1302 Newtonian Mechanics Notes

Based on the 2016 spring lectures by Dr H J Wilson

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L1

If a particle of mass m has a position vector $\underline{r}$

- velocity $= \frac{d\underline{r}}{dt} = \underline{v}$
- acceleration $= \frac{d^2\underline{r}}{dt^2} = \underline{a}$
- momentum $= m\underline{v}$
- angular momentum about the origin $= \underline{L} = \underline{r} \times m\underline{v}$

<u>Systems of forces</u>

$=$ set $\{\underline{F}_1, \underline{F}_2, \underline{F}_3, \ldots, \underline{F}_n\}$ acting at $\{\underline{r}_1, \underline{r}_2, \underline{r}_3, \ldots, \underline{r}_n\}$

<u>Actions that make no physical change</u>

- Add $\alpha \underline{F}_i$ to $\underline{r}_i$ ($\underline{r}_i$ becomes $\underline{r}_i + \alpha \underline{F}_i$).
- Add a pair of equal and opposite forces at a point.

A pair $\underline{F}$ at $\underline{r}_1$ and $-\underline{F}$ at $\underline{r}_2$ has total force $\underline{0}$, and total moment about the origin $(\underline{r}_1 \times \underline{F}) + (\underline{r}_2 \times (-\underline{F})) = (\underline{r}_1 - \underline{r}_2) \times \underline{F}$

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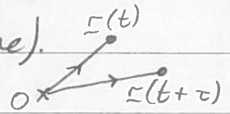
L2 Newtonian Mechanics

Topic 1: Vector Dynamics

1.1 Definitions of basic quantities

- A **particle** is an idealised object having mass but no volume. Good model for eg. planets.
- Its **position** relative to an origin O is the vector $\underset{\sim}{r}(t)$. (t is time).



$$\underset{\sim}{r}(t) = x(t)\underset{\sim}{i} + y(t)\underset{\sim}{j} + z(t)\underset{\sim}{k}$$

- Its **velocity** is the vector $\underset{\sim}{v}(t) = \dot{\underset{\sim}{r}}(t) = \lim_{\tau\to 0}\left[\frac{\underset{\sim}{r}(t+\tau) - \underset{\sim}{r}(t)}{\tau}\right] = \frac{d}{dt}(\underset{\sim}{r}(t))$

$$\underset{\sim}{v}(t) = \frac{dx}{dt}\underset{\sim}{i} + \frac{dy}{dt}\underset{\sim}{j} + \frac{dz}{dt}\underset{\sim}{k}$$

Note: Velocity does not depend on your choice of origin.

- **Speed** is a scalar: $|\underset{\sim}{v}| = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2} \geq 0$.

- **Acceleration** is $\underset{\sim}{a}(t) = \ddot{\underset{\sim}{r}}(t) = \frac{d\underset{\sim}{v}}{dt}$.

- **Momentum** or **linear momentum** $= m\underset{\sim}{v}$ where m is the mass of the particle.

- **Angular momentum** about the origin is $\underset{\sim}{L}_0 = \underset{\sim}{r} \times m\underset{\sim}{v}$.
  Angular momentum about a point A at position $\underset{\sim}{a}$ is $\underset{\sim}{L}_A = (\underset{\sim}{r} - \underset{\sim}{a}) \times m\underset{\sim}{v}$
  Angular momentum about a line $\underset{\sim}{r} = \underset{\sim}{a} + \lambda\hat{\underset{\sim}{s}}$ ($\hat{\underset{\sim}{s}}$ unit) is the scalar $l = \hat{\underset{\sim}{s}} \cdot [(\underset{\sim}{r} - \underset{\sim}{a}) \times m\underset{\sim}{v}]$

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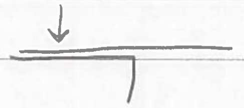
Topic 2: Forces and Moments

2.1 Force as a vector

This force 

is not the same as  this force.

So a force needs a point of application as well as the force itself. This is called a "bound vector".

$\rightarrow$——— is equivalent to ———$\rightarrow$.

So in fact it is the line of application that is important.

2.2 Types of force

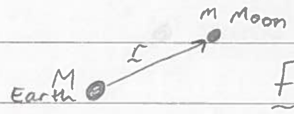
Electrostatic repulsion:

$\leftarrow\ominus\ \ominus\rightarrow$ Equal and opposite forces push these particles apart. Magnitudes depend on the separation between particles.

Gravity:

Locally we use $\underline{F} = -mg\hat{\underline{k}}$ if $\hat{\underline{k}}$ is the unit vector pointing vertically upwards.

On an interplanitary scale e.g. moon and earth

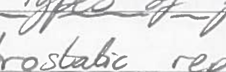


The force acting on the moon is

$$\underline{F} = -\frac{GMm}{r^3}\underline{r}$$

Tension:

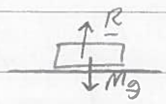
Tension in a spring is a pulling force:



$$\underline{T} = -\frac{k\underline{x}}{l}$$

In a string, $\underline{T} = \begin{cases} -k\underline{x}/l & \text{if length} \geq l \\ \underline{0} & \text{if length} < l. \end{cases}$

Reaction force:

Book on a chair: 

$\underline{g} = -g\hat{\underline{k}}$

The reaction force is equal and opposite to the Weight

If I push down on the book, $|\underline{R}|$ increases.

If I pull up, $|\underset{\sim}{R}|$ decreases, but when $\underset{\sim}{R}$ becomes zero it stays zero (doesn't push down) and the book accelerates upwards.

All we really know is $R = \alpha \underset{\sim}{k}$ with $\alpha \geq 0$.

Friction:

Friction can apply paralle the surface

A smooth surface means no fric , so the only reaction force is perpendicular to the surface.

Reaction force on a general surfa

- If the surface is smoo , R be perpendicular to the surface.
- The surface can only push outwards (stop par cle going throug), not pull , aro sto the partic)

Bead on a wire

The bead is constrained to stay on the wire, reaction force will do whate er is necessary to make t happen.

If the wire is smooth, then has no component parallel to the wire. It can take any value in the plane of vectors perpendicular to the e

2.3 Moments

The moment of a force $\underset{\sim}{F}$ at position $\underset{\sim}{r}$ about the origin is $\underset{\sim}{G}_0 = \underset{\sim}{r} \times \underset{\sim}{F}$.

Note: this force acting at any point $\underset{\sim}{r} + \alpha \underset{\sim}{F}$ gives the same moment.

If there is a fixed point at the origin, then the moment $\underset{\sim}{G}_0$ gives the thrust being applied.

Note: this means angular momentum is "moment of momentum."

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2.4 Conservation Laws

2.4.1 Force and momentum

$\mathbf{F} = m\mathbf{a}$ can be written $\frac{d}{dt}(m\mathbf{v}) = \mathbf{F}$.

ie. if there is no force then momentum is conserved.

2.4.2 Angular momentum and moments

$\mathbf{L} = \mathbf{r} \times (m\mathbf{v})$ is angular momentum

$$\frac{d\mathbf{L}}{dt} = \frac{d}{dt}\left[\mathbf{r} \times (m\mathbf{v})\right] = \underbrace{\underbrace{\frac{d\mathbf{r}}{dt}}_{\mathbf{v}} \times (m\mathbf{v})}_{0} + \mathbf{r} \times \underbrace{\frac{d}{dt}(m\mathbf{v})}_{\mathbf{F}}$$

$$= \mathbf{r} \times \mathbf{F}$$

So if the moment about the origin is zero, then the angular momentum about the origin will be constant.
An example is when $\mathbf{F}$ is parallel to $\mathbf{r}$. The force is pointing directly towards the origin or directly away from the origin. We call it a central force.
Examples include planetary motion.

2.4.3 Angular momentum about a line

We had $l = \hat{\mathbf{s}} \cdot \left[(\mathbf{r} - \mathbf{a}) \times m\mathbf{v}\right]$ $[\hat{\mathbf{s}}$ constant, $\mathbf{a}$ constant$]$

$$\frac{dl}{dt} = \underbrace{\frac{d\hat{\mathbf{s}}}{dt}}_{0} \cdot \left[(\mathbf{r} - \mathbf{a}) \times m\mathbf{v}\right] + \hat{\mathbf{s}} \cdot \frac{d}{dt}\left[(\mathbf{r} - \mathbf{a}) \times m\mathbf{v}\right]$$

$$= \hat{\mathbf{s}} \cdot \left[\overbrace{\overbrace{\frac{d}{dt}(\mathbf{r} - \mathbf{a})}^{\mathbf{v}} \times m\mathbf{v}}^{0} + (\mathbf{r} - \mathbf{a}) \times \overbrace{\frac{d(m\mathbf{v})}{dt}}^{\mathbf{F}}\right]$$

$$= \hat{\mathbf{s}} \cdot \left[(\mathbf{r} - \mathbf{a}) \times \mathbf{F}\right]$$

This is the moment about A, dotted with $\hat{\mathbf{s}}$; or, the component of the moment in the $\hat{\mathbf{s}}$ direction.

Examples: if $\hat{\mathbf{s}} = \mathbf{k}$, vertical:

(coin roll machine)

golf ball spiralling down a hole

2.4.4 Energy

Kinetic energy is energy due to motion

$E_K = \frac{1}{2} m |\underset{\sim}{v}|^2 = \frac{1}{2} m \underset{\sim}{v} \cdot \underset{\sim}{v}$

$$\frac{d}{dt}(E_K) = \frac{d}{dt}\left(\frac{1}{2} m \underset{\sim}{v} \cdot \underset{\sim}{v}\right) = \frac{1}{2} m \left\{ \frac{d\underset{\sim}{v}}{dt} \cdot \underset{\sim}{v} + \underset{\sim}{v} \cdot \frac{d\underset{\sim}{v}}{dt} \right\} = \underset{\sim}{v} \cdot \underbrace{\frac{d(m\underset{\sim}{v})}{dt}}_{\underset{\sim}{F}}$$

$$= \underset{\sim}{v} \cdot \underset{\sim}{F}$$

or $\boxed{\frac{dE_K}{dt} - \dot{\underset{\sim}{r}} \cdot \underset{\sim}{F} = 0}$

This can be very powerful if $\underset{\sim}{F}$ is of the right form.

Local gravity

$\underset{\sim}{F} = -mg\underset{\sim}{k}$ $\qquad \dot{\underset{\sim}{r}} = \dot{x}\underset{\sim}{i} + \dot{y}\underset{\sim}{j} + \dot{z}\underset{\sim}{k}$

$$-\dot{\underset{\sim}{r}} \cdot \underset{\sim}{F} = mg\dot{z} = \frac{d}{dt}(mgz)$$

We have $\frac{d}{dt}\left(\frac{1}{2} m \underset{\sim}{v} \cdot \underset{\sim}{v}\right) + \frac{d}{dt}(mgz) = 0$

So $\underbrace{\frac{1}{2} m \underset{\sim}{v} \cdot \underset{\sim}{v}}_{E_K} + \underbrace{mgz}_{E_p} = E \quad \leftarrow$ constant

Conservative force

We say $\underset{\sim}{F}$ is a conservative force if it can be written as

$$\underset{\sim}{F} = -\frac{\partial V}{\partial x}\underset{\sim}{i} - \frac{\partial V}{\partial y}\underset{\sim}{j} - \frac{\partial V}{\partial z}\underset{\sim}{k}$$ for some function $V(x, y, z)$.

(eg. local gravity $V = mgz$)

[1402]: $\frac{dV}{dt} = \frac{\partial V}{\partial x}\frac{dx}{dt} + \frac{\partial V}{\partial y}\frac{dy}{dt} + \frac{\partial V}{\partial z}\frac{dz}{dt}$

so $$\frac{dV}{dt} = \left(\frac{\partial V}{\partial x}, \frac{\partial V}{\partial y}, \frac{\partial V}{\partial z}\right) \cdot \left(\frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt}\right)$$
$$= \underset{\sim}{v} \cdot (-\underset{\sim}{F})$$

$$\frac{dE_K}{dt} - \underset{\sim}{v} \cdot \underset{\sim}{F} = 0 \Rightarrow \frac{dE_K}{dt} + \frac{dV}{dt} = 0$$

so $\boxed{E_K + V = E}$ $\leftrightarrow$ Conservation of energy.

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2.5 - Systems of forces

A system of forces is a set of forces $\underline{F}$ at positions $\underline{r}$: $\{(\underline{F}_1, \underline{r}_1), (\underline{F}_2, \underline{r}_2), \ldots\}$ which are all acting on the same rigid object.

We say two systems are equivalent if they have the same physical effect.

In particular, there are three changes we can make to a system that don't change the physical effect, these are:

1). Move a force along its line of action, so $(\underline{F}_1, \underline{r}_1) \to (\underline{F}_1, \underline{r}_1 + \alpha \underline{F}_1)$.

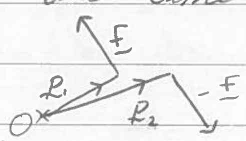
2). Add a pair of equal and opposite forces at a single point, ie. add $(\underline{F}, \underline{r})$ and $(-\underline{F}, \underline{r})$ to the set.

3). Move a pair of equal and opposite forces by the same displacement so $\{(\underline{F}, \underline{r}_1), (-\underline{F}, \underline{r}_2)\} \to \{(\underline{F}, \underline{r}_1 + \underline{x}), (-\underline{F}, \underline{r}_2 + \underline{x})\}$

4). There is in fact one more simple change we can make, so basic it's easy to forget: any two (or more) forces acting at the same point can be added using the standard vector rules.

2.5.1 - Couple

A couple is a pair of equal and opposite forces not acting at the same point.



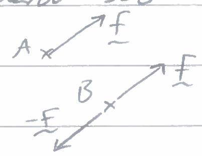
Its moment about the origin is $(\underline{\rho}_1 - \underline{\rho}_2) \times \underline{F}$, which is independent of the choice of origin (ie. we can translate both points together and nothing changes: rule 3). We can move a couple: ie. it is a "normal" (not bound - doesn't matter where you apply it) vector. We can add two couples using vector rules. Its action is a twist.

2.5.2 - Moving a force

This system:

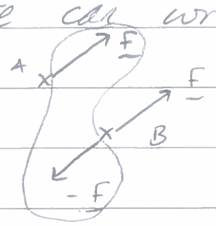
$\{(\underset{\sim}{F}, \underset{\sim}{a})\}$

is equivalent to this system:



$\{(\underset{\sim}{F}, \underset{\sim}{a}), (\underset{\sim}{F}, \underset{\sim}{b}), (-\underset{\sim}{F}, \underset{\sim}{b})\}$

which we can write as:



a force $\underset{\sim}{F}$ acting at B, plus a couple $(\underset{\sim}{a}-\underset{\sim}{b}) \times \underset{\sim}{F}$

We can move any force in our system as long we add the right couple as well:

$\{(\underset{\sim}{F}, \underset{\sim}{a})\} \longrightarrow \{(\underset{\sim}{F}, \underset{\sim}{b})\}$ + couple $(\underset{\sim}{a}-\underset{\sim}{b}) \times \underset{\sim}{F}$

This is the moment of our original system about B

2.5.3 - Reducing a system to a point

Consider a set of forces $\underset{\sim}{F}_1, \underset{\sim}{F}_2, \dots, \underset{\sim}{F}_n$ acting at points $\underset{\sim}{r}_1, \underset{\sim}{r}_2, \dots, \underset{\sim}{r}_n$. We will move each force in turn to a position B with position vector $\underset{\sim}{b}$.

$(\underset{\sim}{F}_i, \underset{\sim}{r}_i) \rightarrow (\underset{\sim}{F}_i, \underset{\sim}{b})$ + couple $(\underset{\sim}{r}_i - \underset{\sim}{b}) \times \underset{\sim}{F}_i$

When we've finished we have a set of forces

$\underset{\sim}{F}_1, \underset{\sim}{F}_2, \dots, \underset{\sim}{F}_n$ all acting at B

along with n couples

$(\underset{\sim}{r}_1 - \underset{\sim}{b}) \times \underset{\sim}{F}_1, (\underset{\sim}{r}_2 - \underset{\sim}{b}) \times \underset{\sim}{F}_2, \dots, (\underset{\sim}{r}_n - \underset{\sim}{b}) \times \underset{\sim}{F}_n$.

We can add the forces (because they all act the same point); we can add the couples.

We get $\underset{\sim}{F} = \sum_{i=1}^{n} \underset{\sim}{F}_i$ acting at B, plus a couple $\underset{\sim}{G}_B = \sum_{i=1}^{n} (\underset{\sim}{r}_i - \underset{\sim}{b}) \times \underset{\sim}{F}_i$.

The moment of our original system about the origin was $\underset{\sim}{G}_0 = \sum_{i=1}^{n} \underset{\sim}{r}_i \times \underset{\sim}{F}_i$

so our new couple can be written as

$$\underset{\sim}{G}_B = \sum_{i=1}^{n} (\underset{\sim}{r}_i - \underset{\sim}{b}) \times \underset{\sim}{F}_i = \sum_{i=1}^{n} \underset{\sim}{r}_i \times \underset{\sim}{F}_i - \sum_{i=1}^{n} \underset{\sim}{b} \times \underset{\sim}{F}_i$$
$$= \underset{\sim}{G}_0 - \underset{\sim}{b} \times \underset{\sim}{F}$$
which is the moment of the original system about B.

2.5.4 - Eliminating the couple

We would like to choose our $\underset{\sim}{b}$ so as to end up with a single force and no couple, ie. set $\underset{\sim}{G}_B = 0$. This means we need to solve $\underset{\sim}{G}_0 - (\underset{\sim}{b} \times \underset{\sim}{F}) = 0$ for $\underset{\sim}{b}$.

Solving $\underset{\sim}{F} \times \underset{\sim}{b} = -\underset{\sim}{G}_0$ for $\underset{\sim}{b}$

In components:

$$\begin{pmatrix} F_2 b_3 - F_3 b_2 \\ F_3 b_1 - F_1 b_3 \\ F_1 b_2 - F_2 b_1 \end{pmatrix} = \begin{pmatrix} -G_1 \\ -G_2 \\ -G_3 \end{pmatrix}$$

So
$$\begin{pmatrix} 0 & -F_3 & F_2 \\ F_3 & 0 & -F_1 \\ -F_2 & F_1 & 0 \end{pmatrix} \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} = \begin{pmatrix} -G_1 \\ -G_2 \\ -G_3 \end{pmatrix}$$

This matrix has zero determinant
($F_1 \times$ row 1 + $F_2 \times$ row 2 + $F_3 \times$ row 3 $= 0$) (no solns or ∞ many)

To solve $\underbrace{\underset{\sim}{F} \times \underset{\sim}{b}}_{(\perp \text{ to } \underset{\sim}{F})} = -\underset{\sim}{G}_0$

We need $\underset{\sim}{F} \cdot \underset{\sim}{G}_0 = 0$ otherwise no solutions.
Assume $\underset{\sim}{F} \cdot \underset{\sim}{G}_0 = 0$, and assume both are nonzero.
(If $\underset{\sim}{F} = \underset{\sim}{0}$ its impossible; if $\underset{\sim}{G}_0 = \underset{\sim}{0}$ we're done).
Then $\{\underset{\sim}{F}, \underset{\sim}{G}_0, \underset{\sim}{F} \times \underset{\sim}{G}_0\}$ is an orthogonal basis.
We can write any vector as a sum of scalar multiples of these three.
In particular, let $\underset{\sim}{b} = \alpha \underset{\sim}{F} + \beta \underset{\sim}{G}_0 + \gamma \underset{\sim}{F} \times \underset{\sim}{G}_0$.

