1401 Mathematical Methods 1 Notes

Based on the 2016 autumn lectures by Prof R Halburd

The Author(s) has made every effort to copy down all the content on the board during lectures. The Author(s) accepts no responsibility whatsoever for mistakes on the notes nor changes to the syllabus for the current year. The Author(s) highly recommends that the reader attends all lectures, making their own notes and to use this document as a reference only.

Wed. 28/09/16

MATH1401 Calculus (Group 1)

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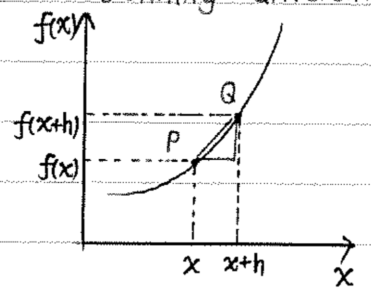
CHAPTER 0. Differential Calculus

Recall that differentiation is about rates of change : speeds, acceleration,et

With distance-time graphs, question about rate of change become question about gradient.

⇨ 0·1 Defining differentiation



gradient of the curve at P

$\approx$ gradient of PQ

$\approx \frac{\Delta y}{\Delta x}$

$\approx \frac{f(x+h)-f(x)}{h}$

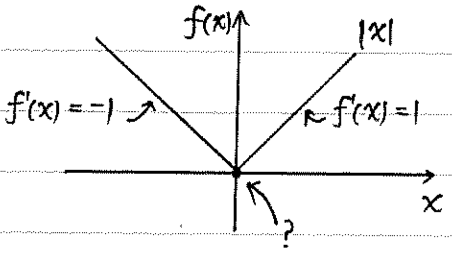
Rigorous Treatment in 1101

∴ If h is small enough,

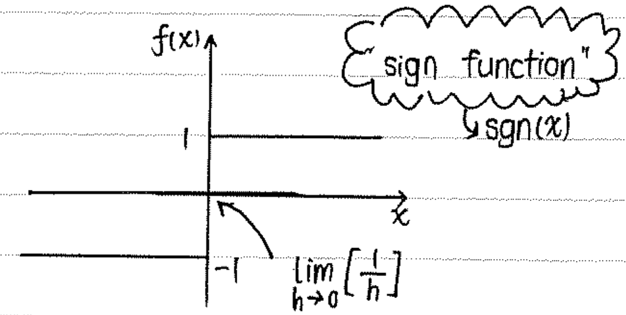
$$\text{gradient at } P = \frac{df}{dx} = \lim_{h\to 0}\left[\frac{f(x+h)-f(x)}{h}\right]$$

so long as the limit exists independently of how $h \to 0$.

i.e. not if

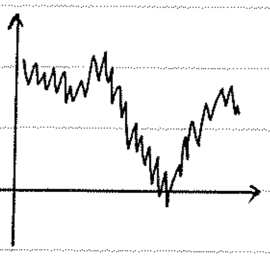


Not smooth



NOT continuous

Differentiable everywhere ⇒ Continuous everywhere



[MATH7102]

Weierstrass equation: continuous everywhere BUT differentiable nowhere!

Use for approximation

If we don't go (all the way) $h \to 0$, then

$$f'(x) \approx \frac{f(x+h)-f(x)}{h}$$

$\Rightarrow$ $\boxed{f(x+h) \approx f(x) + hf'(x)}$ so long as h is small

EXAMPLE: Find $\sqrt[3]{25}$.

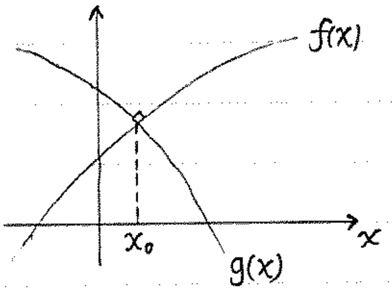
so $f(x) = \sqrt[3]{x}$ and we know $\sqrt[3]{27} = 3$

If $x = 27$, $h = -2$

$\Rightarrow f(25) \approx f(27) + (-2).f'(27) \approx \frac{79}{27}$ (≈ 2.926)

Geometric Application

If you have two curves crossing at right angles,



then locally their gradient are normal to each other, i.e. they're negative reciprocals ← "one over"

$f'(x_0).g'(x_0) = -1$

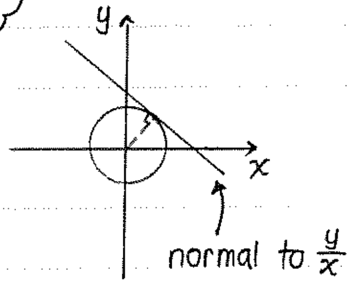
$\Rightarrow$ 0.2 Implicit Differentiation

Sometimes you can't write y as a function of x directly, but you can still differentiate.

EXAMPLE ①: $x^2 + y^2 = r^2$ ← circle

Differentiate: $2x + 2y\frac{dy}{dx} = 0$ ← Chain Rule: $\frac{df}{dx} = \frac{df}{dy}.\frac{dy}{dx}$

$\Rightarrow \frac{dy}{dx} = -\frac{x}{y}$



EXAMPLE ②: $y = x^x$

Natural Log → $\ln y = \ln(x^x) = x\ln x$

Differentiate: $\frac{1}{y}\frac{dy}{dx} = \ln x + x.\frac{1}{x}$

$= \ln x + 1$

$\Rightarrow \frac{dy}{dx} = y\ln x + y$

$\therefore \frac{dy}{dx} = x^x(\ln x + 1)$

⇒ 0·3 Curve Sketching

Rather than being precise, curve sketching is about capturing the essential features of a graph.

Consider
1) odd / even / neither
2.) poles / singularities or crossing axes
3.) behaviour : → +∞, -∞, poles

LAST! ← 4.) stationary points ("extremal") ⇒ $f'(x) = 0$
5.) points of inflection ⇒ $f''(x) = 0$

Recall: odd : $f(-x) = -f(x)$

180° symmetry in origin

even: $f(-x) = f(x)$

reflection in y-axis

"for all functions"

"Most" functions are neither, but ∀F we have

$$f(x) = \frac{1}{2}\underbrace{\left[f(x)+f(-x)\right]}_{\text{even}} + \frac{1}{2}\underbrace{\left[f(x)-f(-x)\right]}_{\text{odd}}$$

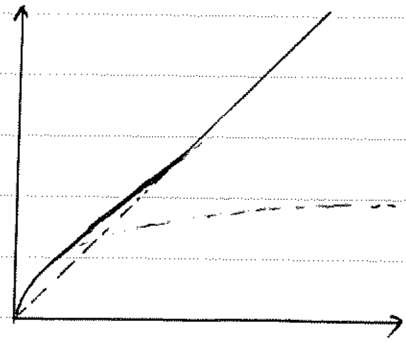
Note: odd × odd = even function

even × even = even function

odd × even = odd function

more like $1 \times (-1)$

EXAMPLE ①: Sketch $y = \sqrt{x^2 + x^{\frac{1}{4}}}$ for $x \geqslant 0$, $y \geqslant 0$



1.) No symmetries

2.) No pole

3.) $x \to +\infty$: $x^2 >> x^{\frac{1}{4}} \Rightarrow y \to \sqrt{x^2} = x$

$x \to -\infty$: $x^{\frac{1}{4}} >> x^2 \Rightarrow y \to \sqrt{x^{\frac{1}{4}}} = x^{\frac{1}{8}}$

4.) $\frac{dy}{dx} = \frac{1}{2}(x^2 + x^{\frac{1}{4}})^{-\frac{1}{2}}(2x + \frac{1}{4}x^{-\frac{3}{4}}) = 0$

$$2x = -\frac{1}{4}x^{-\frac{3}{4}}$$

$$x^{\frac{7}{4}} = -\frac{1}{8}$$

∴ no solution

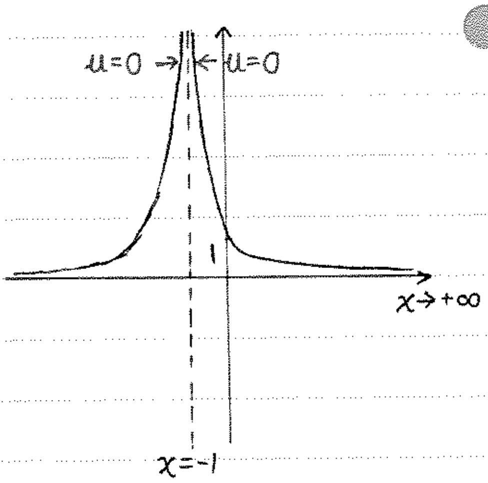
∴ no stationary point ⇒ It just joins up.

EXAMPLE ②: Sketch $y = \frac{\sqrt{x^2+1}}{(x+1)^2}$

1.) No symmetry

2.) poles at $x=-1$

$y(x) \neq 0$, $y(0)=1$



3.) $x \rightarrow +\infty$: $\sqrt{x^2+1} \rightarrow \sqrt{x^2} = |x|$

$$(x+1)^2 \rightarrow x^2$$

$$\therefore y \rightarrow \frac{|x|}{x^2} = \left|\frac{1}{x}\right|$$

$x \rightarrow -\infty$: $\sqrt{x^2+1} \rightarrow \sqrt{x^2} \rightarrow |x|$

$$(x+1)^2 \rightarrow x^2$$

$$\therefore y \rightarrow \frac{|x|}{x^2} = \left|\frac{1}{x}\right|$$

$x \rightarrow -1$: Let $x=-1+u$, what happens as $u \rightarrow 0$?

$$y = \frac{\sqrt{(-1+u)^2+1}}{u^2} \rightarrow \frac{\sqrt{2}}{u^2} \rightarrow +\infty$$ from both sides

4.) $$\frac{dy}{dx} = \frac{\frac{1}{2}(x^2+1)^{-\frac{1}{2}}.2x(x+1)^2 - (x^2+1)^{\frac{1}{2}}.2(x+1).1}{(x+1)^4}$$

$$= \frac{x(x^2+1)^{-\frac{1}{2}}(x+1)^2 - 2(x^2+1)^{\frac{1}{2}}(x+1)}{(x+1)^4}$$

$$= \frac{x(x^2+1)^{-\frac{1}{2}}(x+1) - 2(x^2+1)^{\frac{1}{2}}}{(x+1)^3} = 0$$

$$\therefore (x^2+1)^{-\frac{1}{2}}\left[x(x+1) - 2(x^2+1)\right] = 0$$

$$x(x+1) = 2(x^2+1)$$

$$x^2 - x + 2 = 0$$

No real roots

∴ No stationary points. ⇒ It just joins up.

Thurs. 29/09/16

⇨ 0·4 Elementary Functions

① Exponential Function

{exp(x)}

- 1101 Definition: $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$
$$= \sum_{n=1}^{\infty} \frac{x^n}{n!}$$
Note: $0! = 1$

The sum always converges.

- This gives us $e^x e^y = e^{x+y}$

$$\frac{d}{dx}(e^x) = e^x$$

$$(e^x)^y = e^{xy}$$

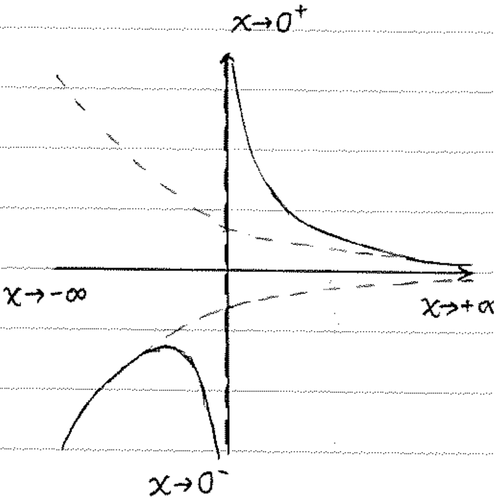
$$\int e^x dx = e^x + c$$

- Exponential growth / decay is faster than algebraic powers.

$\lim_{x\to 0}\left(\frac{x^n}{e^{ax}}\right)\to 0 \quad \forall\, n, a \geq 0$

$\lim_{x\to 0}\left(\frac{e^{ax}}{x^n}\right)\to +\infty \quad \forall\, n, a \geqslant 0$

EXAMPLE : Sketch $y = \frac{e^{2x}+e^{-x}}{e^{3x}-1}$



1.) no symmetry

2.) pole is at $x=0$

$y(x)\neq 0$

3.) $x\to+\infty$: $e^{2x}+e^{-x}\to e^{2x}$

$e^{3x}-1 \to e^{3x}$

$\therefore\ y\to\frac{e^{2x}}{e^{3x}} = e^{-x}$

$x\to-\infty$: $e^{2x}+e^{-x}\to e^{-x}$

$e^{3x}-1 \to -1$

$\therefore\ y\to -e^{-x}$

$x\to 0$: let $x = 0+u$

[Use series expansion !]

$y=\frac{1+2u+1-u+\cdots}{1+3u-1+\cdots} \to \frac{2+u}{3u} = \frac{2}{3u}+\frac{1}{3}$

As $u\to 0^+$: $y\to+\infty$ "as u approaches 0 from the positives"

As $u\to 0^-$: $y\to-\infty$

4.) $\frac{dy}{dx} = \frac{(2e^{2x}-e^{-x})(e^{3x}-1)-(e^{2x}+e^{-x})(3e^{3x})}{(e^{3x}-1)^2} = 0$

stationary point at $x=\frac{1}{3}\ln(-3\pm\sqrt{10})$ (reject $\frac{1}{3}\ln(-3-\sqrt{10})$)

So one stationary point found.

And we connect the curve.

- Inverse of e^x

$\ln(x)$ or $\log(x)$

Satisfy

$\log(xy) = \log x + \log y$

$\log(x^y) = y\log x$

- Find the derivative of $\log x$ (and any inverse function) by implicit differentiation.

$y=\log x \quad\Rightarrow\quad x = e^y$

$$1 = e^y \frac{dy}{dx}$$

$$\frac{dy}{dx} = \frac{1}{e^y} = \frac{1}{x}$$

This means $\int \frac{1}{x}\,dx = \log x + c$

$$\int_1^x \frac{1}{u}\,du = \log x$$

- Logarithmic growth / decay is slower than algebraic.

EXAMPLE: Find derivative of $y = \arcsin x$. ← $\sin^{-1}x$

$$\sin y = x$$

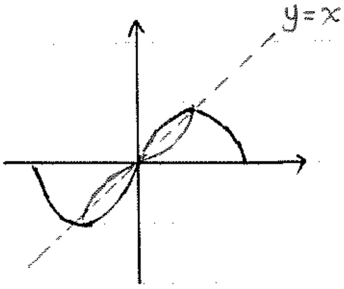
$$\cos y \cdot \frac{dy}{dx} = 1$$

$$\frac{dy}{dx} = \frac{1}{\cos y}$$

$$\therefore \frac{dy}{dx} = \frac{1}{\sqrt{1-\sin^2 y}} = \frac{1}{\sqrt{1-x^2}}$$

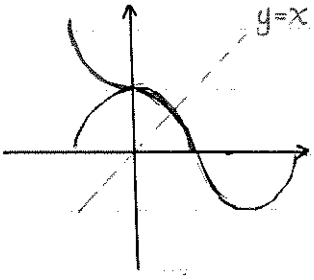
$\sin^2 x + \cos^2 x = 1$

- Inverse Trig Functions



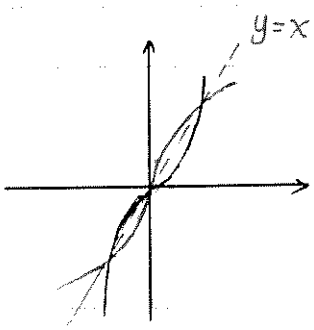
sinx

arcsinx



cosx

arccosx



tanx

arctanx

⇒ 0.5 Leibniz Rule for Differentiation 莱布尼兹定则

- This is the product rule for differentiating n times.

$y = uv$

then $y' = u'v + uv'$

then $y'' = u''v + u'v' + u'v' + uv''$

$= u''v + 2u'v' + uv''$

then $y''' = u'''v + 3u''v' + 3u'v'' + uv'''$

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1

Pascal's Triangle

- Indeed,

$$\underbrace{(uv)^{(n)}}_{n^{th}\text{ derivative}} = \sum_{r=0}^{n} \overbrace{\binom{n}{r}}^{n \text{ choose } r} u^{(n-r)} v^{(r)}$$

- EXAMPLE: Find the n^{th} derivative of $x^3 e^{-x}$

$$\frac{d^n}{dx^n}[\underset{\text{choose to be v so it ends}}{x^3} e^{-x}]$$

⇨ 0·6 Hyperbolic Functions 双曲函数

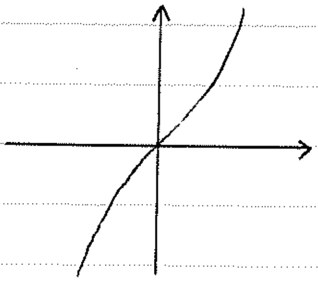
- These are the odd and even part of e^x.

$$\sinh x = \frac{e^x - e^{-x}}{2} \qquad \cosh x = \frac{e^x + e^{-x}}{2}$$

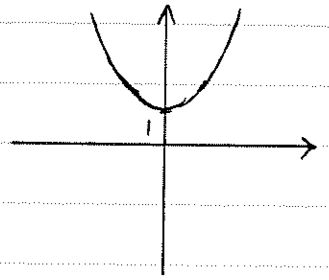
- We can also define other hyperbolic trig functions.

$$\tanh x = \frac{e^x - e^{-x}}{e^x + e^{-x}} = \frac{e^{2x} - 1}{e^{2x} + 1} \qquad \coth x = \frac{1}{\tanh x}$$

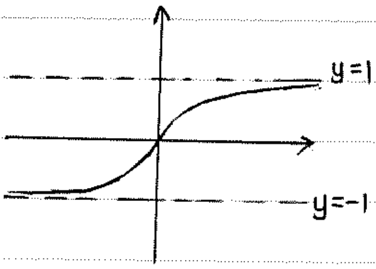
$$\operatorname{sech} x = \frac{1}{\cosh x} \qquad \operatorname{cosech} x = \frac{1}{\sinh x}$$



sinhx



coshx



tanhx

- Behaviour near 0:

$$\sinh x = \frac{e^x - e^{-x}}{2} = \frac{1 + x + \frac{x^2}{2!} - (1 - x + \frac{x^2}{2!} - \cdots)}{2}$$

series form:

$$\sinh x = x + \frac{x^3}{3!} + \cdots \to x \quad \text{when} \quad x \to 0$$

Similarly, $\left.\begin{array}{l}\cosh x \to 1 \\ \tanh x \to x\end{array}\right\}$ for small x

- Behaviour as $x \to \pm\infty$: $\sinh x \to \frac{1}{2} e^{|x|} \operatorname{sgn}(x)$

$\cosh x \to \frac{1}{2} e^{|x|}$

$\tanh x \to \operatorname{sgn}(x)$

- Hyperbolic identities are easier to prove than trig identities but similar.

EXAMPLE: $\cosh^2 x - \sinh^2 x = 1$

$$\begin{aligned} \text{LHS} &= \left(\frac{e^x + e^{-x}}{2}\right)^2 - \left(\frac{e^x - e^{-x}}{2}\right)^2 \\ &= \frac{1}{4}[e^{2x} + 2 + e^{-2x} - e^{2x} + 2 - e^{-2x}] \\ &= \frac{1}{4} . 4 \\ &= 1 = \text{RHS} \end{aligned}$$

$\therefore$ true $\forall\, x \in \mathbb{R}$

Other trig. identities exist of course.

$$\cosh^2 x + \sinh^2 x = \cosh 2x$$

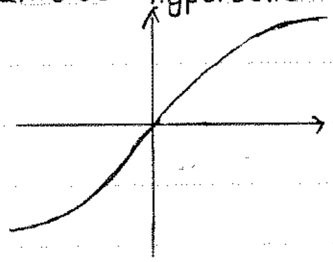
- We can differentiate them:

$\frac{d}{dx}[\sinh x] = \cosh x$ $\frac{d}{dx}[\cosh x] = \sinh x$ $\frac{d}{dx}[\tanh x] = \text{sech}^2 x$

$\frac{d}{dx}[\text{cosech} x] = -\text{cosech} x \coth x$ $\frac{d}{dx}[\text{sech} x] = -\text{sech} x \tanh x$ $\frac{d}{dx}[\coth x] = -\text{cosech}^2 x$

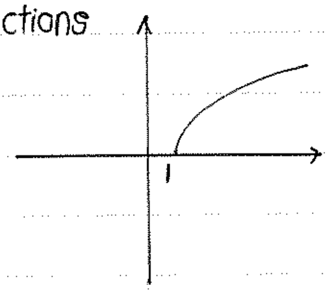
Note: $\int \tanh x \, dx = \ln|\cosh x|$

- Inverse hyperbolic functions



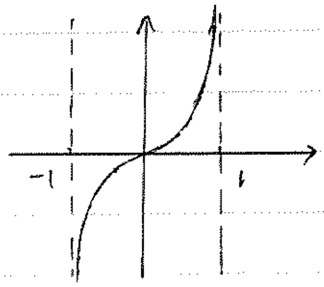
arsinh$x$

$\sinh^{-1} x$



arcosh$x$

$\cosh^{-1} x$



artanh$x$

$\tanh^{-1} x$

(NOT one-to-one function)

$y = \cosh^{-1} x$

$\cosh y = x$

$\frac{e^y + e^{-y}}{2} = x$

$e^{2y} - 2xe^y + 1 = 0$ Quadratic Equation

$e^y = \frac{2x \pm \sqrt{4x^2-4}}{2} = x \pm \sqrt{x^2-1}$

$y = \ln(x \pm \sqrt{x^2-1})$

So we say $\cosh^{-1} x = \ln(x + \sqrt{x^2-1})$

Similarly, $\text{arsinh} x = \ln(x + \sqrt{x^2-1})$

$\text{artanh} x = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right)$

Mon. 03/10/16

Mathematical Methods 1: MATH1401

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Office Hour: Monday at 11am

Assessment
- Calculus Test 5% (in Jan.)
- Problem Sheets [best (n-1) out of n, n ≈ 8 or 9]
  - 10% from problem sheets + vector test (week 5/6) + mid-sessional exam
- Final Exam 85%

Syllabus :
- Vectors
- Complex Numbers
- Taylor Series
- Integration
- Differential Equations
- Probability

⇒ 1 § Vectors §

1·1 Introduction

- A scalar is a quantity, i.e. represented by a single (real) number
eg. temperature, speed, distance

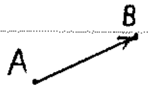
- A vector has magnitude (i.e. length / size) and direction
eg. force, velocity, displacement

- We will represent any vector in $\mathbb{R}^3$ (Euclidean 3-space, i.e. 3-dimensional space) as an arrow, with length represents its magnitude.

1·2 Notation

- Notations:

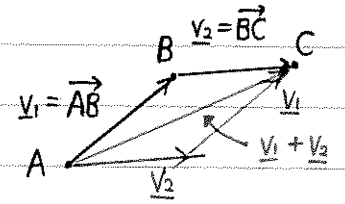
(boldface letter)



$\overrightarrow{AB}$

1·3 Addition & Multiplication by a Scalar

- Consider a displacement from A to B, and from B to C.

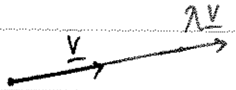


Parallelogram Rule

- Vector addition is commutative $\underline{v} + \underline{u} = \underline{u} + \underline{v}$
associative $(\underline{u} + \underline{v}) + \underline{w} = \underline{u} + (\underline{v} + \underline{w})$

- We denote zero vector by $\underline{0}$
(i.e. the vector with 0 length) $\underline{v} + \underline{0} = \underline{0} + \underline{v} = \underline{v}$

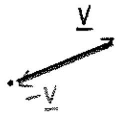
- Multiplication by a scalar :
- Let $\underline{v}$ be a vector and let λ be a non-negative scalar.
We define the vector $\lambda\underline{v}$ to be the vector pointing in the same direction as $\underline{v}$ with length $\lambda|\underline{v}|$, where $|\underline{v}|$ is the length of $\underline{v}$.



- If λ is less than 0, then $\lambda\underline{v}$ points in the opposite direction to $\underline{v}$ and has length $|\lambda||\underline{v}|$.

Let "$-\underline{v}$" denote the unique vector s.t. ← "such that"

$\underline{v} + (-\underline{v}) = \underline{0}$



$-\underline{v} = (-1).\underline{v}$

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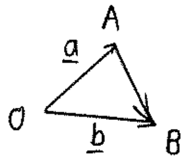
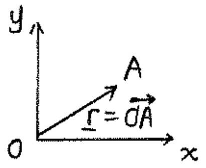


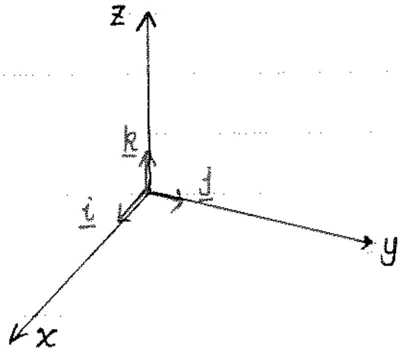
$\vec{AB} = \vec{OB} - \vec{OA}$
$= (-\underline{a}) + \underline{b} = \underline{b} - \underline{a}$



- Position vector of the point A is the vector $\underline{r} = \vec{OA}$

1.4 Cartesian Coordinates

- Any vector $\underline{v}$ can be thought of as the position vector of some points $P = (a, b, c)$: $\underline{v} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$



Right-handed Coordinate System

- A unit vector is a vector of length 1.
- Let $\underline{i}, \underline{j}, \underline{k}$ (or $\underline{e}_1, \underline{e}_2, \underline{e}_3$) be the unit vectors pointing in the direction of increasing x, y and z respectively.

$\underline{i} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \quad \underline{j} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \quad \underline{k} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

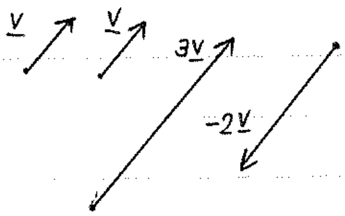
- We essentially have $\underline{i}, \underline{j}, \underline{k}$ all perpendicular to each other.

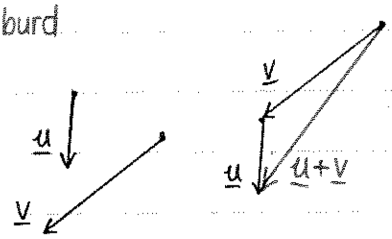
Fri. 07/10/16

MATH1401

Prof. Rod Halburd

Recap:



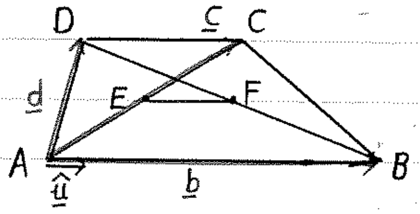


For the length of vector :

$\hat{\underline{v}} = \frac{\underline{v}}{|\underline{v}|} = \left|\frac{\underline{v}}{|\underline{v}|}\right| = \frac{1}{|\underline{v}|}|\underline{v}| = 1$ unit vector of $\underline{v}$ (in the direction of $\underline{v}$)

$$\underline{v} = |\underline{v}| . \frac{\underline{v}}{|\underline{v}|} = \underbrace{|\underline{v}|}_{\text{length}} . \underbrace{\hat{\underline{v}}}_{\text{direction}}$$

EXAMPLE: Given a trapezium ABCD with AB//CD. Show that the line joining the mid-points of the diagonals (EF) is parallel to AB & CD. And show that the length of EF = half the difference of length |AB| & |CD|



Solve: Take the origin at A.

Let $\underline{b} = \vec{AB}$, $\underline{c} = \vec{AC}$, $\underline{d} = \vec{AD}$

$\vec{DC} = \underline{c} - \underline{d}$ is parallel to $\underline{b}$

$\vec{AE} = \frac{1}{2}\underline{c}$ $\quad \vec{FB} = \frac{1}{2}\vec{DB} = \frac{1}{2}(\underline{b} - \underline{d})$

Note that: $\vec{AE} + \vec{EF} + \vec{FB} = \underline{b}$

$$\frac{1}{2}\underline{c} + \vec{EF} + \frac{1}{2}(\underline{b} - \underline{d}) = \underline{b}$$

$$\vec{EF} = \frac{1}{2}[\underline{b} - (\underline{c} - \underline{d})] = \frac{1}{2}\underline{b} - \frac{1}{2}(\underline{c} - \underline{d})$$

$\therefore$ $\vec{EF}$ points in the same direction as $\vec{AB}$ & $\vec{DC}$

Let $\hat{\underline{u}} = \frac{\underline{b}}{|\underline{b}|}$ be the unit vector in direction of $\underline{b}$ & $\vec{DC}$

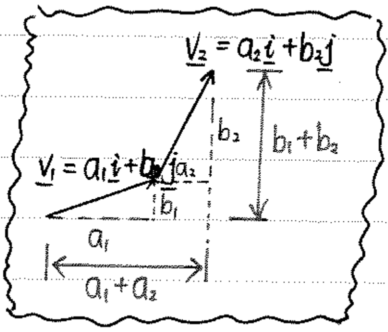
$\vec{EF} = \frac{1}{2}|\underline{b}|\hat{\underline{u}} - \frac{1}{2}|\underline{c} - \underline{d}|\hat{\underline{u}}$

$|\vec{EF}| . \hat{\underline{u}} = \hat{\underline{u}} \left(\frac{1}{2}|\underline{b}| - \frac{1}{2}|\underline{c} - \underline{d}|\right)$

$\therefore |\vec{EF}| = \frac{1}{2}|\underline{b}| - \frac{1}{2}|\underline{c} - \underline{d}|$

$= \frac{1}{2}\left(|\vec{AB}| - |\vec{CD}|\right)$

- $\underline{v} = \begin{pmatrix} a \\ b \\ c \end{pmatrix} = a\underline{i} + b\underline{j} + c\underline{k}$



$\underline{i} = \hat{\underline{i}} = \underline{e}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$

$\underline{j} = \hat{\underline{j}} = \underline{e}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$

$\underline{k} = \hat{\underline{k}} = \underline{e}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

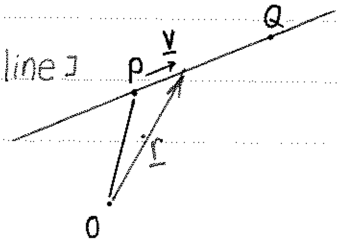
$\lambda\underline{v} = (\lambda a)\underline{i} + (\lambda b)\underline{j} + (\lambda c)\underline{k}$

1.5 Equation of a straight line

- A line through the point P in the direction $\underline{v}$ is given by

$$\underline{r} = \vec{OP} + t\underline{v} \quad \text{[Parametric Equation of a line]}$$

each value of t gives a point on the line



EXAMPLE: Find a parametric equation for the line through (1,2,1) & (0,1,-1)

Solve: Let $P(1,2,1)$, $Q(0,1,-1)$

$$\underline{v} = \overrightarrow{PQ} = \begin{pmatrix}0\\1\\-1\end{pmatrix} - \begin{pmatrix}1\\2\\1\end{pmatrix} = \begin{pmatrix}-1\\-1\\-2\end{pmatrix}$$

<u>OR</u>: $(0+\underline{j}-\underline{k}) - (\underline{i}+2\underline{j}+\underline{k}) = -\underline{i}-\underline{j}-2\underline{k}$

$$\begin{aligned} r(t) &= \overrightarrow{OP} + t\underline{v} \\ &= \begin{pmatrix}1\\2\\1\end{pmatrix} + t\begin{pmatrix}-1\\-1\\-2\end{pmatrix} \\ &= \begin{pmatrix}1-t\\2-t\\1-2t\end{pmatrix} \end{aligned}$$

$$\boxed{r(t) = \begin{pmatrix}x(t)\\y(t)\\z(t)\end{pmatrix} = x(t)\underline{i} + y(t)\underline{j} + z(t)\underline{k}}$$

$$\Rightarrow \left\{\begin{aligned} x &= 1-t \\ y &= 2-t \\ z &= 1-2t \end{aligned}\right] \rightarrow$$ Eliminate t to get a non-parametric form

- Non-parametric Equation

$$(-t) = \quad x-1 = y-2 = \frac{z-1}{2}$$ [Non-parametric Equation]

- Mid-point / Point on lines

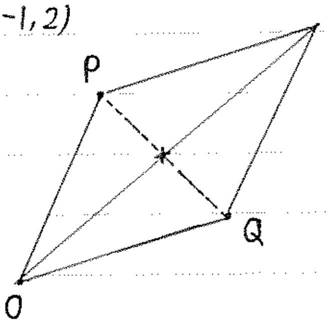
EXAMPLE ①: Find the mid-point of $P(1,2,1)$ & $Q(3,-1,2)$

$\underline{r} = \frac{1}{2}(\overrightarrow{OQ} + \overrightarrow{OP})$ using parallelogram rule

