2101 Analysis 3: Complex Analysis Notes

Based on the 2011 autumn lectures by Prof A Sobolev

2101 COMPLEX ANALYSIS 3/10/2011

A. SOBOLEV, room 710 math's dept
Office hour: monday 1pm

Book: Priestley

Plan

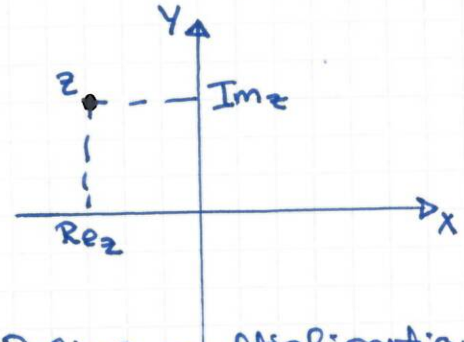
I
1. COMPLEX NUMBERS
2. SETS OF COMPLEX PLANE
3. CONTINUITY

II
4. Differentiability
5. Integration

1. COMPLEX NUMBERS

Let $z \in \mathbb{R}^2$ be a point in the plane.
Then $z = (x, y)$, $x, y \in \mathbb{R}$
notation: x = Real part of z, $\mathrm{Re}\, z$

y = imaginary part of z, $\mathrm{Im}\, z$

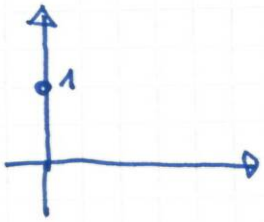


Def 1.1 Define multiplication: let $z_1, z_2 \in \mathbb{R}^2$
If $z_1 = (x_1, y_1)$, $z_2 = (x_2, y_2)$, then $z_1 z_2 = (x_1 x_2 - y_1 y_2,\ x_1 y_2 + x_2 y_2)$
With this multiplication $\mathbb{R}^2$ becomes complex plane.
notation: $\mathbb{C}$

Observe: $z_1 z_2 = z_2 z_1$

$z_1(z_2 + z_3) = z_1 z_2 + z_1 z_3$

GOOD DEFINITION.
Look at $z = (0,1)$



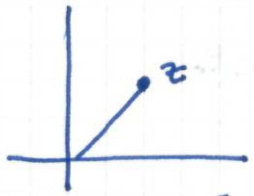
Then $(0,1)^2 = (-1, 0)$

notation: $i = (0,1)$. Then $z = (x, y)$
$= x(1,0) + y(0,1)$
$= x + iy$

standard form of complex numbers.

Complex plane $\mathbb{C}$ = Argand plane

Def 1.2 The modulus (or the absolute value) of $z \in \mathbb{C}$ is
$|z| = \sqrt{x^2+y^2} = \sqrt{(\mathrm{Re}\, z)^2 + (\mathrm{Im}\, z)^2}$



$|z|$ is the distance from z to the origin
$|z_1 - z_2| = \dots$

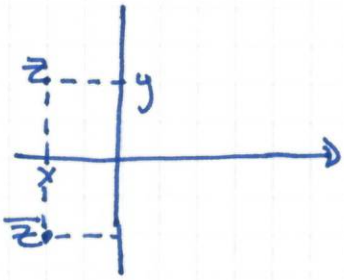
Define $S = \{ z : |z| = 1 \}$ - circle of radius 1
$S_a = \{ z : |z-a| = 1 \}$

circle of radius 1 with center at the point a

<u>notation</u>: $\gamma(a,r) = \{ z : |z-a| = r \}, a \in \mathbb{C}, r \geq 0$
circle of rad r centered at a

Def 1.3 If $z = x+iy \in \mathbb{C}$, then the conjugate of z is defined to be $\bar{z} = x - iy$

note $\bar{\bar{z}} = z$



<u>Prop 1.4</u>

① $\overline{z_1 + z_2} = \bar{z}_1 + \bar{z}_2$

② $\overline{z_1 z_2} = \bar{z}_1 \bar{z}_2$

③ $\bar{z} z = z \bar{z} = |z|^2$

④ $|z_1 z_2| = |z_1| |z_2|$

⑤ $\frac{1}{z} = \frac{\bar{z}}{z \bar{z}} = \frac{\bar{z}}{|z|^2} = \frac{x-iy}{(x^2+y^2)} = \frac{x}{x^2+y^2} - i \frac{y}{x^2+y^2}$

<u>Prop 1.5</u>
Let $z = x+iy$. Then $x = \mathrm{Re}\, z = \frac{z+\bar{z}}{2}$, $y = \mathrm{Im}\, z = \frac{z-\bar{z}}{2i}$

<u>Inequalities</u>

<u>Lemma 1.6</u> Let $z, w \in \mathbb{C}$. Then

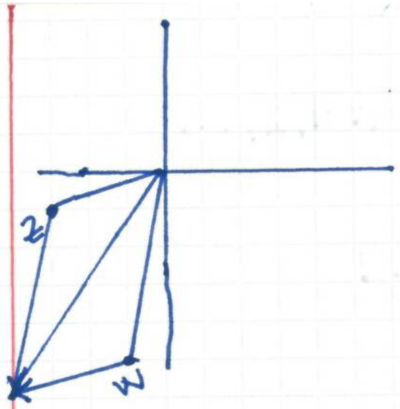
① $|\mathrm{Re}\, z| \leq |z|$, $|\mathrm{Im}\, z| \leq |z|$

② $|z+w| \leq |z| + |w|$, triangle inequality

(3) $|z-w| \geq ||z| - |w||$

PROOF - ①, ③ excercise

②



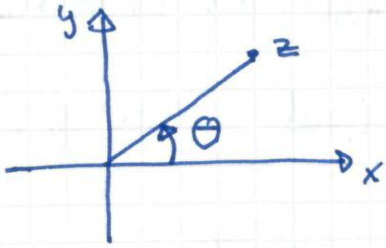
$$|z+w|^2 = (\bar{z}+\bar{w})(z+w)$$
$$= \bar{z}z + \bar{w}z + \bar{z}w + \bar{w}w$$
$$= |z|^2 + 2\mathrm{Re}(\bar{w}z) + |w|^2$$
$$\leq |z|^2 + 2|\bar{w}z| + |w|^2$$
$$= |z|^2 + 2|w||z| + |w|^2$$
$$= (|z|+|w|)^2$$

$\Rightarrow |z+w|^2 \leq (|z|+|w|)^2$

$\Rightarrow |z+w| \leq |z| + |w|$ as required ■

The polar form

Let $z = x+iy \in \mathbb{C}$. Introduce polar coordinates:



sitive irection anti- ockwise

let $r = |z|$

Then $x = r\cos\theta$
$y = r\sin\theta$

Hence $z = r\cos\theta + ir\sin\theta = r(\cos\theta + i\sin\theta)$

Denote: $\cos\theta + i\sin\theta = e^{i\theta}$

The angle θ is called the argument of z, notation: $\theta = \arg z$

Recall: $e^{t+s} = e^t e^s$, $s,t \in \mathbb{R}$

Lemma 1.8 Let $z_1 = r_1 e^{i\theta_1}$, $z_2 = r_2 e^{i\theta_2}$. Then
$z_1 z_2 = r_1 r_2 e^{i(\theta_1+\theta_2)}$

PROOF

Write: $z_1 z_2 = r_1 r_2(\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 + i\sin\theta_2)$
$= r_1 r_2 [\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2 + i(\cos\theta_1\sin\theta_2 + \sin\theta_1\cos\theta_2)]$
$= r_1 r_2 [\cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2)]$
$= r_1 r_2 e^{i(\theta_1+\theta_2)}$ ■

examples

$\arg i = \frac{\pi}{2}$ or $\frac{\pi}{2} + 2\pi$

$\arg -i = \frac{3\pi}{2}$ or $-\frac{\pi}{2}$

Def 1.7
The principal value of the argument is defined as the uniquely defined value of θ in the interval $(-\pi, \pi]$

Notation: $\mathrm{Arg}\, z$

$\mathrm{Arg}(-i) = -\frac{\pi}{2}$, $\mathrm{Arg}(-1) = \pi$

Observe: $e^{2\pi i} = 1$, $e^{2\pi n i} = 1$, $n \in \mathbb{Z}$

Prop 1.9 (De Moivre's formula)

$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta \quad , n=1,2,\ldots$

POWERS OF Z

$z^n = r^n e^{in\theta} \quad , n=1,2,3,\ldots$ from lemma 1.9

Definition: For any $\alpha \in \mathbb{R}$

$z^{\alpha} = r^{\alpha} e^{i\alpha\theta}$.

example: $z^{\frac{1}{2}} = |z|^{1/2} e^{i\frac{1}{2}\arg z}$

$\sqrt{1} = \pm 1 \qquad$ if $\arg 1 = 0$, then $\sqrt{1} = 1$
if $\arg 1 = 2\pi$, then $\sqrt{1} = -1$

$\sqrt{1} = 1$ and $\sqrt{1} = -1$ respect to two different branches of the square root

The value $\sqrt{1} = 1$ is called the principal value of $\sqrt{1}$.

$\sqrt{1} = -1$ is the other value of the root

In general, let $z = re^{i\theta} = re^{i\theta + 2\pi n i} \quad , n \in \mathbb{Z}$

Then $z^{\alpha} = r^{\alpha} e^{i\alpha\theta + i2\pi n\alpha}$

Different values of n represent different branches of z^{α}. The principal value: $z^{\alpha} = |z|^{\alpha} e^{i\alpha \operatorname{Arg} z}$

example

$1^{1/3} = e^{i\frac{2\pi n}{3}} \quad , n \in \mathbb{Z}$

$$1^{1/3} = \begin{cases} 1 & , n=0 \\ e^{i\frac{2\pi}{3}} & , n=1 \\ e^{i\frac{4\pi}{3}} & , n=2 \end{cases}$$

If $n=0 \quad 1^{1/3} = 1$
$n=1 \quad 1^{1/3} = e^{i\frac{2\pi}{3}}$
$n=2 \quad 1^{1/3} = e^{i\frac{4\pi}{3}}$

$z^{\alpha} = r^{\alpha} e^{i\alpha \arg z}$

if $\alpha = \frac{1}{2}$, $z=1$, then

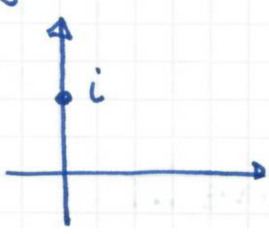
$1^{\frac{1}{2}} = |1|^{1/2} e^{i\frac{1}{2}\arg 1}$

the arithmetic root of 1

05 october 2011

$z = x + iy \quad , x, y \in \mathbb{R} \quad z \in \mathbb{C}$

$i: \quad i^2 = -1$



$\text{Arg } z \in (-\pi, \pi]$

$e^{i\theta} = \cos\theta + i\sin\theta$, $\theta \in \mathbb{R}$

$|e^{i\theta}| = 1$

$\gamma(a, r) = \{z : |z-a| = r\}$, $a \in \mathbb{C}$, $r > 0$

$= \{z = a + re^{i\theta}, \theta \in [0, 2\pi)\}$

GEOMETRY AND TOPOLOGY OF COMPLEX PLANE

sets of complex plane

Def 1.10 Let $z_0 \in \mathbb{C}$, $r > 0$. Then the set $\gamma(z_0, r) = \{z : |z - z_0| = r\}$ is called circle of radius r centered at z_0.

The set $D(z_0, r) = \{z : |z - z_0| < r\}$ is called open disk of radius r centered at z_0.

The set $\bar{D}(z_0, r) = \{z : |z - z_0| \leq r\}$ is called closed disc of radius r centered at z_0.

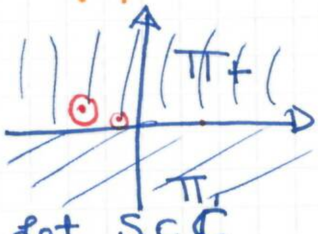
Sometimes we call $D(z_0, r)$ an neighbourhood of z_0.

The set $D'(z_0, r) = \{z : 0 < |z - z_0| < r\}$ is called punctured r-neighbourhood of z_0.

Half-planes

$\Pi_+ = \{z : \text{Im } z > 0\}$ - upper half-plane

$\Pi_- = \{z : \text{Im } z < 0\}$ - lower half-plane



Let $S \subset \mathbb{C}$

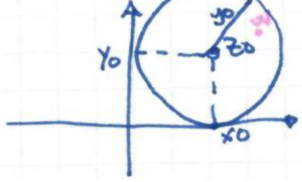
Def 1.11 Let $z \in S$. Then z is said to be an interior point of S if there is a number $r > 0$ s.t $D(z, r) \subset S$

The set of all interior points of S is denoted $\text{int } S$.

Open set We say that S is open if it consists of interior points only, i.e. $\text{int } S = S$, or for any $z \in S$ there is a number $r > 0$ s.t $D(z, r) \subset S$

ex

Π_+ is open. Let $z_0 \in \Pi_+$, i.e $z_0 = x_0 + iy_0$ with $y_0 > 0$.



Let $r = y_0$. Then take $w \in D(z_0, y_0)$.
Let's show that $\text{Im } w > 0$, i.e $w \in \Pi_+$.

Write $w = z_0 + w - z_0$, so $\operatorname{Im} w = \operatorname{Im} z_0 + \operatorname{Im}(w - z_0)$

$= y_0 + \operatorname{Im}(w - z_0)$

Lemma 1.6 $\geq y_0 - |\operatorname{Im}(w - z_0)|$

$\geq y_0 - |w - z_0| \geq y_0 - y_0 = 0$ and hence $\operatorname{Im} w > 0$, as required. ∎

example
Prove that $D(a,r)$, $r>0$, is open

HW1

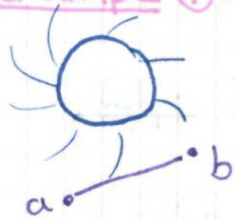

need to show that for any point $w \in D(a,r)$ there is a number $\varepsilon > 0$ s.t. $D(w,\varepsilon) \subset D(a,r)$.

Take $\varepsilon = r - |a-w|$

EXAM: open an closed s definitio

Def 1.12 The set $S^c = \mathbb{C} \setminus S$ is the complement of S. We say that S is closed if complement is open.

example ① $\overline{D}(a,r)$ is closed.



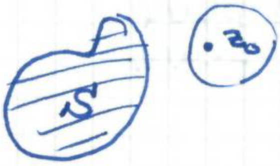
② The interval (segment) $[a,b] = \{(1-t)a + tb,\ t \in [0,1]\}$ is closed

③ $D'(z_0,r)$ is open

④ $S = \{z : 2 \leq |z-1| < 3\}$ is neither closed or open (2 is included, 3 is not)

Def 1.13 (1) A point $z_0 \in \mathbb{C}$ is said to be an accumulation point of the set S if for all $r>0$ we have

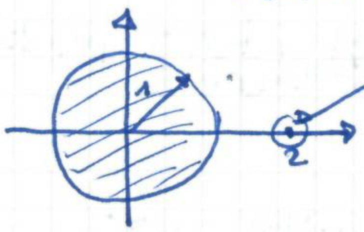
$D'(z_0,r) \cap S \neq \emptyset$



z_0 is not an accumulation point, you can find an r small enough, so the disc doesn't overlap.

z_0 is an accumulation point

Let $T = D(0,1) \cup \{2\}$



Not an accumulation point.

Open set 10/10 - 2011

$D(z,r)$ open disk or radius $r>0$ centered at z.

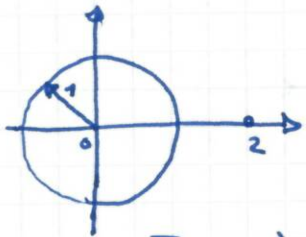
Def 1.13
Let S be a set on $\mathbb{C}$. Let $z \in \mathbb{C}$. We say that z is an accumulation point of the set S iff for all $r>0$ we have $D(z,r) \cap S \neq \emptyset$.
The closure of the set S is the union of the set S and all its accumulation points.
Notation: $\overline{S}$

examples
(1) $S = \Pi_+ = \{z : \operatorname{Im} z > 0\}$
accumulation points of $S = \{z : \operatorname{Im} z \geq 0\}$
$\overline{S} = \{z : \operatorname{Im} z \geq 0\}$

(2) $S = D(0,2)$

accumulation points of $S = \overline{D}(0,2)$
$\overline{S} = \overline{D}(0,2)$

(3) $D(0,1) \cup \{2\} = T$



accumulation points of $S = \overline{D}(0,1)$

?

$D'(2, \frac{1}{4}) \cap T = \emptyset$ so 2 is not an accumulation point.

$\overline{T} = \overline{D}(0,1) \cup \{2\}$

(4) $S = \overline{D}(0,5)$, accumulation points $= \overline{D}(0,5)$
$\overline{S} = \overline{D}(0,5) = S$

THEOREM 1.14
The following statements are equivalent:

1- The set S is closed
2- S contains all its accumulation points
3- $\overline{S} = S$

proposition 1.15 Let $S \subset \mathbb{C}$. Then
① $\overline{S}$ is a closed set,
② $\overline{S}$ is the smallest closed set containing S, i.e. for any closed set $\Omega \supset S$ we have $\overline{S} \subset \Omega$ (they can be equal, but not necessarily)

Def 1.16 The set $\partial S = \overline{S} \setminus \operatorname{int} S$ is called the boundary of S.

Def 1.17 The set S is said to be bounded if there is a number $R>0$ s.t $S \subset D(0,R)$

① Π_+, Π_- are not bounded

② $D(10,1)$ is bounded, as $D(10,1) \subset D(0,100)$

Def 1.17 cont

S is called compact if it is closed and bounded

$D(0,2)$ is not compact since it is not closed.

convexity and connectedness

Def 1.18 The set S is said to be convex if for any two points $a, b \in S$ the segment $[a,b]$ is also in the set

not convex

convex

examples

① $\Pi_- = \{z : \operatorname{Im} z < 0\}$

let $a, b \in \Pi_-$, i.e $\operatorname{Im} a < 0$, $\operatorname{Im} b < 0$

let $z = (1-t)a + tb$, $t \in [0,1]$

Then

$$\operatorname{Im} z = \underbrace{(1-t)}_{\geq 0} \underbrace{\operatorname{Im} a}_{<0} + t \underbrace{\operatorname{Im} b}_{<0} < 0$$

$\Rightarrow z \in \Pi_-$, i.e $[a,b] \subset \Pi_-$, i.e Π_- is convex.

