1301 Applied Mathematics Notes

Based on the 2015 autumn lectures by Prof R Halburd

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1301 - Aplied Mathematics

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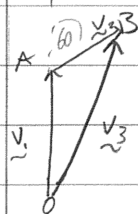
Exam = 90% of module
Coursework = 5% of module
Mid Sessional = 5% of module
Lecture notes on moodle

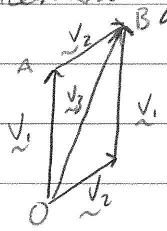
1. Newton's Laws

* Scalar: a quantity described by a single number
  e.g. mass, distance, time, speed, temperature, ...
* Vector: a quantity has magnitude (size) and direction
  e.g. displacement, velocity, force, ...

Example: An ant walks due north for 2cm from its initial starting point (called "O"). The vector $\vec{OA} = \underset{\sim}{v_1}$ describing the displacement can be represented by an arrow 2cm long pointing north.
Suppose the ant now moves 1cm in a direction 60° east of north. 2nd displacement: $\underset{\sim}{v_2} = \vec{AB}$
Final displacement from O: $\vec{OB}$ =



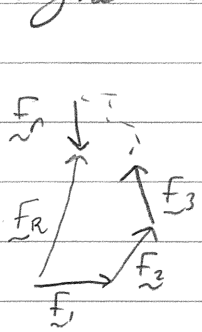
$\underset{\sim}{v_3} = \underset{\sim}{v_1} + \underset{\sim}{v_2}$

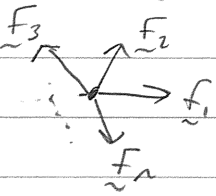


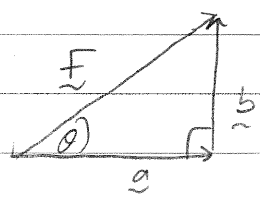
$\underset{\sim}{v_3} = \underset{\sim}{v_1} + \underset{\sim}{v_2} = \underset{\sim}{v_2} + \underset{\sim}{v_1}$

Components of vectors:

Imagine we have forces $\underset{\sim}{F_1}, \dots \underset{\sim}{F_n}$ acting on a point:







↑vert. → horiz.

components

$\underset{\sim}{F} = \underset{\sim}{a} + \underset{\sim}{b}$

$$\Rightarrow \begin{cases} \underset{\sim}{a} = |\underset{\sim}{F}|\cos\theta \\ \underset{\sim}{b} = |\underset{\sim}{F}|\sin\theta \end{cases}$$

where $|\underset{\sim}{F}|$ is the magnitude of $\underset{\sim}{F}$.

Forces and Newton's Laws (point particles)

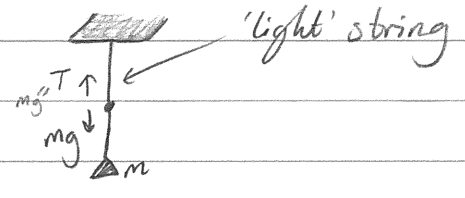
* 1st Law: Every body continues in its state of rest, or uniform motion, unless it is compelled to change that state by forces impressed upon it.

* 2nd Law: The acceleration of a body is parallel and directly proportional to the net force, $\underset{\sim}{F}$, and inversely proportional to the mass, m.
$\underset{\sim}{F} = m\underset{\sim}{a}$ $\quad (1\,N = 1\,kg \times 1\,ms^{-2})$

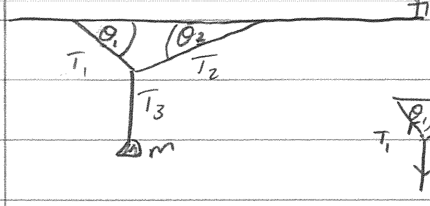
* 3rd Law: To every action there is an equal and opposite reaction.

Near the Earth's surface any mass (in vac) accelerates downwards at a rate $g = 9.8\,ms^{-2}$
Mass, m, is acted on by a force, $F = mg$ called the weight.

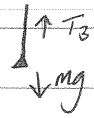
Tension



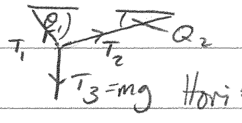
Example: A weight of mass, m, is suspended from a ceiling using 3 light cables as shown in the diagram. find the tensions T_1, T_2, T_3,



Firstly consider m:



so $\underline{T_3 = mg}$



Horiz: $T_1 \cos\theta_1 = T_2 \cos\theta_2$ ①

Ver: $T_1 \sin\theta_1 + T_2 \sin\theta_2 = T_3 = mg$ ②

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① $T_1\cos\theta_1 - T_2\cos\theta_2 = 0$ ③

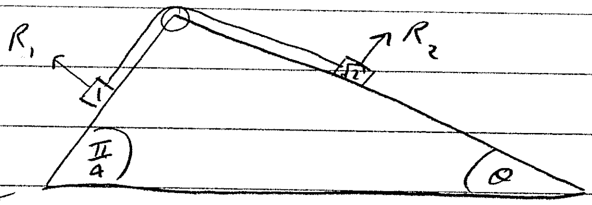
② $\cos\theta_2 + 3\sin\theta_2$

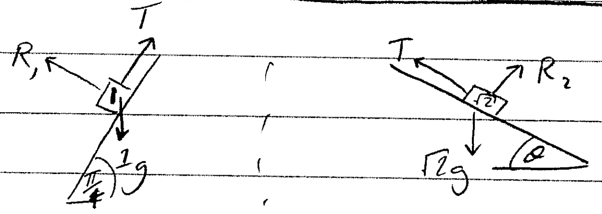
$$T_1(\sin\theta_1\cos\theta_2 + \cos\theta_1\sin\theta_2) = mg\cos\theta_2$$

$$\Rightarrow T_1 = \frac{mg\cos\theta_2}{\sin(\theta_1+\theta_2)}$$

$$\Rightarrow T_2 = \frac{mg\cos\theta_1}{\sin(\theta_1+\theta_2)} \quad \text{(by symmetry)}$$

Example: 2 blocks of mass 1kg and $\sqrt{2}$ kg are connected by a light string, which is then passed over a frictionless pulley and the blocks rest on 2 frictionless inclined planes, as shown in the diagram. Find θ s.t. the system is in equilibrium. Find the magnitudes of the reaction forces exerted on the blocks by the inclined planes.





// to plane:

$T = g\sin\frac{\pi}{4}$

$T = \frac{g}{\sqrt{2}}$

⊥ to plane:

$R_1 = g\cos\frac{\pi}{4}$

$R_1 = \frac{g}{\sqrt{2}}$

// to plane:

$T = \sqrt{2}\,g\sin\theta$

$\Rightarrow \sin\theta = \frac{1}{2} \Rightarrow \theta = \frac{\pi}{6}$

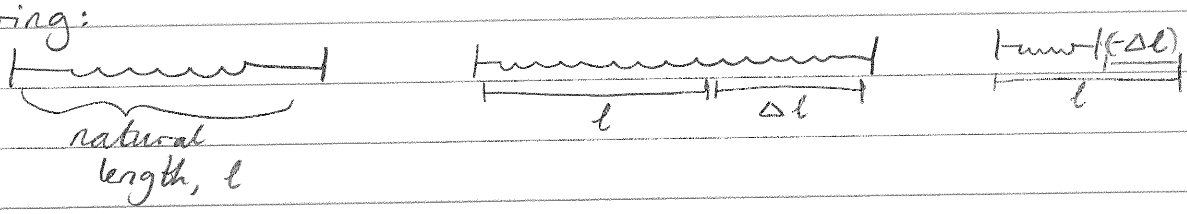
⊥ to plane:

$R_2 = \sqrt{2}\,g\cos\theta = \sqrt{2}\,g\cos\frac{\pi}{6}$

$R_2 = \sqrt{\frac{3}{2}}\,g$

* Hooke's Law

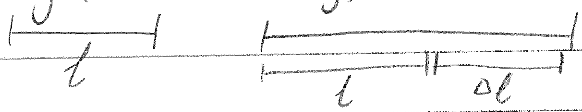
spring:



The extension of a spring is in direct proportion to the load applied to it.

$$\text{Tension} = \underset{\uparrow \text{ spring constant}}{k}\,\Delta l \quad \leftarrow \text{Hooke's Law}$$

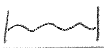
string (elastic string):



Hooke's law : $T = k\,\Delta l$

$= \frac{\lambda}{l}\Delta l$

where λ = modulus of elasticity.



An elastic string can go slack!

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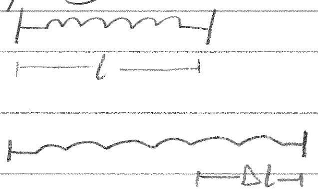
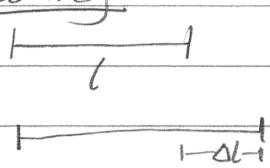
Spring



$T = k\Delta l$ (Hooke's law)
↑spring constant

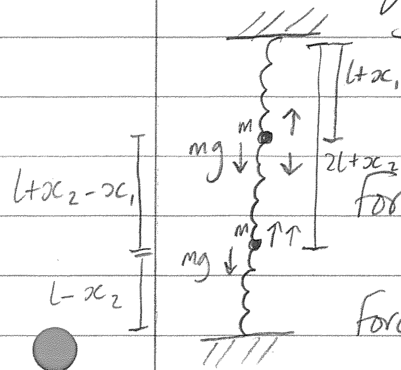
*Force exerted by spring $F = -k\Delta l$

String



$T = k\Delta l = \frac{\lambda}{l}\Delta l$ λ = modulus of elasticity.

Example: Three light springs, each of natural length l and spring constant k, are arranged vertically between 2 points a distance $3l$ apart.
One end of the first spring is fixed to a point on the ceiling. A weight of mass m is connected between the other end of the 1st spring and the upper end of the second. Another weight of mass m is attached between the 2nd and 3rd springs and the lower end of the 3rd spring is attached to the floor. Find equilibrium positions of the weights.



Soln: Let the upper weight be a distance $l + x_1$ below the ceiling and let the lower weight be a distance $2l + x_2$ below the ceiling.

Forces on upper mass: $mg + k(x_2 - x_1) = kx_1$ ①

Forces on lower mass: $mg = k(x_2 - x_1) + kx_2$ ②

①: $2x_1 - x_2 = \frac{mg}{k}$

②: $x_1 - 2x_2 = -\frac{mg}{k}$

①+②: $3x_1 - 3x_2 = 0$, $x_1 = x_2 = \frac{mg}{k}$

<u>One dimensional motion</u>

A force F acting on a particle of mass m at position x.

$F = ma$

Position $= x$, velocity $= v = \dot{x} = \frac{dx}{dt}$,

acceleration $a = \frac{dv}{dt} = \frac{d^2x}{dt^2} = \ddot{x}$

$ma = F$

$m\ddot{x} = F(t, x, \dot{x})$ ← Equation of motion

$a = \frac{dv}{dt} = \frac{dv}{dx} \times \frac{dx}{dt} = v\frac{dv}{dx}$

1. Constant force ($\equiv$ constant acceration), $a = \text{const.}$

$\frac{dv}{dt} = a \Rightarrow v = at + c_1$ (c_1 = initial velocity).

$\therefore$ If at $t=0$, $v=u$, $v = u + at$.

$\frac{dx}{dt} = v = u + at \Rightarrow x = ut + \frac{1}{2}at^2 + c_2$ (*)

$\therefore$ If at $t=0$, the particle is at $x = x_0$, then $c_2 = x_0$ (sub into (*)), so $x = x_0 + ut + \frac{1}{2}at^2$

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2). Force as a function of t only.

Example: The time-dependent force $F = F(t) = mF_0 \sin\omega t$ acts on a particle of mass m, wher F_0 & ω are constants. At time $t=0$, the particle passes $x=0$ with velocity v_0. Show that the particle will eventually move to $+\infty$ if and only if $v_0 > -\frac{F_0}{\omega}$.

Soln: $ma = F$

$$\cancel{m}\frac{dv}{dt} = \cancel{m} F_0 \sin\omega t$$

$$\Rightarrow \quad v = -\frac{F_0}{\omega}\cos\omega t + c$$

At $t=0$, $v = v_0$ i.e. $v_0 = -\frac{F_0}{\omega} + c_1$

$$\Rightarrow v(t) = -\frac{F_0}{\omega}\cos\omega t$$

Integrate: $x = -\frac{F_0}{\omega^2}\sin\omega t + \left(v_0 + \frac{F_0}{\omega}\right)t + c_2$

At $t=0$, $x=0$, $c_2 = 0$.

$$x(t) = -\frac{F_0}{\omega^2}\sin\omega t + \left(v_0 + \frac{F_0}{\omega}\right)$$

Since $-1 \leq \sin\omega t \leq 1$

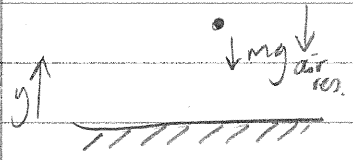
$$(v_0\omega^{-1}F_0)t - \omega^2 F_0 \leq x \leq (v_0 + \omega^{-1} + F_0)t + \omega^{-2}F_0.$$

If $v_0\omega^{-1}F_0 > 0$, first inequality shows that $x \to +\infty$. If $x \to +\infty$, Second inequality shows that $v_0 + \omega^{-1}F_0 > 0$.

3). Velocity dependent forces.

Example: A ball is launched upwards from ground level with initial velocity v_0. The air resistance on the ball is kv^2 per unit mass, where v is the velocity of the ball and k is a constant. Find the maximum height of the ball.

Soln: let y be the height of the ball above the ground.



$$m\ddot{y} = -mg - kv^2 m$$

$$\Leftrightarrow \quad \ddot{y} = -g - kv^2$$

$$v\frac{dv}{dy} = -g - kv^2$$

$$\frac{v}{g+kv^2}\frac{dv}{dy} = -1$$

$$\int \frac{v}{g+kv^2}\frac{dv}{\cancel{dy}}\cancel{dy} = \int -1\, dy$$

$$\int_0^h 1\, dy = \frac{-1}{2k}\int_{v_0}^{0} \frac{2kv}{g+kv^2}\, dv$$

$$\Rightarrow \left[y\right]_0^h = \frac{-1}{2k}\left[\ln\left(g+kv^2\right)\right]_{v_0}^{0}$$

$$h = \frac{1}{kk}\ln\left(\frac{g+kv_0^2}{g}\right)$$

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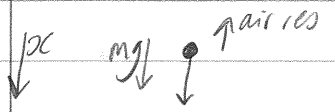
Example: A ball of mass m is thrown downwards at time $t=0$ with velocity $v_0 \geq 0$. Air resistance per unit mass is kv^2. What is the speed at time $t=T$?



Soln: $\cancel{m}\ddot{x} = \cancel{m}g - \cancel{m}kv^2$

$$\ddot{x} = g - kv^2 \Rightarrow \frac{dv}{dt} = g - kv^2$$

let $\alpha = \sqrt{g/k}$. If $v_0 = \alpha$, $v = \alpha$ for all time.
If $v_0 \neq \alpha$, v is a continuous function (small changes in time result in small changes in v)
So $v \neq \alpha$ for small t.

$$\frac{dv}{dt} = g - kv^2$$

$$\int \frac{dv}{g-kv^2} = \int dt \qquad \int \frac{dv}{k(\frac{g}{k} - v^2)}$$

$$\Leftrightarrow \int \frac{dv}{\alpha^2 - v^2} = \int k\, dt$$

$$k\int_0^T dt = \int_{v_0}^{v} \frac{dw}{\alpha^2 - w^2} = \frac{1}{2\alpha}\int_{v_0}^{v} \frac{1}{\alpha - w} + \frac{1}{\alpha + w}\, dw$$

$$\Rightarrow \quad kT = \frac{1}{2\alpha}\Big[\ln|\alpha + w| - \ln|\alpha - w|\Big]_{v_0}^{v}$$

$$= \frac{1}{2\alpha}\left[\ln\left|\frac{\alpha + w}{\alpha - w}\right|\right]_{v_0}^{v} = \frac{1}{2\alpha}\ln\left|\left(\frac{\alpha+v}{\alpha-v}\right)\left(\frac{\alpha - v_0}{\alpha + v_0}\right)\right|$$

$$\Rightarrow \left|\frac{\alpha + v}{\alpha + v_0} \cdot \frac{\alpha - v_0}{\alpha - v}\right| = e^{2\alpha k T}$$

Since $e^{2\alpha kT}$ is finite for all finite T, we see that $v \neq \alpha$ for all finite T.

Initially $v = v_0 \neq \alpha$, $v - \alpha$ has the same sign as $v_0 - \alpha$.

$$\therefore \frac{v+\alpha}{v-\alpha} \cdot \frac{v_0 - \alpha}{v_0 + \alpha} = e^{2\alpha kT}$$ as we can remove the modulus signs.

$$\Rightarrow \underline{\underline{v = \alpha \cdot \frac{(\alpha + v_0)e^{2\alpha kT} + (v_0 - \alpha)}{(\alpha + v_0)e^{2\alpha kt} - (v_0 - \alpha)}}} = \alpha \cdot \frac{1 + \left(\frac{v_0 - \alpha}{v_0 + \alpha}\right)e^{-2\alpha kT}}{1 - \left(\frac{v_0 - \alpha}{v_0 + \alpha}\right)e^{-2\alpha kT}}$$

$\therefore \alpha =$ terminal velocity.

Example: Consider a particle subject to a constant force plus a resistive force proportional to velocity.

$$\ddot{x} = P - \lambda \dot{x}$$

this sign works for $\dot{x} > 0$ and $\dot{x} < 0$.

$$\frac{dv}{dt} = P - \lambda v \qquad \text{terminal velocity } v = \frac{P}{\lambda}$$

$$\int \frac{dv}{v - \frac{P}{\lambda}} = -\int \lambda \, dt$$

$$\Rightarrow -\lambda t = \ln\left|v - \frac{P}{\lambda}\right| + c_1$$

If at $t = 0$, $v = v_0$, $c_1 = -\ln\left|v_0 - \frac{P}{\lambda}\right|$

$$\Rightarrow -\lambda t = \ln\left|\frac{v - \frac{P}{\lambda}}{v_0 - \frac{P}{\lambda}}\right|$$

$$\Rightarrow -\lambda t = \ln\left|\frac{v - \frac{P}{\lambda}}{v_0 - \frac{P}{\lambda}}\right| \Rightarrow \underline{\underline{v = \frac{P}{\lambda} + \left(v_0 - \frac{P}{\lambda}\right)e^{-\lambda t}}}$$

[as $t \to \infty$, $v \to \frac{P}{\lambda}$]

$$\Rightarrow \frac{dx}{dt} = v = \frac{P}{\lambda} + \left(v_0 - \frac{P}{\lambda}\right)e^{-\lambda t}$$

Integrate: $x = \frac{P}{\lambda}t + \frac{1}{\lambda}\left(\frac{P}{\lambda} - v_0\right)e^{-\lambda t} + c_2$

If $x = x_0$ at $t = 0$, then $c_2 = x_0 + \frac{1}{\lambda}\left(v_0 - \frac{P}{\lambda}\right)$

$$x(t) = \frac{P}{\lambda}t + x_0 + \frac{1}{\lambda}\left(v_0 - \frac{P}{\lambda}\right)\left(1 - e^{-\lambda t}\right)$$

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$$ma = F$$

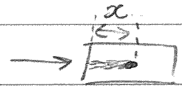
$$a = \frac{dv}{dt} = \frac{d^2x}{dt^2} \quad \text{or} \quad a = v\frac{dv}{dx}$$

Example: (2011-2012)

A bullet is fired horizontally with initial speed u into a block of wood which provides a drag of kv per unit mass, where k is a constant and v is the speed of the bullet. Ignore gravity.

(a) what is the minimum width of block such that the bullet never escapes?

(b) how long does it take the bullet to reach half this distance.



(a) $ma = F$

$ma = -kvm$

$\Rightarrow \frac{dv}{dt} = -kv$

$\Rightarrow \int \frac{dv}{v} = \int -k\,dt$

$\ln|v| = -kt + c$

$\Rightarrow v = Ae^{-kt}$

at $t=0$, $v = u \Rightarrow A = u$

$\Rightarrow v = ue^{-kt}$

$\Rightarrow \frac{dx}{dt} = ue^{-kt}$

$\Rightarrow \int dx = \int ue^{-kt}\,dt$

$\Rightarrow x = -\frac{u}{k}e^{-kt} + c$

$x=0$ at $t=0$ (when bullet enters block), $c = \frac{u}{k}$

$\Rightarrow x = \frac{u}{k}\left(1 - e^{-kt}\right)$

As $t \to \infty$, $x \to \frac{u}{k}$

So $\frac{u}{k}$ is the minimum thickness.

(b) Time T to reach $x = \frac{1}{2}\frac{u}{k}$.

$$\frac{1}{2}\frac{u}{k} = \frac{u}{k}\left(1-e^{-kT}\right)$$

$$\Rightarrow \frac{1}{2} = 1 - e^{-kT}$$

$$\Rightarrow e^{-kT} = \frac{1}{2}$$

$$\Rightarrow e^{kT} = 2$$

$$\Rightarrow kT = \log 2$$

$$\Rightarrow T = \frac{1}{k}\log 2$$

Ex cont: Now suppose that two bullets are fired simultaneously towards each other with initial speeds u_1 and u_2 from opposite sides of the block a distance l apart.

(c) Find a necessary and sufficient condition for the bullets to collide.

$$\left(\frac{u_1}{k} + \frac{u_2}{k}\right) > l$$

(d) If this condition is met, when and where do the particles collide?

collision $\Leftrightarrow$ sum of 2 distances travelled $= l$

$$\frac{u_1}{k}\left(1-e^{-kt}\right) + \frac{u_2}{k}\left(1-e^{-kt}\right) = l$$

$$\Rightarrow 1-e^{-kt} = \frac{kl}{u_1+u_2} \quad \Rightarrow t = \frac{1}{k}\log\left(\frac{u_1+u_2}{u_1+u_2-kl}\right)$$

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① cont.

sub into $x = \frac{u_1}{k}\left(1 - e^{-kt}\right)$

$$x = \frac{u_1 l}{u_1 + u_2} \quad \text{(distance from left side)}$$

2. Position Dependent forces

$$ma = f(x)$$

$$mv\frac{dv}{dx} = f(x)$$

Integrating w.r.t. x:

$$\frac{1}{2}mv^2 = \int f(x)\,dx$$

A potential, V, for a force, F, is any function, V, satisfying $F(x) = -V'(x)$.

