# 1401 Mathematical Methods 1 Notes

Based on the 2015 autumn lectures by Dr C G Böhmer

5/10/15

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# 1401 - Methods 1

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Office hour Mon 2-3pm
Vectors test after R.W.
Drop-box 1 (by Monday)
(last year: ave 65%)

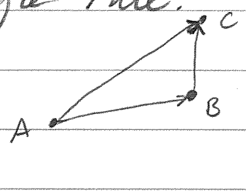
- Vectors
- Complex numbers
- Integral calculus
- 1st order ODE
- 2nd order ODE
- Probability

## 1. Vectors

Introduction: Some things in nature require more than just one number. Temperature, for instance, is a scalar field, however the temperature field on the surface of the Earth requires more information, like the location.
A force on the other hand, has a strength and a direction. Other examples are velocities, accelerations and displacements.
We will describe these types of quantities using 'directed line segments' or arrows.


* Working definition:
Vectors are directed line segments for which we can define a mathematically, physically or geometrically meaningful (and also useful) rule of addition.

We define addition of vectors by the parallelogram or triangle rule.




Experimenting with springs, for instance, shows that forces do behave in this way

Two opposing forces or displacements can cancel each other, therefore we will need to define the concept of a zero vector.

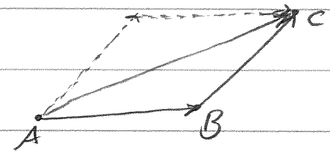
* <u>Note</u>: In Algebra vectors will be studied as 'n-tuples' of numbers $(x_1, x_2, x_3, \dots x_n)$. This is a much more axiomatic approach than the geometrical treatment given here.
Both formulations can be related by writing vectors in a special basis.

We will always work with a special basis.

## *Notation

- Points are denoted by capital Latin letters, $A, B, C, \dots$
- The vector from $A$ to $B$ is denoted by $\vec{AB}$. This vector is equivalent to all vectors obtained by parallel displacement of $\vec{AB}$. All such vectors have the same direction and the same length.
- The vector character is often indicated by an underline, as in $\underset{\sim}{u}$, mainly used on boards and old textbooks. Modern books use $\mathbf{u}$.
- The length or modulus of the vector $\vec{AB}$ is denoted by $|\vec{AB}|$ or $AB$. The length of $\underset{\sim}{u}$ or $\mathbf{u}$ is denoted by $|\underset{\sim}{u}|$ or $u$.
- A vector of length $1$ is called a unit vector and is indicated by a hat. We write $\underset{\sim}{u} = u\hat{\underset{\sim}{u}}$. Notation can be tedious, we try to avoid $|\hat{\underset{\sim}{u}}| = 1$!

## Addition of vectors and multiplication by a scalar

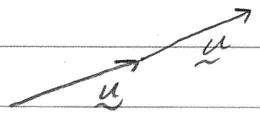



Our definition of vector addition obeys the commutative and associative laws:

$$\underset{\sim}{u} + \underset{\sim}{v} = \underset{\sim}{v} + \underset{\sim}{u}$$

$$(\underset{\sim}{u} + \underset{\sim}{v}) + \underset{\sim}{w} = \underset{\sim}{u} + (\underset{\sim}{v} + \underset{\sim}{w})$$

An important operation on vectors is multiplication by a scalar




$\Rightarrow \quad \underset{\sim}{u} + \underset{\sim}{u} = 2\underset{\sim}{u}$

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We can deduce that $\lambda \underset{\sim}{u}$ is a vector in the direction of $\underset{\sim}{u}$ and $\lambda$- times as long if $\lambda > 0$. For $\lambda < 0$, the vector $\lambda \underset{\sim}{u}$ is in the opposite direction of $\underset{\sim}{u}$ and $|\lambda|$-times its length.

We need to define a zero vector, denoted by $\underset{\sim}{0}$. This vector has zero length and an unspecified direction.

We have the obvious relation

$$\underset{\sim}{0} + \underset{\sim}{u} = \underset{\sim}{u} + \underset{\sim}{0} = \underset{\sim}{u}.$$

The zero vector allows us to define the 'inverse' of a vector.

Let $\underset{\sim}{u}$ and $\underset{\sim}{v}$ be two vectors such that

$$\underset{\sim}{u} + \underset{\sim}{v} = \underset{\sim}{0} = \underset{\sim}{v} + \underset{\sim}{u}$$

then we will write $\underset{\sim}{u} = -\underset{\sim}{v}$.

From our geomebrical approach we have

$$\underset{\sim}{u} = (-1)\,\underset{\sim}{v}.$$

We can call this the negative of the vector.

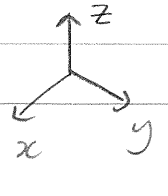
The parallelogram rule also applies to $\lambda(\underset{\sim}{u} + \underset{\sim}{v})$, which means $\lambda(\underset{\sim}{u} + \underset{\sim}{v}) = \lambda \underset{\sim}{u} + \lambda \underset{\sim}{v}$.

Cartesian axes.

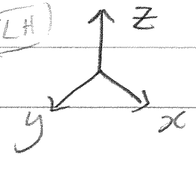
There are two distinct ways of drawing a set of orthogonal (all at right angles) and normal (all of unit length) axes. One speaks of an orthonormal axes.

One can define it in a right-handed and a left-handed way.

RH




LH



We will always use a right-handed system. Along our Cartesian axes we will define unit vectors. They are denoted by $\underset{\sim}{i}, \underset{\sim}{j}, \underset{\sim}{k}$ along the $x, y, z$ axes, respectively.

The translation of the origin $O$ to the point $P$ with coordinates $(x, y, z)$ is given by the vector $x\underset{\sim}{i} + y\underset{\sim}{j} + z\underset{\sim}{k}$.
Sometimes we write

$$x\underset{\sim}{i} + y\underset{\sim}{j} + z\underset{\sim}{k} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

with the identification:

$$\underset{\sim}{i} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad \underset{\sim}{j} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad \underset{\sim}{k} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

The vector $\underset{\sim}{r} = x\underset{\sim}{i} + y\underset{\sim}{j} + z\underset{\sim}{k}$ is generally called the position vector of the point $P$, relative to the origin $O$. By convention we say that the point $A$ has position vector $\underset{\sim}{a}$ and has distance $a$ or $|\underset{\sim}{a}|$ from the origin.

* Note: We could also choose the triple

$$\frac{\underset{\sim}{i} + \underset{\sim}{j}}{\sqrt{2}}, \quad \frac{\underset{\sim}{i} - \underset{\sim}{j}}{\sqrt{2}}, \quad \underset{\sim}{k}.$$

One can check that every vector can be written as a linear combination of this new basis. This basis is not aligned with the Cartesian axes.

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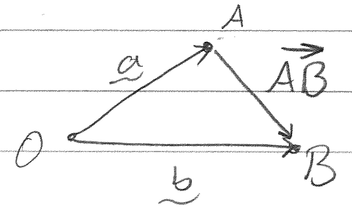
Example 1: Let us try to express the vector joining A and B using the position vectors $\underset{\sim}{a}$ and $\underset{\sim}{b}$, relative to the origin O.




$$\overrightarrow{AB} = \overrightarrow{AO} + \overrightarrow{OB}$$
$$= \overrightarrow{OB} - \overrightarrow{OA}$$
$$= \underset{\sim}{b} - \underset{\sim}{a}$$

So $\underset{\sim}{b} - \underset{\sim}{a}$ is the position vector of B relative to A

Equation of a straight line

We want to write a relation which gives the position vector (relative to some origin) of all points on a given line.

Let us define this simply by the requirement that two points, A and B, are on that line.

We can now define any point on the line by the relation $\underset{\sim}{r} = \overrightarrow{OA} + \lambda \overrightarrow{AB}$ where $\lambda$ is an arbitrary parameter with $\lambda \in \mathbb{R}$. This is the parametric equation of a straight line. As $\lambda$ varies, the position vector $\underset{\sim}{r}$ of a point P moves along the line. We saw $\overrightarrow{AB} = \underset{\sim}{b} - \underset{\sim}{a}$ and hence we can write

$$\underset{\sim}{r} = \overrightarrow{OA} + \lambda \overrightarrow{AB}$$
$$= \underset{\sim}{a} + \lambda(\underset{\sim}{b} - \underset{\sim}{a})$$
$$= (1-\lambda)\underset{\sim}{a} + \lambda \underset{\sim}{b}$$

We could also write

$$\underset{\sim}{r} = \omega \underset{\sim}{a} + \sigma \underset{\sim}{b} \quad , \quad \omega + \sigma = 1$$

Note: that $\underset{\sim}{r} = \underset{\sim}{a} + \lambda \underset{\sim}{b}$ is also a straight line, but not the line through A and B.

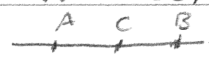
The position vector $\underset{\sim}{r} = \omega \underset{\sim}{a} + \sigma \underset{\sim}{b}$ for arbitrary $\omega$ and $\sigma$ lies anywhere in the plane containing the three points A, B, O. The prescription $\omega + \sigma = 1$ will restrict the position vector $\underline{r}$ to be on the line through A and B.

<u>Exercise 1</u>: Think about the straight line which would be obtained when $\omega + \sigma = n$ where $n$ is a natural number. Look for a pattern.

<u>Example 2</u>: Assume that the three points A, B, C lie on a straight line and consider the relation $\vec{AC} = \mu \vec{CB}$ for some arbitrary (real) $\mu$. What does this mean geometrically?

let $\mu = 1$: $\vec{AC} = \vec{CB}$

$\Rightarrow$ C is the midpoint between A and B.




$\mu = \frac{1}{2}$: $\vec{AC} = \frac{1}{2}\vec{CB} \Rightarrow 2\vec{AC} = \vec{CB}$

$\Rightarrow$ 

C is a third of the way from A to B

$\mu = -2$: $\vec{AC} = -2\vec{CB} = 2\vec{BC}$

$\Rightarrow$ now B is the midpoint between A and C.


<u>Exercise 2</u>: Discuss $\mu = 0$ and $\mu = -1$

It is common common to use the phrase

'C divides AB internally in the ratio $\alpha : \beta$'

to mean $\vec{AC} = \frac{\alpha}{\beta}\vec{CB} \Rightarrow \beta\vec{AC} = \alpha\vec{CB}$. The term 'externally' is used if the ratio is negative.

Let us consider the relation $\vec{AC} = \mu\vec{CB}$ and let us try to solve for the position vector $\underset{\sim}{c}$. We have

$$\vec{AC} = \underset{\sim}{c} - \underset{\sim}{a}$$

$$\vec{CB} = \underset{\sim}{c} - \underset{\sim}{b}$$

$$\vec{AC} = \mu\vec{CB}$$

$$\Leftrightarrow \quad \underset{\sim}{c} - \underset{\sim}{a} = \mu(\underset{\sim}{b} - \underset{\sim}{c})$$

$$\Leftrightarrow \quad \underset{\sim}{c}(1+\underset{\sim}{\mu}) = \underset{\sim}{a} + \mu\underset{\sim}{b}$$

We can divide by $1+\mu$ provided that $\mu \neq -1$, which gives $\underset{\sim}{c} = \frac{1}{1+\mu}\underset{\sim}{a} + \frac{\mu}{1+\mu}\underset{\sim}{b}$.

Using $\mu = \frac{\alpha}{\beta}$ would give

$$\underset{\sim}{c} = \frac{\alpha}{\alpha+\beta}\underset{\sim}{b} + \frac{\beta}{\alpha+\beta}\underset{\sim}{a}$$

which as before we can write

$$\underset{\sim}{c} = \omega\underset{\sim}{a} + \sigma\underset{\sim}{b} \qquad \omega + \sigma = 1.$$

So we are back to the equation of a straight line.

Using Cartesian coordinates and writing vectors as triples we have $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = (1-\lambda)\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} + \lambda\begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$.

This gives us the coordinates $(x, y, z)$ as the parameter varies.

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} - \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} = \lambda \begin{pmatrix} b_1 - a_1 \\ b_2 - a_2 \\ b_3 - a_3 \end{pmatrix}.$$

We can solve these three equations for the parameter $\lambda$.

$$\lambda = \frac{x - a_1}{b_1 - a_1} = \frac{y - a_2}{b_2 - a_2} = \frac{z - a_3}{b_3 - a_3}.$$

Hence there is no need to mention the parameter $\lambda$ which is why it is often not written explicitly. This form of the equation of a straight line is called the implicit form (no explicit $\lambda$).

*A straight like can be written as:

$$\frac{x - a_1}{b_1 - a_1} = \frac{y - a_2}{b_2 - a_2} = \frac{z - a_3}{b_3 - a_3}$$

Example 3: Let a straight line be given by

$$\frac{x-2}{3} = \frac{y-1}{4} = \frac{z-3}{2}$$

This line in parametric form is:

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = (1-\lambda)\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} + \lambda \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$$

$$= \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} + \lambda \begin{pmatrix} b_1 - a_1 \\ b_2 - a_2 \\ b_3 - a_3 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ 4 \\ 2 \end{pmatrix}$$

Note: $(3, 4, 2)$ is not on the line as it is $b-a$.

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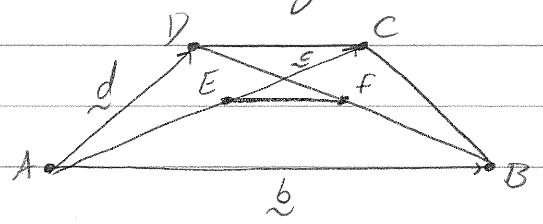
<u>Example 4</u>: Given a trapezium $ABCD$ with $AB \parallel CD$. Show that the line joining the midpoints of the diagonals is parallel to $AB$ and $CD$. Show that its length is the mean of $AB$ and $CD$.




We take $A$ to be the origin.
By assumption $\vec{DC} = \underset{\sim}{c} - \underset{\sim}{d} \parallel \underset{\sim}{b} = \vec{AB}$

From the figure we have:

$$\vec{AE} + \vec{EF} + \vec{FB} + \vec{BA} = \underset{\sim}{0}$$

$$\tfrac{1}{2}\underset{\sim}{c} + \vec{EF} + \tfrac{1}{2}(\underset{\sim}{b} - \underset{\sim}{d}) - \underset{\sim}{b} = \underset{\sim}{0}$$

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$$\Rightarrow \vec{EF} = \underset{\sim}{b} - \tfrac{1}{2}\underset{\sim}{c} + \tfrac{1}{2}(\underset{\sim}{d} - \underset{\sim}{b})$$

$$\Rightarrow \vec{EF} = \tfrac{1}{2}\underset{\sim}{b} - \tfrac{1}{2}\underset{\sim}{c} + \tfrac{1}{2}\underset{\sim}{d}$$

$$\Rightarrow \vec{EF} = \tfrac{1}{2}(\underset{\sim}{b} - \underset{\sim}{c} + \underset{\sim}{d}) = \tfrac{1}{2}\underset{\sim}{b} + \tfrac{1}{2}(\underset{\sim}{d} - \underset{\sim}{c})$$

By assumption, $\underset{\sim}{b}$ is parallel to $(\underset{\sim}{d} - \underset{\sim}{c})$ and thus we have shown the first statement.

Since $\vec{EF}$, $\underset{\sim}{b}$ and $\underset{\sim}{d} - \underset{\sim}{c}$ are parallel let us introduce a unit vector pointing in the direction of $\vec{EF}$. We call this vector $\hat{\underset{\sim}{u}}$.

$$\vec{EF} = \tfrac{1}{2}\underset{\sim}{b} + \tfrac{1}{2}(\underset{\sim}{d} - \underset{\sim}{c})$$

$$\Leftrightarrow |\vec{EF}|\hat{\underset{\sim}{u}} = \tfrac{1}{2}|\vec{AB}|\hat{\underset{\sim}{u}} - \tfrac{1}{2}|\vec{DC}|\hat{\underset{\sim}{u}}$$

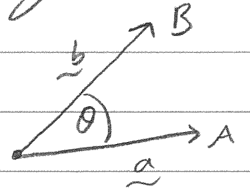
Therefore we find $|\vec{EF}| = \frac{1}{2}(|\vec{AB}| - |\vec{DC}|)$

If the trapezium is changed to a rectangle, the points E and F would coincide which is in agreement with the minus sign.

## The scalar product.

The angle between two vectors $\underset{\sim}{a}$ and $\underset{\sim}{b}$ is defined as the acute angle between the directions of $\underset{\sim}{a}$ and $\underset{\sim}{b}$.

Since vectors obtained by parallel displacement are equivalent, we can always translate both vectors so that they begin at the same point.




We define the scalar product to be the number

$$\underset{\sim}{a} \cdot \underset{\sim}{b} = |a||b| \cos\theta = ab\cos\theta$$

$$\Leftrightarrow \frac{a}{|a|} \cdot \frac{b}{|b|} = \cos\theta$$

$$\Leftrightarrow \hat{\underset{\sim}{a}} \cdot \hat{\underset{\sim}{b}} = \cos\theta$$

This definition implies the following properties:

(i) $\underset{\sim}{a} \cdot \underset{\sim}{b} = \underset{\sim}{b} \cdot \underset{\sim}{a}$

(ii) $(\lambda \underset{\sim}{a}) \cdot b = \underset{\sim}{a} \cdot (\lambda \underset{\sim}{b}) = \lambda (\underset{\sim}{a} \cdot \underset{\sim}{b})$

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Let us now consider the possible scalar products of $\underset{\sim}{i}, \underset{\sim}{j}, \underset{\sim}{k}$

$\underset{\sim}{i} \cdot \underset{\sim}{i} = 1$ $\quad$ $\underset{\sim}{i} \cdot \underset{\sim}{j} = 0$ $\quad$ $\underset{\sim}{i} \cdot \underset{\sim}{k} = 0$

$\underset{\sim}{j} \cdot \underset{\sim}{i} = 0$ $\quad$ $\underset{\sim}{j} \cdot \underset{\sim}{j} = 1$ $\quad$ $\underset{\sim}{j} \cdot \underset{\sim}{k} = 0$

$\underset{\sim}{k} \cdot \underset{\sim}{i} = 0$ $\quad$ $\underset{\sim}{k} \cdot \underset{\sim}{j} = 0$ $\quad$ $\underset{\sim}{k} \cdot \underset{\sim}{k} = 1$

Let us next consider the scalar product of two vectors:

$\underset{\sim}{a} = (a_1, a_2, a_3)$ , $\underset{\sim}{b} = (b_1, b_2, b_3)$

$$(a_1 \underset{\sim}{i} + a_2 \underset{\sim}{j} + a_3 \underset{\sim}{k}) \cdot (b_1 \underset{\sim}{i} + b_2 \underset{\sim}{j} + b_3 \underset{\sim}{k})$$

$$= a_1 b_1 \underset{\sim}{i} \cdot \underset{\sim}{i} + a_1 b_2 \underset{\sim}{i} \cdot \underset{\sim}{j} + \ldots + a_3 b_2 \underset{\sim}{k} \cdot \underset{\sim}{j} + a_3 b_3 \underset{\sim}{k} \cdot \underset{\sim}{k}$$

$$= a_1 b_1 + a_2 b_2 + a_3 b_3$$

This is due to 6 out of the 9 terms vanishing.

