1402 Mathematical Methods 2 Notes

Based on the 2016 spring lectures by Prof E Burman

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E Burman 807b
Office hour: Tues 1-2pm
90% Exam
10% Homework

hw set on Monday (moodle)
due on Thurs (following wk)
8 problem sheets.

L1 Methods 2

Multivariable calculus

[Analysis: $f: \mathbb{R} \to \mathbb{R}$]

Modelling → Partial differential equations

eg. $\begin{cases} \vec{\nabla} \times \vec{\nabla} \times \vec{u} + \vec{\nabla} P = f \\ \vec{\nabla} \cdot \vec{u} = 0 \end{cases}$ $\vec{\nabla}\times$, $\vec{\nabla}\cdot$, $\vec{\nabla}$ all represent differential operators.

[Incompressible flow - Stokes equation, or
Electric field eqn - Maxwell's equation.]

Def

A function is an assignment of every element in a set D (range) to one and only one element in R (range).

Example

$f: \mathbb{R} \to \mathbb{R}$ $f(x) = e^x$

$g: (0, \infty) \to \mathbb{R}$ $g(x) = \log x$

In this course we are interested in the sets (or subsets thereof):

the real plane: $\mathbb{R}^2 := \{(x,y) : x, y \in \mathbb{R}\}$

the real space: $\mathbb{R}^3 := \{(x,y,z) : x,y,z \in \mathbb{R}\}$

We wish to take $D = \mathbb{R}^d$ $(d=1,2,3)$ AND $R = \mathbb{R}^d$ $(d=1,2,3)$.

Scalar functions of several variables

$f: \mathbb{R}^2 \to \mathbb{R}$, a function of two variables, $f(x,y)$

$f: \mathbb{R}^3 \to \mathbb{R}$, a function of three variables, $f(x,y,z)$

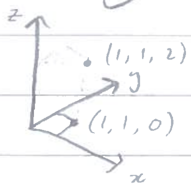
Examples

1) $f:\mathbb{R}^2 \to \mathbb{R}$ $\quad f(x,y) = x+y \quad f(x,y) = 3xy + y^3$

2) $h:\mathbb{R}^3 \to \mathbb{R}$ $\quad h(x,y,z) = x^2 + y^2 + z^2.$

... etc...

Evaluating a multivariable function



$h(1,1,0) = 2$

$h(1,1,2) = 6$

etc.

Graphs of functions

$D = \mathbb{R}$ or $D = \mathbb{R}^2$

Def

For $f:\mathbb{R} \to \mathbb{R}$ the graph consists of all pairs (x,y) s.t. $y = f(x)$.

Def

For $f:\mathbb{R}^2 \to \mathbb{R}$ the graph of f consists of triplets (x,y,z) s.t. $z = f(x,y)$, so $(x,y) \in D$, $z \in R$.

Observation

Graphs are useful for visualisation only for $D = \mathbb{R}$ or $D = \mathbb{R}^2$. If $D = \mathbb{R}^3$ then the graph is $\underbrace{(x,y,z,w)}_{\in \mathbb{R}^4}$ with $w = f(x,y,z)$, this is not practical.

Coordinate planes

Def

A coordinate plane is obtained by setting one of $(x,y,z) \in \mathbb{R}^3$ to zero:

The xy-plane : $(x,y) \in \mathbb{R}^2$, $z = 0$

xz-plane : $(x,z) \in \mathbb{R}^2$, $y = 0$

yz-plane : $(y,z) \in \mathbb{R}^2$, $x = 0$

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Translated coordinate planes are obtained by setting $z = c \in \mathbb{R}$ for the xy-plane case ($y=c$ for xz-plane, $x=c$ for yz plane).

Def (cross-section)

A cross-section of a graph is the intersection of the graph with a given plane.

Example

$z = f(x,y)$ by $ax+by+cz=d$,

find (x,y,z) s.t. $\begin{cases} z = f(x,y) \\ ax+by+cz=d \end{cases}$

$f(x,y) = x^2+y^2$ (x,y,z) $z=f(x,y)$.

Find cross-section of f with the plane $z=2$.

$\begin{cases} z = x^2+y^2 \\ z = 2 \end{cases} \Rightarrow 2 = x^2+y^2$

So the cross-section is a circle of radius 2.

Find cross-section with $x=2$

$\begin{cases} z = x^2+y^2 \\ x = 2 \end{cases} \Rightarrow z = 4+y^2$ (parabola pointing upward).

Special example: the lines on a map.

Over $\mathbb{R}^2$, $f(x,y)$ height over the sea (in z direction).

Take cross-sections with $z = 0m, 1m, 2m, \ldots$

Def

$f : \mathbb{R}^2 \to \mathbb{R}$

The cross-section of f with $z=c$ is called a contour or level curve of f.

3D

Consider $f: \mathbb{R}^3 \to \mathbb{R}$.

We already observed that graphs are impractical

- Restrict the function to a translated ordinate plane $z = c$, so $h(x,y)$ $f(\;, y, \;)$, where (x,y) is function on $\mathbb{R}^2$.

We can get an idea of what f looks like y studying h_c for several .

Def (Level surfaces)

Given $f: \mathbb{R}^3 \to \mathbb{R}$ we define a level surface (level set) of f to be the set $(x, y, \;)$ space such that $f(x,y,z) = c$ for c $\mathbb{R}$ (0 is most important)

Applied example "the level set method"

A surface may be represented y the level set of a function: two phase problem:

$\phi > 0$ in the qud doma

$\phi < 0$ in the gas domain

$\phi = 0$ defines the surface separating the phases.

Alternative coordinate systems

Cartesian coordinates: $\mathbb{R}$ ↳ , $\mathbb{R}$

Depending on the symmetry of the problem (or func) may be useful to consider other ordinat systems.

1). Polar Coordinates

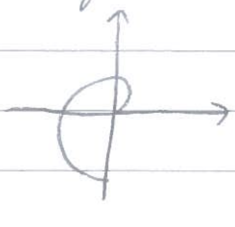
(r, θ) $\begin{cases} r \text{ is the di ce from } p \text{ to } 0 \\ \theta \text{ is the } \quad g \text{ betw } \quad p \text{ ecto and } \quad \text{axis.} \end{cases}$

$$\begin{bmatrix} x = r\cos\theta \\ y = r\sin\theta \end{bmatrix}$$

Example Archimedean spiral.

$r(\theta) = a + b\theta \quad a, b \in \mathbb{R}$

$\Rightarrow \sqrt{x^2+y^2} = a + b\arctan\left(\frac{y}{x}\right)$



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$$f(x,y) = x^2 + y^2 \Rightarrow f(r,\theta) = r^2$$
$$f(x,y) = \frac{y}{x}(x^2+y^2)^{-3/2} \Rightarrow f(r,\theta) = \tan\theta\left(\frac{1}{r^3}\right)$$

2). Cylindrical Coordinates ($\mathbb{R}^3$)

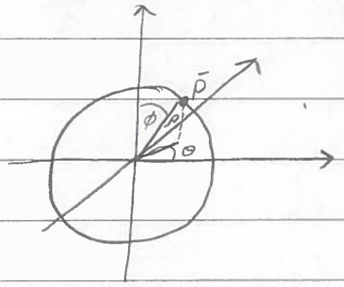
(r, θ, z)

Use polar coordinates in every translated xy-plane and add the z-variable.

Example (The bunsen flame)

The flame is suitable for description in cylindrical coordinates.

3). Spherical coordinates



$$x = \rho\sin\phi\cos\theta$$
$$y = \rho\sin\phi\sin\theta$$
$$z = \rho\cos\phi$$

so $\rho = \sqrt{x^2+y^2+z^2}$ $[dist(\bar{\rho}, \bar{0})]$

$$\theta = \arctan(y/x)$$
$$\phi = \arccos\left(\frac{z}{\sqrt{x^2+y^2+z^2}}\right)$$

$$f(x,y,z) = \frac{1}{2}\ln(x^2+y^2+z^2)$$
$$f(\rho,\theta,\phi) = \frac{1}{2}\ln\rho^2 = \ln\rho.$$

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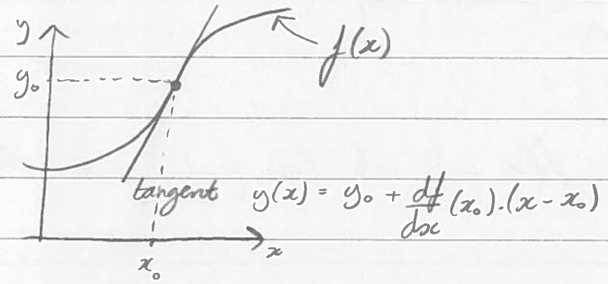
Partial Differentiation

Recall 1D

$f: \mathbb{R} \to \mathbb{R}$

$$\frac{df}{dx}(x) = \lim_{\Delta x \to 0} \left[\frac{f(x+\Delta x) - f(x)}{\Delta x} \right]$$

$\frac{df}{dx}$ is the "slope" of f at x

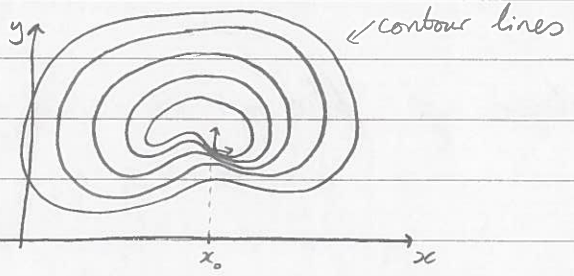


The tangent at $(x_0, f(x_0))$ is the best approximation of f in a neighbourhood of x_0.

Now consider $f: \mathbb{R}^2 \to \mathbb{R}$.
How do we make sense of differentiation now?

Example

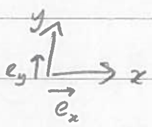
$f(x,y)$



In higher dimensions the slope of f can be different in every direction.

Recall from linear algebra

$\bar{e}_x = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $\bar{e}_y = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$



Reduce $f(x,y)$ to a function of one variable by considering its crossection with $y = y_0$.

$g(x) = f(x, y_0)$

As in the 1D case

$$\frac{dg}{dx} = \lim_{\Delta x \to 0} \left[\frac{g(x+\Delta x) - g(x)}{\Delta x} \right] = \lim_{\Delta x \to 0} \left[\frac{f(x+\Delta x, y_0) - f(x, y_0)}{\Delta x} \right]$$

$$\frac{\partial f}{\partial x}(x_0, y_0) = \frac{dg}{dx}(x_0)$$

$$= \lim_{\Delta x \to 0} \left[\frac{f(x_0+\Delta x, y_0) - f(x_0, y_0)}{\Delta x} \right]$$

$\frac{\partial f}{\partial x}(x_0, y_0)$ is the slope of $f(x, y)$ at (x_0, y_0) in the direction of $\bar{e}_x$.

Similarly, considering the cross-section with $x = x_0$ we have

$$h(y) = f(x_0, y)$$

$$\frac{\partial f}{\partial y}(x_0, y_0) = \frac{dh}{dy}(y_0) = \lim_{\Delta y \to 0} \left[\frac{h(y_0 + \Delta y) - h(y_0)}{\Delta y} \right]$$

$$= \lim_{\Delta y \to 0} \left[\frac{f(x_0, y_0 + \Delta y) - f(x_0, y_0)}{\Delta y} \right]$$

$\frac{\partial f}{\partial y}(x_0, y_0)$ is the slope of $f(x, y)$ at (x_0, y_0) in the direction $\bar{e}_y$.

To summarize: we have shown how to compute the partial derivatives in the directions of the Cartesian axes, $\bar{e}_x$, $\bar{e}_y$.

Do we need to carry out the same argument to obtain the slope in an arbitrary direction $a\bar{e}_x + b\bar{e}_y$?
No. $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ are sufficient to determine the derivative in any direction.

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Functions of two variables

Def

Let $f: \mathbb{R}^2 \to \mathbb{R}$.

The partial derivative of f with respect to x at the point (x_0, y_0) is

$$\frac{\partial f}{\partial x}(x_0, y_0) = \lim_{\Delta x \to 0}\left[\frac{f(x_0+\Delta x, y_0) - f(x_0, y_0)}{\Delta x}\right]$$

The partial derivative of f with respect to y at (x_0, y_0) is

$$\frac{\partial f}{\partial y}(x_0, y_0) = \lim_{\Delta x \to 0}\left[\frac{f(x_0, y_0+\Delta y) - f(x_0, y_0)}{\Delta y}\right]$$

If the limits exist at each point (x_0, y_0) then we can define the part. der. function

$$\frac{\partial f}{\partial x} : \mathbb{R}^2 \to \mathbb{R} \quad , \quad \frac{\partial f}{\partial y} : \mathbb{R}^2 \to \mathbb{R}.$$

Functions of three variables

Def

Let $f: \mathbb{R}^3 \to \mathbb{R}$.

Then the partial derivative of f with respect to z at (x_0, y_0, z_0) is

$$\frac{\partial f}{\partial z}(x_0, y_0, z_0) = \lim_{\Delta z \to 0}\left[\frac{f(x_0, y_0, z_0+\Delta z) - f(x_0, y_0, z_0)}{\Delta z}\right]$$

$\frac{\partial f}{\partial x}(x_0, y_0, z_0)$ and $\frac{\partial f}{\partial y}(x_0, y_0, z_0)$ are as in the 2D case

(keeping x_0, y_0 and z_0 fixed)

Computing partial derivatives

Example

$$f(x, y) = x^2 + \sin\left(\frac{z}{\log y}\right)$$

Let $x_0 = 1$, $y_0 = 2$, $z_0 = 5$

$$\frac{\partial f}{\partial x}(x_0, y_0, z_0) = \lim_{\Delta x \to 0}\left[\frac{(1+\Delta x)^2 + \sin\left(\frac{5}{\log 2}\right) - 1^2 - \sin\left(\frac{5}{\log 2}\right)}{\Delta x}\right]$$

$$= \lim_{\Delta x \to 0}\left[\frac{(\Delta x)^2 + 2\Delta x}{\Delta x}\right]$$

$$= \lim_{\Delta x \to 0}\left[2 + \Delta x\right]$$

$$= 2$$

If $g(x) = f(x, y_0, z_0) = x^2 + c$

$\frac{dg}{dx} = 2x$ $\qquad$ $\frac{dg}{dx}(1) = 2$

The rules of differentiation

Product rule: $(fg)' = f'g + fg'$

Quotient rule: $\left(\frac{f}{g}\right)' = \frac{f'g - fg'}{g^2}$

$$\frac{d}{dx} x^n = n x^{n-1}$$

$$\frac{d}{dx} \sin x = \cos x$$

$$\frac{d}{dx} \log x = \frac{1}{x}$$

$$\frac{d}{dx} e^x = e^x$$

$$\frac{d}{dx} a^x = a^x \log a$$

⋮

etc

⋮

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These rules are still valid for multivariable functions, but we must be careful to keep the right variables constant.

Example

$$f(x, y, z) = x^2 + \sin\left(\frac{z}{\log y}\right)$$

$$m(z) = f(x_0, y_0, z) = x_0^2 + \sin\left(\frac{z}{y_0}\right) = c_1 + \sin\left(\frac{z}{c_2}\right)$$

$$\frac{dm}{dz} = \frac{1}{\log y_0} \cos \frac{z}{\log y_0}$$

$$\text{so } \frac{\partial f}{\partial z}(x_0, y_0, z_0) = \frac{1}{\log y_0} \cos \frac{z_0}{\log y_0}$$

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Partial Differentiation cont.

Todays main topics: - the tangent plane
- directional derivative
- chain rule

Example

$h: \mathbb{R}^3 \to \mathbb{R}$

$h(x, y, z) = x^y + y^z + z^x$ [consider $h(x) = x^{y_0} + y_0^{z_0} + z_0^{x}$]

Compute the partial derivatives of h at (x_0, y_0, z_0)

$$\frac{\partial h}{\partial x}(x_0, y_0, z_0) = y_0 x_0^{y_0 - 1} + 0 + z_0^{x_0} \ln z_0$$
$$= y_0 z_0^{y_0-1} + z_0^{x_0} \ln z_0$$

Higher order derivatives

$f: \mathbb{R}^3 \to \mathbb{R}$, define $g_1 = \frac{\partial f}{\partial x}$, $g_2 = \frac{\partial f}{\partial y}$, $g_3 = \frac{\partial f}{\partial z}$.

If f is smooth we can derive g_1, g_2, g_3 in x, y, z.

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial g_1}{\partial x}, \quad \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial g_2}{\partial x}$$

$$\frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial g_2}{\partial y}, \quad \frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial g_1}{\partial y}$$

In multivariable calculus we get "mixed" derivatives:

$$h_1 = \frac{\partial^2 f}{\partial x \partial y}, \quad h_2 = \frac{\partial^2 f}{\partial y \partial x}$$

If h_1 and h_2 are continuous functions then $h_1 = h_2$.

Theorem

If $\frac{\partial^2 f}{\partial x \partial y}$ and $\frac{\partial^2 f}{\partial y \partial x}$ are continuous functions on all of $\mathbb{R}^2$ then $\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}$.

Example

$f(x,y) = x^5 + ye^x$

$$\frac{\partial f}{\partial x} = 5x^4 + ye^x$$

$$\frac{\partial f}{\partial y} = 0 + e^x$$

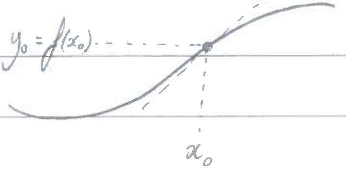
$$\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = 0 + e^x$$

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = e^x$$

So $\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial^2 f}{\partial x \partial y}$ $\square$

Geometrical aspects of partial derivatives

1D $f: \mathbb{R} \to \mathbb{R}$



$t_{x_0}(x) = f(x_0) + f'(x_0)(x - x_0)$

$\frac{\partial f}{\partial x}(x_0)$ is the slope of the tangent line

$t_{x_0}(x)$ is the best affine approximation of locally.
(linear)

$f(x) - t_{x_0}(x) = O((x - x_0)^2)$ as $x \to x_0$

Notation: big Ordo notation

$f(x) = O(g(x))$ as $x \to 0$

then $\exists M > 0, \varepsilon > 0$, s.t. $|f| \leq M|g(x)|$ $\forall x < \varepsilon$

2D $f: R^2 \to R$.

The best linear approximation for f at (x_0, y_0) is a plane: the tangent plane.

The tangent plane must go through $f(x_0, y_0)$.

$$t(x,y) = f(x_0, y_0) + \frac{\partial f}{\partial x}(x_0)(x - x_0) + \frac{\partial f}{\partial y}(y_0)(y - y_0)$$

↑

Equation for the tangent plane.

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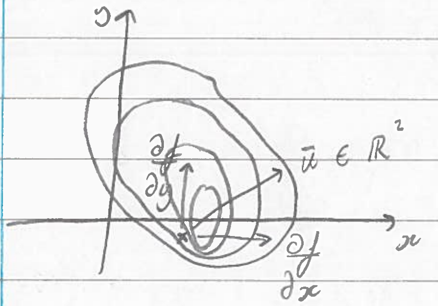
As in 1D, $t(x,y)$ is a good approximation of $f(x,y)$ "close" to (x_0, y_0).

$f(x,y) = t(x,y) + O(|\Delta x|^2)$ as $(x,y) \to (x_0, y_0)$
where $\Delta x = (x - x_0, y - y_0)$.

Directional derivative

So far we have seen $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$

but we said before:
the slope of f is defined in any direction $\vec{n} \in \mathbb{R}^2$.



