MATH0010 Mathematical Methods 1 Notes

Based on the 2018 autumn lectures by Prof R Halburd

MATHEMATICAL METHODS 1

R. Halburd@ucl.ac.uk
Room 703 (Mond: 10–11).

October 1st 2018:

$\boxed{\text{GUIDE}}$ → CALCULUS text books.

1st: Vectors
2nd: Differential and integral calculus.
3rd: Functions of several variables.
4th: Differential equations.
5th: Probability
6th: Complex numbers.

85% {assessment: final exam
10% {problem sheets (best n−1 from n) n=8 (5%) + Midsessional exam (5%)
vectors test (≡ 2 problem sheets).
5% {elementary techniques test.

ELEMENTARY TEST CORRECTION:

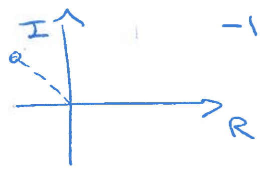
→ $\dfrac{x^2}{\cancel{x}\cdot \ln x}$

→ $\displaystyle\int \frac{(3+2x^2)^2}{\sqrt{x}}\cdot dx = \int x^{-1/2}\cdot(9+12x^2+4x^4) = \int 9x^{-1/2}\ldots$ NOT BY PARTS.

→ $\displaystyle\int \frac{2+3x^2}{x^3+2x-1}\,dx = \int \frac{f'(x)}{f(x)}\,dx = \ln|x^3+2x-1| + C.$

→ $\displaystyle\int \frac{x^2}{x^2+1}\,dx = \int \frac{x^2+1-1}{x^2+1}\,dx = \int 1 - \frac{1}{x^2+1}$

→ COMPLEX Nº:

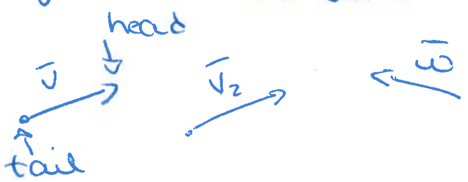


$-1+i \Rightarrow \sqrt{2}\,e^{i\,3\pi/4}$.

$\boxed{\text{VECTORS}}$.

Geometric notion: A vector has length (magnitude) and direction.

Think of a vector as a directed line segment, where 2 such directed line segments represent the same vector if and only if they have the same length and the same direction.

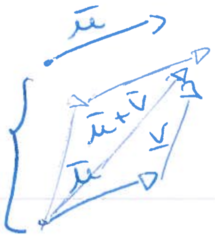


We indicate that something is a vector by writing $\underset{\sim}{v}$, $\vec{v}$

• The vector from some point A to a point B, is written as $\vec{AB}$.

• Think of $\underline{u}$ and $\underline{v}$ as 2 displacements.

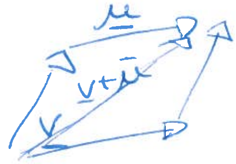
TRIANGLE RULE



addition: $\bar{u}+\bar{v}$.

$$\boxed{\bar{u}+\bar{v} = \bar{v}+\bar{u}}$$

PARALLELOGRAM



$\underline{v}+\underline{u}=\bar{w}$; $\underline{w}-\bar{u}=\underline{v}$

• Scalar multiplication

$\underline{v}$ + $\underline{v}$ = $\underline{v}$ $\underline{v}$ $=2\bar{v}$. $\Big\{$ same direction but more length.

$3\underline{v} = \underline{v}+\underline{v}+\underline{v}$

(greater)

If $\lambda \geq 0$ we define $\lambda\underline{v}$ to be the vector of length $\lambda|\underline{v}|$ in the same direction as $\underline{v}$.

module = length $\underline{v}$

⚠ SPECIAL VECTOR starts and finishes in the same point: vector 0.

$\underline{0}$ = vector of lenght 0 → $0\underline{v}$ for any $\underline{v}$ (vector posición).

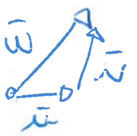
$\underline{0}=(\lambda+(-\lambda))\,\underline{v}$

$\underbrace{\underline{0} = \lambda\underline{v} + (-\lambda)\,\underline{v}}$ → $\underline{v}$ $\lambda\underline{v}$ $-\lambda\underline{v}$

So $(-\lambda)\underline{v}$ has length $\lambda|\underline{v}|$ pointing in the opposite direction to $\underline{v}$.

October 3rd 2018

Addition and inverse



$\bar{w}=\bar{u}+\bar{v}$

$\boxed{\bar{v}=\bar{w}-\bar{u}}$

→ we put $\bar{w}$ and $\bar{u}$ tail to tail instead of head to tail.

$\bar{v}-\bar{u}$.

→ ¡° Donde pongo la head to head.

• Position vector

Let O be de origin (special point) in 3 space. ($\mathbb{R}^3$) → en 3D.

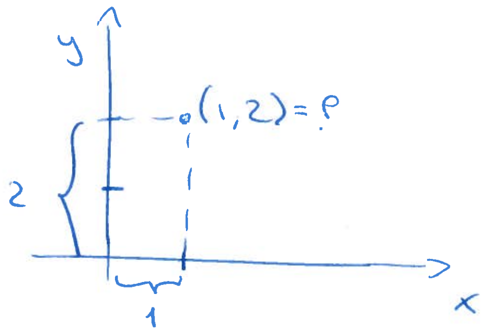
Let P be any point in our 3-space, then the (vector r) $\underline{r}=\underline{OP}$ is called the position vector of P.

Similarly, given any vector $\underline{v}$, we translate it (slide without changing its direction) s.t. its tail is at the origin, then its head is at some pt P.

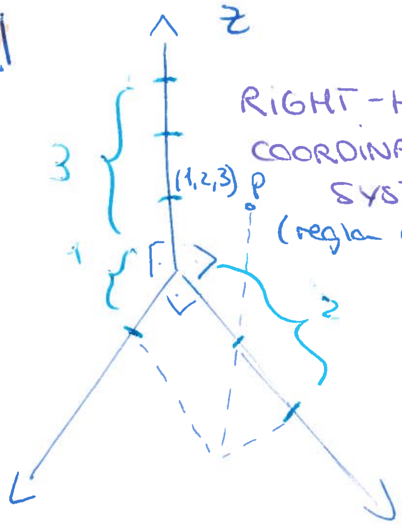
There is therefore a natural one-to-one correspondence between pts and vectors (s.t. the vector is the position vector of the point).

- Cartesian coordinates: labeling points.

$\boxed{2d}$



$\boxed{3d}$



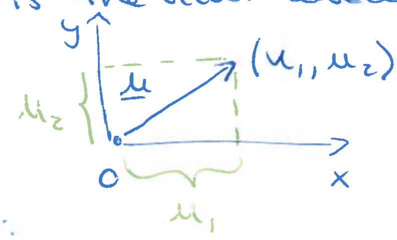
RIGHT-HANDED COORDINATE SYSTEM. (regla mano dcha)

$\boxed{3d}$ Let $\begin{pmatrix} x_0 \\ y_0 \\ z_0 \end{pmatrix}$ be the vector associated with the point (x_0, y_0, z_0).
(i-e the vector is the position vector of the point).

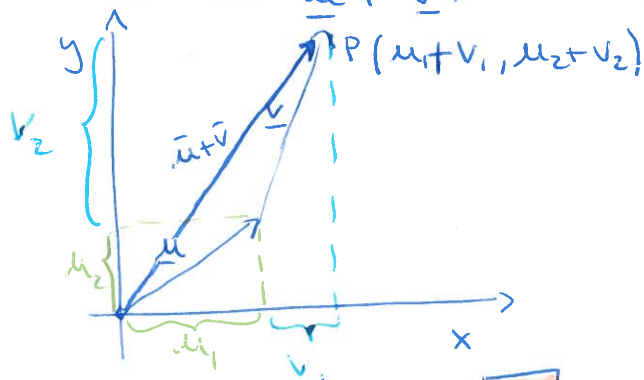
$\boxed{2d}$ Similarly, in 2d, $\begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$ is the vector associated with the point (x_0, y_0)

- Addition $\boxed{2D}$

Let $\underline{u} = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$, $\underline{v} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$

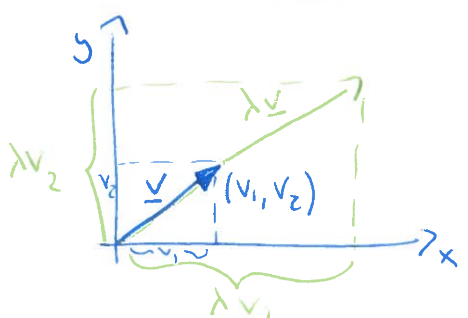


We want to add $\underline{u} + \underline{v}$:



$$\underline{u} + \underline{v} = \begin{pmatrix} u_1 + v_1 \\ u_2 + v_2 \end{pmatrix} = \underbrace{\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} + \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}}_{\text{OPERATION ADDITION}}$$

- Scalar multiplication $\boxed{2D}$



$$\lambda \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} \lambda v_1 \\ \lambda v_2 \end{pmatrix}$$

- Addition

$\boxed{3d}$

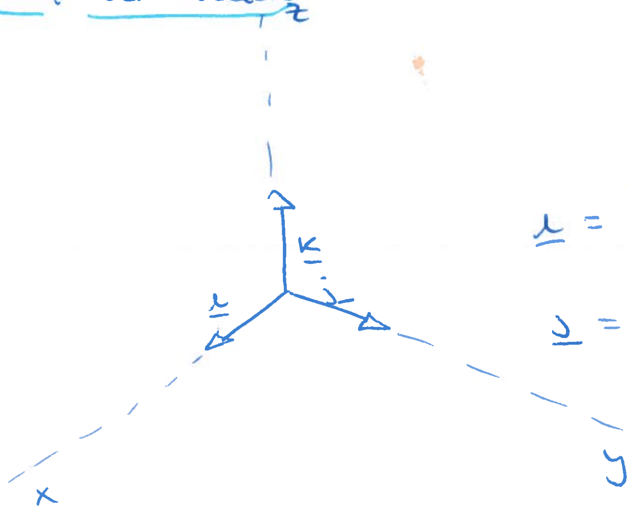
$$\begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} + \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} u_1+v_1 \\ u_2+v_2 \\ u_3+v_3 \end{pmatrix}$$

- Multiplication

$\boxed{3d}$

$$\lambda \cdot \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} \lambda v_1 \\ \lambda v_2 \\ \lambda v_3 \end{pmatrix}$$

- 3 independent vectors



$$\underline{i} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} . \quad \underline{k} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} .$$

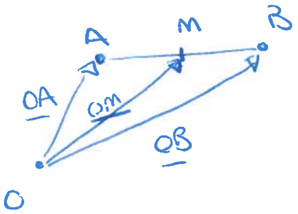
$$\underline{j} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

Any other vector can be defined by this 3 vectors.

Let $\begin{pmatrix} a \\ b \\ c \end{pmatrix}$ be any vector:

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} a \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ b \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ c \end{pmatrix} = a\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + b\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + c\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = a\underline{i} + b\underline{j} + c\underline{k}.$$

Ex: Use vectors to find the mid point of the points $A = (a_1, a_2, a_3)$ and $B = (b_1, b_2, b_3)$.



$\underline{OB} = \underline{OA} + \underline{AB}$

$\underline{OM} = \underline{OA} + \underline{AM}$; $\underline{OM} = \underline{OA} + \frac{1}{2}(\underline{AB}) \Rightarrow \underline{OA} + \frac{1}{2}(\overline{OB} - \overline{OA})$

$\therefore \underline{OM} = \overline{OA} + \frac{1}{2}(\overline{OB} - \overline{OA}) = \frac{1}{2}(\overline{OB} + \overline{OA})$

$\therefore \underline{OM} = \left(\frac{a_1+b_1}{2}, \frac{a_2+b_2}{2}, \frac{a_3+b_3}{2} \right).$

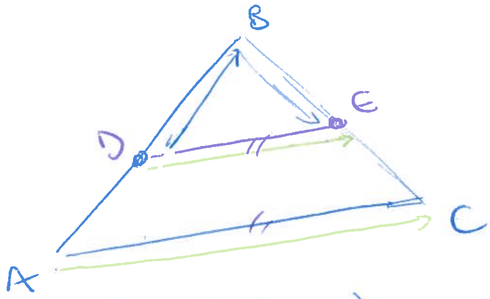
$\underline{OM} = \frac{1}{2}\left((b_1\bar{i} + b_2\bar{j} + b_3\bar{k}) + (a_1\bar{i} + a_2\bar{j} + a_3\bar{k}) \right)$

$\underline{OM} = \left(\frac{a_1+b_1}{2}\right)\bar{i} + \left(\frac{a_2+b_2}{2}\right)\bar{j} + \left(\frac{a_3+b_3}{2}\right)\bar{k}$

$\overline{OM} = \left(\frac{a_1+b_1}{2}, \frac{a_2+b_2}{2}, \frac{a_3+b_3}{2} \right)$

5th October 2018

Ex.: Use vector methods to prove that the line joining the mid points of 2 sides of a triangle is parallel to the third side and half its length.



Let the triangle be ABC and let D and E be the mid points of AB and BC respectively.

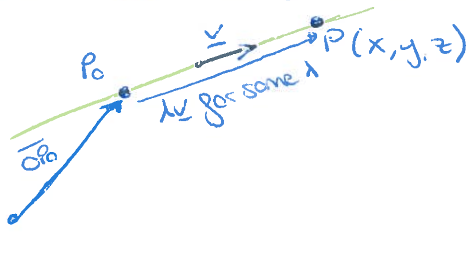
$$\overline{DE} = \overline{DB} + \overline{BE} = \left(\frac{1}{2}\overline{AB}\right) + \left(\frac{1}{2}\overline{BC}\right) = \frac{1}{2}\left(\overline{AB}+\overline{BC}\right) = \boxed{\frac{1}{2}\overline{AC}}.$$

Therefore, $\overline{DE}$ is // to $\overline{AC}$ and $\frac{1}{2}$ its length.

• LINES

↳ Defined by:
- 2 points
- 1 point and a vector

Let L be the line through the point P_0 in direction $\underline{V}$.

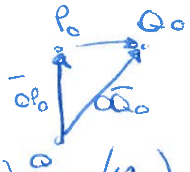


$$\boxed{\overline{OP} = \overline{OP_0} + \overline{P_0P} = \overline{OP_0} + \lambda\overline{V}}$$ VECTOR (PARAMETRIC) EQUATION FOR THE LINE

$$\boxed{(x,y,z) = (P_1,P_2,P_3)+\lambda(V_1,V_2,V_3)}.$$

Example: Find the vector equation for the line through $P_0=(1,2,3)$ and $Q_0(2,1,-1)$.

$$\overline{OP} = \overline{OP_0}+\overline{P_0P} = \overline{OP_0}+t\cdot\overline{P_0Q_0}$$



$$\overline{P_0Q_0} = \overline{OQ_0}-\overline{OP_0}$$

$$\begin{pmatrix}x\\y\\z\end{pmatrix} = \begin{pmatrix}1\\2\\3\end{pmatrix} + t\left(\begin{pmatrix}2\\1\\-1\end{pmatrix}-\begin{pmatrix}1\\2\\3\end{pmatrix}\right) = \begin{pmatrix}1\\2\\3\end{pmatrix}+\lambda\begin{pmatrix}1\\-1\\-4\end{pmatrix} = \begin{pmatrix}1+\lambda\\2-\lambda\\3-4\lambda\end{pmatrix}$$

$$\boxed{\begin{pmatrix}x\\y\\z\end{pmatrix} = \begin{pmatrix}P_1\\P_2\\P_3\end{pmatrix} + \lambda\begin{pmatrix}V_1\\V_2\\V_3\end{pmatrix} = \begin{pmatrix}P_1+\lambda V_1\\P_2+\lambda V_2\\P_3+\lambda V_3\end{pmatrix}}$$

PARAMETRIC EQUATION FOR THE LINE

From this we have the parametric form of the equation

$$\begin{cases} x = 1+t \\ y = 2-t \\ z = 3-4t \end{cases}$$

The cartesian form of the line comes from eliminating t.

$$(t=)\ x-1 = 2-y = \frac{y-2}{-1} = \frac{z-3}{-4}$$

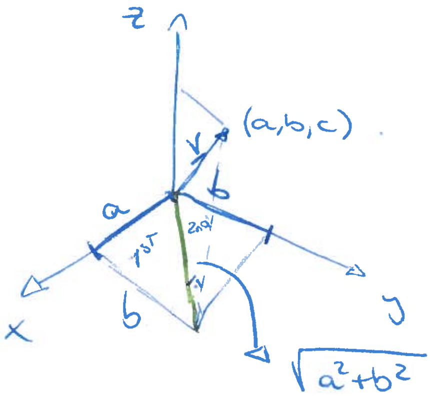
$$\boxed{\frac{x-P_1}{V_1} = \frac{y-P_2}{V_2} = \frac{z-P_3}{V_3}}$$

CARTESIAN EQUATION OF THE LINE = Standard

(ASIDE: In 3 dimensions, the equation $\underbrace{y=2x+1}_{\text{In 2 dimensions this is a line}}$ ⟹ this is a plan).

$$x-1=\frac{y-2}{-1}=\frac{z-3}{-4}$$: It represents 2 equations

• LENGTH of a vector.



$$\underline{v}=a\underline{i}+b\underline{j}+c\underline{k}=\begin{pmatrix}a\\b\\c\end{pmatrix}$$

$$\boxed{|\underline{v}|=\sqrt{\left(\sqrt{a^2+b^2}\right)^2+c^2}=\sqrt{a^2+b^2+c^2}}$$

We often denote a vector of unit length (a unit vector) with a "hat": $\hat{u}$.

let $\underline{v}=\underline{i}+2\underline{j}-2\underline{k}$

$|\underline{v}|=\sqrt{1^2+2^2+(2)^2}=3$

If I want a unit vector in the same direction of $\underline{v}$:

is $\underbrace{\boxed{\hat{v}=\frac{1}{|v|}\cdot\underline{v}}}_{\text{length would be 1}}=\frac{1}{3}(\underline{i}+2\underline{j}-2\underline{k})$.

Proof $\underline{v}=a\underline{i}+b\underline{j}+c\underline{k}$; $\lambda\underline{v}=(\lambda a)\underline{i}+(\lambda b)\underline{j}+(\lambda c)\underline{k}$

$$|\lambda\underline{v}|=\sqrt{(\lambda a)^2+(\lambda b)^2+(\lambda c)^2}=\sqrt{\lambda^2(a^2+b^2+c^2)}=|\lambda|\cdot\sqrt{a^2+b^2+c^2}=|\lambda|\cdot|\underline{V}|$$

⚠ $\sqrt{x^2}=|x|$

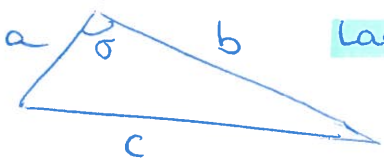
→ Demostración de que si x un $\underline{v}$ por λ su módulo tb se multiplica x λ.

Proof $\hat{v}=\frac{1}{|\underline{v}|}\cdot\underline{v}$

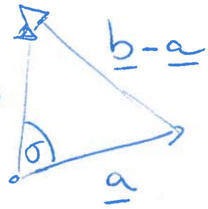
$$\left|\frac{1}{|\underline{v}|}\cdot\underline{v}\right|=\left|\frac{1}{|\underline{v}|}\right|\cdot|\underline{v}|=\left|\frac{1}{|\underline{v}|}\right|\cdot|\underline{v}|=\boxed{1}.$$

• ANGLES of vectors

① Scalar product.



Law of cosines: $\boxed{c^2=a^2+b^2-2ab\cos\sigma}$.



$$|\underline{a}-\underline{b}|^2 = |\underline{a}|^2 + |\underline{b}|^2 - 2\cdot|\underline{a}|\cdot|\underline{b}|\cdot\cos\sigma;$$

as it's a length $|\underline{a}-\underline{b}| = |\underline{b}-\underline{a}|$

$$;(-2)|\underline{a}|\cdot|\underline{b}|\cdot\cos\theta = |\underline{a}-\underline{b}|^2 - |\underline{a}|^2 - |\underline{b}|^2;$$

$$;(-2)|\underline{a}|\cdot|\underline{b}|\cdot\cos\theta = |(a_1-b_1)\underline{i} + (a_2-b_2)\underline{j} + (a_3-b_3)\underline{k}|^2 - |\underline{a}|^2 - |\underline{b}|^2;$$

$$;(-2)\cdot|\underline{a}|\cdot|\underline{b}|\cdot\cos\sigma = (a_1-b_1)^2 + (a_2-b_2)^2 + (a_3-b_3)^2 - (a_1^2+a_2^2+a_3^2) - (b_1^2+b_2^2+b_3^2) =$$

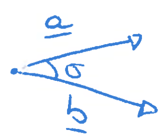
$$= (-2)\, a_1b_1 - 2a_2b_2 - 2a_3b_3;\quad |\underline{a}|\cdot|\underline{b}|\cdot\cos\sigma = a_1b_1 + a_2b_2 + a_3b_3;$$

$$;\ \boxed{\cos\theta = \frac{a_1b_1+a_2b_2+a_3b_3}{\sqrt{a_1^2+a_2^2+a_3^2}\cdot\sqrt{b_1^2+b_2^2+b_3^2}} = \frac{\underline{a}\cdot\underline{b}}{|\underline{A}|\cdot|\underline{B}|}}$$

Definition: The scalar (or dot) product of $\underline{a} = a_1\underline{i} + a_2\underline{j} + a_3\underline{k}$ and $\underline{b} = b_1\underline{i} + b_2\underline{j} + b_3\underline{k}$ is $\boxed{\underline{a}\cdot\underline{b} = a_1\cdot b_1 + a_2\cdot b_2 + a_3\cdot b_3\ (= |\underline{a}|\cdot|\underline{B}|\cdot\cos\sigma)}$.

Length $\to$ $\underline{a}\cdot\underline{a} = a_1^2 + a_2^2 + a_3^2 = |\underline{a}|^2$

Angles $\to$ $\cos\sigma = \dfrac{\underline{a}\cdot\underline{b}}{|\underline{a}|\cdot|\underline{b}|} = \dfrac{\underline{a}\cdot\underline{b}}{\sqrt{\underline{a}\cdot\underline{a}}\cdot\sqrt{\underline{b}\cdot\underline{b}}}$



Properties:

Vector $\underline{a} = a_1\underline{i} + a_2\underline{j} + a_3\underline{k}$, $\underline{b} = b_1\underline{i} + b_2\underline{j} + b_3\underline{k}$ and scalar λ.

* $(\lambda\cdot\underline{a})\cdot\underline{b} = \left((\lambda a_1)\underline{i} + (\lambda a_2)\underline{j} + (\lambda a_3)\underline{k}\right)\cdot(b_1\underline{i} + b_2\underline{j} + b_3\underline{k}) =$

$= (\lambda a_1)\cdot b_1 + (\lambda a_2)\cdot b_2 + (\lambda a_3)\cdot b_3 = \lambda\cdot(a_1b_1 + a_2b_2 + a_3b_3) =$

$= \lambda\cdot(\underline{a}\cdot\underline{b}) = (\lambda\underline{a})\cdot\underline{b}$ it has the same properties as products, that's why we call it a product

* $\underline{a}\cdot\underline{b} = \underline{b}\cdot\underline{a} \to$ symmetric product.

* Lets take $\underline{c} = c_1\underline{i} + c_2\underline{j} + c_3\underline{k}$

$\underline{a}\cdot(\underline{b}+\underline{c}) = (a_1\underline{i} + a_2\underline{j} + a_3\underline{k})\cdot\left((b_1+c_1)\underline{i} + (b_2+c_2)\underline{j} + (b_3+c_3)\underline{k}\right) =$

$= \left(a_1(b_1+c_1) + a_2\cdot(b_2+c_2) + a_3\cdot(b_3+c_3)\right) =$

$= (a_1b_1 + a_2b_2 + a_3b_3) + (a_1c_1 + a_2c_2 + a_3c_3) = \underline{a}\cdot\underline{b} + \underline{a}\cdot\underline{c};$

; $\underline{a}(\underline{b}+\underline{c}) = \underline{a}\cdot\underline{b} + \underline{a}\cdot\underline{c}$: distributive law

Perpendicular vectors.

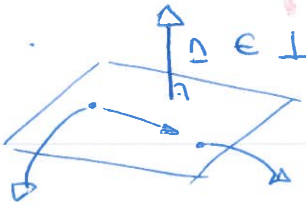
$\underline{a}\cdot\underline{b} = a_1b_1 + a_2b_2 + a_3b_3 = |\underline{a}|\cdot|\underline{b}|\cdot\underbrace{\cos\sigma}_{\text{if } \sigma = 90;\ \cos\sigma = 0}$.

2 non-zero vectors a and b are perpendicular if and only if $\underline{a}\cdot\underline{b} = 0$.

More generally, $\underline{a}$ and $\underline{b}$ are called orthogonal if and only if

$\underline{a}\cdot\underline{b}=0$ ($\iff$ $\underbrace{\text{perpendicular or } \underline{a}\cdot\underline{0} \text{ or } \underline{b}\cdot\underline{0}}_{\text{three options of } \underline{a}\cdot\underline{b}=0}$)

• PLANES . $\underline{n} \in \perp$ to the plane .