Sub: $\underset{\sim}{F} \times [\cancel{\alpha \underset{\sim}{F}} + \beta \underset{\sim}{G}_0 + \gamma \underset{\sim}{F} \times \underset{\sim}{G}_0] = -\underset{\sim}{G}_0$

$\Rightarrow \beta \underset{\sim}{F} \times \underset{\sim}{G}_0 + \gamma \underset{\sim}{F} \times (\underset{\sim}{F} \times \underset{\sim}{G}_0) = -\underset{\sim}{G}_0$

$\Rightarrow \beta \underset{\sim}{F} \times \underset{\sim}{G}_0 + \gamma \underset{\sim}{F} \cancel{(\underset{\sim}{F} \cdot \underset{\sim}{G}_0)} + \gamma \underset{\sim}{G}_0 (\underset{\sim}{F} \cdot \underset{\sim}{F}) = -\underset{\sim}{G}_0$

Equating coefficients of $\underset{\sim}{F}$, $\underset{\sim}{G_0}$ and $\underset{\sim}{F} \times \underset{\sim}{G_0}$ (as they are independent):

$\underset{\sim}{F}$: $\gamma(\underset{\sim}{F} \cdot \underset{\sim}{G_0}) = 0$ ✓ (by assumption)

$\underset{\sim}{G_0}$: $-\gamma(\underset{\sim}{F} \cdot \underset{\sim}{F}) = -1 \Rightarrow \gamma = \frac{1}{(\underset{\sim}{F} \cdot \underset{\sim}{F})}$

$\underset{\sim}{F} \times \underset{\sim}{G_0}$: $\beta = 0$

Solution is $\underset{\sim}{b} = \alpha \underset{\sim}{F} + \frac{\underset{\sim}{F} \times \underset{\sim}{G_0}}{(\underset{\sim}{F} \cdot \underset{\sim}{F})}$ [This is not unique because it is a line of action for $\underset{\sim}{F}$]

<u>Example</u> (Centre of Gravity)

It is unusual to have $\underset{\sim}{F} \cdot \underset{\sim}{G_0} = 0$, but if all forces are parallel it happens.

Let $\underset{\sim}{F_i} = m_i \underset{\sim}{g}$ at position $\underset{\sim}{r_i}$.

Then $\underset{\sim}{F} = \sum_{i=1}^{n} m_i \underset{\sim}{g} = M \underset{\sim}{g}$ where $M = \sum_{i=1}^{n} m_i$

and $\underset{\sim}{G_0} = \sum_{i=1}^{n} r_i \times (m_i \underset{\sim}{g}) = \left(\sum_{i=1}^{n} m_i r_i\right) \times \underset{\sim}{g}$ which is $\perp$ to F.

Then we can move all the weight to a single point $\underset{\sim}{b}$ without adding a couple if

$$\underset{\sim}{b} = \alpha \underset{\sim}{F} + \frac{\underset{\sim}{F} \times \underset{\sim}{G_0}}{|\underset{\sim}{F}|^2} \qquad \underset{\sim}{F} = M\underset{\sim}{g}, \quad \underset{\sim}{G_0} = \left(\sum_{i=1}^{n} m_i \underset{\sim}{r_i}\right) \times \underset{\sim}{g}$$

$$\underset{\sim}{F} \times \underset{\sim}{G_0} = M\underset{\sim}{g} \times \left[\left(\sum_{i=1}^{n} m_i \underset{\sim}{r_i}\right) \times \underset{\sim}{g}\right]$$

$$= \sum_{i=1}^{n} m_i \underset{\sim}{r_i} (M\underset{\sim}{g} \cdot \underset{\sim}{g}) - \underset{\sim}{g}\left(M\underset{\sim}{g} \cdot \sum_{i=1}^{n} m_i \underset{\sim}{r_i}\right)$$

$$|\underset{\sim}{F}|^2 = M^2 \underset{\sim}{g} \cdot \underset{\sim}{g}$$

so $\underset{\sim}{b} = \alpha M \underset{\sim}{g} + \frac{1}{M^2 \underset{\sim}{g} \cdot \underset{\sim}{g}} \left\{ \sum_{i=1}^{n} m_i \underset{\sim}{r_i} (M\underset{\sim}{g} \cdot \underset{\sim}{g}) - \underset{\sim}{g} \underbrace{\left(M\underset{\sim}{g} \cdot \sum_{i=1}^{n} m_i \underset{\sim}{r_i}\right)}_{\text{absorb into } \alpha.} \right\}$

so $\underset{\sim}{b} = \bar{\alpha} \underset{\sim}{g} + \frac{1}{M} \sum_{i=1}^{n} m_i \underset{\sim}{r_i}$

This point $\underset{\sim}{X} = \sum_{i=1}^{n} \frac{m_i}{M} \underset{\sim}{r_i}$ lies on the line of action even when gravity changes (i.e. we rotate the system).

$\underset{\sim}{X}$ is the centre of gravity: it is the weighted mean of the positions.

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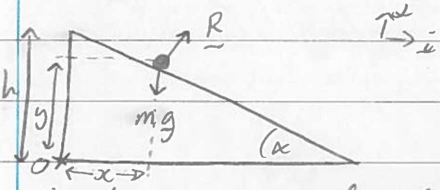
Topic 3 - Particle motion with one degree of freedom

3.1 - Motion in one dimension

Example 1: Ball on an inclined plane



- Particle has mass m.
- Plane is smooth.

Newton's second law for this scenario:

$$\underset{\sim}{R} + m\underset{\sim}{g} = m\ddot{\underset{\sim}{r}}$$

$$\underset{\sim}{r} = x\underset{\sim}{i} + y\underset{\sim}{j}$$
$$\dot{\underset{\sim}{r}} = \dot{x}\underset{\sim}{i} + \dot{y}\underset{\sim}{j}$$
$$\ddot{\underset{\sim}{r}} = \ddot{x}\underset{\sim}{i} + \ddot{y}\underset{\sim}{j}$$

Split into components:

$\underset{\sim}{i}$| $R\sin\alpha = m\ddot{x}$

$\underset{\sim}{j}$| $R\cos\alpha - mg = m\ddot{y}$

We only have two equations, in 3 variables.
We have not captured the geometrical constraint that the particle stays on the plane:

$$\underset{\sim}{r} = h\underset{\sim}{j} + \lambda(\cos\alpha\underset{\sim}{i} - \sin\alpha\underset{\sim}{j}) \quad \leftarrow \text{eqn of a line.}$$

i.e. $x = \lambda\cos\alpha$, $y = h - \lambda\sin\alpha$ (now 4 eqns, 4 unknowns)

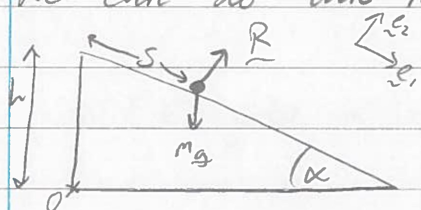
Substitute the last two into the first two:

$$\left.\begin{aligned} R\sin\alpha &= m\ddot{\lambda}\cos\alpha \\ R\cos\alpha - mg &= -m\ddot{\lambda}\sin\alpha \end{aligned}\right\} \text{can now find } R \text{ or } \lambda \text{ in one step: choose } \lambda.$$

So

$$\left.\begin{aligned} R\sin\alpha\cos\alpha &= m\ddot{\lambda}\cos^2\alpha \\ R\sin\alpha\cos\alpha &= -m\ddot{\lambda}\sin^2\alpha + mg\sin\alpha \end{aligned}\right\} \begin{aligned} &m\ddot{\lambda} = mg\sin\alpha \\ &\boxed{\lambda = A + Bt + \tfrac{1}{2}gt^2\sin\alpha} \end{aligned}$$

We can do this more simply!



We still have $\underset{\sim}{R} + m\underset{\sim}{g} = m\ddot{\underset{\sim}{r}}$, but now

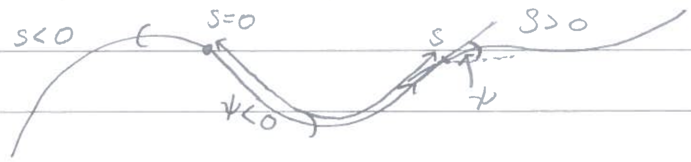
$\underset{\sim}{r} = \underset{\sim}{h} + s\underset{\sim}{e}_1$, so $\ddot{\underset{\sim}{r}} = \ddot{s}\underset{\sim}{e}_1$

Now $\underset{\sim}{R} = R\underset{\sim}{e_2}$, but $\underset{\sim}{g} = g\sin\alpha\,\underset{\sim}{e_1} - g\cos\alpha\,\underset{\sim}{e_2}$

So the $\underset{\sim}{e_1}$ component gives
$$mg\sin\alpha = m\ddot{s}$$
giving $s = A + Bt + \frac{1}{2}gt^2\sin\alpha$ as before ($s = \lambda$).

3.2 - Intrinsic coordinates

Consider a smooth curve in the x,y plane. To specify it in intrinsic coordinates:



- Assign a direction (put an arrow on)
- Choose a point, call it $s=0$
- For any other point on the curve, define s to be the signed arclength distance from the point $s=0$
- At each point, define ψ as the angle between the curve and the x-axis

Now instead of writing $y = f(x)$ to define our curve (or parametrically $x(t), y(t)$) we can write
$$\psi = \psi(s), \quad s \in \mathbb{R}.$$
Depending on the curve, we may be able to use
$s = s(\psi)$ to describe the whole curve or just a segment.

Examples

Constant function, $\psi = \psi_0$
- curve has constant slope so is a straight line.

Linear function $\psi = \alpha s$
- a circle of radius $1/\alpha$

Function increasing faster than linear $\Rightarrow$ spiral
Inward: ◎ (or if $\psi < 0$ but $|\psi|$ grows quickly, ◎)

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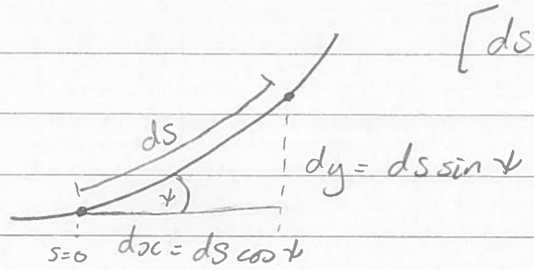
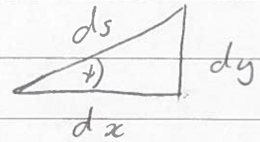
[ds is infinitessimally small]





$$dx^2 + dy^2 = ds^2$$
$$\Rightarrow \left(\frac{ds}{dx}\right)^2 = 1 + \left(\frac{dy}{dx}\right)^2$$

$$\sin\psi = \frac{dy}{ds}, \quad \cos\psi = \frac{dx}{ds}, \quad \tan\psi = \frac{dy}{dx}$$

Example

$$y = \cosh x$$

$$\frac{dy}{dx} = \sinh x = \tan\psi$$

$$\Rightarrow \left(\frac{ds}{dx}\right)^2 = 1 + \sinh^2 x = \cosh^2 x$$

$\Rightarrow \dfrac{ds}{dx} = \cosh x$ so $s = \sinh x$ (choose s s.t. $c = 0$)

so $\sinh x = s = \tan\psi$

$\Rightarrow s = \tan\psi$

Example [UfN]

Cycloid: $x = A(\theta + \sin\theta)$ $\quad \frac{dx}{d\theta} = A(1 + \cos\theta)$

$y = A(1 - \cos\theta)$ $\quad \frac{dy}{d\theta} = A(\sin\theta)$

so $\dfrac{dy}{dx} = \dfrac{\sin\theta}{1+\cos\theta} = \tan\psi$

$$\frac{dy}{dx} = \frac{\sin\theta}{1+\cos\theta} = \tan\psi$$

$$= \frac{\sin\left(\frac{2\theta}{2}\right)}{1+\cos\left(\frac{2\theta}{2}\right)}$$

$$= \frac{\cancel{2}\sin\frac{\theta}{2}\cancel{\cos\frac{\theta}{2}}}{\cancel{1}+\cancel{2}\cos^{\cancel{2}}\frac{\theta}{2}-\cancel{1}}$$

$$= \tan\frac{\theta}{2} = \tan\psi \qquad \text{so } \psi = \frac{\theta}{2}$$

$$\left(\frac{ds}{d\theta}\right)^2 = \left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2$$

$$= A^2\left((1+\cos\theta)^2 + \sin^2\theta\right)$$

$$= A^2(1+2\cos\theta + \cos^2\theta + \sin^2\theta$$

$$= 2A^2(1+\cos\theta)$$

$$\text{so } \frac{ds}{d\theta} = \sqrt{2A^2(1+\cos\theta)}$$

$$\cos\theta = 2\cos^2\frac{\theta}{2} - 1$$

$$= \sqrt{2A^2\left(2\cos^2\frac{\theta}{2}\right)}$$

$$= 2A\cos\frac{\theta}{2}$$

so $s = 4A\sin\frac{\theta}{2}$ (choose const $=0$ by choice of s)

so $s = 4A\sin\psi$

This is valid over $-\frac{\pi}{2} \leq \psi \leq \frac{\pi}{2}$.

If we were to use this formula over a greater range of ψ, we get s decreasing again as we enter the next section of the curve, which is not the way arc length is defined.

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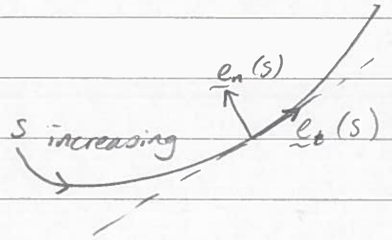
3.3 - Tangents and normals to planar curves

When we looked at a particle sliding on a plane, we saw it was easier to use basis vectors aligned parallel and perpendicular to the plane.

Now suppose we have something sliding along a curved surface in two dimensions. As it moves, the unit vectors $\parallel$ and $\perp$ to the surface will change. We will use intrinsic coordinates and call the vector $\parallel$ to the plane $\underset{\sim}{e}_t$ and $\perp$ to the plane $\underset{\sim}{e}_n$.

$|\underset{\sim}{e}_t| = 1$ and $|\underset{\sim}{e}_n| = 1$.



$\underset{\sim}{e}_t(s) = \cos\psi\, \underset{\sim}{i} + \sin\psi\, \underset{\sim}{j}$

recall: $dx = ds\cos\psi \Rightarrow dx/ds = \cos\psi$

$dy = ds\sin\psi \Rightarrow dy/ds = \sin\psi$

So given a position vector $\underset{\sim}{r} = x\underset{\sim}{i} + y\underset{\sim}{j}$, we can construct

$$\frac{d\underset{\sim}{r}}{ds} = \frac{dx}{ds}\underset{\sim}{i} + \frac{dy}{ds}\underset{\sim}{j} = \cos\psi\,\underset{\sim}{i} + \sin\psi\,\underset{\sim}{j} = \underset{\sim}{e}_t$$

We take this as the definition of the unit tangent

$$\underset{\sim}{e}_t = \frac{d\underset{\sim}{r}}{ds}$$

note $\underset{\sim}{e}_t \cdot \underset{\sim}{e}_t = 1$

$$\frac{d}{ds}(\underset{\sim}{e}_t \cdot \underset{\sim}{e}_t) = 0 = 2\underset{\sim}{e}_t \cdot \frac{d\underset{\sim}{e}_t}{ds}$$

$\Rightarrow \dfrac{d\underset{\sim}{e}_t}{ds}$ is $\perp$ to $\underset{\sim}{e}_t$

$$\frac{d\underset{\sim}{e}_t}{ds} = \frac{d}{ds}\left(\frac{d\underset{\sim}{r}}{ds}\right) = (-\sin\psi\,\underset{\sim}{i} + \cos\psi\,\underset{\sim}{j})\frac{d\psi}{ds}$$

So we define $\underset{\sim}{e}_n = -\sin\psi\,\underset{\sim}{i} + \cos\psi\,\underset{\sim}{j}$

and call $\kappa(s) = \dfrac{d\psi}{ds}$ the curvature of the curve at s.

Note that κ can have either sign. [$\kappa = 0 \Rightarrow$ point of inflection]

So $\frac{d\underset{\sim}{e}_t}{ds} = \kappa \underset{\sim}{e}_n$

Cartesian coords $\rightarrow$ intrinsic coords example:

$y = -\log(\cos x)$

$$\frac{dy}{dx} = \tan\psi = \frac{+1}{\cos x} \cdot \sin x = \tan x$$

$$\text{so } \left(\frac{ds}{dx}\right)^2 = 1 + \left(\frac{dy}{dx}\right)^2$$

$$= 1 + \tan^2 x$$

$$= \sec^2 x$$

$$\Rightarrow \frac{ds}{dx} = \sec x$$

$$\text{so } s = \int \sec x \, dx$$

$$= \int \frac{1+t^2}{1-t^2} \cdot \frac{2}{1+t^2} \, dt$$

$$= \int \frac{2}{1-t^2} \, dt = \log\left(\frac{1+t}{1-t}\right) + C$$

Choose $s = 0$ where $x = 0 \Rightarrow C = 0$

$$\text{so } t = \frac{e^s - 1}{e^s + 1} = \tanh \frac{s}{2}$$

$$\text{so } x(s) = 2\tan^{-1}(\tanh \tfrac{s}{2}) = \psi$$

$$\text{so } \tanh \frac{s}{2} = \tan \frac{\psi}{2}$$

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3.3.1 - Velocity and acceleration in intrinsic coordinates

$$\frac{d\underline{e}_t}{ds} = \kappa \underline{e}_n \quad , \quad \kappa = \frac{d\psi}{ds}$$

As $\underline{e}_t, \underline{e}_n$ form a pair of othonormal vectors we can express any vector $\underline{c}$ as a sum, $\underline{c} = \lambda \underline{e}_t + \mu \underline{e}_n$.
(in fact $\lambda = \underline{c} \cdot \underline{e}_t$, $\mu = \underline{c} \cdot \underline{e}_n$)

$$\underline{v} = \frac{d\underline{r}}{dt} = \frac{d\underline{r}}{ds}\frac{ds}{dt}$$

so $\underline{v} = \dot{s}\underline{e}_t$

So speed $= |\dot{s}|$ and $\underline{v}$ acts tangent to the curve (same direction as $\underline{e}_t$).
[note $\underline{e}_t$ and $\underline{e}_n$ are properties of the curve, but not defined in terms of mechanics on the curve].

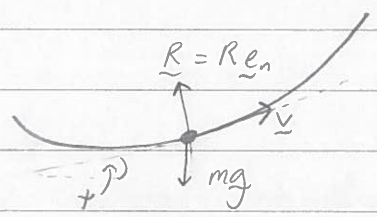
So $\quad \underline{a} = \dot{\underline{v}} = \frac{d}{dt}(\dot{s}\underline{e}_t) = \ddot{s}\underline{e}_t + \dot{s}\frac{d\underline{e}_t}{dt}$

but $\quad \frac{d\underline{e}_t}{dt} = \frac{d\underline{e}_t}{ds} \cdot \frac{ds}{dt} = \dot{s}\kappa\underline{e}_n$

So $\quad \underline{a} = \dot{\underline{v}} = \ddot{s}\underline{e}_t + \kappa\dot{s}^2\underline{e}_n$

[Helen's handy hint #1:
If asked to check / verify that $a = b$, calculate both a and b and demonstrate that they are the same.]