$\bar{u} = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$, $|\bar{u}| = \sqrt{u_1^2 + u_2^2} = 1$

We want to compute the derivative in the direction $\bar{u}$ at (x_0, y_0)

$$\frac{\partial f}{\partial \bar{u}} = \lim_{\Delta u \to 0} \left[\frac{\overbrace{f(x_0 + u_1 \Delta u, y_0 + u_2 \Delta u)}^{(1)} - f(x_0, y_0)}{\Delta u} \right]$$

Use the tangent plane to approximate (1)

$$f(x_0 + u_1 \Delta u, y_0 + u_2 \Delta u) = t(x_0 + u_1 \Delta u, y_0 + u_2 \Delta u) + O(|\Delta u|^2)$$
$$= f(x_0, y_0) + \frac{\partial f}{\partial x}(x_0, y_0)(u_1 \Delta u) + \frac{\partial f}{\partial y}(x_0, y_0)(u_2 \Delta u) + O(|\Delta u|^2)$$

So $$\frac{\partial f}{\partial \bar{u}} = \lim_{\Delta u \to 0} \left[\frac{f(x_0, y_0) + \frac{\partial f}{\partial x}(x_0, y_0) u_1 \Delta u + \frac{\partial f}{\partial y}(x_0, y_0) u_2 \Delta u + O(|\Delta u|^2) - f(x_0, y_0)}{\Delta u} \right]$$

$$= \lim_{\Delta u \to 0} \left[\frac{\partial f}{\partial x}(x_0, y_0) u_1 + \frac{\partial f}{\partial y}(x_0, y_0) u_2 + \underset{\overset{\downarrow}{0}}{O(\Delta u)} \right]$$

Conclusion

$$\frac{\partial f}{\partial \bar{u}}(x_0, y_0) = \frac{\partial f}{\partial x}(x_0, y_0) u_1 + \frac{\partial f}{\partial y}(x_0, y_0) u_2$$

We have seen:

If $\bar{u} \in \mathbb{R}^2$, $|\bar{u}| = 1$, $\bar{u} = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$

$$\frac{\partial f}{\partial \bar{u}} = \frac{\partial f}{\partial x} u_1 + \frac{\partial f}{\partial y} u_2$$

If $\bar{u} \in \mathbb{R}^3$, $|u| = 1$, $u \begin{pmatrix} {}_1 \\ \\ {}_3 \end{pmatrix}$

$$\frac{\partial f}{\partial \bar{u}} = \frac{\partial f}{\partial x} u_1 + \frac{\partial f}{\partial y} u_2 + \frac{\partial f}{\partial z} u_3$$

Recall linear algebra $\bar{u} \in \mathbb{R}^2$, $\bar{v} \in \mathbb{R}^2$

$\bar{u} \cdot \bar{v} = u_1 v_1 + u_2 v_2$

If $\bar{v} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$ with $v_1 = \frac{\partial f}{\partial x}$ and $v_2 = \frac{\partial f}{\partial y}$

Then $\frac{\partial f}{\partial \bar{u}} = \bar{u} \cdot \bar{v}$

Definition (Gradient in Cartesian coordinates)

Let $f : \mathbb{R}^3 \to \mathbb{R}$. The gradient of f is denoted $\bar{\nabla} f$, at the point (x_0, y_0, z_0) is given by the vector

$$\bar{\nabla} f(x_0, y_0, z_0) = \frac{\partial f}{\partial x}(x_0, y_0, z_0) e_x + \frac{\partial f}{\partial y}(x_0, y_0, z_0) \bar{e}_y + \frac{\partial f}{\partial z}(x_0, y_0, z_0) \bar{e}_z$$

Definition (Directional derivative)

Let $f : \mathbb{R}^3 \to \mathbb{R}$. For a given point $(x_0, y_0, {}_0)$ and given vector $\bar{u} = u_1 \bar{e}_x + u_2 \bar{e}_y + u_3 \bar{e}_z$ $[|\bar{u}| = 1]$, the rectiona derivative of f at (x_0, y_0, z_0) in the direction $\bar{u}$ is

$$\frac{\partial f}{\partial \bar{u}}(x_0, y_0, z_0) = \bar{u} \cdot \bar{\nabla} f$$

$$= u_1 \frac{\partial f}{\partial x} + u_2 \frac{\partial f}{\partial y} + u_3 \frac{\partial f}{\partial z}$$

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Example

$$f(x,y) = \sin(xe^{-y})$$

$$(x_0, y_0) = (\pi, 0)$$

$$\bar{u} = \frac{2}{\sqrt{5}}\bar{e}_x + \frac{1}{\sqrt{5}}\bar{e}_y$$

find $\frac{\partial f}{\partial u}(x_0, y_0)$.

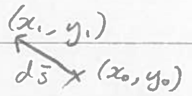
$$\frac{\partial f}{\partial x} = e^{-y}\cos(xe^{-y})$$

$$\frac{\partial f}{\partial y} = e^{-y}x\cos(xe^{-y})$$

$$\bar{u}\cdot\nabla f = \frac{2}{\sqrt{5}}\cos\pi + \frac{1}{\sqrt{5}}(-1\cdot\pi\cos\pi)$$

$$= \frac{-2}{\sqrt{5}} + \frac{\pi}{\sqrt{5}} = \frac{(\pi-2)}{\sqrt{5}}$$

$$df = \nabla f \cdot d\bar{s}$$



$$\left[df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy\right]$$

Asume f is witten in polar coordinates, $f(r,\theta)$,

so $df = \frac{\partial f}{\partial r}dr + \frac{\partial f}{\partial \theta}d\theta$

[The problem is that x,y are lengths but r,θ are not both lengths as θ is an angle.]

$$ds \neq (dr, d\theta)$$

$$\nabla f = \begin{pmatrix} g_1 \\ g_2 \end{pmatrix} \qquad df = \nabla f \cdot d\bar{s}$$

$$= g_1\,dr + g_2\,r\,d\theta$$

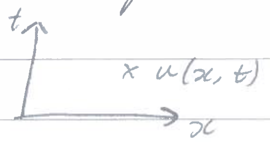
$$= \frac{\partial f}{\partial r}dr + \frac{\partial f}{\partial \theta}d\theta$$

so $g_1 = \frac{\partial f}{\partial r}$, $g_2 r = \frac{\partial f}{\partial \theta}$

so $\nabla f(r,\theta) = \frac{\partial f}{\partial r}\bar{e}_r + \frac{1}{r}\frac{\partial f}{\partial \theta}\bar{e}_\theta$

The chain rule

It is often useful to think of a function as a composition of two functions.



u is a property of some pollutant, a particle will be at position $x(t)$ at time t.

$u(x,t) = u(x(t), t)$

1D $f, g : \mathbb{R} \to \mathbb{R}$

$f \circ g : \mathbb{R} \to \mathbb{R}$

$(f \circ g)(x) = f(g(x))$

Chain rule in 1D: $\frac{d}{dx}(f \circ g)(x) = \frac{df}{dx}(g(x)) \cdot \frac{dg}{dx}$

Example

$f(x) = \sin x \quad , \quad g(x) = \log x$

$(f \circ g)(x) = \sin(\log x)$

$(f \circ g)(x) : (0, \infty) \to \mathbb{R}$

$\frac{df}{dx} = \cos x \quad , \quad \frac{dg}{dx} = \frac{1}{x}$

so $\frac{d}{dx}(f \circ g)(x) = \cos(\log x) \cdot \frac{1}{x}$

Heuristic derivation of the chain rule in $\mathbb{R}^3$.

$f : \mathbb{R}^3 \to \mathbb{R} \quad , \quad f(x, y, t)$

$x : \mathbb{R} \to \mathbb{R} \qquad x = x(t)$

$y : \mathbb{R} \to \mathbb{R} \qquad y = y(t)$

so $f(x, y, t) = f(x(t), y(t), t) = \omega(t)$

$\omega : \mathbb{R} \to \mathbb{R}$

Compute $\frac{d\omega}{dt}$.

$$\frac{d\omega}{dt}(t) = \lim_{\Delta t \to 0}\left[\frac{\omega(t+\Delta t) - \omega(t)}{\Delta t}\right]$$

$$= \lim_{\Delta t \to 0} \left[\frac{f(x(t+\Delta t), y(t+\Delta t), t+\Delta t) - f(x(t), y(t), t)}{\Delta t} \right]$$

First:

Use the tangent line approximation in

$$x(t+\Delta t) = x(t) + \frac{dx}{dt}(t) \cdot \Delta t + O(\Delta t^2)$$

$$y(t+\Delta t) = y(t) + \frac{dy}{dt}(t) \cdot \Delta t + O(\Delta t^2)$$

$$\frac{dw}{dt} = \lim_{\Delta t \to 0} \left[\frac{f(x(t) + \frac{dx}{dt} \cdot \Delta t + O(\Delta t^2), y(t) + \frac{dy}{dt} \cdot \Delta t + O(\Delta t^2), t + \Delta t) - f(x(t), y(t), t)}{\Delta t} \right]$$

This "looks" like a directional derivative in the direction $\left(\frac{dx}{dt}, \frac{dy}{dt}, 1\right)$

Now:

Use the tangent plane approximation

$$f(x+a, y+b, t+c) \approx f(x,y,t) + \frac{df}{dx}(x,y,t) \cdot a + \frac{df}{dy}(x,y,t) \cdot b + \frac{df}{dt}(x,y,t) \cdot c$$

The $f(x(t), y(t), t)$ terms will cancel.

$$\frac{dw}{dt} = \lim_{\Delta t \to 0} \left[\frac{\frac{\partial f}{\partial x} \frac{dx}{dt} \Delta t + O(\Delta t^2) + \frac{\partial f}{\partial y} \frac{dy}{dt} \Delta t + O(\Delta t^2) + \frac{\partial f}{\partial t} \Delta t}{\Delta t} \right]$$

$$= \lim_{\Delta t \to 0} \left[\frac{\partial f}{\partial x} \frac{dx}{dt} + \frac{\partial f}{\partial y} \frac{dy}{dt} + \frac{\partial f}{\partial t} + O(\Delta t) \right]$$

We have shown

$$\frac{df}{dt}(x(t), y(t), t) = \frac{\partial f}{\partial x} \frac{dx}{dt} + \frac{\partial f}{\partial y} \frac{dy}{dt} + \frac{\partial f}{\partial t}$$

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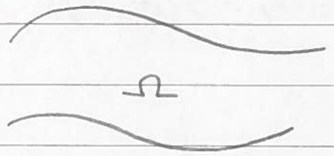
The Chain Rule

Example: modeling

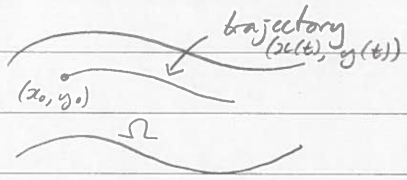
- Contaminated sand
- Strength of contamination of a particle $= f(t)$, we assume $\frac{df}{dt} = 0$ $*$
- The sand is spread over a domain Ω.



The contamination at the point $(x,y) \in \Omega$ at time t is $f(x,y,t)$

$\frac{\partial f}{\partial t} = 0$, $\frac{\partial f}{\partial x} \neq 0$, $\frac{\partial f}{\partial y} \neq 0$.

- Let Ω be the surface of a river at $t=0$, the contamination is distributed as $f(x,y,0)$.



Our contamination function must now be written $f(x(t), y(t), t)$.

- By $(*)$, $0 = \frac{df}{dt}(x(t), y(t), t)$

From last lecture:

$$\frac{df}{dt}(x(t),y(t),t) = \frac{\partial f}{\partial t} + \overbrace{\frac{dx}{dt}}^{u_1}\frac{\partial f}{\partial x} + \overbrace{\frac{dy}{dt}}^{u_2}\frac{\partial f}{\partial y} = 0$$

We know the velocity of the river, $\bar{u} = (u_1, u_2)$.

$$\begin{cases} \frac{\partial f}{\partial t} + \bar{u} + \bar{\nabla} f = 0 & \bar{u} : R^2 \to R^2 \\ f(x,y,0) = f_0(x,y) & \text{flow vector field (incompressible)} \end{cases}$$

This is the transport equation.

Theorem (chain rule)

Let $f : \mathbb{R}^2 \to \mathbb{R}$, $x : \mathbb{R} \to \mathbb{R}$, $y : \mathbb{R} \to \mathbb{R}$

For the function $w : \mathbb{R} \to \mathbb{R}$, $w(t) := f(x(t), y(t))$ we have

$$\frac{dw}{dt}(t) = \frac{\partial f}{\partial x}(x(t), y(t)) \frac{dx}{dt}(t) + \frac{\partial f}{\partial y}(x(t), y(t)) \frac{dy}{dt}(t)$$

On vector form:

$f : \mathbb{R}^d \to \mathbb{R}$ $d \geq 1$, $\bar{g} : \mathbb{R} \to \mathbb{R}^d$, $w(t) := f(\bar{g}(t))$

$$\frac{d\bar{g}}{dt} = \left(\frac{dg_1}{dt}, \frac{dg_2}{dt}, \dots, \frac{dg_d}{dt} \right)$$

Recall the definition of $\bar{\nabla} f$

$$\frac{dw}{dt}(t) = \frac{d\bar{g}}{dt} \cdot \bar{\nabla} f(\bar{g}(t))$$

Example

$f(x,y) = xe^{xy}$

$x(t) = t^2$, $y(t) = t^{-1}$

$w(t) = f(x(t), y(t))$

Compute $\frac{dw}{dt}$ using the chain rule or directly.

$\frac{\partial f}{\partial x} = e^{xy} + xye^{xy}$ $\qquad$ $\frac{dx}{dt} = 2t$

$\frac{\partial f}{\partial y} = x^2 e^{xy}$ $\qquad$ $\frac{dy}{dt} = -\frac{1}{t^2}$

So $\frac{dw}{dt} = \frac{\partial f}{\partial x}(x(t), y(t) \frac{dx}{dt} + \frac{\partial f}{\partial y}(x(t), y(t)) \frac{dy}{dt}$

$$= \left(e^{x(t)y(t)} + x(t)y(t)e^{x(t)y(t)} \right) 2t + (x(t))^2 e^{x(t)y(t)} \left(-\frac{1}{t^2} \right)$$

Replace $x(t)$ by t^2 and $y(t)$ by $1/t$.

So $\frac{dw}{dt} = (e^t + te^t)2t + t^4 e^t \left(-\frac{1}{t^2} \right)$

$$= e^t(2t + 2t^2 - t^2) = e^t(2t + t^2)$$

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Replacing directly in $f(x,y)$ we have

$$w(t) = t^2 e^t$$

$$\frac{dw}{dt}(t) = 2te^t + t^2e^t = e^t(2t+t^2)$$

Example

$$f: \mathbb{R}^2 \to \mathbb{R} \quad , \quad x: \mathbb{R}^2 \to \mathbb{R} \quad , \quad y: \mathbb{R}^2 \to \mathbb{R}$$

$$f(x,y) = x \sin y$$

$$x(s,t) = st$$

$$y(s,t) = s-t$$

$$w(s,t) = f(x(s,t), y(s,t))$$

$$\frac{\partial w}{\partial s} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial s} \quad ①$$

$$\frac{\partial w}{\partial t} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial t} \quad ②$$

$$\frac{\partial f}{\partial x} = \sin y \quad , \quad \frac{\partial f}{\partial y} = x\cos y$$

$$\frac{\partial x}{\partial s} = t \quad , \quad \frac{\partial x}{\partial t} = s \quad , \quad \frac{\partial y}{\partial s} = 1 \quad , \quad \frac{\partial y}{\partial t} = -1$$

① : $\frac{\partial w}{\partial s} = t\sin(s-t) + st\cos(s-t)$

② : $\frac{\partial w}{\partial t} = s\sin(s-t) - st\cos(s-t)$

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Geometric significance of the gradient

Recall:

If $f : \mathbb{R}^3 \to \mathbb{R}$

Gradient

$$\bar{\nabla} f : \mathbb{R}^3 \to \mathbb{R}^3$$

$$\bar{\nabla} f = \frac{\partial f}{\partial x}\bar{e}_x + \frac{\partial f}{\partial y}\bar{e}_y + \frac{\partial f}{\partial z}\bar{e}_z$$

Directional derivative

$v \in \mathbb{R}^3, \ |v| = 1$

$$\frac{\partial f}{\partial v} = \bar{v} \cdot \bar{\nabla} f \qquad \text{[slope in the } \bar{v} \text{ direction]}$$

In what direction is the directional derivative optimum/maximal.

$$\frac{\partial f}{\partial \bar{v}} = \bar{v} \cdot \bar{\nabla} f = |\bar{v}| \cdot |\bar{\nabla} f| \cdot \cos\theta$$

We know $\cos\theta \leq 1$

so $|\bar{v}||\bar{\nabla} f|\cos\theta \leq |\bar{v}||\bar{\nabla} f|$

$= |\bar{v}||\bar{\nabla} f|$ when $\theta = 0$

$\theta = 0 \Rightarrow \bar{v}$ points in the same direction as $\bar{\nabla} f$

So $\bar{v} = \dfrac{\bar{\nabla} f}{|\bar{\nabla} f|} \rightarrow |\bar{u}| = 1$

The maximum is $\bar{v}$ s.t. $\theta = 0$.

$$\frac{\partial f}{\partial \bar{v}} = \frac{\bar{\nabla} f}{|\bar{\nabla} f|} \cdot \bar{\nabla} f = \frac{|\bar{\nabla} f|^2}{|\bar{\nabla} f} = |\bar{\nabla} f|$$

The directional deriviative takes its larges value when it points in the direction $\dfrac{\bar{\nabla} f}{|\bar{\nabla} f|}$.

Its value is $|\bar{\nabla} f|$

Relation between the gradient and tangent planes / contours

Example

Let $f(x, y, z) = x^2 + y^2 + z^2$

Contour lines : $f(x, y, z) = c^2 \Rightarrow x^2 + y^2 + z^2 = c^2$

The contour lines are spheres with radius c.

Normal vector : $\bar{r} = \dfrac{x \underset{\sim}{e_x} + y \underset{\sim}{e_y} + z \underset{\sim}{e_z}}{\sqrt{x^2 + y^2 + z^2}}$

$$\bar{\nabla} f = 2x \bar{e}_x + 2y \bar{e}_y + 2z \bar{e}_z$$

⚠ $\dfrac{\bar{\nabla} f}{|\bar{\nabla} f|} = \dfrac{x \bar{e}_x + y \bar{e}_y + z e_z}{\sqrt{x^2 + y^2 + z^2}}$ ⇐ factor of 2 ?!

Observation

The gradient points in the direction of the normal to the contour.

Example

Graph of $f : \mathbb{R}^2 \to \mathbb{R}$

The graph : $(x, y, f(x, y)) \in \mathbb{R}^3$

The graph ⇒ a surface in $\mathbb{R}^3$.

The tangent plane at (x_0, y_0)

$$t(x, y) = f(x_0, y_0) + \frac{\partial f}{\partial x}(x_0, y_0)(x - x_0) + \frac{\partial f}{\partial y}(x_0, y_0)(y - y_0)$$

Any vector in the tangent plane is tangent to the surface at (x_0, y_0).

Derive the normal of a surface defined by a graph.

Construct two vectors in the plane.

$$\bar{t}_1 = (x_0 + 1, y_0, t(x_0 + 1, y_0)) - (x_0, y_0, t(x_0, y_0))$$
$$= \left(1, 0, \frac{\partial f}{\partial x}(x_0, y_0)\right.$$

$$\bar{t}_2 = \left(0, 1, \frac{\partial f}{\partial y}(x_0, y_0)\right)$$

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<u>The normal</u> (not normalised)

$$\bar{n} = \vec{t}_1 \times \vec{t}_2 = \begin{pmatrix}1\\0\\\frac{\partial f}{\partial x}\end{pmatrix} \times \begin{pmatrix}0\\1\\\frac{\partial f}{\partial y}\end{pmatrix}$$

$$= \bar{e}_x\left(-\frac{\partial f}{\partial x}\right) - \bar{e}_y\left(\frac{\partial f}{\partial y}\right) + \bar{e}_z$$

After normalisation

$$* \quad \bar{n} = \frac{-\frac{\partial f}{\partial x}\bar{e}_x - \frac{\partial f}{\partial y}\bar{e}_y + \bar{e}_z}{\sqrt{\left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2 + 1}}$$

We introduce $G(x,y,z) = z - f(x,y)$

$G : \mathbb{R}^3 \to \mathbb{R}$

$(x,y,z) \in \mathbb{R}^3$ s.t. $G(x,y,z) = 0$

corresponds to the graph $z - f(x,y) = 0 \Rightarrow z = f(x,y)$

$(x,y,z) \in \mathbb{R}^3$ satisfies $(x,y,f(x,y))$

$$\bar{\nabla} G = -\frac{\partial f}{\partial x}\bar{e}_x - \frac{\partial f}{\partial y}\bar{e}_y + \bar{e}_z$$

Compare surface normal.

<u>General result:</u>

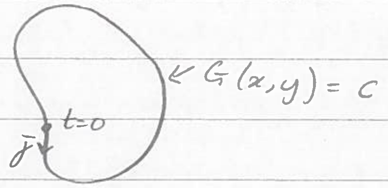
$G : \mathbb{R}^3 \to \mathbb{R}$

$S := \{(x,y,z) : G(x,y,z) = c,\ c \in \mathbb{R}\}$

Then the unit <u>normal</u> to S at some point $\bar{x} \in S$ may be written $\frac{\bar{\nabla} G(\bar{x})}{|\bar{\nabla} G(\bar{x})|}$.

<u>In 2D</u>

$G : \mathbb{R}^2 \to \mathbb{R}$



The contour is parameterised by $\bar{\gamma}(t)$ s.t. $0 \leq t \leq 1$ $\quad \bar{\gamma}(0) = \bar{\gamma}(1)$

Tangent line $\lim_{\Delta t \to 0}\left[\frac{\gamma(t+\Delta t) - \gamma(t)}{\Delta t}\right] = \bar{\gamma}'(t)$. Assume $\bar{\gamma}'(t) \geq \bar{\gamma}_0 > 0$

$$G(\bar{\gamma}(t)) = c$$

$$\frac{d}{dt} G(\bar{\gamma}(t)) = 0$$

$$\bar{\nabla} G(\bar{\gamma}(t)) \cdot \underbrace{\gamma'(t)}_{\text{never zero}} = 0$$

$\bar{\nabla} G \perp \gamma'(t) \Rightarrow \bar{\nabla} G$ normal to the contour.

Summary

$f : \mathbb{R}^3 \to \mathbb{R}$

$\bar{\nabla} f : \mathbb{R}^3 \to \mathbb{R}^3$, $\quad \bar{\nabla} f = \frac{\partial f}{\partial x} \bar{e}_x + \frac{\partial f}{\partial y} \bar{e}_y + \frac{\partial f}{\partial z} \bar{e}_z$

$\bar{v} \in \mathbb{R}^3$, $|\bar{v}| = 1$, $\quad \frac{\partial f}{\partial \bar{v}} = \bar{v} \cdot \bar{\nabla} f$

If $\bar{\nabla} f(\bar{x}_0) \neq \bar{0}$ then (i) $\bar{\nabla} f$ points in the direction of greatest increase of f at $\bar{x}_0$.

(ii) $|\bar{\nabla} f(\bar{x}_0)|$ is the maximum rate of change at $\bar{x}_0$.

(iii) $\bar{\nabla} f(\bar{x}_0)$ is normal to the level surface of f at $\bar{x}_0$.

(iv) the tangent plane at (x_0, y_0, z_0),

$(x, y, z) \in \mathbb{R}^3$ s.t. $\bar{\nabla} f(x_0, y_0, z_0) \cdot (x - x_0, y - y_0, z - z_0) = 0$

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The mean value theorem and taylor's formula

Recall 1D:

$f : \mathbb{R} \to \mathbb{R}$, f is differentiable

$f(x) - f(x_0) = f'(\eta)(x - x_0)$, $\eta \in \mathbb{R}$ between x and x_0.

