2101 Analysis 3: Complex Analysis Notes

Based on the 2016 autumn lectures by Prof M Singer

No lecture notes
Lots of textbook references.

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Office hour: Mon 1-2

03.10.16 Complex Analysis

Introduction

Functions $f(z)$, $z = x + iy$ (x, y real, $i^2 = -1$)
Holomorphic: $f(z)$ is differentiable

Facts:
- f holomorphic $\Rightarrow$ f is infinitely differentiable.
- In fact if f is defined near a point $z_0 \in \mathbb{C}$,
  $f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$ for all $|z - z_0|$ small enough
- (Analytic continuation):
  two holomorphic functions f & g. Suppose
  $f(z) = g(z)$ for all z in $D = \{|z| < 1\}$
  Then actually $f = g$ wherever they are defined.
- All these are completely untrue for real differentiable functions of a real variable.

Fact:
- If f is holomorphic, then $u(x,y) = Re(f(z))$ is harmonic i.e.
  $$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) u(x,y) = 0$$
  $\Rightarrow$ applications of hol (holomorphic) functions in 2D fluid flow.

Shall prove:
Fundamental Thm of Algebra.

Number Theory

$\pi(x)$ = number of primes $\leq x$ ($\pi(12) = 5$).
Prime number theorem: $\pi(x) \sim \frac{x}{\log x}$ for very large x

First proof: 1890's: Made essential use of
$\zeta(z) = \sum_{n=1}^{\infty} \frac{1}{n^z}$ (Riemann zeta function)

Riemann Hypothesis: The only zeros of $\zeta(z)$ in $\{0 < Re(z) < 1\}$ lie on $Re(z) = \frac{1}{2}$.

Chapter 1 Algebra & Geometry of Complex Numbers

1.1 $\mathbb{C}$ = field of complex numbers.

$a = \alpha + i\beta$, $\alpha, \beta \in \mathbb{R}$, $i^2 = -1$

Field $\Rightarrow$ we can add, multiply, have distributive laws etc.

also $a \neq 0 \Rightarrow \exists\, z$ s.t. $az = za = 1$

To find z: suppose $z = x + iy$.

We need $(\alpha + i\beta)(x + iy) = 1$

$\alpha x - \beta y + i(\beta x + \alpha y) = 1$

$\therefore \beta x + \alpha y = 0$

$\alpha x - \beta y = 1$

$$y = -\frac{\beta x}{\alpha}$$

$$\alpha x - \beta\left(-\frac{\beta x}{\alpha}\right) = 1$$

$$(\alpha^2 + \beta^2)x = \alpha$$

Similarly

$$(\alpha^2 + \beta^2)y = -\beta$$

Hence $x + iy = \dfrac{\alpha - i\beta}{\alpha^2 + \beta^2}$ if $\alpha^2 + \beta^2 \neq 0$

Exercise

Show that $3 - 4i$ has a square root in $\mathbb{C}$, by solving $z^2 = (x+iy)^2 = 3 - 4i$.

$x^2 - y^2 + 2ixy = 3 - 4i$

$x^2 - y^2 = 3$, $xy = -2$

so $\dfrac{4}{y^2} - y^2 = 3$

$4 - y^4 = 3y^2$

let $u = y^2 \Rightarrow u^2 + 3u = 4$

$(u+4)(u-1) = 0$

so ~~$u = -4$~~ or $u = 1$

$\Rightarrow y = \pm 1$

$y = 1 \Rightarrow x = -2$

$y = -1 \Rightarrow x = 2$

So $z = \pm 2 \mp i$

Triangle Inequality Proof

We know: 1). $-|z| \le Re(z) \le |z|$

2). $a\bar{b} + \bar{a}b = 2Re(a\bar{b})$

3). $|a+b|^2 = (a+b)(\bar{a}+\bar{b}) = |a|^2 + a\bar{b} + \bar{a}b + |b|^2$

4). $|a-b|^2 = (a-b)(\bar{a}-\bar{b}) = |a|^2 - a\bar{b} - \bar{a}b + |b|^2$

Let $z = a\bar{b}$ $\Rightarrow$ $|z| = \sqrt{a\bar{a}b\bar{b}} = \sqrt{|a|^2|b|^2} = |a||b|$

Using 1)., 2). and $|z|$ we have

$$-2|a||b| \le a\bar{b} + \bar{a}b \le 2|a||b|$$

Adding $(|a|^2 + |b|^2)$ gives

$$|a|^2 - 2|a||b| + |b|^2 \le |a|^2 + a\bar{b} + \bar{a}b \le |a|^2 + 2|a||b| + |b|^2$$

$$\Rightarrow (|a| - |b|)^2 \le |a+b|^2 \le (|a| + |b|)^2$$

By taking (+ve) square roots we get the result:

$$||a| - |b|| \le |a+b| \le |a| + |b|. \quad \square$$

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In fact every complex number has a complex square root, if $(x+iy)^2 = \alpha + i\beta$, then

$$x^2 = \tfrac{1}{2}\left(\alpha + \sqrt{\alpha^2+\beta^2}\right), \quad y^2 = \tfrac{1}{2}\left(-\alpha + \sqrt{\alpha^2+\beta^2}\right)$$

§1.2 <u>Conjugation, absolute value, some inequalities.</u>

If $z = x+iy$, where x & y are real,
the complex conjugate $\bar{z} = x - iy$
$|z|$ = absolute value of z, $|z| = \sqrt{z\bar{z}} = \sqrt{x^2+y^2}$
(note $z\bar{z} = (x+iy)(x-iy) = x^2+y^2$).
Note:
- $|z| = 0$ iff $z = 0$
- $\dfrac{1}{z} = \dfrac{\bar{z}}{|z|^2}$
- In particular $\frac{1}{z} = \bar{z}$ if $|z| = 1$.

Real & Imaginary parts
$Re(z) = \frac{1}{2}(z+\bar{z})$
$Im(z) = \frac{1}{2i}(z-\bar{z})$ NB: <u>Both</u> are <u>real</u> numbers

<u>Triangle Inequality</u>
<u>Proposition</u>
If $a, b \in \mathbb{C}$, $\big||a| - |b|\big| \leq |a+b| \leq |a| + |b|$

<u>Proof</u>
Identities: $|a+b|^2 = (a+b)(\bar{a}+\bar{b}) = |a|^2 + a\bar{b} + \bar{a}b + |b|^2$
$|a-b|^2 = (a-b)(\bar{a}-\bar{b}) = |a|^2 - a\bar{b} - \bar{a}b + |b|^2$
Note $a\bar{b} + \bar{a}b = 2Re(a\bar{b})$
Note for any z, $-|z| \leq Re(z) \leq |z|$

Apply with $z = a\bar{b}$
$$|a|^2 - 2|a||\bar{b}| + |b|^2 \leq |a+b|^2 \leq |a|^2 + 2|a||\bar{b}| + |b|^2$$
$$= |a|^2 + 2|a||b| + |b|^2 = (|a|+|b|)^2$$

So $(|a|-|b|)^2 \leq |a+b|^2 \leq (|a|+|b|)^2$

Result follows by taking (+ve) square roots:
$||a|-|b|| \leq |a+b| \leq |a|+|b|.$

$\square$

Another important inequality: Cauchy-Schwarz:

Proposition

If $a_1, \ldots, a_n, b_1, \ldots, b_n$ are complex numbers, then

$$\left|\sum_{j=1}^{n} a_j b_j\right|^2 \leq \left(\sum_{j=1}^{n} |a_j|^2\right)\left(\sum_{j=1}^{n} |b_j|^2\right)$$

Proof

via 'Lagrange's Identity' is on Problem Set 1.

Exercise 1

Find absolute value of

$$\frac{(3+4i)(-1+2i)}{(1+i)(-3+i)} \quad \text{(don't expand!)}$$

$$\left|\frac{(3+4i)(-1+2i)}{(1+i)(-3+i)}\right| = \frac{|3+4i||-1+2i|}{|1+i||-3+i|}$$

$$= \frac{\sqrt{25}\sqrt{5}}{\sqrt{2}\sqrt{10}} = \frac{5}{2}$$

Exercise 2

If $|z|=2$ find upper bounds for $\frac{1}{z+1}$ and $\frac{1}{z^2+1}$ ie $\left|\frac{1}{z+1}\right|, \left|\frac{1}{z^2+1}\right|$

1). $\left|\frac{1}{z+1}\right| = \frac{1}{|z+1|} \leq \frac{1}{||z|-1|} = \frac{1}{2-1} = 1$

use lower bound as it is a reciprocal.

2). $\left|\frac{1}{z^2+1}\right| = \frac{1}{|z^2+1|} \leq \frac{1}{||z^2|-1|} = \frac{1}{|z||z|-1} = \frac{1}{3}$

$|ab| = |a||b|$

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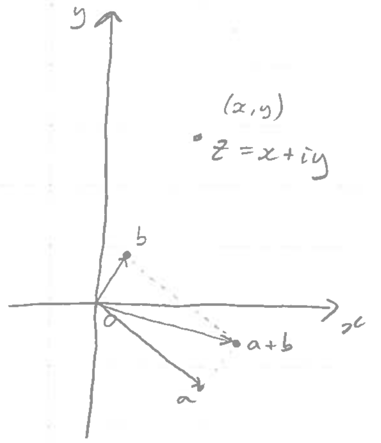
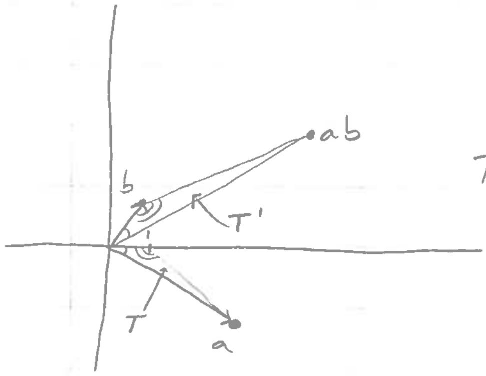
Revise: $\overline{ab} = \bar{a}\bar{b}$, $|ab| = |a||b|$, etc...

§1.3 Geometry of $\mathbb{C}$



Think of the plane $\mathbb{R}^2$, z is identified with the point with coords (x,y). So $a+b$ is represented by vector addition. (parallelogram rule).



T and T' are similar triangles

If $|b| = 1$ then the triangles are the same size and just rotated about the origin. (triangles congruent).

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Warm-up Questions

1) $i^3 = -i$

$i^4 = 1$

$\frac{1}{i} = \frac{i}{i^2} = -i$

i^n (where $n \in \mathbb{Z}$) can take values $i, -1, -i, 1$

2). If $a^2 = 2+i$, what is $|a|$?

$\ulcorner |ab| = |a||b| \lrcorner$

so $|a^2| = |a|^2$

$\Rightarrow \sqrt{4+1} = |a|^2$

so $|a| = \sqrt[4]{5}$

3). What is $\left|\frac{4-i}{1+2i}\right|$?

$= \frac{|4-i|}{|1+2i|} = \frac{\sqrt{17}}{\sqrt{5}}$

4). If $|a|<1$ & $|b|<1$, show that

$\left|\frac{a-b}{1-\bar{a}b}\right| < 1$ $\Rightarrow |a-b| < |1-\bar{a}b|$

Hint: use algebra.

Hint: $\left|\frac{a-b}{1-\bar{a}b}\right| = \frac{|a-b|}{|1-\bar{a}b|}$

square: $\frac{|a-b|^2}{|1-\bar{a}b|^2}$

So we only need to show $\frac{|a-b|^2}{|1-\bar{a}b|^2} < 1$

$\Rightarrow |a-b|^2 < |1-\bar{a}b|^2$

$(|a|-|b|)^2 = |a|^2 + |b|^2 - 2|a|^2|b|^2 < 1$

$|a|<1 \Rightarrow |\bar{a}|<1$

$|\bar{a}||b| = |\bar{a}b| < 1$

$1 - |\bar{a}b| > 0$

$|a-b|^2 = (a-b)(\bar{a}-\bar{b})$

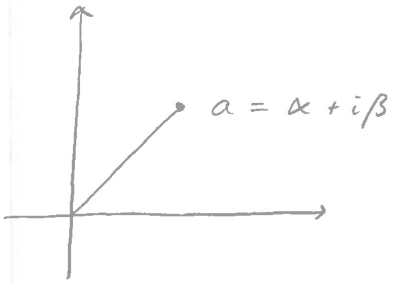
$= a\bar{a} - a\bar{b} - \bar{a}b + b\bar{b}$

$= |a|^2 + |b|^2 - a\bar{b} - \bar{a}b$

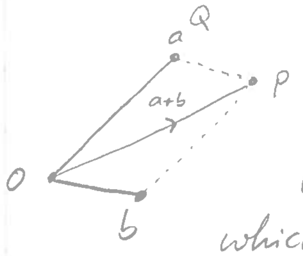
$|1-\bar{a}b|^2 = (1-\bar{a}b)(1-a\bar{b})$

$= 1 - a\bar{b} - \bar{a}b + a\bar{a}b\bar{b}$

§1.3 cont. (Geometry of complex numbers)

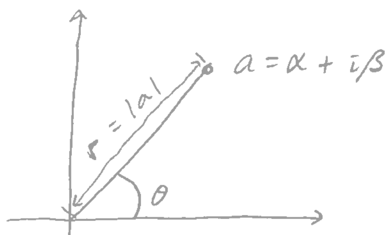


$|a| = \sqrt{a\bar{a}} = \sqrt{\alpha^2+\beta^2}$ = distance of a to O.



In this Δ $\vec{QP} = b$

$|OP| \leq |OQ| + |PQ|$
which is the same as
$|a+b| \leq |a| + |b|$.



Polar form.
$\alpha = r\cos\theta$, $\beta = r\sin\theta$

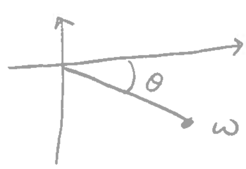
$\alpha + i\beta = r(\cos\theta + i\sin\theta) = re^{i\theta}$

r = modulus, θ = argument of $\alpha + i\beta$ $(\arg(a))$

θ is determined only up to addition of integer multiples of 2π.

Def:
Principle argument, $\mathrm{Arg}(z)$ is the value of $\arg(z)$ lying in $(-\pi, \pi]$

for example, for w as shown,
$\mathrm{Arg}(w) < 0$



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Multiplication:

$z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$, $z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$

then $z_1 z_2 = r_1 r_2(\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 + i\sin\theta_2)$

$= r_1 r_2(\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2 + i(\sin\theta_1\cos\theta_2 + \sin\theta_2\cos\theta_1))$

$= r_1 r_2(\cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2))$

In particular:

$\arg(z_1 z_2) = \arg(z_1) + \arg(z_2) \quad (\text{mod } 2\pi\mathbb{Z})$

For any angle θ,

$|\cos\theta + i\sin\theta| = \cos^2\theta + \sin^2\theta = 1$

In particular

$|z_1 z_2| = r_1 r_2 = |z_1||z_2|$

§1.4 Powers: de Moivre's Theorem

From the product rule:

if $z = r(\cos\theta + i\sin\theta)$

$z^2 = r^2(\cos 2\theta + i\sin 2\theta)$

$\vdots \quad \vdots \quad \vdots$

$z^n = r^n(\cos n\theta + i\sin n\theta)$ (de Moivre)

Hence it is easy to find nth roots of a complex number in polar form.

If $z = r(\cos\theta + i\sin\theta)$, and $z^n = a = R(\cos\varphi + i\sin\varphi)$ then by de Moivre's Thm we must have:

$r^n(\cos n\theta + i\sin n\theta) = R(\cos\varphi + i\sin\varphi)$

Hence $r = R^{1/n}$. Also $n\theta = \varphi + 2k\pi$ for some $k \in \mathbb{Z}$

So $\theta = \frac{\varphi}{n} + \frac{2k\pi}{n}$ for some $k \in \mathbb{Z}$

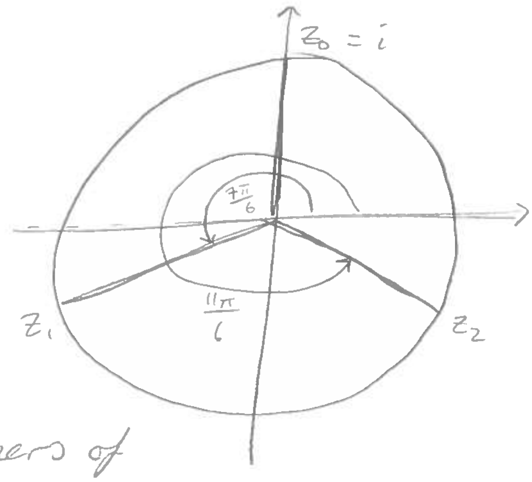
The n nth roots of a are
$$z_k = R^{1/n}\left(\cos\left(\frac{\varphi}{n} + \frac{2k\pi}{n}\right) + i\sin\left(\frac{\varphi}{n} + \frac{2k\pi}{n}\right)\right), \quad k = 0, 1, \dots, n-1$$

Ex:
What are the cube roots of $-i$?

$$z^3 = -i = \cos(3\pi/2) + i\sin(3\pi/2)$$

$$z_k = 1\left(\cos\left(\frac{\pi}{2} + \frac{2k\pi}{3}\right) + i\sin\left(\frac{\pi}{2} + \frac{2k\pi}{3}\right)\right) \qquad k = 0, 1, 2$$

$z_0 = i$



$$\arg(z_1) = \frac{\pi}{2} + \frac{2\pi}{3} = \frac{7\pi}{6}$$

$$\arg(z_2) = \frac{\pi}{2} + \frac{4\pi}{3} = \frac{11\pi}{6}$$

The three roots make the corners of an equilateral triangle.
More generally the n-th roots of $a = R(\cos\varphi + i\sin\varphi)$ are the n corners of a regular polygon with n sides, lying on $|z| = R^{1/n}$ if $R > 0$.

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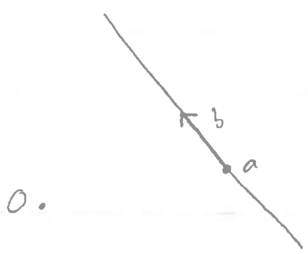
§1.5 Simple geometric figures

• Circle, centre a, radius r is set $\{z : |z-a| = r\}$

$$\Leftrightarrow (z-a)(\bar{z}-\bar{a}) = r^2$$
$$\Leftrightarrow |z|^2 - a\bar{z} - \bar{a}z + |a|^2 - r^2 = 0$$

Conversley any equation of the form
$$\lambda|z|^2 + b\bar{z} + \bar{b}z + c = 0$$
represents a circle if $\lambda \neq 0$ and real, and c is also real. (or point or emptyset)

• Straight line through a, in direction $b \neq 0$ is
$$\{z : z = a + bt : \text{real } t\}$$



(Parametric form of st. line, t=parameter)

§1.6 Extended complex plane and Riemann Sphere.

Introduce ∞, not a number but we define:

$a + \infty = \infty + a = \infty \quad (a \in \mathbb{C})$

$a \cdot \infty = \infty \cdot a = \infty \quad$ if $a \neq 0,\ a \in \mathbb{C}$

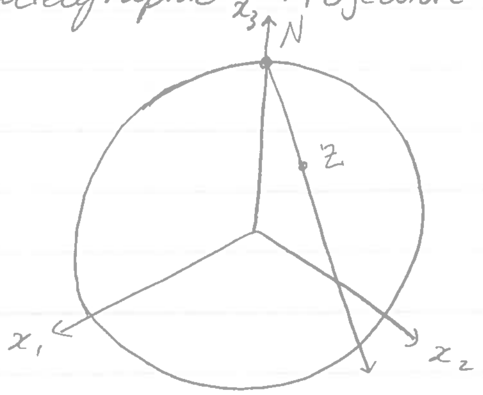
$\frac{a}{0} = \infty$ if $a \neq 0$, $\frac{a}{\infty} = 0$ if $a \in \mathbb{C}$.

By def^n - extended complex plane $\mathbb{C}_\infty = \mathbb{C} \cup \{\infty\}$

Motivation: for example, if
$$f(z) = \frac{1}{z}$$

then natural to regard $f : \mathbb{C} \to \mathbb{C} \cup \{\infty\}$, $f(0) = \infty$.
Even $f : \mathbb{C}_\infty \to \mathbb{C}_\infty$, $f(0) = \infty$.

Stereographic Projection



$S = \{x_1^2 + x_2^2 + x_3^3 = 1\}$

$N = (0, 0, 1)$

Z_1 on S, $Z_1 \neq N$

Join Z_1, N by a straight line : the intersection with $\{x_3 = 0\}$ is called the stereographic projection of Z_1.

• Formula for stereographic projection

Parametric form of st. line joining N to Z_1:

$Z_1 = (a_1, a_2, a_3)$. $\vec{NZ_1} = (a_1, a_2, a_3 - 1)$

General point on st. line:

$$p = (0, 0, 1) + t(a_1, a_2, a_3 - 1) \qquad (t \in \mathbb{R})$$

Meets $\{x_3 = 0\}$ when $1 + t(a_3 - 1) = 0$, i.e. $t = 1/1 - a_3$

Let x and y be the coordinates of the stereographic projection.

$$(x, y, 0) = (0, 0, 1) + \frac{1}{1 - a_3}(a_1, a_2, a_3 - 1)$$

Hence $$\left.\begin{aligned} x &= \frac{a_1}{1 - a_3} \\ y &= \frac{a_2}{1 - a_3} \end{aligned}\right\} z = x + iy = \frac{a_1 + ia_2}{1 - a_3}$$

Note: as $a_3 \to 1$, $z \to \infty$.

From the geometry it is natural to define the abstract '∞' with the point N of S.

Stereographic projection has an inverse.
Given z, want (a_1, a_2, a_3), $a_1^2 + a_2^2 + a_3^2 = 1$
such that $z = \dfrac{a_1 + i a_2}{1 - a_3}$

1). Compute $|z|^2$

$$|z|^2 = z\bar{z} = \frac{a_1 + i a_2}{1 - a_3} \cdot \frac{a_1 - i a_2}{1 - a_3} = \frac{a_1^2 + a_2^2}{(1 - a_3)^2}$$

$$= \frac{1 - a_3^2}{(1 - a_3)^2} = \frac{(1 - a_3)(1 + a_3)}{(1 - a_3)^2} = \frac{1 + a_3}{1 - a_3}$$

$$= \frac{2 - (1 - a_3)}{1 - a_3} = \frac{2}{1 - a_3} - 1$$

so $\boxed{a_3 = \dfrac{|z|^2 - 1}{|z|^2 + 1}}$ $\boxed{1 - a_3 = \dfrac{2}{1 + |z|^2}}$

$a_1 + i a_2 = \dfrac{2z}{1 + |z|^2}$ [as $1 - a_3 = \dfrac{a_1 + i a_2}{z}$]

$a_3 = \dfrac{|z|^2 - 1}{|z|^2 + 1}$

If $|z| \to \infty$ $a_3 \to 1$, $a_1 \to 0$, $a_2 \to 0$
consistent with thinking of N as ∞.

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Recall

$\mathbb{S}$ = Riemann Sphere = $\{x_1^2 + x_2^2 + x_3^2 = 1\} \subset \mathbb{R}^3$

Steregraphic projection: (SP)

$$z = \frac{x_1 + ix_2}{1 - x_3}$$

[N = North Pole = $(0,0,1)$]

Theorem:

SP sets up a 1:1 correspondence between $\mathbb{S} \setminus \{N\}$ and $\mathbb{C}$, with inverse

$$z \longmapsto \left(\frac{2\,\mathrm{Re}(z)}{1+|z|^2}, \frac{2\,\mathrm{Im}(z)}{1+|z|^2}, \frac{|z|^2-1}{|z|^2+1} \right)$$
$$= (x_1, x_2, x_3) \in \mathbb{S}$$

Exs 1

Which point on $\mathbb{S}$ does $0 \in \mathbb{C}$ map to?

Exs 2

What set on $\mathbb{S}$ does the unit circle $|z|=1$ correspond to?

Exs 3

If SP maps (x_1, x_2, x_3) to z and (x_1', x_2', x_3') to z', what is the relation between (x_1, x_2, x_3) & (x_1', x_2', x_3') in $z' = -\frac{1}{z}$

1). $z \longmapsto \left(\frac{2(0)}{1+0^2}, \frac{2(0)}{1+0^2}, \frac{0^2-1}{0^2+1} \right) = (0, 0, -1)$

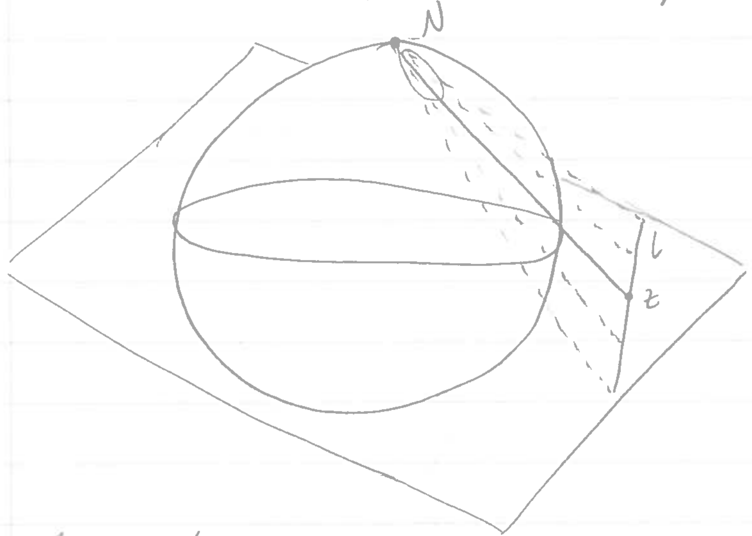
2). $(x_1, x_2, x_3) = (\mathrm{Re}(z), \mathrm{Im}(z), 0)$ (equator on sphere)

3). ?

Remark

If l is a straight line in $\mathbb{C}$ and $z \in l$ is a point, what is $SP^{-1}(z)$?

Note: As z moves on l, the lines joining N to z sweep out a plane.



The intersection of a plane with $\mathbb{S}$ is a circle (if not ϕ or point) and so l corresponds under SP to a circle on $\mathbb{S}$ through N.

Theorem

SP sets up a 1:1 correspondence between:

1). Circles on $\mathbb{S}$

2). Circles and straight lines in $\mathbb{C}$.

Under this correspondence, circles through N on $\mathbb{S}$ go over to straight lines in $\mathbb{C}$.

Proof

Shall start with a circle C on $\mathbb{S}$ and figure out its image under SP.

$$C = \mathbb{S} \cap \{plane\}$$

Can write any plane $\subset \mathbb{R}^3$ in form

$$\alpha_1 x_1 + \alpha_2 x_2 + \alpha_3 x_3 = \alpha_0$$

Can assume $\alpha_1^2 + \alpha_2^2 + \alpha_3^2 = 1$. Also for non-trivial

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intersection with $\mathbb{S}$ can assume $0 \le \alpha_0 < 1$.
(from eqn of plane above)

So if $(x_1, x_2, x_3) \in \mathcal{C}$, we have
$\alpha_1 x_1 + \alpha_2 x_2 + \alpha_3 x_3 = \alpha_0$
and $SP(x_1, x_2, x_3) = z$ satisfies $\quad (z = x + iy)$

$$\alpha_1\left(\frac{2x}{1+|z|^2}\right) + \alpha_2\left(\frac{2y}{1+|z|^2}\right) + \alpha_3\left(\frac{|z|^2-1}{|z|^2+1}\right) = \alpha_0$$

$$\Leftrightarrow 2\alpha_1 x + 2\alpha_2 y + \alpha_3|z|^2 - \alpha_3 = \alpha_0|z|^2 + \alpha_0$$

$$\Leftrightarrow (\alpha_3 - \alpha_0)|z|^2 + 2\alpha_1 x + 2\alpha_2 y = \alpha_0 + \alpha_3 \qquad (*)$$

This is the equation of a circle if $\alpha_3 \neq \alpha_0$ and of a straight line if $\alpha_0 = \alpha_3$.