Eqn of motion:

$$mv\frac{dv}{dx} = -V'(x)$$

$$\Rightarrow \frac{1}{2}mv^2 = -V(x) + E$$ constant energy

Energy equation

$$E = \underbrace{\frac{1}{2}mv^2}_{\text{KE (moving)}} + \underbrace{V(x)}_{\text{potential}}$$

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$$ma = F(t, x, \dot{x})$$

Position-dependent forces.

$$ma = F(x)$$

$$mv\frac{dv}{dx} = F(x) = -V'(x)$$

$$\left(V(x) = -\int F(x)\,dx\right)$$

$$\Rightarrow \frac{1}{2}mv^2 = -V(x) + \underset{\text{energy}}{E}$$

Energy equation: $E = \frac{1}{2}mv^2 + V(x)$

Example: Particle under the influence of gravity.
Let y be the vertical displacement of a particle of mass m, then the gravitational force (weigh) is $-mg$.
Take the potential to be $V = mgy$
$\therefore$ Energy eqn: $E = \frac{1}{2}mv^2 + mgy$.

Example: A heavy ball falls off a table of height h. What is its speed as it hits the ground? (Ignore air resistance).

Soln: At $y = h$, $v = 0$:

$$E = \frac{1}{2}mv^2 + mgy = 0 + mgh$$

At the ground $y = 0$:

$$E = \frac{1}{2}mv^2 + mgy = \frac{1}{2}mv^2 + 0$$

$$\Rightarrow \cancel{m}gh = \frac{1}{2}\cancel{m}v^2$$

$$\Rightarrow v = \sqrt{2gh}$$ (speed as it hits the ground).

Simple harmonic motion

In SHM, the force on a particle acts towards some fixed point (take to be O) with a magnitude proportional to the distance from O.

i.e. $F = -m\omega^2 x$, $\quad \omega > 0$

$$V(x) = \int F \, dx$$

Potential: $V(x) = \frac{1}{2} m\omega^2 x^2$

$$E = \frac{1}{2} m v^2 + \frac{1}{2} m\omega^2 x^2$$

$$v^2 = \frac{2E}{m} - \omega^2 x^2$$

$$\left(\frac{dx}{dt}\right)^2 = \frac{2E}{m}\left(1 - \frac{m\omega^2}{2E} x^2\right)$$

$$\pm \int dt = \int \frac{\sqrt{\frac{m}{2E}} \, dx}{\sqrt{1 - \frac{m\omega^2}{2E} x^2}}$$

let $\quad u = \sqrt{\frac{m\omega^2}{2E}}$

$\quad du = \omega \sqrt{\frac{m}{2E}} \, dx$

$$\Rightarrow \pm \int dt = \omega^{-1} \int \frac{du}{\sqrt{1 - u^2}}$$

$$\pm \int dt = \omega^{-1} \sin^{-1} u = \omega^{-1} \sin^{-1}\left(\sqrt{\frac{m\omega^2}{2E}} \, x\right)$$

let $\varepsilon = \pm 1$, $A = \sqrt{\frac{2E}{m\omega^2}}$

$$x = \varepsilon A \sin(\omega t - c)$$

If $\varepsilon = +1$, choose $\varphi = c$

If $\varepsilon = -1$, choose $\varphi = \pi + c$

$$\Rightarrow \boxed{x = A \sin(\omega t - \varphi)}$$

$\varphi = \text{constant}$

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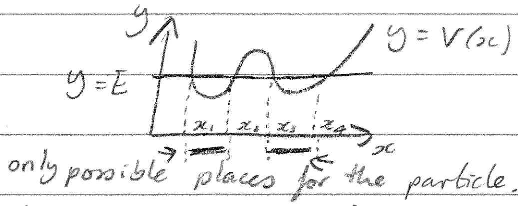
$\underline{A}$ is called the *amplitude*
$\underline{\varphi}$ is called the *phase*
$\underline{\omega}$ is called the *angular frequency*
The *period* $= \dfrac{2\pi}{\omega}$

Potentials:

$$E = \frac{1}{2}mv^2 + V(x)$$



$$v^2 = \frac{2}{m}\left(E - V(x)\right)$$

We must have $E \geq V(x)$ (otherwise $v^2 < 0$)

For E and $V(x)$ as shown in the diagram, the particle can only be found, either in the interval $x_1 \leq x \leq x_2$ or $x_3 \leq x \leq x_4$

Suppose that at some time, the particle is at $x = x_1$

$$v = \pm\sqrt{\frac{2}{m}\left(E - V(x)\right)} = 0,$$

so the particle is momentarily at rest (velocity $= 0$)

$$ma = F = -V'(x)$$

$$\text{acceleration } \ddot{x} = -\frac{1}{m}V'(x_1) > 0$$

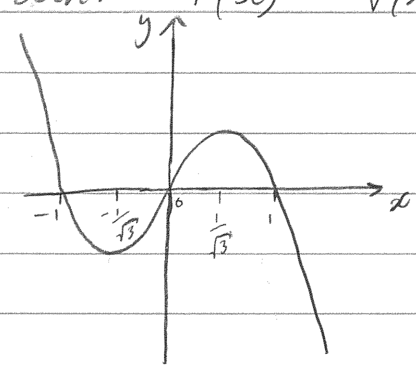
The particle starts moving to the right ($v > 0$). It can only stop or turn around by reaching a points where $v = 0$. It will continue moving to the right until $v = 0$ at $x = x_2$ ($E = V(x_2)$)

At $x = x_2$, acceleration $x = \ddot{x} = -\frac{1}{m} = \frac{1}{m}V'(x_2) < 0$
$\Rightarrow$ particle turns around and moves back to $x = x_1$
and repeats. $\leftarrow$ Periodic motion.

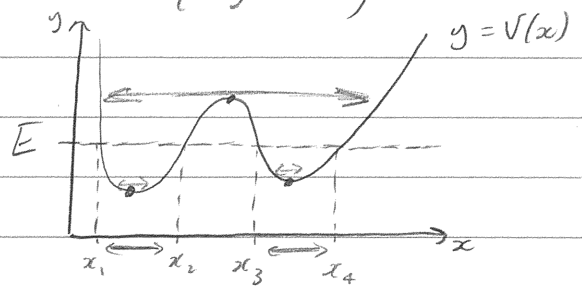
Example: A particle of unit mass moves in a potential given by $V(x) = x - x^3$. $= x(1+x)(1-x^2)$

1. Find the force acting at position x.

Soln: $F(x) = -V'(x) = 3x^2 - 1$



Equilibrium points are points where a particle is momentarily at rest, will stay at rest.
(ie. pts where $a = 0 \Leftrightarrow F = 0 \Leftrightarrow V'(x) = 0$)



If x is a minimum of V we call x a stable equilibrium.

If x is a maximum of V we call x an unstable equilibrium.

Example cont:

2. Classify each equilibrium point as stable or unstable.

Soln: $V'(x) = 1 - 3x^2 = 0$

$\Rightarrow x = \pm \frac{1}{\sqrt{3}}$

$x = -\frac{1}{\sqrt{3}}$ is a local minimum $\Rightarrow$ stable.

$x = +\frac{1}{\sqrt{3}}$ is a local maximum $\Rightarrow$ unstable.

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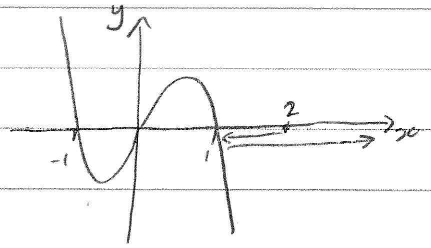
3. A particle is released from $x = 2$ with velocity $v = -2\sqrt{3}$. Describe the subsequent motion.

Soln:

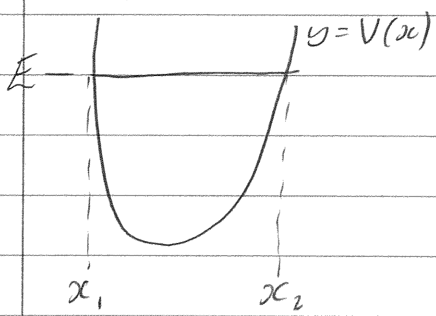
m = 1 (unit mass)

$x = 2$

$$E = \frac{1}{2}v^2 + V = \frac{1}{2}(2\sqrt{3})^2 + V(2) = 0$$



The particle moves to the left until it reaches $x = 1$. It then turns around and accelerates towards $+\infty$ (with increasing and unbounded speed)



Lets calculate the period of motion from x_1 to get back to x_1:

$$E = \frac{1}{2}mv^2 + V(x)$$

$$\left(\frac{dx}{dt}\right)^2 = \frac{2}{m}\left(E - V(x)\right)$$

$$\sqrt{\frac{m}{2}}\frac{dx}{\sqrt{E - V(x)}} = \pm dt$$

Time from x_1 to x_2:

$$T_1 = +\int_0^{T_1} dt = \sqrt{\frac{m}{2}}\int_{x_1}^{x_2} \frac{dx}{\sqrt{E - V(x)}}$$

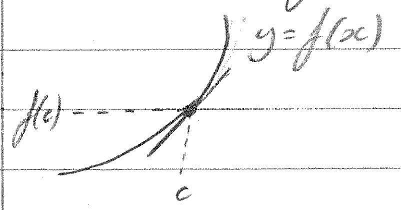
Time back, from x_2 to x_1:

$$T_2 = \int_0^{T_2} dt = -\sqrt{\frac{m}{2}} \int_{x_2}^{x_1} \frac{dx}{\sqrt{E - V(x)}} = T_1$$

Total period:

$$T = T_1 + T_2 = \sqrt{2m} \int_{x_1}^{x_2} \frac{dx}{\sqrt{E - V(x)}}$$

*Linearisation of a differentiable fn f:



$$f(x) \sim L_c(x) = f(c) + f'(c)(x - c)$$

Let c be a stable equilibrium pt of V, s.t.

$V''(c) > 0$

[e.g. $V(x) = x^4$ $V''(x) = 0$!]

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$$ma = F(x)$$

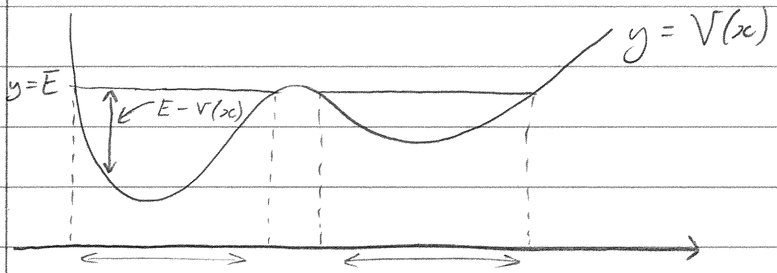
$$m\frac{dv}{dt} = F(x)$$

$$mv\frac{dv}{dx} = F(x) = -V'(x) \qquad \left[V(x) = -\int F(x)\,dx \right]$$

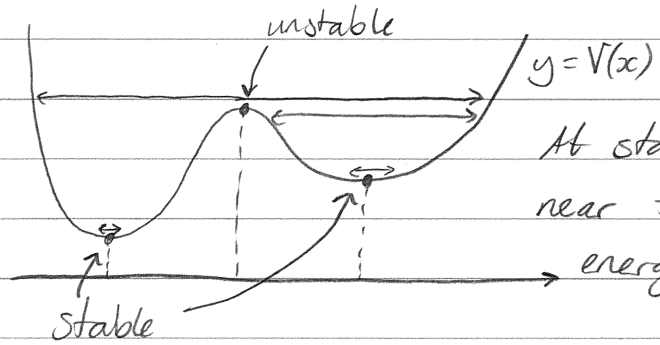
$$\Rightarrow \frac{1}{2}mv^2 = -V(x) + E$$

$$E = \frac{1}{2}mv^2 + V(x)$$

$$v = \pm\sqrt{\frac{2}{m}}\sqrt{E - V(x)}$$ ← particle can only move when $E > V(x)$

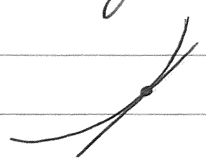


Equilibrium points, $V'(c) = 0$



At stable pts, the particle will stay near the pt even with slightly higher energy. (Imagine a pendulum).

Linearisation of a function f at c.



$$F(x) \approx L(x) = f(c) + f'(c)(x-c)$$

Let c be a stable equilibrium (a minimum) of $V(x)$ s.t. $V''(c) > 0$ (NOT $V(x) = x^4$)

Near $x = c$

$$m\ddot{x} = F(x) = -V'(x)$$
$$\approx -(\underbrace{V'(c)}_{=0} + V''(c)(x-c))$$

$$m\ddot{x} \approx -V''(c)(x-c)$$

$$m\ddot{x} + V''(c)(x-c) = 0$$

Let $X = x - c$

$$\Rightarrow \ddot{X} + \frac{V''(c)}{m} X = 0$$

SHM

$$\ddot{X} + \omega^2 X = 0 \quad , \quad \omega = \sqrt{\frac{V''(c)}{m}}$$

$$X = A\sin(\omega t - \delta)$$

$$\text{period} = \frac{2\pi}{\omega}$$

So the period of small oscillations about $x = c$ is approximately $2\pi\sqrt{\frac{m}{V''(c)}}$

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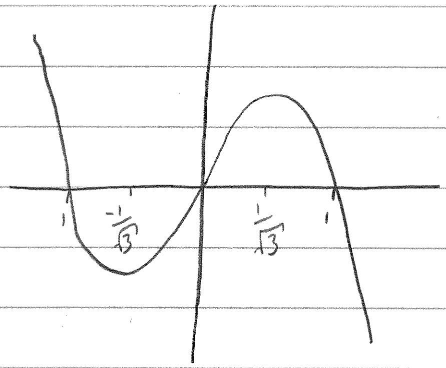
Example: (continued)

$V(x) = x - x^3$, $m=1$.



Find the approximate period of small oscillations about the stable equilibrium point.

$V''(x) = -6x$

At $c = \frac{-1}{\sqrt{3}}$, $V''(c) = 2\sqrt{3}$

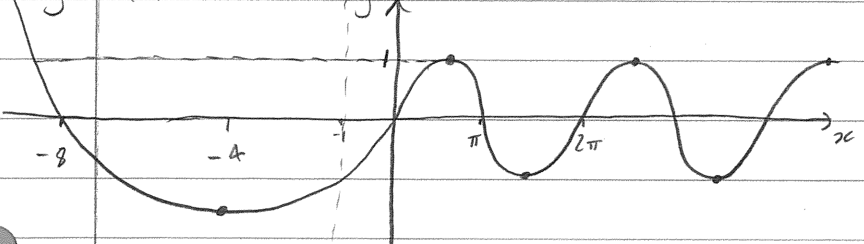
Approx period $= \frac{2\pi}{\sqrt{V''(c)}} = \frac{2\pi}{\sqrt{2\sqrt{3}}} = \frac{2^{\frac{1}{2}}\pi}{3^{\frac{1}{4}}}$

Example:

A particle of mass $m=1$ moves in the potential

$$V(x) = \begin{cases} \sin x & x \geq 0 \\ x + \frac{1}{8}x^2 & x < 0 \end{cases}$$

(i) sketch V and identify the stable and unstable equilibrium points.



$x < 0 \quad V'(x) = 1 + \frac{1}{4}x$

Unstable : $(\frac{1}{2} + 2n)\pi$, $n = 0, 1, \ldots$

Stable : -4 & $(\frac{3}{2} + 2n)\pi$, $n = 0, 1, \ldots$

(ii) Find the largest speed that the particle can have as it passes $x=-1$ if the motion is to remain bounded.

The motion is bounded if and only if $E \leq 1$.

If $E = 1$ at $x = -1$

$$1 = E = \tfrac{1}{2} \underset{\underset{1}{\parallel}}{m} v^2 + V(-1)$$

$$= \frac{v^2}{2} - \frac{7}{8}$$

$$\Leftrightarrow \quad v = \frac{\sqrt{15}}{2}$$

(iii) Find the approx period of small oscillations near $x = \frac{3\pi}{2}$

Soln: $2\pi \sqrt{\dfrac{m}{V''\left(\frac{3\pi}{2}\right)}}$ $\qquad m = 1, \quad \overset{x>0}{V''(x)} = -\sin x$

$$= 2\pi \sqrt{\frac{1}{-\sin\left(\frac{3\pi}{2}\right)}}$$

$$= 2\pi$$

(iv) Suppose that the particle is released from rest at $x = -10$.

(a) What is the max speed in subsequent motion?

(b) How long after the particle is released does it reach this speed?

(a) $E = 0 + V(-10) = -10 + \dfrac{100}{8} = \dfrac{5}{2}$

$$v = \sqrt{\frac{2}{m}\left(E - V(x)\right)} = \sqrt{2\left(\frac{5}{2} - -2\right)} = \sqrt{2\left(9/2\right)} = \underline{\underline{3}}.$$

(b) $\dfrac{dx}{dt} = \sqrt{5 - 2x - \frac{1}{4}x^2}$ ← energy eqn.

$$\Rightarrow \int_0^T dt = \int_{-10}^{-4} \left(-\tfrac{1}{4}(x+4)^2 + 9\right)^{-\frac{1}{2}} dx$$

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(iv) (b) cont.

$$\int_0^T dt = \int_{-10}^{-4} \left(9 - \frac{1}{4}(x+4)^2\right)^{-\frac{1}{2}} dx$$

$$\Rightarrow T = \left[2 \arcsin\left(\frac{\frac{1}{2}(x+4)}{3}\right)\right]_{-10}^{-4}$$

$$T = 0 - 2\arcsin(-1)$$

$$T = -2\left(\frac{-\pi}{2}\right)$$

$$\Rightarrow \underline{\underline{T = \pi}}$$

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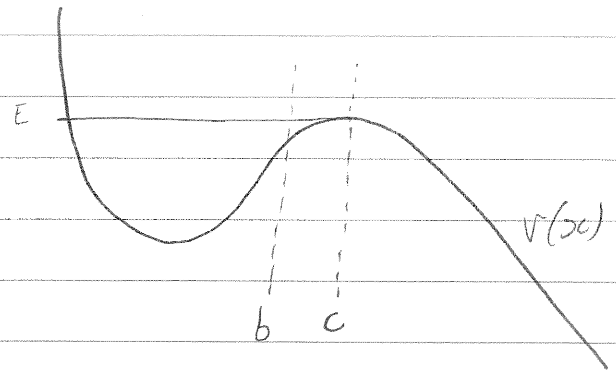

Degenerate Energy Values

Recall: The time for a particle to move from $x = x_1$ to x_2 is

$$T_1 = \sqrt{\frac{m}{2}} \int_{x_1}^{x_2} \frac{dx}{\sqrt{E - V(x)}}$$

Suppose that $x_2 = c$ is a local maximum of V (degenerate energy value).

Does it take a finite of ∞ time to reach $x_2 = c$?

Take $V''(c) = -\alpha^2 < 0$

We only need to consider the time taken to get from a nearby point $x = b < c$ to c

Quadratic approximation (First few terms in the Taylor series).

$$f(x) \approx f(c) + f'(c)(x-c) + \frac{f''(c)}{2}(x-c)^2$$

let $f(x) = \sum a_n (x-c)^n$

$$= a_0 + a_1(x-c) + a_2(x-c)^2 + \dots$$

$x = c$: $a_0 = f(c)$

$f'(x) = a_1 + 2a_2(x-c) + \dots$

$a_1 = f'(c)$

$f''(x) = 2a_2 + 3\times 2(x-c)\dots$

$2a_2 = f''(c)$

$$\Rightarrow a_n = \frac{f^{(n)}(c)}{n!}$$

Degenerate Energy value:

$$E = V(c)$$

$$V(x) = V(c) + V'(c)(x-c) + \frac{V''(c)}{2}(x-c)^2$$

$$\Rightarrow V(x) = E + 0 - \frac{\alpha^2}{2}(x-c)^2$$

Time from $x = b$ to $x = c$:

$$T = \sqrt{\frac{2}{m}} \int_b^c \frac{dx}{\sqrt{E - V(x)}}$$

$$\approx \sqrt{\frac{2}{m}} \int_b^c \frac{dx}{\sqrt{E - \left(E - \frac{\alpha^2}{2}(x-c)^2\right)}}$$

$$= \frac{2}{\sqrt{m}} \int_b^c \frac{dx}{\sqrt{\alpha^2 (x-c)^2}}$$

$\sqrt{(x-c)^2} = |x-c|$
$= c - x$
as x is less than c.

$$= \frac{2}{\alpha\sqrt{m}} \lim_{\varepsilon \to 0^+} \int_b^{c-\varepsilon} \frac{dx}{c - x}$$

$$= \frac{2}{\alpha\sqrt{m}} \lim_{\varepsilon \to 0^+} -\ln|x-c| \Big|_{x=b}^{x=c-\varepsilon}$$

$$= \frac{2}{\alpha\sqrt{m}} \lim_{\varepsilon \to 0^+} \left(-\ln \varepsilon + \ln|b-c|\right)$$

$$= \underline{\underline{\infty}}.$$

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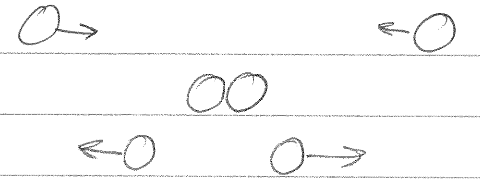
Collisions



* $\underset{\sim}{r}(t)$ - position

* $\underset{\sim}{v} = \frac{d\underset{\sim}{r}}{dt}$ - vecocity

* $\underset{\sim}{a} = \frac{d\underset{\sim}{v}}{dt} = \frac{d^2\underset{\sim}{r}}{dt^2}$ - acceleration

Collision of 2 spheres of mass m_1, m_2 and positions $\underset{\sim}{r}_1(t)$, $r_2(t)$. No forces acting before or after.