To generalise to $\mathbb{R}^3$, let $f : \mathbb{R}^3 \to \mathbb{R}$, $x, x_0 \in \mathbb{R}^3$, f is differentiable ($\bar{\nabla} f$ exists everywhere)

Let $h(t) = f(\bar{x}_0 + t(\bar{x} - \bar{x}_0))$

$= f \circ \bar{g}(t)$, $\bar{g}(t) = \bar{x}_0 + t(\bar{x} - \bar{x}_0)$

Apply MVT to $h : \mathbb{R} \to \mathbb{R}$

$h(1) - h(0) = \frac{dh}{dt}(\eta)$ $\eta \in [0,1]$

$$f(\bar{x}) - f(\bar{x}_0) \underset{\substack{\uparrow \\ \text{chain rule}}}{=} \bar{\nabla} f(\bar{g}(\eta)) \cdot \bar{g}'(t) = \frac{\partial f}{\partial x}(\bar{g}(\eta))(x - x_0)$$

$$+ \frac{\partial f}{\partial y}(\bar{g}(\eta))(y - y_0) + \frac{\partial f}{\partial z}(\bar{g}(\eta))(z - z_0)$$

Mean value theorem for multivariable functions:

$$f(\bar{x}) - f(\bar{x}_0) = \bar{\nabla} f(\bar{x} + \eta(\bar{x} - \bar{x}_0)) \cdot (\bar{x} - \bar{x}_0)$$

Taylor's formula

Recall 1D:

$f : \mathbb{R} \to \mathbb{R}$ f twice differentiable

$f(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{1}{2} f''(\eta)(x - x_0)^2$ η between x and x_0

Generalisation to higher dimensions $f : \mathbb{R}^3 \to \mathbb{R}$ (f twice diff.)

Let $h(t) = f(\bar{x}_0 + t(\bar{x} - \bar{x}_0)$

Given $\bar{x}, \bar{x}_0 \in \mathbb{R}^3$

$\bar{g}(t) = \bar{x}_0 + t(\bar{x} - \bar{x}_0)$ $\bar{g} : \mathbb{R} \to \mathbb{R}^3$

As before, apply the 1D formula to $h(t)$.

$\bar{x}_0 = 0$, $\bar{x} = 1$, $f = h$

$$h(1) = \underbrace{h(0) + h'(0)(1 - 0)}_{\text{compare MVT}} + \frac{1}{2} h''(\eta)(1 - 0)^2 \quad (*)$$

$\eta \in [0, 1]$

$\bar{x} = (x_1, x_2, x_3)$ (ie $x = x_1$, $y = x_2$, $z = x_3$)

Recall from chain rule

$$h'(t) = \nabla f(\bar{\gamma}(t)) \cdot \bar{\gamma}(t)$$

$$= \sum_{j=1}^{3} \frac{\partial f}{\partial x_j}(\bar{\gamma}(t)) \cdot \gamma_j'(t)$$

$\bar{\gamma}(t) = \bar{x}_0 + t(\bar{x} - x_0)$ $\gamma : \mathbb{R} \to \mathbb{R}^3$

$\bar{\gamma}'(t) = \bar{x} - \bar{x}_0$

$\bar{\gamma}'(0) = \bar{x} - \bar{x}_0$

$$h'(0) = \sum_{j=1}^{3} \frac{\partial f}{\partial x_j}(\bar{x}_0)(\bar{x} - \bar{x}_0)$$

$$h''(t) = \frac{d}{dt} h'(t) = \frac{d}{dt} \sum_{j=1}^{3} \frac{\partial f}{\partial x_j}(\bar{\gamma}(t))\gamma_j'(t) \quad (**)$$

Fix j and study

$$\frac{d}{dt}\frac{\partial f}{\partial x_j}(\bar{\gamma}(t))\bar{\gamma}_j'(t) \overset{\text{product rule}}{=} \frac{d}{dt}\frac{\partial f}{\partial x_j}(\bar{\gamma}(t))\gamma_j'(t) + \frac{\partial f}{\partial x_j}(\bar{\gamma}(t))\frac{d}{dt}\bar{\gamma}_j'(t)$$

Since $\bar{\gamma}(t) = \bar{x}_0 + t(\bar{x} - \bar{x}_0)$

$\bar{\gamma}'(t) = \bar{x} - \bar{x}_0$

$\bar{\gamma}''(t) = 0$

set $g_i = \frac{\partial f}{\partial x_j}$

$$\left(\frac{d}{dt} g_j(\bar{\gamma}(t))\right)\bar{\gamma}_j \overset{\text{chain rule}}{=} \left(\sum_{i=1}^{3} \frac{\partial g_j}{\partial x_i}\gamma_i'(t)\right)\gamma_j'(t)$$

$$= \left(\sum_{i=1}^{3} \frac{\partial^2 f}{\partial x_i \partial x_j}\gamma_i'(t)\right)\gamma_j'(t)$$

$\bar{\gamma}(t) = (\gamma_1(t), \gamma_2(t), \gamma_3(t))$

By summing over j

$$(**) = \sum_{j=1}^{3}\sum_{i=1}^{3} \frac{\partial^2 f}{\partial x_i \partial x_j}(\bar{\gamma}(t))\bar{\gamma}_i'(t)\gamma_j'(t)$$

$$= \sum_{j=1}^{3}\sum_{i=1}^{3} \underbrace{\frac{\partial^2 f}{\partial x_i \partial x_j}}_{H_{ij}(\bar{\gamma}(t))}(\bar{\gamma}(t))(x_i - x_{0i})(x_j - x_{0j})$$

$$= \sum_{j=1}^{3}\sum_{i=1}^{3} H_{ij}(\bar{\gamma}(t))(x_i - x_{0i})(x_j - x_{0j}) \quad (***)$$

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Let H be the matrix (3×3) with coefficients $\{H_{ij}\}_{ij=1}^{3}$

Let $\bar{x}-\bar{x}_0 = (x_1 - x_{01}, x_2 - x_{02}, x_3 - x_{03})$

Show that $(***) = (\bar{x}-\bar{x}_0)^T H(\bar{x}-\bar{x}_0)$

Conclusion

$f(\bar{x}) = f(\bar{x}_0) + \bar{\nabla} f(\bar{x}_0)(\bar{x}-\bar{x}_0) + \frac{1}{2}(\bar{x}-\bar{x}_0)^T H(\bar{J}(\eta))(\bar{x}-\bar{x}_0)$
where $\eta \in [0,1]$.

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Taylor's formula

Assume $f: \mathbb{R}^d \to \mathbb{R}$ and f twice differentiable.
Then for $\bar{x} \in \mathbb{R}^d$ and $\bar{y} \in \mathbb{R}^d$
$[\bar{x} = (x_1, x_2, \dots, x_d), \quad \bar{y} = (y_1, y_2, \dots, y_d)]$
there holds

$$f(\bar{x}) = f(\bar{y}) + \bar{\nabla} f(\bar{y}) \cdot (\bar{x} - \bar{y}) + \frac{1}{2}(\bar{x} - \bar{y})^T H(\bar{\xi}(\eta))(\bar{x} - \bar{y})$$

where $\bar{\xi}(\eta) = \bar{y} + \eta(\bar{x} - \bar{y}), \quad \eta \in [0, 1]$.
$H(\bar{\xi}(\eta))$ is the Hessian matrix evaluated at $\bar{\xi}(\eta)$.
H is defined by the coeffs $\{H_{ij}\}_{i,j=1}^{d}$

$$H_{ij}(\bar{\xi}(\eta)) = \frac{\partial^2 f}{\partial x_i \partial x_j}(\bar{\xi}(\eta)).$$

Example

$d = 2, \quad \bar{x} = (x_1, x_2), \quad \bar{y} = (y_1, y_2).$

$$\bar{\nabla} f(y_1, y_2) = \frac{\partial f}{\partial x_1}(y_1, y_2)\bar{e}_x + \frac{\partial f}{\partial x_2}(y_1, y_2)\bar{e}_y$$

$$f(x_1, x_2) = f(y_1, y_2) + \frac{\partial f}{\partial x_1}(y_1, y_2)(x_1 - y_1) + \frac{\partial f}{\partial x_2}(y_1, y_2)(x_2 - y_2)$$
$$+ \frac{1}{2}\frac{\partial^2 f}{\partial x_1^2}(\bar{\xi}(\eta))(x_1 - y_1) + \frac{\partial^2 f}{\partial x_1 \partial x_2}(\bar{\xi}(\eta))(x_1 - y_1)(x_2 - y_2) + \frac{1}{2}\frac{\partial^2 f}{\partial x_2^2}(x_2 - y_2)^2$$

$$\begin{pmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{pmatrix}\begin{pmatrix} x_1 - y_1 \\ x_2 - y_2 \end{pmatrix} = \begin{pmatrix} H_{11}(x_1 - y_1) + H_{12}(x_2 - y_2) \\ H_{21}(x_1 - y_1) + H_{22}(x_2 - y_2) \end{pmatrix}$$

← explanation of last line above.

$$(x_1 - y_1, x_2 - y_2) \cdot \begin{pmatrix} H_{11}(x_1 - y_1) + H_{12}(x_2 - y_2) \\ H_{21}(x_1 - y_1 + H_{22}(x_2 - y_2) \end{pmatrix}$$
$$= H_{11}(x_1 - y_1)^2 + H_{12}(x_2 - y_2)(x_1 - y_1) + H_{21}(x_1 - y_1)(x_2 - y_2) + H_{22}(x_2 - y_2)^2$$

Recall $H_{12} = \frac{\partial^2 f}{\partial x_1 \partial x_2} = \frac{\partial^2 f}{\partial x_2 \partial x_1} = H_{12}$

$$= H_{11}(x_1 - y_1)^2 + 2H_{12}(x_1 - y_1)(x_2 - y_2) + H_{22}(x_2 - y_2)^2$$

Take $\eta = 0$ in Taylor's formula to obtain
$\bar{\xi}(0) = \bar{y} + 0(\bar{x} - \bar{y}) = \bar{y}$.

$$T_{\bar{y}}(\bar{x}) = f(\bar{y}) + \bar{\nabla} f(\bar{y})(\bar{x} - \bar{y}) + \frac{1}{2}(\bar{x} - \bar{y})^T H(\bar{y})(\bar{x} - \bar{y})$$

$T_{\bar{y}}(\bar{x}) \approx f(\bar{x})$ in a neighbourhood of $\bar{y}$.

If f is smooth

$$|f(\bar{x}) - T_{\bar{y}}(\bar{x})| = O(|\bar{x} - \bar{y}|^3)$$

The hidden constant depends on the third order derivatives of f.

The tangent plane revisited: if we drop the 2nd order term in Taylor's formula we get:

$$t(\bar{x}) = f(\bar{y}) + \bar{\nabla} f(\bar{y})(\bar{x} - \bar{y}).$$

We recognise the tangent plane immediately

$$|f(\bar{x}) - t_{\bar{y}}(\bar{x})| = |\tfrac{1}{2}(\bar{x} - \bar{y})^T H(\bar{\xi}(\eta))(\bar{x} - \bar{y})| = (*)$$

The taylor polynomial (2nd order)

How big can $(*)$ be?

$$(*) \leq \max_{(i,j)} \max_{(\eta)} \left| \frac{\partial^2 f}{\partial x_i \partial x_j}(\bar{\xi}(\eta)) \right| |\bar{x} - \bar{y}|^2$$

$$= O(|\bar{x} - \bar{y}|^2)$$

Example: Find the 2nd order Taylor polynomial of $f(x,y) = e^x(1+y)$ at $(x,y) = (0,1)$

$$T_{(0,1)}(x,y) = f(0,1) + \overbrace{\bar{\nabla} f(0,1)\left[(x-0)\bar{e}_x + (y-1)\bar{e}_y\right]}^{(**)}$$
$$+ \underbrace{\frac{1}{2}(x\bar{e}_x + (y-1)\bar{e}_y) \cdot \left[H(0,1)\, x\bar{e}_x + (y-1)e_y\right]}_{(***)}$$

$$f(0,1) = e^0(1+1) = 2$$

$$\bar{\nabla} f(0,1) = \underbrace{e^0(1+1)}_{\frac{\partial f}{\partial x}(0,1)}\bar{e}_x + \underbrace{e^0}_{\frac{\partial f}{\partial y}(0,1)}\bar{e}_y$$

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$$H(x,y) = \begin{pmatrix} e^{x}(1+y) & e^{x} \\ e^{x} & 0 \end{pmatrix}$$

$$H(0,1) = \begin{pmatrix} 2 & 1 \\ 1 & 0 \end{pmatrix}$$

$(**) = 2x + (y-1)$

Note $(***) = \frac{1}{2}\begin{pmatrix} x \\ y-1 \end{pmatrix}^{T} \underbrace{\begin{pmatrix} 2 & 1 \\ 1 & 0 \end{pmatrix}}_{H(\bar{f}(0))} \begin{pmatrix} x \\ y-1 \end{pmatrix}$

$$\begin{pmatrix} 2 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} x \\ y-1 \end{pmatrix} = \begin{pmatrix} 2x + (y-1) \\ x \end{pmatrix}$$

$$\begin{pmatrix} x & y-1 \end{pmatrix} \cdot \begin{pmatrix} 2x + (y-1) \\ x \end{pmatrix} = 2x^{2} + 2x(y-1)$$

So $(***) = \frac{1}{2}(2x^{2} + 2x(y-1) = x^{2} + x(y-1)$

So $T_{(0,1)}(x,y) = \cancel{1} + (2x + y - \cancel{1}) + (x^{2} + xy - x)$

$= x^{2} + xy + x + y + 1$

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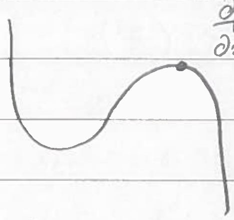
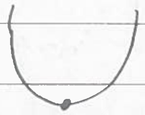
Local Extrema, Saddle Points

In 1D, $f: R \to R$



$\frac{\partial f}{\partial x} = 0, \quad \frac{\partial^2 f}{\partial x^2} < 0$



$\frac{\partial f}{\partial x} = 0, \quad \frac{\partial^2 f}{\partial x^2} > 0$

$\frac{\partial f}{\partial x} = 0, \quad \frac{d^2 f}{d x^2} = 0$

Def

A point $\bar{x} \in R^d$ is called a "critical" (or "stationary") point if $\bar{\nabla} f(\bar{x}) = 0$

Local extrema are critical points (f smooth) but not necessarily vice versa.

Lemma

Suppose $f: R^d \to R$ has a local extremum at $\bar{x}$, then $\bar{\nabla} f(\bar{x}) = 0$.

Proof

Assume $\bar{x}$ local minimum for ϵ sufficiently small.

$f(\bar{y}) \geq f(\bar{x}) \quad \forall \bar{y} \in B_\epsilon(\bar{x})$ $\qquad \boxed{d\bar{x}} \in B_\epsilon(\bar{x})$

Taylor's formula:

$$f(\bar{x}) \leq \underset{(\bar{x}+\bar{s}) \in B_\epsilon(\bar{x})}{f(\bar{x}+\bar{s})} = f(\bar{x}) + \bar{\nabla} f(\bar{x}) \cdot \bar{s} + \frac{1}{2} \bar{s}^T H(\bar{\xi}(\eta)) \bar{s}$$

$$H_{max} = \max_{\bar{z},\, \bar{x}+\bar{t} \in B_\epsilon} \left(\frac{\bar{t}^T H(\bar{z}) \bar{t}}{|\bar{t}|^2} \right)$$

Then $\frac{1}{2}\bar{s}^T H(\bar{j}(\bar{y}))\bar{s} \leq \frac{1}{2}|\bar{s}|^2 H_{max}$

Assume $\bar{\nabla} f(\bar{x}) \neq 0$, pick $\bar{s} = -\mu \bar{\nabla} f(\bar{x})$
where μ is so small that $(\bar{s} + \bar{x}) \in B_\epsilon(\bar{x})$

$$f(\bar{x}) \leq f(\bar{x}) - \mu |\bar{\nabla} f|^2 + \frac{1}{2}\mu^2 |\bar{\nabla} f|^2 H_{max}$$

Choose $\mu < \frac{1}{H_{max}}$

so $f(\bar{x}) \leq f(\bar{x}) - \frac{1}{2}\mu |\bar{\nabla} f|^2 < f(\bar{x})$

Example: Critical points

$f(x,y) = x^2 + \frac{1}{3}y^3 - \frac{1}{2}y^2$

Find critical points, $\bar{\nabla} f = 0$.

$\frac{\partial f}{\partial x} = 2x$, $\frac{\partial f}{\partial x} = 0 \Rightarrow x = 0$

$\frac{\partial f}{\partial y} = y^2 - y = y(y-1)$, $\frac{\partial f}{\partial y} = 0 \Rightarrow y = 0$ or 1

So two critical points: $\bar{x}_1 = (0,0)$ and $\bar{x}_2 = (0,1)$

Do $\bar{x}_1$ and $\bar{x}_2$ correspond to local extrema?
We must study second derivatives, that is: the Hessian.

Assume $\bar{\nabla} f(\bar{y}) = 0$

$$f(\bar{x}) = f(\bar{y}) + \underbrace{\bar{\nabla} f(\bar{y})(\bar{x} - \bar{y})}_{=0} + \frac{1}{2}(\bar{x} - \bar{y})^T H(\bar{j}(\eta))(\bar{x} - \bar{y})$$

1). $\bar{y}$ local max $\rightarrow$ $\bar{x}$ close enough to $\bar{y}$, $(*) < 0$
2). y local min $\rightarrow$ $(*) > 0$
where $(*) = \frac{1}{2}(\bar{x} - \bar{y})^T H(\bar{j}(\eta))(\bar{x} - \bar{y}) < 0$

We will study the eigenvalues of the Hessian at $\bar{y}$,
solve: $\det(H(\bar{y}) - \lambda I) = 0$

Denote the eigenvalues $B\{A_i\}$.

- If $\lambda_i < 0 \ \forall i \in [1, \ldots, d]$ then 1). holds.
- If $\lambda_i > 0 \ \forall i$ then 2). holds.
- If some $\lambda_i > 0$ and some $\lambda_i < 0$ then we have a saddle point. This is not a local extremum
- If some $\lambda_i = 0$ we say that the critical point is degenerate.

Back to example:

$$f(x,y) = x^2 + \frac{1}{3}y^3 - \frac{1}{2}y^2 \ , \quad \frac{\partial f}{\partial x} = 2x \ , \quad \frac{\partial f}{\partial y} = y^2 - y$$

$$H(x,y) = \begin{bmatrix} 2 & 0 \\ 0 & 2y-1 \end{bmatrix}$$

$$\det(H - \lambda I) = \begin{vmatrix} 2-\lambda & 0 \\ 0 & 2y-1-\lambda \end{vmatrix}$$

$$\Rightarrow \lambda_1 = 2, \quad \lambda_2 = 2y - 1$$

Critical points: $\bar{x}_1 = (0,0)$, $\bar{x}_2 = (0,1)$

For $\bar{x}_1$: $\lambda_1 = 2$, $\lambda_2 = -1$ $\Rightarrow$ $\bar{x}_1$ is a saddle point.

For $\bar{x}_2$: $\lambda_1 = 2$, $\lambda_1 = 1$ $\Rightarrow$ $\bar{x}_2$ is a local minimum

For optimisation we want to find

$$\min_{\bar{x} \in \Omega} f(\bar{x}) \ , \quad \Omega \subset \mathbb{R}^d$$

- Find local extrema inside Ω.
- Find extreme points on the boundary

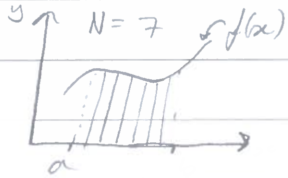
Integrals in higher dimensions

Multiple integrals

Single variable calculus

$f : R \to \mathbb{R}$

$$\int_a^b f(x)dx = \lim_{N\to\infty} \sum_{j=1}^{N} f(x_j)\Delta x$$

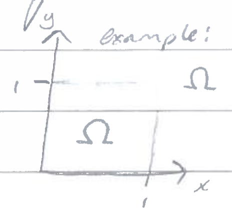


Riemann sum

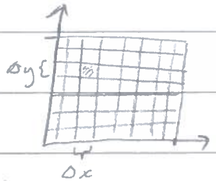
Double integrals

$f : R^2 \to \mathbb{R}$ $\Omega \subset \mathbb{R}^2$, Ω region (or domain) of integration

example:

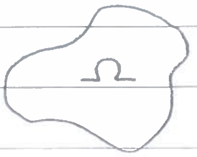


$\Omega = [0,1] \times [0,1]$



$$\iint_\Omega f(x,y)\,dx\,dy = \lim_{N\to\infty} \sum_{i=1}^{N} f(x_i, y_i)\underbrace{\Delta x \Delta y}$$

Can be extended to general domains Ω with piecewise smooth boundary.



