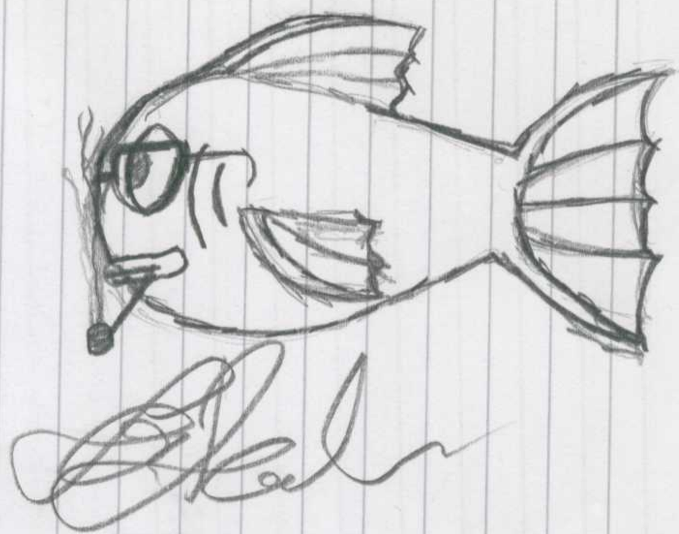
2101 Analysis 3: Complex Analysis Notes
Based on the 2011 autumn lectures by
Prof A Sobolev.



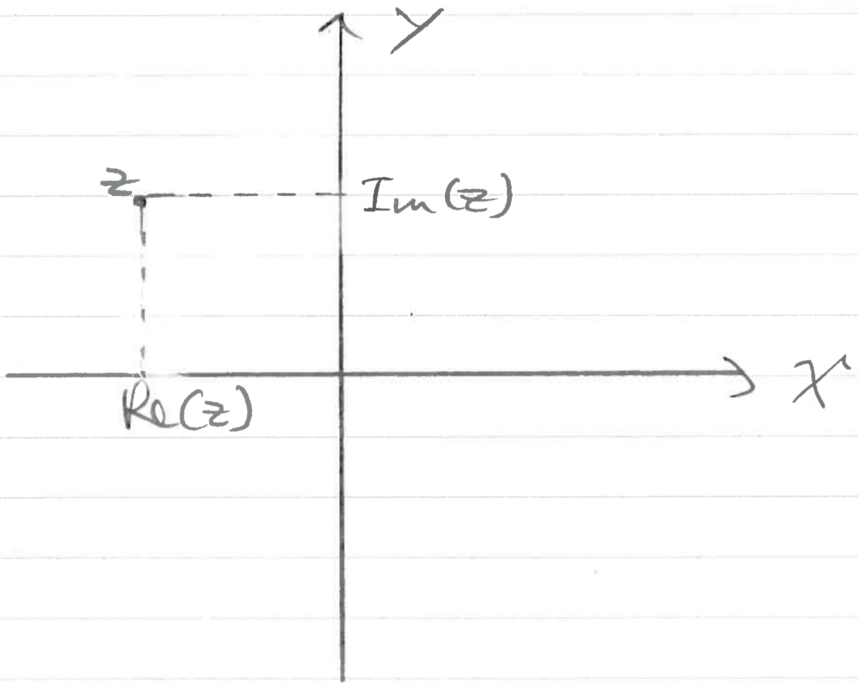
3/10/11

2101 Complex Analyses.
A. Sobolev, 710
1 pm Monday office hour 710.
Homework Due on Wednesday.

Plan:
1) Introduction to Complex numbers
2) Sets of complex plane
3) Continuity
} pt 1

4) Differentiability
5) Integration.
} pt 2

1 Complex numbers
Let $z \in \mathbb{R}^2$ be a point in the plane.
Then $z = (x, y)$, $x, y \in \mathbb{R}$.
Notation $x = \operatorname{Re} z$, real part of z.
$y = \operatorname{Im} z$, imaginary part of z



Def 1.1

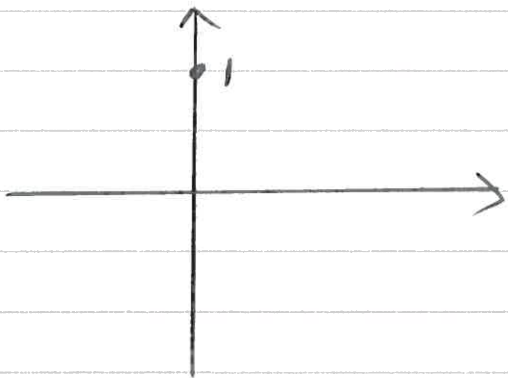
Define multiplication: Let $z_1, z_2 \in \mathbb{R}^2$. If $z_1 = (x_1, y_1)$, $z_2 = (x_2, y_2)$ then

$$z_1 z_2 = (x_1 x_2 - y_1 y_2,\ x_1 y_2 + x_2 y_2)$$

with this multiplication $\mathbb{R}^2$ becomes the complex plane!

Observe: $z_1 z_2 = z_2 z_1$
$z_1(z_2 + z_3) = z_1 z_2 + z_1 z_3$.

"Good" Definition look at $z = (0, 1)$



Then $(0, 1)^2 = (-1, 0)$

Notation $i = (0, +1)$ then

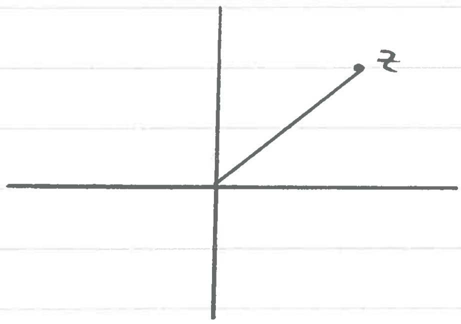
$$z = (x, y) = x(1, 0) + y(0, 1) = x + iy.$$

Standard form of complex numbers

Complex plane $\mathbb{C}$ = Argand plane:

Def 1.2: The modulus (or the absolute value) of $z \in \mathbb{C}$ is:

$$|z| = \sqrt{x^2+y^2} = \sqrt{(\mathrm{Re}\, z)^2 + (\mathrm{Im}\, z)^2}$$



$|z|$ is the distance from z to the origin.

$|z_1 - z_2| = \ldots$

Define $S = \{z : |z| = 1\}$ – circle of radius 1

$S' = \{z : |z-a| = 1\}$ – circle at a and of radius 1.

Notation:

$\gamma(a, r) = \{z : |z-a| = r\}$, $a \in \mathbb{C}$, $r > 0$
circle of rad r centered at a.

Def 1.3 If $z = x + iy \in \mathbb{C}$, then the conjugate of z is defined to be $\bar{z} = x - iy$.

Note $\bar{\bar{z}} = z$.

Propositcon 1.4.

1) $\overline{z_1 + z_2} = \bar{z}_1 + \bar{z}_2$

2) $\overline{z_1 z_2} = \bar{z}_1 \bar{z}_2$

3) $\bar{z} z = z \bar{z} = |z|^2$

4) $|z_1 z_2| = |z_1||z_2|$

5) $\frac{1}{z} = \frac{\bar{z}}{z\bar{z}} = \frac{\bar{z}}{|z|^2} = \frac{x - iy}{(x+y)^2} = \frac{x}{x^2+y^2} - i\frac{y}{x^2+y^2}$.

Propositcon 1.5 - Let $z = x + iy$. Then $x = \mathrm{Re}\, z = \frac{z + \bar{z}}{2}$, $y = \mathrm{Im} = \frac{z - \bar{z}}{2i}$

Inequalites.

Lemma 1.6 : Let $z, w \in \mathbb{C}$ Then.

1) $|\mathrm{Re}\, z| \leq |z|$, $|\mathrm{Im}\, z| \leq |z|$

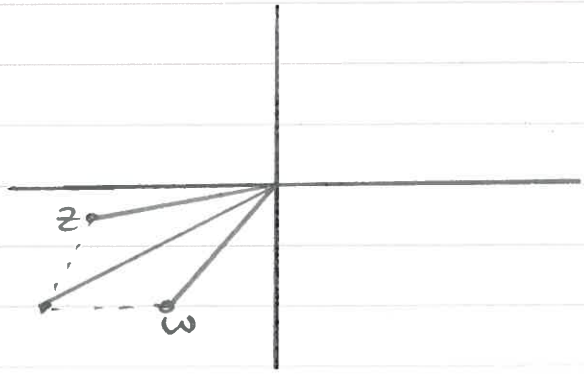
2) $|z + w| \leq |z| + |w|$, triangle ciequalities

3) $|z - w| \geq ||z| - |w||$

Proof:

① - ③ Excer. ise.

2)



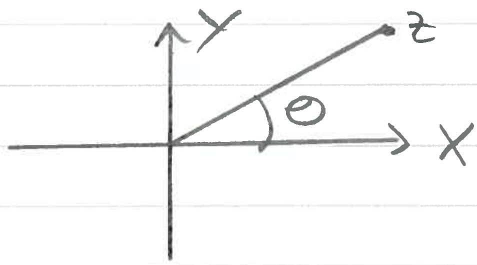
$$
\begin{aligned}
|z+w|^2 &= (\bar{z}+\bar{w})(z+w) \\
&= \bar{z}z + \bar{w}z + \bar{z}w + \bar{w}w \\
&= |z|^2 + 2\mathrm{Re}(\bar{w}z) + |w|^2 \\
&\leq |z|^2 + 2|\bar{w}z| + |w|^2 \\
&= |z|^2 = 2|w||z| + |w|^2 \\
&= (|z| + |w|)^2
\end{aligned}
$$

$\Rightarrow |z+w|^2 \leq (|z|+|w|)^2$

$\Rightarrow |z+w| \leq |z| + |w|$ As required $\square$

<u>The polar form</u>:

Let $z = x + iy \in \mathbb{C}$. Introduce polar coordinates.



Let $r = |z|$.

Then : $x = r\cos\theta$

$y = r\sin\theta$

Hence $z = r\cos\theta + i\sin\theta = r(\cos\theta + i\sin\theta)$

Denote: $\cos\theta + i\sin\theta = e^{i\theta}$
The angle θ is called the arguement of z, notation $\theta = \arg z$.

Recall: $e^{t+s} = e^t e^s$, $s, t \in \mathbb{R}$.

Lemma 1.8: Let $z_1 = r_1 e^{i\theta_1}$, $z_2 = r_2 e^{i\theta_2}$
Then $z_1 z_2 = r_1 r_2 e^{i(\theta_1 + \theta_2)}$

Proof: Write:

$z_1 z_2 = r_1 r_2 (\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 + i\sin\theta_2)$

$= r_1 r_2 [\cos\theta_1 \cos\theta_2 - \sin\theta_1 \sin\theta_2 + i(\cos\theta_1 \sin\theta_2 + \sin\theta_1 \cos\theta_2)]$

$= r_1 r_2 [\cos(\theta_1 + \theta_2) + \sin(\theta_1 + \theta_2)]$

$= r_1 r_2 \; e^{i(\theta_1 + \theta_2)}$ $\square$

Example: $\arg i = \pi/2$ or $\pi/2 + 2\pi$
$\arg -i = 3\pi/2$ or $-\pi/2$

Definition 1.7 The principle value of the arguement is defined as the uniquely defined value of θ the interval $(-\pi, \pi]$.

Notation: $\operatorname{Arg} z$.

$\therefore$ $\text{Arg}(-i) = \pi/2$.

$\text{Arg}(-1) = \pi$

Observe $e^{2\pi i} = 1$, $e^{2\pi n i} = 1$, $n \in \mathbb{Z}$.

Proposition 1.9 (De Moive's formula)

$$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$$

$n = 1, 2, \dots$

Powers of z.

$z^n = r^n e^{in\theta}$, $n = 1, 2, \dots$ from lemma 1.8

Definition: For any $\alpha \in \mathbb{R}$.

$$z^\alpha = r^\alpha e^{i\theta}$$

Example: $z^{1/2} = |z|^{1/2} e^{i\frac{1}{2}\arg z}$

$\sqrt{1} = ?$ where $z = 1$.

If $\arg 1 = 0$, then $\sqrt{1} = 1$

If $\arg 1 = 2\pi$, then $\sqrt{1} = -1$.

$\sqrt{1} = 1$ and $\sqrt{1} = -1$ repersent two different branches of the square root.

The value $\sqrt{1} = 1$ is called the principle value of $\sqrt{1}$.

$\sqrt{1} = -1$ is the other value of the root.

In general, let $z = re^{i\theta} = re^{i\theta + 2\pi i}$
$n \in \mathbb{Z}$.

Then $z^{\alpha} = r^{\alpha} e^{i\alpha\theta + i2\pi n\alpha}$

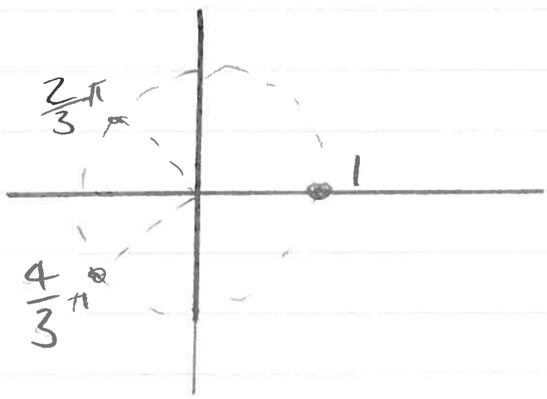
Different values of n repersent different branches of z^{α}. The principle value : $z^{\alpha} = |z|^{\alpha} e^{i\alpha \operatorname{Arg} z}$.

<u>Example</u>:

$$1^{1/3} = e^{i\frac{2\pi n}{3}}, \quad n \in \mathbb{Z}.$$

If $n = 0 \Rightarrow e^{0} = 1$.

$$1^{1/3} = \begin{cases} 1 & n = 0 \\ e^{i2\pi/3} & n = 1 \\ e^{i\frac{4\pi}{3}} & n = 2 \end{cases}$$



$z^{\alpha} = r^{\alpha} e^{i\alpha \arg z}$

If $\alpha = 1/2$, $z = 1$ then $1^{1/2} = |1|^{1/2} e^{i\frac{1}{2}\arg 1}$

The arithmetic root of 1

<u>5/10/11</u>.

$z = x + iy$, $x, y \in \mathbb{R}$.

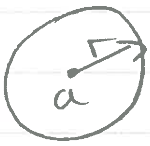
$z \in \mathbb{C}$.

$i : i^2 = -1$

$\text{Arg}\, z \in (-\pi, \pi]$

$|e^{i\theta}| = 1$

$\gamma(a, r) = \{z : |z-a| = r\}$, $a \in \mathbb{C}$, $r > 0$


$= \{z = a + re^{i\theta}, \theta \in [0, 2\pi)\}$

<u>Geometry and topology of complex plane</u>

<u>Sets of complex plane</u>

<u>Defn 1.10</u> Let $z \in \mathbb{C}$, $r > 0$. The the set

$$\gamma(z_0, r) = \{z : |z - z_0| = r\}$$

is called a circle of radius centered at z.

The set

$$D(z_0, r) = \{z : |z - z_0| < r\},$$

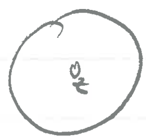
is called <u>open</u> disk of r, radius, centered at z_0.

The set

$$\bar{D}(z_0, r) = \{z : |z - z_0| \leq r\}$$

is called <u>closed</u> disk —— " —— "
—— " ——

Sometimes we called $D(z_0, r)$ an r-neighbourhood of z_0.



The set

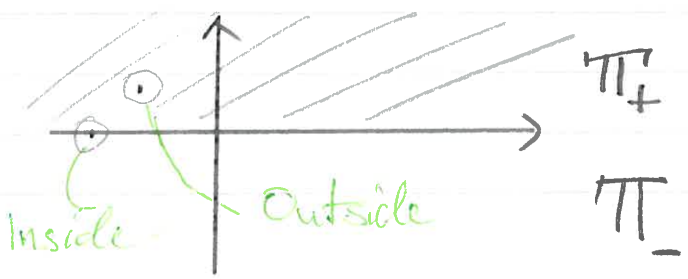
$$D'(z, r) = \{z : |z - z_0| < r\}$$

is called <u>punctured</u> r-neighbourhood at z_0

<u>Half planes</u>.

$$\Pi_+ = \{z : \operatorname{Im} z > 0\}$$ – upper half plane.

$$\Pi_- = \{z : \operatorname{Im} z < 0\}$$ – lower half plane.



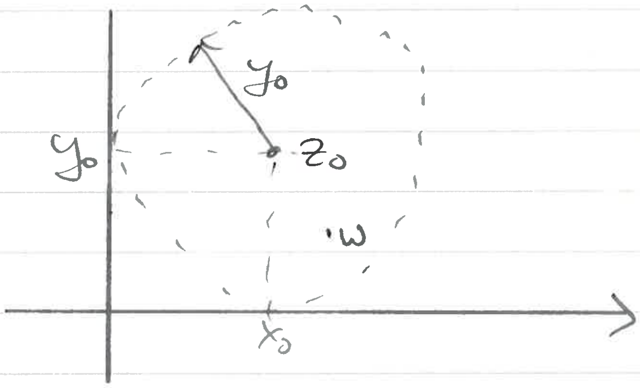
Notation: Let $S \subset \mathbb{C}$.

Definition 1.11: Let $z \in S'$. Then z is said to be an interior point of S' if there is a number $r > 0$ st $D(z, r) \in S'$

The set of all interior points of S' is denoted $\text{int}\, S$.

We say that S is open if consists of interior points only $\text{int}\, S' = S'$ for any $z \in S$ there is number $r > 0$ st $D(z, r) \in S$.

Example: Π_+ is open. Let $z_0 \in \Pi_+$ i.e. $z_0 = x_0 + iy_0$. with $y_0 > 0$.



Let $r = y_0$.

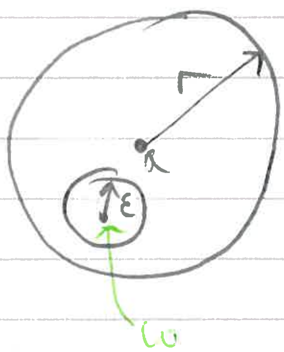
Then take $w \in D(z_0, y_0)$. Let's show that $\text{Im}\, w > 0$ i.e $w \in \Pi_+$.

Write $w = z_0 + w - z_0$.

$$\operatorname{Im} w = \operatorname{Im} z + \operatorname{Im}(w - z_0).$$
$$= y_0 + \operatorname{Im}(w - z_0)$$
$$\geq y_0 - |\operatorname{Im}(w - z_0)|$$
lemma 1.16 $\geq y_0 - |w - z_0|$
$$> y_0 - y_0 = 0.$$

and hence $\operatorname{Im} w > 0$ as required $\square$.

<u>Example</u>: Prove that $D(a,r)$, $r > 0$ is open.



Need to show that for any point $w \in D(a,r)$ there is number $\varepsilon > 0$ st $D(w,\varepsilon) \subset D(a,r)$.

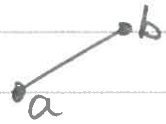
Take $\varepsilon = r - |a - w|$.

<u>Defn 11.2</u>. The set $S^c = \mathbb{C} \setminus S$ is called the complement of S.

We say that S is closed if S^c is open.

<u>Example</u>:

1) $\overline{D}(a,r)$ is closed.



2) The <u>interval</u> (segment)

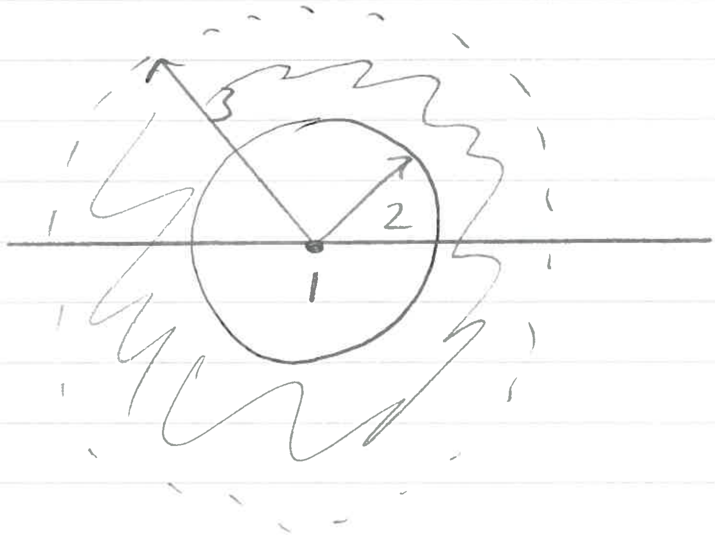
$$[a,b] = \{(1-t)a + tb \,;\, t \in [0,1]\}.$$

is closed, $a, b \in \mathbb{C}$.

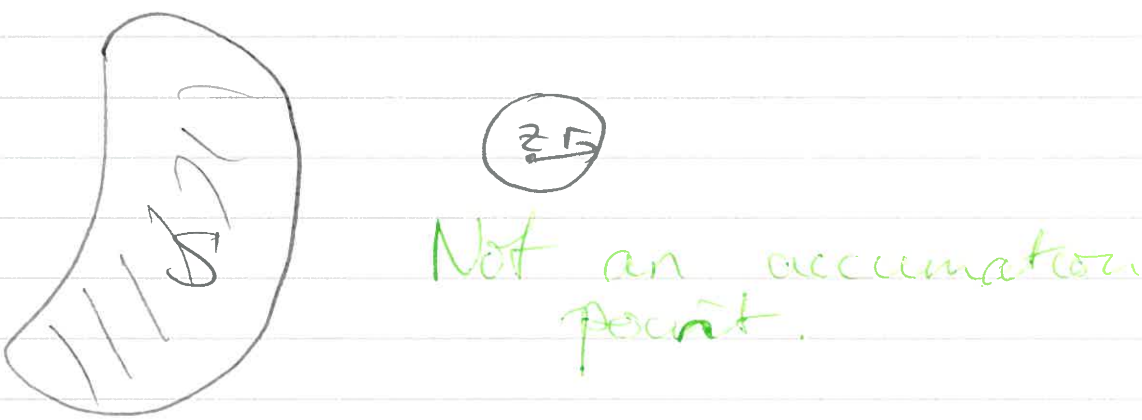
3) $D'(a,r)$ is open.

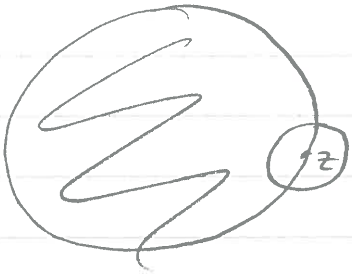
4) $S = \{z : 2 \le |z-1| < 3\}$

Is both open and closed.



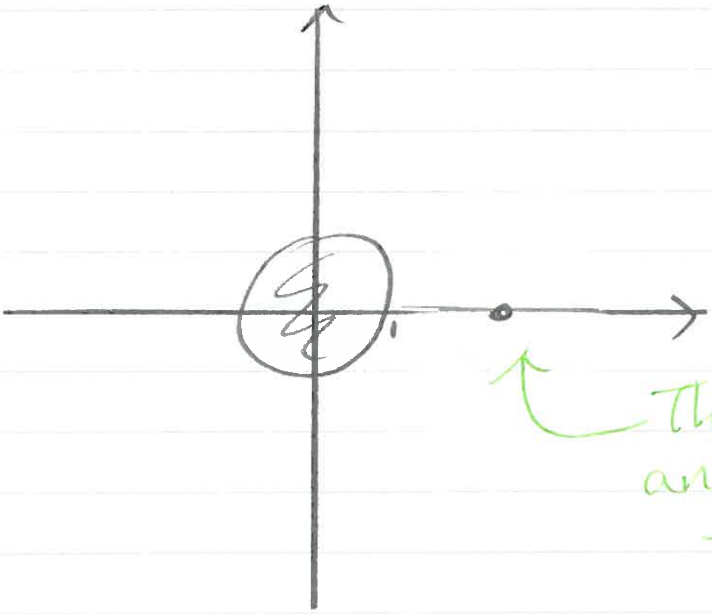
Def 1.13 A point $z_0 \in \mathbb{C}$ is said to be an accumulation point of the set S if for all $r > 0$ we have $D'(z_0, r) \cap S = \emptyset$.





This is an accumation point

Let $T = D(0,1) \cup \{2\}$



This is not an accumation point.

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Sets of Complex plane.

Open set – $D(z,r)$
$D(z,r)$ open disk of radius $r>0$ centered at z.