② $S = D(a,r)$ - convex

let $z_1, z_2 \subset S$, i.e $|z_1 - a| < r$, $|z_2 - a| < r$

let $z = (1-t)z_1 + tz_2$, so need to show $t \in [0,1]$, so need to show that $|z-a| < r$

write $z - a = (1-t)(z_1 - a) + t(z_2 - a)$, so

$$|z-a| = |(1-t)(z_1-a) + t(z_2-a)|$$

$$\overset{\text{triangle}}{\leq} |(1-t)(z_1-a)| + |t(z_2-a)|$$

$$= (1-t)|z_1-a| + t|z_2-a| < (1-t)r + tr = r$$

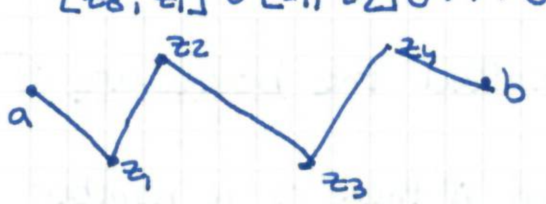
This is exactly what we want, $z \in D(a,r)$, i.e $[z_1, z_2] \subset D(a,r)$ i.e $D(a,r)$ is convex.

③ $\overline{D}(a,r)$ - convex [illegible]

PROVE IT!

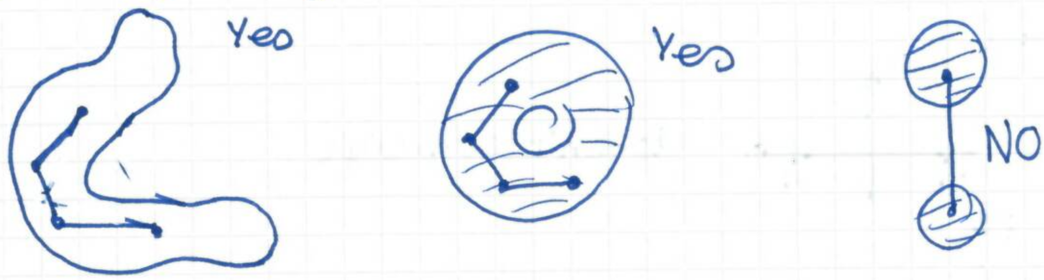
Def 1.19 let $a, b \in \mathbb{C}$ and let $a = z_0, z_1, \ldots, z_n = b$. we call the set

$$[z_0, z_1] \cup [z_1, z_2] \cup \ldots \cup [z_{n-1}, z_n]$$



is called a polygonal path from a to b.

A set S is said to be polygonally conected if for any two points $a, b \in S$ there is a polygonal path joining a and b, which is inside S



Definition 1.20

The set S which is open and connected is called domain (or region)

polygonal connected = connected

examples

1. $\overline{D}(1,2)$ - not a domain
   - connected
   - bounded
   - compact
   - convex
   - closed

2. $T = D(0,1) \cup \overline{D}(3, \frac{1}{2})$
   - not a domain
   - not compact
   - not connected
   - not open
   - not closed
   - bounded, since $T \subset D(0, 100)$

SEQUENCES AND CONVERGENCE

A complex sequence $\{z_n\}$, $n = 1, 2, \dots$ is a collection of complex numbers

Def 1.21 We say that the sequence z_n converges to a number $a \in \mathbb{C}$ if for any number $\varepsilon > 0$ there is a natural number N s.t

$$|z_n - a| < \varepsilon \quad \text{for all } n > N. \qquad N = N_\varepsilon$$

A sequence $\{w_k\}$ is said to be a subsequence of $\{z_n\}$ if there is a sequence of natural numbers $n_1, n_2, \dots, n_k$ s.t $n_k \to \infty$, $k \to \infty$ and $w_k = z_{n_k}$

Lemma 1.22

The sequence z_n converges to a as $n \to \infty$ iff $\operatorname{Im} z_n$ converges to $\operatorname{Im} a$, and $\operatorname{Re} z_n$ converges to $\operatorname{Re} a$ as $n \to \infty$

Follows from: $|\operatorname{Im} z_n - \operatorname{Im} a| \le |z_n - a| = \sqrt{|\operatorname{Im} z_n - \operatorname{Im} a|^2 + |\operatorname{Re} z_n - \operatorname{Re} a|^2}$

and

$$|\operatorname{Re} z_n - \operatorname{Re} a| \le |z_n - a|$$

Moreover, if z_n converges to a, then $|z_n|$ converges to $|a|$

$\overline{z_n}$ converges to $\overline{a}$

Notation: z_n converges to a: $z_n \to a$ as $n \to \infty$

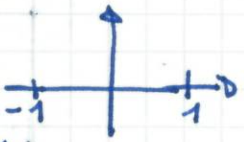
or $\lim\limits_{n \to \infty} z_n = a$

examples

① $z_n = \frac{1}{n} + i \frac{n^2}{n^2+1}$

$\lim_{n \to \infty} z_n = i$

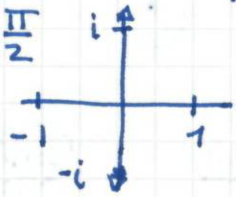
② $w_n = e^{in\pi}$



doesn't converge

$p_k = e^{ik\frac{\pi}{2}}$

no limit



Prop 1.23 - not examined ?

Let z_n be a convergent sequence. Then

① $\{z_n\}$ is bounded

② The limit is unique, i.e if $z_n \xrightarrow[n\to\infty]{} a$ and $z_n \xrightarrow[n\to\infty]{} b \Rightarrow a=b$

③ each subsequence of z_n has the same limit, i.e a

④ $\{z_n\}$ is a Cauchy sequence, i.e $\forall \varepsilon > 0$ there is a number $N - N_\varepsilon$ s.t $|z_n - z_m| < \varepsilon$ for all $n > N$, $m > N$.

Conversely, any Cauchy sequence converges.

THEOREM 1.24 (Bolzano-Weierstraß theorem)

Any bounded sequence contains a convergent subsequen

example

$w_n = e^{in\pi}$

$q_k = w_{2k} = e^{i2k\pi} = 1$ or $s_k = w_{2k+1} = e^{i(2k+1)\pi} = e^{i\pi} = -1$

Corollary 1.25

Any infinite compact set S has the limit point in S will not be in the exam

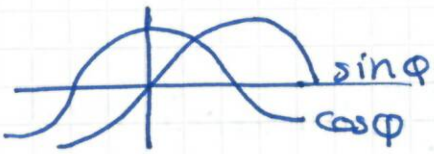
Problem class
Olga Chervova
olgac@math.ucl.ac.uk
o.chervova@ucl.ac.uk
Homework due before problem class on wednesday

- $|z| \geq 0$
- you can NOT compare complex numbers
  $\cancel{z_1 < z_2}$
  $z = w$ or $z \neq w$
- "it is obvious / clear / easy to see" } just one line proof, if longer don't write this
- write down explanations
- answer should be in form $a+ib$, simplify your answer

HW1 Qu 1

$z = 1 = 1 + i \cdot 0$ $\quad 1 = re^{i\varphi}$ $\quad 1 = e^0$

need to find φ st $\cos\varphi = 1$, $\sin\varphi = 0$



$r = \sqrt{1^2 + 0^2} = 1$

$\varphi = 2\pi$ $\quad 1 = e^{2\pi i}$

$z = -1 \cdot \cancel{e^{2\pi i}}$ $\quad z = r(\cos\varphi + i\sin\varphi)$

$\cos\varphi = \cos(-\varphi)$ $\quad \sin\varphi = -\sin(-\varphi)$

$r = \sqrt{(-1)^2 + 0^2} = 1$

φ: $\cos\varphi = -1$, $\sin\varphi = 0$ $\quad \varphi = \pi$

$z = -1 = e^{i\pi}$

$z = i = e^{i\frac{\pi}{2}}$ $\quad z = -i = e^{-i\frac{\pi}{2}}$ or $e^{\frac{3\pi}{2}i}$

$z = 2$

(c)(d) multiply by conjugate to get real

$z \cdot \bar{z} = |z|^2$

(e) need to obtain all of the powers that are divisible by 3

$$\frac{1}{\alpha^2(\alpha-1)^2} \Big| \cdot (\alpha-1)^2$$

Qu 2

$||z| - |w|| \leq |z+w|$ true

- $|z| - |w| \leq |z+w|$
  $|w| - |z| \leq |z+w|$

$z = (z+w) - w$ $\quad$ use triangle inequality

$w = (z+w) - z$ $\quad |a-b| \leq |a| + |b|$

Qu 3

(a) line, mid-point of z_1 and z_2, perpendicular $\perp$ $90°$ bisector

$$\underbrace{|z-z_1|}_{\substack{\text{distance}\\ \text{between}\\ z \text{ and } z_1}} = \underbrace{|z-z_2|}_{\substack{\text{distance}\\ \text{between}\\ z \text{ and } z_2}}$$

(b) $\frac{1}{z} = \underline{\underline{z}}$ circle of radius 1, centre O

$1 = |z|^2$

(c) $\mathrm{Re}(z) = 3$ vertical line

(d) $\mathrm{Re}(z)$ half-plane

(e) $\mathrm{Re}(az+b) > 0$ $a, b \in \mathbb{C}$ $a = x_1 + iy_1$ $b = x_2 + iy_2$

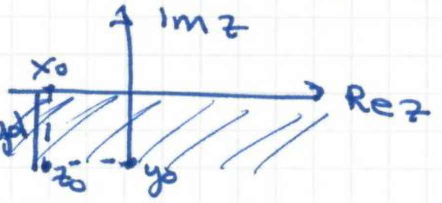
line, mention sign of a, what happens when $\mathrm{Re}(a) \geq 0$

$\mathrm{Re}(a) < 0$

Qu 4

$\{z : \mathrm{Im}\, z > 0\}$ proved

Prove $\{z : \mathrm{Im}\, z < 0\}$ is open



consider $z_0 \in \{z : \mathrm{Im}\, z < 0\}$

$z_0 = x_0 + iy_0$

$y_0 < 0 \Rightarrow -y_0 > 0$

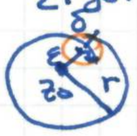
need to show that I can find $r > 0$ s.t

$D(z_0, r) \subset \{z : \mathrm{Im}\, z < 0\}$

$\boxed{\text{or}}$

$\exists\, r > 0$ s.t $\forall z \in D(z_0, r)$ $z \in \{z : \mathrm{Im}\, z < 0\}$

$r = \frac{1}{2}|y_0| = -\frac{y_0}{2} > 0$



fix a point $z \in D(z_0, r)$

$|z_0 - z| = \varepsilon > 0$: $\varepsilon < r$

Need to prove that $z \in \underbrace{\{z : \mathrm{Im}\, z < 0\}}_{S}$ OR $\mathrm{Im}\, z < 0$

~~Choose $D(z, \delta)$ choose $\delta = r - \varepsilon > 0$~~

~~fix $a \in D(z, \delta)$~~

consider ~~$|z_0 - z_0| = |(z-a) + (a - z_0)| \leq |z-a| + |z_0 - a|$~~

$|z_0 - z| < -\frac{y_0}{2}$ $z_0 = x_0 + iy_0$

$z = x + iy$

$$\underbrace{\sqrt{(x-x_0)^2 + (y-y_0)^2}}_{\geq (y-y_0)^2} < -\frac{y_0}{2} \Rightarrow |y - y_0| < -\frac{y_0}{2}$$

$y - y_0 < -\frac{y_0}{2}$

$y < \frac{y_0}{2} < 0$

FUNCTIONS

12/10/2011

Maps defined on sets of complex plano $\mathbb{C}$ with values in $\mathbb{C}$.

Need to know:

1. The set where f is defined called domain of f, $D(f)$
2. The mapping itself

examples

1. $f(z) = z^2$, $D(f) = \mathbb{C}$
   $f(x+iy) = (x+iy)^2 = x^2 - y^2 + 2ixy$

In general, for any function $g: D(g) \to \mathbb{C}$
We write:
$$g(z) = u(x,y) + iv(x,y), \text{ so } u = \operatorname{Re} g, \; v = \operatorname{Im} g$$

2. $h(z) = \frac{1}{z}$, $D(h) = \mathbb{C} \setminus \{0\}$ or $D(h) = D(5,3)$

3. $w(z) = \sin x + i \cos y$, $z \in \mathbb{C}$

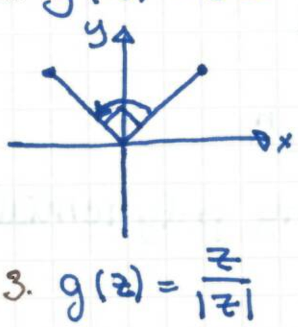
4. $P(z) = a_n z^n + a_{n-1} z^{n-1} + \dots + a_1 z + a_0$, where
   $a_n, a_{n-1}, \dots, a_1, a_0$ are fixed complex numbers
   z is the variable.
   If $a_n \neq 0$, then $P(z)$ is called polynomial of degree n.
   For any two polynomials P, Q, the function $M(z) = \frac{P(x)}{Q(x)}$ is called rational function.

Observe: $D(P) = \mathbb{C}$, $D(M) = \mathbb{C} \setminus \{\text{roots of } Q(z)\}$

Mapping properties

1. $f(z) = z - 1$, $D(f) = \mathbb{C}$
   $\mathbb{C}$ is shifted by 1 to the left

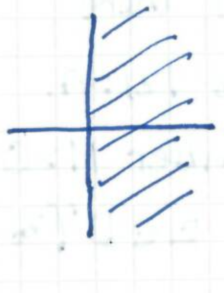
2. $g(z) = iz$



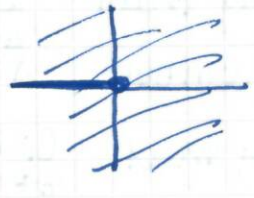
$$g(z) = |z| e^{i\frac{\pi}{2}} e^{i\theta} = |z| e^{i(\theta + \frac{\pi}{2})}$$
rotation by $\frac{\pi}{2}$ counterclockwise

3. $g(z) = \frac{z}{|z|}$ $z \in \mathbb{C} \setminus \{0\}$
   What is the image of $g = \{\text{set of values}\}$?

4. Let $D(h) = \{z : \operatorname{Re} z > 0\}$ and $h(z) = z^2 = |z|^2 e^{i2\theta}$, $\theta = \arg z$



What is the image of h?
Image $= \mathbb{C}$ with a cut along the negative real axis



More precisely, Image $= \{z \in \mathbb{C}\} \setminus \{w : \operatorname{Re} w \leq 0, \operatorname{Im} w = 0\}$

Limits of functions

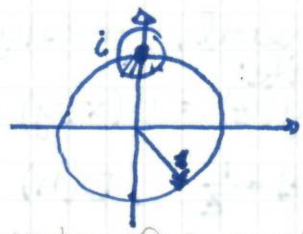
Def 1.26
Let $f: S \to \mathbb{C}$ be a function and $z_0 \in \mathbb{C}$. Then we say that f has a limit at z_0, denoted

$$\lim_{z \to z_0} f(z) = w_0$$

if for any $\varepsilon > 0$ there is a $\delta > 0$ s.t $|f(z) - w_0| < \varepsilon$ as soon as $z \in D'(z_0, \delta) \cap S$

examples

1. $S = D(0,1)$, $z_0 = i$, $f(z) = z$



$\lim_{z \to z_0} f(x) = z_0 = i$

z_0 does not belong to S

$\lim_{z \to 2} f(z) = ?$ doesn't make sense

2 is too far away from the disc

2. $S = D(0,1)$, $z_0 = 0$

$$h(z) = \begin{cases} 1, & z = 0 \\ z, & z \neq 0 \end{cases}$$

$$\lim_{z \to 0} h(z) = 0$$

Properties

1. If $\lim_{z \to z_0} f$ exists, it is unique
2. If $\lim_{z \to z_0} f = w_0$, then $\lim_{z \to z_0} \operatorname{Re} f = \operatorname{Re} w_0$
   $\lim_{z \to z_0} \operatorname{Im} f = \operatorname{Im} w_0$, $\lim_{z \to z_0} \overline{f} = \overline{w_0}$, $\lim_{z \to z_0} |f| = |w_0|$
3. Algebra of limits is applicable (AOL)

__Observe:__ $z \in \mathbb{C}$

$\lim_{z \to z_0} z = z_0$. By AOL $\lim_{z \to z_0} z^n = z_0^{\,n}$

Thus, by AOL $\lim_{z \to z_0} P(z) = P(z_0)$ for any polynomial.
This means that P is continuous on $\mathbb{C}$.

Infinite limits and limits of infinity

Def 1.27

① We say that $\lim_{z \to \infty} f(z) = W$, if for any $\varepsilon > 0$ there is a number A s.t $|f(z) - W| < \varepsilon$ as soon as $|z| > A$.