Two important properties:

1). Linearity in the nt grand , , , f, g $\mathbb{R}$ R

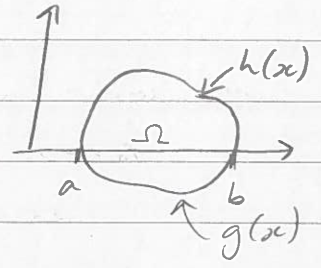
$$\iint_\Omega (k f(x,y) + l g(x,y))\,dx\,dy = k\iint_\Omega f(x,y)\,dx\,dy + l\iint_\Omega g(x,y)\,dx\,dy$$

2). If $\Omega = \Omega_1 \cup \Omega_2$, $\Omega_1 \cap \Omega_2 = \emptyset$

$$\iint_\Omega f(x,y)\,dx\,dy = \iint_{\Omega_1} f(x,y)\,dx\,dy + \iint_{\Omega} f(x,y)\,dx\,dy$$

Evaluating double integrals

Assume the Ω may be written $a \le x \le b$, $g(x) \le y \le h(x)$



Then $$\iint_{\Omega} f(x,y)\,dx\,dy = \int_a^b \left(\int_{g(x)}^{h(x)} f(x,y)\,dy \right) dx$$

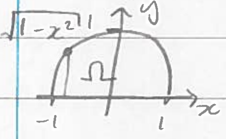
This reduces the evaluation of the double integral to two 1D integrals.

1). Let $F(x) := \displaystyle\int_{g(x)}^{h(x)} f(x,y)\,dy$

We integrate in y, considering x constant.

2). Evaluate $\displaystyle\int_a^b F(x)\,dx$ as usual.

Example



Ω semi disc with radius 1.
Compute the area of Ω.

$$|\Omega| = \iint_{\Omega} dx\,dy = \int_{-1}^{1}\left(\int_0^{\sqrt{1-x^2}} dy\right)dx$$

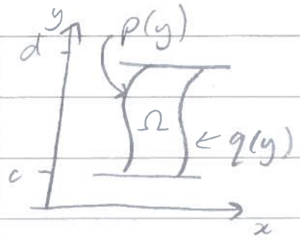
$$F(x) = [y]_0^{\sqrt{1-x^2}} = \sqrt{1-x^2}$$

So $$|\Omega| = \int_{-1}^{1}\sqrt{1-x^2}\,dx = \left[\frac{1}{2}x\sqrt{1-x^2} + \frac{1}{2}\arcsin(x)\right]_{-1}^{1}$$

$$= \frac{1}{2}\left(\frac{\pi}{2} - \left(-\frac{\pi}{2}\right)\right) = \frac{1}{2}\pi$$

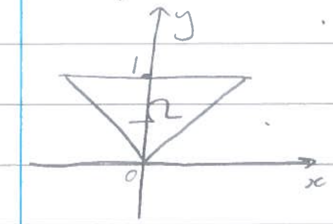
Similarly we may treat domains

Ω s.t. $c \leq y \leq d$, $p(y) \leq x \leq q(y)$



$$\iint_\Omega f(x,y)\,dx\,dy = \int_c^d \left(\int_{p(y)}^{q(y)} f(x,y)\,dx \right) dy$$

Example



$0 \leq y \leq 1$, $-y \leq x \leq y$

$$f(x,y) = x^2 y$$

$$\iint_\Omega f(x,y)\,dx\,dy = \int_0^1 \int_{-y}^{y} x^2 y\,dx\,dy$$

$$= \int_0^1 y\left[\frac{x^3}{3}\right]_{-y}^{y} = \int_0^1 y\left(\frac{y^3}{3} - \left(\frac{-y^3}{3}\right)\right) dy$$

$$= \int_0^1 \frac{2}{3} y^4\,dy = \frac{2}{3}\left[\frac{y^5}{5}\right]_0^1$$

$$= \frac{2}{15}$$

Assume $\Omega = [-1,1] \times [-1,1]$

$$f(x,y) = (x+1)e^{y/2}$$

$$\iint_\Omega (x+1)e^{y/2}\,dx\,dy = \int_{-1}^1 \int_{-1}^1 (x+1)e^{y/2}\,dy\,dx$$

$$F(x) = \int_{-1}^1 (x+1)e^{y/2}\,dy = (x+1)\int_{-1}^1 e^{y/2}\,dy = 2[x+1][e^{\frac{1}{2}} - e^{-\frac{1}{2}}]$$

$$\int_{-1}^1 F(x)\,dx = 2(e^{\frac{1}{2}} - e^{-\frac{1}{2}})\int_{-1}^1 (x+1)\,dx = 2(e^{\frac{1}{2}} - e^{-\frac{1}{2}})\left[\frac{1}{2}x^2 + x\right]_{-1}^1$$

$$= 4(e^{\frac{1}{2}} - e^{-\frac{1}{2}})$$

Observe (in very special cases!)

$$\iint_{\Omega} e^{\frac{y}{2}}(x+1)\,dx\,dy = \int_{-1}^{1}(x+1)\,dx \int_{-1}^{1} e^{\frac{y}{2}}\,dy$$

Change of variables

1D

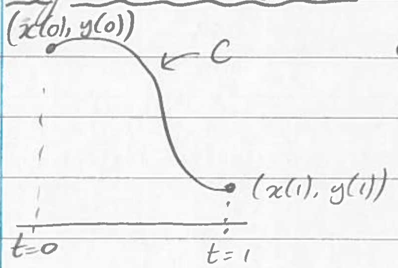
Consider

$$\int_a^b f(x)\,dx$$

Let $x = X(u) \quad X'(u) > 0 \qquad a = X(\alpha),\ b = X(\beta)$

$$\int_a^b f(x)\,dx = \int_{\alpha}^{\beta} f(X(u))\,X'(u)\,du$$

Higher dimensions



$C = \{(x,y):\ x(t),\ y(t)$ are given, $0 \le t \le 1\}$

$$\int_C f(x,y)\,ds$$

$$\int_C f(x,y)\,ds = \lim_{N\to\infty} \sum_{i=1}^{N} f(x_i, y_i)\,\Delta s$$

$$\approx \lim_{N\to\infty} \sum_{i=1}^{N} f(x(t_i), y(t_i))\,\underbrace{|\bar{j}(t+\Delta t) - \bar{j}(t)|}_{|\bar{j}'(\eta)|\,\Delta t}$$

passing to the limit

$$= \int_0^1 f(x(t), y(t))\,|\bar{j}'(t)|\,dt$$

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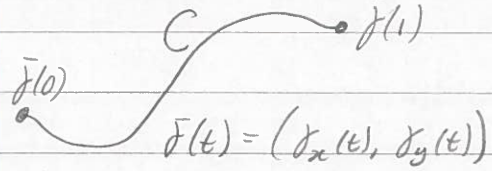
Last time:



$C := \{\bar{\gamma}(t) : 0 \leq t \leq 1\}$

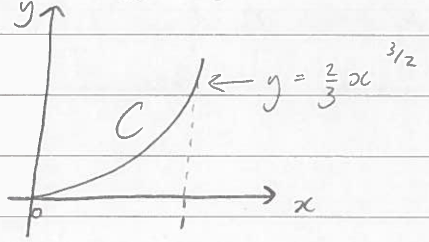
$$\int_C f(x,y)\, ds = \lim_{N\to\infty} \sum_{i=1}^{N} f(x_i, y_i) \Delta s$$

$$\int_0^1 f(\gamma_x(t), \gamma_y(t)) |\bar{\gamma}'(t)|\, dt$$

since $\Delta s \approx |\bar{\gamma}(t+\Delta t) - \bar{\gamma}(t)| = |\bar{\gamma}'(\eta)| \Delta t$

Example

Let $y = \frac{2}{3} x^{3/2}$



Q: What is the length of C?

$$\int ds = \int_0^1 |\bar{\gamma}'(t)|\, dt \qquad (*)$$

$(\bar{\gamma}(t) = (t, \frac{2}{3} t^{3/2}),\ 0 \leq t \leq 1)$

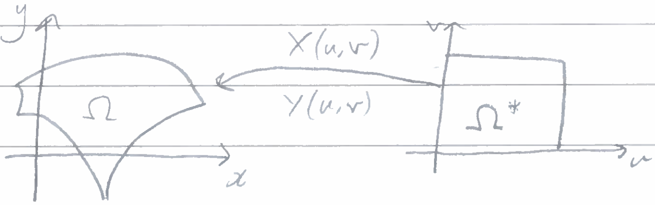
$\bar{\gamma}'(t) = (1, t^{\frac{1}{2}}) \quad |\bar{\gamma}'(t)| = \sqrt{1 + (t^{\frac{1}{2}})^2} = (1+t)^{\frac{1}{2}}$

$$(*) = \int_0^1 (1+t)^{\frac{1}{2}} dt = \left[\frac{2}{3}(1+t)^{3/2}\right]_0^1 = \frac{2}{3}(2\sqrt{2} - 1)$$

What happens in 2D

$$\iint_{\Omega} f(x,y)\,dx\,dy = \lim_{N\to\infty} \sum_{i=1}^{N} f(x_i, y_i)\,\Delta x\,\Delta y$$

The idea is to change the domain of the integration from Ω (in (x,y)) to Ω^* (in (u,v)) through the change of variables, $x = X(u,v)$, y $((,)$



3 Steps

1). Represent Ω by Ω^* using (u,v) coordinates and the mapping $(X(u,v), Y(u,v))$.
2). Transform $f(x,y)$ to $F(u,v) = f(X(,), (,v))$
3). Represent the area element do du, in dudv.

[Recall in 1D $ds = |\gamma'(t)|dt$

In 1D parameterisation : $\gamma : \mathbb{R} \to \mathbb{R}$, $|\gamma'(t)|dt = ds$
In 2D parameterisation: $(X, Y) : \mathbb{R}^2 \to \mathbb{R}^2$ $X(u,v)$, $Y(u,v)$
Here the scaling factor is given by the Jacobian :

$$J(u,v) = \frac{\partial(X,Y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial X}{\partial u} & \frac{\partial X}{\partial v} \\ \frac{\partial Y}{\partial u} & \frac{\partial Y}{\partial v} \end{vmatrix}$$

$$= \frac{\partial X}{\partial u}\frac{\partial Y}{\partial v} - \frac{\partial X}{\partial u}\frac{\partial Y}{\partial v}$$

$$dx\,dy = |J(u,v)|\,du\,dv$$

$$\iint_{\Omega} f(x,y)\,dx\,dy = \iint_{\Omega} f(X(u,v), Y(u,v))\,|J(u,v)|\,du\,dv.$$

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Example (Polar coordinates)

$X(r,\theta) = r\cos\theta \qquad (x,y),\ -1 \le x \le 1,\ 0 \le y \le \sqrt{1-x^2}$

$Y(r,\theta) = r\sin\theta$

$$\iint_{\Omega} f(x,y)\,dx\,dy = \iint_{\Omega^*(r,\theta)} f(X(r,\theta), Y(r,\theta))\,|J(r,\theta)|\,dr\,d\theta$$

$$J(r,\theta) = \begin{vmatrix} \frac{\partial X}{\partial r} & \frac{\partial X}{\partial \theta} \\ \frac{\partial Y}{\partial r} & \frac{\partial Y}{\partial \theta} \end{vmatrix}$$

$$= \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix}$$

$$= r\cos^2\theta - r(-\sin^2\theta)$$

$$= r(\cos^2\theta + \sin^2\theta) = r$$

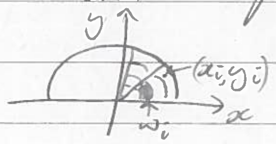
Revisiting area of semi-disc example.

$$\int_{-1}^{1}\int_{0}^{\sqrt{1-x^2}} 1 \cdot dx\,dy = \frac{\pi}{2}$$

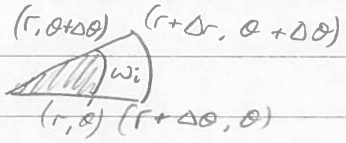
$$\int_{-1}^{1} \sqrt{1-x^2}\,dx$$

in polar coodinates $\displaystyle\int_0^1\int_0^{\pi} r\,d\theta\,dr = \pi\left[\frac{r^2}{2}\right]_0^1 = \frac{\pi}{2}$

Jacobian in polar coords:



$$\iint_{\Omega} f(x,y)\,dx\,dy = \lim_{N\to\infty} \sum_{i=1}^{N} f(x_i,y_i)\,A(w_i)$$



$$A(w_i) = \frac{\Delta\theta}{2}(r+\Delta r)^2 - \frac{\Delta\theta}{2}r^2$$

$$= \frac{\Delta\theta}{2}(\cancel{r^2} + \Delta\theta^2 + 2r\Delta r - \cancel{r^2})$$

$$= \underbrace{r\,\Delta r\,\Delta\theta}_{r\,dr\,d\theta} + \underbrace{\frac{\Delta r^2}{2}\Delta\theta}_{\text{higher order}}$$

Compare $|J(r,\theta)|\,dr\,d\theta$
$= r\,dr\,d\theta$

$\Big[\Delta r, \Delta\theta \to 0 \quad \Delta r^2\Delta\theta \to 0,$
as $N \to \infty\Big]$

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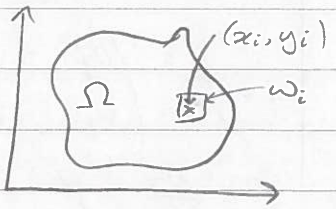
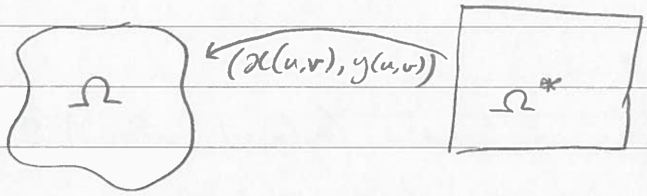
Integration in 2D

$$\iint_{\Omega} f(x,y)\,dx\,dy = \lim_{N\to\infty} \sum_{i=1}^{N} f(x_i, y_i) A(\omega_i)$$



Change of variables

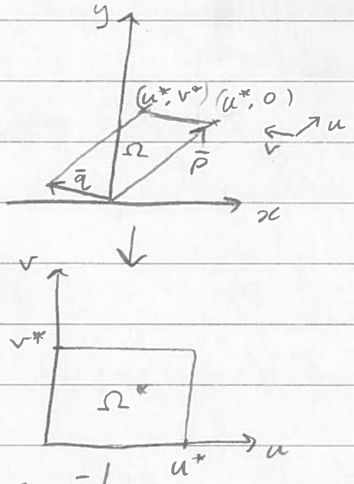


$$\iint_{\Omega} f(x,y)\,dx\,dy = \iint_{\Omega^*} f(x(u,v), y(u,v))\,|J|\,du\,dv$$

J is the Jacobian, $J = \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u}$

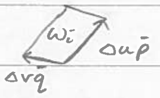
Example

Ω is a parallelogram



$$\begin{pmatrix} x \\ y \end{pmatrix} = u\bar{p} + v\bar{q} = \begin{pmatrix} up_1 + vq_1 \\ up_2 + vq_2 \end{pmatrix}$$

$$\iint_{\Omega} f(x,y)\,dx\,dy = \lim_{N\to\infty} \sum_{i=1}^{N} f(x_i, y_i) A(\omega_i) \quad (*)$$



so $A(\omega_i) = |\Delta v\bar{q} \times \Delta u\bar{p}|$

$= |\bar{q} \times \bar{p}|\,\Delta u \Delta v$ where $\bar{q}\times\bar{p} = p_1 q_2 - p_2 q_1$

so $(*) = \lim_{N\to\infty} \sum_{i=1}^{N} f(x(u_i, v_i), y(u_i, v_i))\,|\bar{q} \times \bar{p}|\,\Delta u \Delta v$

$= \iint_{\Omega^*} F(u,v)\,|\bar{q}\times\bar{p}|\,du\,dv$ where $F(u,v) = f(x(u,v), y(u,v))$

Recall $J = \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u}$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} up_1 + vq_1 \\ up_2 + vq_2 \end{pmatrix}$$

so $\frac{\partial x}{\partial u} = p_1$, $\frac{\partial x}{\partial v} = q_1$, $\frac{\partial y}{\partial u} = p_2$, $\frac{\partial y}{\partial v} = q_2$

so $J = p_1 q_2 - p_2 q_1$.

Why does it work for nonlinear $(x(u,v), y(uv))$?
In this case $A(w_i) = |J| \Delta u \Delta v + (\text{h.o.t. in } \Delta u \Delta v)$
But the h.o.t $\to 0$ so quickly that what works in the linear case is a good approximation for the non linear case.

h.o.t.
= higher
order
terms

<u>Example</u>

Let $\Omega := \{(x,y) : -1 \leq x+y \leq 1 \ , \ -1 \leq x-y \leq 1\}$
Evaluate $\iint_\Omega (x+y)^2 \, dx\, dy$

Natural change of variables: $u = x+y$, $v = x-y$
so $x = \frac{1}{2}(u+v)$, $y = \frac{1}{2}(u-v)$.

$\frac{\partial x}{\partial u} = \frac{1}{2}$, $\frac{\partial x}{\partial v} = \frac{1}{2}$, $\frac{\partial y}{\partial u} = \frac{1}{2}$, $\frac{\partial y}{\partial v} = -\frac{1}{2}$.

so $J = (\frac{1}{2})(-\frac{1}{2}) - (\frac{1}{2})(\frac{1}{2}) = -\frac{1}{4} - \frac{1}{4} = -\frac{1}{2}$

$\Omega^* = \{(u,v) : -1 \leq u \leq 1, \ -1 \leq v \leq 1\}$

$$\iint_\Omega (x+y)^2 \, dx\, dy = \iint_{\Omega^*} u^2 \left|-\tfrac{1}{2}\right| du\, dv$$

$$= \frac{1}{2}\int_{-1}^{1}\int_{-1}^{1} u^2 \, du\, dv = \left[\frac{u^3}{3}\right]_{-1}^{1} = \frac{2}{3}$$

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Triple integrals

$$\iiint_{\Omega} f(x,y,z)\,dx\,dy\,dz = \lim_{N\to\infty} \sum_{i=1}^{N} f(x_i, y_i, z_i) V(\omega_i)$$

Decompose Ω into N volume elements ω_i with volume $V(\omega_i)$, $(x_i, y_i, z_i) \in \omega_i$.
If the volume element is a brick then $V(\omega_i) = \Delta x \Delta y \Delta z$.

To evaluate $\iiint_{\Omega} f(x,y,z)\,dx\,dy\,dz$ use iterated integrals as in 2D, but with an additional dimension.

Change of variable in 3D

$\bar{x} = (x(u,v,r), y(u,v,r), z(u,v,r))$

$\bar{x} : \Omega^* \to \Omega$

so $$\iiint_{\Omega} f(x,y,z)\,dx\,dy\,dz = \iiint_{\Omega^*} f(\bar{x}(u,v,r))\,|J|\,du\,dv\,dr$$

but here $|J|$ is a volume scaling.

$$J := \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial r} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial r} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial r} \end{vmatrix}$$

= volume of the parallelepiped defined by $\bar{\nabla} x, \bar{\nabla} y, \bar{\nabla} z$.

Example

Evaluate the volume of a ball with radius R

$\Omega := \{(x,y,z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 \leq R^2\}$

$$V = \iiint_{\Omega} dx\,dy\,dz$$

Spherical coordinates

$x(\rho, \theta, \phi) = \rho \cos\theta \sin\phi$

$y(\rho, \theta, \phi) = \rho \sin\theta \sin\phi$

$z(\rho, \theta, \phi) = \rho \cos\phi$

$\Omega^* = \{(\rho, \theta, \phi) \in \mathbb{R}^3 : 0 \leq \rho \leq R \, , \, 0 \leq \theta \leq 2\pi \, , \, 0 \leq \phi \leq \pi\}$

Exercise: show that $J = \rho^2 \sin\phi$.

$$V = \iiint_{\Omega^*} |\rho^2 \sin\phi| \, d\theta \, d\phi \, d\rho$$

$$= \int_0^R \int_0^\pi \int_0^{2\pi} \rho^2 \sin\phi \, d\theta \, d\phi \, d\rho$$

$$= 2\pi \int_0^R \left[-\rho^2 \cos\phi \right]_0^\pi d\rho$$

$$= 4\pi \int_0^R \rho^2 \, d\rho = 4\pi \left[\frac{\rho^3}{3} \right]_0^R = \frac{4\pi R^3}{3}$$

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Vector fields

So far mainly $f: \mathbb{R}^3 \to \mathbb{R}$

Definition (Vector function or field)

A vector function is a function $\bar{F}: \mathbb{R}^3 \to \mathbb{R}^3$ (or $\bar{F}: \mathbb{R}^2 \to \mathbb{R}^2$).

