5) = 1-b(0)-b(1) =$

$$= 1-\binom{5}{0}\left(\frac{2}{5}\right)^0\left(\frac{3}{5}\right)^5-\binom{5}{1}\left(\frac{2}{5}\right)^1\left(\frac{3}{5}\right)^4 = 1-\left(\frac{3}{5}\right)^4\left(\frac{3}{5}+2\right) = 1-\left(\frac{3}{5}\right)^4\left(\frac{13}{5}\right)$$

Example

you have a fair coin is tossed $2n$ times what is the prob. that the first n coin tosses and the second n coin tosses contain the same number of heads.



Solution: Fair coin $p=q=\frac{1}{2}$

· First count prob. of 0 heads in n coin tosses

$$\binom{n}{0}\frac{1}{2^n} \quad \left[=\binom{n}{0}\left(\frac{1}{2}\right)^0\left(\frac{1}{2}\right)^{n-0}\right]$$

· By independence of the coin tosses, the prob. of 0 heads in the first n-tosses, and 0 heads in 2^{nd} n is $\binom{n}{0}^2\frac{1}{2^{2n}}$

In above $A=\{0$ heads in first n trials$\}$ $P(A)=\binom{n}{0}\frac{1}{2^n}$

$B=\{0$ heads in 2^{nd} n trials$\}$ $P(B)=\binom{n}{0}\frac{1}{2^n}$

21 $\mathbb{P}(A \cap B) = \mathbb{P}(0 \text{ heads in } 1^{st} n, 0 \text{ heads in } 2^{nd} n)$

independence $\longrightarrow$ $= \mathbb{P}(A) \cdot \mathbb{P}(B) = \binom{n}{0}^2 \frac{1}{2^{2n}}$

. Probability of 1 head in first n is $\binom{n}{1}\left(\frac{1}{2}\right)^1\left(\frac{1}{2}\right)^{n-1} = \binom{n}{1}\frac{1}{2^n}$

So, the prob. of 1 head in first, 1 head in 2^{nd} n is

$$\binom{n}{1}^2 \cdot \frac{1}{2^{2n}}$$

So all together, the answer is

proove on handout 5

$$\frac{1}{2^{2n}}\left[\binom{n}{0}^2 + \binom{n}{1}^2 + \binom{n}{2}^2 + \dots + \binom{n}{n}^2\right] = \frac{1}{2^{2n}} \sum_{r=0}^{n} \binom{n}{r}^2 = \frac{1}{2^{2n}}\binom{2n}{n}$$

Handout 5 Advanced use of the binomial theorem

- A true coin is tossed $2n$ times.
- We were looking for the probability that the first n tosses and the second n tosses results in the same number of heads.

The probability of r heads in the first n and r heads in the second n was

$$\mathbb{P}(r;r) = \binom{n}{r}^2 \frac{1}{2^{2n}}$$

← откуда это?

Altogether, the probability is:

$$\frac{1}{2^{2n}}\left[\binom{n}{0}^2 + \binom{n}{1}^2 + \binom{n}{2}^2 + \cdots + \binom{n}{n}^2\right] = \frac{1}{2^{2n}}\sum_{r=0}^{n}\binom{n}{r}^2.$$

How can we simplify this expression?

Use the binomial theorem in two different ways:

$$(1+x)^n = \sum_{r=0}^{n}\binom{n}{r}x^r \text{ so } (1+x)^{2n} = \sum_{r=0}^{2n}\binom{2n}{r}x^r.$$

$$C_0^3 \quad C_1^3 \quad C_2^3 \quad C_1^3$$
$$C_3^3 \quad C_2^3 \quad C_1^3 \quad C_0^3$$
$$\alpha x^0 \quad \beta x^1 \quad \gamma x^2 + x^3$$

$$(1+x)^n = \sum_{t=0}^{n}\binom{n}{t}x^t = \sum_{t=0}^{n}\binom{n}{n-t}x^t$$

and then if we set $s = n - t$ this becomes

$$(1+x)^n = \sum_{s=0}^{n}\binom{n}{s}x^{n-s}.$$

Now we can use the fact that

$$(1+x)^{2n} = (1+x)^n(1+x)^n$$

to give us

$$\sum_{r=0}^{2n}\binom{2n}{r}x^r = \left[\sum_{t=0}^{n}\binom{n}{t}x^t\right]\left[\sum_{s=0}^{n}\binom{n}{s}x^{n-s}\right]$$

Because this is a finite sum, we can put everything inside the summation without worrying:

$$\sum_{r=0}^{2n}\binom{2n}{r}x^r = \sum_{t=0}^{n}\sum_{s=0}^{n}\binom{n}{t}\binom{n}{s}x^t x^{n-s} = \sum_{t=0}^{n}\sum_{s=0}^{n}\binom{n}{t}\binom{n}{s}x^{n+t-s}.$$

disagree

Now this must be true for all possible values of x, so the coefficients of x^α on both sides of the equation must be equal (for any α). Let us look at the coefficient of x^n:

$$\binom{2n}{n}x^n = \sum_{t=0}^{n}\binom{n}{t}^2 x^n$$

– откуда?

and we have shown that

$$\sum_{t=0}^{n}\binom{n}{t}^2 = \binom{2n}{n}.$$

Thus the probability that the first and second n throws have the same number of heads is

$$\binom{2n}{n}\frac{1}{2^{2n}}.$$

Exercise (just for fun): Can you think of a non-algebraic argument for why this must be the answer?

5. Probability and Continuous Sample Spaces

5.1. Continuous Probability Distribution.

So, before sample points were thing like $r = 0, 1, 2, 3 \ldots$ (poisson)
$r = 1, 2, 3, 4, 5, 6$ (Dice)

However, sample points might lie on an interval on the whole real line.

e.g. Sample points for the temperature of this room
Let us suppose that sample space is $(-\infty, \infty)$

To define a probability distribution on this sample space consider the function

$$\boxed{F(x) = \mathbb{P}(\{x' : x' \leq x\}) \quad , \quad x \in (-\infty, \infty)}$$

$F(x)$ is the prob. that a sample point takes value less than or equal to x.

- $\lim\limits_{x \to \infty} F(x) = F(\infty) = 1 \quad (\mathbb{P}((-\infty, \infty)) = 1)$
- $\lim\limits_{x \to -\infty} F(x) = F(-\infty) = 0$
- let $a < b$

$\mathbb{P}((-\infty; b]) - \mathbb{P}((-\infty; a]) = \mathbb{P}((a; b])$

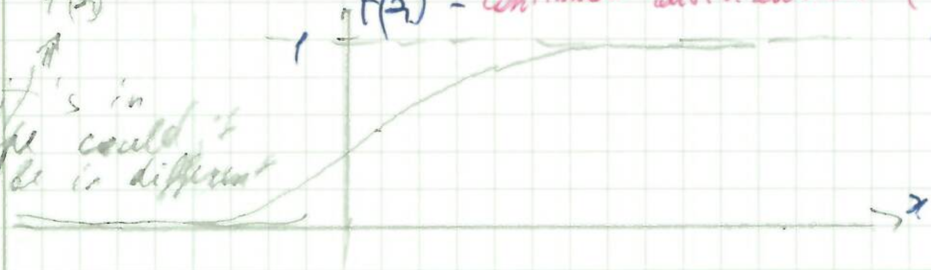
$$F(b) - F(a) = \mathbb{P}(\{x' : x' \leq b\}) - \mathbb{P}(\{x' : x' \leq a\})$$
Axiom 1
$$= \mathbb{P}(\{x' : a < x' \leq b\}) \geq 0$$

$$\therefore \quad \boxed{\text{if } a < b \text{ then } F(a) \leq F(b)}$$

NB! $F(x)$ is defferentiable

Only if it's in this shape could be is different shapes?

$F(x)$ - continuous distribution (cdf)



5.2. Probability Density Function

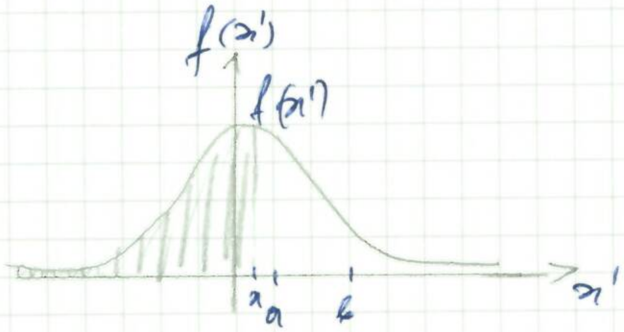
if $F(x)$ is a cdf

let $\boxed{f(x) = \frac{dF}{dx} = F'(x)}$ - probability density function (pdf)

Propositions:

- since $F(x)$ is increasing always

$\Rightarrow$ $F'(x) \geq 0 \Rightarrow f(x) \geq 0$



what is f(x) in terms of probability. Is it a probability f(x) = P(x)

if we're given $f(x)$ and wanted $F(x)$

$$\boxed{F(x) = \int_{-\infty}^{x} f(x')\,dx'}$$

On the graph $F(x) = \mathbb{P}(\{x' : x' \leq x\})$ = shaded area

- Since $F\infty = 1$

$\Rightarrow$ $1 = \int_{-\infty}^{\infty} f(x')\,dx'$

Def The mean of a continuous probability distribution with pdf $f(x)$ is

Why??

$$\boxed{\mu = \int_{-\infty}^{\infty} x\, f(x)\,dx}$$

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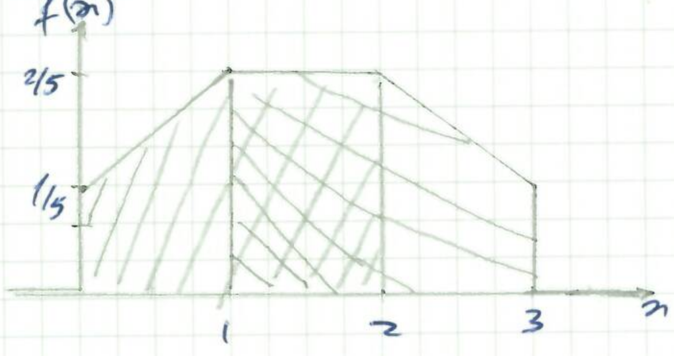
Example

1) Consider a pdf given by
$$f(x) = \begin{cases} 0 & x \leq 0 \\ \frac{1}{5}(x+1) & 0 < x \leq 1 \\ \frac{2}{5} & 1 < x \leq 2 \\ \frac{1}{5}(4-x) & 2 < x \leq 3 \\ 0 & x > 3 \end{cases}$$



Check $f(x)$ is a pdf (1) $f(x) \geq 0$ ✓

(2) $\int_{-\infty}^{\infty} f(x')\,dx' = 1$ ✓

Define two events A & B by
$$A = \{x' : 0 \leq x' \leq 2\}$$
$$B = \{x' : 1 \leq x \leq 3\}$$

$$\mathbb{P}(A) = \int_0^2 f(x')\,dx' = \int_0^1 \frac{1}{5}(1+x')\,dx' + \int_1^2 \frac{2}{5}\,dx' = \frac{7}{10}$$

$\mathbb{P}(A) = \mathbb{P}(B) = \frac{7}{10}$ - symmetrical regions.

Also $\mathbb{P}(A \cap B) = \int_1^2 f(x')\,dx' = \frac{2}{5}$

2) Consider a function $f(x) = \frac{1}{c^2 + x^2}$ for what values of c is $f(x)$ a p.d.f.?

(1) $f(x) \geq 0$ ✓ (2) $\int_{-\infty}^{\infty} f(x)\,dx = 1$

But $\int_{-\infty}^{\infty} \frac{1}{c^2+x^2} = \frac{1}{c}\arctan\left(\frac{x}{c}\right)\Big|_{-\infty}^{\infty} = \frac{\pi}{c}$

So $c = \pi$.

Handout 6 Normal Distribution

Recall that for the normal distribution with mean μ and standard deviation σ, the probability distribution is given by

$$\mathbb{P}(x \le a) = F(a) = \frac{1}{\sqrt{2\pi}\,\sigma}\int_{-\infty}^{a} \exp\left[-\frac{1}{2}\left(\frac{x'-\mu}{\sigma}\right)^2\right] dx'$$

and this can be written as

$$\mathbb{P}(x \le a) = 0.5 + \Phi\left(\frac{a-\mu}{\sigma}\right)$$

where

$$\Phi(z) = \frac{1}{\sqrt{2\pi}}\int_{0}^{z} \exp\left[-\frac{z_1^2}{2}\right] dz_1,$$

and

$$\Phi(-z) = -\Phi(z).$$

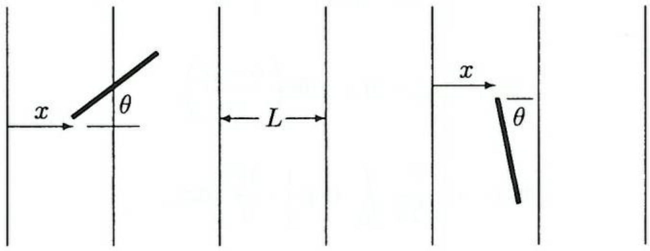
The following table gives values of $\Phi(z)$: select your value of z using the left column and top row; the required value of Φ is given by the contents of the table.

	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.0000	0.0040	0.0080	0.0120	0.0160	0.0199	0.0239	0.0279	0.0319	0.0359
0.1	0.0398	0.0438	0.0478	0.0517	0.0557	0.0596	0.0636	0.0675	0.0714	0.0753
0.2	0.0793	0.0832	0.0871	0.0910	0.0948	0.0987	0.1026	0.1064	0.1103	0.1141
0.3	0.1179	0.1217	0.1255	0.1293	0.1331	0.1368	0.1406	0.1443	0.1480	0.1517
0.4	0.1554	0.1591	0.1628	0.1664	0.1700	0.1736	0.1772	0.1808	0.1844	0.1879
0.5	0.1915	0.1950	0.1985	0.2019	0.2054	0.2088	0.2123	0.2157	0.2190	0.2224
0.6	0.2257	0.2291	0.2324	0.2357	0.2389	0.2422	0.2454	0.2486	0.2517	0.2549
0.7	0.2580	0.2611	0.2642	0.2673	0.2704	0.2734	0.2764	0.2794	0.2823	0.2852
0.8	0.2881	0.2910	0.2939	0.2967	0.2995	0.3023	0.3051	0.3078	0.3106	0.3133
0.9	0.3159	0.3186	0.3212	0.3238	0.3264	0.3289	0.3315	0.3340	0.3365	0.3389
1.0	0.3413	0.3438	0.3461	0.3485	0.3508	0.3531	0.3554	0.3577	0.3599	0.3621
1.1	0.3643	0.3665	0.3686	0.3708	0.3729	0.3749	0.3770	0.3790	0.3810	0.3830
1.2	0.3849	0.3869	0.3888	0.3907	0.3925	0.3944	0.3962	0.3980	0.3997	0.4015
1.3	0.4032	0.4049	0.4066	0.4082	0.4099	0.4115	0.4131	0.4147	0.4162	0.4177
1.4	0.4192	0.4207	0.4222	0.4236	0.4251	0.4265	0.4279	0.4292	0.4306	0.4319
1.5	0.4332	0.4345	0.4357	0.4370	0.4382	0.4394	0.4406	0.4418	0.4429	0.4441
1.6	0.4452	0.4463	0.4474	0.4484	0.4495	0.4505	0.4515	0.4525	0.4535	0.4545
1.7	0.4554	0.4564	0.4573	0.4582	0.4591	0.4599	0.4608	0.4616	0.4625	0.4633
1.8	0.4641	0.4649	0.4656	0.4664	0.4671	0.4678	0.4686	0.4693	0.4699	0.4706
1.9	0.4713	0.4719	0.4726	0.4732	0.4738	0.4744	0.4750	0.4756	0.4761	0.4767
2.0	0.4772	0.4778	0.4783	0.4788	0.4793	0.4798	0.4803	0.4808	0.4812	0.4817
2.1	0.4821	0.4826	0.4830	0.4834	0.4838	0.4842	0.4846	0.4850	0.4854	0.4857
2.2	0.4861	0.4864	0.4868	0.4871	0.4875	0.4878	0.4881	0.4884	0.4887	0.4890
2.3	0.4893	0.4896	0.4898	0.4901	0.4904	0.4906	0.4909	0.4911	0.4913	0.4916
2.4	0.4918	0.4920	0.4922	0.4925	0.4927	0.4929	0.4931	0.4932	0.4934	0.4936
2.5	0.4938	0.4940	0.4941	0.4943	0.4945	0.4946	0.4948	0.4949	0.4951	0.4952
2.6	0.4953	0.4955	0.4956	0.4957	0.4959	0.4960	0.4961	0.4962	0.4963	0.4964
2.7	0.4965	0.4966	0.4967	0.4968	0.4969	0.4970	0.4971	0.4972	0.4973	0.4974
2.8	0.4974	0.4975	0.4976	0.4977	0.4977	0.4978	0.4979	0.4979	0.4980	0.4981
2.9	0.4981	0.4982	0.4982	0.4983	0.4984	0.4984	0.4985	0.4985	0.4986	0.4986
3.0	0.4987	0.4987	0.4987	0.4988	0.4988	0.4989	0.4989	0.4989	0.4990	0.4990

Handout 7 Buffon's needle

This material is non-examinable (but cool).

Suppose you have a needle of length L, and a table covered with lines spaced a distance L apart. You throw the needle onto the table. It will either cross one line or no lines. What is the probability it crosses a line?



Let us look first at the angle where it lands. The needle falls at an angle between 0 and $\pi/2$ to the horizontal: we will call this angle θ.

Next, let us look at the position of the left-hand end of the needle. Suppose the lines are at horizontal positions L, $2L$, etc., and suppose the left end is at a position $nL + x$ with $0 \leq x < L$. Then all values of x between 0 and L are equally likely: the needle doesn't care about where the lines are.

$$\mathbb{P}(0 \leq x < x_0) = \frac{x_0}{L}.$$

Now we can work out whether the needle crosses a line. A needle which has landed at an angle θ to the horizontal covers a horizontal distance of $L\cos\theta$. It will cross a line if its right-hand end is beyond the next line: that is, if $x + L\cos\theta \geq L$. The probability of this (given the value of θ) is

$$\mathbb{P}(x + L\cos\theta \geq L) = \mathbb{P}(x \geq L(1-\cos\theta)) = 1 - \mathbb{P}(0 \leq x < L(1-\cos\theta)) = 1 - \frac{L(1-\cos\theta)}{L} = \cos\theta.$$

So now we have a needle which lands with $0 \leq \theta < \pi/2$, with all angles equally likely. Thus

$$\mathbb{P}(0 \leq \theta < \theta_0) = \frac{\theta_0}{(\pi/2)} = \frac{2\theta_0}{\pi} \equiv F(\theta_0).$$

The pdf of this distribution is $f(\theta) = 2/\pi$.

We can now do the continuous distribution version of conditional probability:

$$\begin{aligned}\mathbb{P}(\text{crosses a line}) &= \int_0^{\pi/2} \mathbb{P}(\text{crosses a line}|\theta = \theta_0) f(\theta_0)\,\mathrm{d}\theta_0 \\ &= \int_0^{\pi/2} \cos\theta_0 \frac{2}{\pi}\,\mathrm{d}\theta_0 = \frac{2}{\pi}[\sin\theta_0]_0^{\pi/2} = \frac{2}{\pi}.\end{aligned}$$

This can be used as a (slow) experimental method for estimating π.

You can try it yourself at `http://www.ms.uky.edu/~mai/java/stat/buff.html`.

21.10.09

Recap Suppose x is a sample from a continuous probability distribution with c.d.f. F.

then $P(x < a) = F(a)$ for each $a \in (-\infty, +\infty)$

In this cas we will assume that F is ~~constant~~ continuous and differentible.

let $f(a) = F'(a)$. then f is a p.d.f.

and $P(x < a) = F(a) = \int_{-\infty}^{a} f(x)\, dx$

in particular $P(a < x \leq b) = F(b) - F(a) = \int_a^b f(x)\, dx$

$$P(a < x \leq b) = \int_{[a,b]} f x\, dx = \int_{[a,b]} f(x)\, dx = \int_{(a,b)} f(x)\, dx = \int_{[a,b)} f(x)\, dx$$

$$\Rightarrow P(a < x \leq b) = P(a \leq x \leq b) = P(a < x < b) = P(a \leq x < b)$$

Conversely: given a function $f(x)$ which satisfies

(1) $f(x) \geq 0$ (2) $\int_{-\infty}^{\infty} f(x)\, dx = 1$

then if we define a function F by

$$F(a) = \int_{-\infty}^{a} f(x)\, dx \ [= P(x \leq a)]$$

then F is called c.d.f. for some continuous prob. distribution

5.3. The Normal Distribution.

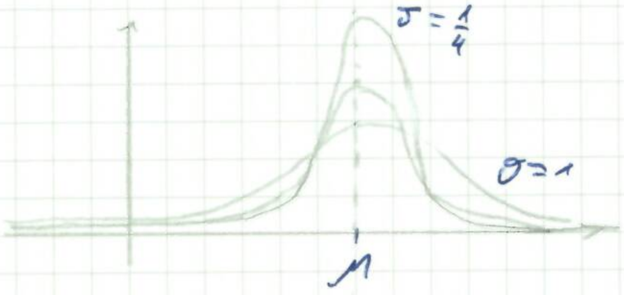
Consider the probability density function:

$$\boxed{f(x) = \frac{1}{\sqrt{2\pi}\,\sigma} \exp\left(-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2\right), \quad \text{for } x \in (-\infty;\infty)}$$

μ & σ are constants, where

μ is the mean of the norm. distr.

σ is the standart deviation of the norm. distr.



task: Have to check $f(x)$ is actually a p.d.f.