② We say that $\lim_{z \to z_0} f(z) = \infty$ if for any number $M > 0$ there is a $\delta > 0$ s.t $|f(z)| > M$ as soon as $z \in D'(z_0, \delta) \cap D($

example

① $\lim_{z\to\infty} \frac{1}{z^2+z} = 0$, $\lim_{z\to\infty} \frac{z^2}{z^2+z} = 1$

$\lim_{z\to i} \frac{1}{z-i} = \infty$, $\lim_{z\to 2} \frac{1}{z-i} = \frac{1}{2-i}$

Problem Class 2

oblem heet 1

4b $D(z_0, r)$ is open $\forall z_0 \in \mathbb{C}$ $r > 0$

$D(z_0, r)$ $|z-z_0| < r$

fix $z \in D(z_0, r)$

a $|z-a| < \varepsilon \Rightarrow a \in D(z_0, r)$

$|z_0 - a| = |z_0 - z + z - a| \le \underbrace{|z_0 - z|}_{<r} + \underbrace{|z-a|}_{\varepsilon}$

$< |z-z_0| + r - |z-z_0| = r$

$\varepsilon < r - |z-z_0|$

3e

$Re(az+b) > 0$, $a, b \in \mathbb{C}$

$a = a_1 + ia_2$
$b = b_1 + ib_2$
$z = x + iy$

$az+b = (a_1+ia_2)(x+iy) + (b_1+ib_2) =$

$= \underbrace{(a_1x - a_2y + b_1)}_{Re} + i(\ldots)$

$Re(az+b) = a_1x - a_2y + b_1 > 0$

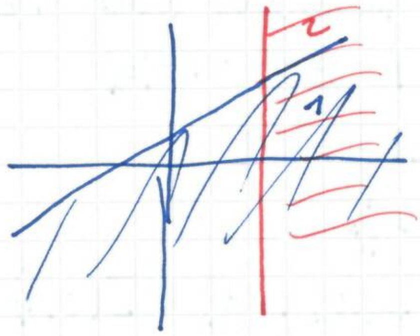
$a_2 y < a_1 x + b_1$

1 assume that $a_2 > 0$

$y < \frac{a_1}{a_2}x + \frac{b_1}{a_2}$

2 assume that $a_2 < 0$

$y > \frac{a_1}{a_2}x + \frac{b_1}{a_2}$



HW 2

① find $|z|$ and $\arg z$ $\quad -\pi < \arg z < \pi$

$\arg z = \arctan \frac{b}{a}$

$z_1 = 3i$
$z_2 = 1+i$
$z_3 = -1-i$
$\vdots$
$z_7 = -bi$
$z_8 = a+ib$ $\quad a, b \in \mathbb{R}$

② 1- $|a+b|^2+|a-b|^2 = 2(|a|^2+|b|^2)$

2- $|1-\bar{a}b|^2 - |\bar{a}-b| = (1+|ab|^2) - (|a|+|b|^2)$

$LHS = (a+b)(\bar{a}+\bar{b}) + (a-b)(\bar{a}-\bar{b}) = \dots$

③ $\underbrace{z \in \mathbb{C} \text{ limit point of } S}_{A} \Leftrightarrow \underbrace{\exists \{z_n\} : z_n \in S, n \in \mathbb{N}}_{B}$

1) $A \to B$
$z \in \mathbb{C}$ - limit point of S
- limit point of S

$\forall n \in \mathbb{N} \quad \exists z_n \in S \quad |z - z_n| < \frac{1}{n} \quad z_n \neq z$

$\{z_n\}, z_n \neq z \quad z_n \xrightarrow[n\to\infty]{} z$

2) $B \to A$

$\begin{cases} \exists \{z_n\}, z_n \in S, n \in \mathbb{N}, z \neq z_n, z_n \xrightarrow[n\to\infty]{} z \\ \forall \delta > 0 \ \exists N \in \mathbb{N} \ \forall n \geq N \quad |z_n - z| < \delta \end{cases}$

$\Rightarrow z$ - limit point

4. $D(z,r)$ - open, $\overline{D}(z,r)$ - closed disc

1) $\underbrace{\bigcup_{n\in\mathbb{N}} \overline{D}(0, 1-\frac{1}{n})}_{A} = \underbrace{D(0,1)}_{B}$

$A \subset B$

fix $z \in A \quad \exists N \ \forall n \geq N \quad z \in \overline{D}(0, 1-\frac{1}{n})$

$|z-0| \leq 1 - \frac{1}{n} < 1$

$A \supset B$ fix $z \in B \quad |z| < 1$

need to show that $|z| \leq 1 - \frac{1}{n} \ \forall n > N, n \in \mathbb{N}$

$|z| < 1 \quad \exists \varepsilon = 1 - |z|$

$|z| = 1 - \varepsilon$

$N: \varepsilon > \frac{1}{N}$

2) $\bigcap_{n\in\mathbb{N}} D(0, 1+\frac{1}{n}) = \overline{D}(0,1)$

$A \subset B \quad |z| < 1 + \frac{1}{n} \quad \forall n$ - take limit

$|z| \leq 1$

$A \supset B \quad |z| \leq 1 < 1 + \frac{1}{n}$

⑥ A, B - closed $\quad A \cup B, A \cap B$ - closed set

hint $\begin{cases} A^c = \mathbb{C} \setminus A \\ B^c = \mathbb{C} \setminus B \end{cases}$ open $\quad (A\cup B)^c = A^c \cap B^c$

$(A \cap B)^c = A^c \cup B^c$

$A \cup B$ - closed $\Leftrightarrow (A \cup B)^c$ is open

$(A\cup B)^c = A^c \cap B^c$ fix $z \in \underbrace{A^c \cap B^c}_{\text{need to find } \delta>0 \text{ s.t}} \Leftrightarrow \underbrace{z \in A^c}_{A^c\text{-open}}$ & $z \in B^c$

A^c-open $\exists \delta_1 > 0$ $D(z,\delta_1) \subset A^c$

B^c open $\exists \delta_2 > 0$ $D(z, \delta_2) \subset B^c$

choose $\delta = \min\{\delta_1, \delta_2\}$

$A \cap B$

Let A_k - closed, $k \in \mathbb{N}$. Is it true

(a) $\bigcup_k A_k$ is closed - false, counter example from Qu 4

(b) $\bigcap_k A_k$ is closed

$(\bigcap_k A_k)^c = \bigcup_k A_k^c$

fix $z \in \bigcup_k A_k^c$ A_k - closed $\Rightarrow A_k^c$ - open

$\exists m \in \mathbb{N}$ $z \in A_m^c \leftarrow$ open $\exists \delta > 0$ $D(z,\delta) \subset A_m^c \subset \bigcup_k A_k^c$

7. $S \subset \mathbb{C}$ ← closed, $z \notin S$

distance between z & S: $d(z,S) = \inf_{w \in S} |z-w|$

$d(z,S) > 0$

S^c - open $z \in S^c$ $D(z,r) \subset S^c$

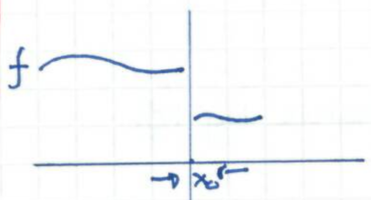
17/10-2011

Limits

$\lim_{z \to z_0} P(z) = P(z_0)$, $z_0 \in \mathbb{C}$

same limit, no matter from where you aproach z_0

real analysis

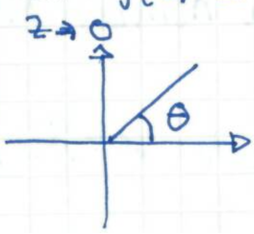


f has no limit at x_0

example

$f(z) = \frac{\operatorname{Im} z}{z}$, $z \neq 0$

$\lim_{z\to 0} f(z) = ?$



Let $z = re^{i\theta}$, $\theta \in [-\pi, \pi]$.

Then $f(z) = \frac{r\sin\theta}{re^{i\theta}} = \frac{\sin\theta}{e^{i\theta}} \xrightarrow[r\to 0]{} \frac{\sin\theta}{e^{i\theta}}$ θ is fixed

$\theta = 0 \Rightarrow \lim = 0$

$\theta = \frac{\pi}{2} \Rightarrow \lim = -i$

The limits are different for different values for θ

$\Rightarrow f$ has no limit at $z_0 = 0$

CONTINUITY

Def 1.28 Function f is continous at z_0 if

① $z_0 \in D(f)$,

② $\lim_{z \to z_0} f(z) = f(z_0)$

f is said to be continous on the set S if f is continuous at every point of S.

Alternative Let $z_0 \in D(f)$. Assume that $D'(z_0, r) \cap D(f) \neq \emptyset$, for any $r > 0$. Then f is continuous at z_0 if $\forall \varepsilon > 0\ \exists \delta$ s.t $|f(z) - f(z_0)| < \varepsilon$ as soon as $|z - z_0| < \delta$, $z \in D(f)$.

Properties

① Polynomials are continuous on $\mathbb{C}$. Rational functions $\frac{P(z)}{Q(z)}$ are continous away from the roots of $Q(z)$

② By Algebra of Limits, if f & g are continous at z_0, then so are

1) $f+g$
2) fg
3) $\frac{f}{g}$, away from the roots of g

③ If $f = u + iv$ is continuous, then so are u, v. And vice versa

④ If f, g are continous, then $f(g(z))$ is also continous.
Notation: $(f \circ g)(z) = f(g(z))$

⑤ If f is continuous, then $|f|$ is continous
The converse (opposite) is not true.

example

$g(z) = e^{\sqrt{1+x^2}} + i \sin(y^3 x)$

Reg and Img are continuous on $\mathbb{R}^2$, and hence by ③ g is continuous.

For a real valued function $h(x)$, $x \in \mathbb{R}$: How to guarantee that h is bounded?

Answer: h is bounded if it is continuous on a closed interval $[a, b]$

Bounded continuous functions

We say that f is bounded on $D(f)$ if there is a number $M > 0$ s.t $|f(z)| \leq M$ when $z \in D(f)$

Theorem 1.29

Suppose that f is continuous on the compact set S. Then

1) f is bounded on S
2) the function $|f|$ attains its max and min values on S

Chapter 2: Derivatives and analytic Functions

Real analysis (remainder):

$$f'(x_0) = \lim_{x \to x_0} \frac{f(x)-f(x_0)}{x-x_0} = \lim_{h\to 0} \frac{f(x_0+h)-f(x_0)}{h}$$

Def 2.1 Suppose that $D(z_0, r) \subset D(f)$ for some $r>0$. Then we say that f is differentiable at z_0 if the limit

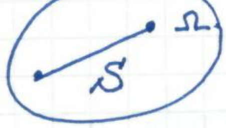
$$f'(z_0) = \lim_{z\to z_0} \frac{f(z)-f(z_0)}{z-z_0} \quad \text{exists.}$$

The limit is called the derivative of f at z_0.

If $D(f)$ is a domain, then if f is differentiable at every $z \in D(f)$, then f is holomorphic on $D(f)$.

$H(\Omega)$ is the set of all holomorphic functions on a domain Ω (open, connected set)

If $S \subset \mathbb{C}$, then we say that f is holomorphic on S if $f \in H(\Omega)$ for some $\Omega \supset S$.



f is holomorphic at z_0 if it is holomorphic on $D(z_0, r)$ with some $r>0$.

If f is analytic on $\mathbb{C}$ we say that f is an <u>entire</u> function.

Rewrite: $f'(z_0) = \lim_{h\to 0} \frac{f(z_0+h)-f(z_0)}{h}$

example

1- let $g(z) = |z|^2 = x^2+y^2$

let's try to find $g'(z_0)$:

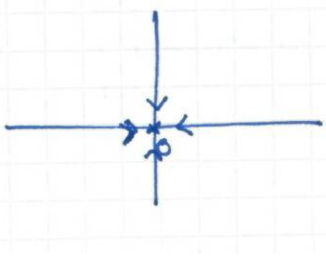
$$\frac{g(z_0+h)-g(z_0)}{h} = \frac{|z_0+h|^2-|z_0|^2}{h} = \frac{(z_0+h)\overline{(z_0+h)} - |z_0|^2}{h}$$

$$= \frac{\cancel{z_0\overline{z_0}} + h\overline{z_0} + z_0\overline{h} + h\overline{h} - \cancel{|z_0|^2}}{h} = \overline{z_0} + \overline{h} + \frac{z_0\overline{h}}{h}$$

? look at $z_0\frac{\overline{h}}{h}$. If $z_0 = 0 \Rightarrow \frac{|\ |^2 - |\ |^2}{h} = \overline{h} \to 0$ as $h \to 0$

$\Rightarrow g'(0) = 0$

suppose $z_0 \neq 0$. Assume first that $h = t \in \mathbb{R}$



Then $z_0 \frac{\overline{h}}{h} = z_0\frac{t}{t} = z_0$

Assume that $h = iu,\ u \in \mathbb{R}$:

$$z_0\frac{\overline{h}}{h} = z_0 \frac{-iu}{iu} = -z_0$$

Thus g is differentiable only at $z_0 = 0$ and $g'(0) = 0$.
Note: g is continuous on $\mathbb{C}$

2- $f(z) = z^2$, $z \in \mathbb{C}$
write:

$$\frac{(z+h)^2 - z^2}{h} = \frac{2zh + h^2}{h} = 2z + h \to 2z \text{ as } h \to 0$$

$\Rightarrow f$ is differentiable on $\mathbb{C}$, i.e f is holomorphic on $\mathbb{C}$ and $f'(z) = 2z$

Lemma 2.2 If f is differentiable at z_0, f is continuous at z_0.

PROOF: Want to show $f(z) \to f(z_0)$ as $z \to z_0$
Write

$$\underbrace{f(z) - f(z_0)} = \frac{f(z) - f(z_0)}{z - z_0}(z - z_0) \xrightarrow[z \to z_0]{\text{by AoL}} f'(z_0) \cdot 0 = \underbrace{0}$$

Thus $\lim_{z \to z_0} f(z) = f(z_0)$, as required ■

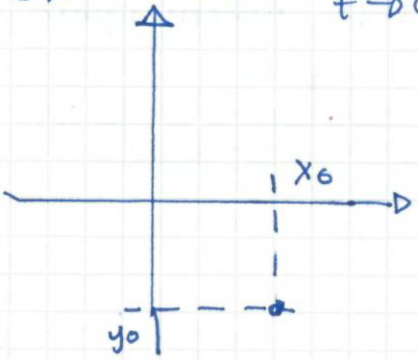
The Cauchy-Riemann equations

We are looking for a link between real and imaginary part of f, which guarantees differentiability.

Partial derivatives (remainder):
look $g(x, y)$

$$\frac{\partial g}{\partial x}(x_0, y_0) = \lim_{t \to 0} \frac{g(x_0 + t, y_0) - g(x_0, y_0)}{t}$$ (fix y, move along x-axis)

$$= g_x(x_0, y_0)$$



$$\frac{\partial g}{\partial y}(x_0, y_0) = \lim_{t \to 0} \frac{g(x_0, y_0 + t) - g(x_0, y_0)}{t} = g_y(x_0, y_0)$$

Theorem 2.3
Suppose that $f(z) = u(x, y) + iv(x, y)$ is differentiable at $z_0 = x_0 + iy_0$. Then the partial derivatives u_x, v_x, u_y, v_y exist at (x_0, y_0) and

$f'(z_0) = u_x + iv_x = v_y - iu_y$ and therefore

$$\begin{cases} u_x = v_y \\ u_y = -v_x \end{cases}$$ Cauchy-Riemann equations

Proof Use $f'(z_0) = \lim_{h \to 0} \frac{f(z_0 + h) - f(z_0)}{h}$

Let $h = t \in \mathbb{R}$. Then $f'(z_0) = \lim_{t \to 0} \frac{f(x_0 + t, y_0) - f(x_0, y_0)}{t}$

$$= \lim_{t \to 0} \left[\frac{u(x_0 + t, y_0) - u(x_0, y_0)}{t} + i \frac{v(x_0 + t, y_0) - v(x_0, y_0)}{t} \right]$$

$= u_x + iv_x$

$$= \lim_{t\to 0} \frac{U(x_0,y_0+t)-U(x_0,y_0)}{it} + i \lim_{t\to 0} \frac{V(x_0,y_0+t)-V(x_0,y_0)}{it}$$

$$= \lim_{t\to 0} \frac{V(x_0,y_0+t)-V(x_0,y_0)}{t} - i \lim_{t\to 0} \frac{U(x_0,y_0+t)-U(x_0,y_0)}{t}$$

Thus limits of re and im parts exist as $t \to 0$ and hence U_x, V_x exist and $f'(z_0) = U_x(x_0,y_0) + iV_x(x_0,y_0)$

Now assume that $h = it$, $t \in \mathbb{R}$

Then

$$f'(z_0) = \lim_{t\to 0} \frac{f(x_0,y_0+t)-f(x_0,y_0)}{it}$$

$$= \lim_{t\to 0} \left[\frac{V(x_0,y_0+t)-V(x_0,y_0)}{t} - i\, \frac{U(x_0,y_0+t)-U(x_0,y_0)}{t} \right]$$

$= V_y - iU_y$ as claimed □

example

$f = z^2$, $z \in \mathbb{C}$

Rewrite: $f(z) = \underbrace{x^2-y^2}_{U} + i\underbrace{2xy}_{V}$ so

$U_x = 2x \qquad V_x = 2y$

$U_y = -2y \qquad V_y = 2x$

$\Rightarrow \begin{cases} U_x = V_y \\ U_y = -V_x \end{cases}$ Cauchy-Riemann equations ✓

let $g(z) = |z|^2 = \underbrace{x^2+y^2}_{U} + i\underbrace{0}_{V}$

$U_x = 2x \qquad V_x = 0$

$U_y = 2y \qquad V_y = 0$

$\begin{cases} U_x = V_y \\ U_y = -V_x \end{cases}$ only at $z = 0$

Remaind: $f' \Rightarrow$ CRE

If CRE are not satisfied, then f is not differentiable.

$f = U + iV$ satisfy CRE

Q: $V + iU$

$\begin{cases} V_x = U_y \\ V_y = -U_x \end{cases}$ NO! does not satisfy CRE

Properties of differentiable functions

① $\frac{d}{dz} c = 0$, c = constant

② $\frac{d}{dz}(cf) = c\frac{df}{dz}$, $\frac{d}{dz} f = f'$

③ $\frac{d}{dz} z^n = nz^{n-1}$, for any $n = 1, 2, \dots$

④ $\frac{d}{dz}(f+g) = \frac{d}{dz} f + \frac{d}{dz} g$,

⑤ $\frac{d}{dz}(fg) = f'g + fg'$

⑥ $\left(\frac{f}{g}\right)' = \frac{f'g - fg'}{g^2}$

⑦ $\frac{d}{dz}(f \circ g)(z) = f'(g(z))\, g'(z)$

19/10-2011

Holomorphic functions are also called analytic.
The function f is said to be <u>entire</u> if it is analytic in $\mathbb{C}$.

example $f(z) = z^2$

$f = u + iv$ f is diff. at z_0
then $\begin{cases} u_x = v_y \\ u_y = -v_x \end{cases}$ CRE

Theorem 2.5

Let f be holomorphic on a domain Ω

① Assume that $f'(z) = 0$ for all $z \in \Omega$. Then $f(z) = \text{const}$ for all $z \in \Omega$

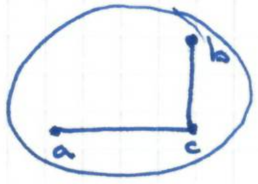
② suppose that $|f|$ is constant on Ω. Then f is constant on Ω.

PROOF EXAM

Write CRE for $f = u + iv$

$(*) \begin{cases} u_x = v_y \\ u_y = -v_x \end{cases}$ From $f' = u_x + iv_x = 0$ we conclude: $u_x = v_x = 0$, and due to $(*)$, $u_y = v_y = 0$

suppose first that $\Omega = D(z_0, r)$, $r > 0$, $z_0 \in \mathbb{C}$. Let $a, b \in D(z_0, r)$
Want $f(a) = f(b)$
Observe a & b can be joined by a polygonal path which consists of the segments, parallel to the coordinate axes

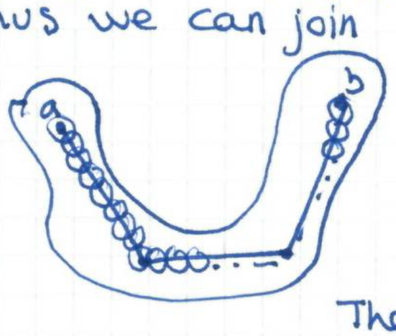


On $[a,c]$ we use $u_x = v_x = 0$, so u, v are constant $\Rightarrow f(a) = f(c)$

On $[c,b]$ we use $u_y = v_y = 0$, so u, v are constant $\Rightarrow f(c) = f(b)$

Thus $f(a) = f(b)$ as required.

Let R be an arbitrary domain, i.e connected and open set.
Thus we can join any $a, b \in \Omega$ with a polygonal path



cover the path with open disk of a suitable radius, $r > 0$.

In every disk f is constant. Due to the overlap, these constants are the same.

Therefore $f(a) = f(b)$, i.e f is constant on Ω

② PROOF EXAM

Assume $|f| = c \geq 0$

If $c=0 \Rightarrow |f|^2 = u^2+v^2 = 0$ and $u=v=0$.