Notation (in cartesian coordinates)

$$\bar{F}(x,y,z) = F_x(x,y,z)\bar{e}_x + F_y(x,y,z)\bar{e}_y + F_z(x,y,z)\bar{e}_z$$

or

$$\bar{F}(x_1,x_2,x_3) = F_1(x_1,x_2,x_3)\bar{e}_1 + F_2(x_1,x_2,x_3)\bar{e}_2 + F_3(x_1,x_2,x_3)\bar{e}_3$$

with index notation $\bar{F} = F_i e_i$

or

$$\bar{F}(x_1,x_2,x_3) = (F_1, F_2, F_3)$$

F_i, $i = 1, 2, 3$ are scalar functions

Examples of vector fields

1). $\bar{F}: \mathbb{R}^3 \to \mathbb{R}^3$, $\bar{F}(x,y,z) = x\bar{e}_x + y\bar{e}_y + z\bar{e}_z$

2). Given $f: \mathbb{R}^3 \to \mathbb{R}$

$\bar{G}_f : \bar{\nabla} f \qquad \bar{G}_f : \mathbb{R}^3 \to \mathbb{R}^3$

3). $\bar{F}: \mathbb{R}^3 \to \mathbb{R}^3$, $\bar{F} := \bar{e}_x + e^{xy}\bar{e}_y + \sin^2 x\, \bar{e}_z$ (arbitrary example)

Practical examples

1). Velocity of a fluid

2). Electric field : the gradient of the potential

Visualisation

1). Pick (x,y)

2). Evaluate $(F_x(x,y), F_y(x,y))$

3). Draw (F_x, F_y) starting in (x,y)

4). Repeat with another point (x,y)

Exercise: plot $\bar{F} = x\bar{e}_x + y\bar{e}_y$

Operations:

The same as for vectors in linear algebra:

1). Norm: $\bar{F} : \mathbb{R}^3 \Rightarrow \mathbb{R}^3$

$|\bar{F}| : \mathbb{R}^3 \to \mathbb{R}$ ← magnitude of the vector

$$|\bar{F}| = \sqrt{\sum_{i=1}^{3} F_i^2}$$

Normalised vector field $\bar{e}_F : \mathbb{R}^3 \to \mathbb{R}^3$

Assume $|\bar{F}| > 0$, $\bar{e}_F = \dfrac{\bar{F}}{|\bar{F}|}$

$\bar{e}_F$ points in the same direction as $\bar{F}$ but $|\bar{e}_F| = 1$

2). Dot-product

$\bar{F} : \mathbb{R}^3 \Rightarrow \mathbb{R}^3$, $\bar{G} : \mathbb{R}^3 \to \mathbb{R}^3$

$d_{Fg}(x, y, z) := \bar{F} \cdot \bar{G}(x, y, z)$, $d_{FG} : \mathbb{R}^3 \to \mathbb{R}$

$$\bar{F} \cdot \bar{G} := \sum_{i=1}^{3} F_i G_i$$

If $d_{FG} = 0$ everywhere then $\bar{F}$ and $\bar{G}$ are $\perp$ everywhere.

Directional derivative $\dfrac{\partial f}{\partial \bar{e}} = \bar{e} \cdot \nabla f$, where $\bar{e}$ is a unit vector

3). Cross product

$\bar{F} : \mathbb{R}^3 \to \mathbb{R}^3$, $\bar{G} : \mathbb{R}^3 \to \mathbb{R}^3$

$$\bar{F} \times \bar{G} := \begin{vmatrix} \bar{e}_1 & \bar{e}_2 & \bar{e}_3 \\ F_1 & F_2 & F_3 \\ G_1 & G_2 & G_3 \end{vmatrix} = (F_2 G_3 - F_3 G_2)\bar{e}_1 - (F_1 G_3 - G_1 F_3)\bar{e}_2 + (F_3 G_2 - F_2 G_1)\bar{e}_3$$

[$\bar{F} : \mathbb{R}^2 \to \mathbb{R}^2$, $\bar{G} : \mathbb{R}^2 \Rightarrow \mathbb{R}^2$

$\bar{F} \times \bar{G} = F_1 G_2 - G_1 F_2$]

Two motivations

1) integration over vector quantities

2). vector operations using differential operators

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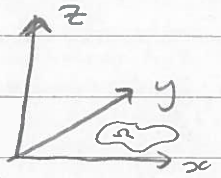
Integration over surfaces

$\Omega \subset \mathbb{R}^2$ can be seen as a 'flat' surface in $\mathbb{R}^3$



We know how to evaluate $\iint_{\Omega} f(x,y)\,dx\,dy$

How can we integrate over some other surface S?

Parameterised surface

$S := \{(x,y,z) \in \mathbb{R}^3 : (x,y,z) = (s_1(u,v), s_2(u,v), s_3(u,v)) = \bar{s}(u,v), (u,v) \in R \subset \mathbb{R}^2\}$

$\bar{s} : \mathbb{R}^2 \to \mathbb{R}^3$

Example: The graph of $f(x,y)$

$\bar{s}_f = (x, y, f(x,y))$

Riemann Sum: decompose S in small area $\overset{\text{(open)}}{\text{elements}}, w_i$, such that $\bigcup_i w_i = S$, $w_i \cap w_j = \phi$ if $i \neq j$

Then $$\iint_S f(\bar{s})\,\underset{\substack{\uparrow \\ \text{area element} \\ \text{on } S}}{ds} = \lim_{N \to \infty} \sum_{i=1}^{N} f(\bar{s}_i)\, \underbrace{A(w_i)}_{\substack{\uparrow \\ \text{area of } w_i}}$$

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Significance of multi-dimensional integrals

Let $V \subset \mathbb{R}^3$

The volume of V is

$$\iiint_V dx\,dy\,dz = \iiint_V dV$$

$f : \mathbb{R}^3 \to \mathbb{R}$

f is a density (mass, charge, etc...)

Then $\iiint_V f\,dV$ is the total mass/charge etc in V.

If $|V| = \iiint_V dV$ and $\frac{1}{|V|}\iiint_V f\,dV$ is the average density in V.

Let S be a surface.

The area of S is $|S| = \iint_S dS$

Order of integration

$F = \{(x,y) : a \leq x \leq b,\ p(x) \leq y \leq q(x)\}$

f, p, q are continuous

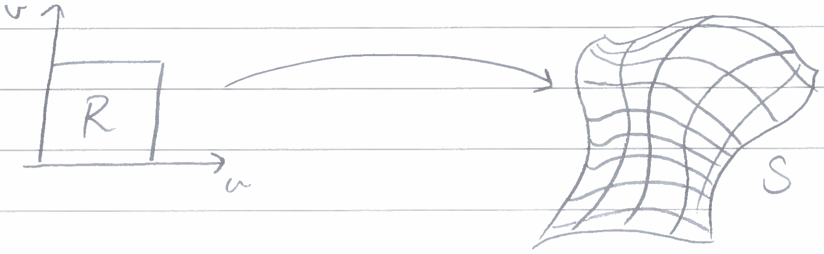
$$\iint_F f(x,y)\,dF = \int_a^b \int_{p(x)}^{q(x)} f(x,y)\,dy\,dx$$

Also if $F = \{(x,y) : c \leq y \leq d,\ r(y) \leq x \leq s(y)\}$

f, r, s continuous

then $\iint_F f(x,y)\,dF = \int_c^d \int_{r(y)}^{s(y)} f(x,y)\,dx\,dy$

Integration on Surfaces



$\bar{s}(u,v) = (s_1(u,v),\ s_2(u,v),\ s_3(u,v))$

$\bar{s} : R \subset \mathbb{R}^2 \longrightarrow S \subset \mathbb{R}^3$

$f : S \to R$, f smooth

$$\iint_S f \, dS = \iint_R f(\bar{s}(u,v)) \left| \frac{\partial \bar{s}}{\partial u} \times \frac{\partial \bar{s}}{\partial v} \right| du\, dv$$

Compare modulus term with $|\gamma'(t)|$ in 1D and $|J|$ in 2 and 3D.

Application of formula area of a sphere with radius ρ

Spherical coordinates

$x(\rho, \theta, \phi) = \rho \cos\theta \sin\phi$

$y(\rho, \theta, \phi) = \rho \sin\theta \sin\phi$

$z(\rho, \theta, \phi) = \rho \cos\phi$

ρ fixed, $\bar{s}(\theta, \phi) = \rho(\cos\theta \sin\phi ,\ \sin\theta \sin \qquad \cos$

$$\frac{\partial \bar{s}}{\partial \theta} = \rho(-\sin\theta \sin\phi ,\ \cos\theta \sin\phi ,\ 0)$$

$$\frac{\partial \bar{s}}{\partial \phi} = \rho(\cos\theta \cos\phi ,\ \sin\theta \cos\phi ,\ -\sin\phi)$$

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$$\frac{\partial \bar{s}}{\partial \theta} \times \frac{\partial \bar{s}}{\partial \phi} = \rho^2 \begin{vmatrix} \bar{e}_x & \bar{e}_y & \bar{e}_z \\ -\sin\theta\sin\phi & \cos\theta\sin\phi & 0 \\ \cos\theta\cos\phi & \sin\theta\cos\phi & -\sin\phi \end{vmatrix}$$

$$= \rho^2\left[\bar{e}_x(-\cos\theta\sin^2\theta) - \bar{e}_y(\sin\theta\sin^2\phi) + \bar{e}_z(-\cos\phi\sin\phi)\right] \quad (*)$$

$$(*) = \rho\sin\phi\left[\bar{e}_x(-\rho\cos\theta\sin\phi) - \bar{e}_y(\rho\sin\theta\sin\phi) + \bar{e}_z(-\cos\phi)\right]$$

$$= \rho\sin\phi \underbrace{\left[-x\bar{e}_x - y\bar{e}_y - z\bar{e}_z\right]}_{\text{normal to sphere!}}$$

so $$\left|\frac{\partial \bar{s}}{\partial \theta} \times \frac{\partial \bar{s}}{\partial \phi}\right| = \rho^2\sin\phi$$

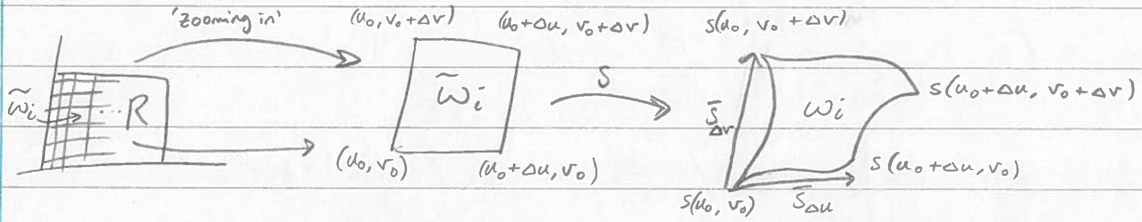
$$R = \{(\theta, \phi) : 0 \leq \theta \leq 2\pi,\ 0 \leq \phi \leq \pi\}$$

$$\iint_S ds = \int_0^{2\pi}\int_0^{\pi} \rho^2 \sin\phi \, d\theta \, d\phi$$

$$= \rho^2 \int_0^{2\pi} d\theta \int_0^{\pi} \sin\phi \, d\phi$$

$$= \rho^2 \, 2\pi \left[-\cos\phi\right]_0^{\pi} = 4\pi\rho^2$$

$$\iint_S f \, ds = \lim_{N\to\infty} \sum_{i=1}^{N} f(\bar{s}_i) A(\omega_i)$$



An appoximation: $A(\omega_i) \cong |\bar{s}_{\Delta u} \times \bar{s}_{\Delta v}|$

$$\bar{s}_{\Delta u} = \bar{s}(u_0 + \Delta u, v_0) - \bar{s}(u_0, v_0)$$

$$\underset{\uparrow \text{MVT}}{=} \left[\frac{\partial s_1}{\partial u}(\eta_1, v_0) \quad \frac{\partial s_2}{\partial u}(\eta_2, v_0) \quad \frac{\partial s_3}{\partial u}(\eta_3, v_0) \right] \Delta u$$

$$\text{so} \quad A(\omega_i) \cong |\bar{s}_{\Delta u} \times \bar{s}_{\Delta v}| \cong \left| \underbrace{\frac{\partial \bar{s}}{\partial u}(u_0, v_0) \times \frac{\partial \bar{s}}{\partial v}(u_0, v_0)}_{(*)} \right|$$

Comment: $(*)$ is the normal of S at (u_0, v_0).

$$\iint_S ds = \lim_{N \to \infty} \sum_{i=1}^{N} f(\bar{s}_i) A(\omega_i)$$

$$\cong \lim_{N \to \infty} \sum_{i=1}^{N} f(\bar{s}(u_i, v_i)) |(*)| \Delta u \Delta v$$

$$= \iint_R f(\bar{s}(u, v)) \left| \frac{\partial s}{\partial u} \times \frac{\partial s}{\partial v} \right| du\, dv$$

$$A(\omega_i) = \left| \frac{\partial s}{\partial u}(u_0, v_0) \times \frac{\partial s}{\partial v}(u_0, V_0) \right| \Delta u \Delta v \left(1 + O\left(\frac{1}{N}\right)\right)$$

<u>Important example</u>

S given by graph $(x, y) \in R \subset \mathbb{R}^2$, $z = g(x, y)$

$s(x, y) = (x, y, g(x, y))$, $(x, y) \in \mathbb{R}$

$$\iint_S f\, ds = \iint_R f(\bar{s}(x, y)) \left| \frac{\partial \bar{s}}{\partial x} \times \frac{\partial \bar{s}}{\partial y} \right| dx\, dy$$

$$\frac{\partial \bar{s}}{\partial x} = \left(1, 0, \frac{\partial g}{\partial x}\right), \quad \frac{\partial \bar{s}}{\partial y} = \left(0, 1, \frac{\partial g}{\partial y}\right)$$

$$\begin{vmatrix} \bar{e}_x & \bar{e}_y & \bar{e}_z \\ 1 & 0 & \frac{\partial g}{\partial x} \\ 0 & 1 & \frac{\partial g}{\partial y} \end{vmatrix} = \bar{e}_x \left(-\frac{\partial g}{\partial x}\right) + \bar{e}_y \left(-\frac{\partial g}{\partial y}\right) + \bar{e}_z$$

$$\left| \frac{\partial \bar{s}}{\partial x} \times \frac{\partial \bar{s}}{\partial y} \right| = \sqrt{\left(\frac{\partial g}{\partial x}\right)^2 + \left(\frac{\partial g}{\partial y}\right)^2 + 1}$$

$$\text{so} \quad \iint_S f\, ds = \iint_R f(\bar{s}(x, y)) \sqrt{\left(\frac{\partial g}{\partial x}\right)^2 + \left(\frac{\partial g}{\partial y}\right)^2 + 1}$$

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Surface Integrals on surfaces given by graphs

$$\iint_S f\,ds = \iint_R f(\bar{s}(u,v)) \left| \frac{\partial \bar{s}}{\partial u} \times \frac{\partial \bar{s}}{\partial v} \right| du\,dv$$

$\bar{s} : \mathbb{R}^2 \to \mathbb{R}^2$ $\quad s = \{ s_1(u,v), s_2(u,v), s_3(u,v) ; (u,v) \in R \subset \mathbb{R}^2 \}$

$s = (x, y, g(x,y)) \quad (x,y) \in R \subset \mathbb{R}^2$

$$\iint_S f\,ds = \iint_R f(x,y,g(x,y)) \sqrt{\left(\frac{\partial g}{\partial x}\right)^2 + \left(\frac{\partial g}{\partial y}\right)^2 + 1}\; dx\,dy$$

$f : \mathbb{R}^3 \to \mathbb{R}$ $\quad \left| \frac{\partial \bar{s}}{\partial x} \times \frac{\partial \bar{s}}{\partial y} \right| = \sqrt{\left(\frac{\partial g}{\partial x}\right)^2 + \left(\frac{\partial g}{\partial y}\right)^2 + 1}$

Example

Area of sphere as graph.

Area of sphere = 2 × area of upper half.

$S_{\frac{1}{2}} = \{ (x, y, \sqrt{\rho^2 - x^2 - y^2}) ; x^2 + y^2 \leq \rho^2 \}$ $\quad \rho = $ radius

$$\text{Area of sphere} = 2 \iint_S ds = 2 \iint_{x^2+y^2 \leq \rho^2} \sqrt{\frac{x^2+y^2}{\rho^2 - x^2 - y^2} + 1}\; dx\,dy$$

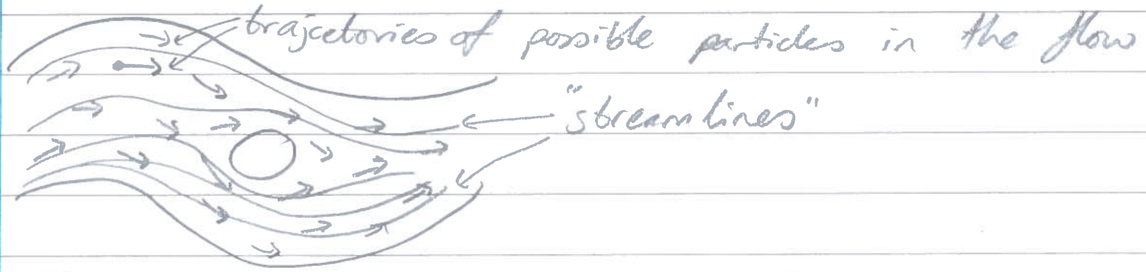
$$\frac{\partial g}{\partial x} = \frac{-x}{\sqrt{\rho^2 - x^2 - y^2}} \quad \frac{\partial g}{\partial y} = \frac{-y}{\sqrt{\rho^2 - x^2 - y^2}}$$

$$= \iint_{x^2+y^2 \leq \rho^2} \frac{\rho}{\sqrt{\rho^2 - x^2 - y^2}}\, dx\,dy$$

In polar coordinates:

$$2 \int_0^{2\pi} \int_0^{\rho} \frac{\rho}{\sqrt{\rho^2 - r^2}}\, r\,dr\,d\theta = 4\pi\rho \left[-\sqrt{\rho^2 - r^2} \right]_0^{\rho} = 4\pi\rho^2$$

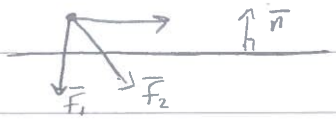
Vector fields and flux



Equation for streamline $\frac{d\bar{x}}{dt} = \underset{\text{(vectorfield)}}{\underline{\bar{F}(\bar{x})}}$

$\bar{x}(0) = \bar{x}_0$

The flux is "the flow across the surface"



$|\bar{n}| = 1$

$\bar{n}$ ⊥ to S $\bar{F} : \mathbb{R}^3 \to \mathbb{R}^3$

Flux : $\bar{F} \cdot \bar{n}$ $\bar{n}$ is the unit normal of the surface S.

Definition (closed surface and normal)
A closed surface separates space into two pieces such that you must cross the surface to go from one to the other. One of the sets is bounded, the interior, the other part is the exterior.
The unit normal is taken to be the outward pointing normal of magnitude 1.
The flux over S of $\bar{F} : \mathbb{R}^3 \to \mathbb{R}^3$

$\iint_S \vec{F} \cdot \bar{n}\, ds$, $\bar{n}$ denotes the unit normal of s.

Example
Body $\Omega \subset \mathbb{R}^3$.
S is the surface of Ω.
ρ denotes some density / concentration
$\rho \sim$ unit / unit volume.

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Vector $\rho\bar{v}$ defines the flow of ρ.
$\uparrow$ velocity vector.

Flux of ρ across $S = \iint_S \rho\bar{v}\cdot n \, ds$

$\frac{\partial \rho}{\partial t}$ is the rate of change of ρ.

$$\iiint_\Omega \frac{\partial \rho}{\partial t} \, d\Omega = -\iint_S \rho\bar{v}\cdot\bar{n} \, ds$$

<u>Example</u>

Compute the flux.

Given vectorfield $F = xy\bar{e}_x + zy\bar{e}_y + xz\bar{e}_z$

$V = \{(x, y, z) : 0 \le x \le 1, \; 0 \le y \le 1, \; 0 \le z \le 1\}$

Find $\iint_S F.\bar{n} \, ds$ where S is the surface enclosing V.

Decompose s into the sides of the cube $S_1, S_2, \ldots S_6$

$$\iint_S \bar{F}\cdot\bar{n} \, ds = \sum_{i=1}^{6} \iint_{S_i} F\cdot\bar{n} \, ds$$

$S_1 = \{y = 0, \; 0 \le x, z \le 1\}, \; \bar{n}|_{S_1} = -\bar{e}_y$

$S_2 = \{y = 1, \; 0 \le x, z \le 1\}, \; \bar{n}|_{S_2} = \bar{e}_y$

$S_3 = \{z = 0, \; 0 \le x, y \le 1\}, \; \bar{n}|_{S_3} = -\bar{e}_z$

$S_4 = \{z = 1, \; 0 \le x, y \le 1\}, \; \bar{n}|_{S_4} = \bar{e}_z$

$S_5 = \{x = 0, \; 0 \le y, z \le 1\}, \; \bar{n}|_{S_5} = -\bar{e}_x$

$S_6 = \{x = 1, \; 0 \le y, z \le 1\}, \; \bar{n}|_{S_6} = \bar{e}_x$

$$\iint_S \bar{F}\cdot\bar{n} \, ds = \overbrace{\iint_{S_1} (-zy) \, ds}^{=0} + \iint_{S_2} zy \, ds + \overbrace{\iint_{S_3} (-xz) \, ds}^{=0} + \iint_{S_4} xz \, ds$$
$$+ \underbrace{\iint_{S_5} (-xy) \, ds}_{=0} + \iint_{S_6} xy \, ds$$

$$= \int_0^1\int_0^1 z\,dx\,dz + \int_0^1\int_0^1 x\,dx\,dy + \int_0^1\int_0^1 y\,dy\,dz$$

$$= 3 \cdot \frac{1}{2} = \frac{3}{2}$$

The general case $S = (s_1(u,v), s_2(u,v), s_3(u,v))$.

Normal of the surface (not unit)

$$\bar{n} = \frac{\partial s}{\partial u} \times \frac{\partial s}{\partial v}$$

unit normal: $\bar{n} = \dfrac{\frac{\partial s}{\partial u} \times \frac{\partial s}{\partial v}}{\left|\frac{\partial s}{\partial u} \times \frac{\partial s}{\partial v}\right|}$

$$\iint_S \bar{F}\cdot\bar{n}\,d\sigma = \iint_R \bar{F}(\bar{s}(u,v)) \cdot \frac{\frac{\partial s}{\partial u} \times \frac{\partial s}{\partial v}}{\cancel{\left|\frac{\partial s}{\partial u} \times \frac{\partial s}{\partial v}\right|}} \cancel{\left|\frac{\partial s}{\partial u} \times \frac{\partial s}{\partial v}\right|}\,du\,dv$$

$$= \iint_R \bar{F}(\bar{s}(u,v)) \cdot \left(\frac{\partial s}{\partial u} \times \frac{\partial s}{\partial v}\right) du\,dv$$

The divergence

Calculus for vector fields, local rate of change of vector fields.