(1) $f x \geq 0$ (2) $\int_{-\infty}^{\infty} f(x)\,dx = 1$

$$\int_{-\infty}^{\infty} f(x)\,dx = \frac{1}{\sqrt{2\pi}\,\sigma}\int_{-\infty}^{\infty} \exp\left(-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2\right)dx = \begin{bmatrix} z = \frac{x-\mu}{\sigma} \\ \sigma dz = dx \end{bmatrix} =$$

$$\to(*402) = \frac{1}{\sqrt{2\pi}} \overbrace{\int_{-\infty}^{\infty} \exp\left(-\frac{z^2}{2}\right) dz}^{\sqrt{2\pi}} = 1$$

Mean

Given by $\int_{-\infty}^{\infty} x\, f(x)\,dx = \frac{1}{\sqrt{2\pi}\,\sigma}\int_{-\infty}^{\infty} x \exp\left(-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2\right)dx =$

where this σ gone?

$$= \left[\frac{1}{\sqrt{2\pi}\,\sigma}\int_{-\infty}^{\infty}(x-\mu)\exp\left(-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2\right)dx\right] + \left[\frac{1}{\sqrt{2\pi}\,\sigma}\int_{-\infty}^{\infty}\mu \exp\left(-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2\right)dx\right]$$

// In the first integral let $z = \frac{x-\mu}{\sigma}$

$$= \frac{\sigma}{\sqrt{2\pi}}\int_{-\infty}^{\infty} z \exp(-z^2/2)\,dz + \frac{\mu}{\sqrt{2\pi}}\overbrace{\int_{-\infty}^{\infty}\exp\left(-z^2/2\right)dz}^{=\sqrt{2\pi}} =$$

$$= \frac{\sigma}{\sqrt{2\pi}}\left[-\exp(-z^2/2)\right]_{-\infty}^{\infty} + \mu = 0 + \mu = \mu$$

The standard deviation of a cont. prob. distr. is given by

$$\boxed{(\text{st. dev.})^2 = \int_{-\infty}^{\infty} (x-\mu)^2 f(x)\, dx}$$

where μ is the mean of the distribution.

For the normal distribution:

(exercise) $$(\text{st. dev.})^2 = \frac{1}{\sqrt{2\pi}\,\sigma} \int_{-\infty}^{\infty} (x-\mu)^2 \exp\left(-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2\right) dx = \sigma^2$$

Probability distribution

$$\boxed{F(a) = \frac{1}{\sqrt{2\pi}\,\sigma} \int_{-\infty}^{a} \exp\left(-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2\right) dx}$$

Probability [represents →] that a sample space point takes value less than a:

$$\mathbb{P}(a < x \le b) = F(b) - F(a)$$

Problem: $$F(b) - F(a) = \frac{1}{\sqrt{2\pi}\,\sigma} \int_{a}^{b} \exp\left(-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2\right) dx$$

why? ← but these numbers can't be found analytically

Answer: use tables

(think $\mu = 0$, $\sigma = 1$)

The tables gives values $\Phi(z) = \frac{1}{\sqrt{2\pi}} \int_{0}^{z} \exp\left(-\frac{x^2}{2}\right) dx$

Also by substitution $\boxed{\Phi(-z) = -\Phi(z)}$

$$\int_{-\infty}^{\frac{a-\mu}{\sigma}} = \int_{-\infty}^{0} + \int_{0}^{\frac{a-\mu}{\sigma}}$$

So suppose I have a sample x from a norm. distr. with mean μ, st. dev. σ.

$$\mathbb{P}(x \le a) = \frac{1}{\sqrt{2\pi}\,\sigma} \int_{-\infty}^{a} \exp\left(-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2\right) dx = \left[z = \frac{x-\mu}{\sigma}\right] =$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\frac{a-\mu}{\sigma}} \exp\left(-\frac{z^2}{2}\right) dz = 0{,}5 + \Phi\left(\frac{a-\mu}{\sigma}\right)$$

Also $\mathbb{P}(a < x \le b) = F(b) - F(a) = \left[0{,}5 + \Phi\frac{b-\mu}{\sigma}\right] -$
$- \left[0{,}5 + \Phi\left(\frac{a-\mu}{\sigma}\right)\right] = \Phi\frac{b-\mu}{\sigma} - \Phi\left(\frac{a-\mu}{\sigma}\right)$

then can use tables.

<u>Example</u> The mean diameters of a sample 400 coins is 22,50 mm and the std dev. is 0,50 mm. The acceptable limits for the diameters are 22,36 ± 0,53 mm.

<u>task</u> Determine the prob. that any one coin has a acceptable diameter.

<u>answer</u> Find the acceptable range

$x_1 = 22{,}36 - 0{,}53 = 21{,}83$

$x_2 = 22{,}36 + 0{,}53 = 22{,}89$

loc var = cont
$\mu = 22{,}5$
$\sigma = 0{,}5$

$\mathbb{P}(\text{acceptable}) = \mathbb{P}(x_1 < x \le x_2) = \Phi\left(\frac{x_2-\mu}{\sigma}\right) - \Phi\left(\frac{x_1-\mu}{\sigma}\right) =$
$= \Phi(0{,}78) - \Phi(-1{,}34) = \Phi(0{,}78) + \Phi(1{,}34) = 0{,}2823 + 0{,}4099 =$
$= 0{,}6922$

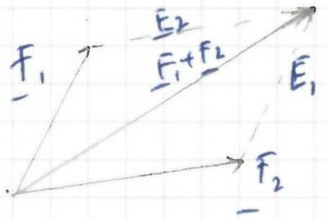
1

+ h. 8

26.10.09. 1301

PART II. Stats

A consequence of Law II is that if two forces $\underline{F}_1$ and $\underline{F}_2$ acting on a particle they can be thought of as a single force $\underline{F}_1 + \underline{F}_2$



This is called the parallelogram of forces rule.

1.2. Definitions

Mass: In the equation $W = mg$ where g is gravitational acceleration ($g \simeq 9,81\ ms^{-2}$), where

m is mass in kg

W weight in N (newtons)

Frame = система

Particle: A scrap of matter which has mass but no size ("a point of mass"). This is of interest when the objects we are considering has insignificant size compared to other properties in the problem. E.g. S

e.g. Size of the earth compared to the distance between the earth and the sun.

cartesian coordinates = декартовые координаты

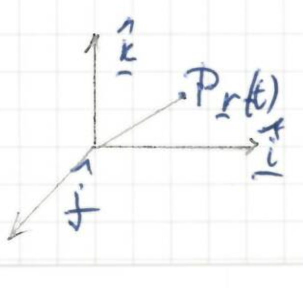
Rigid body: твёрдое тело.

rigid = + негнущийся косный.

Rigid body*: this is an infinite (or finite) number of particles which keep a fixed distance from eachother.

Frame of reference = система отсчёта

Frame of reference*: Time of cartesian* coordinates



Consider a particle P with position vector $\underline{r}(t)$ depending on time t.

2. точки зрения
3. критерий.

Handout 8 Newton's laws of motion = 1

Newton's First Law: Inertia

Every object in a state of uniform motion, relative to a basic frame of reference, tends to remain in that state of motion unless an external force is applied to it.

This is sometimes stated as "a body initially at rest will remain at rest unless an external force is applied to it". It can be used as a definition of the basic frame of reference.

Newton's Second Law: Motion

по отношению к основной системе отсчета

Relative to a basic frame of reference, a particle of mass m subject to a force $\underline{F}$ moves in accordance with the equation $$\underline{F} = m\underline{a}$$ where $\underline{a}$ is its acceleration.

This is the most powerful of Newton's three Laws, because it allows quantitative calculations of dynamics: how velocities change when forces are applied.

Newton's Third Law of Motion: Action and Reaction

For every action there is an equal and opposite reaction.

When two particles exert forces on one another, these forces are equal in magnitude and opposite in sense and act along the line joining the particles.

This law is exemplified by what happens if we step off a boat onto the bank of a lake: as we move in the direction of the shore, the boat tends to move in the opposite direction.

Notes.

Now, $\underline{r}(t) = x(t)\hat{\underline{i}} + y(t)\hat{\underline{j}} + z(t)\hat{\underline{k}}$ – this is position vector of P with respect to our chosen frame.

The velocity with respect of to frame is ~~$\underline{v}(t) = x$~~

$$\underline{v}(t) = x'(t)\hat{\underline{i}} + y'(t)\hat{\underline{j}} + z'(t)\underline{k}$$

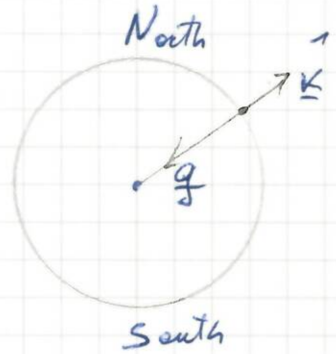
$$\underline{a}(t) = x''(t)\hat{\underline{i}} + y''(t)\hat{\underline{j}} + z''(t)\underline{k} \text{ - acceleration.}$$

Note: the eq. would not work in accelerating frame.

The frame may be moving but we can't assume it is accelerating

e.g. a train pulling out of a station won't do.

1.3. Forces



The most intuitive force is Gravity

$\hat{\underline{k}}$ is a unit vector $\perp$ to surface

g is the force on a unit mass

The force on a particle of mass m

is $$\boxed{g = -mg\hat{\underline{k}}}$$

This force has dimensions

$$\boxed{M \cdot L \cdot T^{-2}}$$

из формулы $F = \frac{m\Delta v}{\Delta t} = ma$

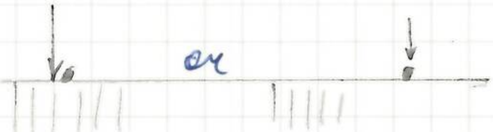
where M is a mass kg

L is length m

T is time sec.

Since force is a vector. It has magnitude & direction. However, the point of application of a force is also important. This is often called a bound vector

e.g.



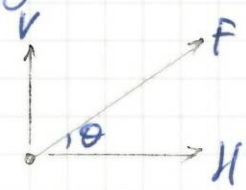
However 

because these two forces have the same line of action

1.3.1. Forces acting on a particle

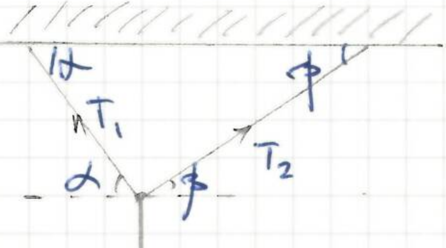
Two force can be added together

More useful to break a force into components
e.g. single force with magnitude F at angle θ to the horizontal



$$\boxed{\begin{array}{ll} V = F\sin\theta & \text{- vertical comp.} \\ H = F\cos\theta & \text{- horiz. comp.} \end{array}}$$

Example



Consider a particle hanging on its own weight which is stationary or static

For the system to be stationary the total force on the point must be zero (by $\underline{F} = m\underline{a}$)

The horizontal comp. of the forces are zero =>

$$0 = T_2 \cos\beta - T_1 \cos\alpha \qquad (1)$$

The vertical comp. of the forces are zero =>

$$0 = T_2 \sin\beta + T_1 \sin\alpha - W \qquad (2)$$

from(1) => $0 \Rightarrow T_1 = T_2 \cos\beta / \cos\alpha \Rightarrow$

$$\Rightarrow W = T_2 \sin\beta + \frac{T_2 \cos\beta \sin\alpha}{\cos\alpha}$$

$$\Rightarrow W = T_2\left(\frac{\cos\alpha\sin\beta + \cos\beta\sin\alpha}{\cos\alpha}\right) = T_2\left(\frac{\sin(\alpha+\beta)}{\cos\alpha}\right)$$

$$\Rightarrow T_2 = \frac{\cos\alpha \cdot W}{\sin(\alpha+\beta)} \Rightarrow T_1 = \frac{\cos\beta \cdot W}{\sin(\alpha+\beta)}$$

moments = момент ~~силы~~:

$\frac{\Delta L}{\Delta t} = \frac{rp}{\Delta t} = \frac{rm v}{\Delta t} = rma = rF_t = rF\sin\alpha$ – изменение момента импульса в течении времени.

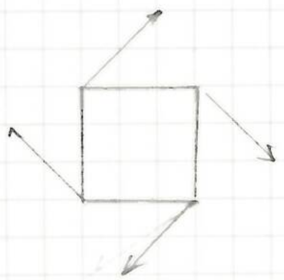
twisting*: кручение.

torque: 1. вращающий момент. 2. закручивать.

1.4. Moments

In the previous example the total forces on the particle were zero and the particle did not move.

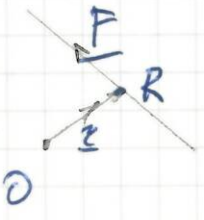
However, just because the total force on an object sum to zero. Does not mean it will not move



The box has no total force, but it will rotate. This is called "twisting*" effect.

Force is called a couple or moment or torque*

Def^n



Let $\underline{F}$ be a force with line of action as shown. The moment of $\underline{F}$ about O, written $\underline{G}_O$, is defined to be

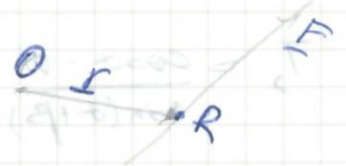
$$\boxed{\underline{G}_O = \underline{r} \wedge \underline{F}}$$

Where $\underline{r}$ is the position vector of <u>any</u> point R on $\underline{F}$'s line of action.

Dano:

1. If ΣF_i in the system $= 0$ it does not mean that the object is in equilibrium.

2. $g_o = \underline{r} \times \underline{F}$

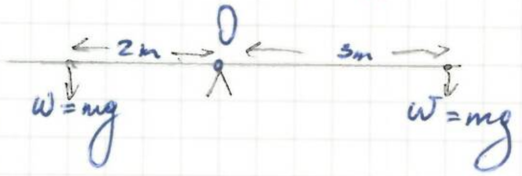


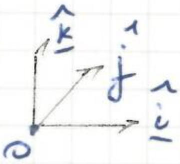
moment of force, torque, couple

3. To body be in equilibrium ~~this~~

$$\sum g_o^i = 0$$

⑤ **Example** Physical meaning



① Pick some coordinates: 

1.1. where here $\hat{j}$ points into the board

1.2. pick Origin O to be the point of the pivot.

$\Rightarrow$ the moment of the right hand force about o (the pivot)

is $G_0 = 3\underline{i} \wedge -mg\hat{k} = 3mg\hat{j}$. — clockwise rotation " vector the board

The moment of the left hand force is

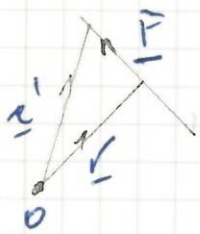
$G_0 = (-2\hat{\underline{i}}) \wedge (-mg\hat{k}) = -2mg\hat{j}$ — axes around what it's rotating " anticlockwise rotation " vector out of the board

Q Is there dependence on point chosen?

Suppose a force acts along a line $\underline{r} + t\underline{F}$

the moment about zero is $G_0 = \underline{r} \wedge \underline{F}$

However if $\underline{r}'$ is another point on the line of action of $\underline{F}$



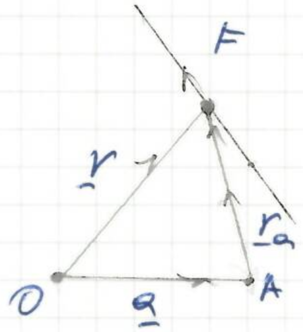
then $\underline{r}' = \underline{r} + s\underline{F}$ for some $s \in \mathbb{R}$

$\Rightarrow \underline{r}' \wedge \underline{F} = (\underline{r} + s\underline{F}) \wedge \underline{F} = \underline{r} \wedge \underline{F} + s\underline{F} \wedge \underline{F} = \underline{r} \wedge \underline{F}$

conclusion
- So the def^n of moment makes sense since $G_0 = \underline{r} \wedge \underline{F}$ is independent of the point $\underline{r}$ on the line of action of $\underline{F}$
- However the moment is not independent of the point O. In fact if A is a point with position vector $\underline{a}$, then the moment of $\underline{F}$ about A

is $\boxed{G_A = \underline{r}_a \wedge \underline{F}}$ where $\underline{r}_a$ is any vector joining A to the line of action of $\underline{F}$.

⑥



$\underline{r}_a = \underline{r} - \underline{a}$, $\underline{G}_0 = \underline{r} \wedge \underline{F}$

So $\underline{G}_A = \underline{r}_a \wedge \underline{F}$
$= (\underline{r} - \underline{a}) \wedge \underline{F}$
$= \underline{r} \wedge \underline{F} - \underline{a} \wedge \underline{F}$

$$\boxed{\underline{G}_A = \underline{G}_0 - \underline{a} \wedge \underline{F}}$$

Example Let $\underline{F} = 2\underline{i} + \underline{j}$ be a force acting at the point $\underline{r} = \underline{i} + \underline{k}$. What is the moment of $\underline{F}$ about the point $A = (0, 2, 0)$?

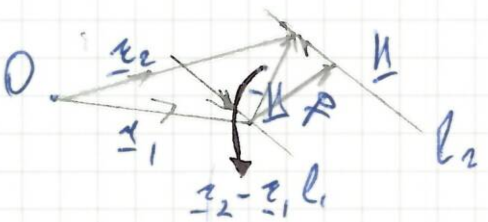
$\underline{a} = 2\underline{j} \Rightarrow \underline{r} - \underline{a} = \underline{i} - 2\underline{j} + \underline{k}$

So the moment of $\underline{F}$ about A is $(\underline{i} - 2\underline{j} + \underline{k}) \wedge \underline{F} =$
$= (\underline{i} - 2\underline{j} + \underline{k}) \wedge (2\underline{i} + \underline{j}) =$

$$= \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 1 & -2 & 1 \\ 2 & 1 & 0 \end{vmatrix} = -\underline{i} + 2\underline{j} + 5\underline{k} = \underline{G}_A$$

1.5. Couples

A **couple** is a pair of equal and opposite forces



The combined moment about O

$$\boxed{\begin{aligned} \underline{G}_0 &= \underline{r}_2 \wedge \underline{H} + (\underline{r}_1 \wedge (-\underline{H})) \\ &= (\underline{r}_2 - \underline{r}_1) \wedge \underline{H} \\ &= \underline{p} \wedge \underline{H} \end{aligned}}$$

$p = (r_2 - r_1) + sH$
$p \wedge \underline{H} = (r_2 - r_1) \wedge H$
since $(\underline{H} \wedge \underline{H} = 0)$
the доказ.
+ можно представить что r_1 и r_2 изображены так где $p(\underline{r}_2 - \underline{r}_1) \perp H$

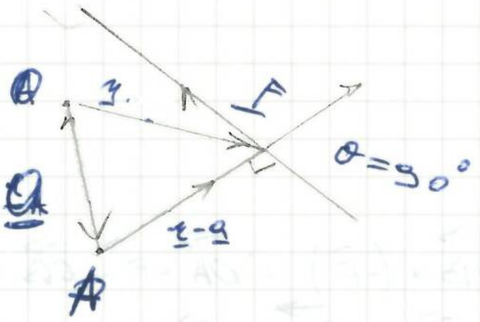
here $\underline{p}$ is the component of $\underline{r}_2 - \underline{r}_1$ $\perp$ to $\underline{H}$

Le Color

(1)

Q How $\wedge$-product is working?

If all the forces act in the same plane

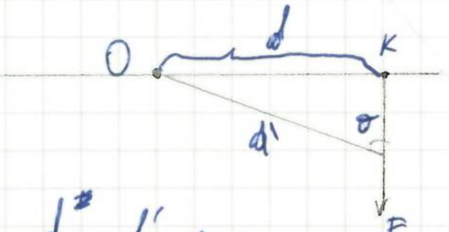


$$\underline{g}_A = (\underline{r} - \underline{a}) \wedge \underline{F}$$

$$\Rightarrow |\underline{g}_A| = |\underline{r} - \underline{a}| \cdot |\underline{F}| \sin\theta$$

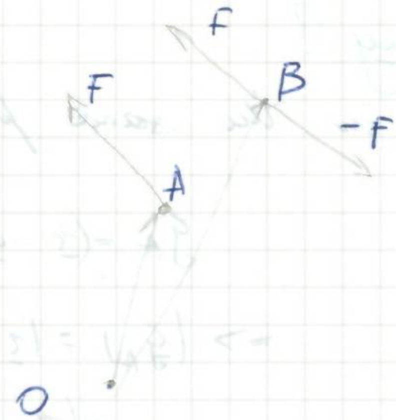
$$= |\underline{r} - \underline{a}| \cdot |\underline{F}|$$

$$= \left(\begin{array}{c}\text{perpend. distance} \\ \text{between A and} \\ \text{line of action of } \underline{F}\end{array}\right) \times \left(\begin{array}{c}\text{magnitude} \\ \text{of} \\ \underline{F}\end{array}\right)$$



$$|m| = |\underline{F}| \cdot |\underline{d}|$$

$$d^{\#} = d' \sin\theta$$



$$G_0 = \vec{OA} \times \vec{F} + \vec{OB} \times (-\vec{F}) = \vec{OA} \times \vec{F} - \vec{OB} \times \vec{F} =$$
$$= (\vec{OA} - \vec{OB}) \times \vec{F} = \vec{BA} \times \vec{F} = G_B$$

(8)

$\Rightarrow G_0 = (\underline{r}_2 - \underline{r}_1) \wedge \underline{H}$ – the total moment of the couple about O.

Also $G_0 = \underline{r} \wedge \underline{H}$, where $\underline{r}$ is any point vector joining

doesn't matter l_1 & l_2 (goes from the $-\underline{H}$ force to the $\underline{H}$ force)

e.g. the total moment $= \underline{p} \wedge \underline{H}$, where $\underline{p}$ is $\boxed{\perp}$ to $\underline{H}$ & $-\underline{H}$.

In particular, the total moment of the couple did not depend on O. So the moment of a couple is the same about any point in space.

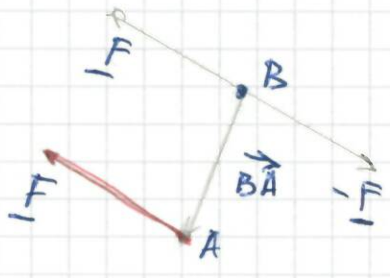
If $\underline{G}^{I}$ & $\underline{G}^{II}$ are couples, then I can make a new couple $\underline{G}^{III}$ by adding them together

$$\boxed{\underline{G}^{III} = \underline{G}^{I} + \underline{G}^{II}}$$

1.5.1. Moving a force.

Take a force $\underline{F}$ acting at a point A. Let B be another point.