Let $c>0$ Then write

$$u^2+v^2 = c^2$$

differenciate wrt x: $2u_x u + 2v_x v = 0$

— // — y: $2u_y u + 2v_y v = 0$

$$\begin{cases} u_x = v_y \\ u_y = -v_x \end{cases}$$

By CRE: $\begin{cases} u_x u - u_y v = 0 \\ u_y u + u_x v = 0 \end{cases}$

Multiply line 1 by u, — // — line 2 by v $\begin{cases} u_x u^2 - u_y v u = 0 \\ u_y u v + u_x v^2 = 0 \end{cases}$

Add up: $u_x u^2 + u_x v^2 = 0 \Leftrightarrow u_x(u^2+v^2) = c^2 u_x = 0$

as $c \neq 0$, we have $u_x = 0$

In the same way you will find $u_y = 0$, therefore by CRE $v_x = v_y = 0$

$\Rightarrow f'(z) = u_x + iv_x = 0 \Rightarrow f$ is constant by part ① $\blacksquare$

Recall: f differentiable $\Rightarrow$ CRE

Theorem 2.6

Let $f = u+iv$ be continuous on a domain Ω, and let u_x, v_x, u_y, v_y be continuous on Ω. If u,v satisfy CRE at some point $z_0 \in \Omega$, then f is differentiable at z_0.

example 2.7

Let $f(z) = e^x(\cos y + i \sin y)$, $z = x+iy$

The real part $u(x,y) = e^x \cos y$ and imaginary part $v(x,y) = e^x \sin y$, are continuous on $\mathbb{C}$, and u_x, u_y, v_x, v_y exist and are continuous on $\mathbb{C}$:

$u_x = e^x \cos y \qquad v_x = e^x \sin y$

$u_y = -e^x \sin y \qquad v_y = e^x \cos y$

CRE hold for all x,y: $u_x = v_y$, $u_y = -v_x$

By theorem 2.6 f is analytic on $\mathbb{C}$, i.e entire.

Moreover $f' = u_x + iv_x = u + iv = f$, i.e $\boxed{f' = f}$

This is why we denote $f(z) = \exp(z) = e^z$

Remark

Define $f_z = \dfrac{\partial f}{\partial z} = \dfrac{1}{2}\left[\dfrac{\partial f}{\partial x} - i\dfrac{\partial f}{\partial y}\right]$,

$f_{\bar{z}} = \dfrac{\partial f}{\partial \bar{z}} = \dfrac{1}{2}\left[\dfrac{\partial f}{\partial x} + i\dfrac{\partial f}{\partial y}\right]$

Assume that f is analytic, i.e $u_x = v_y$, $u_y = -v_x$.

Find $f_{\bar{z}}$ in terms of u,v:

$$f_{\bar{z}} = \frac{1}{2}\left[u_x + iv_x + i(u_y + iv_y)\right] = \frac{1}{2}\left[u_x - v_y + i(u_y + v_x)\right] = 0$$

This "means" that f doesn't depend on $\bar{z}$. To find out if f is differentiable, rewrite it as a function of $z, \bar{z}$, using

$x = \frac{z+\bar{z}}{2}$, $y = \frac{z-\bar{z}}{2i}$

Assuming that f is analytic, what is $\frac{\partial f}{\partial \bar{z}}$?

$\frac{\partial f}{\partial z} = f'$

Problem class 3

PS2

Qu 3

$\underbrace{z \text{ - limit point } S}_{A} \Leftrightarrow \underbrace{\exists \{z_n\}_{n=1}^{\infty} \in S \quad z_n \neq z \ \forall n \quad \lim_{n\to\infty} z_n = z}_{B}$

$A \Leftrightarrow B$

1) $z \in A \Rightarrow z \in D$ $\quad A \subset B$

z - limit point of S

$\forall \varepsilon$ $D(z, \varepsilon)$ contains points from S $\neq z$

$\varepsilon_n = \frac{1}{n}$ $D(z, \frac{1}{n})$ contains $z_n \neq z$,

$z_n \in S$ $\quad n = 1, 2, \dots$ $\{z_n\}_{n=1}^{\infty}$

$D(z, \frac{1}{n})$

$z_n \in D(z, \frac{1}{n})$

$|z - z_n| < \frac{1}{n} \to z_n \underset{n\to\infty}{\to} z$

2) $z \in B \Rightarrow z \in A$ $\quad B \subset A$

$\lim_{n\to\infty} z_n = z$

$|z_n - z| < \varepsilon$

$\exists N \ \forall n > N$

PS3

Qu 1 limits as $z \to 0$? what is the limit as $z \to a$, $a \in \mathbb{R}$

1- $\frac{z}{|z|}$ $\quad$ 2- $\frac{\operatorname{Re} z}{\operatorname{Im} z}$

write $z = x + iy$ or $z = re^{i\phi}$, $r \geq 0$ $\quad z = r(\cos\phi + i\sin\phi)$

1- show for real z, has no limit or show it depends on ϕ, so no limit

2 - from last year, and high school?

Qu 2 f - holomorphic on a domain (open and connected)
USE CRE to show $f = \text{const}$

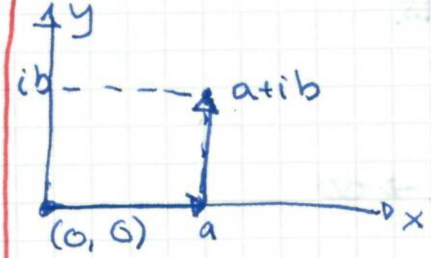
(a) f real-valued , $\operatorname{Im} f = 0$
(b) $\operatorname{Im} f = \text{const}$
(c) $\operatorname{Re} f = \text{const}$

NO integration!

b) $f(z) = u(x,y) + ic$ $\quad c = \text{const}$
$\qquad\qquad\qquad\quad \overset{\shortparallel}{v(x,y)}$

CRE : $u_x = v_y = 0$
$u_y = -v_x = 0$

$U_x = U_y = V_x = V_y = 0$



$0 \to a$

$a \to a+ib$

$U_x = U_y = 0$

$U(x,y) = U(0,y) \quad \forall x \in (0,a)$

$V(x,y) = V(0,y) \qquad V(a,b)$

$D(0,R)$

Think about definition of derivative

Qu 3

a) Verify $\operatorname{Im} z$ & $\bar{z}$ do not satisfy CRE at any point

$f(z) = \operatorname{Im} z = y \qquad z = x+iy$

$g(z) = \bar{z} = x - iy$

write partial derivatives

b) $f(z) = |z| = \sqrt{x^2+y^2}$

$z_0 = 0$ • at z_0 CRE hold

• $f(z)$ is not holomorphic at 0

-use def, consider $\dfrac{f(z+h) - f(z)}{h}$

Qu 4

$f \in H(D(0,R)) \quad R > 0 \qquad g(z) = f(\bar{z})$ is holomorphic at 0

$z \in D(0,R) \quad , \bar{z} \in D(0,R)$

f is holomorphic at $\bar{z}$

$\exists \; f'(\bar{z})$

$$\lim_{h \to 0} \frac{g(z+h) - g(z)}{h} = \overline{\frac{f(\overline{z+h}) - f(\bar{z})}{h}} = \overline{\lim_{h \to 0}\left(\frac{f(\overline{z+h}) - f(\bar{z})}{h}\right)} = \overline{f'(\bar{z})}$$

from PS1 $D(0,R$ - open$)$

$\forall z \in D(0,R)$ we can find h s.t $z+h \in D(0,R)$

Qu 5

Prove $\dfrac{\partial}{\partial z} z^n = nz^{n-1}$ use induction

Binomial

$$\text{Don't } \frac{(z+h)^n - z^n}{h} = \frac{(z+h-z)\left((z+h)^{n-1}z^0 + \dots + (z+h)z^{n-1}\right)}{h}$$

$$= (z+h)^{n-1} \cdot z^0 + (z+h)^{n-2}$$

24/10-2011

3 COMPLEX SERIES

Let a_k, $k = 0, 1, \dots$ be a complex sequence. Then the formal sum $\sum_{k=0}^{\infty} a_k$ is called a complex series

Define $S_n = \sum_{n=0}^{n} a_n$ for finite n, "Partial sums"

If S_n converges as $n \to \infty$, we say that the series $\sum_{k=0}^{\infty} a_k$ converges. So, by definition

$$\sum_{k=0}^{\infty} a_k = \lim_{n\to\infty} \sum_{k=0}^{n} a_k \text{, if the limit exists.}$$

Properties

1. If the series converges then $a_k \to 0$, $k \to \infty$

$\sum_{k=0}^{\infty} (-1)^k$ - doesn't converge, $a_k \not\to 0$ $k\to\infty$

$\sum_{k=0}^{\infty} e^{ik\theta}$, $\theta \in (-\pi, \pi]$ doesn't converge

As a consequence $\{a_n\}$ is a bounded sequence

2. If $\sum a_k$ & $\sum b_k$ are convergent, then $\sum (a_k + A b_k)$ converges as well for any complex A.

3. We say that $\sum a_k$ converges absolutely, if $\sum |a_k|$ converges. If the series converges absolutely, it is convergent.

example

$\sum \frac{(-1)^n}{n}$ converges, but $\sum \frac{1}{n}$ diverges!

Preposition 3.1 (comparison test)
Let $\sum a_k$ be a complex series, and let $\sum b_k$ be a series of non-negative numbers b_k. Assume that for some number $M > 0$ we have $|a_k| \leq M b_k$ for all k. Then if $\sum b_k$ converges, then $\sum a_k$ converges absolutely.

Write $\sum_{k=0}^{\infty} a_k = a_0 + a_1 + a_2 + \dots$

If one deletes finitely many terms, this doesn't affect convergence.

Proposition 3.2 (Ratio test)
Let $\sum a_k$ be a series. Suppose that

$\lim_{k\to\infty} \frac{|a_{k+1}|}{|a_k|} = l$ exists with some $l \geq 0$. Then if $l < 1$ then the series converges absolutely. If $l > 1$ it diverges
if $l = 1$ we don't know

example

$\sum \frac{1}{n^2} \Rightarrow l = 1$ but it converges.

$$\sum_{n=1}^{\infty} \frac{1}{n^\alpha} \to \begin{cases} \alpha > 1 - \text{convergence} \\ \alpha \leq 1 - \text{divergence} \end{cases}$$

Proposition 3.3 (Root test)
Let $\sum a_k$ be a series. Assume that $\lim_{k\to\infty} |a_k|^{\frac{1}{k}} = r$ exists, $r \geq 0$. If $r < 1$, then the series converges absolutely, and if $r > 1$, then it diverges.

$r = 1$ doesn't know

example

$\sum_{k=0}^{\infty} z^k$ geometrical series. Here $z \in \mathbb{C}$.

For what values of z does it converge?

Recall: $\sum_{k=0}^{\infty} z^k = \frac{1}{1-z}$, $|z|<1$

Ratio test:

$$\frac{|z^{k+1}|}{|z^k|} = |z| \xrightarrow[k\to\infty]{} |z| \begin{cases} <1 & \text{converges} \\ >1 & \text{diverges} \end{cases}$$

POWER SERIES

Power series is this:

(*) $\sum_{k=0}^{\infty} a_k (z-z_0)^k$, where a_k, $k=0,1,\dots$, are fixed complex numbers and $z_0 \in \mathbb{C}$ is also fixed.

All def'n and theorems given in Real Analysis are applicable

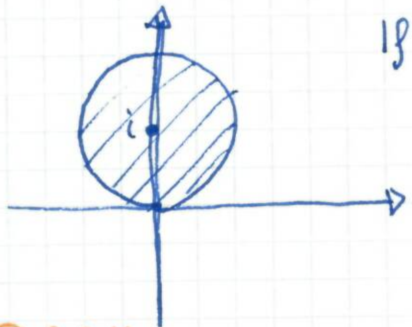
The function f depends on the variable $z \in \mathbb{C}$

example

$\sum_{k=0}^{\infty} (z-i)^k$ For which values of z does this series converge?

Answer: $|z-i|<1$, i.e. $z \in D(i,1)$

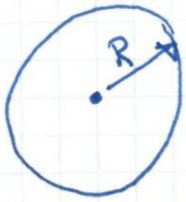
If $|z-i|>1$, then it diverges



Def 3.4

The radius of convergence of (*) is defined to be

$R = \sup\{|z| : \sum_{k=0}^{\infty} |a_k z^k| \text{ converges}\}$ (Note that R can be infinite, then the series converges for all values of z)



Lemma 3.5

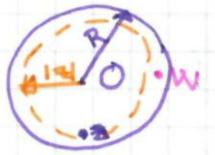
Let R be the radius of convergence of (*)

Then

① If $|z-z_0|<R$, then the series converges absolutely

② If $R<\infty$ and $|z-z_0|>R$, then the series diverges.

Proof Assume $z_0 = 0$. Suppose that $|z| < R$. Pick a number w:



Pick a number w: $|z| < |w| < R$, the series (*) at w, converges absolutely, i.e $\sum_{k=0}^{\infty} |a_k||w|^k$ converges. This is possible due to definition 3.4.

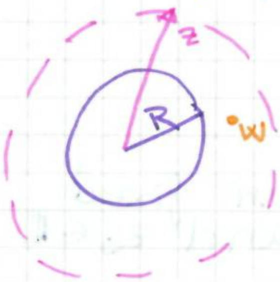
~~Thus~~ Since $|z| < |w|$, we have $|a_k||z|^k < |a_k||w|^k$.

By comparison test, $\sum^{\infty} |a_n||z|^k$ converges, as required.

<u>second part</u>

Suppose that $\sum a_k z^k$ converges and $|z| > R$

Pick a w : $R < |w| < |z|$. Want to show:



$\sum |a_k w^k|$ converges. Indeed, $a_k z^k$ is a bounded sequence by prop 1, so $|a_k z^k| \leq M$, with some $M > 0$.

Thus $|a_k w^k| = |a_k||z^k| \left|\frac{w}{z}\right|^k \leq M \underbrace{\left|\frac{w}{z}\right|^k}_{\text{geometrical series}}$

The series $\left|\frac{w}{z}\right|^k$ converges if $|w| < |z|$

$\Rightarrow$ by comparison principle (this series)

$\sum |a_k w^k|$ converges. This contradicts def 3.4

$\Rightarrow$ Part 2 is prooved. ∎

example

① $\sum z^k$, $R = 1$

$\sum (z-i)^k$, $R = 1$

② $\sum_{k=0}^{\infty} k^{10} (3z)^k$

<u>Ratio test:</u>

$$\frac{(k+1)^{10} |3z|^{k+1}}{k^{10} |3z|^k} = \left(1 + \frac{1}{k}\right)^{10} |3z| \xrightarrow[k\to\infty]{} |3z|$$

By Ratio Test, if $|3z| < 1 \Rightarrow$ convegence

$|3z| > 1 \Rightarrow$ divergence

or $|z| \begin{cases} < 1/3 \Rightarrow \text{conv} \\ > 1/3 \Rightarrow \text{div} \end{cases}$

$\Rightarrow$ Radius of convergence $= \frac{1}{3}$

③ $\sum_{n=0}^{\infty} \frac{n^{150}}{n!} z^n$

Ratio test

$$\frac{(n+1)^{150} |z|^{n+1}}{(n+1)!\, n^{150}} \frac{n!}{|z|^n} = \left(1 + \frac{1}{n}\right)^{150} \frac{1}{n+1} |z| \xrightarrow[n\to\infty]{} 0$$

Since $Q<1$, the series converges for all $z \in \mathbb{C}$, i.e. $R=\infty$

Remark

$\sum \frac{z^k}{k!}$ is defined for all $z \in \mathbb{C}$. It is called the exponential function.

Notation: $\exp(z)$

DIFFERENTIABILITY OF POWER SERIES

Again: $f(z) = \sum_{k=0}^{\infty} a_k (z-z_0)^k$ (*) R_1

Compare f with $y(z) = \sum_{k=0}^{\infty} a_k k (z-z_0)^{k-1}$ (**) R_2

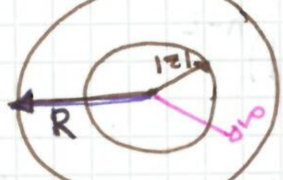
Lemma 3.6

The series (*) & (**) have the same radius of convergence.

Proof Let R_1, R_2 be the radii of convergence for (*) & (**).

Let's prove that $R_1 \le R_2$, i.e assuming that $\sum a_k (z-z_0)^k$ converges absolutely, we'll show that $\sum k |a_k| |z-z_0|^{k-1}$ converges as well. Assume $z_0 = 0$.

Pick a number $\rho > 0$ s.t $|z| < \rho < R$



Write:

$$k |a_k| |z|^{k-1} = \underbrace{\frac{k}{|z|} \left|\frac{z}{\rho}\right|^k}\ \underbrace{|a_k| \rho^k}$$

Observe the series

$\sum k \left|\frac{z}{\rho}\right|^k$ converges since $|z| < \rho$

The series $\sum |a_k| \rho^k$ converges since $\rho < R$, i.e $|a_k \rho^k| \le C$ for some constant $C>0$

so $|a_k \rho^k|$ is a bounded sequence, and hence $k |a_k| |z|^{k-1} \le \underset{\text{constant}}{C} \frac{k}{|z|} \left|\frac{z}{\rho}\right|^k$, and therefore

by comparison test, $\sum k |a_k| |z|^{k-1}$ converges.

Thus $R_1 \le R_2$

Part 2 Let's show that $R_2 \le R_1$, i.e if $\sum k |a_k| |z|^{k-1}$ converges, then $\sum |a_k| |z|^k$ converges too.

Write:

$|a_k| |z|^k \le k |z| |a_k| |z|^{k-1}$ for all $k \ge 1$

By comparison test $\sum |a_k| |z|^k$ converges

$\Rightarrow R_1 = R_2$ ∎

Denote $R = R_1 = R_2$

By lemma 3.6 the series $\sum_{k=0}^{\infty} a_k k(k-1)(z-z_0)^{k-2}$ has the same radius of convergence.

26/10/2011

$$f(z) = \sum_{k=0}^{\infty} a_k (z-z_0)^k, \quad (*)$$

$R_1 = R_2 = R$

$$g(z) = \sum_{k=0}^{\infty} k\, a_k (z-z_0)^{k-1}, \quad (**)$$

Remark $\sum_{k=2}^{\infty} k(k-1) a_k (z-z_0)^{k-2}$ has the same radius of convergence as $(*)$ & $(**)$

To study $f'(z)$ we need to look at

$$\frac{f(z+h)-f(z)}{h} \quad \text{as } h \to 0$$

In other words, need to investigate

$$\frac{(z+h-z_0)^n - (z-z_0)^n}{h} \quad \text{as } h \to 0$$

Important

PS4

Lemma 3.7 Let $z, h \in \mathbb{C}$ and $n \geq 2$. Then

$$\left| \frac{(z+h)^n - z^n}{h} - n z^{n-1} \right| \leq \frac{n(n-1)}{2} |h| \left(|z| + |h|\right)^{n-2}$$

Theorem 3.8

Let $R > 0$ be the radius of convergence of $(*)$. Then $f \in H(D(z_0, R))$, the series $(**)$ converges within the same radius, and $f'(z) = g(z)$ for all $z \in D(z_0, R)$

PROOF

Need to show that

$$\frac{f(z+h)-f(z)}{h} - g(z) \to 0 \quad \text{as } h \to 0$$

We'll show

$$\left| \frac{f(z+h)-f(z)}{h} - g(z) \right| \leq C|h|,$$ with some constant C independent of h.