$P(x,y,z)$.

WAY 1 $\quad P_0=(x_0,y_0,z_0)$:

vector n and a point

$P_0P\cdot n=0$

P is on the plane if and only if $\underbrace{\underline{n}\cdot\overline{P_0P}}=0$.

$\underline{n}$ and $\overline{P_0P}$ are orthogonal .

$$\boxed{\Pi \equiv n_1(x-x_0)+n_2(y-y_0)+n_3(z-z_0)=0}$$

EQUATION OF THE PLANE : CARTESIAN FORM OF THE PLANE .

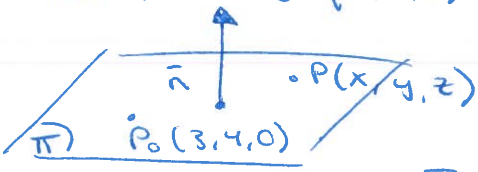
October 8th

Dot product $\rightarrow$ $\underline{a}\cdot\underline{b}=\boxed{|\underline{a}|\cdot|\underline{b}|\cdot\cos\theta}=\underbrace{\frac{1}{2}\left(|\underline{a}|^2+|\underline{b}|^2-|\underline{a}-\underline{b}|^2\right)}_{\text{cos law}}=$

$=\boxed{a_1b_1+a_2b_2+a_3b_3}$ — mostly . $\quad$ cos law : $ab\cos\theta=$

Example : Find the equation of a plane $\perp$ to the vector $\underline{n}=\vec{i}-2\vec{j}+3\vec{k}$

contains the point $P_0=(3,4,0)$.



$\Pi \equiv \underbrace{Ax+By+Cz}_{(A,B,C)=\bar{n}}+D=0$; $\overline{P_0P}\cdot\bar{n}=0$; $n_1\cdot(x-P_{0_1})+n_2(y-P_{0_2})+n_3(z-P_{0_3})=0$;

∴ P is in the plane if and only if $\overline{P_0P}\cdot\bar{n}=0$.

$\Pi \equiv x-3-2y+8+3z=0$; $\Pi \equiv x-2y+3z+5=0$.

Another way of defining a plane $\boxed{\text{WAY 2}}$ < 2 vectors // , a point

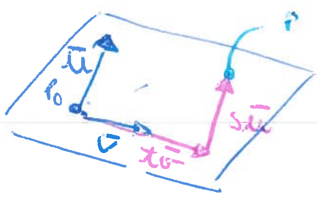
Let P_0 be a point of the plane and let $\bar{u}$ and $\bar{v}$ be vectors parallel to the plane, but not paralel to each other .

Let $P(x,y,z)$ be a general point on the plane

then $\exists$ s, t s.t :

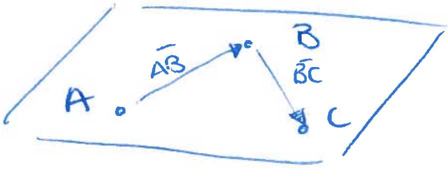
$$\boxed{\overline{OP}=\overline{OP_0}+s\bar{u}+t\bar{v}}$$

PARAMETRIC ~~POINT~~ FORM OF THE PLANE



O .

Ex: Find a parametric form of the plane containing the points $A=(1,2,1)$, $B=(3,1,2)$, $C=(1,0,1)$.



$\bar{AB} = \bar{OB} - \bar{OA} = (3,1,2) - (1,2,1) = (2,-1,1) = 2\vec{i} - \vec{j} + \vec{k}$

$\bar{BC} = \bar{OC} - \bar{OB} = (1,0,1) - (3,1,2) = (-2,-1,-1) = -2\vec{i} - \vec{j} - \vec{k}$

} These 2 are vectors in the plane

$\vec{OP} = \vec{OA} + s\bar{AB} + t\bar{BC}$; $P(x,y,z)$.

$(x,y,z) = (1,2,1) + s(2,-1,1) + t(-2,-1,-1)$.

$(x\vec{i} + y\vec{j} + z\vec{k}) = (\vec{i} + 2\vec{j} + \vec{k}) + s(2\vec{i} - \vec{j} + \vec{k}) + t(-2\vec{i} - \vec{j} - \vec{k}) =$

$= \vec{i} + 2\vec{j} + \vec{k} + 2s\vec{i} - s\vec{j} + s\vec{k} - 2t\vec{i} - t\vec{j} - t\vec{k} =$

$= (1+2s-2t)\vec{i} + (2-s-t)\vec{j} + (1+s-t)\vec{k}$

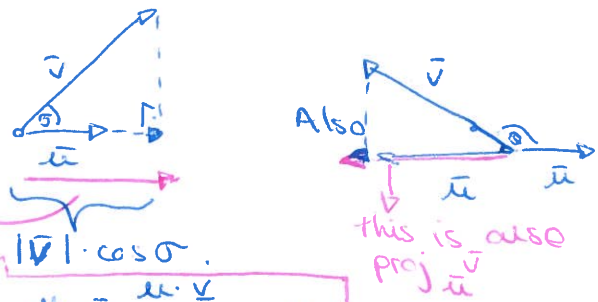
$$\Pi \equiv \begin{cases} x = 1+2s-2t \\ y = 2-s-t \\ z = 1+s-t \end{cases}$$

PARAMETRIC FORM OF THE PLANE.

- PROJECTIONS

Given two vectors $\underline{u}$ and $\underline{v}$, we often want to express $\underline{v}$ as a sum of a vector in a direction of $\bar{u}$ and a vector perpendicular.

The first vector in the sum (i.e. proportional to $\bar{u}$) is called the projection of $\bar{v}$ on $\bar{u}$, and written $\text{Proj}_{\underline{u}}\underline{v}$



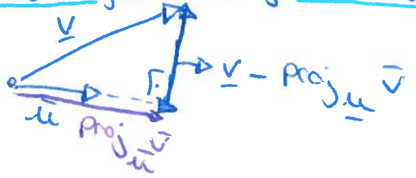
unit vector of $\underline{u}$ (so that the projection has the same direction as $\bar{u}$)

$|\bar{v}| \cdot \cos\theta$.

$$\boxed{\text{Proj}_{\underline{u}}\underline{v} = (|\underline{v}| \cdot \cos\theta) \cdot \frac{\underline{u}}{|\underline{u}|} = \frac{|\underline{u}| \cdot |\underline{v}| \cdot \cos\theta}{|\underline{u}|^2} \cdot \underline{u} = \frac{\underline{u} \cdot \underline{v}}{\underline{u} \cdot \underline{u}} \cdot \underline{u}}.$$

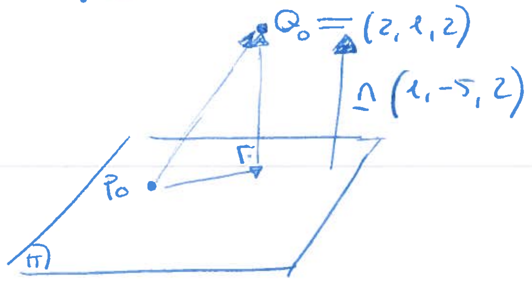
If $\hat{\underline{u}}$ is a unit vector then $\text{proj}_{\hat{\underline{u}}}\underline{v} = (\hat{\underline{u}} \cdot \underline{v}) \cdot \underline{u}$

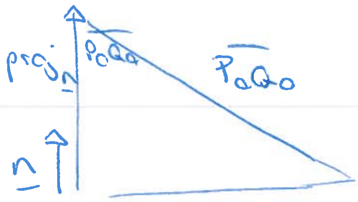
A use of the projection vector.



October 12^{th} 2018

Ex: lets find the distance between the point $(2,1,2)$ and the plane $x-5y+2z=3$.





•

○

$\underline{n} = \underline{i} - 5\underline{j} + 2\underline{k}$ is a normal vector to the plane.

$P_0 = (3,0,0)$ (If $y=0$, $z=0$ then x is ?) is a point on the plane.

$$\text{Distance} = \left|\text{proj}_{\underline{n}} \overline{P_0Q_0}\right| = \underbrace{\left|\hat{n} \cdot \overline{P_0Q_0}\right|}_{\text{absolute value}} = \left|\frac{\underline{n}\cdot\overline{P_0Q_0}}{\underline{n}\cdot\underline{n}}\cdot \underline{n}\right| = \left|\frac{\underline{n}\cdot\overline{P_0Q_0}}{|\underline{n}|}\right| \qquad \hat{n} = \frac{\underline{n}}{|\underline{n}|}$$

$$\left|\frac{\underline{n}\cdot P_0Q_0}{|\underline{n}|^2}\right| \cancel{|\underline{n}|}$$

$|\underline{n}| = \sqrt{1+25+4} = \boxed{\sqrt{30}}$

$$\left|\frac{(\underline{i}-5\underline{j}+2\underline{k})\cdot(-\underline{i}+\underline{j})+2\underline{k}}{\sqrt{30}}\right| = \frac{|-2|}{\sqrt{30}} = \frac{2}{\sqrt{30}} \quad \boxed{\frac{\sqrt{30}}{15}}$$

• VECTOR PRODUCT.

Given two vectors $\underline{A}$ and $\underline{b}$ (not multiple of each others), find a vector $\underline{c}$ perpendicular to both. $(\underline{a} = a_1\underline{i} + a_2\underline{j} + a_3\underline{k})$

$\underline{a}\cdot\underline{c} = 0$; $a_1c_1 + a_2c_2 + a_3c_3 = 0$ ... (1)

$\underline{b}\cdot\underline{c} = 0$; $b_1c_1 + b_2c_2 + b_3c_3 = 0$ ... (2).

b_1 (1): $a_1b_1c_1 + b_1a_2c_2 + b_1a_3c_3 = 0$ (3)

a_1 (2): $a_1b_1c_1 + a_1b_2c_2 + a_1b_3c_3 = 0$ (4).

(4) − (3) $= (a_1b_2 - a_2b_1)c_2 + (a_1b_3 - b_1a_3)c_3 = 0$.

Siempre el following 1, 2, 3

If $a_1b_2 - a_2b_1 \neq 0$; $c_2 = \dfrac{a_3b_1 - a_1b_3}{a_1b_2 - a_2b_1}\cdot c_3$ ... (5).

Similarly:

$c_1 = \dfrac{a_2b_3 - a_3b_2}{a_1b_2 - a_2b_1}\cdot c_3$. (haciendo el mismo proceso con c_2). despejo c_3.

$$\Rightarrow \underline{c} = c_1\underline{i} + c_2\underline{j} + c_3\underline{k} = \frac{c_3}{a_1b_2 - a_2b_1}\underbrace{\left((a_2b_3 - a_3b_2)\underline{i} + (a_3b_1 - a_1b_3)\underline{j} + (a_1b_2 - a_2b_1)\underline{k}\right)}_{\underline{a}\times\underline{b}}$$

$\underline{a} \times \underline{b}$ (vector product of $\underline{a}$ and $\underline{b}$ = cross product = wedge product = $\underline{a} \wedge \underline{b}$)

Properties:

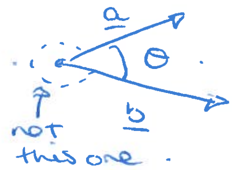
- $\lambda(a \times b) = (\lambda a \times b) = (a \times \lambda b)$.
- Check that $\underline{a} \times \underline{b}$ is orthogonal to $\underline{a}$ and $\underline{b}$ for all $\underline{a}$ and $\underline{b}$.

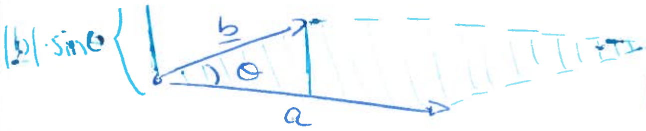
$$\underline{a} \cdot (\underline{a} \times \underline{b}) = 0 \Rightarrow a_1(\cancel{a_2 b_3} - \cancel{a_3 b_2}) + a_2(\cancel{a_3 b_1} - \cancel{a_1 b_3}) + a_3(\cancel{a_1 b_2} - \cancel{a_2 b_1}) = 0$$

$$\underbrace{\underline{b} \cdot (\underline{a} \times \underline{b})}_{\text{producto mixto}} = 0$$

- It can be shown that $|\underline{a} \times \underline{b}|^2 = |\underline{a}|^2 \cdot |\underline{b}|^2 - (\underline{a} \cdot \underline{b})^2 = |\underline{a}|^2 \cdot |\underline{b}|^2 - (|\underline{b}| \cdot |\underline{a}| \cdot \cos\theta)^2 =$
$= |\underline{a}|^2 \cdot |\underline{b}|^2 \cdot (1 - (\cos\theta)^2) = |\underline{a}|^2 \cdot |\underline{b}|^2 \cdot (\sin\theta)^2$

$$|\underline{a} \times \underline{b}| = |\underline{a}| \cdot |\underline{b}| \cdot \sin\theta \qquad \theta \in [0, \pi]$$

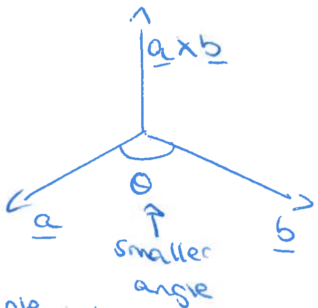




Area $= |\underline{a}| \cdot |\underline{b}| \cdot \sin\theta = |\underline{a} \times \underline{b}|$.

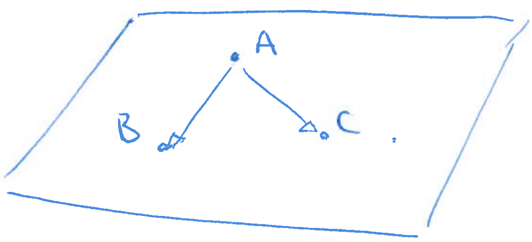
- $\underline{a} \times \underline{b} = \underline{0}$ if and only if $\underline{a} = \lambda \underline{b}$ or $\underline{b} = \lambda \underline{a}$
- $\underline{i} \times \underline{j} = \underbrace{(1\underline{i} + 0\underline{j} + 0\underline{k})}_{a_1 = 1} \times \underbrace{(0\underline{i} + 1\underline{j} + 0\underline{k})}_{b_2 = 1} = \underline{k}$ Regla mano dcha
- $\underline{a} \times \underline{b} = (|\underline{a}| \cdot |\underline{b}| \cdot \sin\theta)\hat{\underline{c}}$ where $\hat{\underline{c}}$ is a unit vector s.t $\underline{a}, \underline{b}, \hat{\underline{c}}$ form a right handed system. $\Leftrightarrow$ $\hat{\underline{c}}$ is given by the right hand rule.

REGLA MANO DCHA:



$$\underline{a} \times \underline{b} = -\underline{b} \times \underline{a}.$$

Example Let $A = (-1, 0, 1)$, $B = (2, -2, 1)$ and $C = (3, -3, 0)$. Find the equation of the plane containing A, B and C. Find the distance from the plane to $O(-3, -1, 2)$.
Find the area of the triangle ΔABC.



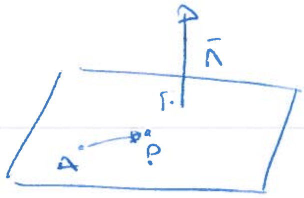
$\overline{AB} = \overline{OB} - \overline{OA} = -3\underline{i} - 2\underline{j} = (3, -2, 0)$.

$\overline{BC} = \overline{OC} - \overline{OB} = (3, -3, 0) - (2, -2, 1) = \underline{i} - \underline{j} - \underline{k} = (1, -1, -1)$

$$A \times b \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

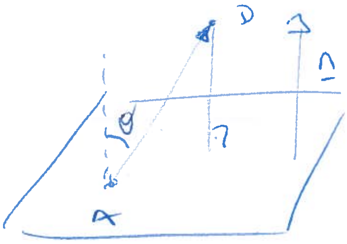
$$\overline{AB}\times\overline{BC} = \begin{vmatrix} i & j & k \\ 3 & -2 & 0 \\ 1 & -1 & -1 \end{vmatrix} = 2\bar{i} - 3\bar{k} + 2\bar{k} + 3\bar{j} = \boxed{2\bar{i} + 3\bar{j} - \bar{k} = \bar{n}}$$

Eqn of plane for $P = (x, y, z)$



$$\underline{n}\cdot\overline{AP} = 0 .$$

$$0 = (2\bar{i} + 3\bar{j} - \bar{k})\cdot\big((x+1)\underline{i} + (y)\underline{j} + (z-1)\underline{k}\big) = 2(x+1) + 3y - (z-1) \iff \boxed{2x+3y-z=}$$



$$\text{distance} = \big|\,|\overline{AD}|\cdot\cos\theta\,\big| = \big|\,|\hat{n}|\cdot|\overline{AD}|\cdot\cos\theta\,\big| =$$

$$\hat{n} = \frac{\bar{n}}{|\bar{n}|}$$

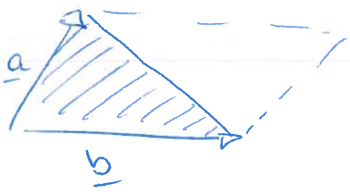
$$= \left|\hat{\underline{n}}\cdot\overline{AD}\right| = \left|\frac{\underline{n}\cdot\overline{AD}}{|\underline{n}|\cdot n}\cdot\underline{n}\right| = \left|\frac{\underline{n}\cdot\overline{AD}}{|\underline{n}|^2}\right|\cdot\cancel{|\underline{n}|} =$$

$$= \left|\frac{\underline{n}\cdot\overline{AD}}{|\underline{n}|}\right|$$

$$|\underline{n}| = \sqrt{2^2+3^2+1^2} = \sqrt{14} .$$

$$d = \left|\frac{1}{\sqrt{14}}(2\underline{i} + 3\underline{j} - \underline{k})\cdot(-2\underline{i} - \underline{j} + \underline{k})\right| = \frac{8}{\sqrt{14}} .$$

Area of triangle $= \frac{1}{2}|\overline{AB}\times\overline{BC}|$



PROPERTIES

- $(\lambda\underline{a})\times\underline{b} = \lambda\cdot(\underline{a}\times\underline{b})$.
- $\underline{a}\times(\underline{b}+\underline{c}) = (\underline{a}\times\underline{b}) + (\underline{a}\times\underline{c})$
- $\underline{a}\times\underline{b} = -\underline{b}\times\underline{a}$ Anti-symmetric
- $\underline{a}\times\underline{b} = (a_1\underline{i} + a_2\underline{j} + a_3\underline{k})\times(b_1\underline{i} + b_2\underline{j} + b_3\underline{k}) =$

$$= a_1b_1\,\underline{i}\times\underline{i} + \underbrace{(a_1b_2 - a_2b_1)}_{a_2b_1\,\underline{j}\times\underline{i}}\,\underline{i}\times\underline{j} + \dots$$

Let $\underline{e}_1 = \underline{i}$, $\underline{e}_2 = \underline{j}$, $\underline{e}_3 = \underline{k}$ $\quad \underline{a} = a_1\underline{e}_1 + a_2\underline{e}_2 + a_3\underline{e}_3 = \sum_{j=1}^{3} a_j\underline{e}_j$

Einstein summation convention

Repeated indices in an object or product of objects implies summation!

$\underline{a} = a_j \underline{e}_j \quad , \quad \underline{b} = b_j \underline{e}_j$

$$\underline{a} \times \underline{b} = (a_j e_j) \times (b_k e_k) \underset{\uparrow}{=} (a_j b_k e_j \times e_k)$$

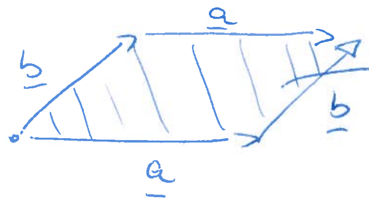
Implied $\sum_{j=1}^{3} \sum_{k=1}^{3}$

October 15th 2018

$$\underline{a}\cdot\underline{b} = \boxed{|\underline{a}|\cdot|\underline{b}|\cdot\cos\theta = a_1b_1 + a_2b_2 + a_3b_3}$$

$$\underline{a}\times\underline{b} = \boxed{(a_2b_3 - a_3b_2)\underline{e}_1 + (a_3b_1 - a_1b_3)\underline{e}_2 + (a_1b_2 - a_2b_1)\,\underline{e}_3 = (|\underline{a}|\,|\underline{b}|\cdot\sin\theta)\cdot\hat{\underline{n}}}$$

Where $\hat{\underline{n}}$ is the unit vector, normal to $\underline{a}$ and $\underline{b}$ ($\perp$) and given by the right hand rule.



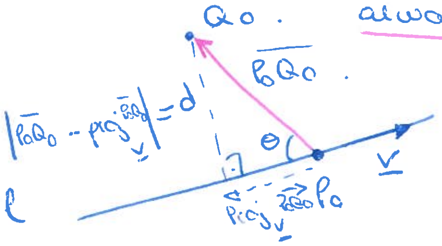
→ área = $|\underline{a}|\,|\underline{b}|\cdot$ seno $= |\underline{a}\times\underline{b}|$

⊛ $\bar{v}$: unit vector : $\hat{\underline{v}} = \dfrac{\underline{v}}{|\underline{v}|}$; $\underline{v} = |\underline{v}|\cdot\hat{\underline{v}}$.

Ex: Find the $\underbrace{\text{distance}}_{\text{always the shortest ⊛}}$ between the points $(-2,2,4)$ and the line $\underbrace{x+3 = \frac{y-1}{2} = 2-z}$.

$2 - z = t$;

; $z = 2 - t$.



$$d = |\overline{P_0Q_0}|\cdot\sin\theta = \frac{|\underline{v}|\cdot|\overline{P_0Q_0}|\cdot\sin\theta}{|\underline{v}|} = \frac{|\underline{v}\times\overline{P_0Q_0}|}{|\underline{v}|}$$

d no es el producto vectorial, pero su length sí que lo es.

Let $Q_0 = (-2,2,4)$, $\underbrace{P_0 = (-3,1,2)}_{\text{Point of the line}}$, $\underbrace{\bar{v} = (1,2,-1)}_{\text{a vector // to the line}}$, $\overline{P_0Q_0} = (1,1,2)$.

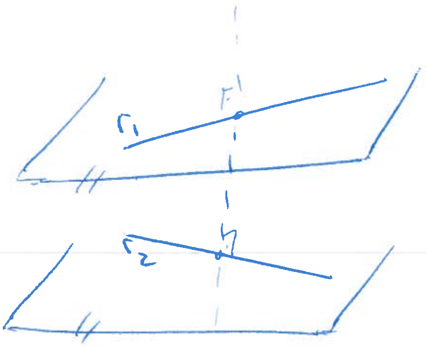
$$\ell \equiv \begin{cases} x = -3 + t \\ y = 1 + 2t \\ z = 2 - t \end{cases}$$

$$\bar{v}\times\overline{P_0Q_0} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ 1 & 2 & -1 \\ 1 & 1 & 2 \end{vmatrix} = 4\bar{i} - \bar{j} + \bar{k} - 2\bar{k} + \bar{i} - 2\bar{j} = 5\bar{i} - 3\bar{j} - \bar{k}$$

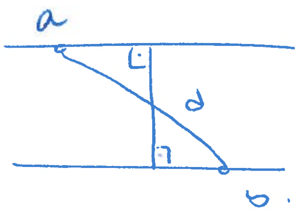
$$\text{Distance} = \frac{|\bar{v}\times\overline{P_0Q_0}|}{|\underline{v}|} = \frac{\sqrt{25+9+1}}{\sqrt{1+4+1}} = \sqrt{\frac{35}{6}} = \frac{\sqrt{6\cdot 35}}{6} = \boxed{\frac{\sqrt{210}}{6}}$$

Ex: Find the distance between the skew lines $\underline{r}_1(t) = \underline{a} + t\,\underline{u}$.
$\underline{r}_2(t) = \underline{b} + t\,\underline{v}$.

not parallel.
(se croton).



Lines are in horizontal planes.



The shortest line connecting the lines L_1 and L_2 is $\perp$ to both, so it's in the direction of $\underline{u} \times \underline{v}$.

The distance is the length of the projection of any vector connecting points on the 2 lines in the direction $\underline{u} \times \underline{v}$.

So distance $= \left| \dfrac{(\underline{u} \times \underline{v}) \cdot (b - a)}{|\underline{u} \times \underline{v}|} \right|$.

• Scalar product:

$$\boxed{\underline{e}_i \cdot \underline{e}_j = \delta_{ij} = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}}$$

$\underline{a} = a_1 \underline{e}_1 + a_2 \underline{e}_2 + a_3 \underline{e}_3 = a_i \underline{e}_i$. Einstein summation convention.

$\underline{b} = b_j \underline{e}_j$

$$\underline{a} \cdot \underline{b} = (a_i e_i) \cdot (b_j e_j) = (a_i b_j) \underbrace{(e_i \cdot e_j)}_{\delta_{ij}} = a_i b_j \delta_{ij} =$$

everything else collapses

$= a_i b_i = a_1 b_1 + a_2 b_2 + a_3 b_3$.