Before collision : $m_1\ddot{\underset{\sim}{r}}_1 = 0$ $\qquad m_2\ddot{\underset{\sim}{r}}_2 = 0$

After " " "

During collision, the 1st particle experiences a force $\underset{\sim}{F}(t)$. The 2nd particle experiences the force $-\underset{\sim}{F}(t)$.

During: $m_1\ddot{\underset{\sim}{r}}_1 = \underset{\sim}{F}$ and $m_2\ddot{\underset{\sim}{r}}_2 = -\underset{\sim}{F}$

So $m_1\ddot{\underset{\sim}{r}}_1 + m_2\ddot{\underset{\sim}{r}}_2 = 0$ (before, during and after collision).

$\Rightarrow m_1\dot{\underset{\sim}{r}}_1 + m_2\dot{\underset{\sim}{r}}_2 = \text{const.}$

i.e. if the first particle initially has velocity u_1 and then v_1 after the collision etc.

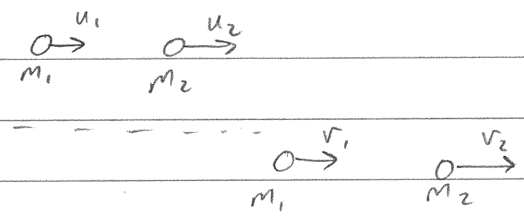
$m_1\underset{\sim}{u}_1 + m_2\underset{\sim}{u}_2 = m_1\underset{\sim}{v}_1 + m_2\underset{\sim}{v}_2$ (conservation of momentum)

(mass x velocity = momentum).

Collisions in one-dimension

$m_1 u_1 + m_2 u_2 = m_1 v_1 + m_2 v_2$

velocity is a scalar here (not vector) as we are in 1 dimension.



In an elastic collision the total kinetic energy is conserved.

Initial K.E. = Final K.E.

$$\Rightarrow \tfrac{1}{2} m_1 u_1^2 + \tfrac{1}{2} m_2 u_2^2 = \tfrac{1}{2} m_1 v_1^2 + \tfrac{1}{2} m_2 v_2^2$$

i.e. $m_1 (v_1^2 - u_1^2) = -m_2 (v_2^2 - u_2^2)$ ①

conservation of momentum:

$$m_1 (v_1 - u_1) = -m_2 (v_2 - u_2) \quad ②$$

One solution: $v_1 = u_1$ & $v_2 = u_2$ (no collision).

Otherwise, both sides of ② are non-zero:

$\frac{①}{②}$: $v_1 + u_1 = v_2 + u_2$

$$\Rightarrow v_2 - v_1 = -(u_2 - u_2)$$

Experiments show that even if K.E. is not conserved, there is a constant e s.t.

$$v_2 - v_1 = -e(u_2 - u_1)$$

e is called the coefficient of restitution.

$0 \le e \le 1$ ($e = 1 \Leftrightarrow$ elastic collision, i.e. K.E. is conserved).

Write conservation of momentum (eqn 2) as

$$m_1 v_1 + m_2 v_2 = m_1 u_1 + m_2 u_2$$
$$-v_1 + v_2 = e(u_1 - u_2)$$

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$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

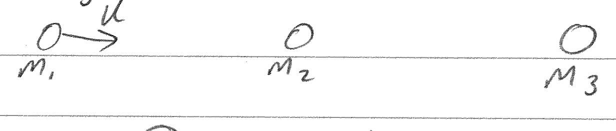
$$\begin{pmatrix} m_1 & m_2 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} m_1 u_1 + m_2 u_2 \\ e(u_1 - u_2) \end{pmatrix}$$

$$\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \frac{1}{m_1+m_2}\begin{pmatrix} 1 & -m_2 \\ 1 & m_1 \end{pmatrix}\begin{pmatrix} m_1 u_1 + m_2 u_2 \\ e(u_1 - u_2) \end{pmatrix}$$

$$* \; v_1 = \frac{(m_1 - e m_2)u_1 + m_2(1+e)u_2}{m_1 + m_2} \qquad (A)$$

$$* \; v_2 = \frac{m_1(1+e)u_1 + (m_2 - e m_1)u_2}{m_1 + m_2} \qquad (B)$$

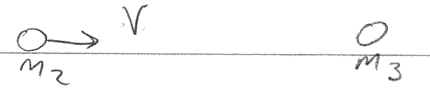
Example: Three balls of masses m_1, m_2, m_3 are arranged in a line in the order given. m_1 initially moves toward m_2 with velocity U, while m_2 & m_3 are at rest. If the coefficient of restitution is e, determine the final speed of the mass m_3.



Soln. Use (B) to find the final velocity of m_2 after the first collision.

$u_1 = U$, $u_2 = 0$

$$v_2 = \frac{m_1(1+e)U}{m_1+m_2} = V$$



2nd collision m_2 moves at speed V and m_3 is stationary. (B): $v_3 = \dfrac{m_2(1+e)V}{m_2+m_3} = \underline{\underline{\dfrac{m_1 m_2 (1+e)^2 U}{(m_1+m_2)(m_2+m_3)}}}$

Example: Consider a ball of mass m bouncing on the ground. (collision between ball & Earth).
u = initial velocity of ball, directed down.
Velocity of Earth ≈ 0
Let M be the mas of the Earth.

Final velocity of ball:

$$v_1 = \frac{(m_1 - e m_2)u + 0}{m_1 + m_2} \quad \leftarrow \text{using } Ⓐ$$

$$= \frac{(m - eM)u}{m + M}$$

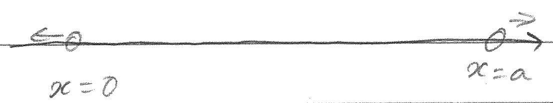
$$= \frac{\left(\frac{m}{M} - e\right)u}{\frac{m}{M} + 1}$$

$$\frac{m}{M} \to 0 \quad \Rightarrow \quad v_1 = -eu$$

Example: A sphere of mass m moves along the +ve x-axis from $x = 0$ until it collides elastically with a stationary sphere of mass $M > m$. If both spheres have unit radii & the centre of the second sphere is at $x = a$ when the centre of the first sphere returns to $x = 0$, find the original location of the centre of the second sphere.

before:

after:



$u_1 = u > 0$, $u_2 = 0$, $e = 1$
$m_1 = m$, $m_2 = M$, $M > m$

$$v_1 = \frac{(m - M)u}{m + M}, \quad v_2 = \frac{2mu}{m + M}$$

Let X be the initial position of the centre of M.
At the collision, the centre of m is $X-2$
(sum of 2 radii = 2)

In the time T that M moves from X to a (with speed v_2), m has moved from $X-2$ to 0 (with speed $-v_1$)
$(\rightarrow +ve)$

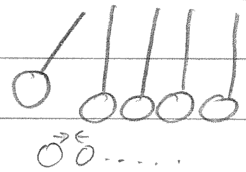
$\boxed{v = \frac{x}{t}}$

$$T = \frac{X-2}{-v_1} = \frac{a-X}{v_2}$$

$$(X-2)v_2 + (a-X)v_1 = 0$$

$$\Leftrightarrow (X-2)(2m\cancel{U}) + (a-X)(m-M)\cancel{U} = 0$$

$$\Leftrightarrow \quad X = \frac{(M-m)a + 4m}{M+m}$$



elastic $e=1$

$m_1 = m_2 = \ldots$

$u_1 = U,\ u_2 = 0,\ e = 1$

$$v_1 = \frac{0+0}{2m} = 0 \quad , \quad v_2 = \frac{m(2)U + 0}{m+m} = U$$

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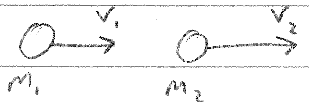
Collisions in 1 dimension:

- Before



- After

* $m_1 \ddot{x}_1 + m_2 \ddot{x}_2 = 0$

* $\Rightarrow m_1 \dot{x}_1 + m_2 \dot{x}_2 = \text{const.}$

* $m_1 v_1 + m_2 v_2 = m_1 u_1 + m_2 u_2$

Elastic collisions:

* $$\frac{1}{2} m_1 v_1^2 + \frac{1}{2} m_2 v_2^2 = \frac{1}{2} m_1 u_1^2 + \frac{1}{2} m_2 u_2^2$$

$$m_2 (v_2^2 - u_2^2) = - m_1 (v_1^2 - u_1^2) \qquad A$$

$$m_2 (v_2 - u_2) = - m_1 (v_1 - u_1) \qquad B$$

$A/B \Rightarrow \quad v_2 + u_2 = v_1 + u_1$

* $v_2 - v_1 = -(u_2 - u_1)$ ← elastic case.

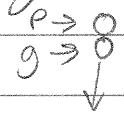
* $v_2 - v_1 = -e(u_2 - u_1)$ ← inelastic case

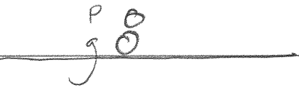
*
$$v_1 = \frac{(m_1 - e m_2) u_1 + m_2 (1+e) u_2}{m_1 + m_2} \qquad (1)$$

$$v_2 = \frac{m_1 (1+e) u_1 + (m_2 - e m_1) u_2}{m_1 + m_2} \qquad (2)$$

Example:

A pingpong ball is held on top of a golf ball and the 2 are dropped onto the ground from height 1m (take radii to be negligible). Assuming the mass of the pingpong ball is negligible, and that all collisions are perfectly elastic (and neglecting air resistance), what is the height reached by the ping pong ball?





Solution:

Look at this as two collisions. The first is between the golf ball and the ground, and then between the golf ball and the pingpong ball.

Initially:

$$E = \frac{1}{2}mv^2 + mgh$$
$$= 0 + mg = mg$$

At ground level

$$E = \frac{1}{2}mv^2 + 0 = mg$$

$$\Rightarrow |v| = \sqrt{2g}$$

$(\uparrow +ve)$

p $\downarrow$ O $u_2 = -\sqrt{2g}$
g $\uparrow$ O $u_1 = \sqrt{2g}$

After the golf ball collides with the ground, its velocity, $u_1 = \sqrt{2g}$ and the pingpong ball is moving down with velocity $u_2 = -\sqrt{2g}$

After the second collision, the velocity of the pingpong ball is $v_2 = \dfrac{m_1(1+e)u_1 + (m_2 - em_1)u_2}{m_1 + m_2}$ using ②

$$= \frac{(2m_1 + (m_1 - m_2))\sqrt{2g}}{m_1 + m_2}$$

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$$\Rightarrow v_2 = \frac{\left(3 - \frac{m_2}{m_1}\right)\sqrt{2g}}{1 + \frac{m_2}{m_1}} \rightarrow 3\sqrt{2g} \quad \text{as } \frac{m_2}{m_1} \rightarrow 0$$

Let H be the height reached by the ping pong ball:

The pingpong ball moves up initially with velocity $U = 3\sqrt{2g}$

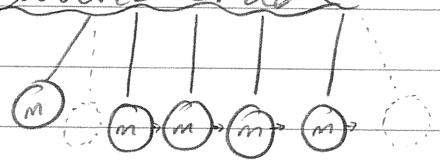
$$E = \frac{1}{2}m_2 U^2 + 0 \quad \text{Before}$$

$$E = 0 + m_2 g H \quad \text{After}$$

$$\Rightarrow \frac{1}{2}\cancel{m_2} U^2 = \cancel{m_2} g H$$

$$\Rightarrow H = \frac{U^2}{2g} = \frac{9 \times 2g}{2g} = \underline{\underline{9m}}$$

Newton's cradle:



$u_1 = U$, $u_2 = 0$ $\quad e = 1$, $m = m = m \ldots = m$

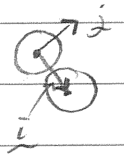
$\downarrow$

$u_1 = 0$, $u_2 = U$

Collisions in 2 dimensions:

Consider the collision between 2 spheres of mass m_1 & m_2 with initial velocities u_1 & u_2 and final velocities v_1 & v_2.

When the 2 spheres collide, let the unit vector $\underset{\sim}{i}$ point in the direction of the line connecting the centres. Let $\underset{\sim}{j}$ be a unit vector ⊥ to $\underset{\sim}{i}$.



$$
\left\{
\begin{aligned}
\underset{\sim}{u}_1 &= u_i^1 \underset{\sim}{i} + u_j^2 \underset{\sim}{j} \\
\underset{\sim}{u}_2 &= u_i^2 \underset{\sim}{i} + u_j^2 \underset{\sim}{j} \\
\underset{\sim}{v}_1 &= v_i^1 \underset{\sim}{i} + v_j^1 \underset{\sim}{j} \\
\underset{\sim}{v}_2 &= v_i^2 \underset{\sim}{i} + v_j^2 \underset{\sim}{j}
\end{aligned}
\right.
$$

$$\underset{\sim}{i} \cdot \underset{\sim}{u}_1 = u_i^1 \underbrace{\underset{\sim}{i} \cdot \underset{\sim}{i}}_{1} + u_j \underbrace{\cancel{\underset{\sim}{j} \cdot \underset{\sim}{i}}}_{0} = u_i^1 \Rightarrow u_i^1 = \underset{\sim}{u}^1 \cdot \underset{\sim}{i}$$

Motion in the $\underset{\sim}{j}$-direction (perpendicular to $\underset{\sim}{i}$) is unaffected by the collision:

$$v_j^1 = u_j^1 \iff \underset{\sim}{j} \cdot \underset{\sim}{v}^1 = \underset{\sim}{j} \cdot \underset{\sim}{u}^1$$

$$v_j^2 = u_j^2 \iff \underset{\sim}{j} \cdot \underset{\sim}{v}^2 = \underset{\sim}{j} \cdot \underset{\sim}{u}^2$$

In the $\underset{\sim}{i}$-direction the $\underset{\sim}{i}$-components of the velocities behave as in the one-dimensional collision case.

$$(\underset{\sim}{v}_2 - \underset{\sim}{v}_1) \cdot \underset{\sim}{i} = -e(\underset{\sim}{u}_2 - \underset{\sim}{u}_1)\underset{\sim}{i} \leftarrow \text{restitution}$$

and conservation of momentum

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$$\underset{\sim}{v}_1 = \frac{(m_1 - e m_2) u_i^1 + m_2 (1+e) u_i^2}{m_1 + m_2} \underset{\sim}{i} + u_j^1 j$$

$$\underset{\sim}{v}_2 = \frac{m_1 (1+e) u_i^1 + (m_2 - e m_1) u_i^2}{m_1 + m_2} \underset{\sim}{i} + u_j^2 j$$

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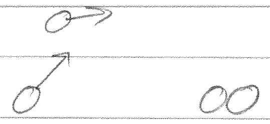


Restitution:

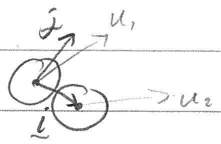
$v_2 - v_1 = -e(u_2 - u_1)$

$e = 1 \Leftrightarrow$ conservation of KE (elastic collision).

e = coefficient of restitution.

$$m_1 \ddot{x}_1 + m_2 \ddot{x}_2 = 0$$

$$\Rightarrow m_1 \underset{\sim}{v}_1 + m_2 \underset{\sim}{v}_2 = m_1 \underset{\sim}{u}_1 + m_2 \underset{\sim}{u}_2$$



$\underset{\sim}{i}$ points between centres of the spheres.
$\underset{\sim}{j}$ points $\perp$ to $\underset{\sim}{i}$.

Components of velocities in the $\underset{\sim}{j}$-direction remain the same:

$$\underset{\sim}{v}_1 \cdot \underset{\sim}{j} = \underset{\sim}{u}_1 \cdot \underset{\sim}{j}$$

$\underset{\sim}{i}$ direction:

$$\underset{\sim}{v}_2 \cdot \underset{\sim}{i} - \underset{\sim}{v}_1 \cdot \underset{\sim}{i} = -e(\underset{\sim}{u}_2 \cdot \underset{\sim}{i} - \underset{\sim}{u}_1 \cdot \underset{\sim}{i}) \quad -①$$

$\underset{\sim}{i}$-component of momentum:

$$m_1 \underset{\sim}{v}_1 \cdot \underset{\sim}{i} + m_2 \underset{\sim}{v}_2 \cdot \underset{\sim}{i} = m_1 \underset{\sim}{u}_1 \cdot \underset{\sim}{i} + m_2 \underset{\sim}{u}_2 \cdot \underset{\sim}{i} \quad -②$$

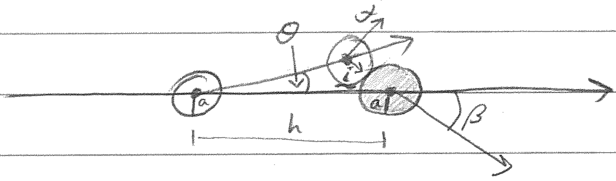
Eqns ① and ② are the same as in the 1D case if we replace $\underset{\sim}{u}_1 \cdot \underset{\sim}{i}$ by u_1, $\underset{\sim}{u}_2 \cdot \underset{\sim}{i}$ by u_2 etc.

$$\Rightarrow \underset{\sim}{v}_1 \cdot \underset{\sim}{i} = \frac{(m_1 - e m_2)\underset{\sim}{u}_1 \cdot \underset{\sim}{i} + m(1+e)\underset{\sim}{u}_2 \cdot \underset{\sim}{i}}{m_1 + m_2}$$

$$\Rightarrow \underset{\sim}{v}_2 \cdot \underset{\sim}{i} = \frac{\quad}{m_1 + m_2}$$

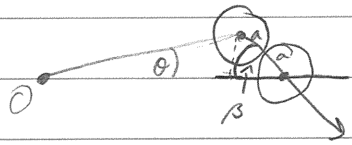
$\underset{\sim}{v}_1 = (\underset{\sim}{v}_1 \cdot \underset{\sim}{i})\underset{\sim}{i} + (\underset{\sim}{v}_1 \cdot \underset{\sim}{j})\underset{\sim}{j}$ etc.

Example



At what angle θ should we send the white ball (in the diagram) so that the red ball moves in the direction shown?
Initially the white ball is at the origin & the red ball is at $(h, 0)$. Both balls have radius a.

Soln: Since the red ball (R) is initially at rest, its final velocity component in the $\underset{\sim}{j}$-direction (tangent to the collision) will be zero, so R will move in the $\underset{\sim}{i}$-direction. So the centre of W and the centre of R should lie on the line through $(h, 0)$ making an angle β as shown.



Coordinates of the centre of W at collision are
$(h - 2a\cos\beta,\ 2a\sin\beta)$
So $\theta = \tan^{-1}\left(\dfrac{2a\sin\beta}{h - 2a\cos\beta}\right)$

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Second order linear differential equations

$$\frac{d^2x}{dt^2} + p(t)\frac{dx}{dt} + q(t)x = f(t) \quad -①$$

(no terms involving products of $x, \dot{x}, \ddot{x}$ together)

Homogeneous Case

$f(t) = 0$ is called the homogenous case.

$$\ddot{x} + p\dot{x} + qx = 0 \quad -②$$

$$\Rightarrow \frac{d^2(cx)}{dt^2} + p\frac{d(cx)}{dt} + q(cx)$$

So if x solves x so does cx for any constant number c.

let x_1 & x_2 be two solutions of ②. Then so is

$x = c_1x_1 + c_2x_2$

for any constants c_1 & c_2.

Proof:

$$\frac{d^2x}{dt^2} + p\frac{dx}{dt} + qx$$

$$= \frac{d^2(c_1x_1 + c_2x_2)}{dt^2} + p\frac{d(c_1x_1 + c_2x_2)}{dt} + q(c_1x_1 + c_2x_2)$$

$$= \left(c_1\frac{d^2x_1}{dt^2} + c_2\frac{d^2x_2}{dt^2}\right) + p\left(c_1\frac{dx_1}{dt} + c_2\frac{dx_2}{dt}\right) + q(c_1x_1 + c_2x_2)$$

$$= c_1\left(\frac{d^2x_1}{dt^2} + p\frac{dx_1}{dt} + qx_1\right) + c_2\left(\frac{d^2x_2}{dt^2} + p\frac{dx}{dt} + qx_1\right)$$

$= c_1 \times 0 + c_2 \times 0 = 0 \Rightarrow x$ is a solution.

Fact:

If x_1 & x_2 are independent (one is not a constant multiple of the other) then
$x = c_1 x_1 + c_2 x_2$ is the general solution of ②
(i.e. every solution is of this form).

Now assume that p and q are constants:

$$\ddot{x} + p\dot{x} + qx + 0 \quad -③$$

constant coefficients

Aside:

1st-order eqn: $\dot{x} = ax \longrightarrow x = Ae^{at}$

$\hookrightarrow \ddot{x} = a\dot{x} = a^2 x$

This suggests that we look for solutions of ③ of the form $x(t) = e^{rt}$, r = constant

sub into ③:

$$r^2 e^{rt} + pre^{rt} + qe^{rt} = 0$$

$$\Rightarrow (r^2 + pr + q)\cancel{e^{rt}} = 0$$

So $x = e^{rt}$ is a solution of ③ if and only if r solves the characteristic eqn:

$$r^2 + pr + q = 0 \quad -④$$

Case 1:

④ has 2 distinct real roots $r_1 \neq r_2$,

$x_1 = e^{r_1 t}$, $x_2 = e^{r_2 t}$ are solutions.

(They are not multiples, if they were:

$e^{r_2 t} = ke^{r_1 t} \Leftrightarrow e^{(r_2 - r_1)t} = k$ ※!)

So x_1 & x_2 are independant,

general solution is

$$x(t) = c_1 e^{r_1 t} + c_2 e^{r_2 t}$$

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Case 2:

④ has 1 repeated (real) root.