We'll show this: for any N

$$\left| \frac{1}{h} \sum_{k=0}^{N} \left[a_k (z+h)^k - a_k z^k \right] - \sum_{k=0}^{N} k a_k z^{k-1} \right| \leq C|h|$$

with constant $C > 0$ independent of N, h.

Rewrite:

$$\left| \sum_{k=0}^{N} a_k \left[\frac{(z+h)^k - z^k}{h} - k z^{k-1} \right] \right| \leq \sum_{k=0}^{N} |a_k| \left| \frac{(z+h)^k - z^k}{h} - k z^{k-1} \right|$$

$$\underset{\text{by lemma 3.7}}{\leq} \frac{|h|}{2} \sum_{k=0}^{N} k(k-1)\left(|z|+|h|\right)^{k-2} |a_k|$$

Due to the remark made earlier,
The series $\sum_{k=0}^{\infty} k(k-1)\,|a_k|\,(|z|+|h|)^{k-2}$ converges and hence the right-hand side is bounded by $C|h|$ with

$$C = \frac{1}{2} \sum_{k=0}^{\infty} k(k-1)\,|a_k|\,(|z|+|h_0|)^{k-2},$$

where h_0 is s.t
$|h_0| = \frac{R-|z|}{2}$

Thus $f'(z) = g(z)$ as required. $\blacksquare$

Note: in the proof we assumed 'without loss of generality' (WLOG) that $z_0 = 0$

Corollary 3.9
The power series $(*)$ is differentiable any number of times in the disk $D(z_0, R)$. Moreover,

$$f'(z_0) = a_1, \quad f''(z_0) = 2a_2, \quad f'''(z_0) = 6a_3, \quad f^{(n)}(z_0) = n!\,a_n$$

$$f'(z) = a_1 + 2a_2(z-z_0) + 3a_3(z-z_0)^2 + \dots$$

$$f''(z) = 2a_2 + 6a_3(z-z_0) + \dots$$

Therefore
$$f(z) = \sum_{k=0}^{\infty} \frac{f^{(k)}(z_0)}{k!}(z-z_0)^k \rightarrow \underline{\text{Taylor series!}}$$

Exponential & trigonometric functions

Define: $\exp(z) = \sum_{k=0}^{\infty} \frac{z^k}{k!} \; (= e^z)$

The radius of convergence $= \infty \Rightarrow$ by theorem 3.8 $\exp(z)$ is an entire function.

Theorem 3.10 (Properties of e^z)
1. $(e^z)' = e^z$
2. $e^0 = 1$
3. $e^{z+w} = e^z \cdot e^w \quad \forall\, z, w \in \mathbb{C}$
4. $e^z \neq 0$ for all $z \in \mathbb{C}$

PROOF 1 By theorem 3.8

$$(e^z)' = \sum_{k=0}^{\infty} k \frac{z^{k-1}}{k!} = \sum_{k=1}^{\infty} k \frac{z^{k-1}}{k!} = \sum_{k=1}^{\infty} \frac{z^{k-1}}{(k-1)!} = \sum_{n=0}^{\infty} \frac{z^n}{n!} = e^z$$

2. $e^0 = 1$ (write out formula, easy)

Problem class 4

PS 2 $z = a+ib$ $\quad \arctan \frac{b}{a}$, $a>0$

$\pi - \arctan \left|\frac{b}{a}\right| \to a<0, b\geq 0$

$a<0, b<0$

$\arctan \left|\frac{b}{a}\right| - \pi$

$z = bi$ mod $= |b|$ $\quad b>0 \quad \arg z = \frac{\pi}{2}$

$\quad b<0 \quad \arg z = -\frac{\pi}{2}$

$\cancel{|a+b|^2 = |a|^2 + 2|a||b|} + |b|^2$

$|a+b|^2 = (a+b)(\bar{a}+\bar{b})$

$= |a|^2 + \underbrace{a\bar{b} + b\bar{a}}_{2\operatorname{Re}(a\bar{b})} + |b|^2$

$d(z,S) = \inf_{w\in S} |z-w|$

3 b

$(0, y_0)$

$\lim_{x\to 0} \frac{f(x_0+h, y_0) - f(x_0, y_0)}{h}$

PS 3

Qu 4 $f \in H(D(0,R))$ $\quad R>0$

$g(z) = \overline{f(\bar{z})} \in H(D(0,R))$

$\lim_{h\to 0} \frac{g(z+h) - g(z)}{h} = \lim_{h\to 0} \frac{\overline{f(\overline{z+h})} - \overline{f(\bar{z})}}{h}$

$= \overline{\underbrace{\lim_{h\to 0}\left(\frac{f(\bar{z}+\bar{h}) - f(\bar{z})}{\bar{h}}\right)}_{\text{this exists since } f \text{ is holomorphic}}} = \overline{f'(\bar{z})}$

PS 2

Qu 7 $\quad S$ - closed $\Rightarrow S^c$ - open

$z \notin S \Rightarrow z \in S^c$ S contains all lim points

$d(z,S) = \inf_{w\in S} |z-w| > 0$ $\qquad d\geq 0$ always by def

show $d>0$, exclude $d=0$

$z \in S^c$ - open $\quad \exists D(z,r) \subset S^c \quad r\geq 0$

$\forall w \in S \quad w \notin S^c \quad w \notin D(z,r) \quad |z-w| \geq r > 0$

Problem Sheet 4

Qu 2

$f = u + iv$ - analytic on D

u_x, u_y, v_x, v_y - continuous on D

Prove $g = \underbrace{(u-v)}_{a} + \underbrace{(u+v)}_{b} i$ - analytic on D

$a_x = b_y$ $\quad a_x = u_x - v_x \quad b_x = u_x + v_x$

$a_y = -b_x$ $\quad a_y = u_y - v_y = -v_x - u_x \quad b_y = u_y + v_y = -v_x + u_x$

since f analytic on D

$u_x = v_y$

$u_y = -v_x$

Qu 3

$z, h \in \mathbb{C}$, $n \geq 2$, $n \in \mathbb{N}$

$$(z+h)^n - z^n = h \sum_{k=0}^{n-1} (z+h)^k z^{n-1-k} \quad (*)$$

$$\left| \frac{(z+h)^n - z^n}{h} - nz^{n-1} \right| \leq \frac{n(n-1)|h|}{2} (|z| + |h|)^{n-2}$$

$$LHS = \Big| \underbrace{\sum_{k=0}^{n-1} [(z+h)^k z^{n-1-k}]}_{z^{n-1} + \sum_{k=1}^{n-1} (\ )^k z^{n+k}} - \underbrace{n z^{n-1}}_{\sum_{k=0}^{n-1} z^{n-1}} \Big|$$

$$= \left| \sum_{k=1}^{n-1} [(z+h)^k z^{n-1-k} - z^{n-1}] \right| = \left| \sum_{k=1}^{n-1} z^{n-1} [(z+h)^k z^{-k} - 1] \right|$$

$$\overset{(*)}{=} \left| \sum_{k=1}^{n-1} z^{n-1-k} \cdot h \cdot \sum_{j=0}^{k-1} (z+h)^{j-1+k} z^k \right|$$

$$\leq |h| \sum_{k=1}^{n-1} \underbrace{|z|^{n-1-k}}_{(|z|+|h|)^{n-1-k}} \cdot \underbrace{\sum_{j=0}^{k-1} (|z|+|h|)^{j-1+k} \overbrace{|z|^j}^{\text{rewrite } (|z|+|h|)^j}}_{(|z|+|h|)^{k-1}}$$

Exponential and trigonometric functions 31/10/2011

Def $e^z = \exp(z) = \sum_{k=0}^{\infty} \frac{z^k}{k!}$, $R = \infty$

entire function

Theorem 3.10

1. $(e^z)' = e^z$
2. $e^0 = 1$
3. $e^z e^w = e^{z+w}$, $\forall z, w \in \mathbb{C}$
4. $e^z \neq 0$, $z \in \mathbb{C}$

Proof

1 & 2 are done

3 Define $f(z) = e^{p-z} e^z$, $p \in \mathbb{C}$

Differetiate $f'(z) = -e^{p-z} e^z + e^{p-z} e^z = 0$, so by theorem 2.5
$f(z) = c$ $\forall z \in \mathbb{C}$

take $f(z) = f(0) = e^p$, and hence $e^{p-z} e^z = e^p$

Now $p = w + z$, so $e^w e^z = e^{w+z}$

4 By part 3
$e^z \cdot e^{-z} = 1$, and thus $e^z \neq 0$ $\forall z \in \mathbb{C}$

Corollary 3.11

Let f be entire, and let $f'(z) = f(z)$, and $f(0) = 1$. Then $f(z) = e^z$

Proof

Let $g(z) = e^{-z} f(z)$. Differenciate:

$g'(z) = -e^{-z} f(z) + e^{-z} f'(z) = -e^{-z} f(z) + e^{-z} f(z) = 0$, and hence by Thm. 2.5 $g(z) = \text{const}$, $z \in \mathbb{C}$.

Thus $g(z) = g(0) = f(0) = 1 \Rightarrow e^{-z} f(z) = 1 \Rightarrow f(z) = e^z$ ∎

Definition

$e^z = \sum_{k=0}^{\infty} \frac{1}{k!} z^k$

Recall example 2.7: $e^z = e^x(\cos y + i \sin y)$, $z = x + iy$

Denote $f(z) = e^x(\cos y + i \sin y)$.

By ex. 2.7, $f'(z) = f(z)$, $f(0) = 1$, and therefore $f(z) = e^z$.

Thus these two definitions give the same exponential function.

Define trigonometric and hyperbolic functions:

$$\sinh z = \frac{e^z - e^{-z}}{2}, \quad \cosh z = \frac{e^z + e^{-z}}{2},$$

$$\sin z = \frac{e^{iz} - e^{-iz}}{2i}, \quad \cos z = \frac{e^{iz} + e^{-iz}}{2}$$

Theorem 3.13

Prove !! $\frac{\partial}{\partial z} \sinh z = \cosh z$, $\frac{\partial}{\partial z} \cosh z = \sinh z$

usual identities for trigonometric functions:

EXERCISE $\frac{\partial}{\partial z} \sin z = \cos z$, $\frac{\partial}{\partial z} \cos z = -\sin z$

$\cos(z+w) = \cos z \cdot \cos w - \sin z \sin w$ $\qquad \cos^2 z + \sin^2 z = 1$

$\sin(z+w) = \sin z \cos w + \cos z \sin w$

NOTE:

$-1 \leq \sin \leq 1$ $\qquad \cosh^2 \cancel{z} - \sinh^2 z = 1$

$? \leq \sin z \leq ?$ $\quad |\sin z| \leq 1$? WRONG!!!

Take $z = it$, $t \in \mathbb{R}$. Then

$\sin(it) = \frac{e^{-t} - e^{t}}{2i} = i \sinh(t)$ $\qquad t \to \infty \Rightarrow e^t \to \infty$

$\sin(it) = i \sinh(t)$

EXERCISE SERIES EXPANSIONS:

Derive the formulaes, might appear in exam.

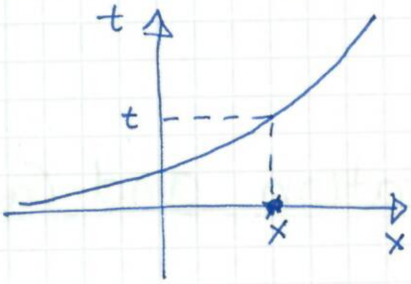
$\sinh z = \sum_{k=0}^{\infty} \frac{z^{2k+1}}{(2k+1)!}$, $\cosh z = \sum_{k=0}^{\infty} \frac{z^{2k}}{(2k)!}$

$\sin z = \sum_{k=0}^{\infty} (-1)^k \frac{z^{2k+1}}{(2k+1)!}$, $\cos z = \sum_{k=0}^{\infty} (-1)^k \frac{z^{2k}}{(2k)!}$

LOGARITHM

Remainder Real Analysis

For any $t > 0$ the number $x = \ln t^{\in \mathbb{R}}$ is defined to be the unique number s.t $e^x = t$.



$\ln t$ is the inverse of e^x.

COMPLEX ANALYSIS

Let us find $w \in \mathbb{C}$ s.t $e^w = z$ for some $z \in \mathbb{C}$.

Then we'll define:

$w = \log z$

Represent $w = u + iv$, so $z = e^{u+iv} = e^u \cdot e^{iv}$ ← polar representation of z

Thus $|z| = e^u$, $\underbrace{v = \arg z}_{\text{is not uniquely defined!}}$, and therefore $u = \ln |z|$.

$\arg z$ is not uniquely defined!

To fix, take $\operatorname{Arg} z \in (-\pi, \pi]$ and define the principal logarithm:

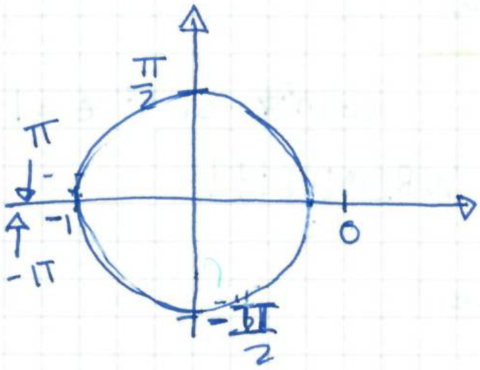
ln – real analysis
log – complex analysis
something new

$$\boxed{\text{Log}\, z = \ln|z| + i\,\text{Arg}\, z}$$

For the other values of the argument, arg z, define the following BRANCHES of log:

$$\boxed{\log_n z = \ln|z| + i(\text{Arg}\, z + 2\pi n), \quad n \in \mathbb{Z}}$$

$\ln|z|$ defined $\forall z \neq 0$



Log z is well defined on $\mathbb{C} \setminus \{z : \text{Im}\, z = 0, \text{Re}\, z \leq 0\} = \Lambda$

Theorem 3.14
Log z is analytic on Λ (Examinable)

Moreover, $\frac{\partial}{\partial z} \log z = \frac{1}{z}$

Powers Already know $z^n, n \in \mathbb{Z}$ and $z^\alpha, \alpha \in \mathbb{R}$
Assume that $\alpha \in \mathbb{C}$

Define: $\boxed{z^\alpha = e^{\alpha \log z}}$ ← have to say which branch of log is used

Principal value: $z^\alpha = e^{\alpha \text{Log}\, z}$

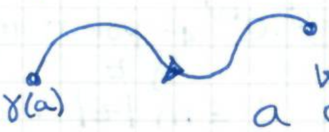
example

$$i^i = e^{i \log_n i} = e^{i(\ln|i| + i\frac{\pi}{2} + 2\pi n i)} = e^{-\frac{\pi}{2} - 2\pi n}$$
$n \in \mathbb{Z}$

Chapter 4: Contour Integration and Cauchy Theor

Curves, paths, contours

Def 4.1
Let $[a,b]$ be an interval of $\mathbb{R}$. Then a continuous functio
$\gamma : [a,b] \to \mathbb{C}$ is called a curve.

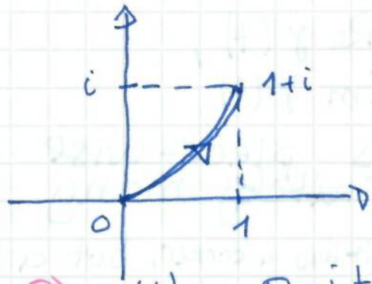


The image: $\gamma^* = \{z \in \mathbb{C} : z = \gamma(t)$ for some $t \in [$
We say that γ parametrises γ^*
a curve has a natural orientation:

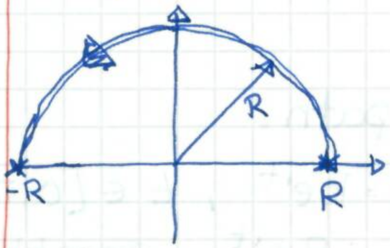
$\gamma(a)$ is the initial point, $\gamma(b)$ is the final point.

examples

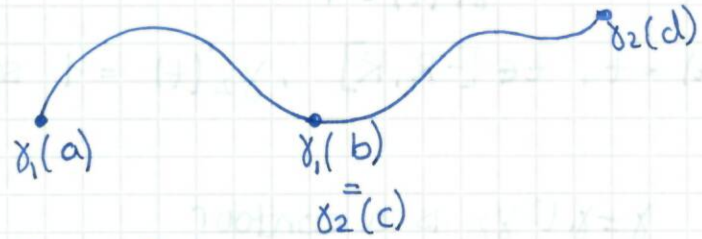
① $\gamma(t) = t + it^2, \ t \in [0,1]$



② $\gamma(t) = Re^{it}$, $\underline{R>0}$, $t \in [0,\pi]$



A different parametrisation:
$w(t) = Re^{it^2}$, $t \in [0,\sqrt{\pi}]$



let $\gamma_1 : [a,b] \to \mathbb{C}$
$\gamma_2 : [c,d] \to \mathbb{C}$
and $\gamma_1(b) = \gamma_2(c)$

<u>Join of two curves:</u>
$\gamma : [a, b+d-c]$

$$\gamma(t) = \begin{cases} \gamma_1(t), & t \in [a,b], \\ \gamma_2(t+c-b), & t \in [b, b+d-c] \end{cases}$$

<u>Notation:</u> $\gamma_1 \cup \gamma_2$

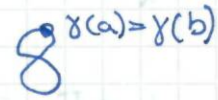
<u>Reverse curve</u>

Let $\gamma : [a,b] \to \mathbb{C}$. Then the curve
$\gamma_1(t) = \gamma(b+a-t)$, $t \in [a,b]$
is called the <u>reverse curve</u> and
$\gamma_2(a) = \gamma(b)$
$\gamma_2(b) = \gamma(a)$

<u>Definition 4.2</u>
Let $\gamma : [a,b] \to \mathbb{C}$ be a curve. Then
① γ is called simple if γ^* doesn't have self-intersections,
i.e. $\gamma(t_1) = \gamma(t_2)$ if $t_1 \neq t_2$ and $|t_1 - t_2| < b-a$

]

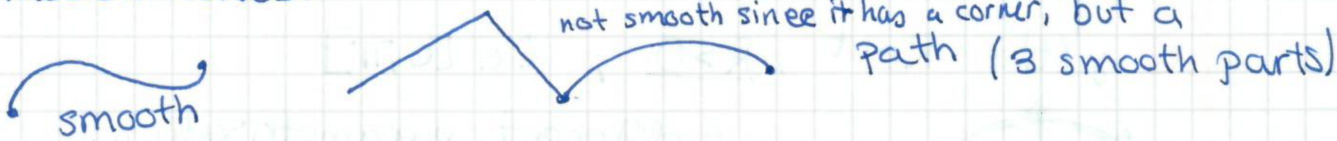
② γ is closed if $\gamma(a) = \gamma(b)$



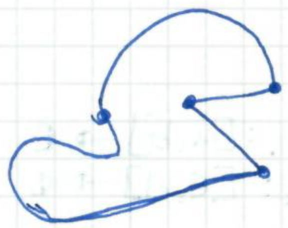
③ γ is closed and simple if it is closed & simple
④ γ is smooth if $\gamma'(t)$ exists on (a,b), here ~~$\gamma'(t)$~~.