3.3.2 - Application: Heavy bead moving on a smooth convex wire.
[using intrinsic coords, $\psi(s)$]



Newton's second law: $\boxed{m\ddot{\underline{r}} = R\underline{e}_n + m\underline{g}}$

$\underline{R}$ is $\perp$ to curve and $\dot{\underline{r}}$ is $\parallel$ to curve, so $\dot{\underline{r}} \cdot \underline{R} = 0$

So dotting the whole governing equation with $\dot{\underline{r}}$ gives

$$m\dot{\underline{r}} \cdot \ddot{\underline{r}} = \dot{\underline{r}} \cdot m\underline{g}$$

Since g points $\downarrow$, we get

$$m\ddot{\mathbf{r}} \cdot \dot{\mathbf{r}} = -mg\dot{y}$$

which is exactly the same as in our original energy conservation discussion:

$$\frac{d}{dt}\left(\tfrac{1}{2}m\dot{\mathbf{r}} \cdot \dot{\mathbf{r}}\right) = -\frac{d}{dt}(mgy)$$

We can integrate to have

$$\tfrac{1}{2}m\dot{\mathbf{r}} \cdot \dot{\mathbf{r}} + mgy = E$$

Since $\mathbf{v} = \dot{\mathbf{r}} = \dot{s}\mathbf{e}_t$,

$$\tfrac{1}{2}m\dot{s} + mgy = E$$

[Helen's Handy Hint #2
On seeing an eqn of the form $\ddot{z} = f(z)$, multiply by $\dot{z}$ and integrate. This gives some form of energy law]

Governing eqns in components

$$\dot{\mathbf{v}} = \ddot{s}\mathbf{e}_t + \kappa\dot{s}^2\mathbf{e}_n$$

we have $m\dot{\mathbf{v}} = \mathbf{F} = \mathbf{R} + m\mathbf{g}$.

Write $\mathbf{R} = R\mathbf{e}_n$ and resolve the gravity vector:

$$\mathbf{g} = (\mathbf{e}_t \cdot \mathbf{g})\mathbf{e}_t + (\mathbf{e}_n \cdot \mathbf{g})\mathbf{e}_n = -g\mathbf{e}_t \sin\psi - g\mathbf{e}_n \cos\psi$$

to obtain

$$m(\ddot{s}\mathbf{e}_t + \kappa\dot{s}^2\mathbf{e}_n) = m\dot{\mathbf{v}} = R\mathbf{e}_n - mg\mathbf{e}_t\sin\psi - mg\mathbf{e}_n\cos\psi.$$

Hence equating coefficients we obtain

$$m\ddot{s} = -mg\sin\psi$$

$$m\kappa\dot{s}^2 = R - mg\cos\psi$$

These compare with the Cartesian coordinate version:

$$m\ddot{x} = R\cos\psi$$

$$m\ddot{y} = -mg + R\sin\psi.$$

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Example

$$y = -\log(\cos x)$$

so $\tanh \frac{s}{2} = \tan \frac{\psi}{2}$

for our eqn for $\ddot{s}$ we need $\sin\psi$:

$$\sin\psi = 2\sin(\tfrac{\psi}{2})\cos(\psi/2) = \frac{2\tan(\psi/2)}{\sec^2(\psi/2)}$$

$$= \frac{2\tanh(s/2)}{1+\tanh^2(s/2)}$$

$$= \frac{2\sinh(s/2)\cosh(s/2)}{\cosh^2(s/2)+\sinh^2(s/2)}$$

$$= \frac{\sinh s}{\cosh s} = \tanh s$$

Thus our governing eqn becomes

$$\ddot{s} = -g\tanh s$$

Using HHH #2 :

$$\ddot{s}\dot{s} = -g\tanh s\,\dot{s} \quad \text{so} \quad \tfrac{1}{2}\dot{s}^2 = -g\log\cosh s + C.$$

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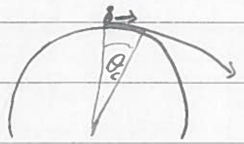
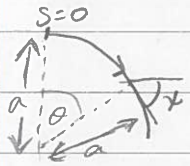
Example

Heavy penguin on a smooth spherical penguin.



Curve: $y = \sqrt{a^2 - x^2}$



Define arc length as $s = a\theta$

Then the angle $\psi = -\theta$ (Negative because the curve slopes downwards)

Our curve is $\boxed{\psi = \frac{-s}{a}}$

Governing equations:

$$m(\ddot{s}\,\underset{\sim}{e}_t + \kappa \dot{s}^2 \underset{\sim}{e}_n) = R\underset{\sim}{e}_n + m\underset{\sim}{g}$$

We need $\kappa = \frac{d\psi}{ds} = \frac{-1}{a}$, and
$$\begin{cases} \underset{\sim}{e}_n = \sin\left(\frac{s}{a}\right)\underset{\sim}{i} + \cos\left(\frac{s}{a}\right)\underset{\sim}{j} \\ \underset{\sim}{e}_t = \cos\left(\frac{s}{a}\right)\underset{\sim}{i} - \sin\left(\frac{s}{a}\right)\underset{\sim}{j} \end{cases}$$

$\underset{\sim}{e}_t$ component:

$$m\ddot{s} = mg\sin\left(\frac{s}{a}\right) \quad (A)$$

$\underset{\sim}{e}_n$ component:

$$m\left(\frac{-1}{a}\right)\dot{s}^2 = R - mg\cos\left(\frac{s}{a}\right) \quad (B)$$

We use (A) to solve for dynamics, then (B) gives R.

$$\dot{s}\ddot{s} = \dot{s}g\sin\left(\frac{s}{a}\right)$$

$$\Rightarrow \frac{1}{2}\dot{s}^2 = \int \left(g\sin\left(\frac{s}{a}\right)\dot{s}\right) dt$$

$$= \int g\sin\left(\frac{s}{a}\right) ds = -ag\cos\left(\frac{s}{a}\right) + C \quad \text{(energy eqn)}$$

To determine C, we use initial conditions:
at $s=0$, $\dot{s}=0$

so $0 = -ag\cos(0) + C$ so $C = ag$

so $\frac{1}{2}\dot{s}^2 = ag(1-\cos(\frac{s}{a}))$

Then (B) becomes
$m\left(\frac{-1}{a}\right)\cdot 2ag(1-\cos(\frac{s}{a})) = R - mg\cos(\frac{s}{a})$

so $R = mg\left[-2 + 2\cos(\frac{s}{a}) + \cos(\frac{s}{a})\right]$

$= mg\left[3\cos(\frac{s}{a}) - 2\right]$.

So penguin 'flies' when $R=0$:
$3\cos(\frac{s}{a}) = 2$
so $\theta_c = \cos^{-1}\left(\frac{2}{3}\right)$

3.4 - Motion in 3D

We can still describe our curve as $\underset{\sim}{r}(s)$ where s is arc length. This will mean
$$\left(\frac{dx}{ds}\right)^2 + \left(\frac{dy}{ds}\right)^2 + \left(\frac{dz}{ds}\right)^2 = 1$$
We can still calculate $\underset{\sim}{e}_t = \frac{d\underset{\sim}{r}}{ds}$ for the unit tangent vector. It is still true that $\frac{d\underset{\sim}{e}_t}{ds}$ is perpendicular to the curve, so we can write
$$\frac{d\underset{\sim}{e}_t}{ds} = \kappa \underset{\sim}{e}_n$$

But we now define $\kappa = \left|\frac{d\underset{\sim}{e}_t}{ds}\right|$ and $\underset{\sim}{e}_n$ follows as $\frac{d\underset{\sim}{e}_t/ds}{|d\underset{\sim}{e}_t/ds|}$.

We also need a third basis vector:
$\underset{\sim}{e}_b = \underset{\sim}{e}_t \times \underset{\sim}{e}_n$: the binormal.

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3.4.1 - Serret-Frenet

We can write any vector in the form $\alpha \underset{\sim}{e}_t + \beta \underset{\sim}{e}_n + \gamma \underset{\sim}{e}_b$, so let's set

$$\frac{d\underset{\sim}{e}_n}{ds} = a\underset{\sim}{e}_t + b\underset{\sim}{e}_n + \tau \underset{\sim}{e}_b$$

$$\frac{d\underset{\sim}{e}_b}{ds} = \alpha \underset{\sim}{e}_t + \beta \underset{\sim}{e}_n + \gamma \underset{\sim}{e}_b$$

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Last time

In 3D we have 3 intrinsic basis vectors

$\underset{\sim}{e}_t = \frac{d\underset{\sim}{r}}{ds}$, $\underset{\sim}{e}_n = \frac{d\underset{\sim}{e}_t}{ds} \cdot \frac{1}{\kappa}$, $\underset{\sim}{e}_b = \underset{\sim}{e}_t \times \underset{\sim}{e}_n$

and we define $\kappa = \left|\frac{d\underset{\sim}{e}_t}{ds}\right|$

$$\frac{d\underset{\sim}{e}_n}{ds} = a\underset{\sim}{e}_t + b\underset{\sim}{e}_n + \tau\underset{\sim}{e}_b$$

$$\frac{d\underset{\sim}{e}_b}{ds} = \alpha\underset{\sim}{e}_t + \beta\underset{\sim}{e}_n + \gamma\underset{\sim}{e}_b$$

① $\underset{\sim}{e}_n \cdot \underset{\sim}{e}_t$ $\quad a = -\kappa$

② $\underset{\sim}{e}_n \cdot \underset{\sim}{e}_n$ $\quad b = 0$

③ $\underset{\sim}{e}_n \cdot \underset{\sim}{e}_b$ $\quad \beta = -\tau$

④ $\underset{\sim}{e}_t \cdot \underset{\sim}{e}_b$ $\quad \alpha = 0$

$$\underset{\sim}{e}_b \cdot \underset{\sim}{e}_b = 1$$

$$\frac{d}{ds}(\underset{\sim}{e}_b \cdot \underset{\sim}{e}_b) = 0$$

$$\Rightarrow \frac{d}{ds}(\underset{\sim}{e}_b) \cdot \underset{\sim}{e}_b + \underset{\sim}{e}_b \cdot \frac{d}{ds}(\underset{\sim}{e}_b) = 0$$

$$\Rightarrow 2\underset{\sim}{e}_b \cdot (\alpha\underset{\sim}{e}_t + \beta\underset{\sim}{e}_n + \gamma\underset{\sim}{e}_b) = 0$$

$$2\gamma = 0 \;,\; \gamma = 0$$

Serret - Frenet

$$\frac{d\underset{\sim}{e}_t}{ds} = \kappa\underset{\sim}{e}_n$$

$$\frac{d\underset{\sim}{e}_n}{ds} = -\kappa\underset{\sim}{e}_t + \tau\underset{\sim}{e}_b$$

$$\frac{d\underset{\sim}{e}_b}{ds} = -\tau\underset{\sim}{e}_n$$

kappa

κ is the curvature

τ is the torsion, also a property of the curve.

3.4.2 – Velocity & Acceleration

$$\underset{\sim}{v} = \frac{d\underset{\sim}{r}}{dt} = \frac{d\underset{\sim}{r}}{ds}\frac{ds}{dt} = \dot{s}\,\underset{\sim}{e}_t$$

$$\underset{\sim}{a} = \frac{d\underset{\sim}{v}}{dt} = \ddot{s}\,\underset{\sim}{e}_t + \dot{s}\frac{d}{dt}(\underset{\sim}{e}_t) = \ddot{s}\,\underset{\sim}{e}_t + \dot{s}\frac{d\underset{\sim}{e}_t}{ds}\frac{ds}{dt}$$

$$= \ddot{s}\,\underset{\sim}{e}_t + \kappa\dot{s}^2\underset{\sim}{e}_n$$ [like in 2D]

3.4.3 – Particle on a smooth wire

We have (Newton's 2nd Law):

$$m(\ddot{s}\,\underset{\sim}{e}_t + \kappa\dot{s}^2\underset{\sim}{e}_n) = \underset{\sim}{R} + m\underset{\sim}{g}$$

for a particle of mass m.

Because the wire is smooth,

$$\underset{\sim}{R} = R_1\underset{\sim}{e}_n + R_2\underset{\sim}{e}_b$$

so we have three equations.

$\underset{\sim}{e}_t$: $m\ddot{s} = m\underset{\sim}{g}\cdot\underset{\sim}{e}_t$

$\underset{\sim}{e}_n$: $m\kappa\dot{s}^2 = R_1 + m\underset{\sim}{g}\cdot\underset{\sim}{e}_n$

$\underset{\sim}{e}_b$: $0 = R_2 + m\underset{\sim}{g}\cdot\underset{\sim}{e}_b$

The dot products will depend on the shape of the curve.

$\underset{\sim}{e}_t$ eq^n $\times\,\dot{s}$: $m\dot{s}\ddot{s} = m\underset{\sim}{g}\cdot(\dot{s}\underset{\sim}{e}_t)$

$$= m\underset{\sim}{g}\cdot\underset{\sim}{v}$$

$$\frac{d}{dt}\left(\tfrac{1}{2}m\dot{s}^2\right) = m\underset{\sim}{g}\cdot\underset{\sim}{v}$$

$$= m\underset{\sim}{g}\cdot\frac{d\underset{\sim}{r}}{dt} = \frac{d}{dt}(m\underset{\sim}{g}\cdot\underset{\sim}{r}) = \frac{d}{dt}(-mgz)$$

so $\frac{1}{2}m\dot{s}^2 + mgz = E$ [energy eq^n]

The usual process to get an energy equation is to dot N2 ($m\ddot{\underset{\sim}{r}} = \underset{\sim}{F}$) with velocity. Here since $\underset{\sim}{v} = \dot{s}\underset{\sim}{e}_t$, this is equivalent to taking the $\underset{\sim}{e}_t$ component and

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multiplying by $\dot{s}$.

<u>Example</u> - Helix

A vertical helix can be parametrised as

$x = a\cos\phi$

$y = a\sin\phi$

$z = -b\phi$

First we need the arc length

$$\left(\frac{ds}{d\phi}\right)^2 = \left(\frac{dx}{d\phi}\right)^2 + \left(\frac{dy}{d\phi}\right)^2 + \left(\frac{dz}{d\phi}\right)^2$$

$$= a^2\sin^2\phi + a^2\cos^2\phi + b^2$$

$$= a^2 + b^2$$

So let $\omega^2 = a^2 + b^2$,

let $s = \omega\phi$ (choose s where $\phi = 0$).

So $x = a\cos(s/\omega)$

$y = a\sin(s/\omega)$

$z = \frac{bs}{\omega}$

The e_t eq^n is

$m\ddot{s} = m\underset{\sim}{g}\cdot\underset{\sim}{e}_t$

but $\underset{\sim}{e}_t = \frac{d\underset{\sim}{r}}{ds} = \frac{dx}{ds}\underset{\sim}{i} + \frac{dy}{ds}\underset{\sim}{j} + \frac{dz}{ds}\underset{\sim}{k}$

$$= -\frac{a}{\omega}\sin\left(\frac{s}{\omega}\right)\underset{\sim}{i} + \frac{a}{\omega}\cos\left(\frac{s}{\omega}\right)\underset{\sim}{j} - \frac{b}{\omega}\underset{\sim}{k}$$

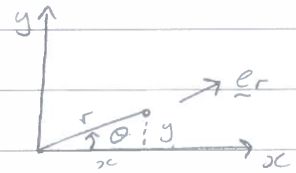
and $\underset{\sim}{g} = -g\underset{\sim}{k}$

$$\Rightarrow \ddot{s} = \frac{gb}{\omega} \Rightarrow s = \frac{gbt^2}{2\omega} + At + B$$

Topic 4 - Motion in Polar Coordinates

Here we (will) know the basis vectors at every point in space, but because they vary with position, a moving particle can see its basis vectors changing.

4.1 - Definition of Plane Polars



HHH: Visually easi to draw angles $\approx 20°-30°$

$x = r\cos\theta$

$y = r\sin\theta$

4.2 - Unit vectors

- If we hold θ fixed and increase r, the point will move in the direction $\underset{\sim}{e}_r$.

  So $\underset{\sim}{e}_r \parallel \frac{d\underset{\sim}{r}}{dr} = \frac{d}{dr}(x\underset{\sim}{i} + y\underset{\sim}{j})$

  so $\boxed{\underset{\sim}{e}_r = \cos\theta\,\underset{\sim}{i} + \sin\theta\,\underset{\sim}{j}}$

- If we hold r fixed and increase θ, the point moves in the direction $\underset{\sim}{e}_\theta$.

  $\underset{\sim}{e}_\theta \parallel \frac{d\underset{\sim}{r}}{d\theta} = \frac{d}{d\theta}(x\underset{\sim}{i} + y\underset{\sim}{j})$

  $= -r\sin\theta\,\underset{\sim}{i} + r\cos\theta\,\underset{\sim}{j}$

  so $\underset{\sim}{e}_\theta = -\sin\theta\,\underset{\sim}{i} + \cos\theta\,\underset{\sim}{j}$

4.3 - motivational and on moodle!

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4.4 – Position, velocity, acceleration

Note that $\dfrac{d\underset{\sim}{e}_r}{d\theta} = \underset{\sim}{e}_\theta$ and $\dfrac{d\underset{\sim}{e}_\theta}{\theta} = -\underset{\sim}{e}_r$

Now $\underset{\sim}{r} = r\underset{\sim}{e}_r$

$$\underset{\sim}{v} = \frac{d}{dt}(r\underset{\sim}{e}_r) = \frac{dr}{dt}\underset{\sim}{e}_r + r\frac{d\underset{\sim}{e}_r}{dt}$$

$$= \dot{r}\underset{\sim}{e}_r + r\frac{d\underset{\sim}{e}_r}{d\theta}\frac{d\theta}{dt}$$

so $\boxed{\underset{\sim}{v} = \dot{r}\underset{\sim}{e}_r + r\dot{\theta}\underset{\sim}{e}_\theta}$

$$\underset{\sim}{a} = \frac{d\underset{\sim}{v}}{dt} = \ddot{r}\underset{\sim}{e}_r + \dot{r}\frac{d}{dt}(\underset{\sim}{e}_r) + \frac{d}{dt}(r\dot{\theta})\underset{\sim}{e}_\theta + r\dot{\theta}\frac{d}{dt}(\underset{\sim}{e}_\theta)$$

$$= \ddot{r}\underset{\sim}{e}_r + \dot{r}\dot{\theta}\underset{\sim}{e}_\theta + (\dot{r}\dot{\theta} + r\ddot{\theta})\underset{\sim}{e}_\theta + r\dot{\theta}(-\underset{\sim}{e}_r)\dot{\theta}$$

$$= (\ddot{r} - r\dot{\theta}^2)\underset{\sim}{e}_r + (2\dot{r}\dot{\theta} + r\ddot{\theta})\underset{\sim}{e}_\theta$$

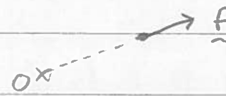
BUT $\dfrac{d}{dt}(r^2\dot{\theta}) = 2r\dot{r}\dot{\theta} + r^2\ddot{\theta}$

$$= r(2\dot{r}\dot{\theta} + r\ddot{\theta})$$

so $\boxed{\underset{\sim}{a} = \underbrace{(\ddot{r} - r\dot{\theta}^2)\underset{\sim}{e}_r}_{\text{radial component}} + \underbrace{\frac{1}{r}\frac{d}{dt}(r^2\dot{\theta})\underset{\sim}{e}_\theta}_{\text{tangential component}}}$

4.5 – Motion under a central force

Suppose the only force acting on our particle acts radially (either towards or away from the origin)



$\underset{\sim}{F} = f(r,\theta)\underset{\sim}{e}_r$

Then Newton's 2nd law is

$$m\ddot{\underset{\sim}{r}} = f(\underset{\sim}{r})\underset{\sim}{e}_r$$

and in components

$$m\left\{(\ddot{r} - r\dot{\theta}^2)\underset{\sim}{e}_r + \frac{1}{r}\frac{d}{dt}(r^2\dot{\theta})\underset{\sim}{e}_\theta\right\} = f(r,\theta)\underset{\sim}{e}_r$$

$\boxed{e_r}$ $m(\ddot{r} - r\dot{\theta}^2) = f(r, \theta)$

$\boxed{e_\theta}$ $\frac{m}{r}\frac{d}{dt}(r^2\dot{\theta}) = 0$

4.5.1 Conservation of angular momentum

The e_θ equation is

$$\frac{m}{r}\frac{d}{dt}(r^2\dot{\theta}) = 0$$

Which we can integrate to give

$r^2\dot{\theta} = h$, a constant.

h is the angular momentum about the origin
(technically it is the angular momentum per unit mass about an axis through the origin, $\perp$ to our plane).