Definition 1.13 : Let S be a set on $\mathbb{C}$. Let $z \in \mathbb{C}$. We can say that z is an accumulation point of the set S iff for all $r>0$ we have $D(z,r) \cap S \neq \phi$

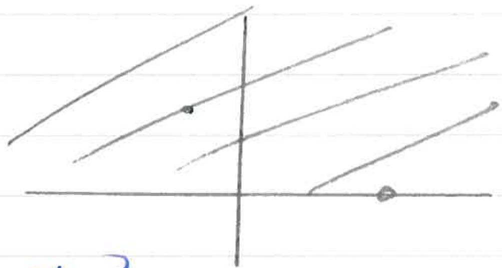
The closure of the set S is the union of the set S and all its accumulation points.

Example.

1) $S = \Pi_+ = \{z : \operatorname{Im} z > 0\}$

$\{\text{Acc points of } S\} = \{z : \operatorname{Im} z \geq 0\}$

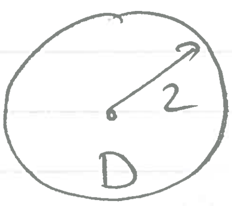
$\bar{S} = \{z : \operatorname{Im} z \geq 0\}$



2) $S = D(0,2)$

$\{\text{Accumulation points of } S\}$

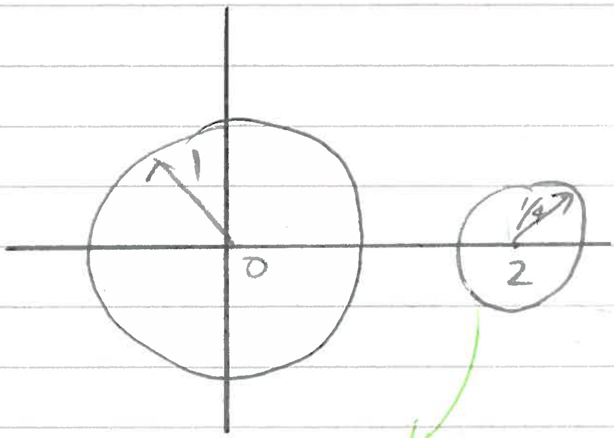
$= \bar{D}(0,2)$



$\bar{S} = \bar{D}(0,2)$

3) $D(0,1) \cap \{2\}$.

$\{\text{Acc. point}\} = \bar{D}(0,1)$



$D'(2, 1/4)$ will not contain the element 2

$T = D(0,1) \cup \{2\}$

$D'(2, 1/4) \cap T = \phi$

$\bar{T} = \bar{D}(0,1) \cup \{2\}$.

4) $S = \bar{D}(0,5)$, $\{\text{Acc. point}\}$
$= \bar{D}(0,5)$

$\bar{S} = \bar{D}(0,5)$
$= S$.

Theorem 1.14: The following statement are equivalent:

1) The set S is closed.

2) S contains all its accumulation points

3) $\overline{S} = S$.

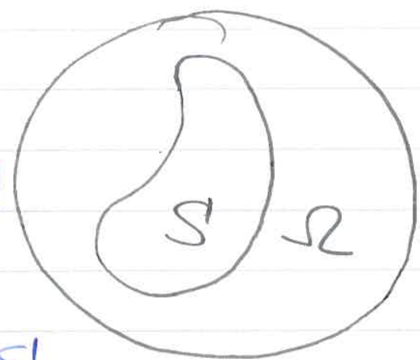
Proposition 1.15 (Not examine) Let $S \subset \mathbb{C}$ Then:

1) $\overline{S}$ is a closed set

2) $\overline{S}$ is the smallest closed set containing S. i.e. for any closed set $\Omega \supset S$ we have $\overline{S} \subset \Omega$

Def: S is called compact if it is closed and bounded.

Definition 1.16: The set $\partial S = \overline{S} \setminus \text{int } S$ is called the boundary of S.



Definition 1.17 - The set S is said to be bounded if there is a number $R > 0$ st $S \subset D(0, R)$.

Examples:

1) Π_+, Π_- are not Bdd.

2) $D(10, 1)$ is bounded, as $D(10, 1) \subset D(0, 100)$.

— / —

S is called compact if it is closed and bounded.

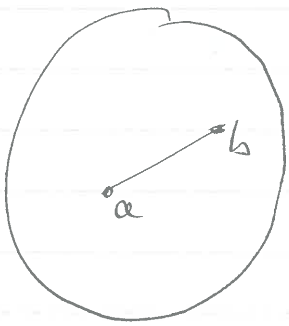
— / —

$D(0,2)$ is not compact, since it is open.

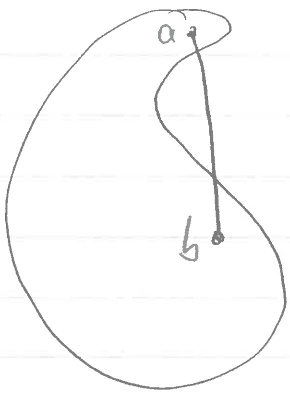
Convexity and connectedness.

Definition 1.18: The set S is said to be convex if any two points $a, b \in S$ the segment $[a,b]$ is also in the set.

Eg:



It is convex



It is not convex.

Example:

1) $\Pi_- = \{z : \operatorname{Im} z < 0\}$

Let $a, b \in \Pi$, i.e. $\operatorname{Im} a < 0$, $\operatorname{Im} b < 0$

Let $z = (1-t)a + tb$, $t \in [0,1]$

Then:

$$\operatorname{Im} z = \underbrace{(1-t)}_{\geq 0}\underbrace{\operatorname{Im} a}_{<0} + \underbrace{t}_{\geq 0}\underbrace{\operatorname{Im} b}_{<0} < 0.$$

$\Rightarrow z \in \Pi_-$ i.e $[a,b] \in \Pi_-$

i.e. Π_- is convex.

2): $S = D(a,r)$ – convex.

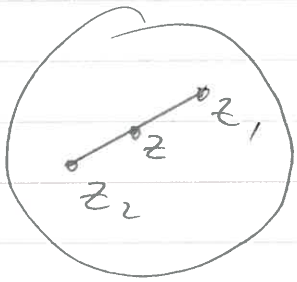
Let $z_1, z_2 \in S$, i.e $|z_1 - a| < r$, $|z_2 - a| <$

Let $z = (1-t)z_1 + z_2$, $t \in [0,1]$, so need to show that $|z-a| < r$.

Write:

$$z - a = (1-t)(z_1 - a) + t(z_2 - a)$$

So:



$$|z-a| = |(1-t)(z_1-a) + t(z_2-a)|$$

Tri inequality

$$\leq |(1-t)(z_1-a)| + |t(z_2-a)|$$

$$= (1-t)|z_1 - a| + t\,|z_2 - a|$$

$$< (1-t)r + tr = r$$

Thus $z \in D(a,r)$, i.e $[z_1, z_2] \subset D(a,r)$
i.e $D(a,r)$ is convex.

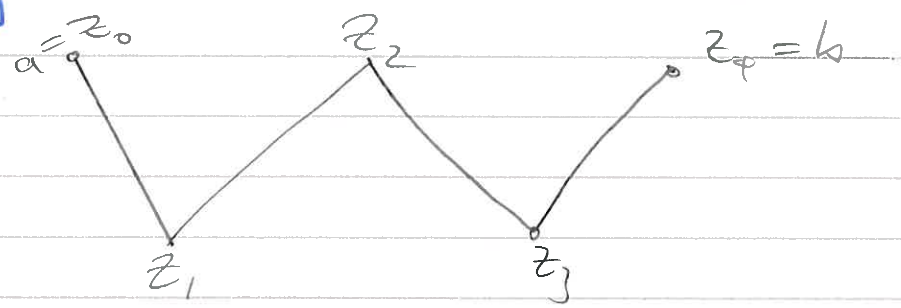
3) $\overline{D}(a,r)$ - convex.

<u>Definition 1.19</u> - Let $a, b \in \mathbb{C}$ and let $a = z_0, z_1, \dots, z_n = b$.

We call the set

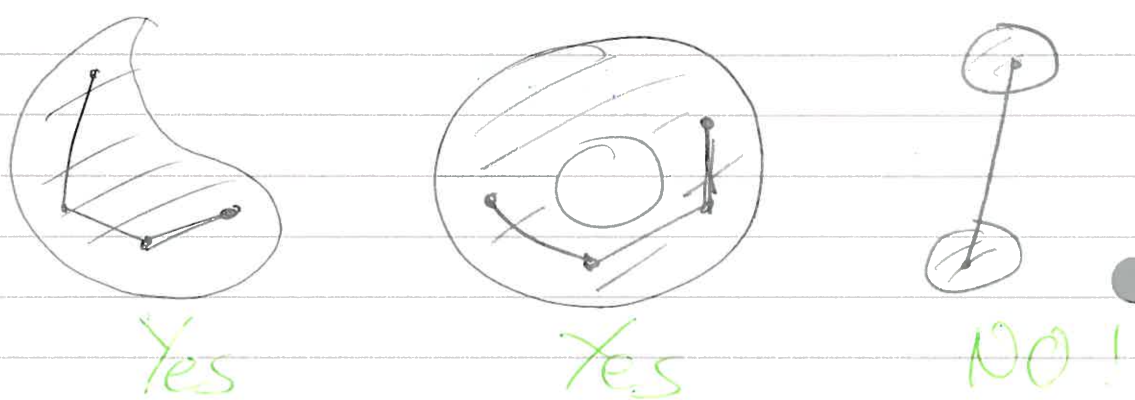
$$[z_0, z_1] \cup [z_1, z_2] \cup \dots \cup [z_{n-1}, z_n].$$

Eg:



A set S is said to be <u>polygonally connected</u> if for any two points $a, b \in S$ there is a polygonal path joining a and b, which is inside S.

Eg:



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Polygonally connected = connected.

— / —

<u>Definition 1.20</u>: The set S which is open and connected is a <u>domain</u> (or region)

<u>Example</u>:

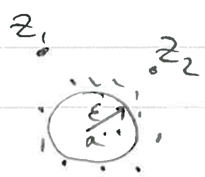
1) $\overline{D}(1,2)$ – not a domain, connected, bounded, compact, convex.

2) $T = D(0,1) \cup \overline{D}(3,1/2)$ not a domain, not connected, not open, not closed, bdd since $T \subset D(0,100)$, not compact.

<u>Sequence and Convergence</u>.

A complex sequence $\{z_n\}$ $n = 1, 2, \ldots,$ is a collection of complex numbers.

<u>Definition 1.21</u>: We say that the sequence z_n converges to any number $\varepsilon > 0$ there is a natural number N st $|z_n - a| < \varepsilon$ for all $n > N$, $N_\varepsilon = N$.



A sequence $\{w_k\}$ is said to be a sequence of $\{z_n\}$ if there is a sequence of natural number $n_1, n_2, \ldots, n_k$ st $n_k \to \infty$, $k \to \infty$ and $w_k = z_{n_k}$.

<u>Lemma 1.22</u>: The sequence z_n converages to a as $n \to \infty$ iff $\operatorname{Im} z_n$ converages to $\operatorname{Im} a$ and $\operatorname{Re} z_n$ converages to $\operatorname{Re} a$ as $n \to \infty$.

Follows from:

$$|\operatorname{Im} z_n - \operatorname{Im} a| \leq |z_n - a|$$

$$= \sqrt{|\operatorname{Im} z_n - \operatorname{Im} a|^2 + |\operatorname{Re} z_n - \operatorname{Re} a|^2}$$

and

$$|\operatorname{Re} z_n - \operatorname{Re} a| \leq |z_n - a|.$$

Moreover, if z_n converages to a, then $|z_n|$ converages to $|a|$, $\overline{z}_0$ converages to $\overline{a}$.

Notation:

z_n converages to a

$z_n \to a$, $n \to \infty$

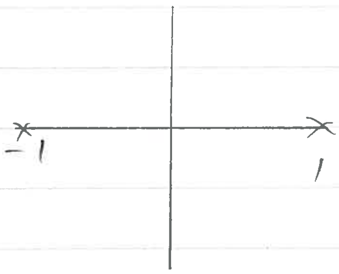
$\lim_{n \to \infty} (z_n) = a$.

Example:

1) $z_n = \frac{1}{n} + i\frac{n^2}{n^2+1}$

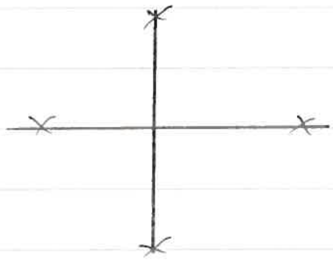
$\lim_{n \to \infty}(z_n) = i$.

2) $\omega_n = e^{in\pi}$



Doesn't converge

3) $p_k = e^{ik\frac{\pi}{2}}$



No limit so it doesn't converge

Proposition 1.23 - Let z_n be a convergent sequence. Then:

1) $\{z_n\}$ is bounded.

2) The limit is unique if $\{z_n\} \to a$ as $n \to \infty$ and $\{z_n\} \to b$ as $n \to \infty$ then $a = b$.

3) Each subsequence of z_n has the

has the same limit i.e. a.

4) $\{z_n\}$ is a Cauchy's sequence i.e. $\forall \varepsilon > 0$ there is a number $N = N_\varepsilon$ st $|z_n - z_m| < \varepsilon$ for all $n, m > N$.

Conversely, any Cauchy sequence converges.

<u>Theorem 1.24</u>: (Not examine (he thinks)) (Bolzano - Weierstass theorem): Any bounded sequence convergent subsequence.

Example: $w_n = e^{int}$

$$q_k = w_{2k} = e^{i2k\pi} = 1.$$

$$\text{or} \quad s_k = w_{2k+1} = e^{i(2k+1)\pi} = e^{i\pi} = -1.$$

<u>Corollary 1.25</u>. Any infinite compact set S has a limit point in S.

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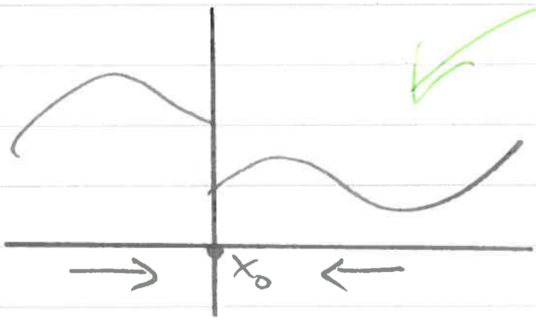
Limits

$$\lim_{z \to z_0} P(z) = P(z_0), \quad z_0 \in \mathbb{C}.$$



In real analysis

f has no limit at x_0

Example: $f(z) = \dfrac{\operatorname{Im} z}{z}, \quad z \neq 0.$

$$\lim_{z \to \infty} f(z) = ?$$

Let $z = re^{i\theta}, \quad \theta \in (-\pi, \pi].$

$$f(z) = \frac{r\sin\theta}{re^{i\theta}} = \frac{\sin\theta}{e^{i\theta}} \longrightarrow \frac{\sin\theta}{e^{i\theta}}$$

$\theta = 0 \Rightarrow \lim = 0.$

$\theta = \pi/2 \Rightarrow \lim = -i.$

θ is fixed.

$\Rightarrow f$ has no limit at $z_0 = 0$

Continuity.

Definition 1.28: Function f is continuous at z_0 if:
1) $z_0 \in D(f)$
2) $\lim_{z \to z_0} f(z) = f(z_0)$.

f is said to be continuous on the set S if f is continuous at every point of S.

Alternative: Let $z_0 \in D(f)$. Assume that $D'(z_0, r) \cap D(f) \neq \phi$, for any $r > 0$.

Then f is continuous at z_0 if $\forall \varepsilon > 0$ $\exists \delta$ st $|f(z) - f(z_0)| < \varepsilon$ as soon as $|z - z_0| < \delta$, $z \in D(f)$.

Properties.
1) Polynomials are continuous on $\mathbb{C}$.
Rational fn's $(P(z)/Q(z))$ are continuous away from roots of $Q(z)$.

2) By AQL if f and g are contiuous at z_0, then so are:

1) $f+g$, 2) fg, 3) f/g ← away from the roots of g.

3) If $f = u + iv$ is contiuous so are u, v and vice varsa.

4) If f, g are contiuous, then $f(g(z))$ is also contiuous notation: $(f \circ g)(z) = f(g(z))$

5) If f is const, then $|f|$ is contiuous. The oppsite is not ture!

Example:

$$g(z) = e^{\sqrt{1+x^2}} + i\sin(y^3 x).$$

$\operatorname{Re} g$ and $\operatorname{Im} g$ are contiuous on $\mathbb{R}^2$, and by 3 g is contiuous.

For a real valued function $h(x)$, $x \in \mathbb{R}$: how to guarnatee that h is bdd?

Answer: h is bdd if it is contiuous on a closed interval $[a, b]$

Bounded continuous function.

We say that f is bounded on $D(f)$ if there is a number $M > 0$ st

$$|f(z)| \leq M \quad \text{when} \quad z \in D(f)$$

Theorem 1.29 (Not examine): Suppose that f is continuous on the compact set S. Then
1) f is bounded on S.
2) the function $|f|$ contains its max and min values of S.

Chapter 2.

Derivaties and analytic function

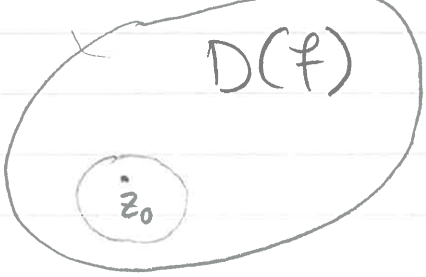
Real analysis (reminder)

$$f'(x_0) = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}$$

$$= \lim_{h \to 0} \frac{f(x_0+h) + f(x_0)}{h}$$

Def^n 2.1: Suppose that $D(z_0, r) \subset D(f)$ for some $r > 0$. Then we say that f is differentiable at z_0, if the limit

$$f'(z) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

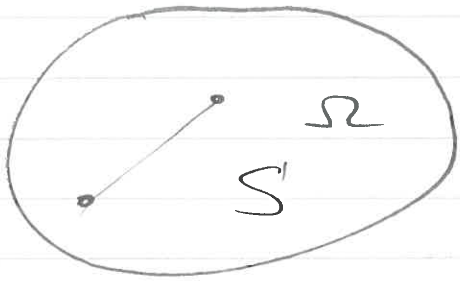


exist. The limit is called the derivative of f at z_0.

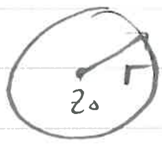
If $D(f)$ is a domain, then if f is diff at every $z \in D(f)$, then f is holomorphic on $D(f)$.

$H(\Omega)$ is the set of all holomorphic functions on the domain Ω (open connected set)

If $S \subset \mathbb{C}$, then we say that f is holomorphic on S if $f \in H(\Omega)$ for some $\Omega \supset S$.



f is holomorphic at z_0 if it is holomorphic on $D(z_0, r)$ with some $r > 0$.



If f is analytic on $\mathbb{C}$ we say f is an entire function

Rewrite:

$$f'(z_0) = \lim_{h \to 0} \frac{f(z_0+h) - f(z_0)}{h}, \quad h \in \mathbb{C}.$$

Example: 1) Let $g(z) = |z|^2 = x^2 + y^2$.

Lets try and find $g'(z_0)$

$$\frac{g(z_0+h)-g(z_0)}{h} = \frac{|z_0+h|^2-|z_0|}{h}$$

$$= \frac{(z_0+h)\overline{(z_0+h)}-|z_0|^2}{h}$$

$$= \frac{\cancel{z_0\bar{z}_0} + h\bar{z}_0 + z_0 h \cancel{-|z_0|^2}}{h}$$

$$= \bar{z}_0 + h + z_0\frac{\bar{h}}{h}.$$

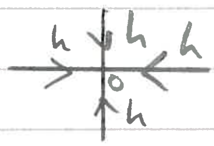
Look at $z_0\bar{h}/h$.

If $z_0 = 0 \Rightarrow \frac{|\ |^2-|\ |^2}{h} = \bar{h} \to 0.$

as $h \to 0 \Rightarrow g'(0) = 0.$

Suppose $z_0 \neq 0$. Assume first that $h = \epsilon \in \mathbb{R}$.

Then: $z_0\frac{\bar{h}}{h} = z_0\frac{\epsilon}{\epsilon} = z_0.$



Suppose that $h = iu$, $u \in \mathbb{R}$.

$$z_0\frac{\bar{h}}{h} = z_0\left(\frac{-iu}{iu}\right) = -z_0.$$

Thus g is differentiable only at $z_0 = 0$ and $g'(0) = 0$.

Note: g is continuous on $\mathbb{C}$.

2) $f(z) = z^2$, $z \in \mathbb{C}$.

Write: $\frac{(z+h)^2 - z^2}{h} = \frac{2zh + h^2}{h} = 2z + h \to 2z$ as $h \to 0$

$\Rightarrow f$ is diff on $\mathbb{C}$ i.e. f is holomorphic on $\mathbb{C}$ and $f'(z) = 2z$.

<u>Lemma 2.2</u> - If f is diff. at z_0, f is continuous at z_0.

<u>Proof</u>: Want: $f(z) \to f(z_0)$ as $z \to z_0$.

Write: $f(z) - f(z_0) = \frac{f(z_0) - f(z_0)}{z - z_0} (z - z_0)$

by AOL:

$\xrightarrow[z \to z_0]{} f'(z_0) \cdot 0 = 0$.

Thus $\lim_{z \to z_0} f(z) = f(z_0)$ as required.

<u>The Cauchy-Riemann equations</u>.

We are looking at a link between real and imaginary part of f which guarantee differentiability.

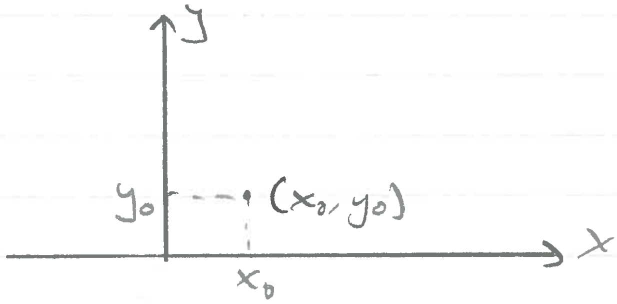
Partial Derivatives (reminder); look at $g(x,y)$.

$$\frac{\partial g}{\partial x}(x_0,y_0) = \lim_{\epsilon\to 0} \frac{g(x_0+\epsilon,y_0)-g(x_0,y_0)}{\epsilon}.$$

$$= g_x(x_0,y_0).$$

$$\frac{\partial y}{\partial x}(x_0,y_0) = \lim_{\epsilon\to 0} \frac{g(x_0,y_0+\epsilon)-g(x_0,y_0)}{\epsilon}.$$

$$= g_y(x_0,y_0).$$



Theorem 2.3: Suppose that $f(z)=u(x,y)+iv(x,y)$. The the partial deriviates u_x, v_x, u_y, u_y exist at (x_0,y_0) and $f'(z) = u_x+iv_x = v_y - iu_y$ and therefore:

$$\begin{cases} u_x = v_y \\ u_y = -v_x \end{cases}$$ (Cauchy's - Riemann equation).

Proof: Use $f(z_0) = \lim_{h\to 0} \frac{f(z_0+h)-f(z_0)}{h}$.

Let $h=\epsilon\in\mathbb{R}$. Then

$$f'(z_0) = \lim_{t \to 0} \left[\frac{f(x_0 + t, y_0) - f(x_0, y_0)}{t} \right]$$

$$= \lim_{t \to 0} \left[\frac{u(x_0 + t, y_0) - u(x_0, y_0)}{t} \right.$$

$$\left. + i \left(\frac{v(x_0 + t, y_0) - v(x_0, y_0)}{t} \right) \right]$$

Thus limits of Re and Im parts exist as $t \to 0$ and have u_x, v_x exist and

$$f'(z_0) = u_x(x_0, y_0) + i v_x(x_0, y_0).$$

Now assume that $h = it$, $t \in \mathbb{R}$.

Then, $$f'(z_0) = \lim_{t \to 0} \left[\frac{f(x_0, y_0 + t) - f(x_0, y_0)}{it} \right]$$

$$= \lim_{t \to 0} \left[\frac{v(x_0, y_0 + t) - v(x_0, y_0)}{t} \right.$$

$$\left. - i \left(\frac{u(x_0, y_0 + t) - u(x_0, y_0)}{t} \right) \right]$$

$$= u_y - i u_y \quad \text{as claimed} \qquad \square.$$

<u>Example</u>: $f(z) = z^2$, $z \in \mathbb{C}$.

Rewrite $f(x) = \underbrace{x^2 + y^2}_{u} + i \underbrace{2xy}_{v}$

So $u_x = 2x$ $\quad v_x = 2y$
$u_y = -2y$ $\quad v_y = 2x$.

$\Rightarrow \left.\begin{array}{l} u_x = +v_y \\ u_y = -v_x \end{array}\right\}$ CRE $\rightarrow$ Cauchy Riemann eq^n.