<u>OR</u>

$$\begin{aligned} \underline{r} &= \overrightarrow{OP} + \tfrac{1}{2}\overrightarrow{PQ} \\ &= \overrightarrow{OP} + \tfrac{1}{2}(\overrightarrow{OQ} - \overrightarrow{OP}) \\ &= \tfrac{1}{2}(\overrightarrow{OP} + \overrightarrow{OQ}) \end{aligned}$$

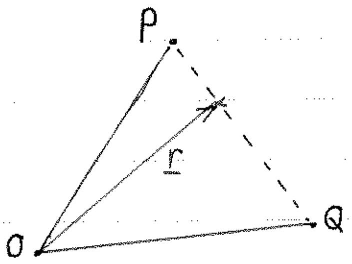
$$\underline{r} = \frac{1}{2}\left(\begin{pmatrix}1\\2\\1\end{pmatrix} + \begin{pmatrix}3\\-1\\2\end{pmatrix}\right) = \begin{pmatrix}3\\1/2\\3/2\end{pmatrix}$$



$\Rightarrow$ mid-point: $(2, \frac{1}{2}, \frac{3}{2})$

EXAMPLE ②: Find the point 1/3 the distance from $P(1,2,2)$ to $Q(-2,1,3)$

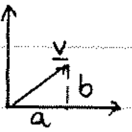
$$\begin{aligned} \underline{r} &= \overrightarrow{OP} + \tfrac{1}{3}\overrightarrow{PQ} \\ &= \overrightarrow{OP} + \tfrac{1}{3}(\overrightarrow{OQ} - \overrightarrow{OP}) \\ &= \begin{pmatrix}1\\2\\2\end{pmatrix} + \tfrac{1}{3}\left(\begin{pmatrix}-2\\1\\3\end{pmatrix} - \begin{pmatrix}1\\2\\2\end{pmatrix}\right) \\ &= \begin{pmatrix}0\\5/3\\7/3\end{pmatrix} \Rightarrow \text{point is } (0, \tfrac{5}{3}, \tfrac{7}{3}) \end{aligned}$$



1·6·1 Length of a vector

$$\underline{v} = a\underline{i} + b\underline{j}$$

$$|\underline{v}| = \sqrt{a^2+b^2}$$



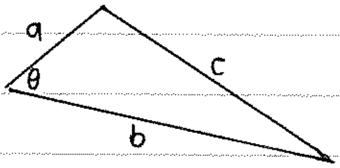
Similarly, $\underline{v} = a\underline{i} + b\underline{j} + c\underline{k}$

$$|\underline{v}| = \sqrt{a^2+b^2+c^2}$$

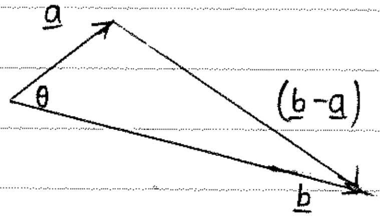
1·6·2 Angles between two vectors

- Law of cosine:

$$c^2 = a^2 + b^2 - 2ab\cos\theta$$



- In terms of vectors,



$$|\underline{b}-\underline{a}|^2 = |\underline{a}|^2 + |\underline{b}|^2 - 2|\underline{a}||\underline{b}|\cos\theta$$

where $\underline{a} = a_1\underline{i} + a_2\underline{j} + a_3\underline{k}$

$\underline{b} = b_1\underline{i} + b_2\underline{j} + b_3\underline{k}$

$\therefore\ \underline{b} - \underline{a} = (b_1-a_1)\underline{i} + (b_2-a_2)\underline{j} + (b_3-a_3)\underline{k}$

sub: $\left(\sqrt{(b_1-a_1)^2+(b_2-a_2)^2+(b_3-a_3)^2}\right)^2 = \left(\sqrt{a_1^2+a_2^2+a_3^2}\right)^2 + \left(\sqrt{b_1^2+b_2^2+b_3^2}\right)^2 - 2|a||b|\cos\theta$

$$-2a_1b_1 - 2a_2b_2 - 2a_3b_3 = -2|a||b|\cos\theta$$

$$\boxed{a_1b_1 + a_2b_2 + a_3b_3 = |a||b|\cos\theta}$$

Mon. 10/10/16

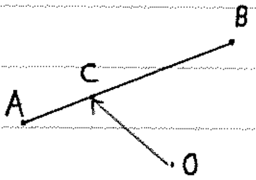
MATH1401: Mathematical Methods 1

Prof. Rod Halburd

EXAMPLE: Let C be a point on the line segment AB

If C divides AB in the ratio $\alpha:\beta$, then

$\vec{AC} = \frac{\alpha}{\beta}\vec{CB} = \mu\vec{CB}$



Let $\underline{a} = \vec{OA}$, $\underline{b} = \vec{OB}$, $\underline{c} = \vec{OC}$

$\therefore\ \underline{c} = \vec{OA} + \vec{AC} = \underline{a} + \mu\vec{CB} = \underline{a} + \mu(\underline{b}-\underline{c})$

$\therefore\ (1+\mu)\underline{c} = \underline{a} + \mu\underline{b}$

$\Rightarrow\ \underline{c} = \frac{1}{1+\mu}\underline{a} + \frac{\mu}{1+\mu}\underline{b}$ OR $\underline{c} = \frac{\beta}{\alpha+\beta}\underline{a} + \frac{\alpha}{\alpha+\beta}\underline{b}$

(If $\mu<0$, C divides AB externally.)

Recap: Let $\underline{a} = a_1\underline{i} + a_2\underline{j} + a_3\underline{k}$

$\underline{b} = b_1\underline{i} + b_2\underline{j} + b_3\underline{k}$

According to law of cosine:

$|\underline{a}-\underline{b}|^2 = |\underline{a}|^2 + |\underline{b}|^2 - 2|\underline{a}||\underline{b}|\cos\theta$

$\boxed{|\underline{a}||\underline{b}|\cos\theta = a_1b_1 + a_2b_2 + a_3b_3 = \underline{a}.\underline{b}}$ Scalar / Dot product

- Scalar λ:

$\lambda\underline{a} = (\lambda a_1)\underline{i} + (\lambda a_2)\underline{j} + (\lambda a_3)\underline{k}$

$\Rightarrow (\lambda\underline{a}).\underline{b} = (\lambda a_1)b_1 + (\lambda a_2)b_2 + (\lambda a_3)b_3$

$= \lambda(a_1b_1 + a_2b_2 + a_3b_3)$

$= \lambda(\underline{a}.\underline{b})$

$= \underline{a}.(\lambda\underline{b})$

※ Answer = real numbers

$\underline{\underline{\text{NO}}}$ $\underline{i}/\underline{j}/\underline{k}$

Let $\underline{c} = c_1\underline{i} + c_2\underline{j} + c_3\underline{k}$, then

$\underline{a}.(\underline{b}+\underline{c}) = (a_1\underline{i} + a_2\underline{j} + a_3\underline{k})[(b_1+c_1)\underline{i} + (b_2+c_2)\underline{j} + (b_3+c_3)\underline{k}]$

$= a_1(b_1+c_1) + a_2(b_2+c_2) + a_3(b_3+c_3)$

$= (a_1b_1 + a_2b_2 + a_3b_3) + (a_1c_1 + a_2c_2 + a_3c_3)$

$= \underline{a}.\underline{b} + \underline{a}.\underline{c}$ Distributivity

$$\boxed{\cos\theta = \frac{\underline{a}.\underline{b}}{|\underline{a}||\underline{b}|} = \frac{a_1b_1 + a_2b_2 + a_3b_3}{\sqrt{a_1^2+a_2^2+a_3^2}.\sqrt{b_1^2+b_2^2+b_3^2}}}$$

Note: $\underline{a}.\underline{a} = |\underline{a}||\underline{a}|\cos\theta = |\underline{a}|^2 = \underline{a}^2$

- $\underline{u}.\underline{v} = 0$ iff $\underline{u}$ and $\underline{v}$ are perpendicular OR $|\underline{u}| = 0$ OR $|\underline{v}| = 0$

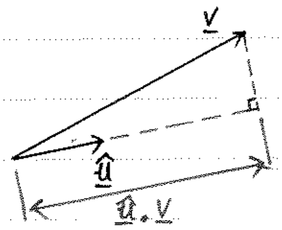
"iff" ≡ "if and only if"

$\Leftrightarrow$ $\underline{u}.\underline{v} = 0$ $\Leftrightarrow$ $\underline{u}$ are $\underline{v}$ are orthogonal (正交)

1.7 Projections (映射)

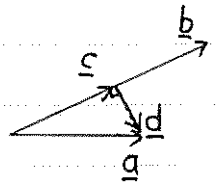
- Let $\underline{v}$ be a vector and let $\hat{\underline{u}}$ be a unit vector.

$\hat{\underline{u}}.\underline{v} = |\underline{v}|\cos\theta$ represents the length of projection of $\underline{v}$ in the direction $\hat{\underline{u}}$.



- We define the projection of $\underline{v}$ in the direction $\hat{\underline{u}}$ to be

$$\mathrm{Proj}_{\hat{\underline{u}}}\underline{v} = (\hat{\underline{u}}.\underline{v}).\hat{\underline{u}}$$



Proof: $\hat{\underline{b}} = \frac{\underline{b}}{|\underline{b}|}$ shows direction of $\underline{b}$

$\therefore$ direction of $\underline{c}$: $\pm\hat{\underline{b}}$

$\underline{c} = \lambda\hat{\underline{b}}$ where λ represents the length of $\underline{c}$

$\boxed{\underline{c}.\underline{d}=0}$ where $\underline{d}=\underline{a}-\underline{c}$

$\underline{c}(\underline{a}-\underline{c})=0$) expand

$\underline{c}.\underline{a}-\underline{c}^2=0$

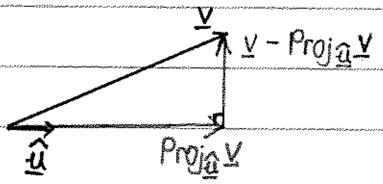
substitute : $\lambda\hat{\underline{b}}.\underline{a}-\lambda^2=0$

$\lambda(\hat{\underline{b}}.\underline{a}-\lambda)=0$

$\lambda=\underline{a}.\hat{\underline{b}}$

$\Rightarrow \boxed{\underline{c}=(\underline{a}.\hat{\underline{b}})\hat{\underline{b}}}$

length direction



- Given a unit vector $\hat{\underline{u}}$, we can uniquely write down $\underline{v}$ as a sum of a vector parallel to $\hat{\underline{u}}$ and a vector orthogonal to $\hat{\underline{u}}$.

$$\underline{v}=(\text{Proj}_{\hat{\underline{u}}}\underline{v})+(\underline{v}-\text{Proj}_{\hat{\underline{u}}}\underline{v})$$
$$=\underbrace{(\hat{\underline{u}}.\underline{v})\hat{\underline{u}}}_{\text{parallel to }\hat{u}}+\underbrace{(\underline{v}-(\hat{\underline{u}}.\underline{v})\hat{\underline{u}})}_{\text{orthogonal to }\hat{u}}$$

- Check that these vectors are orthogonal

$$[(\hat{\underline{u}}.\underline{v})\hat{\underline{u}}][\underline{v}-(\hat{\underline{u}}.\underline{v})\hat{\underline{u}}]=(\hat{\underline{u}}.\underline{v})(\hat{\underline{u}}.\underline{v})-(\hat{\underline{u}}.\underline{v})^2\underbrace{\hat{\underline{u}}.\hat{\underline{u}}}$$

$\hat{\underline{u}}.\hat{\underline{u}}=|\hat{\underline{u}}|=1$

$$=(\hat{\underline{u}}.\underline{v})^2-(\hat{\underline{u}}.\underline{v})^2$$
$$=1$$

$\Rightarrow$ perpendicular to each other

Fri. 14/10/16

MATH1401: Mathematical Methods I

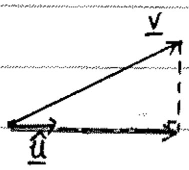
Prof. Rod Halburd

Recap. $\underline{a}.\underline{b}=|\underline{a}||\underline{b}|\cos\theta$

$|\underline{a}|=\sqrt{\underline{a}.\underline{a}}$

$\text{proj}_{\hat{\underline{u}}}\underline{v}=(\hat{\underline{u}}.\underline{v})\hat{\underline{u}}$

length direction



- EXAMPLE: Write $\underline{v}=\underline{i}+2\underline{j}+3\underline{k}$ as the sum of a vector parallel to $\underline{u}=3\underline{i}+4\underline{j}$ and a vector perpendicular to $\underline{u}$.

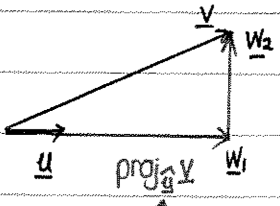
Soln: $|\underline{u}|=\sqrt{3^2+4^2}=5$

"soln" ≡ "solution"

$\hat{\underline{u}}=\frac{\underline{u}}{|\underline{u}|}=\frac{3\underline{i}+4\underline{j}}{5}=\frac{3}{5}\underline{i}+\frac{4}{5}\underline{j}$

Let $\underline{w}_1=\text{proj}_{\hat{\underline{u}}}\underline{v}=(\hat{\underline{u}}.\underline{v})\hat{\underline{u}}$

$$=\left(\frac{3}{5}+\frac{8}{5}\right)\left(\frac{3}{5}\underline{i}+\frac{4}{5}\underline{j}\right)=\frac{11}{25}(3\underline{i}+4\underline{j})$$

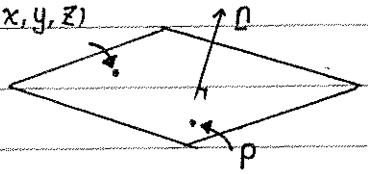


unit vector!

$$\underline{v} = \overset{\text{projection}}{\underline{W_1}} + \underline{W_2} \leftarrow \text{perpendicular vector}$$

$$\text{where } \underline{W_2} = \underline{v} - \underline{W_1} = \begin{pmatrix}1\\2\\3\end{pmatrix} - \frac{11}{25}\begin{pmatrix}3\\4\\0\end{pmatrix} = \begin{pmatrix}-\frac{8}{25}\\ \frac{6}{25}\\ 3\end{pmatrix}$$

1·8 Equation of a plane



- Let $\underline{n}$ be a normal vector to a plane.
  (perpendicular)

  and let P be a point on the plane.

Let $Q \equiv (x, y, z)$ be a general point on the plane.

"orthogonal" $\equiv$ either

Now $\overrightarrow{PQ}$ is orthogonal to $\underline{n}$.

so $\underline{n} \cdot \overrightarrow{PQ} = 0$ where $\underline{n} = n_1\underline{i} + n_2\underline{j} + n_3\underline{k}$

$P = (x_0, y_0, z_0)$

$$\begin{pmatrix}n_1\\n_2\\n_3\end{pmatrix} \cdot \begin{pmatrix}x - x_0\\ y - y_0\\ z - z_0\end{pmatrix} = 0$$

$$n_1 x + n_2 y + n_3 z = n_1 x_0 + n_2 y_0 + n_3 z_0$$

$$= \underline{n} \cdot \overrightarrow{OP}$$

- EXAMPLE: Find a normal vector to the plane

$$3x + 2y - 2z = 0$$

Soln: $(\boxed{\underline{n} \cdot \underline{r} = d})$

$\underline{n} = (3\underline{i} + 2\underline{j} - 2\underline{k})$

$\boxed{\begin{array}{l}\text{Note:}\\ \underline{r} = \underline{a} + \lambda\underline{d_1} + \mu\underline{d_2}\end{array}}$

- Direction Cosines: $\underline{v} = |\underline{v}|\hat{\underline{v}} \Rightarrow \hat{\underline{v}} = \frac{\underline{v}}{|\underline{v}|}$

$\hat{\underline{v}} = u_1\underline{i} + u_2\underline{j} + u_3\underline{k}$

$l = \underline{i} \cdot \hat{\underline{v}} = u_1 = \cos\alpha$

$m = \underline{j} \cdot \hat{\underline{v}} = u_2 = \cos\beta$

$n = \underline{k} \cdot \hat{\underline{v}} = u_3 = \cos\gamma$ where α, β, γ are the angles between $\underline{v}$ & $\underline{i}, \underline{j}, \underline{k}$

↳ direction cosines

EXAMPLE: Let $A = (1,2,1)$ & $B = (3,4,2)$ be two points.

Find the direction cosines of the line AB.

$$\overrightarrow{AB} = \begin{pmatrix}3\\4\\2\end{pmatrix} - \begin{pmatrix}1\\2\\1\end{pmatrix} = \begin{pmatrix}2\\2\\1\end{pmatrix}$$

$$|\overrightarrow{AB}| = \sqrt{2^2 + 2^2 + 1^2} = 3$$

$$\widehat{AB} = \frac{1}{3}\begin{pmatrix}2\\2\\1\end{pmatrix}$$

$$\left.\begin{array}{l}\cos\alpha = \frac{2}{3}\\ \cos\beta = \frac{2}{3}\\ \cos\gamma = \frac{1}{3}\end{array}\right\} l^2 + m^2 + n^2 = 1$$

1·12·2 Distance of a point from a plane

"pt" $\equiv$ "point"

- The distance between a point Q & a plane P is distance between Q and the closest pt to Q on P.

$$\hat{\underline{n}} = \frac{\underline{n}}{|\underline{n}|}$$

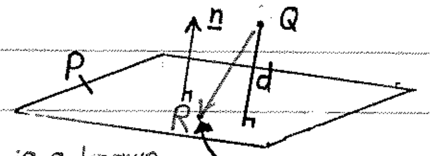
$$d = |\text{proj}_{\hat{\underline{n}}}\overrightarrow{QR}|$$

$$= \boxed{|\hat{\underline{n}}.\overrightarrow{QR}|}$$

Note:

$$d = \frac{|(\underline{a}-\underline{q}).\underline{n}|}{|\underline{n}|}$$

where A is a known point on the plane



Let $R = (x_0, y_0, z_0)$ be a pt on the plane

- EXAMPLE: Find the distance from (1,1,0) to the plane $x+2y-2z=1$

Soln: normal: $\underline{n} = \underline{i} + 2\underline{j} - 2\underline{k}$

$$|\underline{n}| = \sqrt{1^2+2^2+(-2)^2} = 3$$

$$\hat{\underline{n}} = \frac{\underline{n}}{|\underline{n}|} = \frac{1}{3}(\underline{i} + 2\underline{j} - 2\underline{k})$$

$R \equiv (1,0,0)$ is a point on the plane. ← Any point on the plane (that satisfies $x+2y-2z=1$)

$Q \equiv (1,1,0)$

$\Rightarrow \overrightarrow{QR} = -\underline{j}$

distance $d = |\hat{\underline{n}}.\overrightarrow{QR}| = \frac{2}{3}$

1·11 Vector Product / Cross Product / Wedge Product

- Let's try to construct a product of vectors $\underline{u}\times\underline{v}$ giving a vector. We want

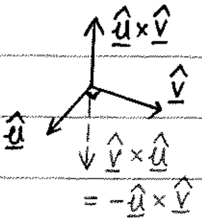
$$(\lambda\underline{u})\times\underline{v} = \lambda(\underline{u}\times\underline{v}) = \underline{u}\times(\lambda\underline{v}) \quad \text{associativity}$$

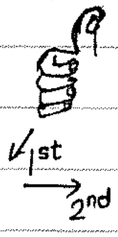
$$\underline{u}\times(\underline{v}+\underline{w}) = \underline{u}\times\underline{v} + \underline{u}\times\underline{w}$$

$$(\underline{v}+\underline{w})\times\underline{u} = \underline{v}\times\underline{u} + \underline{w}\times\underline{u}$$

distributivity

- For two perpendicular unit vectors $\hat{\underline{u}}, \hat{\underline{v}}$, define $\hat{\underline{u}}\times\hat{\underline{v}}$ to be the unit vector perpendicular to $\hat{\underline{u}}$ and $\hat{\underline{v}}$, with direction given by the right hand rule.





not commutative

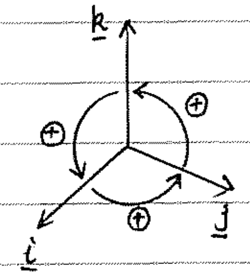
- finally $\boxed{\underline{v}\times\underline{v} = \underline{0}}$

$\underline{i}\times\underline{i} = \underline{j}\times\underline{j} = \underline{k}\times\underline{k} = \underline{0}$

$\underline{i}\times\underline{j}=\underline{k}$ $\qquad$ $\underline{j}\times\underline{i}=-\underline{k}$

$\underline{j}\times\underline{k}=\underline{i}$ $\qquad$ $\underline{k}\times\underline{j}=-\underline{i}$

$\underline{k}\times\underline{i}=\underline{j}$ $\qquad$ $\underline{i}\times\underline{k}=-\underline{j}$



- $\underline{a}=a_1\underline{i}+a_2\underline{j}+a_3\underline{k}$

  $\underline{b}=b_1\underline{i}+b_2\underline{j}+b_3\underline{k}$

- $\underline{a}\times\underline{b}=(a_1\underline{i}+a_2\underline{j}+a_3\underline{k})\times(b_1\underline{i}+b_2\underline{j}+b_3\underline{k})$ $\qquad$ The order cannot be reversed.

$$=a_1b_1\underbrace{\underline{i}\times\underline{i}}_{0}+a_1b_2\underbrace{\underline{i}\times\underline{j}}_{\underline{k}}+a_1b_3\underbrace{\underline{i}\times\underline{k}}_{-\underline{j}}+a_2b_1\underbrace{\underline{j}\times\underline{i}}_{-\underline{k}}+a_2b_2\underbrace{\underline{j}\times\underline{j}}_{0}+a_2b_3\underbrace{\underline{j}\times\underline{k}}_{\underline{i}}+a_3b_1\underbrace{\underline{k}\times\underline{i}}_{\underline{j}}+a_3b_2\underbrace{\underline{k}\times\underline{j}}_{-\underline{i}}+a_3b_3\underbrace{\underline{k}\times\underline{k}}_{0}$$

$$=(a_2b_3-a_3b_2)\underline{i}+(a_3b_1-a_1b_3)\underline{j}+(a_1b_2-a_2b_1)\underline{k}$$

Note: $\underline{n}=\begin{vmatrix}\underline{i}&\underline{j}&\underline{k}\\a_1&a_2&a_3\\b_1&b_2&b_3\end{vmatrix}$

$=(a_2b_3-a_3b_2)\underline{i}\underset{\triangle}{-}(a_1b_3-a_3b_1)\underline{j}$

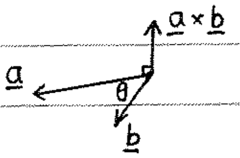
$+(a_1b_2-a_2b_1)\underline{k}$

This defines the vector product

= cross product $\qquad \underline{a}\times\underline{b}$

= wedge product $\qquad \underline{a}\wedge\underline{b}$

- Check:



$\underline{a}\cdot(\underline{a}\times\underline{b})=\underline{b}\cdot(\underline{a}\times\underline{b})=0$

$\Rightarrow \underline{a}\times\underline{b}$ is orthogonal to $\underline{a}$ & $\underline{b}$

$\underline{a}\times\underline{b}$ orthogonal to $\underline{a}$ & $\underline{b}$ and pts in the direction given by the right hand rule. ☆

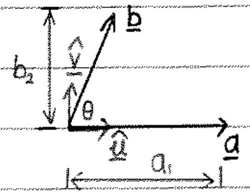
- We have 2 vectors $\underline{a}$ & $\underline{b}$

  Let $\hat{\underline{u}}=\hat{\underline{a}}$ = unit vector in the direction of $\underline{a}$

  Let $\hat{\underline{v}}$ be a unit vector perpendicular to $\hat{\underline{u}}$, and in the plane spanned by $\underline{a}$ & $\underline{b}$

  $\underline{a}=a_1\hat{\underline{u}}$

  $\underline{b}=b_1\hat{\underline{u}}+b_2\hat{\underline{v}}$



$\underline{a}\times\underline{b}=a_1\hat{\underline{u}}\times(b_1\hat{\underline{u}}+b_2\hat{\underline{v}})$

$$=a_1b_1\underbrace{\hat{\underline{u}}\times\hat{\underline{u}}}_{0}+a_1b_2\underbrace{\hat{\underline{u}}\times\hat{\underline{v}}}_{\text{unit vector}}$$

$|a_1|=|\underline{a}|$

$|b_2|=|\underline{b}|\sin\theta$ where $\theta\in[0,\pi]$

$\Rightarrow |\underline{a}\times\underline{b}|=|a_1b_2||\hat{\underline{u}}\times\hat{\underline{v}}|$

$=|a_1b_2|$

$=|a||b|\sin\theta$

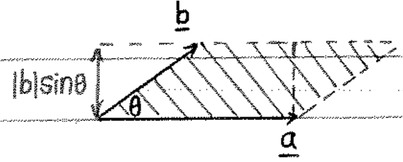
※ scalar product → scalar ; vector product → vector

$\underline{a}\times\underline{b}=-\underline{b}\times\underline{a}$

$\underline{a}\times(\underline{b}\times\underline{c})\neq(\underline{a}\times\underline{b})\times\underline{c}$ in general

eg. $\underline{a}\times(\underline{a}\times\underline{b})\neq(\underline{a}\times\underline{a})\times\underline{b}$

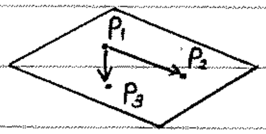
$\boxed{|\underline{a}\times\underline{b}|=|\underline{a}||\underline{b}|\sin\theta=\text{area of parallelogram}}$



EXAMPLE ①.

Find the eqn of the plane containing the (non-collinear) pts P_1, P_2, P_3.

"eqn" ≡ "equation"



Soln: $\overrightarrow{P_1P_3}$ & $\overrightarrow{P_1P_2}$ lie in the plane.

so a normal vector is given by

$$\underline{n} = \overrightarrow{P_1P_3} \times \overrightarrow{P_1P_2}$$

so P_1 is a pt on the plane & $\underline{n}$ is a normal vector

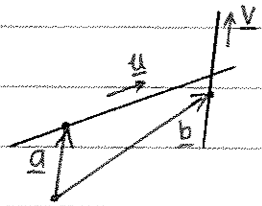
[see previous example]

EXAMPLE ②:

Find the distance between the two skew lines

$$\underline{r} = \underline{a} + \lambda \underline{u}$$

$$\underline{r} = \underline{b} + \mu \underline{v}$$



Mon. 17/10/16

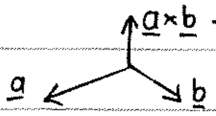
MATH1401: Mathematical Methods 1

Prof. Rod Halburd

- Recap: $\underline{a} \times \underline{b} = (a_2b_3 - a_3b_2)\,\underline{i} + (a_3b_1 - a_1b_3)\,\underline{j} + (a_1b_2 - a_2b_1)\,\underline{k}$

  $|\underline{a} \times \underline{b}| = |\underline{a}||\underline{b}|\sin\theta$ where $\theta \in [0, \pi]$

  = area of parallelogram



direction is perpendicular to $\underline{a}$ & $\underline{b}$ and given by the right hand rule.

- Two alternative way of calculating $\underline{a} \times \underline{b}$:

Method 1:

$$\underline{a} \times \underline{b} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

$$= \underline{i}\begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} - \underline{j}\begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} + \underline{k}\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \quad \text{where} \quad \begin{vmatrix} \alpha & \beta \\ \gamma & \delta \end{vmatrix} = \alpha\delta - \beta\gamma$$

determinant

1.9 Index Notations

Method 2: ε_{ijk} where $i, j, k \in \{1, 2, 3\}$

(i) $\varepsilon_{123} = 1$

(ii) ε_{ijk} changes sign when we swap two indices

For example, $-\varepsilon_{ijk} = \varepsilon_{jik}$

$$\varepsilon_{ijk} = \begin{cases} 0 & \text{if any 2 indices are the same} \\ 1 & \text{for even permutations of } (1,2,3) \\ -1 & \text{for odd permutations of } (1,2,3) \end{cases}$$

e.g. ① $(\overset{\frown}{2,1},3) \to (1,2,3)$ 1 permutation (odd) $\Rightarrow \varepsilon_{213} = -1$

② $(\overset{\frown}{3,1},2) \to (1,\overset{\frown}{3,2}) \to (1,2,3)$ 2 permutations (even) $\Rightarrow \varepsilon_{312} = 1$

- $\underline{e}_1 = \underline{i}$, $\underline{e}_2 = \underline{j}$, $\underline{e}_3 = \underline{k}$

  $\underline{e}_i \times \underline{e}_j = \sum_{k=1}^{3} \varepsilon_{ijk} \underline{e}_k$ under Einstein summation convention

- Einstein Summation Convention:

  repeated indices are summed over (from 1 to 3).

EXAMPLE:

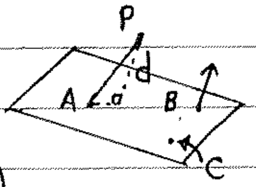
$\underline{a} = a_j \underline{e}_j \ (= a_1\underline{e}_1 + a_2\underline{e}_2 + a_3\underline{e}_3)$

$\underline{a} \times \underline{b} = (a_i\underline{e}_i) \times (b_j \underline{e}_j)$ index notation

$= a_i b_j (\underline{e}_i \times \underline{e}_j)$

$= \varepsilon_{ijk} a_i b_j \underline{e}_k$

- Q. Find the distance from the pt $P(-1,1,-1)$ to the plane contains the pts $A(1,0,0)$, $B(-1,2,1)$ & $C(0,0,1)$



$|\vec{AP} . \underline{\hat{n}}| = d$

unit normal

2 vectors in the plane are $\vec{AB} = \begin{pmatrix} -2 \\ 2 \\ 1 \end{pmatrix}$ & $\vec{AC} = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$

A normal to the plane is $\underline{n} = \vec{AB} \times \vec{AC}$

$= \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ -2 & 2 & 1 \\ -1 & 0 & 1 \end{vmatrix}$

$= \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix}$

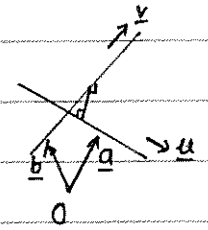
$|\underline{n}| = \sqrt{2^2+1^2+2^2} = 3$

$\therefore \ \underline{\hat{n}} = \frac{\underline{n}}{|\underline{n}|} = \frac{1}{3}\begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix} = \frac{2}{3}\underline{i} + \frac{1}{3}\underline{j} + \frac{2}{3}\underline{k}$

$\vec{AP} = (-2)\underline{i} + \underline{j} - \underline{k}$

$\Rightarrow$ distance $d = |\vec{AP} . \underline{\hat{n}}| = \left| -\frac{4}{3} + \frac{1}{3} - \frac{2}{3} \right| = \frac{5}{3}$

1·12·3 Distance between two (skew) lines

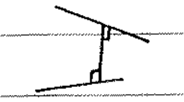


- $\underline{r}_1 = \underline{a} + \lambda \underline{u}$

  $\underline{r}_2 = \underline{b} + \mu \underline{v}$

$\therefore$ $\underline{u} \times \underline{v}$ is perpendicular to both $\underline{u}$ and $\underline{v}$ (i.e. to the lines)

unit vector $\underline{\hat{n}} = \frac{\underline{u} \times \underline{v}}{|\underline{u} \times \underline{v}|}$

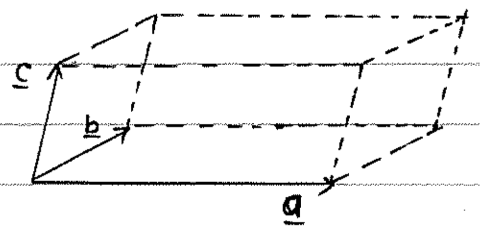


- Let $\underline{w}$ be a vector connecting any pt on the 1st line with any pt on the 2nd, then we want the length of the projection of $\underline{w}$ in the direction $\underline{\hat{n}}$.