The scalar product satisfies a 'product' rule:

$$\underset{\sim}{a} \cdot (\underset{\sim}{b} + \underset{\sim}{c}) = \underset{\sim}{a} \cdot \underset{\sim}{b} + \underset{\sim}{b} \cdot \underset{\sim}{c}$$

Exercise: show this directly using $\underset{\sim}{a} = a_1 \underset{\sim}{i} + a_2 \underset{\sim}{j} + a_3 \underset{\sim}{k}$ ...

## Length of a vector

The scalar product of $\underset{\sim}{a}$ with itself gives

$$\underset{\sim}{a} \cdot \underset{\sim}{a} = a a \cos 0$$
$$= a^2$$

Hence we can write

$$|\underset{\sim}{a}| = \sqrt{\underset{\sim}{a} \cdot \underset{\sim}{a}}$$

We know $\underset{\sim}{a} \cdot \underset{\sim}{a} = a_1^2 + a_2^2 + a_3^2$

$$\Rightarrow a = \sqrt{a_1^2 + a_2^2 + a_3^2}$$

which is Pythagoras' Theorem in three dimentions.

Having defined the length of a vector, we can always define $\hat{\underset{\sim}{a}} = \dfrac{\underset{\sim}{a}}{\sqrt{\underset{\sim}{a} \cdot \underset{\sim}{a}}}$

## The angle between two vectors

We can rearrange the scalar product definition as follows

$$\cos\theta = \frac{\underset{\sim}{a} \cdot \underset{\sim}{b}}{a b} = \frac{\underset{\sim}{a} \cdot \underset{\sim}{b}}{\sqrt{\underset{\sim}{a} \cdot \underset{\sim}{a}}\sqrt{\underset{\sim}{b} \cdot \underset{\sim}{b}}}$$

Example: The angle between the vectors $(1, 1, 1)$ and $(1, 2, 1)$ is

$$\cos\theta = \frac{1.1 + 1.2 + 1.1}{\sqrt{1^2+1^2+1^2}\sqrt{1^2+2^2+1^2}}$$

$$= \frac{4}{\sqrt{3}\sqrt{6}} = \frac{4}{3\sqrt{2}} = \frac{3\sqrt{2}}{2}$$

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## Vector equation of a plane

We have seen previously that the equation of a straight line can be written as
$$\underset{\sim}{r} = \omega \underset{\sim}{a} + \sigma \underset{\sim}{b}$$
with a constraint on $\omega$ and $\sigma$.

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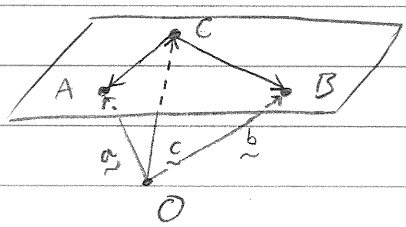
We can 'move' this plane away from the origin $O$ by displacing it by some vector $\underset{\sim}{c}$. Any point on that plane would be given by
$$\underset{\sim}{r} = \underset{\sim}{c} + \omega \underset{\sim}{a} + \sigma \underset{\sim}{b}.$$

Note: This plane does <u>not</u> contain the points $A, B, C$.




The plane containing these points $A, B, C$ can be written as
$$\underset{\sim}{r} = \underset{\sim}{c} + \lambda \vec{CA} + \mu \vec{CB}$$
$$= \underset{\sim}{c} + \lambda(\underset{\sim}{a} - \underset{\sim}{c}) + \mu(\underset{\sim}{b} - \underset{\sim}{c})$$

We can rewrite this as
$$\underset{\sim}{r} = (1 - \lambda - \mu)\underset{\sim}{c} + \lambda \underset{\sim}{a} + \mu \underset{\sim}{b}$$

We could also write this plane as
$$\underset{\sim}{r} = \alpha \underset{\sim}{a} + \beta \underset{\sim}{b} + \gamma \underset{\sim}{c} \quad \text{with} \quad \alpha + \beta + \gamma = 1.$$

For arbitrary $\alpha, \beta, \gamma$, this is any point in 3D space. With one constraint this becomes a plane (2D space).

# Projections and the scalar product

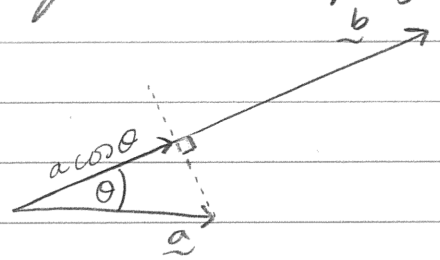
The scalar product is related to the idea of projections. Consider

$$\underset{\sim}{a} = a_1 \underset{\sim}{i} + a_2 \underset{\sim}{j} + a_3 \underset{\sim}{k}$$

$$\underset{\sim}{a} \cdot \underset{\sim}{i} = (a_1 \underset{\sim}{i} + a_2 \underset{\sim}{j} + a_3 \underset{\sim}{k}) \cdot \underset{\sim}{i}$$
$$= a_1$$

Thus we have projected $\underset{\sim}{a}$ along the $x$-axis. The length of the 'shadow' is $a_1$.

However, we can make this more general. Let us find the projection of $\underset{\sim}{a}$ onto $\underset{\sim}{b}$.




The part of $\underset{\sim}{b}$ cut off by the perpendicular is the projection of $\underset{\sim}{a}$ onto $\underset{\sim}{b}$.
Note: this is different to the projection of $\underset{\sim}{b}$ onto $\underset{\sim}{a}$.
This is obvious as the projection of $\underset{\sim}{a}$ onto $\underset{\sim}{b}$ has direction $\underset{\sim}{b}$, while the projection of $\underset{\sim}{b}$ onto $\underset{\sim}{a}$ has the direction of $\underset{\sim}{a}$.

Therefore we have

$$(a\cos\theta)\,\hat{\underset{\sim}{b}} = (\underset{\sim}{a} \cdot \hat{\underset{\sim}{b}})\,\hat{\underset{\sim}{b}}$$

If we subtract the projection from the original vector $\underset{\sim}{a}$, we get the perpendicular which is given by $\underset{\sim}{a} - (\underset{\sim}{a} \cdot \hat{\underset{\sim}{b}})\hat{\underset{\sim}{b}}$.

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Check the angle between the projection and the perpendicular

$$\left[\mathbf{a} - (\mathbf{a}\cdot\hat{\mathbf{b}})\hat{\mathbf{b}}\right]\cdot\left[(\mathbf{a}\cdot\hat{\mathbf{b}})\hat{\mathbf{b}}\right]$$

$$= \mathbf{a}\cdot(\mathbf{a}\cdot\hat{\mathbf{b}})\hat{\mathbf{b}} - \left[(\mathbf{a}\cdot\hat{\mathbf{b}})\hat{\mathbf{b}}\right]\cdot\left[(\mathbf{a}\cdot\hat{\mathbf{b}})\hat{\mathbf{b}}\right]$$

$$= (\mathbf{a}\cdot\hat{\mathbf{b}})\,\mathbf{a}\cdot\hat{\mathbf{b}} - (\mathbf{a}\cdot\hat{\mathbf{b}})^2\,\hat{\mathbf{b}}\cdot\hat{\mathbf{b}}$$

$$= (\mathbf{a}\cdot\hat{\mathbf{b}})^2 - (\mathbf{a}\cdot\hat{\mathbf{b}})^2 = 0$$

Therefore we can decompose any vector $\mathbf{a}$ into two pieces relative to another vector $\mathbf{b}$. One part along $\mathbf{b}$ and one perpendicular to it.

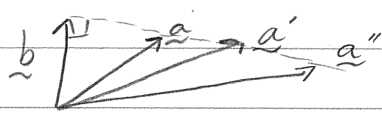
$$\mathbf{a} = \underbrace{(\mathbf{a}\cdot\hat{\mathbf{b}})\hat{\mathbf{b}}}_{/\!/\hat{\mathbf{b}}} + \underbrace{\left[\mathbf{a} - (\mathbf{a}\cdot\hat{\mathbf{b}})\hat{\mathbf{b}}\right]}_{\perp\hat{\mathbf{b}}}$$

__Exercise:__

$\mathbf{a}\cdot(\mathbf{b}+\mathbf{c}) = \mathbf{a}\cdot\mathbf{b} + \mathbf{a}\cdot\mathbf{c}$

Prove this using projections.

__Note__: $\mathbf{a}\cdot\mathbf{b} = \mathbf{a}\cdot\mathbf{c} \not\Rightarrow \mathbf{b} = \mathbf{c}$




Likewise $\mathbf{a}(\mathbf{b}\cdot\mathbf{c}) \neq (\mathbf{a}\cdot\mathbf{b})\mathbf{c}$

## Direction cosines

Let $\hat{\underset{\sim}{a}}$ be a unit vector which we write

$$\hat{\underset{\sim}{a}} = \hat{a}_1 \underset{\sim}{i} + \hat{a}_2 \underset{\sim}{j} + \hat{a}_3 \underset{\sim}{k}$$

We define the direction cosines $l, m, n$ to be the angles $\alpha, \beta, \gamma$ between $\hat{\underset{\sim}{a}}$ and the Cartesian axes. Since $\hat{\underset{\sim}{a}}, \underset{\sim}{i}, \underset{\sim}{j}, \underset{\sim}{k}$ all have unit length we have:

$$l = \hat{\underset{\sim}{a}} \cdot \underset{\sim}{i} = \hat{a}_1 = \cos\alpha$$
$$m = \hat{\underset{\sim}{a}} \cdot \underset{\sim}{j} = \hat{a}_2 = \cos\beta$$
$$n = \hat{\underset{\sim}{a}} \cdot \underset{\sim}{k} = \hat{a}_3 = \cos\gamma$$

## Index notation

The index notation is a very powerful method to handle equations using vectors.
Some identities which take pages to prove can often be proved in one or two lines.

Firstly we will write

$$\underset{\sim}{e}_1 = \underset{\sim}{i}, \quad \underset{\sim}{e}_2 = \underset{\sim}{j}, \quad \underset{\sim}{e}_3 = \underset{\sim}{k}$$

Then we can write

$$\underset{\sim}{a} = a_1 \underset{\sim}{i} + a_2 \underset{\sim}{j} + a_3 \underset{\sim}{k}$$
$$= a_1 \underset{\sim}{e}_1 + a_2 \underset{\sim}{e}_2 + a_3 \underset{\sim}{e}_3$$

$$\Rightarrow \underset{\sim}{a} = \sum_{i=1}^{3} a_i e_i$$

Whenever we have equations involving vectors, there may be many summations involved.
In order to simplify this notation we will will follow the Einstein summation convention.
This states:

We sum over twice repeated indices and drop the summation symbol. We will write $\underset{\sim}{a} = a_i \underset{\sim}{e}_i$.

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The scalar product of the $\underset{\sim}{i}, \underset{\sim}{j}, \underset{\sim}{k}$ looks like an identity matrix.
We define $\underset{\sim}{e_i} \cdot \underset{\sim}{e_j} = \delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$

$\delta$ is called Kronecker delta.

Example the scalar product of two vectors $\underset{\sim}{a}$ and $\underset{\sim}{b}$.
We have $\underset{\sim}{a} = a_i \underset{\sim}{e_i}$ and $\underset{\sim}{b} = b_j \underset{\sim}{e_j}$

$$\underset{\sim}{a} \cdot \underset{\sim}{b} = a_i \underset{\sim}{e_i} \cdot b_j \underset{\sim}{e_j}$$

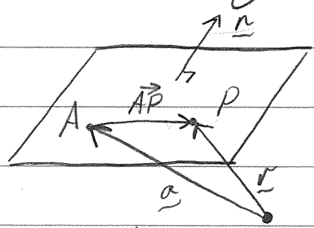
$$\underset{\sim}{a} \cdot \underset{\sim}{b} = a_i b_j (e_i \cdot e_j)$$

$$\underset{\sim}{a} \cdot \underset{\sim}{b} = a_i b_j \delta_{ij} \overset{\text{recall}}{=} \sum_{i=1}^{3} \sum_{j=1}^{3} a_i b_j \delta_{ij}$$

$$\underset{\sim}{a} \cdot \underset{\sim}{b} = a_i b_i \overset{\text{recall}}{=} \sum_{i=1}^{3} a_i b_i = a_1 b_1 + a_2 b_2 + a_3 b_3$$

## Vector equation of the plane II

Let $A$ be a given point on the plane, and let $\underset{\sim}{n}$ be a normal vector to the plane. We can always find $\hat{\underset{\sim}{n}}$.




If $\underset{\sim}{r}$ is the position vector of a point $P$ in the plane, then the vector $\overrightarrow{AP}$ lies in the plane. Therefore it must be normal to $\hat{\underset{\sim}{n}}$. We can define a plane by saying that $\underset{\sim}{r}$ is the position vector of a point $P$ in the plane if

$$\hat{\underset{\sim}{n}} \cdot (\underset{\sim}{r} - \underset{\sim}{a}) = 0$$

This is equivalent to $\hat{\underset{\sim}{n}} \cdot \underset{\sim}{r} = \hat{\underset{\sim}{n}} \cdot \underset{\sim}{a} = c = \text{const.}$ The number $c$ has a neat geometrical interpretation. $\hat{\underset{\sim}{n}} \cdot \underset{\sim}{r}$ is the length of the projection of $\underset{\sim}{r}$ onto $\hat{\underset{\sim}{n}}$. Therefore, $c$ is the minimum distance of the plane from the origin.

Example:

What is the minimum distance from $3x + 2y + z = 1$ from the origin.

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \cdot \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix} = 1$$

$$|\underset{\sim}{n}| = \sqrt{3^2 + 2^2 + 1^2} = \sqrt{14}$$

We divide by $\sqrt{14}$

$$\underbrace{\frac{3}{\sqrt{14}} x + \frac{2}{\sqrt{14}} y + \frac{1}{\sqrt{14}} z}_{\hat{\underset{\sim}{n}} \cdot \underset{\sim}{r}} = \frac{1}{\sqrt{14}}$$

$$\Rightarrow c = \frac{1}{\sqrt{14}}$$

The vector or cross product.

The vector product $\underset{\sim}{a} \times \underset{\sim}{b}$ (sometimes $\underset{\sim}{a} \wedge \underset{\sim}{b}$) is defined to be the vector

$$ab \sin\theta \, \hat{\underset{\sim}{c}}$$

where $\theta$ is the angle between $\underset{\sim}{a}$ and $\underset{\sim}{b}$ and $\hat{\underset{\sim}{c}}$ is a unit vector, normal to both $\underset{\sim}{a}$ and $\underset{\sim}{b}$. The triple $\underset{\sim}{a}, \underset{\sim}{b}, (\underset{\sim}{a} \times \underset{\sim}{b})$ should form a right handed set of vectors.

Some properties are:

(i) $\underset{\sim}{a} \times \underset{\sim}{b}$ is a vector

(ii) $(\lambda \underset{\sim}{a}) \times \underset{\sim}{b} = \underset{\sim}{a} \times (\lambda \underset{\sim}{b}) = \lambda (\underset{\sim}{a} \times \underset{\sim}{b})$

(iii) $\underset{\sim}{a} \times \underset{\sim}{a} = 0$

(iv) $\underset{\sim}{a} \times \underset{\sim}{b} = -\underset{\sim}{b} \times \underset{\sim}{a}$

This final property follows from the fact that we

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require a right-handed set

Let us consider all possible vector products of our basis vectors $\underset{\sim}{i}, \underset{\sim}{j}, \underset{\sim}{k}$.

$\underset{\sim}{i} \times \underset{\sim}{i} = 0 \qquad \underset{\sim}{i} \times \underset{\sim}{j} = \underset{\sim}{k} \qquad \underset{\sim}{i} \times \underset{\sim}{k} = -\underset{\sim}{j}$

$\underset{\sim}{j} \times \underset{\sim}{i} = -\underset{\sim}{k} \qquad \underset{\sim}{j} \times \underset{\sim}{j} = 0 \qquad \underset{\sim}{j} \times \underset{\sim}{k} = \underset{\sim}{i}$

$\underset{\sim}{k} \times \underset{\sim}{i} = \underset{\sim}{j} \qquad \underset{\sim}{k} \times \underset{\sim}{j} = -\underset{\sim}{i} \qquad \underset{\sim}{k} \times \underset{\sim}{k} = 0$

We note that this array of equations is skew-symmetric
Using index notation we can write

$$\underset{\sim}{e}_i \times \underset{\sim}{e}_j = \varepsilon_{ijk} \underset{\sim}{e}_k$$

where $\varepsilon_{ijk}$ is the Levi-Civita symbol (or permutation symbol, antisymmetric symbol or alternating symbol).

It is defined by

$$\varepsilon_{ijk} = \begin{cases} 0 & \text{if any two indices are equal} \\ +1 & \text{if } ijk \text{ is an even permutation of } (ijk) \\ -1 & \text{if } ijk \text{ is an odd permutation of } (ijk) \end{cases}$$

This means $\varepsilon_{123} = \varepsilon_{312} = \varepsilon_{231} = 1$ ← even permutation

swap two indices and change the sign. ↙↗ $\varepsilon_{213} = \varepsilon_{321} = \varepsilon_{132} = -1$ ← odd permutation

and zero otherwise.

The Levi-Civita symbol satisfies the following identities

$$\varepsilon_{ijk}\,\varepsilon_{imn} = \delta_{jm}\delta_{kn} - \delta_{jn}\delta_{km}$$

$$\varepsilon_{imn}\,\varepsilon_{jmn} = 2\delta_{ij}$$

$$\varepsilon_{ijk}\,\varepsilon_{ijk} = 6$$

<u>Exercise</u>: Prove these identities.