• Cross product.

$$\underline{e}_i \times \underline{e}_j = \epsilon_{ij1} \underline{e}_1 + \epsilon_{ij2} \underline{e}_2 + \epsilon_{ij3} \underline{e}_3$$

October 21st 2018

• EINSTEIN SUMMATION CONVENTIONS

$\underline{a} = a_1 \underline{e}_1 + a_2 \underline{e}_2 + a_3 \underline{e}_3 = \sum_{i=1} a_i \cdot \underline{e}_i$

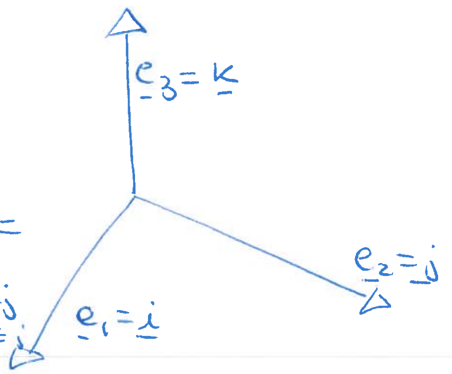
DOT PRODUCT.

$\underline{a} \cdot \underline{b} = (a_i e_i) \cdot (b_j \cdot e_j) = a_i b_j \underbrace{e_i \cdot e_j}_{} =$

$= a_i b_j \delta_{ij} = a_i b_i$.

$\delta_{ij} = \begin{cases} 1 & i = j \\ 0 & \text{if } i \neq j \end{cases}$

only works when $i = j$ ($\delta_{ij} = 1$).
if $i \neq j$, $\delta_{ij} = 0$.



Explanation

$$\sum_{i=1}^{3} a_i b_j \delta_{ij} = a_1 b_1 \cancel{\delta_{11}}^{1} + a_1 b_2 \cancel{\delta_{12}}^{0} + a_1 b_3 \cancel{\delta_{13}}^{0} + a_2 b_1 \cancel{\delta_{21}}^{0} + a_2 b_2 \cancel{\delta_{22}}^{1} +$$
$$+ a_2 b_3 \cancel{\delta_{23}}^{0} + a_3 b_1 \cancel{\delta_{31}}^{0} + a_3 b_2 \cancel{\delta_{32}}^{0} + a_3 b_3 \cancel{\delta_{33}}^{1} =$$
$$= \boxed{a_1 b_1 + a_2 b_2 + a_3 b_3} = \boxed{a_n \cdot b_n}$$

CROSS PRODUCT

$$\underline{a} \times \underline{b} = a_i \underline{e}_i \times b_j \underline{e}_j = a_i b_j \cdot \underbrace{\underline{e}_i \times \underline{e}_j}$$

$$\underline{e}_i \times \underline{e}_j \Rightarrow \underline{e}_1 \times \underline{e}_1 = \underline{0} = \underbrace{0 \underline{e}_1}_{\epsilon_{111} \cdot \underline{e}_1} + \underbrace{0 \underline{e}_2}_{\epsilon_{112} \cdot \underline{e}_2} + \underbrace{0 \underline{e}_3}_{\epsilon_{113} \underline{e}_3}$$: Right hand rule.

$$\underline{e}_1 \times \underline{e}_2 = \underline{e}_3 = \underbrace{0 \underline{e}_1}_{\epsilon_{121} \cdot \underline{e}_1} + \underbrace{0 \underline{e}_2}_{\epsilon_{122} \underline{e}_2} + \underbrace{1 \underline{e}_3}_{\epsilon_{123} \cdot \underline{e}_3}$$

Define ϵ_{ijk} s.t. $\underline{e}_i \times \underline{e}_j = \epsilon_{ij1} \cdot \underline{e}_1 + \epsilon_{ij2} \underline{e}_2 + \epsilon_{ij3} \underline{e}_3 = \epsilon_{ijk} \underline{e}_k$

$$= \sum_{k=1}^{3} \epsilon_{ijk} \cdot \underline{e}_k$$

ϵ_{ijk} is determined uniquely by the following:

$\epsilon_{123} = 1$.

Swapping 2 indices changes the sign $\epsilon_{ijk} = -\epsilon_{jik} = -\epsilon_{ikj} = -\epsilon_{kij}$

* If there are repited indices $\epsilon = 0$.

$\epsilon_{112} = -\epsilon_{112} = 0$.

$1 = \epsilon_{123} = \epsilon_{231} = \epsilon_{312}$
$-1 = \epsilon_{213} = \epsilon_{132} = \epsilon_{221}$

Si para volver al orden normal tengo que hacer
- nº par de cambios = 1.
- nº impar de cambios = -1.

$$\underline{a} \times \underline{b} = a_i \cdot b_j \cdot \epsilon_{ijk} \cdot \underline{e}_k = \left(\underbrace{\epsilon_{231} a_2 b_3}_{1} + \underbrace{\epsilon_{321} a_3 b_2}_{-1}\right) \underline{e}_1 +$$
$$+ \left(\underbrace{\epsilon_{132} a_1 b_3}_{-1} + \underbrace{\epsilon_{312} a_3 b_1}_{1}\right) \underline{e}_2 + \left(\underbrace{\epsilon_{123} a_1 b_2}_{1} + \underbrace{\epsilon_{213} a_2 b_1}_{-1}\right) \underline{e}_3 =$$

I'm only writing the ones in which $\epsilon_{ijk} = 1$.

Ex:

$$(\underline{a} \times \underline{b}) \cdot \underline{c} = (a_i b_j \epsilon_{ijk} \cdot \underline{e}_k) \cdot (c_l \underline{e}_l) = a_i b_j c_l \epsilon_{ijk} \underbrace{\underline{e}_k \cdot \underline{e}_l}_{\delta_{kl}} =$$

$$= a_i b_j c_l \epsilon_{ijk} \cdot \delta_{kl} \underset{\uparrow}{=} a_i b_j c_l \epsilon_{ijk}.$$

Suppose $k = l$.

- Identities:

① $\epsilon_{ijk} \cdot \epsilon_{imn} = \delta_{jm} \delta_{kn} - \delta_{jn} \cdot \delta_{km}$

② $\epsilon_{imn} \cdot \epsilon_{jmn} = 2 \delta_{ij}$

③ $\epsilon_{ijk} \cdot \epsilon_{ijk} = 6$

Ex:

$(\underline{a}\times\underline{b})\times\underline{c}$

$\underline{d}=\underline{a}\times\underline{b} = \epsilon_{ijk}\, a_i b_j\, \underline{e}_k = d_k\, \underline{e}_k$ → I can change k with p;

$\underline{d}\times\underline{c} = \epsilon_{pqr}\, d_p\, c_q\, \underline{e}_r = \epsilon_{pqr}\left(\epsilon_{ijp}\, a_i b_j\right) c_q\, \underline{e}_r =$

$= \epsilon_{pqr}\,\epsilon_{ijp}\, a_i b_j c_q \underline{e}_r = \underset{①}{\epsilon_{pqr}\,\epsilon_{pij}}\, a_i b_j c_q \underline{e}_r = (\delta_{qi}\delta_{rj} - \delta_{qj}\delta_{ri})\, a_i b_j c_q \underline{e}_r$

$= \delta_{qi}\delta_{rj}\, a_i b_j c_q \underline{e}_r - \delta_{qj}\delta_{ri}\, a_i b_j c_q \underline{e}_r = a_i b_j c_i \underline{e}_j - a_i b_j c_j \underline{e}_i =$

① $\boxed{q=i,\ r=j}$ $\boxed{q=j,\ r=i}$ ②

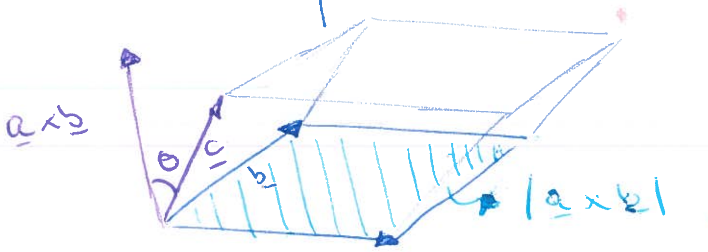
$= a_i c_i\, b_j \underline{e}_j - b_j c_j\, a_i \underline{e}_i = (\underline{a}\cdot\underline{c})\,\underline{b} - (\underline{b}\cdot\underline{c})\,\underline{a}$

- Triple product:

$[a,b,c] = (\underline{a}\times\underline{b})\cdot\underline{c} = \epsilon_{ijk}\, a_i b_j c_k = (\underline{c}\times\underline{a})\cdot\underline{b} = (\underline{b}\times\underline{c})\cdot\underline{a} =$

(swap 2 vectors change sign)

$$= \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$



$(\underline{a}\times\underline{b})\cdot\underline{c} = \underbrace{\text{area base}} \cdot \underbrace{\text{height}} = \text{volume}$.

$= \underbrace{|\underline{a}\times\underline{b}|} \cdot \underbrace{|\underline{c}|\cdot\cos\theta}$.

$|(\underline{a}\times\underline{b})\cdot\underline{c}|$ = volume of the parallel piped.

COMPLEX NUMBERS

$i^2 = -1$

$i = \sqrt{-1}$

$z = x + iy \longrightarrow$ $z_1 = x_1 + iy_1$, $z_2 = x_2 + iy_2$

$z_1 + z_2 = x_1 + x_2 + i(y_1 + y_2)$.

⚠ $-z = -x - (iy) = (-x) + i(-y)$.

$z_1 \cdot z_2 = (x_1 + iy_1)\cdot(x_2 + iy_2) = x_1x_2 + iy_1x_2 + ix_1y_2 + \underbrace{i^2}_{=-1}y_1y_2 =$

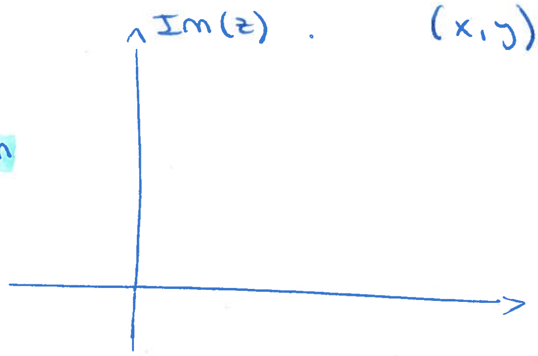
$= (x_1x_2 - y_1y_2) + i\cdot(y_1x_2 + y_2x_1)$.

MULTIPLICO POR EL CONJUGADO $(i \Rightarrow -i)$.

⚠ $\boxed{\frac{1}{z} = \frac{1}{x+iy}} = \frac{1}{(x+iy)}\cdot\frac{(x-iy)}{(x-iy)} = \frac{x-iy}{x^2 - i^2y^2} = \frac{x-iy}{x^2+y^2} =$

$= \frac{x}{x^2+y^2} + i\cdot\left(-\frac{y}{x^2+y^2}\right)$.

⚠ ARGAND DIAGRAM



$(x,y) \sim x+iy$.

$x = Re(z) \to$ real part

$y = Im(z) \to$ imaginary part.

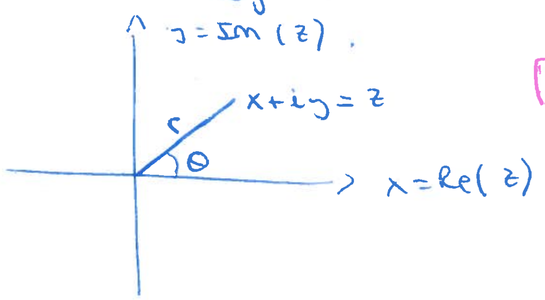
⚠ Define: A complex Nb is a pair of real numbers (x,y).

① $(x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2)$

② $(x_1, y_1)\cdot(x_2, y_2) = (x_1x_2 - y_1y_2, x_1y_2 + x_2y_1)$.

③ $(x, y)^{-1} = \left(\frac{x}{x^2+y^2}, \frac{-y}{x^2+y^2}\right)$.

⚠ We use the notation $x+iy$ to mean (x,y). Real $x \to (x,0)$, $i = (0,1)$.



$\boxed{r =}$ distance from the origin to $z = \boxed{\sqrt{x^2+y^2} = |z|}$ MODULUS

$x = r \cdot \cos\theta$, $y = r \cdot \sin\theta$ $\Big\}$ $\tan\theta = y/x$.

$z = x + iy = r\cdot\cos\theta + i \cdot r\cdot\sin\theta = r\cdot(\cos\theta + i\cdot\sin\theta)$.

$z_1 = r_1 \cdot (\cos\theta_1 + i \sin\theta_1)$

already developped.

$z_2 = r_2 \cdot (\cos\theta_2 + i \cdot \sin\theta_2)$.

$z_1 \cdot z_2 = r_1 \cdot r_2 \cdot \left[(\cos\theta_1 \cdot \cos\theta_2 - \sin\theta_1 \cdot \sin\theta_2) + i\cdot(\cos\theta_1 \sin\theta_2 + \sin\theta_1 \cos\theta_2)\right] =$

$= r_1 \cdot r_2 \cdot \left[\cos(\theta_1 + \theta_2) + i \cdot \sin(\theta_1 + \theta_2)\right]$.

$|z_1 z_2| = |z_1| \cdot |z_2|$.

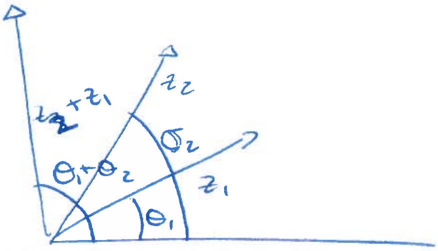
⚠ and add the angles (arguments).

$z = r\cdot(\cos\theta + i\cdot\sin\theta) \to \theta = \text{argument} = \arg(z)$ = angle from the real axis in anti clockwise direction

$\Downarrow$

only defined up to integer multiples of 2π.

$\text{Arg}(z) = \arg(z)$; $-\pi < \text{Arg}(z) < \pi$.



In particular : $z = r\cdot(\cos\theta + i\cdot\sin\theta)$; $z^2 = zz = r^2\cdot(\cos 2\theta + i \sin 2\theta)$ (set $r=1$)

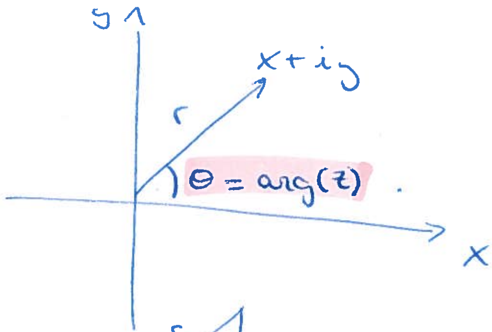
$\left[(\cos\theta + i\sin\theta)^2 = (\cos 2\theta + i \sin 2\theta)\right]$

For integer n:

$(\cos\theta + i\cdot\sin\theta)^n = \cos n\theta + i \sin n\theta$ de Moivre's Th.

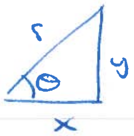
October 22nd 2018

Complex conjugate: $\bar{z} = x - iy$. $(= z^*)$



$r = |z| = \sqrt{x^2 + y^2}$

$\tan\theta = \dfrac{y}{x}$



: STILL THINK ABOUT THE SIGN.

$$z = x + iy = r \cdot (\cos\theta + i\sin\theta)$$

$$(\cos\theta + i\sin\theta)^2 = \cos 2\theta + i\sin 2\theta$$

For integer n: DE MOIVRE

$$\boxed{(\cos\theta + i\cdot\sin\theta)^n = \cos n\theta + i\cdot\sin n\theta}$$

conjugate

$$\boxed{(\cos\theta + i\cdot\sin\theta)^{-n}} = \frac{1}{(\cos\theta + i\cdot\sin\theta)^n} = \frac{1}{(\cos n\theta + i\cdot\sin n\theta)} \cdot \frac{(\cos n\theta - i\sin n\theta)}{(\cos n\theta - i\sin n\theta)} =$$

$$= \frac{\cos n\theta - i\cdot\sin n\theta}{(\cos n\theta)^2 - (i\sin n\theta)^2} = \frac{\cos n\theta - i\sin n\theta}{\cos^2 n\theta + \sin^2 n\theta} = \boxed{\cos(-n)\theta + i\cdot\sin(-n)\theta}$$

Ex: write $\dfrac{\sin 5\theta}{\sin\theta}$ as a polynomial in $\cos\theta$:

De Moivre:

$$\cos 5\theta + i\sin 5\theta = (\cos\theta + i\sin\theta)^5 =$$

$(x+y)^n =$

(Tartaglia).

					1					
				1		1				
			+1		−2		+1			
		1		3		3		1		
	1		4		6		4		1	
+1		−5		+10		−10		+5		−1

EMPIEZO CON $\oplus$.

$i^2 = -1$
$i^3 = -i$
$i^4 = 1$
$i^5 = i$

$$= (\cos\theta)^5 + 5(\cos\theta)^4\cdot(i\sin\theta) + 10(\cos\theta)^3\cdot(i\sin\theta)^2 + 10(\cos\theta)^2\cdot(i\sin\theta)^3 +$$

$$+ 5(\cos\theta)\cdot(i\sin\theta)^4 + (i\sin\theta)^5 .$$

not necessary for the question

$$\cos 5\theta + i\sin 5\theta = \underbrace{\left[(\cos\theta)^5 - (10\cos\theta)^3(\sin\theta)^2 + 5(\cos\theta)(\sin\theta)^4\right]}_{\text{not necessary for the question}} +$$

$$+ i\cdot\left[5(\cos\theta)^4\cdot\sin\theta - 10(\cos\theta)^2\cdot(\sin\theta)^3 + (\sin\theta)^5\right]$$

Take imaginary part:

$$\sin 5\theta = 5\cdot(\cos\theta)^4\cdot\sin\theta - 10(\cos\theta)^2(\sin\theta)^3 + (\sin\theta)^5 .$$

$$\Rightarrow \frac{\sin 5\theta}{\sin\theta} = (5\cdot\cos\theta)^4 - 10(\cos\theta)^2(\sin\theta)^2 + (\sin\theta)^4 =$$

$$= (5\cdot\cos\theta)^4 - 10(\cos\theta)^2\cdot\left[1-(\cos\theta)^2\right] + \left[1-(\cos\theta)^2\right]^2 =$$

$$= \boxed{16\cos^4\theta - 3(\cos\theta)^2 + 1}$$

EULER'S FORMULA

$$\boxed{e^{i\theta} = \cos\theta + i\cdot\sin\theta}$$ equation ①

EXPLANATION: (not really necessary).

$$e^x = \exp(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

$$\cos x = \sum_{k=0}^{\infty} (-1)^k \cdot \frac{x^{2k}}{2k!}$$

$$\sin x = \sum_{k=0}^{\infty} \frac{(-1)^k \cdot x^{2k+1}}{(2k+1)!}$$

$e^{-i\theta}$ → conjugate of $e^{i\theta}$

$e^{-i\theta} = \cos\theta - i\sin\theta$ equation②

$\boxed{\frac{①+②}{2}}$ $\cos\theta = \frac{1}{2}\left(e^{i\theta} + e^{-i\theta}\right)$

$\boxed{\frac{①-②}{2}}$ $\sin\theta = \frac{1}{2i}\left(e^{i\theta} - e^{-i\theta}\right)$.

Ex: Show $(\cos\theta)^6 = \frac{1}{32}\cos 6\theta + \frac{3}{16}\cos 4\theta + \frac{15}{32}\cos 2\theta + \frac{5}{16}$.

Hence evaluate $\int (\cos\theta)^6 \, d\theta$.

$(\cos\theta)^6 = \frac{1}{2^6}\cdot\left(z+\frac{1}{z}\right)^6 =$ | where: $z = e^{i\theta}$ → $z^6 = (e^{i\theta})^6 = e^{i\theta 6}$

$\frac{1}{z} = e^{-i\theta}$.

$$= \frac{1}{2^6}\cdot\left(z^6 + 6z^5\frac{1}{z} + 15z^4\frac{1}{z^2} + 20z^3\frac{1}{z^3} + 15z^2\frac{1}{z^4} + 6z\frac{1}{z^5} + \frac{1}{z^6}\right) =$$

$$= \frac{1}{2^6}\cdot\left((z^6+z^{-6}) + 6(z^4+z^{-4}) + 15(z^2+z^{-2}) + 20\right) = \frac{1}{2^6}\cdot\left(\left(e^{6i\theta}+e^{-6i\theta}\right)\ldots\right) =$$

por la formula $\frac{1+2}{2}$.

$$(\cos\theta)^6 = \frac{1}{2^6}\left(2\cos 6\theta + 12\cos 4\theta + 30\cos 2\theta + 20\right) =$$

$$= \frac{1}{32}\cos 6\theta + \frac{3}{16}\cos 4\theta + \frac{15}{32}\cos 2\theta + \frac{5}{10}$$

So $\int (\cos\theta)^6 \, d\theta = \frac{1}{6\times 32}\sin 6\theta + \frac{3}{64}\sin 4\theta + \frac{15}{64}\sin 2\theta + \frac{5}{16}\theta + C$.

⚠ n^{th} roots of complex numbers.

$z^n = z_0$.

$z = re^{i\theta}$ → it comes from $z = r\cdot(\cos\theta + i\sin\theta)$.

$z_0 = r_0 \cdot e^{i\theta_0}$

$(r\cdot e^{i\theta})^n = r_0\cdot e^{i\theta_0} = r^n e^{ni\theta}$

Modulus

$r^n = r_0 \implies r = r_0^{1/n}$.

Arguments:

$n\theta = \theta_0 + 2k\pi \quad k \in \mathbb{Z}$

$\theta = \frac{\theta_0}{n} + \frac{2k\pi}{n} \quad k = 0, 1, \dots, n-1$

Ex:

$z^4 = -4 = 4e^{i\pi}$.

$r^4 e^{i4\theta} = 4e^{i\pi}$

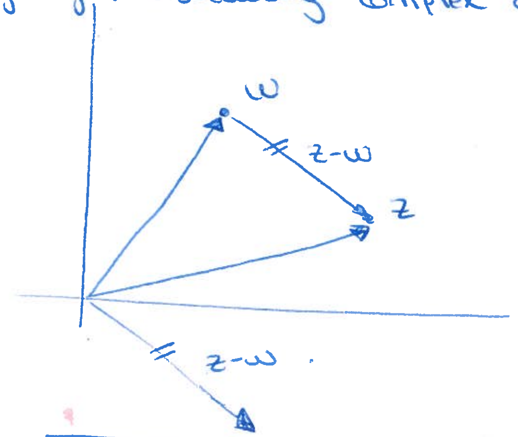
$r^4 = 4 ; \; r^2 = 2 \implies r = \sqrt{2}$.

$4\theta = \pi + 2k\pi = (2k+1)\pi$

$\theta = \frac{(2k+1)\pi}{4}$.

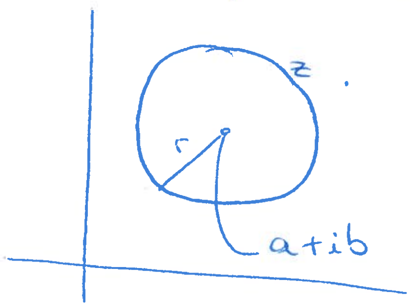
October 26th 2018

Useful for visualizing complex addition: $\begin{cases} z \text{ and } w \text{ are complex nbs.} \\ z = x+iy \end{cases}$



$|z| = \sqrt{x^2+y^2}$ = distance from $z = x+iy$ to 0. So, $|w-z|$ = distance from w to z.

The set of all z s.t $|z-a-ib| = r$ (a, b are real) is a circle with centre (a,b) and radius r. $w = a+ib$



$z = x+iy$.

$|x+iy-a-ib| = r$.

$|(x-a)+i(y-b)|^2 = r^2$;

$r^2 = (x-a)^2 + (y-b)^2$.

Similarly $|z-1| = |z+2-i|$ is a line $\rightarrow$ (claim), we are finding the equation of the line which is the bisector of them 2.

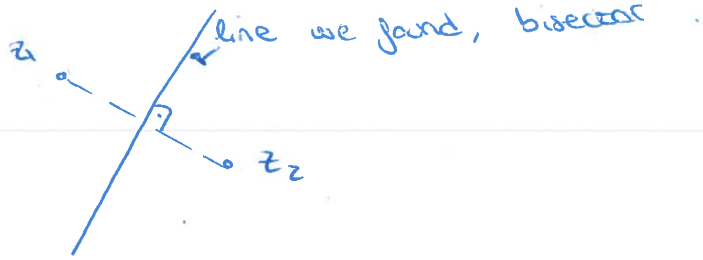
$(1,0)$ $\quad -(a+ib)$ $\quad (-2,1)$

$z = x+iy$.

$|(x-1)+(iy)|^2 = |x+2+i(y-1)|^2 \implies (x-1)^2+y^2 = (x+2)^2+(y-1)^2$;

$2x+1 = 4x+4-2y+1$; $2y = 6x+4$; $\boxed{y = 3x+2}$

$|z-z_1| = |z-z_2|$ perpendicular bisector of z_1 and z_2.



Ex: Find all z s.t:

$$\left|\frac{z-1}{z+3}\right| = a \quad \text{for some constant } a.$$

1st, we have that a is real and $a \geq 0$ or the set is empty.