The divergence is related to "the rate of change" of flux. In 3D we can differentiate a vectorfield using the gradient.

$\bar{F} = (F_1, F_2, F_3)$

$$G = [\bar{\nabla}F_1 \,|\, \bar{\nabla}F_2 \,|\, \bar{\nabla}F_3] = \begin{pmatrix} \partial x_1 F_1 & \partial x_1 F_2 & \partial x_1 F_3 \\ \partial x_2 F_1 & \partial x_2 F_2 & \partial x_2 F_3 \\ \partial x_3 F_1 & \partial x_3 F_2 & \partial x_3 F_3 \end{pmatrix}$$

In 1D, derivative from limit:

$$\frac{\partial f}{\partial x} = \lim_{e \to 0} \left(\frac{f(x+e) - f(x-e)}{2e} \right)$$

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L11

Can we find something similar for vector fields?

Let $\bar{F} : \mathbb{R}^3 \to \mathbb{R}^3$.

Consider (x_0, y_0, z_0).

Let $S_\epsilon(x_0)$ be a ball with radius ϵ around $\bar{x}_0$.

Consider the flux through the surface of $S_\epsilon(x_0)$

$$\iint_{\partial S_\epsilon} \bar{F} \cdot \bar{n} \, ds$$

← "changes the sign"

As ϵ becomes smaller we get info on the local variation of $\bar{F}$ at (x_0, y_0, z_0)

$$\Delta S_\epsilon = \iiint_{S_\epsilon} dV \quad \text{volume of the ball } S_\epsilon.$$

Definition (divergence)

Let $\bar{F} : \mathbb{R}^3 \to \mathbb{R}^3$.

The divergence of $\bar{F}$ is

$$\text{div } \bar{F}(x_0, y_0, z_0) := \lim_{\epsilon \to 0} \frac{1}{\Delta S_\epsilon} \iint_{\partial S_\epsilon} \bar{F} \cdot \bar{n} \, ds$$

Example

$$\bar{F} = a \underline{e}_x + b \underline{e}_y + c \underline{e}_z$$

Consider the flux through the surface of $S_\epsilon(x_0)$

$$\iint_{\partial S_\epsilon} \bar{F} \cdot \bar{n} \, ds = 0 \quad \text{by symmetry.}$$

Remarks

1). We used spheres in the definition but any family of closed surfaces collapsing to (x_0, y_0, z_0) will do.

2). $\text{div } \bar{F} : \mathbb{R}^3 \to \mathbb{R}$ (check that the def. returns a scalar)

3). $\text{div } \bar{F}(\bar{x}_0) > 0$, $\bar{F}$ flows out of any sufficiently small ball around $\bar{x}_0$.

$\text{div } \bar{F}(\bar{x}_0) < 0$, $\bar{F}$ flows into any sufficiently small ball around $\bar{x}_0$.

4). the definition is useful for understanding but unpractical.

Example

Recall $\iiint\limits_V \frac{\partial \rho}{\partial t} dV = -\int\limits_S \rho \bar{v} \cdot \bar{n} ds$

Formally passing to the limit

$$\lim_{\Delta S_\epsilon} \frac{1}{\Delta S_\epsilon} \iiint\limits_{S_\epsilon} \frac{\partial \rho}{\partial t} dV = \lim_{\Delta S_\epsilon \to 0} \frac{1}{\Delta S_\epsilon} \iint\limits_{\partial S_\epsilon} \rho \bar{v} \cdot \bar{n} ds$$

Hence

$\frac{\partial \rho}{\partial t} + div \rho \bar{v} = 0$ (transport equation)

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Divergence

Def

$\vec{F} : \mathbb{R}^3 \to \mathbb{R}^3$

The divergence of $\bar{F}$ at $\bar{x}_0 = (x_0, y_0, z_0)$ is

$$\operatorname{div} \bar{F} := \lim_{\epsilon \to 0} \frac{1}{\Delta S_\epsilon} \iint_{\partial S_\epsilon} \bar{F} \cdot \bar{n} \, dS$$

where S_ϵ is the ball of radius ϵ centered at $\bar{x}_0$. ΔS_ϵ is the volume of S_ϵ, ∂S_ϵ is the surface of S_ϵ and $\bar{n}$ is the outward pointing normal to S_ϵ.

Example

$\bar{F} : x\bar{e}_x + y\bar{e}_y + z\bar{e}_z \ (=: \bar{r})$

Find $\operatorname{div} F$ at $(0,0)$.

S_ϵ is defined by $|\bar{r}| = \epsilon$, $\bar{n} = \frac{\bar{r}}{|\bar{r}|}$

$$\lim_{\epsilon \to 0} \frac{1}{\Delta S_\epsilon} \iint_{\partial S_\epsilon} \bar{F} \cdot \bar{n} \, dS \qquad \bar{F} \cdot \bar{n} = \bar{r} \cdot \frac{\bar{r}}{|\bar{r}|} = |\bar{r}| = \epsilon$$

$$= \lim_{\epsilon \to 0} \frac{1}{\Delta S_\epsilon} \iint_{\partial S_\epsilon} \epsilon \, dS = \lim_{\epsilon \to 0} \frac{\epsilon}{\Delta S_\epsilon} \iint_{\partial S_\epsilon} dS$$

We have computed:

$$\Delta S_\epsilon = \frac{4\pi\epsilon^3}{3} \qquad \partial S_\epsilon = 4\pi\epsilon^2$$

$$\operatorname{div} F = \lim_{\epsilon \to 0} \frac{\epsilon}{\left(\frac{\cancel{4\pi}\epsilon^3}{3}\right)} \cdot \cancel{4\pi}\cancel{\epsilon}^2 = 3$$

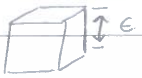
In 2D (Exercise) then $\operatorname{div} F = 2$.

In 2D, ΔS_ϵ is the area and the surface ∂S_ϵ is the curve enclosing S_ϵ.

Differential form of divergence.

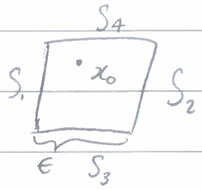
To find the differential form of div we use cubes instead of balls in the definition.

$$C_\epsilon := \{(x,y,z)\in R^3 \text{ s.t. } x_0-\frac{\epsilon}{2}\le x\le x_0+\frac{\epsilon}{2},\ y_0-\frac{\epsilon}{2}\le y\le y_0+\frac{\epsilon}{2},\ z_0-\frac{\epsilon}{2}\le z\le z_0+\frac{\epsilon}{2}\}$$



study $\text{div}\,\bar{F} := \lim_{\epsilon\to 0}\frac{1}{\epsilon^2}\iint_{\partial C_\epsilon}\bar{F}\cdot\bar{n}\,dS$, volume of $C_\epsilon = \epsilon^3$

In 2D: $\text{div}\,\bar{F} := \lim_{\epsilon\to 0}\frac{1}{\epsilon^2}\iint_{C_\epsilon}\bar{F}\cdot\bar{n}\,dS$



$$\iint_{C_\epsilon}\bar{F}\cdot\bar{n}\,dS = \sum_{i=1}^{4}\iint_{S_i}\bar{F}\cdot\bar{n}\,dS \quad (*)$$

$S_1 : \{(x,y) \cdot x = x_0-\frac{\epsilon}{2} ;\ y_0-\frac{\epsilon}{2}\le y\le y_0+\frac{\epsilon}{2}\}$ $\quad \bar{n} = -\bar{e}_x$

$S_2 : \{(x,y)\in \mathbb{R}^2 : x = x_0-\frac{\epsilon}{2} ;\ y_0-\frac{\epsilon}{2}\le y\le y_0+\frac{\epsilon}{2}\}$ $\quad \bar{n} = \bar{e}_x$

S_3 and S_4 similar.

$$(*) = -\int_{y_0-\epsilon/2}^{y_0+\epsilon/2} F_1(x_0-\frac{\epsilon}{2},y)\,dy + \int_{y_0-\frac{\epsilon}{2}}^{y_0+\frac{\epsilon}{2}} F_1(x_0+\frac{\epsilon}{2},y)\,dy$$
$$+\int_{x_0-\epsilon/2}^{x_0+\epsilon/2}(-F_2(x,y_0-\frac{\epsilon}{2}))\,dx + \int_{x_0-\epsilon/2}^{x_0+\epsilon/2} F_2(x,y_0+\frac{\epsilon}{2})\,dx \quad (**)$$

[on S_1 : $\bar{F}\cdot\bar{n} = -F_1(x,y)$, S_2 : $\bar{F}\cdot\bar{n} = F_1(x,y)$, etc...]

LMVT ↓

$$(**) = \int_{y_0-\epsilon/2}^{y_0+\epsilon/2} F_1(x_0+\frac{\epsilon}{2},y) - \underbrace{F_1(x_0-\epsilon/2,y)}_{\exists\,\eta_x\in[x_0-\epsilon/2,\ x_0+\epsilon/2] = \frac{\partial \bar{F}_1}{\partial x}(\eta_x,y)\epsilon}\,dy$$
$$+\int_{x_0-\epsilon/2}^{x_0+\epsilon/2} F_2(x,y_0+\frac{\epsilon}{2}) - \underbrace{F_2(x,y_0-\frac{\epsilon}{2})}_{\exists\,\eta_y\in[y_0-\frac{\epsilon}{2},\ y_0+\frac{\epsilon}{2}] = \frac{\partial \bar{F}_2}{\partial y}(x,\eta_y)\epsilon}\,dx$$

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In 2D $\text{div}\,\bar{F} := \lim_{\epsilon \to 0} \frac{1}{\epsilon^2} \iint_{C_\epsilon} \bar{F} \cdot \bar{n}\, dS$

$$= \lim_{\epsilon \to 0} \frac{1}{\epsilon^2} \left[\epsilon \int_{y_0 - \epsilon/2}^{y_0 + \epsilon/2} \frac{\partial F_1}{\partial x} (\eta_x, y)\, dy + \epsilon \int_{x_0 - \epsilon/2}^{x_0 + \epsilon/2} \frac{\partial F_2}{\partial y} (x, \eta_y)\, dx \right]$$

[Integral Mean Value theorem:

$\exists x_0 \in [a,b] : \int_a^b f(x)\, dx = f(x_0)(b-a)$]

$$= \lim_{\epsilon \to 0} \frac{1}{\epsilon^2} \left[\epsilon^2 \left(\frac{\partial \bar{F}_1}{\partial x} (\eta_x, \eta_y) + \frac{\partial \bar{F}_2}{\partial y} (\tilde{\eta}_x, \tilde{\eta}_y) \right) \right]$$

$\eta_x \to x_0, \quad \eta_y \to y_0, \quad \tilde{\eta}_x \to x_0, \quad \tilde{\eta}_y \to y_0$

$$= \frac{\partial \bar{F}_1}{\partial x} (x_0, y_0) + \frac{\partial \bar{F}_2}{\partial y} (x_0, y_0)$$

Alternative definition

$$\text{div}\,\bar{F} := \sum_{i=1}^{d} \frac{\partial \bar{F}_i}{\partial x_i} \overset{d=3}{=} \frac{\partial F_1}{\partial x_1} + \frac{\partial F_2}{\partial x_2} + \frac{\partial F_3}{\partial x_3}$$

$$\bar{\nabla} = \frac{\partial}{\partial x} \bar{e}_x + \frac{\partial}{\partial y} \bar{e}_y + \frac{\partial}{\partial z} \bar{e}_z$$

This is the 'Dell' operator.

$$\bar{\nabla} \cdot \bar{F} = \frac{\partial}{\partial x} F_1 + \frac{\partial}{\partial y} F_2 + \frac{\partial}{\partial z} F_3$$

Recall gradient $f : R^3 \to R$

$$\bar{\nabla} f = \frac{\partial f}{\partial x} \bar{e}_x + \frac{\partial f}{\partial y} \bar{e}_y + \frac{\partial f}{\partial z} \bar{e}_z$$

If G is the gradient matrix of $\bar{F}$ show that $\text{div}\,\bar{F} = \text{Trace}(G)$. [the trace is the sum of the diagonal elements.]

Fundamental theorem of Calculus

$$f(b) - f(a) = \int_a^b f'(x)\,dx \qquad f: R \to R$$

Can we find "something" similar for vectors in multi dimensions?

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L13

Divergence Recap.

Def

$\bar{F}: \mathbb{R}^3 \to \mathbb{R}^3$, $S_\epsilon \in \mathbb{R}^3$, volume: ΔS_ϵ

$$\text{Div}\,\bar{F}(\bar{x}_0) = \bar{\nabla}\cdot\bar{F}(\bar{x}_0) = \lim_{\epsilon\to 0}\frac{1}{\Delta S_\epsilon}\iint_{S_\epsilon}\bar{F}\cdot\bar{n}\,dS$$

$$\bar{\nabla} = \frac{\partial}{\partial x}\bar{e}_x + \frac{\partial}{\partial y}\bar{e}_y + \frac{\partial}{\partial z}\bar{e}_z$$

$$\bar{\nabla}\cdot\bar{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}$$

Important special case:

Divergence free or Solenoidal

vector fields $\bar{F}: \mathbb{R}^3 \to \mathbb{R}^3$

s.t. $\bar{\nabla}\cdot\bar{F} = 0$

Ex $\bar{F} = F_1(y,z)\bar{e}_x + F_2(x,z)\bar{e}_y + F_3(x,y)\bar{e}_z$

Examples include:

- Incompressible fluids
- Magnetic field

The Divergence Theorem

For a closed surface S (with outward pointing normal, $\bar{n}$) and a vectorfield $\bar{F}$ defined on S and everywhere in its interior V, we have

$$\iint_S \bar{F}\cdot\bar{n}\,dS = \iiint_V \text{div}\,\bar{F}\,dV \left[= \iiint_V \bar{\nabla}\cdot\bar{F}\,dV\right]$$

Example

$$\bar{r} = x\bar{e}_x + y\bar{e}_y + z\bar{e}_z$$

Evaluate $\iint_S \bar{F}\cdot\bar{n}\,dS$

S is a closed surface containing a unit volume.

So $\iint_S \bar{F}\cdot\bar{n}\,dS = \iiint_V \nabla\cdot\bar{r}\,dV$

$$= 3\underbrace{\iiint_V dV}_{1}$$

$$= 3$$

Sketch of Proof

Fundamental Thm of Calculus

$$f(b) - f(a) = \int_a^b f'(x)\,dx$$

$$a = x_0 < x_1 < x_2 < \dots < x_n = b$$

$$x_i - x_{i-1} = \Delta x$$

$$f(b) - f(a) = f(b) - f(x_1) + f(x_1) - f(a)$$

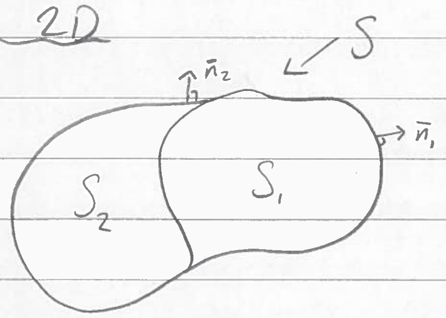
$$= \sum_{i=1}^{n} \left(f(x_i) - f(x_{i-1})\right)$$

$$\overset{\text{LMVT}\downarrow}{=} \sum_{i=1}^{n} f'(\eta_i)\Delta x \qquad \eta_i \in [x_{i-1}, x_i]$$

$$= \lim_{n\to\infty} \sum_{i=1}^{n} f'(\eta_i)\Delta x$$

$$= \int_a^b f'(x)\,dx$$

In 2D



$$\int_{S_1} \bar{F}\cdot\bar{n}_1\, dS_1 + \int_{S_2} \bar{F}\cdot\bar{n}_2\, dS_2$$

$$= \underset{\mathrm{I}}{\iint_{S_1\cap S} \bar{F}\cdot\bar{n}_1\, dS} + \underset{\mathrm{II}}{\iint_{S_1\setminus S} \bar{F}\cdot\bar{n}_1\, dS} + \underset{\mathrm{III}}{\iint_{S_2\cap S} \bar{F}\cdot\bar{n}_2\, dS} + \underset{\mathrm{IV}}{\iint_{S_2\setminus S} \bar{F}\cdot\bar{n}_2\, dS}$$

observe $\mathrm{I} + \mathrm{III} = \iint_S \bar{F}\cdot\bar{n}\, dS$

$$\mathrm{II} + \mathrm{IV} = \iint_{S_1\setminus S} \bar{F}\cdot\bar{n}_1\, dS + \iint_{S_2\setminus S} \bar{F}\cdot(-\bar{n}_1)\, dS = 0$$

here $\bar{n}_1 = -\bar{n}_2$, $S_1\setminus S = S_2\setminus S$

So $\int_{S_1} \bar{F}\cdot\bar{n}_1\, dS_1 + \int_{S_2} \bar{F}\cdot\bar{n}_2\, dS_2 = \iint_S \bar{F}\cdot\bar{n}\, dS$

Create a subdivision $\{S_i\}_{i=1}^{n}$ such that

$$\iint_S \bar{F}\cdot\bar{n}\, dS = \sum_{i=1}^{n} \iint_{S_i} \bar{F}\cdot\bar{n}\, dS$$

$$= \sum_{i=1}^{n} \left(\frac{1}{\Delta V_i} \iint_{S_i} \bar{F}\cdot\bar{n}\, dS\right) \Delta V_i$$

(where each S_i encloses the volume ΔV_i)

(assumption: $n\to\infty \Rightarrow \Delta V_i \to 0$) ↗ $= \lim_{n\to\infty} \sum_{i=1}^{n} \left(\frac{1}{\Delta V_i} \iint_{S_i} \bar{F}\cdot\bar{n}\, dS\right) \Delta V_i \quad (*)$

Assume S_i shrinks to x_i,

so

$$\lim_{\Delta V_i \to 0} \frac{1}{\Delta V_i} \iint_{S_i} \bar{F} \cdot \bar{n} \, dS = \bar{\nabla} \cdot \bar{F}(\bar{x}_i)$$

$$(*) = \lim_{n\to\infty} \sum_{i=1}^{n} \bar{\nabla} \cdot \bar{F}(\bar{x}_i) \Delta V_i$$

$$+ \lim_{n\to\infty} \sum_{i=1}^{n} \left(\frac{1}{\Delta V_i} \iint_{S_i} \bar{F} \cdot \bar{n} \, dS - \bar{\nabla} \cdot \bar{F}(\bar{x}_i) \right) \Delta V_i$$

$$(*) \leq \iiint_V \bar{\nabla} \cdot \bar{F} \, dV + \underbrace{\sup_i \left(\frac{1}{\Delta V_i} \iint_{S_i} \bar{F} \cdot \bar{n} \, dS - \bar{\nabla} \cdot \bar{F}(\bar{x}_i) \right) \Delta V_i}_{\to 0}$$

$$(*) \geq \iiint_V \bar{\nabla} \cdot \bar{F} \, dV - \underbrace{\sup_i \left(\frac{1}{\Delta V_i} \iint_{S_i} \bar{F} \cdot \bar{n} \, dS - \bar{\nabla} \cdot \bar{F}(\bar{x}_i) \right) \Delta V_i}_{\to 0}$$

so $$\iiint_V \bar{\nabla} \cdot \bar{F} \, dV \leq (*) \leq \iiint_V \bar{\nabla} \cdot \bar{F} \, dV$$

so $$(*) = \iiint_V \bar{\nabla} \cdot \bar{F} \, dV .$$

Example

$\bar{F} = xy\bar{e}_x + yz\bar{e}_y + zx\bar{e}_z$

$V = \{(x, y, z) \in \mathbb{R}^3 : 0 \leq x, y, z \leq 1\}$

S is the surface enclosing V.

Compute $\iint_S \bar{F} \cdot \bar{n} \, dS = \iiint_V \bar{\nabla} \cdot \bar{F} \, dV$

$$= \int_0^1 \int_0^1 \int_0^1 (y + z + x) \, dx \, dy \, dz$$

$$= \left[\tfrac{1}{2} y^2\right]_0^1 + \left[\tfrac{1}{2} z^2\right]_0^1 + \left[\tfrac{1}{2} x^2\right]_0^1 = \frac{3}{2}$$

Line integrals with vector fields

Work = $F \cdot \Delta x$

vector form: $\bar{F} \bullet \overline{\Delta x}$

The work on the particle: Line integral of one force component.

Curves and parameterisation

$\bar{r}(t) = x(t)\,\bar{e}_x + y(t)\,\bar{e}_y + z(t)\,\bar{e}_z$

$a \leq t \leq b$

$\bar{r}'(t) = x'(t)\,\bar{e}_x + y'(t)\,\bar{e}_y + z'(t)\,\bar{e}_z$ ← velocity

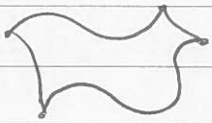
$|\bar{r}'(t)| > 0 \quad , \quad a \leq t \leq b$

$\bar{r} : [a, b] \to \mathbb{R}^3$ traces a curve C in 3D.

The curve is oriented by the growth of t.

If $\bar{r}(a) = \bar{r}(b)$ we say that C is a closed curve.

e.g.



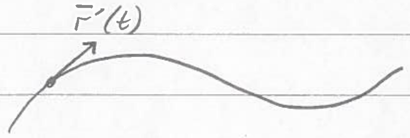
← finite no. of corners is okay.

Recap: scalar functions on a curve

$$\int_C f \, dr =: \int_a^b f(\bar{r}(t))\, |r'(t)| \, dt$$

Integration over vector-fields.