$d = |\underline{\hat{n}} . (\underline{b} - \underline{a})|$

1·13·1 Volume of a parallelopiped

$V = |(\underline{a} \times \underline{b}) . \underline{c}|$



(scalar triple product)
$[\underline{a}, \underline{b}, \underline{c}] = (\underline{a} \times \underline{b}) . \underline{c}$
$= [b, c, a] = [c, a, b] = -[b, a, c]$

- Three Identities:

- $\varepsilon_{ijk}\varepsilon_{imn} = \delta_{jm}\delta_{kn} - \delta_{jn}\delta_{km}$ $\qquad \delta_{ij} = \begin{cases} 1 & i=j \\ 0 & i \neq j \end{cases}$ (summing over i)
- $\varepsilon_{imn}\varepsilon_{jmn} = 2\delta_{ij}$ (summing over m and n)
- $\varepsilon_{ijk}\varepsilon_{ijk} = 6$ (summing over i, j and k)

(Extremely Helpful Notes see 19/10/16 Problem Class, 19/10/16 Applied Tutorial & printed notes 1401 – Vector products and indices)

Fri. 21/10/16

MATH1401: Mathematical Methods 1

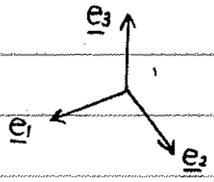
Prof. Halburd

Recap: $(\underline{a} \times \underline{b}) \times \underline{c}$

- $\underline{e}_1 = \underline{i}$, $\underline{e}_2 = \underline{j}$, $\underline{e}_3 = \underline{k}$

$\underline{e}_m \times \underline{e}_n = \varepsilon_{mn1}\underline{e}_1 + \varepsilon_{mn2}\underline{e}_2 + \varepsilon_{mn3}\underline{e}_3 = \sum_{p=1}^{3} \varepsilon_{mnp}\underline{e}_p$ where $m, n, p \in \{1,2,3\}$

↑ ε_{mnp} is the coefficient of $\underline{e}_p$ in the expansion of $\underline{e}_m \times \underline{e}_n$



$\underline{e}_1 \times \underline{e}_2 = \underline{e}_3 = 0\underline{e}_1 + 0\underline{e}_2 + \underline{e}_3$
$= \varepsilon_{121}\underline{e}_1 + \varepsilon_{122}\underline{e}_2 + \varepsilon_{123}\underline{e}_3$ $\Rightarrow$ $\varepsilon_{121} = \varepsilon_{122} = 0$
$\varepsilon_{123} = 1$

- All ε_{ijk} can be deduced from

(i) $\varepsilon_{123} = 1$

(ii) swapping 2 indices changes the sign

eg. ε_{123} (swapping 1 & 2) → $\varepsilon_{213} = -\varepsilon_{123}$

(iii) If 2 indices are the same → 0

e.g. ε_{131} (has 2 1s) → $\varepsilon_{131} = 0$

- Einstein Summation Convention

$\underline{a} = \sum_{j=1}^{3} a_j \underline{e}_j$ repeated "j" means we're summing over "j"

a_j is a scalar (if $j=1$, then $a_j = a_1$)

$\underline{e}_j$ is a vector (if $j=2$, then $\underline{e}_j = \underline{e}_2$)

$\therefore \underline{a} = a_j \underline{e}_j$

Similarly, $\underline{b} = b_k \underline{e}_k$

Therefore, $\underline{a} \times \underline{b} = (a_j \underline{e}_j) \times (b_k \underline{e}_k)$

$$= a_j b_k (\underline{e}_j \times \underline{e}_k)$$

$$= a_j b_k \varepsilon_{jki} \underline{e}_i \qquad \left(\sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3}\right)$$

- Kronecker Delta

$$\delta_{ij} = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$$

$\underline{a} \cdot \underline{b} = a_j b_j$

$\delta_{ij} a_j = a_i$ since we're summing over "j"

$= \delta_{i1} a_1 + \delta_{i2} a_2 + \delta_{i3} a_3$

$\delta_{ij} a_i b_j = a_j b_j = \underline{a} \cdot \underline{b}$

$= a_1 b_1 + a_2 b_2 + a_3 b_3$

- 3 Identities:

$\varepsilon_{ijk} \varepsilon_{imn} = \delta_{jm}\delta_{kn} - \delta_{jn}\delta_{km} \qquad \left(\sum_{i=1}^{3}\right)$

$\varepsilon_{imn} \varepsilon_{jmn} = 2\delta_{ij} \qquad \left(\sum_{m=1}^{3}\sum_{n=1}^{3}\right)$

$\varepsilon_{ijk} \varepsilon_{ijk} = 6 \qquad \left(\sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3}\right)$

- $\underline{a} \times \underline{b} = \varepsilon_{ijk} a_i b_j \underline{e}_k = d_k \underline{e}_k$

where $d_k = \varepsilon_{ijk} a_i b_j$

What is $(\underline{a} \times \underline{b}) \times \underline{c}$?

Anything that has been summed over is a dummy index.

$(\underline{a} \times \underline{b}) \times \underline{c} = \varepsilon_{ijk} d_i c_j \underline{e}_k$

$= \varepsilon_{ijk} (\varepsilon_{\underbrace{mni}} a_m b_n) c_j \underline{e}_k$

$= \varepsilon_{\textcircled{i}jk} \varepsilon_{\textcircled{i}mn} a_m b_n c_j \underline{e}_k$

The 1st index must be the same.

$= (\delta_{jm}\delta_{kn} - \delta_{jn}\delta_{km}) a_m b_n c_j \underline{e}_k$

$= \delta_{jm} a_m \delta_{kn} b_n c_j \underline{e}_k - \delta_{jn} b_n \delta_{km} a_m c_j \underline{e}_k$

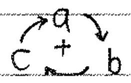
$= (a_j c_j) b_k \underline{e}_k - (b_j c_j) a_k \underline{e}_k$

$= (\underline{a} \cdot \underline{c}) \underline{b} - (\underline{b} \cdot \underline{c}) \underline{a}$

Triple Products

① Scalar triple product

$(\underline{a} \times \underline{b}) \cdot \underline{c} = (\underline{b} \times \underline{c}) \cdot \underline{a} = (\underline{c} \times \underline{a}) \cdot \underline{b}$



(clockwise)

$= -(\underline{c} \times \underline{b}) \cdot \underline{a} = -(\underline{b} \times \underline{a}) \cdot \underline{c} = -(\underline{a} \times \underline{c}) \cdot \underline{b}$

② Vector triple product

- $\underline{a} \times (\underline{b} \times \underline{c}) = (\underline{a} \cdot \underline{c}) \underline{b} - (\underline{a} \cdot \underline{b}) \underline{c}$

Note: $\underline{a} \times (\underline{b} \times \underline{c}) \neq (\underline{a} \times \underline{b}) \times \underline{c}$

- Jacobi Identity: $\underline{a} \times (\underline{b} \times \underline{c}) + \underline{c} \times (\underline{a} \times \underline{b}) + \underline{b} \times (\underline{c} \times \underline{a}) = 0$

Wednesday 12/10/16

MATH1401 Problem Class

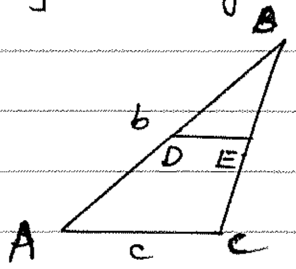
Prof Wilson (Helen)

helen.wilson@ucl.ac.uk QUESTIONS on Moodle

① $|\underline{a}| = \sqrt{\underline{a}\cdot\underline{a}}$ to find the length of $\underline{a}$

because $\underline{a}\cdot\underline{a} = |\underline{a}||\underline{a}|\cos\theta$ where $\theta = 0$

② Show that the line joining the mid-points of two sides of a triangle is paralle to the third side and half its length.



let $\vec{AB} = \underline{b}$, $\vec{AC} = \underline{c}$, then

$\vec{BC} = \underline{c} - \underline{b}$

$\vec{AD} = \frac{1}{2}\underline{b}$, $\vec{EC} = \frac{1}{2}\vec{BC} = \frac{1}{2}(\underline{c} - \underline{b})$

$\because \vec{AD} + \vec{DE} + \vec{EC} = \vec{AC}$

$\therefore \frac{1}{2}\underline{b} + \vec{DE} + \frac{1}{2}\underline{c} - \frac{1}{2}\underline{b} = \underline{c}$

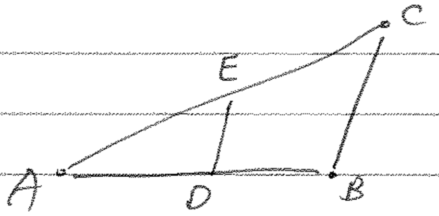
$\vec{DE} = \frac{1}{2}\underline{c}$

$\therefore$ parallel and $\frac{1}{2}$ lengths

[What do we need to show?

$\vec{DE}$ parallel to $\vec{BC}$

and $|\vec{DE}| = \frac{1}{2}|\vec{BC}|$]



- We know:

$\vec{AD} = \frac{1}{2}\vec{AB}$ because D is the midpoint of AB

and $\vec{AE} = \frac{1}{2}\vec{AC}$

So $\vec{DE} = \vec{DA} + \vec{AE}$

$= -\vec{AD} + \vec{AE}$

$= -\frac{1}{2}\vec{AB} + \frac{1}{2}\vec{AC}$

$= \frac{1}{2}(\vec{AC} - \vec{AB})$

$= \frac{1}{2}\vec{BC}$ which shows both results we needed.

③ If $\underline{a} = 3\underline{i} - 2\underline{j} + \underline{k}$

$\underline{b} = 2\underline{i} - 4\underline{j} - 3\underline{k}$

$\underline{c} = -\underline{i} + 2\underline{j} + 2\underline{k}$,

356
121
477

Find the magnitude of $\underline{a} + 2\underline{b} - 3\underline{c}$

$$\underline{a} + 2\underline{b} - 3\underline{c} = \begin{pmatrix} 3 \\ -2 \\ 1 \end{pmatrix} + 2\begin{pmatrix} 2 \\ -4 \\ -3 \end{pmatrix} - 3\begin{pmatrix} -1 \\ 2 \\ 2 \end{pmatrix} = \begin{pmatrix} 10 \\ -16 \\ -11 \end{pmatrix}$$

$$\therefore \left|\underline{a} + 2\underline{b} - 3\underline{c}\right| = \sqrt{100 + 256 + 121} = \sqrt{477}$$

④ Find the angle between the vectors $\overset{\underline{a}}{(2,-3,6)}$ and $\overset{\underline{b}}{(1,2,2)}$

$$\cos\theta = \frac{\begin{pmatrix}2\\-3\\6\end{pmatrix}\cdot\begin{pmatrix}1\\2\\2\end{pmatrix}}{\sqrt{2^2+3^2+6^2}\cdot\sqrt{1^2+2^2+2^2}} = \frac{2-6+12}{7\cdot3} = \frac{8}{21} \Rightarrow \theta = \cos^{-1}\left(\frac{8}{21}\right)$$

$\underline{a}\cdot\underline{b} = |\underline{a}||\underline{b}|\cos\theta$

$\therefore$ The angle is $\cos^{-1}\left(\frac{8}{21}\right)$

⑤ Given two non-zero vectors $\underline{a}$ and $\underline{b}$, ~~to~~ show:

(i) if $\underline{a}+\underline{b}$ and $\underline{a}-\underline{b}$ are perpendicular, then $|\underline{a}|=|\underline{b}|$

(ii) if $|\underline{a}+\underline{b}| = |\underline{a}-\underline{b}|$, then $\underline{a}$ and $\underline{b}$ are perpendicular.

解: (i) $(\underline{a}+\underline{b})\cdot(\underline{a}-\underline{b}) = 0$

$$\underline{a}\cdot(\underline{a}-\underline{b}) + \underline{b}\cdot(\underline{a}-\underline{b}) = 0$$

$$\underline{a}\cdot\underline{a}^{2} \underbrace{- \underline{a}\cdot\underline{b} + \underline{b}\cdot\underline{a}}_{\underline{b}\cdot\underline{a} = \underline{a}\cdot\underline{b}} - \overset{\underline{b}\cdot\underline{b}}{\cancel{\underline{b}^2}} = 0$$

$$\cancel{\underline{a}^2 = \underline{b}^2}$$

$$\underline{a}\cdot\underline{a} = \underline{b}\cdot\underline{b}$$

$\Rightarrow |\underline{a}|^2 = |\underline{b}|^2$

And since both length are non-negative,

$|\underline{a}| = |\underline{b}|$

(ii) $|\underline{a}+\underline{b}| = |\underline{a}-\underline{b}|$

Absolute Value
↓
Square both sides

so $|\underline{a}+\underline{b}|^2 = |\underline{a}-\underline{b}|^2$

$$(\underline{a}+\underline{b})\cdot(\underline{a}+\underline{b}) = (\underline{a}-\underline{b})\cdot(\underline{a}-\underline{b})$$

$$\underline{a}\cdot\underline{a} + 2\underline{a}\cdot\underline{b} + \underline{b}\cdot\underline{b} = \underline{a}\cdot\underline{a} - 2\underline{a}\cdot\underline{b} + \underline{b}\cdot\underline{b}$$

$$4\underline{a}\cdot\underline{b} = 0$$

$\underline{a}\cdot\underline{b} = 0$ so $\underline{a}$ and $\underline{b}$ must be perpendicular

Wednesday 19/10/16

MATH1401 Problem Class — Index Notation

Dr. Wilson

1. $\underline{e}_1 = \underline{i}$, $\underline{e}_2 = \underline{j}$, $\underline{e}_3 = \underline{k}$

$\underline{a} = (a_1, a_2, a_3)$

$= a_1\underline{i} + a_2\underline{j} + a_3\underline{k}$

$= a_1\underline{e}_1 + a_2\underline{e}_2 + a_3\underline{e}_3$

$= \sum_{k=1}^{3} a_k \underline{e}_k$

coordinate vector

2. Einstein Summation Convention:

The notation

$a_k b_k$ means $\sum_{k=1}^{3} a_k b_k$

Write "using the Einstein summation convention"

Always 3D space

3. Dot product:

$\underline{a}\cdot\underline{b} = (a_1, a_2, a_3).(b_1, b_2, b_3)$

$= a_1b_1 + a_2b_2 + a_3b_3$

$= \sum_{k=1}^{3} a_k b_k$

OR with summation convention,

$\boxed{\underline{a}\cdot\underline{b} = a_k b_k}$

4. $\delta_{ij} = \begin{cases} 1 & i=j \\ 0 & i\neq j \end{cases}$

What is $\delta_{ij}a_j$?

$\boxed{\delta_{ij}a_j} = \sum_{j=1}^{3} \delta_{ij} a_j$ means "sum over the repeated index"

$= \delta_{i1}a_1 + \delta_{i2}a_2 + \delta_{i3}a_3$

$\boxed{= a_i}$ It effectively replaces 'j' with 'i'

So, δ_{ij} is the (3×3) identity matrix.

5. Rules:

\#1 A repeated index (in one expression) is called a dummy index. Its name does not matter.

e.g. $a_{ik}c_k = a_{ij}c_j$

And it can only repeat twice. $a_i\cancel{b_i}c_i$

\#2 An index which appears only once is a free index and it represents each of the 3 possible values.

. $a_i = b_{ij}c_j$ (as an example)

What does it mean?

It means 3 things (at once).

- $a_1 = b_{11}c_1 + b_{12}c_2 + b_{13}c_3$ $(i=1)$

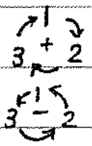
AND $a_2 = b_{21}c_1 + b_{22}c_2 + b_{23}c_3$ $(i=2)$

AND $a_3 = b_{31}c_1 + b_{32}c_2 + b_{33}c_3$ $(i=3)$

i (free index) labels which equation

j (dummy index) is summed within each equation

6. $$\varepsilon_{ijk} = \begin{cases} 1 & \{ijk\} = \{123\} \text{ or } \{231\} \text{ or } \{312\} \\ -1 & \{ijk\} = \{321\} \text{ or } \{213\} \text{ or } \{132\} \\ 0 & \text{otherwise} \end{cases}$$



(clockwise)

(anti-clockwise)

Note:

(I) $\varepsilon_{\overset{\frown}{ijk}} = \varepsilon_{jki}$ take 1st to the last → same group

(II) $\varepsilon_{i\overset{\sim}{jk}} = -\varepsilon_{ikj}$ swap any 2 of them → different groups

(III) Any two the same → given zero eg. $\varepsilon_{ikk} = 0$

7. Cross Product

$\underline{a} \times \underline{b} = \underline{e}_i \varepsilon_{ijk} a_j b_k$

Remember $a_i = \delta_{ij} a_j$

※ We can always use δ_{ij} to 'convert' j to i.

i.e. $\underline{\delta_{ij}} X_{jmnpq} = X_{imnpq}$ (very handy)

↳ it replaces "j" with "i"

8. The ε fact

1) $\underset{\uparrow}{\varepsilon_{ijk}}\underset{\uparrow}{\varepsilon_{ipq}} = \delta_{jp}\delta_{kq} - \delta_{jq}\delta_{kp}$

- Make sure to rotate until we get the same index to the front
- Combine the other two
- Swap one of them

so $\underline{a} \times (\underline{b} \times \underline{c}) = \underline{e}_i \varepsilon_{ijk} a_j (\underline{b} \times \underline{c})_k$

$= \underline{e}_i \varepsilon_{ijk} a_j \varepsilon_{kpq} b_p c_q$

What I mean by $(\underline{b} \times \underline{c})_k = \varepsilon_{kpq} b_p c_q$ is $\underline{b} \times \underline{c} = \underline{e}_k \varepsilon_{kpq} b_p c_q$

$= \underline{e}_i a_j b_p c_q \varepsilon_{ijk} \varepsilon_{kpq}$ Rotate!

$= \underline{e}_i a_j b_p c_q \varepsilon_{kij} \varepsilon_{kpq}$

$= \underline{e}_i a_j b_p c_q (\delta_{ip}\delta_{jq} - \delta_{iq}\delta_{jp})$

$= \underline{e}_i a_j b_p c_q \delta_{ip}\delta_{jq} - \underline{e}_i a_j b_p c_q \delta_{iq}\delta_{jp}$

$\delta_{ip} b_p = b_i$ $\delta_{jq} c_q = c_j$ $\delta_{iq} c_q = c_i$ $\delta_{jp} b_p = b_j$

$\underline{a} \times (\underline{b} \times \underline{c}) = \underline{e}_i a_j b_i c_j - \underline{e}_i a_j b_j c_i$

$= b_i \underline{e}_i a_j c_j - c_i \underline{e}_i a_j b_j = \underline{b}(\underline{a} \cdot \underline{c}) - \underline{c}(\underline{a} \cdot \underline{b})$

Wed. 19/10/16

Applied Tutorial

$[\underline{a}\times\underline{b}]_i = \varepsilon_{ijk}a_jb_k$ [i^{th} entry]

- EXAMPLE ①:

Prove $(\underline{a}\times\underline{b}).(\underline{c}\times\underline{d}) = (\underline{a}.\underline{c})(\underline{b}.\underline{d}) - (\underline{a}.\underline{d})(\underline{b}.\underline{c})$

Proof: $LHS \equiv (\underline{a}\times\underline{b})_i(\underline{c}\times\underline{d})_i$

$$\equiv \varepsilon_{ijk}a_jb_k\varepsilon_{imn}c_md_n$$

$$\equiv a_jb_kc_md_n(\varepsilon_{\underset{\uparrow}{i}jk}\varepsilon_{\underset{\uparrow}{i}mn})$$

$$\equiv a_jb_kc_md_n(\delta_{jm}\delta_{kn} - \delta_{jn}\delta_{km})$$

$$\equiv a_jc_m\delta_{jm}b_kd_n\delta_{kn} - a_jd_n\delta_{jn}b_kc_m\delta_{km}$$

$$\equiv a_jc_jb_kd_k - a_jd_jb_kc_k$$

$$\equiv (\underline{a}.\underline{c})(\underline{b}.\underline{d}) - (\underline{a}.\underline{d})(\underline{b}.\underline{c}) \equiv RHS$$ ▨

- EXAMPLE ②:

Prove $\underline{a}.(\underline{b}\times\underline{c}) = \underline{b}.(\underline{c}\times\underline{a}) = \underline{c}.(\underline{a}\times\underline{b})$

Proof. $\underline{a}\cdot(\underline{b}\times\underline{c}) = a_i(\underline{b}\times\underline{c})_i$

$$= a_i\varepsilon_{ijk}b_j\ c_k$$

$$= \varepsilon_{ijk}a_ib_jc_k$$

$$= b_{\underset{\uparrow}{j}}\varepsilon_{\underset{\uparrow}{j}ki}a_ic_k$$

$$= b_j(\underline{c}\times\underline{a})_{\underset{\uparrow}{j}}$$

$$= \underline{b}.(\underline{c}\times\underline{a})$$

$$= c_{\underset{\uparrow}{k}}\varepsilon_{\underset{\uparrow}{k}ij}a_ib_j$$

$$= c_k(\underline{a}\times\underline{b})_{\underset{\uparrow}{k}}$$

$$= \underline{c}.(\underline{a}\times\underline{b})$$ ▨

Note:

① The 1st index must be corresponding to the entry

② $\varepsilon_{jki}a_ic_k = (\underline{c}\times\underline{a})_j$
$\neq (\underline{a}\times\underline{c})_j$

The order !!!

Fri. 21/10/16 (continued)

MATH1401: Mathematical Methods 1

Chapter 2. § Complex Numbers §

3.1 Introduction

- $i^2 = 1$

$x+iy$, where $x, y \in \mathbb{R}$, is a complex number.

- Addition: $(x_1+iy_1)+(x_2+iy_2) = (x_1+x_2)+i(y_1+y_2)$
- Multiplication: $(x_1+iy_1)(x_2+iy_2) = x_1x_2 + ix_1y_2 + iy_1x_2 + i^2y_1y_2$

$$= (x_1x_2 - y_1y_2) + i(x_1y_2 + y_1x_2)$$

- Division: $\frac{1}{x+iy}$ where $(x,y) \neq (0,0)$

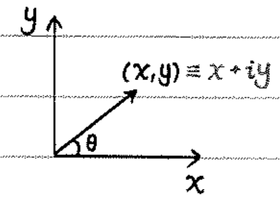
$$= \frac{1}{x+iy} \cdot \frac{x-iy}{x-iy}$$

multiply numerator & denominator by complex conjugate

$$= \frac{x-iy}{x^2+y^2}$$

$$= \frac{x}{x^2+y^2} + i\left(\frac{-y}{x^2+y^2}\right)$$

- Def. A complex number is a pair of real numbers (x,y) with the operations:



$$(x_1, y_1) + (x_2, y_2) = (x_1+x_2,\ y_1+y_2)$$

$$(x_1, y_1)(x_2, y_2) = (x_1x_2 - y_1y_2,\ x_1y_2 + y_1x_2)$$

$$(x, y)^{-1} = \left(\frac{x}{x^2+y^2}, -\frac{y}{x^2+y^2}\right)$$

- Note. Write $(x, y) = x+iy$

$$= x(1,0) + y(0,1)$$

'(1,0)' ≡ 'i'

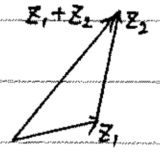
'(0,1)' ≡ 'j'

- $(x,0)$ is a 'copy' of the real numbers.

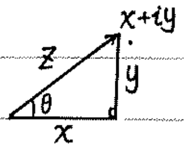
- $z = x+iy$ (assume x, y are real)

where $x = Re(z)$ real part of z

$y = Im(z)$ imaginary part of z



- $\bar{z} = x - iy = z^*$ (complex) conjugate of z
- $|z| = \sqrt{x^2+y^2}$ modulus of z
- $\theta = \arg(z)$ argument of z



(defined up to integer multiples of 2π)

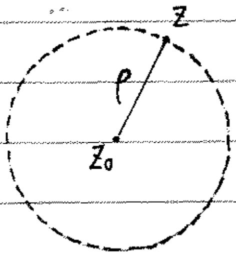
$-\pi < \arg(z) \leq \pi$

- $x = r\cos\theta$, $y = r\sin\theta$

$z = r(\cos\theta + i\sin\theta)$ polar form

3·2 Geometry in the complex plane

3·2·1 Circle:



complex numbers

- (The set of all $z \in \mathbb{C}$ s.t.) $|z-z_0| = \rho$, for same $\rho > 0$
  - ✓ set of pts distant ρ from z_0
  - ✓ circle of radius ρ, centred at z_0
- Algebraic:

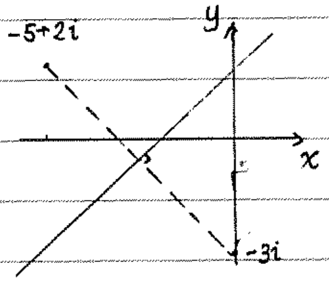
$z = x+iy$ $\qquad z_0 = x_0 + iy_0$

$\rho = |z-z_0| = |(x-x_0)+i(y-y_0)|$

$\rho^2 = (x-x_0)^2+(y-y_0)^2$ which is eqn of a circle

3·2·2 Line:

Note: $|\underset{\uparrow}{z-z_1}| = |\underset{\uparrow}{z-z_2}|$



- $|z+3i| = |z+(5-2i)|$

($|a-b| \equiv$ 'distance from a to b')

✓ pts equal-distant from $-3i$ and $-5+2i$

line, perpendicular to line joining these pts & through the mid-point perpendicular bisector

✓ $z = x+iy$

- Algebraic:

$|x+i(y+3)| = |(x+5)+i(y-2)|$

$x^2+(y+3)^2 = (x+5)^2+(y-2)^2$

$x^2+y^2+6y+9 = x^2+10x+25+y^2-4y+4$

$y = x+2$

3·3 Euler's Formula

- $e^{i\theta} = \cos\theta + i\sin\theta$ $\qquad$ Note. $(e^{i\theta})^n = e^{in\theta}$
- EXAMPLE: (find n^{th} roots)

Given z_0, find all z s.t. $z^n = z_0$.

Since $z = r(\cos\theta + i\sin\theta) = re^{i\theta}$,

$z_0 = r_0 e^{i\theta_0}$,

$\therefore r^n e^{in\theta} = r_0 e^{i\theta_0}$

Take modulus of both sides: $r^n = r_0$

$\Rightarrow r = r_0^{1/n}$

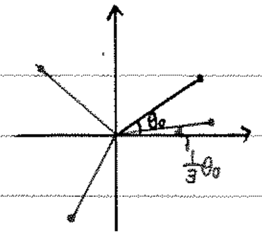
The usual n^{th} root of a non-negative real number

$(re^{i\theta})^n = r_0 e^{i(\theta_0+2k\pi)}$ $\qquad k = \cdots, -2, -1, 0, 1, 2, \cdots$

Equate arguments: $n\theta = \theta_0 + 2k\pi$

$$\theta = \frac{\theta_0}{n} + \frac{2k\pi}{n}$$

e.g. if $n = 3$, $\theta = \frac{1}{3}\theta_0 + \frac{2}{3}k\pi$



3.4 De Moivre's Theorem

- $e^{i\theta} = \cos\theta + i\sin\theta$

$(e^{i\theta})^n = (\cos\theta + i\sin\theta)^n$

$= \cos(n\theta) + i\sin(n\theta)$

$\Rightarrow \boxed{(\cos\theta + i\sin\theta)^n = \cos(n\theta) + i\sin(n\theta)}$

- EXAMPLE: Find $\cos 3\theta$ in terms of $\sin\theta$ & $\cos\theta$.

$\cos 3\theta + i\sin 3\theta = (\cos\theta + i\sin\theta)^3$

$= \cos^3\theta + 3\cos^2\theta(i\sin\theta) + 3\cos\theta(i\sin\ )^2 + (i\sin\theta)^3$ according to Pascal's △

$= (\cos^3\theta - 3\cos\theta\sin^2\theta) + i\,(3\cos^2\theta\sin\theta - \sin^3\theta)$

Real part: $\cos 3\theta = \cos^3\theta - 3\cos\theta\sin^2\theta$

Mon. 24/10/16

MATH1401: Mathematical Methods I

3.4.1 Trignometric Identities Prof. Halburd

- $e^z = e^{x+iy} = e^x e^{iy} = e^x(\cos y + i\sin y)$

$$\begin{cases} e^{i\theta} = \cos\theta + i\sin\theta & ① \\ e^{-i\theta} = \cos\theta - i\sin\theta & ② \end{cases}$$

$\frac{①+②}{2}$: $\cos\theta = \frac{e^{i\theta}+e^{-i\theta}}{2}$

$\sin\theta = \frac{e^{i\theta}-e^{-i\theta}}{2i}$

EXAMPLE:

Find $\cos^4\theta$. [binomial expansion]

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1

$\cos^4\theta = \left[\frac{1}{2}(e^{i\theta}+e^{-i\theta})\right]^4$ pascal's triangle

$= \frac{1}{16}\left[(e^{i\theta})^4 + 4(e^{i\theta})^3(e^{-i\theta}) + 6(e^{i\theta})^2(e^{-i\theta})^2 + 4(e^{i\theta})(e^{-i\theta})^3 + (e^{-i\theta})^4\right]$

$= \frac{1}{16}(e^{4i\theta} + 4e^{2i\theta} + 6 + 4e^{-2i\theta} + e^{-4i\theta})$

$= \frac{1}{16}(e^{4i\theta} + e^{-4i\theta}) + \frac{4}{16}(e^{2i\theta} + e^{-2i\theta}) + \frac{6}{16}$

$= \frac{1}{8}\cos 4\theta + \frac{1}{2}\cos 2\theta + \frac{3}{8}$

3.5 Roots of Unity

- example: Find z s.t. $z^6 = 64i$

$z^6 = \underset{|64i|}{\boxed{64}}\, e^{i(\underset{\arg(64i)}{\boxed{\frac{\pi}{2}}} + 2k\pi)}$

$\Rightarrow z = \sqrt[6]{64}\, e^{i\left(\frac{\pi}{12}+\frac{k\pi}{3}\right)} = 2\left[\cos\left(\frac{\pi}{12}+\frac{k\pi}{3}\right)+i\sin\left(\frac{\pi}{12}+\frac{k\pi}{3}\right)\right]$

when $k=-3$, $z_1 = 2\left[\cos\left(-\frac{11\pi}{12}\right)+i\sin\left(-\frac{11\pi}{12}\right)\right]$

when $k=-2$, $z_2 = 2\left[\cos\left(-\frac{7\pi}{12}\right)+i\sin\left(-\frac{7\pi}{12}\right)\right]$

when $k=-1$, $z_3 = 2\left[\cos\left(-\frac{\pi}{4}\right)+i\sin\left(-\frac{\pi}{4}\right)\right]$

when $k=0$, $z_4 = 2\left[\cos\left(\frac{\pi}{12}\right)+i\sin\left(\frac{\pi}{12}\right)\right]$

when $k=1$, $z_5 = 2\left[\cos\left(\frac{5\pi}{12}\right)+i\sin\left(\frac{5\pi}{12}\right)\right]$

when $k=2$, $z_6 = 2\left[\cos\left(\frac{3\pi}{4}\right)+i\sin\left(\frac{3\pi}{4}\right)\right]$

- Show that the sum of first n^{th} roots of unity is 0.

(see HW 3)

$z^n = 1 = e^{i(2k\pi)}$

$\Rightarrow z = e^{\frac{i(2k\pi)}{n}}$

Denote $\omega = e^{i\left(\frac{2\pi}{n}\right)}$, then

$\omega^0 + \omega^1 + \omega^2 + \cdots + \omega^{n-1} = 0$

where $n = 0, 1, 2, \cdots, n-1$

Wed. 26/10/16

MATH1401 Help Class

Prof. Wilson

§ n^{th} roots of unity §

$z^n = 1$

Let $z = re^{i\theta} = r(\cos\theta + i\sin\theta)$, then

$$(z^n =)\ r^n e^{in\theta} = 1$$

$$r^n[\cos(n\theta) + i\sin(n\theta)] = 1$$

Re: $r^n\cos(n\theta) = 1 \Rightarrow r \neq 0$

Im: $r^n\sin(n\theta) = 0 \Rightarrow \sin(n\theta) = 0$

$$\boxed{\sin x = 0 \text{ iff } x = M\pi \quad M \in \mathbb{Z}}$$

$$\Rightarrow \quad n\theta = M\pi \quad M \in \mathbb{Z} \quad (*)$$

Since $-\pi < \theta \leq \pi$,

$$-n\pi < n\theta \leq n\pi$$

$(*) \Rightarrow \quad -n\pi < M\pi \leq n\pi$

$$-n < M \leq n \qquad (\#)$$

Return to Real Part: $r^n\cos(M\pi) = 1$

$$r^n(-1)^M = 1$$

Since $r > 0$, $(-1)^M = 1 \Rightarrow M$ even

So we get

$$M = 2m, \ r = 1$$

$(\#) \Rightarrow \quad -n < 2m \leq n$

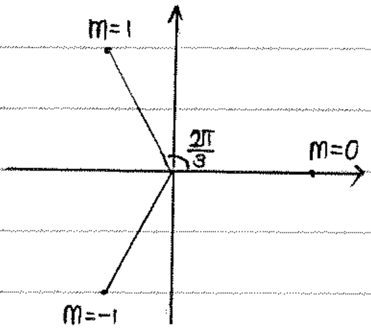
$$-\lfloor \tfrac{n}{2} \rfloor \leq m \leq \lfloor \tfrac{n}{2} \rfloor$$

means "the greatest integer that is smaller than $\frac{n}{2}$"

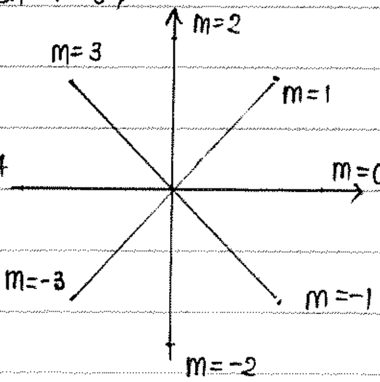
So roots are

$$z = \cos\left(\frac{2m\pi}{n}\right) + i\sin\left(\frac{2m\pi}{n}\right) \qquad -\lfloor \tfrac{n}{2} \rfloor \leq m \leq \lfloor \tfrac{n}{2} \rfloor$$

When $n = 3$,



When $n = 8$,



Wed. 26/10/16

MATH1401: Mathematical Methods I

Applied Tutorial

$$\left|e^{\frac{1}{x+iy}}\right| = a$$

$$\left|e^{\frac{1}{x+iy} \cdot \frac{x-iy}{x-iy}}\right| = a$$

$$\left|e^{\frac{x-iy}{x^2+y^2}}\right| = a$$

$$\left|e^{\frac{x}{x^2+y^2}}\right|\left|e^{-\frac{iy}{x^2+y^2}}\right| = a$$

Since $\left|e^{-\frac{iy}{x^2+y^2}}\right| = 1$, $\leftarrow$ $\left|e^{iy}\right| = 1$ because $\sqrt{\cos^2\theta + \sin^2\theta} = 1$

$$\left|e^{\frac{x}{x^2+y^2}}\right| = a$$

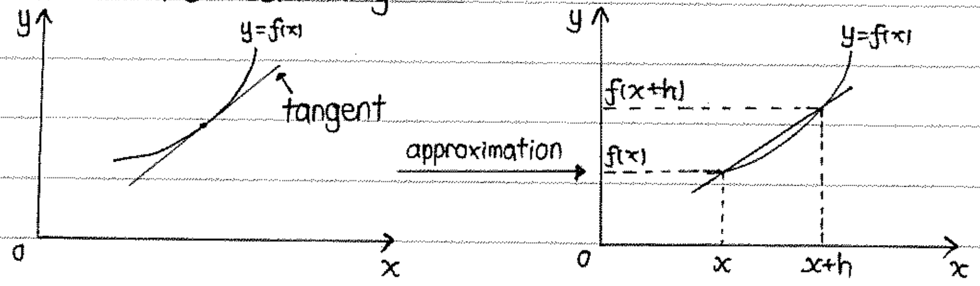
Mon. 24/10/16 (continued)

MATH1401: Mathematical Methods 1

Prof. Halburd

Chapter 3. § Differential Calculus §

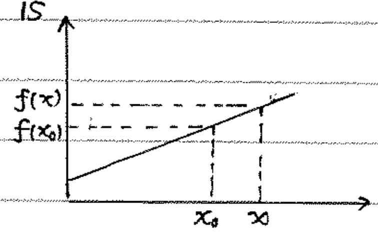
3·1 Differentiation and Continuity



(gradient) $f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$ = slope of the tangent to $y=f(x)$ at point x

- The equation of the tangent to $y=f(x)$ at $x=x_0$ is

$$\frac{y-f(x_0)}{x-x_0} = f'(x_0)$$



$\Rightarrow y = f(x_0) + f'(x_0)(x-x_0)$

✓ Def.