We can rewrite as $\dot{\theta} = h/r^2$ and substitute into the e_r equation $m\left(\ddot{r} - \frac{h^2}{r^3}\right) = f(r, \theta)$.

If f is independent of θ (very common) we now have a single scalar differential equation governing r.

4.5.2 Conservation of Energy

Remember a force is conservative (& gives an energy equation) if $\underline{F} = -\underline{\nabla} V$.

In polars

$$\underline{\nabla} V = \frac{\partial V}{\partial r}\underline{e}_r + \frac{1}{r}\frac{\partial V}{\partial \theta}\underline{e}_\theta$$

So a central force ($f\underline{e}_r$) corresponds to $\frac{\partial V}{\partial \theta} = 0$
i.e. $V(r)$.

Then $\underline{F} = -\underline{\nabla} V = -\frac{\partial V}{\partial r}\underline{e}_r$

So we need the force to be independence of θ

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in order to be conservative.
In fact, a central force is conservative iff it is independent of θ.
Set $\underset{\sim}{F} = f(r)\underset{\sim}{e}_r$
We have
$$m\ddot{\underset{\sim}{r}} = f(r)\underset{\sim}{e}_r \quad , \quad f(r) = -\frac{dV}{dr}$$

To get an energy equation we dot with velocity
$$m\dot{\underset{\sim}{r}}\cdot\ddot{\underset{\sim}{r}} = f(r)\dot{\underset{\sim}{r}}\cdot\underset{\sim}{e}_r$$
$$m\dot{\underset{\sim}{r}}\cdot\ddot{\underset{\sim}{r}} = f(r)\dot{r}$$
$$= -\frac{dV}{dr}\frac{dr}{dt} = -\frac{dV}{dt}$$

so $\frac{d}{dt}\left(\frac{1}{2}m\dot{\underset{\sim}{r}}\cdot\dot{\underset{\sim}{r}}\right) = \frac{-dV}{dt}$

so $\boxed{\frac{1}{2}m|\dot{\underset{\sim}{r}}|^2 + V(r) = E}$

We can rewrite using
$$\dot{\underset{\sim}{r}} = \dot{r}\underset{\sim}{e}_r + r\dot{\theta}\underset{\sim}{e}_\theta$$
$$|\dot{r}|^2 = \dot{r}^2 + (r\dot{\theta})^2 = \dot{r}^2 + r^2\dot{\theta}^2$$

$$\frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) + V(r) = E$$

and remember $\dot{\theta} = \frac{h}{r^2}$

so $\frac{1}{2}m\left(\dot{r}^2 + \frac{h^2}{r^2}\right) + V(r) = E$

If the force is the gravitational law,
then $V(r) = \frac{-GMm}{r}$

4.6 - Circular motion and stability

We'd like to answer two questions:

- at what value of r is it possible to have circular motion?
- is this motion stable to small perturbations?

Circular orbits

We had

$$r^2\dot{\theta} = h$$

$$m\left(\ddot{r} - \frac{h^2}{r^3}\right) = f(r)$$

We want $r=b$ constant, so $\dot{r} = \ddot{r} = 0$

Then $\dot{\theta} = \frac{h}{b^2}$ $\quad -\frac{mh^2}{b^3} = f(b)$

Example

If $f(r) = -\frac{Am}{r^4}$

with $A > 0$ (artificial gravity law) we need

$$-\frac{mh^2}{b^3} = -\frac{Am}{b^4}$$

So $h_b^2 = \frac{A}{b}$ i.e, the radius b defines the permitted angular mom^m, h_b.

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Last time: Motion under a central Force

$m\ddot{\underline{r}} = f(r,\theta)\,\underline{e}_r$

We showed

$$r^2\dot{\theta} = h \quad \text{and} \quad m\left(\ddot{r} - \frac{h^2}{r^3}\right) = f(r,\theta)$$

Example

$$f(r,\theta) = -Am/r^4.$$

Circular orbit possible at $r=b$ where

$b = A/h^2$; define $h_b{}^2 = A/b$, the valid angular mom^m for circular orbit, radius b.

Stability

What happens if I 'kick' the particle off its orbit slightly?

Suppose we move the radius to $r = b+\varepsilon$, $|\varepsilon| \ll b$, but keep the angular momentum the same, $h = h_b$.

The equation $r^2\dot{\theta} = h_b$ will determine the dynamics of θ, but we will only worry about the dynamics of r.

Our governing equation is $m\left(\ddot{r} - \frac{h_b{}^2}{r^3}\right) = -\frac{Am}{r^4}$

Substitute in $r = b+\varepsilon$

$$\cancel{m}\left(\ddot{\varepsilon} - \frac{h_b{}^2}{(b+\varepsilon)^3}\right) = \frac{-A\cancel{m}}{(b+\varepsilon)^4}$$

Taylor expansion

$$\ddot{\varepsilon} - h_b^2\left[b^{-3} - 3\varepsilon b^{-4} + O(\varepsilon^2)\right] = -A\left[b^{-4} - 4b^{-5}\varepsilon + O(\varepsilon^2)\right]$$

Ignore the ε^2 terms: $\ddot{\varepsilon} + \frac{3h_b{}^2}{b^4}\varepsilon - \frac{4A\varepsilon}{b^5} = \frac{h_b{}^2}{b^3} - \frac{A}{b^4} = 0$ as $h_b{}^2 = \frac{A}{b}$

$$\ddot{\varepsilon} - \frac{A}{b^5}\varepsilon = 0$$

Let $q^2 = A/b^5$

so $\ddot{\varepsilon} = q^2\varepsilon$

try $e^{\lambda t}$

$\Rightarrow \lambda^2 - q^2 = 0$

$\lambda = \pm q$

$\varepsilon = Ae^{qt} + Be^{-qt}$ or $\varepsilon = C\cosh qt + D\sinh qt$

Because q grows without bound, we say the orbit is unstable.

What if the 'kick' does change h?

- Remember h is still constant (just a different one!)

Say $h = h_b + \delta$, $r = b + \varepsilon$, $\varepsilon(t)$, but δ constant.

Our governing eqn becomes

$$\left(\ddot{r} - \frac{h^2}{r^3}\right) = -\frac{A}{r^4} \Rightarrow \ddot{\varepsilon} - \frac{(h_b+\delta)^2}{(b+\varepsilon)^3} = \frac{-A}{(b+\varepsilon)^4}$$

Taylor expanded:

$$\ddot{\varepsilon} - (h_b^2 + 2h_b\delta + O(\delta^2))(b^{-3} - 3b^{-4}\varepsilon + O(\varepsilon^2))$$
$$= -A(b^{-4} - 4b^{-5}\varepsilon + O(\varepsilon^2))$$

$$\ddot{\varepsilon} \underbrace{- \frac{h_b^2}{b^3}} + \frac{3h_b^2}{b^4}\varepsilon - \frac{2h_b\delta}{b^3} = \underbrace{-\frac{A}{b^4}} + \frac{4A}{b^5}\varepsilon + \cancel{O(\delta^2)} + \cancel{O(\varepsilon\delta)} + \cancel{O(\varepsilon^2)}$$

$$\ddot{\varepsilon} - \frac{A}{b^5}\varepsilon = \frac{2h_b\delta}{b^3}$$

Complimentary function (CF) for this is solution from before $[e^{qt}]$

Particular integral (PI) is constant. unstable!

General Force

Governing equations are

$$r^2\dot{\theta} = h, \quad m\left(\ddot{r} - \frac{h^2}{r^3}\right) = f(r)$$

Can we have a circular orbit at $r = b$?

- Yes, if $m\left(\frac{-h^2}{b^3}\right) = f(b)$

i.e. it must have angular momentum h_b where

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$$h_b^2 = -\frac{f(b)\,b^3}{m}$$

Is it stable to perturbations that don't change angular momentum?

- Set $r = b + \varepsilon$, Taylor expand and look at the dynamics of ε.

$$\ddot{r} - \frac{h_b^2}{r^3} = \frac{f(r)}{m}$$

Set $r = b + \varepsilon$

$$\ddot{\varepsilon} - \frac{h_b^2}{(b+\varepsilon)^3} = \frac{f(b+\varepsilon)}{m}$$

Taylor expansions:

$$\ddot{\varepsilon} \underbrace{- h_b^2 (b^{-3}}\, - 3b^{-4}\varepsilon + \dots) = \frac{1}{m}(\underbrace{f(b)} + f'(b)\varepsilon + \dots)$$

$$-\frac{h_b^2}{b^3} = \frac{f(b)}{m} \longleftarrow$$

So $\ddot{\varepsilon} + \frac{3h_b^2}{b^4}\varepsilon - \frac{f'(b)}{m}\varepsilon = 0$ (Neglecting terms of order ε^2)

Let $q^2 = \frac{3h_b^2}{b^4} - \frac{f'(b)}{m} = -\frac{3f(b)}{mb} - \frac{f'(b)}{m}$

So $\ddot{\varepsilon} + q^2\varepsilon = 0$ as before

If $q^2 > 0$ soln grows like $e^{qt} \Rightarrow$ unstable

If $q^2 < 0$ let $\omega^2 = -q^2$ and $\ddot{\varepsilon} - \omega^2\varepsilon = 0$ (SHM)

Soln: $\varepsilon = A\cos\omega t + B\sin\omega t$ so ε stays small $\Rightarrow$ stable.

<u>Summary</u>

$-\frac{3f(b)}{b} - f'(b) > 0 \Rightarrow$ orbit is unstable.

$-\frac{3f(b)}{b} - f'(b) < 0 \Rightarrow$ orbit is stable

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(Flipped lectures)

Topic 5 - Orbital Motion

5.1 - Governing equations for motion under a central force

If a particle of mass m is moving under the action of a central force $\underline{F} = f(r)\,\underline{e}_r$, the governing equation for the motion is:

$$m\ddot{\underline{r}} = f(r)\,\underline{e}_r .$$

Since acceleration in plane polar coordinates is given by

$$\ddot{\underline{r}} = (\ddot{r} - r\dot{\theta}^2)\,\underline{e}_r + \frac{1}{r}\frac{d}{dt}(r^2\dot{\theta})\,\underline{e}_\theta$$

the two component equations become

$$m(\ddot{r} - r\dot{\theta}^2) = f(r,\theta) \quad , \quad \frac{m}{r}\frac{d}{dt}(r^2\dot{\theta}) = 0 .$$

Integrating the second gives $r^2\dot{\theta} = h$, and substituting this into the first,

$$m\left(\ddot{r} - \frac{h^2}{r^3}\right) = f(r,\theta).$$

This is a scalar differential equation for $r(t)$, but it's not easy to solve.

5.2 - Change of variables: the u-equation

There is a very useful change of variables. We change from using r to using $u = 1/r$, and treat it as a function of θ instead of a function of t. The derivative is a bit messy but the resultant differential equation is very powerful.

The useful quantity we're going to need is $\frac{d^2u}{d\theta^2}$. We start by finding the first derivative:

$$\frac{du}{d\theta} = \frac{du}{dt} \times \frac{dt}{d\theta} = \frac{d(1/r)}{dt} \times \frac{dt}{d\theta} = \left(\frac{d(1/r)}{dr} \times \frac{dr}{dt}\right) \times \frac{dt}{d\theta}$$

$$= -\frac{1}{r^2} \times \dot{r} \times \frac{1}{\dot{\theta}} = -\frac{\dot{r}}{h}$$

So

$$\frac{d^2u}{d\theta^2} = \frac{d}{d\theta}\left(-\frac{\dot{r}}{h}\right) = \frac{d}{dt}\left(-\frac{\dot{r}}{h}\right) \times \frac{dt}{d\theta} = -\frac{\ddot{r}}{h} \times \frac{1}{\dot{\theta}} = -\frac{\ddot{r}}{h} \times \overbrace{\frac{r^2}{h}}^{\dot{\theta} = \frac{h}{r^2}} = -\frac{r^2\ddot{r}}{h^2}$$

We have the result $\frac{d^2u}{d\theta^2} = \frac{-r^2\ddot{r}}{h^2}$

We can rearrange it to $\ddot{r} = -\frac{h^2}{r^2}\frac{d^2u}{d\theta^2}$

Using this in our r equation:

$m\left(\ddot{r} - \frac{h^2}{r^3}\right) = f(r,\theta)$ becomes $m\left(\frac{-h^2}{r^2}\frac{d^2u}{d\theta^2} - \frac{h^2}{r^3}\right) = f(r,\theta)$.

Using $r = 1/u$ gives

$$m\left(-h^2u^2\frac{d^2u}{d\theta^2} - h^2u^3\right) = f(1/u, \theta)$$

which rearranges to give

$$\boxed{\frac{d^2u}{d\theta^2} + u = -\frac{f(1/u,\theta)}{mh^2u^2}}$$

For the right kind of function f, this might be a linear equation (this is so powerful because gravity gives us the right kind of function).

5.3 - Properties of conics

A conic section is the curve we get when we intersect a cone with a plane. We'll use a cone with an angle $\pi/4$ to keep the equations simple. The cone's surface is given by the equation $z^2 = x^2 + y^2$.

Since the cone is symmetric about the z-axis, we can choose a plane (to slice our cone with) that only slopes in the x-direction (so its normal is in the x-z plane). So our plane can be written as $ex + z = l$ for some constants e and l. It's conventional to take both $e \geq 0$ and $l \geq 0$, which means choosing a plane which slopes downwards to the right, and passes above the origin (so in the downward-pointing section of the cone).

We can put these together and eliminate z, and then convert into polar coordinates, to have

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$$ex + \sqrt{x^2+y^2} = l, \quad er\cos\theta + r = l, \quad \frac{l}{r} = 1 + e\cos\theta.$$

Returning to the first equation of the above line, we can write

$$x^2+y^2 = (l-ex)^2 \quad \Rightarrow \quad (1-e^2)x^2 + 2lex + y^2 = l^2$$

and then complete the square

$$\left(x + \frac{le}{(1-e^2)}\right)^2 + \frac{y^2}{(1-e^2)} = \frac{l^2}{(1-e^2)^2}.$$

Now we look at the different possible values of e.

<u>Case 1</u> : $e = 0$

So $x^2+y^2 = l^2$ which is a circle of radius l and results from taking a horizontal plane.

<u>Case 2</u> : $0 < e < 1$

If we introduce the positive constants $a = l/(1-e^2)$ and $b = l/\sqrt{(1-e^2)}$ we can write the equation as

$$\frac{(x+ea)^2}{a^2} + \frac{y^2}{b^2} = 1$$

which is an ellipse with centre at $x = -ea$, it has the origin at one focus.

<u>Case 3</u> : $e = 1$

He we use the form of the equation before we completed the square:

$$(1-e^2)x^2 + 2lex + y^2 = l^2 \quad \Rightarrow \quad 2lx + y^2 = l^2$$

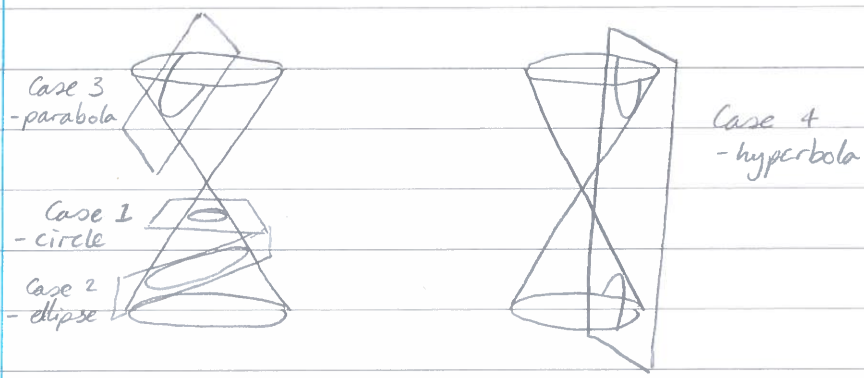
which is a parabola and results from taking a plane parallel to the cone's sides.

<u>Case 4</u> : $e > 1$

We introduce positive constants $a = l/(e^2-1)$ and $b = l/\sqrt{(e^2-1)}$ and write the equation as

$$\frac{(x-ea)^2}{a^2} - \frac{y^2}{b^2} = 1$$

which is a hyperbola and results from taking a plane so steep it intersects both the top and bottom of the cone.



5.4 - Kepler's Laws of Planetary Motion

Kepler's Laws of Motion

1 - Planets move in ellipses with the sun at one focus
2 - The area swept out per unit time by the radi ector joining a planet and the sun is constan.
3 - The ratio of the square of the period of orbit to the cube of the semi-major axis is constant.

We will prove these starting from knowledge we didn hav, of the form of the gravitational force.

5.4.1 - First law

Each planet moves in a plane that contains the sun, so the convenient way to describe the motion is in polar coordinates within that plane, with the sun at the centre. We will use the u-equation, remembering $u = 1/r$, along with the conservation of angular momentum $\dot{\theta} = hu^2$:

$$\frac{d^2u}{d\theta^2} + u = -\frac{f(1/u)}{mh^2u^2}$$

For the gravitational law, the force is $f(r) = -GmM/r^2$ where $G>0$ is the gravitational constant, m is the mass of the planet, and M is the mass of the sun. This gives

$$\frac{d^2u}{d\theta^2} + u = -\frac{(-GMu^2)}{h^2u^2} = \frac{GM}{h^2}$$

The general solution of this is $u = C\cos\theta + D\sin\theta +$ (particular integral), and it's easy to spot the particular integral in this case:

$$u = C\cos\theta + D\sin\theta + \frac{GM}{h^2}.$$

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Alternatively we can write the general solution more conveniently as

$$u = \frac{1}{r} = A\cos(\theta - \delta) + \frac{GM}{h^2}.$$

Now define two new constants:

$$l = \frac{h^2}{GM} \quad , \quad e = Al = \frac{Ah^2}{GM}$$

so that the previous equation becomes

$$\frac{l}{r} = 1 + e\cos(\theta - \delta).$$

Apart from the δ term (which just rotates everything about the origin by an angle δ) this is exactly the conic section equation we have just studied.
For the planets, it turns out that in all cases $e < 1$ and in most, $e << 1$ so that the orbits are nearly circular.
We have shown Kepler's first law: the orbit of each planet is an ellipse with the sun (the origin of our polar coordinates) at a focus.