<u>Ex 4</u>

If $\alpha_0 \neq \alpha_3$ find centre and radius of the circle with equation $(*)$

$$(\alpha_3 - \alpha_0)(x^2 + y^2) + 2\alpha_1 x + 2\alpha_2 y = \alpha_0 + \alpha_3$$

$$x^2 + y^2 + \left(\frac{2\alpha_1}{\alpha_3 - \alpha_0}\right)x + \left(\frac{2\alpha_2}{\alpha_3 - \alpha_0}\right)y = \frac{\alpha_0 + \alpha_3}{\alpha_3 - \alpha_0}$$

$$\left(x + \left(\frac{\alpha_1}{\alpha_3 - \alpha_0}\right)\right)^2 + \left(y + \left(\frac{\alpha_2}{\alpha_3 - \alpha_0}\right)\right)^2 = \frac{\alpha_0 + \alpha_3}{\alpha_3 - \alpha_0} + \frac{\alpha_1^2 + \alpha_2^2}{(\alpha_3 - \alpha_0)^2}$$

$$= \frac{(-\alpha_0^2) + \alpha_3^2 + \alpha_1^2 + \alpha_2^2}{(\alpha_3 - \alpha_0)^2}$$

$$= \frac{1 - \alpha_0^2}{(\alpha_3 - \alpha_0)^2}$$

so centre $= \left(\frac{-\alpha_1}{\alpha_3 - \alpha_0}, \frac{-\alpha_2}{\alpha_3 - \alpha_0}\right) = \left(\frac{\alpha_1}{\alpha_0 - \alpha_3}, \frac{\alpha_2}{\alpha_0 - \alpha_3}\right)$

radius $= \dfrac{\sqrt{1 - \alpha_0^2}}{(\alpha_3 - \alpha_0)}$

§2 Chapter 2

Intro to Holomorphic functions

- For this first 'look', suppose our functions are defined for all $z \in \mathbb{C}$

- Let $f(z)$ be a complex-value function. (Also write $f: \mathbb{C} \mapsto \mathbb{C}$)

Definition:

If $a \in \mathbb{C}$ is a point, then f is said to be differentiable at a if
$$\lim_{h \to 0} \left[\frac{f(a+h) - f(a)}{h} \right] \text{ exists.}$$

Remark:

This copies the real-variable definition:
If $u(x)$ is a function of the real variable x, say u is differentiable at $x = a$ if
$$\lim_{h \to 0} \left[\frac{u(a+h) - u(a)}{h} \right]$$

If $f(z)$ is differentiable at $z = a$, the limit is denoted $f'(a)$ and is called the derivative of f at a.

Recall:

If $G(z)$ is a function of the complex variable z, $\lim_{z \to a} G(z) = A$

means, by definition,
Given $\varepsilon > 0$, $\exists\, \delta > 0$ s.t.
$$|z - a| < \delta \implies |G(z) - A| < \varepsilon.$$

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Definition 2.2

If $f(z)$ is differentiable at every point we say that f is holomorphic.

Proposition 2.3

Suppose that $f(z)$ is holomorphic and that $f(z)$ is real at every point z. Then f must be a constant.

Proof:

We know

$$\lim_{h\to 0}\left[\frac{f(a+h)-f(a)}{h}\right] = A \quad \text{exists for all } a.$$

In particular, letting t be real, we have

$$\lim_{t\to 0}\left(\frac{f(a+t)-f(a)}{t}\right) = A \qquad ①$$

also $$\lim_{t\to 0}\left(\frac{f(a+it)-f(a)}{it}\right) = A \qquad ②$$

Because both are 'instances' of existence of $f'(a) = A$,

LHS of ① is real, so A is real

Multiply ② by i: $\displaystyle\lim_{t\to 0}\frac{f(a+it)-f(a)}{t} = iA$

Hence iA is also real, so A is pure imaginary.
The only number which is real & pure imaginary is 0.
So $f'(a) = 0$, for all a.

Claim: This implies f is constant.

$$f'(a) = \lim_{t\to 0}\left(\frac{f(a+t)-f(a)}{t}\right) = 0$$

the function f on the straight line joining a to c has zero derivative along this line, so $f(c) = f(a)$.

Similarly the derivative of f along the line joining c to b is zero, and so $f(b) = f(c)$

Hence $f(b) = f(a)$ $\square$

<u>Remark:</u>

$f(z) = \text{const}$ is holomorphic (obvious)

$f(z) = z$ is also holomorphic

<u>proof:</u> $\dfrac{f(a+h) - f(a)}{h} = \dfrac{a+h-a}{h} = 1$

Hence $\lim_{h \to 0} \left[\dfrac{f(a+h) - f(a)}{h} \right] = 1$

and so $f(z) = z$ is holomorphic, with derivative 1.

<u>Algebra of limits</u>

(i) If f is holomorphic and g is holomorphic, so is $f+g$ and fg.

(ii) If $g \neq 0$ then $f(z) / g(z)$ is holomorphic

Moreover: $(fg)'(z) = f'(z)g(z) + f(z)g'(z)$

$(f+g)'(z) = f'(z) + g'(z)$

Since z is holomorphic it follows that all powers z^n, $n > 0$ are holomorphic.
By adding a finite number of such powers, with complex coefficients we see that any complex

polynomial
$$P(z) = a_0 + a_1 z + \dots + a_n z^n$$
(when $a_0, \dots, a_n$ are given complex numbers)
is holomorphic.

Note $P'(z) = a_1 + \dots + (n-1)a_{n-1} z^{n-2} + n a_n z^{n-1}$

§2.2 <u>Polynomials</u>

With P as above we say that P has degree $\leq n$.
Shall prove later that any non-constant polynomial has a complex 0.
ie. $\exists \alpha \in \mathbb{C}$ s.t. $P(\alpha) = 0$

By polynomial division, this means we can write
$$P(z) = (z-\alpha) P_1(z), \text{ say}$$
where $\deg(P_1) < \deg(P)$

Either P_1 is constant, in which case
$$P(z) = a_1 (z-\alpha)$$
or it is not, in which case it has another zero, β, say:
$$P(z) = (z-\alpha)(z-\beta) P_2(z)$$

Continuing: (Assuming $a_n \neq 0$)
Every polynomial can be written as a product of factors
$$P(z) = a_n (z-\alpha_1)(z-\alpha_2)\dots(z-\alpha_n)$$
The $\alpha_1, \dots, a_n$ need not all be distinct.
α_i are called <u>roots</u>.
The number of times a particular root occurs is called the order of the root.

For example,

$$P(z) = (z-1)^3(z+2)^2 z$$

1 is a root of order 3,
-2 is a root of order 2,
0 is a root of order 1.

A root of order 1 is called a simple root.

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1). Write $z^n - 1$ as a product of linear factors.

2). A polynomial of degree ≤ 5 has 17 distinct zeros, what can you say about it?

3). If $P(z) = z^4 - 17z^3 + z + 10$ show that $\exists R > 0$ s.t. $|z| > R \Rightarrow |P(z)| > \frac{1}{2}|z|^4$.
[Generalise to an arbitrary polynomial of degree n]

1). $P(z) = a_n(z-\alpha_1)(z-\alpha_2)\dots(z-\alpha_n)$ $\alpha_1, \dots \alpha_n$ are the n zeros (roots) of P.
To factorize, need all solutions of $z^n = 1$.
De Moivre: $\omega = \cos\frac{2\pi}{n} + i\sin\frac{2\pi}{n}$
$\alpha_1 = \omega$, $\omega_2 = \omega^2$, ... , $\omega_{n-1} = \omega^{n-1}$, $\omega^n = 1$
So $z^n - 1 = (z-1)(z-\omega)\dots(z-\omega^{n-1})$
(may help problem 1.2)

2). The polynomial is 0.

Rational Functions

Def^n:
A rational function $R(z)$ is a quotient $R(z) = \frac{P(z)}{Q(z)}$ of two polynomials:

$$R(z) = \frac{a_0 + a_1 z + \dots + a_n z^n}{b_0 + b_1 z + \dots + b_m z^m} \quad (a_n, b_m \neq 0) \quad (Q \text{ not identically zero})$$

Always assume that $P(z)$ and $Q(z)$ have no common factors.

If β is a root of Q $[Q(\beta)=0]$ β is called a pole of R.

Pole β has order (or multiplicity) r if β is a root of Q of order r

Any rational function is holomorphic away from its poles

Extension of R as a map $\mathbb{C} \cup \{\infty\} \to \mathbb{C} \cup \{\infty\}$.
(i) If β is a pole of R, we declare $R(\beta) = \infty$.
(ii) $z = \infty$. Define $R_1(\omega) = R(\frac{1}{\omega})$

$$\begin{aligned} R_1(\omega) &= R(\tfrac{1}{\omega}) \\ &= \frac{a_0 + a_1\omega^{-1} + \dots + a_n\omega^{-n}}{b_0 + b_1\omega^{-1} + \dots + b_m\omega^{-m}} \\ &= \frac{\omega^{-n}(a_n + a_{n-1}\omega + \dots + a_0\omega^n)}{\omega^{-m}(b_m + b_{m-1}\omega + \dots + b_0\omega^m)} \end{aligned}$$

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- If $n > m$, write

$$R_1(\omega) = \frac{a_n + a_{n-1}\omega + \dots + a_0\omega^n}{\omega^{n-m}(b_m + b_{m-1}\omega + \dots + b_0\omega^m)}$$

In this case R_1 has a pole of order $n-m$ at $w = 0$ and we say that R has a zero of order $n-m$ at $\omega = \infty$.

- If $n \le m$, write

$$R_1(\omega) = \frac{\omega^{m-n}(a_n + a_{n-1}\omega + \dots\ a_n\omega^n)}{(b_m + b_{m-1}\omega + \dots + b_0\omega^m)}$$

If $n = m$, $R_1(0) = a_n / b_m$ and we define $R(\infty) = a_n/b_m$.
If $m > n$, we say ∞ is a zero of R of order $m-n$.

<u>Def^n</u>
The degree of our rational function R is defined to be $\max(m, n)$.

<u>Theorem</u>
If w is any point of $\mathbb{C} \cup \{\infty\}$, and R is a rational map of degree d, then $R(z) = w$ has precisely d solutions for $z \in \mathbb{C} \cup \{\infty\}$, counted with multiplicity.

If $R(\alpha) - w = 0$, then α is a root of $S(z) = R(z) - w$. Define the <u>multiplicity</u> of this solution α to be the order of the zero of S at $z = \alpha$.

Remark:

{Rational functions} is a field.

$$R(z)^{-1} = \frac{Q(z)}{P(z)}$$

Partial fractions

Lemma:

Let $R(z)$ be a rational function with a pole at $z = \infty$. Then we can write:

$$R(z) = S(z) + E(z)$$

where S is a polynomial without constant term and E does not have a pole at $z = \infty$.

Proof

Polynomial long division.

$R = P/Q$ so we need

$$P(z) = Q(z)S(z) + E(z)Q(z).$$

Recall: Can find polynomials $T(z)$ and $F(z)$ s.t.

$$P(z) = Q(z)T(z) + F(z), \quad \deg F < \deg Q.$$
$$= Q(z)(T(z) - T(0)) + Q(z)T(0) + F(z).$$

Put $S(z) = T(z) - T(0)$

and $E(z) = T(0) + \dfrac{F(z)}{G(z)}$

to achieve our goal.

Point $E(\infty) = T(0)$, in particular E has no pole at $z = \infty$.

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Thm:
Let R be a rational function with distinct finite poles $\beta_1, \dots, \beta_k$.
Then there exist polynomials $S_1, \dots, S_k$ without constant term and a polynomial $P_\infty(z)$ such that

$$R(z) = S_1\left(\frac{1}{z-\beta_1}\right) + \dots + S_k\left(\frac{1}{z-\beta_k}\right) + P_\infty(z)$$

The decomposition is unique.

Ex:
$$R(z) = \frac{1}{z(z+1)^2}$$

$\beta_1 = 0$, $\beta_2 = -1$
Note β_1 is a simple pole,
β_2 is a pole of order 2

Thm? To use Lemma, consider
$$R_1(\omega) = R\left(\tfrac{1}{\omega}\right), \quad R_2(\omega) = R\left(-1 + \tfrac{1}{\omega}\right)$$
$$R_1(\omega) = \frac{1}{\omega^{-1}(1+\omega^{-1})^2} = \frac{\omega^3}{\omega^2(1+\omega^{-1})^2} = \frac{\omega^3}{(1+\omega)^2}$$
$$R_1(\omega) - \omega = \frac{\omega^3}{(1+\omega)^2} - \omega = \frac{\omega^3 - \omega(\omega^2+2\omega+1)}{(\omega+1)^2}$$
$$= \frac{-2\omega^2 - \omega}{(\omega+1)^2} = E_1(\omega), \quad E_1(\infty) = -2$$

So take $S_1(\omega) = \omega$.

$$R_2(\omega) = \frac{1}{(-1+\frac{1}{\omega})(\omega^{-2})} = \frac{\omega^3}{-\omega+1}$$

$$R_2(\omega) + \omega^2 = \frac{\omega^3 + \omega^2(-\omega+1)}{-\omega+1} = \frac{\omega^2}{-\omega+1}$$

This remainder still has a pole at $\omega = \infty$, so repeat

$$R_2(\omega) + \omega^2 + \omega = \frac{\omega^2}{-\omega+1} + \frac{\omega(-\omega+1)}{-\omega+1}$$

$$= \frac{\omega}{-\omega+1}$$

So set $S_2(\omega) = -\omega^2 - \omega$

Consider:

$$R(z) - S_1\left(\frac{1}{z}\right) - S_2\left(\frac{1}{z+1}\right) = F(z)$$

Where are poles?

No poles at $z=0$ because no pole at ∞ of E_1

Similarly, F has no pot at $z=-1$. So F must be a polynomial.

In our particular case $F=0$.

Hence: $$\frac{1}{z(z+1)^2} = \frac{1}{z} - \frac{1}{z+1} - \frac{1}{(z+1)^2}$$

§ <u>Cauchy - Riemann Equations</u>

$f(z) = f(x+iy) = u(x,y) + iv(x,y)$, u, v real.

<u>Proposition:</u>

If $f(z)$ is holomorphic, then u and v satisfy

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}. \quad (*)$$

Converse: if u and v have continuous first partial derivatives, and satisfy $(*)$ then $f = u + iv$ is holomorphic.

(✱) is the real form of the Cauchy-Riemann Equations.

Remark
(✱) is equivalent to

$$\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y} = 0 \qquad (✱✱)$$

(✱✱) is the complex form of Cauchy-Riemann Equations. (CR)

Proof: (that f holomorphic $\Rightarrow$ CR.
We know that for each $a \in \mathbb{C}$,

$$f'(a) = \lim_{h \to 0}\left[\frac{f(a+h) - f(a)}{h}\right]$$

Let $h = t \in \mathbb{R}$ $\quad (a = \alpha + i\beta)$

$$f'(a) = \lim_{t \to 0}\left(\frac{f(a+t) - f(a)}{t}\right) = \lim_{t \to 0}\left(\frac{f(\alpha+t, \beta) - f(\alpha, \beta)}{t}\right)$$

$$= \frac{\partial f}{\partial x}(\alpha, \beta)$$

Also, letting $h = is$, we have $(s \in \mathbb{R})$

$$f'(a) = \lim_{s \to 0}\left[\frac{f(a+is) - f(a)}{is}\right] = \frac{1}{i}\lim_{s \to 0}\left[\frac{f(\alpha, \beta+s) - f(\alpha, \beta)}{s}\right]$$

$$= \frac{1}{i}\frac{\partial f}{\partial y}(\alpha, \beta)$$

Hence $f'(a) = \frac{\partial f}{\partial x}(a) = \frac{1}{i}\frac{\partial f}{\partial y}(a)$

and so $\frac{\partial f}{\partial x}(a) - \frac{1}{i}\frac{\partial f}{\partial y}(a) = 0.$

This is (✱✱) at a, works at every point a. $\square$

Corollary

If f is holomorphic and real, then f is constant.

$f = u + iv$, $v = 0$.

CR: $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} = 0$

$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} = 0$

Hence $\frac{\partial u}{\partial x} = \frac{\partial u}{\partial y} = 0$ and $u =$ constant.

Holomorphic versus harmonic

Definition:

u with continuous second partial derivatives w.r.t x and y is harmonic if

$\Delta u = 0$ i.e. $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0.$

Proposition:

If $f = u + iv$ is holomorphic and u and v have continuous second partial derivatives then u & v are harmonic.

Proof (u.)

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right) \overset{CR}{=} \frac{\partial}{\partial x}\left(\frac{\partial v}{\partial y}\right) \overset{\text{cont. of partial derivatives}}{=} \frac{\partial}{\partial y}\left(\frac{\partial v}{\partial y}\right)$$

$$\overset{CR}{=} \frac{\partial}{\partial y}\left(-\frac{\partial u}{\partial y}\right) = -\frac{\partial^2 u}{\partial y^2} \quad \square$$

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Thm:

If u is harmonic (in $\mathbb{C}$) then $\exists$ harmonic v, unique up to addition of constant such that $f = u + iv$ is holomorphic.

Def^n:

u and v are called harmonic conjugates.

Proof:

Need to find v such that CR are satisfied: $\dfrac{\partial u}{\partial x} = \dfrac{\partial v}{\partial y}$, $\dfrac{\partial v}{\partial x} = -\dfrac{\partial u}{\partial y}$.

① Define

$$w(x,y) = \int_0^y \frac{\partial u}{\partial x}(x,s)\,ds$$

By fund. thm. of calculus

$$\frac{\partial w}{\partial y} = \frac{\partial u}{\partial x}(x,y)$$

If $\phi(x)$ is any function of x only, then

$$v(x,y) = w(x,y) + \phi(x).$$

We have $\dfrac{\partial v}{\partial y} = \dfrac{\partial v}{\partial x}$.

$$\frac{\partial v}{\partial x} = \frac{\partial}{\partial x}\int_0^y \frac{\partial u}{\partial x}(x,s)\,ds + \phi'(x)$$

$$= \int_0^y \frac{\partial^2 u}{\partial x^2}(x,s)\,ds + \phi'(x)$$

$$= -\int_0^y \frac{\partial^2 u}{\partial y^2}(x,s)\,ds + \phi'(x)$$

$\therefore$ u harmonic

$$= -\frac{\partial u}{\partial y}(x,y) + \frac{\partial u}{\partial y}(x,0) + \phi'(x)$$

Choose $\phi'(x) = -\dfrac{\partial u}{\partial y}(x,0)$ to define v so second CR equation is satisfied.

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$f(z) = \bar{z}$ is not holomorphic

In other words:

$$\lim_{h\to 0}\left(\frac{f(z+h) - f(z)}{h}\right)$$

does not exist for any z.

Difference quotient: $\dfrac{\overline{z+h} - \bar{z}}{h}$

$$= \frac{\bar{z}+\bar{h}-\bar{z}}{h} = \frac{\bar{h}}{h}$$

If $h = t \in \mathbb{R}$, then $\bar{h} = t$, $\dfrac{\bar{h}}{h} = \dfrac{t}{t} = 1$

But if $h = it$, $t \in \mathbb{R}$, $\bar{h} = -it$

$$\frac{\bar{h}}{h} = \frac{-it}{it} = -1$$

These two different values show that $\lim_{h\to 0}\left(\dfrac{\bar{h}}{h}\right)$ does not exist

§3 Chapter 3 Power Series

$$\sum_{n=0}^{N-1} z^n = 1 + z + z^2 + \dots + z^{N-1} = \frac{z^N - 1}{z - 1}$$

If $|z| > 1$, $N \to \infty \Rightarrow \dfrac{z^N - 1}{z-1} \to \infty$ (diverges)

If $|z| < 1$, $N \to \infty \Rightarrow \dfrac{z^N - 1}{z - 1} \to \dfrac{1}{1-z}$ (converges)

$$\sum_{n=0}^{\infty} \frac{z^n}{n!} = 1 + z + \frac{z^2}{2} + \frac{z^3}{6} + \dots = e^z = \exp(z)$$

Convergent for all values of z.

Ratio test: $\left| \dfrac{z^{n+1}}{(n+1)!} \Big/ \dfrac{z^n}{n!} \right| = \left| \dfrac{z}{n+1} \right| \to 0$ as $n \to \infty$ for any fixed $|z|$

so converges $\forall |z|$.

Theorem

Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ (1)

have positive radius of convergence r, and let $D = \{ z : |z| < r \}$.

Then $f(z)$ is holomorphic in D, with derivative $f'(z) = \sum_{n=1}^{\infty} n a_n z^{n-1}$. (2)

The radius of convergence of (2) is the same as (1)

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Proof

Radius of convergence:

Consider, for $s>0$, $\{|a_n|s^n\}$

Either bounded or not (for given s).

If bounded for s and $s'<s$,

$$|a_n(s')^n| = \left|a_n s^n\left(\frac{s'}{s}\right)^n\right|$$

$$= |a_n s^n|\left|\frac{s'}{s}\right|^n \text{ is also bounded.}$$

Radius of convergence:

$$r = \sup\{s \geq 0 : \{|a_n|s^n\} \text{ is bounded}\}.$$

$r=0$ is possible.

$r=\infty$ is also possible if there is no upper bound.

Proposition: (Hadamard)

$$\frac{1}{r} = \lim_{n\to\infty}\left(\sup\{|a_n|^{1/n}\}\right)$$

Claim:

$g(z) = \sum_{n=0}^{\infty} n a_n z^{n-1}$ has same radius of convergence, r, as original series.

Let $b_n = (n+1)a_{n+1}$

$$|b_n|^{1/n} = (n+1)^{1/n}|a_{n+1}|^{1/n}.$$

Fact: $\lim_{n\to\infty}(n+1)^{1/n} = 1$ and $\limsup|a_{n+1}|^{1/n} = \limsup|a_n|^{1/n}$

So $\limsup|b_n|^{1/n} = \frac{1}{r}$. So, by Hadamard, the radius of convergence g is also r.

Now prove that $f'(z) = g(z)$ for $z \in D$

Fix $z_0 \in D$, suppose $|z_0| < r_1 < r$

Need to show: given $\varepsilon > 0$ $\exists \delta > 0 : |z - z_0| < \delta$, we have $\left| \frac{f(z) - f(z_0)}{z - z_0} - g(z_0) \right| < \varepsilon$.

Let $f(z) = f_N(z) + t_N(z)$ where

$$f_N(z) = \sum_{n=0}^{N} a_n z^n \quad \text{and} \quad t_N(z) = \sum_{n=N+1}^{\infty} a_n z^n$$

Similarly, let $g(z) = g_N(z) + s_N(z)$ where

$$g_N(z) = \sum_{n=0}^{N} n a_n z^{n-1} \quad (g_N = f_N') \text{ and } s_N(z) = \sum_{n=N+1}^{\infty} n a_n z^{n-1}$$

Then

$$\frac{f(z) - f(z_0)}{z - z_0} - g(z_0)$$
$$= \left(\frac{f_N(z) - f_N(z_0)}{z - z_0} - g_N(z_0) \right) + \left(\frac{t_N(z) - t_N(z_0)}{z - z_0} - s_N(z_0) \right)$$

Convergence is absolute for $|z|, |z_0| < r_1$ so we can rearrange terms, so

$$t_N(z) - t_N(z_0) = \sum_{n=N+1}^{\infty} a_n (z^n - z_0^n) = \sum_{n=N+1}^{\infty} a_n (z - z_0)(z^{n-1} + z^{n-2} z_0 + \dots + z_0^{n-1})$$

So
$$\left| \frac{t_N(z) - t_N(z_0)}{z - z_0} \right| = \left| \sum_{n=N+1}^{\infty} a_n (z^{n-1} + \dots + z_0^{n-1}) \right|$$
$$\leq \sum_{n=N+1}^{\infty} |a_n| (|z|^{n-1} + \dots + |z_0^{n-1}|)$$
$$\leq \sum_{n=N+1}^{\infty} n |a_n| r_1^{n-1}$$

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g convergent for $|z| < r$
$\Longrightarrow \sum_{n=N+1}^{\infty} |a_n| n r_1^{n-1}$ is convergent

So $\exists N_0$ s.t. $N \geq N_0 : \sum_{n=N+1}^{\infty} n|a_n| r_1^{n-1} < \frac{\varepsilon}{4}$

$S_N(z_0)$ is also a tail of a convergent series
so $\exists N_1 \geq N_0$ s.t.
$$|s_N(z_0)| < \frac{\varepsilon}{4} \text{ if } N \geq N_1$$

Finally since $f_N' = g_N$, we can choose δ so that $|z - z_0| < \delta \Longrightarrow \left| \frac{f_N(z) - f_N(z_0)}{z - z_0} - g_N(z_0) \right| < \frac{\varepsilon}{4}$
for $N = N_1$

Hence:
$$\left| \frac{f(z) - f(z_0)}{z - z_0} - g(z) \right| < \frac{\varepsilon}{4} + \frac{\varepsilon}{4} + \frac{\varepsilon}{4} \text{ if } |z - z_0 < \delta$$
$\square$

Remarks:
Power series based at (centred at) z_0
$$\sum_{n=0}^{\infty} a_n (z - z_0)^n.$$

There is a disk of convergence: $\{z : |z - z_0| < r\}$
Thm works in same way in this setting.

Corollary:
(1) is differentiable to all orders in D, and
$a_n = \frac{f^{(n)}(0)}{n!}$.

$$f'(z) = \sum_{n=1}^{\infty} n a_n z^{n-1} = a_1 + 2a_2 z + \dots$$

$$f'(0) = a_1$$

f' is convergent in $\{z : |z| < r\}$ so it is holomorphic with derivative

$$f''(z) = \sum_{n=2}^{\infty} n(n-1) a_n z^{n-2}$$, convergent in some disc.

$$f''(0) = 2 \cdot 1 \cdot a_2$$

$\vdots$

$f^{(k)}(z)$ is holomorphic in D, represented by a power series convergent in D and

$a_k = \dfrac{f^{(k)}(0)}{k!}$ (Taylor's formula for coefficients).

Exercises

1). Show that if $u : \mathbb{C} \to \mathbb{R}$ has continuous 2nd-order partial derivatives w.r.t. x and y and is harmonic, then $f = \dfrac{\partial u}{\partial x} - i \dfrac{\partial u}{\partial y}$ is holomorphic.

2). Find the radius of convergence of $\sum_{n=0}^{\infty} q^{n^2} z^n$ for fixed $q \in \mathbb{C}$. (Your answer will depend on q.)

3). If $f(z) = \sum_{n=0}^{\infty} a_n z^n$ has radius of convergence $r > 0$,

show that $F(z) = \sum_{n=0}^{\infty} a_n \dfrac{z^{n+1}}{n+1}$ has radius

of convergence r, and $F'(z) = f(z)$

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1). $f(x+iy) = u(x,y) + iv(x,y)$
then $\dfrac{\partial u}{\partial x} = \dfrac{\partial v}{\partial y}$, $\dfrac{\partial v}{\partial x} = -\dfrac{\partial u}{\partial y}$

u harmonic $\Rightarrow \dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0$

u has cont. 2nd-order partial derivatives
$\Rightarrow \dfrac{\partial^2 u}{\partial x \partial y} = \dfrac{\partial^2 u}{\partial y \partial x}$.

Look at $f(z) = a(x,y) + ib(x,y)$
where $a = \dfrac{\partial u}{\partial x}$, $b = -\dfrac{\partial u}{\partial y}$

u harmonic $\Rightarrow \dfrac{\partial^2 u}{\partial x^2} = -\dfrac{\partial^2 u}{\partial y^2}$
$\Rightarrow \dfrac{\partial a}{\partial x} = \dfrac{\partial b}{\partial y}$

Also u cont 2nd order p. derivatives
$\Rightarrow \dfrac{\partial^2 u}{\partial x \partial y} = \dfrac{\partial^2 u}{\partial y \partial x}$
$\Rightarrow -\dfrac{\partial b}{\partial x} = \dfrac{\partial a}{\partial y}$

$\therefore$ u harmonic $\Rightarrow f$ holomorphic.