Let us now consider the vector (cross) product of two vectors $\underset{\sim}{a}$ and $\underset{\sim}{b}$.

$$\begin{aligned}\underset{\sim}{a} \times \underset{\sim}{b} &= a_i \underset{\sim}{e}_i \times b_j \underset{\sim}{e}_j \\ &= (a_i b_j)(\underset{\sim}{e}_i \times \underset{\sim}{e}_j) \\ &= a_i b_j\, \varepsilon_{ijk} \underset{\sim}{e}_k\end{aligned}$$

$$(\underset{\sim}{a} \times \underset{\sim}{b})_k = a_i b_j \varepsilon_{ijk}$$

$$\begin{aligned}(\underset{\sim}{a} \times \underset{\sim}{b})_1 &= a_i b_j \varepsilon_{ij1} \\ &= \varepsilon_{321} a_3 b_2 + \varepsilon_{231} a_2 b_3 \\ &= a_2 b_3 - a_3 b_2\end{aligned}$$

So $(\underset{\sim}{a} \times \underset{\sim}{b}) = \begin{pmatrix} a_2 b_3 - a_3 b_2 \\ a_3 b_1 - a_1 b_3 \\ a_1 b_2 - a_2 b_1 \end{pmatrix}$

Sometimes $\underset{\sim}{a} \times \underset{\sim}{b}$ is expressed using a determinant

$$\begin{vmatrix} i & j & k \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = i \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} - j \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} + k \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}$$

$$= \begin{pmatrix} a_2 b_3 - a_3 b_2 \\ a_3 b_1 - a_1 b_3 \\ a_1 b_2 - a_2 b_1 \end{pmatrix}$$

Example:

Prove $\underset{\sim}{a} \times (\underset{\sim}{b} + \underset{\sim}{c}) = \underset{\sim}{a} \times \underset{\sim}{b} + \underset{\sim}{a} \times \underset{\sim}{c}$ using the index notation.

$$\underset{\sim}{a} \times (\underset{\sim}{b} + \underset{\sim}{c})$$

$$= a_i \underset{\sim}{e}_i \times (b_j \underset{\sim}{e}_j + c_k \underset{\sim}{e}_k)$$

$$= a_i \underset{\sim}{e}_i \times (b_j + c_j) \underset{\sim}{e}_j \qquad \text{(renaming } k \text{ as } j\text{)}$$

$$= a_i (b_j + c_j) \varepsilon_{ijk} \underset{\sim}{e}_k$$

$$= a_i b_j \varepsilon_{ijk} \underset{\sim}{e}_k + a_i c_j \varepsilon_{ijk} \underset{\sim}{e}_k$$

$$= \underset{\sim}{a} \times \underset{\sim}{b} + \underset{\sim}{a} \times \underset{\sim}{c}$$

Exercise:

Re-do the proof without index notation.

Since the vector product gives vectors, we can consider

$$(\mathbf{a}\times\mathbf{b})\times\mathbf{c}$$

$\mathbf{a} = \mathbf{i}$ and $\mathbf{b} = \mathbf{c} = \mathbf{j}$

$$(\mathbf{a}\times\mathbf{b})\times\mathbf{c} = (\mathbf{i}\times\mathbf{j})\times\mathbf{j}$$

$$= \mathbf{k}\times\mathbf{j} = -\mathbf{i}$$

$$\mathbf{a}\times(\mathbf{b}\times\mathbf{c}) = \mathbf{i}\times(\mathbf{j}\times\mathbf{j})$$

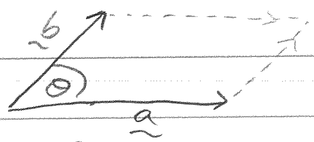
$$= \mathbf{0}$$

When considering the triple vector product, the positions of the bracket are important.

<u>Three applications</u>

Area of a parallelogram.

Consider a parallelogram spanned by two vectors $\mathbf{a}$ and $\mathbf{b}$.




$$\text{Area}_{\square} = |OA||OB|\sin\theta$$

$$= ab\sin\theta = |\mathbf{a}\times\mathbf{b}|$$

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## Distance of a point from a plane

Let a plane be defined by three points A, B, C and let P be a point.
We find the shortest distance between P and the plane as follows:
Find $\vec{AP}$ and project along the normal $\hat{\underset{\sim}{n}}$ which gives

$$d = \vec{AP} \cdot \hat{\underset{\sim}{n}}$$

We can use the vector product to compute $\hat{\underset{\sim}{n}}$. The vectors $\vec{AB}$ and $\vec{AC}$ are in the plane, therefore $\underset{\sim}{n} = \vec{AB} \times \vec{AC}$

$$\Rightarrow \hat{\underset{\sim}{n}} = \frac{\vec{AB} \times \vec{AC}}{|\vec{AB} \times \vec{AC}|}$$

so we have

Learn proof. $*$

$$d = \vec{AP} \cdot \frac{\vec{AB} \times \vec{AC}}{|\vec{AB} \times \vec{AC}|}$$

## Distance between two skew lines

We are considering two non-intersecting (this means skew) lines given by

$$\underset{\sim}{r}_1 = \underset{\sim}{a}_1 + \lambda \underset{\sim}{b}_1 \quad , \quad \underset{\sim}{r}_2 = \underset{\sim}{a}_2 + \mu \underset{\sim}{b}_2$$

The minimum distance of approach is along a direction perpendicular to both lines. These directions are $b_1$ and $b_2$, so

$$\hat{\underset{\sim}{n}} = \frac{\underset{\sim}{b}_1 \times \underset{\sim}{b}_2}{|\underset{\sim}{b}_1 \times \underset{\sim}{b}_2|}$$

Let S and Q be any two points on the two lines so that

$$\underset{\sim}{s} = \underset{\sim}{a}_1 + \lambda^* \underset{\sim}{b}_1$$
$$\underset{\sim}{q} = \underset{\sim}{a}_2 + \mu^* \underset{\sim}{b}_2$$

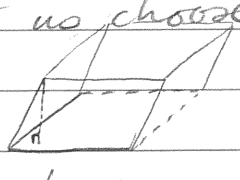
We now project the vector connecting the two lines along the normal.

$$d = \vec{SQ} \cdot \hat{n}$$
$$= (q - s) \cdot \frac{b_1 \times b_2}{|b_1 \times b_2|}$$

$$= (a_2 + \mu^* b_2 - a_1 - \lambda^* b_1) \cdot \frac{b_1 \times b_2}{|b_1 \times b_2|}$$

Using that $b_1 \cdot (b_1 \times b_2) = 0$
and $b_2 \cdot (b_1 \times b_2) = 0$
we have
$$d = (a_2 - a_1) \cdot \frac{b_1 \times b_2}{|b_1 \times b_2|}$$

Triple products.
We have already seen terms like
$(a \times b) \cdot c$ We will call this the scalar triple product
and define $[a, b, c] := (a \times b) \cdot c$
This number represents the volume of the parallelepiped formed by $a, b, c$.
To see this:
let us choose $a$ and $b$ to form the base



The base area is $|a \times b|$ (area of parralelogram).
The normal to the base is $a \times b$ so
$$\hat{n} = \frac{a \times b}{|a \times b|}$$
So we have that
$$\text{height} = c \cdot \frac{a \times b}{|a \times b|}$$

Therefore, we can write the volume as follows:

$$V = \text{base (height)}$$
$$= \cancel{|a\times b|}\left(c \cdot \frac{a\times b}{\cancel{|a\times b|}}\right)$$
$$= (a\times b)\cdot c$$

We can conclude that

$$[a, b, c] = [c, a, b] = [b, c, a]$$

These are the cyclic permutations.

And also

$$[b, a, c] = -[a, b, c]$$
$$[a, c, b] = -[c, a, b]$$
$$[c, b, a] = -[b, c, a]$$

In the index notation this is very clear

$$[a, b, c] = (a\times b)\cdot c$$
$$= a_i b_j \varepsilon_{ijk} e_k \cdot c_m e_m$$
$$= a_i b_j \varepsilon_{ijk} c_m \underbrace{e_k \cdot e_m}_{\delta_{km}}$$
$$= a_i b_j \varepsilon_{ijk} c_k$$
$$= a_i b_j c_k \varepsilon_{ijk} = \sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3} a_i b_j c_k \varepsilon_{ijk} \;?$$

The symmetry properties of [0, 0, 0] are those of $\varepsilon_{ijk}$.

We also have that the positions of the symbols $\cdot$ and $\times$ in this product do not matter.

The orientation of $a, b, c$ matters.

We also encountered the object

$$(\underset{\sim}{a} \times \underset{\sim}{b}) \times \underset{\sim}{c}.$$

We already know that $(a \times b) \times c \neq a \times (b \times c)$

We have the identity

$$\underset{\sim}{a} \times (\underset{\sim}{b} \times \underset{\sim}{c}) = (\underset{\sim}{a} \cdot \underset{\sim}{c})\underset{\sim}{b} - (\underset{\sim}{a} \cdot \underset{\sim}{b})\underset{\sim}{c}$$

Exercise:

Prove this starting with

$$\underset{\sim}{a} = a_1 \underset{\sim}{i} + a_2 \underset{\sim}{j} + a_3 \underset{\sim}{k}$$

We will prove this using the index notation.

let $\underset{\sim}{d} = \underset{\sim}{b} \times \underset{\sim}{c} = b_i c_j \varepsilon_{ijk} \underset{\sim}{e}_k = d_n \underset{\sim}{e}_n$

$$\underset{\sim}{a} \times \underset{\sim}{d} = a_m d_n \varepsilon_{mns} \underset{\sim}{e}_s$$

so, the LHS is

$$\underset{\sim}{a} \times (\underset{\sim}{b} \times \underset{\sim}{c}) = a_m \overbrace{(b_i c_j \varepsilon_{ijn})}^{\text{just components, not full vector}} \varepsilon_{mns} \underset{\sim}{e}_s$$

$$= a_m b_i c_j \varepsilon_{nij} (-\varepsilon_{nms}) \underset{\sim}{e}_s$$

$$= -a_m b_i c_j (\delta_{im}\delta_{js} - \delta_{is}\delta_{jm}) \underset{\sim}{e}_s$$

$$= a_m b_i c_j \delta_{is}\delta_{jm} \underset{\sim}{e}_s - a_m b_i c_j \delta_{im}\delta_{js} \underset{\sim}{e}_s$$

$$= a_m c_m b_i \underset{\sim}{e}_i - a_m b_m c_j \underset{\sim}{e}_j$$

$$= (\underset{\sim}{a} \cdot \underset{\sim}{c})\underset{\sim}{b} - (\underset{\sim}{a} \cdot \underset{\sim}{b})\underset{\sim}{c}$$

The vector triple product also satisfies the Jacobi identity

$$\underset{\sim}{a} \times (\underset{\sim}{b} \times \underset{\sim}{c}) + \underset{\sim}{c} \times (\underset{\sim}{a} \times \underset{\sim}{b}) + \underset{\sim}{b} \times (\underset{\sim}{c} \times \underset{\sim}{a}) = \underset{\sim}{0}$$

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# Complex numbers.

## Introduction

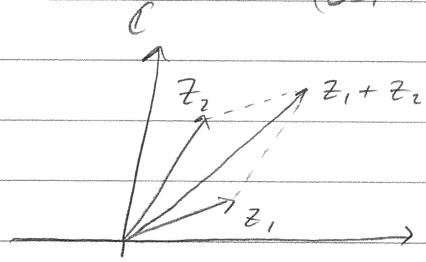
Complex numbers are written as

$$z = x + iy \qquad \text{where } i = \sqrt{-1} \text{ or } i^2 = -1, \ x, y \in \mathbb{R}$$

$x$ is called the real part of $z$ and $y$ is the imaginary part, $x = Re(z)$, $y = Im(z)$.

We can view complex numbers as vectors in the plane. They obey the usual rules of vector addition. We have

$$z_1 + z_1 = (x_1 + iy_1) + (x_2 + iy_2)$$
$$= (x_1 + x_2) + i(y_1 + y_2)$$




We can naturally multiply complex numbers

$$z_1 = x_1 + iy_1$$
$$z_2 = x_2 + iy_2$$

$$z_1 z_2 = (x_1 + iy_1)(x_2 + iy_2)$$
$$= x_1 x_2 + i(x_1 y_2 + x_2 y_1) + i^2 y_1 y_2$$
$$= x_1 x_2 - y_1 y_2 + i(x_1 y_2 + y_1 x_2)$$

We have an additional operation called complex conjugation

$$\bar{z} = x - iy \qquad (\bar{z} = z^*)$$

Geometrically $z \to \bar{z}$ corresponds to reflection along the $x$-axis.

We also note

$$|z_1 + z_2| \leq |z_1| + |z_2|$$

where $|z| = \sqrt{x^2 + y^2}$

and we note that

$$z\bar{z} = (x+iy)(x-iy)$$
$$= (x^2 - iy^2) = x^2 + y^2$$
$$\Rightarrow z\bar{z} = |z|^2$$

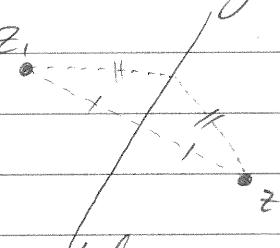
## Geometry in the complex plane

**Circles**: The set of points $z$ satisfying $|z - z_0| = r$ with $r > 0$ are on a circle with centre $z_0$ and radius $r$.

To see this:

$$z = x + iy$$
$$z_0 = x_0 + iy_0$$
$$|x + iy - x_0 - iy_0|^2 = r^2$$
$$|(x - x_0) + i(y - y_0)|^2 = r^2$$
$$\Leftrightarrow (x - x_0)^2 + (y - y_0)^2 = r^2$$

**Lines**: The relation $|z + 3i| = |z + (5 - 2i)|$ defines a straight line.



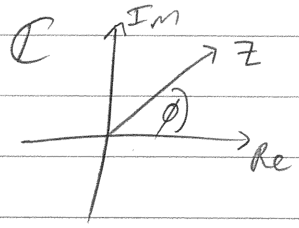

One should read this as 'the distance from $-3i$ equals the distance from $-5+2i$.'

Start with $z = x + iy$

$$|x + iy + 3i| = |x + iy + 5 - 2i|$$
$$|x + i(y+3)| = |(x+5) + i(y-2)|$$
$$x^2 + (y+3)^2 = (x+5)^2 + (y-2)^2$$
$$\cancel{x^2} + \cancel{y^2} + 6y + 9 = \cancel{x^2} + 10x + 25 + \cancel{y^2} - 4y + 4$$
$$10y = 10x + 20$$
$$y = x + 2$$

## Polar Form

Let $z = x + iy$ be a complex number
We define $r = |z| = \sqrt{x^2+y^2}$ and denote the angle between the origin and $z$, and the $x$-axis by $\phi$. We call $\phi$ the argument.




This allows us to write $z = r\cos\phi + ir\sin\phi$
$= r(\cos\phi + i\sin\phi)$

## Euler's formula:

let us consider the series expansion of $e^{i\phi}$ assuming it converges.

$$e^{i\phi} = 1 + i\phi + \frac{(i\phi)^2}{2!} + \frac{(i\phi)^3}{3!} + \frac{(i\phi)^4}{4!} + \ldots$$

$$= 1 - \frac{\phi^2}{2!} + \frac{\phi^4}{4!} + \ldots$$

$$+ i\left[\phi - \frac{\phi^3}{3!} + \frac{\phi^5}{5!} + \ldots\right]$$

$$= \cos\phi + i\sin\phi$$ due to the Taylor series.

Exercise:
Prove Euler's formula by computing
$$\frac{d}{d\phi}\left(e^{-i\phi}(\cos\phi + i\sin\phi)\right)$$

Therefore we can write every complex number as follows:

$$z = re^{i\phi} \quad \text{where} \quad r = |z|$$
$$\phi = \arg(z)$$

If $r=1$ then $e^{i\phi}$ is a complex number on the unit circle.

## De Moivre's Theorem

If $z = re^{i\phi}$ then
$$z^n = r^n (e^{i\phi})^n = r^n (e^{in\phi}) \qquad n \in \mathbb{R}$$

So, using the Euler formula we have
$$(\cos\phi + i\sin\phi)^n = (\cos(n\phi) + i\sin(n\phi))$$

De Moivre's theorem has two main applications:
(i) finding powers of complex numbers.
(ii) proving / finding trigonometric identities.

## Trigonometric identities.

Let us express $\cos(3\phi)$ and $\sin(3\phi)$ in terms of $\cos\phi$ and $\sin\phi$.
Use De Moivre's theorem for $n=3$
$$\cos(3\phi) + i\sin(3\phi) = (\cos\phi + i\sin\phi)^3$$
$$= \cos^3\phi + 3\cos^2\phi\,(i\sin\phi) + 3\cos\phi\,(i\sin\phi)^2 + (i\sin\phi)^3$$
$$= \cos^3\phi - 3\cos\phi\sin^2\phi + 3i\cos^2\phi\sin\phi - i\sin^3\phi$$

$$\Rightarrow \begin{cases} \cos(3\phi) = \cos^3\phi - 3\cos\phi\sin^2\phi \\ \sin(3\phi) = 3\cos^2\phi\sin\phi - \sin^3\phi \end{cases}$$

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Alternatively, let us try to express $\sin^2\phi$ in terms of multiple cosines.

We start with

$$z = e^{i\phi} = \cos\phi + i\sin\phi$$

$$\bar{z} = e^{-i\phi} = \cos\phi - i\sin\phi$$

$$= \frac{1}{z}$$

Therefore

$$\left(z + \frac{1}{z}\right) = 2\cos\phi$$

$$\left(z - \frac{1}{z}\right) = 2i\sin\phi$$

We also have

$$\left(z^n + \frac{1}{z^n}\right) = 2\cos n\phi$$

$$\left(z^n - \frac{1}{z^n}\right) = 2i\sin n\phi$$

Let us start with

$$(2i\sin\phi)^2 = \left(z - \frac{1}{z}\right)^2$$

$$= z^2 - 2z\frac{1}{z} + \frac{1}{z^2}$$

$$= \left(z^2 + \frac{1}{z^2}\right) - 2$$

$$= 2\cos(2\phi) - 2$$

$$\Rightarrow -4\sin^2\phi = 2\cos(2\phi) - 2$$

$$\Rightarrow \sin^2\phi = \frac{1}{2}(1 - \cos(2\phi))$$

$$\sin(2\phi) = 2\sin\phi\cos\phi$$

$$\frac{d}{d\phi} : \cancel{2}\cos(2\phi) = \cancel{2}\left[\cos^2\phi - \sin^2\phi\right]$$

## Roots and roots of unity

Taking roots of complex numbers is best explained using an example.