5.4.2 - Second Law

For Kepler's second law we'll calculate the area swept out in time τ. Suppose the angle goes from θ_0 at $t=0$ and θ_τ at $t=\tau$. The area swept out is just the double integral of 1 over the region swept out:

$$A = \int_{\theta=\theta_0}^{\theta=\theta_\tau} \int_{r'=0}^{r(\theta)} 1.r \, dr \, d\theta$$

This gives $A = \int_{\theta=\theta_0}^{\theta_\tau} \int_{r'=0}^{r(\theta)} r \, dr \, d\theta = \int_{\theta=\theta_0}^{\theta_\tau} \left[\frac{1}{2} r'^2 \right]_{r'=0}^{r(\theta)} d\theta = \frac{1}{2} \int_{\theta=\theta_0}^{\theta_\tau} r^2 \, d\theta$

and changing variables in the integral from θ to t we get

$$A = \int_{t=0}^{\tau} \frac{1}{2} r^2 \dot{\theta} \, dt = \int_{t=0}^{\tau} \frac{1}{2} h \, dt = \frac{1}{2} h\tau.$$

Thus the area swept out in time τ is $\frac{1}{2}h\tau$ and the area swept out in unit time is $\frac{1}{2}h$, a constant.

5.4.3 - Third Law

For Kepler's third law, we begin by calculating the period from the second law. Since the area of an ellipse of semi-major axis a and semi-minor axis b is πab, the period is the length of time the planet takes to sweep out that area: $\text{Period} = \frac{\pi ab}{\frac{1}{2}h} = \frac{2\pi ab}{h}$.

Now, from the sections on conics we know that

$$a = \frac{l}{(1-e^2)}, \quad b = \frac{l}{\sqrt{(1-e^2)}} \quad \text{and thus } b^2 = al,$$

so

$$\text{Period} = \frac{2\pi ab}{h} = \frac{2\pi a^{2/3} l^{1/3}}{h},$$

and we defined our constant l above as $l = h^2/GM$, so

$$\text{Period} = \frac{2\pi a^{3/2}}{(GM)^{1/2}}$$

and the ratio of the square of the period & the cube of the semi-major axis is

$$\frac{(\text{Period})^2}{a^3} = \frac{4\pi^2}{GM}$$

which depends only on the mass of the sun and not on the details of the planet so is the same for all planets in the solar system, which is what we wanted to show.

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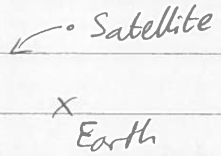
Take Earth as the origin of our coordinate system: use plane polar coordinates.



force is $-\dfrac{G\overset{\text{Earth}}{M}\overset{\text{Sattelite}}{m}}{r^2}\underline{e}_r$

Governing equation:

$$\cancel{m}\ddot{\underline{r}} = -\frac{GM\cancel{m}}{r^2}\underline{e}_r$$

but

$$\ddot{\underline{r}} = (\ddot{r} - r\dot{\theta}^2)\underline{e}_r + \frac{1}{r}\frac{d}{dt}(r^2\dot{\theta})\underline{e}_\theta$$

so $\dfrac{d}{dt}(r^2\dot{\theta}) = 0 \rightarrow r^2\dot{\theta} = h$

$$\ddot{r} - r\dot{\theta}^2 = -\frac{GM}{r^2}$$

$$\ddot{r} - \frac{h^2}{r^3} = -\frac{GM}{r^2}$$

Geostationary orbit at height b must go through angle 2π in a day $= 24h$

$= 1440$ mins $= 86400s$

so we need $\dot{\theta} = \dfrac{2\pi}{86400}$

We also need $\ddot{r} = 0$

$$\Rightarrow \frac{h^2}{r^3} = -\frac{GM}{r^2} \Rightarrow h^2 = GMr$$

$$\Rightarrow r^4\dot{\theta}^2 = GMr$$

$$\Rightarrow r^3 = \frac{GM}{\dot{\theta}^2}$$

Height (above centre of the Earth) is

$$b = \sqrt[3]{\frac{GM}{\dot{\theta}^2}}$$

and speed is

$$|\underline{v}| = |\underbrace{\dot{r}\,\underline{e}_r}_{0} + r\dot{\theta}\,\underline{e}_\theta|$$

$$= b\dot{\theta}$$

A particle of mass m under a gravitational force

$$\underline{F} = -\frac{GMm}{r^2}\underline{e}_r$$

Q: (a) Derive the u-equation for its motion

(b) If, at $\theta = 0$, it is moving in the [illegible] direc[illegible] at speed U, and is at distance a from the [illegible]gi[illegible], find its furthest and closest distances from the origin.

A: (a) We have $\ddot{r} - \frac{h^2}{r^3} = -\frac{GM}{r^2}$

Let $u = \frac{1}{r}$. Then

$$\frac{du}{d\theta} = \frac{du}{dr}\frac{dr}{dt}\frac{dt}{d\theta} = -\frac{1}{r^2}\cdot\dot{r}\cdot\frac{1}{\dot{\theta}} = \frac{-\dot{r}}{r^2\dot{\theta}} = \frac{-\dot{r}}{h}$$

$$\frac{d^2u}{d\theta^2} = \frac{d}{d\theta}\left(\frac{du}{d\theta}\right) = \frac{d}{d\theta}\left(\frac{-\dot{r}}{h}\right) = \frac{d}{dt}\left(\frac{-\dot{r}}{h}\right)\frac{dt}{d\theta}$$

$$= \frac{-\ddot{r}}{h}\cdot\frac{1}{\dot{\theta}} = \frac{-\ddot{r}}{h}\cdot\frac{r^2}{h} = \frac{-\ddot{r}r^2}{h^2} = \frac{-\ddot{r}}{u^2h^2}$$

so $\ddot{r} = -h^2u^2\frac{d^2u}{d\theta^2}$

and our equation becomes

$$-h^2u^2\frac{d^2u}{d\theta^2} - h^2u^3 = -GMu^2$$

$$\Rightarrow \frac{d^2u}{d\theta^2} + u = \frac{GM}{h^2}$$

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Solution for ODE:

CF is $A\cos\theta + B\sin\theta$

PI is $\frac{GM}{h^2}$

$$u = A\cos\theta + B\sin\theta + \frac{GM}{h^2}$$

(b) We have at $\theta = 0$:

$\underset{\sim}{v} = U\underset{\sim}{e}_\theta$, $r = a$

But velocity in polars is

$\underset{\sim}{v} = \dot{r}\underset{\sim}{e}_r + r\dot{\theta}\underset{\sim}{e}_\theta$

so initially $\dot{r} = 0$, $a\dot{\theta} = U$

$$\left.\begin{aligned} u &= \frac{1}{r} = \frac{1}{a} \\ \frac{du}{d\theta} &= -\frac{\dot{r}}{h} = 0 \end{aligned}\right\} \text{at } \theta = 0.$$

$$h = r^2\dot{\theta} = a^2\left(\frac{U}{a}\right) = aU$$

We had $u = A\cos\theta + B\sin\theta + \frac{GM}{h^2}$

so $\frac{du}{d\theta} = -A\sin\theta + B\cos\theta$

and the initial conditions give

$$u = \frac{1}{a} = A + \frac{GM}{h^2} \Rightarrow A = \frac{1}{a} - \frac{GM}{h^2} = \frac{1}{a} - \frac{GM}{a^2U^2}, \quad B = 0$$

$$u = \left(\frac{1}{a} - \frac{GM}{a^2U^2}\right)\cos\theta + \frac{GM}{a^2U^2}$$

Minimum r occurs at maximum u.

Maximum r occurs at minimum u.

Both are when $\frac{du}{d\theta} = 0$.

$$0 = \left(\frac{1}{a} - \frac{GM}{a^2U^2}\right)\sin\theta \quad \Rightarrow \quad \theta = 0, \pi$$
$$\cos\theta = \pm 1$$

Extrema of u are

$$\begin{cases} \left(\frac{1}{a} - \frac{GM}{a^2U^2}\right) + \frac{GM}{a^2U^2} = \frac{1}{a} \\ -\left(\frac{1}{a} - \frac{GM}{a^2U^2}\right) + \frac{GM}{a^2U^2} = \frac{2GM}{a^2U^2} - \frac{1}{a} \end{cases}$$

Extrema of r are

$$\begin{cases} a \\ \dfrac{a}{\left(\frac{2GM}{U^2 a}\right) - 1} \end{cases}$$

What if the second of these is negative?
Then u has been through 0 and the particle escaped
$r \to \infty$ when $u = 0$

$$\cos\theta = \frac{GM/a^2U^2}{GM/a^2U^2 - \frac{1}{2}}$$

Q: A meteorite is approaching the Earth and is first detected a large distance away, moving with speed U. If the meteorite were to continue in the absence of the Earth's gravitational field, its distance of closest approach would be d. Find its closest approach.

A: Particle path $\quad u = A\cos\theta + B\sin\theta + \frac{GM}{h^2}$

If we turn off gravity, then at closest approach we have $r = d$, $|r\dot{\theta}| = U$ so $h = -Ud$ $(\dot{\theta} < 0)$.
But $\frac{d}{dt}$(angular momentum) = moment of force, so with

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no force, the angular momentum does not change.
$\Rightarrow$ we had $h = -Ud$ initialy
and h remains constant when we turn gravity back on.

To determine A and B we will need initial values for u and $\frac{du}{d\theta}$.

Say $r = R$ (very large) ; clearly $\theta \simeq \pi$

$u = \frac{1}{R}$ initially, $\frac{du}{d\theta} = -\frac{\dot{r}}{h}$, but how do we find $\dot{r}$?

? Position (x, d) in Cartesians $\quad r^2 = x^2 + y^2$
Velocity $(u, 0)$

$$2r\dot{r} = 2x\dot{x} + 2y\dot{y}$$
$$\dot{r} = \frac{x\dot{x}}{r} = \frac{x}{r}U \approx ?$$

Initial conditions:
at $\theta = \pi$

$u = 0$ and $\frac{du}{d\theta} = \underset{?}{U} = -\frac{1}{d}$

Substitute in:

$$0 = -A + \frac{GM}{u^2 d^2} \qquad A = \frac{GM}{U^2 d^2}, \quad -\frac{1}{d} = -B \text{ so } B = \frac{1}{d}$$

$$u = \frac{GM}{U^2 d^2}(\cos\theta + 1) + \frac{1}{d}\sin\theta.$$

To find the closest approach (min r) we need max u.
Need $\frac{du}{d\theta} = 0$.

$$\frac{du}{d\theta} = -\frac{GM}{U^2 d^2}\sin\theta + \frac{1}{d}\cos\theta$$

$$\frac{du}{d\theta} = 0 \Rightarrow \frac{\sin\theta}{\cos\theta} = \frac{U^2 d}{GM} = \tan\theta$$

If $\tan\theta = \dfrac{U^2 d}{GM}$

$$\tan^2\theta = \frac{U^4 d^2}{G^2 M^2}$$

$$1 + \tan^2\theta = \sec^2\theta = \frac{G^2M^2 + U^4 d^2}{G^2 M^2} = \frac{1}{\cos^2\theta}$$

so $\cos^2\theta = \dfrac{G^2M^2}{G^2M^2 + U^4 d^2}$, $\sin^2\theta = \dfrac{U^4 d^2}{G^2M^2 + U^4 d^2}$

$$u_{max} = \frac{GM}{U^2 d^2}\left(1 + \frac{GM}{\sqrt{G^2M^2 + U^4 d^2}}\right) + \frac{1}{d}\frac{U^2 d}{\sqrt{G^2M^2+U^4d^2}}$$

and $r_{min} = \dfrac{1}{u_{max}}$

Special cases to check

1). If $d=0$ it is heading directly towards us and we expect $r_{min} = 0$

$$u_{max}\ (d=0) \approx \frac{GM}{U^2 d^2}(1+1) + \frac{u^2}{GM} \leftarrow \text{capital?}$$

$$\rightarrow \infty \text{ as } d \rightarrow 0$$

2). If $u=0$ the meteorite falls to Earth directly

$\Rightarrow r_{min} = 0$, $u_{max} \rightarrow \infty$

$$u_{max}\ (U \rightarrow 0) : \frac{GM}{U^2 d}(1+1) + 0 \rightarrow \infty$$

3). If $G=0$ we need $r_{min} = d$, $u_{max} = \dfrac{1}{d}$

$$u_{max}\ (G=0) : 0 + \frac{U^2}{\sqrt{U^4 d^2}} = \frac{1}{d}.$$

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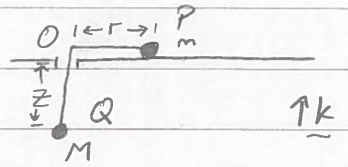
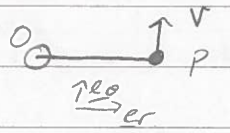
Q from Handout 4 (question on polars - 2005)



Side view

Top view



Initially : speed = V

$|OP| = a$

Particle P.

Use plane polars : r, θ.

Acceleration is $(\ddot{r} - r\dot{\theta}^2)\underline{e}_r + \frac{1}{r}\frac{d}{dt}(r^2\dot{\theta})e_\theta$

In the plane, the only force acting is tension,

$-T\underline{e}_r \quad (T \geq 0)$

so N2 in components gives

① $m(\ddot{r} - r\dot{\theta}^2) = -T$

② $\frac{d}{dt}(r^2\dot{\theta}) = 0$

Particle Q.

Position is $-z\underline{k}$ (relative to O)

So acceleration is $-\ddot{z}\underline{k}$

Forces are weight $-Mg\underline{k}$ and Tension $T\underline{k}$

N2 : ③ $-M\ddot{z} = T - Mg$

Length of string

④ $z + r = l$

Initial conditions

At $t = 0$, $r = a$, $z = l - a$.

Initial velocity of P is $V\underline{e}_\theta$ but $\dot{\underline{r}} = \dot{r}\underline{e}_r + r\theta\underline{e}_\theta$

so initially

$\dot{r} = 0, \quad \dot{\theta} = V/a, \quad \dot{z} = 0$

Eliminate tension T from ① and ③:

$m(\ddot{r} - r\dot{\theta}^2) = M\ddot{z} - Mg$ ✓

Integrate ②: $r^2\dot{\theta} = h$ ✓ initially $r = a$

$\dot{\theta} = V/a$

so $h = a^2(V/a) = aV$

For horizontal circular motion, $\ddot{r} = \ddot{z} = 0$

so we need

$-mr\dot{\theta}^2 = -Mg$ but $r = a$, $\dot{\theta} = \frac{h}{r^2} = \frac{aV}{a^2} = \frac{V}{a}$

so $ma\left(\frac{V}{a}\right)^2 = Mg$

so $V^2 = \frac{Mga}{m}$ ✓

Small perturbations:

$r = a + x$

$z = l - a - x$ (from ④)

$r^2\dot{\theta} = h = aV \Rightarrow V = \sqrt{Mga/m}$ ✓

$m(\ddot{r} - r\dot{\theta}^2) = M(\ddot{z} - g)$

First get rid of $\dot{\theta}$:

$m\left(\ddot{r} - r\left(\frac{h}{r^2}\right)^2\right) = M(\ddot{z} - g)$

$m\left(\ddot{r} - \frac{h^2}{r^3}\right) = M(\ddot{z} - g)$ but $h^2 = a^2V^2 = \frac{Mga^3}{m}$

so $m\left(\ddot{r} - \frac{Mga^3}{mr^3}\right) = M(\ddot{z} - g)$ now $\ddot{z} = -\ddot{x}$

so $m\ddot{r} - \frac{Mga^3}{r^3} - M\ddot{x} - Mg$

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finally $r = a + x$ so $\ddot{r} = \ddot{x}$

$$\Rightarrow m\ddot{x} - \frac{Mga^3}{(a+x)^3} = -M\ddot{x} - Mg$$

$$(M+m)\ddot{x} = -Mg + Mga^3(a+x)^{-3}$$

using Taylor's expansion: $(a+x)^{-3} = a^{-3} - 3a^{-4}x + O(x^2)$

$$(M+m)\ddot{x} = Mg\{\cancel{-1} + a^3(\cancel{a^{-3}} - 3a^{-4}x + O(x^2))$$

$$= -\frac{3Mg}{a}x + O(x^2)$$

so to first order, $\ddot{x} = \frac{-3Mg}{a(M+m)}x$

Note that $h^2 = \frac{Mga^3}{m}$ so $Mg = \frac{mh^2}{a^3}$

so $\ddot{x} = \frac{-3mh^2}{(M+m)a^4}$ ✓

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Topic 6 – Vector Differential Equations

Newton's 2nd law can be written as

$$\frac{d^2\underline{r}}{dt^2} = \frac{\underline{F}}{m}$$

which is a 2nd order differential equation for $\underline{r}$. We've seen a lot of physical arguments for ways to tackle these (eg. conservation laws, u-equations etc.) but in this chapter we will look at them from a more abstract standpoint.

We know lots about scalar differential equations, but what carries over?

We will look at

a). methods from scalar ODEs which work if we're careful.

b). new methods which are purely vectorial.

In general, any scalar method is worth a try as long as it doesn't involve illegal vector operations (e.g. dividing by a vector.)

6.1 – Things that don't work

6.1.1 – Separation of variables

In scalar calculus, if we have

$\frac{dv}{dt} = F(v)G(t)$ we can solve:

$$\int \frac{dv}{F(v)} = \int G(t)\,dt,$$

but if we now have

$$\frac{d\underline{v}}{dt} = g(t)(\underline{v}\cdot\underline{v})\underline{v}$$

we can divide by $\underline{v}\cdot\underline{v}$ (scalar) but not by $\underline{v}$, so we can't use separation of variables.

6.1.2 - Reduction of order

If we have a linear scalar equation with, say, 2^{nd} order derivatives:

$$p(t)\frac{d^2f}{dt^2} + q(t)\frac{df}{dt} \quad r(t)f = s(t)$$

and we know one solution to the homogeneous equation, $f_1(t)$:

$$p(t)\frac{d^2f_1}{dt^2} + q(t)\frac{df_1}{dt} + r(t)f_1 = 0$$

then we can set $f(t) = f_1(t)g(t)$ and substitute:

$$p(t)[f_1''g + 2f_1'g' + f_1g''] + q(t)[f_1'g + f_1g'] + r(t)f_1g = s(t)$$

Because of what we know about f_1, we get

$$p(t)f_1(t)g'' + [2p(t)f_1'(t) + q(t)f_1(t)]g' = s(t)$$

which is a first-order equation for $g'(t)$.

The step "$f(t) = f_1(t)g(t)$" does not carry across to vectors: if we know $\underset{\sim}{f}_1(t)$ there is no guarantee that $\underset{\sim}{f}(t)$ will be parallel to it.

6.2 - Methods from scalar calculus that do carry across

6.2.1 - Just integrate

When we add a constant of integration, it must be a vector.