2). Ratio test: $\left| \dfrac{q^{(n+1)^2} z^{n+1}}{q^{n^2} z^n} \right| = \left| \dfrac{q^{n^2+2n+1}}{q^{n^2}} \right| |z|$
$= |q^{2n+1}| |z|$

fix z: $n \to \infty \Rightarrow |q^{2n+1}| \to \infty$ if $|z| \neq 0$, $|q| > 1$
$|q^{2n+1}| \to 0$, $|q| < 1$

$\therefore$ $|q| < 1 \Rightarrow r = \infty$
$|q| > 1 \Rightarrow r = 0$
$|q| = 1 \Rightarrow r = 1$

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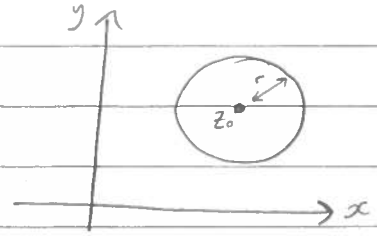
$\sum a_n (z-z_0)^n$ centred at $z=z_0$, will converge in some disc $\{z : |z-z_0| < r\}$



Corollary

Suppose $f(z) = \sum a_n z^n$ as above, radius of convergence > 0.

Let $F(z) = \sum a_n \frac{z^{n+1}}{n+1}$.

Then $F(z)$ has same radius of convergence as $f(z)$ and $F'(z) = f(z)$.

Proof:

Suppose radius of convergence is R. Then by theorem $F'(z) = f(z)$, and $F'(z)$ as same radius of convergence, R, as $F(z)$.

Hence $r = R$. $\square$

[Read main theorem 'backwards']

M-test (Weierstrass)

If $f_n : \Omega \mapsto \mathbb{C}$ (Ω some set)
and $|f_n(z)| \leq M_n$ for all $z \in \Omega$ where $\sum_{n=1}^{\infty} M_n < \infty$ then $\sum f_n(z)$ converges absolutely and uniformly.

Recall:

$\Omega \subset \mathbb{C}$. We say $\sum_{n=0}^{\infty} f_n$ is absolutely convergent on Ω if

$$\sum_{n=0}^{\infty} |f_n(z)| \text{ is convergent of each fixed } \Omega.$$

[Recall that rearrangement of terms is legitimate for absolutely convergent series.]

Say $\sum f_n(z)$ is uniformly convergent if, given $\varepsilon > 0$, $\exists N = N(\varepsilon)$ s.t.

$$\left| \sum_{n=N}^{\infty} f_n(z) \right| < \varepsilon \quad \forall z \in \Omega.$$

Cauchy criterion for uniform convergence:

$\sum f_n$ is uniformly convergent on Ω if given $\varepsilon > 0$, $\exists N = N(\varepsilon)$ such that if $n > m > N$,

$$\left| \sum_{j=m}^{n} f_j(z) \right| < \varepsilon \quad \forall z \in \Omega.$$

Weierstrass M-test follows from this.

Given $\varepsilon > 0$, we use Δ inequality:

$$\left| \sum_{j=m}^{n} f_j(z) \right| \leq \sum_{j=m}^{n} |f_j(z)| \leq \sum_{j=m}^{n} M_j$$

$\sum_{j=m}^{n} M_j$ is convergent. Using Cauchy criterion:

$\exists N_0 = N_0(\varepsilon)$ s.t. $n > m > N_0$, $\sum_{j=m}^{n} M_j < \varepsilon$

Hence if $n > m > N_0$: $\left| \sum_{j=m}^{n} f_j(z) \right| < \varepsilon \quad \forall z \in \Omega.$

Def^n:
Let $F, F_n : \Omega \to \mathbb{C}$ be a sequence of functions.
Say $F_n \to F$ uniformly on Ω if given $\varepsilon > 0$, $\exists\, N_0(\varepsilon)$ st. for $n > N_0(\varepsilon)$ we have

$$|F_n(z) - F(z)| < \varepsilon \quad \forall\, z \in \Omega$$

Def^n of uniform convergence of series $\sum f_n$:
Take

$$F(z) = \sum_{n=1}^{\infty} f_n(z) \qquad F_n(z) = \sum_{j=1}^{n} f_j(z).$$

in above definition.

Exponential Function

$$\exp(z) = \sum_{n=0}^{\infty} \frac{z^n}{n!}$$ is convergent for all $z \in \mathbb{C}$

Properties

$$\frac{d}{dz} \exp(z) = \exp(z)$$

$$\exp(z+w) = \exp(z)\exp(w)$$

[Can be verified by multiplying the power series.
Need to rearrange, using absolute convergence.]

From exp, define

$$\cos z = \frac{e^{iz} + e^{-iz}}{2}, \quad \sin z = \frac{e^{iz} - e^{-iz}}{2}$$

$$\cos z = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \dots = \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n}}{(2n)!}$$

$$\sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \dots = \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n+1}}{(2n+1)!}$$

Trig identities such as
$\cos^2 z + \sin^2 z = 1$ hold.

However it is no longer true that
$|\cos z| \leq 1$, $|\sin z| \leq 1$.

Indeed: $t \in \mathbb{R}$

$$\cos it = \frac{e^{-t} + e^{+t}}{2}$$

as $t \to \pm\infty$. So $\cos it \in \mathbb{R}$ and goes to $+\infty$.

Recall:

$e^{2\pi i} = 1$

Hence e^z is periodic, with period $2\pi i$.

$e^{(z+2\pi i)} = e^z \cdot e^{2\pi i} = e^z \quad \forall z.$

Exercises

1). On what set does $\sum_{n=0}^{\infty} n^2 (z+1)^n$ converge?

2). What is the radius of convergence of $\sum_{n=0}^{\infty} z^{10^n}$?

3). On what set does $\sum_{n=0}^{\infty} \frac{1}{n!} \frac{1}{z^n}$ converge?

4). Expand $f(z) = z^{-2}$ in powers of $z - i$.
On what set does your expansion converge?

1). $\left| \frac{(n+1)^2 (z+1)^{n+1}}{n^2 (z+1)^n} \right| = \left| \frac{(n+1)^2}{n^2} \right| |z+1| \to |z+1|$ as $n \to \infty$

conv. on set $\{ z : |z+1| < 1 \}$

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Logarithms

log = natural logarithm (= ln)

Try to solve $z = \exp(w)$

Any complex number w which solves $z = \exp(w)$ is a <u>choice</u> of $\log z$.

If $z = \exp(w)$ then also $z = \exp(w + 2\pi i)$
So the set of choices of $\log z$ has the form $\{w + 2n\pi i : n \in \mathbb{Z}\}$ and w is a particular solution of $z = e^{w}$.

Note: if $w = u + iv$, we have

$z = \exp(u+iv) = (e^{u})e^{iv}$, $u, v \in \mathbb{R}$

$|z| = e^{u} \Rightarrow u = \log|z|$.

$z = e^{u}(\cos v + i\sin v)$, v is a choice of $\arg(z)$.

Hence:

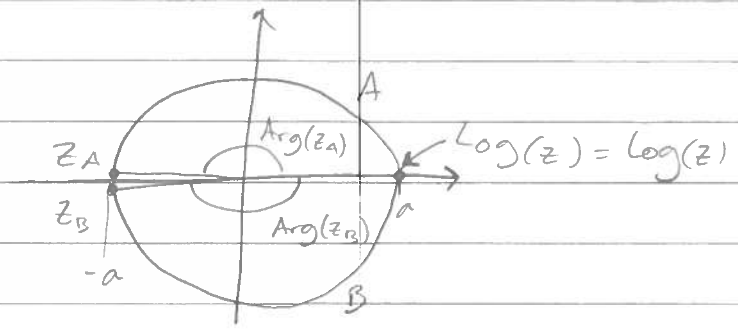
$$\log z = \log|z| + i\arg(z)$$

Note: real part of $\log z$ is uniquely defined.

$\exp(w) \neq 0$ so $\log z$ is only defined for $z \neq 0$.

If z is real and positive, it is natural to choose $\log z$ to be real also. This is the principle log, $\mathrm{Log}(z)$.

$\mathrm{Log}(z) = \log|z| + i\mathrm{Arg}(z)$, $\mathrm{Arg}(z) \in (-\pi, \pi]$.



$\text{Arg}(z_A)$ is just $< \pi$

So $\log(z_A) = \log|z_A| + i\,\text{Arg}(z_A)$
$\approx \log|z_A| + i\pi$

$\text{Arg}(z_B)$ is just $> -\pi$

So $\log(z_B) = \log|z_B| + i\,\text{Arg}(z_B)$
$\approx \log|z_B| - i\pi$

A: $\log(-a) = \log(a) + i\pi$
B: $\log(-a) = \log(a) - i\pi$

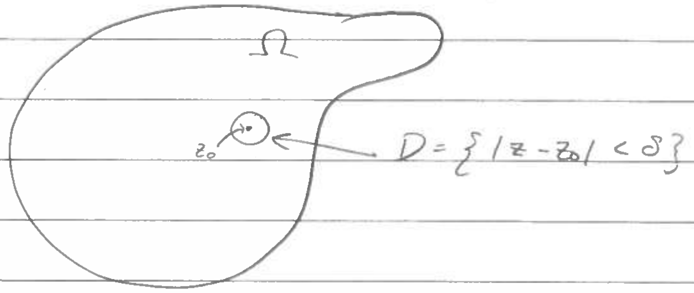
$\nexists$ a continuous choice of $\log(z)$ in $\mathbb{C} \setminus \{0\}$, or any circle centred at $z = 0$

$\Rightarrow$ So need to cut the plane.
(In this case delete the set $S = \{z = x,\ x \leq 0\}$ from $\mathbb{C}$. Then $\text{Log}(z)$ is cont. on $\mathbb{C} \setminus S$).

Definition

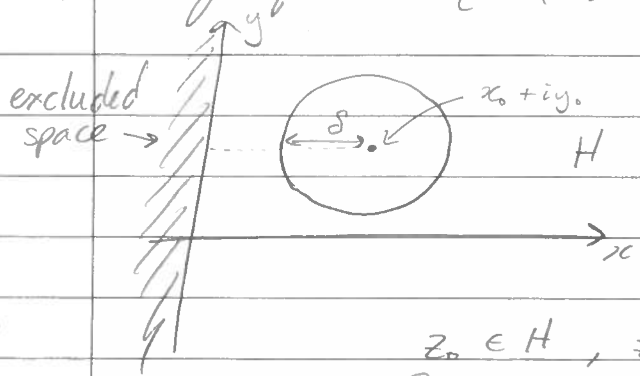
$\Omega \subset \mathbb{C}$ is open if for each $z_0 \in \Omega$, there is an open disc $D = \{|z - z_0| < \delta\}$ contained in Ω.

note: $\emptyset$ is open, as is $\mathbb{C}$.



Examples

1) Half space $\{Re(z) > 0\} = H$



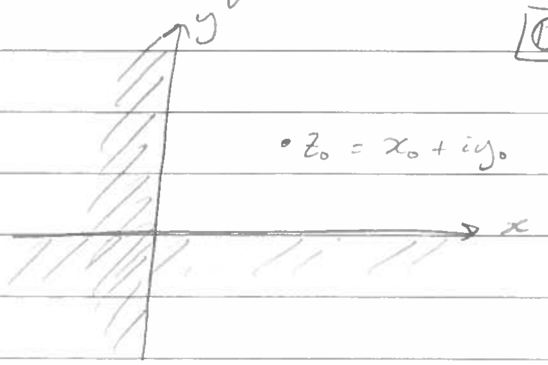
$z_0 \in H$, $z_0 = x_0 + iy_0$, $x_0 > 0$ by def^n

$\{|z - z_0| < \delta\} \subset H$ if $\delta = \frac{1}{2} x_0$ (a choice)

2) $\cap^n$ of two half spaces

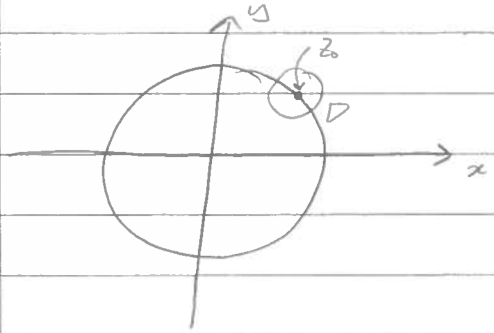
$$Q = \{z : Re(z) > 0 \ \& \ Im(z) > 0\}$$

is also open.



Non-example

$\{|z| \leq 1\} = S$ 'closed unit disc'



Clearly any disc centred at z_0 with $|z| = 1$ will not be contained in S.

Facts:

The union of any family of open sets is again open.

If $\Omega_1, \ldots, \Omega_N$ is any finite collection of open sets then $\Omega_1 \cap \ldots \cap \Omega_N$ will be open.

Notation:

$\mathbb{C} \setminus K = \{z \in \mathbb{C} : z \in K\}$

$\mathbb{C} \setminus K$

WARNING:

'Closed' does NOT mean 'not open'.

What are limit points?

Say $w \in \mathbb{C}$ is a limit point of K if $\exists$ a sequence of points $z_n \in K$ with $z_n \to w$ as $n \to \infty$

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Note: every point in K is a limit point of K (Take constant sequence).
But there may be others.

Take $H = \{ z : Re(z) > 0 \}$

For example $z_n = \frac{1}{n} \in H$ for all $n \geq 1$ but $z_n \to 0 \notin H$.
So 0 is a limit point of H but not in H.
$\mathbb{C} \setminus H = \{ z \in \mathbb{C} : Re(z) \leq 0 \}$ is closed as it contains its 'boundary', the imaginary axis.

Facts:
Any intersection of closed sets is closed
Any finite union of closed sets is closed.

$K_n = \{ Re(z) \geq \frac{1}{n} \}$ $(n = 1, 2, 3, \ldots)$

Each K_n is closed, but $\bigcup K_n = \{ Re(z) > 0 \}$ which is not closed.

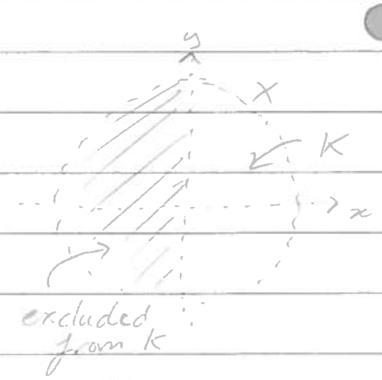
Definition:
$K \subset \mathbb{C}$ is closed if either of the following is true:
- $\mathbb{C} \setminus K$ is open
- K contains all its limit points

Definition

If $X \subset \mathbb{C}$, say U is an open subset of X if $U = X \cap \Omega$, Ω open in $\mathbb{C}$.
$C \subset X$ is a closed subset of X if $X \setminus C$ is open in X
$\Leftrightarrow C = X \cap K$, K closed in $\mathbb{C}$.

Example

$X = \{|z| < 1\}$
$K = \{|z| < 1 \text{ and } Re(z) \geq 0\}$



By definition 'K is closed in X'.
For $K = X \cap \{Re(z) \geq 0\}$
$= X \cap (\text{closed subset of } \mathbb{C})$

But K is neither open nor closed in $\mathbb{C}$.

Open subsets are 'natural homes' for continuous, holomorphic etc functions.

Def^n

If $\Omega \subset \mathbb{C}$ is open and $f : \Omega \mapsto \mathbb{C}$ is a function: say that f is holomorphic in Ω if it is holomorphic (i.e. complex differentiable) at each point $z_0 \in \Omega$.

(Openness guarantees you can approach z_0 in any direction).

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Back to logs and so on

Proposition

If $\mathrm{Arg}(z)$ denotes the value of argument in $(-\pi, \pi]$ then

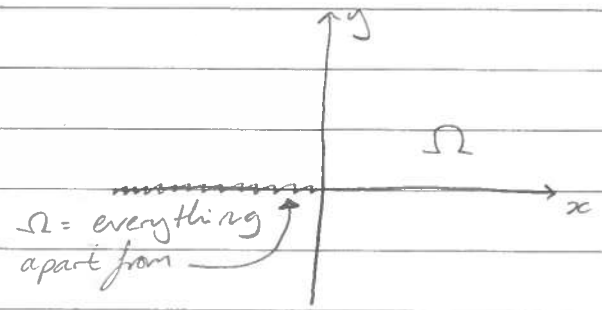
$$\mathrm{Log}(z) := \log|z| + i\,\mathrm{Arg}(z)$$

is holomorphic in the cut plane

$$\Omega = \mathbb{C} \setminus \{z : \mathrm{Im}(z) = 0,\ \mathrm{Re}(z) \leq 0\}$$

with derivative

$$\frac{d}{dz}\mathrm{Log}(z) = \frac{1}{z}$$



Proof

$\mathrm{Log}(z)$ is 'clearly' continuous in Ω and satisfies $z = \exp(\mathrm{Log}(z))$.

Differentiate using chain rule:

$$1 = \exp(\mathrm{Log}(z)) \cdot \frac{d}{dz}(\mathrm{Log}(z))$$

$$= z\frac{d}{dz}(\mathrm{Log}(z)). \qquad \square$$

Complex powers

If $\alpha \in \mathbb{C}$, z^α is defined to be
$$z^\alpha = \exp(\alpha \log(z))$$
for some choice of $\log(z)$.
It is multivalued unless α is an integer.
Different values differ by multiplication by $e^{2n\pi i\alpha}$ for some $n \in \mathbb{Z}$.

Proposition

Let Ω be the cut plane of previous proposition.
Then $f(z) = \exp(\alpha \operatorname{Log}(z))$ is a holomorphic choice of z^α in Ω with derivative
$$f'(z) = \alpha z^{\alpha-1}.$$

Exercises

1). Write down an example of a non-constant holomorphic function which vanishes at $z=0$ and $z=1$.

2). Write down an example of a non-holomorphic function which vanishes at $z=0$ and $z=1$.

3). Write down an example of a non-constant holomorphic function with infinitely many zeros (ie there are infinitely many $z \in \mathbb{C}$ with $f(z)=0$).

1). $f(z) = z(z-1)$
2). $f(z) = z\log(z)$, $f(z) = \mathrm{Re}(z(z-1))$
3). $f(z) = e^{iz} - 1$, $f(z) = \sin z$

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For $\log z$ and z^α, we say that $f(z)$ is a holomorphic branch of one of these functions if it is a holomorphic function defined in some open set $U \subset \mathbb{C}$.

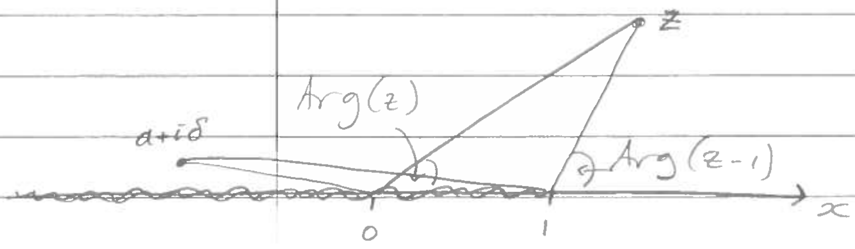
We have seen that holomorphic branches of $\log z$ and z^α exist in
$$\Omega = \mathbb{C} \setminus \{ \mathrm{Im}(z) = 0, \ \mathrm{Re}(z) \leq 0 \}.$$

0 is a branch point of z^α and of $\log z$.
Roughly: you have to cut the plane from branch point to ∞ in order to define a holomorphic branch.

Example with more than one branch point.

Consider $z^{1/2}(z-1)^{1/2}$
Branch points are at zeros of the factors, so at $z=0$ and $z=1$.

Problem: Define a holomorphic branch of $z^{1/2}(z-1)^{1/2}$.
To define $z^{1/2}$ we may cut plane along the negative real axis and define the branch
$$z^{1/2} = |z|^{1/2} \exp(\tfrac{1}{2} i \,\mathrm{Arg}(z)).$$
Similarly a holomorphic branch of $(z-1)^{1/2}$ may be defined as
$$(z-1)^{1/2} = |z-1|^{1/2} \exp(\tfrac{1}{2} i \,\mathrm{Arg}(z-1))$$
defined in plane cut along real axis from $z=1$ to $-\infty$.



So a hol. branch is defined to be

$$f(z) = |z|^{1/2}|z-1|^{1/2} \exp\left(\frac{i}{2}(\mathrm{Arg}(z) + \mathrm{Arg}(z-1))\right)$$

in

$$\Omega_1 = \mathbb{C} \setminus \{z : \mathrm{Im}(z) = 0,\ \mathrm{Re}(z) \leq 1\}$$

Note however that this branch is better than this and is continuous, hence holomorphic in the larger set

$$\Omega_1' = \mathbb{C} \setminus \{z : \mathrm{Im}(z) = 0,\ 0 \leq \mathrm{Re}(z) \leq 1\}$$

Why?

Let $a < 0$ be on negative real axis.

Let $z = a + i\delta$, δ small

If $\delta > 0$ is small, $\mathrm{Arg}(z)$ and $\mathrm{Arg}(z-1)$ are both nearly π (picture) so

$$\exp\left(\frac{i}{2}(\mathrm{Arg}(a+i\delta) + \mathrm{Arg}(a+i\delta-1))\right)$$
$$\approx \exp(\pi i) = -1$$

If $\delta < 0$, $\mathrm{Arg}(z)$ and $\mathrm{Arg}(z-1)$ are both approx $-\pi$ and so

$$\exp\left(\frac{i}{2}(\mathrm{Arg}(z) + \mathrm{Arg}(z-1))\right) \approx \exp\left(\frac{i}{2}(-\pi - \pi)\right)$$
$$= \exp(-\pi i)$$
$$= -1.$$

So $\exp\left(\frac{i}{2}(\mathrm{Arg}(z) + \mathrm{Arg}(z-1))\right)$ is actually continuous at $z = a$ on negative real axis. This implies the claim that our branch

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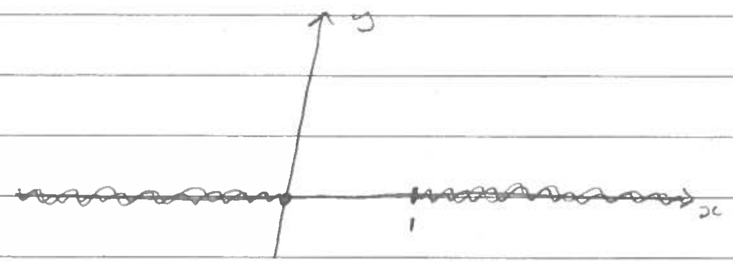
of $z^{1/2}(z-1)^{1/2}$ is holomorphic in
$\mathbb{C} \setminus \{z : Im(z), 0 \le Re(z) \le 1\}$.

<u>Remark</u>

Can instead define a holomorphic branch of $z^{1/2}(z-1)^{1/2}$ on the domain

$$\Omega_2 = \mathbb{C} \setminus \left(\{z : Im(z) = 0, Re(z) \le 0\} \cup \{z : Im(z) = 0, Re(z) \ge 1\} \right)$$



For this choose $\arg(z-1)$ in $[0, 2\pi)$

§4? Conformal Mapping

Conformal means 'angle-preserving'

(precise def^n next time)

<u>Theorem</u>

Let $f : \Omega \to \mathbb{C}$ be holomorphic (Ω open)
The f is a conformal mapping at every point z_0 of Ω with $f'(z_0) \neq 0$.

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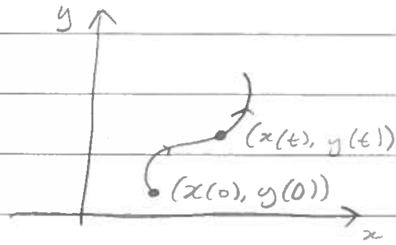
Conformality of holomorphic maps

$\updownarrow$

'angle preservation'

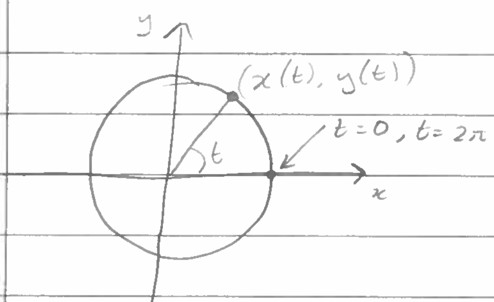
Recall a parameterized curve in $\mathbb{C}$ is just a differentiable mapping $t \longmapsto z(t) = x(t) + iy(t)$, t in the interval $[0,1]$ (say).



Familiar example

$x(t) = \cos t$, $y(t) = \sin t$, $t \in [0, 2\pi]$

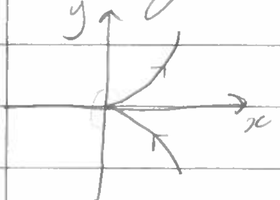
This is the unit circle, $z(t) = e^{it}$



A curve is regular if $z(t)$ is continuously differentiable and $\frac{dz}{dt} = \dot{z}(t) = \dot{x}(t) + i\dot{y}(t)$

is non-zero for all values of t in the parameter interval.

In a general parameterised curve you can have



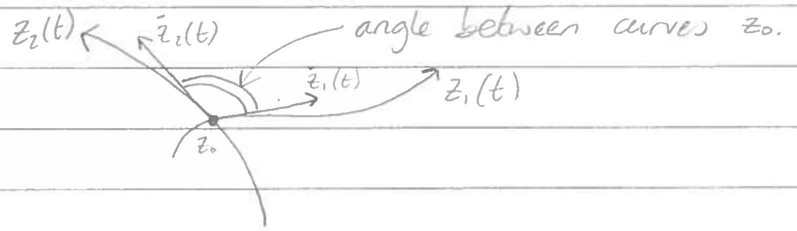
but this is not regular.

Note: $\frac{dz}{dt}$ is the tangent vector to the curve.

Circle: tangent vector: $(-\sin t, \cos t)$ or in $\mathbb{C}$ terms $\dot{z}(t) = ie^{it}$.

Given two regular curves $z_1(t)$, $z_2(t)$ with the same initial point $z_1(0) = z_2(0) = z_0$, define the angle between the curves at z_0 to be the angle between the tangent vectors at $t=0$, i.e.

$$\text{Angle} = \arg(\dot{z}_2(0)) - \arg(\dot{z}_1(0))$$
$$= \arg\left(\frac{\dot{z}_2(0)}{\dot{z}_1(0)}\right)$$



Theorem

$f: \Omega \mapsto \mathbb{C}$ is holomorphic (Ω open) and $f'(z_0) \neq 0$. Let $w_j(t) = f(z_j(t))$ $(j = 1, 2)$.
The angle between $w_1(t)$ and $w_2(t)$ at $w_0 = f(z_0)$ is the same as angle between $z_1(t)$ and $z_2(t)$ at z_0

Proof

[$\because$ provided that
$\therefore$ therefore]

① Chain rule: $\frac{dw_j}{dt} = f'(z_j(t)) \frac{dz_j}{dt}$

Evaluate at $t=0$

$\dot{w}_j(0) = f'(z_0)\dot{z}_j(0)$

$$\arg\left(\frac{\dot{w}_2(0)}{\dot{w}_1(0)}\right) = \arg\left(\frac{f'(z_0)\dot{z}_2(0)}{f'(z_0)\dot{z}_1(0)}\right) = \arg\left(\frac{\dot{z}_2(0)}{\dot{z}_1(0)}\right) \quad \because f'(z_0) \neq 0 \quad \square$$

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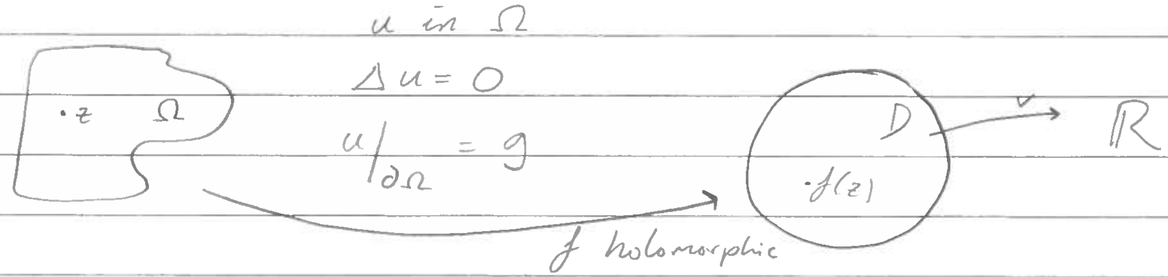
Motivation: In 2D physics (fluid flow) we often need u s.t. $\Delta u = 0$.



Map Ω, 1:1 and onto $D = \{|z|<1\}$ conformally. Solve the problem 'explicitly' in D and transfer back to Ω.

Works because f is holomorphic $f: \Omega \to D$, $v: D \to \mathbb{R}$ is harmonic, then $u(z) = v(f(z))$ will again be harmonic.