Example:

Find $z$ such that $z^6 = 64i$.

We begin with $i = e^{i\frac{\pi}{2}}$

so we can write

$$64i = 64e^{i\frac{\pi}{2}}$$

We can also write $64i = 64e^{i\frac{\pi}{2} + i2\pi m}$.

where $m$ is an integer.

The main point here is to write the complex number in its most general form.

Now we can compute.

$$z = (64)^{\frac{1}{6}} e^{\left(\frac{i\frac{\pi}{2} + i2\pi m}{6}\right)} = re^{i\phi}$$

Comparing arguments and the modulus gives.

$$r = (64)^{1/6} = 2$$

$$i\left(\frac{\pi}{12} + \frac{\pi}{3}m\right) = \phi \qquad m \in \mathbb{Z}$$

Since $m$ is an integer, we must determine the number of different solutions.

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We find:

$m = 0$ $\qquad \phi = \frac{\pi}{12}$

$m = 1$ $\qquad \phi = \frac{5\pi}{12}$

$m = 2$ $\qquad \phi = \frac{3\pi}{4}$

$m = 3$ $\qquad \phi = \frac{13\pi}{12}$

$m = 4$ $\qquad \phi = \frac{17\pi}{12}$

$m = 5$ $\qquad \phi = \frac{21\pi}{12}$

$m = 6$ $\qquad \phi = \frac{\pi}{12} + 2\pi = \frac{\pi}{12}$ as this is an angle

We find six different solutions and for $m \geq 6$ solutions we find will only differ by factors of $2\pi$ from previous solutions.

The fundamental theorem of algebra states that a polynomial of order $n$ with complex coefficients has $n$ complex roots.

Example:

What are the three roots of $z^3 = 1$.

We know that $z = 1$ is one solution to this equation.

$$
\begin{array}{l}
(z^3 - 1) \div (z - 1) = z^2 + z + 1 \\
\underline{-(z^3 - z^2)} \\
\quad z^2 - 1 \\
\quad \underline{-(z^2 - z)} \\
\qquad z - 1 \\
\qquad z - 1
\end{array}
$$

Exercise

$(x^5 + 4x^4 + x^3 + x^2 - x^1 + 2) \div (x^3 - 2x^2 + x - 1)$

$z^3 - 1 = (z^2 + z + 1)(z - 1)$

We can now solve the quadratic:

$z_1 = 1$

$z_{2,3} = -\frac{1}{2} \pm \sqrt{\frac{1}{4} - 1}$

Now we consider the equation $z^n = 1$ $\quad n \in \mathbb{Z}, n > 0$

We write $z = re^{i\theta} \Rightarrow r = 1.$ $\quad [z^n = r^n e^{in\theta}]$

In its most general form we have $1 = 1e^{i2\pi m}$ $\quad m \in \mathbb{Z}$

Then we have

$z^n = r^n e^{i2\pi m} = 1$

so that $\phi = 2\pi \frac{m}{n}.$

If we denote $\omega = e^{\frac{2\pi i}{n}}$

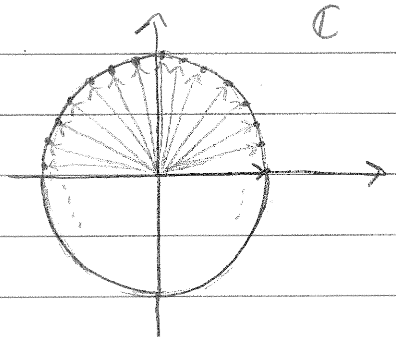
then the roots of the equation are

$1 = \omega^0, \omega^1, \omega^2, \ldots, \omega^{n-1}$

These correspond to $m = 0, 1, \ldots, n-1.$

Exercise

Show $1 + \omega^1 + \omega^2 + \ldots + \omega^{n-1} = 0.$



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## Trigonometric and hyperbolic functions.

We start with Euler's formular

$$e^{i\phi} = \cos\phi + i\sin\phi$$

and have already written that

$$\cos\phi = \frac{1}{2}(e^{i\phi} + e^{-i\phi})$$

$$\sin\phi = \frac{1}{2i}(e^{i\phi} - e^{-i\phi}).$$

This would suggest that we can define these functions with a complex argument as follows

$$\cos(z) = \frac{1}{2}(e^{iz} + e^{-iz})$$

$$\sin(z) = \frac{1}{2i}(e^{iz} - e^{-iz}).$$

Let us consider a purely complex number $z = iy$.

$$\cos z = \cos(iy) = \frac{1}{2}(e^{i(iy)} + e^{-i(iy)})$$

$$= \frac{1}{2}(e^{y} + e^{-y})$$

which is the definition of $\cosh(y)$.

So we have

* $\cos(iy) = \cosh(y)$

Likewise

$$\sin(iy) = \frac{1}{2i}(e^{i(iy)} - e^{-i(iy)})$$

$$= \frac{1}{2i}(e^{-y} - e^{y})$$

$$= -\frac{1}{2i}(e^{y} - e^{-y}) \qquad \left[-\frac{1}{i} \times \frac{i}{i} = \frac{-i}{i^2} = i\right]$$

$$= i\frac{1}{2}(e^{y} - e^{-y}) = i\sinh(y)$$

* so $\sin(iy) = i\sinh(y)$

Example:

Consider $\cos^2 x + \sin^2 x = 1$

and assume this hold for any $x \in \mathbb{C}$.

Set $x = i\phi$ with $\phi \in \mathbb{R}$, then

$$\cos^2(i\phi) + \sin^2(i\phi) = 1$$

$$\Leftrightarrow \cosh^2\phi + i^2 \sinh^2\phi = 1$$

$$\Leftrightarrow \cosh^2\phi - \sinh^2\phi = 1$$

## Maclaurin and Taylor series

Can we approximate a function $f(x)$ near a point $x_0$ using a polynomial?

Let $P(x)$ be a polynomial

$$P(x) = a_0 + a_1(x - x_0) + a_2(x - x_2)^2 + a_3(x - x_0)^3 + \dots$$

We want to choose the numbers $a_0, a_1, a_2, \dots$ such that $P(x_0)$ is $f(x_0)$, and such that all derivatives of $P$ at $x_0$ are those of $f$ at $x_0$

If we put $x = x_0$ then $f(x_0) = P(x_0) = a_0$

Let us differentiate once w.r.t. $x$:

$$\frac{dP}{dx} = a_1 + 2a_2(x - x_0) + 3a_3(x - x_0)^2 + \dots$$

so that $\frac{dP}{dx}(x_0) = a_1 = f'(x_0)$

$$\frac{d^2P}{dx^2}(x_0) = 2a_2 = f''(x_0)$$

$$\frac{d^3P}{dx^3}(x_0) = 3 \times 2\, a_3 = f'''(x_0)$$

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In general we would find

$$f^{(n)}(x_0) = n!\, a_n$$

Having found all $a_i$, we have

$$f(x) = f(x_0) + f'(x_0)(x-x_0) + \frac{f''(x_0)}{2!}(x-x_0)^2$$
$$+ \frac{f'''(x_0)}{3!}(x-x_0)^2 + \dots + \frac{f^{(k)}(x_0)}{k!}(x-x_0)^k + \dots$$

$$\Leftrightarrow f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n$$

This is a power series in $(x-x_0)$ or about the point $x_0$. This is called the Taylor Series. It is called the Maclaurin Series if $x_0 = 0$.

By choosing $x = x_0 + h$
we can also write $f(x_0+h) = \sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!} h^n$

We can show that

$$\sin x = \sum_{m=0}^{\infty} \frac{(-1)^m}{(2m+1)!} x^{2m+1}$$

$$\cos x = \sum_{m=0}^{\infty} \frac{(-1)^m}{(2m)!} x^{2m}$$

$f(x) = \cos x$ $\quad f(0) = 1$
$f'(x) = -\sin x$ $\quad f'(0) = 0$
$f''(x) = -\cos x$ $\quad f''(0) = -1$
$f'''(x) = \sin x$ $\quad f'''(0) = 0$
$f''''(x) = \cos x$ $\quad f''''(0) = 1$

$$\Rightarrow f(x) = 1 + (-1)\frac{x^2}{2!} + (+1)\frac{x^4}{4!} + (-1)\frac{x^6}{6!} + (+1)\frac{x^8}{8!} + \dots$$

$$= 1 + (-1)^1\frac{x^2}{2!} + (-1)^2\frac{x^4}{4!} + (-1)^3\frac{x^6}{6!} + \dots$$

$$= 1 + (-1)^1\frac{x^{2\cdot1}}{(2\cdot1)!} + (-1)^2\frac{x^{2\cdot2}}{(2\cdot2)!} + (-1)^3\frac{x^{2\cdot3}}{(2\cdot3)!} + \dots$$

$$= \sum_{m=0}^{\infty} (-1)^m \frac{x^{2m}}{(2m)!}$$

Example:

Show that $\frac{d}{dx}\cos x = -\sin x$ using the series

$$\frac{d}{dx}\cos x = \frac{d}{dx}\sum_{m=0}^{\infty}(-1)^m\frac{x^{2m}}{(2m)!}$$

$$= \sum_{m=0}^{\infty}(-1)^m\frac{d}{dx}\frac{x^{2m}}{(2m)!}$$

$$= \frac{d1}{dx} + \sum_{m=1}^{\infty}(-1)^m\frac{d}{dx}\frac{x^{2m}}{(2m)!}$$

$$= \sum_{m=1}^{\infty}(-1)^m\frac{2m\,x^{2m-1}}{(2m)!} = \sum_{m=1}^{\infty}(-1)^m\frac{x^{2m-1}}{(2m-1)!}$$

We set $2m-1 = 2\tilde{m}+1 \Rightarrow m = \tilde{m}+1$

$$= \sum_{\tilde{m}=0}^{\infty}(-1)^{\tilde{m}+1}\frac{x^{2\tilde{m}+1}}{(2\tilde{m}+1)!} = -\sum_{\tilde{m}=0}^{\infty}(-1)^{\tilde{m}}\frac{x^{2\tilde{m}+1}}{(2\tilde{m}+1)!}$$

$$= -\sin x$$

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## Binomial theorem for non integers

Consider the function $f(x) = (1+x)^\alpha$ and find its power series expansion.

$$f(x) = (1+x)^\alpha \qquad f(0) = 1$$
$$f'(x) = \alpha(1+x)^{\alpha-1} \qquad f'(0) = \alpha$$
$$f''(x) = \alpha(\alpha-1)(1+x)^{\alpha-2} \qquad f''(0) = \alpha(\alpha-1)$$

For general $n$ we have
$$f^{(n)}(0) = \alpha(\alpha-1)(\alpha-2)\dots(\alpha-n+1)$$

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$$f^{(n)}(0) = \alpha(\alpha-1)(\alpha-2)\ldots\ldots(\alpha-(n-1))$$

Therefore we can write

$$(1+x)^{\alpha} = \sum_{n=0}^{\infty} \frac{\alpha(\alpha-1)\ldots(\alpha-n+1)}{n!} x^n$$

$$= 1 + \alpha x + \frac{\alpha(\alpha-1)}{2!}x^2 + \frac{\alpha(\alpha-1)(\alpha-2)}{3!}x^3 + \ldots$$

If $\alpha$ is a positive integer, this series terminates and we get the binomial theorem.

Example

Find the power series of $\arctan(x)$. We could start differentiating, but this will get hard.

More efficiently, start with

$$\frac{1}{1+x^2} = (1+x^2)^{-1}$$

$$= 1 - x^2 + \frac{(-1)(-2)}{2!}x^4 + \frac{(-1)(-2)(-3)}{3!}x^6 + \ldots$$

$$= 1 - x^2 + x^4 - x^6 + x^8 + \ldots$$

Now let us integrate both sides and we get

$$\arctan(x) + C = x - \frac{1}{3}x^3 + \frac{1}{5}x^5 - \frac{1}{7}x^7 + \frac{1}{9}x^9 + \ldots$$

We can fix $C$ by evaluating both sides at $x=0$. This gives $C=0$.

Series can be composed, multiplied, divided etc... It is often easier to do this than to compute derivatives.

Example

Find the power series of $\arctan(e^x - 1)$ up to $x^4$.

We know $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \ldots$

$$\arctan(y) = y - \frac{1}{3}y^3 + \frac{1}{5}y^5 - \frac{1}{7}y^7 \ldots$$

Combining both series we find

$$\arctan(e^x - 1) = \overbrace{\left(x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots\right)}^{y = e^x - 1}$$

$$- \frac{1}{3}\left(x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots\right)^3$$

$$+ \frac{1}{5}\left(x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots\right)^5 + \ldots$$

$$= x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \ldots$$

$$- \frac{1}{3}\left(x^3 + 3x^2\frac{x^2}{2!} + \ldots\right)$$

$$= x + \frac{1}{2}x^2 + x^3\left(\frac{1}{6} - \frac{1}{3}\right) + x^4\left(\frac{1}{24} - \frac{1}{2}\right) + \ldots$$

$$= x + \frac{1}{2}x^2 - \frac{1}{6}x^3 + \frac{11}{24}x^4 + \ldots$$

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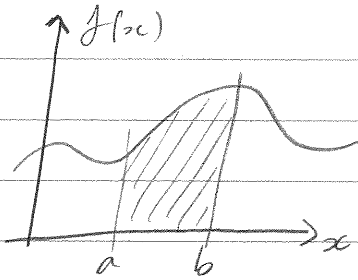
# Integration

## Introduction:

There are two different definitions of integration.
The first relates to integration as the opposite of differentiation.
This means: If $\frac{dF}{dx} = f(x)$ then $F$ is the anti-derivative of $f(x)$.
The second one relates to an area under a curve.




We say $A = \int_a^b f(x)\,dx$ is the area under the curve in the interval $[a, b]$.

Note: An integral is not necessarily related to an area. The usual Riemann integral can be generalised to the Lebesgue integral and one can then integrate functions like

$$f(x) = \begin{cases} 1 & x \in \mathbb{R} \setminus \mathbb{Q} \\ 0 & x \in \mathbb{Q} \end{cases}$$

This function is not Riemann integratable.

Let us consider the function

$$A(x) = \int_a^x f(\tilde{x})\,d\tilde{x}$$

For a specific choice of $x$ we get the area under the curve in the interval $[a, x]$.
Let us make a small change $x + \delta x$ with $|\delta x| << 1$, then $A(x + \delta x) = \int_a^{x+\delta x} f(\tilde{x})\,d\tilde{x}$

$$= \int_a^x f(\tilde{x})\, d\tilde{x} + \int_x^{x+\delta x} f(\tilde{x})\, d\tilde{x} .$$

Next, we will use

$$A(x+\delta x) = A(x) + A'(x)\,\delta x + \ldots$$

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$$A(x) = \int_a^x f(\tilde{x})\,d\tilde{x}$$

make a small change in $x \to x + \delta x$

$$A(x+\delta x) = \int_a^{x+\delta x} f(\tilde{x})\,d\tilde{x}$$

$$= \int_a^x f(\tilde{x})\,d\tilde{x} + \int_x^{x+\delta x} f(\tilde{x})\,d\tilde{x}$$

We use the Taylor series and write

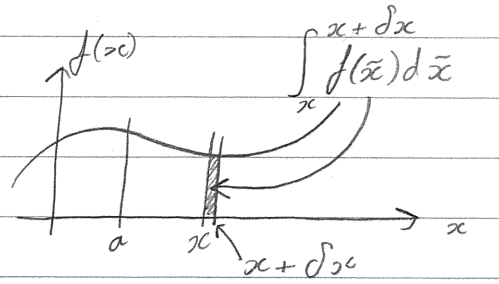
$$A(x+\delta x) = A(x) + A'(x)\delta x + \dots$$

This gives that

$$\cancel{A(x)} + A'(x)\delta x + \dots = \cancel{\int_a^x f(\tilde{x})\,d\tilde{x}} + \int_x^{x+\delta x} f(\tilde{x})\,d\tilde{x}$$

So we get

$$A'(x)\delta x + \dots = \int_x^{x+\delta x} f(\tilde{x})\,d\tilde{x}$$




We can approximate

$$\int_x^{x+\delta x} f(\tilde{x})\,d\tilde{x} \approx f(x)\,dx$$

From this we can deduce

$$A'(x) = f(x)$$

This is sometimes written as the fundamental theorem of calculus

$$F(x) = \int_a^x f(\tilde{x})\,d\tilde{x} \quad \text{with} \quad \frac{dF}{dx} = f(x)$$

## Improper integrals

One frequently encounters integrals of the form $\int_0^{\infty} f(x)\,dx$ or $\int_a^b f(x)\,dx$ where $f(x)$ is singular (ie. infinite) at either a, b, or some $c \in [a, b]$.

In all these cases we must interpret these integrals as follows:

$$\int_0^{\infty} f(x)\,dx = \lim_{L \to 0} \int_0^{L} f(x)\,dx$$

or if $f(a)$ is singular then

$$\int_a^b f(x)\,dx = \lim_{\delta \to 0} \int_{a+\delta}^{b} f(x)\,dx$$

**Example:** (How NOT to integrate!)

$$\int_{-1}^{1} \frac{1}{x^2}\,dx = -\left[\frac{1}{x}\right]_{-1}^{1} = -\left(1 - (-1)\right) = -2$$

**Example:** (Previous example corrected!)

$$\int_{-1}^{1} \frac{1}{x^2}\,dx = \lim_{\delta \to 0^-} \int_{-1}^{\delta} \frac{1}{x^2}\,dx + \lim_{\delta \to 0^+} \int_{\delta}^{1} \frac{1}{x^2}\,dx$$

$$= \lim_{\delta \to 0^{-1}} -\left[\frac{1}{x}\right]_{-1}^{\delta} + \lim_{\delta \to 0^+} -\left[\frac{1}{x}\right]_{\delta}^{1}$$

$$= \lim_{\delta \to 0^-} -\left(\frac{1}{\delta} + 1\right) + \lim_{\delta \to 0^+} -\left(1 - \frac{1}{\delta}\right)$$

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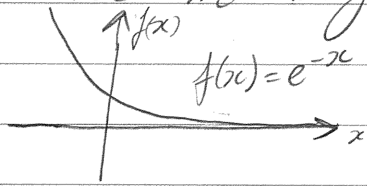
$$= -2 - \lim_{\delta \to 0^-} \frac{1}{\delta} + \lim_{\delta \to 0^+} \frac{1}{\delta}$$

$$\lim_{\delta \to 0^+} \frac{1}{\delta} \to \infty \quad , \quad -\lim_{\delta \to 0^-} \frac{1}{\delta} \to \infty$$

We could say that this integral formally approaches $\infty$. We consider this ill-defined.