Let: $g(z) = |z|^2 = \underbrace{x^2+y^2}_{u} + i\underbrace{(0)}_{v}$.

$u_y = 2x$, $v_x = 0$.
$u_y = 2y$, $v_y = 0$.

$\left\{\begin{array}{l} u_x = v_y \\ u_y = -v_x \end{array}\right.$ only at $z = 0$.

Reminder : $f' \Rightarrow$ CRE.
If CRE are not satisfied, then is not differentiable.

Q: $v + iu$, $f = u + iv$ where u, v satisfy CRE, $\rightarrow$ NO - It can't.

<u>Properties of diff functions.</u>

1) $\frac{d}{dz}(c) = 0$, $\quad c =$ constant.

2) $\frac{d}{dz}(fc) = c\frac{df}{dz}$, $\frac{df}{dz} = f'$

3) $\frac{dz^n}{dz} = nz^{n-1}$ for any $n = 1, 2, \ldots$ (By induction)

4) $\frac{d}{dz}(f+g) = \frac{df}{dz} + \frac{dg}{dz}$

5) $\frac{d}{dz}(fg) = \frac{dg}{dz}f + \frac{df}{dz}g = fg' + f'g$.

6) $\left(\frac{f}{g}\right)' = \frac{f'g - fg'}{g^2}$.

7) $\frac{d}{dz}(f \circ g)(z) = f'(g(z))\,g'(z)$.

19/10/11.

Homomorphic functions are called <u>analytic</u>.

The function f is said to be <u>entire</u> if it analytic on $\mathbb{C}$.

<u>Example</u>: $f(z) = z^2$.

$f = u + iv$, f is diff at z_0.

Then $\begin{cases} u_x = v_y \\ u_y = -v_x \end{cases}$ CRE.

<u>Theorem 2-5</u>: Let f be holomorphic on a domain Ω.

1) Assume that $f'(z) = 0$ for all $z \in \Omega$. The $f(z) = \text{const}$ for all $z \in \Omega$.

2) Suppose that $|f|$ is constant on Ω. The f is constant on Ω.

<u>Proof</u>: Write CRE for $f = u + iv$.

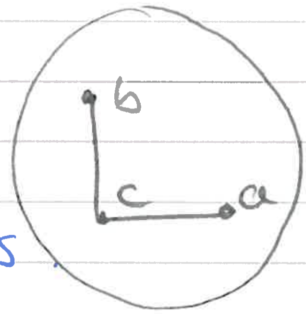
$$(*) \begin{cases} u_x = v_y \\ u_y = -v_x \end{cases}$$

From $f' = u_x + iv_x = 0$ we conclude: $u_x = v_x = 0$ and due to $(*)$, $u_y = v_y = 0$.

Suppose first that $\Omega = D(z_0, r)$, $r > 0$, $z_0 \in \mathbb{C}$.

Let $a, b \in D(z_0, r)$. Want: $f(a) = f(b)$.

Observe: a and b can be joined by a polygonal path, which consists of two segment, parallel to the coordinate axes.



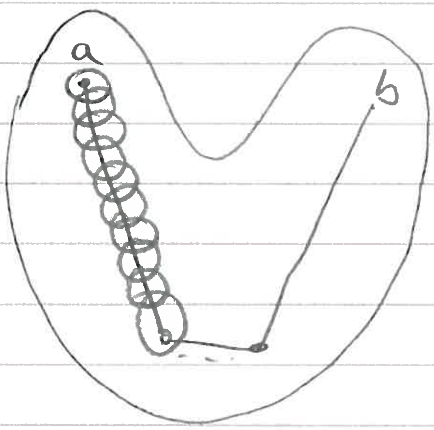
On $[a, c]$ we use $u_x = v_x = 0$, so u, v are constant $\Rightarrow$ $\underline{f(a) = f(c)}$

On $[c, b]$ we use $u_y = v_y = 0$ so, u, v are constant $\Rightarrow$ $\underline{f(c) = f(b)}$

Thus $f(a) = f(b)$ as required.

Let Ω be an arbitrery domain, i.e. conneted and open set. Thus we can join any $a, b \in \Omega$ with a polygonal path

Cover the path with open disk of a suitable radius, $r > 0$.



<u>Priestly</u> (Book for proof).

In every disk f is constant. Due to the overlap, these constants are the same.

Therefore $f(a) = f(b)$ i.e. f is a constant on Ω.

2) Proof: Assume $|f| = c > 0$.
If $c = 0 \Rightarrow |f|^2 = u^2 + v^2 = 0$ and $u = v = 0$.

Let $c > 0$. Then write $\underline{u^2 + v^2 = c^2}$

Differentiate w.r.t x: $2u_x u + 2v_x v = 0$
w.r.t y: $2u_y u + 2v_y v = 0$

By CRE: $\begin{cases} u_x u - u_y v = 0. \\ u_y u + u_x v = 0. \end{cases}$

Multiply: line 1 by u: $\begin{cases} u_x u^2 - u_y v u = 0 \\ u_y u v + u_x v^2 = 0. \end{cases}$
line 2 by v:

Add up: $u_x u^2 + u_x v^2 = 0$.
$\Leftrightarrow u_x(u^2 + v^2) = c^2 u_x = 0$.

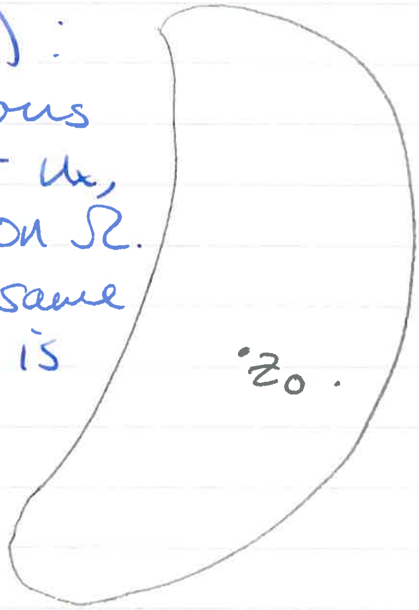
As $c \neq 0$ (IMPORANT IN EXAM: ASK EVERY YEAR).

As $c \neq 0$, we have $u_x = 0$.

In the same way $u_y = 0$. Therefore by CRE $v_x = v_y = 0 \Rightarrow f'(z) = u_y + i v_x \Rightarrow f$ const by part ①. $\square$.

Recall: f is diff $\Rightarrow$ CRE.

Theorem 2.6 (Not proven): Let $f = u + iv$ be continuous on a domain Ω, and let u_x, v_x, u_y, v_y be continuous on Ω. If u, v satisfy CRE at some point $z_0 \in \Omega$, then f is differentiable at z_0.



Example 2.7: Let

$$f(z) = e^x(\cos y + i \operatorname{sen} y), \quad z = x + iy.$$

The real part $u(x,y) = e^x \cos y$ and imaginary part $v(x,y) = e^x \operatorname{sen} y$ are continuous on $\mathbb{C}$ and u_x, u_y, v_x, v_y exist and continuous and are continuous on $\mathbb{C}$:

$$u_x = e^x \cos y, \quad v_x = e^x \operatorname{sen} y$$
$$u_y = -e^x \operatorname{sen} y, \quad v_y = e^x \cos y.$$

CRE hold for all x, y: $u_x = v_y$, $u_y = -v_x$.

By Thm 2.6: f is analytic on $\mathbb{C}$. i.e entire

Moreover: $f' = u_x + i v_x = u + iv = f$ i.e. $\underline{f' = f}$

This is why we denote $f(z) = \exp(z) = e^z$

Remark : Define :

$$f_z = \frac{\partial f}{\partial z} = \frac{1}{2}\left[\frac{\partial f}{\partial x} - i\frac{\partial f}{\partial y}\right], \quad f_{\bar z} = \frac{\partial f}{\partial \bar z} = \frac{1}{2}\left[\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y}\right]$$

Assume that f is analytic i.e. $u_x = v_y$, $u_y = -v_x$

Find $f_{\bar z}$ in terms of u, v:

$$f_{\bar z} = \frac{1}{2}\left[u_x + iv_x + i(u_y + iv_y)\right]$$

$$= \frac{1}{2}\left[u_x - v_y + i(u_y + v_x)\right] = 0.$$

$u_x = v_y$ $\quad$ $u_y = -v_x$

This "means" that f doesn't depend on $\bar z$. To find out if $f = f(x, y)$ is diff, rewrite it as a function of $z, \bar z$, using $x = (z + \bar z)/2$, $y = (z - \bar z)/2i$.

Assuming that f is analytic, what is $\partial f / \partial z$

$$\frac{\partial f}{\partial z} = f' \ldots$$

21/10/11

Functcoin:

Maps defined on sets of complex plane $\mathbb{C}$ with values in $\mathbb{C}$.

Need to know;

1) The set where f is defined called domain of f, $D(f)$.

2) The mapping itself.

Example

1) $f(z) = z^2$, $D(f) = \mathbb{C}$.

$f(x+iy) = x^2 + y^2 + 2xyi$.

In general, for any functcoin $g : D(g) \to \mathbb{C}$, we write.

$g(z) = u(x,y) + v(x,y)$

so $u = \mathrm{Re}\, g$, $v = \mathrm{Im}\, g$.

2) $h(z) = \frac{1}{z}$, $D(h) = \mathbb{C} \setminus \{0\}$.

or $D(h) = D(5,3)$.

3) $w(z) = \sin z + i \cos y \quad , z \in \mathbb{C}$.

4) $P(z) = a_n z^n + a_{n-1} z^{n-1} + \dots + a_1 z + a_0$.
where $a_0, a_1, \dots, a_n$ are fixed complex, z is the variable.

If $a_n \neq 0$ then $P(z)$ is called polynomial of degree n.

For any two polynomial P, Q, the function

$$M(z) = \frac{P(z)}{Q(z)}$$

is called ratcorial.

Observe $D(P) = \mathbb{C}$,

$$D(M) = \mathbb{C} \setminus \{\text{roots of } Q(z)\}.$$

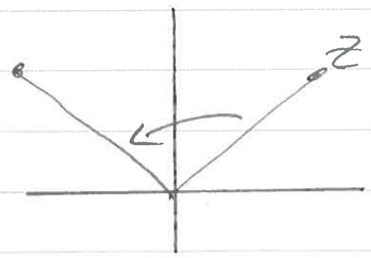
<u>Mapping properties</u>,

1) $f(z) = z - 1 \quad , \quad D(f) = \mathbb{C}$.

$\mathbb{C}$ is shifted by 1 to the left.

2) $g(z) = iz$.

$g(z) = |z| e^{i\frac{\pi}{2}} e^{i\theta} = |z| e^{i(\theta + \frac{\pi}{2})}$.

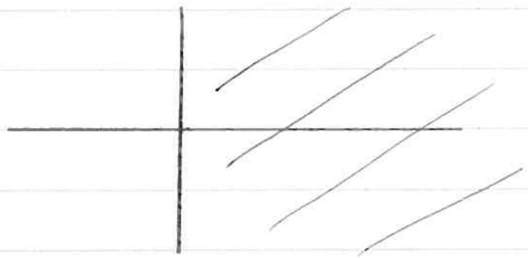


Rotation by $\pi/2$ counterclockwise.

3) $g(z) = \dfrac{z}{|z|}$, $z \in \mathbb{C} \setminus \{0\}$.

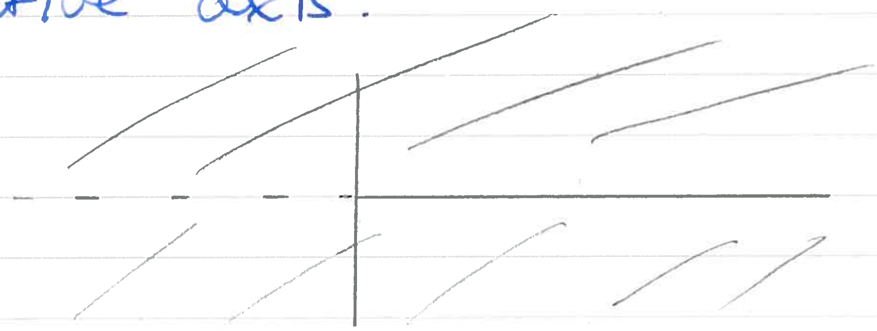
What's is the image of $g = \{$set of values$\}$?

4) Let $D(h) = \{z : \operatorname{Re} z > 0\}$.
and $h(z) = |z|^2 e^{2i\theta}$, $\theta = \arg z$.



What is image of h?

Image $= \mathbb{C}$ with a cut along the negative axis.



More precisely, Image $= \{z \in \mathbb{C}\} \setminus \{w : \operatorname{Re} w \leq 0, \operatorname{Im} w = 0\}$

Limits of functions

Definition 1.26: Let $f : S' \to \mathbb{C}$ be a function and $z_0 \in \mathbb{C}$. Then we say that f has a limit at z_0, denoted

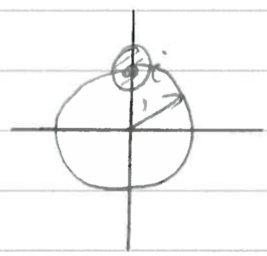
$$\lim_{z \to z_0} f(z) = \omega_0 .$$

if for any $\varepsilon > 0$ there is a $\delta > 0$ st $|f(z) - \omega_0| < \varepsilon$ as soon as $z \in D'(z_0, \delta) \cap S'$.

Examples

1) $S' = D(0,1)$, $z = i$, $f(z) = z$.

$$\lim_{z \to z_0} f(z) = z_0 = i .$$



$$\lim_{z \to 2} f(z) = \text{doesn't make sense} .$$

2) $S' = D(0,1)$, $z_0 = 0$.

$$h(z) = \begin{cases} 1, & z = 0 \\ z, & z \neq 0 . \end{cases}$$

$$\lim_{z \to 0} h(z) = 0 .$$

Properties.

1) If $\lim_{z \to z_0} f$ exist, it is unique.

2) If $\lim_{z \to z_0} f = w$, then.

$$\lim_{z \to z_0} \operatorname{Re} f = \operatorname{Re} w$$

$$\lim_{z \to z_0} \operatorname{Im} f = \operatorname{Im} w.$$

$$\lim_{z \to z_0} \overline{f} = \overline{w_0}$$

$$\lim_{z \to z_0} |f| = |w|.$$

3) AOL is applicable.

Observe: $z \in \mathbb{C}$.

$$\lim_{z \to z_0} z^n = z_0^n.$$

Thus, by AOL $\lim_{z \to z_0} P(z) = P(z_0)$, for any polynomial. This means that P is continuous on $\mathbb{C}$.

Infinite limit and limits at infinite.

Definitions 1.27

1) We say that $\lim_{z \to \infty} f(z) = w$, if for any $\varepsilon > 0$ there is a number A st $|f(z) - w| < \varepsilon$ as soon as $|z| > A$.

2) We say that $\lim_{z \to z_0} f(z) = \infty$ if for

any number $M>0$ there is a $\delta>0$ st $|f(z)|>M$ as soon as $z\in D'(z_0,\delta)\cap D(f)$

Example.

1) $\lim\limits_{z\to\infty}\dfrac{1}{z^2+2}=0.$

2) $\lim\limits_{z\to\infty}\dfrac{z^2}{z^2+2}=1.$

3) $\lim\limits_{z\to i}\dfrac{1}{z-i}=\infty.$

4) $\lim\limits_{z\to 2}\left(\dfrac{1}{z-i}\right)=\dfrac{1}{2-i}$

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3. Complex Series.

Let a_k, $k=0,1,\ldots$ be a complex sequence. Then the formal sum $\sum_{k=0}^{\infty} a_k$ is called a complex series.

Define $S_n = \sum_{k=0}^{n} a_k$ for finite n, "Partial sums".

If S_n converages as $n \to \infty$, we say that the series $\sum_{k=0}^{\infty} a_k$ converages. So by definition

$$\sum_{k=0}^{\infty} a_k = \lim_{n\to\infty} \sum_{k=0}^{n} a_k .$$

if the limit exists.

Properties.

1) If the series converages then $a_k \to 0$, $k \to \infty$

· Eg: $\sum_{k=0}^{\infty} (-1)^k$ does not converage.

· Eg: $\sum_{k=0}^{\infty} e^{ik\theta}$, $\theta \in (-\pi, \pi]$ does not converage

As a consequence, $\{a_k\}$ is a bdd sequence.

2) If $\sum a_k$ and $\sum b_k$ are convergent then $\sum (a_k + A b_k)$ converage as well for any complex A.

3) We say that $\sum a_k$ converage absolutely if $\sum |a_k|$ converages.

If the series absolutely, it is convergent.

Example: $\sum (-1)^n/n$ converages but $\sum 1/n$ diverages.

Proposition 3.1: (Comparsion test) Let $\sum a_k$ be a complex series and let $\sum b_k$ be a series of non-negative b_k and with real terms. Assume that for some number $M > 0$ we have $|a_k| \leq M b_k$ for all k. Then if $\sum b_k$ converages, then $\sum a_k$ converages absolulety

Write: $\sum_{n=0}^{\infty} a_k = a_0 + a_1 + a_2 + \dots$

If one deletes finitely many terms this doesn't affect convergence.

Proposition 3.2 (Root test): Let $\sum a_k$ be a series, suppose that

$$\lim_{k \to \infty} \frac{|a_{k+1}|}{|a_k|} = l.$$

exist with some $l \geq 0$. Then if $l < 1$ then the series converages absolutely. If $l > 1$ it diverages.

Example: $\sum_{n=1}^{\infty} 1/n^2 \Rightarrow l = 1$.

but it converage

$$\sum 1/n^\alpha \Rightarrow \begin{cases} \alpha > 1 - \text{converages} \\ \alpha \leq 1 - \text{diverages}. \end{cases}$$

Proposition 3.3 (Root test): Let $\sum a_k$ be a series. Assume that

$$\lim_{k\to\infty} |a_k|^{1/k} = r.$$

exists, $r \geq 0$.

If $r < 1$, then the series converages absolutel[y] and if $r > 1$, then it diverages.

Example: $\sum_{k=0}^{\infty} z^k$ geometrical series, her[e] $z \in \mathbb{C}$.

For which values of z does it converage?

Recall: $\sum_{k=0}^{\infty} z^k = \frac{1}{1-z}$, $|z| < 1$

Ratio test: $\frac{|z^{k+1}|}{|z_k|} = |z| \to |z|$ As $k \to \infty$

—/—

Power series:
Power series is this:

$$\sum_{k=0}^{\infty} a_k (z - z_0)^k \quad (*)$$

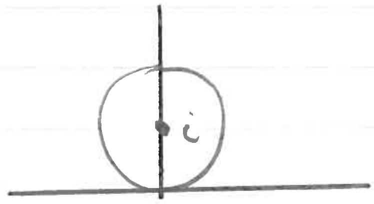
where a_k, $k = 0, 1, 2, \ldots$ are fixed complex numbers, and $z_0 \in \mathbb{C}$ is also fixed.

The function f depends on the variable $z \in \mathbb{C}$.

Example: $\sum_{k=0}^{\infty} (z-i)^k$

For which values of z does this series converage?

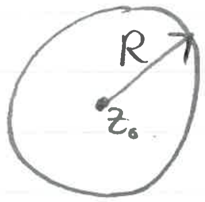
Answer: $|z-1|<1$, i.e. $z \in D(i;1)$.



If $|z-i|>1$ then it diverages.

Def^n 3.4: The radius of convergence of $(*)$ is defined to be:

$$R = \sup\left\{ |z| : \sum_{k=0}^{\infty} |a_k z^k| \text{ converages} \right\}.$$



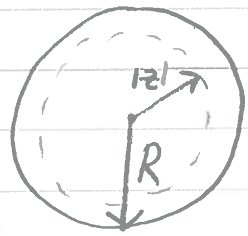
Lemma 3.5: Let R be the radius of converagence of $(*)$. then:

1) If $|z-z_0|<R$, then the series converage absolutely.

2) If $R<\infty$, and $|z-z_0|>R$, then the series diverages.

Proof: Assume $z_0=0$. Suppose that $|z|<R$

Pick a number w: $|z|<|w|<R$ the series (*) at w converages absolutly i.e $\sum_{k=0}^{\infty} |a_k||w|^k$ converages. This is possible due to def 3.4.



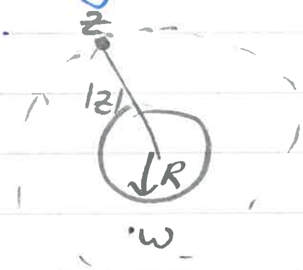
Since $|z|<|w|$ we have:

$$|a_k||z|^k \le |a_k||w|^k$$

By the comparison test, $\sum |a_k||z|^k$ converages as required.

2) Suppose that $\sum a_k z^k$ converages and $|z|>R$.

Pick a w: $R<|w|<|z|$. Want to show $\sum |a_k w^k|$ converages. Indeed; $a_k z^k$ is a bounded sequence by property (c) so $|a_k z^k| \le M$ with some $M>0$.



Thus: $|a_k w^k| = |a_k||z^k|\left|\frac{w}{z}\right|^k$

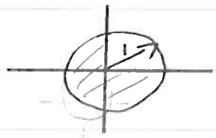
$$\le M\left|\frac{w}{z}\right|^k$$

Thus series $\sum \left|\frac{w}{z}\right|^k$ converages if $|w|<|z|$
$\Rightarrow$ Comparsion principle $\sum |a_k w^k|$ converages
This contradicts def^n 3.4 $\Rightarrow$ Part 2 is proved.

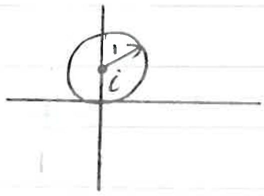
□

Example:

1) $\sum z^k$, $R=1$.



$\sum (z-i)^k$, $R=1$



2) $\sum_{k=0}^{\infty} k^{10}(3z)^k$

Ratio test:

$$\frac{(k+1)^{10}|3z|^{k+1}}{k^{10}|3z|^k} = \left(1+\frac{1}{k}\right)^{10}|3z| \underset{k\to\infty}{\to} |3z|$$

By Ratio test: $\begin{cases} |3z|<1 \Rightarrow \text{convergence.} \\ |3z|>1 \Rightarrow \text{divergence.} \end{cases}$

or $|z| = \begin{cases} < 1/3 \Rightarrow \text{conv} \\ > 1/3 \Rightarrow \text{div.} \end{cases}$

$\Rightarrow$ Radius of convergence $= 1/3$.

3) $\sum_{n=0}^{\infty} n^{150}/n! \; z^n$.

Ratio test:

$$\frac{(n+1)^{150}|z|^{n+1} n!}{(n+1)!\; n^{150}|z|^n} = \left(1+\frac{1}{n}\right)^{150}\frac{1}{n+1}|z| \underset{n\to\infty}{\to} 0.$$

Since $0<1$, the series converges for all $z \in \mathbb{C}$, i.e $R=\infty$.

Remark: $\sum z^k/k!$ is defined for all $z \in \mathbb{C}$. It is called the exponential function.

Notation: $\exp(z)$.

<u>Differentiability of power series.</u>

Again:

$$f(z) = \sum_{k=0}^{\infty} a_k (z - z_0)^k \quad (*).$$

Compare f with:

$$g(z) = \sum_{k=0}^{\infty} a_k k (z - z_0)^{k-1} \quad (**)$$

<u>Lemma 3.6</u>: The series $(*)$ and $(**)$ have the same radius of convergence.

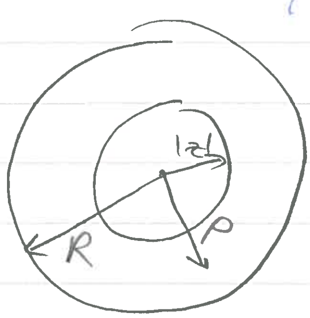
<u>Proof</u>: Let R_1, R_2 be the radii of convergence for $(*)$ and $(**)$.

Let's prove that $R_1 \leq R_2$ i.e assuming that $\sum a_k (z - z_0)^k$ converages absolutely we'll show that $\sum k |a_k| |z - z_0|^{k-1}$ converage as well. Assume $z_0 = 0$.

Pick a number $\rho > 0$ st $|z| < \rho < R_1$

Write:

$$k |a_k| |z|^{k-1} = \frac{k}{|z|} \left|\frac{z}{\rho}\right|^k |a_k| \rho^k$$



Observe, the series:

$\sum k |z/\rho|^k$ converages since $|z| < \rho$.

The series $\sum |a_k|\rho^k$ converges since $\rho < R_1$ so $|a_k \rho^k|$ is a bounded sequence i.e. $|a_k \rho^k| \leq C'$ for some constant $C' > 0$ and hence $k|a_k||z|^{k-1} \leq C'(k/|z|)|z/\rho|^k$, and therefore by comparison test, $\sum k|a_k||z|^{k-1}$ converges. Thus $R_1 \leq R_2$.