We consider the tangential component.



unit tangent = $\dfrac{\bar{r}'(t)}{|\bar{r}'(t)|}$

Tangential component : $\bar{F}(\bar{r}(t)) \cdot \dfrac{\bar{r}'(t)}{|\bar{r}'(t)|}$

Integral of the tangential component:

$$\int_C \bar{F} \cdot d\bar{r} = \int_a^b \left(\bar{F}(\bar{r}(t)) \cdot \frac{\bar{r}'(t)}{\cancel{|\bar{r}'(t)|}} \right) \cancel{|\bar{r}'(t)|} \, dt$$

$$= \int_a^b \bar{F}(\bar{r}(t)) \cdot \bar{r}'(t)\, dt$$

Def

A line (or path) integral of a vector field $\bar{F}$ over a curve C with parameter $\bar{r}(t)$ is defined by

$$\int_C \bar{F}(\bar{r}) \cdot d\bar{r} := \int_a^b \bar{F}(\bar{r}(t)) \cdot \bar{r}'(t)\, dt$$

Other notations

$$\int_C \bar{F} \cdot d\bar{r}$$

Formally

$$d\bar{r} = dx\, \bar{e}_x + dy\, \bar{e}_y + dz\, \bar{e}_z$$

$$\int_C \bar{F} \cdot d\bar{r} = \int_C (F_1\, dx + F_2\, dy + F_3\, dz)$$

$$= \int_a^b (F_1 x' + F_2 y' + F_3 z')\, dt$$

note $dx = x'dt$, $dy = y'dt$, $dz = z'dt$.

Example

$$\bar{F} := -y\,\bar{e}_x + xy\,\bar{e}_y$$

C given by $\bar{r}(t) = \cos t\, \bar{e}_x + \sin t\, \bar{e}_y$, $0 \le t \le \pi$

Compute $\int_C \bar{F} \cdot d\bar{r} = (*)$

$$\bar{r}'(t) = -\sin t\, \bar{e}_x + \cos t\, \bar{e}_y$$

$$\bar{F}(\bar{r}(t)) = -\sin t\, \bar{e}_x + \cos t \sin t\, \bar{e}_y$$

$$\bar{F}(\bar{r}(t)) \cdot \bar{r}'(t) = \sin^2 t + \cos^2 t \sin t$$

$$(*) = \int_0^\pi (\sin^2 t + \cos^2 t \sin t)\, dt = \frac{\pi}{2} + \frac{2}{3}$$

$$\left[\left[-\frac{\cos^3 t}{3} \right]_0^\pi = \frac{2}{3} \right]$$

Alternatively

$F_1 = -y$, $F_2 = xy$

$$\int_C \bar{F} \cdot d\bar{r} = \int_C (-y)\,dx + \int_C (xy)\,dy$$

C may be parameterised

$y = \sqrt{1-x^2}$ $\qquad dy = \frac{-x}{\sqrt{1-x^2}}\,dx$

$$\int_1^{-1} -\sqrt{1-x^2} + x\sqrt{1-x^2}\left(\frac{-x}{\sqrt{1-x^2}}\right) dx$$

$$= \int_1^{-1} -\sqrt{1-x^2} - x^2\,dx$$

Finish as an exercise. $\left[\left[\frac{x^3}{3}\right]_{-1}^{1} = \frac{2}{3}\right]$

The result is independent of the parameterisation, it depends on $\bar{F}$, C, a, b.

Exercise

Do the same, but with

$\bar{r}(t) = \cos(t^2)\bar{e}_x + \sin(t^2)\bar{e}_y$, $0 \le t \le \sqrt{\pi}$

Properties of the line integral

Let $\lambda \in \mathbb{R}$, $\bar{F}, \bar{G} : \mathbb{R}^3 \to \mathbb{R}^3$

C, C_1, C_2 oriented curves

s.t. terminal point of C_1 is the initial point of C_2.

1). $\int_C \lambda\bar{F} \cdot d\bar{r} = \lambda \int_C \bar{F} \cdot d\bar{r}$

2). $\int_C (\bar{F} + \bar{G}) \cdot d\bar{r} = \int_C \bar{F} \cdot d\bar{r} + \int_C \bar{G} \cdot d\bar{r}$

3). $\int_{-C} \bar{F} \cdot d\bar{r} = -\int_C \bar{F} \cdot d\bar{r}$

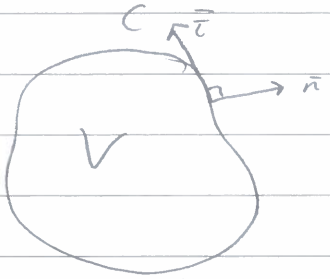
$\nearrow$

change of orientation.

4). $\int_{C_1+C_2} \bar{F}\cdot d\bar{r} = \int_{C_1} \bar{F}\cdot d\bar{r} + \int_{C_2} \bar{F}\cdot d\bar{r}$

Recall divergence thm in 2D

$$\int_S \bar{F}\cdot\bar{n}\, dS = \iint_V \bar{\nabla}\cdot\bar{F}\, dV$$



What about

$$\int_C \bar{F}\cdot d\bar{r} = \int_C \bar{F}\cdot\bar{\tau}\, ds$$

$\bar{n} = \bar{\tau}^{\perp} = (\tau_2 - \tau_1)$

$\bar{F}^{\perp} = (F_2 - F_1)$

$\bar{F}^{\perp}\cdot\bar{\tau}^{\perp} = F_2\tau_2 + F_1\tau_1 = \bar{F}\cdot\bar{\tau}$

$\bar{F}\cdot\bar{\tau} = \bar{F}^{\perp}\cdot\bar{\tau}^{\perp} = \bar{F}^{\perp}\cdot\bar{n}$

$$\int_C \bar{F}\cdot d\bar{r} = \int_C \bar{F}\cdot\bar{\tau}\, ds$$
$$= \int_C \bar{F}^{\perp}\cdot\bar{n}\, ds$$
$$= \iint_V \bar{\nabla}\cdot\bar{F}^{\perp}\, dV$$

so $\int_c \bar{F}\cdot d\bar{r} = \iint_V \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) dV$

C has positive orientation.

Greens' Theorem

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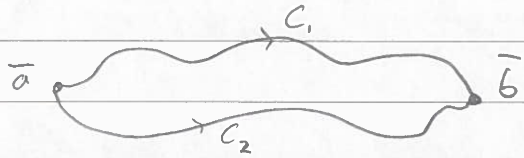
Path independence and the curl

$\bar{F} : \mathbb{R}^3 \to \mathbb{R}^3$

Curve C_1 connects $\bar{a}$ & $\bar{b}$.

Assume that C_2 also connects $\bar{a}$ & $\bar{b}$.

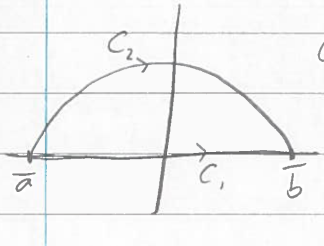


In general $\int_{C_1} \vec{F} \cdot d\bar{r} \neq \int_{C_2} \vec{F} \cdot d\bar{r}$

The integral of $\bar{F}$ from $\bar{a}$ to $\bar{b}$ depends on the path C!

Example

$\bar{F} = -y\,\bar{e}_x + x\,\bar{e}_y$, $\bar{a} = (-1,0)$, $\bar{b}(1,0)$



$C_2 = \{(x,y) : x^2+y^2=1,\ y>0\}$

On C_1: $\begin{cases} \bar{F} = x\,\bar{e}_y \\ d\bar{r} = \bar{e}_x \end{cases}$ so $\bar{F} \cdot d\bar{r} = 0$

$\Rightarrow \int_{C_1} \bar{F} \cdot d\bar{r} = 0$

On C_2: $\begin{cases} \bar{r}(t) = \cos t\,\bar{e}_x + \sin t\,\bar{e}_y \\ \bar{r}'(t) = -\sin t\,\bar{e}_x + \cos t\,\bar{e}_y \end{cases}$, $F(\bar{r}(t)) = -\sin t\,\bar{e}_x + \cos t\,\bar{e}_y$

so $\int_{C_2} \bar{F} \cdot d\bar{r} = \int_{\pi}^{0} \sin^2 t + \cos^2 t\, dt = -\pi$

Defn - Circulation

Let C be a closed path.

The path integral

$$\oint_C \bar{F} \cdot d\bar{r} \qquad (\text{where } C = C_1 - C_2)$$

is called the circulation of $\bar{F}$ arond C.

If $\oint_C \bar{F} \cdot d\bar{r} = 0$ $\forall$ closed curve C, we say that the integral $\oint_C \bar{F} \cdot d\bar{r}$ is path independent

Example

Assume that for $\bar{F} : \mathbb{R}^3 \to \mathbb{R}^3$

$\exists f : \mathbb{R}^3 \to \mathbb{R}$ s.t. $\bar{F} = \bar{\nabla} f$.

Set $f(\bar{r}(t)) : \mathbb{R} \to \mathbb{R}$ where $\bar{r}$ is a parametrisation of C_1, note $\bar{r}(a) = \bar{a}$, $\bar{r}(b) = \bar{b}$

Chain rule:

$$\frac{d}{dt} f(\bar{r}(t)) = \bar{r}'(t) \cdot \bar{\nabla} f(\bar{r}(t))$$

$$= \bar{r}'(t) \cdot \bar{F}(\bar{r}(t))$$

Then

$$\int_{C_1} \bar{F} \cdot d\bar{r} = \int_a^b F(\bar{r}(t)) \cdot \bar{r}'(t)\, dt$$

$$= \int_a^b \frac{d}{dt} f(\bar{r}(t))\, dt$$

fundamental theorem of calculus

$$= f(\bar{r}(b)) - f(\bar{r}(a))$$

$$= f(\bar{b}) - f(\bar{a})$$

$$= \int_{\mathcal{E}} \bar{F} \cdot d\bar{r} \qquad \forall \text{ curve } \mathcal{E} \text{ connecting } \bar{a} \text{ and } \bar{b}.$$

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Flux integral

$$\iint_S \bar F\cdot\bar n\, dS \;+\; \lim \;\Rightarrow\; \text{divercenge.}$$

Circulation integral

$$\int_C \bar F\cdot d\bar r \;+\; \lim \;\Rightarrow\; \text{differential operator.}$$

- Fix $\bar x = (x, y, z) \in \mathbb{R}^3$.
- Fix a plane through $\bar x$ with unit normal $\bar n$.
- Let $\{C_\epsilon\}$ be a set of closed curves "around" $\bar x$ in the plane oriented by $\bar n$.
- C_ϵ "shrinks" to x as $\epsilon \to 0$.
- Let ΔS_ϵ be the surface area enclosed by C_ϵ.

$$\lim_{\epsilon\to 0} \frac{1}{\Delta S_\epsilon} \oint_{C_\epsilon} \bar F\cdot d\bar r =: \text{curl } \bar F \quad \text{in direction } \bar n.$$

Assume the limit exists.

Example

$\bar F = -y\,\bar e_x + x\,\bar e_y$, $\bar n = \bar e_z$, $\bar x = (0,0,0)$

C_ϵ is parameterised by $\bar r = \epsilon(\cos t\,\bar e_x + \sin t\,\bar e_y)$

$S_\epsilon = \pi\epsilon^2$

$$\oint_{C_\epsilon} \bar F\cdot d\bar r \underset{\substack{\uparrow \\ \text{check} \\ \text{orientation}}}{=} \epsilon^2 \int_0^{2\pi} \underbrace{(\sin^2 t + \cos^2 t)\, dt}_{\bar F(\bar r(t))\cdot \bar r'(t)} = 2\pi\epsilon^2$$

curl F, $\bar e_z$ direction $= \lim_{\epsilon\to 0} \frac{1}{\cancel{\pi}\cancel{\epsilon^2}}\cdot 2\cancel{\pi}\cancel{\epsilon^2} = 2$

Observation

Each direction $\bar n$ gives a different value of the curl.

- Let curl F be a vector
  $\overline{\text{curl}}\, F : \mathbb{R}^3 \to \mathbb{R}^3$
- $\overline{\text{curl}}\, \bar F\cdot\bar e_x$ is the curl in the $\bar e_x$ direction, this

is similar for $\bar{e}_y$, $\bar{e}_z$, or in the general case $\text{curl}\,\bar{F}\cdot\bar{n}$.

Definition (curl)

Let $\text{curl}\,\bar{F} : R^3 \to R^3$

$$\text{curl}\,\bar{F}(\bar{x}) := g(\bar{x},\bar{e}_x)\,\bar{e}_x + g(\bar{x},\bar{e}_y)\,\bar{e}_y + g(\bar{x},\bar{e}_z)\,\bar{e}_z$$

where $g(\bar{x},\bar{p})$ is defined by

$$g(\bar{x},\bar{p}) := \lim_{\epsilon\to 0} \frac{1}{\Delta S_\epsilon} \oint_{C_\epsilon} \bar{F}\cdot d\bar{r}$$

and where $\{C_\epsilon\}$ is a family of closed curves around $\bar{x}$ in the plane with normal $\bar{p}$ enclosing area ΔS_ϵ and shrinking to $\bar{x}$ as $\epsilon \to 0$.

How to derive the differential operator

$$\text{curl}\,\bar{F} := \begin{vmatrix} \bar{e}_x & \bar{e}_y & \bar{e}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix}$$

$$\bar{\nabla} = \frac{\partial}{\partial x}\bar{e}_x + \frac{\partial}{\partial y}\bar{e}_y + \frac{\partial}{\partial z}\bar{e}_z \Rightarrow \text{curl}\,\bar{F} = \bar{\nabla}\times\bar{F}.$$

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The curl on differential form

Def

The curl of a vectorfield $\bar{F}$ is a vector valued function given by

$$\text{curl } \bar{F} := \left(\frac{\partial F_3}{\partial y} - \frac{\partial f_2}{\partial z}\right)\bar{e}_x + \left(\frac{\partial f_1}{\partial z} - \frac{\partial f_3}{\partial x}\right)\bar{e}_y + \left(\frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y}\right)\bar{e}_z$$

Memory trick:

$$\text{curl } \bar{F} = \begin{vmatrix} \bar{e}_x & \bar{e}_y & \bar{e}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f_1 & f_2 & f_3 \end{vmatrix}$$

Recall

$$\bar{\nabla} = \frac{\partial}{\partial x}\,\bar{e}_x + \frac{\partial}{\partial y}\,\bar{e}_y + \frac{\partial}{\partial z}\,\bar{e}_z$$

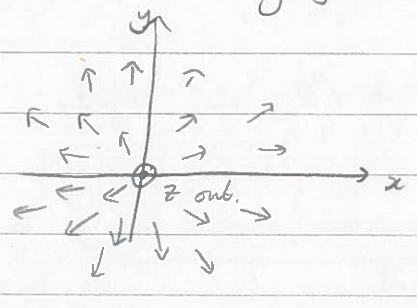
so $\text{curl } \bar{F} = \bar{\nabla} \times \bar{F}$

$$\text{curl } \bar{F} \cdot \bar{n} = \lim_{\epsilon \to 0} \frac{1}{\Delta S_\epsilon} \oint_{C_3} \bar{F} \cdot d\bar{r}$$

Curl measures the local circulation, or rotation of a vectorfield / flow (sometimes "rot").

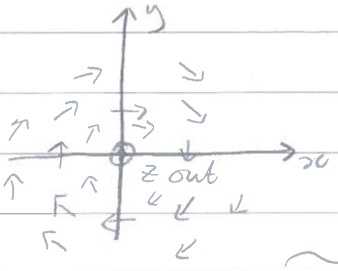
Examples

1). $\bar{F} = x\bar{e}_x + y\bar{e}_y + 0\bar{e}_z$



$$\text{curl } \bar{F} = \overbrace{\left(\frac{\partial 0}{\partial y} - \frac{\partial y}{\partial z}\right)}^{0}\bar{e}_x + \overbrace{\left(\frac{\partial x}{\partial z} - \frac{\partial 0}{\partial x}\right)}^{0}\bar{e}_y + \underbrace{\left(\frac{\partial y}{\partial x} - \frac{\partial x}{\partial y}\right)}_{0}\bar{e}_z = \bar{0}$$

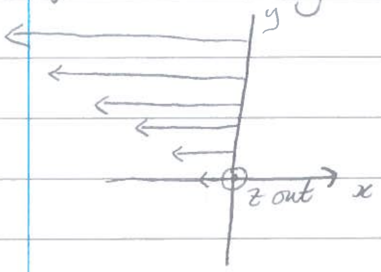
2). $\bar{F} = y\bar{e}_x - x\bar{e}_y$



$$\operatorname{curl} \bar{F} = \overbrace{\left(\frac{\partial 0}{\partial y} - \frac{\partial(-x)}{\partial z}\right)}^{0} \bar{e}_x + \overbrace{\left(\frac{\partial y}{\partial z} - \frac{\partial 0}{\partial x}\right)}^{0} \bar{e}_y + \overbrace{\left(\frac{\partial(-x)}{\partial x} - \frac{\partial y}{\partial y}\right)}^{-2} \bar{e}_z$$

$$= -2\,\bar{e}_z$$

3). $\bar{F} = -(y+1)\bar{e}_x$



guess: $\operatorname{curl} \bar{F} \cdot \bar{e}_z > 0$

$$\operatorname{curl} \bar{F} := 0\bar{e}_x + 0\bar{e}_y + \left(\frac{\partial(0)}{\partial x} - \frac{\partial -(y+1)}{\partial y}\right) \bar{e}_z$$

$$= 1\,\bar{e}_z$$

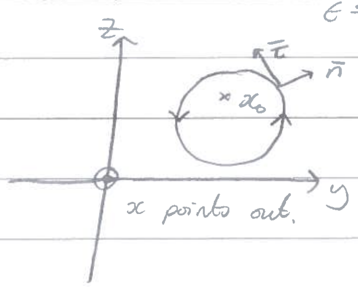
How do we come from

$$\lim_{\epsilon \to 0} \frac{1}{\Delta S_c} \oint_{C_\epsilon} \bar{F} \cdot d\bar{r} \qquad \text{to} \qquad \begin{vmatrix} \bar{e}_x & \bar{e}_y & \bar{e}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix}$$

$$\operatorname{curl} F = g(x_0, \bar{e}_x)\bar{e}_x + g(x_0, \bar{e}_y)\bar{e}_y + g(x_0, \bar{e}_z)\bar{e}_z$$

We only need to consider the directions $\bar{e}_x$, $\bar{e}_y$, $\bar{e}_z$.

$$\operatorname{curl} \bar{F} \cdot \bar{e}_x = \lim_{\epsilon \to 0} \frac{1}{\Delta S_\epsilon} \oint_{C_\epsilon} \bar{F} \cdot d\bar{r}$$



$$\bar{n} = \bar{\tau} \times \bar{e}_x = (0, \tau_z, -\tau_y)$$

$$\bar{F}^{\perp} = (0, F_3, -F_2)$$

$$\bar{F} \cdot \bar{\tau} = \bar{F}^{\perp} \cdot \bar{n}$$

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$$\int_{C_\epsilon} \bar{F}\cdot \underbrace{\bar{\tau}\, dr}_{d\bar{r}} = \int_{C_\epsilon} \bar{F}^{\perp}\cdot \bar{n}\, dr = \iint_{S_\epsilon} \bar{\nabla}\cdot \bar{F}^{\perp}\, dS$$

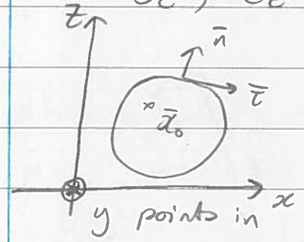
$$\text{so} \quad \text{curl}\, \bar{F}\cdot \bar{e}_x = \lim_{\epsilon\to 0} \frac{1}{\Delta S_\epsilon} \oint_{C_\epsilon} \bar{F}\cdot d\bar{r} = \lim_{\epsilon\to 0} \frac{1}{\Delta S_\epsilon} \iint_{S_\epsilon} \bar{\nabla}\cdot \bar{F}^{\perp}\, dS \quad (*)$$

$$(*) \quad \lim_{\epsilon\to 0} \frac{1}{\Delta S_\epsilon} \iint_{S_\epsilon} \underbrace{\left(\bar{\nabla}\cdot \bar{F}^{\perp} - \bar{\nabla}\cdot \bar{F}^{\perp}(\bar{x}_0)\right)}_{\to 0} dS + \underbrace{\frac{\Delta S_\epsilon}{\Delta S_\epsilon}\, \bar{\nabla}\cdot \bar{F}^{\perp}(\bar{x}_0)}_{\frac{1}{\Delta S_\epsilon} \iint_{S_\epsilon} \bar{\nabla}\cdot \bar{F}^{\perp}(x_0)\, dS}$$

$$= \bar{\nabla}\cdot \bar{F}^{\perp}(\bar{x}_0) = \frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}$$

$\underline{\bar{e}_y}$: $$\text{curl}\, \bar{F}\cdot \bar{e}_y = \lim_{\epsilon\to 0} \frac{1}{\Delta S_\epsilon} \oint_{C_\epsilon} \bar{F}\cdot d\bar{r}$$

S_ϵ, C_ϵ in the $x-z$ plane



$$\bar{n} = \bar{\tau} \times \bar{e}_y = (-\tau_z, 0, \tau_x)$$
$$\bar{F}^{\perp} = (-F_3, 0, F_1)$$

$$\text{curl}\, \bar{F}\cdot \bar{e}_y = \bar{\nabla}\cdot \bar{F}^{\perp}(\bar{x}_0) = \left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\right)$$

$\underline{\bar{e}_z}$: left as an exercise.