Example:
The integral $\int_{-1}^{1} \frac{1}{x}\,dx$ is also undetermined as we would have to subtract two infinities.

Example:
Evaluate the integral $\int_0^\infty e^{-x}\,dx$




$$\int_0^\infty e^{-x}\,dx = \lim_{L \to \infty} \int_0^L e^{-x}\,dx$$

$$= \lim_{L \to \infty} \left[-e^{-x}\right]_0^L$$

$$= \lim_{L \to \infty} \left[-e^{-L} - (-1)\right]$$

$$= 1 - \lim_{L \to \infty} e^{-L} = 1$$

Example:

Evaluate $\int_0^1 x^n \, dx$ for all $n$.

Let us begin with $n \neq -1$.
If $n < 0$ then $x^n$ diverges near $x = 0$.
In this case we should write

$$\lim_{\delta \to 0^+} \int_0^1 x^n \, dx = \lim_{\delta \to 0^+} \left[ \frac{1}{n+1} x^{n+1} \right]_\delta^1$$

$$= \lim_{\delta \to 0^+} \left[ \frac{1}{n+1} - \frac{1}{n+1} \delta^{n+1} \right]$$

$$= \frac{1}{n+1} - \frac{1}{n+1} \lim_{\delta \to 0^+} \delta^{n+1}$$

$$\int_0^1 x^n \, dx = \begin{cases} \frac{1}{n+1} & \text{if } n > -1 \\ \text{diverges} & \text{if } n < -1 \end{cases}$$

If $n = -1$ we have

$$\lim_{\delta \to 0^+} \int_\delta^1 x^{-1} \, dx = \lim_{\delta \to 0^+} \left[ \log|x| \right]_\delta^1$$

$$= - \lim_{\delta \to 0^+} \log|\delta|$$

$$= \text{diverges.}$$

The integral diverges if $n \leq -1$ and converges otherwise.

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## Integration by parts

Starting with the product rule we can write

$$\frac{d}{dx}[uv] = \frac{du}{dx}v + u\frac{dv}{dx}$$

and integrate to find

$$uv = \int \frac{du}{dx} v\, dx + \int u \frac{dv}{dx}\, dx.$$

$$\int u\frac{dv}{dx}\, dx = uv - \int \frac{du}{dx} v\, dx.$$

This identity is useful when integrating products, provided the integral on the right-hand side is 'easier' than the one on the left

Example:

$$\int xe^{-x}\, dx = x(-e^{-x}) - \int 1(-e^{-x})\, dx$$

$$= -xe^{-x} - e^{-x} + C.$$

One could have chosen the functions the other way round and we would have arrived at some identity.

Whenever we wish to integrate an inverse function (log, arcsin, arccos, ....) it turns out that integration by parts is the best approach

This is because we know the derivative of the inverse function

Example:

$$\int \arcsin x \, dx = \int 1 \arcsin x \, dx$$

$$= x \arcsin x - \int \frac{x}{\sqrt{1-x^2}} \, dx$$

$$= x \arcsin x + \sqrt{1-x^2} + C$$

One can derive a host of beautiful identities using integration by parts.
Imagine we want to find $\int \sin^7(x) \, dx$.

We define $I_n = \int \sin^n(x) \, dx$

$$= -\cos x \sin^{n-1} x - \int (-\cos x)(n-1) \sin^{n-2} x \cos x \, dx$$

$$= -\cos x \sin^{n-1} x + \int (n-1) \sin^{n-2} x \cos^2 x \, dx$$

$$= -\cos x \sin^{n-1} x + (n-1) \int \sin^{n-2} x \left(1 - \sin^2 x\right) dx$$

$$= -\cos x \sin^{n-1} x + (n-1) \left[ \int \sin^{n-2} x \, dx - \int \sin^n x \, dx \right]$$

$$= -\cos x \sin^{n-1} x + (n-1) I_{n-2} - (n-1) I_n$$

We can now solve for $I_n$

$$\Rightarrow I_n + (n-1) I_n = -\cos x \sin^{n-1} x + (n-1) I_{n-2}$$

$$\Rightarrow I_n = -\frac{1}{n} \cos x \sin^{n-1} x + \frac{(n-1)}{n} I_{n-2}$$

This is a recursive relationship for the integral $I_n$.
To complete this we need $I_0$ and $I_1$ which are

$I_0 = \int 1 \, dx = x + C$ , $I_1 = \int \sin x \, dx = -\cos x + C$.

Exercise:

Show that $\int_0^{\frac{\pi}{2}} \sin^5 x \, dx = \frac{8}{15}$

Substitution

The idea of substitution is to change the independent variable $x$ in a useful way by introducing a new independent variable $y$ which can be expressed in terms of $x$.

Then $y = y(x)$ $\quad dy = \frac{dy}{dx} dx$

and the original integral $x$ might simplify if expressed in $y$.

Example:

Let us try to find

$$\int \cos x \, e^{\sin x} \, dx$$

let $y = \sin x$

$dy = \cos x \, dx$

$$\Rightarrow \int \cos x \, e^y \frac{dy}{\cos x} = \int e^y \, dy$$

$$= e^y + C$$

$$= e^{\sin x} + C.$$

It is not always easy to guess a useful substitution for a given integral.

Some common substitutions are:

| | |
|---|---|
| $\sqrt{a^2 - x^2}$ | $x = a\sin\theta$ |
| $\sqrt{a^2 + x^2}$ | $x = a\sinh\theta$ |
| $\sqrt{x^2 - a^2}$ | $x = a\cosh\theta$ |
| $\frac{1}{a^2 + x^2}$ | $x = a\tan\theta$ |

## Partial Fractions

We are interested in finding $\int \frac{P_m(x)}{Q_n(x)}\,dx$

where $P_m(x)$ is a polynomial of degree $m$ and $Q_n(x)$ is a polynomial of degree $n$.

Any integral of this type can be found in the following way:

Firstly if $m \geq n$ then one has to start with long division one gets a polynomial plus a fraction where the degree of the numerator is less than the degree of the polynomial in the denominator.

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$$\int \frac{P_m(x)}{Q_n(x)}\,dx$$

If the order (of the polynomial) $m < n$, we can use partial fractions straight away.
If $m > n$ we do polynomial division and then use partial fractions.

The polynomial will have $n$ roots if we allow $x$ to be complex.
All complex roots come in conjugate pairs and thus we can write $Q_n(x)$ in the following way.

$$Q_n(x) = (x - x_1)^{\alpha} \dots (x - x_k)^{\varkappa} \dots (x^2 + p_1 x + q_1)^{\omega} \dots$$
$$\dots (x^2 + p_s x + q_s)^{\sigma}$$

The root $x_1$ has multiplicity $\alpha$ etc...
The quadratics $(x^2 + p_i x + q_i)$ have no real roots which means they all satisfy $p_i^2 < 4q_i$.
They also may have a multiplicity (power).

<u>Example</u>
Consider the polynomial $x^4 + x^3 - x^2 - 5x + 4$
We spot that $x = 1$ is a root of the polynomial.

$$x^4 + x^3 - x^2 - 5x + 4 = (x-1)(x^3 + 2x^2 + x - 4)$$
$$= (x-1)^2(x^2 + 3x + 4)$$

The quadratic has no real roots and we are done.
Now assumming that $m < n$
We can alway achieve the following rewriting:

PTO →

$$\frac{P_m(x)}{Q_n(x)} = \frac{A_1}{(x-x_1)} + \frac{A_2}{(x-x_1)^2} + \dots + \frac{A_\alpha}{(x-x_1)^\alpha}$$

$$+ \frac{B_1}{(x-x_2)} + \dots + \frac{C_\varkappa}{(x-x_k)^\varkappa}$$

$$+ \frac{E_1 x + F_1}{(x^2+p_1x+q_1)} + \dots + \frac{E_\omega x + F_\omega}{(x^2+p_1x+q_1)^\omega}$$

$$+ \dots + \frac{M_\sigma x + N_\sigma}{(x^2+p_s x+q_s)^\sigma}$$

Example

$$\frac{x^3-3}{x^4+x^3-x^2-5x+4}$$

$$= \frac{x^3-3}{(x^2+3x+4)(x-1)^2} = \frac{A_1}{(x-1)} + \frac{A_2}{(x-2)^2} + \frac{Bx+C}{x^2+3x+4}$$

$$= \frac{17}{32(x-1)} - \frac{1}{4(x-1)^2} + \frac{15x+4}{32(x^2+3x+4)}$$

Example

Find the value of the integral

$$I = \int_0^\infty \frac{dx}{(x+2)^2(x^2+1)}$$

The denominator is in the correct form.

We can use partial fractions:

$$\frac{1}{(x+2)^2(x^2+1)} = \frac{A}{x+2} + \frac{B}{(x+2)^2} + \frac{Cx+D}{x^2+1}$$

$$\Rightarrow 1 = A(x+2)(x^2+1) + B(x^2+1) + (Cx+D)(x+2)^2 \quad (*)$$

let $x=-2$ : $1 = 5B \Rightarrow \underline{\underline{B = 1/5}}$

looking at $x^3 \Rightarrow 0 = A + C \Rightarrow A = -C$

let $x=0$ : $1 = 2A + B + 4D \Rightarrow \frac{2}{5} = A + 2D$

differentiating $(*)$: $0 = A[2x(x+2) + x^2+1] + 2xB + 2(Cx+D)(x+2) + C(x+2)^2$

let $x=-2$ : $0 = 5A - 4B \Rightarrow \underline{\underline{A = 4/25}}$ , $\underline{\underline{C = -\frac{4}{25}}}$ , $\underline{\underline{D = \frac{3}{25}}}$

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Therefore our integral becomes

$$I = \frac{1}{25}\int_0^\infty \frac{4}{x+2} + \frac{5}{(x+2)^2} + \frac{3-4x}{x^2+1}\, dx$$

$$= \frac{1}{25}\int_0^\infty \frac{4}{x+2} + \frac{5}{(x+2)^2} - 2 \cdot \frac{2x}{x^2+1} + \frac{3}{x^2+1}\, dx$$

Each part can now be integrated and we arrive at the following

$$I = \frac{1}{25}\left[4\log|x+2| - \frac{5}{x+2} - 2\log|x^2+1| + 3\arctan(x)\right]_0^\infty$$

$$= \frac{1}{25}\left[2\log\left|\frac{(x+2)^2}{x^2+1}\right| - \frac{5}{x+2} + 3\arctan(x)\right]_0^\infty$$

(by combinging logarithms (important!))

$$= \frac{1}{25}\left[\cancel{2\log(1)} - 2\log(4) - \cancel{0} + \frac{5}{2} + 3\frac{\pi}{2} - \cancel{0}\right]$$

$$= \frac{1}{25}\left[\frac{5}{2} + 3\frac{\pi}{2} - 4\ln(2)\right]$$

<u>The universal or tangent half-angle substitution</u>

We want to find integrals of the form

$$\int \frac{d\theta}{2+\sin\theta} \quad \text{or} \quad \int \frac{d\theta}{1+7\cos\theta+3\sin^2\theta}$$

Some of these integrals can be found by using an identity cleverly or by guessing a useful substitution.

However, all rational functions of trigonometric functions can be integrated using the following substitution:

$$t = \tan\left(\frac{\theta}{2}\right) \quad \text{or} \quad \theta = 2\arctan(t)$$

We need to derive some useful identities

$$dt = \frac{1}{2}\sec^2\left(\frac{\theta}{2}\right) d\theta$$

Using $\sin^2 u + \cos^2 u = 1$

$$\Rightarrow \begin{cases} \tan^2 u + 1 = \dfrac{1}{\cos^2 u} \\[2ex] 1 + \dfrac{1}{\tan^2 u} = \dfrac{1}{\sin^2 u} \end{cases}$$

$$\Rightarrow \begin{cases} \cos u = \dfrac{1}{\sqrt{1+\tan^2 u}} \\[2ex] \sin u = \dfrac{\tan u}{\sqrt{1+\tan^2 u}} \end{cases}$$

Using these identities we find

$$\begin{cases} \cos\dfrac{\theta}{2} = \dfrac{1}{\sqrt{1+t^2}} \\[2ex] \sin\dfrac{\theta}{2} = \dfrac{t}{\sqrt{+t^2}} \end{cases} \Rightarrow \begin{cases} \cos\theta = \dfrac{1-t^2}{1+t^2} \\[2ex] \sin\theta = \dfrac{2t}{1+t^2} \end{cases}$$

Therefore we also have

$$d\theta = 2\cos^2\left(\frac{\theta}{2}\right) dt$$

$$= \frac{2}{1+t^2}\, dt$$

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Example

$$I \int \frac{d\theta}{2+\sin\theta}$$

$\sin(2x) = 2\sin x \cos x$

Rewrite the integrals in terms of half-angles.

$$I = \int \frac{d\theta}{2 + 2\sin(\frac{\theta}{2})\cos(\frac{\theta}{2})}$$

Apply the universal substitution, $t = \tan(\frac{\theta}{2})$.
Using the previous identities

$$I = \int \frac{\left(\frac{2}{1+t^2}\right) dt}{2 + 2\left(\frac{t}{\sqrt{1+t^2}}\right)\left(\frac{1}{\sqrt{1+t^2}}\right)}$$

$$= \int \frac{\left(\frac{1}{1+t^2}\right) dt}{1 + \left(\frac{t}{1+t^2}\right)}$$

$$= \int \frac{dt}{t^2+t+1}$$

$$= \int \frac{dt}{(t+\frac{1}{2})^2 + \frac{3}{4}}$$

Next, we can set

$$t+\frac{1}{2} = \frac{\sqrt{3}}{2}\tan u$$

$$dt = \frac{\sqrt{3}}{2}\sec^2 u \, du$$

$$I = \int \frac{\left(\frac{\sqrt{3}}{2}\right)\sec^2 u \, du}{\frac{3}{4}\tan^2 u + \frac{3}{4}}$$

$$= \int \frac{2\sqrt{3}}{3} du = \frac{2}{\sqrt{3}} u + C = \frac{2}{\sqrt{3}}\arctan\left(\frac{2}{\sqrt{3}}(t+\tfrac{1}{2})\right) + C$$

$$= \frac{2}{\sqrt{3}}\arctan\left(\frac{2}{\sqrt{3}}\left(\tan\frac{\theta}{2} + \frac{1}{2}\right)\right) + C$$

Exercise

$$\int_0^{\frac{\pi}{2}} \frac{d\theta}{2+\sin\theta} = \frac{\pi}{3\sqrt{3}}$$

$$\int_0^{\frac{\pi}{2}} \frac{d\theta}{2+\cos\theta} = \frac{2\pi}{\sqrt{3}}$$

First order ordinary differential equations

A first order ordinary differential equation (ODE) is an equation of the form

$$\frac{dy}{dx} = F(x,y)$$

with initial or boundary conditions that specify the value of the function $y(x)$ at some point $x_0$

Seperable equations.

An ODE is seperable if $F(x,y) = f(x)g(y)$ for some functions $f$ and $g$.

For instance $F(x,y) = x^2+y^2$ is not seperable.

All seperable equations can be solved in principle as follows. We have

$$\frac{dy}{dx} = F(x,y) = f(x), g(y)$$

$$\Leftrightarrow \frac{1}{g(y)}\frac{dy}{dx} = f(x)$$

Next we integrate $\displaystyle\int \frac{1}{g(y)} \underbrace{\frac{dy}{dx}\, dx}_{dy \text{ by chain rule.}} = \int f(x)\, dx$

$$\Leftrightarrow \int \frac{1}{g(y)}\, dy = \int f(x)\, dx$$

NOTE: In the exam if you do a question twice with different answers, LEAVE BOTH.

[Normally we write $\frac{dy}{dx} = f(x)g(y)$

$$\frac{dy}{g(y)} = f(x)\,dx \quad ]$$

Provided we can find these integrals we can solve the ODE.
The solution will depend on a constant of integration and so we have a family of solutions.

The initial or boundary condition fixes this constant.

Example
Solve the ODE

$$x\frac{dy}{dx} + 3y = 2$$

with $y = 2$ when $x = 1$ $[y(x=1) = 2]$

Firstly we write

$$\frac{dy}{dx} = \frac{2-3y}{x}$$

so $\int \frac{dy}{2-3y} = \int \frac{dx}{x}$

$$\Rightarrow -\frac{1}{3}\log|2-3y| = \log|x| + C$$

we write

$$\log|2-3y| = -3\log|x| + \hat{C}$$

We apply $\exp(x)$ to both sides

$$2 - 3y = x^{-3}\tilde{C}$$

$$\Leftrightarrow \quad y(x) = \frac{1}{3}\left(2 - \frac{\tilde{c}}{x^3}\right)$$

Lastly, we apply the initial condition

$$y(x=1) = \frac{1}{3}\left(2 - \frac{\tilde{c}}{1^3}\right) = 2$$

$$\Rightarrow \quad 2 - \tilde{c} = 6$$

$$\Rightarrow \quad \tilde{c} = -4$$

Reduction to seperable form.

There are many equations which are not separable, however, some can be transformed into separable form.

Consider the ODE

$$\frac{dy}{dx} = \varphi(y/x)$$

For instance

$$f(x,y) = \frac{x^2+y^2}{xy} \quad \text{is of that form.}$$

$$f(x,y) = \frac{x^2+y^2}{xy}$$

$$= \frac{x^2\left(1 + \frac{y^2}{x^2}\right)}{xy}$$

$$= \frac{x^2\left(1 + y^2/x^2\right)}{x \cdot xy/x} = \frac{1 + y^2/x^2}{y/x}$$

To solve this ODE we introduce a new dependent variable

$$z = \frac{y(x)}{x} \quad \text{or} \quad y(x) = x\,z(x)$$

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Then $\frac{dy}{dx} = 1\, z(x) + x\frac{dz}{dx}$

$\varphi(y/x) = \varphi(z)$

Therefore $\frac{dy}{dx} = \varphi(y/x)$

becomes

$$x\frac{dz}{dx} + z = \varphi(z)$$

$$\Leftrightarrow x\frac{dz}{dx} = \varphi(z) - z$$

$$\Leftrightarrow \frac{dz}{dx} = \frac{\varphi(z) - z}{x}$$

which is seperable.