If I add the same force $\underline{F}$ acting through B and force $-\underline{F}$ acting through B, then the system is unchanged.



Notice that $\underline{F}$ through A and $-\underline{F}$ through B are a couple (equal + opposite). The moment of this couple is $\vec{BA} \wedge \vec{F}$. So a force $\underline{F}$ through A is equivalent to:

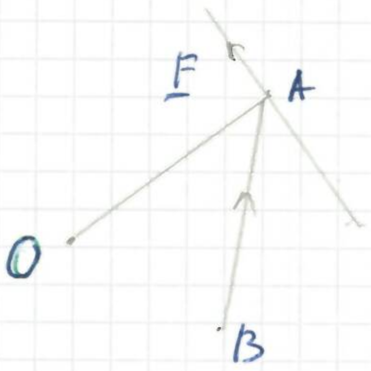
(i) a force $\underline{F}$ through B

(2) a couple with moment $\vec{BA} \wedge \underline{F}$

⑨

Notice that $\vec{BA} \wedge \underline{F} = \underline{g}_B$

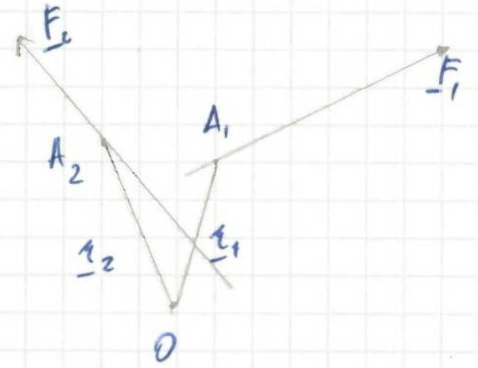
= moment of the force F through the point A about B.



$$\underline{g}_B = \vec{BA} \wedge \underline{F}$$

hand. 9 → 1.6. System of Forces.

Let $\underline{F}_1, \underline{F}_2, \underline{F}_3 \dots \underline{F}_n$ be forces acting at points $A_1, \dots, A_n$, which has position vectors $\underline{r}_1, \underline{r}_2, \dots, \underline{r}_n$.



The total force of the system is $\boxed{\underline{F} = \sum_{i=1}^{n} \underline{F}_i}$

The total moment of the forces about O is $\boxed{\underline{g}_O = \sum_{i=1}^{n} \underline{r}_i \wedge \underline{F}_i}$

Def^n The system is said to be equivalent to zero if the total force is $\underline{0}$ and total moment is $\underline{0}$.

$$\boxed{\text{i.e. } \underline{F} = \underline{0} \ \& \ \underline{g}_O = \underline{0}}$$

Let B be any point in space with position vector $\underline{b}$.
Let's move all the forces to the point B
Each force $\underline{F}_i$ acting through A_i is equivalent to

(1) $\underline{F}_i$ acting through B

(2) A couple $\underline{g}_B^i = \vec{BA_i} \wedge \underline{F}_i = (\underline{r}_i - \underline{b}) \wedge \underline{F}_i$

So doing this for all the forces, gives

10

(1) total force $\underline{F} = \sum_{i=1}^{n} \underline{F}_i$ acting at B

(2) A couple $\underline{G}_B = \underline{G}_B^1 + \dots G_B^n$

$$= \sum_{i=1}^{n} G_B^{i} = \sum_{i=1}^{n} (\underline{r}_i - \underline{b}) \wedge \underline{F}_i$$

$$= \sum_{i=1}^{n} \underline{r}_i \wedge \underline{F}_i - \underline{b} \wedge \sum_{i=1}^{n} F_i$$

$$= G_0 - \underline{b} \wedge \underline{F}$$

So the original system of forces is equivalent to a single force $\underline{F}$ through B plus a couple ($G_B = G_0 - \underline{b} \times \underline{F}$)

$$G_0 = \sum_{i=1}^{n} r_i \times F_i \ , \quad \underline{F} = \sum_{i=1}^{n} \underline{F}_i$$

Def^n A system is reduced to a single force if there exists a point for which $G_b = 0$

(i.e. if $\exists\, \underline{b}$ s.t. $G_0 - \underline{b} \times \underline{F} = 0$)

Lemma Given vectors $\underline{G}_0$ and $\underline{F}$, then the equation $G_0 - \underline{b} \times \underline{F} = 0$ has a solution if & only if

$$\boxed{\underline{F} \cdot \underline{G}_0 = 0}$$

In this case $\underline{b} = \alpha \underline{F} + \frac{1}{|F^2|} \underline{F} \times \underline{G}_0 \quad (\alpha \in \mathbb{R})$

Theorem If $F_1, F_2, \dots, F_n$ act at points $r_1, r_2, \dots, r_n$ and $\sum_{i=1}^{n} F_i = \underline{F}$ & $G_0 = \sum_{i=1}^{n} r_i \times F_i$ then

1) if $\underline{F} \cdot \underline{G}_0 \neq 0$ the system does not reduce to a single force

2) if $\underline{F} \cdot \underline{G}_0 = 0$ the system does reduce to a single force

with line of action $\boxed{\alpha F + \frac{1}{|F|^2} F \times G_0 \ , \ \alpha \in R}$

Example Find the value of 'a' for which $\underline{r} \times (2,1,1) =$
$= (6-a, a, 5-a)$ has a solution.
Find all vectors $\underline{r}$ satisfying equation

Let $\underline{F} = (2,1,1)$, $\underline{G} = (6-a, a, 5-a)$,
then $\underline{F} \cdot \underline{G} = 0$

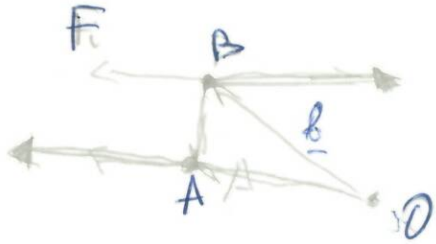
$\Leftrightarrow 12 - 2a + 4 + 5 - a = 0$

$21 = 3a \Rightarrow a = 7$

By the lemma the equation has a solution
(~~i.o.o.i~~ iff) = if & only if $a = 4$
In this case any vector $\underline{r} = \alpha \underline{F} + \frac{1}{|F|^2} \underline{F} \times \underline{G}$, $\alpha \in \mathbb{R}$

Solve equation:
Check $|F|^2 = 6$ $\quad \underline{F} \times \underline{G} = -6\underline{i} + 3\underline{j} + 9\underline{k}$

$$\underline{r} = \alpha \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix} + \frac{1}{6} \begin{pmatrix} -6 \\ 3 \\ 9 \end{pmatrix}, \quad \alpha \in \mathbb{R}$$



Handout 9 Systems of forces

Let's look at a rigid body (remember, idealised as a set of particles staying at fixed distances from one another) with various forces acting at various points on it. Say we have forces $\underline{F}_1$, $\underline{F}_2$, $\ldots \underline{F}_n$ acting at points $\underline{r}_1$, $\underline{r}_2$, $\ldots \underline{r}_n$.

Reducing the system to a single force and couple

We can replace a force $\underline{F}_i$ acting at A by a force $\underline{F}_i$ acting at B along with a couple $\vec{BA} \wedge \underline{F}_i$. Doing this for each force gives us a single force $\underline{F}$ acting at B (a point with position vector $\underline{b}$) plus a single couple $\underline{G}_B$:

$$\underline{F} = \sum_{i=1}^{N} \underline{F}_i \qquad \underline{G}_B = \sum_{i=1}^{N} (\underline{r}_i - \underline{b}) \wedge \underline{F}_i = \sum_{i=1}^{N} \underline{r}_i \wedge \underline{F}_i - \sum_{i=1}^{N} \underline{b} \wedge \underline{F}_i = \underline{G}_0 - \underline{b} \wedge \underline{F}.$$

Special case: total force is zero

If the total force $\underline{F}$ is zero, then we have

$$\underline{G}_B = \underline{G}_0$$

and the moment of the system is the same about any point.

Reducing a system to a single force and no couple

Is it possible to move all the forces to a point B such that $\underline{G}_B = \underline{0}$? If so, the forces reduce to a single force.

To do this, we need to choose $\underline{b}$ so that the corresponding $\underline{G}_B = \underline{0}$: and since $\underline{G}_B = \underline{G}_0 - \underline{b} \wedge \underline{F}$ we need to solve the vector equation

$$\underline{G}_0 = \underline{b} \wedge \underline{F}.$$

The right hand side of this equation is perpendicular to $\underline{F}$ so **the equation only has a solution if** $\underline{F} \cdot \underline{G}_0 = 0$.

If $\underline{F}$ is perpendicular to $\underline{G}_0$ then we can use the three vectors $\underline{F}$, $\underline{G}_0$ and $\underline{F} \wedge \underline{G}_0$ as our axes and write $\underline{b}$ in terms of them:

$$\underline{b} = \alpha \underline{F} + \beta \underline{G}_0 + \gamma \underline{F} \wedge \underline{G}_0$$

Substituting into the original equation gives

$$\begin{aligned} \underline{G}_0 &= [\alpha \underline{F} + \beta \underline{G}_0 + \gamma \underline{F} \wedge \underline{G}_0] \wedge \underline{F} \\ &= \alpha \underline{F} \wedge \underline{F} + \beta \underline{G}_0 \wedge \underline{F} + \gamma (\underline{F} \wedge \underline{G}_0) \wedge \underline{F} \\ &= \beta \underline{G}_0 \wedge \underline{F} + \gamma (\underline{F} \wedge \underline{G}_0) \wedge \underline{F} \end{aligned}$$

Using the result that $(\underline{a} \wedge \underline{b}) \wedge \underline{c} = \underline{b}(\underline{a} \cdot \underline{c}) - \underline{a}(\underline{b} \cdot \underline{c})$ (from MATH1401) gives us

$$\begin{aligned} \underline{G}_0 &= \beta \underline{G}_0 \wedge \underline{F} + \gamma [\underline{G}_0 (\underline{F} \cdot \underline{F}) - \underline{F} (\underline{G}_0 \cdot \underline{F})] \\ &= \beta \underline{G}_0 \wedge \underline{F} + \gamma [\underline{G}_0 (\underline{F} \cdot \underline{F})] - \gamma \underline{F} (\underline{G}_0 \cdot \underline{F})] \end{aligned}$$

and equating coefficients of $\underline{G}_0$, $\underline{F}$ and $\underline{G}_0 \wedge \underline{F}$ gives

$$1 = \gamma F^2, \quad 0 = \gamma (\underline{G}_0 \cdot \underline{F}), \quad 0 = \beta.$$

Hence, if $\underline{F} \cdot \underline{G}_0 = 0$, we have

$$\underline{b} = \alpha \underline{F} + \frac{1}{F^2} \underline{F} \wedge \underline{G}_0.$$

But this is the equation of a line, i.e. $\underline{b}$ is *not* unique, it is a whole line of points parallel to $\underline{F}$, passing through the point $(\underline{F} \wedge \underline{G}_0)/F^2$.

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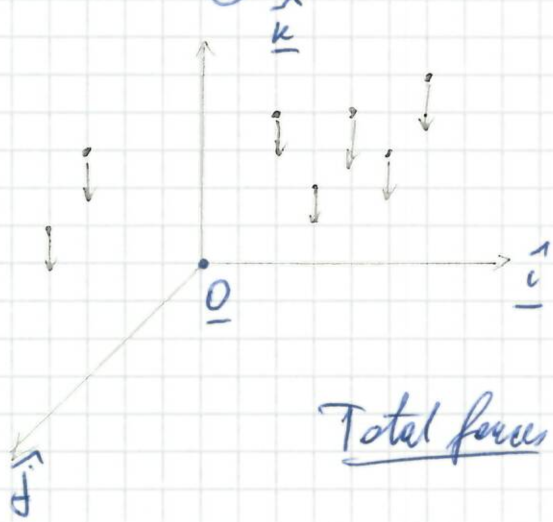
~~Exa~~ Example of moving forces to a single force : Centre of gravity



n masses $m_1, m_2, m_3, \dots, m_n$ at positions $\underline{r}_1, \underline{r}_2, \underline{r}_3, \dots, \underline{r}_n$

Gravity acts in the $-\hat{\underline{k}}$ direction on each of the particles

Total forces: $\underline{F} = -m_1 g\hat{\underline{k}} - m_2 g\hat{\underline{k}} - \dots - m_n g\hat{\underline{k}}$
$= -Mg\underline{k}$,
where $M = m_1 + m_2 + \dots + m_n$ is the total mass.

Total moment about $\underline{0}$: $\underline{G}_0 = \underline{r}_1 \wedge (-m_1 g\hat{\underline{k}}) + \underline{r}_2 \wedge (-m_2 g\hat{\underline{k}}) + \dots$
$+ \underline{r}_n \wedge (-m_n g\hat{\underline{k}}) = -g\left(m_1(\underline{r}_1 \wedge \hat{\underline{k}}) + m_2(\underline{r}_2 \wedge \hat{\underline{k}}) + \dots + m_n(\underline{r}_n \wedge \hat{\underline{k}})\right)$

$$= -g(m_1 r_1 + m_2 r_2 + \dots + m_n r_n) \wedge \hat{\underline{k}}$$

$$= -Mg\left(\frac{m_1 r_1 + m_2 r_2 + \dots + m_n r_n}{M}\right) \wedge \hat{\underline{k}}$$

$$= -Mg(\underline{X} \wedge \hat{\underline{k}}) ,$$

where $\underline{X} = \dfrac{m_1 r_1 + \dots + m_n r_n}{M}$

$\underline{F} = -(Mg)\hat{\underline{k}}$

// From last lecture, the system reduces to a single force iff $\underline{F} \cdot \underline{G}_0 = 0$

In this case $\underline{G}_0 = -Mg(\underline{X} \wedge \hat{\underline{k}})$ so G_0 is $\perp$ to $\hat{\underline{k}}$

Since $\underline{F} = -Mg\hat{\underline{k}}$ this implies that $\underline{G}_0$ is $\perp$ to $\underline{F}$

$\Rightarrow \underline{F} \cdot \underline{G}_0 = 0$

So system ~~of~~ reduces to a single force $\underline{F}$ acting along a line (which we will find)

Let P be a point on this line with position vector $\underline{p}$.

So $0 = \underline{g}_p = \underline{g}_0 - \underline{p} \wedge \underline{F}$ and we need to solve

$$\underline{p} \wedge \underline{F} = \underline{g}_0$$

From last lecture $\underline{p} = \alpha \underline{F} + \frac{1}{|F|^2} \underline{F} \wedge \underline{g}_0 \quad (\alpha \in \mathbb{R})$

$\frac{1}{|F|^2} \underline{F} \wedge \underline{g}_0$

F

Since $\underline{F} = -Mg\hat{\underline{k}}$ and $\underline{g}_0 = -Mg(\underline{x} \wedge \hat{\underline{k}})$

$$\Rightarrow \underline{p} = -\alpha M g \hat{\underline{k}} + \frac{1}{(Mg)^2}\left((-Mg\hat{\underline{k}}) \wedge (-Mg\underline{x} \wedge \hat{\underline{k}})\right) =$$

$$= -\alpha M g \hat{\underline{k}} + \hat{\underline{k}} \wedge (\underline{x} \wedge \hat{\underline{k}})$$

// Using a formula from 1101 : $\cancel{\hat{\underline{k}} \wedge (\underline{x} \wedge \hat{\underline{k}}) = \underline{x}(\hat{\underline{k}} \cdot \hat{\underline{k}}) - \hat{\underline{k}}(\underline{x} \cdot \hat{\underline{k}})}$

$$\hat{\underline{k}} \wedge (\underline{x} \wedge \hat{\underline{k}}) = \underline{x}(\hat{\underline{k}} \cdot \hat{\underline{k}}) - \hat{\underline{k}}(\underline{x} \cdot \hat{\underline{k}}) = \underline{x} - \hat{\underline{k}}(\underline{x} \cdot \hat{\underline{k}})$$

$$\text{So} \quad \underline{p} = \hat{\underline{k}}\underbrace{\left(-\alpha M g - \underline{x} \cdot \hat{\underline{k}}\right)}_{\text{const}} + \underline{x}$$

Since $\underline{F}$ acts in the direction of $-\hat{\underline{k}}$ we can write this as

$$\underline{p} = \alpha' \underline{F} + \underline{x} \quad \text{for some constant } \alpha'$$

Therefore the point with position vector $\underline{x}$ lies on the line of action of the force

Since $\underline{x}$ is independent of $\underline{F}$ (the direction chosen for gravity) the total force of gravity on the system <u>always</u> acts through $\underline{F}$ even if the system of points was rotated.

Hence $\underline{x} = \dfrac{m_1 \underline{r}_1 + \dots + m_n \underline{r}_n}{M}$ is called the centre of gravity

(13) Notice If we think of $\frac{m_i}{M}$ as probability of choosing $\underline{r}_i$

Then since $\sum_{i=1}^{n} \frac{m_i}{M} = 1$, this is a probability distribution

with mean $\underline{x} = \sum_{i=1}^{n} \frac{m_i r_i}{M}$

2. Statics

2.1. Equilibrium of Particle

A particle is in equilibrium if it is not accelerating. It could be moving in a straight line in a constant velocity.

According to Newtons Second Law, need $\underline{F} = \underline{0}$. This is necessary and sufficient for equilibrium of the particle.

hand 10 →

2.2. Equilibrium of a Rigid Body

Def^n A rigid body is a set of n particles keeping a fixed distance from eachothers.

Consider this system of n particles in equilibrium. Look at forces acting on particle i. There are two types of forces (acting):

(i) Forces which are external (e.g. gravity)

Call this force $\underline{F}_i$

(ii) Forces due to the other particles in the body.

Call this force exerted on particle i by particle j $\underline{F}_{ij}$

By Newton's third law $\underline{F}_{ij} = -\underline{F}_{ji}$ ⇒ 1

$$\boxed{F_{ij} + F_{ji} = 0}$$

(14) By Newton's 2^{nd} law, the total force acting each particle sum to zero.

$$\text{Particle } 1: \underline{F}_1 + \underline{0} + \underline{F}_{12} + \underline{F}_{13} + \ldots + \underline{F}_{1n} = \underline{0}$$

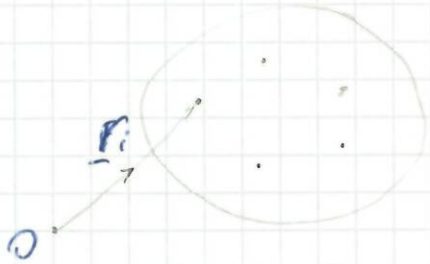
$$2: \underline{F}_2 + \underline{F}_{21} + \underline{0} + \underline{F}_{23} + \ldots + \underline{F}_{2n} = \underline{0}$$

$$\vdots$$

$$n: \underline{F}_n + \underline{F}_{n1} + \underline{F}_{n2} + \ldots \quad + \underline{0} = \underline{0}$$

If I add n equations and use (1) gives

$$\boxed{\underline{F} = \underline{F}_1 + \underline{F}_2 + \ldots + \underline{F}_n = \underline{0}}$$



position of particle i is $\underline{r}_i$

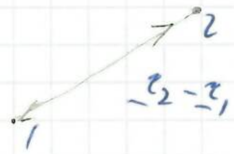
Now look at moments:

For total moment of each particle to be zero we need

$$\underline{r}_1 \wedge \underline{F}_1 + \underline{0} + \underline{r}_1 \wedge \underline{F}_{12} + \underline{r}_1 \wedge \underline{F}_{13} + \ldots + \underline{r}_1 \wedge \underline{F}_{1n} = \underline{0}$$

$$\underline{r}_2 \wedge \underline{F}_2 + \underline{r}_2 \wedge \underline{F}_{21} + \overset{0}{\cancel{\underline{r}_2 \wedge \underline{F}}} + \underline{r}_2 \wedge \underline{F}_{23} + \ldots \underline{r}_2 \wedge \underline{F}_{2n} = \underline{0}$$

Now, $\underline{r}_1 \wedge \underline{F}_{12} + \underline{r}_2 \wedge \underline{F}_{21} = (\underline{r}_1 - \underline{r}_2) \wedge \underline{F}_{12}$



$\underline{r}_2 - \underline{r}_1$ is a vector joining particles 1 & 2

and $\underline{F}_{12}$ acts along the same line. So $\underline{r}_2 - \underline{r}_1$ is $\parallel$ to $\underline{F}_{12}$

$$\text{So } (\underline{r}_1 - \underline{r}_2) \wedge \underline{F}_{12} = \underline{0}$$

$$\Rightarrow \underline{r}_1 \wedge \underline{F}_{12} = -\underline{r}_2 \wedge \underline{F}_{21} \quad (2)$$

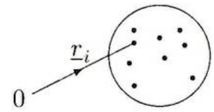
So adding the n eq. as before and use (2)

implies $\boxed{\underline{G}_0 = \underline{r}_1 \wedge \underline{F}_1 + \underline{r}_2 \wedge \underline{F}_2 + \ldots + \underline{r}_n \wedge \underline{F}_n = \underline{0}}$

So if the rigid body is in equilibrium $\Rightarrow \underline{F} = \underline{0}$ & $\underline{G}_0 = \underline{0}$

We will assume these two conditions are necessary and

Handout 10 Equilibrium of a rigid body



We think of a rigid body as an infinite number of particles, all keeping a fixed distance from each other. We will look at a finite system of n particles in equilibrium; to make them into a rigid body we will allow each pair of particles to exert a force on each other along the line joining them.

[Think of a well-oiled meccano structure with lots of triangulating crossbars so it can't flex.]

Look at the i^{th} particle (at $\underline{r}_i$). Think of it as being acting on by two sorts of forces. Firstly the sum of all the external forces (e.g. gravity). We shall call this $\underline{F}_i$.

Secondly, the forces on it due to the other particles. We shall call these $\underline{F}_{ij}$ (representing the force on particle i due to the j^{th} particle). The force acts along the line $\underline{r}_i - \underline{r}_j$, and (from Newton 3), $\underline{F}_{ij} + \underline{F}_{ji} = \underline{0}$.