$Lf = f(x_0) + f'(x_0)(x-x_0)$ is the linearisation of f at $x = x_0$

Lf is an approximation to f for x near x_0.

✓ EXAMPLE.

Use a linearisation to estimate $\sqrt{1\cdot 02}$.

Soln: Let $f(x) = \sqrt{x}$.

We want $f(1\cdot 02)$ and we know $f(1)=1$

$f'(x) = \frac{1}{2\sqrt{x}}$

$\therefore f'(1) = \frac{1}{2}$

$Lf(x) = f(1) + f'(1)(x-1)$ In other words, I've chosen $x_0 = 1$

$\therefore Lf(1\cdot 02) = 1 + \frac{1}{2}(1\cdot 02 - 1)$

$= 1\cdot 01$

$\approx f(1\cdot 02) = \sqrt{1\cdot 02}$

3·2 Taylor Series

- ✓ Using the definition of the derivative, we were able to find an approximation of a function near a point. We suspect that we can get better approximations by taking into account higher derivatives.

$$f(x)=a_0+a_1(x-x_0)+a_2(x-x_0)^2+\dots+a_n(x-x_0)^n+\dots$$

$$f(x)=\sum_{n=0}^{\infty}a_n(x-x_0)^n$$

differentiate wrt x

$$f'(x)=a_1+2a_2(x-x_0)+3a_3(x-x_0)^2+\dots+na_n(x-x_0)^{n-1}+\dots$$

$$f''(x)=2a_2+2.3a_3(x-x_0)+\dots+n(n-1)a_n(x-x_0)^{n-2}+\dots$$

$$f'''(x)=2.3a_3+\dots$$

For x near x_0, $f(x_0)=a_0$

$f'(x_0)=a_1$

$f''(x_0)=2a_2$

$f'''(x_0)=6a_3$

$\vdots$

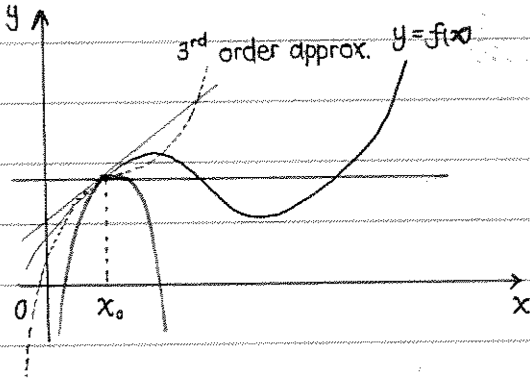
$f^{(n)}(x_0)=n!a_n$

✓ Def.

If f and all its derivatives at x_0 exist, then the Taylor Series of f at $x=x_0$ is $\displaystyle\sum_{n=0}^{\infty}\frac{f^{(n)}(x_0)}{n!}(x-x_0)^n$

✓
$$f(x)=\sum_{n=0}^{\infty}\frac{f^{(n)}(x_0)}{n!}(x-x_0)^n \quad \text{tangent}$$
$$=\underbrace{\overbrace{f(x_0)+f'(x_0)(x-x_0)}+\frac{f''(x_0)}{2!}(x-x_0)^2}_{\text{parabola}}+\dots$$

[approximate a fn locally (close to the pt $x=$)]



0 order approx. $f(x)=f(x_0)$ → const. fn

1st order approx.

$f(x)=f(x_0)+f'(x_0)(x-x_0)$ → tangent at $x=x_0$

2nd order approx.

$$f(x)=f(x_0)+f'(x_0)(x-x_0)+\frac{f''(x_0)}{2!}(x-x_0)^2+\dots$$

↓

parabola at $x=x_0$

✓
$$f(x)=f(x_0)+f'(x_0)(x-x_0)+\frac{f''(x_0)}{2!}(x-x_0)^2+\underline{\frac{f'''(x_0)}{3!}(x-x_0)^3+\frac{f^{(4)}(x_0)}{4!}(x-x_0)^4+\dots}$$

remainder is bounded by

$c.|x-x_0|^3$ where c is a const

e.g. 4.12796438 ...		error
	4.1	≈0.1
	4.13	≈0.01
	4.128	≈0.001

✓ EXAMPLE ① :

$$f(x) = e^x$$

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$
$$= 1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\dots$$

Soln: $f^{(n)}(x) = e^x$

at $x=0$, $f^{(n)}(0)=1$

Therefore, Taylor Series of e^x at $x=0$ is

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

Note: $0! = 1$

- Def.

A Taylor Series about $x=0$ (i.e. $x_0=0$) is also called a Maclaurin Series.

✓ EXAMPLE ② :

Find the Maclaurin Series of $f(x) = \sin x$.

Soln:
$f(x) = \sin x$ $\qquad$ $f(0)=0$
$f'(x) = \cos x$ $\qquad$ $f'(0)=1$
$f''(x) = -\sin x$ $\qquad$ $f''(0)=0$
$f'''(x) = -\cos x$ $\qquad$ $f'''(0)=-1$
$f^{(4)}(x) = \sin x$ $\qquad$ $f^{(4)}(0)=0$

repeats + all the even powers give 0 terms.
$f^{(n)}(0)=0$, n even

$\Rightarrow f^{(n)}(0) = 0$ if n is even

$\Leftrightarrow f^{(2k)}(0) = 0$

* when n is odd, $\boxed{n=2k+1}$

$$\sin x = \sum_{k=0}^{\infty} \frac{f^{(2k+1)}(0)}{(2k+1)!} x^{2k+1}$$

check where it starts

$$\Rightarrow f^{(2k+1)}(0) = \begin{cases} 1 & k \text{ even} \\ -1 & k \text{ odd} \end{cases}$$

$$\Leftrightarrow f^{(2k+1)}(0) = (-1)^k$$

$$\sin x = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} x^{2k+1}$$
$$= x - \frac{x^3}{3!} + \frac{x^5}{5!} + \dots$$

Therefore, Maclaurin Series of $\sin x$ is

$$\sum_{k=0}^{\infty} \frac{f^{(2k+1)}(0)}{(2k+1)!} (x-0)^{2k+1} = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} x^{2k+1}$$

Fri. 28/10/16

MATH1401: Mathematical Methods 1

Prof. Halburd

✓ EXAMPLE ③:

$$f(x) = \cos x$$

Soln:

$$f(x) = \cos x \qquad\qquad f(0) = 1$$
$$f'(x) = -\sin x \qquad\qquad f'(0) = 0$$
$$f''(x) = -\cos x \qquad\qquad f''(0) = -1$$
$$f'''(x) = \sin x \qquad\qquad f'''(0) = 0$$

$$\Rightarrow f^{(n)}(0) = \begin{cases} 0 & n \text{ odd} \\ (-1)^k & n = 2k \text{ (even)} \end{cases}$$

$$\Rightarrow \cos x = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n$$
$$= \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} x^{2k}$$

$$\cos x = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} x^{2k}$$
$$= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots$$

✓ EXAMPLE ④:

$$z \in \underset{\text{complex numbers}}{\mathbb{C}}, \quad \exp(z) = \sum_{n=0}^{\infty} \frac{z^n}{n!}$$

Since $e^{i\theta} = \exp(i\theta)$,

$$\exp(z) = e^{i\theta} = \sum_{n=0}^{\infty} \frac{(i\theta)^n}{n!}$$
$$= \sum_{n=0}^{\infty} \frac{i^n \theta^n}{n!}$$
$$= \sum_{n \text{ even}} + \sum_{n \text{ odd}} \quad n = 2k+1$$
$$\overset{n=2k}{=} \sum_{k=0}^{\infty} \frac{i^{2k}\theta^{2k}}{(2k)!} + \sum_{k=0}^{\infty} \frac{i^{2k+1}\theta^{2k+1}}{(2k+1)!}$$
$$= \sum_{k=0}^{\infty} \frac{(-1)^k \theta^{2k}}{(2k)!} + i \sum_{k=0}^{\infty} \frac{(-1)^k \theta^{2k+1}}{(2k+1)!}$$
$$= \cos\theta + i\sin\theta$$

▨

3.3 Binomial Expansion

- ✓

$$f(x) = (1+x)^n \qquad\qquad f(0) = 1$$
$$f'(x) = n(1+x)^{n-1} \qquad\qquad f'(0) = n$$
$$f''(x) = n(n-1)(1+x)^{n-2} \qquad\qquad f''(0) = n(n-1)$$
$$\vdots \qquad\qquad \vdots$$
$$f^{(n)}(x) = n(n-1)\dots(n-r+1)(1+x)^{n-r} \qquad f^{(n)}(0) = n(n-1)(n-2)\dots(n-r+1)$$

Therefore, the maclaurin series of $(1+x)^n$ is

$$(1+x)^n = \sum_{r=0}^{\infty} \frac{f^{(r)}(0)}{r!} x^r$$
$$= \sum_{r=0}^{\infty} \binom{n}{r} x^r \quad \text{where} \quad \underset{\uparrow}{\binom{n}{r}} = \frac{n(n-1)(n-2)\dots(n-r+1)}{r!}$$

"n choose r"

This is called binomial expansion.

✓ This expansion converges to $(1+x)^n$ for $|x|<1$ even when n is not a positive integer.

$n \in \mathbb{R}$.

If n is a positive integer, the series is finite & $\binom{n}{r} = \frac{n!}{r!(n-r)!}$

✓ EXAMPLE ①:

Find the Maclaurin series of $\frac{1}{\sqrt{1+x}}$ to order 3. (x^3)

Soln: $\frac{1}{\sqrt{1+x}} = (1+x)^{-\frac{1}{2}}$

$$= 1 + \binom{-1/2}{1}x + \binom{-1/2}{2}x^2 + \binom{-1/2}{3}x^3 + \dots$$

$$= 1 + \frac{(-1/2)}{1!}x + \frac{(-1/2)(-3/2)}{2!}x^2 + \frac{(-1/2)(-3/2)(-5/2)}{3!}x^3 + \dots$$

$$= 1 - \frac{1}{2}x + \frac{3}{8}x^2 - \frac{5}{16}x^3 + \dots \qquad (*)$$

✓ EXAMPLE ②:

Find the Taylor Series of $\frac{x}{\sqrt{1-x^2}}$ about $x=0$ up to the x^7 term.

In this case, the fn is relatively complicated. Thus, use previous result from EXAMPLE ①.

Soln: $\frac{x}{\sqrt{1-x^2}} = x.\frac{1}{\sqrt{1-x^2}}$

Change x to $(-x^2)$ in $(*)$.

$$\frac{1}{\sqrt{1-x^2}} = 1 - \frac{1}{2}.(-x^2) + \frac{3}{8}.(-x^2)^2 - \frac{5}{16}.(-x^2)^3 + \dots$$

$$= 1 + \frac{1}{2}x^2 + \frac{3}{8}x^4 + \frac{5}{16}x^6 + \dots$$

So, $\frac{x}{\sqrt{1-x^2}} = x + \frac{1}{2}x^3 + \frac{3}{8}x^5 + \frac{5}{16}x^7 + \dots$

✓ EXAMPLE ③.

Find the series expansion of $\arctan x$.

Soln. We know that $\int \frac{1}{1+x^2}dx = \arctan x + c$

And $\frac{1}{1+x^2} = (1+x^2)^{-1}$

$$= 1 + \frac{(-1)}{1!}x^2 + \frac{(-1)(-2)}{2!}(x^2)^2 + \frac{(-1)(-2)(-3)}{3!}(x^2)^3 + \dots$$

$$= 1 - x^2 + x^4 - x^6 + \dots$$

So, $\arctan x = \int(1 - x^2 + x^4 - x^6 + \dots)\,dx$

$$= x - \frac{1}{3}x^3 + \frac{1}{5}x^5 - \frac{1}{7}x^7 + \dots$$

3·4 Implicit Differentiation

✓ EXAMPLE ①:

Suppose that y is defined implicitly by

$$x^7 + 3xy + 2x^2y^5 = 6$$

Find $\frac{dy}{dx}$ at $x=1$, $y=1$. Note: this pt is on the curve.

Soln: Take $\frac{d}{dx}$ of both sides:

$$7x^6 + 3y + 3x\frac{dy}{dx} + 4xy^5 + 10x^2y^4\frac{dy}{dx} = 0$$

sub $x=1$, $y=1$:

$$7 + 3 + 3\frac{dy}{dx} + 4 + 10\frac{dy}{dx} = 0$$

$$13\frac{dy}{dx} = -14$$

$$\frac{dy}{dx} = -\frac{14}{13}$$

✓ EXAMPLE ②:

Find $\frac{d}{dx}(x^x)$

Soln: Let $y = x^x$

Natural logarithm: 'ln' ≡ 'log' ≡ '$\log_e$'

Take ln( ) of both sides

$$\ln y = \ln x^x$$

$$\ln y = x\ln x$$

Differentiate wrt x:

$$\frac{d}{dx}(\ln y) = \ln x + x.\frac{1}{x} = \ln x + 1$$

$\frac{dy}{dx}.\frac{d(\ln y)}{dy}$

$$\frac{1}{y}.\frac{dy}{dx} = \ln x + 1$$

$$\Rightarrow \frac{d}{dx}(x^x) = \frac{dy}{dx} = (\ln x + 1)\, x^x$$

Alternative Method:

$$x^x = (e^{\ln x})^x = e^{x\ln x}$$

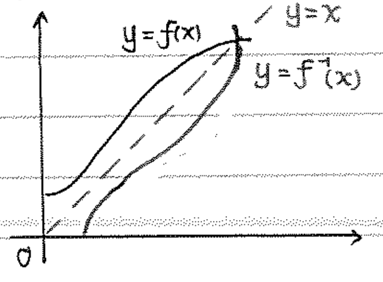
$$\Rightarrow \frac{d}{dx}(x^x) = \frac{d}{dx}(e^{x\ln x})$$

$$= (\ln x + 1)\, e^{x\ln x}$$

$$= (\ln x + 1)\, x^x$$

3·5 Inverse Functions

- How to find the inverse?

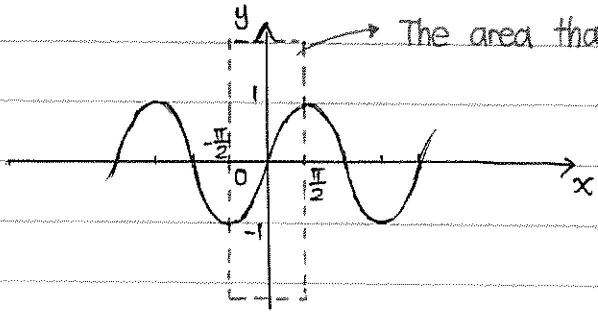


$y = f(x) \Leftrightarrow x = f^{-1}(y)$

↑ only works if the fn is bijective (one-to-one fn)

$y=f^{-1}(x)$ is the reflection of $y=f(x)$ in the line $y=x$.

- $y=\sin x$



The area that can produce an inverse

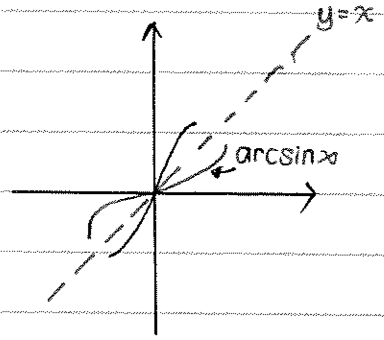
CANNOT find the inverse without restrictions

Note: $(\sin x)^{-1} = \frac{1}{\sin x}$

$\sin^{-1}x = \arcsin x$

- EXAMPLE:

Find $\frac{d}{dx}\arcsin x$.



Soln: Let $y=\arcsin x$

i.e. $\sin y = x$

Differentiate wrt x

$$\cos y \frac{dy}{dx} = 1$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{\cos y}$$

$$\Leftrightarrow \frac{d}{dx}(\arcsin x) = \frac{\pm 1}{\sqrt{1-x^2}}$$ since $\cos y = \pm\sqrt{1-\sin^2 y}$

$$= \frac{1}{\sqrt{1-x^2}}$$ since $\arcsin x$ is increasing

3.6 Hyperbolic Functions

- $e^{i\theta} = \cos\theta + i\sin\theta$
  $e^{-i\theta} = \cos\theta - i\sin\theta$

$$\Rightarrow \cos\theta = \frac{e^{i\theta}+e^{-i\theta}}{2}, \quad \sin\theta = \frac{e^{i\theta}-e^{-i\theta}}{2i}$$

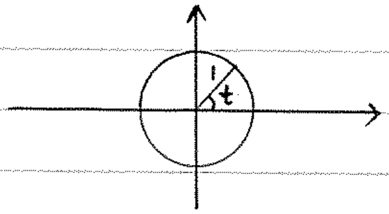
- Def.

$$\cosh x = \frac{e^{x}+e^{-x}}{2}$$

$$\sinh x = \frac{e^{x}-e^{-x}}{2}$$

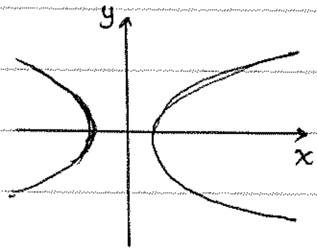
✓ If $(x,y) = (\cos t, \sin t)$, we have

$$x^2+y^2 = \cos^2 t + \sin^2 t = 1$$



If $(x,y) = (\cosh t, \sinh t)$, we have

$$x^2-y^2 = \cosh^2 t - \sinh^2 t = 1$$



proof : $\cosh^2 x = \frac{1}{4}(e^{2x} + 2 + e^{-2x})$

$\sinh^2 x = \frac{1}{4}(e^{2x} - 2 + e^{-2x})$

$\Rightarrow \cosh^2 x - \sinh^2 x = 1$

✓ We also have

$$\cosh^2 x = \frac{1}{4}(e^{2x} + 2 + e^{-2x})$$
$$= \frac{1}{4}(e^{2x} + e^{-2x}) + \frac{1}{2}$$
$$= \frac{1}{2}\cosh 2x + \frac{1}{2}$$

$\Rightarrow \cosh 2x = 2\cosh^2 x - 1$

✓ $\sinh^2 x + \cosh^2 x = \cosh 2x$

$2\sinh^2 x + 1 = \cosh 2x$

$\operatorname{sech}^2 x = 1 - \tanh^2 x$

$-\operatorname{cosech}^2 x = 1 - \coth^2 x$

- Differentiation

$\frac{d}{dx}\sinh x = \cosh x$

$\frac{d}{dx}\cosh x = \sinh x$

$\frac{d}{dx}\tanh x = \operatorname{sech}^2 x$

$\frac{d}{dx}\coth x = \underset{\blacktriangle}{-}\operatorname{cosech}^2 x$

$\frac{d}{dx}\operatorname{sech} x = \underset{\blacktriangle}{-}\operatorname{sech} x \tanh x$

$\frac{d}{dx}\operatorname{cosech} x = \underset{\blacktriangle}{-}\operatorname{cosech} x \coth x$

- Taylor Series

$$\cosh x = \sum_{k=0}^{\infty} \frac{x^{2k}}{(2k)!}$$

$$\sinh x = \sum_{k=0}^{\infty} \frac{x^{2k+1}}{(2k+1)!}$$

- Differences between hyperbolic fns & trig fns :

✓ $\cos x = \frac{1}{2}(e^{ix} + e^{-ix})$

$\Rightarrow \cos(ix) = \frac{1}{2}(e^{-x} + e^{x}) = \cosh x$

✓ $\sin x = \frac{1}{2i}(e^{ix} - e^{-ix})$

$-\frac{1}{i} = -\frac{i}{i^2} = -\frac{i}{-1} = i$

$\Rightarrow \sin(ix) = \frac{1}{2i}(e^{-x} - e^{-x}) = \underset{\sim\sim}{-\frac{1}{i}} \cdot \frac{1}{2}(e^{x} - e^{-x}) = i\sinh x$

- Even / Odd Fn ?

✓ $\cosh x = \frac{1}{2}(e^{x} + e^{-x})$

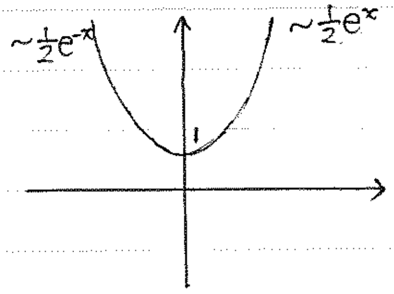
When $x >> 1$, $\cosh x \sim \frac{1}{2}e^{x}$

$x \ll -1$, $\cosh x \sim \frac{1}{2}e^{-x}$

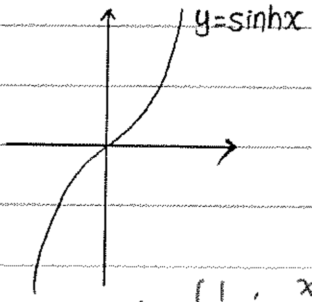
i.e. $\boxed{|x| >> 1,\ \cosh x \sim \frac{1}{2}e^{|x|}}$

$\Rightarrow \cosh x$ is an even function

$f(-x) = f(x)$



and it has a minimum of 1 at $x=0$.



✓ $\sinh x = \frac{1}{2}(e^{x}-e^{-x})$

$$= \begin{cases} \frac{1}{2}e^{x} & x>>1 \\ -\frac{1}{2}e^{-x} & x<<-1 \end{cases}$$

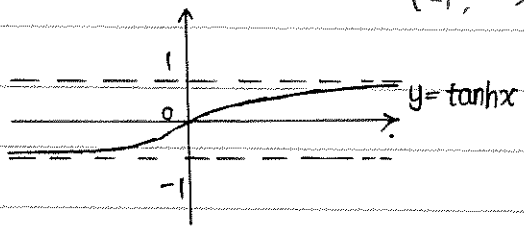
$$\sim \frac{1}{2}e^{|x|}\text{sgn}(x)$$

$$|x| = \begin{cases} x, & x\geqslant 0 \\ -x, & x<0 \end{cases}$$

$$\text{sgn}(x) = \begin{cases} 1, & x>0 \\ -1, & x<0 \end{cases}$$

✓ $\tanh x = \frac{\sinh x}{\cosh x}$

$$= \frac{e^{x}-e^{-x}}{e^{x}+e^{-x}} = \frac{e^{2x}-1}{e^{2x}+1}$$



Mon. 31/10/16

MATH1401: Mathematical Methods 1

Prof. Halburd

· Inverse Fns:

$$y=\sinh^{-1}x = \text{arsinh}\,x$$

$$\sinh y = x$$

$$\frac{e^{y}-e^{-y}}{2} = x$$

$$(e^{y})^{2}-2x(e^{y})-1=0$$

$$e^{y} = \frac{2x\pm\sqrt{4x^{2}+4}}{2} = x\pm\sqrt{x^{2}+1} \longrightarrow$$

(We need to have $x\sim\frac{1}{2}e^{x}$ for $y\to+\infty$)

We need to choose 'x'

$$e^{y} = x+\sqrt{x^{2}+1}$$

$$\Rightarrow y = \text{arsinh}\,x = \ln\left(x+\sqrt{x^{2}+1}\right)$$

3.7 Manipulating Series

① Multiplying Series

$$(a_0+a_1x+a_2x^{2}+\dots)(b_0+b_1x+b_2x^{2}+\dots)$$

$$= \underset{\substack{\uparrow \\ \text{constant term}}}{a_0b_0} + \underset{\substack{\uparrow \\ \text{linear term}}}{(a_0b_1+a_1b_0)x} + \underset{\substack{\uparrow \\ \text{quadratic term}}}{(a_0b_2+a_1b_1+a_2b_0)x^{2}} + \dots$$

★ DO NOT expand all the terms out. Collect the terms as you go.

② Dividing Series

EXAMPLE: $\frac{1}{2+3x+4x^2+\dots}$

Find the series up to the x^2 term

Soln: We know that $\frac{1}{1+x} = 1-x+x^2-x^3+x^4-\dots$

$$\text{So } \frac{1}{2+3x+4x^2+\dots} = \frac{1}{2} \cdot \frac{1}{1+\boxed{(\frac{3}{2}x+2x^2+\dots)}} \text{ 'x'}$$

cubic terms

$$\text{sub: } = \frac{1}{2}\left\{1-(\frac{3}{2}x+2x^2+\boxed{\dots})+(\frac{3}{2}x+2x^2+\boxed{\dots})^2-\dots\right\}$$

$$= \frac{1}{2}\left\{1-\frac{3}{2}x+(\frac{9}{4}-2)x^2+\dots\right\}$$

$$= \frac{1}{2}-\frac{3}{4}x+\frac{1}{8}x^2+\dots$$

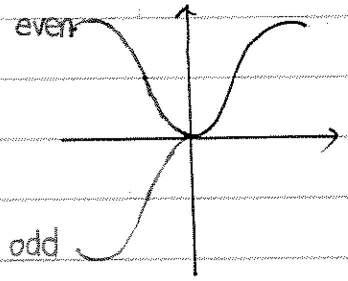
3.8 Curve Sketching

- Basic Steps:

① Basic symmetries : even / odd / neither

✓ even function: $f(-x) = f(x)$ [reflection in the y-axis]

✓ odd function: $f(-x) = -f(x)$ [reflection in the origin]



② Behaviour of $x \to +\infty$ & $x \to -\infty$

③ Vertical asymptotes / pts

where the fn is not defined

④ Extreme pts (minima / maxima) & stationary pts

$f'(x) = 0$

- EXAMPLE:

Sketch $y(x) = \frac{\sqrt{x^2+1}}{(x+1)^2}$.

Soln: No obvious symmetries.

$x \to +\infty$: $\sqrt{x^2+1} \longrightarrow x$

$(x+1)^2 \longrightarrow x^2$

so $y(x) \to \frac{x}{x^2} = \frac{1}{x}$, $x >> 1$

$x \to -\infty$: $\sqrt{x^2+1} \to -x$ $\sqrt{x^2} = |x|$

$(x+1)^2 \to x^2$

so $y(x) \to \frac{-x}{x^2} = -\frac{1}{x}$, $x << -1$

For x near -1,

$$y(x) = \frac{\sqrt{(-1)^2+1}}{(x+1)^2} = \frac{\sqrt{2}}{(x+1)^2} > 0$$

since $y(x)$ is not defined at $x=-1$

So,

$y(x) \to +\infty$ as $\boxed{x \to 1^-}$ & $\boxed{x \to 1^+}$

x approaches 1 from below ·· from above

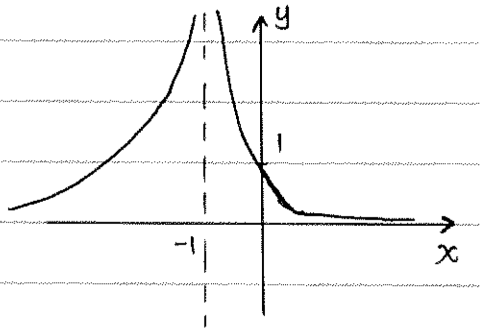
$$y(x) = \frac{\sqrt{x^2+1}}{(x+1)^2}$$

$$y'(x) = \frac{\frac{1}{2} \cdot 2x(x^2+1)^{-\frac{1}{2}}(x+1)^2 - \sqrt{x^2+1} \cdot 2(x+1)}{(x+1)^4}$$

$$= (x+1)^{-3}(x^2+1)^{-\frac{1}{2}}\left[x(x+1) - 2(x^2+1)\right]$$

$$= -(x+1)^{-3}\underbrace{(x^2+1)}_{\text{no real roots}}{}^{-\frac{1}{2}}\underbrace{(x^2-x+2)}_{\Delta = -7 < 0 \Rightarrow \text{no real roots}}$$

$\Rightarrow$ No stationary pts. (i.e. $f'(x) \neq 0$ $\forall x$)



Wed. 02/11/16

MATH1401: Help Class

Prof. Wilson

§ Hyperbolic Functions §

1. Sketch $y = f(x)$ where $f(x) = \dfrac{\sinh x}{\cosh x - 2}$.

- What happens to $y = f(x)$ as $x \to +\infty$?

$e^{-x} \to 0$

$$f(x) = \frac{\frac{1}{2}(e^{x} - e^{-x})}{\frac{1}{2}(e^{x} + e^{-x}) - 2} \xrightarrow{\text{looks similar to}} \frac{\frac{1}{2}e^{x}}{\frac{1}{2}e^{x} - 2} = \frac{\frac{1}{2}}{\frac{1}{2} - 2e^{-x}} \to \frac{\frac{1}{2}}{\frac{1}{2}} = 1$$

Similarly, as $x \to -\infty$,

$e^{x} \to 0$

$$f(x) \to \frac{-\frac{1}{2}e^{-x}}{\frac{1}{2}e^{-x} - 2} = \frac{-\frac{1}{2}}{\frac{1}{2} - 2e^{x}} \to \frac{-\frac{1}{2}}{\frac{1}{2}} = -1$$

- Singularities ? $[\ |f(x)| \to +\infty\ ]$ (asymptotes)

This happens where $\cosh x - 2 = 0$

$$x = \pm \cosh^{-1}(2)$$

- Zeros of $f(x)$ [pts $y = 0$]

✓ For us, $\sinh x = 0$

$\Rightarrow \quad x = 0$

Only one zero. (*)

✓ In $x < -\cosh^{-1}(2)$, we must get from -1 to $\pm\infty$.

However, from (*), we know that it cannot pass through 0

$\Rightarrow f(x) \to -\infty$ as $x^{-} \to -\cosh^{-1}(2)$

Similarly, $f(x) > 1$ in $\cosh^{-1}(2) < x < \infty$.

- $$\frac{df}{dx} = \frac{\cosh x(\cosh x - 2) - \sinh x . \sinh x}{(\cosh x - 2)^2}$$

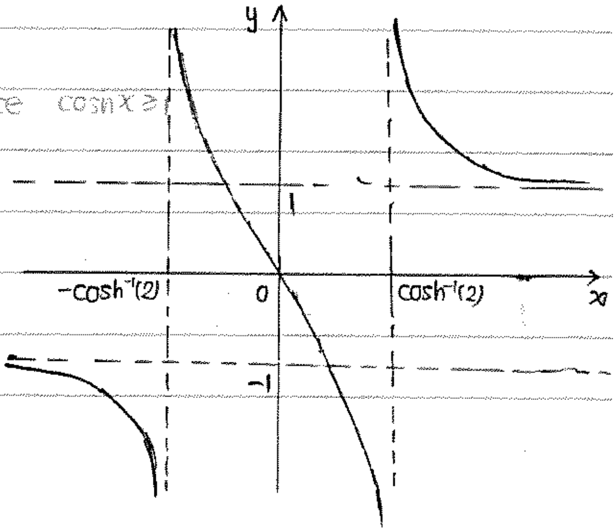
$$= \frac{\cosh^2 x - \sinh^2 x - 2\cosh x}{(\cosh x - 2)^2}$$

$$= \frac{1 - 2\cosh x}{(\cosh x - 2)^2}$$

always -ve since $\cosh x \geq 1$

always +ve

$\Rightarrow$ gradient always -ve.