④ $\Leftrightarrow$ $\left(r + \frac{p}{2}\right)^2 = 0$

$$(\Delta = p^2 - 4q = 0) \qquad \left[r = (-p \pm \sqrt{\Delta})/2 \quad \Delta = p^2 - 4q \right]$$

One solution $= e^{rt}$

It turns out that te^{rt} is also a solution in this case.

$$x_2 = te^{-\frac{p}{2}t} \Rightarrow \dot{x}_2 = \left(1 - \frac{p}{2}t\right)e^{-\frac{p}{2}t}$$

$$\ddot{x} = \left(\frac{p^2}{4}t - p\right)e^{-\frac{p}{2}t}$$

Sub in ③

$$\left(\frac{p^2}{4}t - p + p\left(1 - \frac{p}{2}t\right) + \frac{p^2}{4}t\right)e^{-\frac{p}{2}t} = 0$$

General solution is

$$x = (c_1 + c_2 t)e^{-\frac{p}{2}t}$$

Case 3:

2 complex conjugate solutions $(\Delta < 0)$

$$r = \frac{-p \pm i\sqrt{-\Delta}}{2} = \mu \pm i\nu \quad (\mu, \nu \text{ real})$$

$$\Rightarrow x_1 = e^{(\mu + i\nu)t}, \quad x_2 = e^{(\mu - i\nu)t} \qquad \boxed{\text{note: } e^{i\theta} = \cos\theta + i\sin\theta}$$

$$x_1 = e^{\mu t}e^{i\nu t} = e^{\mu t}(\cos\nu t + i\sin\nu t)$$

$$x_2 = e^{\mu t}e^{-i\nu t} = e^{-\mu t}(\cos\nu t - i\sin\nu t)$$

Take the real linear combinations:

$$X_1 = \frac{x_1 + x_2}{2} \left(= Re(x_1)\right) = e^{\mu t}\cos\nu t$$

$$X_2 = \frac{x_1 - x_2}{2i} \left(= Im(x_1)\right) = e^{\mu t}\sin\nu t$$

General solution of ③:

$$x = c_1 X_1 + c_2 X_2$$

$$\Rightarrow x = e^{\mu t}(c_1 \cos\nu t + c_2 \sin\nu t)$$

Examples:

1). $y'' + 3y' + 2y = 0$ $\qquad y = e^{rt}$

$\qquad y' = re^{rt}$

2). $y'' + 2y' + y = 0$ $\qquad y'' = r^2 e^{rt}$

3). $y'' + y' + y = 0$

solutions:

1). $r^2 + 3r + 2 = 0$

$(r+1)(r+2) = 0$

$\Rightarrow r = -1, \; r = -2$

general solution: $y(t) = c_1 e^{-t} + c_2 e^{-2t}$

2). $r^2 + 2r + 1 = 0$

$(r+1)^2 = 0$

$\Rightarrow r = -1$

general solution: $y(t) = (c_1 + c_2 t)e^{-t}$

3). $r^2 + r + 1 = 0$

$(r + \frac{1}{2})^2 = -1 + \frac{1}{4}$

$r = -\frac{1}{2} \pm \sqrt{-\frac{3}{4}}$

$\Rightarrow r = -\frac{1}{2} \pm \frac{\sqrt{3}}{2} i$

general solution: $y(t) = e^{-\frac{t}{2}}\left(c_1 \cos\left(\frac{\sqrt{3}}{2}t\right) \pm c_2 \sin\left(\frac{\sqrt{3}}{2}t\right)\right).$

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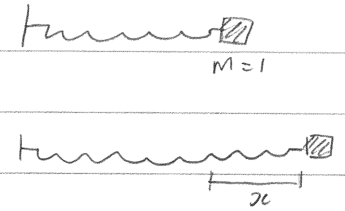
The damped oscillator



If we include friction proportional to velocity when we study a unit mass moving in a straight line at the end of a spring, we get an eqn of the form

$$\ddot{x} = -c\dot{x} - kx$$

friction   Hooke's law.

Equation of motion:

Δ = discriminant

$$\ddot{x} + c\dot{x} + kx = 0$$
$$\Delta = c^2 - 4k$$

1). $\Delta > 0$, 2 real solutions.

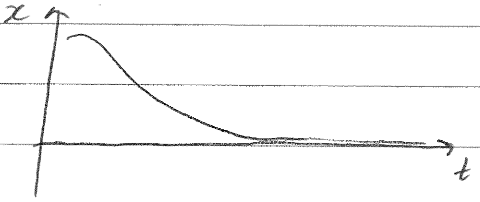
characteristic equation: $r = \dfrac{-c \pm \sqrt{c^2 - 4k}}{2}$

$$0 < c^2 - 4k < c^2$$

$\Rightarrow r_1$ & r_2 are negative.

$r_1 = -\mu_1$, $r_2 = -\mu_2$ $\mu_1, \mu_2 > 0$

$\Rightarrow x = c_1 e^{-\mu_1 t} + c_2 e^{-\mu_2 t} \to 0$ very quickly



over damped

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$\ddot{x} + p\dot{x} + qx = 1$

$x = c_1 x_1(t) + c_2 x_2(t)$ ← general solution.

p, q constant
Look for soln $x = e^{rt}$
$r^2 + pr + q = 0$

Special case: SHM
$\ddot{x} + \omega^2 x = 0$, $\omega > 0$
$\Rightarrow (r^2 + \omega^2)e^{rt} = 0$
$\Rightarrow r^2 + \omega^2 = 0$
$\Rightarrow r = \pm i\omega$
$x_1 = e^{i\omega t} = \cos\omega t + i\sin\omega t$
$x_2 = e^{-i\omega t} = \cos\omega t - i\sin\omega t$

$\frac{1}{2}(x_1 + x_2) = \cos\omega t$, $\frac{1}{2i}(x_1 - x_2) = \sin\omega t$

usually nicer to find c_1, c_2
$x(t) = c_1\cos\omega t + c_2\sin\omega t$ ←

$= A\sin(\omega t - \delta)$ ← more geometric

Damped oscillator
Unit mass on a spring with frictional force proportional to velocity: $\ddot{x} + c\dot{x} + kx = 0$, $c, k > 0$ constants.
Look for solns $x = e^{rt}$ ($\dot{x} = re^{rt}$, $\ddot{x} = r^2e^{rt}$)
characteristic eqn: $r^2 + cr + k = 0$
$r = \dfrac{-c \pm \sqrt{\Delta}}{2}$, where $\Delta = c^2 - 4k$

Case: 1 $\Delta > 0$
2 distinct real roots
$0 < c^2 - 4k < c^2 \Rightarrow 0 < \sqrt{\Delta} < c$
$\underset{-\mu_1}{\overset{}{r_1}} = \dfrac{-c + \sqrt{\Delta}}{2} < 0$, $\underset{-\mu_2}{r_2} = \dfrac{-c - \sqrt{\Delta}}{2}$

general solution:

$$x(t) = c_1 e^{-\mu_1 t} + c_2 e^{-\mu_2 t}$$

$x \to 0$ as $t \to \infty$

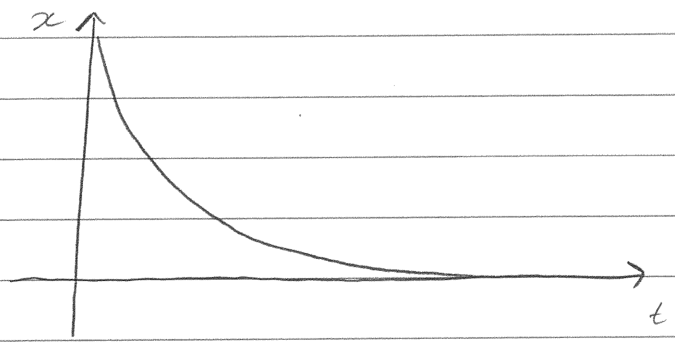
Not oscillating.

Let's check when $x = 0$:

$$0 = c_1 e^{-\mu_1 t} + c_2 e^{-\mu_2 t}$$

$$\Leftrightarrow \quad c_1 e^{(\mu_2 - \mu_1)t} = -c_2$$

Either 0 or 1 solution.



$\boxed{\text{Overdamped case } (\Delta > 0)}$

Case 2: $\Delta = 0$

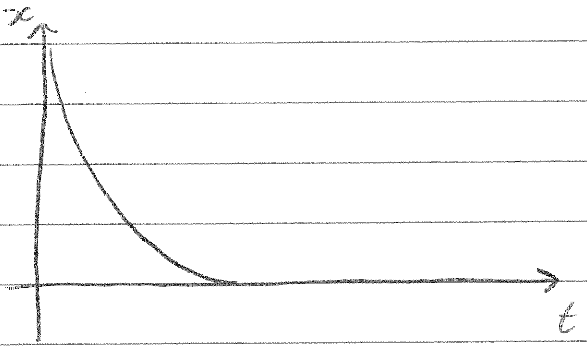
$r = \frac{-c}{2}$ (repeated root as $\pm\sqrt{\Delta} = 0$)

general soln:

$$x(t) = c_1 e^{-\frac{c}{2}t} + c_2 t e^{-\frac{c}{2}t} = (c_1 + c_2 t) e^{-\frac{c}{2}t}$$

$x \to 0$ as $t \to \infty$

Crosses $x = 0$ at most once



$\boxed{\text{Critical damping } (\Delta = 0)}$

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Case 3: $\Delta < 0$

2 complex conjugate roots:

$$r = \frac{-c \pm \sqrt{\Delta}}{2} = -\frac{c}{2} \pm i \frac{\sqrt{-\Delta}}{2} = -\mu \pm i\nu$$

← real

$$x_1 = e^{(-\mu + i\nu)t} = e^{-\mu t} e^{i\nu t}$$
$$= e^{-\mu t}(\cos\nu t + i \sin\nu t)$$

$$x_2 = e^{(-\mu - i\nu)t} = e^{-\mu t} e^{-i\nu t}$$
$$= e^{-\mu t}(\cos\nu t - i \sin\nu t)$$

$$X_1 = \frac{x_1 + x_2}{2} = e^{-\mu t}\cos\nu t$$

$$X_2 = \frac{x_1 - x_2}{2i} = e^{-\mu t}\sin\nu t$$

general solution:

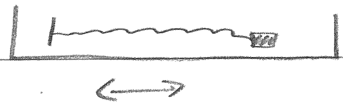
$$x(t) = c_1 X_1 + c_2 X_2$$
$$= e^{-\mu t}(c_1 \cos\nu t + c_2 \sin\nu t) = Ae^{-\mu t}\sin(\nu t - \delta)$$

for some A and δ

(both solns of $\ddot{X} + \nu^2 X = 0$)

$\boxed{\text{Underdamped } \Delta > 0}$ (light damping)

Forced Oscillators



$$m\ddot{x} + c\dot{x} + kx = F(t)$$ ← forcing term.

This leads to eqns of the form

$$\ddot{x} + p\dot{x} + qx = F(t) \quad (2)$$

Let x_p be a particular solution of $(\ddot{x} + p\dot{x} + qx = F(t))$ and write the general soln as $x(t) = x_h(t) + x_p(t)$, where we will determine x_h.

Sub x in (2):

$$\frac{d^2}{dt^2}(x_h + x_p) + p\frac{d}{dt}(x_h + x_p) + q(x_h + x_p) = F(t).$$

$$\ddot{x}_h + \ddot{x}_p + p\dot{x}_h + p\dot{x}_p + qx_h + qx_p = F$$

$$\Leftrightarrow (\ddot{x}_h + p\dot{x}_h + qx_h) + \underbrace{(\ddot{x}_p + p\dot{x}_p + qx_p)}_{=F} = F.$$

$$\Leftrightarrow \ddot{x}_h + p\dot{x} + qx_h = 0 \quad (3)$$ as $x_p(t)$ is a particular soln of (2).

So the general soln of (2) is the general soln of (3) plus a particular soln of (2)

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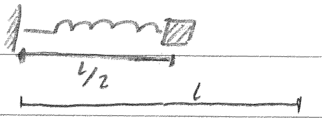
Example:

One end of a spring is attached to the base of the inner wall of a tank filled with oil. A small block is attached to the other end. The spring remains horizontal and the motion of the block is in a straight line. Suppose that the spring const. and drag from the oil are such that the eqn of motion is

$$\frac{d^2x}{dt^2} + 2\frac{dx}{dt} + 5x = 0 \qquad —(*)$$

where x is the displacement from the equilibrium position. At time $t=0$, the block is released from rest when the spring has been compressed to half its length, l.



What is the largest distance from the wall that the mass reaches? At what time does this occur?

Soln:

To solve (*) we consider the characteristic eqn.

$[x = e^{rt}]$ $\quad r^2 + 2r + 5 = 0$

$$\Rightarrow r = \frac{-2 \pm \sqrt{-16}}{2} = -1 \pm 2i$$

$$e^{(-1\pm 2i)t} = e^{-t}e^{\pm 2it} = \begin{cases} e^{-t}(\cos 2t + i\sin 2t) \\ e^{-t}(\cos 2t - i\sin 2t) \end{cases}$$

using linear combinations of the above,

$x(t) = e^{-t}(c_1\cos 2t + c_2\sin 2t)$ $\qquad \dot{x} = e^{-t}(2c_2\cos 2t - 2c_2\sin 2t) - (c_1\cos 2t + c_2\sin 2t)e^{-t}$

Initially $(t=0)$, $x(0) = -\frac{l}{2}$, $\dot{x} = 0$

$x(0) = e^{-0}(c_1 + 0) \Rightarrow c_1 = -\frac{l}{2}$

$\dot{x}(0) = e^{-0}(2c_2 - 0) - e^{-0}(c_1 + 0) \Rightarrow 2c_2 - c_1 = 0$

$\Rightarrow c_2 = -\frac{l}{4}$

So $x(t) = -\frac{l}{4} e^{-t}(2\cos 2t + \sin 2t)$

$$\dot{x}(t) = -\frac{l}{4} e^{-t}(-4\sin 2t + \cancel{2\cos 2t} - \cancel{2\cos 2t} - \sin 2t)$$

$$= \frac{5l}{4} e^{-t} \sin 2t$$

x is maximised at the first positive t s.t. $\dot{x}(t) = 0$. [all local maxima occur on $x = e^{-t}$ which is decreasing].

i.e. $t = \frac{\pi}{2}$

$\Rightarrow x(\frac{\pi}{2}) = \frac{l}{2} e^{-\frac{\pi}{2}}$

Largest distance from the wall is $l + x$
$= \underline{\underline{l(1 + \frac{1}{2}e^{-\frac{\pi}{2}})}}$ which occurs at $t = \underline{\underline{\frac{\pi}{2}}}$.

Forced oscillators:

$$m\ddot{x} + c\dot{x} + kx = F(t) \quad (1)$$

Any soln of (1) has the form

$$x(t) = x_p(t) + x_h(t),$$

where x_p is any particular soln of (1) & x_h is the general soln of the homogeneous eqn $m\ddot{x} + c\dot{x} + kx = 0$.

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Example:

$$\ddot{x} + \omega_0^2 x = f\cos\omega t \quad (3) \qquad \omega_0>0,\ \omega>0,\ f \text{ constants}$$
$$\omega \neq \omega_0$$

The general homogeneous solution x_h satisfies
$$\ddot{x} + \omega_0^2 x = 0$$
characteristic eqn: $r^2 + \omega_0^2 = 0$
$$\Rightarrow \quad x_h(t) = c_1\cos\omega_0 t + c_2 \sin\omega_0 t \qquad c_1, c_2 \text{ constants}$$

We look for a solution of (3) of the form
$$x_p(t) = A\cos(\omega t)$$
$$\ddot{x}_p(t) = -\omega^2 A\cos(\omega t)$$

(3): $(-\omega^2 A + \omega_0^2 A)\cos\omega t = f\cos\omega t$
$$\Leftrightarrow \quad A = \frac{f}{\omega_0^2 - \omega^2}$$

So the general solution of (3) is:
$$x(t) = x_p(t) + x_h(t)$$
$$= \frac{f}{\omega_0^2-\omega^2}\cos\omega t + c_1\cos\omega_0 t + c_2\sin\omega_0 t$$

Suppose that the mass is at rest at $x=0$ initially.

So $x(0) = \dot{x}(0) = 0$
$$0 = x(0) = \frac{f}{\omega_0^2-\omega^2} + c_1 \quad \Rightarrow \quad c_1 = \frac{f}{\omega^2-\omega_0^2}$$
$$\dot{x}(t) = \frac{-\omega f}{\omega_0^2-\omega^2}\sin\omega t - c_1\omega_0\sin\omega_0 t + c_2\omega_0\cos\omega_0 t$$
$$\dot{x}(0) = 0 = c_2\omega_0 \quad \Rightarrow \quad c_2 = 0$$
$$\Rightarrow \quad x(t) = \frac{f}{\omega_0^2-\omega^2}\left(\cos\omega t - \cos\omega_0 t\right) \qquad (\omega \neq \omega_0)$$

$$\left[\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B\right]$$

$$A = \left(\frac{\omega + \omega_0}{2}\right)t, \quad B = \left(\frac{\omega - \omega_0}{2}\right)t$$

$$A + B = \omega t, \quad A - B = \omega_0 t$$

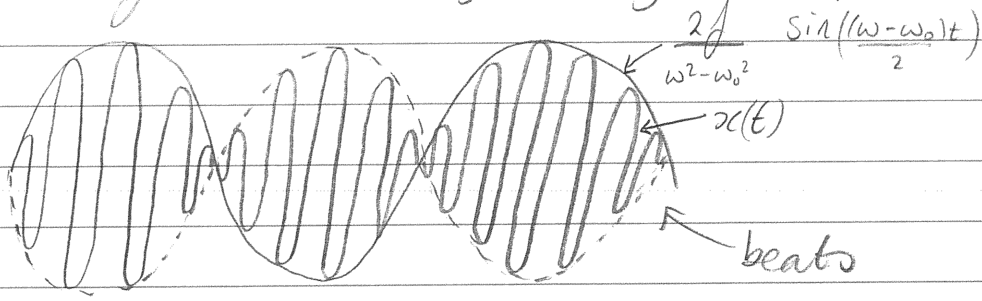
$$\Rightarrow x(t) = \frac{2f}{\omega^2 - \omega_0^2}\left(\sin\left(\left(\frac{\omega - \omega_0}{2}\right)t\right)\sin\left(\left(\frac{\omega + \omega_0}{2}\right)t\right)\right) \quad (4)$$

What happens when ω is close to ω_0:

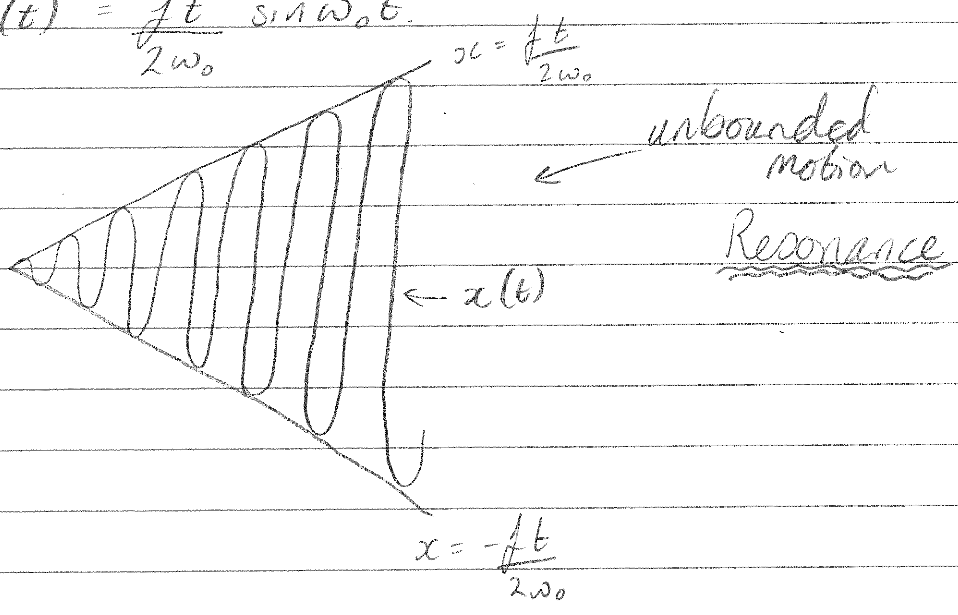
$$|\omega - \omega_0| \ll \omega + \omega_0$$

The period of $\sin\left(\left(\frac{\omega - \omega_0}{2}\right)t\right)$ is much bigger than the period of $\sin\left(\left(\frac{\omega + \omega_0}{2}\right)t\right)$.

So we can consider (4) to be a sine $\underbrace{\text{function}}_{\left[\sin\left(\frac{(\omega+\omega_0)}{2}t\right)\right]}$ oscillating in a slowly varing envelope $\left[\frac{2f}{\omega^2 - \omega_0^2}\sin\left(\frac{(\omega + \omega_0)t}{2}\right)\right]$.



Remark: if $\omega = \omega_0$, the soln of (3) is
$x(t) = \frac{ft}{2\omega_0}\sin \omega_0 t$.



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Mathematical Models

Models using 1st order eqns.

Simple population growth model:

$$\frac{dP}{dt} = kP$$ (rate of increase in population is proportional to the population).

$\Rightarrow P(t) = P_0 e^{kt}$ where $P_0 = P(0)$ = initial population.

To stop the population increasing to ∞ in the model, a standard modification is

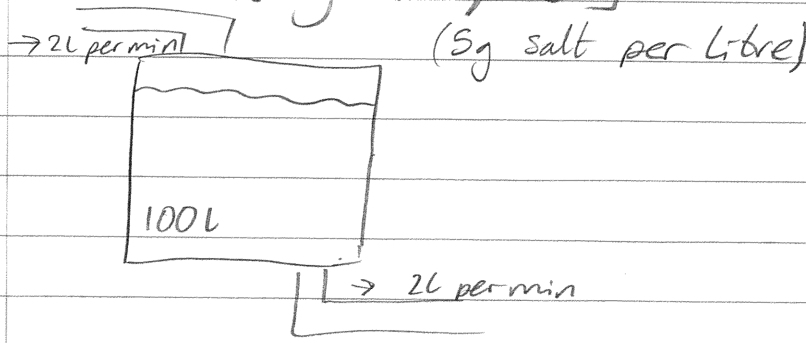
$$\frac{dP}{dt} = kP\left(1 - \frac{P}{N}\right)$$ N is a constant.