Because each particle is in equilibrium, the sum of the forces on it is zero:

$$\begin{array}{llccccccccccccc}
\text{For first particle} & : & \underline{F}_1 & + & \underline{0} & + & \underline{F}_{12} & + & \underline{F}_{13} & + & \cdots & + & \underline{F}_{1n} & = & \underline{0} \\
\text{For second particle} & : & \underline{F}_2 & + & \underline{F}_{21} & + & \underline{0} & + & \underline{F}_{23} & + & \cdots & + & \underline{F}_{2n} & = & \underline{0} \\
\text{For } n^{\text{th}} \text{ particle} & : & \underline{F}_n & + & \underline{F}_{n1} & + & \underline{F}_{n2} & + & \underline{F}_{n3} & + & \cdots & + & \underline{0} & = & \underline{0}
\end{array}$$

Total force

Adding all these equations together makes the internal forces cancel. We obtain $\underline{F}_1 + \underline{F}_2 + \cdots + \underline{F}_n = \underline{0}$ or $\underline{F} = \underline{0}$ where $\underline{F}$ is the total external force. So, if a rigid body is in equilibrium, the sum of the external forces is $\underline{0}$.

Total moment of the external forces

For each of the equations above, we take $\underline{r}_i \wedge$ the equation of particle i. This gives:

$$\begin{array}{ccccccccccccc}
\underline{r}_1 \wedge \underline{F}_1 & + & \underline{0} & + & \underline{r}_1 \wedge \underline{F}_{12} & + & \underline{r}_1 \wedge \underline{F}_{13} & + & \cdots & + & \underline{r}_1 \wedge \underline{F}_{1n} & = & \underline{0} \\
\underline{r}_2 \wedge \underline{F}_2 & + & \underline{r}_2 \wedge \underline{F}_{21} & + & \underline{0} & + & \underline{r}_2 \wedge \underline{F}_{23} & + & \cdots & + & \underline{r}_2 \wedge \underline{F}_{2n} & = & \underline{0} \\
\underline{r}_n \wedge \underline{F}_n & + & \underline{r}_n \wedge \underline{F}_{n1} & + & \underline{r}_n \wedge \underline{F}_{n2} & + & \underline{r}_n \wedge \underline{F}_{n3} & + & \cdots & + & \underline{0} & = & \underline{0}
\end{array}$$

Now $\underline{r}_1 \wedge \underline{F}_{12} + \underline{r}_2 \wedge \underline{F}_{21} = (\underline{r}_1 - \underline{r}_2) \wedge \underline{F}_{12}$. Because the internal force between particles 1 and 2 acts along the line between them, $\underline{r}_1 - \underline{r}_2$ is parallel to $\underline{F}_{12}$ and $(\underline{r}_1 - \underline{r}_2) \wedge \underline{F}_{12} = \underline{0}$; similar equations apply to other pairs. Then when we add up all the equations we get:

$$\underline{r}_1 \wedge \underline{F}_1 + \underline{r}_2 \wedge \underline{F}_2 + \cdots + \underline{r}_n \wedge \underline{F}_n = \underline{0}.$$

We have shown that $\underline{G} = \underline{0}$ where $\underline{G}$ is the total moment of the external forces. Remember, if the total of those forces is zero, then their moment about any point is the same.

Physical meaning of this result for two-dimensional systems

It is necessary that $\underline{F} = \underline{G} = \underline{0}$ for equilibrium. We will assume that this is also sufficient.

Now suppose we're considering points, and forces, which all lie in a single plane (i.e. a system we can draw properly on paper). Then the condition $\underline{F} = \underline{0}$ is actually two scalar equations: we can resolve all the forces horizontally and vertically, to have

$$\boxed{\sum F_i^{\text{horizontal}} = 0} \qquad \boxed{\sum F_i^{\text{vertical}} = 0.}$$

The condition $\underline{G} = \underline{0}$ only has components perpendicular to the plane, so it is just one scalar equation:

$$\boxed{\sum \text{Moment of } \underline{F}_i = 0}$$

so we can choose a point and take the moment of all our forces about it; the clockwise and anticlockwise moments must balance.

These are the **only three equations we can derive** for the equilibrium of a two-dimensional set of forces: resolving forces in another direction, or taking moments about another point, will not give us any more information.

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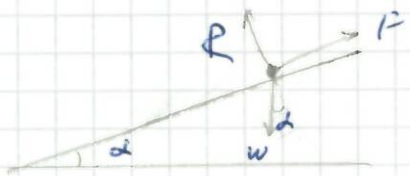
2.4. Friction.

Easy case Friction on a particle

Suppose a particle with weight W rests on a plane inclined at angle α to the horizontal. The plane is rough and the particle experiences friction



The friction force F acts in the direction opposing motion or likely motion

Suppose particle is in equilibrium:

Resolve $\parallel$ to the plane: $F = W \sin\alpha$ (1)

Resolve $\perp$ to the plane: $R = W \cos\alpha$ (2)

Dividing the equations gives $F/R = \tan\alpha$

As $\alpha \to \frac{\pi}{2}$ $\tan\alpha \to \infty$

So for the particle not to slip the ratio F/R must also increase for the particle to remain in equilibrium. In practice there is a limit to the amount of friction the plane can produce.

So in fact there exists a number $\mu \geq 0$ for which $F/R \leq \mu$. μ is called the coefficient of friction

In the above problem we need $\tan\alpha \leq \mu$ for this particle not to slip down the plane.

verge =
1. край
2. грань
3. граница

If $\tan\alpha = \mu$ the particle is in equilibrium i.e. it is on the verge* of slipping

If $\tan\alpha > \mu$ then particle slips down the plane

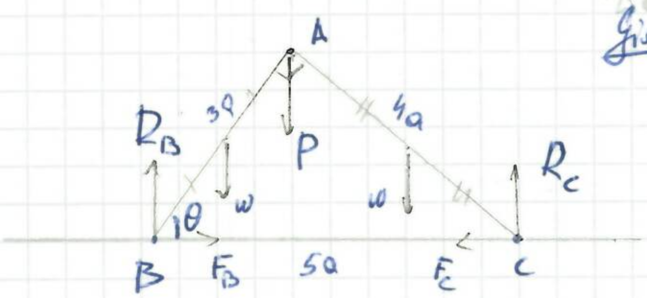
Unreal : If $\mu = 0$ then the surface is said to be smooth

and there is no friction force.

Some typical values of μ:

wood on stone	$\mu \simeq 0{,}4$
wood on wood	$\mu \simeq 0{,}5$ to $0{,}2$
metal on wood	$\mu \simeq 0{,}25$ to $0{,}15$

Example Two uniform rods AB and AC having equal weight W and lengths 3a and 4a respectively are linked at a smoth hinge A. They are resting on the ends B and C on a rough horizontal surface. The linking point A is in the same vertical plane as B and C. The angle at A is a right angle. The friction coefficient μ between each of the two rods and the plane satisfies $\mu > \frac{24}{34}$. If a ~~w~~ weight P is hug from A. So the system is in limiting equilibrium



Given: $AB = 3a$, $AC = 4a$

$\Rightarrow BC = 5a$

$\mu > \frac{24}{34}$

1) $\cos\theta = 3/5$, $\sin\theta = \frac{4}{5}$ $\Rightarrow$ $\tan\theta = \frac{4}{3}$

2) Resolve ~~vertically~~ horizontally

$F_B = F_C$

let $F = F_B = F_C$

3) Resolve vertically:

$P + 2W = R_B + R_c \quad (1)$

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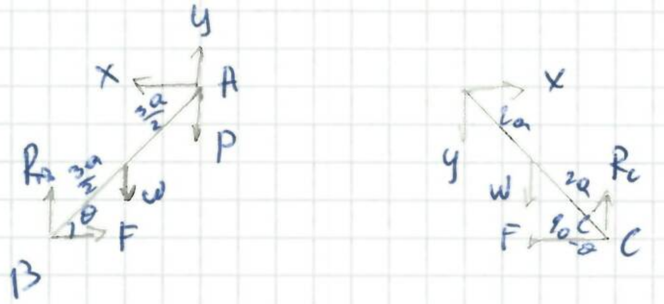


The friction conditions give

$$F \le \mu R_B \quad \text{and} \quad F \le \mu R_C$$

• If $R_B > R_C$ then $F/R_B < F/R_C \Rightarrow$ the system will slip at C
$\Rightarrow F = \mu R_C$

• If $R_C > R_B$ then the system slips at B $\to F = \mu R_B$

5) Take moments of Rod AB and A :

$$W \cdot \frac{3a}{2} \cos\theta + 3a F \sin\theta = R_B \cdot 3a \cos\theta$$

$$\Rightarrow \frac{3W}{2} + 3F \tan\theta = 3R_B$$

$$\Rightarrow \frac{3W}{2} + 4F = 3R_B \quad (2)$$

6) Take moments for rod AC about A

$$2a \cdot W \cdot \cos(90-\theta) + 4a F \sin(90-\theta) = 4a \cdot R_C \cdot \cos(90-\theta)$$

$$\Rightarrow 2W + 4F \tan(90-\theta) = 4R_C$$

$$2W + \frac{4F}{\tan\theta} = 4R_C \quad \Rightarrow \quad 2W + 3F = 4R_C \quad (3)$$

7) From (2) and (3) :

$$R_B = \frac{W}{2} + \frac{4F}{3} \quad , \quad R_C = \frac{W}{2} + \frac{3F}{4} \quad (4)$$

Since $F > 0 \Rightarrow R_B > R_C$

$\Rightarrow$ the system slips at C $\Rightarrow F = \mu R_C \quad (5)$

8) From (4) & (5)

$$R_C = \frac{W}{2} + \frac{3\mu R_C}{4} \Rightarrow R_C = \frac{2W}{4-3\mu}$$

$$R_B = \frac{W}{2} + \frac{4F}{3} = \frac{W}{2} + \frac{4\mu R_C}{3} = \frac{W}{2} + \frac{8\mu W}{3(4-3\mu)}$$

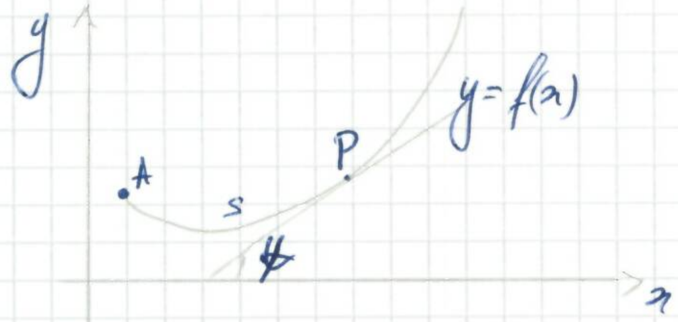
g) From (1)

$$P = R_B + R_C - 2W$$

$$= \frac{-3W}{2} + \frac{8\mu W}{3(4-3\mu)} + \frac{2W}{(4-3\mu)} = \frac{W}{6(4-3\mu)}\left(-3\cdot 3(4-3\mu) + 16\mu + 12\right)$$

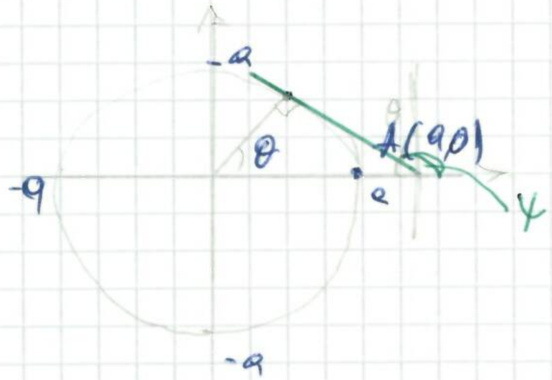
$$= \frac{W}{6(4-3\mu)}\left(43\mu - 24\right)$$

2.5. Properties of Plane Curves.



Pick a point A on the curve (fixed)
Let s be the distance (= arclength along the curve from A to P)
Draw the tangent to the curve at P.
Let ψ be the angle (measured anticlockwise) between the tangent and the x-axis.
If I can find a function g such that $s = g(\psi)$ then this is called the Intrinsic equation of the curve

Example A circle



So $\psi = \pi - \left(\frac{\pi}{2} - \theta\right) = \frac{\pi}{2} + \theta \quad (*)$

In cartesian coordinates: top half of circle $y = \sqrt{a^2 - x^2}$ $(0 \le x \le a)$
bottom half $y = -\sqrt{a^2 - x^2}$

Parametric form: $x = a\cos\theta$, $y = a\sin\theta$, where $0 \le \theta \le 2\pi$

What this mean

Could parametrise in terms of arclength s measured from $A = (a, 0)$ anticlockwise

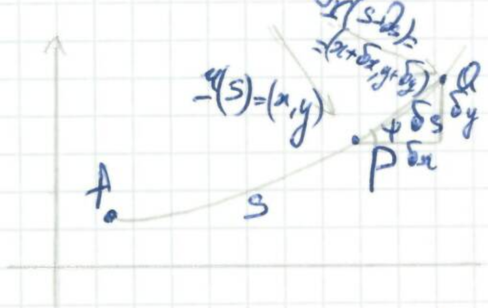
$\Rightarrow s = a\theta \Rightarrow x = a\cos\left(\frac{s}{a}\right),\ y = a\sin\left(\frac{s}{a}\right) \quad 0 \le s < 2\pi a$

From (*) $s = a\theta = a\left(\psi - \frac{\pi}{2}\right)$

So $$\boxed{s = a\left(\psi - \frac{\pi}{2}\right)}$$

is the intrinsic eq. of the circle

Q If we are given a curve $y = f(x)$, how can we calculate arclength?



Point P : $\underline{r}(s) = (x,y)$

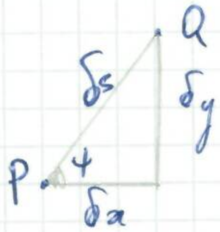
Q : $\underline{r}(s+\delta s) = (x+\delta x, y+\delta y)$

δs - arclength PQ

As $Q \to P$, the chord QP approaches the tangent to the curve at P

Also the ratio of the length of the chord PQ to the length of curve between P and Q tends to 1 as $Q \to P$

i.e. $\dfrac{\text{length of chord}}{\text{length of curve}} \to 1$ as $Q \to P$



By trig. $\dfrac{dy}{dx} = \tan\psi$, $\dfrac{dx}{ds} = \cos\psi$, $\dfrac{dy}{ds} = \sin\psi$

$(\cos^2 + \sin^2 = 1) \Rightarrow \left(\dfrac{dx}{ds}\right)^2 + \left(\dfrac{dy}{ds}\right)^2 = 1$

just believe that it works → mult. by $\left(\dfrac{ds}{dx}\right)^2$ $\Rightarrow 1 + \left(\dfrac{dy}{dx}\right)^2 = \left(\dfrac{ds}{dx}\right)^2$

$\Rightarrow \dfrac{ds}{dx} = \sqrt{1 + f'(x)^2}$ since $y = f(x)$

So, for ex., if $A = (x_0, y_0)$ and $P = (x, y)$ then the arclength from A to P = $\cup AP$

$$\boxed{\cup AP = \int_{x_0}^{x_1} \sqrt{1 + f'(x)^2}\, dx}$$

24 Q what does $\frac{d\psi}{ds}$ measure if we have an intrinsic eq. $s = g(\psi)$

A $\frac{d\psi}{ds}$ measures how quickly ψ changes with respect s.

if $\frac{d\psi}{ds} >$ then curve bends more

It is called the curvature $K = \frac{d\psi}{ds}$

The radius of curvature is the inverse of K:

$$\boxed{\rho = \frac{ds}{d\psi}}$$

For the circle $s = a(\psi - \pi/2)$

$\Rightarrow$ radius of curvature $= \rho = ds/d\psi = a$

In this example ρ is const

The radius of curvature is the radius of the circle which is the closest fit to the curve.

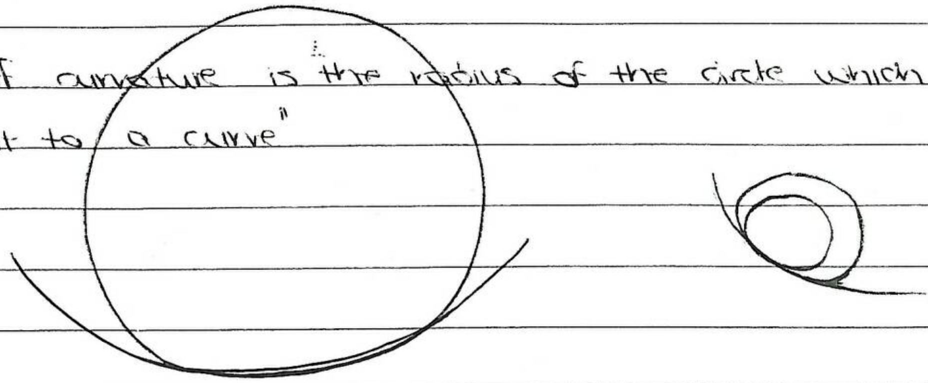
For the circle $s = a\left(\psi - \frac{\pi}{2}\right)$

$\Rightarrow$ radius of curvature, $\rho = \frac{ds}{d\psi} = a$

In this example ρ is constant.

"The radius of curvature is the radius of the circle which is the closest fit to a curve"

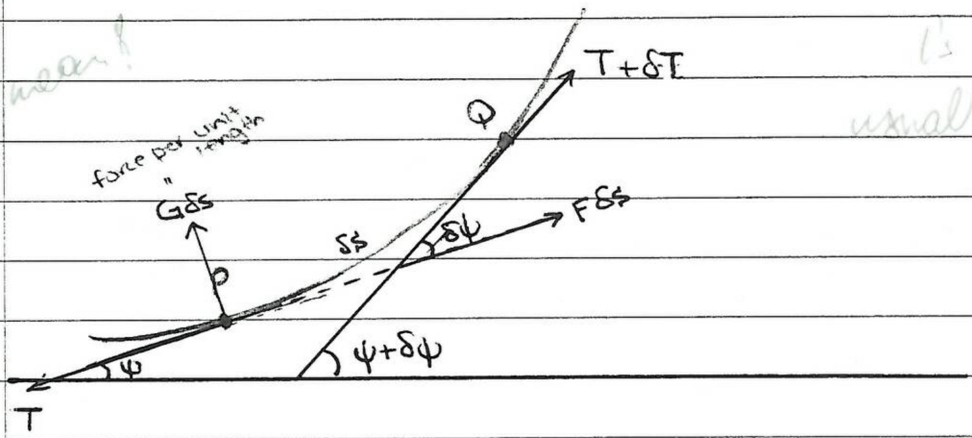
e.g.



18/11/09 2.6 <u>Equilibrium of a plane curve.</u>

what does it mean?

Is it usual case?



We are looking at a section of the curve PQ of length δs

Forces (1) Tension forces T at P

T+δT at Q

acting tangentially to the curve

<u>External forces</u>

(1) force F per unit length acts tangentially to the curve (in the direction of increasing s)

(2) Force G per unit length acts perpendicularly to F (acts into the curve)

Ultimately we will let $Q \to P$, so we assume the external forces act through P.

Suppose the section of the curve is in equilibrium.

Then the total forces acting on the section sum to zero.

Resolve tangentially at P:

<u>parallel</u>: $F\delta s - T + (T+\delta T)\cos\delta\psi = 0$ . . . (1)

<u>perpendicular</u>: $G\delta s + (T+\delta T)\sin\delta\psi = 0$ . . . (2)

* <u>For small angles $\delta\psi$</u>:

$\cos\delta\psi = 1 - \dfrac{(\delta\psi)^2}{2}$, $\sin\delta\psi = \delta\psi$

Actually, we will ignore squares and products of small quantities

From (1) $F\delta s - T + (T+\delta T)\times 1 = 0$

$\Rightarrow \boxed{F\delta s + \delta T = 0}$

From (2) $G\delta s + (T+\delta T)\delta\psi = 0$ ignore product of small terms $\delta T\,\delta\psi$

$\Rightarrow \boxed{G\delta s + T\delta\psi = 0}$

So, using these two equations and dividing by δs

$\Rightarrow F + \dfrac{\delta T}{\delta s} = 0$ and $G + T\dfrac{\delta\psi}{\delta s} = 0$

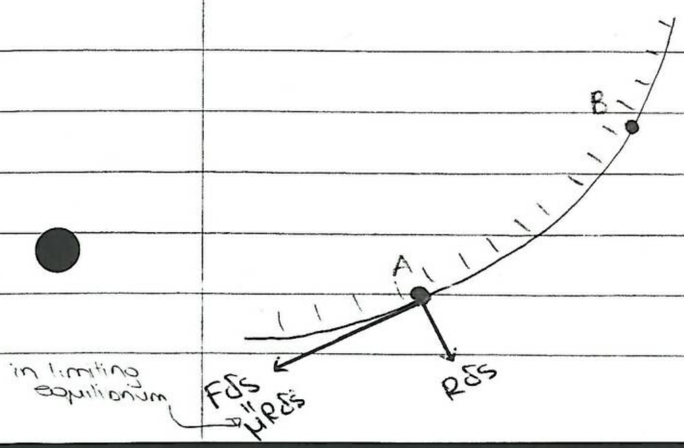
Let $Q \to P$, then $\delta s \to 0$ gives

$\boxed{F + \dfrac{dT}{ds} = 0}$ and $\boxed{G + T\dfrac{d\psi}{ds} = 0}$

[Remember F is a tangential force in the direction of increasing s
G is a force into the curve]

<u>EXAMPLE</u>: Rope around a bollard

This is a light string, wrapped horizontally around a circular bollard. Suppose the string is in limiting equilibrium. The external forces are friction and the reaction force of the bollard.



Suppose the rope is about to slip from A to B.