Summary:
- $x \to \pm\infty$, $x = 0$
- 0s of function
- singularities (asymptotes)
- turning pts / slope
- sign of function
- specific values

振动函数

2. $g(x) = \boxed{\frac{\cos x}{x}} \leftarrow$ the envelope of an oscillating function

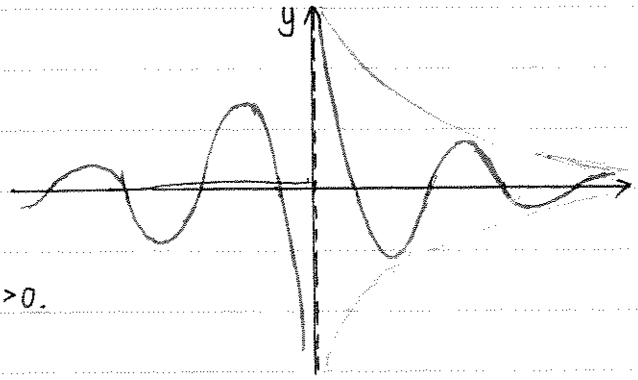
- singularity at $x=0$
- 0s at $x = (2k+1)\frac{\pi}{2}$
- as $|x| \to +\infty$, $g(x) \to 0$

  (with oscillation)
- observation:

  $g(x)$ is odd, so start with $x>0$.
- trick: to sketch the envelope

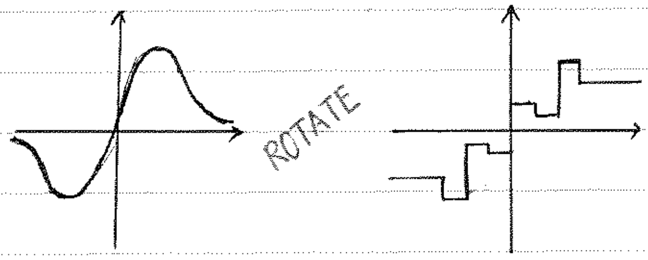
  If $\cos x = 1$, $g(x) = \frac{1}{x}$

  If $\cos x = -1$, $g(x) = -\frac{1}{x}$



Note:

Odd fn: $f(-x) = -f(x)$



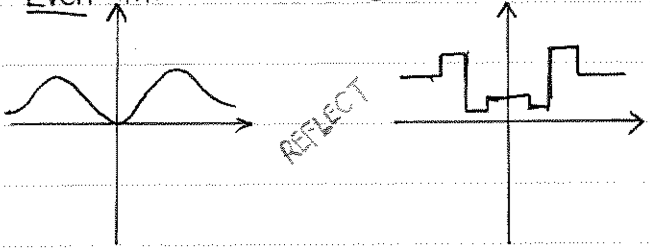
examples:

$\sin x$

$\sinh x$

x^3

Even fn: $f(-x) = f(x)$



examples:

$\cos x$

$\cosh x$

x^2

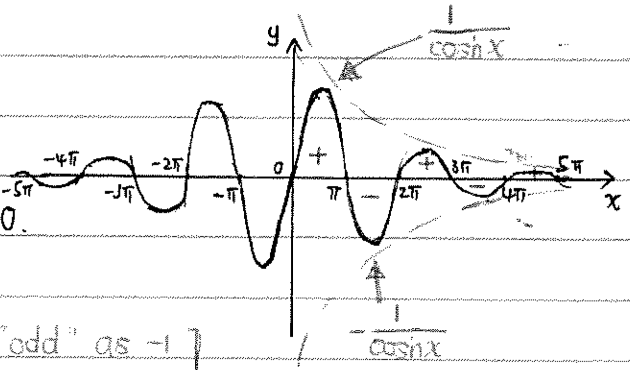
odd

3. $h(x) = \frac{\sin x}{\cosh x} \leftarrow$ even

- zeros where $\sin x = 0$, i.e. $x = m\pi$

- no singularities since $\cosh x \neq 0$.
- as $x \to +\infty$, $\cosh x \approx \frac{1}{2}e^x$

  so $\frac{\sin x}{\frac{1}{2}e^x} \to 0$.
- $\text{sign}(h(x)) = \text{sign}(\sin x)$ since $\cosh x > 0$.



For functions,

even × odd = odd

odd × odd = even

even × even = even

[think of "odd" as −1
& "even" as +1]

4. $\tanh x = \frac{\sinh x}{\cosh x}$

- 0s at $\sinh x = 0$

  $\Rightarrow \quad x = 0$
- singularity at 0
- $x \to +\infty$: $\tanh x \to 1$ [divide top & bottom by e^x]

  $x \to -\infty$: $\tanh x \to -1$ [divide top & bottom by e^{-x}]
- turning pts: $\frac{d}{dx}(\tanh x) = \frac{1}{\cosh^2 x} > 0$

  $\Rightarrow$ no turning pts

Show that $\binom{n}{r-1} + \binom{n}{r} = \binom{n+1}{r}$.

$$\text{LHS} = \frac{n!}{(r-1)!\,(n-r+1)!} + \frac{n!}{r!\,(n-r)!}$$

$$= \frac{n!}{(r-1)!\,(n-r)!}\left[\frac{1}{n-r+1} + \frac{1}{r}\right]$$

$$= \frac{n!}{(r-1)!\,(n-r)!} \cdot \frac{n+1}{(n-r+1)r}$$

$$= \frac{(n+1)!}{r!\,(n-r+1)!} = \binom{n+1}{r} = \text{RHS}$$

▨

Fri. 04/11/16

MATH1401. Mathematical Methods I

Prof. Halburd

Chapter 4. § Functions of 2 & 3 Variables §

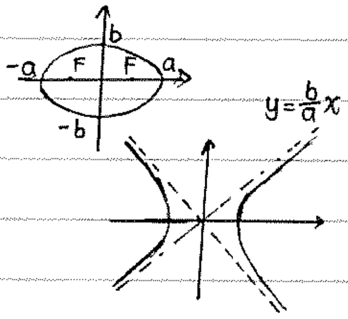
4·1. Conic Sections 圆锥曲线

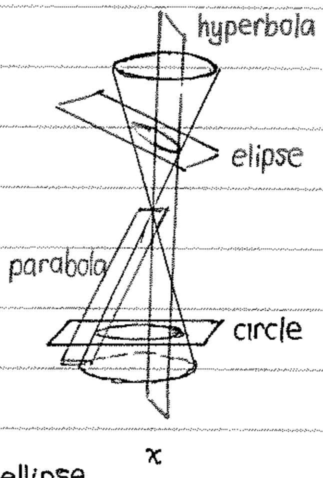
$x^2+y^2=a^2$ circle

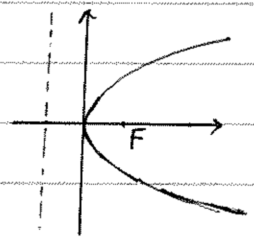
$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ ellipse 椭圆

$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ hyperbola 双曲线

$x=ky^2$ parabola 抛物线







Note: ellipse

$$\left(\frac{x-d}{a}\right)^2+\left(\frac{y-f}{b}\right)^2=1 \quad (a>b>0)$$

Then, focus:

$(c+d, f)$ and $(-c+d, f)$

where $c^2=a^2-b^2$

4·2 Level Set

- Def.

$\mathbb{R}^2 \equiv$ '2D' Let $f(x,y)$ be a function of 2 variables.

A set of the form $\{(x,y)\in\mathbb{R}^2 : f(x,y)=c\}$ for some given constant c is called a level set $\overset{(\equiv)}{,}$ level curve $\overset{(\equiv)}{\text{or}}$ contour.

✓ $\mathbb{R}^2=\{(x,y) : x,y\in\mathbb{R}\}$ plane

✓ $\mathbb{R}^3=\{(x,y,z) : x,y,z\in\mathbb{R}\}$ 3-space

- EXAMPLE ①:

$$f(x,y)=x^2+y^2$$

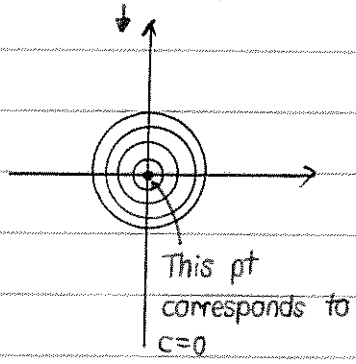
The level sets are $f(x,y)=c$.

$$\Leftrightarrow x^2+y^2=c$$

This is $\begin{cases} \text{a circle}, & \text{if } c>0 \\ \{(0,0)\}, & \text{if } c=0 \\ \emptyset, & \text{if } c<0 \end{cases}$ (radius $\sqrt{c}$ centred at $(0,0)$)

empty set → $\emptyset$

A contour plot



This pt corresponds to c=0

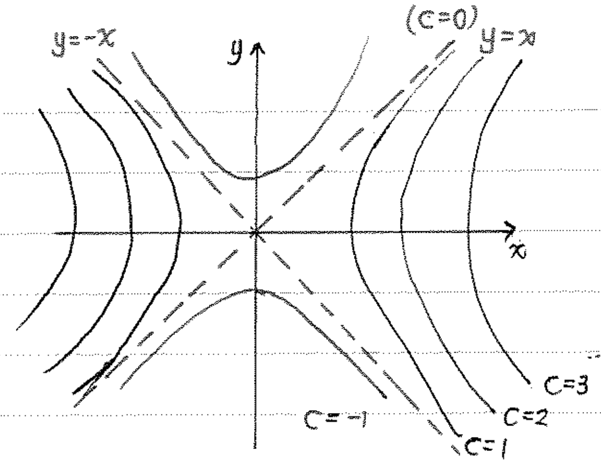
- EXAMPLE ②:

$$f(x,y) = x^2 - y^2$$

The level sets are $f(x,y) = c$

$$\Leftrightarrow x^2 - y^2 = c$$

This is $\begin{cases} \text{a hyperbola}, & \text{if } c>0 \\ y=\pm x, & \text{if } c=0 \\ y^2+c = x^2, & \text{if } c<0 \end{cases}$



4·3 Graph in 3 Dimensions

- ✓ The graph of a function $f(x,y)$ in 3d ($\mathbb{R}^3$) is the set of all (x,y,z) s.t.

$$z = f(x,y)$$

✓ Also, for functions $f(x,y,z)$, we can plot level sets, level curves, etc.

- Recall:

$ax+by+cz=1$ is a plane in $\mathbb{R}^3$.

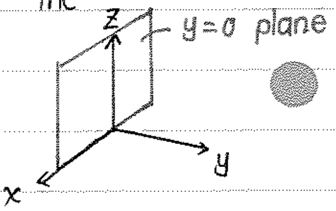
In particular, $y=2x+1$ is a plane in $\mathbb{R}^3$.

↗ the line (in $\mathbb{R}^2$) $y=2x+1$ being streched up in z direction.

(since z can be anything you like).

- ✓ The 3 special planes $x=0$, $y=0$, and $z=0$ are called the coordinite planes.

↗ y-z plane ↑ x-z plane ↖ x-y plane



✓ In plots of $f(x,y,z)=0$, we intersect the graph with planes parallel to the coordinate planes.

(i.e. $x=k$, or $y=k$, etc.)

- EXAMPLE ①:

$$z = x^2+y^2$$

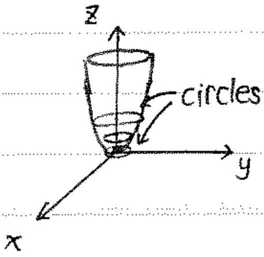
Take slices Intersect this with a plane of the form $z=k$.

$$\begin{cases} z = x^2+y^2 \\ z=k \end{cases} \Rightarrow x^2+y^2=k$$

$\therefore$ $k>0$ circles

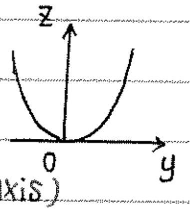
$k=0$ $(0,0)$

$k<0$ $\emptyset$



Now intersect surface with the coordinate plane $x=0$.

$$\begin{cases} x=0 \\ z=x^2+y^2 \end{cases} \Rightarrow z=y^2 \quad \text{parabola}$$



旋转抛物面 So we have a $\boxed{\text{paraboloid}}$ $az=x^2+y^2$ (parabola rotated around an axis)

- EXAMPLE ②.

$$\frac{x^2}{4}+y^2+z^2=1$$

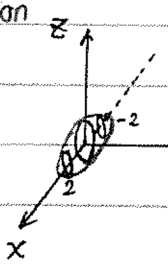
Intersect with plane $x=k$. take slices in x direction (set x as a constant)

$$\Rightarrow y^2+z^2=1-\frac{k^2}{4}$$

$\therefore$ a circle $-2<k<2$

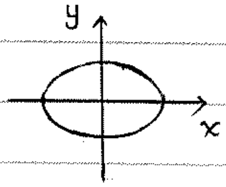
a point $k=\pm 2$

$\varnothing$ $|k|>2$



Now intersect with $z=0$

$$\Rightarrow \frac{x^2}{4}+y^2=1$$ an ellipse (focus on x axis since $4>1$)



椭球 So we have an $\boxed{\text{ellipsoid}}$ $\frac{x^2}{a}+\frac{y^2}{b}+\frac{z^2}{c}=1$

- EXAMPLE ③.

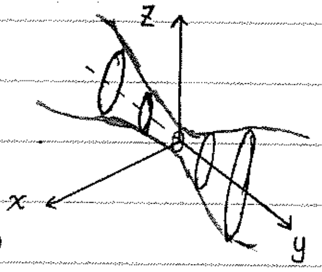
$$x^2-y^2+z^2=1$$

Intersect with plane $y=k$.

$$\Rightarrow x^2+z^2=1+k^2$$

$\therefore$ a circle $\forall k$

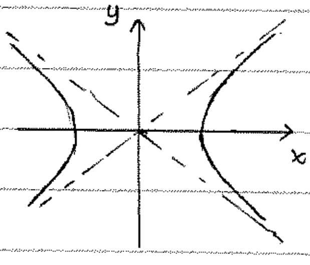
$\Rightarrow$ always circles with the smallest circle at $k=0$



Now intersect with $z=0$.

$$\Rightarrow x^2-y^2=1$$

$\therefore$ a hyperbola in x-y plane



单叶双曲面 So we have an $\boxed{\text{1-sheeted hyperboloid}}$, $\frac{x^2}{a}+\frac{y^2}{b}-\frac{z^2}{c}=1$

Advice: Go with circles first!

Let's revisit this example using an alternative method.

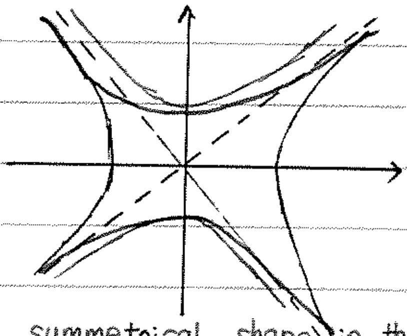
Suppose we take slices $z=k$ first

$$\Rightarrow x^2-y^2=1-k^2$$

$\therefore$ hyperbola if $|k|<1$

line $y=\pm x$ if $|k|=1$

hyperbola if $|k|>1$



We cannot take $x=0/y=0$ since we do not have a symmetrical shape in this case.

Take $y=c$.

……

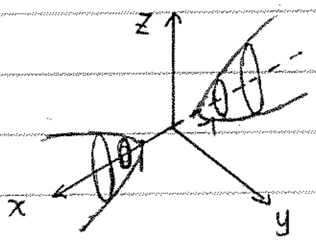
· <u>EXAMPLE ④</u>:

$$x^2-y^2-z^2=1$$

Intersect with $x=k$.

This is rotational symmetric about x-axis.

$$\Rightarrow y^2+z^2=k^2-1 \quad \begin{cases} \text{circle}, & \text{if } |k|>1 \\ \text{point } \{(0,0)\}, & \text{if } k=\pm 1 \\ \emptyset, & \text{if } |k|<1 \end{cases}$$



Now intersect with $z=0$.

$$\Rightarrow x^2-y^2=1 \quad \text{hyperbola}$$

双叶双曲面

So we have a $\boxed{\text{2-sheeted hyperbaloid}}$. $-\frac{x^2}{a}-\frac{y^2}{b}+\frac{z^2}{c}=1$

4.4 Partial Derivative

· <u>Def.</u>

The partial derivative of a function $f(x,y)$ is

pronounced '<u>partial / curly</u> df by dx'

$$\frac{\partial f}{\partial x}=\lim_{h\to 0}\frac{f(x+h,y)-f(x,y)}{h}$$

$$\frac{\partial f}{\partial y}=\lim_{h\to 0}\frac{f(x,y+h)-f(x,y)}{h} \qquad \text{(if they exist)}$$

✓ Treat y as a constant if partial derivative wrt x.

Treat x as a constant if partial derivative wrt y.

✓ EXAMPLE ①:

$$f(x,y)=x^2+y^2$$

$$\frac{\partial f}{\partial x}=\lim_{h\to 0}\frac{f(x+h,y)-f(x,y)}{h}$$

$$=\lim_{h\to 0}\frac{[(x+h)^2+y^2]-[x^2+y^2]}{h}$$

$$=\lim_{h\to 0}(2x+h)$$

$$=2x$$

✓ EXAMPLE ②:

$$g(x,y)=xe^y+x+2y$$

$$\frac{\partial g}{\partial x}=e^y+1 \qquad \text{(treat } y \text{ as if it is a constant)}$$

$$\frac{\partial g}{\partial y} = xe^{y}+2 \qquad \text{(treat } x \text{ as if it is a constant)}$$

- Other notation:

$$\frac{\partial f(x,y)}{\partial x} = f_x(x,y)$$
$$\frac{\partial f(x,y)}{\partial y} = f_y(x,y)$$

Note: $\cancel{f'(x,y)}$ means nothing !

- Also, $\frac{\partial f}{\partial x}$ can be differentiated again

$$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial x^2} = (f_x)_{\textcircled{x}} = f_{xx} \leftarrow \text{subscript means wrt } x$$
$$\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial y \partial x} = (f_x)_y = f_{xy}$$

Similarly, $\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial x \partial y} = (f_y)_x = f_{yx}$

$$\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial y^2} = (f_y)_y = f_{yy}$$

✓ EXAMPLE ①.

$$f(x,y) = y^2e^x + x^2 + 3y$$
$$\frac{\partial f}{\partial x} = y^2e^x + 2x$$
$$\frac{\partial f}{\partial y} = 2ye^x + 3$$
$$\therefore \frac{\partial^2 f}{\partial x^2} = y^2e^x + 2$$
$$\frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = 2e^x$$
$$\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = 2ye^x$$
$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = 2ye^x$$

✓ In general, if $\frac{\partial^2 f}{\partial x \partial y}$ and $\frac{\partial^2 f}{\partial y \partial x}$ are continuous, then they're equal.

✓ EXAMPLE ②:

Find h_x, h_y & h_z where

$$h(x,y,z) = x^y + y^z + z^x$$

[same def. for fns of 3 variables] 'fns' ≡ 'functions'

$$h_x = \frac{\partial h}{\partial x} = yx^{y-1} + (\ln z)z^x$$
$$h_y = \frac{\partial h}{\partial y} = (\ln x)x^y + zy^{z-1}$$
$$h_z = \frac{\partial h}{\partial z} = (\ln y)y^z + xz^{x-1}$$

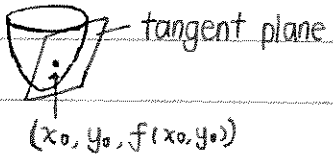
$$\left[\begin{array}{l} z^x = e^{x\log z} \\ \text{differentiate wrt } x: \ \frac{\partial z^x}{\partial x} = (\ln z)z^x \end{array}\right.$$

4·4·1 Equation of Tangent Planes

- Graph $z = f(x,y) \leftarrow z$ as a function of x and y.

Ques. Find the tangent plane at (x_0, y_0)

– We want to find 2 tangent vectors in the plane.



- Intersect $z=f(x,y)$ with the plane $y=y_0$.

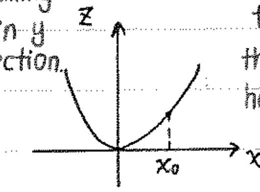
$$z=f(x,y_0)$$

Slope of $z=f(x,y_0)$ at $x=x_0$ (in x-z plane) is $f_x(x_0,y_0)$

So a tangent vector is $\underline{i}+f_x(x_0,y_0)\underline{k}$ ← no $\underline{j}$ ∵ taking the slice in y direction.

- Same game with $x=x_0$ (plane) gives the 2nd vector

$$\underline{j}+f_y(x_0,y_0)\underline{k}$$



tangents in the plane $y=y_0$ has slope $\frac{\partial f}{\partial x}(x_0,y_0)$

Note.

A tangent to a surface is a tangent to any curves in the surface.

Mon. 14/11/16

MATH1401: Mathematical Methods I

Prof. Halburd

- Taking the slice $y=y_0$ gives a tangent vector

$$\underline{v_1}=\underline{i}+\frac{\partial f(x_0,y_0)}{\partial x}\underline{k}$$

Taking the slice $x=x_0$ gives a second tangent vector

$$\underline{v_2}=\underline{j}+\frac{\partial f(x_0,y_0)}{\partial y}\underline{k}$$

- $\underline{v_1}$ & $\underline{v_2}$ lie in the tangent plane to $z=f(x,y)$ at $(x_0,y_0,f(x_0,y_0))$.

So a normal vector to the plane is

$$\underline{n}=\underline{v_1}\times\underline{v_2}$$

$$=\begin{vmatrix}\underline{i} & \underline{j} & \underline{k}\\ 1 & 0 & f_x\\ 0 & 1 & f_y\end{vmatrix}$$

$$=-f_x\underline{i}-f_y\underline{j}+\underline{k}$$

- If we have a particular pt in plane $P_0(x_0,y_0,f(x_0,y_0))$ and a general pt $P(x,y,z)$, equation of the plane is

$$\overrightarrow{P_0P}.\underline{n}=0$$

$$[(x-x_0)\underline{i}+(y-y_0)\underline{j}+(z-f(x_0,y_0))\underline{k}]\cdot(-f_x\underline{i}-f_y\underline{j}+\underline{k})=0$$

$$-f_x(x_0,y_0)(x-x_0)-f_y(x_0,y_0)(y-y_0)+(z-f(x_0,y_0))=0$$

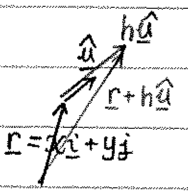
$$\Rightarrow \boxed{z=f(x_0,y_0)+f_x(x_0,y_0)(x-x_0)+f_y(x_0,y_0)(y-y_0)}$$ eqn of a tangent plane

4.5 Directional Derivative

- $\frac{\partial f(x,y)}{\partial x}\equiv$ rate of change of f in the x-direction (by held fixed)

- **Def.**

Let $\hat{\underline{u}}$ be a unit vector, the directional derivative of $f(x,y)$ in the direction $\hat{\underline{u}} = u_1\underline{i} + u_2\underline{j}$ is



$$D_{\underline{\hat{u}}}f(x,y) = \lim_{h\to 0}\frac{f(\underline{r}+h\hat{\underline{u}})-f(\underline{r})}{h}$$

$$= \lim_{h\to 0}\frac{\boxed{f(x+hu_1,y+hu_2)}-f(x,y)}{h}$$

partial derivative wrt x (i.e. y held constant)

$$= \lim_{h\to 0}\left[\frac{f(x+hu_1,y+hu_2)-f(x,y+hu_2)\textcircled{u_1}}{h\textcircled{u_1}} + \frac{f(x,y+hu_2)-f(x,y)\textcircled{u_2}}{h\textcircled{u_2}}\right]$$

multiply top & bottom by u_1 in order to make hu_1 the denominator (because $\Delta x = hu_1$)

$$= \frac{\partial f}{\partial x}u_1 + \frac{\partial f}{\partial y}u_2$$

$$= (\nabla f).\hat{\underline{u}}$$ where $\nabla f = \frac{\partial f}{\partial x}\underline{i} + \frac{\partial f}{\partial y}\underline{j}$ = gradient of f

✓ In $\mathbb{R}^3$ (3d),

$$D_{\underline{\hat{u}}}f = \frac{\partial f}{\partial x}u_1 + \frac{\partial f}{\partial y}u_2 + \frac{\partial f}{\partial z}u_3 = (\nabla f)\cdot\hat{\underline{u}}$$

where $\nabla f = \frac{\partial f}{\partial x}\underline{i} + \frac{\partial f}{\partial y}\underline{j} + \frac{\partial f}{\partial z}\underline{k}$

✓ EXAMPLE:

Find the directional derivative of $f(x,y,z) = (x+2y)^3 + e^{-z}$ at $(1,-1,0)$ in the direction $\hat{\underline{u}} = \frac{1}{3}(-2\underline{i}+\underline{j}+2\underline{k})$.

$$\nabla f = \frac{\partial f}{\partial x}\underline{i} + \frac{\partial f}{\partial y}\underline{j} + \frac{\partial f}{\partial z}\underline{k}$$

$$= 3(x+2y)^2\underline{i} + 6(x+2y)^2\underline{j} - e^{-z}\underline{k}$$

'$\nabla f|_{(x_0,y_0,z_0)}$' $\equiv$ '∇f at (x_0,y_0,z_0)'

$$\nabla f|_{(1,-1,0)} = 3\underline{i} + 6\underline{j} - \underline{k}$$

$$\therefore D_{\underline{\hat{u}}}f = (\nabla f)\cdot\hat{\underline{u}}$$

$$= \begin{pmatrix}3\\6\\-1\end{pmatrix}\cdot\begin{pmatrix}-\frac{2}{3}\\ \frac{1}{3}\\ \frac{2}{3}\end{pmatrix}$$

$$= -2+2-\frac{2}{3}$$

$$= -\frac{2}{3} < 0$$ ← This implies that f decreases in this direction.

- Recall:

$$D_{\underline{\hat{u}}}f = \hat{\underline{u}}(\nabla f)$$

$$= |\nabla f|\cos\theta$$ ← since $|\hat{\underline{u}}|=1$

where θ is the angle between $\hat{\underline{u}}$ & ∇f.

- ✓ Fix f. (since $D_{\underline{\hat{u}}}f$ depends on both function f and direction $\hat{\underline{u}}$)

✓ $D_{\underline{\hat{u}}}f$ has a max, $|\nabla f|$, when $\theta = 0$

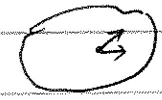
$\Rightarrow \hat{\underline{u}}$ points in the same direction as ∇f.

✓ $D_{\underline{\hat{u}}}f$ has a minimum, $-|\nabla f|$, when $\theta = \pi$

$\Rightarrow \hat{\underline{u}}$ points in the opposite direction as ∇f.

✓ $D_{\hat{u}} f = 0 \Leftrightarrow \hat{\underline{u}}$ is orthogonal to ∇f.

- Consider a level surface $f(x,y,z) = c$, for some constant c.



✓ $D_{\hat{u}} f = 0$ for $\hat{\underline{u}}$ tangent to the surface

$\Rightarrow (\nabla f)$ is normal to the surface.

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MATH1401: Mathematical Methods I

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Recap: ✓ $D_{\underline{u}} f(x,y) = \lim_{h \to +\infty} \frac{(x+hu_1, y+hu_2) - f(x,y)}{h}$

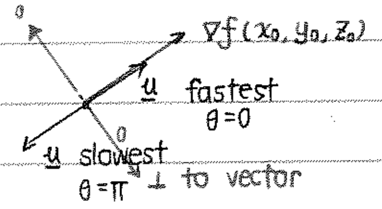
$= (\nabla f) . \hat{\underline{u}}$ where $\hat{\underline{u}} = u_1 \underline{i} + u_2 \underline{j}$ (unit vector)

$\nabla f = \frac{\partial f}{\partial x} \underline{i} + \frac{\partial f}{\partial y} \underline{j} + \left(\frac{\partial f}{\partial z} \underline{k}\right) \rightarrow$ if $f(x,y,z)$ is a function of 3 variables

(gradient of f)

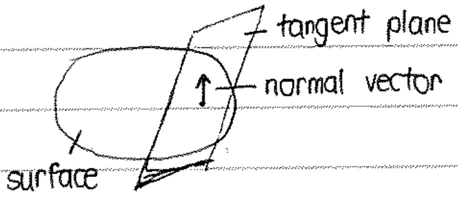
✓ Given $f(x,y,z)$.

∇f is some vector at $(x_0, y_0, z_0) \rightarrow$ consider different directions $\underline{u}$



✓ If we have a surface given as a level set $f(x,y,z) = c$

direction of tangent does not change



tangent to surface are in direction $\hat{\underline{u}}$ s.t. $D_{\hat{u}} f = 0$

$\Leftrightarrow (\nabla f) . \hat{\underline{u}} = 0$

If $\nabla f(x_0, y_0, z_0) \neq 0$, then it is a normal vector to the surface at (x_0, y_0, z_0).

✓ EXAMPLE:

Find the tangent plane to $x^2 + 2y^2 + 3z^2 = 6$ at $(1,1,1)$.

Soln: The surface is $f(x,y,z) = 6$ where $f(x,y,z) = x^2 + 2y^2 + 3z^2$

$\nabla f = 2x \underline{i} + 4y \underline{j} + 6z \underline{k}$

$\nabla f(1,1,1) = 2\underline{i} + 4\underline{j} + 6\underline{k} = 2(\underline{i} + 2\underline{j} + 3\underline{k})$ ← coefficient are all +ve

⇒ continue to move in the same direction

So a normal vector at (1,1,1) is

$$\underline{n} = \underline{i} + 2\underline{j} + 3\underline{k}$$

The tangent plane is $\underline{n} \cdot \overrightarrow{P_0P} = 0$

where $P_0 = (1,1,1)$, $P = (x,y,z)$.

$$\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \cdot \begin{pmatrix} x-1 \\ y-1 \\ z-1 \end{pmatrix} = 0$$

$$x - 1 + 2(y-1) + 3(z-1) = 0$$

$$\Leftrightarrow x + 2y + 3z = 6$$

Wed. 16/11/16

MATH1401: Mathematical Methods 1

Applied Tutorial

1. level set: ① $\{(x,y) : f(x,y)=c\}$ where $f(x,y)=x^2+2y^2$

when $c=1$, $\{(x,y) : x^2+2y^2=1\} \longrightarrow$ ellipse

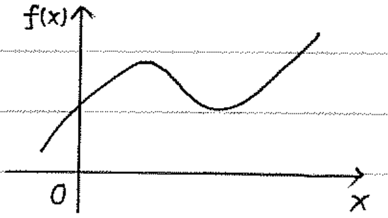
$c=0$, $\{(x,y) : x^2+2y^2=0\} \longrightarrow$ point $(0,0)$

② $g : \mathbb{R}^2 \to \mathbb{R}$

$\text{graph}(g) = \{(x,y,z) : z=f(x,y)\} \longrightarrow$ surface

③ $f : \mathbb{R} \to \mathbb{R}$

$\text{graph}(f) = \{(x,f(x)) : x \in \mathbb{R}\}$



2. Functions of Multiple Variables

$\frac{\partial g(x_0,y_0)}{\partial x}$

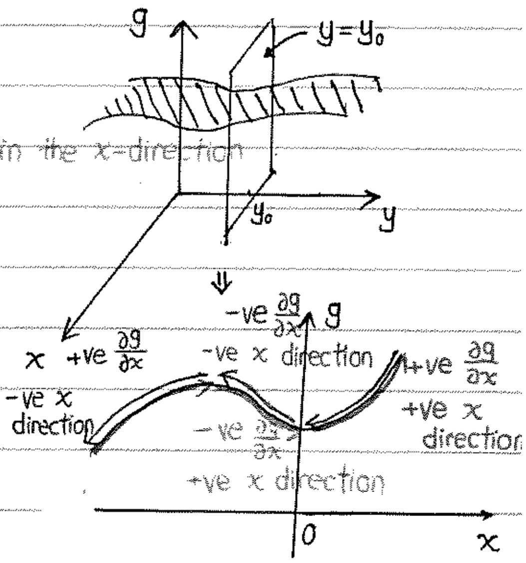
- gradient of a function means

$$(\nabla g)(x,y) = \begin{pmatrix} \frac{\partial g}{\partial x}(x,y) \\ \frac{\partial g}{\partial y}(x,y) \end{pmatrix} \in \mathbb{R}^2$$

rate of change of f in the x-direction

means 2D

rate of change of f in the y-direction



Fix $y=y_0$

⊕ve $\frac{\partial g}{\partial x}$ & moves in ⊕ve x direction $\Rightarrow g\uparrow$

−ve $\frac{\partial g}{\partial x}$ & moves in +ve x direction $\Rightarrow g\downarrow$

+ve $\frac{\partial g}{\partial x}$ & moves in −ve x direction $\Rightarrow g\downarrow$

⊖ve $\frac{\partial g}{\partial x}$ & moves in ⊖ve x direction $\Rightarrow g\uparrow$

$\Rightarrow$ gradient points into direction of greater increase of g

(same sign $\Rightarrow g\uparrow$)

- gradient : perpendicular to the level set

Any point on the same level set has the same $g(x,y)$

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MATH1401: Mathematical Methods 1

Help Class

Prof. Wilson

§ gradient §

1. "$\underline{\nabla} f$" $\equiv \frac{\partial f}{\partial x}\underline{i} + \frac{\partial f}{\partial y}\underline{j} + \frac{\partial f}{\partial z}\underline{k}$

"grad f"

- Q1. Suppose $\underline{v} = 3x^2\underline{i} + z\underline{j} + (y+z^3)\underline{k}$

  Find f s.t. $\underline{v} = \underline{\nabla} f$. "constant" (depends on y & z, but not x)

Soln: $\frac{\partial f}{\partial x} = 3x^2 \Rightarrow f = x^3 + \boxed{A(y,z)}$ ①

$\frac{\partial f}{\partial y} = z \Rightarrow f = yz + B(x,z)$ ②

$\frac{\partial f}{\partial z} = y + z^3 \Rightarrow f = yz + \frac{1}{4}z^4 + C(x,y)$ ③

②−③: $0 = B(x,z) - \frac{1}{4}z^4 - C(x,y)$

$C(x,y) = \underbrace{-\frac{1}{4}z^4 + B(x,z)}_{\text{no } y \text{ dependence}}$

$\Rightarrow C(x,y) = C(x)$ and

$B(x,z) = \frac{1}{4}z^4 + C(x)$

$f = x^3 + A(x,z)$

$f = yz + \frac{1}{4}z^4 + C(x)$

$\Rightarrow f = x^3 + yz + \frac{1}{4}z^4 + D$ general solution

- Q2. $\underline{v} = yz\underline{j} + y\underline{k}$

Soln: $\left.\begin{aligned} \frac{\partial f}{\partial y} &= yz \\ \frac{\partial f}{\partial z} &= y \\ \frac{\partial f}{\partial x} &= 0 \end{aligned}\right\} \begin{aligned} f &= \tfrac{1}{2}y^2z + A(z) \quad ① \\ f &= zy + B(y) \quad ② \end{aligned}$

a function of Z

①−②: $0 = yz(\frac{1}{2}y - 1) + A(z) - B(y)$

$A(z) = -yz(\frac{1}{2}y - 1) + B(y)$ cannot get zy out of $A(z)$

Not possible.