Soln: $P(t) = \dfrac{NP_0}{P_0 + (N-P_0)e^{-kt}}$ $P_0 = P(0)$

as $t \to \infty$, $P \to N$

A mixing problem:

A vat initially contains 100 litres of pure water. A brine consisting of 5g of salt per litre is pumped in at 2 litres per minute and the mixture is pumped out at the same rate. Find the concentration of salt in the vat at time t. [Assume that the vat is well mixed - i.e. concentration doesn't vary in space.]



Let $x(t)$ be the amount of salt (grams) in the vat at time t.
Let δt be a small interval of time.
Let $\delta(x)$ be the increase in x between times t and $t + \delta t$.

$$\delta(x) = \underbrace{(2 \times 5)}_{\text{coming in per min}} \delta t - \frac{2x}{100} \delta t \Leftrightarrow \frac{dx}{dt} = 10 - \frac{x}{50}$$

$$\downarrow$$

$$\frac{dx}{dt} = 10 - \frac{x}{50}.$$

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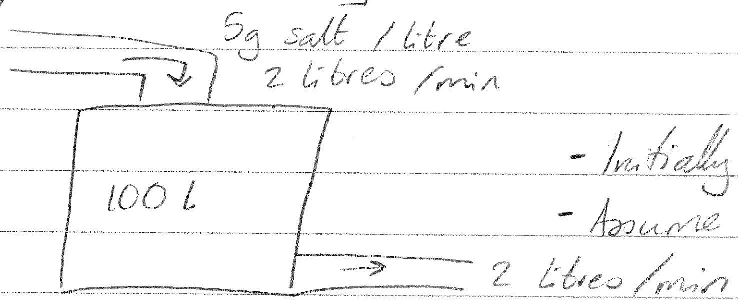
[Q from last lecture]



- Initially pure water
- Assume vat is well-mixed

concentration same at each pt. in vat

Find concentration at time t.

$x(t)$ = mass of salt in vat at time t.

During a short interval of time from t to $t+\delta t$, $x(t)$ increases by $\delta x(t)$.

$$\delta x \approx 5 \times 2 \times \delta t - \frac{x}{100} \times 2\delta t$$

concentration = # grams per litre

$$\frac{\delta x}{\delta t} = 10 - \frac{x}{50}$$

[$\delta x = x(t+\delta t) - x(t)$

$$\frac{\delta x}{\delta t} = \frac{x(t+\delta t) - x(t)}{\delta t} \approx 10 - \frac{x}{50}$$

$\to \delta t \to 0$

$$\frac{dx}{dt} = 10 - \frac{x}{50}$$]

$$\Rightarrow \frac{dx}{dt} = 10 - \frac{x}{50} \quad (*)$$

$$\int \frac{dx}{x-500} = -\int \frac{dt}{50}$$

$$\Rightarrow \ln|x-500| = -\frac{1}{50}t + c \quad , c \text{ const.}$$

$$\Rightarrow |x-500| = e^{-\frac{t}{50}+c} = e^{-\frac{t}{50}}e^{c}$$

$$\Rightarrow x = Ae^{-\frac{t}{50}} + 500 \quad [A = \pm e^{c}]$$

No salt at $t=0 \Rightarrow x(0)=0$

$0 = x(0) = 500 + A$
$\Rightarrow A = -500$

$x(t) = 500\left(1 - e^{-t/50}\right)$ ← amount of salt in vat.

so the concentration is

$$\frac{x(t)}{100} = \underline{\underline{5\left(1 - e^{-t/50}\right)}}$$

Another way to solve $(*)$:

$\frac{dx}{dt} + \frac{x}{50} = 10$ (first order inhomogeneous linear ODE)

$x = x_h + x_p$ where x_p is a particular soln and x_h solves $\dot{x} + x/50 = 0$.

$x_p = 500$ is a particular soln.

$\frac{dx}{dt} + \frac{x}{50} = 0$

$x_h = Ae^{-\frac{t}{50}}$

$x = 500 + Ae^{-t/500}$.

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Epidemics

Split a population into 3 groups:

S = # of susceptibles.

I = # of infectives.

R = # of removals. (removed / isolated / recovered ...)

$S \rightarrow I \rightarrow R$ (SIR models)

$$\frac{dS}{dt} = -\alpha S I \quad (1) \qquad \alpha, \beta > 0 \text{ const.}$$

$$\frac{dI}{dt} = \alpha S I - \beta I \quad (2)$$

Note: Total population $N = S + I + R = \text{const.}$

$$\frac{dR}{dt} = \beta I \quad (3) \qquad \frac{dN}{dt} = \frac{dS}{dt} + \frac{dI}{dt} + \frac{dR}{dt} = 0$$

Initially $S = S_0$, $I = I_0$, $R = 0$.

$S_0, I_0 > 0$

$N = S + I + R = S_0 + I_0$

Eqns (1) & (2) depend on S & I but not R.

$\frac{(2)}{(1)}$: $\frac{(dI/dt)}{(dS/dt)} = -1 + \frac{\beta}{\alpha}\frac{1}{S}$

$$\Rightarrow \frac{dI}{dS} = -1 + \frac{\beta}{\alpha}\frac{1}{S}$$

integrating:

$$I = -S + \frac{\beta}{\alpha}\ln S + c \quad (4)$$

At $t = 0$, $I = I_0$, $S = S_0$ $\Rightarrow I_0 + S_0 = \frac{\beta}{\alpha}\ln S_0 + c$

$$\Rightarrow c = N - \frac{\beta}{\alpha}\ln S_0$$

$(4): \quad I = N - S + \frac{\beta}{\alpha} \ln\left(\frac{S}{S_0}\right) \qquad (5)$

let $\rho = \frac{\beta}{\alpha}$

Outbreak (i.e. epidemic) is at its worst when I has its maximum. from (2) : $\frac{dI}{dt} = 0$

this implies $S = \frac{\beta}{\alpha} = \rho$

$I_{max} = N - \rho + \rho \ln\left(\frac{\rho}{S_0}\right)$.

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$S \rightarrow I \rightarrow R$ $\qquad N = S + I + R = S_0 + I_0$

initially $\begin{cases} S = S_0 > 0 \\ I = I_0 > 0 \\ R = 0 \end{cases}$

$$\frac{dS}{dt} = -\alpha S I \quad -①$$

$$\frac{dI}{dt} = \alpha S I - \beta I \quad -②$$

$$\frac{dR}{dt} = \beta I \quad -③$$

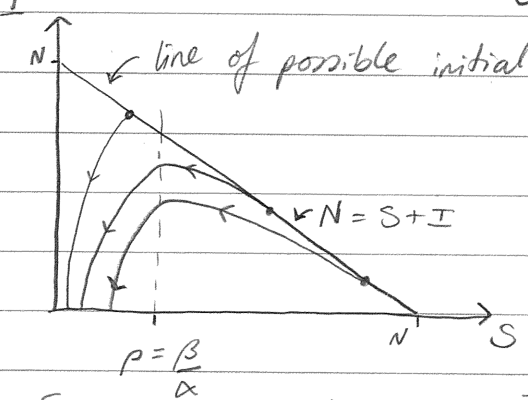
②/① : $\dfrac{dI}{dS} = -1 + \underbrace{\dfrac{\beta}{\alpha}}_{\rho} S^{-1}$

$$\Rightarrow \quad I = -S + \rho \ln S + C$$

$$\Rightarrow \quad I = N - S + \rho \ln\left(\frac{S}{S_0}\right) \quad -④$$

② $\Rightarrow$ I_{max} occurs at $S = \rho = \dfrac{\beta}{\alpha}$

④ $\Rightarrow$ $I_{max} = N - \rho + \rho \ln\left(\dfrac{\rho}{S_0}\right)$



$I(0) = I_0$, $S(0) = S_0$

$\Rightarrow N = S_0 + I_0$

[curved lines with arrows represent different solutions.]

[I_{max} occurs at $S = \rho = \dfrac{\beta}{\alpha}$]

$\frac{①}{③}$: $\frac{dS}{dR} = -\frac{\alpha S}{\beta} = -\rho^{-1} S$

$\Rightarrow S = Ae^{-\rho^{-1}R}$

at $t=0$, $S=S_0$ & $R=0$ $\Rightarrow A = S_0$

$\Rightarrow S = S_0 e^{-\rho^{-1}R}$ — ⑤

$R < S + I + R = N$

Note $S = S_0 e^{-\rho^{-1}R} > S_0 e^{-\rho N} > 0$

So some of the population will never become infected.

Sub ⑤ in ④ & then in ③

$$\frac{dR}{dt} = \beta\left(N - S_0 e^{-\rho^{-1}R} + \rho \ln\left(e^{-\rho^{-1}R}\right)\right)$$

$$= \beta\left(N - S_0 e^{-\rho^{-1}R} - R\right) \quad — ⑥$$

We can't explicitly do the integral from eqn ⑥ but often R/ρ is small, in which case we use the first few terms in the Taylor series of e^x:

$e^x = 1 + x + \frac{x^2}{2}$

⑥ becomes $\frac{dR}{dt} = \beta\left[(N - S_0) + (\rho^{-1}S - 1)R + (2\rho^2)^{-1} S_0 R^2\right]$

$\left[e^{-R/\rho} = 1 - \frac{R}{\rho} + \frac{1}{2}\left(\frac{R}{\rho}\right)^2\right]$

$$\Rightarrow R(t) = \frac{\rho^2}{S_0}\left[(\rho^{-1}S_0 - 1) + \alpha \tanh\left(\frac{\hat{\alpha}\beta t}{2} - \varphi\right)\right]$$

$\hat{\alpha}$, φ = explicit constants.

$$\frac{dR}{dt} = \frac{\beta^3}{2S_0} \operatorname{sech}^2\left(\frac{\hat{\alpha}\beta t}{2} - \varphi\right)$$

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Predator - prey models

let $\begin{cases} x = \text{population of predators (eg. fox)} \\ y = \text{population of prey (e.g. rabbit)} \end{cases}$

$$\frac{dx}{dt} = -ax + bxy$$

$$\frac{dy}{dt} = cy - dxy$$

$$\left.\begin{aligned} \frac{dx}{dt} &= x(-a+by) \quad —① \\ \frac{dy}{dt} &= y(c-dx) \quad —② \end{aligned}\right\} \leftarrow \text{Lotka - Volterrra model.}$$

$$\frac{②}{①} : \frac{dy}{dx} = \frac{y}{x}\left(\frac{c-dx}{-a+by}\right)$$

$$= \frac{cx^{-1}-d}{-ay^{-1}+b}$$

$$\Leftrightarrow (ay^{-1}-b)\frac{dy}{dx} + (cx^{-1}-d) = 0$$

$$\Leftrightarrow a\ln y - by + c\ln x - dx = K_1 \quad \text{(const.)}$$

Take the exponential:

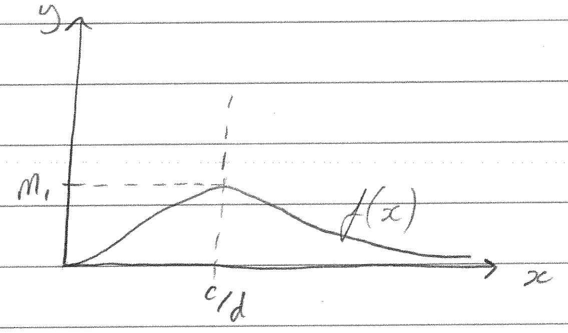
$$(y^a e^{-by})(x^c e^{-dx}) = K \quad \left[= e^{K_1}\right]$$

Let $f(x) = x^c e^{-dx}$ & $g(y) = y^a e^{-by}$

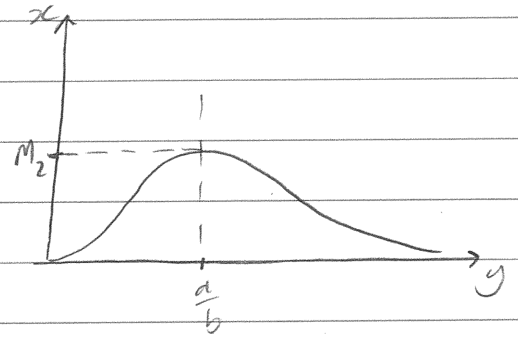
$$\Rightarrow f(x)g(y) = K$$

Graph $f(x)$ for $x \geq 0$ [note $a, b, c, d > 0$]

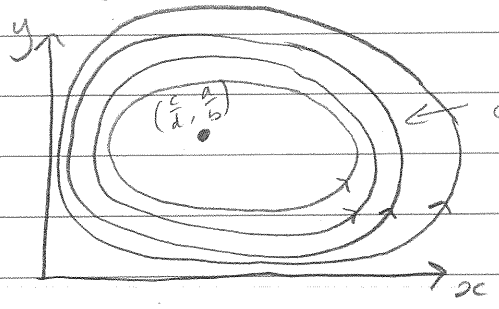
$f(0) = 0$, $\lim_{x \to \infty} f(x) = 0$ $\qquad f'(x) = x^{c-1} e^{-dx}(c - dx) = 0$
$\Rightarrow x = c/d$



So f has one maximum at $x = c/d$.
$M_1 = \left(\frac{c}{d}\right)^c e^{-c}$



$f(x)\, g(y) = K$, No soln unless $K \leq M_1 M_2$



corresponding to different initial conditions.

periodic behaviour.

[Atto-fox problem! (Atto = 10^{-14})]

Average number of predators over a period is

$$\bar{x} = \frac{1}{T} \int_0^T x(t)\, dt$$

where T is the (time) period (i.e. time to return to initial populations)

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$$\bar{x} = \frac{1}{T}\int_0^T x(t)\,dt$$

$$= \frac{1}{T}\int_0^T \left[\frac{c}{d} - \frac{1}{d}\frac{1}{y}\frac{dy}{dt}\right] dt$$

$$= \frac{1}{T}\left[\frac{c}{d}T - \frac{1}{d}\left(\ln y(t)\right)\Big|_0^T\right]$$

$$= \frac{c}{d} - \frac{1}{dT}\left(\cancel{\ln(y(T)) - \ln(y(0))}\right) = \frac{c}{d}$$

as $y(T) = y(0)$

So $\bar{x} = \frac{c}{d}$. Similarly $\bar{y} = \frac{a}{b}$

Now we include the effects of harvesting.
Assume that a certain fraction of each population is constantly removed.
This modifies our system

$$\frac{dx}{dt} = x(-a + by - h_1)$$

$$\frac{dy}{dt} = y(c - dx - h_2)$$

This system is the same as ① & ② with a & c replaced by $(a+h_1)$ & $(c-h_2)$ respectively.
So the average predator population has decreased from $\frac{c}{d}$ to $\frac{c-h_2}{d}$ & the prey population has increased from $\frac{a}{b}$ to $\frac{a+h_1}{b}$.

Recall: $\frac{dP}{dt} = kP\left(1 - \frac{P}{N}\right)$

$\uparrow$ limiting term

Nullclines

A model of two competing spieces with limiting population growth is

$$\frac{dx}{dt} = x(5 - x - y)$$

$$\frac{dy}{dt} = y(8 - 2y - x)$$

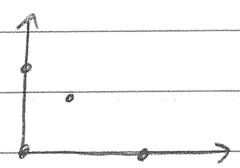
Equlibrium solutions (solns where x & y are constant)

$$\frac{dx}{dt} = 0 = \frac{dy}{dt}$$

$x(5-x-y)=0 \iff x=0$ or $x+y=5$

$y(8-2y-x)=0 \iff y=0$ or $x+2y=8$

$(0,0)$, $(0,4)$, $(5,0)$, $(2,3)$



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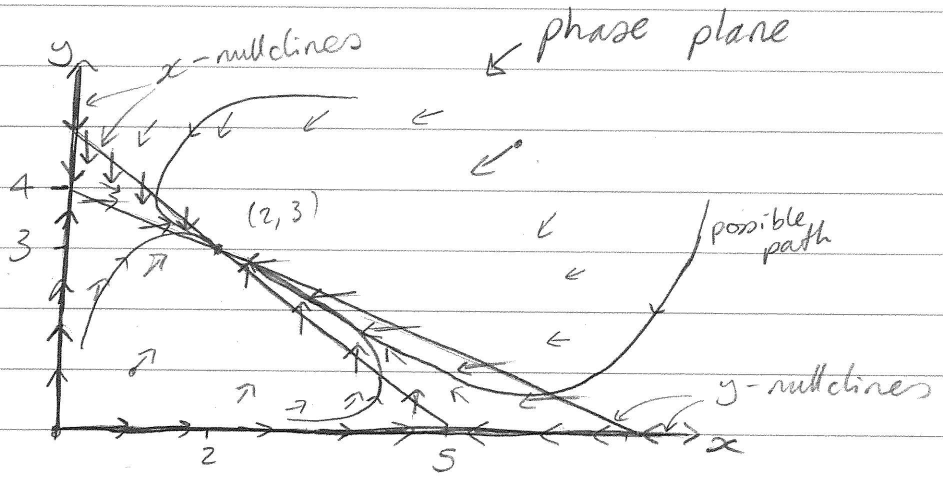
Example

2 competing species

$$\dot{x} = x(5 - x - y) \quad -①$$

$$\dot{y} = y(8 - 2y - x) \quad -②$$



Equilibrium solns $(\dot{x} = 0, \ \dot{y} = 0)$

$x(5 - x - y) = 0$ & $y(8 - 2y - x) = 0$

$\Rightarrow \left\{ \begin{array}{l} x = 0 \\ \text{or} \\ x + y = 5 \end{array} \right\}$ and $\left\{ \begin{array}{l} y = 0 \\ \text{or} \\ x + 2y = 8 \end{array} \right\}$

$\Rightarrow (0,0), (5,0), (0,4), (2,3)$

In ①, ② think of (x,y) as the position of some particle at time t.
Then its velocity is $(\dot{x}, \dot{y})$. So eqns ① and ② let us calculate the velocity given the position.
The velocity vector is tangent to the curve along which the particle moves.
So we draw arrows representing the direction of the velocity at any pt. (i.e, all of unit length).
To help this we start by drawing curves at which $\dot{x} = 0$ (x-nullclines) & curves on which $\dot{y} = 0$ (y-nullclines). The velocity vectors will be vertical

(respectively horizontal) on these curves.

x-nullclines: $\dot{x} = 0$

$x = 0$, $x + y = 5$

y-nullclines: $\dot{y} = 0$

$y = 0$, $x + 2y = 8$

At $(1, 1)$

$\dot{x} = 3$, $\dot{y} = 5$

$\dot{x}, \dot{y}$ vary continuously.
The sign of $\dot{x}$ cannot change without $\dot{x}$ first becoming 0, i.e. unless we cross an x-nullcline.
Similarly for y.

For large x, y, $\dot{x} < 0$, $\dot{y} < 0$.

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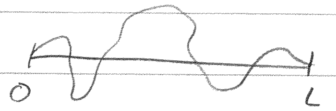
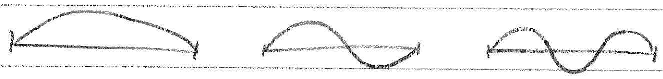
Waves and Fourier Series





Fourier sine series of $f(x)$ on $[0, L]$

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right) \quad -① $$

$$\int_0^L \sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{m\pi x}{L}\right) dx \qquad n, m > 0$$

$n \neq m$

$$= \frac{1}{2}\int_0^L \cos\left(\frac{(n-m)\pi x}{L}\right) - \cos\left(\frac{(n+m)\pi x}{L}\right) dx$$

$$= \frac{L}{2}\left[\frac{1}{(n-m)\pi}\sin\left(\frac{(n-m)\pi x}{L}\right) - \frac{1}{(n+m)\pi}\sin\left(\frac{(n+m)\pi x}{L}\right)\right] = 0 \quad , n \neq m$$

If $n = m$

$$= \int_0^L \sin\left(\frac{n\pi x}{L}\right)^2 dx = \frac{L}{2}$$

$$\Rightarrow \int_0^L \sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{m\pi x}{L}\right) dx = \frac{L}{2}\delta_{mn} = \begin{cases} L/2 & n = m \\ 0 & n \neq m \end{cases}$$

① $\times \sin\left(\frac{m\pi x}{L}\right)$: $\quad f(x)\sin\left(\frac{m\pi x}{L}\right) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{m\pi x}{2}\right)$

$$\int_0^L f(x)\sin\left(\frac{m\pi x}{L}\right) dx = \sum_{n=1}^{\infty} b_n \int_0^L \sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{m\pi x}{L}\right) dx$$

$$= \sum_{n=1}^{\infty} b_n \delta_{mn}\frac{L}{2} = \frac{L}{2} b_m$$

$$b_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)dx$$

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Stability

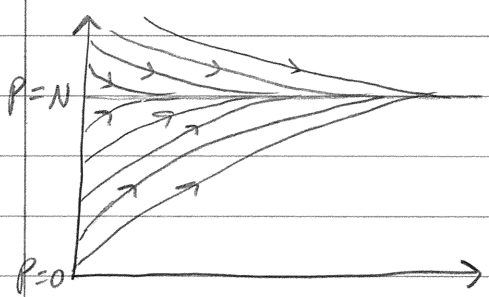
Example:

$$\frac{dP}{dt} = kP\left(1 - \frac{P}{N}\right)$$

Equilibrium solns: $\frac{dP}{dt} = 0$ (constant solns)

2 equilibrium solns here: $P=0$, $P=N$

If nearby initial conditions to an equilibrium soln, lead to solns that stay close to the equilibrium soln, then the equilibrium soln is called stable.



If P_0 statisfies $0 < P_0 < N$, then $\frac{dP}{dt} > 0$.

If $P_0 > N$, $\frac{dP}{dt} < 0$.

$P = N$ is a stable soln

For P_0 just above 0, the soln moves away from 0 so $P=0$ is an unstable soln.