2nd: if $a=0$ then the set is $\{1\}$.

if $a>0$ then $z = x+iy$

$$|(x-1)+iy| = a\,|(x+3)+iy|$$

$$|(x-1)+iy|^2 = a^2\,|(x+3)+iy|^2 \qquad z\bar{z}$$

$z\bar{z}$ = parte real · parte real + parte imag · parte imag; como $(x-1)$ es parte real y $\bar{z} = z$ $(x-1)^2$

$$(x-1)^2 + y^2 = a^2\left[(x+3)^2 + y^2\right] \quad \Delta\,(-y)^2 = y^2$$

$$(1-a^2)(x^2+y^2) - 2(1+3a^2)x + (1-9a^2) = 0 \quad \boxed{\text{This is a circle}}$$

If $a \neq 1 \rightarrow$ $x^2+y^2-2\left(\frac{1+3a^2}{1-a^2}\right)x + \frac{1-9a^2}{1-a^2} = 0$; COMPLETING THE SQUARE.

$\hookrightarrow x^2+ax+b = \left(x+\frac{a}{2}\right)^2 - \left(\frac{a}{2}\right)^2 + b$

$x^2-ax+b = \left(x-\frac{a}{2}\right)^2 - \left(\frac{a}{2}\right)^2 + b$

$$\left(x-\frac{1+3a^2}{1-a^2}\right)^2 + y^2 + \frac{1-9a^2}{1-a^2} - \left(\frac{1+3a^2}{1-a^2}\right)^2 = 0$$

$$\rightarrow \left(x-\frac{1+3a^2}{1-a^2}\right)^2 + y^2 = \frac{1}{(1-a^2)^2}\left[1+6a^2+9a^4-(1-10a^2+9a^4)\right] = \frac{16a^2}{(1-a^2)^2}$$

$\Rightarrow$ circle with centre $\left(\frac{1+3a^2}{1-a^2}, 0\right)$ and radius $= \frac{4a}{|1-a^2|}$

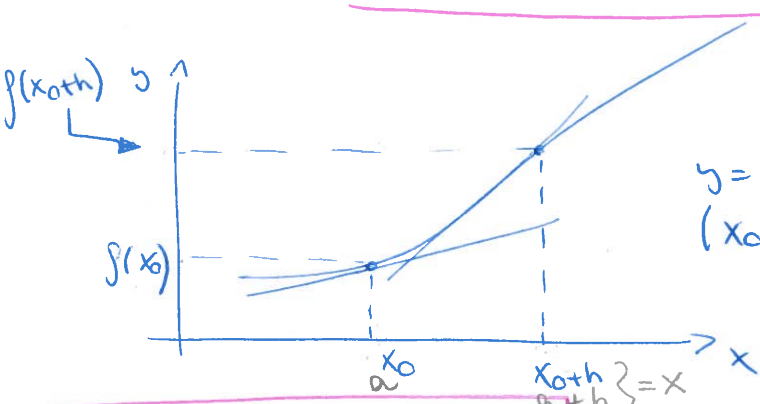
If $a=1 \rightarrow$ $x=-1$.

EQUATION OF THE CIRCLE: $(x-a)^2 + (y-b)^2 = c$

$c = (a,b)$: centre

$r = \sqrt{c}$: radius.

DIFFERENTIAL CALCULUS AND TAYLOR SERIES



$(y-y_a) = m\cdot(x-x_0)$,

$y - f(a) = f'(a)\cdot(x - f(a))$

$$\boxed{f'(x) = \lim_{h\to 0} \frac{f(x_0+h) - f(x_0)}{h}}$$

Equation of the tangent at $x = a$ is: $\left[f'(a) = \frac{y - f(a)}{x-a}\right]$

$$\boxed{y = f(a) + f'(a)\cdot(x-a)}$$

→ this is also called the linearization of f at a : $L(x)$

So we have the approximation

$f(x) \sim L(x)$.

we get the same approximation by writing $f'(a) \sim \frac{f(a+h) - f(a)}{h}$, $h \to x - a$.

<u>Ex</u>: Find an approximate value for $\sqrt[5]{30}$.

let $f(x) = x^{1/5}$ $\quad\left\{\begin{array}{l} x = 30 \\ h = (-2) \\ a = 32 \end{array}\right.$

$f(30) = f(32 + (-2))$

$f(a+h) \sim f(a) + h f'(a) \sim a^{1/5} + \frac{h}{5} a^{-4/5} = 32^{1/5} + \frac{(-2)}{5}\cdot\frac{1}{32^{4/5}} = 2 - \frac{1}{40} =$

$= 1.975 = \sqrt[5]{30} = 1.9744\ldots$ ✓

The linear approximation has the form

$$\boxed{f(x) = c_0 + c_1(x-a)}$$

$c_0 \Uparrow f(a)$, $\quad c_1 \Uparrow f'(a)$

we could look for more terms:

$$f(x) \sim c_0 + c_1(x-a) + c_2(x-a)^2 + \ldots + c_n(x-a)^n.$$

For x near a, suppose we can expand some function $f(x)$ as:

$$f(x) = c_0 + c_1(x-a) + \ldots = \underbrace{\sum_{n=0}^{\infty} c_n (x-a)^n}_{\text{power series}}$$

$$f(x) = c_0 + c_1(x-a) + c_2(x-a)^2 + c_3(x-a)^3 + \dots + c_n(x-a)^n + \dots$$

so $f(a) = c_0$.

$$f'(x) = c_1 + 2c_2(x-a) + 3c_3(x-a)^2 + \dots + nc_n(x-a)^{n-1} + \dots$$

$$f'(a) = c_1$$

$$f''(x) = 2\cdot 1 c_2 + 3\cdot 2 c_3(x-a) + \dots + n(n-1)(x-a)^{n-2} + \dots$$

$$f''(a) = 2\cdot 1\cdot c_2$$

$$f'''(a) = 3\cdot 2\cdot 1\cdot c_3$$

$$f^{(n)}(a) = n!\, c_n$$

$$\boxed{c_n = \frac{f^{(n)}(a)}{n!}} \quad (0! = 1) \quad : \boxed{\text{TAYLOR SERIES}}$$

- If f is infinitely differentiable at $x=a$ (i.e. $f^{(n)}(a)$ exists for all n) then the Taylor series of f at a is

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

- If $a=0$, the Taylor series is called the Maclaurin series.

Ex: $f(x) = e^x$, $f'(x) = e^x$ $\qquad a=0$, $\frac{f^{(n)}(0)=1}{n!}(x-0)^n$

$f^{(n)}(x) = e^x$, $f^{(n)}(0) = 1$.

The Taylor series of e^x is $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$ for all $x \in \mathbb{R}$ =

$$= 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \dots$$

Ex: $f(x) = \cos x$ (maclaurin series).

$$f'(x) = -\sin x$$

$$f''(x) = -\cos x \quad k=1$$

$$f'''(x) = \sin x$$

$$f^{IV}(x) = \cos x \quad k=2$$

even nbs

$$f^{(2k)}(x) = (-1)^k \cos x \quad \text{when } (n=2k) \text{ is even}.$$

$$f^{(2k+1)}(x) = (-1)^{k-1} \sin x \quad \text{when } (n=2k+1) \text{ is odd}.$$

If $x=0$ $f^{(2k)}(0) = (-1)^k$ $\xrightarrow{} \cos 0 = 1$, $f^{(2k+1)}(0) = 0$.

The Maclaurin series of $\cos x$ is : $\cos x = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n =$

$$= \left(\sum_{n \text{ even}} \frac{f^{(n)}(0)}{n!} x^n \right) + \left(\sum_{n \text{ odd}} \frac{f^{(n)}(0)}{n!} x^n \right)$$

$f^{(n)}(0) = 0$ if n is odd.

$$\cos x = \sum_{k=0}^{\infty} \frac{f^{(2k)}(0)}{(2k)!} x^{2k} = \boxed{(n=2k)}$$

$$= \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} x^{2k} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots$$

Ex : $f(x) = \sin x$

$f'(x) = \cos x$	$f(0) = 0$
$f''(x) = -\sin x$	$f'(0) = 1$
$f'''(x) = -\cos x$	$f''(0) = 0$
$f^{IV}(x) = \sin x$	$f'''(0) = -1$

$f^{(2k)}(0) = 0$ $(\sin 0 = 0)$ for $n = \text{even} = 2k$.

$f^{(2k+1)}(x) = (-1)^k$ for $n = \text{odd} = 2k+1$.

$$\sin x = \sum_{n \text{ odd}} \frac{f^{(n)}(0)}{n!} x^n = \sum_{k=0}^{\infty} \frac{f^{(2k+1)}(0)}{(2k+1)!} \cdot x^{(2k+1)} =$$

$$= \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} \cdot x^{(2k+1)} = x - \frac{x^3}{3!} + \frac{x^5}{5!} + \dots$$

- BINOMIAL EXPANSION:

$f(x) = (1+x)^\alpha$ α real

$f'(x) = \alpha \cdot (1+x)^{\alpha-1}$

$f''(x) = \alpha \cdot (\alpha-1) \cdot (1+x)^{\alpha-2}$.

$f'''(x) = \alpha \cdot (\alpha-1) \cdot (\alpha-2) \cdot (1+x)^{\alpha-3}$.

$$f^{n}(x) = \alpha \cdot (\alpha-1) \cdot (\alpha-2) \cdots (\alpha - \overset{-n+1}{n+1}) \cdot (1+x)^{\alpha-n}$$

siempre es un nº menos que aquí

$$(1+x)^{\alpha} = \sum_{r=0}^{\infty} \frac{f^{(r)}(0)}{r!} x^{r} = \boxed{\sum_{r=0}^{\infty} \binom{\alpha}{r} x^{r}}, \text{ where } \binom{\alpha}{r} = \frac{\alpha \cdot (\alpha-1)\cdot(\alpha-2)\cdots(\alpha-r+1)}{r!}.$$

$$\frac{f^{(r)}(0)}{r!} \to \binom{\alpha}{r}$$

If $\alpha = n$, a positive integer: $\binom{n}{r} = \left[\frac{n\cdot(n-1)\cdots(n-r+1)}{r!}\right] \frac{(n-r)\cdots 3\cdot 2\cdot 1}{(n-r)\cdots 3\cdot 2\cdot 1} =$

$$= \frac{n!}{r!\,(n-r)!}$$

For $r > n$: $\binom{n}{r} = 0$.

- **IMPLICIT DIFFERENTIATION**

Ex: Find the equation of the tangent to $xy^{7} + y^{4} + x^{3}y^{2} + 3y^{2} - 5x - 4 = 0$ at the point $(x,y) = (2,1)$.

$\boxed{1^{st}}$ Differentiate the eqn but remember, y depends on x

$$\left((1 \cdot y^{7}) + 7xy^{6}\frac{dy}{dx}\right) + 4y^{3}\frac{dy}{dx} + \left(3x^{2}y^{2} + 2x^{3}y\frac{dy}{dx} + 6y\frac{dy}{dx} - 5\right) = 0$$

$$* \quad \frac{dy^{7}}{dx} = \frac{dy^{7}}{dy} \cdot \frac{dy}{dx}.$$

at $x=2, y=1$.

$$1 + 14y' + 4y' + 12 + 16y' + 6y' - 5 = 0$$

$$\Rightarrow \left.\frac{dy}{dx}\right|_{(x=2,y=1)} = -1/5.$$

29th October 2018

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^{n} \longrightarrow |x-a| > r.$$

$\boxed{0! = 1}$

$$e^{x} = \sum_{n=0}^{\infty} \frac{x^{n}}{n!}$$

$\sin x$

$\cos x$

} Already done (26th October).

BINOMIAL EXPANSION →

$$(1+x)^{\alpha} = \sum_{r=0}^{\infty} \binom{\alpha}{r} x^r$$

$$\boxed{\binom{\alpha}{0} = 1} \qquad \binom{\alpha}{r} = \frac{\alpha \cdot (\alpha-1) \cdots (\alpha - r + 1)}{r!}$$

If $\alpha = n$ positive integer $\Rightarrow$ $\boxed{\binom{n}{r} = 0 \quad \forall\, r > n}$.

$$\boxed{r \leq n \quad \binom{n}{r} = \frac{n!}{r!\,(n-r)!}}$$

Ex: According to Einstein, the energy E of a particle of mass m is $E = mc^2$ and the relativistic mass is:

$$m = \frac{m_0}{\sqrt{1 - \frac{v^2}{c^2}}} \qquad m_0 = \text{rest mass}.$$

$$E = \frac{m_0 \cdot c^2}{\sqrt{1 - \frac{v_0^2}{c^2}}}$$

Consider $\dfrac{1}{\sqrt{1+x}} = (1+x)^{-1/2} = 1 + \dfrac{(-1/2)}{1!} x + \dfrac{(-1/2)(-3/2)}{2!} x^2 + \dots =$

$\simeq 1 - 1/2\, x + \dots$

$$E = m_0 \cdot c^2 \cdot \left(1 - \frac{v^2}{c^2}\right)^{-1/2} = m_0 \cdot c^2 \cdot \left(1 + 1/2 \frac{v^2}{c^2} + \dots\right) = m_0 \cdot c^2 + \frac{1}{2} m v^2 + \dots$$

Ex: Find the Maclaurin series (Taylor Series about $a=0$).

for $\dfrac{\cos(x^3)}{(1+x^6)^{1/3}}$ up to and including x^{12} terms.

$$\cos x = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{2k!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \dots$$ → we know this from last class.

$$\Rightarrow (\cos(x^3)) = 1 - \frac{x^6}{2} + \frac{x^{12}}{24} + \dots$$

$$(1+x)^{-1/3} = 1 + \frac{(-1/3)}{1!} x + \frac{(-1/3)(-4/3)}{2!} x^2 + \dots$$

$$\Rightarrow (1+x^6) = 1 - \frac{1}{3}x^6 + \frac{2}{9}x^{12} + \dots$$

$$\Rightarrow \frac{\cos(x^3)}{(1+x^6)^{1/3}} = \left(1 - \frac{x^6}{2} + \frac{x^{12}}{24} + \dots\right)\left(1 - \frac{1}{3}x^6 + \frac{2}{9}x^{12} + \dots\right) =$$

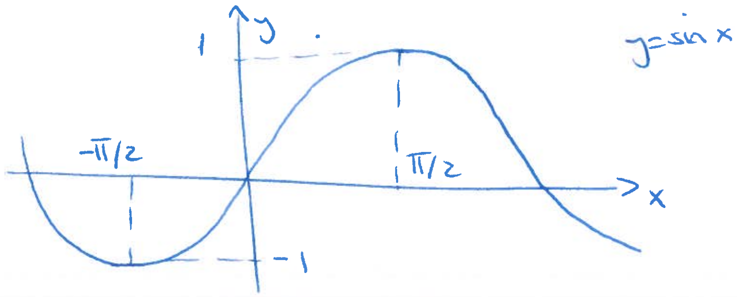
$$= 1 + \left(-\frac{1}{3} - \frac{1}{2}\right)x^6 + \left(\frac{2}{9} + \left(-\frac{1}{2}\right)\left(-\frac{1}{3}\right) + \frac{1}{24}\right)x^{12} + \dots$$

$$= 1 - \frac{5}{6}x^6 + \frac{31}{72}x^{12} + \dots$$

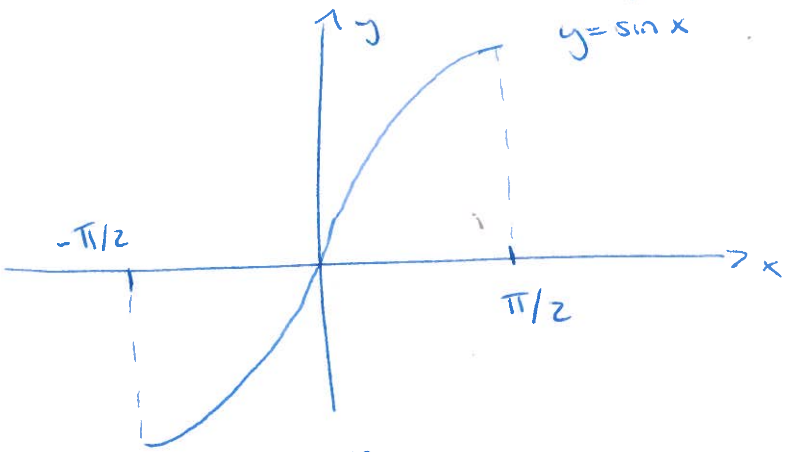
$$\left[\begin{array}{l} \frac{1}{1+x} = (-x + x^2 - x^3 + \dots) \\ \frac{1}{\cos x} = \frac{1}{1 + \left(-\frac{x^2}{2} + \frac{x^2}{24} + \dots\right)} = 1 - \left(-\frac{x^2}{2} + \frac{x^2}{24} + \dots\right) + \left(-\frac{x^2}{2} + \dots\right)^2 + \dots \end{array}\right]$$

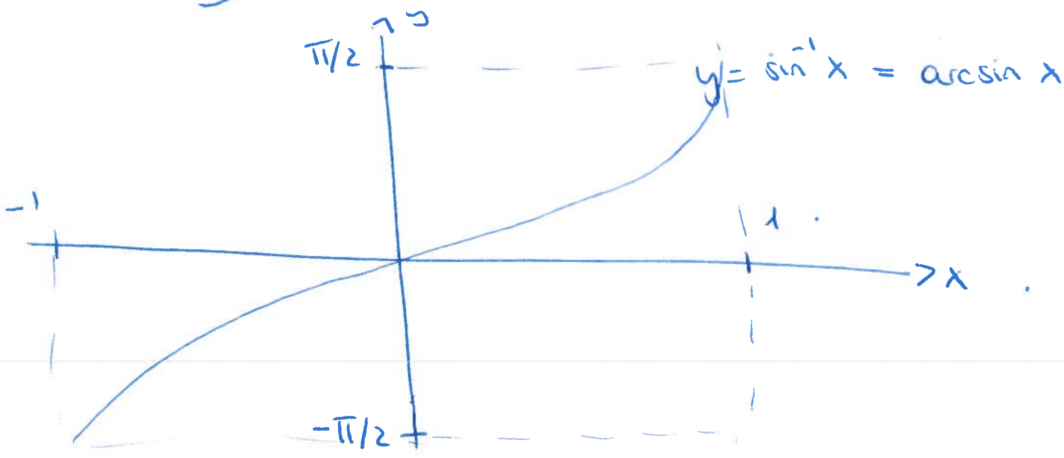
INVERSE FUNCTIONS AND THEIR DERIVATIVES

$\sin : \mathbb{R} \longrightarrow [-1, 1]$.



To get an inverse, we restrict the domain of sin to $[-\pi/2, \pi/2]$.





~~(y=)~~

Let $y = \sin^{-1} x$

$\Rightarrow \quad x = \sin y$.

- I differentiate both sides by $\frac{dx}{dy}$.

$$\frac{dx}{dy} = \cos y .$$

- But what I want is $\frac{dy}{dx}$:

$$\frac{dy}{dx} = \frac{1}{\cos y} = \frac{1}{\pm\sqrt{1-(\sin y)^2}}$$

We know it can only be the $\oplus$, because looking at the graph we see it is always increasing! ($\sin^{-1} x$ graph) .

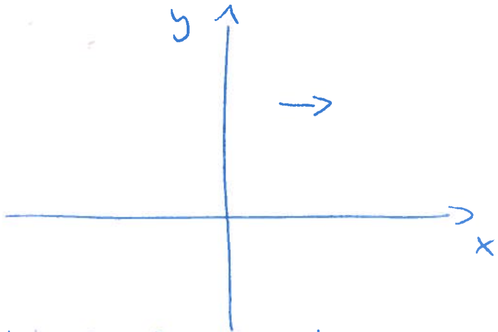
$\cos^2 y = 1 - \sin^2 y$.

$$\frac{dy}{dx} = \frac{1}{\sqrt{1-\sin^2 y}} = \frac{1}{\sqrt{1-x^2}}$$

$$\Rightarrow \boxed{\frac{d\sin^{-1}x}{dx} = \frac{1}{\sqrt{1-x^2}} .}$$

FUNCTIONS OF SEVERAL VARIABLES

$$f(x,y) : \mathcal{U} \longrightarrow \mathbb{R}$$
$$\mathcal{U} \subset \mathbb{R}^2$$



The rate of change of f in the x direction is given by the partial derivative of f with respect to x .

$$\boxed{\frac{\partial f}{\partial x} = \lim_{h \to 0} \frac{f(x+h, y) - f(x,y)}{h}}$$

November 2nd 2018

Wednesday November 21ST

FUNCTIONS OF SEVERAL VARIABLES:

$\underbrace{\text{SEVERAL}}_{\text{anything } \geq 1}$

Ex

$f(x,y) = \sin(x^2y) + xy$:

$\boxed{\frac{\partial f}{\partial x}} = \lim_{h \to 0} \frac{f(x+h, y) - f(x,y)}{h}$ → rate of change of f in the x-direction

→ partial derivative in terms of x, y is constant.

" ∂ " = back-to-front "6".

$\boxed{\frac{\partial f}{\partial y}} = \lim_{h \to 0} \frac{f(x, y+h) - f(x,y)}{h}$ → rate of change of f in the y-direction

→ partial derivative in terms of y, x is constant

In our example: $f(x,y) = \sin(x^2y) + xy$.

→ treat it as a number in $\frac{\partial f}{\partial x}$

$$\frac{\partial f}{\partial x} = 2xy\cos(x^2y) + y.$$

$$\frac{\partial f}{\partial y} = x^2 \cdot \cos(x^2y) + x.$$

We also use the notation:

$$f_x(x,y) = \frac{\partial f}{\partial x} \qquad f_y(x,y) = \frac{\partial f}{\partial y}.$$

Higher derivatives:

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial\left(\frac{\partial f}{\partial x}\right)}{\partial x} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = f_x(f_x(x,y)) = (f_x)_x = f_{xx}.$$

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial\left(\frac{\partial f}{\partial y}\right)}{\partial x} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = f_x\left(f_y(x,y)\right) = (f_y)_x = f_{yx}.$$

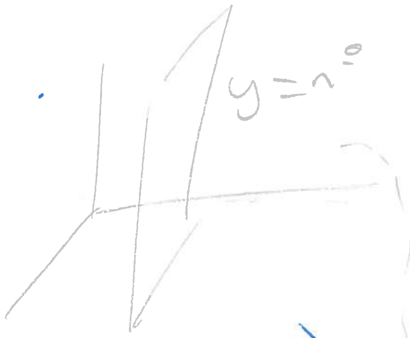
Continuing our example:

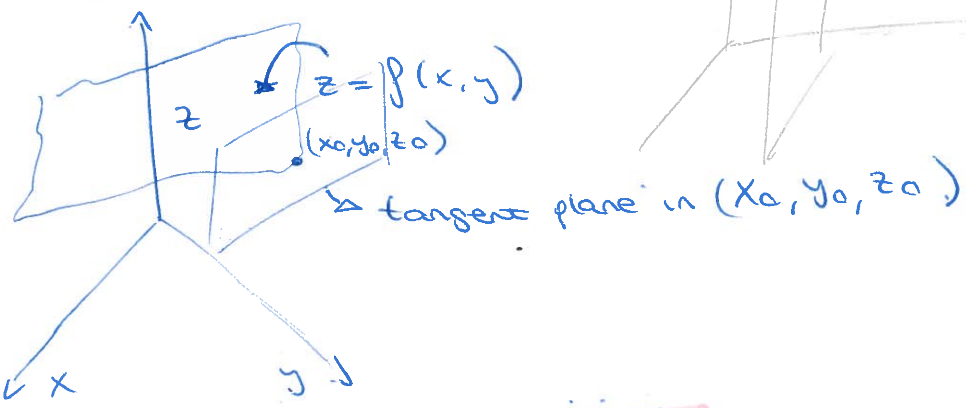
$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = 2y\cos(x^2y) - 4x^2y\sin(x^2y).$$

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = 2x\cos(x^2y) - 2x^3y\sin(x^2y) + 1.$$

$$\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = 2x\cos(x^2y) - 2x^3y\sin(x^2y) + 1 = \frac{\partial^2 f}{\partial x \partial y}.$$

$$\frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = -x^4 \cdot \sin(x^2y).$$



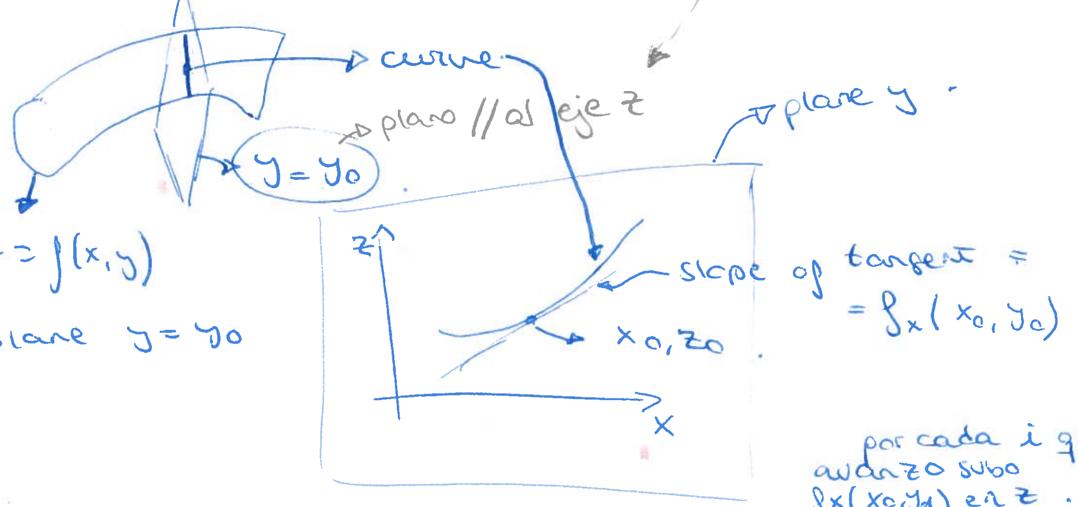


Tangent planes and linear approximations

We want the tangent plane to $z=f(x,y)$ at some point (x_0, y_0, z_0).
where $z_0 = f(x_0, y_0)$.