Not every vector held $\underline{v}(x, y, z)$ is a gradient $\underline{\nabla} f$.

Summary: Mixed Derivative Theorem

For well-behaved functions $f(x,y,z)$,

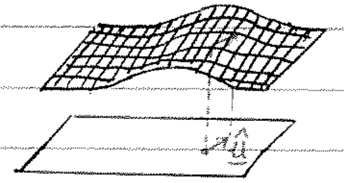
$$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)$$

so if $\underline{v} = v_1\underline{i} + v_2\underline{j} + v_3\underline{k}$ is a gradient, then

$$\left.\begin{aligned}\frac{\partial}{\partial y}(V_1) &= \frac{\partial}{\partial x}(V_2)\\ \frac{\partial}{\partial z}(V_1) &= \frac{\partial}{\partial x}(V_3)\\ \frac{\partial}{\partial z}(V_2) &= \frac{\partial}{\partial y}(V_3)\end{aligned}\right\}$$

These are sufficient conditions
(if all is smooth, differentiable, etc.)

2. $z = f(x, y)$ [2D]



✓ $\underline{\hat{u}} \cdot \underline{\nabla} f$ is the slope I feel if I walk in the direction of $\underline{\hat{u}}$.

✓ The direction of $\underline{\nabla} f$ is in the direction of steepest increase of f.

✓ $|\underline{\nabla} f|$ is the max slope.

✓ perpendicular to $\underline{\nabla} f$ is the line/ contour $f = \text{constant}$.

. $g(x, y, z)$ [3D]

$\underline{\nabla} g$ points in the direction of increasing g.

$|\underline{\nabla} g|$ tells us how fast g increases in that direction.

. A surface can be defined as $g = \text{constant}$.

$\underline{\nabla} g$ is perpendicular to the surface.

Fri. 18/11/16

MATH1401: Mathematical Methods I

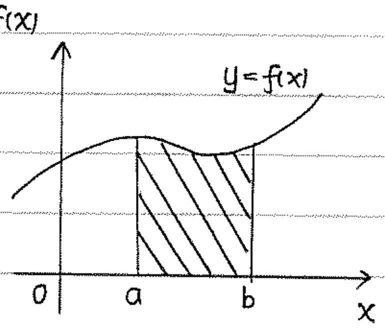
Prof. Halburd

Chapter 5. § Integration §

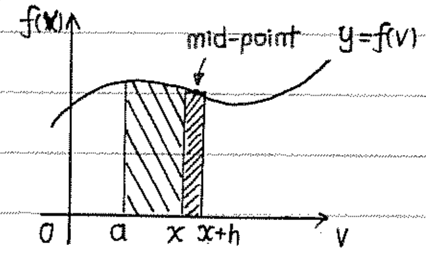
4.1 Introduction : Integral Calculus

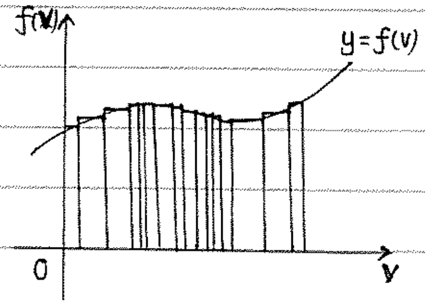
- Def.

  Area $\int_a^b f(x)\,dx$ defines integral.



- Riemann Integrals





✓ Let $F(x) = \int_a^x f(v)\,dv$.

$$F'(x) = \lim_{h\to 0}\frac{F(x+h)-F(x)}{h}$$

$$= \lim_{h\to 0}\frac{\int_a^{x+h} f(v)\,dv - \int_0^x f(v)\,dv}{h}$$

$$= \lim_{h\to 0}\frac{\boxed{\int_x^{x+h} f(v)\,dv}}{h} \quad \text{area of } ▨ = h.f(\text{mid-point})$$

$$= \lim_{h\to 0}\frac{\boxed{hf(x)}}{h} \xleftarrow[\text{approx.}]{} h.f(\text{mid-point})$$

$$= f(x)$$

- The number on top is approx. the area of triangle. ($\because$ mid-point)
- The partition does not need to be regular. The approximation works as long as the width of each subdivision tends to 0.

- Fundamental Theorem of Calculus

  $\frac{d}{dx}\int f(x)\,dx = f(x)$

4.2 Improper Integrals

- Suppose that f is defined in $[a,b]$ except at $c \in (a,b)$, then

$$\int_a^b f(x)\,dx = \underset{\boxed{\varepsilon\to 0^+}}{\lim}\int_a^{c-\varepsilon} f(x)\,dx + \lim_{\varepsilon\to 0^+}\int_{c+\varepsilon}^b f(x)\,dx$$

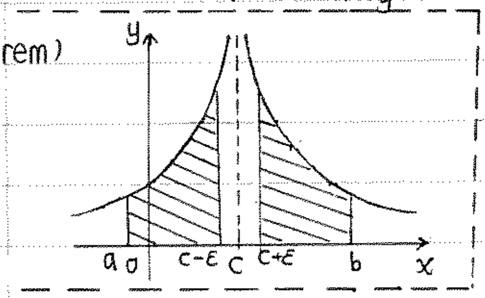
ε approaches 0 from the +ve side

Both limits must exist.

- EXAMPLE ①:

How to integrate $f(x)=\frac{1}{x^2}$?

(for theorem)



$$\left[-\frac{1}{x}\right]_{-1}^{1}$$

$$\int_{-1}^{1}\frac{1}{x^2}\,dx=\boxed{-\frac{1}{x}\Big|_{-1}^{1}}=(-1)+(-1)=-2$$

And this is WRONG.

<u>Right way</u>: Consider $\lim\limits_{\varepsilon\to a}\int_{-1}^{a-\varepsilon}\frac{1}{x^2}\,dx$

$$=\lim_{\varepsilon\to 0^+}(-x^{-1})\Big|_{-1}^{-\varepsilon}$$

$$=\lim_{\varepsilon\to 0^+}(\varepsilon^{-1}-1)$$

$$=+\infty$$

Also, $\lim\limits_{\varepsilon\to 0^+}\int_{\varepsilon}^{1}\frac{1}{x^2}\,dx=\infty$

Hence, integral $\int_{1}^{-1}\frac{1}{x^2}\,dx$ doesn't converge.

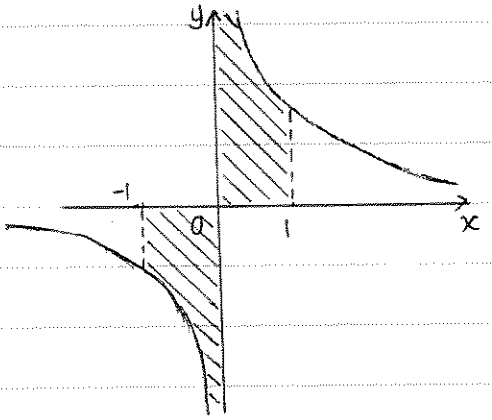
✓ EXAMPLE ②:

$$\int_{-1}^{1}\frac{1}{x}\,dx=\lim_{c\to 0^-}\int_{-1}^{c}\frac{1}{x}\,dx+\lim_{c\to 0^+}\int_{c}^{1}\frac{1}{x}\,dx$$

not continuous
⇓
must integrate separately

$$=\lim_{c\to 0^-}\ln|x|\Big|_{-1}^{c}+\lim_{c\to 0^+}\ln|x|\Big|_{1}^{c}$$

$$=\underbrace{\lim_{c\to 0^-}(\ln|c|-\ln|-1|)}_{-\infty}+\underbrace{\lim_{c\to 0^+}(\ln 1-\ln c)}_{+\infty}$$



indeterminate
(doesn't converge)

✓ EXAMPLE ③:

$$\int_0^{\infty}e^{-x}\,dx=\lim_{L\to+\infty}\int_0^{L}e^{-x}\,dx$$

$$=\lim_{L\to+\infty}(-e^{-x})\Big|_0^{L}$$

$$=1-\lim_{L\to+\infty}e^{-L}=1$$

- Recap: $y = f(x)$

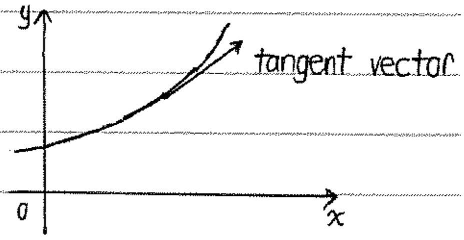
$g(x,y) = f(x) - y = 0$ $\quad \frac{\partial g}{\partial y}$

gradient of g is the derivative wrt x $\rightarrow \boxed{\nabla g} = f'(x)\underline{i} - \boxed{1}\underline{j}$ in this case

$\left(\nabla g = \frac{\partial g}{\partial x}\underline{i} - \frac{\partial g}{\partial y}\underline{j} \right.$ in general)

$\underline{T} . (\nabla g) = 0$

$\Rightarrow \underline{T} = \underline{i} + f'(x)\underline{j}$ tangent vector



4·3 Integration by Parts

- $\frac{d(uv)}{dx} = \frac{du}{dx}.v + u\frac{dv}{dx}$

$\Rightarrow \int u . \frac{dv}{dx} dx = uv - \int v \frac{du}{dx} dx$

e.g. differentiate the polynomial & integrate trig.

- Especially useful for 'inverse fns'

e.g. ① $\int \ln x \, dx = \int 1 . \ln x \, dx$

$u = \ln x \qquad v' = 1$

② $\int \arcsin x \, dx = \int 1 . \arcsin x \, dx$

$u = \arcsin x \qquad v' = 1$

- Iterated Integration (Reduction Formula)

EXAMPLE:

✓ $I_n = \int \sin^n x \, dx$

$= \int \sin x . \sin^{n-1} x \, dx$

$u = \sin^{n-1} x \qquad v' = \sin x$

$u' = (n-1)\sin^{n-2} x \cos x \qquad v = -\cos x$

$I_n = -\sin^{n-1} x \cos x + \int \cos^2 x \, (n-1) \sin^{n-2} x \, dx$

$= -\sin^{n-1} x \cos x + \int (1 - \sin^2 x)(n-1)\sin^{n-2} x \, dx$

$= -\sin^{n-1} x \cos x + (n-1) \int (\sin^{n-2} x - \sin^n x) \, dx$

$= -\sin^{n-1} x \cos x + (n-1) \underbrace{\int \sin^{n-2} x \, dx}_{I_{n-2}} - (n-1) \underbrace{\int \sin^n x \, dx}_{I_n}$

$\therefore nI_n = (n-1) I_{n-2} - \sin^{n-1} x \cos x$

✓ Use this iteratively to reduce I_n to evaluating

$I_0 = \int dx = x + c$

$I_1 = \int \sin x \, dx = -\cos x + c$

4.4 Substitution Method

- $\int f(x)\,dx = \int f(y)\frac{dx}{dy}\,dy$
- $\int f'(x)e^{f(x)}dx = e^{f(x)} + C$

$\int \frac{f'(x)}{f(x)}\,dx = \ln[f(x)] + C$

$\int f'(x)[f(x)]^n dx = \frac{[f(x)]^{n+1}}{n+1} + C$

✓ EXAMPLE:

Find $\int e^{\sin x}\cos x\,dx$

Soln: $u = \sin x \quad u' = \cos x$

$$\int \cos x e^{\sin x} dx = \int \cos x e^{u}\frac{dx}{du}\,du$$
$$= \int e^{u}du$$
$$= e^{u} + C$$
$$= e^{\sin x} + C$$

- $\sqrt{a^2-x^2}$ $\quad x = a\sin u$

$\sqrt{a^2+x^2}$ $\quad x = a\sinh u$

$\sqrt{x^2-a^2}$ $\quad x = a\cosh u$

$\frac{1}{a^2+x^2}$ $\quad x = a\tan u$

- Some FP3 Formulae:

$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \arcsin(\frac{x}{a}) + c$

$\int \frac{1}{\sqrt{x^2+a^2}}\,dx = \mathrm{arsinh}(\frac{x}{a}) + c$

$\int \frac{1}{\sqrt{x^2-a^2}}\,dx = \mathrm{arcosh}(\frac{x}{a}) + c$

$\int \frac{1}{a^2-x^2}\,dx = \frac{1}{a}\mathrm{artanh}(\frac{x}{a}) + c$

$\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\arctan(\frac{x}{a}) + c$

$\int \frac{1}{x^2-a^2}\,dx = \frac{1}{2a}\ln\left|\frac{x+a}{x-a}\right| + c$

4.5 Integrating Rational Functions (Partial Fractions)

- $R(x) = \frac{P_m(x)}{Q_n(x)}$

where $P_m(x)$ is a polynomial of degree m,

$Q_n(x)$ is a polynomial of degree n.

- How to find $\int R(x)\,dx$?

① If $m \geq n$, use polynomial division (long division) to obtain a rational function with $m < n$.

② Factorise $Q_n(x)$ (over $\mathbb{R}$)

$$Q_n(x) = (x-a_1)^{\text{\textcircled{r_1}}}(x-a_2)^{r_2} \cdots (x-a_\mu)^{r_\mu}(x^2+b_1x+c_1)^{s_1} \cdots (x^2+b_\nu x+c_\nu)^{s_\nu}$$

③ The ration function $R(x)$ can be expressed in terms of partial fractions as

$$R(x) = \frac{P_m(x)}{Q_n(x)}$$

$$= \overbrace{\left[\frac{A_1}{x-a_1} + \frac{A_2}{(x-a_1)^2} + \cdots + \frac{A_{r_1}}{(x-a_1)^{\text{\textcircled{r_1}}}}\right]}^{r_1 \text{ terms}} + \cdots + \overbrace{\left[\frac{B_1}{x-a_\mu} + \cdots + \frac{B_{r_\mu}}{(x-a_\mu)^{r_\mu}}\right]}^{r_\mu \text{ terms}}$$

$$+ \left[\frac{\overbrace{C_1x+D_1}^{\text{degree } 1}}{\underbrace{x^2+b_1x+C_1}_{\text{degree } 2}} + \cdots + \frac{C_1x+D_1}{(x^2+b_1x+C_1)^{s_1}}\right] + \cdots + \left[\frac{C_\nu x+D_\nu}{(x^2+b_\nu x+C_\nu)} + \cdots + \frac{C_\nu x+D_\nu}{(x^2+b_\nu x+C_\nu)^{s_\nu}}\right]$$

The degree on numerator is one more than that on denominator

✓ EXAMPLE:

Find $\int_0^\infty \frac{dx}{(x+2)^2(x^2+1)}$.

Soln: $$\frac{1}{(x+2)^{\text{\textcircled{2}}}(x^2+1)} \equiv \underbrace{\frac{A}{x+2} + \frac{B}{(x+2)^2}}_{\text{repeated terms}} + \frac{\underline{Cx+D}}{x^2+1}$$ → degree 1 less than the denominator

$$1 \equiv A(x+2)(x^2+1) + B(x^2+1) + (Cx+D)(x+2)^2$$

④ Systematic way is to expand out and equate coefficients.

e.g. sub $x=-2$: $B=\frac{1}{5}$

equate coefficients:

$$1 = (A+C)x^3 + (2A+\tfrac{1}{5}+4C+D)x^2 + (A+4C+4D)x + (2A+\tfrac{1}{5}+4D)$$

x^3: $A+C=0$

x^2: $2A+\frac{1}{5}+4C+D=0$

x : $A+4C+4D=0$

x^0: $2A+\frac{1}{5}+4D=1$

$\therefore A = \frac{4}{25}$

$B=\frac{1}{5}$

$C = -\frac{4}{25}$

$D = \frac{3}{25}$

Mon. 21/11/16

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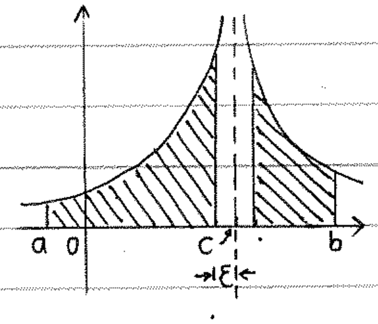
Recap:

4.2 Improper Integral



① f continuous on [a,b] except at x=c.

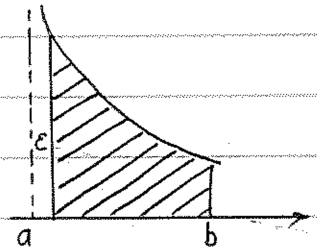
$$\int_a^b f(x)dx = \underbrace{\lim_{\varepsilon\to0^+}\int_a^{c-\varepsilon} f(x)dx}_{} + \underbrace{\lim_{\varepsilon\to0^+}\int_{c+\varepsilon}^{b} f(x)dx}_{}$$

both limits must exist

② f continuous on (a,b] but not at x=a.

$$\int_a^b f(x)dx = \lim_{\varepsilon\to0^+}\int_{a+\varepsilon}^{b} f(x)\,dx$$

if this limit exists



③ f continuous on [a,+∞), then

$$\int_a^{\infty} f(x)dx = \lim_{b\to+\infty}\int_a^b f(x)dx$$

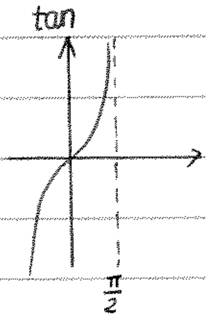
4·5 Partial Fractions (Cont.)

✓ EXAMPLE: (cont.)

$$\int_0^{\infty}\frac{dx}{(x+2)^2(x^2+1)} = \int_0^{\infty}\left[\tfrac{4}{25}(x+2)^{-1}+\tfrac{1}{5}(x+2)^{-2}+(-\tfrac{4}{25}x+\tfrac{3}{25})(x^2+1)^{-1}\right]dx$$

$$= \lim_{b\to+\infty}\int_0^b\left[\tfrac{4}{25}(x+2)^{-1}+\tfrac{1}{5}(x+2)^{-2}-\frac{4x}{25(x^2+1)}+\frac{3}{25(x^2+1)}\right]dx$$

$$= \lim_{b\to\infty}\left\{\left[\tfrac{4}{25}\ln|x+2|-\tfrac{1}{5}(x+2)^{-1}-\tfrac{2}{25}\ln(x^2+1)+\tfrac{3}{25}\arctan x\right]_0^b\right\}$$



$$= \lim_{b\to\infty}\left\{\tfrac{2}{25}\ln\frac{|b+2|}{b^2+1}-\frac{1}{5(b+2)}+\tfrac{3}{25}\arctan b\right\}$$

$$\to 1 \text{ since } \frac{(1+\frac{2}{b})^2}{1+\frac{1}{b^2}}\to 1$$

Always remember to take out constant terms esp. when dealing with log

$$= \tfrac{3}{25}.\tfrac{\pi}{2}-\tfrac{4}{25}\ln 2+\tfrac{1}{10}$$

$$= \tfrac{3\pi}{50}-\tfrac{4}{25}\ln 2+\tfrac{1}{10}$$

4·6 Double Angle Formulae

- $\cos^2u+\sin^2u=1$

$$1+\tan^2u=\sec^2u=\frac{1}{\cos^2u}$$

$$\Rightarrow \cos u=\frac{1}{\sqrt{1+\tan^2u}} \qquad (\cos u>0)$$

Similarly, $\sin u = \frac{\tan u}{\sqrt{1+\tan^2 u}}$

So, $\cos 2u = \cos^2 u - \sin^2 u = \frac{1-\tan^2 u}{1+\tan^2 u}$

$\sin 2u = 2\sin u\cos u = \frac{2\tan u}{1+\tan^2 u}$

Put $u = \frac{\theta}{2}$, $t = \tan u = \tan(\frac{\theta}{2})$, then.

$$\cos\theta = \frac{1-t^2}{1+t^2}, \quad \sin\theta = \frac{2t}{1+t^2}$$

$$\cos\frac{\theta}{2} = \frac{1}{\sqrt{1+t^2}}, \quad \sin\frac{\theta}{2} = \frac{t}{\sqrt{1+t^2}}$$

$$t = \tan\frac{\theta}{2} \Rightarrow \theta = 2\arctan t$$

$$\Rightarrow d\theta = 2\cos^2\frac{\theta}{2}dt$$

$$\Rightarrow d\theta = \frac{2}{1+t^2}dt$$

✓ EXAMPLE:

Find $\int_0^{\pi/2} \frac{d\theta}{2+\sin\theta}$.

Soln: change the limits

Let $t = \tan\frac{\theta}{2}$, then $\frac{dt}{d\theta} = \sec^2(\frac{\theta}{2})$

$$\int_0^1 \frac{\left(\frac{2dt}{1+t^2}\right)}{\left(2+\frac{2t}{1+t^2}\right)} = \int_0^1 \frac{2}{2+2t^2+2t}dt$$

$$= \int_0^1 \frac{1}{t^2+t+1}dt$$

$$= \int_0^1 \frac{1}{(t+\frac{1}{2})^2+\frac{3}{4}}dt$$

change the limits

Let $t+\frac{1}{2} = \frac{\sqrt{3}}{2}\tan u$, then $\frac{dt}{du} = \frac{\sqrt{3}}{2}\sec^2 u$

$$I = \int_{\arctan\frac{\sqrt{3}}{3}}^{\arctan\sqrt{3}} \frac{1}{\frac{3}{4}(\tan^2 u+1)} \cdot \frac{\sqrt{3}}{2}\sec^2 u \, du$$

$$= \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{4}{3} \cdot \frac{\sqrt{3}}{2} du$$

$$= \frac{2\sqrt{3}}{3}[u]_{\frac{\pi}{6}}^{\frac{\pi}{3}}$$

$$= \frac{\sqrt{3}}{9}\pi$$

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. Indeterminate forms of limits (facts) : $\frac{0}{0} / \frac{\infty}{\infty}$.

e.g. $\lim_{x\to 0^+} x^a(\log x)^b$ $\quad a, b > 0$

$\lim_{x\to 0^+} x^a(e^x)^b$ $\quad a, b > 0$

✓ exponential grows faster than power

power grows faster than log.

✓ L'Hôpital's Rule

$$\lim_{x\to x_0}\frac{f(x)}{g(x)} = \lim_{x\to x_0}\frac{f'(x)}{g'(x)} \quad \text{if} \quad f(x_0) = g(x_0) = 0 \text{ or } \pm\infty.$$

EXAMPLES:

$$\lim_{x\to 0}\frac{\sin x}{x} = \lim_{x\to 0}\frac{\cos x}{1} = 1$$

$$\lim_{x\to\infty}\frac{\ln x}{x^n} = \lim_{x\to\infty}\frac{\frac{1}{x}}{nx^{n-1}} = \lim_{x\to\infty}\frac{1}{nx^n} = 0$$

Fri. 25/11/16 (cont.)

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Chapter 6. § (Ordinary) Differential Equations (ODEs) §

- Def.

  The order of an ODE is the order of the highest derivative appearing in the equation.

- Def.

  The most general form of an n^{th} order ODE is

$$F\left(x, y, \frac{dy}{dx}, \dots, \frac{d^n y}{dx^{(n)}}\right) = 0$$

6·1 First Order Differential Equations

6·1·1 Separable Equations

- If an ODE is separable, then

$$\frac{dy}{dx} = f(x)g(y)$$

Note: $\frac{dy}{dx} = x^2+y^2$ is not separable

$$\Leftrightarrow \frac{1}{g(y)}.\frac{dy}{dx} = f(x)$$

integrate wrt x :

$$\int \frac{1}{g(y)}.\frac{dy}{dx}\,dx = \int f(x)\,dx$$

$$\Leftrightarrow \int \frac{dy}{g(y)} = \int f(x)\,dx$$

This gives a one-parameter family of solns.

Note: The parameter is an integration constant.

- EXAMPLE:

  Solve the ODE $x\frac{dy}{dx}+3y=2$, given the initial condition $y(1)=2$.

  (i.e. $y=2$ when $x=1$)

Soln:

✓ $x\frac{dy}{dx} = 2-3y$

$\int \frac{dy}{2-3y} = \int \frac{dy}{x}$ ← don't forget the absolute value

$-\frac{1}{3}\ln|2-3y| = \ln|x|+c$

$\ln|2-3y| = \ln|x|^{-3}\cdot\enclose{circle}{\hat{c}}$ ← another constant e^{-3c}

$2-3y = c_0 x^{-3}$ (where $c_0 = \pm e^{-3x}$)

✓ sub $y(1)=2$: $2-6=c_0=-4$

$\Rightarrow y = \frac{1}{3}(2+\frac{4}{x^3}) = \frac{2}{3}(1+\frac{2}{x^3})$

6·1·2 General First Order Linear Equations

$$\frac{dy}{dx} + a(x)y = \boxed{f(x)} \leftarrow \text{doesn't include any } y \text{ terms} \qquad (*)$$

- Idea :

✓ We multiply both sides of $(*)$ by a function $I(x)$ (called an integrating factor) to make LHS an exact derivative (of a product).

$$I(x)\frac{dy}{dx} + a(x)I(x)y = f(x)I(x) \qquad (\#)$$

✓ We want the LHS to be

$$\frac{d}{dx}(Iy) = I\frac{dy}{dx} + \frac{dI}{dx}y$$

✓ Then, we want: $\frac{dI}{dx} = aI$ (by comparing what we have and what we want)

integrate wrt x:

$$\int \frac{1}{I}.\frac{dI}{dx}\,dx = \int a(x)dx$$

$$\Leftrightarrow \quad \int \frac{dI}{I} = \int a\,dx \quad \text{'a' represents 'a(x)'}$$

$$\ln|I| = \int a(x)\,dx + c$$

$$I = \hat{c}\exp\left(\int a(x)dx\right) \qquad (\hat{c} = e^c)$$

Choose $\hat{c}$.

$$I(x) = \exp\left(\int a(x)dx\right)$$

✓ Then, $(\#)$ becomes

$$\frac{d}{dx}(Iy) = I(x)f(x)$$

(Integrate): $\Rightarrow I(x)\,y(x) = \int I(x)f(x)\,dx$

$$\Rightarrow \quad y(x) = \frac{1}{I(x)}\int I(x)f(x)\,dx$$

✓ To summarize,

$$b(x)\frac{dy}{dx} + c(x)y = g(x)$$

① divide by $b(x)$ to get a standard eqn

② compute the integrating factor $I(x)$.

③ multiply the eqn. by $I(x)$ s.t. the LHS is $\frac{d}{dx}(Iy)$.

④ integrate to find $g(x)$.

⑤ find particular soln.

- ✓ EXAMPLE ①:

Solve $x\frac{dy}{dx} + 2y = \frac{\cos x}{x}$ ... ①

$$y(\pi)=1.$$

Soln: $$\frac{dy}{dx}+\frac{2}{x}y=\frac{\cos x}{x^2} \qquad \cdots ②$$

Then $I(x)=\exp(\int \frac{2}{x}dx) = \exp(2\ln x) = x^2$

Multiply ② by $I=x^2$:

$$x^2\frac{dy}{dx}+2xy=\cos x$$

$$\Rightarrow \frac{d}{dx}(x^2y)=\cos x$$

$$\Rightarrow \quad x^2y \overset{\downarrow \text{integrate}}{=} \sin x + C$$

$y(\pi)=1$: $\pi^2=C$

$$\Rightarrow y=\frac{\sin x+\pi^2}{x^2}$$

✓ EXAMPLE ②.

Solve $x^2\frac{dy}{dx}+(1+x)y=\frac{1}{x}$

Soln: $$\frac{dy}{dx}+\left(\frac{1+x}{x^2}\right)y=\frac{1}{x^3} \qquad \cdots ①$$

Then

$$\begin{aligned} I(x) &= \exp\left(\int \frac{1+x}{x^2}dx\right) = \exp\left(\int(\tfrac{1}{x^2})dx+\int(\tfrac{1}{x})dx\right) \\ &= \exp\left(-\tfrac{1}{x}+\ln|x|\right) \\ &= xe^{-\frac{1}{x}} \end{aligned}$$

Multiply ① by $I(x)$:

$$xe^{-\frac{1}{x}}\frac{dy}{dx}+(\tfrac{1}{x}+1)e^{-\frac{1}{x}}y=\frac{1}{x^2}e^{-\frac{1}{x}}$$

$$\Rightarrow \frac{d}{dx}(xe^{-\frac{1}{x}}y) = \frac{1}{x^2}e^{-\frac{1}{x}} \qquad f'(x)e^{f(x)}$$

$$\Rightarrow \quad xe^{-\frac{1}{x}}y = \int\frac{1}{x^2}e^{-\frac{1}{x}}dx = e^{-\frac{1}{x}}+C$$

$$\Rightarrow y=\frac{1}{x}+\frac{C}{xe^{-1/x}}$$

- Bernoulli's Equation

$$\frac{dy}{dx}+yP(x)=y^nQ(x), \qquad n\neq 1$$

✓ This can be reduced to a linear equation by introducing

$$z=y^{1-n}$$

$$\Rightarrow \frac{dz}{dx}=(1-n)y^{-n}\frac{dy}{dx}$$

(substitute) $$\Rightarrow \frac{dz}{dx}=(1-n)y^{-n}\left(y^nQ(x)-yP(x)\right)$$

$$= (n-1)y^{-n}.\,yP(x) - (n-1)Q(x)$$
$$= (n-1)y^{1-n}P(x) - (n-1)Q(x)$$
$$= (n-1)zP(x) - (n-1)Q(x) \longleftarrow \text{general } 1^{st} \text{ order linear ODE}$$

6·2 Second Order Differential Equations

$$\frac{d^2y}{dx^2} + a(x)\frac{dy}{dx} + b(x)y = f(x) \quad \cdots (1)$$

✓ If $f(x)=0$, we have

$$y'' + a(x)y' + b(x)y = 0 \quad \cdots (2)$$

which is said to be homogeneous.

✓ $f(x)$ is sometimes called a forcing function.

✓ Suppose that y_1 & y_2 are solns of (2).

Let $y(x) \equiv \alpha y_1(x) + \beta y_2(x)$ where α, β are constants.

Then,

$$y'' + ay' + by = (\alpha y_1 + \beta y_2)'' + a(\alpha y_1 + \beta y_2)' + b(\alpha y_1 + \beta y_2)$$
$$= \alpha\{y_1'' + ay_1' + by_1\} + \beta\{y_2'' + ay_2' + by_2\} = 0$$

$\Rightarrow y$ solves (2).

✓ If y_1 & y_2 are independent solns,

(i.e. one is not a multiple of the other for 2nd order ODEs)

then $y = \alpha y_1 + \beta y_2$ is the general soln.