Examples of Conformal mapping:

1). Möbius transformations (Fractional linear transformations)

$$T(z) = \frac{az+b}{cz+d} \quad (ad-bc \neq 0) \quad a,b,c,d \in \mathbb{C}$$

Extends to $\mathbb{C} \cup \{\infty\}$ or equivalently to $\mathbb{S}$ by defining $T(-d/c) = \infty$, $T(\infty) = \frac{a}{c}$

T is a bijective map $\mathbb{S} \to \mathbb{S}$ with inverse $T^{-1}(z) = \dfrac{dz-b}{-cz+a}$

Theorem

Any Möbius transformation, T, is everywhere conformal.
T maps circles and straight lines in $\mathbb{C}$ to circles and straight lines.
Given any two triples (z_1, z_2, z_3) and (w_1, w_2, w_3) of pairwise distinct points, there is a unique Möbius T with $T(z_1) = w_1$, $T(z_2) = w_2$, $T(z_3) = w_3$.

Exercises

4.4). Consider the mapping $w = 1/_{z-1}$. Where is this mapping conformal? What is the image of the real axis? What is the image of the imaginary axis?

4.5). Write down the Möbius transformation mapping i to 1, 1 to 0, ∞ to -17.

4.5).

$$\frac{ai+b}{ci+d} = 1$$

$$\frac{a+b}{c+d} = 0 \Rightarrow a = -b$$

$$\frac{a\infty+b}{c\infty+d} = \frac{a}{c} = -17$$

$$a = -17c, \quad b = 17c$$

$$\frac{-17i+17}{i+d} = 1$$

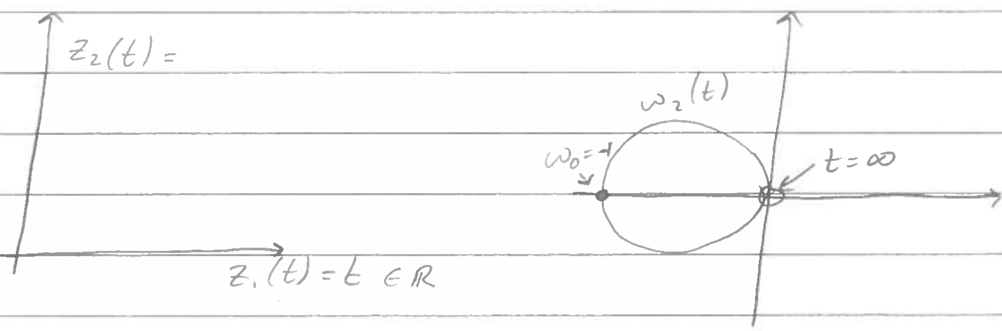
$$-17i+17 = i+d$$

$$17-18i = d$$

$$\text{so } T = \frac{-17z+17}{z+17-18i}$$

$$= \frac{17z-17}{-z+18i-17}$$

4.4).



$$w_1 = \frac{1}{t-1} \Rightarrow \text{real axis} \quad w_1 \to 0 \text{ as } t \to \infty$$
$$w_1 \to \infty \text{ as } t \to 1$$

$$w_2 = \frac{1}{it-1} = \frac{-1}{1+t^2} - \frac{it}{1+t^2}$$

Definition
A map $f : \Omega \to \mathbb{C}$ is conformal at $z_0 \in \Omega$ if: for any pair of regular curves $z_1(t)$, $z_2(t)$ with $z_1(0) = z_2(0) = z_0$.
the angle between image curves $w_j(t) = f(z_j(t))$ at $w_0 = f(z_0)$ is equal to the angle between $z_1(t)$ & $z_2(t)$ at z_0

Theorem (paraphrase - from before)
If f is holomorphic and $f'(z_0) \neq 0$ then f is conformal at z_0.

Proof of theorem (Möbius)
1). Any Möbius T is a composition of the basic transformations:
(*) $z \mapsto z + c$ (translation with vector c)
(**) $z \mapsto az$ (enlargement, scale factor $|a|$ together with rotation through $\arg(a)$)
(***) $z \mapsto 1/z$ (inversion)

It is clear that circles and straight lines are mapped to circles and straight lines by (*) and (**)

To understand $T(z) = 1/z$ consider $\mathbb{S}$:
$$z = \frac{x_1 + i x_2}{1 - x_3} \quad \text{(Stereographic Projection)}$$

We saw: circles and straight lines in $\mathbb{C}$ all become circles in $\mathbb{S}$.

Note that

$$\frac{1}{z} = \frac{1 - x_3}{x_1 + ix_2}$$

$$= \frac{(1-x_3)(x_1 - ix_2)}{x_1^2 - x_2^2}$$

$$= \frac{(1-x_3)(x_1 - ix_2)}{1 - x_3^2}$$

$$= \frac{x_1 - ix_2}{1 + x_3}$$

Therefore, transformation $z \to 1/z$ corresponds to mapping $(x_1, x_2, x_3) \mapsto (x_1, -x_2, -x_3)$ of S

This is a $180°$ rotation of S around the x_1 axis

- maps circles to circles.

What about mapping (z_1, z_2, z_3) to (w_1, w_2, w_3)?

It is enough to find T_{z_1, z_2, z_3} mapping (z_1, z_2, z_3) to $(1, 0, \infty)$.

For then, required Möbius will be

$$(T_{w_1, w_2, w_3})^{-1} \circ (T_{z_1, z_2, z_3})$$

Construction of T_{z_1, z_2, z_3}?

$z_2 \mapsto 0$ so must have $\dfrac{z - z_2}{cz + d}$

$z_3 \mapsto \infty$ implies $\lambda . \left(\dfrac{z - z_2}{z - z_3}\right)$

$z_1 \mapsto 1$ means $\lambda = \dfrac{z_1 - z_3}{z_1 - z_2}$

So $T_{z_1, z_2, z_3}(z) = \left(\dfrac{z_1 - z_3}{z_1 - z_2}\right) . \left(\dfrac{z - z_2}{z - z_3}\right)$

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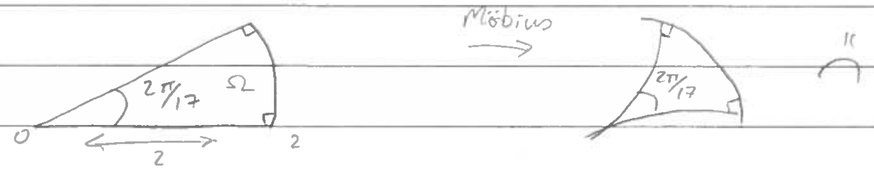
Example

Find a conformal mapping of the region
$$\Omega = \{ z \in \mathbb{C} : 0 < |z| < 2 ,\ 0 < \arg(z) < \tfrac{2\pi}{17} \}$$
onto (and 1:1) the upper half space
$$H = \{ \omega \in \mathbb{C} : \operatorname{Im}(\omega) > 0 \}$$



We need to go beyond Möbius transformations (Too many curves)

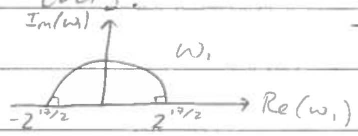
Step 1: 'Open up' the angle at $z=0$ with a map of the form $\omega_1 = z^a = |z|^a \exp(ia\arg z)$

Image of Ω by this map is the set
$$\Omega_1 = \{ \omega_1 : 0 < |\omega_1| < 2^a \text{ and } 0 < \arg(\omega_1) < 2\pi a/17 \}$$

If we choose $a = 17/2$, this is
$$\Omega_1 = \{ \omega_1 : 0 < |\omega_1| < 2^{17/2} ,\ 0 < \arg(\omega_1) < \pi \}.$$

implicit in definition

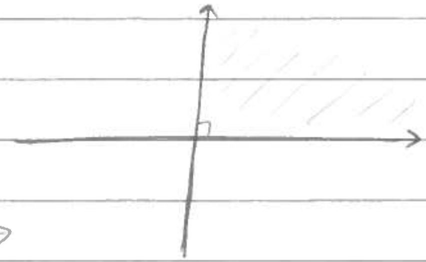
NB: because a is not an integer, we need to make a choice of $\arg(z)$ continuous on Ω, could use $\operatorname{Arg}(z)$ for this.



Step 2:

Map to a reigion bounded by straight lines. Use a Möbius, for example
$$\omega_2 = \frac{\omega_1 - 2^{17/2}}{\omega_1 + 2^{17/2}}$$
Conformal everywhere and maps circles and straight lines to circles and straight lines.

Call image Ω_2.
Real w_1 axis maps to real w_2 axis



$$\left[w_2 = \frac{(w_1 - 2^{17/2})^2}{w_1^2 - 2^{17/2}} = \frac{w_1^2 - 2w_1 2^{17/2} + 2^{17}}{w_1^2 - 2^{17/2}} \quad ? \right]$$

Ω_2 is one of the four quadrants in the w_2-plane
(2nd quadrant)

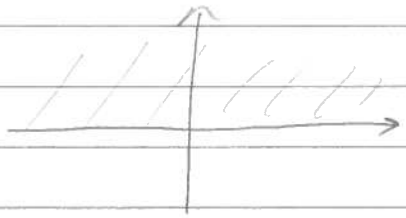
After multiplication by $i, -1, -i$ or 1 we may assume Ω_2 is the first quadrant. ($-i$ in this case)

$$\Omega_2 = \{ w_2 \in \mathbb{C} : 0 < \arg(w_2) < \pi/2 \}$$

Step 3

Then if $w_3 = w_2^2$ and Ω_3 is the image of Ω_2 by this map, we see:

$$\Omega_3 = \{ w_3 \in \mathbb{C} : 0 < \arg(w_3) < \pi \}$$



So the required conformal mapping is the composite

$$z \to w_1 \to w_2 \to w_3$$

$$w_3 = w_2^2 = \left(\frac{-i(w_1 - 2^{17/2})}{w_1 + 2^{17/2}} \right)^2 = -\left(\frac{z^{17/2} - 2^{17/2}}{z^{17/2} + 2^{17/2}} \right)^2$$

All maps have inverses where defined so this is 1:1 and onto (ie bijective).

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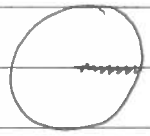
Suppose

$z \in \Omega$

$w = z^{17}$ $\quad (*)$

Image Ω' of Ω by $(*)$:

$\Omega' = \{ w \in \mathbb{C} : 0 < |w| < 2^{17},\ 0 < \arg(w) < 2\pi \}$



non-examinable.

<u>Why can we expect to find conformal mappings?</u>

<u>Riemann Mapping Theorem</u>

Let $\Omega \subset \mathbb{C}$, $\Omega \neq \mathbb{C}$, Ω open subset of $\mathbb{C}$.
Also suppose that Ω is connected and simply connected. Then $\exists$ a conformal map $f : \Omega \to \{|z| < 1\}$ which is 1:1 and onto.

Connected:

Can join any two points of Ω by continuous curve (Not: Ω) in Ω.

Simply connected:

Any closed curve in Ω can be 'continuously deformed to a point' (within Ω)
(Basic non-example is $\mathbb{C} \setminus \{0\}$.)

Integration

Definition

$f : [a,b] \to \mathbb{C}$, continuous, then

$$\int_a^b f(t)\,dt = \int_a^b \mathrm{Re}(f(t))\,dt + i\int_a^b \mathrm{Im}(f(t))\,dt.$$

Proposition

Let $M = \sup\{|f(t)| : a \le t \le b\}$

Then: $\left|\int_a^b f(t)\,dt\right| \le M(b-a)$

Proof

Let $\alpha = \arg\left(\int_a^b f(t)\,dt\right)$

Then $e^{-i\alpha}\int_a^b f(t)\,dt$ is real by definition.

Now: $$\left|\int_a^b f(t)\,dt\right| = \left|e^{-i\alpha}\int_a^b f(t)\,dt\right|$$
$$= \left|\int_a^b e^{-i\alpha} f(t)\,dt\right|$$
$$= \left|\int_a^b \mathrm{Re}\left(e^{-i\alpha} f(t)\right) dt\right|$$
$$\le \int_a^b \left|\mathrm{Re}\left(e^{-i\alpha} f(t)\right)\right| dt$$
$$\le \int_a^b |f(t)|\,dt$$
$$\le \int_a^b M\,dt = M(b-a) \qquad \square$$

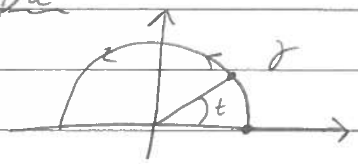
Integration along curves in $\mathbb{C}$

A *curve* γ is a C' map

$$\gamma : [a,b] \longrightarrow \mathbb{C}$$

[C' meaning continuously differentiable]

Example



Take parameter to be angle $t : 0 \le t \le \pi$
and the point at angle t is $Re^{it} = R\cos t + iR\sin t$.
So $\gamma(t) = Re^{it}, \quad 0 \le t \le \pi$

Definition

If $f : \Omega \to \mathbb{C}$ (where Ω is open) is continuous and $\gamma : [a,b] \longrightarrow \Omega$ is a curve, then we define:

$$\int_\gamma f(z)\,dz \left[= \int_\gamma f\right]$$

$$= \int_a^b f(\gamma(t))\,\gamma'(t)\,dt$$

this is the integral of a complex-valued function of a real variable and hence covered by the previous definition.

Example

If $f(z) = \frac{1}{1+z^2}$ and γ is the semicircle as before.

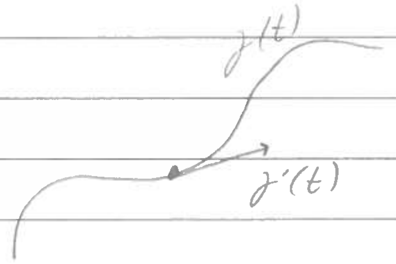
Then

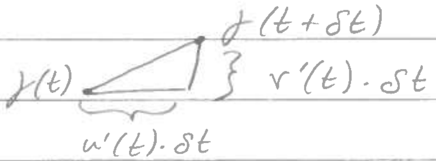
$$\int_\gamma f = \int_0^\pi \frac{1}{1+R^2e^{2it}} (iRe^{it})\,dt$$

$$= \int_0^\pi \frac{iRe^{it}\,dt}{1+R^2e^{2it}}$$

Definition

If $\gamma : [a,b] \to \mathbb{C}$ is a C^1 curve then we define $\text{length}(\gamma) = \int_a^b |\gamma'(t)|\,dt$





$\gamma(t) = u(t) + iv(t)$

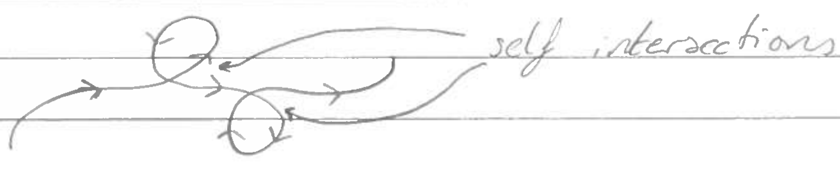
Hypotenuse: $\sqrt{u'(t)^2 + v'(t)^2}\ \delta t$
$= |\gamma'(t)|\,\delta t.$

Definition

γ is simple:
$\gamma(t_1) = \gamma(t_2) \implies t_1 = t_2$

[i.e. $\gamma : [a,b] \to \mathbb{C}$ is an injective map.]

Not allowed:



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Properties

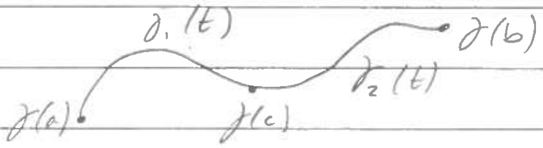
Linearity:

$$\int_\gamma (c_1 f(z) + c_2 g(z))\, dz = c_1 \int_\gamma f(z)\, dz + c_2 \int_\gamma g(z)\, dz$$

Additivity:

if $a < c < b$, $\gamma : [a,b] \to \Omega$ and $\gamma_1 = \gamma$ in $[a,c]$, $\gamma_2 = \gamma$ in $[c,b]$ then

$$\int_{\gamma_1} f(z)\,dz + \int_{\gamma_2} f(z)\,dz = \int_\gamma f(z)\,dz$$



Sign-change under path reversal:

Let $\gamma^{opp}(s) = \gamma(-s)$, $-b \le s \le -a$.

Then

$$\int_{\gamma^{opp}} f(z)\,dz = -\int_\gamma f(z)\,dz$$

Comment

Often γ^{opp} is denoted by $-\gamma$.

There are good reasons for this but there is room for confusion:

$-\gamma(t)$ might be $\gamma(t)$ rotated through $180°$.

Reparameterisation invarience

If $a' < b'$ and $\varphi : [a', b'] \to [a, b]$ is C',
$\varphi(a') = a$, $\varphi(b') = b$, and if
$\delta = \gamma \circ \varphi : [a', b'] \to \Omega$, then

$$\int_\gamma f(z)\,dz = \int_\delta f(z)\,dz$$

Proof

$$\int_\gamma f(z)\,dz = \int_a^b f(\gamma(t))\gamma'(t)\,dt$$

$$\int_\delta f(z)\,dz = \int_{a'}^{b'} f(\delta(\tau)\,\delta'(\tau)\,d\tau$$

$\delta(\tau) = \gamma(\varphi(\tau))$. $\delta'(\tau) = \gamma'(\varphi(t))\dfrac{d\varphi}{d\tau}$

$$\int_\delta f(z)\,dz = \int_{a'}^{b'} f(\gamma(\varphi(\tau)))\,\gamma'(\varphi(\tau))\,\varphi'(\tau)\,d\tau$$

Use change of variables $t = \varphi(\tau)$:

$$\int_\delta f(z)\,dz = \int_a^b f(\gamma(t))\gamma'(t)\,dt \qquad dt = \varphi'(\tau)\,d\tau$$

$$= \int_\gamma f(z)\,dz$$

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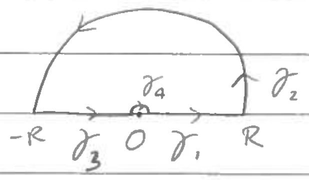
Piecewise C^1 curves

Very often, we shall want to integrate along curves like this:



Definition

Suppose $\gamma : [a,b] \to \mathbb{C}$ is a continuous curve.

Say γ is piecewise C^1 if

$$\exists\, a = a_0 < a_1 < a_2 < \dots < a_{n-1} < a_n = b$$

such that if

$$\gamma_j(t) = \gamma(t) \quad \text{for} \quad a_{j-1} \le t \le a_j$$

then γ_j is C^1 for all $j = 1, \dots, n$.

For continuity we shall need

$$\gamma_1(a_1) = \gamma_2(a_1)$$
$$\gamma_2(a_2) = \gamma_3(a_2)$$
$$\vdots$$
$$\gamma_{n-1}(a_{n-1}) = \gamma_n(a_{n-1})$$

Extend definition of $\int_\gamma f$ to piecewise C^1 curves.

If γ is piecewise C^1 and is decomposed into C^1 curves as in definition:

$$\int_\gamma f(z)\,dz = \sum_{j=1}^{n} \int_{\gamma_j} f(z)\,dz$$

$$= \sum_{j=1}^{n} \int_{a_{j-1}}^{a_j} f(\gamma_j(t))\,\gamma_j'(t)\,dt.$$

We didn't have to assume that the decomposition of the interval is unique because of the additivity property of $\int_I f$.

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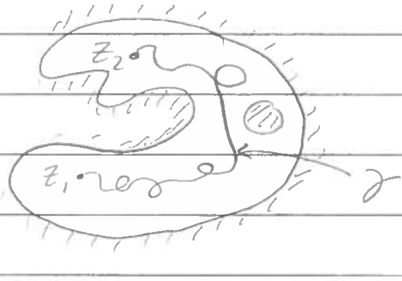
Definition

We say that an open set $\Omega \in \mathbb{C}$ is path-connected if:

Given any 2 points z_1 & z_2 in Ω, $\exists$ a continuous curve $\gamma : [a,b] \to \Omega$, $\gamma(a) = z_1$, $\gamma(b) = z_2$



Examples

- Any disc is path connected.
- Any half plane is path connected

Non-example

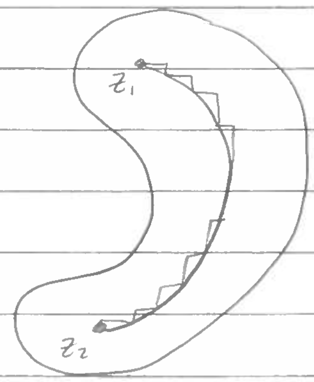
$U = \{|z+17| < 1\} \cup \{|z-17| < 1\}$

This is not path-connected.

Remark

If Ω is open and path-connected, then the curve γ connecting z_1 & z_2 can always be chosen to consist of a sequence of straight line segments parallel to either Re or Im axes.

Essential that Ω be open here.



Definition
Ω is called a domain (or 'region') if it is open and path connected.

Theorem
Let $f: \Omega \mapsto \mathbb{C}$ be holomorphic, let Ω be a domain and suppose $f'(z) = 0$ at all points of Ω.
Then f is a constant.

Proof
Remark: we do need Ω to be path connected, else there are simple counter examples.

We've seen $f' = 0$
$$\Rightarrow \frac{\partial f}{\partial x} = 0 \ , \ \frac{\partial f}{\partial y} = 0$$

Select any $z_1 \in \Omega$
Need to show $f(z_2) = f(z_1)$ for any other $z_2 \in \Omega$.
See picture (above): the vanishing of both partial derivatives $\Rightarrow$ value of f of successive corners of curve are the same. $\square$

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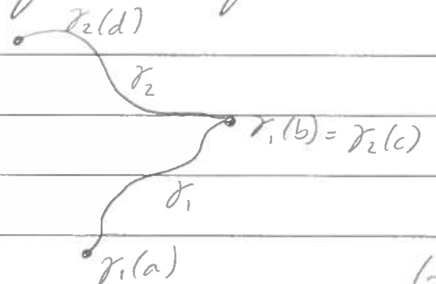
Last time:

$\int_\gamma f = \int_\gamma f(z)\,dz$, γ a piecewise C^1 curve

Addition

If γ_1 and γ_2 are curves, $\gamma_1 : [a,b] \to \mathbb{C}$, $\gamma_2 : [c,d] \to \mathbb{C}$



Suppose $\gamma_1(b) = \gamma_2(c)$.

Define

$$(\gamma_1 + \gamma_2)(t) = \begin{cases} \gamma_1(t) & a \le t \le b \\ \gamma_2(t+c-b), & b \le t \le b+d-c \end{cases}$$

where $(\gamma_1 + \gamma_2) : [a, b+d-c] \to \mathbb{C}$

$$\int_{\gamma_1+\gamma_2} f = \int_{\gamma_1} f + \int_{\gamma_2} f.$$

In the picture from before, if γ_j is the j-th line segment in the zig-zag curve, the total curve $\tilde{\gamma} = \gamma_1 + \gamma_2 + \ldots + \gamma_N$
and

$$\int_{\tilde{\gamma}} f = \sum_{j=1}^{N} \int_{\gamma_j} f.$$

* Terminology

From now on, 'curve' will mean 'piecewise C^1 curve' unless otherwise stated.

Proposition

Let $\gamma : [a,b] \longrightarrow \Omega$ be a curve, $f : \Omega \to \mathbb{C}$ a continuous function. Then

$$\left| \int_\gamma f(z)\, dz \right| \leq \text{length}(\gamma) \cdot \sup_{z \text{ on } \gamma} \{ |f(z)| \}.$$

$$\sup_{z \text{ on } \gamma} |f(z)| = \sup \{ |f(\gamma(t))| : a \leq t \leq b \} =: M$$

Proof

By Δ inequality, it is enough to consider case of γ being C^1. Then

$$\left| \int_\gamma f(z)\, dz \right| = \left| \int_a^b f(\gamma(t)) \cdot \gamma'(t)\, dt \right|$$

$$\leq \int_a^b |f(\gamma(t))| \cdot |\gamma'(t)|\, dt$$

$$\leq M \int_a^b |\gamma'(t)|\, dt$$

Last time, we defined

$$\text{length}(\gamma) = \int_a^b |\gamma'(t)|\, dt$$

$\boxtimes$

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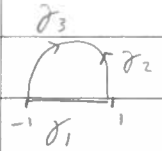
Exercises

1). Let $\gamma(t) = t + it^2$, $0 \le t \le 1$. Calculate
$$\int_\gamma z\,dz.$$
What is the value of this integral along the curve γ^{opp} which is traversed in the opposite direction.

$$\int_\gamma z\,dz = \int_0^1 (t+it^2)(1+2it)\,dt$$
$$= \int_0^1 (t-2t^3) + i(t^2+2t^2)\,dt$$
$$= \left[\left(\tfrac{1}{2}t^2 - \tfrac{1}{2}t^4\right) + i\left(\tfrac{1}{3}t^3 + \tfrac{2}{3}t^3\right)\right]_0^1$$
$$= \left[\left(\tfrac{1}{2} - \tfrac{1}{2}\right) + i\left(\tfrac{1}{3} + \tfrac{2}{3}\right)\right] - 0$$
$$= i \qquad \text{so} \int_{\gamma^{opp}} z\,dz = -i$$

2). Let γ be the piecewise C' curve $\gamma_1 + \gamma_2$, where γ_1 is a part of the real axis from -1 to 1 and $\gamma_2(t) = e^{it}$, $0 \le t \le \pi$, is the semi-circular arc joining 1 to -1 in the upper half plane.
Given that $\int_\gamma f(z)\,dz = 0$ for some function f, what can you say about the values of $\int_{\gamma_2} f(z)\,dz$ and $\int_{\gamma_3} f(z)\,dz$ where $\gamma_3(t) = e^{-it}$, $-\pi \le t \le 0$.



Proposition

Let $f_n : \Omega \to \mathbb{C}$ be continuous and let $\gamma : [a,b] \to \Omega$ be a curve. Suppose $f_n \to f$ uniformly on γ.
Then $\int_\gamma f_n(z)\,dz \to \int_\gamma f(z)\,dz$ as $n \to \infty$.

Proof

Let $M_n = \sup_{z \in \gamma} |f(z) - f_n(z)|$.

Uniform convergence $\iff M_n \to 0$

$$\left|\int_\gamma f(z)\,dz - \int_\gamma f_n(z)\,dz\right| = \left|\int_\gamma (f(z) - f_n(z))\,dz\right|$$
$$\leq \text{length}(\gamma) \cdot M_n \to 0 \quad \text{as } n \to \infty$$

Theorem (Fundamental Thm of Calculus - Complex version).

Suppose $F : \Omega \to \mathbb{C}$ is holomorphic (& F' is continuous).
Then if $\gamma : [a,b] \to \Omega$ is any curve, then
$$\int_\gamma F'(z)\,dz = F(\gamma(b)) - F(\gamma(a)).$$

Proof

Calculate.

$$\int_\gamma F'(z)\,dz = \int_a^b F'(\gamma(t))\gamma'(t)\,dt$$
$$= \int_a^b \frac{d}{dt} F(\gamma(t))\,dt \quad \text{(Chain rule)}$$
$$= F(\gamma(b)) - F(\gamma(a))$$ by Fundamental Theorem of Calculus for $\mathbb{C}$-valued functions of a real variable. $\square$

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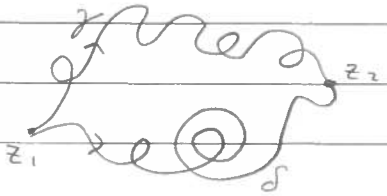
Remark

The RHS depends only on endpoints, so we have "Path - independence" of integral.



$$\int_\gamma F'(z)\,dz = \int_\delta F'(z)\,dz.$$

Exercises

3). If $\gamma(t) = Re^{it}$, where $R > 0$ is a constant and $0 \le t \le 2\pi$, calculate

$$\int_\gamma z^n\,dz, \quad n \in \mathbb{Z}.$$

Further, calculate

$$\int_\gamma z^n z^m\,dz, \quad n, m \in \mathbb{Z}.$$

$$\int_\gamma z^n\,dz = \int_0^{2\pi} R^n e^{int} \cdot Rie^{it}\,dt$$

$$= \int_0^{2\pi} iR^{n+1} e^{i(nt+t)}\,dt$$

$$= \left[\frac{iR^{n+1} e^{i(nt+t)}}{i(n+1)}\right]_0^{2\pi}$$

$$= \frac{R^{n+1} e^{2\pi i(n+1)}}{n+1} - \frac{R^{n+1}}{n+1} = 0 \quad \text{when } n \neq -1$$

$n = -1$: $\displaystyle\int_\gamma z^{-1}\,dz = \int_0^{2\pi} R^{-1} e^{-it} \cdot Rie^{it}\,dt$

$$= \int_0^{2\pi} i\,dt = \Big[it\Big]_0^{2\pi} = 2\pi i$$

Corollary
There does not exist a holomorphic function $F : \mathbb{C}\setminus\{0\} \mapsto \mathbb{C}$ s.t. $F'(z) = \frac{1}{z}$.