Next, let us consider the equation

$$\frac{dy}{dx} = \frac{x - y - 5}{x + y - 1}$$

We can transform this into seperable form by introducing a new dependent and a new independent variable.

$x = u + a$

$y = v(u) + b$

Firstly, we write $\frac{dy}{dx}$ in terms of $v(u)$ and $u$.

$$\frac{dy}{dx} = \frac{d\,(v(u))}{dx} = \frac{dv}{du}\frac{du}{dx} = \frac{dv}{du}$$

Now our equation becomes

$$\frac{dv}{du} = \frac{u - v + a - b - 5}{u + v + a + b - 1}$$

Since we can choose $a$ and $b$, we will choose them such that

$$a - b - 5 = 0 \quad \text{and} \quad a + b - 1 = 0$$

We get $a = 3$ and $b = -2$

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$$\frac{dy}{dx} = \frac{x-y-5}{x+y+1}$$

$x = u + a$

$y = v(u) + b$

$$\frac{dy}{dx} = \frac{d}{dx}(v(u)+b)$$

$$= \frac{d}{dx} v(u) = \frac{d}{dx} v(u(x))$$

$$= \frac{dv}{du}\frac{du}{dx}$$

$u = x - a \Rightarrow \frac{du}{dx} = 1$

$$= \frac{dv}{du}$$

$a - b - 5 = 0$

$a + b + 1 = 0 \quad \Rightarrow a = 3 \ \& \ b = -2$

Now our equation becomes $\frac{dv}{du} = \frac{u-v}{u+v}$

$$\Rightarrow \frac{dv}{du} = \frac{1 - \frac{v}{u}}{1 + \frac{v}{u}}$$

which is of the form previously discussed.
Introducing $z = \frac{v}{u}$ transforms this into a separable equation which in turn can be solved.
If the two linear equations have no solutions, then there exists a single substitution for a new dependent variable.

<u>Exercise</u>

Solve $\frac{dy}{dx} = \frac{x+y-1}{2x+2y+4}$. (i) start as before, there is no solution.

(ii) introduce $z = x+y$ and solve the equation.

Exercise

Solve $\frac{dy}{dx} = \frac{1}{2}(x^2 + 4xy + 4y^2) + \frac{3}{2}$

You get $y = \tan(2x + C) - \frac{x}{2}$

The general first order linear equation.

The general first order linear equation is

$$\frac{dy}{dx} + a(x)y = f(x)$$

If $f = 0$ the equation is called homogeneous and it is separable.

Let us multiply our equation by some $Q(x)$

$$\Rightarrow Q\frac{dy}{dx} + Qay = Qf$$

Consider the expression

$$\frac{d}{dx}(Qy) = Q'y + Qy' = Q\frac{dy}{dx} + Qay$$

Since $Q$ is arbitrary we can choose $Q$ such that $Q\cancel{y} = Qa\cancel{y}$

This equation is separable and so $\frac{dQ}{dx} = Qa(x)$

$$\Rightarrow \int \frac{dQ}{Q} = \int a(x)\,dx$$

$$\Rightarrow \ln Q + C = \int a(x)\,dx$$

$$\Rightarrow Q\tilde{C} = \exp\left(\int a(x)\,dx\right)$$

We can set $\tilde{C} = 1$

The function $Q$ chosen in this way is called the integrating factor. (Can be quoted.)
Then we can write

$$\frac{d}{dx}(Qy) = Qf$$

$$\Rightarrow \quad Q(x)y(x) = \int^x Q(t)f(t)\,dt + C$$

Then $y(x)$ is given by

$$y(x) = \frac{1}{Q(x)}\int^x Q(t)f(t)\,dt + \frac{C}{Q(x)}$$

To summarise this procedure

$$b(x)\frac{dy}{dx} + c(x)y = g(x)$$

(i) Divide by $b(x)$ to get

$$\frac{dy}{dx} + a(x)y = f(x)$$

(ii) Find $Q = \exp \int a(x)\,dx$

(iii) Multiply your equation by $Q$ and we get $\frac{d}{dx}(Qy) = Qf$

(iv) Integrate to find $y(x)$

(v) Apply initial / boundary conditions to fix the constant of integration.

Example

$$x\frac{dy}{dx} + 2y = \frac{\cos(x)}{x} \quad \text{with} \quad y(\pi)=1$$

(i) $$\frac{dy}{dx} + \frac{2}{x}y = \frac{\cos x}{x^2}$$

(ii) $$Q = \exp\int \frac{2}{x}\,dx$$

$$= \exp(2\log x)$$

$$= x^2$$

(iii) $$\frac{d}{dx}(x^2 y) = x^2 f = \cos x$$

check: $$2xy + x^2\frac{dy}{dx} = x^2\left(\frac{dy}{dx} + \frac{2}{x}y\right) \checkmark$$

(iv) $$x^2 y = \int \cos x\,dx$$

$$= \sin x + C$$

$$\Rightarrow y(x) = \frac{\sin x}{x^2} + \frac{C}{x^2}$$

(v) $$y(\pi) = \frac{\sin\pi}{\pi^2} + \frac{C}{\pi^2} = 1$$

$$\Rightarrow C = \pi^2$$

$$\Rightarrow y(x) = \frac{\sin x}{x^2} + \frac{\pi^2}{x^2}$$

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Example

$$x^2 \frac{dy}{dx} + (1+x)y = \frac{1}{x}$$

$$\frac{dy}{dx} + \left(\frac{1}{x^2} + \frac{1}{x}\right)y = \frac{1}{x^3}$$

$$Q = \exp \int \frac{1}{x^2} + \frac{1}{x} \, dx$$

$$= \exp\left[-\frac{1}{x} + \log x\right] = x e^{-\frac{1}{x}}$$

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$$\frac{dy}{dx} + \frac{1+x}{x^2} y = \frac{1}{x^3}$$

The integrating factor is

$$Q = \exp\int \frac{1+x}{x^2}\,dx$$

$$= \exp\left(\frac{-1}{x} + \log x\right)$$

$$= xe^{-\frac{1}{x}}$$

Therefore

$$\frac{d}{dx}\left(xe^{-\frac{1}{x}}y\right) = xe^{-\frac{1}{x}}\frac{1}{x^3}$$

$$= \frac{1}{x^2}e^{-\frac{1}{x}}$$

$$\Rightarrow \quad xe^{-\frac{1}{x}}y = e^{-\frac{1}{x}} + C$$

$$\Rightarrow \quad y(x) = \frac{1}{x} + \frac{C}{xe^{-\frac{1}{x}}}$$

$$= \frac{1}{x} + \frac{Ce^{\frac{1}{x}}}{x}$$

## Bernoulli's equation

The non linear equation

$$\frac{dy}{dx} + P(x)y = y^n Q(x) \qquad n \neq 1$$

is called Bernoulli's equation

This can be reduced to a linear equation by introducing the new dependent variable $z = y^{1-n}$

We have $\quad \frac{dz}{dx} = (1-n)y^{-n}\frac{dy}{dx}$

$$\Leftrightarrow \quad \frac{dy}{dx} = y^n \frac{1}{(1-n)} \frac{dz}{dx}$$

We put this into our equation

$$y^n \frac{1}{(1-n)} \frac{dz}{x} + P(x)y = y^n Q(x)$$

$$\Rightarrow \quad \frac{1}{(1-n)} \frac{dz}{dx} + P(x)y^{1-n} = Q(x)$$

$$\Rightarrow \quad \frac{1}{(1-n)} \frac{dz}{dx} + zP(x) = Q(x)$$

$$\Leftrightarrow \quad \frac{dz}{dx} + (1-n)P(x)z = (1-n)Q(x)$$

This equation can now be solved using the integrating factor method.

## Interchanging variables

Let us consider the equation

$$\frac{dy}{dx} = \frac{y\log(y)}{\log(y) - x}$$

This is non-linear in $y(x)$ but linear in $x(y)$.

We make $y$ the independent variable and $x(y)$ the dependent variable.

This means

$$\frac{dx}{dy} = \frac{\log(y) - x}{y\log(y)}$$

$$= \frac{1}{y} - \frac{x}{y\log(y)}$$

$$\Leftrightarrow \frac{dx}{dy} + \frac{x}{y\log y} = \frac{1}{y}$$

The integrating factor contains the integral

$$\int \frac{1}{y}\frac{1}{\log y}\,dy$$

let $u = \log y$

$du = \frac{1}{y}\,dy$

$$= \int \frac{1}{\log(y)} \cdot \frac{dy}{y}$$

$$= \int \frac{1}{u}\,du = \log u = \log\log(y)$$

Therefore $Q = \exp(\int\ldots)$

$$= \exp(\log\log(y))$$

Now our differential equation becomes

$$\frac{d}{dy}(\log(y)) = \frac{\log(y)}{y}$$

$$\log(y)x = \int^y \frac{\log(\tilde{y})}{\tilde{y}}\, d\tilde{y} + C$$

$$= \int^y \log(\tilde{y}) [\log(\tilde{y})]^3\, d\tilde{y} + C$$

$$= \frac{1}{2}[\log y]^2 + C.$$

$$\Rightarrow x(y) = \frac{1}{2}\log(y) + C\log(y).$$

Going back one equation:

$$[\log(y)]^2 - 2\log(y)x + 2C = 0$$

$$\Rightarrow q^2 - 2qx + \underbrace{2C}_{\hat{C}} = 0 \qquad \text{where } q = \log(\hat{y})$$

$$\Rightarrow q_{1,2} = x \pm \sqrt{x^2 - \hat{C}}$$

$$\log(y_{12}) = x \pm \sqrt{x^2 - \hat{C}}$$

$$\Rightarrow y = \exp(x \pm \sqrt{x^2 - \hat{C}})$$

Initial boundary conditions will fix the sign and value of $\hat{C}$

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# Second order ODEs

## Introduction

The general linear second order ODE is given by

$$y'' + a(x)y' + b(x)y = f(x)$$

We will concentrate on equations where a and b are constants.
There are some equations which reduce to this.
If $f \equiv 0$ the equation is said to be homogeneous.
$f(x)$ is often called the forcing term of the equation.

## Constant coefficient homogeneous equations

Consider the equation

$$y'' + ay + by = 0$$

We know that $y' + ay = 0$ is solved by $y = Ae^{-ax}$, therefore we begin with the ansatz

$$y(x) = Ae^{\lambda x}$$
$$y'(x) = A\lambda e^{\lambda x}$$
$$y''(x) = A\lambda^2 e^{\lambda x}$$

Substituting this into the ODE

$$A\lambda^2 e^{\lambda x} + aA\lambda e^{\lambda x} + bAe^{\lambda x} = 0$$
$$\Leftrightarrow \quad Ae^{\lambda x}(\lambda^2 + a\lambda + b) = 0$$

Assuming $A \neq 0$

$$\Rightarrow \lambda^2 + a\lambda + b = 0$$

This is called the auxiliary equation of the ODE.
In general we have two solutions to the auxiliary equation, unless there are repeated roots.
Assuming we have two roots $\lambda_1$ and $\lambda_2$ the general solution is $y(x) = Ae^{\lambda_1 x} + Be^{\lambda_1 x}$.

Initial / Boundary conditions fix the values of $A$ and $B$.

If we have a pair of complex conjugate roots, we write

$$\lambda_1 = p + iq$$
$$\lambda_2 = p - iq \qquad p, q \in \mathbb{R}$$

Then our solution is

$$\begin{aligned}
y(x) &= Ae^{(p+iq)x} + Be^{(p-iq)x} \\
&= Ae^{px}e^{iqx} + Be^{px}e^{-iqx} \\
&= e^{px}(Ae^{iqx} + Be^{-iqx}) \\
&= e^{px}(A\cos(qx) + iA\sin(qx) + B\cos(qx) - iB\sin(qx)) \\
&= e^{px}((A+B)\cos(qx) + (iA - iB)\sin(qx)) \\
&= e^{px}(E\cos(qx) + F\sin(qx))
\end{aligned}$$

where $E = A + B$, $F = i(A - B)$

Lastly, if $\lambda = \lambda_1 = \lambda_2$ we know that one solution is given by $y = Ae^{\lambda x}$ where $\lambda = -\frac{a}{2}$ $\left(\lambda^2 + a\lambda + b = 0,\ \lambda_{1,2} = -\frac{a}{2} \pm \underbrace{\sqrt{\frac{a^2}{4} - b}}_{=0}\right)$ [for a repeated root]

To find the other solution, we try

$$\begin{aligned}
y(x) &= f(x)e^{\lambda x} \\
y'(x) &= f'e^{\lambda x} + f\lambda e^{\lambda x} \\
y''(x) &= f''e^{\lambda x} + f'\lambda e^{\lambda x} + f'\lambda e^{\lambda x} + f\lambda^2 e^{\lambda x} \\
&= f''e^{\lambda x} + 2\lambda f'e^{\lambda x} + f\lambda^2 e^{\lambda x}
\end{aligned}$$

Then we substitute into our equation and get

$$y'' + ay' + by = 0$$

$$\Rightarrow (f''e^{\lambda x} + 2\lambda f'e^{\lambda x} + f\lambda^2 e^{\lambda x}) + a(f'e^{\lambda x} + f\lambda e^{\lambda x}) + bfe^{\lambda x} = 0$$

$$\Leftrightarrow e^{\lambda x}\left[f'' + f'(2\lambda + a) + f\underbrace{(\lambda^2 + a\lambda + b)}_{0\ !!!}\right] = 0$$

$$\Leftrightarrow e^{\lambda x}\left[f'' + f'(2\lambda + a)\right] = 0$$

because $\lambda$ is a solution of the auxiliary equation.

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$2\lambda + a = 0$ because $\lambda$ is a repeated root $(\lambda = -\frac{a}{2})$
Therefore we are left to solve

$$e^{\lambda x} f'' = 0$$

$$\Rightarrow f(x) = Ax + B$$

And hence

$$y(x) = (Ax + B)\, e^{\lambda x}.$$

<u>Note:</u>
This method of finding the second solution to an ODE using $y = f(x) e^{\lambda x}$ where $e^{\lambda x}$ is a homogeneous solution is very powerful. It also works for inhomogeneous equations and can be very efficient.

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## 2nd order ODEs

**Example**

$$y'' + y' + y = 0 \quad \text{with} \begin{cases} y(0) = 1 \\ y'(0) = 3 \end{cases}$$

The auxiliary equation is

$$\lambda^2 + \lambda + 1 = 0$$

$$\lambda_{1,2} = -\frac{1}{2} \pm \sqrt{\frac{1}{4} - 1}$$

$$= -\frac{1}{2} \pm \frac{1}{2}\sqrt{-3}$$

$$= -\frac{1}{2} \pm \frac{1}{2}\sqrt{3}\, i$$

Therefore the general solution is

$$y(x) = A e^{-\frac{x}{2}} \cos\left(\frac{\sqrt{3}}{2} x\right) + B e^{-\frac{x}{2}} \sin\left(\frac{\sqrt{3}}{2} x\right)$$

Now we apply the initial conditions

$$y(x=0) = A.1 + B.0 = A = 1$$

$$y'(x) = A\left(-\frac{1}{2} e^{-\frac{x}{2}}\right)\cos\left(\frac{\sqrt{3}}{2} x\right) + A e^{-\frac{x}{2}}\left(-\sin\left(\frac{\sqrt{3}}{2} x\right) \frac{\sqrt{3}}{2}\right)$$
$$+ B\left(-\frac{1}{2} e^{-\frac{x}{2}}\right)\sin\left(\frac{\sqrt{3}}{2} x\right) + B e^{-\frac{x}{2}}\left(\cos\left(\frac{\sqrt{3}}{2} x\right) \frac{\sqrt{3}}{2}\right)$$

$$y'(x=0) = -\frac{1}{2} A + A.0 + B.0 + \frac{\sqrt{3}}{2} B = 3$$

$$\Rightarrow B \frac{\sqrt{3}}{2} = 3 + \frac{1}{2}$$

$$\Rightarrow B = \frac{7}{\sqrt{3}}$$

Hence our solution is

$$y(x) = e^{-\frac{x}{2}} \cos\left(\frac{\sqrt{3}}{2} x\right) + \frac{7}{\sqrt{3}} e^{-\frac{x}{2}} \sin\left(\frac{\sqrt{3}}{2} x\right)$$

## Inhomogeneous equations

We are now considering the equation

$$y''(x) + a y'(x) + b y(x) = f(x) \quad a, b \in \mathbb{R}.$$

Assume we have two distinct solutions $y_1$ and $y_2$ which satisfy our equations

$$y_1'' + ay_1' + by_1 = f(x)$$

$$y''_2 + ay'_2 + b_2 = f(x)$$

Let us subtract both equations, and we find

$$(y_2 - y_1)'' + a(y_2 - y_1)' + b(y_2 - y_1) = 0$$

Therefore, we can conclude that the difference

$$y_{hom} = y_2 - y_1$$

satisfies the homogeneous equation which we can solve.

Therefore we also have

$$y_2 = y_1 + y_{hom}$$

so that any two solutions only differ by a homogeneous solution.

If we can find one particular $y_1$ then we have found all the solutions since we can solve the homogeneous equation.

In other words

$$y_1 + y_{hom} = (\text{particular integral}) + (\text{complimentary function})$$

is the general solution.

Hence, we need to find a solution to the inhomogeneous ODE. The standard technique used for finding such solutions is the 'guided trial and error' approach.

The following table contains some common forms of $f(x)$ and the corresponding trial function.