$\gamma'(t) = \gamma_1'(t) + i\gamma_2'(t)$, when $\gamma_1(t) = \operatorname{Re}\gamma(t)$,
$\gamma_2(t) = \operatorname{Im}\gamma(t)$

(5) We say that γ is a path if γ is piece-wise smooth, i.e. γ is a join of finitely many smooth curves.

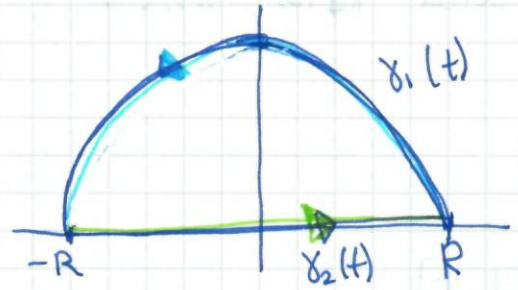


(6) A contour is a closed simple path.



example 1) Let $\gamma_1(t) = Re^{it}$, $t \in [0, \pi]$
$\gamma_1'(t) = iRe^{it}$, smooth

$\gamma_2(t) = t$, $t \in [-R, R]$, $\gamma_2'(t) = 1$ smooth



$\gamma = \gamma_1 \cup \gamma_2$ is a contour

γ_1, γ_2 are smooth curves

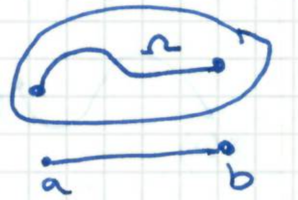
Theorem 4.3 (JORDAN CURVE THEOREM)
Let γ be a contour. The complement of γ^* is the union of two open sets, denoted $\operatorname{Int}\gamma$ and $\operatorname{Ext}\gamma$, where $\operatorname{Int}\gamma$ is bounded, and $\operatorname{Ext}\gamma$ is unbounded, so $\mathbb{C} = \gamma^* \cup \operatorname{Int}\gamma \cup \operatorname{Ext}\gamma$

2/11 | 11

Integration
Let $F(t) = A(t) + iB(t)$ be a complex valued function, with real valued $A(t)$ & $B(t)$.

Then by definition:
$$\int_a^b F(t)\,dt = \int_a^b A(t)\,dt + i\int_a^b B(t)\,dt$$

Def 4.4 Let f be defined on some domain $\Omega \subset \mathbb{C}$, and let $\gamma : [a,b] \to \Omega$ be a path.
Then we define the integral of f along γ:
$$\int_\gamma f(z)\,dz := \int_a^b f(\gamma(t))\,\gamma'(t)\,dt.$$



example
Let $\gamma(t) = re^{it}$, $t \in [0, 2\pi]$. Let $f(z) = z^n$, $n \in \mathbb{Z}$. Find $I_n = \int_\gamma z^n\,dz$. $I_n = \int_0^{2\pi} (re^{it})^n \cdot ire^{it} = ir^{n+1}\int_0^{2\pi} e^{i(n+1)t}\,dt$

case 1: suppose $n+1 \neq 0$. Then $I_n = i r^{n+1} \left. \dfrac{e^{i(n+1)t}}{i(n+1)} \right|_0^{2\pi} = 0$

case 2: let $n+1 = 0 \Rightarrow n = -1$.

Then $I_{-1} = i \int_0^{2\pi} dt = 2\pi i$

Take 2 cases as $\int_c^d e^{at} = \left. \dfrac{e^{at}}{a} \right|_c^d$ and we can't divide by 0

so $I_n = \begin{cases} 0 & , n \neq -1 \\ 2\pi i & , n = -1 \end{cases}$

(Observe: so we can take r as big/small as we like, and I_n will still be the same)

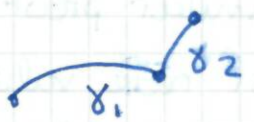
Def 4.5 Let $\gamma : [a,b] \to \mathbb{C}$ be a path. Then the lenght of γ is defined to be

$$L(\gamma) = \int_a^b |\gamma'(t)| dt$$

Example 4.6
Let $\gamma = re^{it}$, $t \in [0, 2\pi]$. Then

$$L(\gamma) = \int_0^{2\pi} |ire^{it}| dt = r \int_0^{2\pi} dt = 2\pi r \text{ as expected.}$$

Theorem 4.7
Let $\gamma, \gamma_1, \gamma_2$ be paths. Then:



① $\int_{\gamma} f(z)\, dz = - \int_{-\gamma} f(z)\, dz$

② If $\gamma = \gamma_1 \cup \gamma_2$, then $\int_{\gamma} f(z)\, dz = \int_{\gamma_1} f(z)\, dz + \int_{\gamma_2} f(z)\, dz$

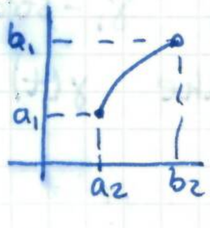
(Helpful, as we can parametrise curves differently, to something suitable)

③ $\int_{\gamma} c f(z)\, dz = c \int_{\gamma} f(z) = dz$ for any constant $c \in \mathbb{C}$

④ $\int_{\gamma} (f_1(z) + f_2(z))\, dz = \int_{\gamma} f_1(z)\, dz + \int_{\gamma} f_2(z)\, dz$

⑤ The integral doesn't depend on parametrisation
Suppose that γ^* is parametrised by two functions
$\gamma_1 : [a_1, b_1] \to \gamma^*$ and $\gamma_2 : [a_2, b_2] \to \gamma^*$
Suppose that there is a function
$\psi : [a_2, b_2] \to [a_1, b_1]$ with a positive derivative
$\psi(t)$ such that $\gamma_2 = \gamma_1 \circ \psi$
so we can get γ_2 from $\underbrace{[a_1, b_1] \to [a_2, b_2]}_{\psi} \underbrace{\to \gamma^*}_{\gamma_1}$

Then $\int_{\gamma_1} f(z)\, dz = \int_{\gamma_2} f(z)\, dz$



can't have this: This is using we find $\psi(t)$ with positive derivative.

⑥ The lenght of the path doesn't depend on parametrisation

⑦ Suppose $\sup\limits_{z \in \gamma^*} |f(z)| \le M$. Then,

$$\left|\int_\gamma f(z)\,dz\right| \le M\,L(\gamma)$$ very important   call this the "ML-result"

PROOF

①,②,③,④ are proved the same as real analysis

⑤ Write $\int_{\gamma_2} f(z)\,dz = \int_{a_2}^{b_2} f(\gamma_2(t))\,\gamma_2'(t)\,dt$

Recall: $\gamma_2(t) = \gamma_1(\psi(t))$ so $\gamma_2'(t) = \gamma_1'(\psi(t))\,\psi'(t)$

so $\int_{\gamma_2} f(z)\,dz = \int_{a_2}^{b_2} f(\gamma_2(t))\,\gamma_2'(t)\,dt$

$= \int_{a_2}^{b_2} f(\gamma_1(\psi(t)))\,\gamma_1(\psi(t))\,\psi'(t)\,dt = \int_{a_1}^{b_1} f(\gamma_1(s))\,\gamma_1'(s)\,ds$ using $s = \psi(t)$

$= \int_{\gamma_1} f(z)\,dz$ as claimed. QED

⑥ Similar proof

⑦ For real valued functions: $\left|\int_a^b F(t)\,dt\right| \le \sup\limits_{t\in[a,b]} |F(t)|\,(b-a)$ and we kn

$\left|\int_a^b f(t)\,dt\right| \le \int_a^b |f(t)|\,dt$

Let $I = \int_\gamma f(z) = dt = \int_a^b f(\gamma(t))\,\gamma'(t)\,dt$, write I in polar form i.e $I = |I|e^{i\theta}$

Then $|I| = \int e^{-i\theta} f(\gamma(t))\,\gamma'(t)\,dt$

$= \mathrm{Re} \int_a^b (f(\gamma(t))\,dt = \int_a^b \mathrm{Re}\,\gamma'(t)\,dt \le \int_a^b |f(\gamma(t))|\,|\gamma'(t)|\,dt$

$\le \sup\limits_{z\in\gamma^*} |f(z)| \int_a^b |\gamma'(t)|\,dt \le ML(\gamma)$

QED

14/11/11

PATH INTEGRALS

Let $\gamma : [a,b] \to \mathbb{C}$ be a path. Let f be a function. Then

$$\int_\gamma f(z)\,dz = \int_a^b f(\gamma(t))\,\gamma'(t)\,dt$$

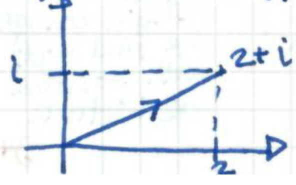
Theorem 4.7 (7)

$$\left|\int_\gamma f(z)\,dz\right| \le \sup_{z\in\gamma} |f(z)|\,L(\gamma)$$

example

Find $I_1 = \int_{\gamma_1} z^2\,dz$ γ_1 - straight path joining 0 and $2+i$

Take $\gamma(t) = t + \frac{1}{2}it = \frac{t}{2}(2+i)$ $t \in [0,2]$



Then

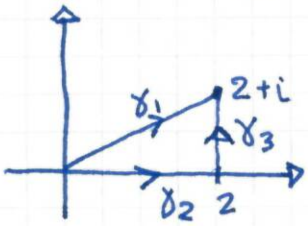
$$I_1 = \int_0^2 \left(\frac{t}{2}(2+i)\right)^2 \frac{2+i}{2}\, dt = \frac{(2+i)^3}{2^3}\int_0^2 t^2\, dt$$

$$= \left(\frac{2+i}{2}\right)^3 \frac{2^3}{3} = \frac{(2+i)^3}{3}$$

$$I_1 = \frac{(2+i)^3}{3}$$

Find

$$I_2 = \int_{\gamma_2} z^2\, dz \ , \text{ where } \gamma_2(t) = 2t\ ,\ t\in[0,1]$$



Then

$$I_2 = \int_0^1 (2t)^2\, 2\, dt = 8\int_0^1 t^2\, dt$$

$$= \frac{8}{3}$$

Find

$$I_3 = \int_{\gamma_3} z^2\, dz \ , \text{ where } \gamma_3(t) = 2+it\ ,\ t\in[0,1]$$

Then $I_3 = i\int_0^1 (2+it)^2\, dt = i\int_0^1 (4+4it-t^2)\, dt$

$$= 4i - 2 - \frac{i}{3} = -2 + \frac{11}{3}i$$

Compute

$$I_2 + I_3 = \frac{8}{3} - 2 + \frac{11}{3}i = \frac{2}{3} + \frac{11}{3}i$$

On the other hand

$$I_1 = \frac{1}{3}(8 - i + 12i - 6) = \frac{1}{3}(2 + 11i) = I_2 + I_3$$

ANTIDERIVATIVES (or PRIMITIVES)

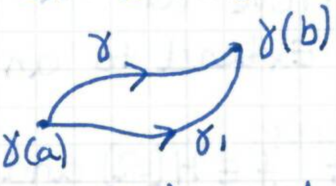
Def 4.8

Let f be continuous on a domain Ω, and let F be a function, analytic on Ω, s.t. $F'(z) = f$. Then F is called an antiderivative (or a primitive of f).

Theorem 4.9 ("Fundamental Theorem of Calculus")

Let f have an antiderivative F on Ω. Let $\gamma : [a,b] \to \Omega$ be a path. Then

$$\int_\gamma f(z)\, dz = F(\gamma(b)) - F(\gamma(a))$$



Observe:

if γ_1 is another path joining $\gamma(a)$ and $\gamma(b)$, then $\int_{\gamma_1} f(z)\, dz = \int_\gamma f(z)\, dz$

PROOF Write

$$\int_\gamma f(z)\, dz = \int_a^b f(\gamma(t))\gamma'(t)\, dt = \int_a^b F'(\gamma(t))\gamma'(t)\, dt$$

HL

by Chain Rule:

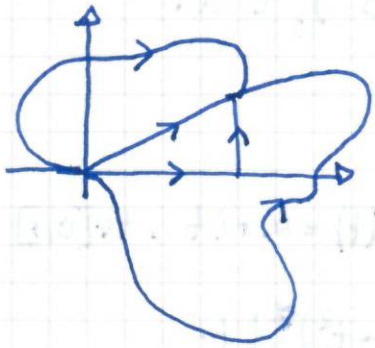
$$= \int_a^b \frac{d}{dt} F(\gamma(t))\, dt = F(\gamma(b)) - F(\gamma(a)) \quad \blacksquare$$

Back to example:

$$\int_{\gamma_1} z^2 dz$$

If $f(z) = z^2$, then $F(z) = \frac{z^3}{3}$ is a primitive

Therefore $I_1 = I_2 + I_3$

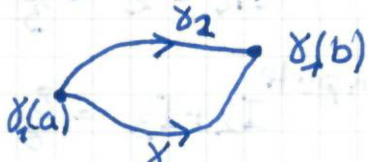


(all paths give the same value)

The Cauchy-Goursat Theorem

Reformulate: take two paths γ_1, γ_2 s.t

$\gamma_1(a) = \gamma_2(a)$, $\gamma_1(b) = \gamma_2(b)$



Consider $\gamma_1 \cup (-\gamma_2) = \gamma$
This is a contour. Under what conditions on f

$$\int_\gamma f(z)\, dz = 0 \; ?$$

Theorem 4.10

Let $f \in H(\Omega)$ and let γ be a contour, s.t $\text{int}\, \gamma \subset \Omega$.
Then

$$\int_\gamma f(z)\, dz = 0$$

Plan:

① Prove for a triangular contour 

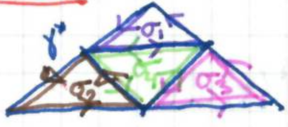
② Extend to an arbitrary contour γ.

Theorem 4.11

Let $f \in H(\Omega)$ and let γ be a triangular contour s.t $\text{int}\, \gamma \subset \Omega$. Then

$$\int_\gamma f(z)\, dz = 0$$

Proof



Denote: $\Delta = \gamma^* \cup \text{int}\, \gamma$.
Let $L(\gamma)$ be the lenght of the contour.

split it in four smaller ~~[illegible]~~ triangles by joining the middle points of each side.

Let $\sigma_1, \sigma_2, \sigma_3, \sigma_4$ be the resulting contours. Thus

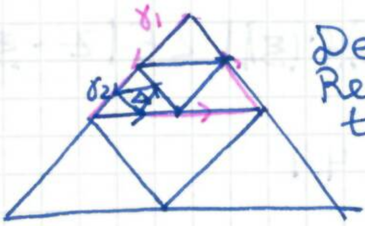
$$\int_\gamma f(z)\,dz = \sum_{j=1}^{4} \int_{\sigma_j} f(z)\,dz$$

Let $\left|\int_{\sigma_1} f(z)\,dz\right|$ be the largest. Denote

$$I_1 = \int_{\sigma_1} f(z)\,dz \text{. Therefore } \underline{\left|\int_\gamma f(z)\,dz\right| \leq 4\,|I_1|}$$

Observe:

$$L(\sigma_1) = \frac{L(\gamma)}{2}$$



Denote $\gamma_1 = \sigma_1$

Repeat the partition procedure with the triangle $\Delta_1 = \gamma_1^* \cup \operatorname{int}\gamma_1$

Thus we can find a contour γ_2 s.t $\underline{|I_1| \leq 4|I_2|}$, where

$$I_2 = \int_{\gamma_2} f(z)\,dz,$$

and $L(\gamma_2) = \frac{1}{2}L(\gamma_1) = \frac{1}{4}L(\gamma)$

Note:
$$\left|\int_\gamma f(z)\,dz\right| \leq 4\,|I_1| \leq \underbrace{16}_{4^2}\,|I_2|$$

Keep repeating the same construction:
we get a sequence of contours γ_k and of triangles $\Delta_k = \gamma_k^* \cup \operatorname{int}\gamma_k$ s.t

① $\Delta_{k+1} \subset \Delta_k$,

② $L(\gamma_k) = 2^{-k}L(\gamma)$,

③ $\left|\int_\gamma f(z)\,dz\right| \leq 4^k \left|\int_{\gamma_k} f(z)\,dz\right|$

<u>Note:</u> The set $\bigcap_{k=1}^{\infty} \Delta_k$ is not empty.

Indeed, let $z_k \in \Delta_k$ be an arbitrary point. The sequence z_k is bounded, since $z_k \in \Delta$.
Thus, by Bolzano-Weierstraß there is a convergent subsequence, z_{k_j}.
Let $\xi = \lim_{j\to\infty} z_{k_j}$

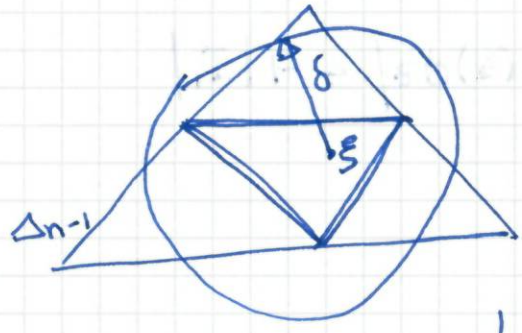
For any n one can find J s t $z_{k_j} \in \Delta_n$ $\forall j \geq J$

Since Δ_n is closed, and ξ is an accumulation point, we can claim that $\xi \in \Delta_n$

Thus $\xi \in \Delta_n$ for all n, and therefore $\xi \in \bigcap_{k=1}^{\infty} \Delta_k$.