Example

$$\frac{d\underset{\sim}{v}}{dt} = 2\underset{\sim}{b}t \Rightarrow \underset{\sim}{v} = \underset{\sim}{b}t^2 + \underset{\sim}{c}$$

6.2.2 - Integrating factors

$$\frac{d\underset{\sim}{v}}{dt} + 3t^2\underset{\sim}{v} = t^2\underset{\sim}{b}$$

Here the integrating factor is $I = e^{\int 3t^2 dt} = e^{t^3}$

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$$e^{t^3}\frac{d\underset{\sim}{v}}{dt} + 3t^2 e^{t^3}\underset{\sim}{v} = t^2 e^{t^3}\underset{\sim}{b}$$ ← multiplying by a scalar is OK

$$I\frac{d\underset{\sim}{v}}{dt} + \frac{dI}{dt}\underset{\sim}{v} = t^2 e^{t^3}\underset{\sim}{b}$$

← product rule for a scalar times a vector

$$\frac{d}{dt}\left[I\underset{\sim}{v}\right] = t^2 e^{t^3}\underset{\sim}{b}$$

finally $e^{t^3}\underset{\sim}{v} = \int t^2 e^{t^3}\underset{\sim}{b}\,dt = \frac{1}{3}e^{t^3}\underset{\sim}{b} + \underset{\sim}{c}$

so $\underset{\sim}{v} = \frac{1}{3}\underset{\sim}{b} + \underset{\sim}{c}e^{-t^3}$

6.2.3 - Linear equations

To solve a linear equation, we first find the CF (general soln to the homogeneous eqn) then the PI (any valid soln to the whole equation) and finally apply initial conditions to determine the constants.

For vectors, the underlying principle, linear superposition, is still valid so this will work.

Example

Particle moving under gravity and air resistance.

$\ddot{\underset{\sim}{x}} = -k\dot{\underset{\sim}{x}} + \underset{\sim}{g}$

where, at $t=0$, $\underset{\sim}{x} = \underset{\sim}{0}$ and $\dot{\underset{\sim}{x}} = \underset{\sim}{u}$.

We can rewrite this as

$$\frac{d^2\underset{\sim}{x}}{dt^2} + k\frac{d\underset{\sim}{x}}{dt} = \underset{\sim}{g}$$

This is now more clearly a differential equation.

CF: Try to solve

$$\frac{d^2\underset{\sim}{x}}{dt^2} + k\frac{d\underset{\sim}{x}}{dt} = 0$$

We would usually try $e^{\lambda t}$. We need a vector: try $\underset{\sim}{A}e^{\lambda t}$.

If $\underset{\sim}{x} = \underset{\sim}{A}e^{\lambda t}$, $\frac{d\underset{\sim}{x}}{dt} = \underset{\sim}{A}\lambda e^{\lambda t}$, $\frac{d^2\underset{\sim}{x}}{dt^2} = \underset{\sim}{A}\lambda^2 e^{\lambda t}$

so $\underset{\sim}{A}e^{\lambda t}(\lambda^2 + k\lambda) = \underset{\sim}{0}$

Solutions: $\lambda = 0$, $\lambda = -k$

$$x_{CF} = A e^{-kt} + B$$

PI :

We need one solution.
We would usually try a constant as that is the RHS; but there is a constant in the CF so try Ct instead.

$$x = Ct, \quad \frac{dx}{dt} = C, \quad \frac{d^2x}{dt^2} = 0$$

$$\Rightarrow kC = g$$

$$x_{PI} = g\frac{t}{k}$$

So the general soln is

$$x = x_{CF} + x_{PI}$$
$$= Ae^{-kt} + B + \frac{1}{k} g t$$

HHH

NEVER apply the initial conditions to the CF alone!

Initial conditions:

$t = 0$, $x = 0$, $\dot{x} = u$

$$\left.\begin{aligned} x &= Ae^{-kt} + B + \frac{1}{k} g t \\ \dot{x} &= -kAe^{-kt} + \frac{1}{k} g \end{aligned}\right\} \forall t$$

$$x(t=0) = A + B = 0$$
$$\dot{x}(t=0) = -kA + \frac{1}{k} g = u$$

so $A = -\frac{1}{k} u + \frac{1}{k^2} g$, $B = -A = \frac{1}{k} u - \frac{1}{k^2} g$

so $x = \left(\frac{1}{k^2} g - \frac{1}{k} u\right)\left(e^{-kt} - 1\right) + \frac{1}{k} g t.$

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Example: Coefficients powers of t

$$t^2\frac{d^2\underline{x}}{dt^2} - 4t\frac{d\underline{x}}{dt} + 6\underline{x} = 2\underline{b}t + \underline{c}t^2$$

this also fits the CF, PI model.

CF:

$$t^2\frac{d^2\underline{x}}{dt} - 4t\frac{d\underline{x}}{dt} + 6\underline{x} = \underline{0}$$

our trial function (if scalar) would be t^λ so we use $\underline{C}t^\lambda$

$$\underline{x}_{trial} = \underline{C}t^\lambda, \quad \frac{d\underline{x}_{trial}}{dt} = \underline{C}\lambda t^{\lambda-1}, \quad \frac{d^2\underline{x}_{trial}}{dt^2} = \underline{C}\lambda(\lambda-1)t^{\lambda-2}$$

so $\underline{C}t^\lambda\{\lambda(\lambda-1) - 4\lambda + 6\} = \underline{0}$

need $\lambda(\lambda-1) - 4\lambda + 6 = 0$

$$\lambda^2 - 5\lambda + 6 = 0$$

$$(\lambda-2)(\lambda-3) = 0$$

$$\underline{x}_{CF} = \underline{A}t^2 + \underline{B}t^3$$

we could find the PI by trial and error (try $\underline{C}t$, $\underline{D}t^2\log t$ ...) or we could set

$\underline{x} = t^3\underline{g}$ (note: $t^2\underline{g}$ works too - scalar function from CF).

Here $\underline{g}(t)$ is an unknown function defined as

$$\underline{g}(t) = t^{-3}\underline{x}(t).$$

$$\underline{x} = t^3\underline{g}, \quad \frac{d\underline{x}}{dt} = 3t^2\underline{g} + t^3\frac{d\underline{g}}{dt}$$

$$\frac{d^2\underline{x}}{dt^2} = 6t\underline{g} + 6t^2\frac{d\underline{g}}{dt} + t^3\frac{d^2\underline{g}}{dt^2}$$

Substituting:

$$\cancel{6t^3\underline{g}} + 6t^4\frac{d\underline{g}}{dt} + t^5\frac{d^2\underline{g}}{dt^2} - 4\left(3t^3\underline{g} + t^4\frac{d\underline{g}}{dt}\right) + \cancel{6t^3\underline{g}} = 2\underline{b}t + \underline{c}t^2$$

so $t^5\dfrac{d^2\underline{g}}{dt^2} + 2t^4\dfrac{d\underline{g}}{dt} = 2\underline{b}t + \underline{c}t^2$

Put $y = \frac{dg}{dt}$ and divide by t^5:

$$\frac{dy}{dt} + \frac{2}{t} y = \frac{2\underline{b}}{t^4} + \frac{\underline{c}}{t^3} .$$

This is first-order and linear so we can use an integrating factor:

$$I = e^{\int 2/t \, dt} = e^{2 \log t} = t^2$$

so $$t^2 \frac{dy}{dt} + 2t y + \frac{2\underline{b}}{t^2} + \frac{\underline{c}}{t}$$

$$\frac{d}{dt}\left[t^2 y\right] = \frac{2\underline{b}}{t^2} + \frac{\underline{c}}{t}$$

$$t^2 y = -\frac{2\underline{b}}{t} + \underline{c} \log t + \underline{\alpha}$$

$$y = \frac{-2\underline{b}}{t^3} + \frac{\underline{c} \log t}{t^2} + \frac{\underline{\alpha}}{t^2}$$

but $y = \frac{dg}{dt}$:

$$\frac{dg}{dt} = \frac{2\underline{b}}{t^3} + \frac{\underline{c} \log t}{t^2} + \frac{\underline{\alpha}}{t^2}$$

$$g = \frac{\underline{b}}{t^2} + \underline{c}\left(-\frac{1}{t} - \frac{\log t}{t}\right) - \frac{\underline{\alpha}}{t} + \underline{\beta}$$

but $\underline{x} = t^3 g$:

$$\underline{x} = \underline{b}\, t + \underline{c}\left(-t^2 - t^2 \log t\right) - \underline{\alpha}\, t^2 + \underline{\beta}\, t^3$$

6.3 - Techniques specific to vectors

Broadly speaking, these can be described as "use the dot product or cross product to improve the equation".

Example: Dot product

$$\frac{d\underline{v}}{dt} = g(t)(\underline{v}\cdot\underline{v})\underline{v}$$

We can't do separation of variables but we can dot with $\underline{v}$:

$$\underline{v}\cdot\frac{d\underline{v}}{dt} = g(t)(\underline{v}\cdot\underline{v})(\underline{v}\cdot\underline{v})$$

Now spot that

$$\frac{d}{dt}(\underline{v}\cdot\underline{v}) = \frac{d\underline{v}}{dt}\cdot\underline{v} + \underline{v}\cdot\frac{d\underline{v}}{dt} = 2\underline{v}\cdot\frac{d\underline{v}}{dt}$$

$\Rightarrow \frac{d}{dt}\left(\frac{1}{2}(\underline{v}\cdot\underline{v})\right) = g(t)(\underline{v}\cdot\underline{v})^2$ and if we let $\phi = \underline{v}\cdot\underline{v}$

then we have a scalar equation to solve for ϕ:

$$\frac{1}{2}\frac{d\phi}{dt} = g(t)\,\phi^2$$

which we can do by separation of variables.
Once solved, substitude in:

$$\frac{d\underline{v}}{dt} = g(t)\phi(t)\underline{v}$$

which we can do with an integrating factor.
We need to check that

$$\underline{v}\cdot\underline{v} = \phi$$

at the end, which will determine one of our two constants of integration (as with a first-order differential equation, we only need one).

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Example: Dot product for energy equation.
Suppose our force is conservative, $\underline{F} = -\underline{\nabla} V$

Then the differential equation that comes from Newton's 2nd Law responds really well to a dot product with velocity:

$$m\frac{d^2\underline{r}}{dt^2} = -\underline{\nabla} V$$

$$\Rightarrow m\underline{\dot{r}} \cdot \underline{\ddot{r}} = -\underline{\dot{r}} \cdot \underline{\nabla} V$$

(and we know $m\underline{\dot{r}} \cdot \underline{\ddot{r}} = \frac{d}{dt}\left(\frac{1}{2}m\underline{\dot{r}} \cdot \underline{\dot{r}}\right)$

$$-\underline{\dot{r}} \cdot \underline{\nabla} V = -\left(\frac{dx}{dt}\frac{\partial V}{\partial x} + \frac{dy}{dt}\frac{\partial V}{\partial y} + \frac{dz}{dt}\frac{\partial V}{\partial z}\right)$$

$$= -\frac{dV}{dt}$$

so $\frac{d}{dt}\left(\frac{1}{2}m\underline{\dot{r}} \cdot \underline{\dot{r}}\right) = -\frac{dV}{dt}$

Integrating: $\frac{1}{2}m|\underline{\dot{r}}|^2 + V(\underline{r}) = E$

Example: Energy equation with a constraint force.
Now suppose there is an additional force $\underline{R}$ whose only purpose is to constrain our particle to move in a subset of space (e.g. a given surface or curve). The force $\underline{R}$ will not have a component parallel to the surface or curve (because that is not needed for its purpose). This means that $\underline{\dot{r}} = \underline{v}$ (which is along the curve / within the surface) is perpendicular to $\underline{R}$.

"Because the surface / curve is smooth, $\underline{R}$ is $\perp$ to it. The velocity is parallel to the surface / curve, so $\underline{\dot{r}} \cdot \underline{R} = 0$."

The equation of motion is

$$m\ddot{\underline{r}} = -\nabla V + \underline{R}$$

Dotting with $\dot{\underline{r}}$: $m\dot{\underline{r}}\cdot\ddot{\underline{r}} = -\dot{\underline{r}}\cdot\underline{\nabla}V + \cancel{\dot{\underline{r}}\cdot\underline{R}}$

and the energy equation follows as before.

Example: Cross product for angular momentum.

If we have a central force,

$F = \alpha(\underline{r})\underline{r}$

then our differential eqn is

$$m\ddot{\underline{r}} = \alpha(\underline{r})\underline{r}.$$

Taking the cross product with $\underline{r}$:

$$m\underline{r}\times\ddot{\underline{r}} = \alpha(\underline{r})\underline{r}\times\underline{r} = \underline{0} \quad \text{(by properties of '}\times\text{')}$$

Spot that $\frac{d}{dt}(m\underline{r}\times\dot{\underline{r}}) = \underbrace{m\dot{\underline{r}}\times\dot{\underline{r}}}_{0} + m\underline{r}\times\ddot{\underline{r}}$

so we have

$$\frac{d}{dt}(m\underline{r}\times\dot{\underline{r}}) = \underline{0}$$

$$\Rightarrow m\underline{r}\times\dot{\underline{r}} = L \quad \text{, constant.}$$

Example: Air Resistance

$$\frac{d\underline{v}}{dt} = -\mu\underline{v} + \frac{\alpha\underline{v}}{|\underline{v}|}$$

dot with $\underline{v}$:

$$\underline{v}\cdot\frac{d\underline{v}}{dt} = -\mu\underline{v}\cdot\underline{v} + \alpha|\underline{v}|$$

Introduce the scalar function $f(t) = |\underline{v}(t)|$

$$\frac{d}{dt}\left(\tfrac{1}{2}f^2\right) = -\mu f^2 + \alpha f$$

$$\Rightarrow f\frac{df}{dt} = -\mu f^2 + \alpha f$$

$$\Rightarrow \frac{df}{dt} + \mu f = \alpha$$

$$\Rightarrow f = \frac{\alpha}{\mu} + \frac{k}{\mu}e^{-\mu t}$$ where k is an unknown constant.

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Cunning way:

Observe that $\frac{d\underline{v}}{dt}$ is $\parallel$ to $\underline{v}$.

Thus if $\underline{v} = v_0\underline{e}$ initially ($\underline{e}$ is a unit vector), $\frac{d\underline{v}}{dt}$ will always be in the $\underline{e}$ direction and so will $\underline{v}$: $\underline{v} = \left(\frac{\alpha}{\mu} + \frac{k}{\mu}e^{-\mu t}\right)\underline{e}$

Less cunning

We now know $|\underline{v}| = \frac{\alpha}{\mu} + \frac{k}{\mu}e^{-\mu t}$

Substitute in:

$$\frac{d\underline{v}}{dt} = \left(-\mu + \frac{\alpha}{\alpha/\mu + \frac{k}{\mu}e^{-\mu t}}\right)\underline{v}$$

$$= \mu\left(-1 + \frac{\alpha}{\alpha + ke^{-\mu t}}\right)\underline{v}$$

$$= \mu\left(\frac{-ke^{-\mu t}}{\alpha + -ke^{-\mu t}}\right)\underline{v}$$

$$\frac{d\underline{v}}{dt} + \frac{\mu ke^{-\mu t}}{\alpha + ke^{-\mu t}}\underline{v} = \underline{0}$$

$$I = \exp\left[\int \frac{\mu ke^{-\mu t}}{\alpha + ke^{-\mu t}}\,dt\right]$$

$$= \exp\left[-\log(\alpha + ke^{-\mu t})\right] = \frac{1}{\alpha + ke^{-\mu t}}$$

$$\Rightarrow \frac{d}{dt}\left[\frac{\underline{v}}{\alpha + ke^{-\mu t}}\right] = \underline{0}$$

so $\underline{v} = \underline{A}(\alpha + ke^{-\mu t})$ we have 2 unkown constants!

So to complete this, make sure $f = |\underline{v}|$.

$$\frac{\alpha}{\mu} + \frac{k}{\mu}e^{-\mu t} = |\underline{A}|(\alpha + ke^{-\mu t})$$

so $\frac{1}{\mu} = |\underline{A}|$

If we set $\underline{A} = \frac{1}{\mu}\underline{e}$ we are done:

$\underline{v} = \frac{1}{\mu}(\kappa + ke^{-\mu t})\underline{e}$

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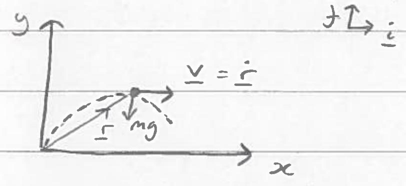
Topic 7 - Projectiles

Motion with 2 degrees of freedom, one of the forces is gravity.

7.1 Pure projectile: gravity only



N2: $m\ddot{\underline{r}} = m\underline{g}$

$\int$: $m\dot{\underline{r}} = m\underline{g}t + \underline{c}$

If initial velocity is $\underline{U}$, $\dot{\underline{r}} = \underline{U} + \underline{g}t$

$\int$: $\underline{r} = \underline{U}t + \frac{1}{2}\underline{g}t^2 + \underline{c}'$

If it starts from the origin, $\underline{c}' = \underline{0}$ and $\underline{r} = \underline{U}t + \frac{1}{2}\underline{g}t^2$

Components

Suppose $\underline{U} = U\cos\alpha\,\underline{i} + U\sin\alpha\,\underline{j}$,

i.e. speed U at angle α to the horizontal. Then

$x = U\cos\alpha\, t$, $y = U\sin\alpha\, t - \frac{1}{2}gt^2$

Horizontal range

This is the value of $x > 0$ at which $y = 0$, at time T.

$0 = U\sin\alpha\, T - \frac{1}{2}gT^2$.

$T = 0$ or $T = 2U\sin\alpha / g$

$x = 0$ (initial point) or $x = \frac{2U^2}{g}\sin\alpha\cos\alpha = \frac{U^2}{g}\sin 2\alpha$

Exercise

Set $\alpha = \frac{\pi}{2}$, find maximum height.

Cartesian Path

We can eliminate t to get a curve described by x, y.

$t = x / u\cos\alpha$

$\Rightarrow y = \frac{\cancel{u}\sin\alpha \, x}{\cancel{u}\cos\alpha} - \frac{gx^2}{2u^2\cos^2\alpha}$

$\Rightarrow y = x\tan\alpha - (gx^2/2u^2)(1+\tan^2\alpha)$

Safety zone

What reigion can the projectile reach if u is fixed, α free?
OR: for a point (X, Y), can I reach it by varying $\tan\alpha$?

Does $Y = X\tan\alpha - \left(\frac{gX^2}{2u^2}\right)(1+\tan^2\alpha)$ have a solution?

Answer:

Treat as a quadratic equation in $\tan\alpha$:

$$\frac{gX^2}{2u^2}\tan^2\alpha - X\tan\alpha + \left(Y + \frac{gX^2}{2u^2}\right) = 0$$

Real solutions if $b^2 - 4ac > 0$ (discriminant > 0).

$$X^2 - 4\left(\frac{gX^2}{2u^2}\right)\left(Y + \frac{gX^2}{2u^2}\right) > 0$$

$$X^2 > \frac{2gX^2}{u^2}\left(Y + \frac{gX^2}{2u^2}\right)$$

$$\frac{u^2}{2g} > Y + \frac{gX^2}{2u^2}$$

so $Y < \frac{u^2}{2g} - \frac{gX^2}{2u^2}$ ← this is the danger region.

The edge of the region, $Y = \frac{u^2}{2g} - \frac{gX^2}{2u^2}$

is called the <u>parabola of safety</u>.

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In 3D, we use cylindrical polar coordinates.
Height: $Y \to z$
Horizontal distance: $X \to r$
So $z = \frac{U^2}{2g} - \frac{gr^2}{2U^2}$ is the paraboloid of safety.

7.2 - Linear air resistance

Suppose we also have air resistance $\lambda \underline{v}$ per unit mass, $\lambda > 0$.