Fourier Series

The Fourier series expansion of f on $[-L, L]$ is:

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right) \quad -①$$

Note: $[n = 1, 2, \ldots]$ $\int_{-L}^{L} \cos\left(\frac{n\pi x}{L}\right) dx$

$$= \frac{L}{n\pi} \sin \frac{n\pi x}{L} \Big|_{-L}^{L}$$

$$= \frac{L}{n\pi} (\sin n\pi - \sin n\pi) \qquad (\sin n\pi = 0)$$

$$= 0$$

$$\int_{-L}^{L} \sin\left(\frac{n\pi x}{L}\right) dx$$

$$= \frac{-L}{n\pi} (\cos n\pi - \cos(-n\pi))$$

$$= 0$$

$$\int_{-L}^{L} f(x)\, dx = \frac{a_0}{2} \int_{-L}^{L} dx + \sum_{n=1}^{\infty} a_n \underbrace{\int_{-L}^{L} \cos\left(\frac{n\pi x}{L}\right) dx}_{0} + \sum_{n=1}^{\infty} b_n \underbrace{\int_{-L}^{L} \sin\left(\frac{n\pi x}{L}\right) dx}_{0}$$

$$= a_0 L$$

$$\int_{-L}^{L} \cos\left(\frac{n\pi x}{L}\right) \cos\left(\frac{m\pi x}{L}\right) dx = L\delta_{mn} = \begin{cases} L & m = n \\ 0 & m \neq n \end{cases}$$

$$\int_{-L}^{L} \sin\left(\frac{n\pi x}{L}\right) \cos\left(\frac{m\pi x}{L}\right) dx = 0 \quad \left[\int_{-L}^{L} (\text{odd function}) = 0\right]$$

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$$\int_{-L}^{L} ① \cos\left(\frac{m\pi x}{L}\right) dx$$

$$= \int_{-L}^{L} f(x) \cos\left(\frac{m\pi x}{L}\right) dx$$

$$= \int_{-L}^{L} \left[\frac{a_0}{2} \cos\left(\frac{m\pi x}{L}\right) + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right) \cos\left(\frac{m\pi x}{L}\right) + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L} \cos\left(\frac{m\pi x}{L}\right)\right] dx$$

$$= \frac{a_0}{2} \cdot 0 + \sum_{n=1}^{\infty} a_n L \delta_{mn} + \sum_{n=1}^{\infty} b_n \cdot 0$$

$$= L a_m \qquad \left(\text{as } \delta = \begin{cases} 1, & n = m \\ 0, & n \neq m \end{cases}\right)$$

So $a_n = \int_{-L}^{L} f(x) \cos\left(\frac{n\pi x}{L}\right) dx \qquad [n = 0, 1, 2, \ldots]$

Similarly, if we multiply ① by $\sin\left(\frac{m\pi x}{L}\right)$ and integrate, we get

$$b_n = \frac{1}{L} \int_{-L}^{L} f(x) \sin\left(\frac{n\pi x}{L}\right) dx. \qquad [n = 0, 1, 2, \ldots]$$

[$f(x)$ = const., polynomial, sin/cos, e^{ax}]

If $f(x)$ is even $(f(-x) = f(x))$
then $b_n = \frac{1}{L} \int_{-L}^{L} f(x) \sin\left(\frac{n\pi x}{L}\right) dx = 0$

$$a_n = \int_{-L}^{L} f(x) \cos\left(\frac{n\pi x}{L}\right) dx = 2 \int_{0}^{L} f(x) \cos\left(\frac{n\pi x}{L}\right) dx$$

This gives the Fourier cosine series of f on $[0, L]$.

$$= \left(\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right)\right), \quad a_n = \frac{2}{L}\int_0^L f(x)\cos\left(\frac{n\pi x}{L}\right)dx$$

If $f(x) = x$ on $[0, L]$ then its Fourier cosine series extends to the even, $2L$-periodic extension.

[F has period T if $F(x+T) = F(x) \quad \forall x$]

Example

Find the Fourier series for $f(x) = x^2$ on $[-\pi, \pi]$. Assuming that the series converges to the 2π-periodic extension of f, evaluate

$$\sum_{n=1}^{\infty} \frac{1}{n^2}$$

$f(x)$ is even, so $b_n = 0$

$$a_0 = \frac{2}{\pi}\int_0^{\pi} x^2\,dx = \frac{2}{3}\pi^2, \quad a_n = \frac{2}{\pi}\int_0^{\pi} x^2 \cos nx\,dx \quad n = 1, 2, 3, \ldots$$

$$= \frac{4\cos n\pi}{n^2} = \frac{4}{n^2}(-1)^n$$

$$f(x) = \frac{a_0}{2} + \sum_{n=1} a_n \cos nx$$

$$x^2 = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty}(-1)^n \frac{1}{n^2}\cos nx \qquad -\pi \le x \le \pi$$

at $x = \pi$:

$$\pi^2 = \frac{\pi^3}{3} + 4\sum_{n=1}^{\infty} \frac{(-1)^n}{\left(\frac{1}{n^2}\right)} \underbrace{\cos n\pi}_{(-1)^n}$$

$$\Rightarrow \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$$

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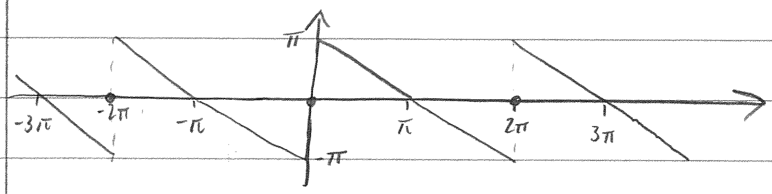
__Example__

Find the Fourier sine series & the Fourier cosine series on $0 \le x \le \pi$ for $f(x) = \pi - x \ (0 < x < \pi)$

Plot the sum of each series for $-3\pi < x \ 3\pi$

Sine series: (odd 2π - periodic extension)



← Fourier sine series converges to this

$$\sum_{n=1}^{\infty} b_n \sin nx \quad \text{where} \quad b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin(nx)\,dx$$

$$b_n = \frac{2}{\pi}\int_0^{\pi} (\pi - x)\sin(nx)\,dx$$

$$= \frac{2}{n\pi}\int_0^{\pi} (x-\pi)\frac{d\cos nx}{dx}\,dx$$

$$= \frac{2}{n\pi}\left[(x-\pi)\cos nx\Big|_0^{\pi} - \int_0^{\pi}\cos nx\,dx\right]$$

$$= \frac{2}{n\pi}\left[\pi - \frac{1}{n}\sin nx\Big|_0^{\pi}\right]$$

$$= \frac{2}{n}$$

Fourier sine series is

$$\sum_{n=1}^{\infty} b_n \sin nx = 2\sum_{n=1}^{\infty}\frac{1}{n}\sin(nx) \quad (= \pi - x \text{ for } 0 < x < \pi)$$

Fourier cosine series:

$$a_n = \frac{2}{\pi}\int_0^{\pi} (\pi - x)\cos nx \, dx \qquad [n = 0, 1, \ldots]$$

$$a_0 = \frac{2}{\pi}\int_0^{\pi} (\pi - x)\, dx = \frac{2}{\pi}\left(\pi x - \frac{1}{2}x^2\right)\Big|_0^{\pi}$$

$$= \pi$$

$n = 1, 2, \ldots$

$$a_n = \frac{2}{n\pi}\int_0^{\pi} (\pi - x)\frac{d \sin nx}{dx}\, dx$$

$$= \frac{2}{n\pi}\left[\cancel{(\pi - x)\sin nx\Big|_0^{\pi}} + \int_0^{\pi} \sin nx \, dx\right]$$

$$= \frac{2}{n\pi}\left(\frac{-1}{n}\cos nx\Big|_0^{\pi}\right)$$

$$= \frac{2}{n^2\pi}\left(1 - \cos n\pi\right)$$

$\Rightarrow$ $n = 1, 2, \ldots$

$$a_n = \frac{2}{n^2\pi}\left(1 - (-1)^n\right) = \begin{cases} 0 & n \text{ even} \\ \dfrac{4}{n^2\pi} & n \text{ odd} \end{cases}$$

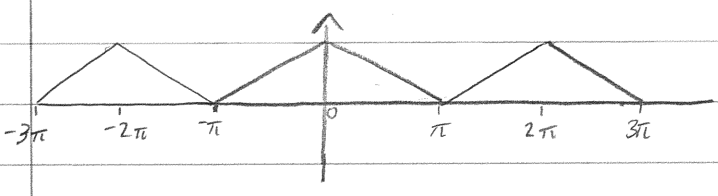
Fourier cosine series:

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx = \frac{\pi}{2} + \sum_{k=1}^{\infty} \frac{4}{(2k-1)^2\pi}$$

(only the odd part is required as even part $= 0$.)

[where $n = 2k - 1$]

$$= \frac{\pi}{2} + \frac{4}{\pi}\sum_{k=1}^{\infty}\frac{1}{(2k-1)^2}$$



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Waves

Two important classes of waves.

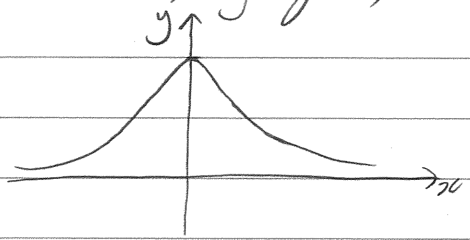
Travelling (or progressive) waves

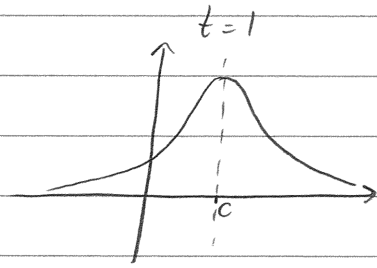
These waves maintain a fixed shape and move at a fixed speed.

We represent the vertical displacement of these waves as $y = f(x - ct)$.

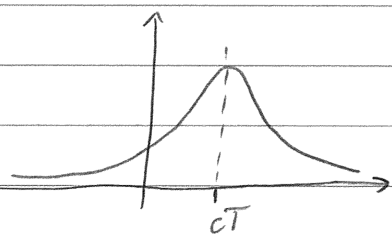
x ↑ space, t ↑ time

At $t = 0$, $y = f(x)$





At time $t = T$



$f(x - ct)$ represents a wave moving to the right (if $c > 0$) with speed c.

(If $c < 0$, it is moving left with speed $|c|$).

Standing waves

Each "particle" moves up & down, but not horizontally.

Any wave of the form $y = X(x)\,T(t)$ is a standing wave.

eg. $y(x,t) = \cos(\alpha x)\sin(\beta t)$

$$f(x) = \underbrace{F(x)}_{\text{even}} + \underbrace{G(x)}_{\text{odd.}} \quad -①$$

$$f(-x) = F(-x) + G(-x)$$
$$= F(x) - G(x) \quad -②$$

$$\frac{①+②}{2} : F(x) = \frac{1}{2}\left(f(x) + f(-x)\right)$$

$$\frac{①-②}{2} : G(x) = \frac{1}{2}\left(f(x) - f(-x)\right)$$

$$F(x) + G(x) = f(x)$$

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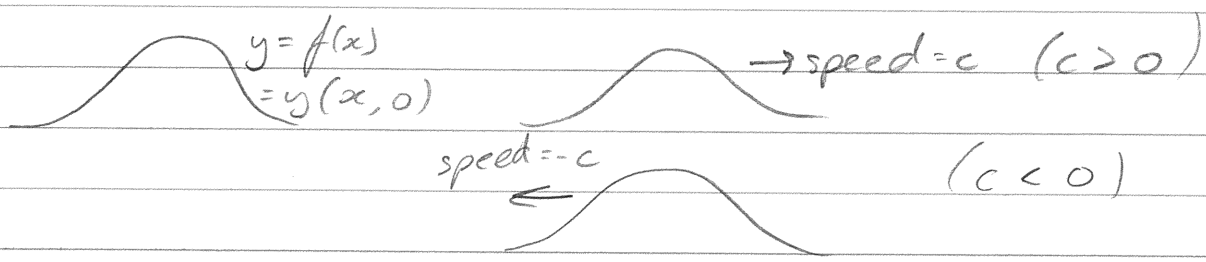
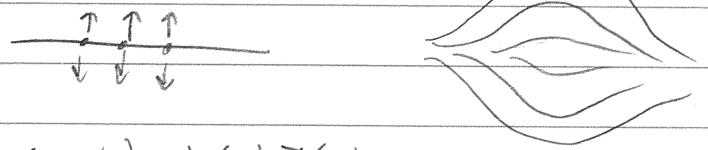
Travelling waves

$$y(x,t) = f(x - ct)$$



Standing waves



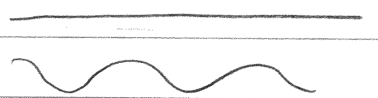
$y = X(x)$

$$y(x,t) = X(x)\,T(t)$$

$$y(x,t) = \cos(\alpha x)\sin(\beta t)$$

$t = 0$

$t = c$



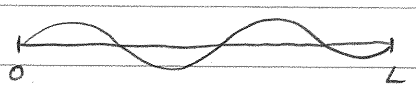
Waves in an elastic string

The vertical displacement of an elastic string at time t is given by $y(x,t) = f(x - ct) + g(x + ct)$ —① where c is a constant depending on the string. (f and g are arbitrary).

Now consider a string stretched between $(x,y) = (0,0)$ and $(x,y) = (L,0)$.

$y(0,t) = 0$ —②

$y(L,t) = 0$ —③



② in ①:

$$0 = y(0,t) = f(-ct) + g(ct) \quad \forall t$$

so $\quad g(z) = -f(-z) \quad \forall z \quad — ④$

③ in ①:

$$0 = y(L,t) = f(L-ct) + g(L+ct)$$

$$= f(L-ct) - f(-L-ct)$$

(Let $u = -L-ct$)

$$f(u+2L) = f(u) \quad [\Rightarrow f \text{ is } 2L\text{-periodic}]$$

So $y(x,t) = f(x-ct) - f(-x-ct)$, where f is $2L$-periodic. ⑤

Suppose now that initially $(t=0)$ the shape of the string is given by

$$y(x,0) = F(x) \qquad 0 < x < L$$

⑤: $\quad f(x) - f(-x) = F(x), \quad 0 < x < L \quad$ ⑥

Choose $x_0 \in (0,L)$
and define $Y(t) = y(x_0,t) = f(x_0 - ct) - f(-x_0 - ct)$
so $Y(t)$ represents the vertical displacement of the 'particle' on the string directly above (or below) $x = x_0$.
So its vertical velocity is $\dfrac{dY(t)}{dt}$.

$$Y'(t) = -cf'(x_0 - ct) + cf'(-x_0 - ct)$$

Now impose the condition that the string is realeased from rest.

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So $Y'(t) = 0 \quad \forall x_0$

$\Leftrightarrow \quad f'(x_0 - ct) = f'(-x_0 - ct)$ at $t = 0$.

$f'(x_0) = f'(-x_0) \quad \Leftrightarrow \quad f'(z) = f'(-z)$ for $z \in (0, L)$

(integrating:)

$\Leftrightarrow \quad -f(-z) = f(z) + C$ ⑦ for some constant, C.

Let $h(z) = 2f(z) + C$

⑥ : $h(x) - h(-x) = F(x)$ — ⑧

⑦ : $h(x) + h(-x) = 0$ — ⑨

⑤ : $y(x,t) = \frac{1}{2}\{h(x-ct) - c\} - \frac{1}{2}\{h(-x-ct) - c\}$

$= \frac{1}{2}\left(h(x-ct) - h(-x-ct)\right)$

⑨: (h is odd)

$y(x,t) = \frac{1}{2}\{h(x-ct) + h(x+ct)\}$

⑧ (use h odd) : $h(x) + h(x) = 2F(x)$

$\Leftrightarrow \quad h(x) = F(x) \qquad 0 < x < L$

In summary:

- $y(x,t) = \frac{1}{2}\left(h(x-ct) + h(x+ct)\right)$
- where $h(x) = F(x) \qquad 0 < x < L$
- $h(x)$ is odd $\quad (h(-x) = -h(x))$
- h is $2L$-periodic.

So $h(x)$ has a Fourier sine series representation:

$h(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right), \quad b_n = \frac{2}{L}\int_0^L F(x) \sin\left(\frac{n\pi x}{L}\right) dx.$

So $y(x,t) = \frac{1}{2}\left\{h(x-ct) + h(x+ct)\right\}$

$$y(x,t) = \frac{1}{2}\sum_{n=1}^{\infty} b_n \left\{\sin\frac{n\pi}{L}(x-ct) + \sin\frac{n\pi}{L}(x+ct)\right\}$$

$$= \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right)\cos\left(\frac{n\pi ct}{L}\right)$$

(∞ sum of standing waves).

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Systems with changing mass

Last remark on Fourier Series

$$\sum_{n=0}^{\infty} a^{-n} \cos b^n x \qquad b \geq a > 1$$

converges to a continuous function that is differentiable nowhere.

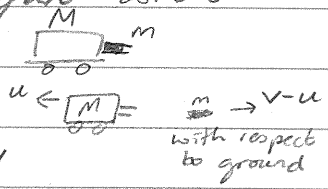
Systems with changing mass

eg. rocket using up fuel, etc.

Examples of conservation of momentum.

Example 1

Consider a gun of mass M firing a shell of mass m such that the shell leaves the gun with speed v relative to the barrel.



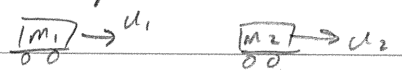
The gun is on wheels and is free to move without friction. When the gun fires the shell it leaves the barrel horizontally and the gun recoils with speed u. Find u.

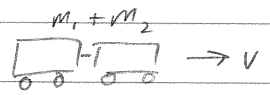
Initial momentum = final momentum

$$\Rightarrow \quad 0 = M(-u) + m(v-u) \qquad \Rightarrow \quad u = \frac{mv}{M+m}$$

Example 2

2 train carriages of mass m_1 & m_2 move on track with speeds u_1 and u_2 $(u_1 > u_2)$ when they meet they couple & move together with speed v. Find v.





Initial momentum = final momentum

$$m_1 u_1 + m_2 u_2 = (m_1 + m_2) v$$

$$\Rightarrow \quad v = \frac{m_1 u_1 + m_2 u_2}{m_1 + m_2}$$

Changing mass systems.

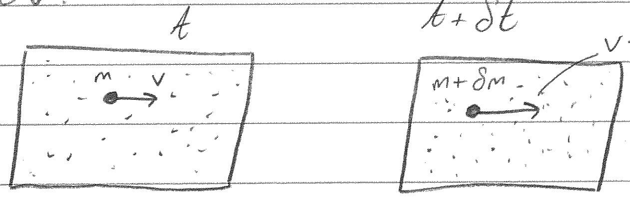
We must be careful how we state Newton's 2nd law for systems

$F =$ rate of change of momentum

(if $m = \text{const}$, $\frac{d(mv)}{dt} = m\frac{dv}{dt} = ma$)

Example

Consider a particle moving through a cloud of stationary dust. The particle has mass m at time t & the total mass of the dust cloud is $M(t)$. Let v be the velocity of the particle at time t & at time $t+\delta t$, the velocity is $v+\delta v$.



Change in momentum = 'final' − 'initial'

$$= \left[(m+\delta m)(v+\delta v) - (M-\delta m)\cdot 0\right] - \left[mv - M\cdot 0\right]$$

$$= \cancel{mv} - \cancel{mv} + m\delta v + v\delta m + \delta m\delta v$$

Rate of change of momentum $= \lim_{\delta t \to 0}\left(\frac{m\delta v + v\delta m + \delta m\delta v}{\delta t}\right)$

$$= \lim_{\delta t\to 0}\left(m\frac{\delta v}{\delta t} + v\frac{\delta m}{\delta t} + \delta m\frac{\delta v}{\delta t}\right)$$

$$= m\frac{dv}{dt} + v\frac{dm}{dt} = \frac{d(mv)}{dt}$$

$$F = \frac{d(mv)}{dt}$$

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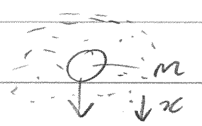
<u>Example</u>

Falling raindrop.

A raindrop falls through a cloud and accumulates mass at a rate kmv, wher $k>0$ is a constant, m is its mass & v is its speed.

What is the speed of the raindrop at a given time t if it starts from rest. & what is its mass?



Only force is gravity.

Take displacement x pointing down:

$$mg = F = \frac{d(mv)}{dt} = m\frac{dv}{dt} + v\frac{dm}{dt}$$

Now $\frac{dm}{dt} = kmv$

$$\Rightarrow \quad mg = m\frac{dv}{dt} + kmv^2$$

$$\Leftrightarrow \quad g = \frac{dv}{dt} + kv^2$$

$$\Leftrightarrow \quad \frac{dv}{dt} = g - kv^2$$

$$= k(\alpha^2 - v^2) \quad \text{where } \alpha^2 = \frac{g}{k}$$

$$dt = \frac{dv}{k(\alpha^2 - v^2)}$$

$$\int_0^t dT = \frac{1}{2\alpha k}\int_0^V \left(\frac{1}{(\alpha+v)} + \frac{1}{(\alpha - v)}\right) dv$$

$$t = \frac{1}{2\alpha k}\left(\log(\alpha + v) - \log(\alpha - v)\right)$$

$$\Rightarrow t = \frac{1}{2\alpha k} \log\left(\frac{\alpha+v}{\alpha-v}\right)$$

$$(\alpha - v)e^{2\alpha k t} = \alpha + v$$

$$\Rightarrow v = \alpha\left(\frac{e^{2\alpha k t} - 1}{e^{2\alpha k t} + 1}\right) = \alpha\left(\frac{e^{\alpha k t} - e^{-\alpha k t}}{e^{\alpha k t} + e^{-\alpha k t}}\right)$$

$$\Rightarrow v = \alpha \tanh(\alpha k t)$$

$$\Rightarrow v = \sqrt{\frac{g}{k}} \tanh(\sqrt{gk}\, t)$$

Now $\frac{dm}{dt} = kmv$

$$\Rightarrow \frac{1}{m}\frac{dm}{dt} = kv = \sqrt{gk} \tanh(\sqrt{gk}\, t)$$

integrating:

$$\Rightarrow \log m - \log \underset{\text{initial mass}}{m_0} = \log \cosh(\sqrt{gk}\, t)$$

$$\Rightarrow m = m_0 \cosh(\sqrt{gk}\, t)$$

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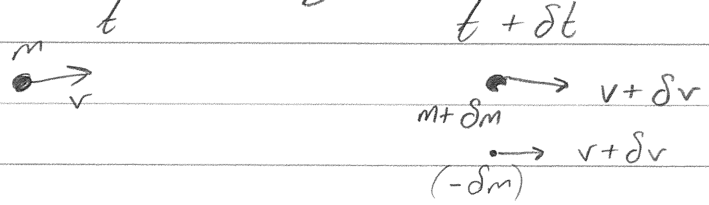
Mass lost or gained at 0 relative velocity



Momentum at t: mv

" " $t+\delta t$: $(m+\delta m)(v+\delta v) + (-\delta m)(v+\delta v)$

Rate of change of momentum:

$$\lim_{\delta t \to 0} \left\{ (\delta t)^{-1} \left[\left(\cancel{mv} + m\delta v + \cancel{v\delta m} + \underbrace{\delta m \delta v}_{0} - \cancel{v\delta m} - \underbrace{\delta m \delta v}_{0} \right) - \cancel{mv} \right] \right\}$$

$$= m\frac{dv}{dt}$$

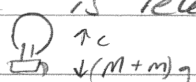
Newton's 2nd Law:

$$F = m\frac{dv}{dt} \qquad (*)$$

Example

A balloon of constant mass M containing a bag of sand of mass m_0 experiences a constant upward thrust, c. Initially it is in equilibrium & then the sand is released at a constant rate so that it is all released in time t_0. Find the height of the balloon & its velocity when all the sand is released.