This is the same as before with

$F = -\mu R$ and $G = -R$

F is in direction of increasing arc length

now force acts out of curve

The above equations give $-\mu R + \frac{dT}{ds} = 0$

and $-R + T\frac{d\psi}{ds} = 0$

If we eliminate R we get $\frac{dT}{ds} = \mu T \frac{d\psi}{ds}$

$\div \frac{d\psi}{ds} \quad \Rightarrow \quad \frac{dT}{d\psi} = \mu T$

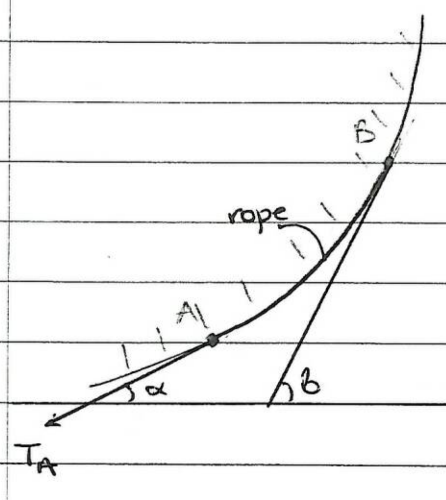
separate variables $\Rightarrow \int \frac{dT}{T} = \int \mu \, d\psi$

$\Rightarrow \ln T = \mu\psi + a$ (a is constant of integration)

$\Rightarrow T = e^{a} e^{\mu\psi} = T_0 e^{\mu\psi}$ $\quad (T_0 = e^{a})$

So $\boxed{T = T_0 e^{\mu\psi}}$

Suppose I stand at A and pull the rope with a tension T_A and the angle at A is $\psi = \alpha$



Then $T_A = T_0 e^{\mu\alpha} \Rightarrow T_0 = T_A e^{-\mu\alpha}$

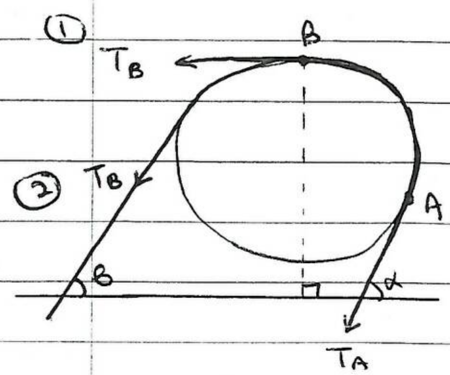
Suppose now that the angle at B is $\psi = \beta$

Then $T_B = T_0 e^{\mu\beta} = T_A e^{\mu(\beta - \alpha)}$

So the ratio $\frac{T_B}{T_A} = e^{\mu(\beta - \alpha)}$

which increases exponentially with β

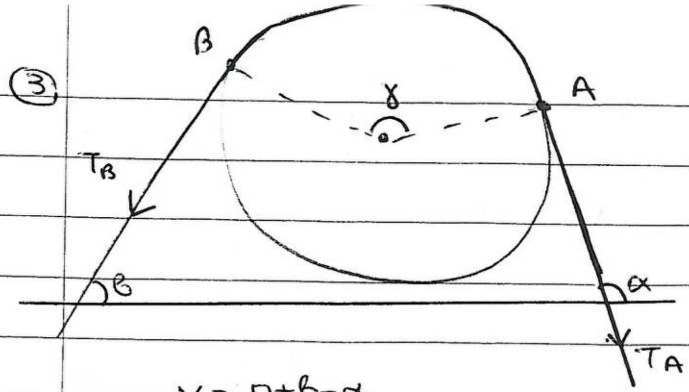
Examples:



① If the rope is about to slip from A to B then $\frac{T_B}{T_A} = e^{\mu(\pi - \alpha)}$

② If the rope is about to slip from A to B then $\frac{T_B}{T_A} = e^{\mu(\pi + \beta - \alpha)}$

3



$\gamma = \pi + \beta - \alpha$

In general, if γ is the angle of the sector where the rope touches the bollard.

Then $\frac{T_B}{T_A} = e^{\mu\gamma}$

e.g. Suppose a rope with coefficient of friction $\mu = \frac{1}{2}$ is wrapped around a bollard 4 times. Find the ratio of tensions

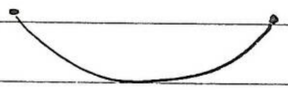
many times $\Downarrow$ increases T to a huge amount

4 times $\Rightarrow$ rope touches the bollard on a sector of angle

$\gamma = 4 \times 2\pi = 8\pi$

$\Rightarrow \frac{T_B}{T_A} = e^{\mu\gamma} = e^{\frac{8\pi}{2}} = e^{4\pi} \approx 287\,000$

23/11/09 Example: Suppose we have a heavy uniform chain hanging under its own weight.

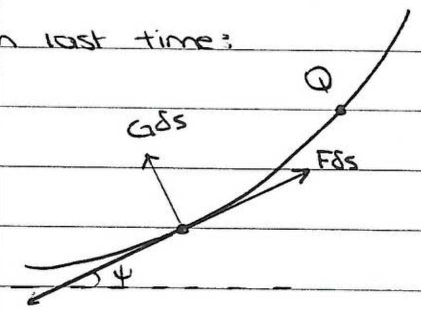


QUESTION: What is the equation which describes the curve in which the chain hangs?

SOLUTION:

First look for the intrinsic equation of the curve.

Recall from last time:

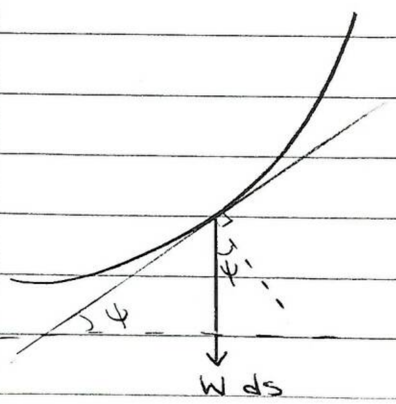


took 2 external forces:

(1) force F per unit length acting tangentially in the direction of increasing s

(2) Force G per unit length pointing into the circle

letting $Q \to P$ we derived (i) $F + \frac{dT}{ds} = 0$ (ii) $G + T\frac{d\psi}{ds} = 0$



Let W be the weight per unit length of the chain

The component of the weight pointing into the curve is $G = -W\cos\psi$

The component of the weight acting tangentially is $F = -W\sin\psi$

Using (i) and (ii) $\quad \frac{dT}{ds} = W \sin\psi$

$$T\frac{d\psi}{ds} = W\cos\psi \quad (+)$$

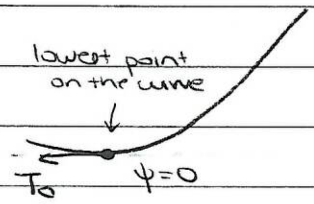
Divide the two equations to eliminate W.

$$\tan\psi = \frac{dT/ds}{T\, d\psi/ds} = \frac{1}{T}\frac{dT}{d\psi}.$$

Separate variables and integrate: $\quad \int \tan\psi \, d\psi = \int \frac{dT}{T}.$

$$\Rightarrow -\ln|\cos\psi| = \ln T + c$$

To deal with the constant c we make a choice



Assume that the tension at the bottom of the curve is T_0.
So when $\psi=0$, $T=T_0$.

Substitute into the equations $\quad -\ln(\cos(0)) = \ln T_0 + c$

$$\Rightarrow \quad 0 = -\ln(1) = \ln T_0 + c$$

$$c = -\ln T_0$$

$$\Rightarrow \quad -\ln(\cos\psi) = \ln T - \ln T_0$$

$$\Rightarrow \quad \ln\sec\psi = \ln\left(\frac{T}{T_0}\right)$$

$$\Rightarrow \quad \boxed{T = T_0\sec\psi}$$

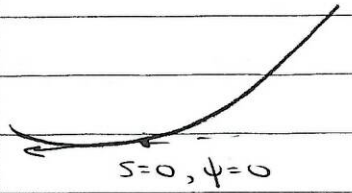
Using $(+)$ $\Rightarrow T_0\sec\psi \frac{d\psi}{ds} = W\cos\psi$

$$T_0\sec^2\psi\frac{d\psi}{ds} = W$$

Separate variables and integrate: $\quad T_0\int\sec^2\psi\, d\psi = W\int 1 \, ds$

$$T_0\tan\psi = Ws + k$$

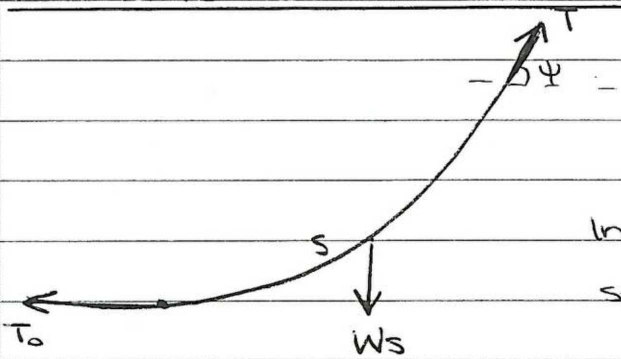
Again assume that $s=0$ at the bottom of the curve.



So $\psi=0$ when $s=0 \Rightarrow k=0$

Hence $\boxed{ws = T_0 \tan\psi}$ This is the intrinsic equation of the curve describing the heavy chain.

<u>Alternative method to derive the intrinsic equation</u>



w = weight per unit length

In equilibrium the forces on this section of length s balance.

Resolve horizontally: $T_0 = T\cos\psi$

Resolve vertically: $ws = T\sin\psi$

Dividing the two equations gives $\tan\psi = \frac{ws}{T_0}$

$\Rightarrow \boxed{ws = T_0 \tan\psi}$ This is the same intrinsic equation derived by the first method

* <u>Converting Intrinsic to Cartesian Coordinates</u>

Have an intrinsic equation $s = g(\psi)$ e.g. $s = \frac{T_0}{w}\tan\psi$

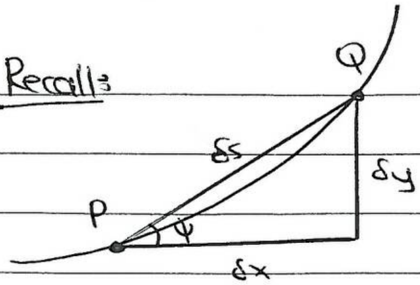
Q How can we convert this to a cartesian equation $y=f(x)$?

We will work through the chain example.

Let $c = \frac{T_0}{w}$ Then I have 2 equations: (1) $s = c\tan\psi$

(2) $T = T_0 \sec\psi$

Recall:



So: $\frac{dx}{ds} = \cos\psi \qquad \frac{dy}{ds} = \sin\psi$

① Calculate y (in terms of ψ)

$$\frac{dy}{d\psi} = \frac{dy}{ds}\cdot\frac{ds}{d\psi} = \sin\psi\, c\cdot\sec^2\psi$$

$$\Rightarrow \frac{dy}{d\psi} = c\cdot\tan\psi\sec\psi$$

Separate variables and integrate: $\int 1\, dy = \int c\sec\psi\cdot\tan\psi\, d\psi$

$$y = c\sec\psi + a$$

To deal with a, take $y=c$ when $\psi=0$ ($y=c$ at the lowest point of the curve)

substitute in: $c = c + a \Rightarrow a = 0$

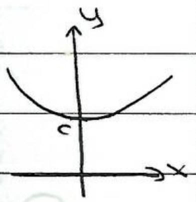
$$\Rightarrow \boxed{y = c\cdot\sec\psi}$$

② Calculate x (in terms of ψ)

$$\frac{dx}{d\psi} = \frac{dx}{ds}\cdot\frac{ds}{d\psi} = \cos\psi\, c\cdot\sec^2\psi = c\cdot\sec\psi$$

Separate variables and integrate: $\int 1\, dx = \int c\sec\psi\, d\psi$

$$\Rightarrow x = c\ln|\sec\psi + \tan\psi| + b$$



take $x=0$ when $\psi=0$ $\Rightarrow 0 = c\cdot\ln(1) + b \Rightarrow b = 0$

$$\Rightarrow \boxed{x = c\cdot\ln|\sec\psi + \tan\psi|}$$

Also, we had $y = c\cdot\sec\psi$

③ Eliminate ψ

$$\frac{x}{c} = \ln|\sec\psi + \tan\psi| \Rightarrow e^{x/c} = \sec\psi + \tan\psi$$

$$\Rightarrow e^{-x/c} = \frac{1}{\sec\psi + \tan\psi}$$

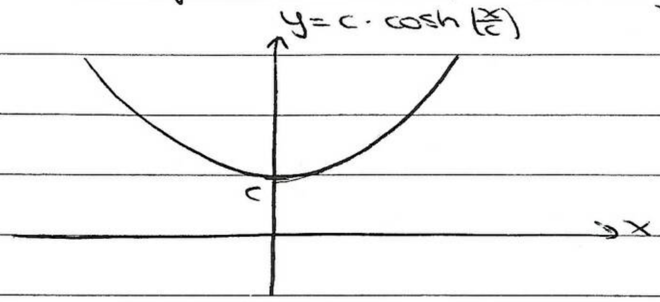
$$\Rightarrow e^{-x/c} = \frac{\sec\psi - \tan\psi}{(\sec\psi + \tan\psi)(\sec\psi - \tan\psi)} = \frac{\sec\psi - \tan\psi}{\sec^2\psi - \tan^2\psi}$$

(know $\sec^2\psi = 1 + \tan^2\psi$
$\sec^2\psi - \tan^2\psi = 1$) $\Rightarrow e^{-\frac{x}{c}} = \sec\psi - \tan\psi$

Hence $e^{\frac{x}{c}} + e^{-\frac{x}{c}} = 2\sec\psi = 2\frac{y}{c}$

$$y = c \cdot \frac{e^{x/c} + e^{-x/c}}{2} \Rightarrow y = \cosh\left(\frac{x}{c}\right) \cdot c$$

This is the equation of a catenery.



The assumptions we made about the constants of integration were EQUIVALENT to assuming the lowest point at the curve had Cartesian coordinates $(0, c)$

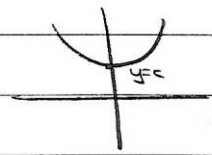
For converting to Cartesian: 3 steps

1. Find y in terms of ψ by using $\frac{dy}{d\psi} = \frac{dy}{ds} \cdot \frac{ds}{d\psi}$
2. Find x in terms of ψ by using $\frac{dx}{d\psi} = \frac{dx}{ds} \cdot \frac{ds}{d\psi}$.
3. Then have $y = f_1(\psi)$ and $x = f_2(\psi)$ eliminate ψ to get y in terms of x.

Return to the chain example: Can I write T and s in terms of y and x?

Tension: $T = T_0 \sec\psi = \frac{T_0 y}{c}$

Assumed that $c = \frac{T_0}{w} \Rightarrow \boxed{T = wy}$

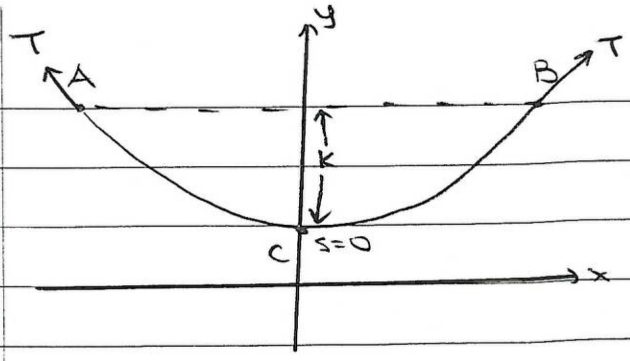


Arclength: $s = c \cdot \tan\psi = \frac{c}{2}\left(e^{x/c} - e^{-x/c}\right) = c \cdot \sinh\left(\frac{x}{c}\right)$

$$\Rightarrow \boxed{s = c \cdot \sinh\left(\frac{x}{c}\right)}$$

EXAMPLE: A uniform chain has length L and total weight W. The chain hangs between two points A and B at the same level.

The distance from the lowest point of the chain to the line AB is k



① Find the Tension of the points A and B

Solution: We know the curve has equation $y = c\cdot\cosh\left(\frac{x}{c}\right)$

and $s = c\cdot\sinh\left(\frac{x}{c}\right)$

At the end points $y = c + k$. So the tension at the end points is

(Recall: $T = wy$) $\quad T = \left(\frac{W}{L}\right)(c+k)$

We don't know the constant c which come from $T_0 = c\cdot\left(\frac{W}{L}\right)$

Using $s = c\cdot\sinh\left(\frac{x}{c}\right)$: at the point B, $s = \frac{L}{2}$

so $\frac{L}{2} = c\cdot\sinh\left(\frac{x_B}{c}\right)$

Also $c + k = c\cdot\cosh\left(\frac{x_B}{c}\right)$

Since $\cosh^2\left(\frac{x_B}{c}\right) - \sinh^2\left(\frac{x_B}{c}\right) = 1$

$$\Rightarrow \left(\frac{c+k}{c}\right)^2 - \left(\frac{L}{2c}\right)^2 = 1$$

$$\Rightarrow \cancel{c^2} = (c+k)^2 - \left(\frac{L}{2}\right)^2 = \cancel{c^2} + 2ck + k^2 - \frac{L^2}{4}$$

$$\Rightarrow 2kc = \frac{L^2}{4} - k^2 \quad \Rightarrow \quad c = \frac{1}{2k}\left(\frac{L^2}{4} - k^2\right)$$

Since $T = \left(\frac{W}{L}\right)(k+c) = \frac{W}{L}\left(k + \frac{1}{2k}\left(\frac{L^2}{4} - k^2\right)\right)$

$$\Rightarrow T = \frac{W}{2kL}\left(2k^2 + \frac{L^2}{4} - k^2\right)$$

$$\Rightarrow \boxed{T = \frac{W}{2kL}\left(\frac{L^2}{4} + k^2\right)}$$

Note that as $k \to 0$ then $T \to \infty$

So to get the chain truely horizontal we need infinite tension at the end points.

substitute – 1. замена 2. суррогат

Handout 11 Mechanics glossary

These are not intended to be formal definitions or to be a substitute for the lectures; they're just a quick reference to allow you to "translate" some of the loaded phrases we use in statics and mechanics.

Force

The force acting on an object is a **bound vector**: both the vector itself and its line of application are important.

hinge = 1. петля (дверная) 2. стержень

Hinge

A hinge between two objects is the same as a smooth joint or pin.

Hooke's law

An elastic string which obeys Hooke's law has a natural length, and if it is stretched beyond that length it will be under tension. If a string with natural length l is stretched an additional length – the extension – e, this causes a tension force T to act along the string. The magnitude of the force, T, is given by

$$\boxed{T = \frac{\lambda e}{l},} \quad \text{where } \lambda \text{ is the modulus of elasticity.}$$

pin = 1. гвоздь 2. штифт 3. болт 4. ось

Moment

A force $\underline{F}$ acting at a point $\underline{r}$ has a moment $\underline{r} \wedge \underline{F}$ about the origin.

Pin

A pin between two rods is a joint which can transmit force (in any direction) but not torque.

Rigid joint

Two rods which are **fixed** or **joined rigidly together** at a point act as if they were part of the same object. If you consider them as separate objects then the joining point can transmit both force and torque.

Rough contact

A rough contact between two objects can transmit force both normal to the contacting surfaces and parallel to them.

Smooth contact

A smooth contact between two objects can only transmit force normal to the contacting surfaces. The force parallel to the surfaces is zero.

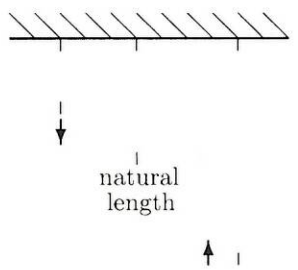
Smooth joint

A smooth joint is the same as a **pin**.

Springs and Strings

A spring exerts a force to resist either extension or compression from its natural length: see figure.

A string, on the other hand, will exert a force to resist extension beyond its natural length, but will compress (or go slack) with no resistance.



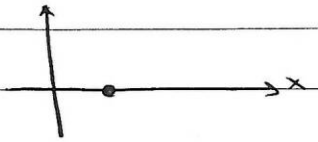
Torque

The torque acting on an object about a point within the object is the same as the total moment acting on it about that point (i.e. the sum of the moments of all the forces acting on it). Loosely, it is the tendency of the forces acting on it to cause it to rotate. The vector points along the axis of rotation.

III MECHANICS: MOTION IN A STRAIGHT LINE

1/ **Introduction:** A point mass m is moving along the x-axis under a FORCE $F(x, \dot{x}, t)$

what does it mean?



By Newton's Second Law

$$\boxed{m\ddot{x} = F(x, \dot{x}, t)}$$

$\ddot{x}$ - acceleration

The particle is in EQUILIBRIUM if the acceleration is zero: it may still be moving in a straight line with constant speed. The force being zero is necessary and sufficient for equilibrium of the particle.

2/ **Constant Acceleration:**

If F is constant then acceleration is constant

so $\ddot{x} = \frac{d^2x}{dt^2} = a$, for some constant a

Suppose initially that at time $t=0$, $x = x_0$ and $\dot{x} = u_0$

Integrating gives $\dot{x} = at + c$

Using initial values: $u_0 = c$

$$\Rightarrow \quad \dot{x} = \boxed{v = at + u_0} \quad \left(v = \dot{x} = \text{velocity of the particle}\right)$$

To find the position x of the particle at time t, integrate again: $x = \frac{at^2}{2} + u_0 t + c'$

Use initial values $(t=0, x=x_0) \Rightarrow c' = x_0$

So $$\boxed{x(t) = \frac{at^2}{2} + u_0 t + x_0}$$

So we have found velocity v and position x in terms of t. It would be possible to use these two equations to eliminate t and get a relationship between v and x.