We will find tangent vectors to 2 curves in the surface.

We intersect the surface with the plane $y=y_0$ to get the curve $z=f(x,y_0)$.



So in the plane $y=y_0$

por cada i que avanzo subo $f_x(x_0,y_0)$ en z.

So a tangent vector to the curve at (x_0, y_0, z_0) is $\underline{i} + f_x(x_0, y_0)\underline{k}$

Playing the same game with the plane $x=x_0$ gives $\underline{v_2} = \underline{j} + f_y(x_0, y_0)\underline{k}$.

The tangent plane is the plane through $(x_0, y_0, f(x_0, y_0))$ with normal vector $\underline{n} = \underline{v}_1 \times \underline{v}_2 = -f_x(x_0,y_0)\,\underline{i} - f_y(x_0,y_0)\,\underline{j} + \underline{k}$.

So if $P = (x,y,z)$ is a point on the plane, we have:

$$0 = \underline{n} \cdot \overline{P_0P} = \left(-f_x \underline{i} - f_y \underline{j} + \underline{k}\right) \cdot \left((x-x_0)\underline{i} + (y-y_0)\underline{j} + (z - f(x_0,y_0))\underline{k}\right)$$

$$\Rightarrow z = \underbrace{f(x_0,y_0)}_{Cz} + \underbrace{f_x(x_0,y_0)(x-x_0)}_{Ax} + \underbrace{f_y(x_0,y_0)(y-y_0)}_{By}$$

↳ sobrante términos indep $= D$.

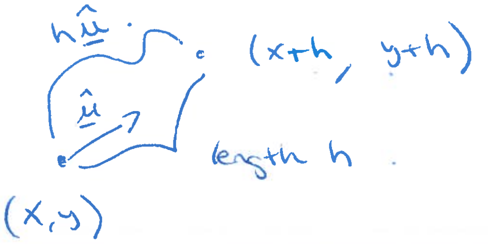
For (x,y) near (x_0,y_0) we have the approximation:

$$\boxed{f(x,y) \approx f(x_0,y_0) + f_x(x_0,y_0)\cdot(x-x_0) + f_y(x_0,y_0)(y-y_0)}.$$

Directional derivatives

Let $\hat{u} = u_1 \bar{i} + u_2 \bar{j}$.

Find the rate of change of $f(x,y)$ at (x,y) in the direction $\hat{u}$.



The directional derivative of $f(x,y)$ at (x_0,y_0) in the direction $\hat{u}$ is

$$\boxed{D_{\hat{u}} f(x_0,y_0) = \lim_{h\to 0} \frac{f(x_0+hu_1, y_0+hu_2) - f(x_0,y_0)}{h}}.$$

Linear approximation:

$$f(x_0+hu_1, y_0+hu_2) \approx f(x_0,y_0) + f_x(x_0,y_0)\,hu_1 + f_y(x_0,y_0)\,hu_2.$$

so $\boxed{D_{\hat{u}} f(x_0,y_0) = f_x(x_0,y_0)u_1 + f_y(x_0,y_0)u_2}$.

$$\frac{f(x_0+hu_1, y_0+hu_2) - f(x_0,y_0)}{h}$$

$$= \boxed{\hat{u} \cdot \nabla f \Big|_{(x_0,y_0)}} \to$$

(vector)

$$\nabla f(x_0,y_0) = \nabla f \Big|_{(x_0,y_0)} = \frac{\partial f}{\partial x}\cdot(x_0,y_0)\,\underline{i} + \frac{\partial f}{\partial y}(x_0,y_0)\,\underline{j} =$$

$=$ "gradient of f".

Let θ be the angle between $\nabla f(x_0,y_0)$ and $\hat{u}$. $\nabla f(x_0,y_0) = \nabla f|_{(x_0,y_0)}$.

$|\hat{u}| = 1$ unit vector.

$$\boxed{D_{\hat{u}} f(x_0,y_0)} = \hat{u}\cdot \nabla f|_{(x_0,y_0)} = |\hat{u}|\cdot|\nabla f|\cdot\cos\theta = \boxed{|\nabla f|\cdot\cos\theta}.$$

The $D_{\hat{u}} f(x_0,y_0)$ is the biggest when it $= |\nabla f|$ because $|\cos\theta| \le 1$.

Then $D_{\hat{u}} f$ is the largest when $\theta = 0 \to D_{\hat{u}} f = |\nabla f|$.

$D_{\hat{u}} f$ is the most negative when $\theta = \pi \to D_{\hat{u}} f = -|\nabla f|$.

$D_{\hat{u}} f$ is 0 when $\theta = \pi/2$

So at (x_0,y_0) f increases most rapidly in the direction of ∇f and its rate of change (i.e. direction derivative) in that direction is $|\nabla f|$.

f decreases most rapidly in the direction $-\nabla f$, with rate of change $-|\nabla f|$

f is "most constant" in direction $\perp$ to ∇f.

Ex: Find the rate of change of $f(x,y) = x^2 + 2y^2$ in the direction of the vector $u = i + j$ at $P(1,1)$.

i) Find the direction in which f decreases most rapidly.

$$\nabla f = f_x\, i + f_y\, j = 2x i + 4y j = 2(x i + 2y j).$$

$$\nabla f(1,1) = 2(i + 2j).$$

Unit vector in direction of u:

$$\hat{u} = \frac{u}{|u|} = \frac{1}{\sqrt{2}}\cdot(i+j).$$

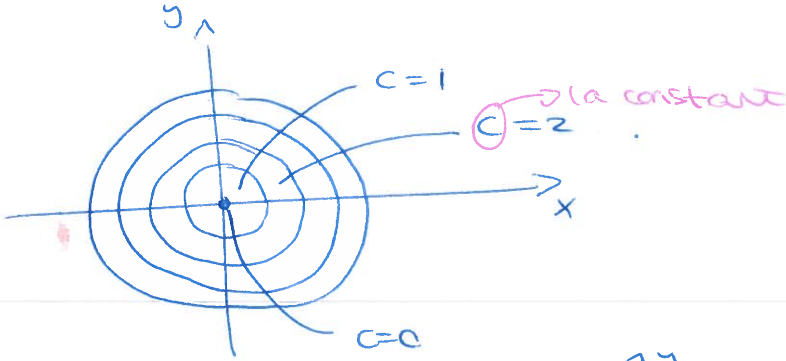
Rate of change:

$$D_{\hat{u}} f = \hat{u}\cdot\nabla f = \frac{2}{\sqrt{2}}\cdot(i+j)\cdot(i+2j) = \boxed{3\sqrt{2}}.$$

It is increasing then.

i) It decreases most rapidly in the direction of $-\nabla f = -2\cdot(i+2j)$

So i.e. in the direction: $-\frac{1}{\sqrt{5}}\cdot(i+2j) = -\frac{\sqrt{5}}{5}\cdot(i+2j) = -\widehat{\nabla f}$ (unit vector)

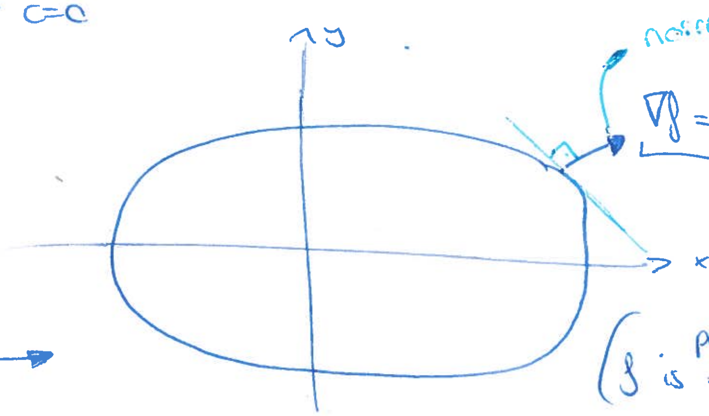


Level sets of $f(x,y)$:

$$\boxed{f(x,y) = \text{constant}}$$

$x^2 + 2y^2 = \text{constant}$

$\underbrace{x^2 + 2y^2 = 3}_{\text{ellipse}}$ →



normal to the tangent

$\nabla f = \underbrace{2(\underline{i} + 2\underline{j})}$

normal vector to the ellipse

(f is mostly constant)

point where the f is mostly constant

✗ If someone gives me the eq. of the elipse and asks me to find the normal vector of the elipse I find ∇f.

In 3 dimensions

$$f(x,y,z) = f(x_0,y_0,z_0) + f_x(x_0,y_0,z_0)\cdot(x-x_0) + f_y(x_0,y_0,z_0)(y-y_0) + f_z(x_0,y_0,z_0)(z-z_0)$$

$$\boxed{D_{\hat{\underline{u}}} f = \hat{\underline{u}} \cdot \boxed{\nabla f}} \rightarrow \left[\nabla f = \frac{\partial f}{\partial x}\,\underline{i} + \frac{\partial f}{\partial y}\,\underline{j} + \frac{\partial f}{\partial z}\,\underline{k}\right]$$

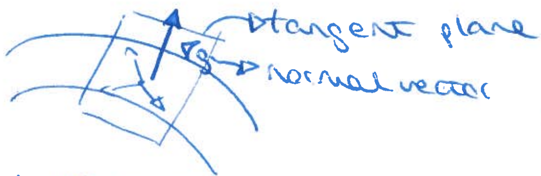
∇f is direction of the greatest increase. f is most constant in directions $\perp$ to ∇f.

Ex: Find a vector normal to the surface $\underbrace{x^2 + 2y^3 + 3z^2 = 6}$ at $(1,1,1)$.

$g(x,y,z) = x^2 + 2y^2 + 3z^2$ | Level surface $g(x,y,z) = 6$

$$\nabla g = 2x\underline{i} + 4y\underline{j} + 6z\underline{k} = 2(x\underline{i} + 2y\underline{j} + 3z\underline{k})$$

$$\nabla g = 2\cdot(\underline{i} + 2\underline{j} + 3\underline{k})$$



So $\underline{i} + 2\underline{j} + 3\underline{k}$ is a normal vector

November 12th 2018

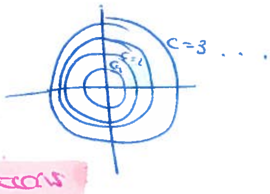
$f(x,y)$.

Level curve (level set) → todos los ptos de uno de los ⊙ dependiendo de la c

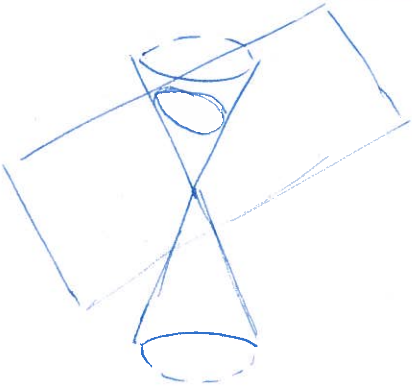
Sets of points of the form $f(x,y) = c$. for some constant c.

Ex: $f(x,y) = x^2 + y^2$.

$\underbrace{x^2 + y^2 = c}_{\text{bunch of concentric circles}}$.

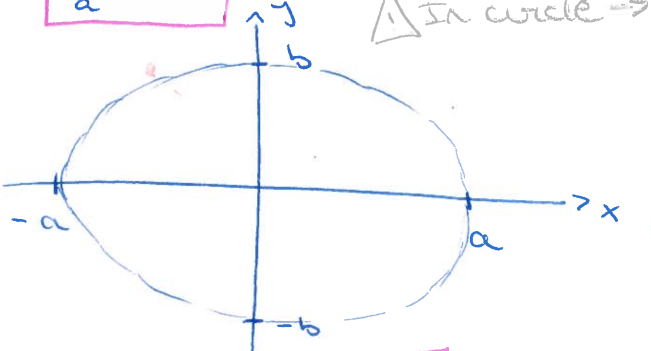


✳ Special section: conic sections



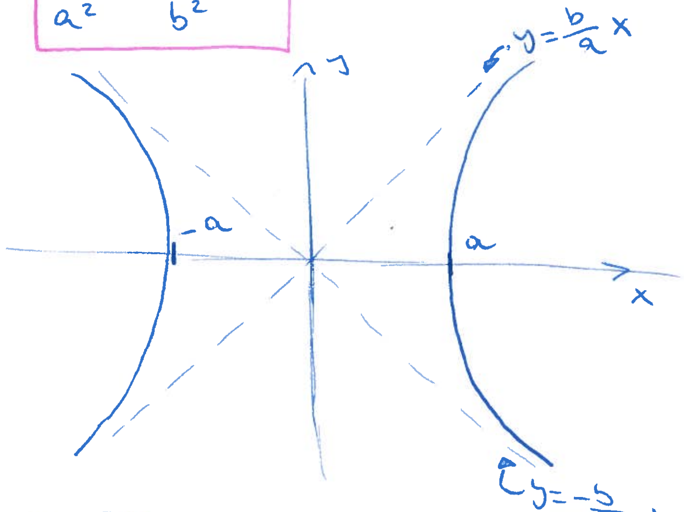
① $\boxed{\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1}$. ELIPSE

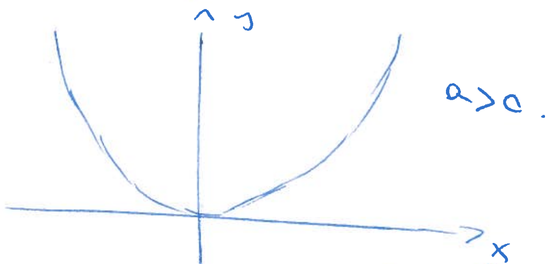
⚠ In circle → a=1 and b=1

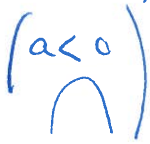


② $\boxed{\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1}$ HYPERBOLA .

③ $\boxed{y = ax^2}$ PARABOLA .







Plots in 3D .

Ex: We intersect unknown surface with (simple) planes:

$z = x^2 + y^2$.

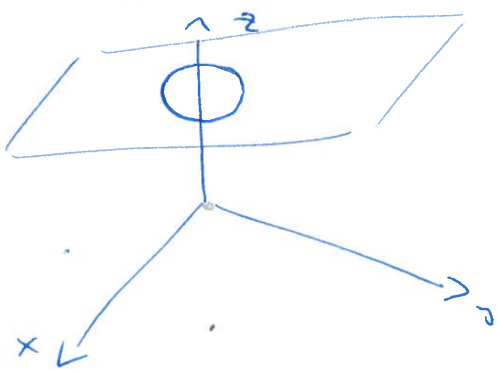
• Intersect the surface with the plane $z = k$.

$k = x^2 + y^2$.

$k < 0$ no intersection

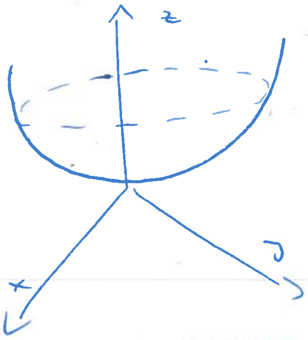
$k = 0$ origin $(0,0)$.

$k > 0$ circle (radius $\sqrt{k}$)



- Intersect surface with $x=0$.

$z = y^2$ (parabola).



paraboloid

Ex: We intersect a $\boxed{x^2 + \frac{y^2}{4} + z^2 = 1}$.

- Intersect with the plane $y=k$

$$x^2 + z^2 = 1 - \frac{k^2}{4}$$

$|k| > 2$, empty.

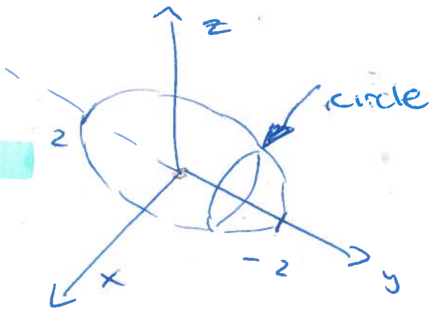
$|k| = 2$ $\{0,0\}$ $\longrightarrow$ cut the solid in the origin.

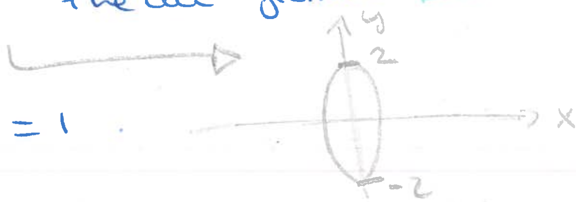
$\underbrace{\{0,0\}}_{x=0,\ z=0}$

$|k| < 1$ circle.

- Intersect surface with plane: $z=0$.

$$x^2 + \frac{y^2}{4} = 1$$ $\underbrace{\text{ellipse}}_{\text{the cut}}$ from an ellipsoid





Ex: $x^2 - \frac{y^2}{4} - \frac{z^2}{9} = 1$

- Intersect with $x=k$.

$$\frac{y^2}{4} + \frac{z^2}{9} = k^2 - 1.$$

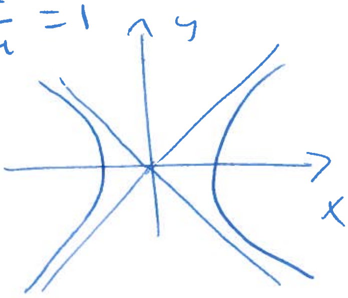
$|k| < 1$ $\emptyset$

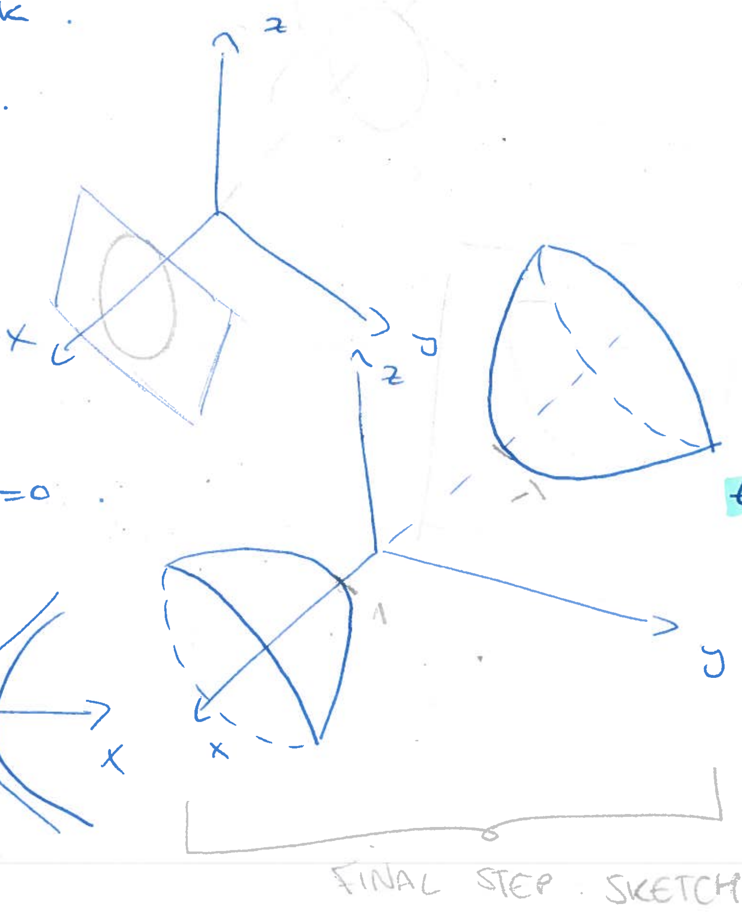
$k = \pm 1$ $\{(0,0)\}$.

$|k| > 1$ ellipse

- Intersect with plane $z=0$.

$$x^2 - \frac{y^2}{4} = 1$$





two-sheeted hyperboloid.

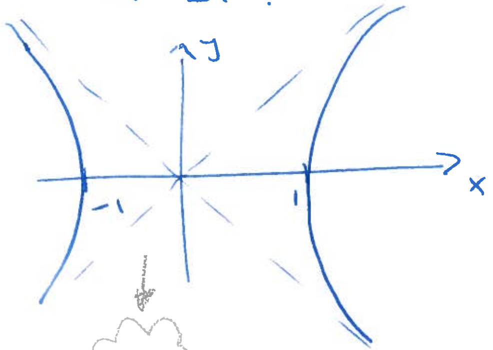
Ex : $x^2+y^2-z^2=1$.

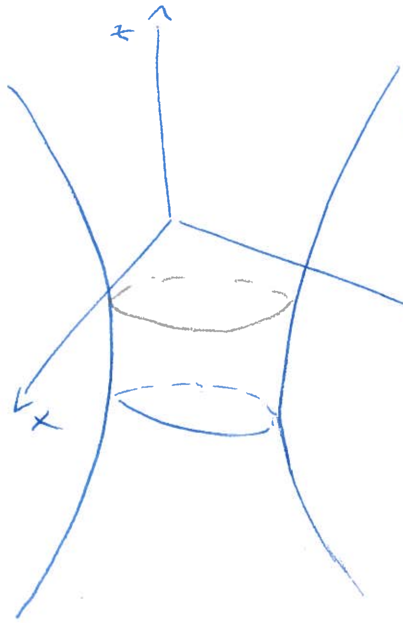
- Intersect with $z=k$

$$x^2+y^2=1+k^2 \quad : \text{circle}.$$

- Intersect with $y=0$

$$x^2-z^2=1.$$

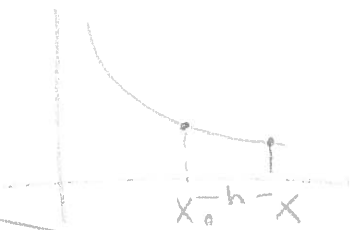




1-sheeted hyperboloid

$$f(x,y)\simeq f(x_0,y_0)+f_x(x_0,y_0)(x-x_0)+f_y(x_0,y_0)(y-y_0).$$

Chain rule



Ex : $f(x,y)$, $\boxed{x=x(t)}$, $\boxed{y=y(t)}$.

$$g(t)=f(x(t),y(t)).$$

Find $f'(t)$

$$g(t+h)=f(x(t+h),y(t+h))\simeq f(x(t)+h\cdot x'(t),\,y(t)+hy'(t))\simeq$$

$$\simeq f(\underbrace{x(t),y(t)}_{})+f_x(x(t),y(t))\,h\cdot x'(t)+f_y(x(t),y(t))\,hy'(t)$$

no pongo +h, porque es f(x0) que no incluye h

→ Tienes que hacer el todo y tb las derivadas de las funciones peques

so

$$g'(t)=\lim_{h\to 0}\frac{g(t+h)-g(t)}{h}=f_x(x(t),y(t))\,x'(t)+f_y(x(t),y(t),y'(t)).$$

$$g(t)=f(x,y),\quad x=x(t);\quad y=y(t).$$

$$\boxed{\frac{df}{dt}(x,y)=\frac{\partial f}{\partial x}\frac{dx}{dt}+\frac{\partial f}{\partial y}\frac{dy}{dt}}$$

→ when I have a function of functions and these have a variable.

$$f(x,y);\quad x=x(u,v),\; y(u,v).$$

$$\frac{\partial f}{\partial u}=\frac{\partial f}{\partial x}\cdot\frac{\partial x}{\partial u}+\frac{\partial f}{\partial y}\frac{\partial y}{\partial u}.$$

November 16th 2018

INTEGRAL CALCULUS

Aim: review some techniques you already know, and study some advance ones

Preparation: hyperbolic functions.

Recall: $\boxed{\cos x = \frac{1}{2}(e^{ix} + e^{-ix})}$

$\boxed{\sin x = \frac{1}{2i}(e^{ix} - e^{-ix})}$

Now define: $\boxed{\cosh x = \frac{1}{2}(e^{x} + e^{-x})}$ $\forall x \in \mathbb{R}$.

HYPERBOLIC COSINE

$\boxed{\sinh x = \frac{1}{2}(e^{x} - e^{-x})}$ $\forall x \in \mathbb{R}$

HYPERBOLIC SINE

Also, $\boxed{\tanh x := \frac{\sinh x}{\cosh x}} = \frac{\frac{1}{2}(e^{x} + e^{-x})}{\frac{1}{2}(e^{x} - e^{-x})} = \boxed{\frac{e^{x} + e^{-x}}{e^{x} - e^{-x}}}$

$\boxed{\operatorname{sech} = \frac{1}{\cosh x}}$.