6·2·1 Constant Coefficient Homogeneous Linear Second Order ODEs

$$y'' + ay' + by = 0 \quad \cdots (3) \qquad a, b \text{ constant}$$

✓ We want to look for solns of (3) of the form

$$y(x) = e^{\lambda x} \quad \cdots (4)$$

$$\Rightarrow y' = \lambda e^{\lambda x}$$
$$y'' = \lambda^2 e^{\lambda x}$$

substitute into (3): $(\lambda^2 + a\lambda + b)e^{\lambda x} = 0$

$$\Leftrightarrow \lambda^2 + a\lambda + b = 0 \quad \cdots (5)$$

Characteristic Eqn.

Auxiliary Eqn.

✓ Case 1: (5) has 2 different real roots λ_1 & λ_2.

Then, eqn (3) has 2 roots

$$e^{\lambda_1 x} \quad \& \quad e^{\lambda_2 x}$$

(not multiple of each other)

So the general soln is

$$\boxed{y(x) = \alpha e^{\lambda_1 x} + \beta e^{\lambda_2 x}} \qquad \alpha, \beta \text{ constant}$$

✓Case 2: (5) has 1 real repeated root

$$\lambda = \frac{-a \pm \sqrt{a^2-4b}}{2} = -\frac{a}{2} \qquad \text{since } a^2-4b=0$$

due to 1 repeated root

Only 1 soln of eqn (4).

Let's look for another soln of the form

$$y = g(x)e^{\lambda x} \qquad \left(\lambda = -\frac{a}{2}\right)$$

(5) ⇔ $(\lambda+\sqrt{b})^2=0$

so $a = 2\sqrt{b}$

⇔ $a^2=4b$

$$\Rightarrow y' = (g'+\lambda g)e^{\lambda x}$$

$$y'' = (g''+2\lambda g'+\lambda^2 g)e^{\lambda x}$$

So eqn (3) becomes

$$(g''+2\lambda g'+\lambda^2 g)+a(g'+\lambda g)e^{\lambda x}+bge^{\lambda x}=0$$

$$\{g\underbrace{(\lambda^2+a\lambda+b)}_{\text{disappears because this is a characteristic eqn.}}+g'\underbrace{(2\lambda+a)}_{\text{disappears because } \lambda=-\frac{a}{2}}+g''\}e^{\lambda x}=0$$

$$g''e^{\lambda x}=0$$

$$\Leftrightarrow g''=0$$

$$\Rightarrow g(x)=\alpha x+\beta$$

Conclusion: repeated root

$$\boxed{y(x) = (\alpha x+\beta)e^{\lambda x}}$$

✓Case 3: (5) has 2 (distinct) complex roots λ_i where

$$\left.\begin{aligned}\lambda_1 &= p+iq\\ \lambda_2 &= p-iq\end{aligned}\right\} \text{they must be complex conjugate}$$

General soln is

$$c_1e^{(p+iq)x}+c_2e^{(p-iq)x}$$

by Euler's Formula $e^{i\varphi}=\cos\varphi+i\sin\varphi$

$$\text{where} \quad e^{(p+iq)x} = e^{px}e^{iqx} = e^{px}(\cos qx+i\sin qx)$$

$$e^{(p-iq)x} = e^{px}e^{-iqx} = e^{px}(\cos qx-i\sin qx)$$

So,

$$\frac{1}{2}\left[e^{(p+iq)x}+e^{(p-iq)x}\right] = e^{px}\cos qx \ \left[=Re\left(e^{(p+iq)x}\right)\right]$$

$$\frac{1}{2i}\left[e^{(p+iq)x}+e^{(p-iq)x}\right] = e^{px}\sin qx \ \left[=Im\left(e^{(p+iq)x}\right)\right]$$

So, the general soln is

$$\boxed{y(x) = e^{px}(\alpha\cos qx + \beta\sin qx)}$$

- EXAMPLE:

Solve $y'' + y' + y = 0$ where $y(0) = 0$
$y'(0) = 1$

Soln: Characteristic Eqn:

$$\lambda^2 + \lambda + 1 = 0$$

$$(\lambda + \tfrac{1}{2})^2 = -\tfrac{3}{4}$$

$$\lambda = -\tfrac{1}{2} \pm \tfrac{\sqrt{3}}{2}$$

Then,

$$e^{\lambda x} = e^{-\frac{1}{2}x} e^{i\frac{\sqrt{3}}{2}x} = e^{-\frac{1}{2}x}(\cos\tfrac{\sqrt{3}}{2}x + i\sin\tfrac{\sqrt{3}}{2}x)$$

$$\Rightarrow Re(e^{\lambda x}) = e^{-\frac{1}{2}x}\cos\tfrac{\sqrt{3}}{2}x$$

$$Im(e^{\lambda x}) = e^{-\frac{1}{2}x}\sin\tfrac{\sqrt{3}}{2}x$$

General soln:

$$y(x) = e^{-\frac{x}{2}}[\alpha\cos\tfrac{\sqrt{3}}{2}x + \beta\sin\tfrac{\sqrt{3}}{2}x]$$

Since $y(0) = 0$,

$$0 = \alpha$$

$$\Rightarrow y(x) = \beta e^{-\frac{x}{2}}\sin\tfrac{\sqrt{3}}{2}x$$

$$y'(x) = -\tfrac{1}{2}\beta e^{-\frac{x}{2}}\sin\tfrac{\sqrt{3}}{2}x + \tfrac{\sqrt{3}}{2}\beta e^{-\frac{x}{2}}\cos\tfrac{\sqrt{3}}{2}x$$

Since $y'(0) = 1$,

$$\beta = \tfrac{2\sqrt{3}}{3}$$

- $\Rightarrow$ P.S. $y(x) = \frac{2\sqrt{3}}{3}e^{-\frac{x}{2}}\sin(\frac{\sqrt{3}}{2}x)$

6·2·2 Inhomogeneous Equations

- $$y'' + ay' + by = f(x) \qquad (*)$$

✓ Suppose that y_1 & y_2 are solns.

$$y_1'' + ay_1' + by_1 = f(x) \qquad (1)$$

$$y_2'' + ay_2' + by_2 = f(x)$$

Take the difference: Note: $(y_1 \pm y_2)'' = y_1'' \pm y_2''$

$$(y_1'' - y_2'') + a(x)(y_1' - y_2') + b(x)(y_1 - y_2) = 0$$

$$(y_1 - y_2)'' + a(x)(y_1 - y_2)' + b(x)(y_1 - y_2) = 0$$

✓ So the difference between any two solns of the inhomogeneous eqn solves the corresponding homogeneous eqn.

✓ Any soln. of (✱) is the sum of any particular soln. of (✱) plus a soln. of the homogeneous eqn.

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✓ Any soln. of (✱) is of the form

$$\underset{\swarrow}{y_{PI}} + \underset{\searrow}{y_{hom}}$$

particular integral (PI) $\equiv$ 'a particular soln. of (1)'

complementary function (CF) $\equiv$ 'soln. of the homogeneous eqn'

• EXAMPLE:

Find the general soln of $y''-3y'+2y = x+\sin x$.

Soln: <u>Step 1:</u> Solve the homogeneous eqn.

$$y_h''-3y_h'+2y_h=0$$

characteristic eqn:

$$\lambda^2-3\lambda+2=0$$

$$(\lambda-1)(\lambda-2)=0$$

$$\lambda=1,\ 2$$

$\therefore$ C.F. $y(h)=\alpha e^{x}+\beta e^{2x}$

<u>Step 2:</u> Find PI.

$y(x) = c_0+c_1x+c_2\sin x+c_3\cos x$, then

$y'(x) = c_1 + c_2\cos x - c_3\sin x$

$y''(x) = -c_2\sin x - c_3\cos x$

substitute:

$$(-c_2\sin x - c_3\cos x)-3(c_1+c_2\cos x-c_3\sin x)+2(c_0+c_1x+c_2\sin x+c_3\cos x)=x+\sin x$$

equate coefficients:

x^0: $-3C_1+2C_0=0$ $\quad C_1=\frac{1}{2}$

x^1: $2C_1=1$ $\quad\Rightarrow\quad C_2=\frac{3}{2}$

$\sin x$: $-C_2+3C_3+2C_2=1$ $\quad C_0=\frac{3}{4}$

$\cos x$: $-C_3-3C_2+2C_3=0$ $\quad C_3=\frac{3}{10}$

So the general soln is

$$y(x) = y_p(x) + y_h(x)$$
$$= \frac{3}{4} + \frac{1}{2}x + \frac{1}{10}(\sin x + 3\cos x) + \alpha e^{x} + \beta e^{2x}$$

- Trial Function (ansatz → guess)

$f(x)$	Trial Function
ae^{bx}	$c_1 e^{bx}$ if b is not a soln. of AE $c_1 x e^{bx}$ if b is a non-repeated root, i.e. e^{bx} solves homogeneous eqn $c_1 x^2 e^{bx}$ if b is a repeated root
polynomial in x of degree n	general polynomial of degree n
$a\cos(bx)$ and/or $a\sin(bx)$	$\alpha\cos(bx) + \beta\sin(bx)$ provided this does not solve the homogeneous eqn. $x[\alpha\cos(bx) + \beta\sin(bx)]$ if $\alpha\cos(bx) + \beta\sin(bx)$ solves the homogeneous eqn
e^{ax}. $\sin(bx)$ $\cos(bx)$	Write a Re / Im part of $e^{(a+ib)x}$. Soln solve with this & take Re or Im part
$\cos^2 x$	→ Rewrite as sums of trig fns
$\cosh^2 x$	→ Rewrite in terms of exponentials

Fri. 02/12/16

MATH1401 : Mathematical Methods 1

Prof. Halburd

6·2·3 Euler's Equation

$$x^2 y'' + axy' + by = f(x) \qquad (\#)$$

✓ For homogeneous case , i.e. $f(x) = 0$

look for solns of the form $y(x) = x^{\lambda}$.

Then,

$$y' = \lambda x^{\lambda-1} , \qquad y'' = \lambda(\lambda-1)x^{\lambda-2}$$

Then ,

$$x^2 y_h'' + axy_h' + by_h = 0$$

$$\Leftrightarrow \quad [\lambda(\lambda-1)+a\lambda+b]x^{\lambda}=0$$

$$\Leftrightarrow \quad \lambda^2+(a-1)\lambda+b=0 \quad \text{(characteristic eqn)}$$

Alternative Method: change variable / substitution

Let $x=e^t \Leftrightarrow t=\log x$, then

$$\frac{dy}{dx}=\frac{dy}{dt}\cdot\frac{dt}{dx}=x^{-1}\frac{dy}{dt}$$

Then $\frac{d^2y}{dx^2}=\frac{d}{dx}\left(x^{-1}\frac{dy}{dt}\right)=-\frac{1}{x^2}\frac{dy}{dt}+x^{-1}\boxed{\frac{d}{dx}}\left(\frac{dy}{dt}\right)$ ← differentiate wrt x

$$=-x^{-2}\frac{dy}{dt}+x^{-1}\boxed{\frac{dt}{dx}\cdot\frac{d}{dt}}\left(\frac{dy}{dt}\right)$$

$$=-x^{-2}\frac{dy}{dt}+x^{-2}\frac{d^2y}{dt^2}$$

$$=x^{-2}\left(\frac{d^2y}{dt^2}-\frac{dy}{dt}\right)$$

✓ Therefore,

$$(\#):\quad \left(\frac{d^2y}{dt^2}-\frac{dy}{dt}\right)+a\frac{dy}{dt}+by=f(e^t)$$

$$\Leftrightarrow \quad \frac{d^2y}{dt^2}+(a-1)\frac{dy}{dt}+by=f(e^t)$$

which is a constant coefficient linear 2^{nd} order ODE.

6·3 First Order Differential Eqn (Not in exam)

- $$\frac{dy}{dx}=\varphi\left(\frac{y}{x}\right)$$

Let $z(x)=\frac{y}{x}$, then

$$y=xz(x)$$

Separate variables.

✓ EXAMPLE. $\frac{dy}{dx}=\frac{x+3y}{2x+7y}$

$$\Leftrightarrow \frac{dy}{dx}=\frac{1+3\left(\frac{y}{x}\right)}{2+7\left(\frac{y}{x}\right)}=\varphi\left(\frac{y}{x}\right)$$

- Slightly different forms:

$$\frac{dy}{dx}=\frac{x+3y+1}{2x+7y+2} \qquad (*)$$

Transform using $x=u+a$

$y=v(u)+b$

Since $\frac{dy}{dx}=\frac{dy}{du}\cdot\frac{du}{dx}=\frac{dv}{du}\cdot 1=\frac{dv}{du}$,

$$(*):\quad \frac{dv}{du}=\frac{u+3v+(a+3b+1)}{2u+7v+(2a+7b+2)}$$

Choose a,b s.t. $\begin{cases} a+3b+1=0 \\ 2a+7b+2=0 \end{cases}$

Then it is transformed into the form $\frac{dy}{dx} = \varphi(\frac{y}{x})$.

Wed. 30/11/16

MATH1401 Help Class

Prof. Wilson

- linear

1st order 2nd order

$$g(x) = f_0(x)y + f_1(x)\frac{dy}{dx}\Big] + f_2(x)\frac{d^2y}{dx^2}\Big]$$

homogeneous: $g(x) = 0$

for a homogenous eqn, if $y = h(x)$ is a soln, then $y = Ah(x)$ is also a soln

1. 1st order linear integrating factor

- EXAMPLE ①.

$$\frac{dy}{dx} + \frac{3}{x}y = \sin x \qquad \text{when } x = \pi,\ y = 1$$

Integrating factor

$$I(x) = e^{\int p(x)dx}$$

[Here $p(x) = \frac{3}{x}$]

$$I(x) = \exp\left(\int \frac{3}{x}dx\right) = \exp(3\ln x) = x^3$$

Then

$$x^3\frac{dy}{dx} + 3x^2y = x^3\sin x$$

$$\frac{d}{dx}(x^3y) = x^3\sin x$$

$$\Rightarrow x^3y = \int x^3\sin x\,dx$$

$u = x^3$ $v' = \sin x$

$u' = 3x^2$ $v = -\cos x$

$$= -x^3\cos x + \int 3x^2\cos x\,dx$$

$u = 3x^2$ $v' = \cos x$

$u' = 6x$ $v = \sin x$

$$= -x^3\cos x + 3x^2\sin x - \int 6x\sin x\,dx$$

$u = 6x$ $v' = \sin x$

$u' = 6$ $v = -\cos x$

$$= -x^3\cos x + 3x^2\sin x + 6x\cos x - 6\int\cos x\,dx$$

$$= -x^3\cos x + 3x^2\sin x + 6x\cos x - 6\sin x + C$$

Therefore,

$$x^3y = -x^3\cos x + 3x^2\sin x + 6x\cos x - 6\sin x + c$$

$$y = -\cos x + \frac{3}{x}\sin x + \frac{6}{x^2}\cos x - \frac{6}{x^3}\sin x + \frac{c}{x^3}$$

$y(\pi) = 1$:

$$1 = 1 - \frac{6}{\pi^2} + \frac{C}{\pi^3}$$

$$\Rightarrow C = 6\pi$$

$$\therefore\ y = -\cos x + \frac{3}{x}\sin x + \frac{6}{x^2}\cos x - \frac{6}{x^3}\sin x + \frac{6\pi}{x^3}$$

The very last thing we do is use the initial condition

2. 2nd order (constant coefficients)

(i.e. f_0, f_1, f_2 do not depend on x)

- EXAMPLE:

$$\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = xe^x \qquad y(0)=2$$
$$y'(0)=1$$

"CF & PI" method :

① CF is the soln of the homogeneous eqn.
$\uparrow$
complementary function

If $y = e^{\lambda x}$, then $\frac{dy}{dx} = \lambda e^{\lambda x}$
$$\frac{d^2y}{dx^2} = \lambda^2 e^{\lambda x}$$

sub: $\lambda^2 e^{\lambda x} - 3\lambda e^{\lambda x} + 2e^{\lambda x} = 0$
$$e^{\lambda x}(\lambda^2 - 3\lambda + 2) = 0$$
$$e^{\lambda x}(\lambda - 2)(\lambda - 1) = 0$$
$$\lambda = 1 \text{ or } 2$$

So 2 solns, e^x & e^{2x}, to the hom eqn.

$\Rightarrow$ general soln of the homogeneous eqn:
$$y(x) = Ae^x + Be^{2x}$$

② PI: find any one soln. to the whole eqn.
$\uparrow$
particular integral

Trial & error: try a y that "looks like" the RHS

Our RHS is xe^x. So try $y = \alpha x e^x$
$$\Rightarrow \frac{dy}{dx} = \alpha e^x + \alpha x e^x$$
$$\Rightarrow \frac{d^2y}{dx^2} = \alpha e^x + \alpha e^x + \alpha x e^x = 2\alpha e^x + \alpha x e^x$$

substitute:
$$2\alpha e^x + \alpha x e^x - 3(\alpha e^x + \alpha x e^x) + 2\alpha x e^x = xe^x$$
$$2\alpha e^x - 3\alpha e^x = xe^x$$

This DOES NOT work !!!

- Systematic method:

CF: e^x, e^{2x} pick <u>one</u> and factor it out, i.e.
$$y = e^x f(x), \quad \text{start again.}$$
$$\frac{dy}{dx} = e^x f(x) + e^x \frac{dy}{dx}$$
$$\frac{d^2y}{dx^2} = e^x f(x) + 2e^x \frac{dy}{dx} + e^x \frac{d^2y}{dx^2}$$

Sub: $\left[e^x f(x) + 2e^x \frac{dy}{dx} + e^x \frac{d^2y}{dx^2}\right] - 3\lambda\left[e^x f(x) + e^x \frac{dy}{dx}\right] + 2e^x f(x) = xe^x$

$$f'' - f' = x$$

Now set $g = \frac{df}{dx}$

$\frac{dg}{dx} - g = x$. This is a 1^{st} order.

$I(x) = e^{-x}$ is the integrating factor.

Then,

$$e^{-x}\frac{dg}{dx} - e^{-x}g = xe^{-x}$$

$$\frac{d}{dx}(e^{-x}g) = xe^{-x}$$

$$e^{-x}g = \int xe^{-x}dx$$

$u = x$ $v' = e^{-x}$

$u' = 1$ $v = -e^{-x}$

$$e^{-x}g = -xe^{-x} + \int e^{-x}dx$$

$$e^{-x}g = -xe^{-x} - e^{-x} + c$$

$$g = -x - 1 + ce^{x}$$

$\Rightarrow \frac{df}{dx} = -x - 1 + ce^{x}$

$\Rightarrow f(x) = -\frac{1}{2}x^2 - x + ce^{x} + D$

$\Rightarrow y = \underbrace{-\frac{1}{2}x^2e^{x} - xe^{x}}_{PI} + \underbrace{Ce^{2x} + De^{x}}_{CF}$

③ initial conditions (LAST step!)

$y(0) = 2$, $y'(0) = 1$

... $\Rightarrow C = 0$, $D = 2$

- Trial & error : αxe^{x} failed

Next try $\alpha x^2e^{x} + \beta xe^{x}$

- If λ is repeated in CF, then use

$$y_{CF} = Ae^{\lambda x} + Bxe^{\lambda x} = (A + Bx)e^{\lambda x}$$

If λ is a pair of complex roots, i.e. $\lambda = a \pm ib$, then

$$y_{CF} = e^{ax}(A\cos bx + B\sin bx)$$

Wed. 30/11/16

Applied Tutorial

1. Find the soln of $\frac{dy}{dx} = \frac{4x+y}{x+y}$ that passes through $(1,1)$.

General Method: $\frac{dy}{dx} = \varphi\left(\frac{y}{x}\right) \leftarrow$ have a fn that only depends on $\frac{y}{x}$

Let $z = \frac{y}{x}$, then

$$y = zx$$

$$\Rightarrow \frac{dy}{dx} = z + x\frac{dz}{dx} = \varphi\left(\frac{y}{x}\right)$$

2. $\frac{dy}{dx} = \varphi(xy)$

Let $z = xy$, then

$$\frac{dz}{dx} = y + x\frac{dy}{dx}$$

$$\Leftrightarrow \frac{dy}{dx} = \frac{1}{x}\left(\frac{dz}{dx} - y\right)$$

$$= \frac{1}{x}\frac{dz}{dx} - \frac{z}{x^2}$$

3. $\frac{dy}{dx} = \frac{x+y+3}{x-y+5} = \frac{(x+4)+(y-1)}{(x+4)-(y-1)}$

Let $u = x+a$, $v = y+b$.

So $\frac{dv}{dx} = \frac{dy}{dx}$

Fri. 02/12/16 (cont.)

MATH1401: Mathematical Methods I

Prof. Halburd

Chapter 7. § Probability §

7·1 Sample Space & Set Operations

7·1·1 Introduction

- Def.

A sample space is a set representing the possible outcomes of an experimental / trial etc.

✓ EXAMPLE.

Two coins are tossed. The possible sample space is

$$\{H\&H, H\&T, T\&T\}$$

Another possibility is

$$\{HH, \underset{\uparrow}{HT}, TH, TT\}$$

This means the 1st coin = H & the 2nd coin = T

- Def.

Subsets of a sample space are called events. They represent collection of outcome

✓ EXAMPLE:

Consider the set $S = \{0,1,2\}$

The elements of S are 0, 1 and 2.

Subsets of S are

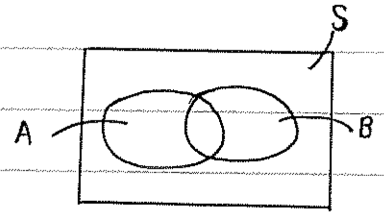
$$\underset{\nwarrow}{\emptyset}, \{0\}, \{1\}, \{2\}, \{0,1\}, \{0,2\}, \{1,2\}, \underset{\uparrow}{\{0,1,2\}}$$

empty set is a subset of every set   every set is a subset of itself

Note: $\underset{\nearrow}{\emptyset}$, $\underset{\uparrow}{0}$ and $\underset{\nwarrow}{\{0\}}$ are completely different objects.

empty set   a number   a set consisting of one element (0).

7·1·2 Set Operations



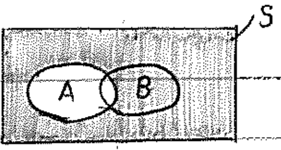
'is an element of'

① Intersection: $A \cap B = \{x : x \in A \ \& \ x \in B\}$

② Union: $A \cup B = \{x : x \in A \text{ or } x \in B\}$

③ Complement of A:

$$A^c \equiv A' \equiv \overline{A} = \{x : x \in S,\ x \notin A\}$$



everything that is not in A

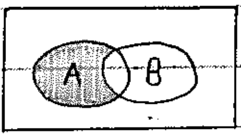
Note: $S^c = \emptyset$, $\emptyset^c = S$

④ A and B are disjoint (mutually exclusive) if $A \cap B = \emptyset$.

⑤ The relative complement

'A∖B' ≡ 'A-B' ≡ 'A without B'

$$A \setminus B = \{x \in A,\ x \notin B\}$$



· Def.

Given a sample space S.

A probability on S is a fn that assigns a number P(A) to each subset $A \subseteq S$.
'is a subset of'

✓ Note: $\boxed{\text{Don't distinguish between } \subset \text{ and } \subseteq .}$

✓ Properties:

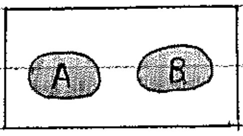
1) $P(A) \geq 0 \quad \forall A \subset S$

2) $P(S) = 1$

3) If $A \cap B = \emptyset$, then

$$P(A \cup B) = P(A) + P(B)$$

'A or B'



✓ Lemma:

Let A and B be events in S (not necessarily disjoint), then

(1) $P(A') = 1 - P(A)$ 'not A'

(2) $P(\emptyset) = 0$

(3) $P(A \cup B) = P(A) + P(B) - P(A \cap B)$ 'A and B'

Proof of (1):

$A \cap A' = \emptyset$

By 2), $1 = P(S)$

$= P(A \cup A')$

$= P(A) + P(A')$ by 3)

$\Rightarrow P(A') = 1 - P(A)$

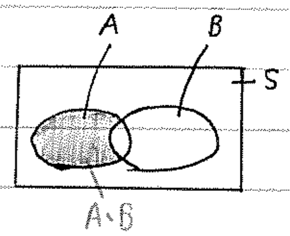
Proof of (2):

$P(\emptyset) = 1 - P(\emptyset^c)$

$= 1 - P(S)$

$= 1-1$)

$= 0$



Proof of (3):

$A \setminus B$ and B are disjoint.

Therefore,

$$P(A \cup B) = P\big((A \setminus B) \cup B\big)$$
$$= P(A \setminus B) + P(B) \quad \text{by } 3)$$

$A \setminus B$ and $A \cap B$ are disjoint.

Therefore,

$$P(A) = P\big((A \setminus B) \cup (A \cap B)\big)$$
$$= P(A \setminus B) + P(A \cap B) \quad \text{by } 3)$$

Eliminate: $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

7·1·3 Equally-likely Outcomes

- ✓ Let S be a finite sample space.

  $S = \{s_1, s_2, \dots, s_n\}$, $n = |S|$ = number of elements = cardinality

  Let $S_1 = \{s_1\}$, $S_2 = \{s_2\}$, ..., $S_n = \{s_n\}$, then

  $S = S_1 \cup S_2 \cup S_3 \cup \dots \cup S_n$ where $S_i \cap S_j = \emptyset \Leftrightarrow i \neq j$ (mutually exclusive)

  $\Rightarrow P(S) = P(S_1) + P(S_2) + \dots + P(S_n)$

  ✓ Now suppose that each of the events S_j is equally-likely.

  Then $P(S_i) = P(S_j)$

  Since $P(S) = 1$,

  $$1 = nP(S_j)$$
  $$\Rightarrow P(S_j) = \frac{1}{n} = \frac{1}{|S|}$$

- Def.

  Let A be (a non-empty) event $A \subset S$.

  $A = \{S_{i_1}; S_{i_2}, S_{i_3}, \dots, S_{ik}\}$, $k = |A|$.

  $$P(A) = P(S_{i_1} \cup S_{i_2} \cup S_{i_3} \cup \dots \cup S_{ik})$$
  $$= P(S_{i_1}) + P(S_{i_2}) + \dots + P(S_{ik}) \quad \text{since they are disjoint}$$
  $$= \frac{k}{n}$$
  $$= \frac{|A|}{|S|}$$

  no. of elements in A

  no. of elements in S

✓ EXAMPLE ①:

A fair coin is tossed twice. We use the sample space

$S = \{HH, HT, TH, TT\}$. (equally-likely)

Soln: 1^{st}: use words to describe events ; 2^{nd}: use sets

$P(\text{Two heads appear}) = P(\{HH\}) = \frac{|\{HH\}|}{|S|} = \frac{1}{4}$

$P(\text{One head and one tail appear}) = P(\{HT, TH\}) = \frac{|\{HT, TH\}|}{|S|} = \frac{2}{4} = \frac{1}{2}$

$$\begin{aligned} P(\text{At least one tail}) &= P(\{HT, TH, TT\}) \\ &= \frac{|\{HT, TH, TT\}|}{|S|} \\ &= \frac{3}{4} \end{aligned}$$

$$\begin{aligned} \text{Or } &= 1 - P(\text{no tail}) \\ &= 1 - P(\{HH\}) \\ &= 1 - \frac{1}{4} \\ &= \frac{3}{4} \end{aligned}$$

✓ EXAMPLE ②:

A fair die is rolled twice and the numbers are recorded.

Let A = first roll is a 5.

B = the largest number shown is 4.

C = the sum of numbers is prime.

Calculate P(A), P(B) and P(C).

Soln:

Sample space

$$S = \begin{cases} (1,1), (1,2), \cdots, (1,6) \\ (2,1), (2,2), \cdots, (2,6) \\ \vdots \\ (6,1), (6,2), \cdots, (6,6) \end{cases} \qquad |S| = 36$$

Then,

$P(A) = P(\{(5,1), (5,2), (5,3), (5,4), (5,5), (5,6)\}) = \frac{|A|}{|S|} = \frac{6}{36} = \frac{1}{6}$

$P(B) = P(\{(1,4), (2,4), (3,4), (4,4), (4,3), (4,2), (4,1)\}) = \frac{|B|}{|S|} = \frac{7}{36}$

$$\begin{aligned} P(C) &= P(\{(1,1), (1,2), (1,4), (1,6), (2,1), (2,3), (2,5), \\ &\qquad (3,2), (3,4), (4,3), (4,1), (5,2), (5,6), (6,1), (6,5)\}) \\ &= \frac{|C|}{|S|} \\ &= \frac{15}{36} \\ &= \frac{5}{12} \end{aligned}$$

7·1·4 Discrete Sample Space

- Def. ①:

A sample space is discrete if it is finite or it has a countable infinite number of elements.

- Def. ②:

✓ A set is countably infinite if you can list the elements

$$\begin{matrix} 1.\ x_1 \\ 2.\ x_2 \\ \vdots \end{matrix} \longrightarrow \sum_{j=1}^{\infty} P(x_j) = 1$$

i.e. there is a one-to-one correspondence between the elements of S and $\mathbb{N}$

✓ If S is countable, we can write

sum over everything in set S $\longrightarrow \sum_{x \in S} P(x) = 1$

Mon. 05/12/16

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✓ EXAMPLE:

— Consider a game in which a fair coin is tossed until the first head appears when the game ends.

— $S = \{H, TH, TTH, TTTH, \dots\}$

This is countably infinite because we can list the elements.

— Let P_n = probability that the game ends on the n^{th} row, then

$$P_n = \underbrace{\left(\tfrac{1}{2} \cdot \tfrac{1}{2} \cdot \tfrac{1}{2} \dots \tfrac{1}{2}\right)}_{\substack{(n-1) \text{ terms} \\ \text{corresponding to} \\ \text{the first } (n-1) \text{ rolls}}} \times \tfrac{1}{2} = \left(\tfrac{1}{2}\right)^n$$

↑ the last roll is a head

The sum of probabilities of all events (subsets) adds up to 1. ← Check:

$$\sum_{n=1}^{\infty} P_n = \sum_{n=1}^{\infty} \left(\tfrac{1}{2}\right)^n = \frac{1/2}{1-1/2} = 1$$

A geometric series with first term $\frac{1}{2}$ and ratio $r = \frac{1}{2}$.

For geometric series, $|r| < 1 \Rightarrow \lim \longrightarrow \frac{a}{1-r}$

— Aside, a coin flipped infinitely many times.

$S = \{HTTHHTHT\dots\}$

This is uncountable / not discrete.

7.2 Conditional Probability

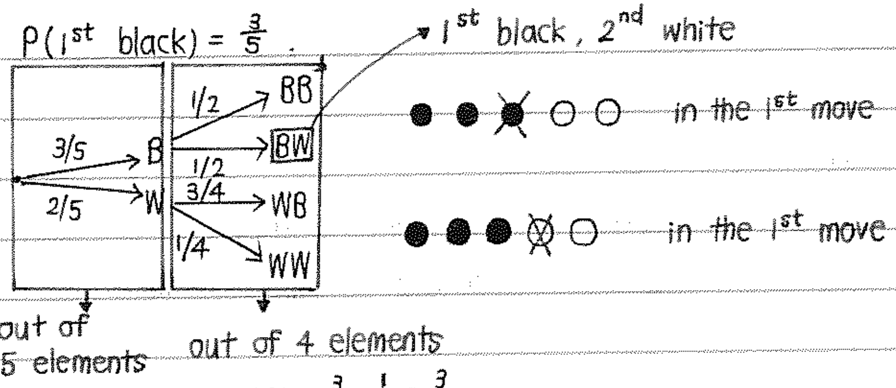
✓ EXAMPLE:

An urn contains 3 black balls and 2 white balls. Two balls are removed in order (without being put back). Find the probability that

1.) The first ball is black.
2.) The second ball is black.
3.) The two balls have the same colour.

Soln: ● ● ● ○ ○

$P(1^{st}\ \text{black}) = \frac{3}{5}$.