However, we obtain such an equation by using the formula

$$\ddot{x} = \frac{d^2x}{dt^2} = \frac{dv}{dt} = \frac{dv}{dx}\,\frac{dx}{dt} = v\frac{dv}{dx}$$

7

So acceleration has two useful equations

$$\boxed{\ddot{x} = \frac{d^2x}{dt^2} = v\frac{dv}{dx}}$$

$$\ddot{x} = a \Rightarrow v\frac{dv}{dx} = a$$

$$\Rightarrow \int v\,dv = \int a\,dx + \alpha$$

$$\Rightarrow \frac{1}{2}v^2 = ax + \alpha$$

Initially $v = u_0$, $x = x_0$ $\Rightarrow$ $\alpha = \frac{1}{2}u_0^2 - ax_0$

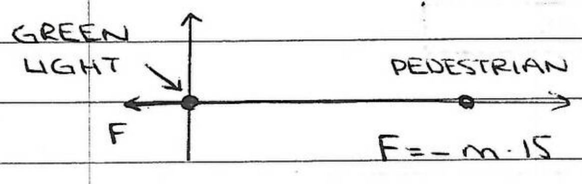
$$\Rightarrow \frac{1}{2}v^2 = ax + \frac{1}{2}u_0^2 - ax_0$$

$$\boxed{v^2 = 2ax + u_0^2 - 2ax_0}$$

<u>Example:</u> A car passes a green light at speed 20 m/s. The driver sees a pedestrian and applies the brakes which exert a decelerating force of 15N per unit mass

Q. How long does it take the car to stop? And how far must the pedestrian stand away to be safe?

To total acceleration Tou



Initial conditions

$t = 0$, $x = 0$, $\dot{x} = v = 20\,ms^{-1}$

Using $F = m\ddot{x} \Rightarrow m\ddot{x} = -15m$

$$\Rightarrow \ddot{x} = -15$$

$$\Rightarrow \dot{x} = -15t + C$$

$t = 0, \dot{x} = 20 \Rightarrow C = 20 \Rightarrow \boxed{\dot{x} = -15t + 20}$

The car has stopped when $\dot{x} = v = 0$

i.e. when $-15t + 20 = 0 \Rightarrow 15t = 20 \Rightarrow t = 4/3$

i.e. it takes 4/3 seconds to stop

We can integrate again but there is another method as well.

To find distance use $v\frac{dv}{dx} = -15 = \ddot{x}$

$$\Rightarrow \frac{1}{2}v^2 = -15x + B$$

③

$t=0,\ v=20,\ x=0 \Rightarrow \frac{1}{2}(20)^2 = \delta \Rightarrow \delta = 200$

$$\Rightarrow \boxed{\frac{1}{2}v^2 = 200 - 15x}$$

So distance taken to stop when $(v=0) \Rightarrow 0 = 200 - 15x$

$\Rightarrow 15x = 200 \Rightarrow x = \frac{40}{3}$ m

i.e. The pedestrian must stand $\frac{40}{3}$ m away to be safe

3// <u>Acceleration is a given function of time</u>

$m\ddot{x} = F(t)$ F varies ($\Leftrightarrow$ acceleration varies) as t varies

<u>Example:</u> A particle moves under a force $F(t) = mF_0 \sin(pt)$ where F_0 and p are positive constants.
When $t=0$ the particle goes through the origin with velocity V_0 [Note: V_0 can be positive or negative]
Show that the particle will eventually move to $+\infty$ if and only if $V_0 > -\frac{F_0}{p}$

Soln// By Newton's Second Law $m\ddot{x} = mF_0 \sin(pt)$

$\Rightarrow \ddot{x} = F_0 \sin(pt)$

I need a relationship with x and t

<u>Integrate:</u> $\dot{x} = -\frac{F_0}{p}\cos(pt) + c$

when $t=0$ $V_0 = -\frac{F_0}{p} + c \Rightarrow c = V_0 + \frac{F_0}{p}$

$$\boxed{\dot{x} = -\frac{F_0}{p}\cos(pt) + V_0 + \frac{F_0}{p}}$$

<u>Integrate:</u> $x = -\frac{F_0}{p^2}\sin(pt) + \left(V_0 + \frac{F_0}{p}\right)t + c'$

when $t=0,\ x=0 \Rightarrow 0 = \sin 0 + 0 = c' \Rightarrow c' = 0$

So $$\boxed{x = -\frac{F_0}{p^2}\sin(pt) + \left(V_0 + \frac{F_0}{p}\right)t}$$

For any t, $\left|-\frac{F_0}{p^2}\sin(pt)\right| \leq \frac{F_0}{p^2}$ which for large t is small in comparison to $\left(V_0 + \frac{F_0}{p}\right)t$

So the particle moves from 0 to $+\infty$ iff $V_0 + \frac{F_0}{p} > 0$

$\Rightarrow V_0 > -F_0/p$

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3.4. Acceleration a function of velocity.

$$m\ddot{x} = F(\dot{x})$$

where $\cdot$ means differentiation with respect to time t

This is a second order ODE

However since velocity $v = \frac{dx}{dt} = \dot{x} \rightarrow \dot{v} = \frac{dv}{dt} = \ddot{x}$

Hence, the second order problem, becomes

$$m\dot{v} = F(v)$$

which is first order O.D.E.

We solve $m\dot{v} = F(v)$ be separating values.

Suppose at $t=0$, $v = v_0$

$$m\frac{dv}{dt} = F(v)$$

So $m\int_{v_0}^{v} \frac{dv_1}{F(v_1)} = \int_0^t dt_1$

Or you could the indefinite integral

$$m\int \frac{dv}{F(v)} = \int dt + c$$

then work out c using the initial conditions $t=0$, $v=v_0$

task: If we wanted to find a relation between v and x in this case use

$$\ddot{x} = v\frac{dv}{dx}$$

So $mv\frac{dv}{dx} = F(v)$ $\Rightarrow$ solve by separating variables

So if $x = x_0$ when $v = v_0$ we have

$$m\int_{v_0}^{v} \frac{v_1\, dv_1}{F(v_1)} = \int_{x_0}^{x} dx_1$$

[or alternatively solve $m\int \frac{v\,dv}{F(v)} = \int dx + c$
and find c using $x = x_0$ when $v = v_0$]

Example 2 A particle acts under a const force P per unit mass and is subject to air resistance which we assume to be proportional to the velocity of the particle.

The resistance force R always opposes motion

Is it always? or here just for conditions

So $\boxed{R = -m\lambda\dot{x}}$ where $\lambda > 0$ is a const

The total force on the particle is $mP - m\lambda\dot{x}$

So by Newton's 2nd Law $m\ddot{x} = mP - m\lambda\dot{x}$

So $\ddot{x} = P - \lambda\dot{x}$ where $\lambda > 0$

Suppose we have initial conditions at $t=0$, $x=0$, $v=v_0>0$

Q (a) Find v in terms of t

(b) Find displacement of x in terms of t

(c) Find v in terms of x.

~~Solution~~

Solution (a) Writing $v = \dot{x}$ we need to solve

$$\dot{v} = P - \lambda v$$

$$\Rightarrow \int_{v_0}^{v} \frac{dv_1}{P-\lambda v_1} = \int_0^t dt_1$$

$$\Rightarrow \left[\left(-\frac{1}{\lambda}\right)\ln|P-\lambda v_1|\right]_{v_0}^{v} = t$$

where we need modulus signs since $\ln(x)$ doesn't make sense for $x \le 0$

$$\Rightarrow -\frac{1}{\lambda}\ln|P-\lambda v| + \frac{1}{\lambda}\ln|P-\lambda v_0| = t$$

$$\Rightarrow \frac{1}{\lambda}\ln\left|\frac{P-\lambda v_0}{P-\lambda v}\right| = t$$

Suppose $P-\lambda v_0$ & $P-\lambda v$ have the same sign.

Then $\frac{P-\lambda v_0}{P-\lambda v} > 0$

So we can remove modulus sign to give

$$\ln\left(\frac{P-\lambda v_0}{P-\lambda v}\right) = \lambda t$$

$$\Rightarrow \quad \frac{P-\lambda v_0}{P-\lambda v} = e^{\lambda t}$$

$$v = P/\lambda - \left(P/\lambda - v_0\right)e^{-\lambda t}$$

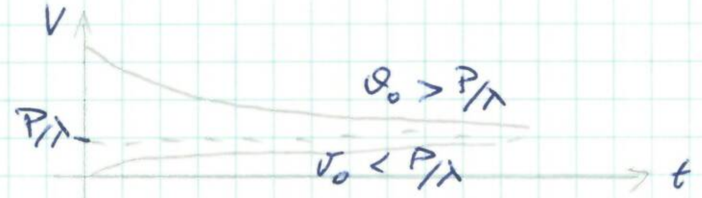
As $t \to \infty$ $e^{-\lambda t} \to 0$

so $v \to P/\lambda$

This is called the terminal velocity

If $v_0 > P/\lambda \Rightarrow P/\lambda - v_0 < 0$

$\Rightarrow -\left(P/\lambda - v_0\right)e^{-\lambda t} > 0$



$$V = P/\lambda - \left(P/\lambda - v_0\right)e^{-\lambda t}$$

The assumption at the start was that $P-\lambda v_0$ & $P-\lambda v$ have the same sign. From the diagram this corresponds ~~the~~ to velocity never crossing the line $v = P/\lambda$. which is always true.

b) $\frac{dx}{dt} = \frac{P}{\lambda} - \left(\frac{P}{\lambda} - v_0\right)e^{-\lambda t}$

$\Rightarrow x = \frac{Pt}{\lambda} + \frac{1}{\lambda}\left(\frac{P}{\lambda} - v_0\right)e^{-\lambda t} + c$

when $t=0,\ x=0 \Rightarrow c = -\frac{1}{\lambda}\left(\frac{P}{\lambda} - v_0\right)$

$\Rightarrow x = \frac{Pt}{\lambda} + \frac{1}{\lambda}\left(P/\lambda - v_0\right)\left(e^{-\lambda t} - 1\right)$

as $t \to \infty$ $\quad x \to \infty$

1

(c) To find v in terms of x

$$v\frac{dv}{dx} = P - \lambda v$$

$$\Rightarrow \int_{v_0}^{v}\frac{v\,dv}{P-\lambda v_1} = \int_0^x dx_1$$

Now $\frac{v_1}{P-\lambda v_1} = \frac{1}{\lambda}\left(\frac{\lambda v_1}{P-\lambda v_1}\right) = \frac{1}{\lambda}\left(-\frac{P-\lambda v_1}{P-\lambda v_1} + \frac{P}{P-\lambda v_1}\right) =$

$$= \frac{1}{\lambda}\left(-1 + \frac{P}{P-\lambda v_1}\right)$$

$$\Rightarrow \frac{1}{\lambda}\left(-v_1 - \frac{P}{\lambda}\ln|P-\lambda v_1|\right)\Big|_{v_0}^{v} = x$$

$$\Rightarrow -v - \frac{P}{\lambda}\ln|P-\lambda v| + v_0 + \frac{P}{\lambda}\ln|P-\lambda v_0| = \lambda x$$

$$\Rightarrow (v_0 - v) + \frac{P}{\lambda}\ln\left|\frac{P-\lambda v_0}{P-\lambda v}\right| = \lambda x$$

where the modulus signs have been removed by the previous assumption.

Example Suppose now that air resistance is proportional to the velocity squared.

For convenience assume that the const force acting on ~~a po~~ the particle is P^2 per unit mass.

Assume the magnitude of the resistance force is $m\lambda^2\dot{x}^2$ opposing motion.

Since velocity squared $\dot{x}^2$ is always positive we need two governing eq.

governing =
1. руководящий
2. контролирующий
3. главный
4. основной

$$m\ddot{x} = mP^2 - m\lambda^2\dot{x}^2 \qquad (\dot{x} > 0)$$

$$m\ddot{x} = mP^2 + m\lambda^2\dot{x}^2 \qquad \dot{x} < 0$$

Assume initially:

Case 1 : $v_0 > 0$

a) So need to solve $\ddot{x} = P^2 - \lambda^2\dot{x}^2$ as long as $\dot{x} > 0$

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Let $v = \dot{x} \Rightarrow \frac{dv}{dt} = P^2 - \lambda^2 v^2$ ~~+~~ $v > 0$

So
$$\int_{v_0}^{v} \frac{dv_1}{P^2 - \lambda^2 v_1^2} = \int_0^t dt_1$$

Now, $\frac{1}{p^2 - \lambda^2 v_1^2} = \frac{1}{\lambda^2}\left(\frac{1}{\frac{P^2}{\lambda^2} - v_1^2}\right) = \frac{1}{\lambda^2} \frac{1}{(P/\lambda - v_1)(P/\lambda + v_1)} =$

$= \frac{1}{\lambda^2}\left(\frac{1}{P/\lambda - v_1} + \frac{1}{P/\lambda + v_1}\right) \cdot \frac{\lambda}{2P} = \frac{1}{2P\lambda}\left(\frac{1}{P/\lambda + v_1} + \frac{1}{P/\lambda + v_1}\right) =$

Assumptions Since we are assuming $v > 0 \Rightarrow P/\lambda + v > 0$

Additions we will assume that $P/\lambda - v_0$ & $P/\lambda + v_0$ have the same sign

$$\Rightarrow t = \int_{v_0}^{v} \frac{dv_1}{P^2 - \lambda^2 v_1^2} = \frac{1}{2\lambda P}\int_{v_0}^{v} \left(\frac{1}{(P/\lambda - v_1)} + \frac{1}{P/\lambda + v_1}\right) dv_1 =$$

$$= \frac{1}{2\lambda P}\left[\ln\left(\frac{P/\lambda + v_1}{P/\lambda - v_1}\right)\right]_{v_0}^{v} = \frac{1}{2\lambda P} \ln\left(\left(\frac{P/\lambda + v}{P/\lambda - v}\right)\left(\frac{P/\lambda - v_0}{P/\lambda + v_0}\right)\right)$$

Multiply by $2\lambda P$ and take exponentials

$$\Rightarrow \left(\frac{P/\lambda + v}{P/\lambda - v}\right) \frac{(P/\lambda - v_0)}{(P/\lambda + v_0)} = e^{2\lambda P t}$$

Solve for v:

$$v = P/\lambda \left[\frac{e^{2\lambda P t}(P/\lambda + v_0) - (P/\lambda - v_0)}{e^{2\lambda P t}(P/\lambda + v_0) + (P/\lambda - v_0)}\right]$$

as $t \to \infty$ $\quad v \to P/\lambda$

Could this assumption influence our result

We assumed two things: (1) $v > 0$

(2) the sign of $P/\lambda - v$ did not change.

Check (1) Is v ever zero?

If $v = 0$ then using this implies

$$e^{2\lambda P t}(P/\lambda + v_0) = P/\lambda - v_0 \Rightarrow e^{2\lambda P t} = \frac{P/\lambda - v_0}{P/\lambda + v_0}$$

The LHS > 1 & RHS < 1

So v is never zero

(2) The sign of $P/\lambda - v$ changes if $P/\lambda = v$ ever occurs

what if it initially is negative

$P/\lambda = v$ if $e^{2\lambda Pt}(P/\lambda + v_0) - (P/\lambda - v_0) = e^{2\lambda PT}(P/\lambda + v_0) + (P/\lambda - v_0)$

$\Rightarrow 2(P/\lambda - v_0) = 0 \Rightarrow v_0 = P/\lambda$

So unless $V_0 = P/\lambda$ our assumptions were justified [if $V_0 = P/\lambda$ then $\dot{v} = 0$ and $V = V_0$ for any time t]

b) For displacement in terms of $\frac{v\,dv}{dx} = P^2 - \lambda v^2$

Exercise: $\frac{P^2 - \lambda^2 v^2}{P^2 - \lambda v_0^2} = e^{-2\lambda^2(x - x_0)}$

[initial conditions $x = x_0$, $v = v_0$]

Case 2 $V_0 < 0$: we would expect the particles to eventually stop after which we have case 1 again.

$$\bullet\longrightarrow P^2 + m\lambda^2 \dot{x}^2$$

$\overleftarrow{v_0}$

Q At what time t_0 does the particle stop?

For this solve $\ddot{x} = P^2 + \lambda^2 \dot{x}^2$

$$\frac{dv}{dt} = P^2 + \lambda^2 v^2$$

$$\int_{v_0}^{0} \frac{dv_1}{P^2 + \lambda^2 v_1^2} = \int_0^{t_0} dt$$

$$t = \frac{1}{\lambda^2}\int_{v_0}^{0} \frac{dv_1}{(P/\lambda)^2 + v_1^2} = \frac{1}{\lambda^2}\left[\frac{\lambda}{P}\arctan\left(\frac{\lambda v_1}{P}\right)\right]_{v_0}^{0} =$$

$$= -\frac{1}{\lambda P}\arctan\left(\frac{\lambda v_0}{P}\right)$$

[Note: since $V_0 < 0$, $-\frac{1}{\lambda P}\arctan\left(\frac{\lambda v_0}{P}\right) > 0$]

3.5. Acceleration as a function of displacement.

$m\ddot{x} = F(x)$

Suppose with initial conditions $t=0,\ x=x_0,\ v=v_0$

write $\ddot{x} = v\frac{dv}{dx} = \frac{d}{dx}\left(\frac{1}{2}v^2\right) = \frac{d}{dx}\left(\frac{1}{2}v^2\right)$

Integrating wrt x : $\frac{1}{2}mv^2 - \frac{1}{2}mv_0^2 = \int_{x_0}^{x} F(x_1)\,dx_1$

// $\frac{d}{dx}\left(\frac{1}{2}v^2\right) = \frac{dv}{dx} \cdot \frac{d}{dv}\left(\frac{1}{2}v^2\right) = \frac{dv}{dx}(v)$

$\Rightarrow$ $\frac{1}{2}mv^2$ is the kinetic energy of the particle

$\int_{x_0}^{x} F(x_1)\,dx$ is the work done by the force

as the particle moves from position x_0 to x.

by (+) So "change in kinetic energy" = work done by the force acting on the particle

In order to integrate the right hand side (+) we define a potential function $V(x)$ by

V - is a function not velocity)

$$F(x) = -m\frac{dV}{dx}$$

If this is the case (+) $\left(\frac{1}{2}mv^2 - \frac{1}{2}mv_0^2 = \int_{x_0}^{x} F(x_1)\,dx_1\right)$

$\Rightarrow$ gives $\frac{1}{2}mv^2 - \frac{1}{2}mv_0^2 = -mV(x) + mV(x_0)$

$\Rightarrow$ $\frac{1}{2}mv^2 + mV(x) = \frac{1}{2}mv_0^2 + mV(x_0)$

So $\boxed{\frac{1}{2}v^2 + V(x) = \frac{1}{2}v_0^2 + V(x_0)}$ (++) per unit mass

$V(x)$ is called the potential energy per unit mass of the particle

$\Rightarrow mV(x)$ is the potential energy

(++) $\Rightarrow$ that kinetic energy + P.E. = constant

Eq. (**) is called the energy equation of the particle

The total energy per unit mass on the RHS of (**) is a const

If we denote this const by E then energy eq. becomes

$$\boxed{\frac{1}{2}v^2 + V(x) = E}$$ per unit mass

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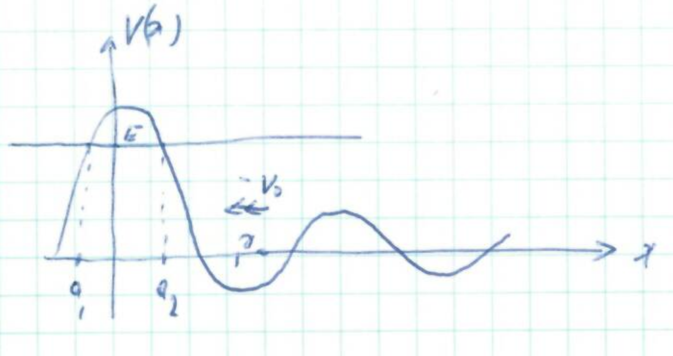
Suppose that a particle moves under a potential $V(x)$ this means $\ddot{x} = -\frac{dV}{dx}$. So 1

$$\frac{1}{2}v^2 = E - V(x) \quad \leftarrow \text{Energy Equation; } E \text{ - energy of the particle.}$$



Since $\frac{1}{2}v^2 = E - V(x)$

$v = 0$ exactly when $E = V(x)$

So since $E = V(a_2)$, the particle stop at a_2

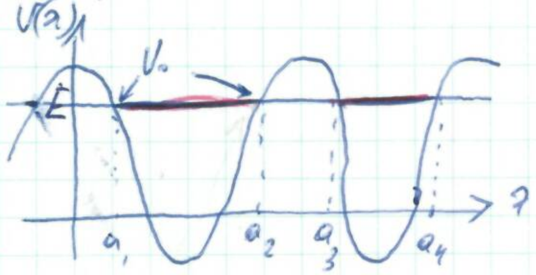
What happen next?

Since $\ddot{x} = -V'(x)$ the acceleration

particle

So the acceleration of the particle at a_2 is positive so the particle turns around and moves to the right. The particle will never stop since to the right a_2 $V(x) < E$. So it moves $+\infty$

Suppose a particle moves under this potential



In this case the particle can only be at points which are directly below the red sections on the diagram.

i.e. it can be between a_1 & a_2 or a_3 & a_4

But for example between a_2 & a_3 is not posible

If the particle starts between a_1 & a_2 it will remain in this range forever

It stops ($v = \phi$) at the endpoints a_1 & a_2

As before the acceleration at a_1 is positive and the acceleration at a_2 is negative, so at each point the particle is pulled towards to minimum of $V(x)$

So the particle osillates between a_1 & a_2

Q Time taken (what is the) to get from a_1 to a_2 and then back to a_1?

Use $\frac{1}{2} v^2 = E - V(x)$

$$\left(\frac{dx}{dt}\right)^2 = 2\left(E - V(x)\right)$$

Starting from a_1 the velocity is positive so

$$\frac{dx}{dt} = \sqrt{2(E - V(x))}$$

So time T_1 take to reach a_2 is given by T_1

$$T_1 = \int_0^{T_1} dt = \int_{a_1}^{a_2} \frac{dx}{\sqrt{2(E - V(x))}}$$

The particle is now at a_2 and will move back to a_1 so the velocity will be negative

$$\frac{dx}{dt} = -\sqrt{2(E-V(x))}$$

So time T_2 ~~total~~ taken to get back to a_1 from a_2 is

$$T_2 = \int_0^{T_2} dt = -\int_{a_2}^{a_1} \frac{dx}{\sqrt{2(E-V(x))}} = \int_{a_1}^{a_2} \frac{dx}{\sqrt{2(E-V(x))}} = T_1$$

So total time taken

$$T = 2T_1 = 2\int_{a_1}^{a_2} \frac{dx}{\sqrt{2(E-V(x))}}$$

So the motion is period with period T.