If such an F exists, apply the F.T.C. to conclude
$\int_\gamma \frac{dz}{z} = 0$, γ any closed curve.

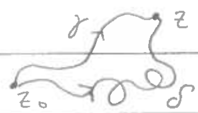
But this is a contradiction if $\gamma(t) = Re^{it}$, $0 \le t \le 2\pi$.

Conversely:
Theorem (Converse of F.T.C.)
Let $f : \Omega \to \mathbb{C}$ be continuous in a domain Ω. Then if $\int_\gamma f(z)\,dz = 0$ for all closed curves

$\gamma : [a,b] \to \Omega$ $[\gamma(a) = \gamma(b)]$.

Then $\exists F : \Omega \to \mathbb{C}$, F holomorphic, $F' = f$.

Proof:



Choose $z_0 \in \Omega$. For $z \in \Omega$, define
$F(z) = \int_\gamma f(w)\,dw$ where $\gamma : [a,b] \to \Omega$, $\gamma(a) = z_0$, $\gamma(b) = z$

1). If δ is another curve joining z_0 to z
$$\int_\gamma f(w)\,dw - \int_\delta f(w)\,dw$$
$$= \int_{\gamma + \delta^{opp}} f(w)\,dw = 0$$
$\because \gamma + \delta^{opp}$ is a closed curve.

$\because$ means 'because'

So $F(z)$ is well defined, independent of choice of curve γ.

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2). Need to show $F'(z) = f(z)$. Pick $z_1 \in \Omega$
Consider $F(z_1+h) - F(z_1)$ where $|h|$ is sufficiently small, $z+h \in \Omega$.
Ω is open so suppose
$\{|z - z_1| < \delta\} \subset \Omega$
then $F(z_1+h) = \int_\gamma f(w)\,dw + \int_{z_1}^{z_1+h} f(w)\,dw$

$\int_{z_1}^{z_1+h}$ = integral along straight line from z_1 to z_1+h.

$$\begin{aligned} F(z_1+h) - F(z_1) - hf(z_1) &= \int_{z_1}^{z_1+h} f(w)\,dw - hf(z_1) \\ &= \int_{z_1}^{z_1+h} (f(w) - f(z_1))\,dw \end{aligned}$$
$$\left[\because \int_{z_1}^{z_1+h} C\,dz = C \cdot h\right]$$

Estimate RHS by length × sup:
$$|F(z_1+h) - F(z_1) - hf(z_1)| \leq |h| \sup_{0 \leq t \leq 1} |f(z_1+th) - f(z_1)|$$

So $\left|\dfrac{F(z_1+h) - F(z_1)}{h} - f(z_1)\right| \leq \sup_{0 \leq t \leq 1} |f(z_1+th) - f(z_1)|$

Given $\varepsilon > 0$, $\exists \delta > 0$ s.t. $|f(z_1+th) - f(z_1)| < \varepsilon \ \forall t$ by continuity of f at z, if $|h| < \delta$.

If $|h| < \delta$ we have

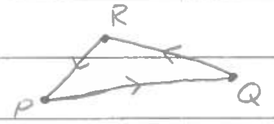
$$\left|\frac{F(z_1+h) - F(z_1)}{h} - f(z_1)\right| < \varepsilon$$

$\square$

Lemma

Let D be an open disc,
f is continuous $: D \mapsto \mathbb{C}$ and

$$\int_{\partial\Delta} f(z)\,dz = 0 \quad \text{for any triangle, } \Delta \in \Omega$$



Then $\exists\, F : D \mapsto \mathbb{C}$, F holomorphic, $F' = f$.

Proof

Same line as above.

z_0 = centre of disc.

$$F(z_1) = \int_{z_0}^{z_1} f(w)\,dw$$ = integral along line segment which joins z_0 to z_1.

Proof that $F'(z_1) = f(z_1)$ uses

$$\int_{\partial\Delta} f(w)\,dw = 0$$ and so goes through.

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Thm Cauchy's Thm for a disc.
Let $D \subset \mathbb{C}$ be an open disc, and let $f: D \mapsto \mathbb{C}$ be holomorphic.

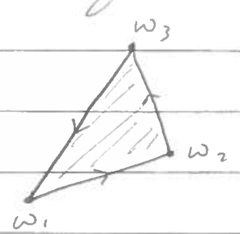
Then for every closed curve $\gamma: [t_0, t_1] \to D$
$$\int_\gamma f(z)\,dz = 0.$$

Proposition Cauchy. for triangles.
Let $\Omega \subset \mathbb{C}$ be an open set and let $f: \Omega \mapsto \mathbb{C}$ be holomorphic. Then for any triangle $\Delta \subset \Omega$
$$\int_{\partial\Delta} f(z)\,dz = 0.$$
$\partial\Delta$ ← boundary of triangle

Cauchy for triangles

Δ = {all points inside the union of 3 line segments in $\mathbb{C}$ }.

More precision: see notes.

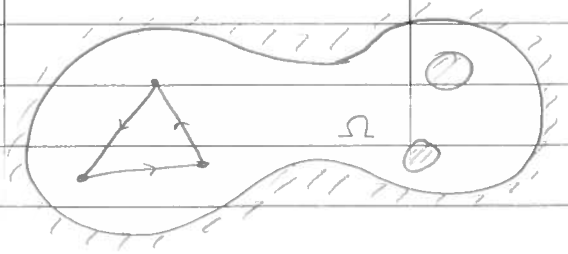


$\partial\Delta = [w_1, w_2] + [w_2, w_3] + [w_3, w_1]$

[Here for any two complex numbers a, b, $[a, b]$ denotes the segment starting at a & finishing at b.]

Assume that w_1, w_2, w_3 are oriented (as in picture) so $\partial\Delta$ is traversed anti-clockwise.
[It is allowed for w_1, w_2, w_3 to be collinear, so Δ collapses to a line segment.]

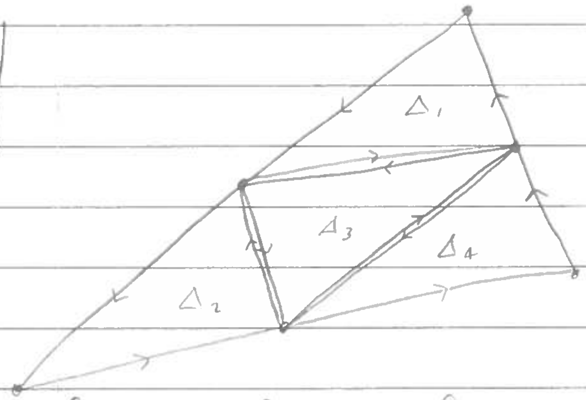
for proposition:



Proof

Subdivision argument.

$$I = \left| \int_{\partial\Delta} f(z)\, dz \right|$$



$$\int_{\partial\Delta} f = \int_{\partial\Delta_1} f + \int_{\partial\Delta_2} f + \int_{\partial\Delta_3} f + \int_{\partial\Delta_4} f$$

[Because the contributions along the interior edges cancel in pairs.]

By the triangle inequality

$$I = \left|\int_{\partial\Delta} f\right| \leq \left|\int_{\partial\Delta_1} f\right| + \left|\int_{\partial\Delta_2} f\right| + \left|\int_{\partial\Delta_3} f\right| + \left|\int_{\partial\Delta_4} f\right|$$

Not possible that $\left|\int_{\partial\Delta_i} f\right| < \dfrac{I}{4}$ for all $i = 1, 2, 3, 4$.

So it follows that at least one of these integrals is $\geq I/4$ (in modulus).

After renaming the Δ_i, suppose it is Δ_1.

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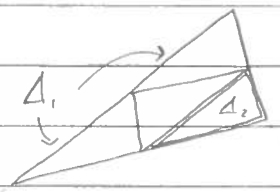
Let $I_1 = \left|\int_{\partial\Delta_1} f\right|$, know $I_1 \geq \frac{1}{4} I$.



Subdivide Δ_1. Find a triangle Δ_2, say s.t.
$\left|\int_{\partial\Delta_2} f\right| =: I_2 \geq \frac{1}{4} I_1$.

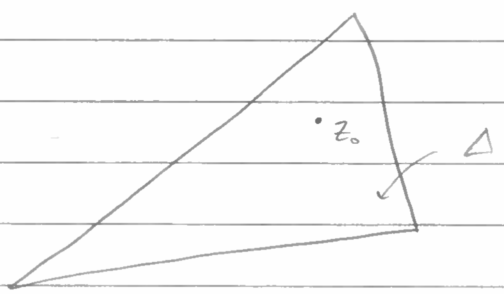
Continue process of subdivision:
Get:

- $\Delta \supset \Delta_1 \supset \Delta_2 \supset \Delta_3 \supset \dots$
- Δ_{n+1} is 'half size in linear dimension' of Δ_n
  (quater of area) length$(\partial\Delta_n) = \frac{1}{2}$ length$(\partial\Delta_{n-1})$
- $I_n = \left|\int_{\partial\Delta_n} f\right| \geq \frac{1}{4} I_{n-1}$

Claim

$\bigcap_{n=1}^{\infty} \Delta_n = \{z_0\}$ is a single point

To see this, pick $z_n \in \Delta_n$ and show that (z_n) is a Cauchy sequence. [Proof - see notes/books]

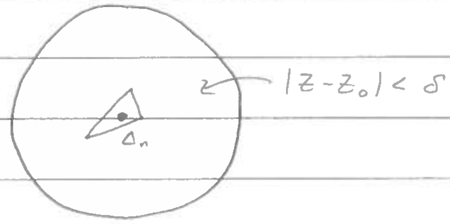


Shall show $I = 0$ by using fact that f is holomorphic at z_0.

Pick $\varepsilon > 0$. By differentiability we know $\exists \delta > 0$ such that

$$\left| \frac{f(z+h) - f(z)}{h} - f'(z_0) \right| < \varepsilon \quad \text{if } |h| < \delta.$$

Suppose n is so large that Δ_n is contained in $\{z : |z - z_0| < \delta\}$.



We want to look at

$$\int_{\partial \Delta_n} f(z)\, dz.$$

We know $\int_{\partial \Delta_n} (\text{const})\, dz = 0$, $\int_{\partial \Delta_n} z\, dz = 0$

by $\mathbb{C}$ Fundamental theorem of Calculus.

$$\left| f(z) - f(z_0) - f'(z_0)(z - z_0) \right| < \varepsilon \quad [z_0 + h = z]$$

$$\int_{\partial \Delta_n} f(z)\, dz$$
$$= \int_{\partial \Delta_n} \left(f(z) - f(z_0) - f'(z_0)(z - z_0) \right) dz$$

by Fund. Thm. of Algebra.

By choice of δ

$$\left| f(z) - f(z_0) - f'(z_0)(z - z_0) \right| \le \varepsilon |z - z_0|$$

By the length-sup estimate,

$$\left| \int_{\partial \Delta_n} \left(f(z) - f(z_0) - f'(z_0)(z - z_0) \right) dz \right|$$
$$\le \text{length}(\partial \Delta_n)\, \varepsilon \cdot \sup_{z \in \partial \Delta_n} |z - z_0| \le \text{length}(\partial \Delta_n)^2\, \varepsilon$$

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By process of subdivision,
$\text{Length}(\partial\Delta_n) = \left(\frac{1}{2}\right)^n \text{Length}(\partial\Delta)$

So $I_n \geq \left(\frac{1}{4}\right)^n I$

Combine:

$$I_n = \left|\int_{\partial\Delta_n} f\right| \leq \varepsilon \,\text{Length}(\partial\Delta_n)^2$$

$$\leq \varepsilon\, 4^{-n}\, \text{Length}(\partial\Delta)^2$$

So $4^{-n} I \leq \varepsilon\, 4^{-n}\, \text{Length}(\partial\Delta)^2$

So $I \leq \varepsilon\, \text{Length}(\partial\Delta)^2$

Since $\varepsilon > 0$ was arbitrary, it follows that $I = 0$.

$\square$

Back to thm

Recall: If $f(z)$ is continuous in Ω and $F : \Omega \mapsto \mathbb{C}$ is holomorphic, with $F' = f$, then $\int_\gamma f(z)\,dz = 0$ for any closed curve γ.

Proof

$\gamma : [t_0, t_1] \to \Omega$

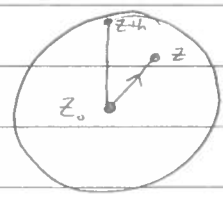
$$\int_\gamma f(z)\,dz = \int_{t_0}^{t_1} f(\gamma(t))\gamma'(t)\,dt$$

$$= \int_{t_0}^{t_1} F'(\gamma(t))\gamma'(t)\,dt$$

$$= \int_{t_0}^{t_1} \frac{d}{dt} F(\gamma(t))\,dt = F(\gamma(t_1)) - F(\gamma(t_0))$$

$$= 0 \text{ if } \gamma(t_1) = \gamma(t_0)$$

Given $f : D \mapsto \mathbb{C}$, hol.
Define $F(z) = \int_{[z_0, z]} f(\omega)\,d\omega$



Claim $F'(z) = f(z)$.

$$F(z+h) - F(z) = \int_{[z_0, z+h]} f - \int_{[z_0, z]} f$$

Apply Cauchy to Δ with corners $z_0, z, z+h$

$$\int_{[z_0, z]} f + \int_{[z, z+h]} f + \int_{[z+h, z_0]} f = 0$$

$$\therefore \int_{[z, z+h]} f = \int_{[z_0, z+h]} f - \int_{[z_0, z]} f$$

$$= F(z+h) - F(z)$$

So $F(z+h) - F(z) - hf(z) = \int_{[z, z+h]} (f(\omega) - f(z))\,d\omega$

Now use continuity of f at z to deduce

$$\left| \frac{F(z+h) - F(z)}{h} - f(z) \right| \to 0 \quad \text{as } |h| \to 0.$$

(Omitting ε, δ proof)

We have used Cauchy for Δ to prove $F'(z) = f(z)$ and then FTC $\Rightarrow \int_\gamma f(z)\,dz = 0$

for any closed curve.

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<u>Remark</u>

<u>Same argument</u> $\Rightarrow \int_\gamma f(z)\,dz = 0$, γ closed curve

for any open set Ω with property:

$\exists z_0 \in \Omega$, s.t. $[z_0, z] \subset \Omega$ for any $z \in \Omega$.

$z_0 = -1$

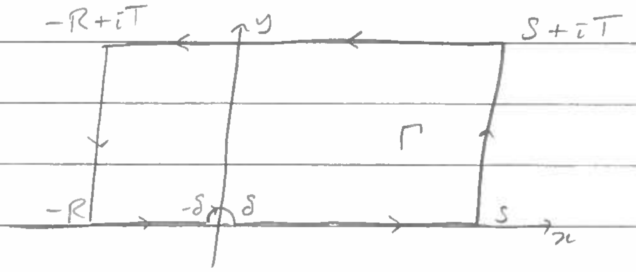
$\Omega = \mathbb{C} \setminus \{ Im(z) = 0,\ Re(z) \geq 0 \}$
works, taking $z_0 = -1$ or any point on negative real axis.

<u>Terminology</u> — "Contour" $\longleftrightarrow$ Piecewise C^1 <u>closed</u> curve.

<u>Example</u>

$$\int_{-\infty}^{\infty} \frac{\sin x}{x}\,dx = \pi$$

Integrate $\frac{e^{iz}}{z}$ around Γ



Let $f(z) = \frac{e^{iz}}{z}$

① By Cauchy's Theorem, $\int_\Gamma f(z)\,dz = 0$

<u>Why?</u> $f(z)$ is holomorphic in $\Omega = \mathbb{C} \setminus \{ Im(z) \leq 0,\ Re(z) = 0 \}$

which is star-like.
Γ is a closed curve contained in Ω, so Cauchy's Thm applies.

② What does $\int_\Gamma f$ have to do with $\int_{-\infty}^{\infty} \frac{\sin x}{x} dx$?

$\int_\Gamma f$ = sum of terms

$$\int_{-R}^{-\delta} = \int_{-R}^{-\delta} \frac{e^{ix}}{x} dx = \int_{-R}^{-\delta} \left(\frac{\cos x}{x} + i \frac{\sin x}{x} \right) dx \quad (i)$$

Similarly

$$\int_{\delta}^{S} f(z)\, dz = \int_{\delta}^{S} \left(\frac{\cos x}{x} + i \frac{\sin x}{x} \right) dx \quad (ii)$$

Semicircle:

$\gamma(t) = \delta e^{-it}$, $-\pi \le t \le 0$ (note sign $\Rightarrow$ clockwise)

$$\int_\gamma f(z)\, dz = \int_{-\pi}^{0} \frac{\exp(i\delta e^{-it})}{\delta e^{-it}} (-i\delta e^{-it}) dt$$

$$= -i \int_{-\pi}^{0} \exp(i\delta e^{-it}) dt$$

$$= -i\pi + O(\delta) \quad \text{for small } \delta \quad (iii)$$

Combining (i), (ii) & (iii) with $\int_\Gamma f = 0$, we get:

$$\int_{-R}^{-\delta} \frac{\sin x}{x} dx + \int_{\delta}^{S} \frac{\sin x}{x} dx - \pi + O(\delta)$$

$$= -\,\mathrm{Im} \left[\int_{[S,\, S+iT]} f + \int_{[S+iT,\, -R+iT]} f + \int_{[-R+iT,\, -R]} f \right]$$

<u>Idea</u>: Estimate RHS, and show that the modulus of each part $\to 0$ as $R, S, T \to \infty$.

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Last time:

$$\int_{-R}^{-\delta} \frac{\sin x}{x}\,dx + \int_{\delta}^{S} \frac{\sin x}{x}\,dx \quad -\pi + O(\delta)$$

$$= -\operatorname{Im} \int_{\gamma_1+\gamma_2+\gamma_3} \frac{e^{iz}}{z}\,dz$$

Where $\begin{cases} \gamma_1 \text{ is line } S \to S+iT \\ \gamma_2 \text{ is line } S+iT \to -R+iT \\ \gamma_3 \text{ is line } -R+iT \to -R \end{cases}$

Want to show

$$\left|\int_{\gamma_1} f\right|,\ \left|\int_{\gamma_2} f\right|,\ \left|\int_{\gamma_3} f\right| \to 0 \text{ as } R,S,T \to \infty.$$

Try length-sup estimate on γ_2:

length$(\gamma_2) = R+S$

on γ_2, $z = x+iT$

$$|f(z)| = \frac{|e^{i(x+iT)}|}{|x+iT|}$$

$$= \frac{e^{-T}}{\sqrt{x^2+T^2}}$$

So $\left|\int_{\gamma_2} \frac{e^{iz}}{z}\,dz\right| \le (R+S)\frac{e^{-T}}{T}$

so for fixed R,S, this goes to zero as $T\to\infty$.

Try length-sup estimate on γ_1:

On γ_1, $z = S+iy$, $0\le y\le T$

$$\left|\frac{e^{iz}}{z}\right| = \left|\frac{e^{i(S+iy)}}{S+iy}\right| = \frac{e^{-y}}{\sqrt{S^2+y^2}} \le \frac{1}{S}$$

length$(\gamma_1) = T$

So $\left|\int_{\gamma_1}\right| \le \frac{T}{S}$ this doesn't work as from above

we decided to let $T\to\infty$ first.

But:

$$\left|\int_{\gamma_1} f(z)\,dz\right| = \left|\int_0^T \frac{e^{i(S+iy)}}{S+y}\, i\,dy\right|$$

$$\leq \int_0^T \frac{e^{-y}}{\sqrt{S^2+y^2}}\,dy$$

$$\leq \int_0^T \frac{e^{-y}}{S}\,dy = \left[\frac{-e^{-y}}{S}\right]_0^T$$

$$= \frac{1}{S}\left(1-e^{-T}\right) \leq \frac{1}{S}.$$

Similarly $\left|\int_{\gamma_3} \frac{e^{iz}}{z}\,dz\right| \leq \frac{1}{R}.$

Let $T \to \infty$

So $\int_{-R}^{-\delta} \frac{\sin x}{x}\,dx + \int_{\delta}^{S} \frac{\sin x}{x}\,dx - \pi + O(\delta)$

$$= -\operatorname{Im}\left(\int_{\gamma_1} f(z)\,dz + \int_{\gamma_3} f(z)\,dz\right) \qquad (T = +\infty)$$

$$\leq O\left(\frac{1}{R}\right) + O\left(\frac{1}{S}\right)$$

Let $\delta \to 0$.

Then $\left|\int_{-R}^{S} \frac{\sin x}{x}\,dx - \pi\right| \leq \frac{1}{R} + \frac{1}{S} \to 0$ as $R, S \to \infty$

$$\Rightarrow \int_{-\infty}^{\infty} \frac{\sin x}{x}\,dx = \pi$$

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Thm (4.5.1) Cauchy's Integral Formula

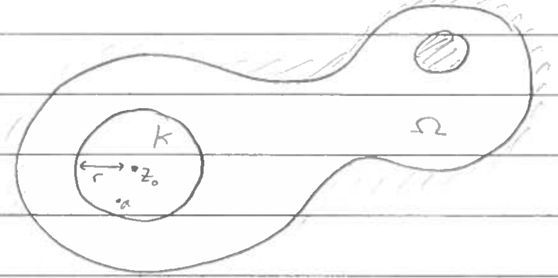
$\Omega \subset \mathbb{C}$ open, $f : \Omega \mapsto \mathbb{C}$ holomorphic.

$K = \{|z - z_0| \le r\} \subset \Omega$.

Then if a is interior of K $(|a - z_0| < r)$ we have

$$f(a) = \frac{1}{2\pi i} \int_{|z - z_0| = r} \frac{f(z)}{z-a} \, dz$$

means traversed once, anticlockwise

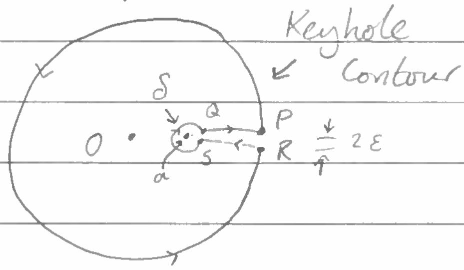


Proof

w.l.o.g. $z_0 = 0$ and a is on the positive real axis.

For keyhole contour Γ shown →

$$\int_\Gamma \frac{f(z)}{z-a} \, dz = 0 \text{ by Cauchy.}$$



Indeed $\frac{f(z)}{z-a}$ is hol. in $\Omega \setminus \{a\}$, in particular $f(z)/(z-a)$ is holomorphic in a cut disc.

$\Omega' = \{|z| < r'\} \setminus \{z : Im(z) = 0, Re(z) \ge a\}$ and $r' > r$, but such that $\Omega' \subset \Omega$.

Ω' is starlike in sense that if $z \in \Omega'$ it can be connected to 0 by a straight line contained in Ω'.

$\Gamma \subset \Omega'$ and $f(z)/(z-a)$ is hol in Ω'

Hence $\int_\Gamma \frac{f(z)}{z-a} \, dz = 0$ by Cauchy.

Claims

1). For fixed δ

$$\lim_{\varepsilon\to 0}\left(\int_{QP}\frac{f(z)}{z-a}\,dz+\int_{RS}\frac{f(z)}{z-a}\,dz\right)\to 0$$

Hence

$$\int_{|z|=r}\frac{f(z)}{z-a}\,dz-\int_{|z-a|=\delta}\frac{f(z)}{z-a}=0$$

2). $\displaystyle\lim_{\delta\to 0}\int_{|z-a|<\delta}\frac{f(z)}{z-a}\,dz=2\pi i f(a)$

Proof of ②

The circle can be parameterised as $\gamma(t)=a+\delta e^{it}$

$0\le t<2\pi$.

$$\int_{|z-a|=\delta}\frac{f(z)}{z-a}\,dz=\int_0^{2\pi}\frac{f(a+\delta e^{it})}{\delta e^{it}}\,i\delta e^{it}\,dt$$

$$=i\int_0^{2\pi}f(a+\delta e^{it})\,dt$$

$$=i\int_0^{2\pi}f(a)\,dt+i\int_0^{2\pi}\left(f(a+\delta e^{it})-f(a)\right)dt$$

By uniform continuity $\sup|f(a+\delta e^{it})-f(a)|\to 0$ as $\delta\to 0$, $0\le t\le 2\pi$.

Let $\delta\to 0$, then $\displaystyle\int_{|z-a|=\delta}\frac{f(z)}{z-a}\,dz=2\pi i f(a)$. $\square$

Proof of ① is similar

$$\int_{QP}\frac{f(z)}{z-a}\,dz=\int_{x_1}^{x_2}\frac{f(x+i\varepsilon)}{x+i\varepsilon-a}\,dx=I(\varepsilon),\quad Q=x_1+i\varepsilon,\quad P=x_2+i\varepsilon$$

$$I(\varepsilon)-I(0)=\int_{x_1}^{x_2}\left(\frac{f(x+i\varepsilon)}{x+i\varepsilon-a}-\frac{f(x)}{x-a}\right)dx$$

Again $\displaystyle\sup_{x\in[x_1,x_2]}\left|\frac{f(x+i\varepsilon)}{x+i\varepsilon-a}-\frac{f(x)}{x-a}\right|\longrightarrow 0$ as $\varepsilon\to 0$.

$\propto$ uniform cont.

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So $I(\varepsilon) \to I(0)$ as $\varepsilon \to 0$

Similarly $I(-\varepsilon) \to I(0)$ as $\varepsilon \to 0$.

$$\int_{QP} + \int_{RS} = I(\varepsilon) - I(-\varepsilon) \to I(0) - I(0) = 0$$

as $\varepsilon \to 0$.

This proves the Thm. $\square$

Example

$$\int_0^{2\pi} \sin^{2n} t \, dt = I_n$$

Reverse-engineer as a contour integral

$z = e^{it} = \cos t + i \sin t$

$\sin t = \frac{1}{2i}(z - z^{-1})$

$dz = e^{it} i\,dt = iz\,dt$

$dt = \frac{dz}{iz}$

$$I_n = \int_{|z|=1} \left(\frac{1}{2i}(z - z^{-1})\right)^{2n} \frac{dz}{iz}$$

$$= \frac{1}{(2i)^{2n} i} \int_{|z|=1} (z - z^{-1})^{2n} \frac{dz}{z}$$

$$= \frac{1}{2^{2n}(-1)^n i} \int_{|z|=1} \sum_{r=0}^{2n} \binom{2n}{r} z^r (-1)^{2n-r} z^{-(2n-r)} \frac{dz}{z}$$

$$= \frac{1}{2^{2n}(-1)^n i} \sum_{r=0}^{2n} \int_{|z|=1} \binom{2n}{r} (-1)^r z^{2r-2n-1} \, dz$$

By FTC of direct computation

$$\int_{|z|=1} z^p \, dz = \begin{cases} 0 & \text{if } p \neq -1 \\ 2\pi i & \text{if } p = -1 \end{cases}$$

Hence $I_n = \dfrac{1}{2^{2n} \cancel{(-1)^n} \cancel{i}} \binom{2n}{n} \cancel{(-1)^n} 2\pi \cancel{i}$

$$= \frac{2\pi}{2^{2n}} \binom{2n}{n}$$

Thm (4.5.3) Hol. fns are analytic

$\Omega \subset \mathbb{C}$ open, $f : \Omega \longmapsto \mathbb{C}$ hol.

$z_0 \in \Omega$ s.t. $K = \{|z - z_0| \leq r\} \subset \Omega$.