Therefore, $P(BB) = \frac{3}{5} \times \frac{1}{2} = \frac{3}{10}$

$P(BW) = \frac{3}{5} \times \frac{1}{2} = \frac{3}{10}$

$P(WB) = \frac{2}{5} \times \frac{3}{4} = \frac{3}{10}$

$P(WW) = \frac{2}{5} \times \frac{1}{4} = \frac{1}{10}$

Then clearly,

$P(2^{nd}\ \text{ball is black}) = P(\{BB, WB\})$

$= P(\{BB\}) + P(\{WB\})$ since they're disjoint

$= \frac{3}{10} + \frac{3}{10}$

$= \frac{3}{5}$

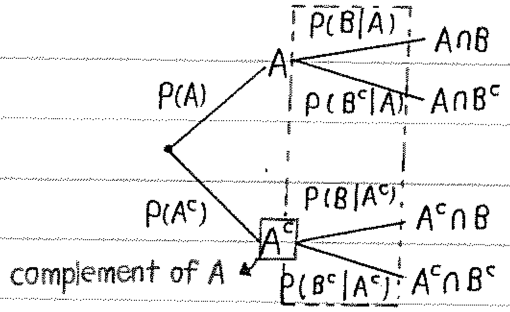
$P(\text{same colour}) = P(\{BB, WW\})$

$= P(\{BB\}) + P(\{WW\})$ since disjoint

$= \frac{3}{10} + \frac{1}{10}$

$= \frac{2}{5}$

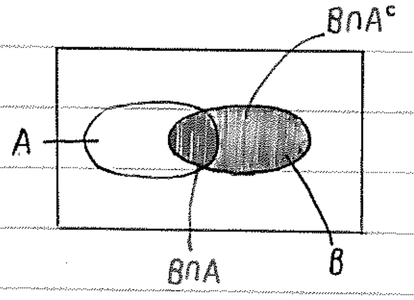
✓ General Case:



- Def Conditional Probabilities

$P(B|A)$ is the conditional probability that B occurs, given that A has occurre

$$P(B|A) = \frac{P(B\cap A)}{P(A)}$$



✓ Clearly, we can write

$$B = (B\cap A)\cup(B\cap A^c)$$

Since $(B\cap A)\cap(B\cap A^c) = \emptyset$, we have

$$P(B) = P(B\cap A) + P(B\cap A^c)$$

$$\boxed{P(B) = P(B|A)P(A) + P(B|A^c)P(A^c)}$$ Important!

7.3 Counting

Let S be a set with n elements.

- ① Ordered samples, repetition allowed

If the number of samples is r, then there are n^r different orderings of length r from n objects.

✓ EXAMPLE:

How many 4-digit numbers can be constructed using the digits 1, 2 & 3? (Repeats allowed)

Soln: $3\times3\times3\times3 = 3^4$

- ② Ordered samples, with no repeats

Choose any of the n objects … 1st

Choose any of the n-1 remaining objects … 2nd

⋮

Therefore, there are

$$\boxed{n(n-1)(n-2)\ldots(n-r+1)}$$

called number (#) of permutations of length r

Notation: ${}^nP_r = n(n-1)(n-2)\ldots(n-r+1) = \frac{n!}{(n-r)!}$

③ Unordered samples, with no repeats

Consider a set S of n elements.

Let ${}^nC_r$ be the number of subsets of r elements.

Note: sets & subsets do not have orders

Each such subset could be written in ${}^rP_r = r!$ different ordered ways.

e.g. $\underbrace{\{1,2,3\}, \{1,3,2\}, \{2,3,1\}, \cdots}_{3!\ \text{ways}}$

So $(r!)\,{}^nC_r =$ number of permutations of length r, chosen from n objects.

$$\Rightarrow (r!)\,{}^nC_r = {}^nP_r = \frac{n!}{(n-r)!}$$

$$\Rightarrow \boxed{{}^nC_r} = \frac{n!}{(n-r)!\,r!} = \binom{n}{r}$$

'n choose r' : the order does not matter

Fri. 09/12/16

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Recap: $S = \{1,2,3,4\}$

① find the number of all 3-digit numbers, using S where we can repeat.

$$4 \times 4 \times 4 = 64$$

② find the number of all 3-digit numbers, using S where we cannot repeat.

$$4 \times 3 \times 2 = \frac{4!}{1!} = {}^4P_3$$

order matters $\Rightarrow$ ${}^nP_r = \frac{n!}{(n-r)!}$

③ order does not matter (combination)

$${}^nC_r = \frac{n!}{(n-r)!\,r!} = \binom{n}{r}$$

Note: $\binom{n}{r}$ = binomial coefficients

$$(1+x)^n = \sum_{r=0}^{n} \binom{n}{r} x^r$$

Pascal's Triangle

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1

✓ EXAMPLE ①:

Find the total number of subsets of a set of size n.

$$S = \{1, 2, 3, \ldots, n\}$$

For each element in S , for in given subset A of S , I put a tick or a cross depending on whether the element is in A.

1st approach : $A = \{1, 2, 4\}$

$S = \{1, 2, 3, 4, 5, 6, \cdots, n\}$ list all the elements

$\checkmark \checkmark \times \checkmark \times \times \ldots \times$

counting subsets = counting ticks & crosses

2^n = # of subsets

2nd approach: subsets with 0 element : $\binom{n}{0} = 1$

1 element: $\binom{n}{1} = n$

2 elements: $\binom{n}{2} = \frac{n(n-1)}{2}$

3 elements: $\binom{n}{3} = \frac{n(n-1)(n-2)}{3}$

total # of subsets $= \sum_{r=0}^{n} \binom{n}{r} 1^r \rightsquigarrow$ binomial expansion

$= (1+1)^n$

$= 2^n$

✓EXAMPLE ②:

In a group of r people , find the probability that at least 2 share the same birthday (ignore 29th Feb.).

Soln: prob = 1 - prob (all b'days are different)

Since P(all b'days are different) $= \frac{365}{365} \cdot \frac{364}{365} \cdot \frac{363}{365} \cdot \cdots \cdot \frac{365-r+1}{365}$

$= \frac{1}{(365)^r} \cdot \frac{365!}{(365-r)!}$,

we have P(r) [at least 2 share b'days]

$= 1 - \frac{365!}{(365)^r (365-r)!}$

Then,

$P(2) = 0.003$, $P(3) = 0.005$, $P(\ 22) = 0.476$, $P(23) = 0.507$

7.4 Independence & Bayes' Formula

7.4.1 Independence

- Def.

2 events are independent if and only if

$P(A \cap B) = P(A)P(B)$

Recall: $P(B|A) = \frac{P(A \cap B)}{P(A)}$

So independent $\Rightarrow P(B|A) = P(B)$

✓ EXAMPLE:

A fair coin is tossed 3 times.

Consider the 2 events

(a) throwing at least one head & one tail

(b) throwing at most one head

Q. (1) Are these events independent ?

(2) Are they still independent if the coin is tossed 4 times?

Soln: (1) $S = \{HHH, HHT, HTH, \cdots\}$ $|S| = 2^3 = 8$

$A = \{\text{at least } 1H \,\&\, 1T\}$

$= \{HHT, HTH, THH, TTH, THT, HTT\}$ $|A| = 6$

$= S \setminus \{HHH, TTT\} \longleftarrow$ 'S without $\{HHH, TTT\}$'

$B = \{\text{at most } 1H\}$

$= \{TTT, TTH, THT, HTT\}$ $|B| = 4$

$\Rightarrow P(A\cap B) = \frac{|A\cap B|}{|S|} = \frac{3}{8}$

$P(A)P(B) = \frac{|A|}{|S|} \cdot \frac{|B|}{|S|} = \frac{6}{8} \cdot \frac{4}{8} = \frac{3}{8}$

(2) $S = \{HHHH, \cdots\}$ $|S| = 2^4 = 16$

$A = S \setminus \{HHHH, TTTT\}$ $|A| = 16 - 2 = 14$

$B = \{TTTT, HTTT, THTT, TTHT, TTTH\}$ $|B| = 5$

$A\cap B = \{HTTT, THTT, TTHT, TTTH\}$ $|A\cap B| = 4$

$\Rightarrow P(A\cap B) = \frac{1}{4}$

$P(A)P(B) = \frac{14}{16} \times \frac{5}{16} \neq \frac{1}{4}$

So A & B are dependent.

7·4·2 Bayes' Formula

- Recall: Conditional Probability

$P(B|A) = \frac{P(A\cap B)}{P(A)}$ (*)

$P(B) = P(B|A)P(A) + P(B|A^c)P(A^c)$ (#)

- Note: $P(A|B) = \frac{P(A\cap B)}{P(B)}$
- Def.

$$P(A|B) = \frac{P(B|A)P(A) \xleftarrow{} \text{from } (*)}{P(B|A)P(A) + P(B|A^c)P(A^c) \xleftarrow{} \text{from } (\#)}$$

This is called Bayes' Formula

✓ EXAMPLE:

This problem involves 2 coins : one is fair & the other has two heads. A coin is selected at random and tossed , and the result is a head. Find the probability that the coin was a fair coin.

Soln: Let F be the event that the fair coin was selected.

Let H_1 be the event that a head is chosen.

$$P(F|H_1) = \frac{P(H_1|F)P(F)}{P(H_1|F)P(F) + P(H_1|F^c)P(F^c)}$$

$$= \frac{\frac{1}{2}.\frac{1}{2}}{\frac{1}{2}.\frac{1}{2} + 1.\frac{1}{2}} = \frac{1}{3}$$

If the coin is tossed a 2nd time & again reveals a H, what is the probability that it is fair ?

Soln: Let H_2 = event that 2H's are shown.

$$P(F|H_2) = \frac{P(H_2|F)P(F)}{P(H_2|F)P(F) + P(H_2|F^c)P(F^c)}$$

$$= \frac{\frac{1}{4}.\frac{1}{2}}{\frac{1}{4}.\frac{1}{2} + 1.\frac{1}{2}} = \frac{1}{5}$$

7.5 Binomial Distribution $X \sim B(n,p)$ [n × Bernoulli Trials]

- ✓ a fixed number of independent trials.
- ✓ on each trial , there are two outcomes.
- ✓ the probability of success remains constant.
- $\boxed{\text{prob} = {}^nC_x\, p^x (1-p)^{n-x}}$

} A-level

✓ EXAMPLE:

A multiple-choice exam consists of 25 questions , each with 4 possible answers. A student guesses at random. Find the probability that

(1) every answer is correct

(2) exactly one answer is correct

(3) at least two answers are correct

(4) exactly 7 answers are correct

(5) the student passes , given the pass mark is 40%.

Soln: (1) $(\frac{1}{4})^{25} \approx 8.88 \times 10^{-16}$

(2) $$P(\text{exactly } 1 \text{ correct}) = \underset{\binom{25}{1}}{25}.\underset{\text{the correct answer}}{(\tfrac{1}{4})}.\underset{\text{the remaining 24 questions are wrong}}{(\tfrac{3}{4})^{24}}$$

(3) P (at least 2 correct)

$= 1 - P(0 \text{ correct}) - P(\text{exactly } 1 \text{ correct})$

$= 1 - (\frac{3}{4})^{25} - 25(\frac{1}{4})(\frac{3}{4})^{24}$

$= 0.993$

(4) $P(\text{exactly } 7 \text{ correct}) = \binom{25}{7}(\frac{1}{4})^7(\frac{3}{4})^{18} = 0.165$

↑ # of ways of selecting which 7 questions are correct

(5) 40% of 25 (questions) is 10.

$P(\text{pass}) = \text{prob}(10 \text{ or more correct})$

$= 1 - \text{prob}(\text{at most } 9 \text{ correct})$

$= 1 - [P(0 \text{ correct}) + P(1 \text{ correct}) + \dots + P(9 \text{ correct})]$

$= 1 - \binom{25}{0}(\frac{3}{4})^{25} - \binom{25}{1}(\frac{1}{4})(\frac{3}{4})^{24} - \binom{25}{2}(\frac{1}{4})^2(\frac{3}{4})^{23} - \dots - \binom{25}{9}(\frac{1}{4})^9(\frac{3}{4})^{16}$

$= 0.0713$

7.6 Bernoulli Trials

- Def.

A repeated event/ experiment with 2 possible outcomes (success/ failure) is called a Bernoulli Trial

If p is the probability of success and $q = 1-p$ is the probability of failure, then r successes in n trials is

$$b(r) = \binom{n}{r} p^r q^{n-r}$$

- Mean Value

✓ Def.

Suppose that the outcomes of a sequence of experiments or events are numbers (e.g. rolling a die)

If each outcome is x_i, and occurs with probability $P(x_i)$, then the mean is

$$\bar{x} = \sum_i x_i P(x_i)$$

✓ EXAMPLE:

Fair die. $x_1 = 1$, $x_2 = 2$, $\dots$, $x_6 = 6$

Soln: $P(x_j) = \frac{1}{6}$.

Then the mean is

$$\sum_{i=1}^{6} x_i P(x_i) = (1+2+3+4+5+6) \cdot \frac{1}{6} = \frac{7}{2}$$

· EXAMPLE:

– To any sequence of n Bernoulli trials, we associate the probability of r successes. $b(r) = \binom{n}{r} p^r q^{n-r}$

– The mean (or average) of successes is

$$\sum_{r=0}^{n} \underset{\uparrow}{r} . \boxed{b(r)} \longleftarrow \text{probability of } r \text{ successes}$$

# of successes

$$\sum_{r=0}^{n} r.b(r) = \sum_{r=0}^{n} r\binom{n}{r} p^r \boxed{q^n}^{-r}$$

take out q^n ∵ it does not contain r (sum on r)

$$= \boxed{q^n} \sum_{r=0}^{n} r\binom{n}{r}\left(\frac{p}{q}\right)^r$$

looks similar

– Recall: $(1+x)^n = \sum_{r=0}^{n} \binom{n}{r} x^r$

differentiate wrt x

$$\Rightarrow n(1+x)^{n-1} = \sum_{r=0}^{n} r\binom{n}{r} x^{r-1}$$

$$\Rightarrow \sum_{r=0}^{\infty} r\binom{n}{r} x^r = nx(1+x)^{n-1}$$

– Mean $= q^n . n\left(\frac{p}{q}\right)\left(1+\frac{p}{q}\right)^{n-1}$ $\quad (x = \frac{p}{q})$

$$= \cancel{q^n} . n . \frac{p}{q} . \frac{1}{\cancel{q^{n-1}}} (q+p)^{n-1}$$

$$= np\,(\underbrace{q+p}_{=1})^{n-1}$$

$$= np$$

7.7 Poisson Distribution

We want to approximate the binomial distribution for large n, where the mean $\lambda = np$ is held fixed (so p is small).

Mon. 12/12/16

MATH1401: Mathematical Methods 1

Prof. Halburd

Recall:

· Bernoulli Trials:

✓ probability of success: p

✓ probability of failure: $q = rp$

✓ probability of r successes from n trials: $b(r) = \binom{n}{r} p^r q^{n-r}$

· mean or average # of successes for binomial distribution:

$$\lambda = \sum_{r=0}^{n} r.b(r) = np \qquad (*)$$

7·7 Poisson Distribution

- λ fixed, consider the limit as $n \to +\infty$.

$$(*): \; p = \frac{\lambda}{n} \to 0$$

Then, we have

$$b(r) = \binom{n}{r} p^r q^{n-r}$$

$$= \binom{n}{r} \left(\frac{\lambda}{n}\right)^r \left(1-\frac{\lambda}{n}\right)^{n-r}$$

$$= \frac{\boxed{n!}}{r!\boxed{(n-r)!}} \left(\frac{\lambda}{n}\right)^r \left(1-\frac{\lambda}{n}\right)^{n-r}$$

So,

$$b(r) = \frac{\lambda^r}{r!} . \boxed{\frac{n!}{(n-r)!}} . \frac{1}{n^r} \boxed{\left(1-\frac{\lambda}{n}\right)^{n-r}}$$

$$= \frac{\lambda^r}{r!} . \frac{\overset{r \text{ terms}}{\underbrace{n(n-1)(n-2)(n-3)\ldots}}}{n.n.n\ldots,n} . \left(1-\frac{\lambda}{n}\right)^n . \left(1-\frac{\lambda}{n}\right)^{-r}$$

$$= \frac{\lambda^r}{r!} . \; 1 \; \underset{\downarrow \atop 1}{\left(1-\frac{1}{n}\right)} \underset{\downarrow \atop 1}{\left(1-\frac{2}{n}\right)} \underset{\downarrow \atop 1}{\left(1-\frac{3}{n}\right)} \ldots \underset{\downarrow \atop 1}{\left(1-\frac{r-1}{n}\right)} . \left(1-\frac{\lambda}{n}\right)^n \left(1-\frac{\lambda}{n}\right)^{-r}$$

$n \to \infty$

$$\frac{n!}{(n-r)!} \xrightarrow{n \to \infty} 1$$

Also, similarly, $\lim\limits_{n\to\infty} \left(1-\frac{\lambda}{n}\right)^r = 1$

Let $a_n = \left(1-\frac{\lambda}{n}\right)^n$, then

$$\log a_n = n \log\left(1-\frac{\lambda}{n}\right) \qquad \log(1+x) = x + ?\,x^2 + \ldots$$

$$= n\left(-\frac{\lambda}{n} + ?\left(\frac{1}{n^2}\right)\right) \xrightarrow{n \to +\infty} -\lambda$$

$$\Rightarrow \lim_{n\to\infty} a_n = e^{-\lambda}$$

So, $\lim\limits_{n\to\infty} b(r) = \frac{\lambda^r . e^{-\lambda}}{r!} = P(r)$ Poisson Distribution

✓ EXAMPLE:

An insurance company pays £500,000 to each client who experiences a fire. The company has 5,000 clients. The probability of a client having a fire in one year is 10^{-4}.

Find the probability that the company pays at least £2000,000 in a year.

Soln: Assume no client has more than 1 fire.

$p = 10^{-4}$ $n = 5000 \longleftarrow$ this is large

So the mean number of fires is

$$\lambda = np = 0{\cdot}5$$

£2,000,000 = 4 fires

prob (≥ £2,000,000) = prob (at least 4 fires)

$$= 1 - P(0) - P(1) - \underset{\substack{\uparrow \\ \text{prob (2 fires)}}}{P(2)} - P(3)$$

Since $P(r) = \frac{e^{-\lambda}\lambda^r}{r!} = e^{-\frac{1}{2}} \cdot \frac{(\frac{1}{2})^r}{r!}$,

$$\text{prob } (\geq £2,000,000) = 1 - e^{-\frac{1}{2}} . \left[1 + \frac{(\frac{1}{2})^1}{1!} + \frac{(\frac{1}{2})^2}{2!} + \frac{(\frac{1}{2})^3}{3!} \right]$$

$$= 1 - \frac{79}{48} e^{-\frac{1}{2}}$$

$$= 0.00175 \approx 0.2\%$$

- Check $\sum_{r=0}^{\infty} P(r) = \sum_{r=0}^{\infty} \frac{e^{-\lambda}\lambda^r}{r!}$

$$= e^{-\lambda} \sum_{r=0}^{\infty} \frac{\lambda^r}{r!}$$

$$= e^{-\lambda} . e^{\lambda}$$

$$= 1$$

- Events occurring over intervals of time or space

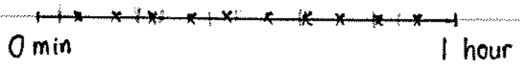
✓ EXAMPLE:

cars passing at a pt X over an hour.

(Let λ = average # passing in an hour)

Assume ① no 2 cars pass X at the same time

② the time that a car passes is independent of the other cars.



- Place a mark on the line for each time a car passes. For sufficiently large (small enough) n, split the interval into n sub-intervals (s.t.) containing 0 or 1 mark.
- View that as Bernoulli trials with probability $p = \frac{\lambda}{n}$ that a given sub-interva has a mark, then prob (r cars pass in the hour) = prob (r successes)

$$b(r) = \binom{n}{r} p^r q^{n-r} = \binom{n}{r} \left(\frac{\lambda}{n}\right)^r \left(1 - \frac{\lambda}{n}\right)^{n-r}$$

- To make the subintervals arbitrarily small (i.e. to allow the time to be arbitrarily close), take $\lim_{n \to \infty}$. This is the calculation we just did. So we have the Poisson distribution $P(r) = e^{-\lambda} . \frac{\lambda^r}{r!}$.

Fri. 16/12/16

MATH1401: Mathematical Methods I

Prof. Halburd

- Def.

Poisson Distribution

The probability of r events occurring in some interval (of space or time) with an average of λ events in the interval is

$$P(r) = e^{-\lambda} \frac{\lambda^r}{r!}$$

(independent events)

✓ EXAMPLE ①.

An office receives on average 3 calls per hour. Find the probability that in a particular hour

(a) no calls are received and

(b) exactly 3 calls are received.

Soln: average calls per hour $= \lambda = 3$

(a) prob (no calls) $= P(0)$

$$= e^{-3} \cdot \frac{3^0}{0!}$$

$$= e^{-3} = 0{\cdot}0498 \quad (5\%)$$

(b) prob (3 calls) $= P(3)$

$$= e^{-3} \cdot \frac{3^3}{3!}$$

$$= \frac{9}{2} e^{-3} \quad (22\%)$$

✓ EXAMPLE ②:

A roll of ribbon contains one defect per metre on average. A 50 cm piece is cut. What is the probability that it contains at least one defect ?

Soln: $\lambda = \frac{1}{2}$ defect per metre

prob (at least one defect) = 1 − prob (no defect)

$$= 1 - P(0)$$

$$= 1 - e^{-\frac{1}{2}} \cdot \frac{(\frac{1}{2})^0}{0!}$$

$$= 1 - \frac{1}{\sqrt{e}}$$

$$= 0{\cdot}39$$

7·8 Continuous Probability Distribution

- Def.

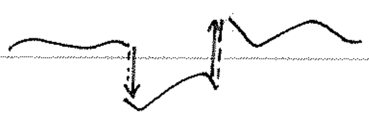
Suppose that a particle can be anywhere on the real line $\mathbb{R}$, we represent the probability that the particle is between $x=a$ and $x=b$ as

$P(a<x<b) = \int_a^b f(x)\,dx$, where $f(x)$ is called a probability density

- Def.

$f(x)$ is a piecewise continuous function.

i.e. $x_1 < x_2 < x_3 < \dots$ s.t. $f(x)$ is continuous on (x_n, x_{n+1})



← $f(x)$ can have several jumps

compared to $f(x) = \begin{cases} 1 & x \text{ irrational} \\ 0 & x \text{ rational} \end{cases}$ is not piecewise continuous.

✓ Properties.

① $f(x) \geq 0$ for all x

② $\int_{-\infty}^{\infty} f(x)\,dx = 1$

③ The mean of probability distribution is $\mu = \int_{-\infty}^{\infty} x f(x)\,dx$

(like the discrete case $\mu = \sum_j x_j P(x_j)$)

✓ EXAMPLE :

The probability density describing the location of a particle is

$$f(x) = \begin{cases} C(x-x^3) & 0<x<1 \\ 0 & \text{otherwise} \end{cases}$$

Find (i) the normalisation constant c.

(ii) the mean.

(iii) the probability that the particle is between $x=0$ and $x=\frac{1}{2}$.

Soln: (i) $\int_{-\infty}^{\infty} f(x)\,dx = 1$

$c\int_0^1 (x-x^3)\,dx = 1$ since $f(x)=0$ otherwise

$c\left(\frac{1}{2}-\frac{1}{4}\right) = 1$

$C=4$

Therefore, when $0<x<1$,

$$f(x) = 4(x-x^3)$$

$$= 4x(1-x^2)$$

$$= 4x(1+x)(1-x) > 0$$

(ii) $\mu = \int_{-\infty}^{\infty} x f(x)\,dx$

$= 4\int_0^1 (x^2 - x^4)\,dx$

$= 4\left(\frac{1}{3} - \frac{1}{5}\right)$

$= \frac{8}{15}$

(iii) prob $(0 < X < \frac{1}{2}) = \int_0^{1/2} f(x)\,dx$

$= 4\int_0^{1/2} (x - x^3)\,dx$

$= 4\left[(\frac{1}{2})(\frac{1}{2})^2 - (\frac{1}{4})(\frac{1}{2})^4\right]$

$= \frac{7}{16}$

Note: Suppose f(x) is (piecewise) continuous, then

$P(x=c) = 0 \quad \forall c \in (a,b)$

(probability at a specific point is 0)

7.9 Normal Distribution

- Def. Normal Distribution

The normal distribution of mean μ and standard deviation σ is

$$P(a<x<b) = \frac{1}{\sqrt{2\pi}.\sigma}\int_a^b \exp\left(-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2\right)dx$$

✓ Def.

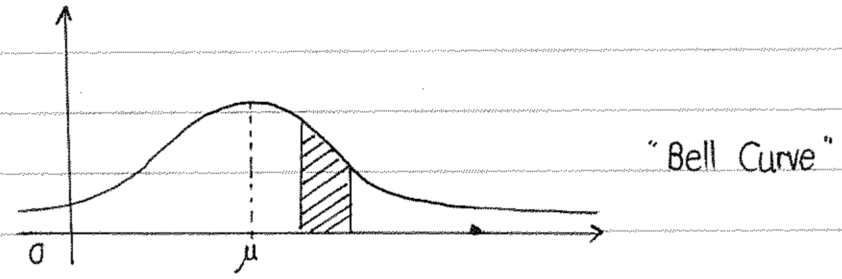
a measure of dispersion (spread)

Standard deviation $\sigma = \sqrt{\sigma^2}$ where $\sigma^2 =$ variance

$$= \int_{-\infty}^{\infty} (x-\mu)^2 \underset{\text{probability density}}{f(x)}\,dx$$

✓ Plot integrand (the thing that you integrate)

$$f(x) = \frac{1}{\sqrt{2\pi}\,\sigma}\exp(\ldots) = \text{probability density}$$



$$\Phi(z) = \frac{1}{\sqrt{2\pi}}\int_0^z \exp\left(-\frac{\zeta}{2}\right)d\zeta$$

table

greek letter "zeta"

$z = \frac{a-\mu}{\sigma}$ $\quad P(X \leq a) = 0.5 + P(Z \leq \frac{a-\mu}{\sigma}) = 0.5 + \phi\left(\frac{a-\mu}{\sigma}\right)$ $\quad$ if $z=0 \Rightarrow \phi(0)=0$

$\phi(-z) = -\phi(z)$

Wed. 07/12/16

MATH1401 Help Class

Prof. Wilson

§ Probability §

Choosing r from n

(1) Replacement allowed, order matters.

$$\text{Number of ways} = \underset{\underset{1^{st}}{\uparrow}}{n} \times \underset{\underset{2^{nd}}{\uparrow}}{n} \times \underset{\underset{3^{rd}}{\uparrow}}{n} \times \cdots \times \underset{\underset{r^{th}}{\uparrow}}{n} = n^r$$

(2) No replacement, order matters.

$$\text{Number of ways} = \underset{\underset{1^{st}}{\uparrow}}{n} \times \underset{\underset{2^{nd}}{\uparrow}}{(n-1)} \times \underset{\underset{3^{rd}}{\uparrow}}{(n-2)} \times \cdots \times \underset{\underset{r^{th}}{\uparrow}}{(n+1-r)} = \frac{n!}{(n-r)!} = {}^nP_r$$

(3) No replacement, order does not matter

If there are M ways to do it, each gives $r!$ ordered sets. So $M \times r! = {}^nP_r = \frac{n!}{(n-r)!}$

Therefore,

$$M = \frac{n!}{(n-r)!\,r!} = {}^nC_r = \binom{n}{r} \qquad \text{"n choose r"}$$

HW8-9

Q10. • Each possible ordering is equally-likely.

There are $n!$ ordering.

So the probability of each occurring is $\frac{1}{n!}$.

Therefore,

P(Exactly r are in the right place)

$$= \frac{\text{Number of orderings in which exactly r are in the right place}}{n!}$$

① $\underline{r = n}$.

no. of orderings = 1

$\Rightarrow P(n \text{ correct}) = \frac{1}{n!}$

② $\underline{r = n-1}$.

It is impossible to get just one book wrong. (where would it go?)

$\Rightarrow P((n-1) \text{ correct}) = 0$

③ $r = n-2$:

Number of ways to get exactly 2 wrong

= number of ways to choose 2 books (which we swap)

$= {}^nC_2$

$= \frac{n!}{2!(n-2)!}$

Therefore,

$$P(2 \text{ wrong}) = \frac{{}^nC_2}{n!} = \frac{1}{2!(n-2)!} = \frac{1}{(n-2)!\,2}$$

④ $r = n-3$:

Number of ways to get exactly 3 wrong

$= \boxed{\text{number of ways to choose 3 books}} \times \boxed{\text{number of ways to get these 3 wrong}}$

$\binom{n}{3}$

~~A B C~~ x all in right places

$\underline{A}$ C B x A right

B A $\underline{\underline{C}}$ x C right

B C A } ✓ works
C A B }

C $\underline{\underline{B}}$ A x B right

Therefore, $P(3 \text{ wrong}) = \binom{n}{3} \times 2 \times \frac{1}{n!} = \frac{n!}{3!(n-3)!} \times \frac{2}{n!} = \frac{1}{(n-3)!\,3}$

⑤ $r = n-4$:

Number of ways to get exactly 4 wrong

$= \boxed{\text{number of ways to choose 4 books}} \times \boxed{\text{number of ways to get these 4 wrong}}$

$\binom{n}{4}$

$9 = 4! - 1 - 6 - 8 = 4! - {}^4C_0 - {}^4C_2 - {}^4C_3$

0 wrong → 1

4! permutations → 4!

2 wrong → 6

3 wrong → 8

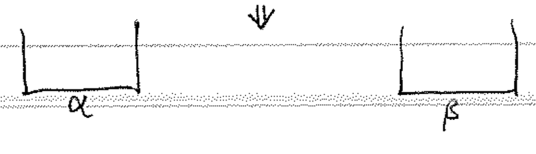
0 wrong → ~~ABCD~~	~~ABDC~~	~~ADBC~~	DABC
2 wrong → ~~ACBD~~	~~ACDB~~	~~ADCB~~	~~DACB~~
~~BACD~~	BADC	BDAC	~~DBAC~~
3 wrong → ~~BCAD~~	BCDA	~~BDCA~~	~~DBCA~~
~~CABD~~	CADB	CDAB	DCAB
~~CBAD~~	~~CBDA~~	CDBA	DCBA

Therefore, $P(4 \text{ wrong}) = \binom{n}{4} \times 9 \times \frac{1}{n!} = \frac{n!}{4!(n-4)!} \times \frac{9}{n!} = \frac{3}{(n-4)!\,8}$

- QUESTION 1.

50 red balls & 50 black balls

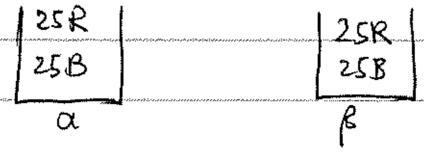
⇓

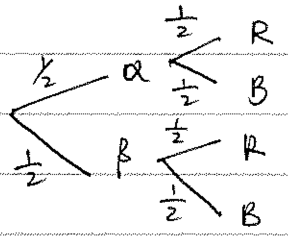


Want to maximise chance of getting a red

Choose urn → choose ball → maximise $P(\text{red})$

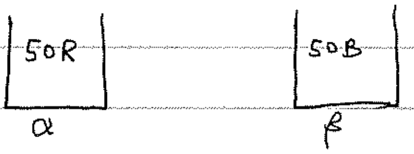
①

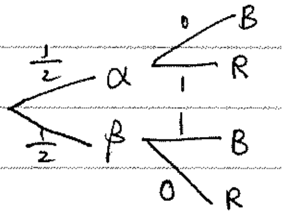




$$P(R) = P(\alpha \cap R) + P(\beta \cap R)$$
$$= 1/2 \times 1/2 + 1/2 \times 1/2$$
$$= 1/2$$

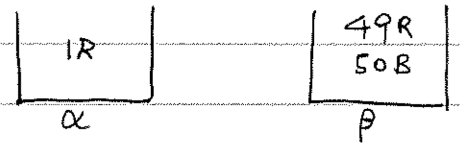
②

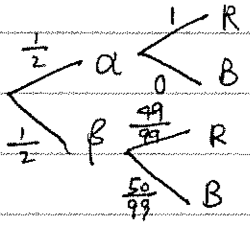




$$P(R) = P(\alpha \cap R) = \frac{1}{2} \times 1 = \frac{1}{2}$$

③

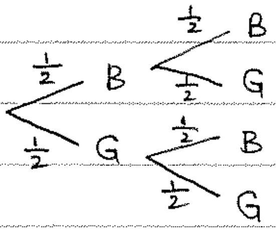




$$P(R) = P(\alpha \cap R) + P(\beta \cap R)$$
$$= \frac{1}{2} \times 1 + \frac{1}{2} \times \frac{49}{99}$$
$$= \frac{74}{99}$$

- QUESTION 2

Given that I have a boy, what's the probability that the other child is a girl?

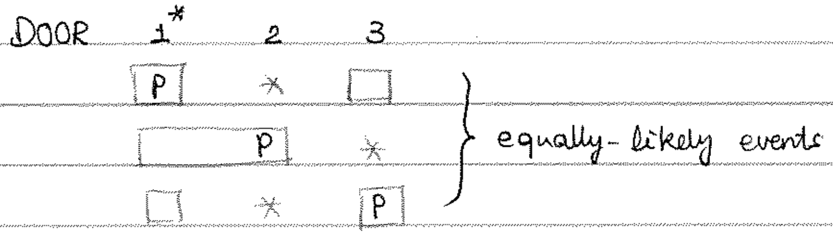


Since a boy is given

$$\{(B,B), (B,G), (G,B), \cancel{(G,G)}\}$$

$P(\text{the other child is a girl} \mid \text{I have a boy}) = \frac{2}{3}$

- QUESTION 3.



$P(\text{win} \mid \text{switch}) = \frac{2}{3}$

Wed 12/12/16

MATH1401 Help Class

Dr. Rila

HW8-9

Q13. 15. 16. 17. 19