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| $f(x)$ | trial function $y(x)$ |
| --- | --- |
| $Ae^{bx}$ | $ae^{bx}$ if b is not a root of the auxiliary equation. |
| | $axe^{bx}$ if b is a root of the auxiliary equation. |
| | $ax^2e^{bx}$ if b is a repeated root of the auxiliary equation |
| polynomial of degree n eg. $x^2$ | a polynomial of degree n $ax^2+bx+c$ |
| $A\cos(\beta x)$ or $C\sin(Dx)$ | $\alpha\cos(\beta x)+\gamma\sin(\beta x)$ this does not work if $\pm iB$ (or $\pm iD$) is the root of the auxiliary equation. otherwise try: $x(\alpha\cos(\beta x)+\gamma\sin(\beta x))$ |
| products of $\sin()$ & $\cos()$ | turn the product into sums using trigonometric identities and then proceed as before. |
| hyperbolic functions | rewrite as exponentials and guess accordingly |
| $e^{ax}\cos(bx)$ | write this expression as $Re(e^{(a+ib)x})$ and treat it as exponentials |
| linear combinations of the above | also linear combinations |

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## Example

Solve the differential equation

$$y'' - 3y' + 2y = e^{4x} + e^{3x} + e^{2x}$$

We start with the complementary function (solution to the homogeneous equation). The auxiliary equation is

$$\lambda^2 - 3\lambda + 2 = 0$$

$$\Rightarrow (\lambda - 2)(\lambda - 1) = 0 \quad \Rightarrow \lambda = 1, \lambda = 2$$

Hence we find

$$y_{hom} = Ae^{2x} + Be^{x}$$

For the particular integral we begin with

$$ae^{4x} + be^{3x}.$$

Since $e^{2x}$ is part of the homogeneous solution we use $cxe^{2x}$.

We guess

$$y_p = ae^{4x} + be^{3x} + cxe^{2x}$$

$$y_p' = 4ae^{4x} + 3be^{3x} + c(e^{2x} + 2xe^{2x})$$

$$y_p'' = 16ae^{4x} + 9be^{3x} + c(2e^{2x} + 2(e^{2x} + 2xe^{2x}))$$

We substitute back into our ODE and get

$$16ae^{4x} + 9be^{3x} + 4c(e^{2x} + xe^{2x}) - 3(4ae^{4x} + 3be^{3x} + c(e^{2x} + 2e^{2x}))$$
$$+ 2(ae^{4x} + be^{3x} + cxe^{2x}) = e^{4x} + e^{3x} + e^{2x}$$

$e^{4x}$: $16a - 12a + 2a = 1 \quad \Rightarrow a = \frac{1}{6}$

$e^{3x}$: $9b - 9b + 2b = 1 \quad \Rightarrow b = \frac{1}{2}$

$xe^{2x}$: $4c - 6c + 2c = 0$

$e^{2x}$: $4c - 3c = 1 \quad \Rightarrow c = 1$

We expect the $xe^{2x}$ terms to give an identity since they arise from differentiating the $e^{2x}$ part of that product. This is the homogeneous equation and so we expect zero.

Therefore our particular integral is given by

$$y_p = \frac{1}{6}e^{4x} + \frac{1}{2}e^{3x} + xe^{2x}$$

and hence the general solution is

$$y(x) = Ae^{2x} + Be^{x} + \frac{1}{6}e^{4x} + \frac{1}{2}e^{3x} + xe^{2x}$$

Now we could apply initial / boundary conditions.

<u>Example</u>

Solve $y'' + 4y = x^2 + \cos(x)$

The auxiliary equation is $\lambda^2 + 4 = 0$

and we get $\lambda = \pm 2i$

So the complimentary function is given by

$$y_{hom} = A\cos(2x) + B\sin(2x)$$

For the particular we begin with

$$y_{PI} = ax^2 + bx + c + m\cos x + n\sin x$$

$$y'_{PI} = 2ax + b - m\sin x + n\cos x$$

$$y''_{PI} = 2a - m\cos x - n\sin x$$

We substitute back into the ODE and get

$$2a - m\cos x - n\sin x + 4(ax^2 + bx + c + m\cos x + n\sin x) = x^2\cos x$$

next we compare coefficients

$\cos x: -m + 4m = 1 \Rightarrow m = \frac{1}{3}$

$\sin x: -n + 4n = 0 \Rightarrow n = 0$

$x^2: 4a = 1 \Rightarrow a = \frac{1}{4}$

$x: 4b = 0 \Rightarrow b = 0$

$1: 2a + 4c = 0 \Rightarrow c = -\frac{1}{8}$

Therefore, our general solution is

$$y(x) = A\cos(2x) + B\sin(2x) + \frac{1}{3}\cos x + \frac{1}{4}x^2 - \frac{1}{8}$$

At this point we could apply initial / boundary conditions.

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Example

Solve the equation

$$y'' - y = x + e^x$$

The auxiliary equation is $\lambda^2 - 1 = 0$ so that $\lambda = \pm 1$.
So $e^x$ is a solution of the homogeneous equation.
We will try solving this equation using

$$y(x) = f(x)e^x.$$

$$y'(x) = f'e^x + fe^x$$

$$y''(x) = f''e^x + 2f'e^x + fe^x$$

Substitution gives

$$f''e^x + 2f'e^x \cancel{+ fe^x} \cancel{- fe^x} = x + e^x$$

$$\Rightarrow f''e^x + 2f'e^x = x + e^x$$

Set $g = f'$ and get

$$g'e^x + 2ge^x = x + e^x$$

$$\Rightarrow g' + 2g = xe^{-x} + 1$$

$$Q = \exp\left(\int 2\,dx\right) = e^{2x}$$

$$\Rightarrow \frac{d}{dx}\left(e^{2x}g\right) = xe^x + e^{2x}$$

$$\Rightarrow e^{2x}g = xe^x - e^x + \tfrac{1}{2}e^{2x} + C$$

$$g = xe^{-x} - e^{-x} + \tfrac{1}{2} + Ce^{-2x}$$

Since $f' = g$ we find

$$f = \tilde{C}e^{-2x} + \tfrac{1}{2}x - e^{-x}x + D$$

Therefore

$$y = fe^x = \tilde{C}e^{-x} + De^x + \tfrac{1}{2}xe^x - x$$

Example

$$y'' - 3y' + 2y = xe^x$$

The auxiliary equation is

$$\lambda^2 - 3\lambda + 2 = 0$$

$$(\lambda - 2)(\lambda - 1) = 0$$

$$y_{hom} = Ae^{2x} + Be^{x}$$

We cannot guess a suitable particular integral and so begin with

$$y = fe^x$$

$$y' = f'e^x + fe^x$$

$$y'' = f''e^x + fe^x$$

Substitution into the ODE gives

$$(f''e^x + 2f'e^x + fe^x) - 3(f'e^x + fe^x) + 2fe^x = xe^x$$

$$f'' + 2f' + \cancel{f} - 3f' - \cancel{3f} + \cancel{2f} = x$$

$$\Leftrightarrow \quad f'' - f' = x$$

Set $g = f'$

$$g' - g = x$$

$$Q = \exp\left(\int - dx\right) = e^{-x}$$

$$\frac{d}{dx}(e^{-x}g) = e^{-x}g$$

$$e^{-x}g = -e^{-x}x - e^{-x} + C$$

$$\Rightarrow \quad g = -x - 1 + Ce^x$$

$$\Rightarrow \quad f = -\tfrac{1}{2}x^2 - x + Ce^x + D$$

Therefore our final solution is

$$y = fe^x = Ce^{2x} + De^x - e^x\left(\tfrac{1}{2}x^2 + x\right)$$

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## Euler's equation

The Euler or Cauchy-Euler equation is

$x^2 y'' + axy' + by = f(x)$ with $a, b \in \mathbb{R}$

## Exercise

Show that introducing the new independent variable $x = e^t$ or $t = \log(x)$ transforms this equation into a constant coefficient equation.

The hardest part is to show that

$$\frac{d^2y}{dx^2} = \frac{1}{x^2}\frac{d^2y}{dt^2} - \frac{1}{x^2}\frac{dy}{dt}$$

This should be done first and only requires careful use of the chain rule

$x = e^t$ $\qquad y(x) = y(x(t))$

$dx = e^t dt$ $\qquad \frac{dy}{dt} = \frac{dy}{dx}\frac{dx}{dt} = \frac{dy}{dx}e^t$

However, we can solve this differently by noting its particular structure

$x^2 \times$ second derivative

$x^1 \times$ first derivative

$x^0 \times$ function

This suggests we try $y = x^\lambda$ to find a solution to the homogeneous equation.

## Example

$x^2y'' + xy' - 4y = x^2 + x^4$

Let us try using $y = x^\lambda$

$y' = \lambda x^{\lambda-1}$ $\qquad xy' = \lambda x^\lambda$

$y'' = \lambda(\lambda-1)x^{\lambda-2}$ $\qquad x^2y'' = \lambda(\lambda-1)x^\lambda$

Therefore, the homogeneous equation becomes

$$\lambda(\lambda-1)x^\lambda + \lambda x^\lambda - 4x^\lambda = 0$$

$$\Leftrightarrow [\lambda(\lambda-1) + \lambda - 4]x^\lambda = 0$$

$$\Leftrightarrow [\lambda^2 - \lambda + \lambda - 4]x^\lambda = 0$$

$$\Rightarrow \lambda = \pm 2$$

Hence $y_{hom} = Ax^2 + Bx^{-2}$

Now, we can again try a solution of the form

$$y = fx^2$$
$$y' = f'x^2 + 2fx$$
$$y'' = f''x^2 + 4f'x + 2f$$

We substitute

$$x^2(f''x^2 + 4f'x + \cancel{2f}) + x(f'x^2 + \cancel{2fx}) - 4(\cancel{fx^2}) = x^2 + x^4$$

$$\Rightarrow x^2(f''x^2 + 4f'x) + x(f'x^2) = x^2 + x^4$$

Divide by $x^4$:

$$f'' + \frac{4}{x}f' + \frac{1}{x}f' = 1 + \frac{1}{x^2}$$

set $g = f'$

$$\text{so } g' + \frac{5}{x}g = 1 + \frac{1}{x^2}$$

$$Q = \exp\left(\int \frac{5}{x}\,dx\right) = x^5$$

$$\frac{d}{dx}(x^5 g) = x^5 + x^3$$

$$x^5 g = \frac{1}{6}x^6 + \frac{1}{4}x^4 + C$$

So $f' = g = \frac{1}{6}x + \frac{1}{4}x^{-1} + Cx^{-5}$

$$\Rightarrow f = \frac{1}{12}x^2 + \frac{1}{4}\log|x| + \tilde{C}x^{-4} + D$$

Hence $y = x^2 f = \frac{1}{12}x^4 + \frac{1}{4}x^2 \cdot \log|x| + \tilde{C}x^{-2} + Dx^2$

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## Probability

### Basics

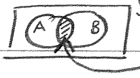
The starting point of a probability model is a sample space or state space) which is representing all possible outcomes of an experiment, trial, game etc.

### Example

Two coins are tossed.
One possible sample space is
$\{HH, HT, TH, TT\}$
another one is
$\{HH, HT, TT\}$ (order irrelevant).

An event $A$ is a subset of our sample space $S$.
Set operations:

- intersection: $A \cap B = \{x : x \in A \text{ and } x \in B\}$
  (x belongs to both A and B)


$S$ $A \cap B$

- union : $A \cup B = \{x : x \in A \text{ or } x \in B\}$
  (x belongs to A or B or both)


$S$ $A \cup B$

- complement: $A^c = \{x : x \notin A\}$
  (the complement is denoted by $A^c$ or $A'$)


$S$ $A^c$

- relative compliment: $A \setminus B = \{x : x \in A \text{ and } x \notin B\}$
  $A \setminus B$ is the set of all points which are in A but not in B.


$A \setminus B$

- disjoint (or mutually exclusive sets):
  $A \cap B = \emptyset$ where $\emptyset$ is the empty set.
  A and B are disjoint if they have no element in common.

**Example**
Consider the set $S = \{0, 1, 2\}$
The elements of $S$ are $0, 1, 2$.
We write $1 \in S$.
The subsets of $S$ are $\{0\}$, $\{1\}$, $\{2\}$, $\{0,1\}$, $\{0,2\}$, $\{1,2\}$, $\{0,1,2\}$, $\emptyset = \{\}$

Definition of a probability:
To each event $A \subset S$ we assign a number, $P(A)$ called the probability of $A$ which satisfies the following conditions:
(i) $P(A) \geq 0$ for all $A$
(ii) $P(S) = 1$
(iii) If $A \cap B = \emptyset$ then
$$P(A \cup B) = P(A) + P(B)$$

Two useful identities that we will need are:
$$(A \setminus B) \cup (A \cap B) = A$$
$$(A \setminus B) \cup B = A \cup B$$

**Lemma**
(a) $P(A^c) = 1 - P(A)$
(b) $P(\emptyset) = 0$
(c) $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

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Proof of (a)

We have $A \cap A^c = \emptyset$

and also $A \cup A^c = S$

(ii) stated $P(S) = 1$

$$1 = P(S) = P(A \cup A^c)$$
$$= P(A) + P(A^c) \quad \text{by (iii)}$$
$$\Rightarrow P(A^c) = 1 - P(A)$$

(b) We know that $S^c = \emptyset$

Then by (a) $P(\emptyset) = P(S^c) = 1 - P(S)$
$$= 0 \quad \text{by (ii)}$$

(c) We have that $A \setminus B$ and $A \cap B$ are disjoint

$(A \setminus B) \cap (A \cap B) = \emptyset$

Also $(A \setminus B) \cup (A \cap B) = A$

Then $P(A) = P((A \setminus B) \cup (A \cap B))$
$$= P(A \setminus B) + P(A \cap B) \quad \text{by (iii)}$$

Likewise

$(A \setminus B) \cap B = \emptyset$

$(A \setminus B) \cup B = A \cup B$

Therefore

$P(A \cup B) = P((A \setminus B) \cup B)$
$$= P(A \setminus B) + P(B)$$

We can eliminate $P(A \setminus B)$ and get

$P(A \cup B) = P(A) - P(A \cap B) + P(B)$
$$= P(A) + P(B) - P(A \cap B)$$

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## Equally Likeley Events

Consider a sample with a finite number of elements
$$S = \{s_1, \dots, s_n\}$$
where $n = |S|$.

Consider the events consisting of only one element of $S$ (simple events). Then
$$S_1 = \{s_1\} \quad ; \quad S_2 = \{s_2\} \; ; \dots$$

Since all $S_i$ are disjoint we have
$$1 = P(S) = P(S_1 \cup S_2 \cup S_3 \cup \dots \cup S_n)$$
$$= P(S_1) + P(S_2) + \dots + P(S_n)$$

If each event $S_j = \{s_j\}$ is equally likely, then
$$1 = n\,P(S_j) \quad \text{for one fixed } j,$$
then $P(S_j) = \frac{1}{n}$.

Any event $A$ which is a union of disjoint events $S_j$, for instance
$$A = \{s_{i_1}, s_{i_2}, \dots, s_{i_k}\}$$
with $k = |A|$

Then
$$P(A) = P(s_{i_1}) + P(s_{i_2}) + \dots + P(s_{i_k})$$
$$= \underbrace{\frac{1}{n} + \frac{1}{n} + \dots + \frac{1}{n}}_{k \text{ times}}$$
$$= \frac{k}{n} = \frac{|A|}{|S|}$$

### Example

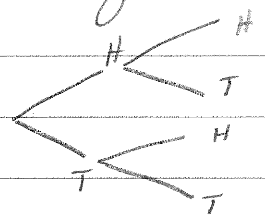
A fair coin is tossed twice. Our sample space is $S = \{HH, HT, TH, TT\}$ and all simple events are equally likely.

$P(HH) = \frac{1}{4}$

$P(\text{one head and one tail}) = P(HT) + P(TH) = \frac{1}{2}$

$P(\text{at least one tail appears}) = 1 - P(HH) = \frac{3}{4}$

Tree diagram:




Example

A fair die is rolled twice and the numbers are recorded. Our sample space is

$$S = \begin{Bmatrix} 11, & 12, & \ldots, & 16 \\ 21, & 22, & \ldots & \\ \vdots & & & \\ & & \ldots 65, & 66 \end{Bmatrix}$$

so that $|S| = 36$

Every event is equally likely.

A: the first roll is a 5.

B: the largest number shown on either roll of the die is 4.

C: the sum is a prime.

We find

$$A = \{51, 52, 53, \ldots, 56\} \Rightarrow P(A) = \frac{6}{36} = \frac{1}{6}$$

$$B = \{14, 24, 34, 44, 41, 42, 43\} \Rightarrow P(B) = \frac{7}{36}$$

$$C = \{11, 12, 21, 14, 41, 16, 61, 23, 32, 25, 52, 34, 43, 56, 65\} \Rightarrow P(C) = \frac{15}{36} = \frac{5}{12}$$

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## Discrete Sample Space

A sample space is said to be discrete if it is either finite or countably infinite (the elements can be listed one after the other).
If $S$ is countably infinite then there is a one-to-one correspondance between $S$ and the natural numbers $\mathbb{N}$.
Therefore we can sum over all probabilities and get

$$\sum_{x \in S} P(x) = 1$$

### Example

Consider a game involving a fair coin tossed until the first time we throw a head, when the game ends.
A possible sample space is

$$S = \{H, TH, TTH, TTTH, TTTTH, \ldots\}$$

The game ends on the $n$-th throw if and only if $n-1$ tails have been thrown in a row.

$$p_n = \left(\tfrac{1}{2}\right)^{n-1}\left(\tfrac{1}{2}\right) = \left(\tfrac{1}{2}\right)^n$$

Let us compute

$$\sum_{n=1}^{\infty} p_n = \sum_{n=1}^{\infty} \left(\tfrac{1}{2}\right)^n = \frac{\frac{1}{2}}{1-\frac{1}{2}} = 1$$

Conditional Probability

Example

An urn contains 3 black balls and 2 white balls. Two balls are removed (without putting them back).