Then $f(z) = \displaystyle\sum_{n=0}^{\infty} a_n (z - z_0)^n$

where $a_n = \dfrac{f^{(n)}(z_0)}{n!} = \dfrac{1}{2\pi i} \displaystyle\int_{|w - z_0| = r} \dfrac{f(w)}{(w - z_0)^{n+1}} \, dw$

is convergent for $|z - z_0| < r$.

Corollary

If f is holomorphic in Ω then so is f'.
In particular f' is continuous.

Key: $|w - z_0| = r$, $|z - z_0| < r$

$$\Rightarrow \frac{1}{w - z} = \frac{1}{w - z_0} \sum_{n=0}^{\infty} \frac{(z - z_0)^n}{(w - z_0)^n}$$

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Proof of Thm

Have seen:

$$f(z) = \frac{1}{2\pi i}\int_{|w-z_0|=r} \frac{f(w)}{w-z}\,dw \quad \text{for } |z-z_0|<r$$

$$\frac{1}{w-z} = \frac{1}{(w-z_0)-(z-z_0)} = \frac{1}{w-z_0}\left(1-\frac{z-z_0}{w-z_0}\right)^{-1}$$

$$= \frac{1}{w-z_0}\sum_{n=0}^{\infty}\left(\frac{z-z_0}{w-z_0}\right)^n = \sum_{n=0}^{\infty}\frac{(z-z_0)^n}{(w-z_0)^{n+1}}$$

Insert this expansion in CIF.

$$f(z) = \frac{1}{2\pi i}\int_{|w-z_0|=r}\left(\sum_{n=0}^{\infty}\frac{(z-z_0)^n}{(w-z_0)^{n+1}}\right) f(w)\,dw$$

Uniform convergence of series for $|z-z_0|<|w-z_0|$
+ length-sup estimate

$$\Rightarrow f(z) = \sum_{n=0}^{\infty}\left[(z-z_0)^n \cdot \frac{1}{2\pi i}\int_{|w-z_0|=r}\frac{f(w)}{(w-z_0)^{n+1}}\,dw\right]$$

Define $a_n = \frac{1}{2\pi i}\int_{|w-z_0|=r}\frac{f(w)}{(w-z_0)^{n+1}}\,dw$

$\square$

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Thm 4. Hol fn's are analytic

$\Omega \subset \mathbb{C}$ open, $f : \Omega \mapsto \mathbb{C}$ hol.

$z_0 \in \Omega$ st. $K = \{ |z - z_0| \leq r \} \subset \Omega$

Then

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$$

where $a_n = \dfrac{f^{(n)}(z_0)}{n!} = \dfrac{1}{2\pi i} \displaystyle\int_{|w - z_0| = r} \dfrac{f(w)}{(w - z_0)^{n+1}} \, dw$

is convergent for $|z - z_0| < r$

Key: $|w - z_0| = r$, $|z - z_0| < r$

$$\Rightarrow \frac{1}{w - z} = \frac{1}{w - z_0} \sum_{n=0}^{\infty} \frac{(z - z_0)^n}{(w - z_0)^n}$$

So if f is holomorphic on Ω then so is f'.
In particular f has continuous partial derivatives.
In fact all partial derivatives of all orders are continuous.

Lemma (Cauchy's Inequalities)

In the convergent power series expansion of f about $z = z_0$, we have

$$|a_n| \leq \frac{1}{r^n} \sup \{ |f(w)| : |w - z_0| = r \}.$$

Proof

Length-sup estimate in formula for a_n

Length of contour is $2\pi r$

For $|w - z_0| = r$,

$$\left|\frac{f(w)}{(w-z_0)^{n+1}}\right| = \frac{|f(w)|}{r^{n+1}} \leq \frac{1}{r^{n+1}} \sup\left\{|f(w)| : |w-z_0|=r\right\}$$

Hence $|a_n| \leq \frac{1}{2\pi} \cdot 2\pi r \cdot \frac{1}{r^{n+1}} \sup\left\{|f(w)| : |w-z_0|=r\right\}$

$\square$

<u>Theorem 5</u> Liouville's Thm

If $f : \mathbb{C} \mapsto \mathbb{C}$ is holomorphic and bounded, then f must be a constant.

<u>Proof:</u>

$\Omega = \mathbb{C}$, $z_0 = 0$, $r > 0$ in previous Thm.

If $|z| < r$, we have $f(z) = \sum_{n=0}^{\infty} a_n z^n$

where $|a_n| \leq \frac{1}{r^n} \sup\left\{|f(w)| : |w| = r\right\}$.

If $|f(z)| \leq M \quad \forall z \in \mathbb{C}$, then

the above inequality gives $|a_n| \leq \frac{1}{r^n} M$.

'r' is arbitrary. So if we fix n and let $r \to \infty$, we learn $|a_n| = 0$ for $n > 0$ ($|a_0| \leq M$).

Hence the power series expansion of f reduces to $f(z) = a_0$ $\square$.

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Corollary (Fundamental Thm of Algebra)
Let $P(z)$ be a non-constant poynomial.
Then $\exists\ \alpha \in \mathbb{C}$ s.t. $P(\alpha) = 0$.

Proof (by contradiction)
Suppose $P(z) \neq 0$ for all $z \in \mathbb{C}$.
Then $f(z) = \dfrac{1}{P(z)}$ is holomorphic in $\mathbb{C}$

Claim: f is bounded.

Idea: $f(z) = \dfrac{1}{a_n z^n + \dots + a_0}$

$a_n \neq 0 \implies |P(z)| \geq \frac{1}{2}|a_n||z|^n$ if $|z| \geq R$.

$$\text{So } |f(z)| = \frac{1}{|P(z)|} \leq \frac{2}{|a_n|}|z|^{-n} \text{ for } |z| \geq R$$
$$\leq \frac{2}{|a_n|} R^{-n}$$

Moreover, $\{|z| \leq R\}$ is closed and bounded, f is continuous on this set so it is bounded.

So $|f(z)| \leq M$ for $|z| \leq R$

So $|f(z)| \leq \max\left(M, \frac{2}{|a_n|}R^{-n}\right)\ \forall z \in \mathbb{C}$.

Liouville's Thm $\rightarrow f(z)$ is constant.
$\Rightarrow P = \frac{1}{f}$ is also constant

So $P = $ const, contradiction $\square$.

If $|z|$ is large

$|a_n z^n + \dots + a_0|$

$$= |a_n||z|^n \left| 1 + \frac{a_{n-1}}{a_n z} + \frac{a_{n-2}}{a_n z^2} + \dots + \frac{a_0}{a_n z^n} \right|$$

Choose R so that

$$\left| \frac{a_j}{a_n z^{n-j}} \right| < \frac{1}{3n} \quad \forall \; j = 0, 1, \dots, n-1, \text{ for } |z| > R$$

Then

$$|a_n z^n + \dots + a_0| \geq |a_n z^n| \Big(1 \underbrace{- \frac{1}{3n} - \frac{1}{3n} - \dots - \frac{1}{3n}}_{n \text{ times}} \Big)$$

$$\geq \frac{1}{2} |a_n||z|^n$$

<u>Theorem</u> (Cauchy's Integral Formula for derivatives)

Let $f: \Omega \mapsto \mathbb{C}$ be holomorphic, Ω a domain.

Let $\bar{D} = \{ z \in \mathbb{C} : |z - z_0| \leq r \} \subset \Omega$

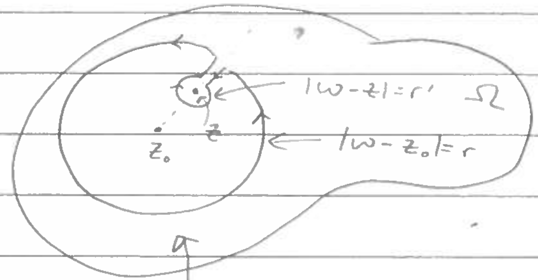
Then, for $|z - z_0| < r$,

$$f^{(n)}(z) = \frac{n!}{2\pi i} \int_{|w - z_0| = r} \frac{f(w)\, dw}{(w - z)^{n+1}}$$

<u>Proof</u>

Keyhole contour method

$$\frac{f^{(n)}(z)}{n!} = \int_{|w - z| = r'} \frac{f(w)\, dw}{(w - z)^{n+1}} \quad \text{provided } \{ w : |w - z| \leq r' \} \subset \Omega.$$



Γ keyhole countour

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$$\int_\Gamma \frac{f(\omega)\,d\omega}{(\omega-z)^{n+1}} = 0$$ by Cauchy's Thm (CIF proof)

Let distance between straight segments go to zero: contributions to integral cancel.

So $$\int_{|\omega - z| = r'} \frac{f(\omega)}{(\omega-z)^{n+1}}\,d\omega = \int_{|\omega - z_0| = r} \frac{f(\omega)}{(\omega-z)^{n+1}}\,d\omega$$ $\square$

Morera's Theorem

Suppose $f: \Omega \mapsto \mathbb{C}$ is continuous, where Ω is a domain (open & path connected).

If $\int_\gamma f(z)\,dz = 0$ for all closed curves γ in Ω, then f is holomorphic.

Proof

Pick $z_0 \in \Omega$

$$F(z) = \int_\Gamma f(\omega)\,d\omega$$

where Γ is a C' curve connecting z_0 to z.

$F(z)$ is well defined (independent of choice of connecting path Γ), because

$\int_\gamma f(z)\,dz = 0$ if γ is closed.

Have seen $F(z)$ is holomorphic and $F'(z) = f(z)$. (Fundamental Theorem of Calculus).

Analyticity of hol fns says that F' is holomorphic if f is hol.

(Thm 4 (4.5.3) of notes). $\square$

Chapter 5 Cauchy's Theorem + Green's Theorem

Green's Thm in plane:

- Bounded domain D with piecewise C^1 boundary
- $P(x,y)dx + Q(x,y)dy$ where P and Q are continuous on $\bar{D}$ and $\frac{\partial P}{\partial x}, \frac{\partial P}{\partial y}, \frac{\partial Q}{\partial x}, \frac{\partial Q}{\partial y}$ are continuous in D

$\bar{D}$ = closure of D i.e. D + boundary

$$\int_{\partial D} P\,dx + Q\,dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy \quad (*)$$

Relation to Cauchy's Thm?
No problem extending to P and Q complex-valued.
In particular since

$$f(z)\,dz = f(z)(dx + i\,dy) = f(z)\,dx + i f(z)\,dy$$

Let $P = f(z)$, $Q = i f(z)$ is Green's Thm $(*)$, LHS is $\int_{\partial D} f(z)\,dz$.

$$RHS = \iint_D \left(i\frac{\partial f}{\partial x} - \frac{\partial f}{\partial y}\right) dx\,dy$$

If f is hol. the integrand vanishes by Cauchy-Riemann eqns $f = u + iv$.

CRE: $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$, $\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$

$$i\frac{\partial f}{\partial x} - \frac{\partial f}{\partial y} = i\left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right) - \left(\frac{\partial u}{\partial y} + i\frac{\partial v}{\partial y}\right)$$

$$= i\left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right) - \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) = 0 \text{ if } f \text{ hol.}$$

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Hence Green's Thm $\Rightarrow$ Cauchy's Thm.

Comments

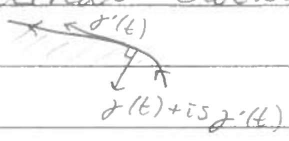
Objection: if 'holomorphic' means 'complex derivative exists', we do not have f is C' and so we can't apply Green's Thm.

Following Thm 4 we now know that f holomorphic $\Rightarrow$ f is C'. Now we can use Green's Thm to get more general versions of Cauchy.

Definition

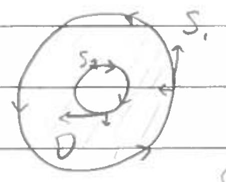
A bounded domain D has piecewise C' boundary if, for each path connected subset S of ∂D there is a piecewise C' closed curve $\gamma : [t_0, t_1] \to \mathbb{C}$ such that γ is a bijection $\gamma : [t_0, t_1) \to S$.
Say also that γ agrees with standard orientation of the boundary (∂D) if at all points t with $\gamma'(t) \neq 0$, $\gamma(t) + is\gamma'(t) \in D$ for all small positive s.

Intuitively: 'domain is on left' as you traverse boundary.
(external boundary: anticlockwise, internal: clockwise)



Example

$D = \{z \in \mathbb{C} : a < |z| < b\}$



$\partial D = S_1 \cup S_2$

$S_1 = \{|z| = b\}$

$S_2 = \{|z| = a\}$

S_2 is traversed clockwise to agree with orientation.

Definition / Notation

Let D be a bounded domain with piecewise C^1 boundary. If there are a finite number of path-connected boundary components $S_1, \dots, S_n$, and γ_j a parameterisation of S_j agreeing with standard orientation, then we write

$$\partial D = S_1 \cup \dots \cup S_n$$
$$= \gamma_1 + \gamma_2 + \dots + \gamma_n$$

Green's Thm

With above conventions and in this situation:

$$\int_{\partial D} P\,dx + Q\,dy = \sum_{j=1}^{n} \int_{\gamma_j} (P\,dx + Q\,dy)$$

and we have $\displaystyle\int_{\partial D} P\,dx + Q\,dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy.$

Generalisation of Cauchy's Thm

$f : \Omega \longmapsto \mathbb{C}$ is holomorphic

(Ω an open set)

Let D be a domain st.

$\bar{D}$ = closure of $D \subset \Omega$

Then $\displaystyle\int_{\partial D} f(z)\,dz = 0.$

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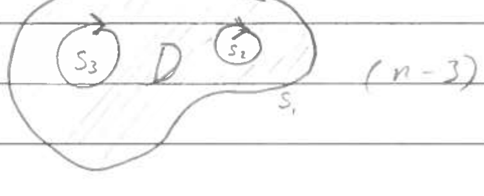
[Green's Theorem]

D bounded domain with C'-boundary ∂D,
$\partial D = \gamma_1 + \dots + \gamma_n$.



Green's Theorem

If $P(x,y)$, $Q(x,y)$ are C' (first partial derivatives are continuous).

$$\int_{\partial D} P(x,y)\,dx + Q(x,y)\,dy = \iint_{\partial D} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy.$$

Thm (Generalisation of Cauchy)

Suppose $D \cup \partial D \subset \Omega$, where Ω is an open subset of $\mathbb{C}$.
Suppose $f : \Omega \mapsto \mathbb{C}$ is holomorphic.
Then $\displaystyle\int_{\partial D} f(z)\,dz = \int_{\gamma_1+\gamma_2+\dots+\gamma_n} f(z)\,dz = 0.$

Proof

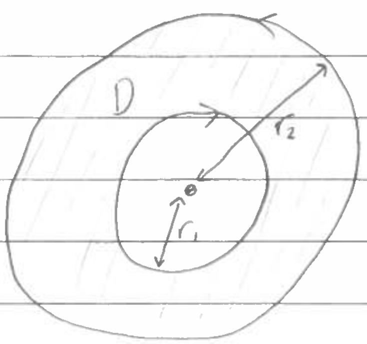
By Thm 4 (holomorphic fns are expandable in power series) it follows that f is C' in Ω and hence also in $D \cup \partial D$.
So we can apply Green to calculate $\displaystyle\int_{\partial D} f(z)dz = \int_{\partial D} f(z)(dx + i\,dy)$

(i.e. $P = f$, $Q = if$).

Then $\displaystyle\int_{\partial D} f(z)\,dz = \int_D \left(i\frac{\partial f}{\partial x} - \frac{\partial f}{\partial y}\right) dx\,dy = 0$ by Cauchy Rieman equations.
$\square$.

Remark

$\Omega = \mathbb{C} \setminus \{0\}$
$f : \Omega \mapsto \mathbb{C}$ holomorphic
$D = \{r_1 < |z| < r_2\}$

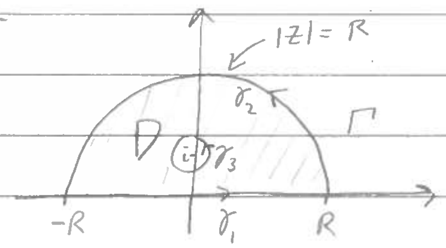


Cauchy:

$$\int_{|z|=r_2} f(z)\,dz = \int_{|z|=r_1} f(z)\,dz$$

Intuition: 'Can move contour through region where f is holomorphic, without changing value of the integral.'

Example



$$f(z) = \frac{1}{z^2+1}$$

Want:

$$\int_\Gamma \frac{1}{z^2+1}\,dz \qquad (R>1)$$

Let D = interior of half disc of radius $R \setminus \{z \in \mathbb{C} : |z-i| = \frac{1}{2}\}$

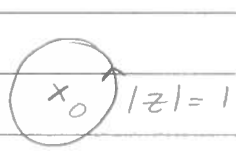
$f(z)$ is holomorphic in $\mathbb{C} \setminus \{i, -i\}$ which contains $D \cup \partial D$. So Cauchy:

$$\int_\Gamma f(z)\,dz = \int_{\gamma_3} f(z)\,dz \qquad (\partial D = \Gamma - \gamma_3)$$

[Orientation of γ_3 is opposite to standard for convienience here.]

Example

$$\int_{|z|=1} \frac{dz}{z} = 2\pi i$$



Here $|z|=1$ is not the boundary of a bounded domain D with $D \cup \partial D \subset \Omega$ where f is hol on Ω as the origin is removed.

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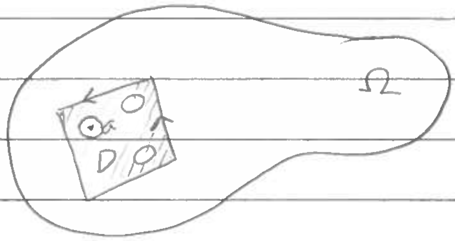
Similar version of Cauchy's Integral formula.
Ω is an open subset of $\mathbb{C}$,
$f : \Omega \mapsto \mathbb{C}$.
D - bounded domain with $D \cup \partial D \subset \Omega$.
Suppose $a \in D$.

Then
$$f(a) = \frac{1}{2\pi i} \int_{\partial D} \frac{f(z)dz}{z-a}.$$



Proof
Let $\varepsilon > 0$ and consider $D_\varepsilon = D \setminus \{|z-a| \leq \varepsilon\}$.
$(|z-a| < \varepsilon \Rightarrow z \in D)$.
$\partial D_\varepsilon = \partial D - \gamma$, where $\gamma(t) = \varepsilon e^{it} + a \quad 0 \leq t \leq 2\pi$

Cauchy applies to $\frac{f(z)}{z-a}$ and D_ε,
$$\int_{\partial D_\varepsilon} \frac{f(z)}{z-a} = 0 = \int_{\partial D_\varepsilon} \frac{f(z)\,dz}{z-a} - \underbrace{\int_{|z-a|=\varepsilon} \frac{f(z)}{z-a}\,dz}_{I_\varepsilon}$$

Parameterise the second integral
$$I_\varepsilon = \int_0^{2\pi} \frac{f(a+\varepsilon e^{it}) i\varepsilon e^{it}}{\varepsilon e^{it}}\,dt$$
$$= i \int_0^{2\pi} f(a+\varepsilon e^{it})\,dt$$

Now $I_\varepsilon - 2\pi i f(a) = i \int_0^{2\pi} (f(a+\varepsilon e^{it}) - f(a))\,dt$

length -sup: $|I_\varepsilon - 2\pi i f(a)| \leq 2\pi \sup \{|f(a+\varepsilon e^{it}) - f(a)| : 0 \leq t \leq 2\pi\}$

$\rightarrow 0$ by continuity of f at a

$$\therefore \int_{\partial D} \frac{f(z)\,dz}{z-a} = 2\pi i f(a) \qquad \square$$

Exercises

Use Cauchy's Integral Formula (for derivatives) to calculate

1). $\displaystyle\int_{-\infty}^{\infty} \frac{dx}{(x^2+1)^2}$

2). $\displaystyle\int_{-\infty}^{\infty} \frac{x \sin x}{(x^2+1)^2}\,dx$

3). $\displaystyle\int_{-\infty}^{\infty} \frac{\log|x|\,dx}{4+x^2}$

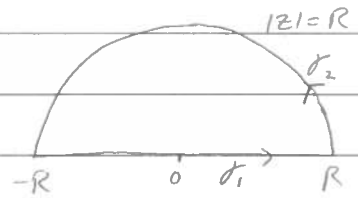
(Large semicircular contour in each case).

Need:

① Holomorphic function (not nec. in whole of $\mathbb{C}$)

② Contour (piecewise C^1 closed curve)

Q 2 answer



Need to choose a hol. fn. to be equal (or very closely related to) the real integral we are after.

Could try $f(z) = \dfrac{z e^{iz}}{(1+z^2)^2}$

when $z = x$, $f(x) = \dfrac{x(\cos x + i \sin x)}{(x^2+1)^2}$

Could try $g(z) = \dfrac{z \sin z}{(z^2+1)^2}$

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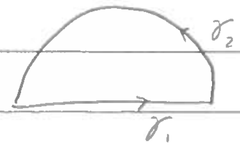
$I_1 \displaystyle\int_{\gamma_1+\gamma_2} \frac{ze^{iz}}{(z^2+1)^2}\, dz$



$I_2 \displaystyle\int_{\gamma_1+\gamma_2} \frac{z\sin z}{(z^2+1)^2}\, dz$

In each case we have a singularity at $z=i$.

CIF for first deriv.

$$I_1 = \int_{\gamma_1+\gamma_2} \frac{ze^{iz}}{(z-1)^2(z+1)^2}\, dz = \int_{\gamma_1+\gamma_2} \frac{ze^{iz}/(z+i)^2}{(z-i)^2} = 2\pi i \frac{d}{dz}\left[\frac{ze^{iz}}{(z+i)^2}\right]\Bigg|_{z=i}$$

$$= \left(\frac{e^{iz}+ize^{iz}}{(z+i)^2} - \frac{2ze^{iz}}{(z+i)^3}\right)\Bigg|_{z=i} \times 2\pi i$$

$$= 2\pi i\left(\frac{\cancel{e^{-1}-e^{-1}}}{\cancel{(2i)^2}} - \frac{2ie^{-1}}{(2i)^3}\right)$$

$$= \frac{4\pi e^{-1}}{8(-i)} = \frac{\pi i e^{-1}}{2}$$

[I_2 similar.]

$$\int_{-R}^{R} \frac{xe^{ix}}{(1+x^2)^2}\, dx + \int_{\gamma_2} \frac{ze^{iz}}{(1+z^2)^2}\, dz = \frac{\pi i}{2e}$$

$$R\to\infty, \quad \mathrm{Im}\int_{-R}^{R} \frac{xe^{ix}}{(1+x^2)^2} \longrightarrow \int_{-\infty}^{\infty} \frac{x\sin x}{(1+x^2)}\, dx$$

Claim

$$\int_{\gamma_2} \frac{ze^{iz}}{(1+z^2)^2}\, dz \longrightarrow 0 \text{ as } R\to\infty.$$

Try Length - sup estimate:

Length πR.

If $|z|=R$, $\left|\dfrac{ze^{iz}}{(z^2+1)^2}\right| = \dfrac{|z||e^{iz}|}{|z^2+1|^2} = \dfrac{Re^{-y}}{|z^2+1|^2}$ as $iz = ix - y$

Note $|z^2+1| \geq |z|^2 - 1 = R^2 - 1$

So $\left|\frac{ze^{iz}}{(z^2+1)^2}\right| \leq \frac{Re^{-y}}{(R^2-1)^2}$

$\leq \frac{R}{(R^2-1)^2} \quad \because y > 0$

So $\text{leng} \times \text{sup} \leq \frac{\pi R^2}{(R^2-1)^2} \to 0$ as $R \to \infty$.

Hence $\int_{-\infty}^{\infty} \frac{xe^{ix}}{(x^2+1)^2}\,dx = \frac{\pi i}{2e}$

$\int_{-\infty}^{\infty} \frac{x\sin x}{(x^2+1)^2}\,dx = \frac{\pi}{2e}$ (This is imaginary part)

What if we had used $\frac{z\sin z}{(1+z^2)^2}$?

$\int_{\Gamma} \frac{z\sin z}{(1+z^2)^2}\,dz = C$ by CIF for deriv.

$\int_{\gamma_2} \frac{z\sin z}{(1+z^2)^2} \not\to 0.$

$\sin z = \frac{e^{iz} - e^{-iz}}{2i}$

$= \frac{e^{ix-y} - e^{-ix+y}}{2i}$

which cannot be controlled on γ_2 because $e^y \to \infty$

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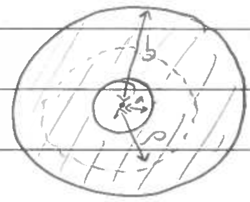
Singularities of holomorphic functions

- Laurent's Theorem
- Isolated singularities
- Residue Theorem

Let A be an annulus

$A = \{z \in \mathbb{C} : a < |z| < b\}$

a, b are real, $a < b$; $a = 0$ is allowed,
in this case we get $D^*(0,b) = \{z \in \mathbb{C} : 0 < |z| < b\}$



Laurent's Theorem

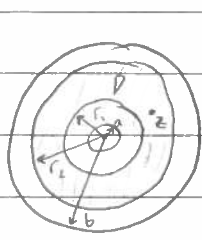
Let A be an annulus, and $f : A \mapsto \mathbb{C}$ be holomorphic.
Then $\exists c_n \in \mathbb{C}$, such that $f(z) = \sum_{n=-\infty}^{\infty} c_n z^n$, $\forall z \in A$.

Moreover if $\rho \in (a,b)$ we have

$$c_n = \frac{1}{2\pi i} \int_{|z|=\rho} z^{-n-1} f(z)\,dz \quad \forall n \in \mathbb{Z}.$$

Remark:
Note formula for c_n has flexibility to vary ρ.

Proof
Idea: use CIF.
Pick $z \in A$, also pick r_1, r_2 st.
$a < r_1 < |z| < r_2 < b$.



Then $D = \{ r_1 < |z| < r_2 \}$
is contained in A and CIF gives

$$f(z) = \frac{1}{2\pi i} \int_{\partial D} \frac{f(\omega)}{\omega - z} \, d\omega$$

$$= \underbrace{\frac{1}{2\pi i} \int_{|\omega| = r_2} \frac{f(\omega) \, d\omega}{\omega - z}}_{\text{outer circle}} - \underbrace{\frac{1}{2\pi i} \int_{|\omega| = r_1} \frac{f(\omega) \, d\omega}{\omega - z}}_{\text{inner circle}}$$

Idea: expand $\frac{1}{\omega - z}$ by binomial theorem:

compare with proof that 'hol' $\Rightarrow$ 'analytic'.

In $|\omega| = r_2$ integral, we have $|\omega| > |z|$ and so
we can expand

$$\frac{1}{\omega - z} = \frac{1}{\omega} \left(1 - \frac{z}{\omega} \right)^{-1} = \frac{1}{\omega} \sum_{n=0}^{\infty} \left(\frac{z^n}{\omega^n} \right)$$

By uniform convergence, we can switch Σ and $\int$ and

so: $$\frac{1}{2\pi i} \int_{|\omega| = r_2} \frac{f(\omega)}{\omega - z} \, d\omega = \frac{1}{2\pi i} \int_{|\omega| = r_2} f(\omega) \sum_{n=0}^{\infty} \left(\frac{z^n}{\omega^{n+1}} \right) d\omega$$

$$= \frac{1}{2\pi i} \sum_{n=0}^{\infty} \left(\int_{|\omega| = r_2} f(\omega) \omega^{-n-1} \, d\omega \right) z^n$$

Let $$C_n = \frac{1}{2\pi i} \int_{|\omega| = r_2} \omega^{-n-1} f(\omega) \, d\omega.$$

In $|\omega| = r_1$ integral, we have $|z| > |\omega|$, so

$$\frac{1}{\omega - z} = \frac{-1}{z} \left(1 - \frac{\omega}{z} \right)^{-1} = \frac{-1}{z} \sum_{n=0}^{\infty} \left(\frac{\omega^n}{z^n} \right)$$ is valid for $|z| > r_1$, $|\omega| = r_1$.

Apply same 'moves' to this integral.