$m\ddot{\underline{r}} = m\underline{g} - m\lambda\underline{v}$

Rewrite

$$\dot{\underline{v}} + \lambda\underline{v} = \underline{g}$$

Integrating factor $e^{\int \lambda \, dt}$

$$\frac{d}{dt}(e^{\lambda t}\underline{v}) = \underline{g}e^{\lambda t}$$

$$e^{\lambda t}\underline{v} = \frac{1}{\lambda}\underline{g}e^{\lambda t} + \underline{c}$$

$$\underline{v} = \frac{1}{\lambda}\underline{g} + \underline{c}e^{-\lambda t}$$

$\underline{v} = \underline{U}$ at $t = 0 \Rightarrow \underline{c} = \underline{U} - \frac{1}{\lambda}\underline{g}$

$$\underline{v} = \underline{U}e^{-\lambda t} + \frac{1}{\lambda}\underline{g}(1 - e^{-\lambda t})$$

$\dot{\underline{r}} = \underline{v}$ so integrate:

$$\underline{r} = -\frac{1}{\lambda}\underline{U}e^{-\lambda t} + \frac{1}{\lambda}\underline{g}\left(t + \frac{1}{\lambda}e^{-\lambda t}\right) + \underline{B}$$

If we start from the origin,

$$\underline{0} = -\frac{1}{\lambda}\underline{U} + \frac{1}{\lambda}\underline{g}\left(\frac{1}{\lambda}\right) + \underline{B}$$

so $\underline{r} = \frac{1}{\lambda}\underline{U}(1 - e^{-\lambda t}) + \frac{1}{\lambda^2}\underline{g}(e^{-\lambda t} - 1 + \lambda t)$

Limit of no air resistance

$\lambda t \to 0$

Taylor series: $e^{-\lambda t} = 1 - \lambda t + \frac{\lambda^2 t^2}{2} + O(\lambda^3)$

$$\underline{r} = \frac{1}{\lambda}\underline{u}\left(\lambda t - \tfrac{1}{2}\lambda^2 t^2 + O(\lambda^3)\right) + \frac{1}{\lambda^2}\underline{g}\left(\tfrac{1}{2}\lambda^2 t^2 + O(\lambda^3)\right)$$

so $\underline{r} = \underline{u}t + \frac{1}{2}\underline{g}t^2$ ✓

Valid only not-too-long times $(t \ll \lambda^{-1})$

Long times (any λ)

$e^{-\lambda t} \to 0$ as $t \to \infty$

$$\underline{r} = \frac{1}{\lambda}\underline{u} + \frac{1}{\lambda^2}\underline{g}(\lambda t - 1)$$

$\underline{v} = \frac{1}{\lambda}\underline{g}$ ← terminal velocity.

The terminal velocity is the velocity at which there is no acceleration $\ddot{\underline{r}} = \underline{g} - \lambda\underline{v}$ here, so need $\underline{v} = \frac{1}{\lambda}\underline{g}$

Scalings

Suppose a cubic object falls from a great height.

It has sides of length L, and density ρ.

Weight $= \rho g L^3$, $m = \rho L^3$.

Air resistance $\propto$ area

coefficient $m\lambda$ $\quad L^2$

Scalings with L

$m \sim L^3$, $m\lambda \sim L^2 \Rightarrow \lambda \sim L^{-1}$

$\Rightarrow v_T = \frac{1}{\lambda}g \sim L$

So, large objects fall faster.

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7.3 – More realistic air resistance

In reality, at fast flow speeds,
air resistance $\propto |\underline{v}|^2$, so we have

$$m\ddot{\underline{r}} = m\underline{g} - \lambda m|\underline{v}|\underline{v}$$

or

$$\dot{\underline{v}} = \underline{g} - \lambda|\underline{v}|\underline{v} \quad \leftarrow \text{Non linear!}$$

In general we can't solve this analytically, but if the particle is projected vertically, than all the motion is vertical.

Purely vertical:

$$\underline{r} = y\underline{j}$$
$$\underline{v} = \dot{y}\underline{j}$$

$\underline{j}$| $\ddot{y} = -g - \lambda\dot{y}^2$

If $u = \dot{y}$,

$$\dot{u} = -g - \lambda u^2$$

Separation of variables:

$$\int \frac{du}{g+\lambda u^2} = \int -1\, dt$$

$$\Rightarrow \frac{1}{g}\arctan\left(u\sqrt{\lambda/g}\right) = -t + A \quad \text{(maybe?)}$$

$$\frac{dy}{dt} = u = \sqrt{\frac{g}{\lambda}}\tan\left(\sqrt{g\lambda}(A-t)\right)$$

which can, in theory, be integrated to give $y(t)$.
We can also find $u(y)$ using

$$\frac{du}{dt} = \frac{du}{dy}\frac{dy}{dt} = u\frac{du}{dy}$$

$$u\frac{du}{dy} = -g - \lambda u^2$$

+ seperation of variables.
Hence find max height.

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Topic 8 - Motion in a Surface of Revolution

A surface of revolution is the surface obtained by rotating a curve $y(x)$ $(x \geq 0)$ around the y axis.

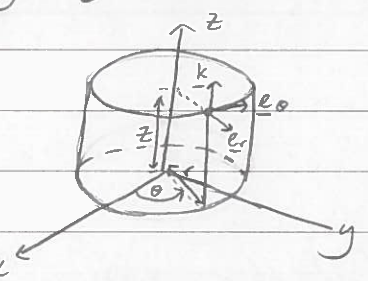
- e.g. rotating a parabola gives a paraboloid
- rotating a half circle gives a sphere.

They are best described in cylindrical coordinates, in which they are given by

$$z = f(r) \quad \text{for any } f.$$

8.1 - Cylindrical polar coordinates

We define $\{r, \theta, z\}$ from the cartesian coordinates $\{x, y, z\}$ as $x = r\cos\theta$, $y = r\sin\theta$, $z = z$



The unit basis vectors are

$\{\underline{e}_r, \underline{e}_\theta, \underline{k}\}$ as shown:

$\underline{e}_r$ and $\underline{e}_\theta$ have the same form as in plane polars.

$\underline{e}_r = \cos\theta\, \underline{i} + \sin\theta\, \underline{j}$

$\underline{e}_\theta = -\sin\theta\, \underline{i} + \cos\theta\, \underline{j}$

and $\underline{k}$ is out of the plane.

The position vector is

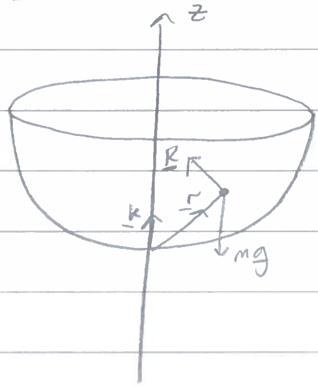
$$\underline{r} = r\underline{e}_r + z\underline{k}$$

which means (drawing on our knowledge from plane polars)

$$\underline{v} = \underline{\dot{r}} = \dot{r}\underline{e}_r + r\dot{\theta}\underline{e}_\theta + \dot{z}\underline{k}$$

$$\underline{a} = \underline{\ddot{r}} = (\ddot{r} - r\dot{\theta}^2)\underline{e}_r + \frac{1}{r}\frac{d}{dt}(r^2\dot{\theta})\underline{e}_\theta + \ddot{z}\underline{k}$$

8.2 - Motion on a smooth surface of revolution



Because the surface is smooth, the reaction force is perpendicular to it.

This means: $\dot{\underline{r}} \cdot \underline{R} = 0$

because velocity is always parallel to the surface.

But also $\underline{e}_\theta \cdot \underline{R} = 0$

by symmetry of the surface.

Alternatively:

It is intuitively obvious that the vector $\underline{R}$ will intersect the z axis which means

$\underline{k}$, $\underline{r}$ and $\underline{R}$

lie in a plane.

i.e. $\underline{R} = \alpha \underline{k} + \beta(r \underline{e}_r + z \underline{k})$

$= a\underline{k} + b\underline{e}_r$ so $\underline{e}_\theta \cdot \underline{R} = 0$

This also means

$\underline{r} \times \underline{R}$ is perpendicular to that plane, so

$\underline{k} \cdot (\underline{r} \times \underline{R}) = 0$

8.2.1 - Angular momentum

Newtons 2nd Law

$m\ddot{\underline{r}} = m\underline{g} + \underline{R}$.

Take $\underline{e}_\theta$ component:

$$\frac{m}{r} \frac{d}{dt}(r^2\dot{\theta}) = m\underline{g} \cdot \underline{e}_\theta + \underline{e}_\theta \cdot \underline{R} = 0$$

$$\Rightarrow \quad r^2\dot{\theta} = h$$

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Or: cross with $\underline{r}$

$$m\underline{r}\times\ddot{\underline{r}} = m\underline{r}\times g + \underline{r}\times\underline{R}$$
$$= m(r\underline{e}_r + \cancel{z\underline{k}})\times g + \underline{r}\times\underline{R}$$

and dot with $\underline{k}$:

$$m\underline{k}\cdot(\underline{r}\times\ddot{\underline{r}}) = m\underline{k}\cdot(r\underline{e}_r\times \overset{\parallel \text{ to } \underline{k}}{g}) + \underbrace{\underline{k}\cdot(\underline{r}\times\underline{R})}_{0}$$

$$\Rightarrow \underline{k}\cdot(\underline{r}\times\ddot{\underline{r}}) = 0$$

But $\frac{d}{dt}(\underline{k}\cdot(\underline{r}\times m\dot{\underline{r}})) = \underline{k}\cdot\frac{d}{dt}(\underline{r}\times m\dot{\underline{r}})$

$$= \underline{k}\cdot\left\{\underbrace{\dot{\underline{r}}\times m\dot{\underline{r}}}_{0} + \underbrace{\underline{r}\times m\ddot{\underline{r}}}_{0}\right\}$$

so $\underline{k}\cdot(\underline{r}\times m\dot{\underline{r}}) = mh$ (constant)

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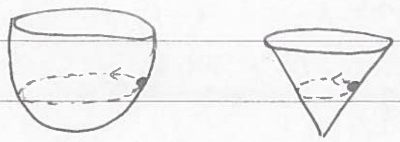

$m\ddot{\underline{r}} = m\underline{g} + \underline{R}$

Smooth surface $\Rightarrow$ $\underline{R}$ is $\perp$ to the surface

Last time: $r^2\dot{\theta} = h$ constant

OR $\underline{k}\cdot(\underline{r} \times \dot{\underline{r}})$ constant.

8.2.2 – Energy Conservation

From Newton's 2nd law

$m\ddot{\underline{r}} = m\underline{g} + \underline{R}$

Dot with velocity to obtain

$m\dot{\underline{r}}\cdot\ddot{\underline{r}} = m\dot{\underline{r}}\cdot\underline{g} + \dot{\underline{r}}\cdot\underline{R}$

LHS is $\frac{d}{dt}\left(\frac{1}{2}m\dot{\underline{r}}\cdot\dot{\underline{r}}\right)$,

ie. rate of change of kinetic energy.

To simplify $m\dot{\underline{r}}\cdot\underline{g}$, we use

$\dot{\underline{r}} = \dot{r}\underline{e}_r + r\dot{\theta}\underline{e}_\theta + \dot{z}\underline{k}$

and $\underline{g} = -g\underline{k}$

so $m\dot{\underline{r}}\cdot\underline{g} = -mg\dot{z} = \frac{d}{dt}(-mgz)$

ie. (-1) x rate of change of potential energy due to gravity.

For a smooth surface, $\underline{R}$ is normal to the surface.
The velocity $\underline{r}$ is always parallel to the surface so for a smooth surface

$\underline{r}\cdot\underline{R} = 0$

So $\frac{d}{dt}\left(\frac{1}{2}m\dot{\underline{r}}\cdot\dot{\underline{r}}\right) = \frac{d}{dt}(-mgz) + 0$ ← 1st order ODE for z.

$$\Rightarrow \quad \frac{1}{2}m|\dot{\underline{r}}|^2 + mgz = E \leftarrow \text{constant}$$

$\leftarrow$ Energy conservation

[If the surface were not smooth, we can still define

$$E = \frac{1}{2}m|\dot{\underline{r}}|^2 + mgz$$

but we will have

$$\frac{dE}{dt} = \dot{\underline{r}} \cdot \underline{R},$$

the friction dissipates energy]

To return to our energy conservation equation, recall $\dot{\underline{r}} = \dot{r}\underline{e_r} + r\dot{\theta}\underline{e_\theta} + \dot{z}\underline{k}$ to have

$$\frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2 + \dot{z}^2) + mgz = E$$

Because we know $r^2\dot{\theta} = h$, we can eliminate $\dot{\theta}$:

$$\frac{1}{2}m\left(\dot{r}^2 + \frac{h^2}{r^2} + \dot{z}^2\right) + mgz = E$$

We can't go further without specifying the shape of the surface.

8.3 - Confined Motion

Let us now define our surface as $r = \omega(z)$.
We have derived conservation of angular momentum and of energy and that is all we will need.

Angular momentum:

$$r^2\dot{\theta} = h \Rightarrow \dot{\theta} = h/\omega^2(z)$$

Energy: if $r = \omega(z)$

then $$\dot{r} = \frac{dr}{dt} = \frac{dr}{dz}\frac{dz}{dt} = \omega'(z)\dot{z}$$

So we have

$$\frac{1}{2}m\left[(\omega'(z))^2\dot{z}^2 + \frac{h^2}{\omega^2(z)} + \dot{z}^2\right] + mgz = E$$

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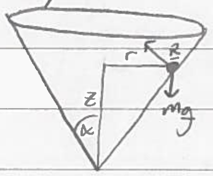
$$\frac{1}{2}m\left(1+w'(z)^2\right)\dot{z}^2 = E - mgz - \frac{mh}{2w^2(z)}$$

The LHS is ≥ 0 always, so we can deduce

$$E - mgz - \frac{mh^2}{2w^2(z)} \geq 0$$

This will constrain the possible values of z.

Example - Motion on the inner surface of a cone



We consider a vertical cone with half-angle α.

This means $r = z\tan\alpha$.

Suppose initially our particle is at height $z = z_0$ and moving horizontally with speed U.
So initial position is $r = z_0 \tan\alpha$, $z = z_0$
and initial velocity is $U\underline{e}_\theta$
(no $\underline{e}_r$ component because that would require a $\underline{k}$ component) so initially

$$\dot{r} = 0, \quad r\dot{\theta} = U \Rightarrow \dot{\theta} = U/(z_0\tan\alpha), \quad \dot{z} = 0.$$

Angular momentum

$$r^2\dot{\theta} = h.$$

Initial conditions $\Rightarrow h = (z_0\tan\alpha)^2 U/(z_0\tan\alpha)$

$$\Rightarrow h = Uz_0\tan\alpha.$$

Energy:

$$\frac{1}{2}m\left(\dot{r}^2 + \frac{h^2}{r^2} + \dot{z}^2\right) + mgz = E$$

Initial conditions:

$$\frac{1}{2}m\left(0 + \frac{U^2 \cancel{z_0^2\tan^2\alpha}}{\cancel{z_0^2\tan^2\alpha}} + 0\right) + mgz_0 = E$$

so $E = \frac{1}{2}mU^2 + mgz_0$

Now use energy equation:

$$\frac{1}{2}m\left(\dot{r}^2 + \frac{U^2 z_0^2 \tan^2\alpha}{r^2} + \dot{z}^2\right) + mgz = \frac{1}{2}mU^2 + mgz$$

and eliminate r using

$r = z\tan\alpha$

$\dot{r} = \dot{z}\tan\alpha$ (note: α = constant)

so $\frac{1}{2}\left(\dot{z}^2\tan^2\alpha + \frac{U^2 z_0^2}{z^2} + \dot{z}^2\right) + gz = \frac{1}{2}U^2 + gz_0$

$$\Rightarrow \frac{\dot{z}^2}{2}(\tan^2\alpha + 1) = \frac{1}{2}U^2\left(1 - \frac{z_0^2}{z^2}\right) + g(z_0 - z)$$

LHS ≥ 0 so RHS ≥ 0.

So $U^2\left(1 - \frac{z_0^2}{z^2}\right) + 2g(z_0 - z) \geq 0$

Since $z \neq 0$ (by angular momentum) we can multiply by z^2 safely.

$$U^2(z^2 - z_0^2) + 2g(z_0 - z)z^2 \geq 0$$

Because our initial conditions gave $\dot{z} = 0$ when $z = z_0$, we know $z = z_0$ must be a root of this cubic.

$$(z - z_0)\{U^2(z + z_0) - 2gz^2\} \geq 0$$

Call $f(z) = (z - z_0)\{-2gz^2 + U^2 z + U^2 z_0\}$ and plot it.

We need $f(z) \geq 0$.

Roots of $f(z)$, $z = z_0$ and

$$z = \frac{-U^2 \pm \sqrt{U^4 - 4(-2g)(U^2 z_0)}}{-4g}$$

$$= \frac{U^2}{4g} + \frac{U}{4g}\sqrt{U^2 + 8gz_0} = \frac{U^2}{4g}\left(1 \pm \sqrt{1 + \frac{8gz_0}{U^2}}\right)$$

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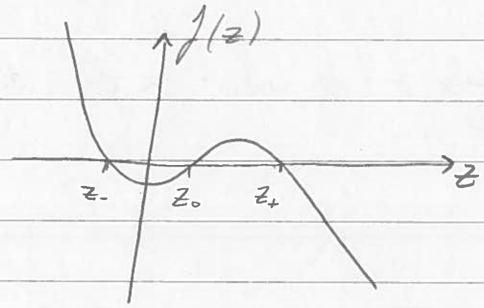
Behaviour for large n:

$|z| \to \infty$, $f(z) \sim -2g z^3$.

Behaviour for small n:

$z \to 0$, $f(z) = -U^2 z_0^2$

Because $\left(1 + \frac{8gz_0}{U^2}\right) > 1$ we z_- is negative and z_+ is positive.

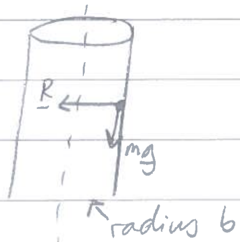


We need the region where $f(x) \geq 0$,
i.e. $z_0 \leq z \leq z_+$ from the diagram.
(The negative z region is not accessible because we would have to move through forbidden z to get there)
Note: we don't actually know $z_+ > z_0$.
There are two possible heights at which $\dot{z} = 0$.
One is z_0, the other is $\frac{U^2}{4g}\left(1 + \sqrt{1 + \frac{8gz_0}{U^2}}\right)$

The particle moves in the region between them, but we can't know whether $z_0 > z_+$ or $z_0 < z_+$.

If $z_0 = z_+$, i.e. $gz_0 = U^2$, the particle moves in a horizontal circle.

Example – Vertical cylinder



Here we can write $\underline{R} = -R\underline{e}_r$ (with $R \geq 0$) and $\underline{g} = -g\underline{k}$

$m\ddot{\underline{r}} = -mg\underline{k} - R\underline{e}_r$

Three components:

$\boxed{\underline{e}_r}$ $m(\ddot{r} - r\dot{\theta}^2) = -R$

$\boxed{\underline{e}_\theta}$ $\frac{m}{r}\frac{d}{dt}(r^2\dot{\theta}) = 0 \longrightarrow r^2\dot{\theta} = h, \quad \dot{\theta} = h/b^2$

$\boxed{\underline{k}}$ $m\ddot{z} = -mg$

$\underline{e}_r$ eq^n: $m\left(\ddot{r} - r\left(\frac{h}{b^2}\right)^2\right) = -R$

but $\ddot{r} = 0$ because r is constant.

$\Rightarrow R = \frac{mh^2}{b^3}$

Finally, $\ddot{z} = -g$ gives

$z = -\frac{1}{2}gt^2 + At + B$

$r = b$

$\theta = C + \frac{h}{b^2}t$

Motion is completely determined.