Soln.



Sand is realeased with (approx) 0 relative velocity.

From $(*)$: $F = m\frac{dv}{dt}$

Take x to be vertical displacement $v = \dot{x}$ ↑ etc.

$$c - (m+M)g = (m+M)\frac{dv}{dt}$$

Sand is released at a constant rate.

$$m(t) = m_0 - \lambda t \qquad \left[\frac{dm}{dt} = -\lambda \right]$$

$$= m_0 - \frac{m_0}{t_0} t$$

so $$\frac{dv}{dt} = \frac{c}{(m+M)} - g = \frac{c}{(M + m_0 - \lambda t)} - g$$

integrating

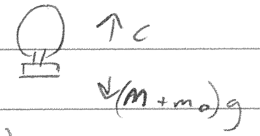
$$\Rightarrow \quad v = -\frac{c}{\lambda} \log(M + m_0 - \lambda t) - gt + K$$

At $t=0$, $v=0$

$$\Rightarrow K = \frac{c}{\lambda} \log(M + m_0)$$

$$\Rightarrow v = -gt - \frac{c}{\lambda} \log\left(\frac{M + m_0 - \lambda t}{M + m_0}\right)$$

Initially : equilibrium



$$\Rightarrow \quad c = (M + m_0) g$$

At $t = t_0$, velocity is

$$v(t_0) = -g t_0 - \left(\frac{M + m_0 g t_0}{m_0}\right) \log\left(\frac{M + m_0 - \frac{m_0}{t_0} t_0}{M + m_0}\right)$$

$$= -g t_0 - \left(\frac{M + m_0 g t_0}{m_0}\right) \log\left(\frac{M}{M + m_0}\right)$$

$$x = \int_0^{t_0} v\, dt = -\frac{1}{2} g t_0^2 - \frac{c}{\lambda} \int_0^{t_0} \log\left(1 - \frac{\lambda}{M+m_0} t\right) dt$$

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let $u = 1 - \frac{\frac{m_0}{t_0} t}{M + m_0}$

$du = \frac{-\frac{m_0}{t_0}}{M + m_0} dt$

$x = \frac{1}{2} g t_0^2 - \frac{c}{\lambda} \int_0^{m_0/M+m_0} \log u \, du . \frac{M + m_0}{\left(\frac{m_0}{t_0}\right)}$

$\int \log u \, du = u \log u - \int du = u \log u - u + \text{const.}$

So $x(t_0) = x(0) + \frac{g t_0^2}{2 m_0^2} \left(4M^2 + 6M m_0 + 3 m_0^2\right) - \frac{g t_0^2}{2} \log\left(\frac{M}{M + m_0}\right)$

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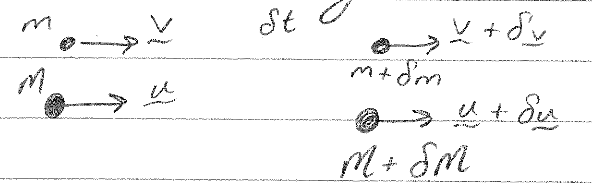
Variable Mass Systems

F = rate of change of momentum.

General variable mass problems

Mass m moves with velocity v and mass M moves with velocity u. Suppose that a small amount of mass is exchanged between these.



Mass is exchanged (no net gain or loss) so $\delta M = -\delta m$

(δm could be -ve or +ve).

Change in momentum = final − initial

$$= [(M+\delta M)(\underset{\sim}{u}+\delta \underset{\sim}{u}) + (m+\delta m)(\underset{\sim}{v}+\delta \underset{\sim}{v})] - [m\underset{\sim}{v} + M\underset{\sim}{u}]$$

$$= \cancel{M\underset{\sim}{u}} + \overset{-\delta m}{\delta M \underset{\sim}{u}} + M\delta \underset{\sim}{u} + \underbrace{\delta M \delta \underset{\sim}{u}}_{\text{will} \to \text{ to } 0} + \cancel{m\underset{\sim}{v}} + \delta m \underset{\sim}{v} + m\delta \underset{\sim}{v} + \delta m \delta \underset{\sim}{v} - \cancel{m\underset{\sim}{v}} - \cancel{M\underset{\sim}{u}}$$

$$= m\delta \underset{\sim}{v} + \underset{\sim}{v}\delta m + M\delta \underset{\sim}{u} - \underset{\sim}{u}\delta M + (\delta M \delta \underset{\sim}{u} + \delta m \delta \underset{\sim}{v})$$

$$F = \lim_{\delta t \to 0} \frac{\text{change in momentum}}{\delta t}$$

$$= \lim_{\delta t \to 0} \left(m\frac{\delta \underset{\sim}{v}}{\delta t} + \underset{\sim}{v}\frac{\delta m}{\delta t} + M\frac{\delta \underset{\sim}{u}}{\delta t} - \underset{\sim}{u}\frac{\delta m}{\delta t} + \delta M \frac{\delta \underset{\sim}{u}}{\delta t} + \delta m \frac{\delta \underset{\sim}{v}}{\delta t} \right.$$

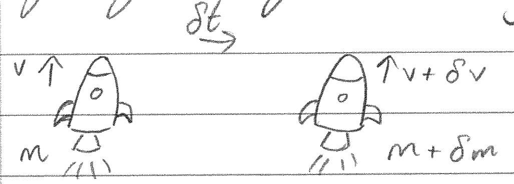
$$\Rightarrow F = m\frac{d\underset{\sim}{v}}{dt} + \underset{\sim}{v}\frac{dm}{dt} + M\frac{d\underset{\sim}{u}}{dt} - \underset{\sim}{u}\frac{dm}{dt}$$

$$= \frac{d}{dt}(m\underset{\sim}{v}) + M\frac{d\underset{\sim}{u}}{dt} - \underset{\sim}{u}\frac{dm}{dt}$$

Examples

Rockets 1

A rocket of mass m emits mass backwards at speed relative to the rocket, at a constant rate k. Ignoring gravity and air resistance, find its speed v at time t if at $t=0$ it has speed v_0 and mass $M + m_0$, where m_0 is the amount of fuel for burning.



No external forces.

fuel $\rightarrow$ $-\delta m$

$(v + \delta v - u)$

$$0 = \lim_{\delta t \to 0}\left[\frac{(m+\delta m)(v+\delta v) + (-\delta m)(v + \delta v - u) - mv}{\delta t}\right]$$

$$= \lim_{\delta t \to 0}\left[\frac{\cancel{mv} + m\delta v + \cancel{\delta m v} + \cancel{\delta m \delta v} - \cancel{\delta m v} - \cancel{\delta m \delta v} + u\delta m - \cancel{mv}}{\delta t}\right]$$

$$= \lim_{\delta t \to 0}\left[m\frac{\delta v}{\delta t} + u\frac{\delta m}{\delta t}\right]$$

$$\Rightarrow 0 = m\frac{dv}{dt} + u\frac{dm}{dt}$$

mass ejected at rate k: $\frac{dm}{dt} = -k$

$$m(t) = \overset{\text{const}}{K} - kt \quad (K = m(0) = M + m_0 = M_0)$$

$$= M + m_0 - kt$$

$$= M_0 - kt$$

$$0 = m\frac{dv}{dt} + u\frac{dm}{dt}$$

$$= m\frac{dv}{dt} + u(-k)$$

$$= m\frac{dv}{dt} - ku$$

$$\Rightarrow \frac{dv}{dt} = \frac{ku}{m} = \frac{ku}{M_0 - kt}$$

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$$\frac{dv}{dt} = \frac{ku}{M_0 - kt} \Rightarrow v = -u\log(M_0 - kt) + C$$

at $t=0$, $v = v_0$

$$v_0 = -u\log(M_0) + C$$
$$\Rightarrow C = v_0 + u\log(M_0)$$

$$v = -u\log(M_0 - kt) + v_0 + u\log(M_0)$$
$$= v_0 + u\log\left(\frac{M_0}{M_0 - kt}\right) \quad \text{or} \quad = v_0 - u\log\left(1 - \frac{k}{M_0}t\right), \quad t < \frac{M_0}{k}$$

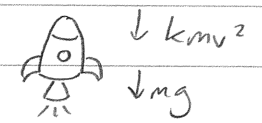
(up to where the rocket runs out of fuel).

<u>Rockets 2</u>

Suppose that the rocket now moves under gravity and air resistance kmv^2, and the fuel is burned so that $m = m_0 e^{-bt}$ where $b > \frac{g}{u}$ is a constant. The rocket starts from rest.

From first example:

$$F = m\frac{dv}{dt} + u\frac{dm}{dt} = -mg - kmv^2$$



$$\Leftrightarrow \frac{dv}{dt} = -g - kv^2 - \frac{u}{m}\frac{dm}{dt}$$

Now $\frac{dm}{dt} = -bm_0e^{-bt} = -bm$

$$\Rightarrow \frac{1}{m}\frac{dm}{dt} = -b$$

$$\frac{dv}{dt} = (ub - g) - kv^2 \qquad \text{let } \lambda^2 = ub - g > 0 \ \left(b > \frac{g}{u}\right)$$

$$k = \alpha^2$$

$$= \lambda^2 - \alpha^2 v^2$$

$$\Rightarrow dt = \frac{dv}{\lambda^2 - \alpha^2 v^2}$$

$$2\lambda\, dt = \left(\frac{1}{\lambda + av} + \frac{1}{\lambda - av}\right) dv$$

$$\Rightarrow \log\left(\frac{\lambda + \alpha v}{\lambda - \alpha v}\right) = 2\lambda\alpha t + c'$$

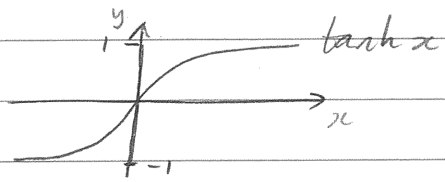
At $t = 0,\ v = 0 \Rightarrow c' = 0$

$$\Rightarrow \log\left(\frac{\lambda + \alpha v}{\lambda - \alpha v}\right) = 2\lambda\alpha t$$

$$\Rightarrow \frac{\lambda + \alpha v}{\lambda - \alpha v} = e^{2\lambda\alpha t}$$

$$\Rightarrow v(t) = \frac{\lambda}{\alpha} \tanh(\lambda\alpha t)$$

$$t \to \infty \qquad v(t) \to \frac{\lambda}{\alpha} = \sqrt{\frac{ub - g}{k}}$$



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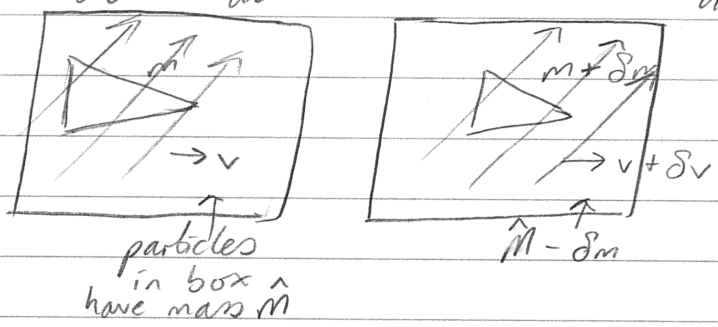
Example

The mass of a spacecraft at time t is $m(t)$ & its velocity is $\underset{\sim}{V}(t)$. For $t<0$ $m(t)=m$ and $\underset{\sim}{V}=U\underset{\sim}{i}$ where M & U are const. and $\underset{\sim}{i}$ is a constant unit vector.

For $0<t<T$, the craft encounters a stream of particles which have velocity $w(\cos\alpha\underset{\sim}{i}+\sin\alpha\underset{\sim}{j})$, where w & α are constants & j is a unit vector $\perp$ to $\underset{\sim}{i}$. A constant mass ρ of the particles enter the craft per unit time & these particles are thereafter stationary relative to the craft.

(a) If $\underset{\sim}{V}=u\underset{\sim}{i}+v\underset{\sim}{j}$, show that

$$m\frac{du}{dt}+\frac{dm}{dt}u=\rho w\cos\alpha,\quad m\frac{dv}{dt}+\frac{dm}{dt}v=\rho w\sin\alpha,\quad \frac{dm}{dt}=\rho.$$



Initial momentum:

$$m\underset{\sim}{v}+\hat{M}w(\cos\alpha\underset{\sim}{i}+\sin\alpha\underset{\sim}{j})$$

Final momentum:

$$(m+\delta m)(\underset{\sim}{v}+\delta\underset{\sim}{v})+(\hat{M}-\delta m)w(\cos\alpha\underset{\sim}{i}+\sin\alpha\underset{\sim}{j})$$

No external forces:

$$0=(\text{final momentum})-(\text{initial momentum})$$
$$=m\delta\underset{\sim}{v}+(\delta m)\underset{\sim}{v}-(\delta m)w(\cos\alpha\underset{\sim}{i}+\sin\alpha\underset{\sim}{j})+\delta m\,\delta v$$

$$\lim_{\delta t\to 0}\left(\frac{1}{\delta t}\right)\Rightarrow 0=m\frac{d\underset{\sim}{v}}{dt}+\frac{dm}{dt}\underset{\sim}{v}-\frac{dm}{dt}w(\cos\alpha\underset{\sim}{i}+\sin\alpha\underset{\sim}{j})$$

Now $\underset{\sim}{v} = u\underset{\sim}{i} + v\underset{\sim}{j}$:

$$0 = m\left(\frac{du}{dt}\underset{\sim}{i} + \frac{dv}{dt}\underset{\sim}{j}\right) + \frac{dm}{dt}(u\underset{\sim}{i} + v\underset{\sim}{j}) - w\frac{dm}{dt}(\cos\alpha\underset{\sim}{i} + \sin\alpha\underset{\sim}{j})$$

coefficients of $\underset{\sim}{i}$:

$$m\frac{du}{dt} + \frac{dm}{dt}u = w\frac{dm}{dt}\cos\alpha \quad ①$$

coefficients of j:

$$m\frac{dv}{dt} + \frac{dm}{dt}v = w\frac{dm}{dt}\sin\alpha \quad ②$$

Craft mass increases by ρ per unit time

$$\Rightarrow \frac{dm}{dt} = \rho.$$

Sub into RHSs of ① and ② to get required equations.

(b) Solve these equations to show that at some time T the direction of the craft has been turned through an angle β, where

$$\tan\beta = \frac{\rho w T\sin\alpha}{MU + wT\cos\alpha}$$

Rewrite eqns as $\frac{d}{dt}(mu) = \rho w\cos\alpha$

$$\Rightarrow mu = \rho w t\cos\alpha + c_1 \quad (*)$$

$$\frac{dm}{dt} = \rho \Rightarrow m = m_0 + \rho t \quad (\text{at } t=0,\ m = M \Rightarrow m_0 = M)$$

$$\Rightarrow m = M + \rho t.$$

$(M+\rho t)u = \rho w t\cos\alpha + c_1$ (subbing into *)

Initially $\underline{v} = U\underline{i}$, so $u(0) = U$

$\Rightarrow c_1 = MU$

$$\Rightarrow u(t) = \frac{\rho w t\cos\alpha + MU}{M+\rho t}$$

Also $\frac{d}{dt}(mv) = \rho w\sin\alpha$

$\Rightarrow (M+\rho t)v = \rho w t\sin\alpha + c_2$

$v(0) = 0 \Rightarrow c_2 = 0$

$$v(t) = \frac{\rho w t\sin\alpha}{M+\rho t}$$

Initially velocity is in the $\underline{i}$ direction. After time T it has moved through an angle β:

$$\tan\beta = \frac{V(T)}{U(T)} = \frac{\rho w T\sin\alpha}{\rho w T\cos\alpha + MU}$$

First-order linear ODEs

$$\frac{dy}{dt} + p(t)y = q(t)$$

$$I: \quad I(t)\frac{dy}{dt} + p(t)I(t)y(t) = q(t)I(t)$$

Want $\frac{dI}{dt} = pI \Rightarrow I = \exp\int p(t)\,dt$

$$\Rightarrow y(t) = \frac{1}{I(t)}\int q(t)I(t)\,dt$$

Example (also Q5, Hw)

A hailstone falls from rest through a cloud under gravity. Initially it is spherical with radius a. As it falls it accumulates mass at rate $\pi\rho r^2$, where ρ is its constant density, but remains spherical.

(a) Find radius of the hailstone at time t.

$$\frac{dm}{dt} = \pi\rho r^2$$

$$m = \text{density} \times \text{volume} = \frac{4}{3}\rho\pi r^3$$

$$\frac{d}{dt}\left(\frac{4}{3}\pi\rho r^3\right) = \pi\rho r^2$$

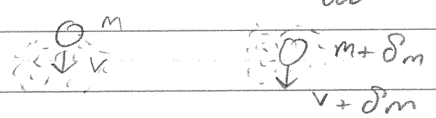
$$\Rightarrow 4\pi\rho r^2 \frac{dr}{dt} = \pi\rho r^2$$

$$\Rightarrow \frac{dr}{dt} = \frac{1}{4} \quad \Rightarrow r = \frac{t}{4} + c_1$$

at $t = 0$, $r = a \Rightarrow c_1 = a$

$$r = \frac{t}{4} + a$$

(b) Show that $\frac{dv}{dt} + \frac{3v}{t+4a} = g$



Cloud is at zero velocity (no momentum before or after)

$$\left[F = \lim_{\delta t \to 0} \frac{1}{\delta t}\left((m+\delta m)(v+\delta v) - mv\right) = \frac{d}{dt}(mv)\right]$$

$$mg = \frac{d}{dt}(mv)$$

$$m = \frac{4}{3}\pi\rho r^3 = \frac{4}{3}\pi\rho\left(\frac{t}{4} + a\right)^3$$

$$mg = \frac{dm}{dt}\cdot v + m\cdot\frac{dv}{dt}$$

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$$\Rightarrow \quad g = \left(\frac{1}{m}\frac{dm}{dt}\right) \cdot v + \frac{dv}{dt}$$

$$\frac{1}{m}\frac{dm}{dt} = 3\left(\frac{t}{4} + a^{-1}\right) \cdot \frac{1}{4}$$

$$\Leftrightarrow \quad \frac{dv}{dt} + \frac{3}{4\left(\frac{t}{4}+a\right)} v = g \quad \Rightarrow \quad \frac{dv}{dt} + \frac{3v}{t+4a} = g$$

ⓒ Find $v(t)$

$$I = \exp\left(\int \frac{3}{t+4a} dt\right)$$

$$= \exp\left(3\log(t+4a)\right) \qquad \text{'const = 0'}$$

$$= (t+4a)^3$$

$$(t+4a)^3 \frac{dv}{dt} + 3(t+4a)^2 v = g(t+4a)^3$$

$$\frac{d}{dt}\left((t+4a)^3 v\right) = g(t+4a)^3$$

$$\Rightarrow \quad (t+4a)^3 v = \frac{g}{4}(t+4a)^4 + c$$

$$v=0 \text{ at } t=0 \Rightarrow c = -\frac{g}{4}(4a)^4 = -g4^3a^4$$

$$\Rightarrow \quad v = \frac{g}{4}(t+4a) - \frac{4^3 g a^4}{(t+4a)^3}$$

$$\Rightarrow \quad v = \frac{g}{4}(t+4a)\left[1 - \frac{4^4 a^4}{(t+4a)^4}\right]$$

(d) If at $t = t_1$, the hailstone emerges from the cloud into the sunshine and continues to fall, but now loses its mass due to evaporation at rate $\pi \rho r^2$, while remaining sherical, find its radius at time $t_2 > t_1$.

So at time t_1, radius, $r = \frac{t_1}{4} + a$

$$\frac{dm}{dt} = -\pi \rho r^2 \quad , \quad m = \rho \frac{4}{3} \pi r^3$$

$$\frac{dr}{dt} = -\frac{1}{4}$$

$$r = c - \frac{t}{4}$$

at $t = t_1$, $r = \frac{t_1}{4} + a$

$$\Rightarrow c = \frac{t_1}{2} + a$$

$$r(t_2) = a + \frac{t_1}{2} - \frac{t_2}{4}$$

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Systems of linear equations with constant coefficients

$$\frac{dx}{dt} = ax + by \quad ①$$

$$\frac{dy}{dt} = cx + dy \quad ②$$

If $b \neq 0$,

①: $y = \frac{1}{b}(\dot{x} - ax)$ ③

② $\Rightarrow$ $\frac{1}{b}(\ddot{x} - a\dot{x}) = cx + \frac{d}{b}(\dot{x} - ax)$ ④

which is a 2nd-order constant coeff ODE.