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We get

$$\frac{-1}{2\pi i}\int_{|w|=r_1} \frac{f(w)}{w-z}\,dw$$
$$= \frac{1}{2\pi i}\int_{|w|=r_1}\left(\sum_{n=0}^{\infty} f(w)w^n z^{-n-1}\right)dw$$
$$= \sum_{n=0}^{\infty} D_n z^{-n-1}$$

where $D_n = \frac{1}{2\pi i}\int_{|w|=r_1} f(w)w^n\,dw$

This gives the expansion:
$$f(z) = \sum_{n=0}^{\infty} C_n z^n + \sum_{n=0}^{\infty} D_n z^{-n-1}$$

Formulae for c_n?
The function $F(w) = f(w)w^n$ $(n \in \mathbb{Z})$ is holomorphic in A.

By Cauchy's Thm $\int_{|w|=\rho} f(w)w^n\,dw$
$$= \int_{|w|=\rho'} f(w)w^n\,dw$$

for any radii ρ, ρ' s.t. $a < \rho, \rho' < b$.
[Apply Cauchy to $D = \{\rho < |w| < \rho'\}$ for example)

Formula given $c_n = \frac{1}{2\pi i}\int_{|w|=\rho} f(w)w^{-n-1}\,dw \quad \forall\ a<\rho<b.$

But by above Cauchy Thm argument,
$$C_n = \frac{1}{2\pi i}\int_{|w|=r_2} f(w)\,w^{-n-1}\,dw = \frac{1}{2\pi i}\int_{|w|=\rho} f(w)\,w^{-n-1}\,dw = c_n$$
for $n = 0, 1, 2, \dots$

Similarly
$$D_m = \frac{1}{2\pi i}\int_{|w|=r_1} f(w)w^m\,dw = \frac{1}{2\pi i}\int_{|w|=\rho} f(w)w^m\,dw =: c_{-m-1}.$$
$(m = -n-1,\ m = 0, 1, 2, \dots)$

$\square$

Remarks

1). Analogues of Cauchy inequalities:

If $M_\rho = \sup\{|f(z)| : |z| = \rho\}$,

length-sup gives:

$$|C_n| = \left|\frac{1}{2\pi i}\int_{|w|=\rho} f(w)w^{-n-1}\,dw\right| \leq \frac{1}{2\pi}\cdot 2\pi\rho \cdot M_\rho \cdot \rho^{-n-1}$$

$$= M_\rho \rho^{-n} \quad (*)$$

This can give useful information, especially if $a = 0$.

2). The coefficients are unique.

For if not and $f(z) = \sum_{n=-\infty}^{\infty} c_n z^n = \sum_{n=-\infty}^{\infty} c_n' z^n$

then $0 = \sum_{n=-\infty}^{\infty} (c_n - c_n') z^n$

In this case $M_\rho = 0$, so plugging in to estimate for c_n, find $c_n - c_n' = 0 \ \forall n$.

Example

Consider $f(z) = \exp(\frac{1}{z})$ which is holomorphic except where $z = 0$, in particular in any punched disc, $D^*(0,b)$.

If $z \neq 0$, $\frac{1}{z} \in \mathbb{C}$, and $\exp(\frac{1}{z}) = \sum_{n=0}^{\infty} \frac{1}{n!}\left(\frac{1}{z}\right)^n = \sum_{n=0}^{\infty} \frac{1}{n!} z^{-n}$.

So this <u>is</u> the Laurent expansion and

$c_0 = 1$, $c_{-n} = \frac{1}{n!}$ for $n = 1, 2, \ldots$

and $c_n = 0$ if $n = 1, 2, 3, \ldots$

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Laurent series v. Fourier series.

Suppose $f(z)$ is holomorphic in an annulus A of the form $\{1-\varepsilon < |z| < 1+\varepsilon\}$ $(\varepsilon > 0)$.

Laurent gives the expansion

$$f(z) = \sum_{n=-\infty}^{\infty} c_n z^n \text{ for } 1-\varepsilon < |z| < 1+\varepsilon,$$

where $c_n = \frac{1}{2\pi i} \int_{|w|=1} f(w) w^{-n-1} \, dw$

When $|z| = 1$, $z = e^{i\theta}$

and we get

$$f(e^{i\theta}) = \sum_{n=-\infty}^{\infty} c_n e^{in\theta}$$

where $c_n = \frac{1}{2\pi i} \int_0^{2\pi} f(e^{i\theta}) e^{-(n+1)i\theta} i e^{i\theta} \, d\theta$

$$= \frac{1}{2\pi i} \int_0^{2\pi} f(e^{i\theta}) e^{-in\theta} \, d\theta$$

$(w = e^{i\theta}, \ dw = ie^{i\theta})$ Cf Fourier series

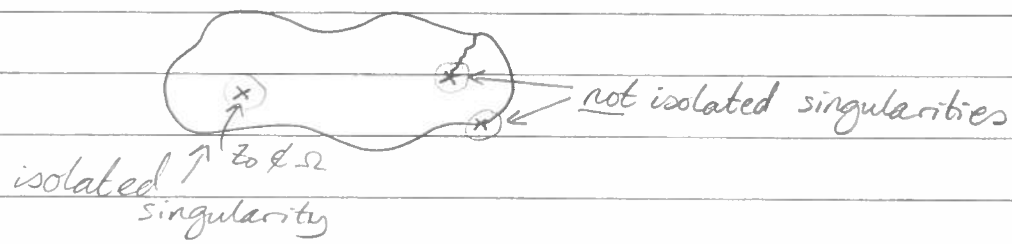
write $F(\theta) = f(e^{i\theta})$ to tie in with previous course.

Isolated singularities

Def^n

Suppose $f : \Omega \mapsto \mathbb{C}$ is holomorphic, where Ω is an open set.

A point $z_0 \notin \Omega$ is called an isolated singularity of f if $\exists \delta > 0$ s.t. $D^*(z_0, \delta) = \{z : 0 < |z - z_0| < \delta\}$ is contained in Ω.



Examples

Any rational function $\frac{P(z)}{Q(z)}$ has isolated singularities at the zeros of Q.

$\frac{e^{iz}}{(z^2+1)(z^2+9)}$ is holomorphic in $\mathbb{C} \setminus \{i, -i, 3i, -3i\}$.

These removed points are all isolated singularity.

Let $z_0 \notin \Omega$ be an isolated singularity of f.
In particular, f is holomorphic in a punctured disc $D^* = \{z \in \mathbb{C} : 0 < |z - z_0| < \delta\}$.
So $f(z)$ has a Laurent expansion in powers of $z - z_0$: $f(z) = \sum_{n=-\infty}^{\infty} c_n (z-z_0)^n$, for $z \in D^*$.

$$f(z) = \dots + c_{-2}(z-z_0)^{-2} + c_{-1}(z-z_0)^{-1} + c_0 + c_1(z-z_0) + \dots$$

Definition

(i) z_0 is a removable singularity if $c_n = 0 \ \forall n < 0$.

(ii) z_0 is a pole of order $m > 0$ if $c_{-m} \neq 0$ but $c_n = 0 \ \forall n < -m$.

(iii) Otherwise if $c_n \neq 0$ for infinitely many $n < 0$, z_0 is an essential singularity.

Note: (i), (ii) and (iii) are mutually exclusive possibilities.

Definition

If z_0 is an isolated singularity of f, then the coefficient c_{-1} in the Laurent expansion is called the residue of f at z_0.

$$c_{-1} = \mathrm{Res}_{z_0}(f).$$

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Example 1

$f(z) = \frac{\sin z}{z}$ is hol if $z \neq 0$, $z_0 = 0$ is an isolated singularity.

$$f(z) = \frac{1}{z}\left(z - \frac{z^3}{3!} + \frac{z^5}{5!} - \ldots\right) = 1 - \frac{z^2}{3!} + \frac{z^4}{5!} - \ldots$$

This is the Laurent expansion of $\frac{\sin z}{z}$ in $D^*(0,\delta)$.

No negative powers of $z \Rightarrow z = 0$ is a removable singularity.

$\frac{\sin z}{z}$, $z \neq 0$ extends holomorphically to $\mathbb{C}$, at 0, defined to be 1.

(Previous) example 2

$\exp\left(\frac{1}{z}\right)$: essential singularity at $z = 0$.

example 3

$$\frac{1}{e^{2\pi i z} - 1} = f(z)$$

This has singularities at $z = n$, any $n \in \mathbb{Z}$.

$z = n + h \quad (z_0 = n)$

$$f(n+h) = \frac{1}{e^{2\pi i(n+h)} - 1} = \frac{1}{e^{2\pi i h} - 1}$$

$$= \frac{1}{\left[1 + 2\pi i h + \frac{1}{2!}(2\pi i h)^2 + \ldots\right] - 1}$$

$$= \frac{1}{2\pi i h + \frac{1}{2!}(2\pi i h)^2 + \ldots}$$

$$= \frac{1}{2\pi i h}\left(1 + \frac{1}{2!}2\pi i h + O(h^2)\right)^{-1}$$

If $|h|$ is small we can expand binomially:

$$\frac{1}{2\pi i h}\left(1 - \pi i h + O(h^2)\right)$$

$$= \frac{1}{2\pi i(z-n)} - \frac{1}{2} + O(z-n) \qquad (h = z-n)$$

So $c_{-1} \neq 0$, $c_n = 0$ for $n < -1$

$\therefore$ Pole of order 1 (aka 'simple pole')

Theorem

Suppose z_0 is an isolated singularity of f.
Then z_0 is removable iff
$\lim_{z \to z_0} (z-z_0) f(z) = 0.$

Proof

Isolated singularity, so
$$f(z) = \sum_{n=-\infty}^{\infty} c_n (z-z_0)^n$$

$\Rightarrow$: if removable, $c_n = 0$ for $n < 0$.
So $f(z) = c_0 + c_1(z-z_0) + \ldots$
is valid for all $0 < |z-z_0| < \delta$.
Then $(z-z_0) f(z) = c_0(z-z_0) + c_1(z-z_0)^2 + \ldots$
$f(z) \to c_0$ as $z \to z_0$ so $|f(z)(z-z_0)| = |f(z)||z-z_0| \to 0$
as $z \to z_0$.
$\Leftarrow$: Need to use $\lim_{z \to z_0} (z-z_0) f(z) = 0$

to prove $c_n = 0$ for all $n < 0$. Use $(*)$ $[|c_n| \leq M_\rho \rho^{-n}]$.
Pick $\varepsilon > 0$.
Then $\exists\, r > 0$ s.t. $|z-z_0||f(z)| < \varepsilon$ if $|z-z_0| < r < \delta$.
Hence if $|z-z_0| = \rho < r$,
$\rho|f(z)| < \varepsilon$, $M_\rho < \varepsilon/\rho$.

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$$|c_n| \leq \frac{\varepsilon}{\rho} \cdot \rho^{-n} = \varepsilon \rho^{-1-n}$$

So if $n < 0$

$$|c_n| < \varepsilon \rho^{|n|-1}$$

If $n = -1$, this is ε

If $n < -1$ it is $\varepsilon \rho$ (positive)

So $|c_n|$ is less than any given positive number, hence 0.

$\square$

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Definitions

1). $f: \mathbb{C} \mapsto \mathbb{C}$ holomorphic is called 'entire'.

2). If f is holomorphic apart from isolated singularities in an open set Ω, then f is meromorphic if all singularities are (removable or) poles.

3). If z_0 is an isolated singularity of f with Laurent expansion $\sum_{n=-\infty}^{\infty} c_n (z-z_0)^n$, then the principle part of at z_0 is

$$p(z) = \sum_{n=-\infty}^{-1} c_n (z-z_0)^n.$$ (sometimes called 'singular part')

The residue $Res_{z_0}(f) := c_{-1}$

Recall

If z_0 is an isolated singularity of f, it means $\exists$ a punctured disc $D^*(z_0, r) = \{z : 0 < |z-z_0| < r\}$ such that $f: D^*(z_0, r) \mapsto \mathbb{C}$ is holomorphic.
z_0 is a removable singularity if $c_n = 0 \; \forall n < 0$
$\Leftrightarrow$ the principle part of f is zero.

Proposition

Suppose z_0 is a removable singularity of f.
Then $\exists \; \tilde{f} : D(z_0, r) \mapsto \mathbb{C}$, holomorphic and $\tilde{f}(z) = f(z)$ if $z \in D^*(z_0, r)$.

Proof

We know $f(z) = \sum_{n=0}^{\infty} c_n (z-z_0)^n = c_0 + c_1(z-z_0) + \ldots$
for $0 < |z-z_0| < r$.

Define:

$$\tilde{f}(z) = \begin{cases} f(z) & \text{if } z \neq z_0 \\ c_0 & \text{if } z = z_0 \end{cases}$$

Then $\tilde{f}(z)$ is holomorphic in $D(z_0, r)$ as it is given by the convergent power series $\sum_{n=0}^{\infty} c_n (z-z_0)^n$.

And $f = \tilde{f}$ if $z \neq z_0$.

$\square$

Notational remark

When z_0 is a removable singularity of f we shall usually denote by the same symbol f the extended hol. fn. you get by removing the singularity.

Further Remarks

1). Let z_0 be an isolated singularity of f.
Let p_f = singular part of f.
Then $p_f(z) = \sum_{n=-\infty}^{-1} c_n (z-z_0)^n$ is holomorphic in

$\mathbb{C} \setminus \{z_0\}$.
[Follows from the ratio test and estimates for $|c_n|$ from last time.]

2). The following are equivalent:
(i) z_0 is a pole of order m
[ie. $f(z) = c_{-m}(z-z_0)^{-m} + c_{1-m}(z-z_0)^{1-m} + \dots \quad c_{-m} \neq 0$]
(ii) $\exists F : D(z_0, r) \to \mathbb{C}$ holomorphic, $F(z_0) \neq 0$, such that
$$f(z) = \frac{F(z)}{(z-z_0)^m} \qquad [F(z) = c_{-m} + c_{1-m}(z-z_0) + \dots]$$
(iii) $h(z) := \frac{1}{f(z)}$ has a zero of order m at $z = z_0$.
$$\left[h(z) = \frac{(z-z_0)^m}{F(z)} \right]$$

3). Branch points are not isolated singularities.

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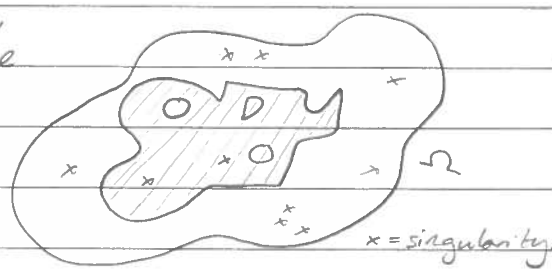
Residue Theorem

Suppose f is holomorphic in Ω (open) apart from isolated singularities.
Let D be a closed and bounded domain with piecewise smooth boundary ∂D, $D \cup \partial D \subset \Omega$.
Suppose there are no singularities of f on ∂D.
Then $\displaystyle\int_{\partial D} f(z)\,dz = 2\pi i \sum_{w \in D} Res_w(f)$.

Here $Res_w(f) = 0$ if w is not a singular point of f and the sum is automatically finite.

More explicitly, if $z_1, \ldots, z_N$ are the singularities of f in D,

$$\int_{\partial D} f(z)\,dz = 2\pi i \sum_{i=1}^{N} Res_{z_i}(f).$$



Lemma

Let z_0 be an isolated singularity of f.
$f: D^*(z_0, r) \longrightarrow \mathbb{C}$.
Then if $0 < \rho < r$

$$\int_{|z-z_0|=\rho} f(z)\,dz = 2\pi i c_{-1} = 2\pi i Res_{z_0}(f)$$

Proof

Parameterise circle $z = z_0 + \rho e^{it}$, $0 \le t \le 2\pi$

$$\int_{|z-z_0|=\rho} f(z)\,dz = \int_0^{2\pi} f(z_0 + \rho e^{it}) i\rho e^{it}\,dt$$

$$= \int_0^{2\pi} \sum_{n=-\infty}^{\infty} c_n (\rho e^{it})^n i\rho e^{it}\,dt$$

$$= \sum_{n=-\infty}^{\infty} \int_0^{2\pi} c_n \rho^{n+1} e^{i(n+1)t}\, i\,dt \quad \text{by uniform convergence.}$$

$$= 2\pi i c_{-1}$$

This uses $\displaystyle\int_{|z-z_0|=\rho} (z-z_0)^n \, dz = \begin{cases} 0 & \text{if } n \neq -1 \\ 2\pi i & \text{if } n = -1 \end{cases}$

<u>Proof of Residue theorem</u>

1). Suppose (z_n) is an infinite sequence of singularities of f, $z_n \in D$. Bolzano-Weierstrass: $\exists$ a convergent subsequence $z_{n_j} \to z_\infty$ as $j \to \infty$ ($D \cup \partial D$ is closed and bounded).

Claim: $z_\infty \in D \cup \partial D \subset \Omega$

By hypothesis, f is hol at z_∞ or z_∞ is an isolated singularity.

But this is not possible as $z_{n_j} \in D^*(z_\infty, r)$ for any small $r > 0$.

2). Enumerate the singularities of f in D, $z_1, \dots, z_N$.

$\exists$ $D_i = D(z_i, r)$, $0 < r << 1$ so that z_i is the only singular point of f in D_i. Also $\overline{D}_i = D_i \cup \partial D_i \subset D$.

Let $E_r = D \setminus (\overline{D}_1 \cup \overline{D}_2 \cup \dots \cup \overline{D}_N)$.

Now f is hol in an open set containing E_r so

Cauchy: $\displaystyle\int_{\partial E_r} f(z)\,dz = 0$.

But also
$$\int_{\partial E_r} f(z)\,dz = \int_{\partial D} f(z)\,dz - \sum_{i=1}^{N} \int_{|z-z_i|=r} f(z)\,dz$$
$$= \int_{\partial D} f(z)\,dz - \sum_{j=1}^{N} 2\pi i \, \mathrm{Res}_{z_j}(f)$$

applying Lemma to each term. $\square$

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Exercises

1). What type of singularity does

$$f(z) = \frac{\pi^2}{\sin^2(\pi z)} - \frac{1}{z^2}$$

have at $z = 0$?

2). What type of singularity does $\frac{1}{(e^z - 1)^2}$ have at $z = 0$?

3). What is the residue of $\frac{f(z)}{(z - z_0)^{n+1}}$ if f is holomorphic in some disc $D(z_0, r)$?

Type — removable singularity: $c_n = 0$ if $n < 0$
pole of order m : $c_{-m} \neq 0$, $c_n = 0$ $\forall n < -m$, $m > 0$
essential singularity: $c_n \neq 0$ for infinitely many $n < 0$

Had a formula for c_n as an integral around a circle. Not usually a good way to calculate c_n! Best to use Taylor expansion where possible.

Note $\frac{1}{(e^z - 1)^2}$ - can't expand as power series.
But can expand e^z.

So $e^z - 1 = z + \frac{z^2}{2!} + \dots = z\left(1 + \frac{z}{2!} + \frac{z^2}{3!} + \dots\right)$

So $\frac{1}{(e^z - 1)^2} = \frac{1}{z^2}\left(1 + \left(\frac{z}{2!} + \frac{z^2}{3!} + \dots\right)\right)^{-2}$ can expand binomially.

Contour integration via Residue theorem.

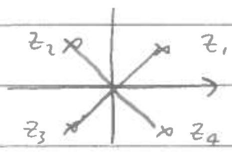
Evaluate $\int_{-\infty}^{\infty} \frac{dx}{1+x^4}$.

Let $f(z) = \frac{1}{1+z^4}$

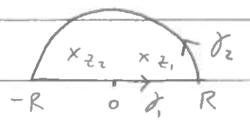
Singularities at $z^4 = -1 = e^{\pi i}$

So $z_j = \exp\left(\frac{\pi i + 2k\pi i}{4}\right)$, $k \in \mathbb{Z}$

$z_1 = e^{\pi i/4}$, $z_2 = e^{3\pi i/4}$, $z_3 = e^{5\pi i/4}$, $z_4 = e^{7\pi i/4}$



$\Gamma = \gamma_1 + \gamma_2$, $\gamma_1 = [-R, R]$, $\gamma_2 = Re^{it}$ $0 \le t \le 2\pi$



Cauchy Residue Thm:

$$\int_\Gamma f(z)dz = 2\pi i\left(Res_{z_1}(f) + Res_{z_2}(f)\right)$$

Fact: (see Problem Set 8)

If $f(z) = \frac{p(z)}{q(z)}$ both hol

and z_0 is simple zero of q $(p(z_0) \neq 0)$.

$Res_{z_0}(f) = \frac{p(z_0)}{q'(z_0)}$

$z_1 = e^{\pi i/4}$, $z_2 = e^{3\pi i/4}$

$p = 1$, $q = z^4 + 1$, $q' = 3z^3$

$Res_{z_1}(f) = \frac{1}{4z_1^3} = \frac{1}{4}e^{-3\pi i/4}$

$Res_{z_2}(f) = \frac{1}{4z_2^3} = \frac{1}{4}e^{-9\pi i/4}$

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So $\int_{\Gamma} f(z)\,dz = \frac{2\pi i}{4}\left(e^{-3\pi i/4} + e^{-9\pi i/4}\right)$

<u>Claim:</u>

$$\left|\int_{\gamma_2} f(z)\,dz\right| \to 0 \quad \text{as } R \to \infty$$

Try length-sup first.

If $|z| = R$,

$$|f(z)| = \frac{1}{|z^4+1|}$$

$$\leq \frac{1}{|z|^4+1} = \frac{1}{R^4-1} \quad (R > 1)$$

$$\text{length} \times \text{sup} = \frac{\pi R}{R^4-1} \to 0 \text{ as } R \to \infty$$

(claim proved).

$$\int_{-\infty}^{\infty} \frac{dx}{1+x^4} = \frac{\pi i}{2}\left(e^{-3\pi i/4} + e^{-\pi i/4}\right)$$

$$= \frac{\pi i}{2}\left(\cos\frac{\pi}{4} - i\sin\frac{\pi}{4} + \cos\frac{3\pi}{4} - i\sin\frac{3\pi}{4}\right)$$

$$= \frac{\pi}{2}\left(\sin\frac{\pi}{4} + \sin\frac{3\pi}{4}\right) + \frac{\pi i}{2}\underbrace{\left(\cos\frac{\pi}{4} + \cos\frac{3\pi}{4}\right)}_{=0}$$

$$= \frac{\pi}{2}\left(\sin\frac{\pi}{4} + \sin\frac{3\pi}{4}\right)$$

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Chapter 7 Analytic Continuation

Ex

$f(z) = \sum_{n=0}^{\infty} z^n$ Defines a hol function

in $D = \{z : |z| < 1\}$.

Series is definitely divergent if $|z| \leq 1$.

However: for $|z| < 1$, $\sum_{n=0}^{\infty} z^n = \frac{1}{1-z}$

In other words, in this case,

$\exists\ F(z) = \frac{1}{1-z}$, hol in $\Omega = \mathbb{C} \setminus \{1\}$ and

such that $F(z) = f(z)$ for all $z \in D$.

[Restriction of F to D is equal to f]

We say that F is an analytic continuation of f.

Main fact:

Analytic continuation is unique.

More generally, suppose D is a domain $\subset \mathbb{C}$ and $f : D \mapsto \mathbb{C}$ is holomorphic.

Suppose $F : \Omega \mapsto \mathbb{C}$ is holomorphic, $D \subset \Omega$, (Ω a domain) and $F(z) = f(z)$ for z in D.

We say F is an analytic continuation of f to Ω.

Uniqueness Thm

If Ω is path connected and F_1 and $F_2 : \Omega \to \mathbb{C}$ are analytic continuations of f, then $F_1(z) = F_2(z)$ $\forall\ z \in \Omega$.

Compare with real functions

$f(x) = \frac{1}{x}$, $0 < x < 1$ differentiable

$F_1(x) = \frac{1}{x}$ $\forall x > 0$ is a differentiable extension of f to $(0, \infty)$

$$F_2(x) = \begin{cases} 1/x, & 0 < x < 1 \\ 2 - x, & x \geq 1 \end{cases}$$ is also differentiable.

Note: Differentiable continuation is not unique for real functions.

§ Isolated zeros of holomorphic functions.

Definition

Suppose $f : D = \{z : |z - z_0| < r\} \mapsto \mathbb{C}$ is holomorphic, $f(z_0) = 0$

1). z_0 has order m if $f(z_0) = f'(z_0) = \dots = f^{(m-1)}(z_0) = 0$ but $f^{(m)}(z_0) \neq 0$.

2). otherwise say the zero is of infinite order.

[NB: (1) and (2) are mutually exclusive and exhaustive]

3). Say z_0 is an <u>isolated zero</u> of f if $\exists$ $D' = \{|z - z_0| < r'\}$ $(r' \leq r)$ such that $f(z) \neq 0$ $\forall z$ s.t. $0 < |z - z_0| < r'$.

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Theorem

f and D as above.
z_0 is of finite order if and only if z_0 is an isolated zero of f.
Moreover z_0 is infinite order $\Leftrightarrow$ it is not isolated $\Leftrightarrow$ $\exists D' = \{z : |z - z_0| < r'\}$ such that $f(z) = 0 \;\; \forall z \in D'$.

Proposition

z_0 is not an isolated zero of f if and only if $\exists \, z_n \to z_0$ s.t. $f(z_n) = 0$.

Proof

Suppose we have such a sequence, but $D' = \{z : |z - z_0| < r'\}$ has property $0 < |z - z_0| < r'$ $\Rightarrow f(z) \neq 0$.
This $D' \Rightarrow |z_n - z_0| \geq r'$ so z_n cannot converge to z_0.

Proof of Thm

Taylor series.

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n \text{ is convergent in } D.$$

$f(z_0) = 0 \Rightarrow a_0 = 0$

- All $a_n = 0$, by definition this is a zero of ∞ order and then $f(z) \equiv 0$.

Hence z_0 is of ∞ order $\Leftrightarrow f(z) = 0$ in D.

- If not all $a_n = 0$, $\exists! \; m > 0$ s.t.

$$a_0 = a_1 = \ldots = a_{m-1} = 0, \quad a_m \neq 0.$$

Then
$$\begin{aligned} f(z) &= a_m (z - z_0)^m + a_{m+1}(z - z_0)^{m+1} + \ldots \\ &= (z - z_0)^m [a_m + a_{m+1}(z - z_0) + \ldots] \\ &= (z - z_0)^m g(z) \end{aligned}$$

where g is given by a convergent power series in

$D(z_0, r)$, so is hol and $g(z_0) = a_m \neq 0$.

By continuity of g at z_0, $\exists r' \leq r$ s.t.
$|z - z_0| < r' \Rightarrow |g(z)| > \frac{|a_m|}{2}$.

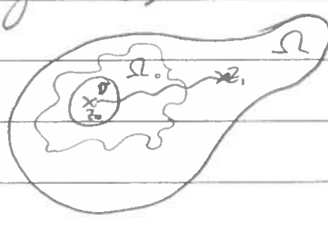
Now z_0 is the only zero of f in
$D' = \{|z - z_0| < r'\}$
$f(z) = 0 \Leftrightarrow (z - z_0)^m g(z) = 0$
$\Leftrightarrow z = z_0$ <u>or</u> $g(z) = 0$
but second doesn't happen for z in D'.

This argument shows:
finite order zero $\Rightarrow$ isolated zero
infinite order zero $\Leftrightarrow$ $f \equiv 0$.
isolated zero $\Rightarrow$ finite order ?
Suppose not of finite order. Then z_0 is infinte order zero, and so $f \equiv 0$, so not isolated.

§ Unique Continuation Thm
(Identity Thm)

Theorem
Suppose $f: \Omega \to \mathbb{C}$ is holomorphic, where Ω is a path-connected open set.
Suppose $z_0 \in \Omega$ is a zero of infinite order of f. Then $f(z) = 0 \ \forall z \in \Omega$.

Proof (Note: problem is to show $f(z) = 0 \ \forall z$ in $D' = \{z : |z - z_0| < r'\} \Rightarrow f(z) = 0 \ \Omega$: Topological argument).



Let $\Omega_0 = \{z \in \Omega : f(z) = 0\}$

$f(z) = 0 \ \forall z \in \Omega \Leftrightarrow \Omega = \Omega_0$

Suppose $\Omega \neq \Omega_0$ and $z_1 \notin \Omega_0$

If $z_1 \notin \Omega_0$, choose a curve $\gamma : [t_0, t] \mapsto \Omega$ such that $\gamma(t_0) = z_0$, $\gamma(t_1) = z_1$

Consider $T \in \mathbb{R}$

$\gamma[t_0, T) \subset \Omega_0$.