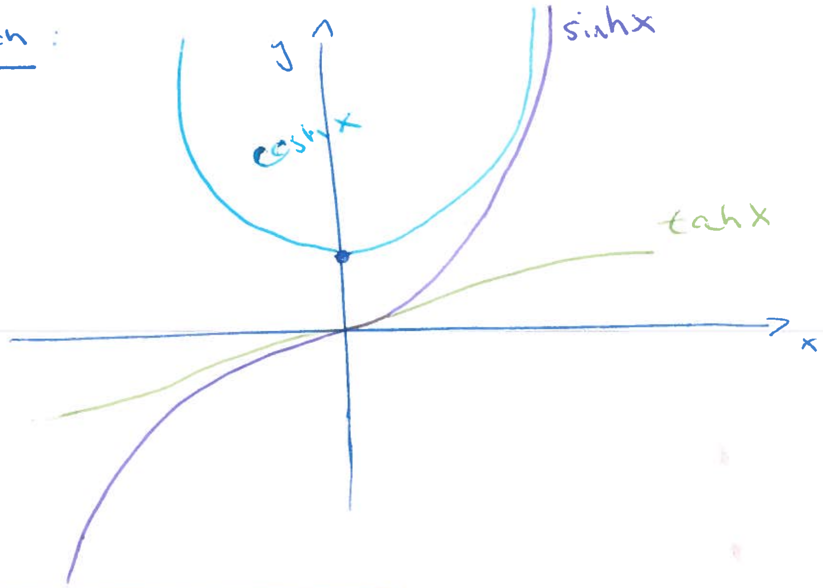
Properties:

- cos and cosh are $\boxed{\text{even}}$ functions ($\cos(-x) = \cos x$, $\cosh(-x) = \cosh(x)$).
- sin and sinh are $\boxed{\text{odd}}$ functions ($\sin(-x) = -\sin x$, $\sinh(-x) = -\sinh(x)$)
- $e^{ix} = \cos x + i\sin x$, $e^{x} = \cosh x + \sinh x$.
- $\sinh x = -i \sin(ix)$, $\cosh x = \cos ix$.
- $(\cos x)^2 + (\sin x)^2 = 1$, $(\cosh x)^2 - (\sinh x)^2 = 1$.
- $\frac{d}{dx}(\sin x) = \cos x$, $\frac{d}{dx}(\cos x) = -\sin x$ No $\ominus$ sign!

  $\frac{d}{dx}(\sinh x) = \cosh x$, $\frac{d}{dx}(\cosh x) = \sinh x$.
- $\frac{d}{dx}(\tanh x) = (\operatorname{sech} x)^2 = \frac{1}{(\cosh x)^2}$, $\frac{d}{dx}(\tan x) = (\sec x)^2$ ✱ $\int \frac{1}{x^2+a^2} = \frac{1}{a}\operatorname{arctg}\left(\frac{x}{a}\right) + C$
- $\frac{d}{dx}(\underbrace{\operatorname{arcsinh} x}_{\text{inverse of sinh}}) = \frac{1}{\sqrt{1+x^2}}$, $\frac{d}{dx}(\arcsin x) = \frac{1}{\sqrt{1-x^2}}$ signo distinto

$\frac{d \sec x}{dx} = \sec x \cdot \tan x$ $\rightarrow (\tan x)^2 = (\sec x)^2 - 1$

Sketch:



Integration by substitution:

For integrals of the form: $I = \int F(f(x)) \cdot f'(x) \cdot dx$, substituting $u = f(x)$ gives $I = \int F(u) \frac{du}{dx} \cdot dx = \int F(u) \cdot du$, which may be easier to evaluate.

It's not always easy to see what F and f are.

Integrand involves	Suggested substitution
$\sqrt{a^2 - x^2}$	$x = a \cdot \sin u$.
$\sqrt{a^2 + x^2}$	$x = a \cdot \sinh u$.
$\sqrt{x^2 - a^2}$	$x = a \cdot \cosh u$.
$a^2 + x^2$	$x = a \tan u$.
$a^2 - x^2$	$x = a \cdot \tanh u$.

Example: Find $I = \int \frac{dx}{(1+x^2)^{3/2}} = (\sqrt{1+x^2})^3$

Solution: Integrand involves $\sqrt{1+x^2}$ so try $x = \sinh u$.
Then $dx = \cosh u \cdot du$ and $1 + x^2 = 1 + (\sinh u)^2 = (\cosh u)^2$.

So $I = \int \frac{\cosh u}{(\cosh u)^3} du = \int \frac{du}{(\cosh u)^2} = \int (\operatorname{sech} u)^2 du = \tanh u + C$

But $\tanh u = \frac{\sinh u}{\cosh u} = \frac{x}{\sqrt{1+x^2}}$, so $I = \frac{x}{\sqrt{1+x^2}} + C$.

Integration by parts : ILATE

Integrating the product rule $(uv)' = u'v + u \cdot v'$ gives

$$\boxed{\int u \frac{dv}{dx} \cdot dx = uv - \int v \frac{du}{dx} \cdot dx}$$

Example: $\displaystyle\int \arcsin x \cdot dx = \int \underbrace{\arcsin x}_{u} \cdot \underbrace{1}_{\frac{dv}{dx}} \cdot dx = x \cdot \arcsin x - \int \frac{x \cdot dx}{\sqrt{1-x^2}} =$

so $v = x$

eg: by $x = \sin u$.

$dx = \cos u \, du$

$\displaystyle\int \frac{\sin u \cos u}{\sqrt{1-\sin^2 u}} du = \int \frac{\sin u \cos u}{\cos u} du$

$= -\cos u$

$\cos u = \sqrt{1-\sin^2 u} = \sqrt{1-x^2}$

$= x \cdot \arcsin x + \sqrt{1-x^2} + C$.

Tabular integration

Integrating by parts $n+1$ times gives

$$\int u \cdot \frac{dv}{dx} \cdot dx = u \cdot v - u' v_1 + u'' v_2 - u''' v_3 + \dots + (-1)^n u^{(n)} v_n + (-1)^{n+1} \int u^{(n+1)} v_n \, dx$$

where $u^{(k)}$ is the k^{th} derivative of u and v_k is the k^{th} integral of v.

(i.e. $v_k^{(k)} = v$).

If u is a polynomial of degree n, then $u^{(n+1)} \equiv 0$.

so the remainder integral is a constant, giving

$$\int u \frac{dv}{dx} \cdot dx = u \cdot v + \sum_{k=1}^{n} (-1)^k \cdot u^k \cdot v_k + C.$$

[Useful if we can easily evaluate v_k — true for exponentials, sine, cosine, etc]

Example: $\displaystyle I = \int \underbrace{(x^4 + 8x + 1)}_{u} \cdot \underbrace{e^{2x}}_{\frac{dv}{dx}} \cdot dx$.

Solution:

$u = x^4 + 8x + 1$. $\qquad v = \frac{1}{2} e^{2x}$.

$u' = 4x^3 + 8$. $\qquad v_1 = \frac{1}{4} \cdot e^{2x}$.

$u'' = 12x^2$. $\qquad v_2 = \frac{1}{8} e^{2x}$.

$u''' = 24x$. $\qquad v_3 = \frac{1}{16} \cdot e^{2x}$.

$u^{iv} = 24$ $\qquad v_4 = \frac{1}{32} \cdot e^{2x}$. $\qquad$ and $u^{(5)} \equiv 0$.

So $I = u \cdot v - u' v_1 + u'' v_2 - u''' v_3 + u^{iv} v_4 = \frac{1}{4} \left(2x^4 - 4x^3 + 6x^2 + 10x + 13\right) e^{2x} + C$.

Recursion formulas:

Sometimes a family of integrals satisfy a useful recursion:

Example: let $I_n = \int_0^{\pi/3} (\sec x)^n \, dx$ for $n \in \mathbb{Z}$.

① Show $(n-1)I_n = \sqrt{3} \; 2^{(n-2)} + (n-2) I_{n-2}$ for $\forall n \in \mathbb{Z}$.

② Use this to evaluate $\int_0^{\pi/3} (\sec x)^6 \, dx$.

Solution:

① $I_n = \int_0^{\pi/3} \underbrace{(\sec x)^{n-2}}_{u} \underbrace{(\sec x)^2}_{\frac{dv}{dx}} dx =$, where $V = \tan x$.

[NOTE: $u' = \tan x \cdot (\sec x)^{n-2} (n-2)$] $\to (n-2) \cdot (\sec x)^{n-3} \cdot \sec x \cdot \tan x$

$= \left[(\sec x)^{n-2} \cdot \tan x \right]_0^{\pi/3} - (n-2) \int_0^{\pi/3} \underbrace{(\tan x)^2 \cdot (\sec x)^{n-2}}_{\frac{\sin x}{\cos x} \cdot \sec x^{n-2}} \cdot dx =$

$= \sqrt{3} \cdot 2^{n-2} - (n-2) \cdot \int_0^{\pi/2} \left((\sec x)^n - (\sec x)^{n-2} \right) dx =$

$= \sqrt{3} \cdot 2^{n-2} - (n-2) (I_n - I_{n-2})$

and rearranging gives the equation we have to show.

② $I_6 = \frac{1}{5} \cdot \left(\sqrt{3} \cdot 2^4 + 4 I_4 \right)$, $I_4 = \frac{1}{3} \left(\sqrt{3}\, 2^2 + 2 I_2 \right)$, and

$I_2 = \sqrt{3}$ (since $n-2=0$ $\forall n=2$).

So $I_6 = \frac{\sqrt{3}}{5} \left(2^4 + \frac{4}{3} (2^2 + 2) \right) = \frac{24}{5} \sqrt{3}$.

Partial fractions useful for integrating rational functions $\frac{P(x)}{Q(x)}$, where $P(x)$ and $Q(x)$ are polynomials. We can assume w.l.o.g. that $Q(x)$ is "monic", i.e. the coefficient of the highest power of x in $Q(x)$ is $= 1$.

Step 1: If degree $(P) \geq$ degree (Q), rewrite $\frac{P(x)}{Q(x)} = A(x) + \frac{\tilde{P}(x)}{Q(x)}$ (by long division of polynomials: $D = C \cdot d + r \Rightarrow \frac{D}{d} = C + \frac{r}{d}$) where $A(x)$ and $\tilde{P}(x)$ are polynomials with degree $(\tilde{P})$ < degree (Q).

Step 2: factorise $Q(x)$ over $\mathbb{R}$ into a product of linear factors $(x-a)$ for $a \in \mathbb{R}$ and irreducible quadratic factors (x^2+bx+c) (since Q is monic I have a 1 in the front) where $b, c \in \mathbb{R}$ and $b^2 - 4c < 0$.

Step 3: Write down an "ansatz" (assumed form) for the partial fractions expansion of $\frac{\tilde{P}(x)}{Q(x)}$, as follows.

for each factor $(x-a)^p$ in $Q(x)$, include a contribution of the form

$$\frac{A_1}{x-a} + \frac{A_2}{(x-a)^2} + \dots + \frac{A_p}{(x-a)^p}$$

for each factor $(x^2+bx+c)^q$ in $Q(x)$, include:

$$\frac{B_1x+C_1}{(x^2+bx+c)} + \frac{B_2x+C_2}{(x^2+bx+c)^2} + \dots + \frac{B_qx+C_q}{(x^2+bx+c)^q}.$$

Step 4: Determine the unknown coefficients in our ansatz, eg multiply both sides by $Q(x)$ and equate coefficients of each power of x in the resulting polynomial equation, then solve the resulting linear equations

Example: Calculate $\int F(x)\,dx$ where $F(x) = \dfrac{x^4+2x^2-2x+3}{x^4-2x^3+2x^2-2x+1}$

$Q(x)$ is monic. But degree (P) = degree (Q) so we do step 1.

F

November 19th 2018

Ex: Find the partial fraction decomposition of $f(x) = \dfrac{x^4+2x^2-2x+3}{x^4-2x^3+2x^2-2x+1}$

$$F(x) = \frac{(x^4-2x^3+2x^2-2x+1)+2x^3+2}{x^4-2x^3+2x^2-2x+1} = 1 + G(x).$$

where $G(x) = \dfrac{2x^3+2}{x^4-2x^3+2x^2-2x+1} = \dfrac{2x^3+2}{(x-1)^2\,(x^2+1)} =$

$$= \frac{A}{(x-1)} + \frac{B}{(x-1)^2} + \frac{(Cx+D)}{(x^2+1)}$$

if it were $(x-1)^3 \rightarrow \dfrac{A}{(x-1)} + \dfrac{B}{(x-1)^2} + \dfrac{C}{(x-1)^3}$

$$2x^3+2 = A(x-1)^2(x^2+1) + B(x^2+1) + (Cx+D)(x-1)^2$$

$$2x^3+2 = (A+C)x^3 + (-A+B-2C+D)x^2 + (A+C-2D)x + (-A+B+D)$$

Equate coefficients:

$x^3 \Rightarrow A + C = 2$.
$x^2 \Rightarrow -A + B - 2C + D = 0$.
$x \Rightarrow A + C - 2D = 0$.
$x^0 \Rightarrow -A + B + D = 2$.

$A=1$, $B=2$, $C=1$, $D=1$.

so $f(x) = 1 + \dfrac{1}{x-1} + \dfrac{2}{(x-1)^2} + \dfrac{x+1}{x^2+1}$

$$\int f(x)\,dx = \int 1 + \frac{1}{x-1} + \frac{2}{(x-1)^2} + \frac{1}{2}\cdot\frac{2x}{x^2+1} + \frac{1}{x^2+1}\,dx =$$

$$= \boxed{x + \log|x-1| - \frac{2}{x-1} + \frac{1}{2}\log(x^2+1) + \tan^{-1}x + C}$$

- The $\tan(\theta/2)$ substitution:

① $\sin 2u = 2\sin u\cdot\cos u = 2\tan u\cdot(\cos u)^2 =$ $\quad \frac{\cos u}{\cos u}$

$$= 2\cdot\frac{\tan u}{(\sec u)^2} = 2\cdot\frac{\tan u}{1+(\tan u)^2}$$

$$\left(\frac{\cos u^2}{\cos u}\right)^2 \quad (\cos u)^2 - \left(\frac{\sin u}{\cos u}\right)^2(\cos u)^2 = (\cos u)^2 - (\tan^2 u\cdot\cos u^2)$$

② $\cos 2u = (\cos u)^2 - (\sin u)^2 = (\cos u)^2\cdot(1-(\tan u)^2) =$

$$= \frac{1-(\tan u)^2}{(\sec u)^2} = \frac{1-(\tan u)^2}{1+(\tan u)^2}$$

let $u = \theta/2$, $t = \tan\theta/2$ $(t = \tan u)$

$\arctan t = \frac{\theta}{2} \to \theta = 2\arctan t$

$$\sin\theta = \frac{2t}{1+t^2} \qquad \to d\theta = \frac{2\,dt}{1+t^2}$$

$$\cos\theta = \frac{1-t^2}{1+t^2}$$

Ex: evaluate $\displaystyle\int_0^{\pi/2}\frac{d\theta}{2+\sin\theta}$

let $t = \tan\theta/2$, $\theta = 0 \to t = 0$; $\theta = \pi/2 \to t = 1$

$$\int_0^1 \frac{1}{2+\frac{2t}{1+t^2}}\cdot\frac{2\,dt}{1+t^2} = \int_0^1\frac{dt}{1+t+t^2} =$$

$$\left(1+\frac{t}{1+t^2}\right)\cdot(1+t^2) = \left(\frac{1+t^2+t}{1+t^2}\right)(1+t^2) =$$

$$= \frac{1+t^2+t}{1+t^2} + \frac{t^2+t^4+t^3}{1+t^2} =$$

$$= \frac{t^4+t^3+2t^2+t+1}{1+t^2} = t^2+t+\frac{t^2+1}{t^2+1} = t^2+t+1$$

$D = dc + r$, $\frac{D}{d} = c + \frac{r}{d}$

completing the □

$$= \int_0^1\frac{dt}{\left(t+\frac{1}{2}\right)^2+\frac{3}{4}}$$

HOJA JORGE

let $t + \frac{1}{2} = \frac{\sqrt{3}}{2}\tan u$.

$dt = \frac{\sqrt{3}}{2}(\sec u)^2\,du$.

$t = 0 \Rightarrow u = \pi/6$, $t = 1$, $u = \pi/3$.

$ax^2 + bx + c \Rightarrow \left(\sqrt{a}x + \frac{b}{2}\right)^2 +$ lo faltante

$$\int_0^1 \frac{dt}{\left(t+\frac{1}{2}\right)^2+\frac{3}{4}}$$

$$\int_0^1 \frac{dt}{\left(\frac{\sqrt{3}}{2}\tan u\right)^2+\frac{3}{4}} = \int_{0=\pi/6}^{1=\pi/3} \frac{\boxed{dt}}{\frac{3}{4}(1+\tan^2 u)}$$

$$\boxed{t+\frac{1}{2} = \cancel{1+\tan}\ \frac{\sqrt{3}}{2}\tan u}$$

$$0+\frac{1}{2} = \frac{\sqrt{3}}{2}\tan u$$

$$\tan u = \frac{1}{\sqrt{3}} \qquad u=\pi/6$$

$$1+\frac{1}{2} = \frac{\sqrt{3}}{2}\tan u \qquad\qquad t+\frac{1}{2} = \frac{\sqrt{3}}{2}\tan u$$

$$\frac{3}{\sqrt{3}} = \tan u \qquad\qquad \frac{dt}{du} = \frac{\sqrt{3}}{2}\sec^2 u$$

$$u=\pi/3 \qquad\qquad dt = \boxed{\frac{\sqrt{3}}{2}\sec^2 u\, du}$$

$$\int_{\pi/6}^{\pi/3} \frac{\frac{\sqrt{3}}{2}\cancel{\sec^2 u}}{\frac{3}{4}\cancel{(1+\tan^2 u)}}\, du \qquad\qquad \int \frac{1}{2}\, dx = \frac{x^2}{4}$$

$$\int_{\pi/6}^{\pi/3} \frac{2\sqrt{3}}{3}\, du = \frac{2\sqrt{3}}{3}\, u \Big|_{\pi/6}^{\pi/3}$$

$$= \frac{2\sqrt{3}}{3}\left[\frac{\pi}{3}-\frac{\pi}{6}\right]$$

$$\int_0^{\pi/2} \frac{d\theta}{2+\sin\theta} \Rightarrow \theta = 2\arctan t$$

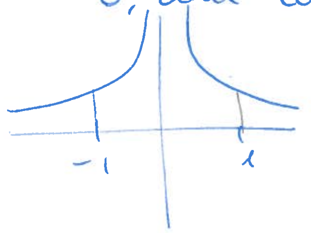
$$= \int_{\pi/6}^{\pi/3} \frac{\frac{\sqrt{3}}{2}(\sec u)^2\, du}{\frac{3}{4}\left(1+(\tan u)^2\right)} = \frac{2}{\sqrt{3}} \cdot \frac{\pi}{6} = \frac{\pi}{3\sqrt{3}}.$$

- IMPROPER INTEGRALS

Ex(!)

$$\int_{-1}^{1} \frac{dx}{x^2} = -\frac{1}{x}\Big|_{-1}^{1} = -1-\left(\frac{-1}{-1}\right) = -2$$

. This is wrong because it includes 0, and that is not defined.



An improper integral is an integral.

1. Over an ∞ domain $[0,\infty) \rightarrow$ ex $\int_0^\infty$
2. Over a domain where the integrand is not defined (or ∞).

en el a no está definida

Integrals over infinite domains:

$$\int_a^\infty f(x)\,dx = \lim_{b\to\infty} \int_a^b f(x)\,dx$$

Integral converges if this limit exists, otherwise it diverges.

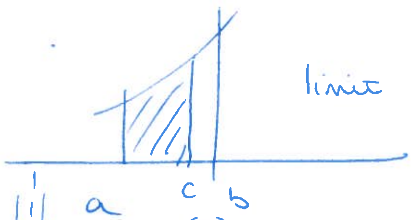
Ex: $\int_0^\infty e^{-x}\,dx = \lim_{b\to\infty} \int_0^b e^{-x}\,dx = \lim_{b\to\infty} -e^{-x}\Big|_{x=0}^{x=b} =$

$\int_0^\infty e^{-x}\,dx = \text{area}$

$= \lim_{b\to\infty} (1-e^{-b}) = 1$ (Integral converges)

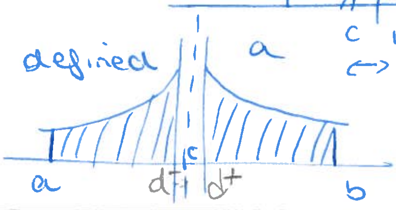
- Suppose that $f(x)$ is defined on $[a,b) = \{x : a \leq x < b\}$

$$\int_a^b f(x)\,dx = \lim_{c\to b^-} \int_a^c f(x)\,dx.$$



limit as c approaches b from below

- Suppose that $f(x)$ is defined on $[a,c) \cup (c,b]$



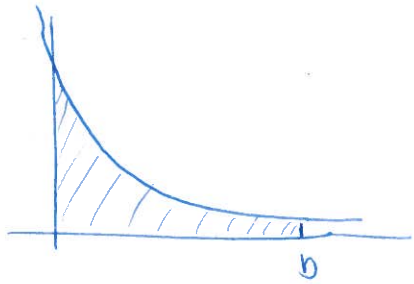
Check that

$$\lim_{d \to c^-} \int_a^d f(x)\, dx \quad \text{and} \quad \lim_{d \to c^+} \int_d^b f(x) \cdot dx \quad \text{exist}.$$

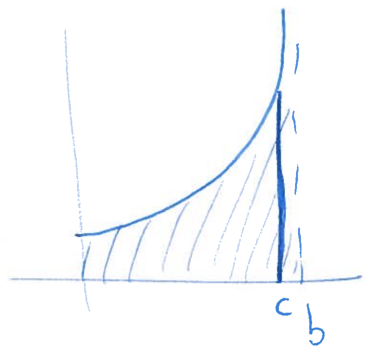
- If not, $\int$ diverges.
- If yes integral converges to the sum

November 23rd 2018.

Ex: ① $\displaystyle\int_0^{\infty} f(x) \cdot dx = \lim_{b \to \infty} \int_0^b f(x) \cdot dx$



② $\displaystyle\int_0^b f(x)\, dx = \lim_{c \to b^-} \int_0^c f(x) \cdot dx.$

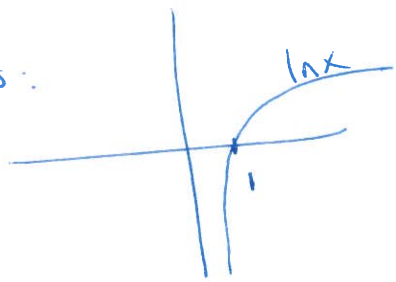


Ex: $\displaystyle\int_0^{\pi/2} \tan\theta \, d\theta$. Does it diverge? If not, to what value does it converge?

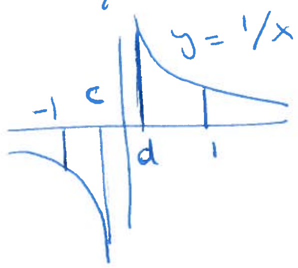
$\tan\theta = \dfrac{\sin\theta}{\cos\theta}$

Consider the $\displaystyle\lim_{b \to \left(\frac{\pi}{2}\right)^-} \int_0^b \tan\theta \cdot d\theta = \lim_{b \to (\pi/2)^-} \Big[-\ln|\cos\theta|\Big]_{\theta=0}^{\theta=b} =$

$\displaystyle= \lim_{b \to (\pi/2)^-} \left(-\ln|\cos b|\right) = \infty$ ∴ integral converges.



Ex: $\displaystyle\int_{-1}^{1} \frac{dx}{x}$



- We need to check the integrals

$$\lim_{c \to 0^-} \int_{-1}^{c} \frac{dx}{x} \quad \text{and} \quad \lim_{d \to 0^+} \int_d^1 \frac{dx}{x}$$

$$\lim_{d \to 0^+} \int_d^1 \frac{dx}{x} = \lim_{d \to 0^+} \ln|x| \Big]_{x=d}^{x=1} = \lim_{d \to 0^+} (-\ln d) = \infty .$$

* Integral diverges.