We can check energy conservation

$$\left.\begin{array}{l} \dot{z} = -gt + A \\ \dot{\theta} = h/b^2 \\ \dot{r} = 0 \end{array}\right\} \dot{\underline{r}} = r\dot{\theta}\underline{e}_\theta + \dot{z}\underline{k} = \frac{h}{b}\underline{e}_\theta + (A - gt)\underline{k}$$

so $|\dot{\underline{r}}|^2 = \frac{h^2}{b^2} + (A - gt)^2$

and $\frac{1}{2}m|\dot{\underline{r}}|^2 + mgz = \frac{mh^2}{2b^2} + \frac{m}{2}(A - gt)^2 + mg\left(-\frac{1}{2}gt^2 + At + B\right)$

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$$= \frac{mh^2}{2b^2} + \frac{mA^2}{2} \cancel{-Amgt} + \cancel{\frac{mg^2t^2}{2}} - \cancel{\tfrac{1}{2}mg^2t^2} + \cancel{mgAt} + mgB$$

$$= \frac{mh^2}{2b^2} + \frac{mA^2}{2} + mgB = \text{constant} \checkmark$$

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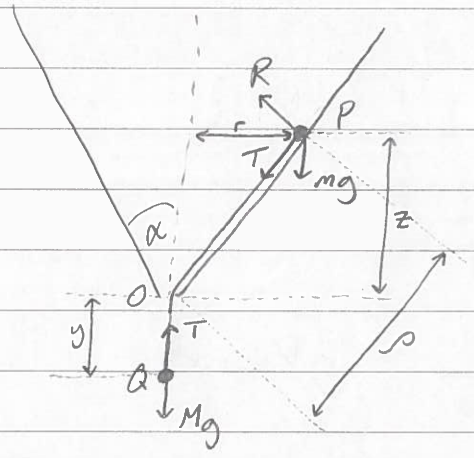


ⓐ $\underline{r}_P = r\underline{e}_r + z\underline{k}$
$\underline{r}_Q = -y\underline{k}$

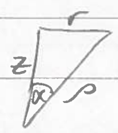
ⓑ N2 (P) : $m\ddot{\underline{r}}_P = \underline{R} + \underline{T} + m\underline{g}$
Taking components:
$\underline{e}_r$/ $m(\ddot{r} - r\dot{\theta}^2) = -R\cos\alpha - T\sin\alpha$ (1)
$\underline{e}_\theta$/ $\frac{m}{r}\frac{d}{dt}(r^2\dot{\theta}) = 0$ (2)

$\underline{k}$/ $m\ddot{z} = R\sin\alpha - T\cos\alpha - mg$ (3)

Aside: count unknowns: $r, z, \theta, y, \rho, R, T \Rightarrow$ need 7 scalar equ's)

N2(Q) : $M\ddot{\underline{r}}_Q = T\underline{k} - Mg\underline{k}$
$-M\ddot{y} = T - Mg$ (4)

Geometrically the cone gives



$z = \rho\cos\alpha$ (5)
$r = \rho\sin\alpha$ (6)

string length: $y + \rho = l$ (7)

ⓒ Initially, P is projected in $\underline{e}_\theta$ direction so
$\dot{\underline{r}}_P = r\dot{\theta}\underline{e}_\theta = V\underline{e}_\theta$
Initially $\rho = a$ and from 5 & 6) $z = a\cos\alpha$, $r = a\sin\alpha$

Initially, $\dot{\theta} = \dfrac{V}{a\sin\alpha}$

and $\dot{r} = \dot{z} = \dot{\rho} = \dot{y} = 0$

also $y = l - a$ (from 7).

ⓐ Integrate (2)

$$r^2\dot{\theta} = h = (a\sin\alpha)^2 \frac{V}{a\sin\alpha} = aV\sin\alpha$$

ⓒ (6) $\rho = \dfrac{r}{\sin\alpha}$

(7) $y = l - \rho = \lambda - \dfrac{r}{\sin\alpha}$

(5) $z = \rho\cos\alpha = \dfrac{r\cos\alpha}{\sin\alpha}$

Angular momentum: $\dot{\theta} = \dfrac{h}{r^2} = \dfrac{aV\sin\alpha}{r^2}$

Left with (1) $m\left(\ddot{r} - \dfrac{a^2V^2\sin^2\alpha}{r^3}\right) = -R\cos\alpha - T\sin\alpha$

(3) $m\ddot{r}\dfrac{\cos\alpha}{\sin\alpha} = R\sin\alpha - T\cos\alpha - mg$

(4) $\dfrac{M\ddot{r}}{\sin\alpha} = T - Mg$

$\Rightarrow$ 3 equs in r, R and T.

Use (1) and (3) to eliminate R.

[$\sin\alpha \times (1) + \cos\alpha \times (3)$]

$$\Rightarrow m\left(\ddot{r}\sin\alpha - \frac{a^2V^2\sin^3\alpha}{r^3} + \frac{\ddot{r}\cos^2\alpha}{\sin\alpha}\right) = -T\sin^2\alpha - T\cos^2\alpha - mg\cos\alpha$$

$$\Rightarrow m\left(\frac{\ddot{r}}{\sin\alpha} - \frac{a^2V^2\sin^3\alpha}{r^3}\right) = -T - mg\cos\alpha \qquad (8)$$

Now use (4) + (8) to eliminate T.

$$\frac{m\ddot{r}}{\sin\alpha} - \frac{ma^2V^2\sin^3\alpha}{r^3} + \frac{M\ddot{r}}{\sin\alpha} = -Mg - mg\cos\alpha$$

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$$\Rightarrow (M+m)\ddot{r} - \frac{ma^2V^2\sin^4\alpha}{r^3} = -(M+m\cos\alpha)g\sin\alpha$$

(f) For circular motion, need $\ddot{r}=0$ so

$$V^2 = \frac{(M+m\cos\alpha)g\sin\alpha\, r^3}{ma^2\sin^4\alpha} = \frac{M+m\cos\alpha\, gr^3}{ma^2\sin^3\alpha}$$

Stability:

put $r = r_0 + \varepsilon$ and linearise.

$$(M+m)\ddot{\varepsilon} - \underbrace{ma^2V^2\sin^4\alpha\,(r_0^{-3}}\, - 3\varepsilon r_0^{-4} + \dots) = \underbrace{-(M+m\cos\alpha)g\sin\alpha}$$

so $$(M+m)\ddot{\varepsilon} + \frac{3ma^2V^2\sin^4\alpha}{r_0^4}\varepsilon = 0$$

looks like $\ddot{\varepsilon} + \omega\varepsilon = 0$
SHM $\Rightarrow$ stable

(g) Now given $V^2 = 4ag\left(\frac{M}{m} + \cos\alpha\right)$

$$\Rightarrow (M+m)\ddot{r} = \frac{ma^2}{r^3}\left[4ag\left(\frac{M}{m}+\cos\alpha\right)\right]\sin^4\alpha - (M+m\cos\alpha)g\sin\alpha$$

$$= (M+m\cos\alpha)g\sin\alpha\left\{\frac{4a^3\sin^3\alpha}{r^3} - 1\right\}$$

To get constraints, we will need an energy equation, so multiply by $\dot{r}$ and integrate.

$$(M+m)\ddot{r}\dot{r} = (M+m\cos\alpha)g\sin\alpha\left\{\frac{4a^3\sin^3\alpha}{r^3} - 1\right\}\dot{r}$$

$$\frac{1}{2}(M+m)\dot{r}^2 = (M+m\cos\alpha)g\sin\alpha\left\{\frac{-2a^3\sin^3\alpha}{r^2} - r\right\} + E$$

Initially $\dot{r}=0$, $r = a\sin\alpha$

so

$$-E = (M+m\cos\alpha)g\sin\alpha\left\{-2a\sin\alpha - a\sin\alpha\right\}$$

Energy equ:

$$\frac{1}{2}(M+m)\dot{r}^2 = (M+m\cos\alpha)g\sin\alpha\left\{\frac{-2a^3\sin^3\alpha}{r^2} - r + 3a\sin\alpha\right\}$$

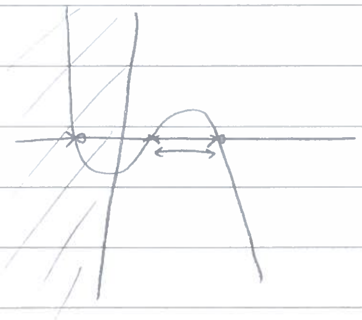
We know LHS ≥ 0 so RHS ≥ 0

$\Rightarrow \dfrac{-2a^3\sin^3\alpha}{r^2} - r + 3a\sin\alpha \geq 0$

$\Rightarrow -2a^3\sin^3\alpha - r^3 + 3ar^2\sin\alpha \geq 0$

$(r - a\sin\alpha)(-r^2 - ar\sin\alpha + 2a^2\sin^2\alpha) \geq 0$

roots: $r = a\sin\alpha$, $\dfrac{a\sin\alpha \pm \sqrt{a^2\sin^2\alpha + 8a^2\sin^2\alpha}}{-2}$



$\Rightarrow a\sin\alpha \leq r \leq -\frac{1}{2}a\sin\alpha + \frac{1}{2}\sqrt{9a^2\sin^2\alpha}$

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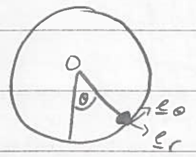
Q3



ⓐ cylinder smooth $\underline{R} = -R\underline{e_r}$

N2: $m\ddot{\underline{r}} = \underline{R} + m\underline{g}$

Energy eqn: dot with velocity

$$m\dot{\underline{r}}\cdot\ddot{\underline{r}} = \cancel{\dot{\underline{r}}\cdot\underline{R}} + \dot{\underline{r}}\cdot m\underline{g}$$

$\underline{r} = a\underline{e_r}$ $\quad\quad$ $\underline{e_r} = \sin\theta\,\underline{i} - \cos\theta\,\underline{j}$

$\quad\quad\quad\quad\quad\quad\quad$ $\underline{e_\theta} = \cos\theta\,\underline{i} + \sin\theta\,\underline{j}$

$$\dot{\underline{r}} = a\frac{d\underline{e_r}}{dt} = a\frac{d\underline{e_r}}{d\theta}\frac{d\theta}{dt}$$

$$= a\dot{\theta}\underline{e_\theta}$$

$$\ddot{\underline{r}} = \frac{d}{dt}(\dot{\underline{r}}) = a\frac{d}{dt}(\dot{\theta}\underline{e_\theta}) = a\ddot{\theta}\underline{e_\theta} + a\dot{\theta}\frac{d}{dt}(\underline{e_\theta})$$

$$= a\ddot{\theta}\underline{e_\theta} + a\dot{\theta}\frac{d}{d\theta}(\underline{e_\theta})\frac{d\theta}{dt}$$

$$= -a\dot{\theta}^2\underline{e_r} + a\ddot{\theta}\underline{e_\theta}$$

Energy: $\quad m\dot{\underline{r}}\cdot\ddot{\underline{r}} = \dot{\underline{r}}\cdot m\underline{g}$

$$\frac{d}{dt}\left(\frac{1}{2}m\dot{\underline{r}}\cdot\dot{\underline{r}}\right) = a\dot{\theta}\underline{e_\theta}\cdot(-mg\underline{j})$$

$$= -mga\dot{\theta}\sin\theta$$

$$= \frac{d}{dt}(mga\cos\theta)$$

$$\Rightarrow \frac{1}{2}m|\dot{\underline{r}}|^2 = mga\cos\theta + E$$

$$\Rightarrow \frac{1}{2}ma^2\dot{\theta}^2 = mga\cos\theta + E$$

Initial conditions: $\theta = 0$ and speed $= U$ so $a\dot{\theta} = U$, $\dot{\theta} = U/a$

$$\Rightarrow \frac{1}{2}mU^2 = mga + E, \quad E = \frac{1}{2}mU^2 - mga$$

so $\frac{1}{2}ma^2\dot{\theta}^2 - mga\cos\theta = \frac{1}{2}mu^2 - mga$

(b) We need the $R > 0 \ \forall \theta$.

Returning to N2 and dotting with $\underline{e}_r$:

$$-ma\dot{\theta}^2 = -R + m\underline{g}\cdot\underline{e}_r$$
$$= -R + mg\cos\theta$$

so $R = m(a\dot{\theta}^2 + g\cos\theta)$

but $a\dot{\theta}^2 = \frac{2}{ma}\left(\frac{1}{2}mu^2 - mga + mga\cos\theta\right)$

$$= \frac{u^2}{a} - 2g + 2g\cos\theta$$

so $R = m\left(\frac{u^2}{a} - 2g + 2g\cos\theta + g\cos\theta\right)$

We need $\frac{u^2}{ag} - 2 + 3\cos\theta \geq 0$

Minimum value is at top, $\cos\theta = -1$

so need $\frac{u^2}{ag} - 2 - 3 \geq 0$

$$\Rightarrow u \geq \sqrt{5ag}$$

(c) $m\ddot{\underline{r}} = -R\underline{e}_r - \mu R\underline{e}_\theta + m\underline{g}$

(i) $m(-a\dot{\theta}^2\underline{e}_r + a\ddot{\theta}\underline{e}_\theta) = -R\underline{e}_r - \mu R\underline{e}_\theta + mg\cos\theta\,\underline{e}_r - mg\sin\theta\,\underline{e}_\theta$

$\underline{e_r}|$ $-am\dot{\theta}^2 = -R + mg\cos\theta \Rightarrow R = am\dot{\theta}^2 + mg\cos\theta$

$\underline{e_\theta}|$ $ma\ddot{\theta} = -\mu R - mg\sin\theta$

$\underline{e_\theta}| \Rightarrow ma\ddot{\theta} = -\mu(am\dot{\theta}^2 + mg\cos\theta) - mg\sin\theta$.

(d) Set $E(t) = \frac{1}{2}ma^2\dot{\theta}^2 - mga\cos\theta$

then $\frac{dE}{dt} = ma^2\dot{\theta}\ddot{\theta} + mga\sin\theta\,\dot{\theta}$

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Q3(d) cont.

$$\frac{dE}{dt} = \dot{\theta}(ma^2\ddot{\theta} + mga\sin\theta)$$

$$= a\dot{\theta}(ma\ddot{\theta} + mg\sin\theta)$$

$$= a\dot{\theta}(-\mu(am\dot{\theta}^2 + mg\cos\theta) - \cancel{mg\sin\theta} + \cancel{mg\sin\theta})$$

$$= -\mu am\dot{\theta}(a\dot{\theta}^2 + g\cos\theta)$$

4). $\underline{F} = -mf(r)\underline{e}_r$

(a) N2: $m\underline{\ddot{r}} = -mf(r)\underline{e}_r$

where $\underline{\ddot{r}} = (\ddot{r} - r\dot{\theta}^2)\underline{e}_r + \frac{1}{r}\frac{d}{dt}(r^2\dot{\theta})\underline{e}_\theta$

$\underline{e_\theta}$) $\frac{m}{r}\frac{d}{dt}(r^2\dot{\theta}) = 0 \Rightarrow r^2\dot{\theta} = h$

$\underline{e_r}$) $m(\ddot{r} - r\dot{\theta}^2) = -mf(r)$

$$\Rightarrow \ddot{r} - \frac{h^2}{r^3} = -f(r)$$

(b) $f(r) = Ar^n$

If $r = b$ then $\dot{r} = \ddot{r} = 0$ so need

$$-\frac{h^2}{b^3} = -f(b) = -Ab^n \Rightarrow h^2 = Ab^{n+3}$$

$$h = A^{1/2}b^{(n+3)/2}$$

(c) Put $r = b + \varepsilon$, $h^2 = Ab^{n+3}$

$$\ddot{r} - \frac{h^2}{r^3} = -Ar^n$$

$$\ddot{\varepsilon} - \frac{h^2}{(b+\varepsilon)^3} = -A(b+\varepsilon)^n$$

Taylor expansion

$$\ddot{\varepsilon} - h^2(b^{-3} - 3b^{-4}\varepsilon + \dots) = -A(b^n + nb^{n-1}\varepsilon + \dots)$$

$$\ddot{\varepsilon} - \cancel{\frac{h^2}{b^3}} + \frac{3h^2}{b^4}\varepsilon = \cancel{-Ab^n} - Anb^{n-1}\varepsilon + O(\varepsilon^2)$$

$$\ddot{\varepsilon} + \left(\frac{3h^2}{b^4} + Anb^{n-1}\right)\varepsilon = O(\varepsilon^2)$$

$$\Rightarrow \quad \ddot{\varepsilon} + (3Ab^{n-1} + Anb^{n-1})\varepsilon = 0 \quad \text{to order } \varepsilon.$$

$$\ddot{\varepsilon} + Ab^{n-1}(n+3)\varepsilon = 0$$

ⓓ Stability?

If eq^n looks like $\ddot{\varepsilon} - q^2\varepsilon = 0$

$\Rightarrow$ solution looks like e^{qt}, e^{-qt}

which is unstable.

If eqn looks like $\ddot{\varepsilon} + \omega^2\varepsilon = 0$

$\Rightarrow$ solution looks like $\cos(\omega t), \sin(\omega t)$

which are bounded $\Rightarrow$ stable.

So stable $\Leftrightarrow Ab^{n-1}(n+3) > 0$

$A, b > 0$

so stable $\Leftrightarrow n+3 > 0 \Leftrightarrow n > -3$

2). ⓐ · Move a force along its line of action.

· Add a pair of equal and opposite forces acting at the same point.

ⓑ For systems to be equivalent, we need $\sum_i \underline{F}_i$ and $\underbrace{\sum_i \underline{r}_i \times \underline{F}_i}_{= \underline{G}_0}$ to match

In each case here, $\sum_i \underline{F}_i = \underline{i} - \underline{k}$

System I

$\underline{r}_1 \times \underline{F}_1 = -2\underline{k}$

$\underline{r}_2 \times \underline{F}_2 = -2\underline{i} \qquad \Rightarrow \quad \underline{G}_0 = -2\underline{i} - 2\underline{k}$

System II

$\underline{r}_1 \times \underline{F}_1 = 0$

$\underline{F}_2 = -\underline{F}_3$ so $\underline{r}_2 \times \underline{F}_2 + \underline{r}_3 \times \underline{F}_3 = (\underline{r}_2 - \underline{r}_3) \times \underline{F}_2$

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Q 2(b) cont

system II

$(\underline{r}_2 - \underline{r}_3) \times \underline{F}_2 = -2\underline{i} - 2\underline{k}$ ← matches system I.

(c) $\underline{F} = \underline{i} - \underline{k}$

$\underline{G}_0 = -2\underline{i} - 2\underline{k}$

Can reduce to a single force if $\underline{F} \cdot \underline{G}_0 = 0$ ✓

Seek $\underline{b}$ s.t. $\underline{b} \times \underline{F} = \underline{G}_0$: Then our system is equivalent to a force $\underline{F} = \underline{i} - \underline{k}$ at position $\underline{b}$.

let $\underline{b} = \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix}$

$$\underline{b} \times \underline{F} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ \alpha & \beta & \gamma \\ 1 & 0 & -1 \end{vmatrix}$$

$$= -\beta \underline{i} + (\alpha + \gamma)\underline{j} - \beta \underline{k} = \underline{G}_0 = -2\underline{i} - 2\underline{k}$$

so $\beta = 2$ and $\alpha + \gamma = 0$

System becomes force $\underline{i} - \underline{k}$ at position

$2\underline{j} + \lambda(\underline{i} - \underline{k})$

↖ line of action