Let $t_* = \sup\{T : \gamma[t_0, T) \subset \Omega_0\}$

Intuitively: $\gamma(t) \in \Omega_0 \ \forall t < t_*$, $\gamma(t) \notin \Omega_0$ for $t > t_*$, $t - t_*$ sufficiently small.

t_* does exist because $f(z) = 0 \ \forall z$ sufficiently close to z_0.

Let $z_* = \gamma(t_*)$.

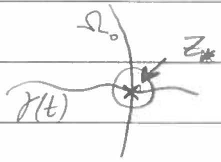
f is continuous at z_* so

$$f(z_*) = \lim_{t \to t_*^-} f(\gamma(t)) = 0.$$

So $f(z_*) = 0$

Two possibilities: — finite order $\Leftrightarrow$ isolated

— infinite order $\Leftrightarrow$ non-isolated $\Leftrightarrow f(z) = 0$ $\forall |z - z_*|$ sufficiently small



But $f(\gamma(t)) = 0$ by definition for $t \leq t_*$

$\gamma(t) \to \gamma(t_*)$ as $t \to t_*^-$ so

z_* cannot be an isolated zero of f.

By previous Thm it follows $\exists r'$ s.t.

$|z - z_*| < r' \Rightarrow f(z) = 0$.

In particular $f(\gamma(t)) = 0 \ \forall t$:

$|\gamma(t) - z_*| < r'$ and in particular for small $t > t_*$

This contradicts maximality of t_*.
This implies no z_1 with with $f(z_1) \neq 0$.
i.e. $f(z) = 0 \quad \forall z \in \Omega$. $\square$

Remark

Uniqueness of analytic continuation follows:

Proposition

Suppose Ω is a domain
$D \subset \Omega$ is open $f : D \mapsto \mathbb{C}$ is hol. and
$F_1, F_2 : \Omega \longmapsto \mathbb{C}$ are both hol,
with $F_1(z) = F_2(z) = f(z)$ for all $z \in D$.
Then $F_1(z) = F_2(z) \quad \forall z \in \Omega$.

Proof

Let $G(z) = F_1(z) - F_2(z)$
Then $G : \Omega \mapsto \mathbb{C}$ is holomorphic and $G(z) = 0$
$\forall z \in D$.
In particular G has zeros of infinite order,
hence $G(z) = 0 \quad \forall z \in \Omega$. $\square$

Remarks

Different formulation:
If f is hol and non-constant in a domain then every zero of f is isolated and hence of finite order.

Follows that if $z_1, \dots, z_n$ is a set of zeros of a non-constant hol function, then can write
$f(z) = (z - z_1)^{m_1} g_1(z)$ where g_1 is hol and $g_1(z_1) \neq 0$.
Continuing: $f(z) = (z - z_1)^{m_1} \dots (z - z_N)^{m_N} g_N(z)$ where
g is holomorphic where f was and $g_N(z_j) \neq 0$
for $j = 1, \dots, N$.

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Maximum Principle

(Maximum Modulus Theorem)

Theorem

Let Ω be a bounded domain (connected open set)
let $f: \Omega \mapsto \mathbb{C}$ be holomorphic.
Suppose also f is continuous on $\bar{\Omega} = \Omega \cup \partial\Omega$.
Then the maximum value of $|f(z)|$ is attained on the boundary of Ω, $\partial\Omega$.

Proof

Since $\Omega \cup \partial\Omega$ is closed and bounded and $|f(z)|$ is continuous on this set, $\exists z_0 \in \Omega \cup \partial\Omega$ st. if
$M := \max\{|f(z)| : z \in \Omega\}$, $|f(z_0)| = M$.

If $z_0 \in \partial\Omega$ then we are done.
Suppose not.
Then z_0 is an interior point and for small enough
$r > 0$, $\bar{D} = \{z : |z - z_0| \leq r\}$

Step 1: Claim that f is constant on D
Apply C.I.F.

$$f(z_0) = \frac{1}{2\pi i} \int_{|z - z_0| = r} \frac{f(z)}{z - z_0}\, dz$$

$|z - z_0| = r$ is parameterised as $\gamma(t) = z_0 + re^{it}$

$$f(z_0) = \frac{1}{2\pi i} \int_0^{2\pi} \frac{f(z_0 + re^{it})\, ire^{it}}{re^{it}}\, dt$$

$$= \frac{1}{2\pi} \int_0^{2\pi} f(z_0 + re^{it})\, dt$$

Applying basic estimate for integrals:

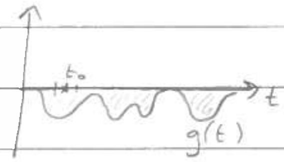
$$|f(z_0)| \leq \frac{1}{2\pi} \int_0^{2\pi} |f(z_0 + re^{it})|\, dt$$

Noting that $\frac{1}{2\pi} \int_0^{2\pi} |f(z_0)|\, dt = |f(z_0)|$

So we obtain

$$\int_0^{2\pi} \underbrace{\left\{ |f(z_0 + re^{it})| - |f(z_0)| \right\}}_{g(t)} dt \geq 0$$

By definition of z_0, $g(t) \leq 0$



So $\int_0^{2\pi} g(t)\, dt \leq 0$.

If $g(t_0) < 0$, by continuity $g(t) < 0$ for $|t - t_0|$ sufficiently small and so $\int_0^{2\pi} g(t) < 0$.

Hence $g(t) = 0 \ \forall t$.

Hence $|f(z_0)| = |f(z_0 + re^{it})| \ \forall t$
Also true for all r sufficiently small and hence for all z in $\bar{D}$
Conclusion is that $|f(z)|$ is constant on
$D = \{z : |z - z_0| < r\}$

Application of Cauchy-Riemann Equations gives that f is constant for z in D.

Step 2 $f(z)$ constant in $D \Rightarrow f(z)$ constant in Ω
Follows by applying identity theorem to $F(z) = f(z) - f(z_0)$.
For $F(z)$ is holomorphic and vanishes identically in an open disc. Identity Thm gives $F(z) = 0$ in Ω.

Since f is constant $|f(z)| = M$ is constant and so max is achieved at a boundary point. $\square$

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Corollary: Fundamental Thm of Algebra

Suppose $p(z) = z^n + a_{n-1}z^{n-1} + \dots + a_0$
has no complex root

Let $f(z) = \dfrac{1}{p(z)}$.

This is holomorphic in $\mathbb{C}$.

In particular it is holomorphic in $D(0,R) = \{z : |z| < R\}$.

For given R, let $M(R) = \max\{|f(z)| : |z| \leq R\}$

Notice that if $S > R$, $M(R) \leq M(S)$.

But max principle says:

$0 \leq M(R) = \max\{|f(z)| : |z| = R\}$.

But $|p(z)| \approx |z|^n$ if $|z|$ is large and so for $|z| = R$, $|p(z)| \approx R^n$.

So $|f(z)| \approx R^{-n} \to 0$ as $R \to \infty$.

Hence $M(R) \to 0$ as $R \to \infty$.

So $0 \leq M(R) \leq M(S)$, if $S > R$, but $M(R) \to 0$.

Only possibility is $M(R) = 0$.

Contradiction.

$\square$

Application

Consider hol functions $f : D \longmapsto D$ with hol. inverse.

$\exists$ Möbius transformation with this property:

$$e^{i\theta}\left(\frac{z+a}{1+\bar{a}z}\right), \quad |a| < 1.$$

Max principle: Can show these are the only such holomorphic mappings.

Computation of residues

Suppose z_0 is an isolated singularity, in fact a pole of f. In particular the Laurent series has the form:

$$\underbrace{\frac{c_{-m}}{(z-z_0)^m} + \frac{c_{-m+1}}{(z-z_0)^{m-1}} + \dots + \frac{c_{-1}}{z-z_0}}_{\text{principle or singular part.}} + c_0 + \dots \qquad (c_{-m} \neq 0)$$

$c_{-n} = 0$ for $n > m$ corresponds to z_0, a pole of order m.

c_{-1} = residue

Formula for c_{-1}

1). If $m = 1$ (simple pole), $c_{-1} = \lim_{z \to z_0} (z - z_0) f(z)$

Proof

$m = 1$, the series collapses to

$f(z) = \frac{c_{-1}}{z - z_0} + O(1)$ for $|z - z_0|$ small.

So $(z - z_0) f(z) = c_{-1} + O(|z - z_0|)$.

Now take limit to get:

$$\lim_{z \to z_0} \left((z - z_0) f(z) \right) = c_{-1}$$

$\square$

2). If $m \geq 1$: $c_{-1} = \lim_{z \to z_0} \left(\frac{d}{dz} \right)^{m-1} \left((z - z_0)^m f(z) \right)$.

Proof

Calculate from Laurent expansion step by step:

$(z - z_0)^m f(z) = c_{-m} + c_{-m+1}(z - z_0) + \dots + c_{-1}(z - z_0)^{m-1} + c_0 (z - z_0)^m + \dots$

Differentiate $m-1$ times, kills all terms where $(z - z_0)^j$ with $j < m-1$, and we are left with

$$\left(\frac{d}{dz} \right)^{m-1} \left((z - z_0)^m f(z) \right) = c_{-1}(m-1)! + c_0 m(m-1)\dots(2)(z - z_0) + \dots$$

Hence,

$$\lim_{z \to z_0} \left(\frac{d}{dz} \right)^{m-1} \left((z - z_0)^m f(z) \right) = (m-1)!\, c_{-1}$$

$\square$

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Worked Example

Calculate $\displaystyle\int_{-\infty}^{\infty} \frac{\cos ax}{\cosh x}\,dx \qquad a > 0.$

Need a holomorphic (or meromorphic) function

try $f(z) = \dfrac{e^{iaz}}{\cosh z}$

singularities are those z with $\cosh z = 0$

So $\frac{1}{2}(e^{z} + e^{-z}) = 0$

$\iff \quad e^{2z} = -1$

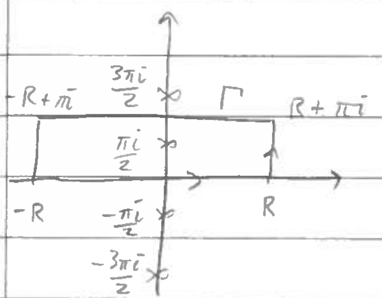
$-1 = e^{\pi i}$ (de Moivre)

So a solution is $2z = \pi i$

But $-1 = e^{\pi i + 2k\pi i} \quad \forall k \in \mathbb{Z}$

So $\cosh z = 0 \iff 2z = \pi i + 2k\pi i$

$\iff z = \frac{\pi i}{2} + k\pi i$



Let Γ be rectangular contour shown, consisting of segments $[-R, R]$, $[R, R+\pi i]$, $[R+\pi i, -R+\pi i]$, $[-R+\pi i, -R]$

$$\int_{\Gamma} f(z)\,dz = 2\pi i \sum \text{Residues} = 2\pi i \operatorname{Res}_{\frac{\pi i}{2}}(f) \qquad (*)$$

Also $\displaystyle\int_{\Gamma} = \int_{\gamma_1} + \int_{\gamma_2} + \int_{\gamma_3} + \int_{\gamma_4}$

$$\int_{\gamma_1} f(z)\,dz = \int_{-R}^{R} \frac{e^{iax}}{\cosh x}\,dx \longrightarrow \int_{-\infty}^{\infty} \frac{\cos ax + i\sin ax}{\cosh x}\,dx \quad \text{as } R \to \infty$$

On γ_3, $z = x + \pi i$

So $$\int_{\gamma_3} f(z)\,dz = -\int_{-R}^{R} \frac{e^{ia(x+\pi i)}}{\cosh(x+\pi i)}\,dx$$
$$= -e^{-\pi a}\int_{-R}^{R} \frac{e^{iax}}{(-\cosh x)}\,dx$$
$$= e^{-\pi a}\int_{-R}^{R} \frac{e^{iax}}{\cosh x}\,dx$$

$$\left[\begin{aligned} &\text{note } \cosh(x+\pi i) \\ &= \tfrac{1}{2}\left(e^{x+\pi i} + e^{-x-\pi i}\right) \\ &= -\tfrac{1}{2}\left(e^{x} + e^{-x}\right) = -\cosh x \end{aligned}\right]$$

So $$\int_{\Gamma} f(z)\,dz = \left(1+e^{-\pi a}\right)\int_{-R}^{R} \frac{e^{iax}}{\cosh x}\,dx + \int_{\gamma_2} f(z)\,dz + \int_{\gamma_4} f(z)\,dz$$
$$= 2\pi i \,\mathrm{Res}_{\frac{\pi i}{2}}(f)$$

Remaining steps

- use length-sup estimate to show $\int_{\gamma_2} \to 0$ and $\int_{\gamma_4} \to 0$ as $R \to \infty$.
- compute the residue at $\frac{\pi i}{2}$.
  can use $\lim_{z \to \frac{\pi i}{2}} \left(\left(z - \frac{\pi i}{2}\right) \frac{e^{iaz}}{\cosh z} \right)$ — (simple pole)

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The Argument Principle

Theorem:

Let $\Omega \subseteq \mathbb{C}$ be an open subset.
Let $f: \Omega \mapsto \mathbb{C}$ be a meromorphic function (i.e. f holomorphic function away from a set of poles).
Let $D \subseteq \Omega$ be a closed disc

$$D = \{z : |z - z_0| \leq r\}.$$

Suppose that none of the zeros or poles of f lie on ∂D.

Let's label the zeros $z_1, z_2, \ldots, z_m$,
poles $p_1, \ldots, p_n$.

Let's say z_i has order k_i i.e.

$$f(z) = (z - z_i)^{k_i} g(z) \quad \text{with } g(z_i) \neq 0$$

& say p_i has order l_i i.e.

$$f(z) = (z - p_i)^{-l_i} g(z) \quad , g(p_i) \neq 0$$

Then if

$$N = \sum_{i=1}^{n} k_i \quad , \quad P = \sum_{i=1}^{n} l_i$$

we have $$N - P = \frac{1}{2\pi i} \int_{|z - z_0| = r} \frac{f'(z)}{f(z)}\, dz$$

This will follow from the residue theorem

$$\frac{1}{2\pi i} \int_C F(z)\, dz = \sum_{w \in D} \mathrm{Res}_w(F) \qquad \text{where } C = \partial D$$

Proof of Thm:

Set $$F(z) = \frac{f'(z)}{f(z)} \quad \left[= \frac{d}{dz} \log f(z) \right]$$

So it's sufficient to prove that

$$\mathrm{Res}_{z_i}(F) = k_i \quad \& \quad \mathrm{Res}_{p_i}(F) = -l_i$$

① ②

Note:

If $f(z_i) = 0$ then F may have a pole at z_i.
Once we know ① & ② we get from residue thm

$$\frac{1}{2\pi i}\int F(z)\,dz = \sum k_i - \sum l_i = N - P$$

So let's calculate

$\mathrm{Res}_{z_i}(F)$ ie. coefficient of $1/z - z_i$ in Laurent expansion of F at z_i.

f has a zero of order k_i at z_i means

$$f(z) = (z - z_i)^{k_i} g(z)$$

$$\Rightarrow f'(z) = k_i (z - z_i)^{k_i - 1} g(z) + (z - z_i)^{k_i} g'(z) \quad \text{(product rule)}$$

$$\Rightarrow F(z) = \frac{f'(z)}{f(z)} = \frac{k_i}{z - z_i} + \frac{g'(z)}{g(z)}$$

$$\Rightarrow \mathrm{Res}_{z_i}(F) = k_i \quad \Rightarrow ①$$

f has a pole of order l_i at p_i then

$$f(z) = (z - p_i)^{-l_i} g(z)$$

$$f'(z) = -l_i (z - p_i)^{-l_i - 1} g(z) + (z - p_i)^{-l_i} g'(z)$$

$$\Rightarrow F(z) = \frac{-l_i}{z - p_i} + \frac{g'(z)}{g(z)}$$

$$\Rightarrow \mathrm{Res}_{p_i}(F) = -l_i \quad \Rightarrow ②$$

So overall $\displaystyle \frac{1}{2\pi i}\int \frac{f'(z)}{f(z)}\,dz$

$$= \frac{1}{2\pi i}\int F(z)\,dz = \sum \mathrm{Res} = \sum k_i - \sum l_i$$

$\square$

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Lecture plan
1). Argument Principle ✓
2). Topological interpretation of this thm
3). Consequences: Rouche's theorem
Fundamental theorem of algebra

Topological interpretation
WINDING NUMBER

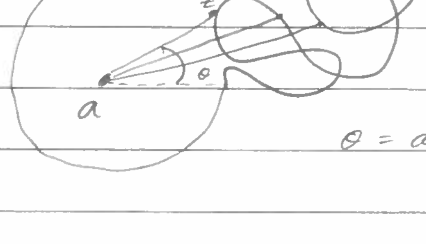
Let $\Gamma(t)$ be a piecewise C^1 closed curve in $\mathbb{C}$ & let $a \in \mathbb{C}$ be a point not on Γ.

Def
The winding number of Γ around a is the following integral

$$n(\Gamma, a) := \frac{1}{2\pi i}\int_\Gamma \frac{dz}{z-a}$$

$$\text{i.e.} = \frac{1}{2\pi i}\int_{t_0}^{t_1} \frac{\Gamma'(t)}{\Gamma(t)-a}\, dt$$

Properties of $n(\Gamma, a)$:
- $n(\Gamma, a) \in \mathbb{Z}$
- counts the number of times Γ "winds" around a i.e. $n(\Gamma, a)$ is a "topological quantity".
- $n(\Gamma, a)$ measures the change of $\arg(z-a)$ as z moves around Γ.



$\theta = \arg(z-a)$

Example:

$\Gamma(t) = e^{int}$, $a = 0$

$$n(\Gamma, 0) = \frac{1}{2\pi i}\int_0^{2\pi} \frac{\Gamma'(t)}{\Gamma(t)-a}\,dt$$

$$= \frac{1}{2\pi i}\int_0^{2\pi} \frac{ine^{int}}{e^{int}}\,dt = \frac{n}{2\pi}\int_0^{2\pi} dt = n$$

Example:

Let $\varepsilon(t) = e^{it}$

Let f be a meromorphic function.

Let $\Gamma(t) = f(\varepsilon(t))$

chain rule

$$n(\Gamma, 0) = \frac{1}{2\pi i}\int_0^{2\pi} \frac{f'(\varepsilon(t))\,\varepsilon'(t)}{f(\varepsilon(t))}\,dt$$

$$= \frac{1}{2\pi i}\int_\varepsilon \frac{f'(z)}{f(z)}\,dz \quad \text{change of variables } z = \varepsilon(t)$$

$= N - P$ by argument principle.

So arg. principle says

$n(f\circ\varepsilon, 0) = N - P$.

Prop

$n(\Gamma, a) \in \mathbb{Z}$

Proof:

$$L(t) := \int_{t_0}^{t} \frac{\Gamma'(s)\,ds}{\Gamma(s)-a}$$

$L(t_1) = 2\pi i\, n(\Gamma, a)$

$$L'(t) = \frac{d}{dt}\int_{t_0}^{t} \frac{\Gamma'(s)}{\Gamma(s)-a}\,ds = \frac{\Gamma'(t)}{\Gamma(t)-a} \quad \text{by F.T.C.}$$

Claim

$$e^{L(t)} = \frac{\Gamma(t)-a}{\Gamma(t_0)-a}$$

Assuming this claim, note that

$\underset{2\pi i n(\Gamma, a)}{e^{L(t_1)}} = (\Gamma(t_1)-a)/(\Gamma(t_0)-a) = 1$ as $\Gamma(t_0) = \Gamma(t_1)$ as Γ is closed curve.

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$\Rightarrow n(\Gamma, a) \in \mathbb{Z}$

Proof of claim:

$$\frac{d}{dt}\left(\exp(-L(t))(\Gamma(t)-a)\right)$$

$$= -L'(t)\exp(-L(t))(\Gamma(t)-a) + \exp(-L(t))\Gamma'(t)$$

$$= \exp(-L(t))\left[\Gamma'(t) - L'(t)(\Gamma(t)-a)\right]$$

$$= e^{-L(t)}\left[\Gamma'(t) - \frac{\Gamma'(t)}{\Gamma(t)-a}(\Gamma(t)-a)\right] = 0$$

$\Rightarrow e^{-L(t)}(\Gamma(t)-a)$ is const.

At $t = t_0$: $e^{-L(t_0)}(\Gamma(t_0)-a)$

$L(t_0) = 0$ so this is just $(\Gamma(t_0)-a)$.

$$\Rightarrow e^{-L(t)}(\Gamma(t)-a) = \Gamma(t_0) - a$$

$$\Rightarrow e^{L(t)} = \frac{\Gamma(t)-a}{\Gamma(t_0)-a}. \quad \square$$

Lemma

$$n(\Gamma, a) = \frac{\text{change in } \arg(z-a)}{2\pi}$$ as z runs around Γ.

Proof

$$e^{L(t)} = \frac{\Gamma(t)-a}{\Gamma(t_0)-a}$$

$\Rightarrow L(t) = \log(\Gamma(t)-a) - \log(\Gamma(t_0)-a)$

$\mathrm{Im}(L(t)) = \arg(\Gamma(t)-a) - \arg(\Gamma(t_0)-a)$

as $\log(re^{i\theta}) = \log r + i\theta$

$L(t_1) = 2\pi i\, n(\Gamma, a)$

$\Rightarrow \mathrm{Im}\, L(t_1) = 2\pi n(\Gamma, a) = \arg(\Gamma(t)-a) - \arg(\Gamma(t_0)-a)$

$\square$

__Theorem__ (Rouché)

Let $\Omega \subseteq \mathbb{C}$ be an open set.

Let $f, g : \Omega \mapsto \mathbb{C}$ be holomorphic functions

Let $D \subseteq \Omega$ be a disc & suppose that

$$|f(z)| > |g(z)| \quad \forall z \in \partial D$$

Then f & $f+g$ have the same number of zeros (counted with multiplicity) inside D.

Equivalently

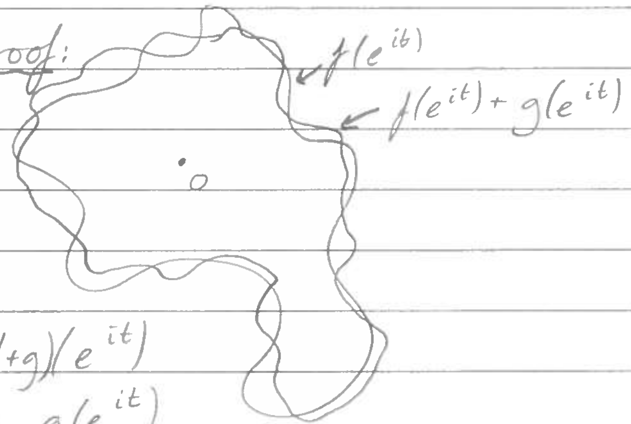
$$n(f(e^{it}), 0) = n((f+g)(e^{it}), 0)$$

There are no poles so $\underset{\uparrow \\ \#\text{ poles of } f}{P} = 0$ & $\underset{\uparrow \\ \#\text{ poles of } (f+g)}{\tilde{P}} = 0$

so arg. princ $\Rightarrow$ $n(f(e^{it}), 0) = N \leftarrow \#$ zeros of f

$n((f+g)(e^{it}), 0) = \tilde{N} \leftarrow \#$ zeros of $(f+g)$

__Picture proof:__



$f(e^{it})$ & $(f+g)(e^{it})$
differ by $g(e^{it})$
& $|g(e^{it})| < |f(e^{it})|$

So the curves wind the same number of times around 0 (they're close to each other). $\square$

__Example__

Let $p(z) = z^7 - 4z^3 + z - 1$.

p has 7 zeros in $\mathbb{C}$. How many of these live inside $\{z : |z| \leq 1\}$

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Take $f(z) = -4z^3$ & $g(z) = z^7 + z - 1$

$f + g = p$

Rouché's Thm applies as

$|f(z)| = 4$

$|g(z)| \leq 1 + 1 + 1 = 3 < 4$

So f and $f+g$ have same number of zeros counted with multiplicity inside $D = \{|z| \leq 1\}$.

f has a unique zero of order 3 in D

$\Rightarrow f+g$ has 3 zeros.

What about inside $\tilde{D} = \{z : |z| \leq 2\}$?

Note that $2^7 = 128$, so if we let

$f(z) = z^7$, $g(z) = -4z^3 + z - 1$

then $|f(z)| = 128$ & $|g(z)| \leq 4 \times 2^3 + 2 + 1 = 35 < 128$ on $|z| = 2$

So Rouché $\Longrightarrow$ $f+g$ has 7 zeros inside $\tilde{D}$ of radius 2.

Proof of Rouche's Thm

N = # zeros of f inside D

$\tilde{N}$ = # zeros of $f+g$ in D

No poles, so arg principle:

$$N = \frac{1}{2\pi i} \int_{\partial D} \frac{f'(z)}{f(z)} \, dz$$

$$\tilde{N} = \frac{1}{2\pi i} \int_{\partial D} \frac{f'(z) + g'(z)}{f(z) + g(z)} \, dz$$

$$\tilde{N} - N = \frac{1}{2\pi i} \int_{\partial D} \left[\frac{f'(z) + g'(z)}{f(z) + g(z)} - \frac{f'(z)}{f(z)} \right] dz$$

$$= \frac{1}{2\pi i} \int_{\partial D} \frac{ff' + fg' - ff' - gf'}{(f+g)f} \, dz$$

So $\tilde{N} - N = \frac{1}{2\pi i} \int_{\partial D} \frac{fg' - gf'}{f(f+g)} dz$

$= \frac{1}{2\pi i} \int_{\partial D} \frac{d}{dz} \left[\log\left(1 + \frac{g}{f}\right) \right] dz$

$\swarrow$ by F.T.C.

$= \frac{1}{2\pi i} \left[\log\left(1 + \frac{g(re^{i2\pi})}{f(re^{i2\pi})}\right) - \log\left(1 + \frac{g(re^{i0})}{f(re^{i0})}\right) \right] = 0$

log is not a well-defined function on $\mathbb{C}$, rather I need to make a branch cut:
get a "branch" of log defined away from the branch cut.

In this proof we're okay because we're taking $\log\left(1 + \frac{g}{f}\right)$ & $\left(1 + g/f\right)$ is in the half plane $\{Re(z) > 0\}$.

Also $|g| < |f|$ on ∂D
$\Rightarrow \frac{|g|}{|f|} < 1$
So $1 + \frac{|g(z)|}{|f(z)|}$ always lies in a ball of radius 1 around 1 so never crosses into $\{Re\ z \leq 0\}$.
So we just pick a branch of log & argument works.

Note: the argument principle requires that none of the zeros of f or $f+g$ lie on ∂D.
Note:
$|f(z)| > |g(z)|$ for $z \in \partial D$

So $|f(z)| > |g(z)| \geq 0$

So $|f(z)| > 0$ on ∂D

$|f(z)+g(z)| \geq |f(z)| - |g(z)| > 0$ by assumption

$\square$

Fundamental Thm of Algebra

$$p(z) = z^n + \sum_{k=0}^{n-1} a_k z^k$$

$$f(z) = z^n \quad , \quad g(z) = \sum_{k=0}^{n-1} a_k z^k$$

Apply Rouché, let $D = \{z : |z| \leq R\}$

If $z \in \partial D$, $|f(z)| = R^n$

$|g(z)| \leq M(1 + R + R^2 + \dots + R^{n-1})$

where $|a_k| \leq M \quad \forall k$.

$$|g(z)| \leq MR^{n-1}\left(1 + \frac{1}{R} + \frac{1}{R^2} + \dots + \frac{1}{R^{n-1}}\right)$$

$$\leq MR^{n-1}\left(1 + \frac{1}{R} + \frac{1}{R^2} + \dots\right)$$

$$= \frac{MR^{n-1}}{1 - \frac{1}{R}} \quad \text{(geometric series)}$$

So $|f| = R^n \quad , \quad |g| \leq \dfrac{MR^{n-1}}{1 - \frac{1}{R}}$

If $R > M+1$ then $|f| > |g|$

So $(1 - \frac{1}{R})R > M \Rightarrow R > M+1$.

$\Rightarrow$ Rouché applies & f & p have same # of zeros (namely n).