→ If one of the 2 diverges, game over, you don't have to try the other one

DIFFERENTIAL EQUATIONS

- **First order equations** → busco $\boxed{y=?}$

$$F\left(x, y, \frac{dy}{dx}\right) = 0 .$$

<u>Separable</u> $F \equiv \frac{dy}{dx} = f(x)\cdot g(y)$ → Separo para tener y con dy and x con dx.

$$\frac{1}{g(y)} \cdot \frac{dy}{dx} = f(x) \longrightarrow \int \frac{1}{g(y)} \cdot \frac{dy}{dx}\, dx = \int f(x)\, dx \; ;$$

$$\longrightarrow \int \frac{dy}{g(y)} = \int f(x)\cdot dx$$

cuando soluciono la $\int$ solo pongo $+C$ en uno de los lados

<u>Ex</u>: Solve the initial value problem (IVP)

$y(x) = y$ → cuando $x=-1$ → $y(-1) = -6$

$$\boxed{x \cdot y \cdot \frac{dy}{dx} = 1 \; , \; \underbrace{y(-1) = -6}_{\text{condición}}} \; ; \quad \int y\, dy = \int \frac{dx}{x} = \frac{1}{2}y^2 = \ln|x| + C .$$

↳ enunciado

$y\, dy = \frac{1}{x}\cdot dx$

$$y(-1) = -6 \Rightarrow \frac{1}{2}(-6)^2 = 0 + C \; ; \quad C = 18 \; ; \quad y^2 = 2\cdot \ln|x| + 36 \; ; \quad y = \pm\sqrt{2\ln|x| + 36}$$

$\frac{1}{2}y^2 = \ln|x| + 18$

Recall: $y(-1) = -6 < 0 \Rightarrow \boxed{y(x) = -\sqrt{2\ln|x| + 36}}$

- **Linear equations**

$$\int \frac{dy}{y} = \int -p(x)\, dx \rightarrow \ln|y| = \int -p(x)\, dx, \quad y = \exp\left(-\int p(x)\, dx\right)$$

$$\frac{dy}{dx} + p(x)\, y = f(x)$$

$\frac{dy}{dx}$ → siempre tiene que estar solo

$$\text{If } f(x) \equiv 0 \Rightarrow \frac{dy}{y} = (-)\, p(x)\cdot dx \Rightarrow y = \exp\left(-\int p(x)\cdot dx\right)$$

si es $y = e^{\left(\int p(x)\, dx\right)}$ se llama integrating factor, en cuanto lo tengo, lo multiplico a los dos lados de $\frac{dy}{dx} + p(x)y = f(x)$

Ex: solve $\underbrace{x^2 y' + 2xy = 1}_{\text{enunciado}}$; $(x^2 y)' = 1$; $x^2 y = x + C$; $y(x) = \frac{1}{x} + \frac{C}{x^2}$

($y' = \frac{dy}{dx}$)

More generally, we wish to find $I(x)$ s.t multiplying (1) by $I(x)$ gives a left side that is an exact derivative

$$\overbrace{I(x)}^{\text{IF}} \cdot \frac{dy}{dx} + \overbrace{p(x)\, I(x)}^{dI\,(*_1)}\, y = f(x)\, I(x)$$

want: $\underbrace{\frac{d}{dx}\left(I(x)\, y\right)}_{I(x)\frac{dy}{dx} + \frac{dI(x)}{dx}\, y} = f(x)\cdot I(x) .$

1er paso: divido todo por lo de al lado de x^2

2º paso: encuentro el I.F

3º multiplico $\frac{dy}{dx} + p(x)y = f(x)$ por IF en los 2 lados

4º. Hago el product rule

$-\ln|\cos x| = \ln|\sec x|$

So we want:

(1) $\frac{dI}{dx} = p(x) I$ ∴ por la product rule $\left(\frac{dI}{dx}(2^{o}\text{ term}) = p(x) I\right)$

$\Rightarrow I(x) = \exp\left(\int p(x)\, dx\right)$.

5º Cojo dx y lo pongo al otro lado e integro los 2 lados. (Solo pongo C en uno de los lados)

6º Si tengo condition busco C, pero sino isolate y.

$\frac{d}{dx}(Iy) = f(x) I(x)$.

$\Rightarrow y(x) = \frac{1}{I(x)} \int f(x) I(x)\, dx$.

__Ex__: $\frac{dy}{dx} + (1+6x) y = x \cdot e^{-x}$ .... (2).

$p(x) = 1+6x$.

$I(x) = \exp\left(\int (1+6x)\, dx\right) = \exp(x+3x^2)$ $\left(\underbrace{\text{set int const} = 0}_{\text{set integration constant } (C) = 0}\right)$.

(2) $\Rightarrow$
$\exp(x+3x^2) y' + (1+6x) \exp(x+3x^2) y = x \cdot e^{-x} (\exp(x+3x^2))$.

$\frac{d}{dx}\left(\exp(x+3x^2) y\right) = x \cdot e^{3x^2}$.

$\Rightarrow \exp(x+3x^2) y = \int x \cdot e^{3x^2}\, dx = \frac{1}{6} e^{3x^2} + c$;

$\Rightarrow y(x) = e^{-x}\left(\frac{1}{6} + c \cdot e^{-3x^2}\right)$.

__Ex__: Solve: $xy' + (6x^3+1) y = 4x^2$.

$y' + \left(6x^2 + \frac{1}{x}\right) y = 4x$; .... (3).

$p(x) = 6x^2 + \frac{1}{x}$.

$I(x) = \exp\left(\int \left(6x^2 + \frac{1}{x}\right) dx\right) = \exp(2x^3 + \ln|x| + c)$.

no es importante, la ignoramos

$I(x) = e^c \cdot e^{\ln|x|} \cdot e^{2x^3} = e^c |x| e^{2x^3} = \pm e^c x \cdot e^{2x^3}$.

Choose: $I(x) = x \cdot e^{2x^3}$.

Multiply (3) by $I(x)$: $x \cdot e^{2x^3} y' + (6x^3+1) e^{2x^3} y = 4x^2 \cdot e^{2x^3}$

$\frac{d}{dx}\left[(x \cdot e^{2x^3}) y\right] = 4x^2 \cdot e^{2x^3}$.

1st order:

$$x^2 \frac{dy}{dx} + 2xy = 1$$

1º $\frac{dy}{dx} + \frac{2x}{x^2} y = \frac{1}{x^2}$,

$\frac{dy}{dx} + \frac{2}{x} y = \frac{1}{x^2}$, $\frac{dy}{dx} = \boxed{\frac{1}{x^2} - \frac{2}{x}} y$,

2º $y = e^{\int \frac{2}{x} dx} = e^{2\ln|x|} = e^{\ln x^2} = x^2$

3º $x^2 \frac{dy}{dx} + \frac{2}{x} x^2 y = \frac{1}{x^2} x^2$,

, $x^2 \frac{dy}{dx} + 2x \cdot y = 1$

4º $\frac{d}{dx}(x^2 y) = 1$

5º $\int d(x^2 y) = \int 1 \, dx$, $x^2 y = x + C$

6º $y(x) = -\frac{1}{x} + \frac{C}{x}$ Él no hace pasos porque es muy fácil

Integrate:

$$x \cdot e^{2x^3} \cdot y = \frac{2}{3} e^{2x^3} + c$$

$$\Rightarrow y(x) = \frac{1}{x}\left(\frac{2}{3} + c \cdot e^{-2x^3}\right).$$

What we are doing:

$$I' = pI$$

$$I = \exp\left(\int p \cdot dx\right)$$

$$I(x) = \exp\left(\int p(x) \cdot dx\right).$$

$$y' + py = f.$$

$$Iy' + pIy = fI.$$

$$Iy' + I'y = fI$$

$$(Iy)' = fI.$$

Bernoulli's equation:

$$\frac{dy}{dx} + p(x)y = Q(x)y^n \quad n \neq 1.$$

Let $w = y^{1-n}$

$$\frac{dw}{dx} = (1-n)y^{-n}y' = (1-n)y^{-n}(Qy^n - Py) = (1-n)(Q - Py^{1-n}) =$$

$$= (1-n) \cdot (Q(x) - P(x)w).$$

So w solves a linear eqn.

- Second-order differential equations

A linear second-order ordinary differential equation (ODE) has the form:

$$\frac{d^2y}{dx^2} + p(x)\frac{dy}{dx} + q(x)y = f(x) \quad \ldots (1).$$

If $f(x) \equiv 0$ the equation is called homogeneous.

$$y'' + p(x)y' + q(x)y = 0 \quad \ldots (2).$$

Let y_1 and y_2 be 2 solutions of (2).
Then for any constants c_1 and c_2,

(3)... $y(x) := c_1 y_1(x) + c_2 y_2(x)$ is also a solution

$n \neq 1$

$$y''+py'+q = (c_1y_1+c_2y_2)''+p(c_1y_1+c_2y_2)'+q(c_1y_1+c_2y_2) =$$

$$= (c_1y_1''+c_2y_2'')+p(c_1y_1'+c_2y_2')+q(c_1y_1+c_2y_2) =$$

$$= c_1(y_1''+py_1'+qy_1)+c_2(y_2''+py_2'+qy_2) =$$

$$= c_1(y_1''+py_1'+qy_1)+c_2(y_2''+py_2'+qy_2) = c_1 0 + c_2 0 = 0$$

Fact: If y_1 and y_2 are solutions of (2) (and not constant multiples of each other) then any solution has the form (3).

We call (3) the general solution of (2).

• Second-order constant coefficient homogeneous (ODES):

(p, q constant)

$$y''+py'+qy=0 \quad \dots (4) \qquad p, q \text{ constant.}$$

[Aside: 1^{st} order $y'+py=0 \Rightarrow y=c\,e^{-px}$]

$\frac{dy}{dx}+py=0$, $dy=-p\,y\,dx$, $\int\frac{1}{y}dy=\int -p\,dx$, $\ln|y| = \int -p\,dx = -px+\ln C$ ($C=\ln C$), $y=e^{(-px+\ln C)} = e^{-px}e^{\ln|C|} = e^{-px}C$

Look for solutions of (4) of the form $y(x)=e^{\lambda x}$

$\Rightarrow y'(x)=\lambda e^{\lambda x}$ and $y''(x)=\lambda^2 e^{\lambda x}$.

(4) $\iff (\lambda^2+p\lambda+q)e^{\lambda x}=0$ $\;\to \lambda^2+p\lambda+q=0$, $e^{\lambda x}\neq 0$

$\iff \lambda^2+p\lambda+q=0 \quad \dots (5)$.

(5) is called the characteristic equation

3 cases: (based on the roots λ_1, λ_2 of (5)).

① 2 Real distinct roots $\lambda_1 \neq \lambda_2$.

So (4) has 2 independent solutions:

$$y_1(x)=e^{\lambda_1 x} \quad \text{and} \quad y_2(x)=e^{\lambda_2 x}$$

So the general solution of (4) is:

$$y(x)=c_1 e^{\lambda_1 x}+c_2 e^{\lambda_2 x}$$

② One repeated (real) root $\lambda_1=\lambda_2=\lambda$ $(p^2-4q=0)$.

One solution of (4) is $e^{\lambda x}$. Another solution is $xe^{\lambda x}$.

So the General solution is

$$y(x) = (c_1+c_2x)\cdot e^{\lambda x}.$$

2^{nd} order constant coefficient homogeneous

$$\frac{d^2y}{dx^2} + p\frac{dy}{dx} + qy = 0 \rightarrow p \text{ and } q \text{ are constant}$$

Case 1 : 2 real distinct roots: $\lambda_1 \neq \lambda_2$

$$\frac{d^2y}{dx^2} + 5\frac{dy}{dx} + 6y = 0$$

$$\lambda^2 + 5\lambda + 6 = 0$$

$$(\lambda+3)(\lambda+2) = 0$$

$$\lambda = -3, \ \lambda = -2$$

$$y = c_1 e^{-3x} + c_2 e^{-2x}$$

$$\boxed{\text{General form: } y = c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x}}$$

Case 2 : One repeated real root:

$$\frac{d^2y}{dx^2} + p\frac{dy}{dx} + qy = 0$$

$$\frac{d^2y}{dx^2} + 10\frac{dy}{dx} + 25y = 0$$

$$\lambda^2 + 10\lambda + 25 = 0$$

$$\lambda = \frac{-10 \pm \sqrt{(10)^2 - 4(1)25}}{2(1)} = \begin{cases} -5 \\ -5 \end{cases}$$

$$(\lambda+5)^2 = 0$$

$$\lambda_1 = -5 = \lambda_2 = -5$$

$$y(x) = (c_1 + c_2 x)e^{\lambda x}$$

$$y(x) = (c_1 + c_2 x)e^{-5x}$$

Bárbara Nieto Aguirre

SN: 18067032.

Case 3 2 complex roots (Imaginary $\lambda \neq 0$).

$$\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 13y = 0$$

$$\lambda^2 + 4\lambda + 13 = 0$$

$i^2 = (-1)$

$$\lambda = \frac{-4 \pm \sqrt{16-52}}{2} = \frac{-4 \pm \sqrt{-36}}{2} = \frac{-4 \pm \sqrt{36i^2}}{2} =$$

$$= \frac{-4 \pm 6i}{2} = \left< \begin{matrix} -2+3i \\ -2-3i \end{matrix} \right\} \lambda = \mu \pm i\nu$$

$$y = e^{\mu x}(c_1 \cos \nu x + c_2 \sin \nu x)$$

$$y = e^{-2x}(c_1 \cos 3x + c_2 \sin 3x)$$

always the positive

Case 3 (3). complex roots (strictly) (Imaginary $\lambda \neq 0$).
siempre va a haber 2, ±

$$\lambda = \frac{-p \pm \sqrt{p^2-4q}}{2}$$ complex conjugates: $\lambda_1 = \lambda$, $\lambda_2 = \bar{\lambda}$.

$$\lambda = \mu + i\nu$$

2 solutions to (4)

$$y_1(x) = e^{\lambda x} = e^{(\mu+i\nu)x} = e^{\mu x} \cdot e^{i\nu x} = e^{\mu x}(\cos\nu x + i\sin\nu x).$$

$$y_2(x) = e^{\bar{\lambda} x} = e^{(\mu-i\nu)x} = e^{\mu x}(\cos\nu x - i\sin\nu x).$$

$$y_3(x) = \frac{1}{2}\left(y_1(x) + y_2(x)\right] = e^{\mu x} \cdot \cos\nu x$$ is a solution of (4)

$$y_4(x) = \frac{1}{2i}\left[(y_1(x) - y_2(x)\right] = e^{\mu x} \sin\nu x$$ is a solution of (4).

So the general solution of (4)

is $$y(x) = c_1 y_3(x) + c_2 y_4(x) = e^{\mu x} \cdot (c_1 \cos\nu x + \underbrace{c_2 \sin\nu x)}_{}$$

this is real, no i.

November 23rd 2018

$$y'' + p(x)y' + q(x)y = 0.$$

If a solution is $y_1(x)$, I can also multiply it by α and it will still be a solution: $\alpha y_1(x)$.

$$\alpha y_1(x) + \beta y_2(x) = \text{general solution}.$$

If we get rid of $p(x)$ and $q(x)$ and we have constants instead, we look for specific solutions:

$$y'' + py' + qy = 0$$

Look for $y = e^{\lambda x}$

$$\lambda^2 + p\lambda + q = 0.$$

Examples: Solve the initial value problem (IVP)

(1) $y'' + 2y' - 3y = 0$, $y(0) = 1$, $y'(0) = 5$.

Characteristic eqn: $\lambda^2 + 2\lambda - 3 = 0$ $(\lambda-1)(\lambda+3) = 0 \Rightarrow \lambda = 1$ or $\lambda = -3$

General solution of (1):

$$y(x) = c_1 e^{x} + c_2 e^{-3x}.$$

$1 = y(0) = C_1 + C_2$.

$y'(x) = C_1 e^{x} - 3C_2 e^{-3x}$.

$\Rightarrow 5 = y'(0) = C_1 - 3C_2$.

$-4 = 4C_2 \Rightarrow C_2 = -1$; $C_1 = 2$.

$\Rightarrow y(x) = 2e^{x} - e^{-3x}$.

Ex: Find the general solution of

(a) $y'' + 4y' + 4y = 0$.

(b) $y'' + 6y' + 13y = 0$.

(a) characteristic equation: $\lambda^2 + 4\lambda + 4 = 0 \Leftrightarrow (\lambda+2)^2 = 0$.

$\lambda = -2, -2$.

So general solution of a is:

$$y(x) = (c_1 + c_2 x)e^{-2x}.$$

(b) Characteristic equation: $\lambda^2 + 6\lambda + 13 = 0$.

$$\lambda = \frac{-6 \pm \sqrt{36-52}}{2} = -3 \pm \frac{4i}{2} = -3 \pm 2i$$

$$Y = e^{(-3+2i)x} = e^{-3x} e^{2ix} = e^{-3x}(\cos 2x + i \sin 2x).$$

So the general solution of (b) is

$$y(x) = c_1 \operatorname{Re}(Y) + c_2 \operatorname{Im}(Y) = e^{-3x}(c_1 \cos(2x) + c_2 \sin 2x).$$

- In homogeneous linear ODEs

$$y'' + p(x)y' + q(x)y = f(x) \quad \dots (2)$$

Let $y_p(x)$ be a particular solution of (2) and let y be any other solution of (2) → (the ones I'm trying to find).

So $y_p'' + py_p' + q y_p = f(x) \cdots (3)$.

$(2) - (3) = (y'' - y_p'') + p(x)(y' - y_p') + q(y - y_p) = 0$.

let $y_h = y - y_p$. then y_h solves the homogeneous eqn.

$$y_h'' + py_h' + qy_h = 0 \quad \dots (4)$$

So the general solution of (2) is

$$y(x) = y_h(x) + y_p(x)$$

Where y_h is the general solution of the homogeneous eqn (4) and

and y_p is any particular solution of 2.

· Method of undetermined coefficients

Ex: find the general solution of $y'' - 4y = 2e^{3x} \dots (5)$.

1^{st} consider the homogeneous case.

The general solution of the homogeneous eqn:

$$y_h'' - 4y_h = 0 \qquad \left(\begin{array}{l} \text{characteristic eqn: } \lambda^2 - 4 = 0, \\ \therefore \lambda = \pm 2 \end{array} \right)$$

$$y_h = c_1 e^{2x} + c_2 e^{-2x}.$$

2^{nd} Find a particular solution for the equation (5)

↳ choose y to have sthing proportional to (5) → that's why he guessed $9Ae^{3x}$ and $4Ae^{3x}$.

Look for a particular solution of (5) of the form: $y_p(x) = Ae^{3x}$

$$9Ae^{3x} - 4Ae^{3x} = 2e^{3x} \Rightarrow A = 2/5.$$

3^{rd} General solution of (5)

$$y(x) = y_h(x) + y_p(x) = c_1 e^{2x} + c_2 e^{-2x} + \frac{2}{5} e^{3x}.$$

CHOICES FOR PARTICULAR SOLUTIONS:

$f(x)$	Choice for y_p.
$K \cdot e^{\alpha x}$	$A e^{\alpha x}$
$K_1 \cdot \cos\alpha x + K_2 \sin \alpha x$	$A \cdot \cos\alpha x + B \sin \alpha x$.
Polynomial of degree n	General polynomial of degree n.
$K_1 e^{\alpha x} \sin\beta x + K_2 e^{\alpha x}\cos\beta x$	$Ae^{\alpha x}\cos\beta x + Be^{\alpha x}\sin\beta x$.

These substitutions will work provided no part of the choice solves the homogeneous. Otherwise, multiply the corresponding choice by x.

If the RHS is a sum of different terms $f_1 + \dots + f_n$, $y = y_1 + \dots + y_n$, where y_i solves $y_i'' + py_i' + qy_i = f_i$ $\boxed{y'' + 2y' + 3y = \sin x + 2e^{-x}.}$

Ex: Solve: $y'' + 5y' - 6y = \sinh x + \sin 2x$.

Homogeneous solution: $y_h(x) = c_1 e^{x} + c_2 e^{-6x}$.

$$y'' + 5y' - 6y = \frac{1}{2} e^{x} - \frac{1}{2} e^{-x} + \sin 2x .$$

$$y_p = Axe^{x} + Be^{-x} + C\sin 2x + D\cos 2x .$$

Ex: $y'' + 5y' - 6y = \sinh x + \sin(2x)$.

$$y'' + 5y' - 6y = \frac{1}{2}(e^{x} - e^{-x}) + \sin(2x);$$

Characteristic equation: $y_h = c_1 e^{x} + c_2 e^{-6x}$.

$$\lambda^2 + 5\lambda - 6 = 0$$

$$\lambda = \frac{-5 \pm \sqrt{25+24}}{2} = \begin{cases} \frac{-5+7}{2} = \frac{2}{2} = 1 \\ \frac{-5-7}{2} = \frac{-12}{2} = -6 \end{cases} \quad \lambda_1 = 1, \ \lambda_2 = -6 .$$

Particular solution of (1):

$$y_p(x) = \underbrace{\alpha x e^{x}}_{\text{because } e^{x} \text{ solves the homogeneous eq}^n \text{ we add } x} + \beta e^{-x} + \underbrace{\gamma \sin(2x) + \delta \cos(2x)}.$$

In some cases I don't have to do the combination of sin and cos, that would be if for ex. I don't have middle term (y').

I substitute them in the first equation
$$\begin{cases} y_p' = \alpha(x+1)e^{x} - \beta e^{-x} + 2\gamma\cos(2x) - 2\delta\sin(2x) \\ y_p'' = \alpha(x+2)e^{x} + \beta e^{-x} - 4\gamma\sin(2x) - 4\delta\cos(2x) \end{cases}$$

$$\alpha\big((2+5) + (1+5-6)x\big)e^{x} + \beta(1-5-6)e^{-x} + (-4\gamma - 10\delta - 6\gamma)\sin(2x) -$$
$$- (-4\delta + 10\gamma - 6\delta)\cos(2x) = \frac{1}{2}e^{x} - \frac{1}{2}e^{-x} + \sin(2x).$$

Equate coefficients

e^{x}: $7\alpha = \frac{1}{2}$

e^{-x}: $-10\beta = -1/2$.

$\sin(2x)$: $-10\gamma - 10\delta = 1$

$\cos(2x)$: $-10\delta + 10\gamma = 0$.

PROBABILITY

The set of all possible outcomes in some situation to be described is called a sample space.

Any subset of a sample space is called an event

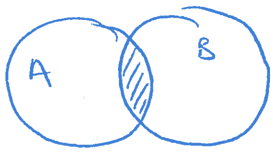
Ex: Two coins are tossed. One possible sample space is {H and H, HT, TT} (ignoring the order).

Another choice records which coin is a head and which is a tail.

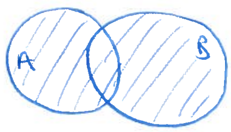
$$\underbrace{\{HH, HT, TH, TT\}}_{\text{all have the same probability}}$$

- SET OPERATIONS

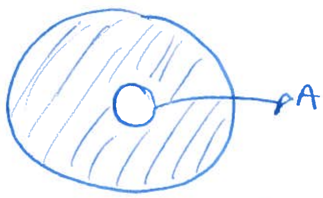
① Intersection $A \cap B$ (and)



② Union $A \cup B$ (or)

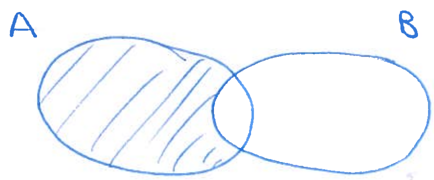


③ Complement: $A^c = A' = \bar{A} = \{x \in S : x \notin A\}$ — S: sample space.



④ Relative complement: $\underbrace{A \setminus B}$. (Everything in A but without B)

A without B
A minus B



A and B are said to be disjoint (mutually exclusive) — disjoint: no compatibles — if $A \cap B = \emptyset$ (empty set).

Ex: Consider the set $S = \{0,1,2\}$.

Elements of S: $0, 1, 2$

$\underbrace{\text{Subsets of } S}$ are: $\{0,1,2\}, \{0,1\}, \{1,2\}, \{2,0\}, \{0\}, \{1\}, \{2\}, \emptyset$.

we include the set

and the empty set

$\boxed{\text{Note}}$: 0, $\{0\}$, $\emptyset$ are completely different things!

0 → number

$\{0\}$ → set with 1 element

$\emptyset$ → set with no element

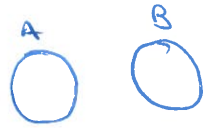
Definition. To each event $\boxed{A \subset S}$ we assign a number $P(A)$ called the probability of A which satisfies:

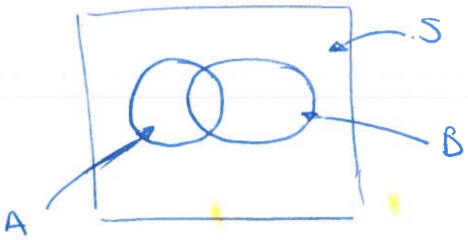
$\boxed{\text{Subset which could possibly be } S. \text{ (never writing } \subseteq) \;\; \} \; A \text{ is a subset of } S, \text{ possibly } S \text{ itself.}}$

① $P(A) \geq 0 \quad \forall A \subset S$

② $P(S) = 1$.

③ If $A \cap B = \emptyset$ then the $P(A \cup B) = P(A) + P(B)$ →





$A \cap B$ = "A and B".

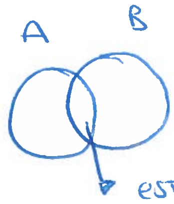
$A \cup B$ = "A or B".

Lemma: Let A and B be events in S (not necessarily disjoint) Then:

a) $P(A^c) = 1 - P(A)$

b) $P(\emptyset) = 0$.

c) $P(A \cup B) = P(A) + P(B) - P(A \cap B)$ →



esta se repite, así que la resto.

Proof:

a) $A \cap A^c = \emptyset$.

$1 = P(S) = P(A \cup A^c) = P(A) + P(A^c)$.

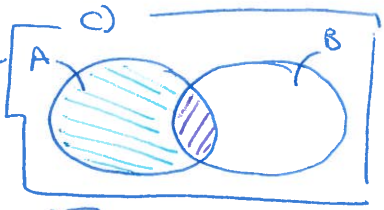
$\Rightarrow P(A^c) = 1 - P(A)$.

b) $P(\emptyset) = 1 - P(\emptyset^c) \Rightarrow$ from the just proved a.

$P(\emptyset) = 1 - P(S) = 1 - 1$ (from property 2 we know that $P(S) = 1$).

so $P(\emptyset) = 0$.

c) $P(A \cup B) = P(\underbrace{(A \setminus B)} \cup B) =$



[We know $((A \setminus B) \cap B = \emptyset)$]

$= P(A \setminus B) + P(B)$ (*) from (3).