Let's show that $R_2 \leq R_1$ i.e. if $\sum k|a_k||z|^{k-1}$ converges then $\sum |a_k||z|^k$ converges too.

Write: $|a_k||z|^k \leq |z|(k|a_k||z|^{k-1})$ for all $k \geq 1$

By the comparison test $\sum |a_k||z|^k$ converges $\Rightarrow R_1 = R_2$.

Denote $R = R_1 = R_2$

By lemma 3.6 the series

$$\sum_{k=1}^{\infty} a_k k(k-1)(z-z_0)^{k-2}$$

has the same radius of convergence.

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$$f(z) = \sum_{k=0}^{\infty} a_k (z - z_0)^k \quad (*)$$

R_1
$\|$
R_2
$\|$
R

$$g(z) = \sum_{k=0}^{\infty} k a_k (z - z_0)^k \quad (**)$$

Remark: $\sum_{k=2}^{\infty} k(k-1) a_k (z - z_0)^{k-2}$ has the same radius of convergence $(*)$ and $(**)$.

The study $f'(z)$ we need to look at

$$\frac{f(z+h) - f(z)}{h} \quad \text{as } h \to 0.$$

In order words, need to investigate:

$$\frac{(z+h-z_0)^k - (z-z_0)^k}{h} \quad \text{as } h \to 0.$$

Important:
Lemma 3.7: Let $z, h \in \mathbb{C}$ and $n \geq 2$. Then

$$\left| \frac{(z+h)^n - z^n}{h} - n z^{n-1} \right| \leq \frac{n(n-1)}{2} |h| (|z| + |h|)^{n-2}$$

Theorem 3.8: Let $R > 0$ be the radius of convergence of $(*)$. Then $f \in H(D(z_0, R))$, the series $(**)$ converages within the same radius, and $f'(z) = g(z)$ for all $z \in D(z_0, R)$.

<u>Proof</u>: Need to show that

$$\frac{f(z+h)-f(z)}{h} - g(z) \to 0 \quad \text{as } h \to 0.$$

We'll show: $\left|\frac{f(z+h)-f(z)}{h} - g(z)\right| \leq C'|h|$

with some constant $C'>0$ independent of h.

We'll show this: for any N

$$\left|\frac{1}{h}\sum_{k=0}^{N}\left[a_k(z+h)^k - a_k z^k\right] - \sum_{k=0}^{N} k a_k z^{k-1}\right| \leq C|h|$$

with a constant $C>0$ independent of N, h.

Rewrite:

$$\left|\sum_{k=0}^{N} a_k\left[\frac{(z+h)^k - z^k}{h} - k z^{k-1}\right]\right|$$

$$\leq \sum_{k=0}^{N} |a_k| \left|\frac{(z+h)^k - z^k}{h} - k z^{k-1}\right|$$

$$\leq \frac{|h|}{2}\sum_{k=0}^{\infty} |a_k| k(k-1)\left[|z|+|h|\right]^{k-2}$$

By lemma 3.7.

Due to a remark made earlier, the series $\sum_{k=0}^{\infty} k(k-1)|a_k|(|z|+|h|)^{k-2}$ converges and hence the right-hand side is bounded by $C|h|$ with:

$$C = \frac{1}{2}\sum_{k=0}^{\infty} k(k-1)|a_k|(|z_0|+|h_0|)^{k-2}$$

where h_0 is st $|h_0| = (R-|z|)/2$.

Thus $f'(z) = g(z)$ as required $\square$.

Note: In the proof we assume without loss of generality that $z_0 = 0$ or "WOLG".
("wlog")

Corollary 3.9: The power series (*) is differentiable any number of times in the disk $D(z_0, R)$. Moreover,

$$f'(z) = a_1 + 2a_2(z-z_0) + 3a_3(z-z_0)^2 + \dots$$
$$\therefore f'(z_0) = a_1$$
$$f''(z_0) = 2a_2$$
$$f'''(z_0) = 6a_3$$
$$\vdots$$
$$f^n(z_0) = n!a_n$$

Therefore: $f(z) = \sum_{k=0}^{\infty} \frac{f^{(k)}(z_0)(z-z_0)^k}{k!}$

Taylor's series

Exponential and trigonometric functions.

Define: $\exp(z) = \sum_{k=0}^{\infty} \frac{z^k}{k!} = e^z$.

The radius of convergence $= \infty \Rightarrow$ by theorem 3.8 $\exp(z)$ is an entire function.

Theorem 3.10 (Properties of e^z).

1) $(e^z)' = e^z$
2) $e^0 = 1$
3) $e^{z+w} = e^z e^w \quad \forall z, w \in \mathbb{C}$.
4) $e^z \neq 0$ for all $z \in \mathbb{C}$.

Proof 1) By Theorem 3.8

reindex

$$(e^z)' = \sum_{k=0}^{\infty} \frac{kz^{k-1}}{k!} = \sum_{k=1}^{\infty} \frac{kz^{k-1}}{k!}$$

$$= \sum_{k=1}^{\infty} \frac{z^{k-1}}{(k+1)!}$$

$$= \sum_{k=0}^{\infty} \frac{z^n}{n!} = e^z.$$

2) $e^0 = 1$ - easy!!!

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Exponentenials and tri functions

Defenition:

Entire function

$$e^z = \exp(z) = \sum_{k=0}^{\infty} \frac{z^k}{k!} \quad , R = \infty.$$

Theorem 3.10

1) $(e^z)' = e^z$
2) $e^0 = 1$
3) $e^z e^w = e^{z+w} \quad \forall z, w \in \mathbb{C}.$
4) $e^z \neq 0 \quad , z \in \mathbb{C}.$

Proof: ① and ② are done.

3) Define the following functions.

$$f(z) = e^{p-z} e^z \quad , p \in \mathbb{C}.$$

Diff: $f'(z) = -e^{p-z}e^z + e^{p-z}e^z = 0$, so by theorem 2.5; $f(z) = $ const for all $z \in \mathbb{C}$

$f(z) = f(0) = e^p$, and hence $e^{p-z}e^z = e^p$
Now $p = w + z$ so $e^w e^z = e^{w+z}$.

4) By part ③ $e^z e^{-z} = 1$ and thus $e^z \neq 0$
$z \in \mathbb{C}$ □.

Corollary 3.11: Let f be entire and let $f'(z) = f(z)$, and $f(0) = 1$. Then $f(z) = e^z$.

Proof: Let $g(z) = e^{-z}f(z)$.

Diff; $g(z) = -e^{-z}f(z) + e^{-z}f'(z)$
$= -e^{-z}f(z) + e^{-z}f(z) = 0$.

and hence by theorem 2.5 $g(z) = \text{const}$, $\forall z \in \mathbb{C}$. Thus $g(z) = g(0) = f(0) = 1 \Rightarrow$ $e^{-z}f(z) = 1 \Rightarrow f(z) = e^{z}$. $\square$.

Def: $e^{x} = \sum_{k=0}^{\infty} \frac{1}{k!} z^{k}$.

Recall Example 2.7:

$e^{z} = e^{x}(\cos y + i \sin y)$, $z = x + iy$.

Denote: $f(z) = e^{x}(\cos y + i \sin y)$. By Ex 2.7 $f'(z) = f(z)$, $f(0) = 1$ and then $f(z) = e^{z}$. Thus these two definitions give the same expontenial functions.

—/—

Define:

$\sinh z = \frac{e^{z} - e^{-z}}{2}$, $\cosh z = \frac{e^{z} + e^{-z}}{2}$

$\sin z = \frac{e^{iz} - e^{-iz}}{2i}$, $\cos z = \frac{e^{iz} + e^{-iz}}{2}$

Theorem 3.13:

$$\frac{d}{dz}(\sinh z) = \cosh z, \quad \frac{d}{dz}(\cosh z) = \sinh z.$$

usual identites for tri functions:

$$\frac{d}{dz}(\sin z) = \cos z, \quad \frac{d}{dz}(\cos z) = -\sin z.$$

(*) $\cos(z+w) = \cos z \cos w - \sin z \sin w$
$\sin(z+w) = \sin z \cos w + \cos z \sin w.$
$\sin^2 z + \cos^2 z = 1.$

WRONG! $|\sin z| \leq 1,$

So its $|\sin z| \nleq 1$, $z \in \mathbb{C}$. Note: Take $z = it$, $t \in \mathbb{R}$. Then

$$\sin(it) = \frac{e^{-t} + e^{t}}{2i} = i\sinh(t).$$

Series expansion:

$$\sinh z = \sum_{k=0}^{\infty} \frac{z^{2k+1}}{(2k+1)!}, \quad \cosh z = \sum_{k=0}^{\infty} \frac{z^{2k}}{(2k)!}$$

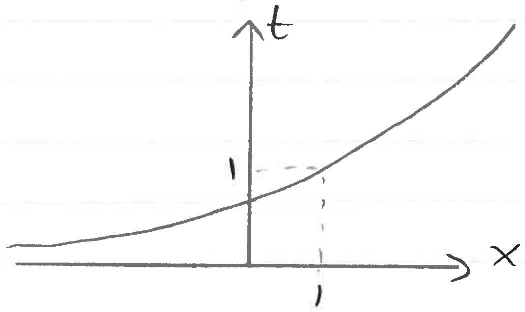
(*) $\sin z = \sum_{k=0}^{\infty} \frac{(-1) z^{2k+1}}{(2k+1)!}, \quad \cos z = \sum_{k=1}^{\infty} \frac{(-1)^k z^{2k}}{(2k)!}$

(*) Exercise

Logarithm.

Reminder: Real Analysis.

For any $t > 0$ the number $x = \ln t \in \mathbb{R}$. to be unique number st $e^x = t$.



Ln t is inverse of e^x

For Complex Analysis: Let us find $w \in \mathbb{C}$. st $e^w = z$ for some $z \in \mathbb{C}$. Then will define

$$w = \log z$$ (Does not work).

Represent $w = u + iv$, so.

$$z = e^{u+vi} = e^u e^{iv}$$ ← Polar representation of z

Thus $|z| = e^u$, $v = \arg z$ and therefore $u = \ln |z|$

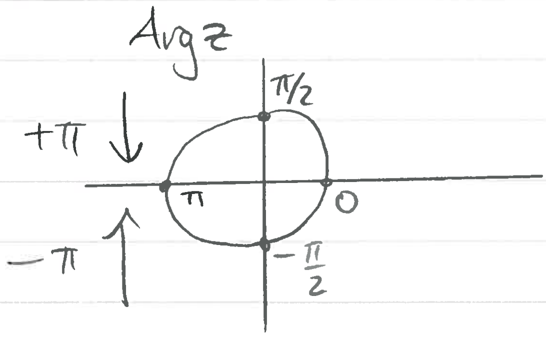
Problem: $\arg z$ is not uniquely defined!

To fix, take $\operatorname{Arg} z \in (-\pi, \pi]$ and define the principal Logarithm.

$$\boxed{\operatorname{Log} z = \ln |z| + i \operatorname{Arg}(z)}$$

For the other values of the arg z define the following BRANCHES.

$$\boxed{\mathrm{Log}_n z = \ln|z| + i(\mathrm{Arg}\, z + 2\pi n), \quad n \in \mathbb{Z}}$$



<u>Theorem 3.14</u> (Examinable) Log z is analytic. Moreover:

$$\frac{d}{dz}(\mathrm{Log}\, z) = \frac{1}{z}$$

<u>Powers</u>: Already know z^n, $n \in \mathbb{Z}$ and z^α, $\alpha \in \mathbb{R}$.

Assume that $\alpha \in \mathbb{C}$.

$$z^\alpha = e^{\alpha \log z}$$ ← have to say which branch is used.

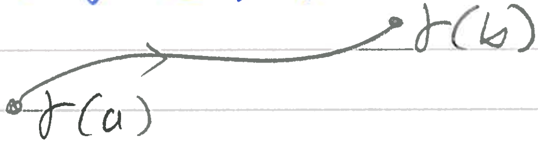
Principle value: $z^\alpha = e^{\alpha \mathrm{Log}\, z}$

<u>Example</u>: $i^i = e^{i \mathrm{Log}\, i} = e^{i(\ln|i| + i\pi/2 + i2\pi n)}$
$= e^{-\frac{\pi}{2} - 2\pi n}, \quad n \in \mathbb{Z}.$

Chapter 4: Contour Integration and Cauchy Theorem

Curves, paths, contours:

Definition 4.1 Let $[a,b]$ be an interval of $\mathbb{R}$. Then a continuous function $\gamma : [a,b] \to \mathbb{C}$ is a curve.



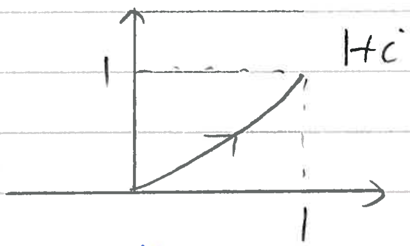
The image:

$$\gamma^* = \{ z \in \mathbb{C} : z = \gamma(t) \text{ for some } t \in [a,b] \}.$$

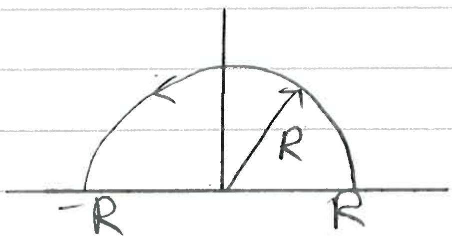
We say that γ parametries γ^* a curve has a natural orientation.
$\gamma(a)$ is the initial point
$\gamma(b)$ is the final point.

Examples:

1) $\gamma(t) = t + it^2$, $t \in [0,1]$

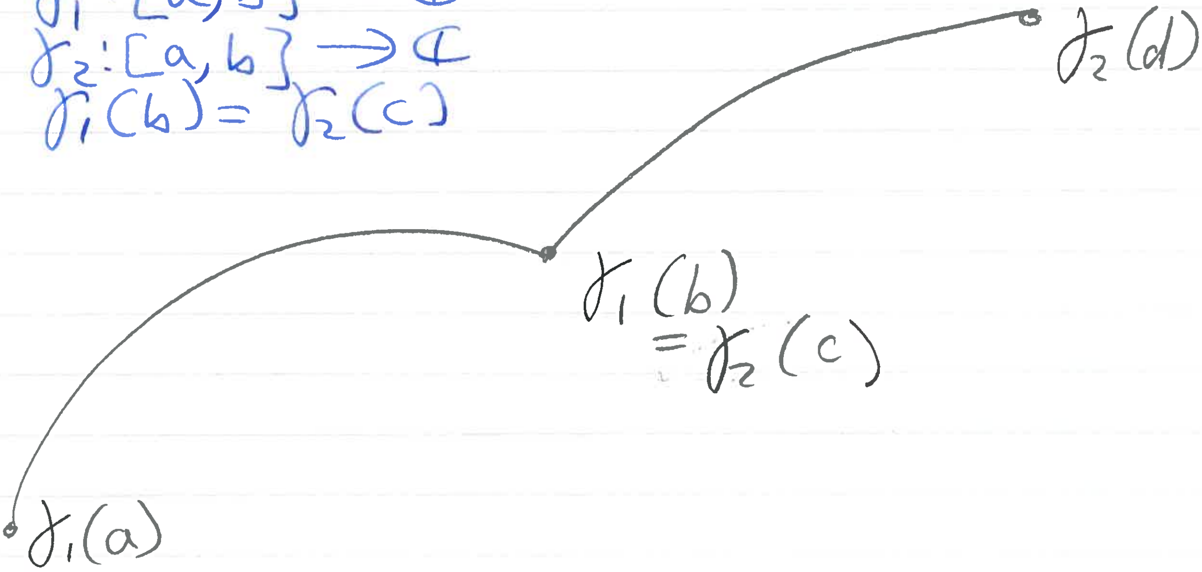


2) $\gamma(t) = Re^{it}$, $R > 0$, $t \in [0, \pi]$.



A different parametrisation
$w(t) = Re^{it^2}$, $t \in [0, \pi]$

Let $\gamma_1 : [a,b] \to \mathbb{C}$
$\gamma_2 : [a,b] \to \mathbb{C}$
and $\gamma_1(b) = \gamma_2(c)$



Join of two curves: $\gamma : [a, b+d-c]$

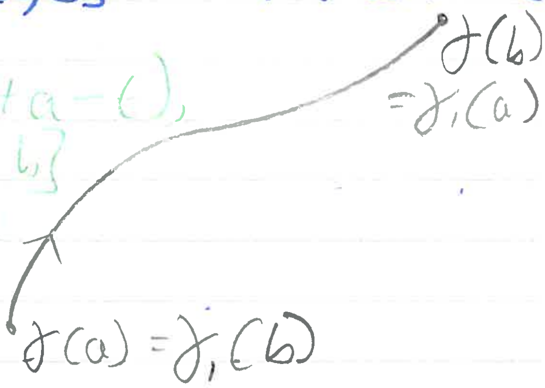
$$\gamma(t) = \begin{cases} \gamma_1(t), & t \in [a,b] \\ \gamma_2(t+c-b) & t \in [b, b+d-c] \end{cases}$$

Notation: $\gamma_1 \cup \gamma_2$.

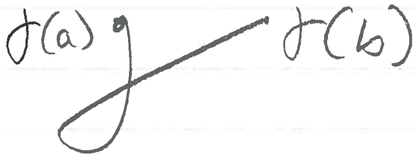
Reverse curve: Let $\gamma : [a,b] \to \mathbb{C}$. Then the curve is called the reverse curve

$\gamma_1(t) = \gamma(b+a-t)$, $t \in [a,b]$

$\gamma_1(a) = \gamma(b)$,
$\gamma_1(b) = \gamma(a)$



Definition 4.2: Let $\gamma : [a,b] \to \mathbb{C}$ be a curve. Then:
1) γ is called simple if γ^* doesn't have self-intersection, i.e. $\gamma(t_1) = \gamma(t_2)$ if $t_1 \neq t_2$ and $|t_1 - t_2| < b-a$.



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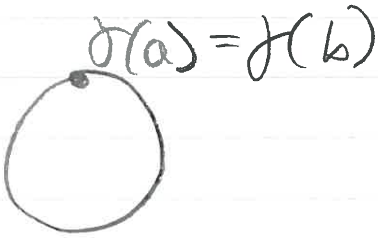
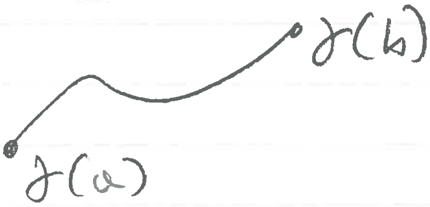
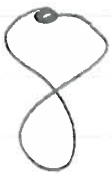


Allowed



Allowed

2) γ is closed if $\gamma(a) = \gamma(b)$

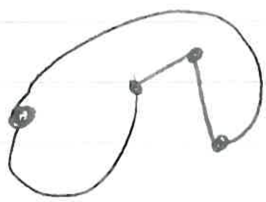


3) γ is closed simple if it is closed and simple
4) γ is smooth if $\gamma'(t)$ exist on (a, b)
here $\gamma'(t) = \gamma_1'(t) + i\,\gamma_2'(t)$ where
$\gamma_1(t) = \mathrm{Re}\,\gamma(t)$
$\gamma_2(t) = \mathrm{Im}(\ t)$.
5) We say that γ is a path if γ is piece-wise smooth i.e γ is a join of finitely smooth curve.

Smooth

Path.

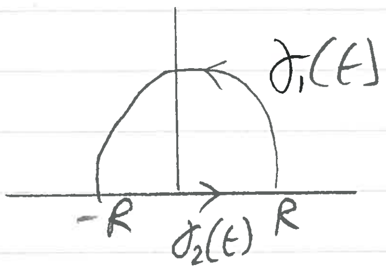
6) A contour is closed simple path.



Example: 1) Let:

$$\gamma_1(t) = Re^{it} \quad t \in [0, \pi]$$
$$\gamma_1'(t) = Ri\,Re^{it}$$

$$\gamma_2(t) = t \quad , \quad t \in [-R, R]$$
$$\gamma_2'(t) = 1.$$



$\gamma = \gamma_1 \cup \gamma_2$ is a contour

γ_1, γ_2 are smooth curve.

<u>Theorem 4.3</u>: (Jordan curve theorem) Let γ be a contour. The complement of γ^* is the union of two open sets, denoted $\text{Int}\,\gamma$ and $\text{Ext}\,\gamma$, where $\text{Int}\,\gamma$ is bounded and $\text{Ext}\,\gamma$ is unbounded, so

$$\mathbb{C} = \gamma^* \cup \text{Int}\,\gamma \cup \text{Ext}\,\gamma.$$

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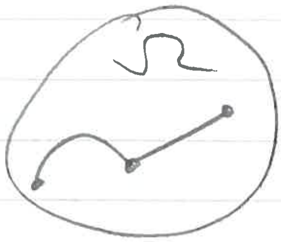
Integration

Let $F(t) = A(t) + iB(t)$ be a complex-valued function with real-valued $A(t)$ and $B(t)$. Then by definition:

$$\int_a^b F(t)\,dt = \int_a^b A(t)\,dt + i\int_a^b B(t)\,dt$$

Definition 4.4: Let f be defined on some domain $\Omega \subset \mathbb{C}$, and let $\gamma : [a,b] \to \Omega$ be a path. Then we define the integral of f along γ:

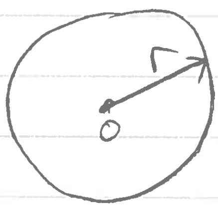
$$\int_\gamma f(z)\,dz = \int_a^b f(\gamma(t))\gamma'(t)\,dt$$



Example:

Let $\gamma(t) = re^{it}$ $t \in [0, 2\pi]$

Let $f(z) = z^n$, $n \in \mathbb{Z}$.



Find $I_n = \int_\gamma z^n\,dz$

$$I_n = \int_0^{2\pi} (re^{it})^n ire^{it}\,dt$$

$$= ir^{n+1}\int_0^{2\pi} e^{i(n+1)t}\,dt$$

Suppose $n \neq -1$. Then:

$$I_n = ir^{n+1} \left. \frac{e^{i(n+1)t}}{i(n+1)} \right|_0^{2\pi} = 0.$$

Let $n = -1$:

$$I_{-1} = i \int_0^{2\pi} dt = 2\pi i.$$

$$I_n = \begin{cases} 0, & n \neq -1 \\ 2\pi i, & n = -1 \end{cases}$$

<u>Definition 4.5</u>: Let $\gamma : [a,b] \to \mathbb{C}$ be a path. Then the length of γ is defined to be $L(\gamma) = \int_a^b |\gamma'(t)| \, dt$

<u>Example 4.6</u>: Let $\gamma(t) = re^{it}$, $t \in [0, 2\pi]$

$$L(\gamma) = \int_0^{i\pi} |ire^{it}| \, dt.$$

$$= r \int_0^{2\pi} dt$$

$$= 2\pi r \quad \text{as expected!!!}$$

Theorem 4.7: Let γ, γ_1, γ_2 be paths

1) $\int_{\gamma} f(z)\, dz = -\int_{-\gamma} f(z)\, dz.$

2) If $\gamma = \gamma_1 \cup \gamma_2$ then

$$\int_{\gamma} f(z)\, dz = \int_{\gamma_1} f(z)\, dz + \int_{\gamma_2} f(z)\, dz$$

3) $\int_{\gamma} c f(z) dz = c \int_{\gamma} f(z)\, dz$

for any constant $c \in \mathbb{C}$.

4) $\int_{\gamma} (f_1(z) + f_2(z))\, dz = \int_{\gamma} f_1(z)\, dz + \int_{\gamma} f_2(z)\, dz$

5) The integral doesn't depend on parameterisation. Suppose that γ^* is parametrised by two functions:

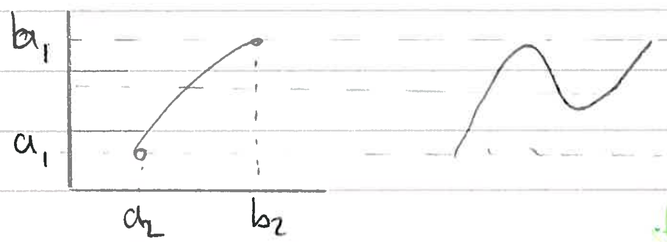
$\gamma_1 : [a_1, b_1] \to \gamma^*$ and $\gamma_2 : [a_2, b_2] \to \gamma^*$

Suppose that these is a function

$$\psi : [a_2, b_2] \to [a_1, b_1].$$

with a positive derivative $\psi'(t)$ such that $\gamma_2 = \gamma_1 \circ \psi$.

Then: $\int_{\gamma_1} f(z)\,dz = \int_{\gamma_2} f(z)\,dz$.