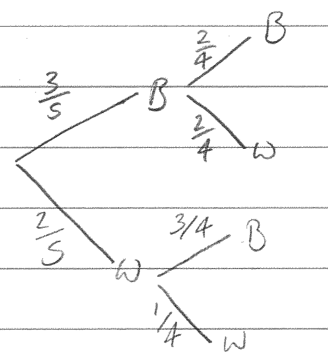
Find the following:

A - the first ball is black

B - the second ball is black

C - the two balls have the same colour

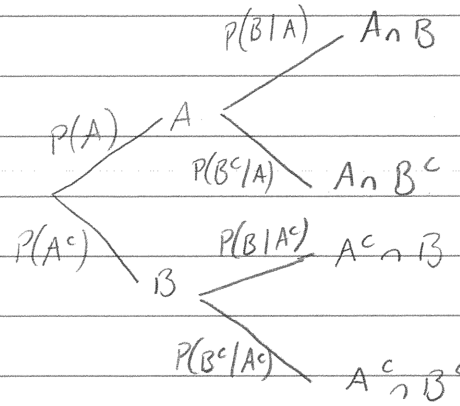
$$P(A) = 3/5$$




$$P(B) = P(BB) + P(WB)$$
$$= \frac{3}{5} \times \frac{2}{4} + \frac{2}{5} \times \frac{3}{4} = \frac{12}{20} = \frac{3}{5}$$

$$P(C) = P(BB) + P(WW)$$
$$= \frac{3}{5} \times \frac{2}{4} + \frac{2}{5} \times \frac{1}{4} = \frac{2}{5}$$

This tree diagram is a special case of a more general one:




We define
$$P(B|A) = \frac{P(A \cap B)}{P(A)}$$
to be the probability that an event happens, given that A has already happened. In the example,

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A was the event that the first ball was black and B was the event that the second ball was black.
We note that

$$B = (B \cap A) \cup (B \cap A^c)$$

and also

$$(B \cap A) \cap (B \cap A^c) = \phi$$

Therefore we can write

$$\begin{aligned} P(B) &= P((B \cap A) \cup (B \cap A^c)) \\ &= P(B \cap A) + P(B \cap A^c) \\ &= P(B|A)P(A) + P(B|A^c)P(A^c) \end{aligned}$$

Therefore we could write

$$* \quad P(A|B) = \frac{P(B \cap A)}{P(B)} = \frac{P(B \cap A)}{P(B|A)P(A) + P(B|A^c)P(A^c)}$$

Counting

Consider a set S with $n$ objects.
When considering ordered samples, repetition is allowed. The number of samples of size $r$ is given by $n^r$.
In an ordered sample, with no repetitions we have $n$ choices for the first, $n-1$ choices for the second, and so on, and $n-(r-1)$ choices for the $r$th. This gives

$$\begin{aligned} n(n-1)(n-2)\dots(n-(r-1)) &= n(n-1)(n-2)\dots(n-r+1) \\ &= \frac{n!}{(n-r)!} \end{aligned}$$

If we take $r=n$ we get $n!$ which is the number of different ways to arrange $n$ elements.

In an unordered sample, with no repetitions we want to determine the number of subsets containing $r$ elements of a set containing $n$ elements. We call this ${}^nC_r$.
Each set contains $r$ elements which can be arranged in $r!$ ways.
Therefore, we have ${}^nC_r\, r!$ ordered samples

$$\Rightarrow \quad {}^nC_r\, r! = \frac{n!}{(n-r)!}$$

$$\text{Hence } {}^nC_r = \frac{n!}{(n-r)!\,r!} = \binom{n}{r}$$

This is the number of combinations of $r$ elements that can be taken from a set of $n$ elements.
We note that $\binom{n}{r}$ is the binomial coefficient which we see in the expansion of $(a+b)^n$ for instance.

Example
Find the total number of subsets of a set of size $n$.

subset of zero : $\binom{n}{0}$ empty subset
subsets of size $r$ : $\binom{n}{r}$
subset of size $n$ : $\binom{n}{n}$

The total number would be

$$\binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \dots + \binom{n}{n} = \sum_{r=0}^{n} \binom{n}{r}$$

$$= \sum_{r=0}^{\infty} \binom{n}{r} 1^r 1^{n-r}$$

$$= (1+1)^n = 2^n$$

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Example

In a group of $r$ people, find the probability that at least two people have the same birthday.

[not nessessarily in the same year, also we neglect the leap year].

The first person can choose $365/365$ days.

The second person can choose $364/365$ days.

$\vdots$

The $r^{th}$ person can choose $\frac{365-(r-1)}{365}$ days.

We can compute the probability of $r$ people having different birthdays by

$$\frac{365}{365} \cdot \frac{364}{365} \cdot \frac{363}{365} \cdot \ldots \cdot \frac{365-r+1}{365}$$

$$= \frac{365!}{(365-r)!(365)^r}$$

So the probability that at least two people have the same birthday is

$$P(r) = 1 - \frac{365!}{(365-r)!(365)^r}$$

$P(22) \approx 47.6\%$

$P(23) \approx 50.7\%$

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## Independence

Two events are independent if

$P(A \cap B) = P(A)P(B)$

which also means $P(B|A) = P(B)$

## Binomial distribution

A Bernoulli trial is a repeated independent experiment or event with only two possible outcomes:

- success with probability $p$
- failure with probability $q = 1-p$

These probabilities must be the same for each trial.

The probability of $r$ successes from $n$ Bernoulli trials is given by the binomial distribution

$$\boxed{b(r) = \binom{n}{r} p^r q^{n-r}}$$

## Example

Letter written in 1693 from Pepys to Newton.

Question: Which of the following propositions has the greatest chance of success?

A: Six fair dice are tossed independently and at least one six appears.

B: Twelve fair dice... and at least two sixes appear.

C: Eighteen fair dice... and at least three sixes appear.

A: P(at least one 6) = 1 - P(no 6 from six dice)

$$= 1 - P(\text{no 6 from one die})^6$$

$$= 1 - \left(\frac{5}{6}\right)^6 \approx 0.665 \approx 66.5\%$$

B: P(at least two 6's from 12 dice) = 1 - {P(no 6's) + P(one 6)}

$$= 1 - \left\{\left(\frac{5}{6}\right)^{12} + \binom{12}{1}\left(\frac{1}{6}\right)\left(\frac{5}{6}\right)^{11}\right\}$$

$$= 1 - \left(\frac{5}{6}\right)^{12} - \binom{12}{1}\left(\frac{1}{6}\right)\left(\frac{5}{6}\right)^{11} \approx 0.619 \approx 61.9\%.$$

C; P(at least three 6's from 18 dice)

= 1 - {P(no 6's) + P(one 6) + P(two 6's)}

$$= 1 - \left(\frac{5}{6}\right)^{18} - \binom{18}{1}\left(\frac{1}{6}\right)\left(\frac{5}{6}\right)^{17} - \binom{18}{2}\left(\frac{1}{6}\right)^2\left(\frac{5}{6}\right)^{16}$$

$$\approx 0.597 \approx 59.7\%$$

Mean Value

Suppose that the possible outcomes of some series of experiments is a number. If these are denoted by $x_i$, then the mean or expectation value is defined by

$$\sum x_i P(x_i) = \mu$$

Example - Fair dice.

$$\mu = 1 \times \frac{1}{6} + 2 \times \frac{1}{6} + \dots + 6 \times \frac{1}{6} = \frac{7}{2}$$

Example - Binomial distribution

$$b(r) = \binom{n}{r} p^r q^{n-r}$$

So the mean value of successes (expectation value of success)

$$\sum_{r=0}^{n} r\, b(r) = \sum_{r=0}^{n} r \binom{n}{r} p^r q^{n-r}$$

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$$= q^n \sum_{r=0}^{n} r \binom{n}{r} \left(\frac{p}{q}\right)^r$$

Recall $\sum_{r=0}^{n} \binom{n}{r} x^r = (1+x)^n$

differentiate with respect to $x$

$$\sum_{r=0}^{n} r \binom{n}{r} x^{r-1} = n(1+x)^{n-1}$$

Multiply by $x$

$$\sum_{r=0}^{n} r \binom{n}{r} x^r = nx(1+x)^{n-1}$$

Therefore we find

$$\sum_{r=0}^{n} r\, b(r) = q^n n \left(\frac{p}{q}\right)\left(1+\frac{p}{q}\right)^{n-1}$$

$$= np(q+p)^{n-1}$$

$$= np \qquad \text{because } p+q=1$$

Poisson distribution

The binomial distribution is very useful but it can involve large factorials which are computationally expensive to compute.

We will define the Poisson distribution to be the limit of the binomial distribution as $n$ becomes very large while keeping the mean $\lambda = np$ constant. We want $n \to \infty$, $p \to 0$ in such a way that $\lambda = np$ does not change.

$$b(r) = \binom{n}{r} p^r q^{n-r}$$

$$= \frac{n!}{r!\,(n-r)!} p^r q^{n-r}$$

$$= \frac{n!}{r!(n-r)!} \left(\frac{\lambda}{n}\right)^{r} \left(1 - \frac{\lambda}{n}\right)^{n-r}$$

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Example

For dice (fair)

$$\mu = 1 \times \frac{1}{6} + 2 \times \frac{1}{6} + \dots + 6 \times \frac{1}{6} = 7/2$$

Example

Binomial distribution

$$b(r) = \binom{n}{r} p^r q^{n-r}$$

So the mean value of successes (expectation) is

$$\sum_{r=0}^{n} r\, b(r) = \sum_{r=0} r \binom{n}{r} p^r q^{n-r}$$

$$= q^n \sum_{r=0}^{n} r \binom{n}{r} \left(\frac{p}{q}\right)^r$$

Recall: $\sum_{r=0}^{n} \binom{n}{r} \lambda^r = (1+\lambda)^n$

$$\sum_{r=0}^{N} r \binom{N}{r} \lambda^{r-1} = N(1+\lambda)^{N-1} \quad \text{(by differentiating w r t } \lambda)$$

$$\sum_{r=0}^{N} r \binom{N}{r} \lambda^r = N\lambda(1+\lambda)^{N-1} \quad \text{(multiplying by } \lambda)$$

Therefore we find:

$$\sum_{r=0}^{N} r\, p(r) = q^N N\left(\frac{p}{q}\right)\left(1 + \frac{p}{q}\right)^{N-1}$$

$$= q^{N-1} Np\left(1 + \frac{p}{q}\right)^{N-1}$$

$$= np(q+p)^{n-1} = np \quad \text{(note: } p+q=1)$$

The Poisson Distribution

The Binomial Distribution is very useful but it can involve large factorials which are computationally expensive to compute. We will define the poisson distribution to be the limit of the Binomial Distribution as $n$ becomes very large, while keeping the mean $\lambda = np$ constant.

We want $n \to \infty$, $p \to 0$ in such a way that $\lambda = np$ does not change

$$b(r) = \binom{n}{r} p^r q^{n-r}$$

$$= \frac{n!}{r!(n-r)!} p^r q^{n-r} = \frac{n!}{r!(n-r)!} \left(\frac{\lambda}{n}\right)^r \left(1 - \frac{\lambda}{n}\right)^{n-r}$$

$$= \frac{\lambda^r}{r!} \frac{n(n-1)\dots(n-r+1)}{n^r} \left(1 - \frac{\lambda}{n}\right)^{-r} \left(1 - \frac{\lambda}{n}\right)^n$$

$$= \frac{\lambda^r}{r!} \Big[ \underbrace{1\left(1 - \frac{1}{n}\right) \dots \left(1 - \left(\frac{r-2}{n}\right)\right)\left(1 - \frac{(r-1)}{n}\right)}_{\text{fixed no. of terms, all of which } \to 1 \text{ as } n \to \infty.} \Big] \left(1 - \frac{\lambda}{n}\right)^{-r} \left(1 - \frac{\lambda}{n}\right)^n$$

<u>Summary</u>

$\left(1 - \frac{\lambda}{n}\right)^{-r} \to 1$ as $n \to \infty$

We only have to be careful with

$$a_n = \left(1 - \frac{\lambda}{n}\right)^n, \quad \lim_{n \to \infty} a_n.$$

To find this limit we start with

$$\log a_n = n \log\left(1 - \frac{\lambda}{n}\right)$$

$$\lim_{n \to \infty} (\log a_n) = \lim_{n \to \infty} \left(n \log\left(1 - \frac{\lambda}{n}\right)\right) = \lim_{n \to \infty} \left[\frac{\log(1 - \lambda/n)}{1/n}\right]$$

<u>L'Hopital's Rule</u>

If $f(\lambda_0) = 0 = g(\lambda_0)$, $\lim_{\lambda \to \lambda_0} \left(\frac{f(\lambda)}{g(\lambda)}\right) = \lim_{\lambda \to \lambda_0} \left(\frac{f'(\lambda)}{g'(\lambda)}\right)$

By L'Hopital's rule: $\lim_{n \to \infty} \left[\frac{\left(\frac{\lambda/n^2}{1 - \lambda/n}\right)}{-1/n^2}\right] = \lim_{n \to \infty} \left(\frac{-\lambda}{1 - \lambda/n}\right) = -\lambda$

Therefore: $\lim_{n \to \infty} \left[n \log\left(1 - \frac{\lambda}{n}\right)\right] = \lim_{n \to \infty} \left[\log\left(1 - \frac{\lambda}{n}\right)^n\right] = -\lambda$

$$\Rightarrow \lim_{n \to \infty} \left(1 - \frac{\lambda}{n}\right)^n = e^{-\lambda}$$

Finally we obtain the Poisson Distribution as the large $n$ limit of the Binomial Distribution.

$$P(r) = \lim_{n \to \infty} (b(r)) = \frac{\lambda^r}{r!} e^{-\lambda}.$$

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Example

An insurance company pays £500k to each client who has a fire. The company has 5000 clients, and the probability of a fire is $10^{-4}$ per year.
Find the probability of the company paying out £2m in a single year.

We have that $p = 10^{-4}$ is small and $n = 5000$ is large, $\lambda = np = \frac{1}{2}$.
[We assume: no client has two fires in the same year]

£2M corresponds to 4 fires in this year.
Hence we compute the probability of 0, 1, 2, 3 fires and have

$$P(r \geq 4) = 1 - \{P(0) + P(1) + P(2) + P(3)\}$$

$$= 1 - e^{-\frac{1}{2}}\left\{1 + \frac{(1/2)}{1!} + \frac{(1/2)^2}{2!} + \frac{(1/2)^3}{3!}\right\}$$

$$\approx 0.00175 \approx 0.2\%$$

Example

An office recieves three calls per hour on average.
Find: (a) No calls are recieved in a given hour.
(b) Exactly three calls are recieved in a given hour.

(a) : $P(0) = e^{-3}$

(b) : $P(3) = \frac{\lambda^3}{3!}e^{-\lambda} = \frac{3^3}{3!}e^{-3} \approx 22\%$

We already know the mean is

$$\mu = \sum_i x_i P(x_i)$$

We are often interested in how much the outcome differs from the mean.
One measure of this is the variance

$$\sigma^2 = \sum_i (x_i - \mu_i)^2 P(x_i)$$

and the standard deviation

$$\sigma = \sqrt{\sigma^2}$$

<u>Continuous probability distributions.</u>
So far we have considered discrete sample spaces.
Now we want to define probabilities over $\mathbb{R}$
We define the probability using a probability density function $f(x)$ s.t.

$$P(a < x \leq b) = \int_a^b f(\tilde{x})\, d\tilde{x}$$

We need to impose conditions on $f(x)$ so that $P$ is a probability consistant with our previous definition.

(i) $f(x) \geq 0 \quad \forall x$

(ii) $\int_{-\infty}^{+\infty} f(x)\, dx = 1$ (normalisation condition).

We can also define the mean by

$$\mu = \int_{-\infty}^{+\infty} x f(x)\, dx$$

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and the variance by

$$\sigma^2 = \int_{-\infty}^{+\infty} (x-\mu)^2 f(x)\,dx$$

$$= \int_{-\infty}^{+\infty} (x^2 - 2x\mu + \mu^2) f(x)\,dx$$

$$= \int_{-\infty}^{+\infty} x^2 f(x)\,dx - 2\mu \int_{\infty}^{+\infty} x f(x)\,dx + \mu^2 \int_{-\infty}^{+\infty} f(x)\,dx$$

$$= \int_{-\infty}^{+\infty} x^2 f(x)\,dx - 2\mu^2 + \mu^2$$

$$= \int_{-\infty}^{+\infty} x^2 f(x)\,dx - \mu^2$$

We call $\sigma$ the standard deviation.

<u>Example</u>

The probability density function describing the location of a particle is

$$f(x) = \begin{cases} c(x - x^3), & 0 < x < 1 \\ 0, & \text{otherwise.} \end{cases}$$

(i) We fix the constant, $c$, using the normalisation condition.

$$1 = \int_{-\infty}^{+\infty} f(x)\,dx$$

$$= c\int_0^1 x - x^3\,dx$$

$$= c\left[\tfrac{1}{2}x^2 - \tfrac{1}{4}x^4\right]_0^1 = \tfrac{1}{4}c \Rightarrow c = 4$$

(ii) Let us compute the mean:

$$\mu = \int_{-\infty}^{+\infty} x f(x)\,dx$$

$$\Rightarrow \mu = \int_0^1 4x(x - x^3)\,dx$$

$$= 4\int_0^1 x^2 - x^4 \,dx$$

$$= 4\left[\frac{1}{3}x^3 - \frac{1}{5}x^5\right]_0^1$$

$$= 4\left(\frac{1}{3} - \frac{1}{5}\right) = \frac{8}{15}$$

(iii) The standard deviation is $\sigma$.

$$\sigma^2 = \int_{-\infty}^{+\infty} x^2 f(x)\,dx - \mu^2$$

$$= 4\int_0^1 x^3 - x^5 \,dx - \mu^2$$

$$= 4\left[\frac{1}{4}x^4 - \frac{1}{6}x^6\right]_0^1 - \mu^2$$

$$= \frac{1}{3} - \left(\frac{8}{15}\right)^2$$

$$= \frac{11}{225}$$

(iv) What is the probability of the particle being in the interval $(0, \frac{1}{2})$?

$$P(0 < x < \tfrac{1}{2}) = \int_0^{\frac{1}{2}} 4(\tilde{x} - \tilde{x}^3)\,d\tilde{x}$$

$$= 4\left[\frac{1}{2}\tilde{x}^2 - \frac{1}{4}\tilde{x}^4\right]_0^{\frac{1}{2}}$$

$$= 2\left[\left(\frac{1}{2}\right)^2 - \frac{1}{2}\left(\frac{1}{2}\right)^4\right]$$

$$= \frac{1}{2} - \left(\frac{1}{2}\right)^4 = \frac{1}{2} - \frac{1}{16} = \frac{7}{16}$$

# Exam

Vectors - 2 questions
Complex numbers
Taylor series 1 or 2 questions
Integration
1st. ODE 1 or 2 questions
2nd. ODE
Probability - 1 question