$(A \setminus B) \cap (A \cap B) = \emptyset$.

$P(A) = P((A \setminus B) \cup (A \cap B)) =$

$= P(A \setminus B) + P(A \cap B)$ (**) (from 2)

Eliminate $P(A \setminus B)$ from (*) and (**) → Despejar $P(A \setminus B)$ en (**) y sustituir. Then $\boxed{//}$

- Equally likely events

Consider a finite sample space $S = \{s_1, s_2, \dots, s_n\}$.

$|S| = n = \#$ of elements in S. (cardinality of S).

Consider the (simple) events

$S_1 = \{s_1\}, \dots, S_n = \{s_n\}$.

Assume that each event is equally probable.

$S_i \cap S_j = \emptyset \quad i \neq j$.

$$S = \bigcup_{j=1}^{n} S_j = S_1 \cup S_2 \cup \dots \cup S_n$$

$1 = P(S) = P(S_1 \cup S_2 \cup S \dots S_n)$.

$1 = P(S_1) + P(S_2) + \dots + P(S_n)$

But $P(S_1) = P(S_2) = \dots = P(S_n) \Rightarrow n \cdot P(S_j) = P(S)$ for any j.

$$\boxed{P(S_j) = \frac{1}{n} = \frac{1}{|S|}}$$

Let $A \subset S$ be any event. Then

$A = \{s_{i_1}, s_{i_2}, \dots, s_{i_k}\}$ $\quad k \leq n$.

* Doble index, because I don't know which elements belongs to the event.

$$A = S_{i_1} \cup S_{i_2} \cup \dots \cup S_{i_k}.$$

$$P(A) = P(S_{i_1}) + P(S_{i_2}) + \dots + P(S_{i_k}) =$$

$$= \underbrace{\frac{1}{|S|} + \frac{1}{|S|} + \dots + \frac{1}{|S|}}_{k \text{ terms}} = k \cdot \frac{1}{|S|}.$$

So $\boxed{P(A)} = k \cdot \frac{1}{|S|} = \boxed{\frac{|A|}{|S|}}$

$k = n^o$ elements in $A = |A|$.

For equally likely outcomes $\boxed{\text{prob (event)} = \frac{\#\text{ of ways the event happens}}{\text{total }\#\text{ of posibilities}}}$

<u>Ex</u>: A fair coin is tossed twice. Use the sample space

$$S = \{HH, HT, TH, TT\}$$

Each outcome is equally likely

$$P(\text{Two heads are thrown}) = \frac{|\{HH\}|}{|S|} = \boxed{\frac{1}{4}}.$$

$$P(\text{One head and one tail appears}) = \frac{|\{HT, TH\}|}{|S|} = \frac{2}{4} = \boxed{\frac{1}{2}}.$$

$$P(\text{At least one tail appears}) = \frac{|\{HT, TH, TT\}|}{|S|} = \boxed{\frac{3}{4}} =$$

$$= 1 - P(\text{no tail appears}) = 1 - \frac{1}{4} = 1 - \frac{|\{HH\}|}{|S|} = \boxed{3/4}$$

<u>Discrete sample spaces</u>

A sample space is called discrete if either it has a finite number of elements or it is countably infinite.

(i.e. the elements can be listed one after the other $\Leftrightarrow$ there is a one-to-one mapping to the integers).

For a discrete sample space, we can write $\boxed{\sum_{x \in S} P(x) = 1}$.

Ex: Consider a game in which a fair coin is thrown until the first time a head shows (and the game stops).

Describe a suitable sample space.

Find the probability P_n that the game ends on the n^{th} throw.

Verify $\sum_{n=1}^{\infty} P_n = 1$.

Sample space: $S = \{H, TH, TTH, TTH, TTTH, \dots, \overbrace{T \cdots T}^{n-1 \text{ tails}}H, \dots\}$.

n^{th} element is $\underbrace{(n-1)}_{\text{porque el } 1^{er} \text{ termino es } 0 \text{ tails}}$ tails followed by a head.

$$P_n = P(\underbrace{T \cdots T}_{n-1 \text{ terms}}H) = \underbrace{\left(\frac{1}{2}\right)^{n-1}}_{\text{tantas T}} \cdot \underbrace{\frac{1}{2}}_{\text{Tantas H}} = \left(\frac{1}{2}\right)^n.$$

$$\sum_{n=1}^{\infty} P_n = \sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n = \left(\begin{array}{l}\text{geometric serie,} \\ 1^{st} \text{ term} = 1/2 \\ \text{common ratio} = 1/2 < 1\end{array}\right) = \frac{1/2}{1-1/2} = 1.$$

December 3^{rd} 2018

Conditional probability

Ex: An urn contains 3 black balls and 2 white balls. Two balls are removed in order (without being put back). Find the probability that

1. The first ball is black.
2. The second ball is black.
3. Both have the same colour.

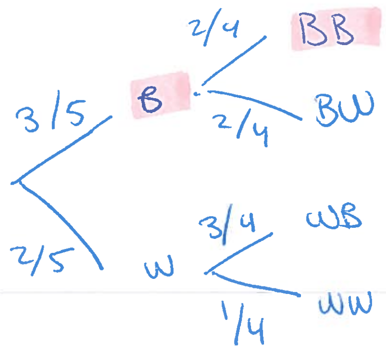
Solution:

1. Probability that the 1^{st} ball is black is:

black $= \frac{3}{5}$.

Suppose this actually happens, then there are 2 black and 2 white balls left.

Then the probability that next we get a black ball $= 2/4$ and a white ball $= 2/4$.



→ we multiply them in order to get $P(BB)$: PROPERTY.

2. $P(\text{second ball is black}) = P(BB \cup WB) = P(BB) + P(WB) =$

$= \frac{3}{5} \cdot \frac{2}{4} + \frac{2}{5} \cdot \frac{3}{4} = \boxed{3/5}$.

they are disjoint

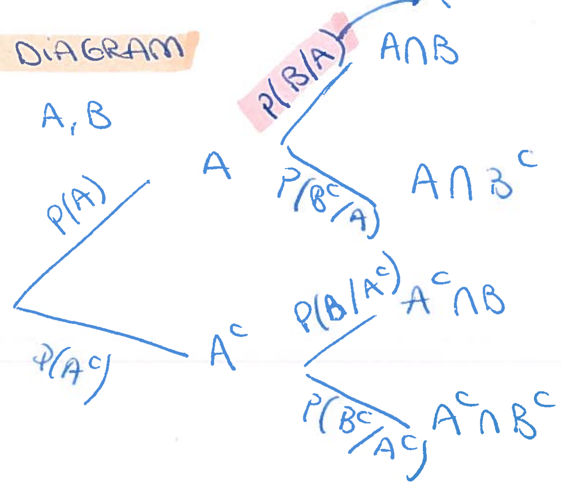
3. $P(\text{Both same colour}) = P(BB \cup WW) = P(BB) + P(WW) =$

$= \frac{3}{5} \cdot \frac{2}{4} + \frac{2}{5} \cdot \frac{1}{4} = \boxed{\frac{2}{5}}$.

probability of B, given A: CONDITIONAL PROBABILITY

GENERAL DIAGRAM

2 events A, B



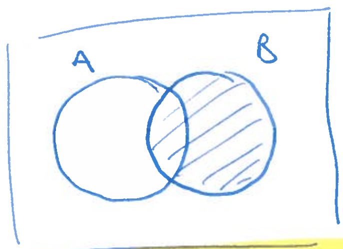
In our example:

$A = 1^{st}$ ball black.

$B = 2^{nd}$ ball black.

$P(B/A)$ is called the conditional probability that B occurs given that A has occurred.

$$\boxed{P(B/A) = \frac{P(A \cap B)}{P(A)}}$$



Clearly, $B = (B \cap A) \cup (B \cap A^c)$

and $(B \cap A) \cap (B \cap A^c)$ are disjoint.

$$P(B) = P((B \cap A) \cup (B \cap A^c)) = P(B \cap A) + P(B \cap A^c) = P(B/A)P(A) + P(B/A^c)P(A^c)$$

Counting A set S with n elements.

1. Ordered samples, repetition allowed.

Number of samples of size r is n^r.

Ex : How many 4-digit numbers can be made using digits 1, 2 and 3 as many times as we like.

$$3 \times 3 \times 3 \times 3 = 3^4 = \boxed{81}$$

2. Ordered samples, no repetition allowed.

1 2 --- r

n $\quad n-1$ $\quad$ ---- $\quad n-r+1$

\# of samples in this case is $n\cdot(n-1)\cdot(n-2)\cdots(n-r+1) =$

$$= \frac{[n\cdot(n-1)\cdots(n-r+1)]\cdot(n-r)\cdots 2\cdot 1}{(n-r)\cdot(n-r-1)\cdots 2\cdot 1} = {}^nP_r = \frac{n!}{(n-r)!}$$

(permutations)

\# Reorderings of n objects :

$$= {}^nP_n = n!$$

$$\left.\begin{array}{ccc} 3 & 2 & 1 \\ 3 & 1 & 2 \\ 1 & 2 & 3 \\ 1 & 3 & 2 \\ 2 & 3 & 1 \\ 2 & 1 & 3 \end{array}\right\} 1,2,3$$

3. Unordered samples, no repetition

\# of samples of size r :

$$ {}^nC_r = \binom{n}{r} = \frac{n!}{(n-r)!\,r!} = \{2,1\}$$

December 7^{th} 2018

$S = \{1, \ldots, n\}$.

# of collections of length r.

① ORDERED samples, repetition

n^r

② ORDERED samples, no repetition

1 5 7

~~1 5 5~~

$n \cdot (n-1) \cdots (n-r+1) = \frac{n!}{(n-r)!}$ $n!$ ways of rewriting the n elements

③ UNORDERED samples, no repetition

e.g. Find subsets of cardinality r from a set S of cardinality n.

Let ${}^nC_r$ be the number of unordered samples.

For any list of r objects, there are $r!$ re-orderings, so

$r! \; {}^nC_r = \#$ of ordered samples $= \frac{n!}{(n-r)!}$

$$\Rightarrow \boxed{{}^nC_r = \frac{n!}{r!\,(n-r)!} = \binom{n}{r} = \text{(“}n \text{ choose } r\text{”)}}$$

↳ we use this for $(1+x)^n = \sum_{r=0}^{n} \binom{n}{r} x^r$.

- Independence

Two elements are called independent if $P(A \cap B) = P(A) \cdot P(B)$.

This is equivalent to saying $P(B/A) = P(B)$.

$$\left(P(B/A) = \frac{P(A \cap B)}{P(A)} \right)$$

Ex: A coin is tossed 3 times. Consider 2 events:

a) Throwing at least one head and one tail.

b) Throwing at most one head

1. Are these events independent ?

2. Are they independent if the coins were tossed 4 times instead of 3?

1→ Obvious sample space:

$S = \{HHH, HHT, HTH, HTT, THH, THT, TTH, TTT\}$

Let A and B be the events in (a) and (b) respectively

$A = \{HHT, \ldots, TTH\} = S \setminus \{TTT, HHH\}$

↳ without

$B = \{HTT, THT, TTH, TTT\}$

$A \cap B = \{HTT, THT, TTH\}$

$|S| = 8, \quad |A| = 6, \quad |B| = 4, \quad |A \cap B| = 3$

$P(A) = \frac{|A|}{|S|} = \frac{3}{4}, \quad P(B) = \frac{|B|}{|S|} = \frac{1}{2}$

$P(A \cap B) = \frac{|A \cap B|}{|S|} = \frac{3}{8} = \frac{3}{4} \cdot \frac{1}{2} = P(A) \cdot P(B)$

⇒ so A and B are independent

2. $S = \{HHHH, \ldots\}$

$|S| = 2^4 = 16$

$A = S \setminus \{HHHH, TTTT\}$

$B = \{HTTT, THTT, TTHT, TTTH, TTTT\}$

$A \cap B = \{HTTT, THTT, TTHT, TTTH\}$

$|A| = 14, \quad |B| = 5, \quad |A \cap B| = 4$

$P(A) = \frac{|A|}{|S|} = \frac{14}{16} = \frac{7}{8}$

$P(B) = \frac{|B|}{|S|} = \frac{5}{16}$

$P(A \cap B) = \frac{|A \cap B|}{|S|} = \frac{1}{4}$

$P(A \cap B) \neq P(A) \cdot P(B)$ $\boxed{\text{dependent}}$

- **Bayes' Formula:**

$$\boxed{P(A|B) = \frac{P(A\cap B)}{P(B)}} = \frac{P(A)\cdot P(B|A)}{P(A)\cdot P(B|A) + P(A^c)\cdot P(B|A^c)}.$$

Ex: This problem involves 2 coins: one fair and one with 2 heads. One coin is chosen at random and tossed. The result is a head. Find the probability that the fair coin was selected.

Sol: Let F be the event that the coin is fair.
Let H be the event that the first toss shows heads.

$$P(F|H_1) = \frac{P(H_1,F)\cdot P(F)}{P(H_1,F)\cdot P(F) + P(H_1,F^c)P(F^c)} = \frac{1/2\cdot 1/2}{1/2\cdot 1/2 + 1\cdot 1/2} = \boxed{1/3}$$

Now suppose the coin is tossed again and we get heads. What is the probability now that the coin is fair.

Let H_2 = event of both throws show heads.

$$P(F/H_2) = \frac{P(H_2/F)\cdot P(F)}{P(H_2|F)\cdot P(F) + P(H_2/F^c)\cdot P(F^c)} = \frac{\frac{1}{4}\cdot 1/2}{1/4\cdot 1/2 + 1\cdot 1/2} = \boxed{1/5}$$

- **Bernoulli trials:**

A Bernoulli trial is a repeated independent experiment or event with only 2 possible outcomes: "success" with probability p and 'failure' with probability $q = 1-p$. The probabilities must be the same for each trial.

The probability of r successes from n Bernoulli trials is:

$$b(r) = \binom{n}{r}\cdot p^r\cdot q^{n-r}.$$

Binomial distribution

1	2	3	....	n
H	T	T	H	H

r successes $\rightarrow p^r\cdot q^{n-r}$

Ex: Which of the following is more likely?

A. 6 fair dice and you get at least one 6?

B. 12 fair dice and you get at least 2 '6's?

C. 18 fair dice and you get at least 3 '6's?

Solution:

A. P(at least one 6 from 6 dice) = 1 - P(no 6 from 6 dice) =

$= 1 - \left(P(\text{no 6 from one dice})\right)^6 = 1 - \left(\frac{5}{6}\right)^6 \simeq \boxed{0.665}$.

B. P(at least two 6s from 12 dice) = 1 - P(no 6's from 12 dice) -

P(exactly one 6 from 12 dice) $= 1 - \left(5/6\right)^{12} - \binom{12}{1}\left(\frac{1}{6}\right)^1 \cdot \left(\frac{5}{6}\right)^{11} =$

$\simeq 0.619$.

C. P(at least 3 6's from 18 dice) = 1 - P(no 6) - P(1, 6) - P(2 6's) =

$= 1 - \binom{18}{0}\left(\frac{1}{6}\right)^0 \cdot \left(\frac{5}{6}\right)^{18} - \binom{18}{1}\left(\frac{1}{6}\right)^1 \cdot \left(\frac{5}{6}\right)^{17} - \binom{18}{2}\left(\frac{1}{6}\right)^2 \cdot \left(5/6\right)^{16} = 0.597$.

- **Mean value**

Suppose that the possible outcome for each experiment in a series of experiments is a number x_i. If $P(x_i)$ is the probability of x_i, the mean (or average) value of a series of experiments is.

$$\bar{x} = \sum_i x_i \; P(x_i)$$

Ex: Each side of a fair die has probability 1/6 of appearing

mean/average/expected value is:

$$1 \cdot \frac{1}{6} + 2 \cdot 1/6 + 3 \cdot 1/6 + \dots + 6 \cdot 1/6 = \frac{1+2+\dots+6}{6} = 7/2.$$

Ex: (The binomial distribution)

To any sequence of Bernoulli trials, we can associate the number r of successes, each with probability b(r).

$$b(r) = \binom{n}{r} p^r \cdot q^{n-r}$$

Expected # of successes is $\sum_{r=0}^{n} r \cdot b(r) = \sum_{r=0}^{n} r \cdot \binom{n}{r} p^r \cdot q^{n-r} =$

$$= q^n \cdot \sum_{r=p}^{n} r \cdot \binom{n}{r} \left(\frac{p}{q}\right)^r$$

$$(1+x)^n = \sum_{r=0}^{n} \binom{n}{r} x^r \quad : \text{I differentiate}$$

$$\Rightarrow n \cdot (1+x)^{n-1} = \sum_{r=0}^{n} r \cdot \binom{n}{r} \cdot x^{r-1} \quad \text{: Add } x' \text{ in both sides}$$

$$\Rightarrow nx(1+x)^{n-1} = \sum_{r=0}^{n} r \binom{n}{r} x^r .$$

So average $= q^n \cdot n \left(p/q\right) \cdot \left(1 + p/q\right)^{n-1} = n \cdot p \cdot q^{n-1} \left(\frac{q+p}{q}\right)^{n-1} =$

$= n \cdot p (q+p)^{n-1}$.

- Poisson distribution

We want to consider $b(r)$ as for large n.
We will consider a limit in which $n \to \infty$ in such a way.
that the mean $\lambda = np$ remains fixed ($\Rightarrow p \to 0$).

$$b(r) = \binom{n}{r} \cdot p^r \cdot q^{n-r} \qquad \lambda = np, \; p+q = 1 .$$

$$= \frac{n!}{r!(n-r)!} \left(\frac{\lambda}{n}\right)^r \cdot \left(1 - \frac{\lambda}{n}\right)^{n-r} \qquad \left(\begin{array}{l}\text{fix } \lambda, r \\ \text{Take } n \to \infty\end{array}\right)$$

$$= \frac{\lambda^r}{r!} \underbrace{\frac{n \cdot (n-1) \cdot (n-2) \cdots (n-r+1)}{n \cdot n \cdots n}}_{r \text{ terms}} \cdot \left(1 - \frac{\lambda}{n}\right)^{n-r}$$

We have r terms of the form $\left(1 - \frac{k}{n}\right) \to 1$ as $n \to \infty$.

Also $\left(1 - \frac{\lambda}{n}\right)^{-r} \to 1$

So we only need $\lim_{n \to \infty} a_n$, where $a_n = \left(1 - \frac{\lambda}{n}\right)^n$.

$$\ln a_n = n \cdot \ln\left(1 - \frac{\lambda}{n}\right) = n \cdot \left(-\frac{\lambda}{n} + \dots\right) = -\lambda + \dots$$

$$\ln(1+x) = x + \cdot x^2 + \dots$$

$$\lim_{n \to \infty} \ln a_n = -\lambda .$$

$$\ln\left(\lim_{n \to \infty} a_n\right) = -\lambda \Rightarrow \lim_{n \to \infty} \left(1 - \frac{\lambda}{n}\right)^n = \lim_{n \to \infty} a_n = e^{-\lambda} .$$

The poisson distribution is defined by:

$$P(r) = \lim_{n \to \infty} b(r) = e^{-\lambda} \cdot \frac{\lambda^r}{r!}$$

<u>Ex</u> : An insurance company pays £500 k to each client who experiences a fire at their house. The company has 5000 clients. Given that the probability of a client having a fire in a 12 month period is 10^{-4}, find the prob. that the company would pay out at least £2000000 in a single year.

<u>December 10th</u>

$$\binom{n}{r} p^r q^{n-r} = b(r) = \frac{n!}{r!(n-r)!}$$

$\lambda = np$.

$n \to \infty$, λ fixed, $p \to 0$.

$P(r) =$ poisson distribution $= e^{-\lambda} \dfrac{\lambda^r}{r!}$

<u>Ex</u>: from before

£500k.

5.000 clients

Prob (1 fire) $= 10^{-4}$.

Pay $\geq$ £2000k.

Average # of fires per year: $\lambda = np = 5000 \cdot 10^{-4} = 1/2$.

Prob of 1 fire per customer $p = 10^{-4}$.

Use the poisson distribution.

We want the probability of at least 4 fires.

P(at least 4 fires) = 1 − prob(no fire) − prob(1 fire) − prob(2 fires) − prob(3 fires) =

$$= 1 - e^{-1/2} \cdot \left(\left(\frac{(1/2)^0}{0!}\right) + \left(\frac{(1/2)^1}{1!}\right) + \left(\frac{(1/2)^2}{2!}\right) + \left(\frac{(1/2)^3}{3!}\right) \right) =$$

$$= 1 - \frac{79}{48} e^{-1/2} \approx 0.00175 \sim 0.2\%$$

Events distributed over intervals of time or space.

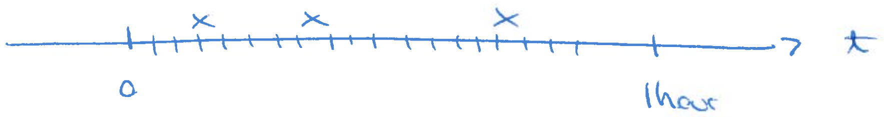
Ex: Imagine that cars pass a point on the road an average of λ times per hour.

Assume cars pass independently and no 2 cars pass at the same time.

Divide the hour into n equal subintervals of time $\frac{1}{n}$ hours.
Now, choose n sufficiently large, so that in any subinterval, either no car passes or exactly one car passes.

The probability that a car passes during a given subinterval is

$p = \frac{\lambda}{n}$.



Probability that exactly r cars pass during the hour is

$$b(r) = \binom{n}{r} p^r q^{n-r} = \binom{n}{r} \cdot \left(\frac{\lambda}{n}\right)^r \cdot \left(1 - \frac{\lambda}{n}\right)^{n-r}.$$

To allow for the intervals between cars to be arbitrary small, we must take the limit $n \to \infty$.

$\xrightarrow{n \to \infty}$ Poisson distribution $= e^{-\lambda} \frac{\lambda^r}{r!}$

Ex: An office received on average 3 calls per hour. Find the probability that in a given hour

(a) no calls are received.

(b) exactly 3 calls are received

Solutions:

Average $\lambda = 3$

(a) Probability (no calls) = poisson distribution $= e^{-\lambda} \frac{\lambda^0}{0!} = e^{-3}$.

(b) Prob (3 calls) = Poiss(3) $= e^{-\lambda} \frac{\lambda^3}{3!} = e^{-3} \frac{3^3}{3!} = \boxed{\frac{9}{2} e^{-3}}$

Check:

$$\sum_{r=0}^{\infty} e^{-\lambda} \frac{\lambda^r}{r!} = e^{-\lambda} \sum_{r=0}^{\infty} \frac{\lambda^r}{r!} = e^{-\lambda} \cdot e^{\lambda} = \boxed{1}.$$

Continuous probability distributions

Consider the probability that a particle is between positions $x=a$ and $x=b$, is given by $P(a < x < b)$.

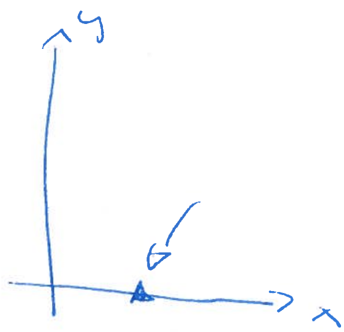
Where $P(a<x<b) = \int_a^b f(x) \cdot dx$,

where $f(x)$ is called a probability density.

f must satisfy: $\underbrace{\int_{-\infty}^{\infty} f(x) \cdot dx = 1.}_{\text{IMPORTANT}}$

(Note general $\int_{\text{all possible values of } x} f(x) \cdot dx = 1$)

$f(x) \geq 0 \quad \forall x.$



We define the mean to be $\mu = \int_{-\infty}^{\infty} f(x) \cdot x \cdot dx$

Ex: The probability density describing the location of a particle is

$$f(x) = \begin{cases} c(x - x^3), & 0 < x < 1. \\ 0, & \text{otherwise}. \end{cases}$$

Find

(a) the normalisation constant c.

(b) the mean.

(c) the probability that the particle is in the interval $(0, 1/2)$.

(a) $1=\int_{-\infty}^{\infty} f(x)\cdot dx = \int_0^1 c(x-x^3)\,dx = c\cdot\int_0^1 (x-x^3)\cdot dx =$

$= c\cdot\left(\frac{1}{2}-\frac{1}{4}\right)=\frac{c}{4} \Rightarrow c=4.$

(b) the mean $=\mu = \int_{-\infty}^{\infty} x\cdot f(x)\cdot dx =$

$=\int_0^1 x\cdot\left(c\cdot(x-x^3)\right)\cdot dx = \int_0^1 x\cdot(4x-4x^3)\,dx=$

$=4\cdot\int_0^1 (x^2-x^4)\cdot dx = 4\cdot\left(\frac{1}{3}-\frac{1}{5}\right)= 8/15$

(c) Probability $(0<x<1/2)=\int_0^{1/2} f(x)\cdot dx=$

$=4\cdot\int_0^{1/2}(x-x^3)\cdot dx = 4\cdot\left(\frac{1}{2}\left(\frac{1}{2}\right)^2-\frac{1}{4}\left(\frac{1}{2}\right)^4\right)=\frac{7}{16}$