Can't have this, This is unique, we find $\varphi(t)$ with positive derivative

6) The length of the path doesn't depend on parametrisation.

7) Suppose that $\sup_{x \in \gamma^*} |f(z)| \leq M$. Then

$$\left| \int_\gamma f(z)\,dz \right| \leq M L(\gamma)$$ "ML – result"

<u>Proof</u>: ① – ④ as in Real Analysis:

5) Write: $\int_\gamma f(z)\,dz = \int_{a_1}^{b_1} f(\gamma_2(t))\gamma_2'(t)\,dt$

Recall: $\gamma_2(t) = \gamma_1(\varphi(t))$

$\gamma_2'(t) = \gamma_1'(\varphi(t))\varphi'(t)$ (Chain rule).

$$= \int_{a_2}^{b_2} f(\gamma_1(\varphi(t)))\gamma'(\varphi'(t))\underbrace{\varphi'(t)}_{s=\varphi(t)} \quad dt$$

$$= \int_{a_1}^{b_1} f(\gamma_1(s))\gamma_1'(s)\, ds$$

$$= \int_{\gamma_1} f(z)\, dz \quad \text{as claimed.}$$

6) Similar proof

7) $\left|\int_a^b F(t)\, dt\right| \leq \sup_{t\in[a,b]} |F(t)| (b-a)$ know

$$\left|\int_a^b F(t)\, dt\right| \leq \int_a^b |F(t)|\, dt$$

Let $I = \int_\gamma f(z)\, dz = \int_a^b (f(\gamma(t))\gamma'(t)\, dt$

Let $I = \int_\gamma f(z)\, dz = \int_a^b (f(\gamma(t))\gamma'(t)\, dt$

Write $I = |I|e^{i\theta}$ – polar form then

$$|I| = \left|\int_a^b e^{-i\theta} f(\gamma(t))\gamma'(t)\, dt\right|$$

$$= \mathrm{Re}\int_a^b f(\gamma(t))\gamma'(t)dt = \int_a^b \mathrm{Re}\,(f(\gamma(t))\gamma'(t))\, dt$$

$$\leq \int_a^b |f(\gamma(t))||\gamma'(t)|$$

$$\leq \sup_{z \in \gamma^*} |f(z)| \int_a^b |\gamma'(t)| \, dt$$

$$\leq M \, L(\gamma) \qquad \square$$

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Path integral

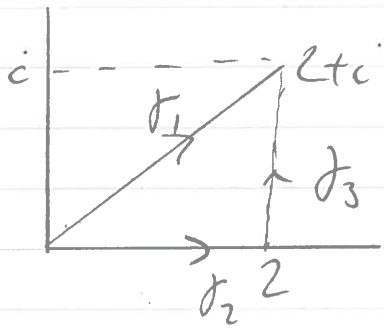
Let $\gamma : [a,b] \to \mathbb{C}$ be a path

Let f be a function then

$$\int_\gamma f(z)\, dz = \int_a^b f(\gamma(t))\gamma'(t)\, dt.$$

Theorem 4.7(7): $\left|\int_\gamma f(z)\, dz\right| \leq \sup_{x \in \gamma^*} |f(z)|\, L(\gamma)$

Example: Find $I_1 = \int_{\gamma_1} z^2\, dz$ — straight path joining 0 and $2+i$



Take: $\gamma(t) = t + \frac{1}{2}it$

$$= \frac{t(2+i)}{2}$$

where $t \in [0,2]$

Then:

$$I_1 = \int_0^2 \left(\frac{t}{2}(2+i)\right)^2 \frac{(2+i)}{2}\, dt$$

$$= \frac{(2+i)^3}{2^3} \int_0^2 t^2\, dt$$

$$= \left(\frac{2+i}{2}\right)^3 \frac{2^3}{3} = \frac{(2+i)^3}{3}$$

$$I_1 = \frac{(2+i)^3}{3}$$

Find $I_2 = \int_{\gamma_2} z^2 \, dz$

where $\gamma_2(t) = 2t$, $t \in [0,1]$

Then: $I_2 = \int_0^1 (2t)^2 \, 2 \, dt$

$$= 8 \int_0^1 t^2 \, dt$$

$$= \frac{8}{3}$$

Find $I_3 = \int_{\gamma_3} z^2 \, dz$, where $\gamma_3(t) = 2+it$
$t \in [0,1]$.

Then: $I_3 = i \int_0^1 (2+it)^2 \, dt$

$$= i \int_0^1 (4 - 4it - t^2) \, dt$$

$$= 4i - 2 - \frac{i}{3}$$

$$= -2 + \frac{11}{3} i$$

Compute: $I_2 + I_3 = \frac{8}{3} = 2 + \frac{11}{3} i = \frac{2}{3} + \frac{11}{3} i$

On the other hand

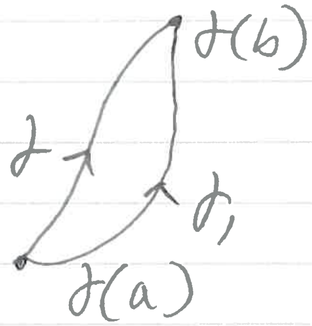
$$I_1 = \frac{1}{3}(8 - i + 12i - 6)$$
$$= \frac{1}{3}(2 + 11i) = I_2 + I_3$$

Antiderivatives (or Primitives)

Defⁿ: 4.8 - Let f be continuous on a domain Ω, and let F, be a function analytic on Ω st $F'(z) = f$. Then F is called the antiderivative (or a primitive) of f.

Theorem 4.9: (Fundamental Thm of Calculus): Let f have an antiderivative F on Ω. Let $\gamma : [a,b] \to \Omega$ be a path then:

$$\int_\gamma f(z)\,dz = F(\gamma(b)) - F(\gamma(a)).$$



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Observe: If γ_1 is another path joining $\gamma(a)$ and $\gamma(b)$ then:

$$\int_{\gamma_1} f(z)\,dz = \int_\gamma f(z)\,dz.$$

—/—

<u>Proof</u>: Write:

$$\int_\gamma f(z)\,dz = \int_a^b f(\gamma(t))\,\gamma'(t)\,dt$$
$$= \int_a^b F'(\gamma(t))\,\gamma'(t)\,dt$$

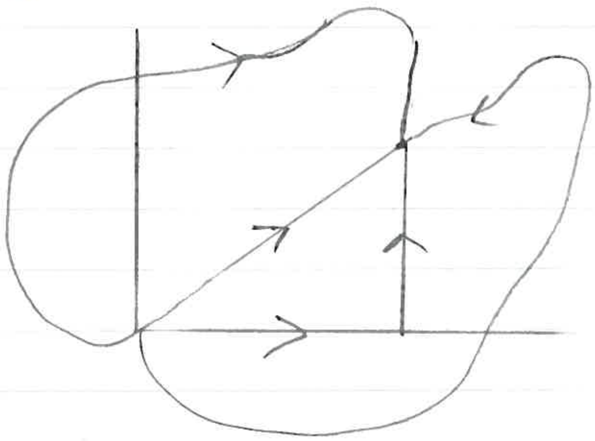
By the chain rule:

$$= \int_a^b \frac{d}{dt} F(\gamma(t))\,dt = F(\gamma(b)) - F(\gamma(a)) \quad \square.$$

Back to example: $\int_\gamma z^2\,dz$

If for z^2, then $F(z) = z^3/3$ is a primitive

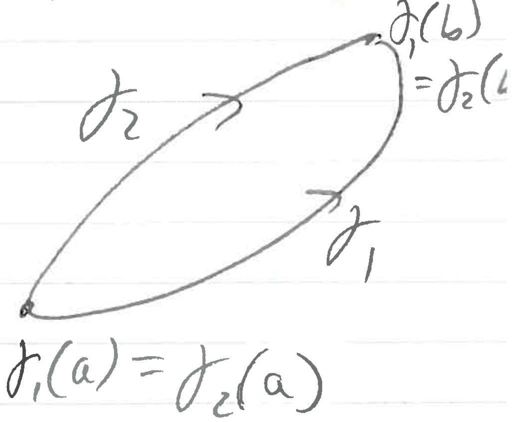
Therefore $I_1 = I_2 + I_3$.



(All paths will give the same value)

The Cauchy – Goursat Theorem

Reformulate: take path γ_1, γ_2 st $\gamma_1(a) = \gamma_2(a)$, $\gamma_1(b) = \gamma_2(b)$



Consider: $\gamma_1 \cup (-\gamma_2)$

This is a contour. Under what conditions of f

$$\int_\gamma f(z)\,dz = 0 \ ?$$

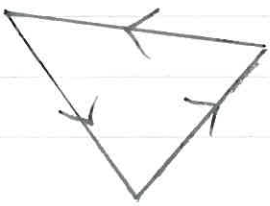
<u>Theorem 4.10</u>: (The Cauchy – Goursat Theorem)
Let $f \in H(\Omega)$ and let γ be a contour st $\text{int}\,\gamma \subset \Omega$. Then:

$$\int_\gamma f(z)\,dz = 0.$$

Plan:

1) Prove for trianglular contour

2) Extend to an arbitiary contour γ.

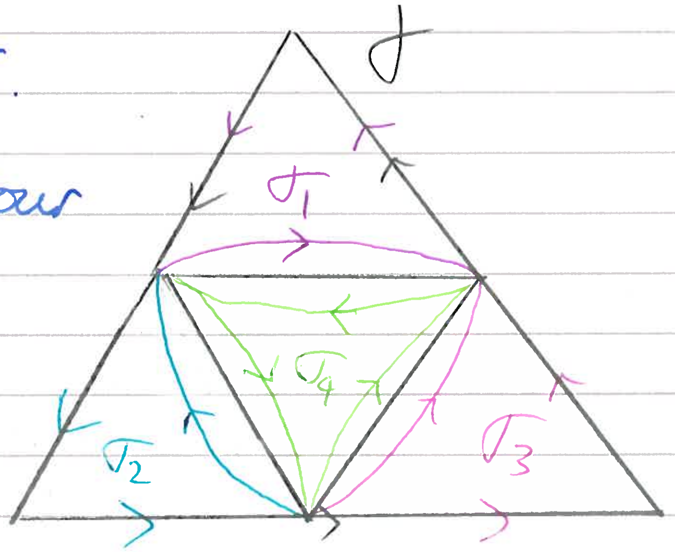


Theorem: 4.11 Let $f \in H(\Omega)$, and let γ be a triangular contour st $\text{int}\,\gamma \subset \Omega$. Then:

$$\int_\gamma f(z)\,dz = 0.$$

Recall: $\gamma : [a,b] \to \mathbb{C}$
$\gamma^* = \{ z \in \mathbb{C} :$
$z = \gamma(t)$ for
some $t \in [a,b] \}$

Proof: Denote
$\Delta = \gamma^* \cup \text{int}\,\gamma$.
Let $L(\gamma)$ be the length of the contour



Split it in four smaller triangles by joining the middle of each side. Let σ_1, σ_2, σ_3, σ_4 be the resulting cotour. Thus

$$\int_\gamma f(z)\,dz = \sum_{j=1}^{4} \int_{\sigma_j} f(z)\,dz$$

Let $\left| \int_{\sigma_1} f(z)\,dz \right|$ be the largest.

Denote: $I_1 = \int_{\sigma_1} f(z)\,dz$

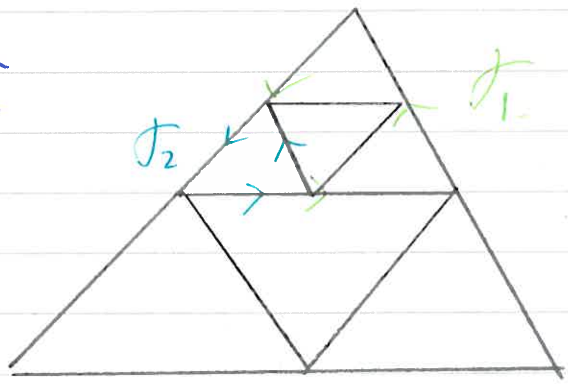
Therefore $\left| \int_\gamma f(z)\,dz \right| \leq 4\,|I_1|$

Observe $L(\sigma_1) = \dfrac{L(\gamma)}{2}$

Denote: $\gamma_1 = \sigma_1$

Repeat the partition procedure with the triangle $\Delta_1 = \gamma_1^* \cup \text{int}\, \gamma_1$

Thus we find a contour γ_2 st



$$|I_1| \leq 4|I_2|$$

where:

$$I_2 = \int_{\gamma_2} f(z)\, dz$$

and $L(\gamma_2) = \frac{1}{2} L(\gamma_1) = \frac{1}{4} L(\gamma)$

4^2

Note: $\left| \int_\gamma f(z)\, dz \right| \leq 4 |I_1| \leq 16 |I_2|$

Keep reepeating the same construction; we get a sequence of contours γ_k and of triangles $\Delta_k = \gamma_k^* \cup \text{Int}\, \gamma_k$ st

1) $\Delta_{k+1} \subset \Delta_k$

2) $L(\gamma_k) = 2^{-k} L(\gamma)$

3) $\left| \int_\gamma f(z)\, dz \right| \leq 4^k \left| \int_{\gamma^k} f(z)\, dz \right|$

Note: The set $\cap_{k=1}^{\infty} \Delta_k$ is non-empty. Indeed, let $z_k \in \Delta_k$ be an arbitrary point. The sequence z_k is bounded, since $z_k \in \Delta$. Thus by Bolzano-Weierstass there is a convergent subsequence z_{k_j}. Let $\xi = \lim_{j\to\infty} z_{k_j}$. For any n, one can find J st $z_{k_j} \in \Delta_n$ for all $j \geq J$.

Since Δ_n is closed and ξ is an accumulation point, we can claimed that $\xi \in \Delta_n$

Thus $\xi \in \Delta_n$ for all n, and therefore $\xi \in \cap_{k=1}^{\infty} \Delta_k$

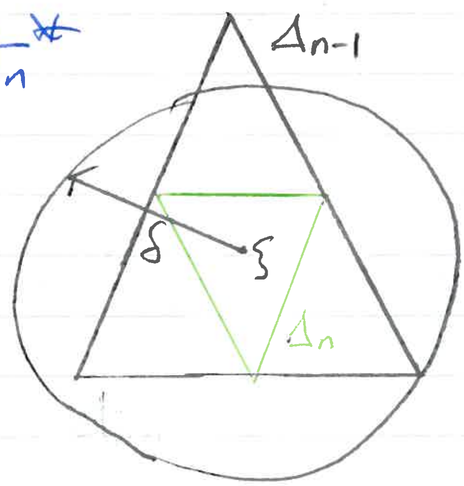
Recall that f is holomorphic on Ω so $\forall \varepsilon > 0 \ \exists \delta > 0$ st

$$\left| \frac{f(z) - f(\xi)}{z - \xi} - f'(\xi) \right| < \varepsilon$$

if $|z - \xi| < \delta$ i.e $z \in D(\xi, \delta)$ (*)

Observe for any $z \in \partial_n^*$

$$|z - \xi| \leq \tfrac{1}{2} L(\partial_n)$$
$$= 2^{-n-1} L(\partial)$$



Thus one can find n st $\Delta_n \subset D(\xi, \delta)$.

Rewrite (*)

$$|f(z)-f(\xi)-(z-\xi)f'(\xi)|<\varepsilon|z-\xi|$$

for $z \in D(\xi, \delta)$

Note $\int_{\gamma_n} f(\xi)\,dz = 0$ by theorem 4.9

and $\int_{\gamma_n} (z-\xi)f'(\xi)\,dz = 0$ by theorem 4.9.

Therefore: $\int_{\gamma_n} f(z)\,dz = \int_{\gamma_n} [f(z)-f(\xi)-(z-\xi)f'(\xi)]\,dz$

and hence $\left|\int_{\gamma_n} f(z)\,dz\right| \leq \frac{\varepsilon}{2} L^2(\gamma_n) = \frac{\varepsilon}{2} L^2(\gamma_n)$

By theorem 4.7(7)

Therefore $\left|\int_{\gamma} f(z)\,dz\right| \leq 4^n \frac{\varepsilon}{2} L^2(\gamma_n)$

$$= 4^n \frac{\varepsilon}{2} 4^{-n} L^2(\gamma) = \frac{\varepsilon}{2} L^2(\gamma)$$

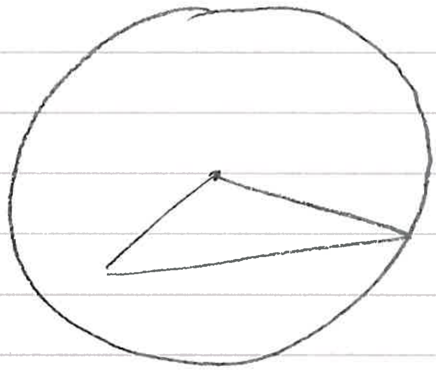
As $\varepsilon > 0$ is arbitrary, $\int_{\gamma} f(z)\,dz = 0$.

as claimed.

$\square$.

Theorem 4.12 (Antiderivate Thm) Let Ω be a convex domain and let f be continuous on Ω and for any trianglar contour γ inside Ω,

$$\int_\gamma f(z)\, dz = 0.$$



Then f has an antiderivative in Ω. More precisely, for any point $a \in \Omega$ the function:

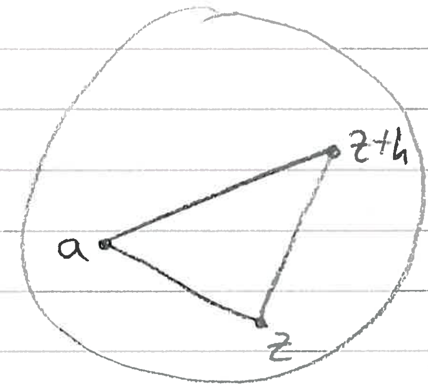
$$F(z) = \int_{[a,z]} f(w)\, dw$$

is an antiderivative of f i.e $F'(z) = f(z)$

Proof: Write:

$$\frac{F(z+h) - F(z)}{h} = \frac{1}{h}\left[\int_{[a,z+h]} f(w)\, dw - \int_{[a,z]} f(w)\, dw\right]$$

$$= \frac{1}{h}\int_{[z,z+h]} f(w)\, dw$$



To be continued...

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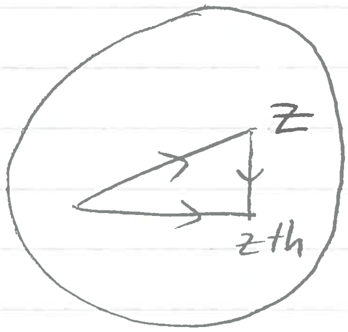
Theorem 4.12: Let Ω be convex and let f be continuous on Ω and $\int_\gamma f(z)\,dz = 0$ for any triangluar contour γ in Ω. Then $F(z) = \int_{[a,z]} f(w)\,dw$ is an antiderivative of f for any $a \in \Omega$.

Proof: cont'd:

$$\frac{F(z+h) - F(z)}{h} = \frac{1}{h}\int_{[z,\,z+h]} f(w)\,dw.$$

Note:

$$f(z) = \frac{1}{h}\int_{[z,\,z+h]} f(w)\,dw$$



Thus: $\dfrac{F(z+h) - F(z)}{h} - f(z) = \dfrac{1}{h}\displaystyle\int_{[z,\,z+h]} [f(w) - f(z)]\,dw$

Let fix $\varepsilon > 0$. Then due to the continuity of f, there is a $\delta > 0$ st $|f(z) - f(w)| < \varepsilon$ if $|z - w| < \delta$.

Assume that $|h| < \delta$. Therefore:

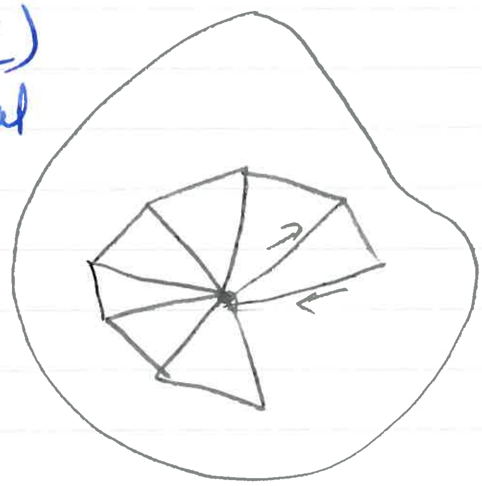
$$\left|\frac{1}{h}\int_{[z,\,z+h]} [f(h) - f(z)]\,dw\right| < \frac{\varepsilon}{|h|}|h| = \varepsilon$$

This means that for any $\varepsilon > 0$ $\exists \delta > 0$ st

$$\left| \frac{F(z+h)-F(z)}{h} - f(z) \right| < \varepsilon$$

if $|h| < \delta$

By def^n of limit, $F'(z) = f(z)$ as claimed $\square$



Remark: Let $f \in H(\Omega)$ and let γ be a polygonal contour s.t. $\operatorname{Int} \gamma \subset \Omega$

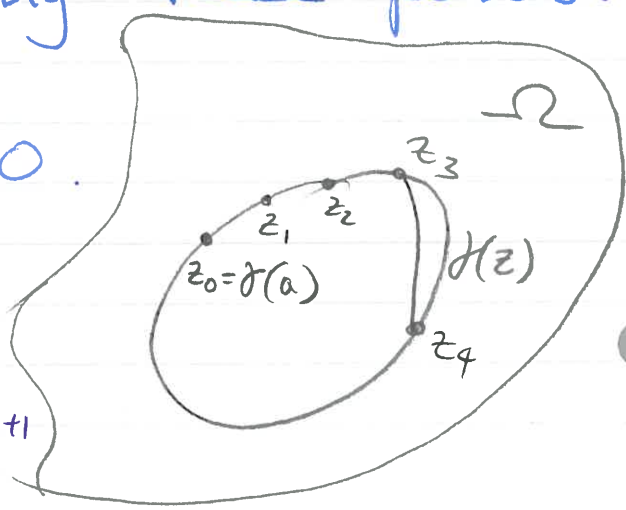
Then: $\displaystyle\int_\gamma f(z)\, dz = 0$.

This follows from theorem 4.11.

Proof of theorem 4.10: Let γ be a contour s.t. $\operatorname{Int} \gamma \subset \Omega$. Pick a sequence of points $\gamma(a) = z_0, z_1, z_2, \dots, z_n = z_0$

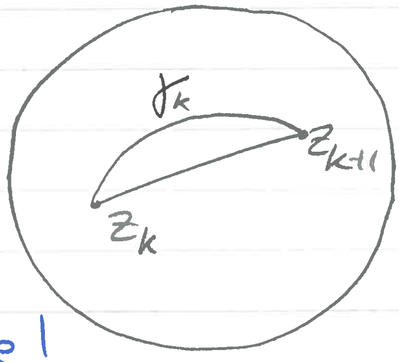
Let σ be the polygonal contour obtained by joining these points. Then

$$\int_\sigma f(z)\, dz = 0.$$



Let γ_k be the part of γ between z_k and z_{k+1}

Assume that z_k and z_{k+1} are so closed that there is a $\delta > 0$ st $[z_k, z_{k+1}] \subset D(z_k, \delta)$.



$\gamma_k \in D(z_k, \delta)$ and $D(z_k, \delta) \subset \Omega$. Then $f \in H(D(z_k, \delta))$

By Thm's 4.11 and 4.12 f has an antiderivative!

By Th 4.9: $\int_{\gamma_k} f(z)\,dz = \int_{[z_k, z_{k+1}]} f(z)\,dz$.

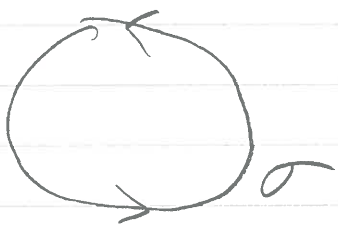
Add them up: $\int_{\gamma} f(z)\,dz = \int_{\sigma} f(z)\,dz = 0$

As required $\square$

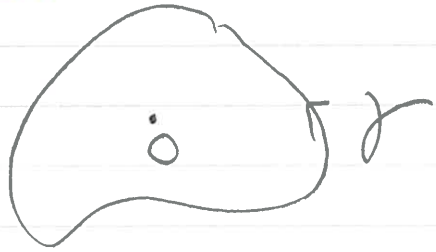
<u>Example:</u>

1) $\int_{\gamma} e^z\,dz = 0$ for any contour γ.