5.2. Simple harmonic motion.

This is the case where $\ddot{x} = -w^2 x$ for a const w

$$\Rightarrow \quad v\frac{dv}{dx} = -w^2 x$$

Suppose $t=0$, $x=x_0$, $v=v_0$. Apply the same technique to get

$$\frac{1}{2}v^2 = \frac{1}{2}v_0^2 - \frac{w^2x^2}{2} + \frac{w^2 x_0^2}{2} = -\frac{w^2x^2}{2} + E$$

where $E = \frac{1}{2}v_0^2 + \frac{w^2x_0^2}{2}$

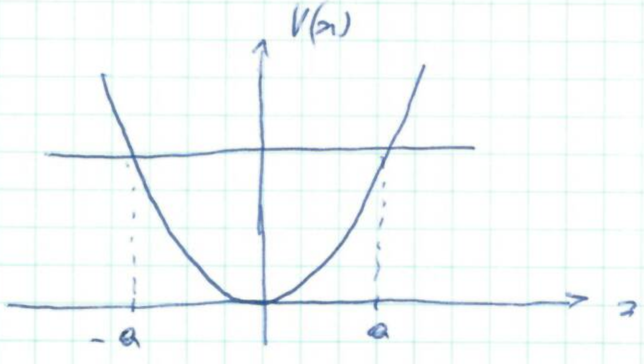
Now choose a const a such that

$$E = \frac{1}{2}w^2a^2 \qquad \left[a^2 = \frac{v_0^2}{w^2} + x_0^2\right]$$

So $\quad v^2 = w^2(a^2 - x^2)$

and the potential $V(x) = \frac{1}{2}\omega^2 x^2$

$$" \frac{1}{2} v^2 = E - V(x) "$$



So the particle oscillates between $-a$ and a.

$\ddot{x} = -w^2 x$ eq. for SHM with const. w

We assumed that at $t=0$, $v=v_0$, $x=x_0$ and integrated

$$\frac{1}{2}v^2 = -\frac{w^2x^2}{2} + \left(\frac{1}{2}v_0^2 + \frac{1}{2}w^2x_0^2\right)$$

This was the energy eq..

$\frac{1}{2}v^2 = -V(x) + E$ – energ. eq. in general

So (in this case) the potential $V(x) = \frac{1}{2}w^2x^2$

and the total energy $E = \frac{1}{2}v_0^2 + \frac{1}{2}w^2x_0^2$

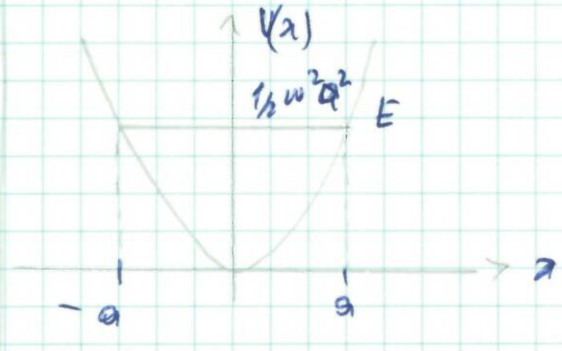
we assume $E = \frac{1}{2}w^2a^2$ for a

const $a^2 = \frac{v_0^2}{w} + x_0^2$

So (in this case) the energy eq. is

$$\boxed{v^2 = w^2(a^2 - x^2)}$$

Plotting $V(x) = \frac{1}{2}w^2x^2$



Since $-V(x) + E = \frac{1}{2}v^2 \geq 0$

possible motion is only when $V(x) \leq E$

So in this case between $-a$ and a

when $x=-a$ or $x=0$ the particle stops

17 but acceleration acts to pull it back towards the point $x=0$

Q what is the period of the oscillation between $-a$ and a?

From before $T = 2\int_{-a}^{a} \frac{dx}{\sqrt{2(E-V(x))}} = 2\int_{-a}^{a} \frac{dx}{\sqrt{w^2(a^2-x^2)}} =$

$= \frac{2}{w}\int_{-a}^{a} \frac{dx}{\sqrt{a^2-x^2}} = \left| \begin{array}{l} x = ay \\ dx = a\,dy \end{array} \right| = \frac{2}{w}\int_{-1}^{1} \frac{a\,dy}{\sqrt{a^2(1-y^2)}} = \frac{2}{w}\int_{-1}^{1} \frac{dy}{\sqrt{1-y^2}} =$

$= \frac{2}{w}\left[\arcsin(y) \right]_{-1}^{1} = \frac{2}{w}\left(\frac{\pi}{2} + \frac{\pi}{2}\right) = \frac{2\pi}{2w}$

So $\boxed{T = \frac{2\pi}{w}}$

Q Can we find an expression for x in terms of t?

$V = \frac{dx}{dt} = \sqrt{2(E-V(x))} = w(a^2-x^2)^{1/2}$

Under the assumptions the particle is moving in the positive direction.

So we can separate variables to get

$$\int \frac{dx}{\sqrt{a^2-x^2}} = \int w\,dt$$

$\Rightarrow \arcsin\left(\frac{x}{a}\right) = wt - \delta$ (where $-\delta$ is the const of integration)

$\Rightarrow \boxed{x = a\sin(wt-\delta)}$, where

- a - the amplitude (maximum displacement from $x=0$)
- $T = \frac{2\pi}{w}$ is the period

$x\left(t + \frac{2\pi}{w}\right) = a\sin\left(w\left(t + 2\frac{\pi}{w}\right) - \delta\right) = a\sin(wt - \delta + 2\pi) =$

$= a\sin(wt-\delta) = x(t)$

18

- ω is frequency of the oscillation
- δ is the phase angle.

<u>An easier approach:</u>

Since $\ddot{x} = -\omega x$ is just a second order linear ODE
So it has auxiliary eq.

$$x^2 + \omega^2 = 0 \rightarrow \lambda = \pm \omega$$

Hence the eq. has solution

$$x(t) = \alpha \cos(\omega t) + \beta \sin(\omega t)$$

for some const α & β. However,

since $\frac{-\alpha}{\sqrt{\alpha^2+\beta^2}} \in (-1, 1)$

So there exists δ s.t. $\sin\delta = \frac{-\alpha}{\sqrt{(\alpha^2+\beta^2)}}$

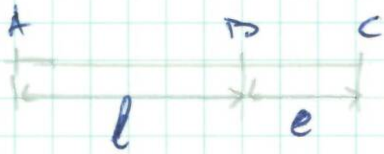
Then $\cos\delta = \frac{\beta}{\sqrt{(\alpha^2+\beta^2)}}$.

Let $a = \sqrt{\alpha^2+\beta^2}$

Then $\alpha = -a\sin\delta$ and $\beta = a\cos\delta$

So $x(t) = \alpha\cos(\omega t) + \beta\sin(\omega t) = a(-\sin\delta\cos(\omega t) + \cos\delta\sin(\omega t))$
$= a\sin(\omega t - \delta)$

Elastic strings



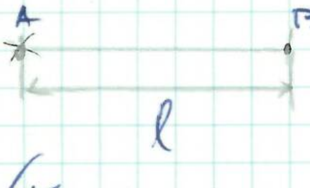
We have an elastic string of natural length l fixed at A. We stretch the string from point B to point C so that it has an extension e. Once the string is stretched there is a tension T in the string.

$$\boxed{T = \frac{\lambda e}{l}}$$ - tension is given by Hooke's law

λ is the modulus of elasticity

__Remark__ For __strings__ the tension T is zero if e is negative.
For __springs__ it is alright to use the formula $T = \frac{\lambda e}{l}$ for negative extension e (we'll get a thrust)

__Example__



A particle of mass m is attached to a string of natural length l at the point B (The point A is fixed). The particle is projected with speed V_0 away from B. Find the maximum length of the string in the subseq. motion.

__Solution:__ Measure x from B. The governing eq. is

why it's here minus sign

$$m\ddot{x} = -T = -\frac{\lambda x}{l} \Rightarrow \ddot{x} = \frac{-\lambda}{ml}x$$

Integrating $\frac{1}{2}v^2 - \frac{1}{2}v_0^2 = \frac{-\lambda x^2}{2ml}$

20 The maximum extension is when the particle comes to rest

$v=0$

i.e. in this case $x = \sqrt{\dfrac{m l v_0^2}{\lambda}}$. So

max length $l + \sqrt{\dfrac{m l v_0^2}{\lambda}}$

5.3. Equilibrium and Stability

We will assume that a particle is moving under a potential $V(x)$, i.e.

$$\ddot{x} = -\frac{dV}{dx}$$

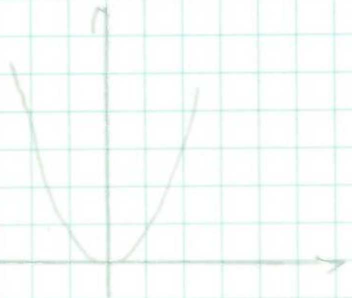
Then as usual we have the energy eq.

$$\frac{1}{2} v^2 = E - V(x)$$

Equilibrium points are where the particle could be at rest and stay at rest.

i.e. when $\ddot{x} = 0$ which occurs, when

$$V'(x) = 0$$



e.g.

We want to analyse stability of the eq. points
To do this we will look at the Taylor series for $V'(x)$ near an equilibrium point

Remember that for a sufficiently well behaved function $f(x)$ near a point $x = a$ we can write (for x close to a)

$$\boxed{f(x) \simeq f(a) + (x-a)f'(a) + \frac{(x-a)^2}{2!} f''(a) + \dots}$$

Let $x = c$ be a point of equilibrium. So $V'(c) = 0$
Apply the Taylor's series with $f = V'$ and $a = c$
So for x near to c:

other terms are very small

$$\boxed{V'(x) = V'(c) + (x-c)V''(c) + \frac{(x-c)^2}{2!} V'''(c) + O\left((x-c)^3\right)}$$

Actually, let us neglect $(x-c)^2$ and higher terms.
Using $V'(c)=0$:

$$\Rightarrow \quad \boxed{V'(x) \simeq (x-c)\,V''(c)}$$

$$\Rightarrow \quad \ddot{x} = -V'(x) = -(x-c)V''(c)$$

Now let $X = x-c$. Then $\ddot{X} = \ddot{x} \Rightarrow \ddot{X} = -X\,V''(c)$

Case 1 So if $V''(c) > 0$ then $\ddot{X} = -\left(\sqrt{V''(c)}\right)^2 X$ -
this is just SHM with period $T = \frac{2\pi}{\sqrt{V''(c)}}$

Def^n An equilibrium point is stable if the particle stays close to that point which is clearly the case in this situation $(V''(0) > 0)$

Case 2 if $V''(c) < 0$ then $\ddot{x}$ has the same sign x and the particle will accelerate away from the equilibrium point. In this case the equilibrium point is called unstable

Def^n A maximum of $V(x)$ is an unstable equilibrium point. $(V''(c)<0)$
A minimum of $V(x)$ is a stable eq. point $(V''(c) > 0)$

Example Let $V(x) = \frac{kx}{x^2+b^2}$, where $k > 0$

Find the period of small oscillations about a stable equilibrium point of $V(x)$.
Give also an expression for the period of an oscillation of the particle starting from rest at position $x = -b/2$

Solution: 1) sketch the potential $V(x)$:

$$V'(x) = \frac{k(x^2+b^2)-2kx^2}{(x^2+b^2)^2} = k\frac{(b^2-x^2)}{(x^2+b^2)^2} = k\frac{(b-x)(b+x)}{(x^2-b^2)^2}$$

So $V'(x) = 0 \iff x = \pm b$

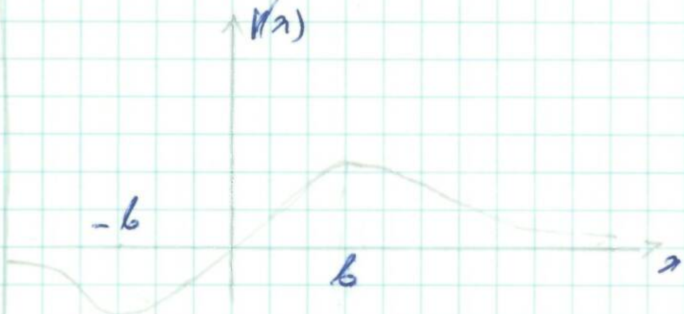
To deduce whether these points are max's or min's.

Note that $V(0) = 0$ and $V(x) > 0$ whenever $x > 0$.

Also $V(x) < 0$ when $x < 0$ and

$V(x) \to 0$ from above as $x \to +\infty$

$V(x) \to 0$ from below as $x \to -\infty$



So $V(x)$ has a min at $x = -b$

max at $x = b$

Hence $x = -b$ is a point of stable equilibrium

The practical could obey SHM about $x = -b$ wi with $T = \frac{2\pi}{\sqrt{V''(-b)}}$ (= period of oscillation)

An alternative to calculate $V''(-b)$ is to use a small perturbation of the equilibrium point $x = -b$.

To do this write $x = -b + X$ where $|X|$ is small compared to b

$$\text{So} \quad \ddot{x} = -V'(x) = -k\frac{(b-x)(b+x)}{(b^2+x^2)^2} = \frac{-k(2b-X)\cdot X}{(b^2+X^2-2bX+b^2)^2}$$

$$\text{So } \ddot{X} = \ddot{x} = \frac{-k(2b-X)X}{(2b^2+X^2-2bX)^2}$$

Now use that $|X|$ is small compared to b. So $2b - X \simeq 2b$

and $(2b^2+X^2-2bX)^2 \simeq (2b^2)^2 = 4b^4$

$$\Rightarrow \ddot{X} \simeq \frac{-k \cdot X \cdot 2b}{4b^4} = -\left(\frac{k}{2b^3}\right)X$$

This is just SHM with $\omega = \sqrt{\frac{k}{2b^3}}$

which has a period $T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{2b^3}{k}}$

24 Check that $V''(-b) = \frac{k}{2b^3}$

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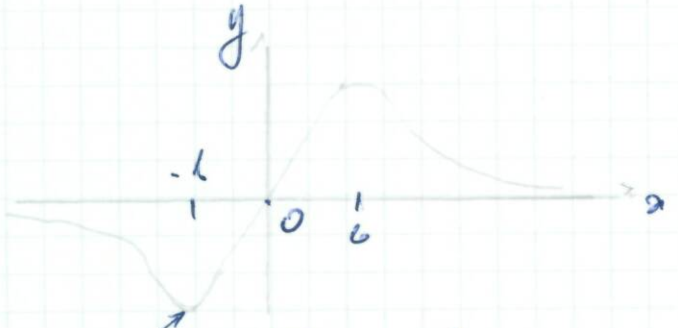
Potential $V(x) = \frac{kx}{x^2+b^2}$, $k>0$

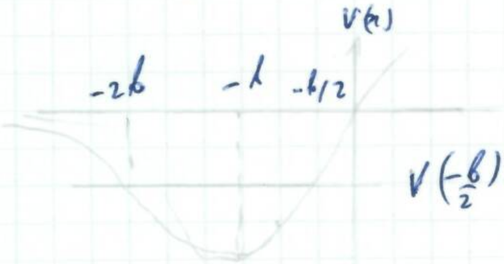


$T_{SHM} = \frac{2\pi\sqrt{2b^2}}{\sqrt{k}}$ -

period of small oscillation about $x=-b$.

point of stable equilibrium

ii) Suppose the particle realised from rest at $x = -\frac{b}{2}$ what happens next?



$V\left(-\frac{b}{2}\right) = E$

The particle moves to the left and stops again when

$$V(x) = V\left(-\frac{b}{2}\right) = -\frac{2k}{b}$$

$$\Leftrightarrow \frac{kx}{x^2+b^2} = \frac{-2k}{5b}$$

$$\Leftrightarrow 5bx = -2x^2 - 2b^2$$

$$\Leftrightarrow 2x^2 + 5bx + 2b^2 = 0$$

$$\Leftrightarrow (2x+b)(x+2b) = 0$$

$$\Leftrightarrow x_1 = -\frac{b}{2} \text{ or } x_2 = -2b$$

The particle oscillates between $-2b$ and $-b/2$

5.4. Phase Plane

Suppose we are looking at the system

$$m\ddot{x} = F(x) = -m\frac{dV}{dx}$$

We integrate above to get the energy eq^n.

$$\frac{1}{2}v^2 = E - V(x)$$

Since $v = \dot{x}$ =>

$$\Rightarrow \quad \frac{1}{2}\dot{x}^2 = E - V(x)$$

Def^n The Phase Plane for this system is given by setting $y(x) = \dot{x}$ and plotting the values in an (x, y) plane.

Example 1) $\ddot{x} = a$ where $a > 0$

integrating $\frac{1}{2}v^2 = ax + E$ $\qquad (V(x) = -ax)$

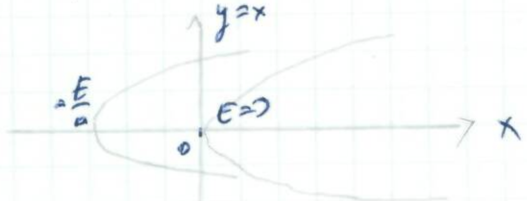
E is a const depending on the initial values (x_0, v_0)

i.e. $E = \frac{1}{2}v_0^2 - ax_0$

$y = \dot{x} = v \Rightarrow \frac{1}{2}y^2 = ax + E$

$$\Rightarrow \quad x = \frac{1}{2a}y^2 - \frac{E}{a}$$

For different values of E the eq^{ns} are parabolas

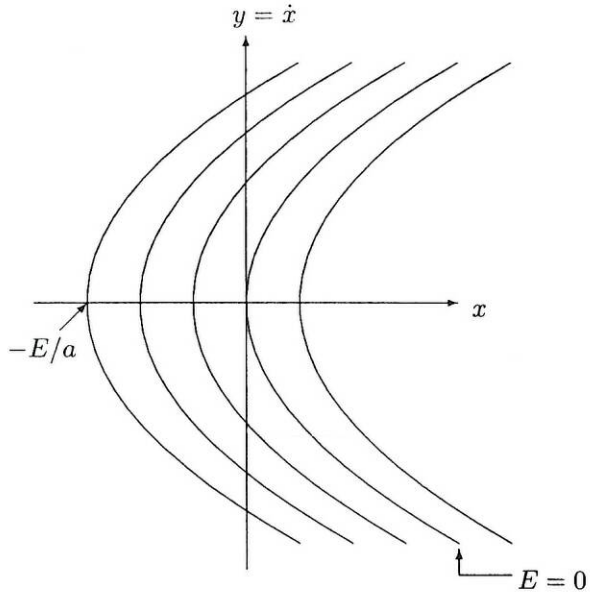


They are similar, just shifted laterally, depending on E.

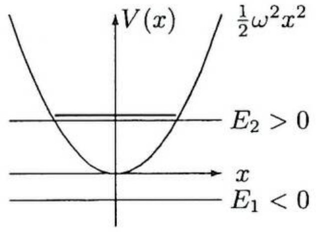
Handout 12: Phase Planes

Constant Acceleration

$\ddot{x} = a$ where a is a constant and $a > 0$.

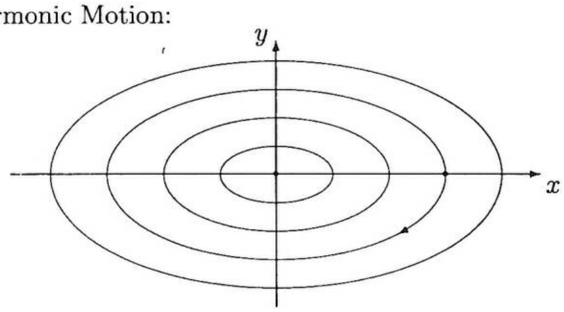


Simple Harmonic Motion



$$\ddot{x} = -\omega^2 x = -\frac{\mathrm{d}V}{\mathrm{d}x}$$
$$\tfrac{1}{2}v^2 = -\tfrac{1}{2}\omega^2 x^2 + E = -V(x) + E$$
$$y^2 = -\omega^2 x^2 + 2E$$

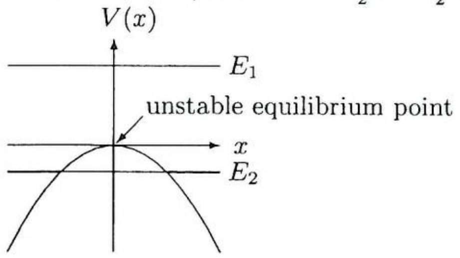
Phase Plane for Simple Harmonic Motion:

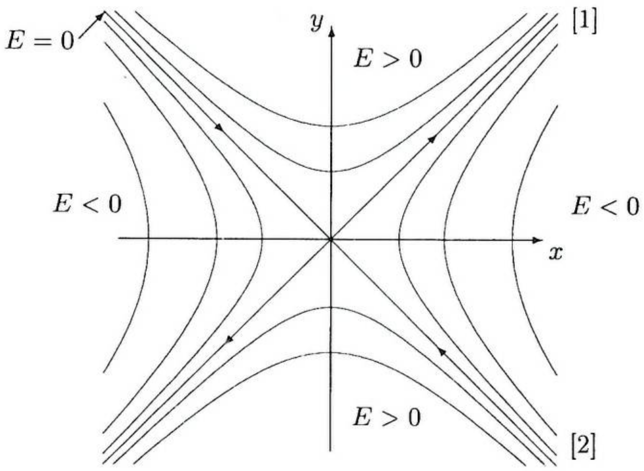


This is a **CENTRE** and is **STABLE**. The behaviour of a system close to a stable equilibrium always looks like this.

Hyperbolic Motion.

$\ddot{x} = \omega^2 x = -\mathrm{d}V/\mathrm{d}x, \qquad \frac{1}{2}v^2 = \frac{1}{2}\omega^2 x^2 + E = -V(x) + E.$





This is a **SADDLE POINT** and is **UNSTABLE**. The behaviour of a system close to an unstable equilibrium always looks like this.