Recall that f is holomorphic on Ω, so
$\forall\, \varepsilon \; \exists\, \delta > 0$ s.t

$$\left| \frac{f(z)-f(\xi)}{z-\xi} - f'(\xi) \right| < \varepsilon \quad \text{if} \quad |z-\xi| < \delta, \text{ i.e } z \in D(\xi,\delta) \qquad (*)$$



Observe:
for any $z \in \gamma_n^*$

$$|z-\xi| \le \frac{1}{2} L(\gamma_n) = 2^{-n-1} L(\gamma)$$

Thus one can find n s.t $\Delta_n \subset D(\xi,\delta)$

Rewrite $(*)$:

$$|f(z)-f(\xi)-(z-\xi)f'(\xi)| < \varepsilon|z-\xi|$$

for $z \in D(\xi,\delta)$

Note: $\int_{\gamma_n} f(\xi)\,dz = 0$ by Theorem 4.9

and $\int_{\gamma_n} (z-\xi) f'(\xi)\,dz = 0$ by Thm 4.9.

$F'(z) = z$
$F(z) = \frac{z^2}{2}$

Therefore $\int_{\gamma_n} f(z)\,dz = \int_{\gamma_n} \left[f(z) - f(\xi) - (z-\xi) f'(\xi) f'(\xi) \right] dz$

and hence

$$\left| \int_{\gamma_n} f(z)\,dz \right| \underset{\text{by Thm 4.7 (2)}}{\le} \varepsilon \frac{1}{2} L^2(\gamma_n) = \frac{\varepsilon}{2} L^2(\gamma_n)$$

Therefore

$$\left| \int_{\gamma} f(z)\,dz \right| \le 4^n \frac{\varepsilon}{2} L^2(\gamma_n) = \cancel{4^n} \frac{\varepsilon}{2} \cdot \cancel{4^{-n}} L^2(\gamma) = \frac{\varepsilon}{2} L^2(\gamma)$$

as $\varepsilon > 0$ is arbitrary, $\int_{\gamma} f(z)\,dz = 0$ as claimed ∎

Theorem 4.12 (Antiderivative theorem) (No holes since it is convex)

Let Ω be a convex domain, and let f be continuous on Ω, and for any triangular contour γ inside Ω.

$$\int_{\gamma} f(z)\,dz = 0$$

Then f has an antiderivative in Ω.
More precisely, for any point $a \in \Omega$ the function

$$F(z) = \int_{[a,z]} f(w)\,dw$$

is an antiderivative of f
i.e $F'(z) = f(z)$

Note by convexity, $[a,z] \subset \Omega$ for all $z \in \Omega$

Proof

Write $\frac{F(z+h)-F(z)}{h} = \frac{1}{h}\left[\int_{[a,z+h]} f(w)\,dw - \int_{[a,z]} f(w)\,dw\right]$

16/11/2011

Proof continued

$$\frac{F(z+h)-F(z)}{h} = \frac{1}{h}\int_{[z,z+h]} f(w)\,dw$$

note:

$$f(z) = \frac{1}{h}\int_{[z,z+h]} f(w)\,dw \quad .\text{Thus}$$

$$\frac{F(z+h)-F(z)}{h} - f(z) = \frac{1}{h}\int_{[z,z+h]}\left[f(w)-f(z)\right]dw$$

Let's fix an $\varepsilon>0$. Then due to continuity of f, there is a $\delta>0$ s.t.

$$|f(z)-f(w)| < \varepsilon \quad \text{if} \quad |z-w|<\delta$$

Assume that $|h|<\delta$. Therefore

$$\left|\frac{1}{h}\int_{[z,z+h]}\left[f(w)-f(z)\right]dw\right| < \frac{\varepsilon}{|h|}\cdot \underbrace{|h|}_{\text{lenght of the path}} = \varepsilon$$

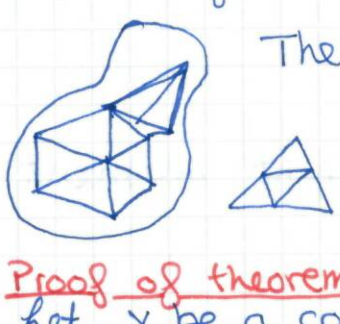
This means that for any $\varepsilon>0$ $\exists$ $\delta>0$ s.t

$$\left|\frac{F(z+h)-F(z)}{h} - f(z)\right| < \varepsilon \quad \text{if} \quad |h|<\delta$$

By def. of limit, $F'(z) = f(z)$ as claimed. ∎

Remark

Let $f \in H(\Omega)$ and let γ be a polygonal contour s.t $\operatorname{Int}\gamma \subset \Omega$

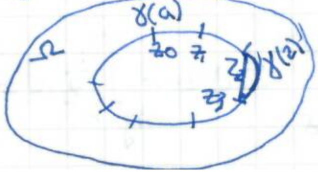


Then $\int_\gamma f(z)\,dz = 0$

This follows from Theorem 4.11.

Proof of theorem 4.10

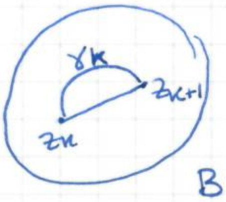
Let γ be a contour, s.t $\operatorname{Int}\gamma \subset \Omega$. Pick a sequence of point $\gamma(a)=z_0, z_1, z_2, \dots, z_n = z_0$



Let σ be the polygonal contour obtained by joining these points.

Then $\displaystyle\int_\sigma f(z)\,dz = 0$

Let γ_k be the part of γ between z_k and z_{k+1}.
Assume that z_k and z_{k+1} are so close, that there is a $\delta>0$ s.t $[z_k, z_{k+1}] \subset D(z_k,\delta)$



$\gamma_k \in D(z_k,\delta)$ and $D(z_k,\delta)\subset\Omega$

Then $f \in H(D(z_k,\delta))$

By Theorems 4.11 and 4.12 f has an antiderivative!

By Thm 4.9

$$\int_{\gamma_k} f(z)\,dz = \int_{[z_k, z_{k+1}]} f(z)\,dz$$

Add them up:

$$\int_\gamma f(z)\,dz = \int_\sigma f(z)\,dz = 0 \text{ as required } \blacksquare$$

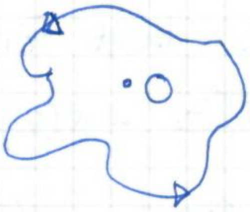
Examples

1. $\displaystyle\int_\gamma e^z\,dz = 0$ for any contour γ.

2. $\displaystyle\int_\sigma \frac{1}{z}\,dz = 0$ for

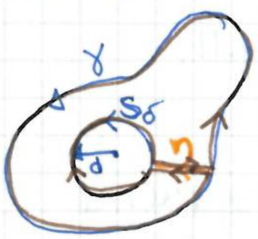
$$\int_\gamma \frac{1}{z}\,dz =$$

$$\int_{|z|=1} \frac{1}{z}\,dz = 2\pi i$$



Example 4.14
Let γ be a contour s.t $0\in \operatorname{Int}\gamma$
Then $\displaystyle\int_\gamma \frac{1}{z}\,dz = 2\pi i$

Proof
Let $\delta>0$ be s.t $D(0,\delta)\subset \operatorname{Int}\gamma$



- Join γ and S_δ with a straight segment

Define the contour:

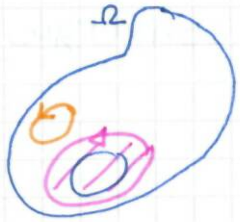
$$\tilde\gamma = \gamma \cup \eta \cup (-S_\delta) \cup (-\eta)$$

By Cauchy-Goursat (Thm 4.10)

$$\int_{\tilde\gamma} \frac{1}{z}\,dz = 0, \text{ i.e } = \int_\gamma \frac{1}{z}\,dz + \cancel{\int_\eta \frac{1}{z}\,dz} - \int_{S_\delta} \frac{1}{z}\,dz - \cancel{\int_\eta \frac{1}{z}\,dz}$$

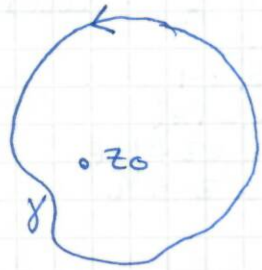
Thus $\displaystyle\int_\gamma \frac{1}{z}\,dz = \int_{S_\delta} \frac{1}{z}\,dz = 2\pi i$ as claimed ∎

Definition 4.13
A domain Ω is said to be simply connected, if for any closed simple curve γ we have $\operatorname{Int}\gamma \subset \Omega$



Lemma 4.15
Let γ be a contour s.t $z_0 \in \operatorname{Int}\gamma$

$$\boxed{\int_\gamma \frac{1}{z-z_0}\,dz = 2\pi i}$$

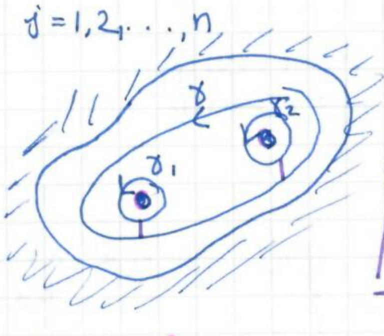


Proof Let $\tilde{\gamma} = \gamma - z_0$. Then

$$\int_\gamma \frac{1}{z-z_0}\,dz = \int_{\tilde{\gamma}} \frac{1}{z}\,dz = 2\pi i \quad ∎$$

Theorem 4.16 (Cauchy-Goursat for multiply connected domains)

Let Ω be a domain, $f \in H(\Omega)$. Let γ be a contour in Ω, and let $\gamma_1, \gamma_2, \gamma_3, \dots, \gamma_n$ in Ω be continuous s.t $\operatorname{Int}\gamma_j \cap \operatorname{Int}\gamma_k = \emptyset$ for $j \neq k$, and $\operatorname{Int}\gamma_j \subset \operatorname{Int}\gamma$, $j = 1, 2, \dots, n$



Do as in ex 4.14 "cut the holes out"

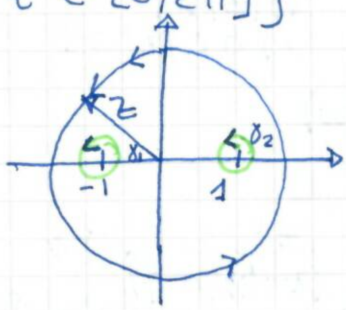
Suppose that
$$f \in H\left(\operatorname{Int}\gamma \setminus \bigcup_{j=1}^{n} \overline{\operatorname{Int}\gamma_j}\right)$$

$$\boxed{\text{Then } \int_\gamma f(z)\,dz = \sum_{j=1}^{n} \int_{\gamma_i} f(z)\,dz}$$

Example

$$\int_\gamma \frac{1}{z^2-1}\,dz \qquad \gamma = \{z : z = 2e^{it}, t \in [0, 2\pi]\}$$

$$= \int_{\gamma_1} \frac{1}{z^2-1}\,dz + \int_{\gamma_2} \frac{1}{z^2-1}\,dz$$



Problem class PS4

① $\frac{\partial f}{\partial x}$ $\quad f=f(x,y) \quad (x,y)\to(r,\theta) \quad f=f(r,\theta)$

$$\frac{\partial f}{\partial r}=\frac{\partial f}{\partial x}\cdot\frac{\partial x}{\partial r}+\frac{\partial f}{\partial y}\cdot\frac{\partial y}{\partial r}$$

on bound there are points for which Σ converges and $\exists$ points Σ diverges

$\Sigma \frac{1}{n}$ - diverges $\quad \Sigma \frac{(-1)^n}{n}$ - converges

~~$R=-1$~~
~~$R=1+i$~~
~~$1-e<z<1+e$~~
~~$z\in\mathbb{C}$~~

Problem sheet 5

Qu 3

$$I_{m,n}=\iint_{|z|<1} z^m\bar{z}^n\,\partial x\partial y \qquad m,n=0,1,2\ldots$$

- $z=re^{i\theta} \to \bar{z}=re^{-i\theta}$
- consider 2 cases: $m=n$ and $m\neq n$, $m>n$
- $\partial x\partial y=r\,\partial r\,d\varphi$

$$I_{m,n}=\begin{cases} m=n & \int_0^{2\pi}\int_0^1 r^{2m}r\,\partial r\,\partial\varphi=2\pi\int_0^1 r^{2m+1}\partial r=\frac{2\pi}{2m+2}=\frac{\pi}{m+1} \\ m>n & \int_0^{2\pi}\int_0^1 r^{m+n}e^{(m-n)i\varphi}r\,\partial r\,d\varphi=\underbrace{\int_0^{2\pi}e^{(m-n)i\varphi}\partial\varphi}_{=0}=0 \end{cases}$$

$$|J|=\det J=\det\begin{pmatrix}\frac{\partial x}{\partial r} & \frac{\partial x}{\partial\varphi}\\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial\varphi}\end{pmatrix}=r \qquad \begin{matrix}x=r\cos\varphi\\ y=r\sin\varphi\end{matrix}$$

Problem sheet 6

1. Find

$$h(z)=\int_{S(0,1)}\frac{\partial w}{w(w-z)}$$

for 1) $|z|>1$
2) $0<|z|<1$

$f(w)$

$$\frac{1}{2\pi i}\int_{S(z_0,r)}\frac{f(z)}{z-a}=\begin{cases}0 & \text{if } a\notin \text{int } S(z_0,r) \text{ by C-G thm}\\ f(a), & \text{if } a\in\text{int } S(z_0,r) \text{ CIF}\end{cases}$$

example

$$\int_{S(0,1)}\frac{z^2}{(z-\frac{1}{2})(z-\frac{3}{2})}\partial z$$

$$\text{CIF}=\int_{S(0,1)}\frac{\overbrace{\frac{z^2}{z-\frac{3}{2}}}^{f(z)}}{z-\underbrace{\tfrac{1}{2}}_{a}}\partial z=2\pi i\,f(a)=\frac{(\frac{1}{2})^2}{\frac{1}{2}-\frac{3}{2}}=\frac{\frac{1}{4}}{-1}=-\frac{1}{4}$$

Consider

$$\int_{S(0,1)} \frac{z^2}{\underset{a}{(z-\frac{1}{2})}\underset{b}{(z+\frac{1}{2})}}\,dz = \int_{S(0,1)} \frac{\overset{f(z)}{z^2}}{z-\underset{a_1}{\frac{1}{2}}}\,dz - \int_{S(0,1)} \frac{\overset{f(z)}{z^2}}{z+\underset{a_2}{\frac{1}{2}}} \underset{CIF}{=} 2\pi i\left(f(a_1) - f(a_2)\right) = 0$$

apply partial fractions $\frac{1}{(z-\frac{1}{2})(z+\frac{1}{2})} = \frac{A}{z-\frac{1}{2}} + \frac{B}{z+\frac{1}{2}}$ $\quad a_1 = \frac{1}{2}$, $a_2 = -\frac{1}{2}$

$$= \frac{z(A+B) + \frac{1}{2}(A-B)}{(z-\frac{1}{2})(z+\frac{1}{2})} \qquad \begin{matrix} A = -B \\ A - B = 2 \end{matrix} \qquad -A = B = -1$$

$$\int_{S(0,1)} \frac{g(z)}{z-5}\,dz = 0 \qquad g(z) \text{ is holomorphic inside } S(0,1)$$

by Cauchy-Goursat Thm

Back to Qu 1

i) $f(w)$

1) $|z| > 1$ $\quad h(z) = \int_{S(0,1)} \frac{1\,dw}{w(w-z)}$ is holomorphic at every point

you can write $w = w - 0$

z is fixed

Use CIF

Qu 2

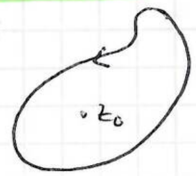
Evaluate $\frac{1}{2\pi i} \int_\gamma \frac{e^z}{z^2+a^2}\,dz$ $\qquad a > 0$ - real number

$\gamma : \overline{D}(0,a) \subset \operatorname{Int}\gamma$

21/11/11

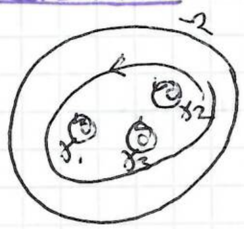
Lemma 4.15

γ:

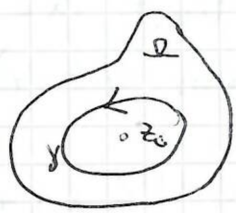


$$\int_\gamma \frac{1}{z-z_0}\,dz = 2\pi i \qquad f(z) = 1$$

Theorem 4.16



$$\int_\gamma f(z)\,dz = \sum_{j=1}^{N} \int_{\gamma_j} f(z)\,dz$$



The Cauchy Integral Formula

NB!!!