2) $\int_{\sigma} \frac{1}{z}\,dz = 0$ for $\dot{0}$



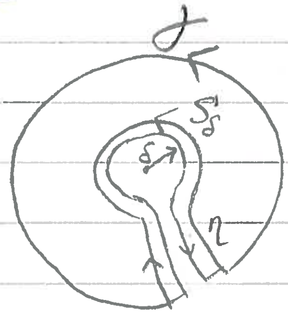
$$\int_{\gamma} \frac{1}{z}\,dz = \int_{|z|=1} \frac{1}{z}\,dz = 2\pi i$$



Example 4.14: Let γ be a contour st $0 \in \operatorname{Int}\gamma$. Then

$$\int_\gamma \frac{1}{z}\,dz = 2\pi i.$$

Proof: Let $\delta > 0$ be st $D(0,\delta) \subset \operatorname{Int}\gamma$



Join γ and S_δ with a straight segment η.

Define the contour

$$\tilde{\gamma} = \gamma \cup \eta \cup (-S_\delta) \cup (-\eta)$$

By Cauchy-Goursat (Thm 4.10)

$$\int_{\tilde{\gamma}} \frac{1}{z}\,dz = 0$$

$$\text{i.e} = \int_\gamma \frac{1}{z}\,dz + \cancel{\int_\eta \frac{1}{z}\,dz} - \int_{S_\delta} \frac{1}{z}\,dz - \cancel{\int_\eta \frac{1}{z}\,dz}$$

Thus:

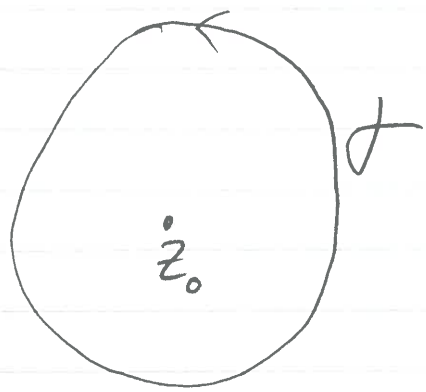
$$\int_\gamma \frac{1}{z}\,dz = \int_{S_\delta} \frac{1}{z}\,dz = 2\pi i \quad \text{As claimed}$$

$\square$

Definition 4.13: A domain Ω is said to be simply connected, if any closed simple curve γ we have $\operatorname{Int}\gamma \subset \Omega$.

Lemma 4.15: Let γ be a contour st $z_0 \in \operatorname{Int}\gamma$. Then:

$$\boxed{\int_\gamma \frac{1}{z-z_0}\,dz = 2\pi i}$$



Proof: Let $\tilde{\gamma} = \gamma - z_0$.
Then

$$\int_\gamma \frac{1}{z-z_0}\,dz = \int_{\tilde{\gamma}} \frac{1}{z}\,dz = 2\pi i.$$

$\square$

Theorem 4.16: (Cauchy–Goursat for multiply connected domains) Let Ω be a domain, $f \in H(\Omega)$. Let γ be a contour in Ω, and let $\gamma_1, \gamma_2, \gamma_3, \dots, \gamma_n$ in Ω be continuous st $\operatorname{Int}\gamma_j \cap \operatorname{Int}\gamma_k = \emptyset$

for $j \neq k$ and $\operatorname{Int} \gamma_j \subset \operatorname{Int} \gamma$, $j = 1, 2, \dots, n$. Suppose that

$$f \in H\left(\operatorname{Int} \gamma \setminus \bigcup_{j=1}^{n} \overline{\operatorname{Int} \gamma_j}\right)$$

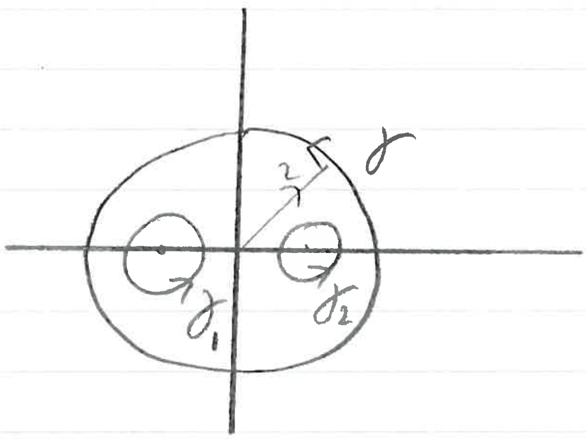
Then:

$$\int_{\gamma} f(z)\, dz = \sum_{j=1}^{n} \int_{\gamma_j} f(z)\, dz$$

Example

$$\gamma = \{z : z = 2e^{it},\ t \in [0, 2\pi]\}$$

$$\int_{\gamma} \frac{1}{z^2-1}\, dz = \int_{\gamma_1} \frac{1}{z^2-1}\, dz + \int_{\gamma_2} \frac{1}{z^2-1}\, dz$$



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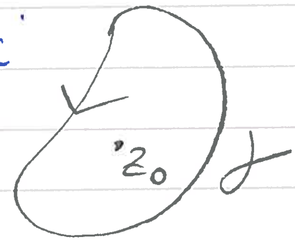
Wednesday

Lecture

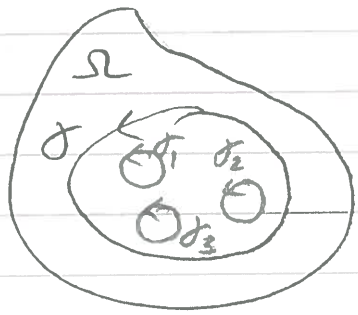
Problem Class

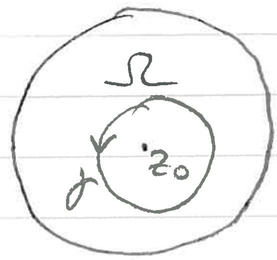
$f(z_0) = 1$

Lemma 4.15.: $\displaystyle\int_\gamma \frac{1}{z-z_0}\,dz = 2\pi i$



Theorem 4.16: $\displaystyle\int_\gamma f(z)\,dz = \sum_{j=1}^{N} \int_{\gamma_j} f(z)\,dz$





The Cauchy Integral Formula: (Very important!!!)

Theorem 4.17 – Assume that Ω is a simply connected domain and let $f \in H(\Omega)$. Let γ be a contour in Ω st $z_0 \in \operatorname{Int}\gamma$. Then

$$f(z_0) = \frac{1}{2\pi i}\int_\gamma \frac{f(z)}{z-z_0}\,dz.$$

Or,

$$\int_\gamma \frac{f(z)}{z-z_0}\,dz = 2f(z_0)\pi i.$$

Proof of theorem 4.17: Write: (Comes up in the exam)

$$\frac{1}{2\pi i}\int \frac{f(z)}{z-z_0}\,dz = \frac{1}{2\pi i}\int \frac{f(z)-f(z_0)}{z-z_0}\,dz$$
$$+\frac{f(z_0)}{2\pi i}\int_\gamma \frac{1}{z-z_0}\,dz.$$

by lemma 4.15:

$$= \frac{1}{2\pi i}\int_\gamma \frac{f(z)-f(z_0)}{z-z_0} + f(z_0)$$

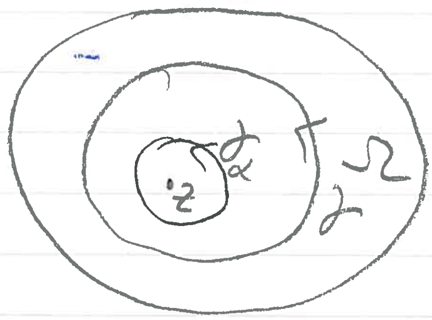
It remains to show that: $\int_\gamma g(z)\,dz = 0$.

with:

$$g(z) = \frac{f(z)-f(z_0)}{z-z_0}.$$

If g were analytic on Ω, this would be true to Cauchy-Goursat

<u>But</u> g is analytic on $\Omega - \{z_0\}$



Let $\alpha > 0$ be number st $\gamma_\alpha = S(z_0, \alpha)$ is inside Int γ. By thm 4.16:

$$\int_\gamma g(z)\,dz = \int_{\gamma_\alpha} g(z)\,dz$$

Since f is diff on Ω, $\forall \varepsilon > 0\ \exists \delta > 0$ s.t. $|g(z) - f'(z_0)| < \varepsilon$ if $|z_0 - z| < \delta$

Use this with $\varepsilon = 1$

$(*)\ |g(z)| < \varepsilon + |f'(z_0)|,\ |z - z_0| < \delta.$

Assume that $\alpha < \delta$ so $(*)$ holds for $z \in \gamma_\alpha$

By Thm 4.7(7)

$$\left|\int_{\gamma_\alpha} g(z)\,dz\right| \leq \overbrace{(1 + |f(z)|)}^{\text{Does not depend on } \alpha}\, 2\pi\alpha$$

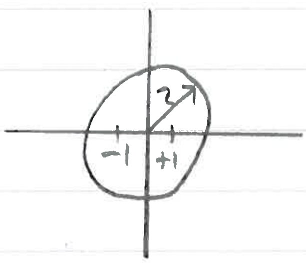
As $\alpha > 0$ is arbitrary,

$$\int_\gamma g(z)\,dz = \int_{\gamma_\alpha} g(z)\,dz = 0.$$

As required $\square$.

Example:

1) $\displaystyle\int_{S(0,2)} \frac{1}{z^2-1}\,dz$



by thm 4.16

$$\ldots = \int_{S(-1,\frac{1}{2})} \frac{1}{z^2-1}\,dz + \int_{S(1,\frac{1}{2})} \frac{1}{z^2-1}\,dz.$$

Take: $$\int_{S(-1,\frac{1}{2})} \frac{1}{z^2-1}\,dz = \int_{S(-1,\frac{1}{2})} \frac{\overbrace{\left(\frac{1}{z-1}\right)}^{=f(z)}}{z+1}\,dz$$

$$= 2\pi i f(-1)$$

$$= 2\pi i\left(-\frac{1}{2}\right)$$

$$= -\pi i.$$

In the same

$$\int_{S(1,\frac{1}{2})} \frac{1}{z^2-1}\,dz = \int_{S(1,\frac{1}{2})} \frac{\overbrace{\left(\frac{1}{z+1}\right)}^{=g(z)}}{z-1}\,dz$$

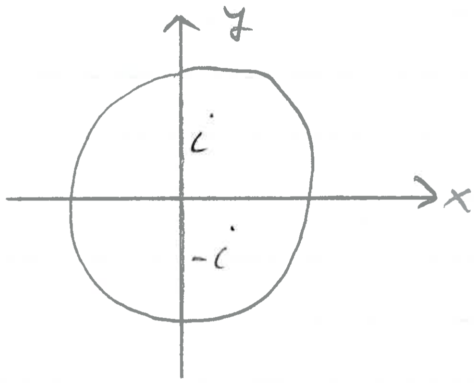
$$= 2\pi i g(1)$$

$$= \pi i$$

Thus: $$\int_{S(0,1)} \frac{1}{z^2+1} = -\pi i + \pi i$$

$$= 0.$$

2) $\displaystyle\int_{S(0,2)} \frac{\operatorname{sen} z}{z^2+1}\, dz$



By Thm 4.16,

$$I = \overbrace{\int_{S(i,\frac{1}{2})} \frac{\operatorname{sen} z}{z^2+1}\, dz}^{I_1}$$

$$+ \underbrace{\int_{S(-i,\frac{1}{2})} \frac{\operatorname{sen} z}{z^2+1}\, dz}_{I_2}$$

$$I_1 = \int_{S(i,\frac{1}{2})} \frac{\frac{\operatorname{sen} z}{z-i}}{z+i}\, dz = 2\pi i \,\frac{\operatorname{sen} i}{i+i} = \pi \operatorname{sen} i$$

$$I_2 = \int_{S(-i,\frac{1}{2})} \frac{\frac{\operatorname{sen} z}{z-i}}{z+i}\, dz = 2\pi i \,\frac{\operatorname{sen}(-i)}{-i-i} = \pi \operatorname{sen} i$$

Therefore $I = I_1 + I_2 = 2\pi \operatorname{sen} i$.

Alternative method: use partial fraction.

$$\frac{\operatorname{sen} z}{z^2+1} = \frac{\operatorname{sen} z}{2i(z-i)} - \frac{\operatorname{sen} z}{2i(z-i)}$$

continue...

Applicatcoin.

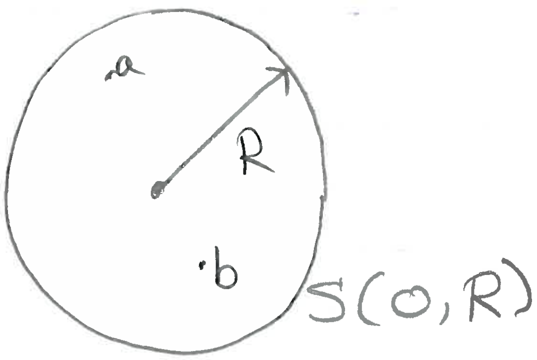
Theorem 4.18 (Liouville's Thm) Let f be an entire functcoin, st there exist a number

$$|f(z)| \leq M \quad \text{for all } z \in \mathbb{C}$$

Then $f = \text{const}$ for all $z \in \mathbb{C}$.

Proof: Let $a, b \in \mathbb{C}$, and let's show that $f(a) = f(b)$

Let $R > 0$ be such that $|z-a| \geq R/2$ and $|z - b_0| \geq R/2$ for all $z \in S(0,R)$



By Cauchy Formula:

$$f(a) - f(b) = \frac{1}{2\pi i} \int_{S(0,R)} \left[\frac{f(z)}{z-a} - \frac{f(z)}{z-b} \right] dz$$

$$= \frac{1}{2\pi i} \int_{S(0,R)} f(z) \left(\frac{a-b}{(z-a)(z-b)} \right) dz$$

By Thm 4.7(7):

$$|f(a)-f(b)| \leq \frac{1}{\cancel{2\pi}} M \cdot \frac{|a-b| \cancel{2\pi} \cancel{R}}{\frac{R}{2} \cdot \frac{\cancel{R}}{2}}$$

$$= \frac{4M|a-b|}{R}$$

As R is arbitiary; $f(a)-f(b)=0$. as required. $\square$

<u>Theorem 4.19</u> (The fundamental of Algebra)
Let p be a polynomial of degree n, $p=p(z)$. Then it has exactly n roots in $\mathbb{C}$ counting multiplicities.

<u>Proof</u>: We'll show that p has at least one root.

Assume that p has no roots, therefore $1/p(z)$ is entire. Write

$$p(z) = a_n z^n + a_{n-1} z^{n-1} + \dots + a_0$$

$$= z^n \underbrace{(a_n + a_{n-1} z^{-1} + a_{n-2} z^{-2} + \dots + a_0 z^{-n})}_{\to a_0 \text{ as } z \to \infty}$$

$$\Rightarrow |p(z)| \to \infty \text{ as } z \to \infty$$

Thus $\dfrac{1}{|p(z)|} \to 0$ as $z \to \infty$

In other words, $\exists R > 0$ st $|1/p(z)| < 1$ if $|z| > R$.

At the same time $1/p(z)$ is continuous on $\overline{D}(0,R)$ so $1/p(z)$ is bounded on $\overline{D}(0,R)$, by theorem 1.19. Thus $|1/p(z)| \leq M$ for all $z \in \mathbb{C}$ with some $M > 0$. By Liouville's theorem $1/p(z) = \text{const} \Rightarrow p(z) =$ constant. A contradiction. $\square$

<u>Cauchy Formula for the derivatives</u>

Write:

$$f(w) = \frac{1}{2\pi i} \int_\gamma \frac{f(w)}{z - w} \, dz$$

Differentiate (formally)

$$f'(w) = \frac{1}{2\pi i} \int_\gamma \frac{f(z)}{(z-w)^2} \, dz$$

Again:

$$f''(w) = \frac{2}{2\pi i} \int_\gamma \frac{f(z)}{(z-z_0)^3} \, dz .$$

Theorem 4.20 (The Cauchy Formula for higher derivates): Suppose Ω is simply connected and $f \in H(\Omega)$. Let γ be a contour in Ω st $\operatorname{Int}\gamma \ni z_0$. Then f is differentiable inside Ω and infinitely:

$$f^{(n)}(z_0) = \frac{n!}{2\pi i} \int_\gamma \frac{f(z)}{(z-z_0)^{n+1}}\, dz$$

Example:

$$I = \int_{S(0,1/3)} \frac{\cos z}{z^2(z-1)}\, dz = ?$$

$$I = \int_{S(0,1/3)} \frac{f(z)}{z^2}\, dz, \quad f(z) = \frac{\cos z}{z-1}$$

By Thm 4.20:

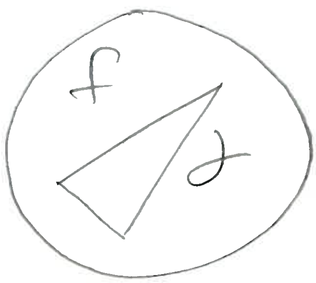
$$I = 2\pi i f'(0)$$

$$\int_{S(i,\frac{1}{2})} \frac{\cos z}{z^2(z-1)}\, dz = 0.$$

these $f'(z) = -\dfrac{\sin z}{z-1} - \dfrac{\cos z}{(z-1)^2}$

so $f'(0) = -1 \Rightarrow I = -2\pi i$

—/—



$\int f\, dz = 0 \Rightarrow f$ has an antiderivati
i.e $\exists F : F' = f$

—/—

Theorem 4.21 (Morera's Thm): Let f be continuous on Ω and assume that $\int_\gamma f(z)\,dz = 0$ for every contour $\gamma \subset \Omega$. Then $f \in H(\Omega)$

Proof: (Proof in online notes – Might be examined).

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Chapter 5: Series expansion for holomorphic functions

Aim: To show that every analytic function can be expanded in a Taylor series.

i.e $f(z) = \sum_{k=0}^{\infty} a_k (z - z_0)^k$

Theorem 5.1: Suppose that $f \in H(D(z_0, R))$ with some $z_0 \in \mathbb{C}$, $R > 0$. Then for every $z \in D(z_0, R)$ we have:

$$f(z) = \sum_{k=0}^{\infty} \frac{f^{(k)}(z_0)}{k!} (z - z_0)^k \quad (*)$$

The series $(*)$ is called Talyor series of f about z_0.

Comments and examples:
Let R_0 be the radius of convergence of $(*)$. Note: Thm 5.1 doesn't say that $R = R_0$. It does say that $R \leq R_0$.

Example:
1) e^z is entire, i.e $R = \infty$

e^z is analytic on $D(0,1) \Rightarrow (*)$ holds.

2) $g(z) = \frac{1}{z^2+3}$. Find its series about $z_0 = ($

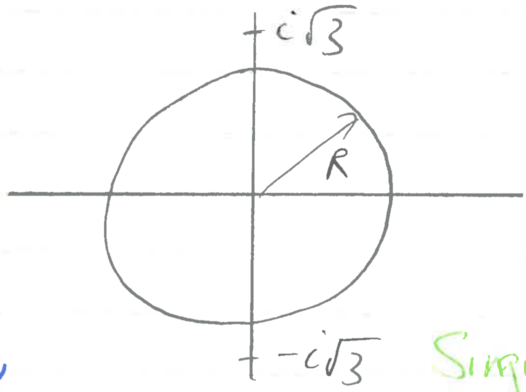
(radius of convergence)

Find R_0, By Thm 5.1 we have

$$g(z) = \sum_{k=0}^{\infty} a_k z^k$$

g is holomorphic on $D(0, \sqrt{3})$

Thus $(*)$ holds for $R = \sqrt{3}$.



Singularities at $\pm i\sqrt{3}$

We know $R_0 \geq R = \sqrt{3}$

On the other hand, $R_0 \leq \sqrt{3}$ so $R_0 = \sqrt{3}$

If $z_0 = 0$, the series is called a <u>Maclarin</u> series

<u>Example</u>: $e^z = \sum_{k=0}^{\infty} \frac{z^k}{k!}$

Question: find Taylor's Series for e^z at $z_0 \in \mathbb{C}$.

$$e^z = e^{z_0} \sum_{k=0}^{\infty} \frac{(z-z_0)^k}{k!},$$

$$e^z = e^{z_0} e^{z - z_0}$$

(EXAM QUESTION)

<u>Exercise</u>: Find Taylor's series for $\sin z$ at z_0.

Laurent Series:

Let $g(z) = \sum_{k=-\infty}^{\infty} a_k (z - z_0)^k = g_1(z) + g_2(z)$

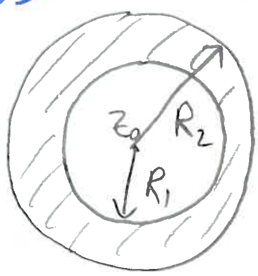
with $g_1(z) = \sum_{k=-\infty}^{-1} a_k (z - z_0)^k$

$g_2(z) = \sum_{k=0}^{\infty} a_k (z - z_0)^k$

g_2: Let R_2 be its radius of convergence, g_2 converages for $|z - z_0| < R_2$

g_1: $g_1(z) = \sum_{k=-\infty}^{-1} a_k (z - z_0)^k$

$= \sum_{k=1}^{\infty} a_{-k} (z - z_0)^{-k}$



Rewrite: $w = (z - z_0)^{-1}$

$\sum_{k=1}^{\infty} a_{-k} w^k \longrightarrow$ converages within its radius of convergences r_1.

$$|w| < r_1 \iff |z - z_0|^{-1} < r_1$$

$$\iff |z - z_0| > \frac{1}{r_1} = R_1$$

If $R_1 < R_2$, then g converges in the ring!

$D_{R_1, R_2}(z_0) = \{ z : R_1 < |z - z_0| < R_2 \}$.

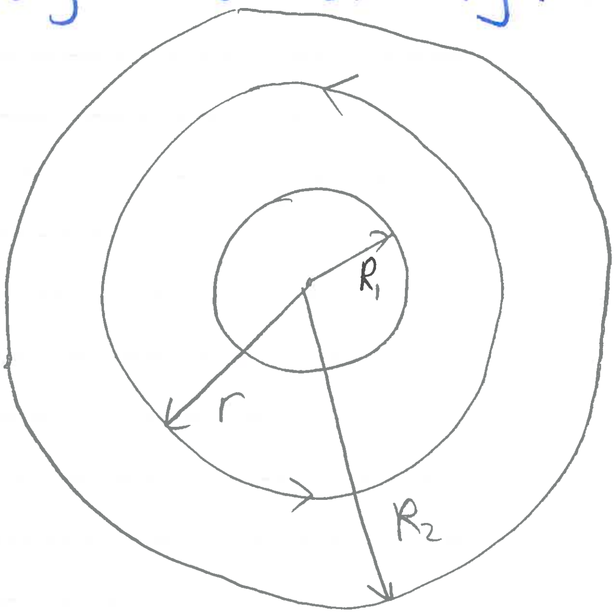
Theorem 5.2 (Laurent's Theorem) Assume that $f \in H(D_{R_1,R_2}(z_0))$. Then for every $z \in D_{R_1,R_2}(z_0)$ we have:

$$f(z) = \sum_{k=-\infty}^{\infty} a_k (z - z_0)^k$$

with

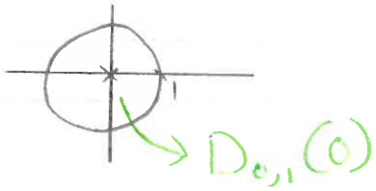
$$a_k = \frac{1}{2\pi i} \int_{S(z_0,r)} \frac{f(w)}{(w-z)^{k+1}}\, dw \quad , k \in \mathbb{Z}.$$

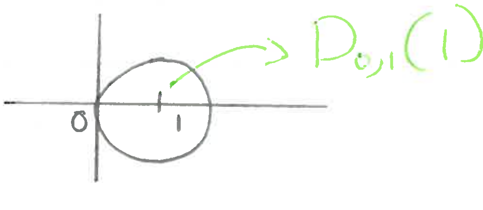
with $r \in (R_1, R_2)$. Moreover, the series converages absolutely.



Example: $f(z) = \dfrac{1}{z(z-1)}$

Find the Laurent expansion about about $z_0 = 0$, and about $z_0 = 1$





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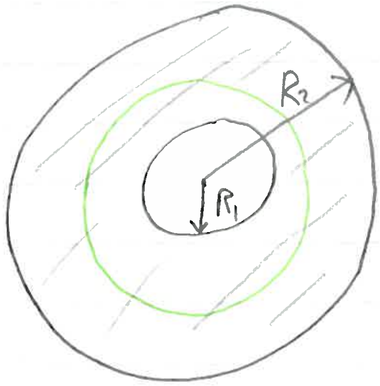
$$g(z) = \sum_{k=-\infty}^{\infty} a_k (z - z_0)^k$$

$$D_{R_1,R_2}(z_0) = \{z \in \mathbb{C} : R_1 < |z - z_0| < R_2\}$$



Theorem 5.2: Suppose $f \in H(D_{R_1,R_2}(z_0))$
Then:

$$f(z) = \sum_{k=-\infty}^{\infty} a_k (z - z_0)^k,$$

Laurent expansion about z_0.