Irrotational vectorfields and path independant integrals

Def

A vectorfield $\bar{F}$ is called irrotational if $\bar{\nabla}\times\bar{F} = 0$ everwhere

Example

$$\bar{F} = F_1(x)\bar{e}_x + F_2(y)\bar{e}_y + F_3(z)\bar{e}_z$$

Suppose $\int_C \bar{F}\cdot d\bar{r}$ is path independant.

$$\Rightarrow \oint_C \bar{F}\cdot d\bar{r} = 0 \qquad \text{circulation} = 0 \Rightarrow \bar{\nabla}\cdot\bar{F} = 0$$

(check using definition).

For any vector field $\bar{F}$ whose path integrals are path independent, we have

$$\bar{\nabla} \times \bar{F} = 0.$$

Important special case

$$\exists f : \mathbb{R}^3 \to R \quad \text{s.t.} \quad \bar{F} = \bar{\nabla} f$$

Hence $\bar{\nabla} \times (\bar{\nabla} f) = 0$

$$\bar{\nabla} \times (\bar{\nabla} f) = \begin{vmatrix} e_x & e_y & e_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} & \frac{\partial f}{\partial z} \end{vmatrix} = 0$$

$$\bar{\nabla} \cdot (\bar{\nabla} \times \bar{F}) = \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix}$$

$$= \frac{\partial}{\partial x}\left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right) - \frac{\partial}{\partial y}\left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\right) + \frac{\partial}{\partial z}\left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) = 0$$

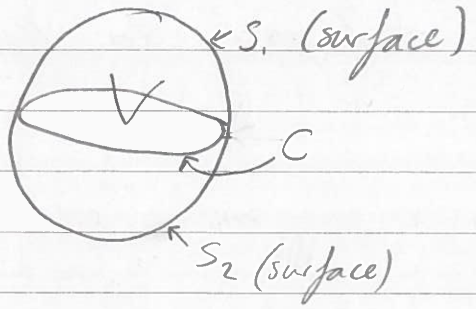
Lemma

If $\bar{\nabla} \times \bar{F} \neq 0$ at some point $\bar{x}_0$ then $\bar{F}$ does not have path independent integrals.

Observation: for any volume V enclosed by a smooth S

$$0 = \iiint_V \nabla \cdot (\bar{\nabla} \times \bar{F})\, dV \underset{\substack{\uparrow \\ \text{divergence} \\ \text{theorem}}}{=} \iint_S (\bar{\nabla} \times \bar{F}) \cdot \underset{\substack{\uparrow \\ \text{outward.}}}{\bar{n}}\, dS$$

$\bar{\nabla} \times \bar{F}$ has zero flux over closed surfaces provided well defined $\bar{F}$ in interior of V and on S.



S is cut by C into two parts S_1 and S_2

$$\iint_{S_1} (\bar{\nabla} \times \bar{F}) \cdot \bar{n}_1 \, dS = -\iint_{S_2} (\bar{\nabla} \times \bar{F}) \cdot \bar{n}_2 \, dS = G(C)$$

$$\iint_{S_1} (\bar{\nabla} \times \bar{F}) \cdot \bar{n}_1 \, dS = \iint_{S_2} (\bar{\nabla} \times \bar{F}) \cdot \bar{n}_1 \, dS = G(C)$$

Fundamental theorem of calculus

$$f(b) - f(a) = \int_a^b f'(x) \, dx$$

We are looking for something of this form where the RHS = $\iint_S (\bar{\nabla} \times \bar{F}) \cdot \bar{n} \, dS$.

Stoke's Theorem

"The fundamental theorem of calculus" for curl.

Def (capping surface)

Given a closed curve C, a capping surface for C is any smooth surface with C as its boundary

Three steps

1). Decompose S into $\{S_i\}$ leading to a telescoping sum. $\oint_C \bar{F} \cdot d\bar{r} = \oint_{C_1} \bar{F} \cdot d\bar{r} + \oint_{C_2} \bar{F} \cdot d\bar{r}$

2). Differential quotient in 1D here use for small enough C_ϵ $\oint_{C_\epsilon} \bar{F} \cdot d\bar{r} \approx \Delta S_\epsilon (\bar{\nabla} \times \bar{F}) \cdot \bar{n}$

3) write a Riemann sum and pass to the limit.

Stoke's theorem

Given a curve C and a capping surface S, if $\bar{F}$ is a smooth vector field defined on C and S then

$$\oint_C \bar{F}\cdot d\bar{r} = \iint_S (\bar{\nabla}\times\bar{F})\cdot \underset{\uparrow}{\bar{n}}\, dS$$

$\bar{n}$ normal given by right hand rule and orientaton of C.

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Theorem (Stoke's)

Given a closed curve ∂S and a capping surface S. If $\bar{F}$ is a smooth vector field defined on ∂S and S then

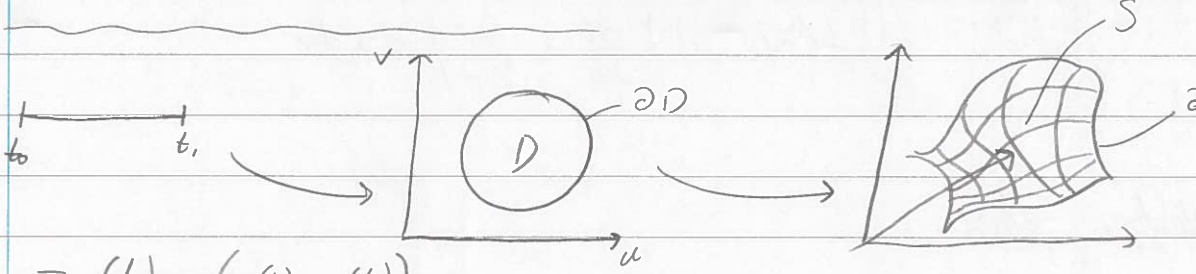
$$\oint_{\partial S} \bar{F} \cdot d\bar{r} = \iint_{S} (\bar{\nabla} \times \bar{F}) \cdot \bar{n} \, dS$$

Theorem (Green's)

Given a closed curve ∂D in the xy-plane, anticlockwise orientation, around D let $\bar{G} = P(x,y)\,\bar{e}_x + Q(x,y)\,\bar{e}_y$ be smooth, defined on ∂D, D. Then

$$\oint_{\partial D} \bar{G} \cdot d\bar{r} = \iint_{D} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx \, dy$$

Proof: Divergence theorem in 2D.



$\bar{r}_{\partial D}(t) = (u(t), v(t))$

$\partial D = \{ (u(t), v(t)) \in \mathbb{R}^2 \, ; \, t_0 \leq t \leq t \}$

$\bar{r}(u,v) = x(u,v)\,\bar{e}_x + y(u,v)\,\bar{e}_y + z(u,v)\,\bar{e}_z$

$S = \{ \bar{r}(u,v) \in \mathbb{R}^3 \, ; \, (u,v) \in D \}$

$\partial S = \{ \bar{r} \circ \bar{r}_{\partial D} \in \mathbb{R}^3 \, ; \, t_0 \leq t \leq t_1 \}$

Integral on a curve

Scalar, f:

$$\int_{\partial D} f \, dr = \int_{t_0}^{t_1} f(\bar{r}_{\partial D}(t)) \, |\bar{r}'_{\partial D}(t)| \, dt$$

Vector, $\bar{F}$:

$$\int_{\partial D} \bar{F} \cdot d\bar{r} = \int^{t_1} \bar{F}(\bar{r}_{\partial D}(t)) \frac{\bar{r}'_{\partial D}(t)}{|\bar{r}'_{\partial D}(t)|} \, |\bar{r}'_{\partial D}(t)| \, dt$$

$$= \int_{t_0}^{t_1} \bar{F}(\bar{r}_{\partial D}(t)) \cdot \bar{r}'(t) \, dt$$

Integral on a surface

Scalar, f:

$$\iint_S f \, dS = \iint_D f(\bar{r}(u,v)) \left| \frac{\partial \bar{r}}{\partial u} \times \frac{\partial \bar{r}}{\partial v} \right| du \, dv$$

Vector flux:

$$\iint_S F \cdot \bar{n} \, dS = \iint_D \bar{F} \cdot \left(\frac{\partial \bar{r}}{\partial u} \times \frac{\partial \bar{r}}{\partial v} \right) du \, dv$$

$$\int_{t_0}^{t_1} \bar{F}(\bar{r}(u(t), v(t))) \cdot \left| \frac{\partial \bar{r}}{\partial u} \frac{\partial u}{\partial t} + \frac{\partial \bar{r}}{\partial v} \frac{\partial v}{\partial t} \right| dt$$

$$= \int_{\partial S} \bar{F} \cdot d\bar{r} = \int_S (\bar{\nabla} \times \bar{F}) \cdot \bar{n} \, ds$$

$$= \iint_D (\bar{\nabla} \times \bar{F}(\bar{r}(u,v))) \cdot \left(\frac{\partial \bar{r}}{\partial u} \times \frac{\partial \bar{r}}{\partial v} \right) du \, dv$$

$$\int_{t_0}^{t_1} \bar{F}(\bar{r}(u(t), v(t)) \cdot \frac{dr}{dt}$$

chain rule

$$= \int_{t_0}^{t_1} \bar{F}(\bar{r}(u(t), v(t))) \cdot \left(\frac{\partial r}{\partial u}\frac{\partial u}{\partial t} + \frac{\partial r}{\partial v}\frac{\partial v}{\partial t}\right) dt$$

$\bar{G} = G_1 \bar{e}_x + G_2 \bar{e}_y$

$$\oint_{\partial D} \bar{G}\cdot d\bar{r} = \int_{t_0}^{t_1}\left(G_1 \frac{\partial u}{\partial t} + G_2\frac{\partial v}{\partial t}\right)dt$$

$$= \int_{t_0}^{t_1}\left(\underbrace{\bar{F}\cdot\frac{\partial \bar{r}}{\partial u}}_{G_1}\frac{du}{dt} + \underbrace{\bar{F}\cdot\frac{\partial\bar{r}}{\partial v}}_{G_2}\frac{dv}{\partial t}\right)dt$$

Green: $\oint_{\partial D}\bar{G}\cdot d\bar{r} = \iint_D \left(\frac{\partial G_2}{\partial u} - \frac{\partial G_1}{\partial v}\right)du\,dv$

$$= \iint_D \underbrace{\left(\frac{\partial \bar{F}}{\partial u}\cdot\frac{\partial\bar{r}}{\partial v} - \frac{\partial\bar{F}}{\partial v}\cdot\frac{\partial\bar{r}}{\partial u}\right)} du\,dv$$

$$\left(\frac{\partial\bar{F}}{\partial u}\frac{\partial\bar{r}}{\partial v} + \bar{F}\cancel{\frac{\partial^2 \bar{r}}{\partial u\partial v}} - \frac{\partial\bar{F}}{\partial v}\cdot\frac{\partial \bar{r}}{\partial u} - \bar{F}\cancel{\frac{\partial^2\bar{r}}{\partial v\partial u}}\right)$$

$$= \iint_D \left(\frac{\partial\bar{F}}{\partial u}\cdot\frac{\partial\bar{r}}{\partial v} - \frac{\partial F}{\partial v}\cdot\frac{\partial\bar{r}}{\partial u}\right)du\,dv$$

<u>Consider one component</u>

$$\frac{\partial F_1}{\partial u} = \frac{\partial F_1}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial F_1}{\partial y}\frac{\partial y}{\partial u} + \frac{\partial F_1}{\partial z}\frac{\partial z}{\partial u}$$

$$\frac{\partial F_1}{\partial u}\frac{\partial x}{\partial v} - \frac{\partial F_1}{\partial v}\frac{\partial x}{\partial u} = \left(\cancel{\frac{\partial F_1}{\partial x}\frac{\partial x}{\partial u}\frac{\partial x}{\partial v}} + \frac{\partial F_1}{\partial y}\frac{\partial y}{\partial u}\frac{\partial x}{\partial v} + \frac{\partial F_1}{\partial z}\frac{\partial z}{\partial u}\frac{\partial x}{\partial v}\right)$$
$$-\left(\cancel{\frac{\partial F_1}{\partial x}\frac{\partial x}{\partial v}\frac{\partial x}{\partial u}} + \frac{\partial F_1}{\partial y}\frac{\partial y}{\partial v}\frac{\partial x}{\partial u} + \frac{\partial F_1}{\partial z}\frac{\partial z}{\partial v}\frac{dx}{du}\right)$$

$$\left.\begin{aligned} &= \left(\frac{\partial F_1}{\partial y}\frac{\partial y}{\partial u} + \frac{\partial F_1}{\partial z}\frac{\partial z}{\partial u}\right)\frac{\partial x}{\partial v} \\ &\quad - \left(\frac{\partial F_1}{\partial y}\frac{\partial y}{\partial v} + \frac{\partial F_1}{\partial z}\frac{\partial z}{\partial v}\right)\frac{\partial x}{\partial u} \end{aligned}\right\} = (*)$$

$$\iint_D (\bar{\nabla} \times \bar{F}) \cdot \left(\frac{\partial \bar{r}}{\partial u} \times \frac{\partial \bar{r}}{\partial v}\right) du\, dv$$

$$(\bar{\nabla} \times \bar{F}) \cdot \left(\frac{\partial \bar{r}}{\partial u} \times \frac{\partial \bar{r}}{\partial v}\right) = \begin{vmatrix} \bar{e}_x & \bar{e}_y & \bar{e}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix} \cdot \begin{vmatrix} \bar{e}_x & \bar{e}_y & \bar{e}_z \\ \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} & \frac{\partial z}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} & \frac{\partial z}{\partial v} \end{vmatrix}$$

$$\begin{aligned} &= \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right)\left(\frac{\partial y}{\partial u}\frac{\partial z}{\partial v} - \frac{\partial z}{\partial u}\frac{\partial y}{\partial u}\right) + \left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\right)\left(\frac{\partial z}{\partial u}\frac{\partial x}{\partial v} - \frac{\partial x}{\partial u}\frac{\partial z}{\partial v}\right) \\ &\quad + \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)\left(\frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial y}{\partial u}\frac{\partial x}{\partial v}\right) \end{aligned}$$

1st component:

$$\left.\begin{aligned} &\frac{\partial F_1}{\partial z}\frac{\partial z}{\partial u}\frac{\partial x}{\partial v} - \frac{\partial F_1}{\partial z}\frac{\partial x}{\partial u}\frac{\partial z}{\partial v} \\ &\quad + \frac{\partial F_1}{\partial y}\frac{\partial y}{\partial u}\frac{\partial x}{\partial v} - \frac{\partial F_1}{\partial y}\frac{\partial y}{\partial v}\frac{\partial x}{\partial u} \end{aligned}\right\} = (*)$$

Thm (stoke's)

Given a closed curve C, a capping surface S, a smooth vector field $\underline{F}: \mathbb{R}^3 \to \mathbb{R}^3$ defined on S and C, then

$$\oint_C \bar{F} \cdot d\bar{r} = \iint_S (\bar{\nabla} \times \bar{F}) \cdot \bar{n} \, dS$$

where $\bar{n}$ is the unit normal of S consistent with the orientation of C (right hand rule)

Example (Electromagnetics)

- A current $\bar{I}$ flows along the z-axis
- The induced magnetic field is:

$$\bar{B}(x,y,z) = \frac{2|\bar{I}|}{c}\left(\frac{-y\bar{e}_x + x\bar{e}_y}{x^2+y^2}\right)$$

Exercise: show that $\bar{\nabla} \times \bar{B} = 0 \quad \forall (x,y,z) \in \mathbb{R}^3 \setminus (0,0,z)$

- $C_2 = \{\bar{r}(t) = 3\cos t \, \bar{e}_x + \sin \bar{e}_y + z\bar{e}_z, \; 0 \leq t \leq 2\pi\}$

$x^2 + 9y^2 = 9$

Compute the circulation of $\bar{B}$ around C_2 anti-clockwise.

Ampere's Law

$$\oint_C \bar{B} \cdot d\bar{r} = \iint_S \bar{I} \cdot \bar{n} \, dS$$

Warmup

Circulation of $\bar{B}$ around unit circle

$\bar{r} = \cos t \, \bar{e}_x + \sin t \, \bar{e}_y$

$\bar{B}(\bar{r}) = \frac{2|\bar{I}|}{c}(-\sin t \, \bar{e}_x + \cos t \, \bar{e}_y)$

$$\int_0^{2\pi} \bar{B}(\bar{r}(t)) \cdot \bar{r}'(t) \, dt = \frac{2|\bar{I}|}{c} \int_0^{2\pi} (\sin^2 t + \cos^2 t) \, dt = \frac{4\pi|\bar{I}|}{c}$$

$$\int_S (\bar{\nabla} \times \bar{B}) \cdot \bar{n} \, dS = \int_{C_1} \bar{B} \cdot d\bar{r} + \int_{-C_2} \bar{B} \cdot d\bar{r}$$

$$\Rightarrow \int_{C_2} \bar{B} \cdot d\bar{r} = \int_{C_1} \bar{B} \cdot dr = \frac{4\pi |\bar{I}|}{c}$$

Green's Theorem

Restrict C to xy-plane

Normal: $\underline{e}_r$

C encloses R.

By Stoke's Thm

$$\oint_C \bar{F} \cdot d\bar{r} = \iint_R (\bar{\nabla} \times \bar{F}) \cdot \bar{e}_z \, dR = \iint_R \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) dx \, dy$$

Computing the area of the enclosed R

$$\oint_C \bar{F} \cdot d\bar{r} = \iint_R \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) dx \, dy$$

Area of $R = \iint_R dx \, dy$

Choose $\bar{F}$ s.t. $\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} = 1$

Choose $\bar{F} = x \, \bar{e}_y$

Area of $R = \oint_C (r_1(t) \bar{e}_y) \cdot (r_1'(t) \bar{e}_x + r_2'(t) \bar{e}_y) \, dt$

$$= \oint_C r_1(t) \, r_2'(t) \, dt$$

Example - Ellipses

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \qquad \bar{r}(t) = a\cos(t)\bar{e}_x + b\sin(t)\bar{e}_y$$
$$\bar{r}'(t) = -a\sin(t)\bar{e}_x + b\cos(t)\bar{e}_y$$

$$\int_0^{2\pi} a\cos t \, b\cos t \, dt = ab\int_0^{2\pi} \cos^2 t \, dt = ab\pi$$

Path independence

- If for $\bar{F}: \mathbb{R}^3 \to \mathbb{R}^3$ $\exists \phi$ s.t. $\bar{F} = \nabla\phi$ then $\int_C \bar{F} \cdot d\bar{r}$ is path independent

- If $\int_C \bar{F} \cdot d\bar{r}$ is path independent then $\bar{\nabla} \times \bar{F} = 0$.

Important:
For stokes theorem, $\bar{F}$ must be defined everywhere in the capping surface.

Define domains that are 'OK'

Def
A closed curve C in a domain D is contractable if we can shrink C to a point in D without leaving D.

Def
A domain D is called simply connected if every closed curve in D is contractable.

If $C \subset D$, D simply connected, $\bar{F}$ is well defined in D $\Rightarrow$ stokes theorem OK.

Important observation:

In a simply connected domain every closed curve has a capping surface lying entirely in D.

$$\oint_C \bar{F}\cdot d\bar{r} = \iint_S (\bar{\nabla}\times\bar{F})\cdot\bar{n}\, dS$$

for all C in D

$$\oint_C \bar{F}\cdot d\bar{r} = 0 \iff \bar{\nabla}\times\bar{F} = 0$$ in D (simply connected)

Path independence diagram

$$\begin{tikzcd}
& \boxed{\oint_C \bar{F}\cdot d\bar{r} \quad \text{Path independent}} \arrow[dl, shift right] \arrow[dr, shift right, "\text{Simply connected domains}"] & \\
\boxed{\bar{F} = \bar{\nabla} f} \arrow[ur, shift right] \arrow[rr, leftrightarrow, "\text{simply connected}", "\text{domains}"'] & & \boxed{\bar{\nabla}\times\bar{F} = 0} \arrow[ul, shift right]
\end{tikzcd}$$

Modelling

Lemma (Raymond - Dubois')

$$\left.\begin{array}{l} f : \bar{\Omega} \to \mathbb{R} \quad \text{if } f \in C^0(\bar{\Omega}) \\ \text{and } \iiint_\omega f\, dV = 0 \quad \forall\, \omega \subset \Omega \end{array}\right\} \Rightarrow f = 0 \text{ in } \Omega$$

Conservation Laws (stationary)

The flux of some quantity over $\partial\omega$ equals the production / destruction in ω.

Example (Heat)

$\bar{q}$ = Heat flow

f = Heat source in $C^0(\Omega)$

$$\iint_{\partial\omega} \bar{q}\cdot\bar{n}\, dS = \iiint_\omega f\, dV$$

$$\underset{\text{divergence theorem}}{\Rightarrow} \iiint_{w} (\bar{\nabla}\cdot\bar{q} - f)\, dV = 0 \qquad \forall w \subset \Omega$$

$$\Rightarrow \quad \bar{\nabla}\cdot\bar{q} = f \quad \text{in } \Omega$$

$$\bar{q} : \bar{\Omega} \to R^3 \qquad \bar{q} = (q_1, q_2, q_3)$$

Constitutive Law

Fourier's Law: $\bar{q} = -\overset{\substack{\text{heat}\\\text{conductivity}}}{\overset{\downarrow}{\lambda}}\, \bar{\nabla} T$

$$\Rightarrow \quad -\bar{\nabla}\cdot(\lambda \bar{\nabla} T) = f \quad \text{in } \Omega$$

Boundary conditions. $T = 0$ on $\partial\Omega$

$\bar{q} = 0$ on $\partial\Omega$ $\quad \bar{\nabla} T \cdot \bar{n} = 0$