Look for soln of form $x = e^{rt}$ of ④

③ shows us $y = ke^{rt}$

①, ② can be written in matrix form

$$\frac{d}{dt}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} \quad ⑤$$

so look for solns of the form

$$\begin{pmatrix} x \\ y \end{pmatrix} = \underset{\sim}{v} e^{\lambda t} \qquad \lambda = r \quad , \underset{\sim}{v} \neq 0$$

sub in ⑤:

$$\lambda \underset{\sim}{v} e^{\lambda t} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \underset{\sim}{v} e^{\lambda t}$$

$e^{\lambda t} \neq 0$: $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \underset{\sim}{v} = \lambda \underset{\sim}{v}$ ← eigenvalues

eigenvectors

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \underset{\sim}{v} - \lambda I \underset{\sim}{v} = \underset{\sim}{0}$$

$$\Leftrightarrow \underbrace{\left[\begin{pmatrix} a & b \\ c & d \end{pmatrix} - \lambda I\right]}_{\text{singular matrix (determinant = 0)}} \underset{\sim}{v} = \underset{\sim}{0}$$

$$\underbrace{\begin{pmatrix} a & b \\ c & d \end{pmatrix}}_{A} - \lambda I = \begin{pmatrix} a-\lambda & b \\ c & d-\lambda \end{pmatrix} \quad \text{is singular}$$

$$\Rightarrow (a-\lambda)(d-\lambda) - bc = 0$$

$$\Rightarrow \lambda^2 - \underbrace{(a+d)}_{Tr(A)}\lambda + \underbrace{ad-bc}_{det(A)} = 0 \leftarrow \text{characteristic eqn}$$

"trace of A.

In general we have 2 solns, $\lambda_{1,2}$ and corresponding eigenvectors $\underset{\sim}{v}_{1,2}$.

General soln of ⑤

$$\begin{pmatrix} x \\ y \end{pmatrix} = c_1 \underset{\sim}{v}_1 e^{\lambda_1 t} + c_2 v_2 e^{\lambda_2 t}$$

2 real solns $\lambda_1, \lambda_2 \Rightarrow$ exponential behaviour

If $\lambda_1 < 0$, $\lambda_2 < 0$

$\begin{pmatrix} x \\ y \end{pmatrix} \to 0 \Rightarrow \underset{\sim}{0}$ is a stable equilibrium point.

If $\lambda_1 > 0$, $\lambda_2 > 0 \Rightarrow$ unstable.

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When looking at Past Papers:
probability ×

+ nullclines
+ mixing problems
+ variable mass (← older 1302)

Final exam 14-15

3(a) A ball of mass m is launched upwards from the ground with a speed u. The ball experiences air resistance kv^2, where $k>0$ is a const & v is the speed of the ball.

(i) Let x be the upward displacement of the ball. Write down the eqn of motion

(I) in the case where the ball is moving upwards

(II) " " " " downwards

(I) $x\uparrow$ (↑v, ↓mg, ↓kv^2) (II) $x\uparrow$ (↑kv^2, ↓mg, ↓v)

$m\ddot{x} = -mg - kv^2$ $m\ddot{x} = kv^2 - mg$

(ii) Find the speed of the ball as it hits the groud.

Let h be the max height

(I) $\Leftrightarrow$ $mv\dfrac{dv}{dx} = -mg - kv^2$

$$\Rightarrow \int_0^h dx = -\int_u^0 \frac{mv\,dv}{kv^2+mg}$$

$$\Rightarrow h = -\frac{m}{2k}\log(kv^2+mg)\Big|_u^0 = \frac{m}{2k}\log\left(\frac{ku^2+mg}{mg}\right)$$

$$\Rightarrow h = \frac{m}{2k}\log\left(1+\frac{ku^2}{mg}\right)$$

As the ball falls

(II) $\Leftrightarrow$ $mv\dfrac{dv}{dx} = kv^2 - mg$

$$\Rightarrow \int_h^0 dx = \int_0^V \frac{mv\,dv}{kv^2 - mg}$$

$$-h = \frac{m}{2k} \log\left|kv^2 - mg\right|_0^V$$

$$-h = \frac{m}{2k} \log\left|\frac{kV^2 - mg}{-mg}\right|$$

$$-h = \frac{m}{2k} \log\left|\underbrace{1 - \frac{kV^2}{mg}}_{>0}\right|$$

$$e^{-\frac{2kh}{m}} = 1 - \frac{k}{mg} V^2$$

$$V^2 = \frac{mg}{k}\left(1 - e^{-\frac{2kh}{m}}\right) = \frac{mg}{k}\left(1 - \left(\frac{1}{1 + \frac{ku^2}{mg}}\right)\right)$$

$$\Rightarrow V = \sqrt{\frac{mg}{k}\left(\frac{\frac{ku^2}{mg}}{1 + \frac{ku^2}{mg}}\right)}$$

$$\Rightarrow V = \frac{u}{\sqrt{1 + \frac{k}{mg}u^2}}$$

(b) Two light elastic strings, each of natural length l, lie along the x-axis. The first string has one end fixed at $x = -l$ & the other attached to a small ball of mass m, which can move without friction along the x-axis. The same ball is attached to the end of the 2nd string. & the other end is fixed at $x = l$. First string has modulus of elasticity λ_1 & the second has λ_2.

(i) Find a potential for the force on the ball at position $x \in [-l, l]$.

$$F(x) = \begin{cases} -\frac{\lambda_2}{l} x, & x < 0 \\ -\frac{\lambda_1}{l} x, & x > 0 \end{cases}$$

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Integrate: $F(x) = -V'(x)$

$$V(x) = \begin{cases} \frac{\lambda_2}{2l} x^2, & x < 0 \\ \frac{\lambda_1}{2l} x^2, & x > 0 \end{cases}$$

(ii) If $\lambda_2 > \lambda_1$ & the ball is released from rest at $x = -l$, with what speed will it hit $x = l$?

$$E = \frac{1}{2}mv^2 + V(x)$$

$x = -l$, $v = 0$

$$\Rightarrow E = 0 + V(-l)$$

$$\Rightarrow E = \frac{\lambda_2 l}{2}$$

$x = l$, $E = \frac{1}{2}mv^2 + V(l)$

$$\Rightarrow \frac{\lambda_2 l}{2} = \frac{1}{2}mv^2 + \frac{\lambda_1 l}{2}$$

$$\Rightarrow v = \sqrt{\frac{l}{m}(\lambda_2 - \lambda_1)}$$

<u>Midsessional 14-15</u>

3. A unit mass moves subject to

$$V(x) = x^4 + ax^3 + bx^2 + cx$$

ⓐ What is the force acting on the particle

$$F(x) = -V'(x) = -(4x^3 + 3ax^2 + 2bx + c)$$

ⓑ Is there a max value for the energy s.t. the motion is bounded?

<u>Aside:</u> $m\ddot{x} = F(x)$

$$mv\frac{dv}{dx} = -V'(x) = -\frac{dV(x)}{dx}$$

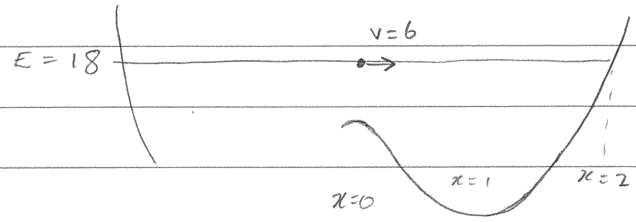
$$\Rightarrow \frac{1}{2}mv^2 = -V(x) + E$$

$$\Rightarrow v^2 = \frac{2}{m}\underbrace{(E - V(x))}_{\geq 0}$$

Since $V \to \infty$ as $|x| \to \infty$, motion is always bounded.

(c) If initially at rest, slightly to the right of $x = 0$, it moves right.
If initially at rest, slightly to the left of $x = 0$, it moves left.
If released from $x = 0$ with velocity 6 it initially speeds up until it passes $x = 1$ & then slows down until it changes direction at $x = 2$.
Find a, b, c.

Soln:
So $x = 0$ is an unstable equilibrium point



$V'(0) = 0$
$\Rightarrow c = 0$

$m = 1$, $E = \frac{1}{2}v^2 + V(x)$
$= 18 + V(0) = 18$

Speeds up then slows down as it passes $x = 1$
$\Rightarrow x = 1$ is a local min of $V(x)$:

$V'(1) = 0 = 4 + 3a + 2b$
$V(2) = 18 = 16 + 8a + 4b$

so $3a + 2b = -4$
$4a + 2b = 1$

$\Rightarrow a = 5$
$b = -\frac{19}{2}$

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Final 2014-15

4). $\ddot{x} + x = f\cos\omega t$ — ①, $\omega \neq 1$, $\omega, f > 0$

ⓐ Find A s.t. $A\cos\omega t$ is a soln of ①

$$-\omega^2 A\cos\omega t + A\cos\omega t = f\cos\omega t$$
$$\Rightarrow A = \frac{f}{1-\omega^2}$$

ⓑ Hence find a soln satisfying $x(0)=0$, $\dot{x}(0)=0$

$$\ddot{x}_h + x_h = 0$$
$$r^2+1=0 \Rightarrow r = \pm i$$

$$x_h(t) = c_1\cos t + c_2\sin t$$

general soln: $x(t) = \frac{f}{1-\omega^2}\cos\omega t + c_1\cos t + c_2\sin t$

$$0 = x(0) = \frac{f}{1-\omega^2} + c_1 \Rightarrow c_1 = \frac{f}{\omega^2-1}$$

$$0 = \dot{x}(0) = c_2 \Rightarrow c_2 = 0 \qquad \left(\dot{x} = -\frac{f\omega}{1-\omega^2}\sin\omega t - c_1\sin t + c_2\cos t\right)$$

$$\Rightarrow x(t) = \frac{f}{\omega^2-1}\left(\cos t - \cos\omega t\right)$$

ⓒ Show this soln can be written as
$K_1\sin K_2 t\sin K_3 t$

$$x(t) = \frac{f}{\omega^2-1}\left(\cos t - \cos\omega t\right)$$

$$\cos(\underbrace{A-B}_{t}) - \cos(\underbrace{A+B}_{\omega t}) = 2\sin A\sin B$$

$$A = \frac{(\omega+1)}{2}t, \quad B = \left(\frac{\omega-1}{2}\right)t$$

$$x(t) = \frac{2f}{\omega^2-1}\left[\sin\left(\frac{\omega-1}{2}\right)t\,\sin\left(\frac{\omega+1}{2}\right)t\right]$$

or d?! → ⓔ By taking the limit $\omega \to 1$, show that this soln becomes $\frac{f}{2} t \sin t$ $\left(\lim_{\theta \to 0} \frac{\sin\theta}{\theta} \right) = 1$

$$x = \frac{2f}{(\omega+1)(\omega-1)} \sin\left(\frac{\omega-1}{2}.t\right) \sin\left(\frac{\omega+1}{2}.t\right)$$

$$= \frac{f}{\omega+1} t \frac{\sin\left(\frac{\omega-1}{2}.t\right)}{\left(\frac{\omega-1}{2}.t\right)} \sin\left(\frac{\omega+1}{2}.t\right)$$

$$\lim_{\omega \to 1} x(t) = \frac{f}{1+1}.t.1.\sin\left(\frac{1+1}{2}.t\right)$$

$$= \frac{ft}{2} \sin t$$

6). m_1, m_2 collide ... u_1 u_2 v_1 v_2 etc.

ⓐ Show:
$$\begin{cases} v_1 = \dfrac{(m_1 - e m_2) u_1 + m_2 (1-e) u_2}{m_1 + m_2} \\ v_2 = \dfrac{m_1 (1+e) u_1 + (m_2 - e m_1) u_2}{m_1 + m_2} \end{cases}$$

where $e = -\frac{v_2 - v_1}{u_2 - u_1}$

Soln: conservation of mom:.

$$\left.\begin{aligned} m_1 v_1 + m_2 v_2 &= m_1 u_1 + m_2 u_2 \\ v_1 - v_2 &= e(u_2 - u_1) \end{aligned}\right\} \text{solve for } v_1, v_2$$

ⓑ 3 spheres in a line. The outer spheres have mass M & are initially at rest. The coefficient of restitution between all spheres is $e > 0$. The middle sphere has mass m & initially moves with speed $U > 0$ towards the right sphere. Show that the middle sphere will collide a second time with the rightmost sphere if & only if

$$e^2 r^2 - 3er - e - r > 0 \quad \text{, where } r = \frac{M}{m}$$

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part b cont.

<u>First collision:</u>

Middle ball has initial velocity $u_1 = U$, right ball $u_2 = 0$

middle ball $v_1 = \dfrac{(m_1 - e m_2)U}{m_1 + m_2}$

$= \dfrac{(m - eM)U}{m + M}$ with $m - eM < 0$

right ball $v_2 = \dfrac{m(1+e)U}{m+M}$

<u>Second collision</u>
(between left and middle balls).

Initially: left: $\tilde{u}_1 = 0$, middle: $\tilde{u}_2 = v_1$

$$\tilde{v}_2 = \frac{(m_2 - e m_1)\tilde{u}_2}{m_1 + m_2}$$

$$= \frac{(m - eM)v_1}{m + M}$$

$$= \frac{(m - eM)^2 U}{(m+M)^2}$$

A second collision between the middle and right balls occurs if and only if

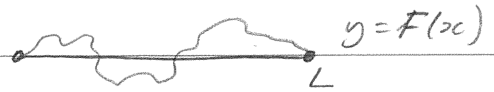
$$\tilde{v}_2 > v_2$$

$$\frac{(m - eM)^2 \cancel{U}}{(m+M)^{\cancel{2}}} > \frac{m(1+e)\cancel{U}}{\cancel{m+M}}$$

$$(m - eM)^2 > m(1+e)(m+M)$$

[expand, rearrange, divide by m^2]

5). (b) In lectures...



$f(x)$ = odd $2L$-periodic extension of F

$$u(x,t) = \tfrac{1}{2}\left(f(x-ct) + f(x+ct)\right)$$

Show that u can be written as

$$u(x,t) = \sum_{n=1}^{\infty} b_n \cos\left(\frac{n\pi ct}{L}\right)\sin\left(\frac{n\pi ct}{L}\right)$$

& determine b_n's in terms of F.

Use $\int_0^{\pi} \sin mx \sin nx \, dx$

Soln:

$f(x)$ = odd $2L$-periodic extension

$$= \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L} \quad \text{— (1)}$$

$$u(x,t) = \tfrac{1}{2}\left(f(x+ct) + f(x-ct)\right)$$

$$= \frac{1}{2}\sum_{n=1}^{\infty} b_n \left(\sin\frac{n\pi}{L}(x+ct) + \sin\frac{n\pi}{L}(x-ct)\right)$$

$$= \frac{1}{2}\sum_{n=1}^{\infty} b_n \left\{\sin\frac{n\pi x}{L}\cos\frac{n\pi ct}{L} + \cos\frac{n\pi x}{L}\sin\frac{n\pi ct}{L}\right\}$$

$$= \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L}\cos\frac{n\pi x}{L}$$

(1) $\cdot \sin\frac{m\pi x}{L}$

$$f(x)\sin\frac{m\pi x}{L} = \sum_{n=1}^{\infty} b_n \sin\frac{m\pi x}{L}\sin\frac{n\pi x}{L}$$

$$\int_0^L f(x)\sin\frac{m\pi x}{L} = \sum_{n=1}^{\infty} b_n \int_0^L \sin\frac{m\pi x}{L}\sin\frac{n\pi x}{L}\,dx$$

let $u = \frac{\pi}{L}x$, $x = \frac{L}{\pi}u$

$dx = \frac{L}{\pi}du$

$[x = L \Leftrightarrow u = \pi]$

$$= \frac{L}{\pi}\sum_{n=1}^{\infty} b_n \int_0^{\pi} \sin mu \sin nu \, du$$

$$= \frac{L}{\pi}\Big(0 + 0 + \dots + \underbrace{\tfrac{\pi}{2} b_m}_{n=m} + 0 + \dots\Big)$$

$$= \frac{L}{2} b_m$$

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5.part b cont

$$\Rightarrow \quad b_m = \frac{2}{L}\int_0^L f(x) \sin \frac{m\pi x}{L} \, dx$$

$$\Rightarrow \quad b_n = \frac{2}{L}\int_0^L F(x) \sin \frac{m\pi x}{L} \, dx \qquad f(x) = F(x) \text{ on } 0 < x < L$$

5). (a) modified SIR model.

$$\frac{dS}{dt} = -\alpha S I - \gamma S$$

$$\frac{dI}{dt} = (\alpha S - \beta) I$$

(i) Find eqn for $\frac{dR}{dt}$

Soln:

Total pop: $N = S + I + R = \text{const.}$

$$\frac{dR}{dt} = \frac{d}{dt}\left[N - S - I\right] = -\frac{dS}{dt} - \frac{dI}{dt}$$

$$= \cancel{\alpha S I} + \gamma S - (\cancel{\alpha S} - \beta) I$$

$$= \gamma S + \beta I$$

(ii) What is the number of susceptibles at the height of the epidemic.

Soln: corresponds to I having a max.

$$\Rightarrow \frac{dI}{dt} = 0 = (\alpha S - \beta) I$$

$$\Rightarrow S = \beta/\alpha$$

(iii) Initially $S(0) = S_0$, $I(0) = I_0$, $R(0) = 0$.

By eliminating dependence on t in eqn's & integrating, show that

$$\beta \ln\left(\frac{S}{S_0}\right) - \gamma \ln\left(\frac{I}{I_0}\right) + \alpha R = 0$$

Ratio of eqns in 5(a).

$$\frac{dS}{dI} = \frac{(dS/dt)}{(dI/dt)}$$

$$= \frac{-(\alpha I + \gamma) S}{(\alpha S - \beta) I}$$

$$\int \frac{\alpha S - \beta}{S}\, dS = \int -\frac{\alpha I + \gamma}{I}\, dI$$

$$\int \alpha + \beta S^{-1}\, dS + \int \alpha + \gamma I^{-1}\, dI = 0$$

$$\Rightarrow \alpha S - \beta \ln S + \alpha I + \gamma \ln I = C \qquad C = \text{const.}$$

at $t = 0$, $S = S_0$, $I = I_0$

$$c = \alpha(S_0 + I_0) - \beta \ln S_0 + \gamma \ln I_0$$

$$\Rightarrow \alpha(S + I) - \beta \ln S + \gamma \ln I = \alpha(S_0 + I_0) - \beta \ln S_0 + \gamma \ln I_0$$

note: $N = S + I + R$

$$\Rightarrow \alpha(\cancel{N} - R) - \beta \ln\left(\frac{S}{S_0}\right) + \gamma \ln\left(\frac{I}{I_0}\right) = \alpha(\cancel{N} - \underbrace{R_0}_{=0})$$

$$\Rightarrow \beta \ln\left(\frac{S}{S_0}\right) - \gamma \ln\left(\frac{I}{I_0}\right) + \alpha R = 0$$

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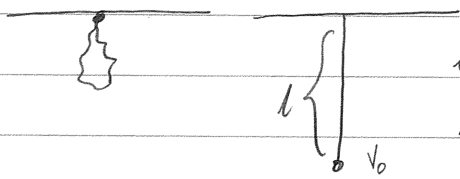
4) (a) One end of a light elastic string with modulus of elasticity λ is attached to a high ceiling & the other end is attached to a weight of mass m.

T_{max} = largest tension before breaking.

If the weight is dropped from rest from the point at which the string is attached to the ceiling, show that the largest value of m s.t. the string doesn't break is

$$m = \frac{T_{max}^2}{2g(T_{max} + \lambda)} \qquad \text{(ignore air res.)}$$

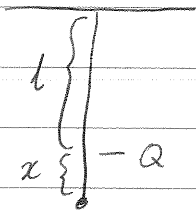
Soln



let v_0 be the velocity of the mass when the string starts to stretch.

$$E = \frac{1}{2}mv^2 + mgx$$

$$E = 0 + mgl = \frac{1}{2}mv_0^2 + 0 \quad \Rightarrow v_0 = \sqrt{2gl}$$



x = downward displacement from Q.

$$m\ddot{x} = mg - \frac{\lambda}{l}x$$

$$mv\frac{dv}{dx} = mg - \frac{\lambda}{l}x$$

$$\frac{1}{2}mv^2 = mgx - \frac{1}{2}\frac{\lambda}{l}x^2 + E \qquad (1)$$

at $x=0$ (particle at Q) $v = v_0$

$$\Rightarrow E = \frac{1}{2}mv_0^2 = mgh$$

furthest distance downwards reached $\Leftrightarrow v=0$ (if string doesn't break).

$$T_{max} = \frac{\lambda}{l}x_{max} \Rightarrow x = x_{max} = \frac{l}{\lambda}T_{max}$$

$\Rightarrow$ ①: $0 = mg\left(\frac{l}{\lambda}T_{max}\right) - \frac{1}{2}\frac{\lambda}{l}\left(\frac{l}{\lambda}T_{max}\right)^2 + mgl$

Rearrange to $m = \dfrac{T_{max}^2}{2g(T_{max}+\lambda)}$

ⓑ A tank initially contains 250g of salt dissolved in 100l of water. Salt water containing 3g of salt per litre is pumped in at 2l per minute & the mixture is pumped ot at the same rate. (The solution is well mixed). volume of tank const.
Find the mass of salt in the mixture after 6 minutes.

Soln:

$x(t)$ = salt in mixture at t.
In the short interval δt, the amound of salt changes from x to $x+\delta x$, where

$$\delta x = 3\times 2\times \delta t - 2\times \underbrace{\frac{x}{100}}_{\text{\# grams of salt per litre}}\delta t$$

$$\lim_{\delta t\to 0}\frac{1}{\delta t}(\;''\;) = \frac{dx}{dt} = 6 - \frac{x}{50}$$

$$x(t) = 300 + Ae^{-\frac{t}{50}}$$

$$250 = x(0) = 300 + A$$

$$\Rightarrow A = -50$$

$$\Rightarrow x(t) = 300 - 50e^{\frac{-t}{50}} = 50(6 - e^{-\frac{t}{50}})$$