~~for~~ for each $z \in D_{R_1,R_2}(z)$ where

$$a_k = \frac{1}{2\pi i} \int_{S(z_0,r)} \frac{f(w)}{(w-z_0)^{k+1}} \, dw$$

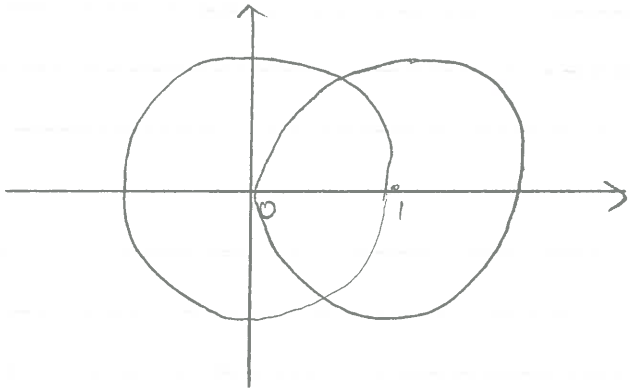
where $r \in (R_1, R_2)$.

Example: $g(z) = \dfrac{1}{z(z-1)}$

where is g analytic?

On $D_{0,1}(0)$, $D_{0,1}(1)$

On $D_{1,\infty}(0)$, $D_{1,\infty}(1)$



Rewrite : $g(z) = -\frac{1}{z} + \frac{1}{z-1}$

Let : $0 < |z| < 1$

$\frac{1}{z}$ is already good

Look at $\frac{1}{z-1}$!

Geom series

$$\frac{1}{z-1} = -\frac{1}{1-z} = \sum_{k=0}^{\infty} z^k$$

$$\Rightarrow g(z) = -\frac{1}{z} - \sum_{k=0}^{\infty} e^k$$

$$= -\sum_{k=-1}^{\infty} z^k$$

Let $|z|<1$.

Write: $\frac{1}{z-1} = \frac{1}{z\left(1-\frac{1}{z}\right)} = \frac{1}{z}\sum_{k=0}^{\infty}\frac{1}{z^k}$

$$= \sum_{k=-\infty}^{-1} z^k$$

Thus $g(z) = -\frac{1}{z} + \sum_{k=-\infty}^{-1} z^k = \sum_{k=-\infty}^{-2} z^k$

Let: $0<|z-1|<1$

$\frac{1}{z-1}$ is good.

Write: $\frac{1}{z} = \frac{1}{1+(z-1)} = \sum_{k=0}^{\infty}(-1)^k(z-1)^k$

Thus:

$$g(z) = -\sum_{k=0}^{\infty}(-1)^k(z-1)^k + \frac{1}{z-1}$$

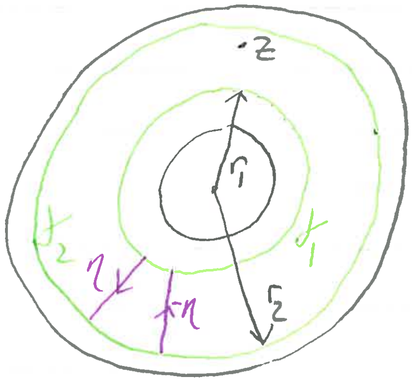
$$= \sum_{k=-1}^{\infty}(-1)^{k+1}(z-1)^k$$

<u>Proof of theorem 5.2</u>: Looking at $\tilde{f}(z) = f(z+z_0)$, assume $z_0 = 0$.

Pick a $z \in D_{R_1,R_2}(0)$.

Let r_1, r_2 be st

$0 < R_1 < r_1 < |z| < r_2 < R_2$



Denote $\gamma_1 = S(0, r_1)$,
$\gamma_2 = S(0, r_2)$

Connect γ_1 and γ_2 with segment η

Define: $\gamma = \gamma_2 \cup (-\eta) \cup (-\gamma_1) \cup \eta$

Observe: $z \in \text{Int}\,\gamma$,

$\text{Int}\,\gamma \subset D_{R_1, R_2}(0)$

By Cauchy formula:

$$f(z) = \frac{1}{2\pi i} \int_\gamma \frac{f(s)}{s-z}\, ds$$

$$= \frac{1}{2\pi i} \int_{\gamma_2} \frac{f(s)}{s-z}\, ds - \frac{1}{2\pi i} \int_{\gamma_1} \frac{f(s)}{s-z}\, ds$$

Note: if $s \in \gamma_2$ then $|s| > |z|$
$s \in \gamma_1$ then $|s| < |z|$

Look at $s \in \gamma_2$:

$$\frac{1}{s-z} = \frac{1}{s\left(1 - \frac{z}{s}\right)} = \frac{1}{s} \sum_{k=0}^{\infty} \frac{z^k}{s^k}$$

Look at $s \in \gamma_1$:

$$\frac{1}{s-z} = -\frac{1}{z\left(1-\frac{s}{z}\right)}$$

$$= -\frac{1}{z}\sum_{k=0}^{\infty}\frac{s^k}{z^k}$$

Thus: $f(z) = \frac{1}{2\pi i}\int_{\gamma_2} f(s) \sum_{k=0}^{\infty} \frac{z^k}{s^{k+1}}\, ds$

$$+ \frac{1}{2\pi i}\int_{\gamma_1} f(s) \sum_{k=0}^{\infty} \frac{s^k}{z^{k+1}}\, ds$$

Exchange $\int$ and Σ (not justified!)

$$f(z) = \sum_{k=0}^{\infty} z^k \frac{1}{2\pi i}\int_{\gamma_2} \frac{f(s)}{s^{k+1}}\, ds$$

$$+ \sum_{k=0}^{\infty} \frac{1}{z^{k+1}} \frac{1}{2\pi i}\int_{\gamma_1} f(s) s^k\, ds$$

$$= \sum_{k=0}^{\infty} z^k \frac{1}{2\pi i}\int_{\gamma_2} \frac{f(s)}{s^{k+1}}\, ds + \sum_{k=-\infty}^{-1} z^k \frac{1}{2\pi i}\int_{\gamma_1} \frac{f(s)}{s^{k+1}}\, ds$$

via letting $m = -(k+1)$

By Cauchy-Goursat theorem for multiply connected domains:

$$\int_{\gamma_2} \frac{f(s)}{s^{k+1}}\, ds = \int_{\gamma_1} \frac{f(s)}{s^{k+1}}\, ds = \int_{S(0,r)} \frac{f(s)}{s^{k+1}}\, ds$$

This gives the required formula for a_k $\square$

<u>Proof of Thm 5.1</u> (Dec Test ends) Assume $f \in H(D(z_0, R))$

Want: $f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!} (z - z_0)^n$

Use theorem 5.2. Obviously, $f \in H(D'(z_0, R))$ and $D'(z_0, R) = D_{0,R}(z_0)$. Thus $f(z)$ is repersented by:

$$f(z) = \sum_{k=-\infty}^{-1} a_k (z - z_0)^k + \sum_{k=0}^{\infty} a_k (z - z_0)^k$$

Need to show that $a_k = 0, \forall k \leq 1$ and

$$a_k = \frac{f^{(k)}(z_0)}{k!} \quad \forall k \geq 0.$$

Write for $m \geq 1$:

By Gauchy - Goursat

$$a_{-m} = \frac{1}{2\pi i} \int_{S(z_0, r)} \frac{f(w)}{(w - z)^{-m+1}} = \int_{S(z_0, r)} f(w)(w - z)^{m-1}\, dw = 0$$

Write: $a_k = \frac{1}{2\pi i} \int_{S(z_0, r)} \frac{f(w)}{(w - z_0)^{k+1}}\, dw$

By theorem 4.20

$= \frac{f^{(k)}(z_0)}{k!}$, as required □

Look at consequences:

Theorem 5.3: Let f be entire and assume $|f(z)| < C|z|^k$ for all $|z| \geq 1$ and some $k \in \mathbb{N}$.

— / —

Example: $|z^3 + 1| \leq C|z|^3$, $|z| \geq 1$ with some constant.

Example: $|e^z| \nleq |z|^k$

Example: $\sin z$ is not bdd by $|z|^k$

— / —

The $f(z)$ is a polynomial of degree at most k.

Proof: For all z:

$$f(z) = \sum_{k=0}^{\infty} a_n z^n$$

and then the series conv. absolutely by Thm 5.1.

Need to show that $a_n = 0$ for $n > k$.

By Thm 5.1, 5.2

$$a_n = \frac{1}{2\pi i}\int_{S(0,r)} \frac{f(z)}{z^{n+1}}\, dz.$$

for any $r > 0$.

Let $r > 1$. Then by Theorem 4.7(7)

$$|a_n| \leq \frac{C}{\cancel{2\pi}} \frac{r^k}{r^{n+\cancel{1}}} \cancel{2\pi r} = C r^{k-n}$$

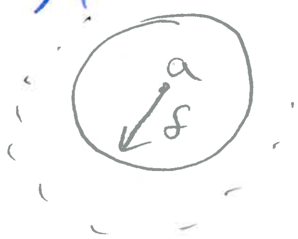
Note : $k - n < 0$ if $n > k$.

as $r > 1$ is arbitrary, $a_n = 0$, $n > k$ as claimed $\square$

Zero and Singularities of analytic functions

<u>Definition 5.4</u> : Let $f \in H(\Omega)$. Then a point $a \in \Omega$ st $f(a) = 0$ is called a zero of f.

We say a zero a is isolated if there is a number $\delta > 0$ st $f(z) \neq 0$, for all $z \in D'(a, \delta)$



An isolated zero is said to have order m if in the Taylor's series: $f(z) = \sum_{k=0}^{\infty} c_k (z-a)^k$ we have $c_0 = c_1 = \dots = c_{m-1} = 0$ and $c_m \neq 0$

<u>Example</u>: $(z-1)^5 - a = 1$ is a zero of order 5.

In other words

$$f(z) = \sum_{k=m}^{\infty} c_k (z-a)^k$$

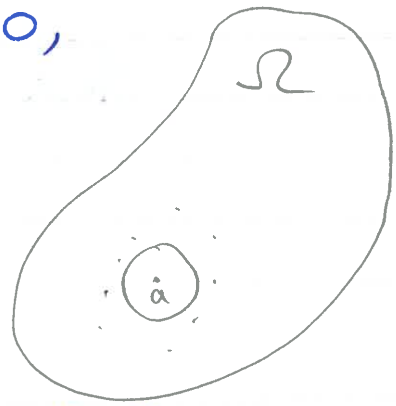
$$= (z-a)^m \left[\overset{\neq 0}{c_m} + c_{m+1}(z-a) + c_{m+2}(z-a)^2 + \dots \right]$$

$$= (z-a)^m \, g(z)$$

where g is analytic at a and $g(a) \neq 0$

Notation: $Z(f)$ is the set of all zero's of f.

<u>Theorem 5.5</u>: Suppose that $f \in H(\Omega)$. Assume that $Z(f)$ has an accummalator point in Ω. Then $f(z) = 0$, $\forall z \in \Omega$.



<u>Proof</u>: Assume that $f \neq 0$ on Ω. Let $a \in \Omega$ be an accummalation point of $Z(f)$. By continuity of f a is also a zero.

Thus for some $r > 0$, the function f can be repersented in $D(a, r)$ by the serie

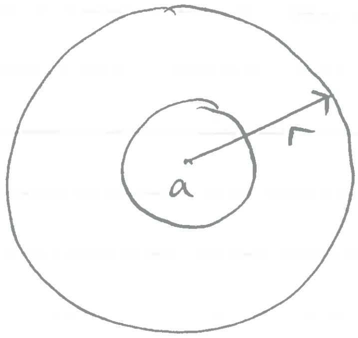
$$f(z) = \sum_{k=0}^{\infty} c_k (z-a)^k$$

Since $f \neq 0$, there is an m st $c_0 = c_1 = c_2 = \dots = c_{m-1} = 0$, and $c_m \neq 0$. Hence

$$f(z) = (z-a)^m (g(z))$$

where g is analytic on $D(a,r)$ and $g(a) \neq 0$.

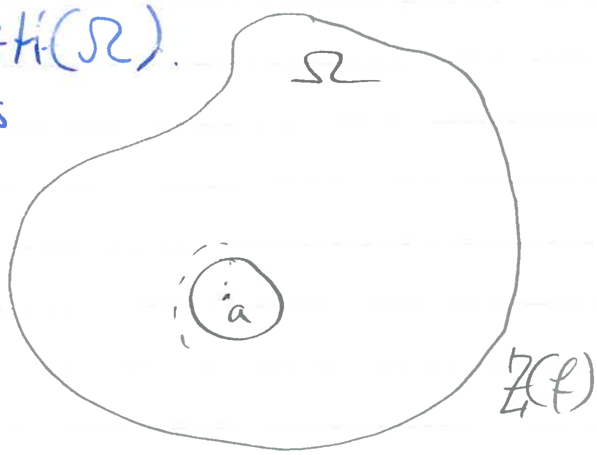
By continuity of g, $g(z) \neq 0$ for all $z \in D(a,\delta)$ with some $\delta > 0$.



This means that a is isolated ~~root~~ zero of f in $D(0,\delta)$. Thus it cannot be all accumation point of $Z(f)$. We have a contradiction, which shows that $f(z)=0$ on $D(0,\delta)$

5/12/11

Theorem 5.5: Let $f \in H(\Omega)$. Assume that $Z(f)$ has a accumulation point in Ω. Then $f(z) = 0$ for all $z \in \Omega$.

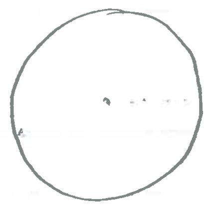


Corollary 5.6 (The unique continuation Thm) Assume that $f, g \in H(\Omega)$. Assume also that $f(z) = g(z)$ on a set $S \subset \Omega$ which has an accumulation point on Ω. Then $f(z) = g(z)$ for all $z \in \Omega$.

Proof: Let $h = f - g$. Use Thm 5.5. $\square$

Example: $\Omega = D(0,1)$. Is there a function $f \in H(\Omega)$ st:

$$f\left(\tfrac{1}{n}\right) = \frac{n}{n+1} \quad \text{for all } n = 1, 2, \ldots ?$$



Let $g(z) = \dfrac{1}{1+z}$.

so $g\left(\tfrac{1}{n}\right) = \dfrac{1}{1+\frac{1}{n}} = \dfrac{n}{1+n}$ for all $n \in \mathbb{N}$.

Note: $g \in H(\Omega)$. Clearly, $f(\frac{1}{n}) = g(\frac{1}{n})$.
Since $1/n \xrightarrow{n \to \infty} 0 \subset D(0,1)$. By Corollary 5.6:

$$f(z) = g(z) = \frac{1}{z+1}$$

<u>Singularities</u>:

<u>Definition 5.7</u>: We say that f has an isolated singularity at $a \in \mathbb{C}$ if f is holomorphic on $D'(a,r)$ with some $r > 0$.

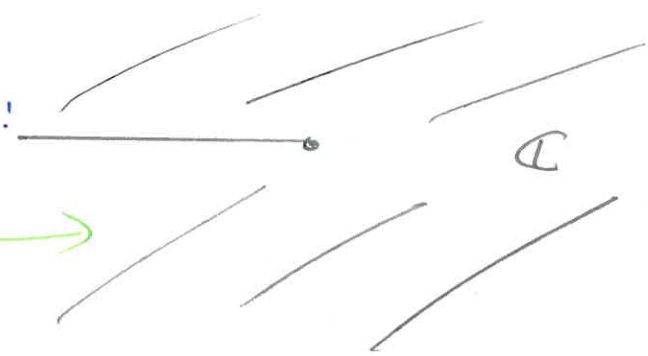
<u>Example</u>:

1) $1/z$ has an isol. sing at $a = 0$.

2) $1/z^2(z-1)$ isol. sing at $a_1 = 0$ and $a_2 = 1$

3) Log z:

Log z is analytic on:



$\Rightarrow$ it does not have isolated singularity

Classification of singularities

as $f \in H(D'(a,r))$, by Laurent's Thm:

$$f(z) = \underbrace{\sum_{k=-\infty}^{-1} a_k (z-a)^k}_{\text{Principal part of Laurent Exp (p.p)}} + \underbrace{\sum_{k=0}^{\infty} a_k (z-a)^k}_{\text{Regular part}}$$

Three type of singularities:

Type(1) A pole at a: If the pp of f contains finitely many terms, then we say that f has a pole at the pole point a. More precisely, if there is number $M \in \mathbb{N}$ st $a_k = 0$ for all $k < -M$ and $a_m \neq 0$, then f is said to have a pole of order M at a.

This means:

$$PP \text{ of } f = \sum_{k=-M}^{-1} a_k (z-a)^k$$

$$= \frac{a_{-m}}{(z-a)^m} + \frac{a_{-m+1}}{(z-a)^{m-1}} + \dots$$

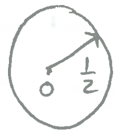
$a_{-m} \neq 0$.

Examples:

1) $1/z$ - pole or order 1 at $a=0$, or a simple pole.

2) $$\frac{1}{z^2(1-z)} = \frac{g(z)}{z^2} = \frac{1}{z^2}\left(1+z+z^2+z^3+\dots\right)$$

$$= \frac{1}{z^2} + \frac{1}{z} + 1 + z + \dots$$



Pole of order 2 at $a=0$.
A simple pole at $a=1$

3) $$\frac{\sin z}{z^2} = \left(z - \frac{z^3}{6} + \dots\right)\frac{1}{z^2} = \underbrace{\frac{1}{z}}_{PP} - \frac{z}{6} + \dots$$

a simple pole $a=0$.

Type (2) Essential singularity at a: If there is no number $M \in \mathbb{N}$ st $a_k = 0$ for $k < -M$, then f is said to have an essential singularity at a.

Example: $g(z) = \sin\left(\frac{1}{z}\right)$

$$= \sum_{k=0}^{\infty} (-1)^k \frac{1}{z^{2k+1}} \frac{1}{(2k+1)!}$$

$\Rightarrow a=0$ is an essential singularity.

Type 3

$f(z) = z, \quad f \in H(D'(0,1))$

A removable singularity at $a=0$.

If pp of $f = 0$, then f is said to have a removable singularity at a. (In other words, if $a_k = 0$, $k \leq -1$). Then f becomes analytic at a if one defines $f(a) = a_0$

Example 1) $g(z) = \frac{\sin z}{z}$ - an isol sing at $a=0$

$$\frac{\sin z}{z} = \frac{z - \frac{z^3}{6} + \ldots}{z} = \overbrace{1}^{= a_0} - \frac{z^2}{6} + \ldots$$

pp $= 0 \Rightarrow$ a removable singularity

To make it analytic at $a=0$ define $g(0) = 1$.

Now:

$$\tilde{g}(z) = \begin{cases} \frac{\sin z}{z}, & z \neq 0 \\ 1, & z = 0. \end{cases}$$

Mixed Example:

2) $h(z) = \dfrac{1 - \cos z}{z^2} = \dfrac{1 - \left(1 - \frac{z^2}{2} + \frac{z^4}{24} - \cdots\right)}{z^2}$

$= \dfrac{1 - 1 + \frac{z^2}{2} - \frac{z^4}{24} + \cdots}{z^2}$

$= \dfrac{1}{2} - \dfrac{z^2}{24} + \cdots$

Removable

$\dfrac{1 - \cos z}{z^2}$ – a simple pole.

3) $\dfrac{z - 3}{(z-4)^3}$ pole of order 3 at $a = 4$.

$= \dfrac{(z - 4 + 1)}{(z-4)^3} = \dfrac{1}{(z-4)^3} - \dfrac{1}{(z-4)^2}$

Theorem 5.8 1) the function $f \in H(D(a,R))$ has a zero of order m at a iff

$$\lim_{z \to a} (z-a)^{-m} f(z) = B$$

with some $B \neq 0$.

2) The function $g \in H(D'(a,R))$ has a pole of order m at a iff

$$\lim_{z \to a} (z-a)^m f(z) = A$$

with some $A \neq 0$.

<u>Proof</u> of ①: Suppose f has a zero of order m. Then by def^n:

$$f(z) = \sum_{k=m}^{\infty} c_k (z-a)^k, \quad c_m \neq 0, \text{ i.e.}$$

$$= (z-a)^m g(z)$$

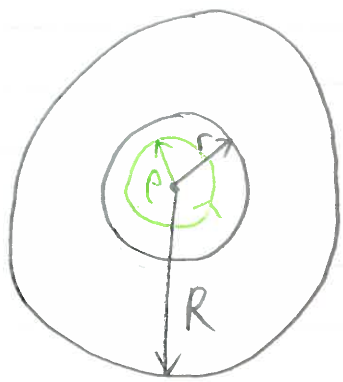
where $g \in H(D(a,R))$ and $g(a) = c_m \neq 0$

Therefore:

$$(z-a)^{-m} f(z) = g(z) \Rightarrow g(z) \xrightarrow[z \to a]{} g(a) = c_m \neq 0$$

Suppose the limit exists and $B \neq 0$. This means that:

$$g(z) = (z-a)^{-m} f(z)$$



is bounded on $D'(a,r)$ with some $r > 0$.

Compute c_k, $k = 0, 1, 2, \ldots$

Want: $c_0 = c_1 = \ldots = c_{m-1} = 0$ and $c_m \neq 0$.

Write: (want a contour inside)

$$c_k = \frac{1}{2\pi i}\int_{S(a,\rho)} \frac{f(z)}{(z-a)^{k+1}}\,dz = \frac{1}{2\pi i}\int_{S(a,\rho)} \frac{g(z)}{(z-a)^{k-m+1}}\,dz$$

$0<\rho<r$

By theorem 4.7:

$$|c_k| \leq \frac{1}{\cancel{2\pi}} \cdot \frac{C}{\rho^{k-m+\cancel{1}}} \cdot \cancel{2\pi\rho} = C\rho^{m-k} \xrightarrow[\rho\to 0]{} 0$$

ρ is arbitrary

$m>k$

As $\rho>0$ is arbitrary, $c_k=0$ for all $k=0,1,\dots,m-1$.

This means that:

$$f(z) = c_m(z-a)^m + c_{m+1}(z-a)^{m+1} + \dots$$

and hence: $\dfrac{f(z)}{(z-a)^m} \xrightarrow{z\to a} c_m = B \neq 0.$

(as limit exist and not equal to zero)

As required $\square$ 1

<u>Example</u>: $f(z) = \dfrac{1}{z^2(z-1)}$

Note: $z^2 f(z-1) = \dfrac{1}{z-1} \xrightarrow[z\to 0]{} -1$

By Thm 5.8(2), $a=0$ is a pole of order 2. On the other hand,

$$(z-1)f(z) = \frac{1}{z^2} \to 1$$

$\Rightarrow$ By Thm 3.8(2), $a=1$ is a simple pole

<u>Corollary 3.9</u>: The function $f \in H(D(a,R))$ has a zero of order m at a iff $1/f$ has a pole of order m at a.

— / —

$(z-a)^m$ — root (zero)

$(z-a)^{-m}$ —

— / —

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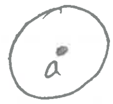
① Poles
② Essential Singularites
③ Removable Singularites.

Residues.

Def^n 5.10 : Assume that f has an isolated singaularity at a and let

$$f(z) = \sum_{k=-\infty}^{\infty} a_k (z-a)^k$$



be its Laurent's Expansion. Then the coefficient a_{-1} is called the residue of f at the point a.