2.7 Using Energy equation

$$v^2 = -2V(x) + 2E$$

$$= \frac{-2kx}{x^2+b^2} - \frac{4k}{5b} = \frac{(-2k)(5bx + 2x^2 + 2b^2)}{(5b)(x^2+b^2)}$$

$$\Rightarrow v = \frac{dx}{dt} = \pm\sqrt{\frac{(-2k)(5bx + 2x^2 + 2b^2)}{(5b)(x^2+b^2)}}$$

Supposing particle is travelling to the right we take the positive velocity

$$T = 2\int_{-2b}^{-b/2} \sqrt{\frac{5b}{-2k}\,\frac{x^2+b^2}{5bx + 2x^2 + 2b^2}}\,dx \quad \text{- period of oscillation}$$

let $x = by$

$\Rightarrow dx = b\,dy$

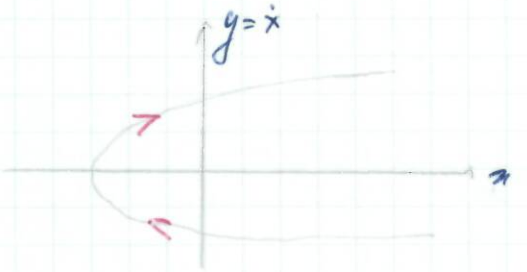
$$\Rightarrow T = 2\int_{-2}^{-1/2} \sqrt{\frac{5b}{-2k}}\sqrt{\frac{-(1+y^2)b^2}{(5y+2y^2+2)b^2}}\,b\,dy =$$

$$= \frac{2\sqrt{5}\cdot b^{3/2}}{\sqrt{2k}} \int_{-2}^{-1/2} \sqrt{\frac{-(1+y^2)}{(5y+2y^2+2)}}\,dy \simeq \frac{11.5\sqrt{b^3}}{\sqrt{k}}$$

compared with to $T_{SHM} = \frac{2\pi\sqrt{2b^2}}{\sqrt{k}} \simeq \frac{8.89\sqrt{b^3}}{\sqrt{k}}$

1) Since $a > 0$, the force is always positive $\Rightarrow \ddot{x} > 0$

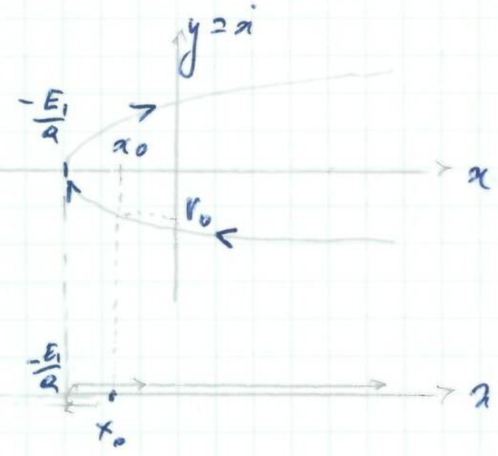
So velocity increase with time.



The arrows point to the left in the lower half plane and to the right in the upper half.

Suppose initially $v_0 < 0$ and the particle starts at x_0

This gives a particular Energy $E_1 = \frac{1}{2} v_0^2 - a x_0$



at $t = 0$, particle at x_0

as t increase it follows the parabola corresponding to energy E_1

. Projection of the parabola.

2) Simple Harmonic Motion

Recall that for $E_1 \leq 0$ there is no motion while for energy $E_2 > 0$ the particle oscillates on the bold line. (K. 12.1)

Energy equation for SHM gives

$$\frac{1}{2} v^2 = -\frac{1}{2} \omega^2 x^2 + E$$

$$\Rightarrow v = y = \dot{x} \quad \text{gives} \quad \dot{x}^2 + \omega^2 x^2 = 2E$$

Plotting the phase plane for various Energies $E \geq 0$ this gives ellipses centred at $(0,0)$.

Q Do arrows go clockwise or anticlockwise?

Suppose that $\dot{x} = 0$ when $x = a$

$E = \frac{1}{2} \omega^2 a^2$. The particle will then move back towards $x = 0$ $\Rightarrow$ arrows $\rightarrow$ clockwise

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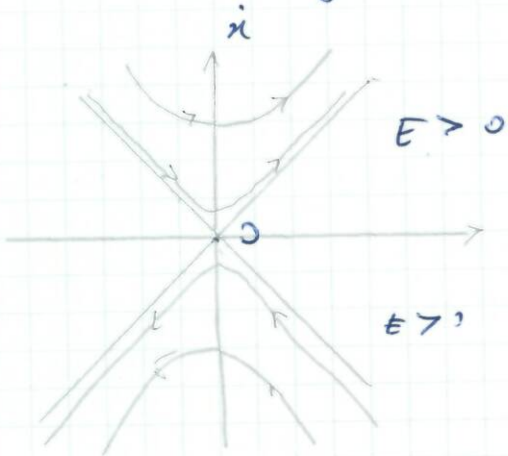
Phase Plane near a point of unstable equilibrium

Case 1 Assume $E > 0$

(i) If the initial velocity is positive (>0) the particle travels to ∞.

ii) If the in. velocity is negative (<0) the particle travels to $-\infty$

Plot $v = \dot{x}$ against x



$$\dot{x}^2 = \omega^2 x^2 + 2E$$

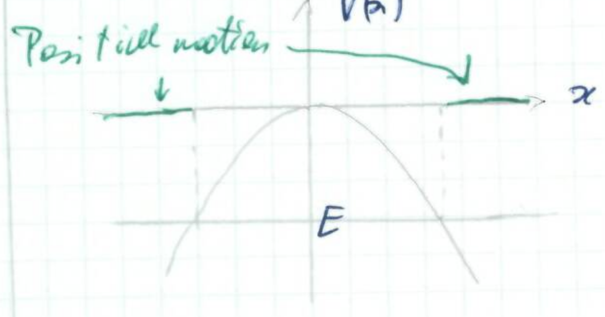
$$\Rightarrow \dot{x} = \pm\sqrt{\omega^2 x^2 + 2E}$$

Positive energy

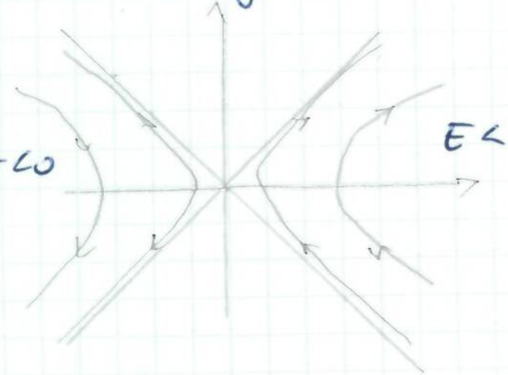
Case 2 Assume $E < 0$

(i) Depending on initial position, the particle either travels to $-\infty$ or $+\infty$

(ii) It cannot pass the point $x = 0$



3b) Plot $v = \dot{x}$ against x



$$\dot{x}^2 = \omega^2 x^2 + 2E$$

$$\Rightarrow \dot{x} = \pm\sqrt{\omega^2 x^2 + 2E}$$

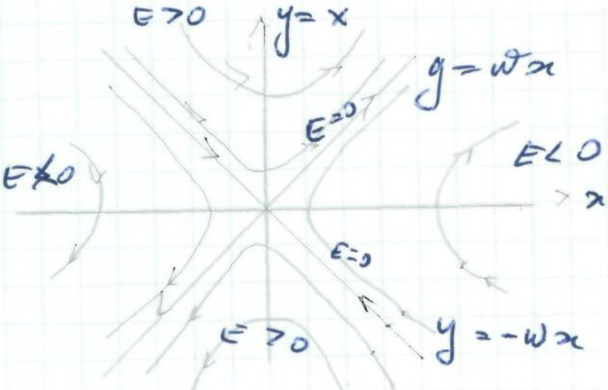
if $E < 0$, then $\dot{x} = 0$, when

$$x^2 = \frac{-2E}{\omega^2} > 0$$

Case 3 Assume $E = 0$

since $E = 0$, $\frac{1}{2}\dot{x}^2 = \frac{1}{2}\omega^2 x^2 \Rightarrow \dot{x} = \pm\omega x$

positive part: $x(t) = Ae^{\omega t} \Rightarrow \dot{x}(t) = A\omega e^{\omega t} = \omega x(t)$



The particle accelerates away from the equilibrium point $x = 0 = \dot{x}$.

negative part: $x(t) = Be^{-\omega t} \Rightarrow \dot{x}(t) = -B\omega e^{-\omega t} = -\omega x(t)$

As $t \to \infty$, x and $\dot{x} \to 0$

$$E = \frac{1}{2}v_0^2 - \frac{1}{2}\omega^2 x_0^2$$

To tend to zero need $\omega^2 x_0^2 = v_0^2$ for say $x_0 > 0$ and $v_0 < 0$

This is ~~known~~ as a saddle point and the phase plane near any point of unstable equilibrium looks like this.

5.5. Further SHM

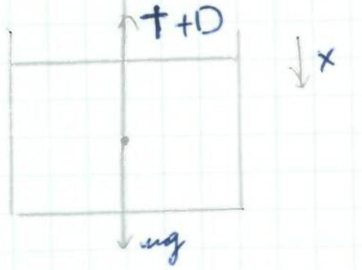
SHM was just $\ddot{x} + \omega^2 x = 0$

$$\Rightarrow x = A \sin(\omega t - \delta)$$

Example Damped SHM

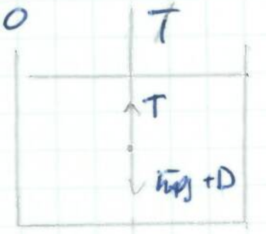
Suppose a particle of mass m is suspended on the end of a light elastic string. Also assume the particle is immersed in a bath of oil providing a resistance proportional velocity.

$\dot{x} > 0$



Oil produces drag force $D \propto$ to $\dot{x}$ where x is the displacement from the equilibrium position

$\dot{x} < 0$



In equilibrium $(x = 0)$

$T = mg \Rightarrow \frac{\lambda e}{l} = mg$

$\Rightarrow$ extension e in equilibrium is $\frac{l\,mg}{\lambda}$

For a general position x, the tension at x is

$$T = \frac{\lambda(\text{extension at } x)}{l} = \frac{\lambda}{l}\left(\frac{l\,mg}{\lambda} + x\right) = mg + \frac{\lambda x}{l}$$

The equation of motion (using $F = ma$) is

32 is $m\ddot{x} = -T + mg - k\dot{x}$ (for $k>0$)

$$= \frac{-\lambda x}{l} - mk\dot{x}$$

$$\Rightarrow \ddot{x} + k\dot{x} + \frac{\lambda}{ml} x = 0$$

Write $\frac{\lambda}{ml} = \omega^2 > 0$

So $\ddot{x} + k\dot{x} + \omega^2 x = 0$

To solve this write down the auxiliary equation ~~μ^2+~~

$$\mu^2 + k\mu + \omega^2 = 0$$

$$\Rightarrow \mu = -\frac{k}{2} \pm \frac{\sqrt{k^2-4\omega^2}}{2} = -\frac{k}{2} \pm \sqrt{\frac{k^2}{4} - \omega^2}$$

case 1 weak damping $\omega^2 > \frac{k^2}{4}$ let $r^2 = \omega^2 - \frac{k^2}{4} > 0$

then $\mu = \frac{-k}{2} \pm ir$ and $x(t) = e^{-\frac{kt}{2}}(A\cos rt + B\sin rt)$

$$= A' e^{-\frac{kt}{2}} \sin(rt+\delta)$$

where $A' = \sqrt{A^2+B^2}$

Case 2 (3?) Strong Damping:

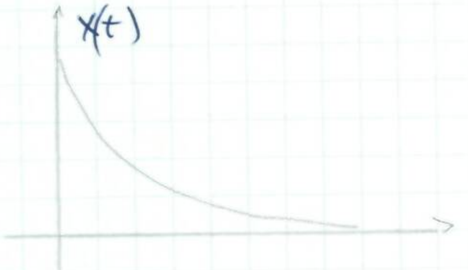
$\omega^2 < \frac{k^2}{4}$ let $s^2 = \frac{k^2}{4} - \omega^2 > 0$

So $\mu = \frac{-k}{2} \pm s$ and notice that μ is always -ve ~~(<0)~~

$s^2 < \frac{k^2}{4}$

$\Rightarrow x(t) = \alpha e^{-(\frac{k}{2} - s)t} + \beta e^{-(k/2 + s)t}$ for constants α, β

$s < \frac{k}{2}$



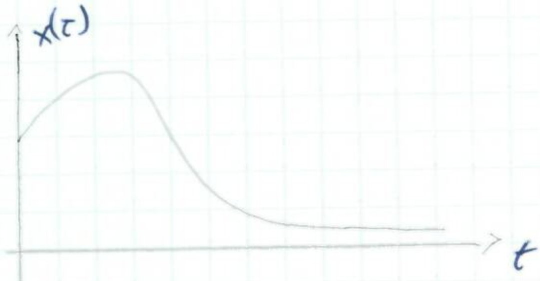
There's no oscillation in this case.

Case 3 Critical Damping

$\omega^2 = \frac{k^2}{4}$

The solution is $x(t) = (A + Bt) e^{-\frac{kt}{2}}$

$\mu = \frac{-k}{2}$



Tends to zero, with no oscillation but not as fast as strong damping.

Forced SHM and Resonance

Look at the SHM with an external force (periodic)

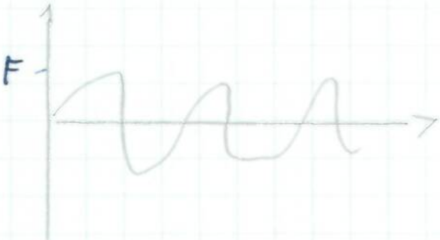
$$\boxed{\ddot{x} + \omega^2 x = F\cos(pt)}$$

This has solution $x(t) = A\sin(\omega t - \delta) + \dfrac{F\cos(pt)}{\omega^2 - p^2}$

for some constants A and δ

As long as $\omega \neq p$, i.e. the natural frequency ω does not equal the frequency (forcing) p the solution works and $x(t)$ is bounded for all times t

$$|x(t)| \leq |A| + \frac{|F|}{|\omega^2 - p^2|}$$



If $\omega = p$ the solution doesn't work. Instead we can solve

$$\dot{x} + \omega^2 x = Fe^{i\omega t}$$

Try $x(t) = \gamma t e^{i\omega t}$

$$\Rightarrow \dot{x}(t) = \gamma e^{i\omega t}$$

$$\Rightarrow \ddot{x}(t) = 2i\omega\gamma e^{i\omega t} - \omega^2 \gamma t e^{i\omega t}$$

So $\ddot{x} + \omega^2 x = 2i\omega\gamma e^{i\omega t}$

So we should choose $\gamma = \dfrac{F}{2i\omega} = \dfrac{-iF}{2\omega}$

The original problem is $\ddot{x} + \omega^2 x = \mathrm{Re}(Fe^{i\omega t})$

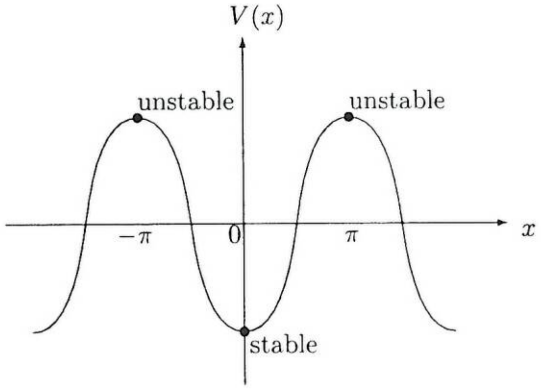
So the solution is $x(t) = A'\sin(\omega t - \delta) + \mathrm{Re}\left(\dfrac{-iF}{2\omega} e^{i\omega t}\right)$

$$\Rightarrow x(t) = A'\sin(\omega t - \delta) + \frac{Ft}{2\omega}\sin(\omega t)$$

This solution is unbounded: Resonance.

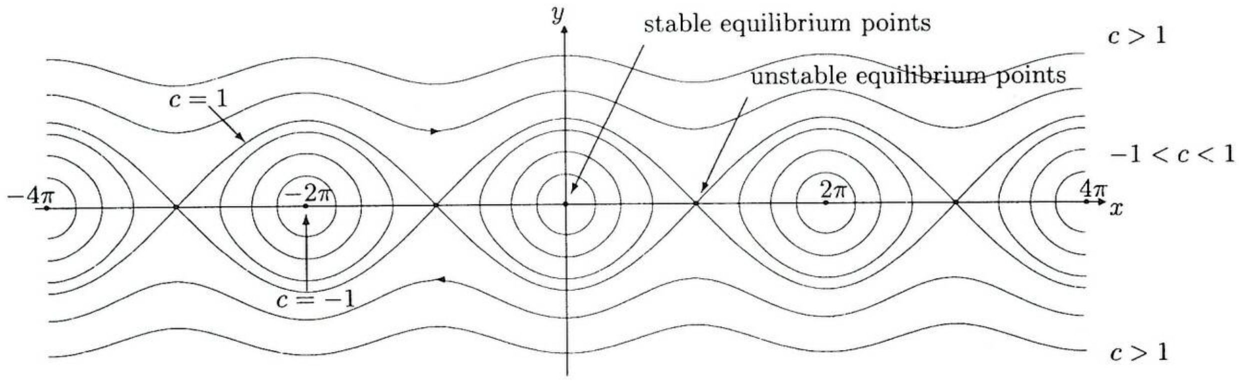
Handout 13: Pendulum Phase Plane

The potential function for the pendulum (using $\theta = x$) is $V(x) = -(g/l)\cos x$:



For the phase plane, if we are plotting $y = \dot{\theta}$ against $x = \theta$, we had derived:

$$y = \pm 2\sqrt{\frac{g}{l}}\,(c + \cos x)\,.$$



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Total force $\underline{F} = \underline{j}\,(T\cos\theta - mg) - \underline{i}\,T\sin\theta$

Now use $\underline{F} = m\underline{a}$ and compare components:

$$\underline{i}: \quad -T\sin\theta = m\ddot{\theta}\, l\cos\theta - m\dot{\theta}^2 l\sin\theta \qquad (1)$$

$$\underline{j}: \quad T\cos\theta - mg = m\ddot{\theta}\, l\sin\theta + m\dot{\theta}^2 l\cos\theta \qquad (2)$$

$$(1)\ \cos\theta \Rightarrow -T\sin\theta\cos\theta = m\ddot{\theta}\, l\cos^2\theta - m\dot{\theta}^2 l\sin\theta\cos\theta$$

$$(2)\ \sin\theta \Rightarrow T\sin\theta\cos\theta - mg\sin\theta = m\ddot{\theta}\, l\sin^2\theta + m\dot{\theta}^2 l\sin\theta\cos\theta$$

adding (1) and (2):

$$-mg\sin\theta = m\ddot{\theta} l\,(\cos^2\theta + \sin^2\theta)$$

$$\Rightarrow \ddot{\theta} = -\left(g/l\right)\sin\theta$$

Similarly can eliminate $\ddot{\theta}$ to get

$$T = ml\dot{\theta}^2 + mg\cos\theta$$

$$\ddot{\theta} = \left(-g/l\right)\sin\theta$$

If θ is very small then $\sin\theta \approx \theta$

So the equation of motion for small angles is roughly

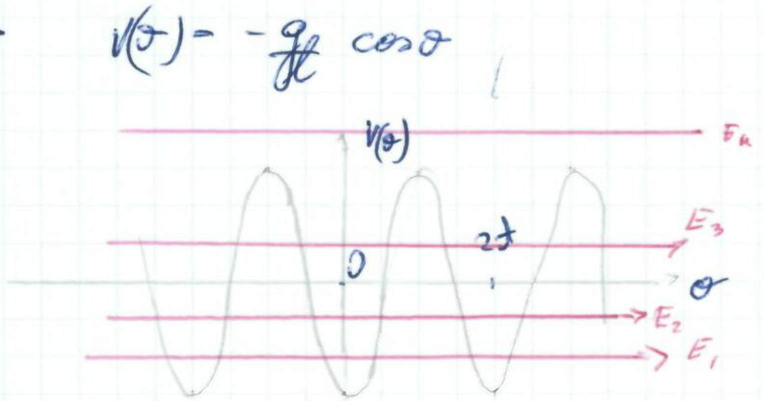
$$\ddot{\theta} = -\frac{g}{l}\theta$$

which is SHM with period $T = \frac{2\pi}{\sqrt{g/l}}$

$$\ddot{\theta} = -g/l\,\sin\theta$$

$$\Rightarrow \frac{1}{2}\dot{\theta}^2 = \frac{g}{l}\cos\theta + E = -V(\theta) + E$$

$$\Rightarrow \quad V(\theta) = -\frac{g}{l}\cos\theta$$



- At E_1 we have small oscillations which is nearly SHM
- E_2 gives larger oscillations but not SHM.
- At E_3 we still have large oscillation but as $\theta > \frac{\pi}{2}$ in some part of the motion we should check $T > 0$.
- E_4 = full rotations

Phase plane.

$$\frac{1}{2}\dot{\theta}^2 = g/l \cos\theta + E$$

So we plot $\frac{1}{2} y^2 = g/l \cos x + E = g/l\,(\cos x + c)$

$$c = El/g$$

If $\underline{c=-1}$ $\quad \frac{1}{2} y^2 = g/l\,(\cos x - 1)$

$\cos x - 1 \geq 0 \iff x = 0, \pm 2\pi, \pm 4\pi, \dots$

in which case $y = 0$

$\underline{c = 1}$ $\quad \frac{1}{2} y^2 = g/l\,(\cos x + 1)$

$\Rightarrow y = 0 \iff \cos x = -1 \iff x = \pm\pi, \pm 3\pi, \dots$

$\cancel{y = \sqrt{\frac{2g}{l}}\sqrt{\cos x + c}}$

$$\frac{1}{2} y^2 = \frac{g}{l}(\cos x + 1)$$

$$= \frac{2g}{l}\left(\cos\left(\frac{x}{2}\right)\right)$$

$$\Rightarrow y = \pm 2\sqrt{\frac{g}{l}} \cos\left(\frac{x}{2}\right)$$