Notation: $a_{-1} = \text{Res}(f, a)$

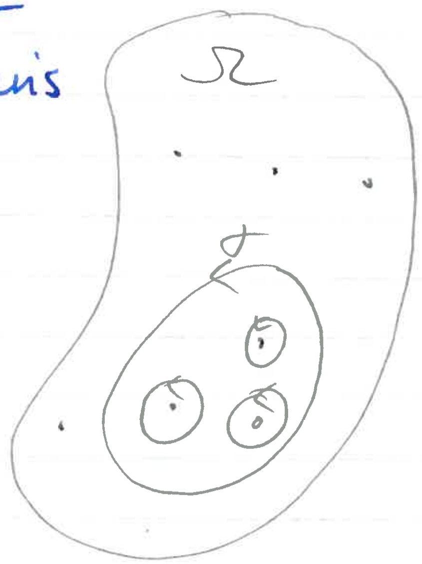
By Laurent's Theorem:

$$a_{-1} = \frac{1}{2\pi i} \int_{S(a,r)} f(z)\, dz$$

Theorem 5.11 (The Cauchy's residue theorem). Assume that f is holomorphic on Ω except for finitely many isolated singularities. Let $\gamma \subset \Omega$ be a contour st Int γ contains sigularties $p_1, p_2 \ldots p_n$
Then:

$$\int_\gamma f(z)\, dz = 2\pi i \sum_{k=1}^{n} \text{Res}(f, p_k)$$

<u>Proof</u>: By Cauchy - Goursat for multiply connected domains (Thm 4.16)



$$\int_\gamma f(z)\,dz = \sum_{k=1}^{N} \int_{S(p_k, r)} f(z)\,dz$$

$$= \sum_{k=1}^{N} 2\pi i \operatorname{Res}(f, p_k)$$

□

<u>Rule for finding residues.</u>

<u>Rule I</u>: Suppose that a is a simple pole i.e

$$f(z) = \frac{a_{-1}}{z-a} + a_0 + a_1(z-a) + \dots$$

Multiply:

$$(z-a)f(z) = a_{-1} + a_0(z-a) + a_1(z-a)^2 + \dots$$

Thus

$$a_{-1} = \operatorname{Res}(f, a) = \lim_{z \to a} (z-a)f(z).$$

<u>Rule II</u>: Suppose f has a pole of order m at a:

$$f(z) = \frac{a_{-m}}{(z-a)^m} + \frac{a_{-m+1}}{(z-a)^{m-1}} + \dots + \frac{a_{-1}}{z-a} + a_0 + a_1(z-a) + \dots$$

Multiply:

$$(z-a)^m f(z) = a_{-m} + a_{-m+1}(z-a) + \dots + a_{-1}(z-a)^{m-1}$$

$$= g(z)$$

Thus; $a_{-1} = \frac{1}{(m-1)!} \frac{d^{m-1}}{dz^{m-1}} g(z) \Big|_{z=a}$

Rule III: Expand f in its Laurent series and take a_{-1}.

Example

1) $f(z) = \frac{\sin z}{z^4} = \dfrac{z - \frac{z^3}{6} + \frac{z^5}{120} - \dots}{z^4}$

$$= \frac{1}{z^3} - \frac{1}{6z} + \frac{z}{160} + \dots$$

$$\Rightarrow \operatorname{Res}(f, 0) = -\frac{1}{6}.$$

2) $f(z) = \frac{1}{z^2(z-1)}$

$$\operatorname{Res}(f, 1) \underset{\text{Rule I}}{=} \lim_{z \to -1} (z-1) f(z)$$

$$= \lim_{z \to 1} \frac{1}{z^2} = 1$$

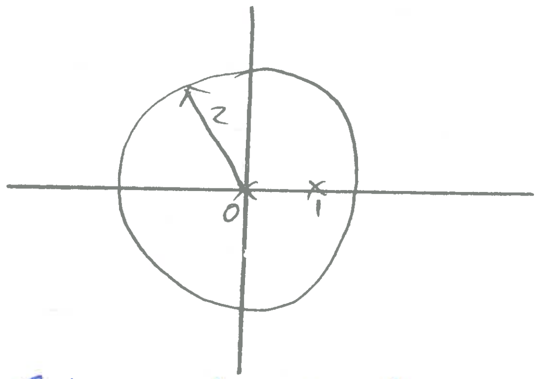
$$\mathrm{Res}(f,0) \underset{\text{Rule II}}{=} \frac{d}{dz}\left(z^2 f(z)\right)\Big|_{z=0}$$

$$= \frac{d}{dz}\frac{1}{z-1}\Big|_{z=0}.$$

$$= -\frac{1}{(z-1)^2}\Big|_{z=0}$$

$$= -1$$

$$\int_\Gamma f(z)\,dz = 2\pi i\left(\mathrm{Res}(f,1) + \mathrm{Res}(f,0)\right) = 0$$



3) $f(z) = z^2 \sin\left(\frac{1}{z}\right) = z^2\left(\frac{1}{z} - \frac{1}{6z^3} + \frac{1}{120z^5} - \dots\right)$

$$= z - \frac{1}{6z} + \frac{1}{120z^3} - \dots$$

$\mathrm{Res}(f,0) = -1/6$.

<u>Question</u>: What is $\mathrm{Res}(f,a)$ if f has a removable singularity at a? (zero)

Removable singularity $\iff$ pp of $f = 0$

In particular $a_{-1} = 0 \Rightarrow \mathrm{Res}(f,a) = 0$

<u>12/12/11</u>

<u>Trigonometric integrals</u>

Looking at : $\int_0^{2\pi} f(\cos\theta, \sin\theta)\, d\theta$.

<u>Example</u>: $I = \int_0^{2\pi} \frac{1}{5 - 4\cos\theta}\, d\theta$

Define: $z = e^{i\theta}$

$dz = ie^{i\theta} d\theta = iz\, d\theta$

$\Rightarrow d\theta = \frac{dz}{iz}$

Thus: $\int_0^{2\pi} f(\cos\theta, \sin\theta)\, d\theta = \int_{S(0,1)} f\left(\frac{1}{2}\left(z+\frac{1}{z}\right), \frac{1}{2i}\left(z-\frac{1}{z}\right)\right) \frac{dz}{iz}$

$\cos\theta = \frac{1}{2}\left(z + \frac{1}{z}\right), \sin\theta = \frac{1}{2i}\left(z - \frac{1}{z}\right)$

<u>Example</u>: $I = \int_{S(0,1)} \frac{1}{5 - 4\cdot\frac{1}{2}(z+z^{-1})} \frac{dz}{iz}$

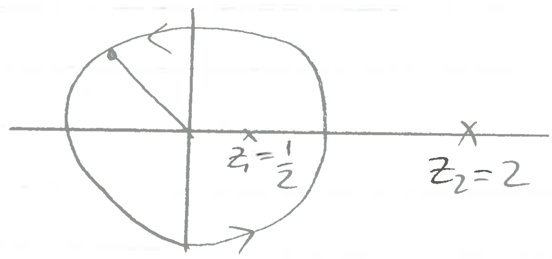
$= -i\int_{S(0,1)} \frac{1}{5z - 2z^2 - 2}\, dz$

$= \int_{S(0,1)} \frac{i}{2z^2 - 5z + 2}\, dz$

$= \int_{S(0,1)} \frac{i}{(2z-1)(z-2)}\, dz$

Two singularities : $z_1 = 1/2$, $z_2 = 2$.

Therefore: $I = 2\pi i \left[\text{Res}(g, \tfrac{1}{2}) + \cancel{\text{Res}(g, 2)} \right]$ ← No need



$$g(z) = \frac{i}{(2z-1)(z-2)}$$

$\{1/2, 2\}$ are simple poles, so:

Rule I

$$\text{Res}(g, \tfrac{1}{2}) = \lim_{z \to \frac{1}{2}} (z - \tfrac{1}{2}) \frac{i}{(2z-1)(z-2)}$$

$$= \frac{i}{2} \lim_{z \to \frac{1}{2}} \frac{1}{z-2} = -\frac{i}{3}$$

NO NEED

$$\cancel{\text{Res}(g, \tfrac{1}{2})} = \lim_{z \to 2} (z-2) \frac{i}{(2z-1)(z-2)} = \frac{i}{3}$$

its outside the contour so we shouldn't consider it

$$\text{Res}(g, \tfrac{1}{2}) + \text{Res}(g, 2) = 0.$$

$$\Rightarrow \cancel{I = 0}$$ ← WRONG ANS.

$$I = 2\pi i \left(-\frac{i}{3} \right) = \frac{2}{3}\pi .$$

Try $\quad I_1 = \displaystyle\int_0^{2\pi} \frac{1}{(5-4\cos\theta)^2}\, d\theta.$ DO IT!

$\quad \| $

$I = \displaystyle\int_{S(0,1)} \frac{i}{(2z-1)^2(z-2)^2}\, dz$

poles of order 2

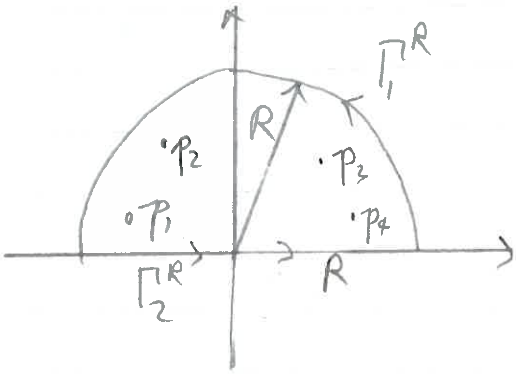
—/—

Improper real integrals:

Integrals of the type:

$$I = \int_{-\infty}^{\infty} f(t)\, dt.$$

Assume: f has a finitely many isolated singularities in the upper half-plane: $p_1, p_2, \dots, p_N$.



Represent: $I = \displaystyle\lim_{R\to\infty} \int_{-R}^{R} f(t)\, dt.$

$$\Gamma_1^R = \{ z \in \mathbb{C} : \operatorname{Im} z \geqslant 0,\ |z| = R \}$$

$$\Gamma_2^R = \{ z \in \mathbb{C} : \operatorname{Im} z = 0,\ |z| \leqslant R \}$$

$\Gamma^R = \Gamma_1^R \cup \Gamma_2^R$

Then: $\displaystyle\int_{\Gamma^R} f(z)\, dz = 2\pi i \sum_{k=1}^{N} \operatorname{Res}(f, p_k)$

Want: $\displaystyle\int_{\Gamma_1^R} f(z) \longrightarrow 0$ as $R \to \infty$

so: $I = \lim_{R \to \infty} \int_{\Gamma^R} f(z)\,dz = 2\pi i \sum_{k=1}^{N} \mathrm{Res}(f, p_k)$

<u>Lemma 5.12</u>: Suppose that f is continuous in $\Pi_+ \setminus D(0, R_0)$ with some $R_0 > 0$ and that:

$$\max_{z \in \Gamma_1^R} |f(z)| \leq \frac{C}{R^{\varkappa}}, \quad \varkappa > 1$$

Then $\int_{\Gamma_1^R} f(z)\,dz \to 0, \quad R \to \infty$.

<u>Proof</u>: By Thm 4.7(7).

$$\left| \int_{\Gamma_1^R} f(z)\,dz \right| \leq \frac{C}{R^{\varkappa}} \pi R = C \pi R^{1-\varkappa} \to 0$$

Important that $\varkappa > 1$

As $R \to \infty$

<u>Example</u>:

$$I = \int_{-\infty}^{\infty} \frac{x^2}{x^4 + 5x^2 + 4}\,dx$$

Define $= \dfrac{z^2}{z^4 + 5z^2 + 4}$

$$= \frac{z^2}{(z^2+1)(z^2+4)}$$

$$= \frac{z^2}{(z-i)(z+i)(z+2i)(z-2i)}$$

Two simple poles: $z_1 = i$, $z_2 = 2i$, $\operatorname{Im} z_1 > 0$

Therefore, by Cauchy Residue Theorem,

$$\int_{\Gamma^R} f(z)\,dz = 2\pi i\left[\operatorname{Res}(f,i) + \operatorname{Res}(f,2i)\right]$$

By Rule I;

$$\operatorname{Res}(f,i) = \lim_{z\to i}(z-a)f(z) = \lim_{z\to i}\frac{z^2}{(z+i)(z^2+4)} = \frac{-1}{(2i)3}$$

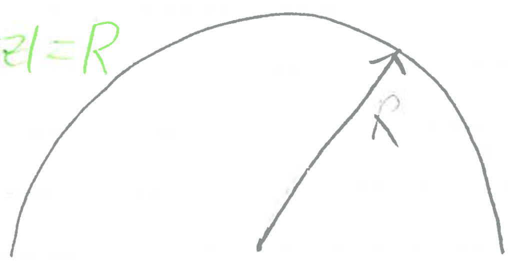
$$= \frac{i}{6}$$

$$\operatorname{Res}(f,2i) = \lim_{z\to 2i}(z-2i)f(z) = \lim_{z\to 2i}\frac{z^2}{(z^2+1)(z+2i)} = \frac{-4}{(-3)(4i)}$$

$$= -\frac{i}{3}$$

Thus: $\int_{\Gamma^R} f(z)\,dz = 2\pi i\left[-\frac{i}{3} + \frac{i}{6}\right] = \frac{\pi}{3}$

Estimate f on Γ_1^R: Note $|z| = R$



$$\left|\frac{z^2}{(z^2+1)(z^2+4)}\right| = \frac{|z|^2}{|z^2+1||z^2+4|}$$

$$\dots \leq \frac{R^2}{(|z|^2-1)(|z|^2-4)} = \frac{R^2}{(R^2-1)(R^2-4)}$$

$$\leq \frac{C' R^2}{R^2 \cdot R^2} = \frac{C'}{R^2} \quad \text{with some constant } C > 0$$

Now use lemma 5.12 $\Rightarrow \int_{\Gamma_1^R} f(z)\,dz \to 0, \ R \to \infty$

Put everything together

$$I = \lim_{R\to\infty} \int_{\Gamma^R} f(z)\,dz = 2\pi i\left(\mathrm{Res}(f, i) + \mathrm{Res}(f, 2i)\right) = \frac{\pi}{3}$$

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MUST QUOTE: Cauchy Residue Theorem and lemma 5.12 (explain why $\int_{\Gamma_1^R} f(z)\,dz \to 0$) Will always come up in exams.

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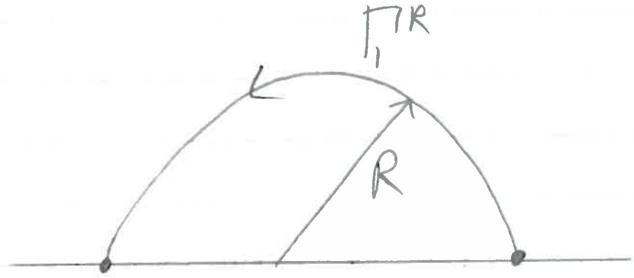
<u>Recall that</u>: $\max_{z\in\Gamma_1^R} |f(z)| \leq \frac{C}{R^{\alpha}}, \ \alpha > 1$

<u>Integral containing exponentials</u>:

$$J = \int_{-\infty}^{\infty} e^{i\lambda z} f(z)\,dz$$

Lemma 5.13 (Jordan's lemma) Suppose that f is continuous in $\Pi_+ \setminus D(0, R_0)$ with some $R_0 > 0$ and let

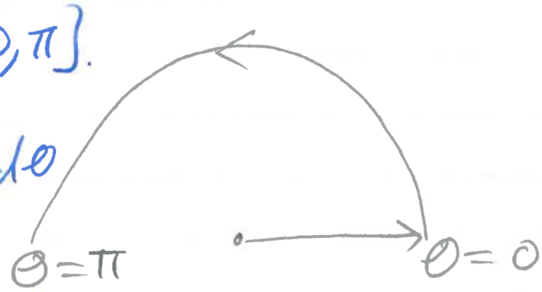


$$M(R) = \max_{z \in \Gamma_1^R} |f(z)| \to 0, \quad R \to \infty$$

If $a > 0$, then

$$\int_{\Gamma_1^R} e^{iaz} f(z)\, dz \to 0, \quad R \to \infty$$

Proof: Let $z = Re^{i\theta}$. $\theta \in [0, \pi]$.

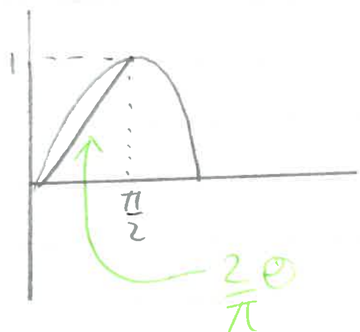


Then $I = \int_0^\pi e^{iaRe^{i\theta}} f(Re^{i\theta})\, iRe^{i\theta}\, d\theta$

$$= iR \int_0^\pi e^{-aR\sin\theta} e^{iaR\cos\theta} f(Re^{i\theta}) e^{i\theta}\, d\theta$$

Therefore: $|I| \leq R \int_0^\pi e^{-aR\sin\theta} |f(Re^{i\theta})|\, d\theta$

$$\leq M(R) R \int_0^\pi e^{-aR\sin\theta}\, d\theta.$$



Observe: $\sin\theta \leq \frac{2}{\pi}\theta$, so

$$|I| \leq 2RM(R) \int_0^{\frac{\pi}{2}} e^{-a\frac{2}{\pi}R\theta}\, d\theta$$

Using the fact $a > 0$

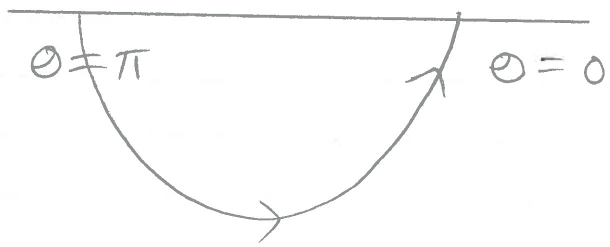
$$\ldots \leq 2RM(R)\int_0^\infty e^{-\frac{2a}{\pi}R\theta}\,d\theta$$ Using the fact $a>0$

$$= \frac{\pi}{a}M(R) \to 0 \quad \text{As } R\to\infty$$

as required. $\square$

Remark: If $a<0$, then lemma 5.13 still holds if one replaces Γ_1^R by the path

$$\tilde{\Gamma}_1^R = \{z = Re^{i\theta} : \theta\in[-\pi, 0]\}$$



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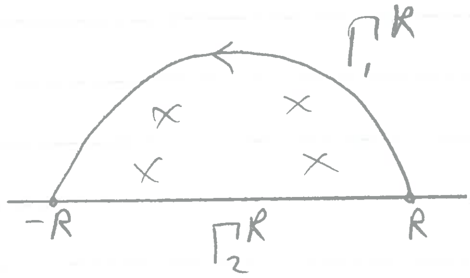
$$\int_{-\infty}^{\infty} f(x)\,dx, \quad \int_{-\infty}^{\infty} e^{iax} f(x)\,dx$$

$\Gamma_1^R = \{z \in \mathbb{C} : z = Re^{i\theta},\ \theta \in [0,\pi]\}$



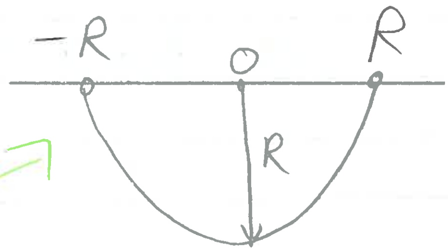
Important:

$$\int_{\Gamma_1^R} f(z)\,dz \to 0, \quad R \to \infty$$

Lemma 5.13 (Jordan's lemma)

$$M(R) = \max_{z \in \Gamma_1^R} |f(z)| \to 0, \quad R \to \infty$$

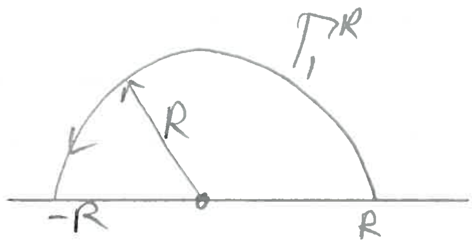
If $a > 0$ then $\displaystyle\int_{\Gamma_1^R} e^{iaz} f(z)\,dz \xrightarrow{R\to\infty} 0$



If $a < 0$ look at the lower half-plane.

Example: $I_1 = \displaystyle\int_{-\infty}^{\infty} \frac{x \sin x}{(x^2+a^2)^2}\,dx$, $I_2 = \displaystyle\int_{-\infty}^{\infty} \frac{x e^{ix}}{(x^2+a^2)^2}\,dx$, $x >$

$f(z) = \dfrac{z}{(z^2+a^2)^2}$. Then $I_2 = \displaystyle\int_{-\infty}^{\infty} f(x) e^{ix}\,dx$ and $I_1 = \mathrm{Im}(I_$



Rewrite: $I_2 = \displaystyle\lim_{R\to\infty} \int_{-R}^{R} f(x) e^{ix}\,dx$

Need to do: ① : find singularities of f in the upper half plane and evaluate the residues.
② Show that $\int_{\Gamma_1^R} f(z)e^{iz}\,dz \to 0$ as $R \to \infty$

② : $a = 1 > 0$ and $\max_{|z|=R} |f(z)| = \max_{|z|=R} \frac{|z|}{|z^2+a^2|^2}$

$\leq \frac{R}{|R^2-a^2|} \leq CR^{-3} \to 0,\ R \to \infty$

By Jordan's Lemma:

$$\int_{\Gamma_1^R} f(z)e^{iz}\,dz \to 0$$

As $R \to \infty$

MUST QUOTE THIS AND ALL THINGS IN THE EXAM LIKE ALL THE CONTOURS

① Residue of f: two singluar points: $+ia$, $-ia$ only $p = ia$ is in the upper half plane. Write:

$$f(z) = \frac{z}{(z+ia)^2(z-ia)^2}$$

$p = ai$ is a pole of order 2.

Thus Rule II

$$\operatorname{Res}(fe^{ia}, p) = \frac{d}{dz}\left[(z-ia)^2 f(z) e^{iz}\right]\Big|_{z=ia}.$$

$$= \frac{d}{dz}\left(\frac{ze^{iz}}{(z+ia)^2}\right)\Big|_{z=ia}.$$

$$= \left[\frac{e^{iz}}{(z+ia)} - \frac{2ze^{iz}}{(z+ia)^3} + \frac{ize^{iz}}{(z+ia)^2}\right]\Big|_{z=ia}.$$

$$= \frac{-e^{-a}}{4a^2} + \frac{2iae^{-a}}{8ia^3} + \frac{ae^{-a}}{4a}.$$

Therefore:

$$I_2 = \lim_{R\to\infty}\int_{-R}^{R} f(x)e^{ix}\,dx = 2\pi i \operatorname{Res}(fe^{iz}, ia)$$

$$= \frac{2\pi i e^{-a}}{4a} = \frac{\pi i}{2a}e^{-a}.$$

Also : $I_2 = \frac{\pi}{2a}e^{-a}$.

<u>The indentation trick</u>: $I = \int_{-\infty}^{\infty} \frac{\sin x}{x}\,dx$

$\int_0^1 \frac{1}{x}\,dx$ - Not good , Trig: $I = \operatorname{Im}\int_{-\infty}^{\infty}\frac{e^{ix}}{x}\,dx$?? (Not good)

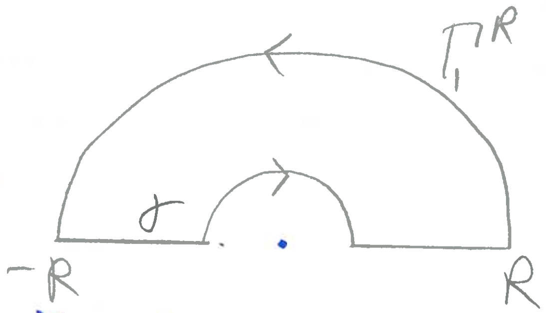
Let $f(z) = \frac{\sin z}{z}$, Know: f is entire.

Therefore: $I = \int_\gamma f(z)\,dz = \underbrace{\frac{1}{2i}\int_\gamma \frac{e^{iz}}{z}\,dz}_{I_1} - \underbrace{\frac{1}{2i}\int_\gamma \frac{e^{-iz}}{z}\,dz}_{I_2}$

$I_1 = ?$

Need to show: $\int_{\Gamma_1^R} \frac{e^{iz}}{z}\,dz \to 0,\ R \to \infty$

By Jordan's Lemma it is the case e^{iz}/z has no singularities inside the contour and hence

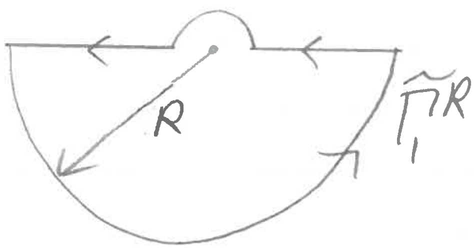


$$\int_{-\infty}^{\infty} \frac{e^{iz}}{z}\,dz = \lim_{R\to\infty} \int \frac{e^{iz}}{z}\,dz = 0.$$

$$I_2 = \frac{1}{2i}\int_\gamma \frac{e^{-iz}}{z}\,dz = ?$$

By Jordan's lemma

$\int_{\tilde{\Gamma}_1^R} \frac{e^{-iz}}{z}\,dz \to 0,\ R \to \infty$



$$\operatorname{Res}\left(\frac{e^{-iz}}{z}, 0\right) = \lim_{z\to 0}\left(\frac{z\,e^{-iz}}{z}\right) = 1.$$

Thus $\displaystyle\int_{-\gamma} \frac{e^{-iz}}{z}\,dz = \lim_{R\to\infty} \int \frac{e^{-iz}}{z}\,dz = 2\pi i$

Therefore :

$$I_2 = \frac{1}{2i}\int_{\gamma} \frac{e^{-iz}}{z}\,dz = \frac{-1}{2i}\int_{-\gamma} \frac{e^{-iz}}{z} = \frac{-2\pi i}{2i} = -\pi$$

$I_1 = 0.$

Thus $I = I_1 - I_2 = \pi.$

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Try

$$\int_{-\infty}^{\infty} \frac{\cos x - 1}{x^2}\,dx$$

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