Very important

Theorem 4.17

Assume that Ω is a simply connected domain, and let $f \in H(\Omega)$ Let γ be a contour in Ω, s.t $z_0 \in \operatorname{Int}_\gamma$

Then

$$f(z_0) = \frac{1}{2\pi i} \int_\gamma \frac{f(z)}{z-z_0}\,dz$$

Or,

$$\int_\gamma \frac{f(z)}{z-z_0}\,dz = 2\pi i f(z_0)$$

EXAM Qu

Proof

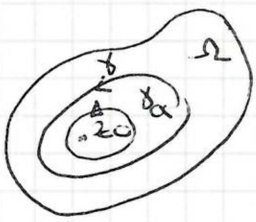
Write
$$\frac{1}{2\pi i}\int_\gamma \frac{f(z)}{z-z_0}\,dz = \frac{1}{2\pi i}\int_\gamma \frac{f(z)-f(z_0)}{z-z_0}\,dz + \frac{1}{2\pi i} f(z_0)\int \frac{1}{z-z_0}\,dz$$

by lemma 4.15: $= \frac{1}{2\pi i}\int \frac{f(z)-f(z_0)}{z-z_0}\,dz + f(z_0)$

It remains to show that
$$\int_\gamma g(z)\,dz = 0, \text{ with } g(z) = \frac{f(z)-f(z_0)}{z-z_0}$$

If g were analytic on Ω, this would be true due to Cauchy-Goursat

But g is not analytic on $\Omega \setminus \{z_0\}$



~~By theorem 4.16~~

Let $\alpha > 0$ be a number s.t $\gamma_\alpha = S(z_0, \alpha)$ is inside Int γ. By Theorem 4.16
$$\int_\gamma g(z)\,dz = \int_{\gamma_\alpha} g(z)\,dz$$

$$\int_\gamma g(z)\,dz = \int_{\gamma_\alpha} g(z)\,dz$$

Since f is diff on Ω, $\forall\, \varepsilon > 0\ \exists\, \delta > 0$ s.t
$$|g(z) - f'(z_0)| < \varepsilon \text{ if } |z - z_0| < \delta$$

Use this with $\varepsilon = 1$:

$(*)$ $|g(z)| < 1 + |f'(z_0)|$, $|z - z_0| < \delta$

Assume that $\alpha < \delta$, so $(*)$ holds for $z \in \gamma_\alpha$

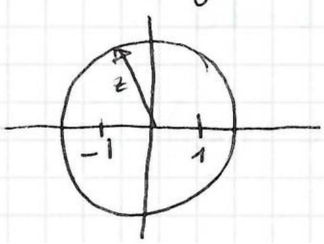
By theorem 4.7(7)
$$\left|\int_{\gamma_\alpha} g(z)\,dz\right| \le \underbrace{\left(1 + |f'(z_0)|\right)}_{\text{does not depend on } \alpha} \cdot 2\pi\alpha$$

as $\alpha > 0$ is arbitrary, $\int_\gamma g(z)\,dz = \int_{\gamma_\alpha} g(z)\,dz = 0$ as required

Take α arbitrary small, then RHS $\to 0$

Examples

1. $$\int_{S(0,2)} \frac{1}{z^2-1}\,dz$$



by thm 4.16

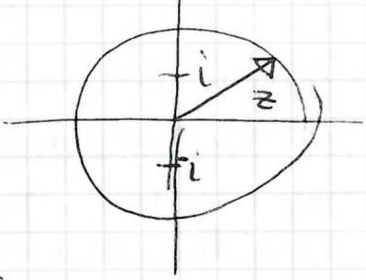
$$\int_{S(0,2)} \frac{1}{z^2-1}\,dz = \int_{S(-1,\frac{1}{2})} \frac{1}{z^2-1}\,dz + \int_{S(1,\frac{1}{2})} \frac{1}{z^2-1}\,dz$$

$$\int_{S(-1,\frac{1}{2})} \frac{1}{z^2-1}\,dz = \int_{S(-1,\frac{1}{2})} \frac{\overbrace{\frac{1}{z-1}}^{=f(z)}}{z+1}\,dz = 2\pi i\, f(-1) = 2\pi i\left(-\tfrac{1}{2}\right) = \underline{-\pi i}$$

In the same way,

$$\int_{S(1,\frac{1}{2})} \frac{1}{z^2-1}\,dz = \int_{S(1,\frac{1}{2})} \frac{\overbrace{\frac{1}{z+1}}^{=g(z)}}{z-1}\,dz = 2\pi i\, g(1) = \underline{\pi i}$$

Thus $\displaystyle\int_{S(0,2)} \frac{1}{z^2-1}\,dz = -\pi i + \pi i = \underline{\underline{0}}$



② $\displaystyle\int_{S(0,2)} \frac{\sin z}{z^2+1}\,dz$

By Thm 4.16,

Cauchy-Goursat

$$I = \underbrace{\int_{S(i,\frac{1}{2})} \frac{\sin z}{z^2+1}\,dz}_{I_1} + \underbrace{\int_{S(-i,\frac{1}{2})} \frac{\sin z}{z^2+1}\,dz}_{I_2}$$

$$I_1 = \int_{S(i,\frac{1}{2})} \frac{\frac{\sin z}{z+i}}{z-i}\,dz = 2\pi i\,\frac{\sin i}{i+i} = \underline{\pi \sin i}$$

- related to hyperbolic sin, find it!

$$I_2 = \int_{S(-i,\frac{1}{2})} \frac{\frac{\sin z}{z-i}}{z+i}\,dz = 2\pi i\,\frac{\sin(-i)}{-i-i} = \underline{\pi \sin i}$$

Therefore, $I = I_1 + I_2 = 2\pi \sin i$

Alternative method: use partial fractions:

$$\frac{\sin z}{z^2+1} = \frac{\sin z}{2i(z-i)} - \frac{\sin z}{2i(z+i)}$$

you can apply CIF to this and get the same answer!

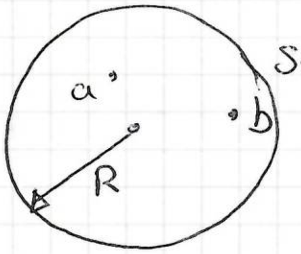
Applications

Theorem 4.18 (Liouville's Theorem)
Let f be an entire function, s.t. there exists a number $M>0$: $|f(z)| \leq M \quad \forall z \in \mathbb{C}$

Then $f(z) = \text{const} \quad \forall z \in \mathbb{C}$

Proof

Let $a, b \in \mathbb{C}$, and let's show that $f(a) = f(b)$.
Let $R > 0$ be such that $|z-a| \geq \frac{R}{2}$ and $|z-b| \geq \frac{R}{2}$ for all $z \in S(0,R)$



By Cauchy Formula:

$$f(a) - f(b) = \frac{1}{2\pi i} \int_{S(0,R)} \left[\frac{f(z)}{z-a} - \frac{f(z)}{z-b} \right] dz$$

$$= \frac{1}{2\pi i} \int_{S(0,R)} f(z) \frac{a-b}{(z-a)(z-b)} dz$$

By Thm 4.7(7),

$$|f(a) - f(b)| \leq \frac{1}{\cancel{2\pi}} M \frac{|a-b|}{\frac{R}{2} \cdot \frac{\cancel{R}}{2}} \cancel{2\pi R} = \frac{4M|a-b|}{R}$$

Take R arbitra small so $\to 0$

As R is arbitrary, $f(a) - f(b) = 0$ as required

Theorem 4.19 (The Fundamental Theorem of Algebra)

Let p be a polynomial of degree n $p = p(z)$. Then it has exactly n roots in $\mathbb{C}$ counting multiplicities.

Proof

We'll show that p has at least one root.

Assume that p has no roots, therefore $\frac{1}{p(z)}$ is entire. Write:

$$p(z) = a_n z^n + a_{n-1} z^{n-1} + \ldots + a_0$$

$$= z^n \underbrace{(a_n + a_{n-1} z^{-1} + a_{n-2} z^{-2} + \ldots + a_0 z^{-n})}_{\to a_n \text{ as } z \to \infty}$$

$\to |p(z)| \to \infty$ as $z \to \infty$

Thus $\frac{1}{|p(z)|} \to 0$ as $z \to \infty$

In other words, $\exists\, R > 0$ s.t $\left|\frac{1}{p(z)}\right| < 1$ if $|z| > R$

At the same time

$\frac{1}{p(z)}$ is continuous on $\overline{D}(0,R)$, so $\frac{1}{p(z)}$ is bounded on $\overline{D}(0,R)$, by theorem 1.29.

Thus $\left|\frac{1}{p(z)}\right| \leq M$ $\forall z \in \mathbb{C}$ with some $M > 0$. By Liouville's theorem $\frac{1}{p(z)} = \text{const} \Rightarrow p(z) = \text{const}$.

We have a contradiction ▨

Cauchy Formula for the derivatives

Write:

$$f(w) = \frac{1}{2\pi i} \int_\gamma \frac{f(z)}{z-w}\, dz$$

Differentiate (formally)

$$f'(w) = \frac{1}{2\pi i} \int \frac{f(z)}{(z-w)^2}\, dz$$

$$f''(w) = \frac{2}{2\pi i} \int_\gamma \frac{f(z)}{(z-w)^3}\, dz$$

Theorem 4.20 (The Cauchy Formula for higher derivatives)
Suppose Ω is simply connected, and $f \in H(\Omega)$.
Let γ be a contour in Ω, s.t. $\operatorname{Int}\gamma \ni z_0$.
Then f is infinitely differentiable inside Ω and

$$f^{(n)}(z_0) = \frac{n!}{2\pi i} \int \frac{f(z)}{(z-z_0)^{n+1}}\, dz$$

example

$$I = \int_{S(0,\frac{1}{3})} \frac{\cos z}{z^2(z-1)}\, dz = ?$$

$S(0,\frac{1}{3})$

$\cdot 1$ far away, so analytic.

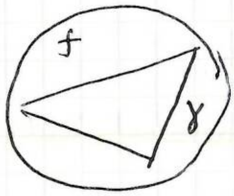
$$I = \int_{S(0,\frac{1}{3})} \frac{f(z)}{z^2}\, dz, \quad f(z) = \frac{\cos z}{z-1}$$

By Thm 4.20,

$$I = 2\pi i\, f'(0). \text{ Here } f'(z) = -\frac{\sin z}{z-1} - \frac{\cos z}{(z-1)^2}$$

so $f'(0) = -1 \Rightarrow I = -2\pi i$

$$\int_{S(i,\frac{1}{2})} \frac{\cos z}{z^2(z-i)}\, dz = 0$$



$\int_\gamma f\, dz = 0 \Rightarrow f$ has an antiderivative
i.e $\exists F : F' = f$.

Theorem 4.21 (Morera's Theorem)
Let f be continuous on Ω, and assume that
$\int_\gamma f(z)\, dz = 0$ for every contour $\gamma \subset \Omega$.

Then $f \in H(\Omega)$. (Proof in online notes)
Might be examined

23/11/11

Chapter 5: Series expansions of holomorphic functions

Aim: to show that every analytic function can be expanded in a Taylor series

i.e $f(z) = \sum_{k=0}^{\infty} a_k (z-z_0)^k$

Theorem 5.1
Suppose that $f \in H(D(z_0,R))$, with some $z_0 \in \mathbb{C}$, $R>0$. Then for every $z \in D(z_0,R)$ we have;

$$f(z) = \sum_{k=0}^{\infty} \frac{f^{(k)}(z_0)}{k!} (z-z_0)^k ; \quad (*)$$

and the series converges absolutely

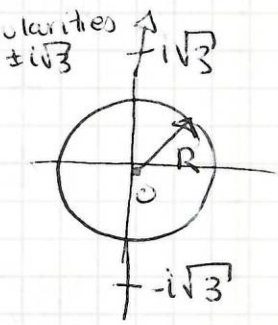
The series $(*)$ is called the Taylor series of f about z_0.

Comments and examples
Let R_0 be the radius of convergence of $(*)$. Note: Theorem 5.1 doesn't say that $R = R_0$.. it does say that $R \le R_0$

example ①
e^z is entire, i.e $R_0 = \infty$
e^z is analytic on $D(0,1) \Rightarrow (*)$ holds

② $g(z) = \frac{1}{z^2+3}$. Find its series about $z_0 = 0$

singularities at $\pm i\sqrt{3}$



Find R_0. By Thm 5.1, we have

$$g(z) = \sum_{k=0}^{\infty} a_k z^k$$

g is holomorphic on $D(0,\sqrt{3})$
Thus $(*)$ holds for $R = \sqrt{3}$
We know $R_0 \ge R = \sqrt{3}$

On the other hand, $R_0 \le \sqrt{3}$, so $R_0 = \sqrt{3}$.

If $z_0 = 0$, the series is called Maclaurin series.

example $\quad e^z = \sum_{k=0}^{\infty} \frac{z^k}{k!}$

Question: Find Taylor series for e^z at $z_0 \in \mathbb{C}$

$$e^z = e^{z_0} \sum_{k=0}^{\infty} \frac{(z-z_0)^k}{k!}, \quad e^z = e^{z_0} e^{z-z_0}$$

Past exam question !!!

Exercise

Find Taylor series for $\sin z$ at z_0.

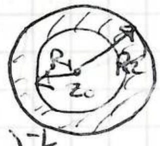
Laurent Series

Let $g(z) = \sum_{k=-\infty}^{\infty} a_k (z-z_0)^k = g_1(z) + g_2(z)$

with $g_1(z) = \sum_{k=-\infty}^{-1} a_k (z-z_0)^k, \quad g_2(z) = \sum_{k=0}^{\infty} a_k (z-z_0)^k$

g_2: Let R_2 be its radius of convergens

g_2 converges for $|z-z_0| < R_2$



g_1: $g_1(z) = \sum_{k=-\infty}^{-1} a_k (z-z_0)^k = \sum_{k=1}^{\infty} a_{-k} (z-z_0)^{-k}$

rewrite: $w = (z-z_0)^{-1}$

$\sum_{k=1}^{\infty} a_{-k} w^k \Rightarrow$ converges within its radius of convergence, r_1.

$|w| < r_1 \Leftrightarrow |z-z_0|^{-1} < r_1 \Leftrightarrow |z-z_0| > \frac{1}{r_1} = R_1$

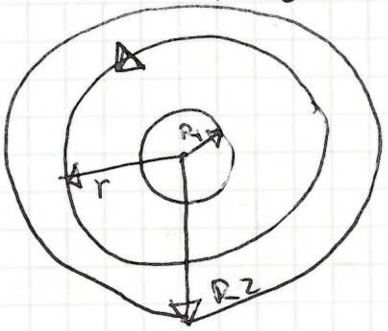
If $R_1 < R_2$, then g converges in the ring

$D_{R_1,R_2} = \{ z : R_1 < |z-z_0| < R_2 \}$

Theorem 5.2 (Laurent's Theorem)

Assume that $f \in H(D_{R_1,R_2}(z_0))$. Then for every $z \in D_{R_1,R_2}(z_0)$ we have

$$f(z) = \sum_{r=-\infty}^{\infty} a_k (z-z_0)^k, \text{ with } a_k = \frac{1}{2\pi i} \int_{S(z_0,r)} \frac{f(w)}{(w-z_0)^{k+1}} dw \quad k \in \mathbb{Z}$$



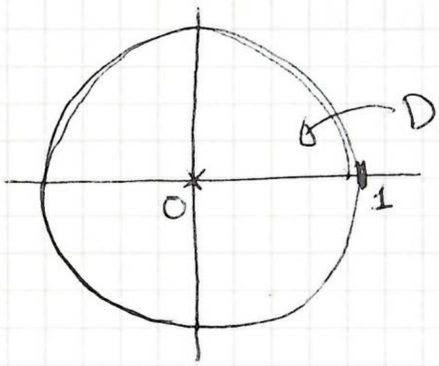
where $r \in (R_1, R_2)$. Moreover, the series converges absolutely.

Example

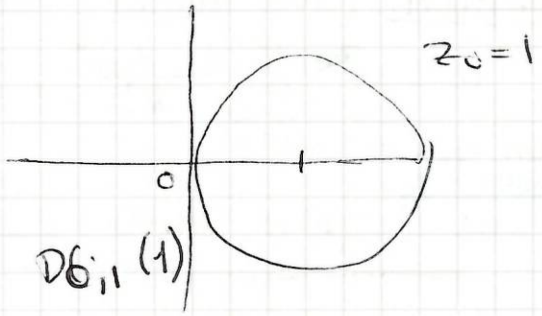
$f(z) = \frac{1}{z(z-1)}$ Find Laurent's expansion about $z_0 = 0$, $z_0 = 1$

(points where the function is not analytic)

$z_0 = 0$



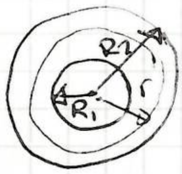
$z_0 = 1$



28|11|11

$$g(z) = \sum_{k=-\infty}^{\infty} a_k (z - z_0)^k$$ ← Laurent expansion about z_0

$$D_{R_1, R_2}(z_0) = \{ z \in \mathbb{C} : R_1 < |z - z_0| < R_2 \}$$



Theorem 5.2

Suppose $f \in H(D_{R_1, R_2}(z_0))$. Then

$$f(z) = \sum_{k=-\infty}^{\infty} a_k (z - z_0)^k,$$

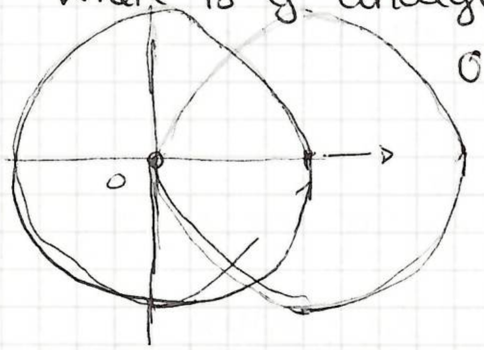
for each $z \in D_{R_1, R_2}(z_0)$, where

$$a_k = \frac{1}{2\pi i} \int_{S(z_0, r)} \frac{f(w)}{(w - z_0)^{k+1}} \, dw \qquad \text{arbitrary } r \in (R_1, R_2)$$

example

$$g(z) = \frac{1}{z(z-1)}$$

Where is g analytic?



On $D_{0,1}(0)$, $D_{0,1}(1)$

(we can center the disc at $1/2$ and make a circle with a big radius so that $z = 0$ and $z = 1$ is inside)

on $D_{1,\infty}(0)$, $D_{1,\infty}(1)$

Rewrite: $g(z) = -\frac{1}{z} + \frac{1}{z-1}$

Let $0 < |z| < 1$

$-\frac{1}{z}$ is already good

$$\frac{1}{z-1} = \frac{-1}{1-z} \underset{\text{geometric series}}{=} -\sum_{k=0}^{\infty} z^k \Rightarrow g(z) = -\frac{1}{z} - \sum_{n=0}^{\infty} z^k = -\sum_{k=-1}^{\infty} z^k$$

Let $|z| > 1$

Write:

$$\frac{1}{z-1} = \frac{1}{z(1 - \frac{1}{z})} = \frac{1}{z} \sum_{n=0}^{\infty} \frac{1}{z^k} = \sum_{k=-\infty}^{-1} z^k$$

Thus $g(z) = -\frac{1}{z} + \sum_{k=-\infty}^{-1} z^k = \sum_{k=-\infty}^{-2} z^k$

Let $0 < |z-1| < 1$

$\frac{1}{z-1}$ is good. Write $\frac{1}{z} = \frac{1}{1+(z-1)} = \sum_{k=0}^{\infty} (-1)^k (z-1)^k$

you leave the answer in two bits in the exam, but you get extra credit for just one element

Thus
$$g(z) = -\sum_{k=0}^{\infty} (-1)^k (z-1)^k + \frac{1}{z-1} = \sum_{k=-1}^{\infty} (-1)^{k+1} (z-1)^k$$

Proof of theorem 5.2

Pick a $z \in D_{R_1,R_2}(z_0)$

Looking at $\tilde{f}(z) = f(z+z_0)$, assume $z_0 = 0$

$\Rightarrow z \in D_{R_1,R_2}(0)$

Let r_1, r_2 be s.t. $0 < R_1 < r_1 < |z| < r_2 < R_2$

I will denote $\gamma_1 = S(0, r_1)$, $\gamma_2 = S(0, r_2)$

Connect γ_1 & γ_2 with a segment η

Define: $\gamma = \gamma_2 \cup (-\eta) \cup (-\gamma_1) \cup \eta$. Observe that $z \in \operatorname{Int}\gamma$, $\operatorname{Int}\gamma \subset D_{R_1,R_2}(0)$.

By Cauchy Formula:
$$f(z) = \frac{1}{2\pi i} \int_\gamma \frac{f(s)}{s-z} ds = \frac{1}{2\pi i} \int_{\gamma_2} \frac{f(s)}{s-z} ds - \frac{1}{2\pi i} \int_{\gamma_1} \frac{f(s)}{s-z} ds$$

note: if $s \in \gamma_2$, then $|s| > |z|$

if $s \in \gamma_1$, then $|s| < |z|$

Look at $s \in \gamma_2$:
$$\frac{1}{s-z} = \frac{1}{s(1-\frac{z}{s})} = \frac{1}{s} \sum_{k=0}^{\infty} \frac{z^k}{s^k}$$

Look at $s \in \gamma_1$: $\frac{1}{s-z} = -\frac{1}{z(1-\frac{s}{z})} = -\frac{1}{z} \sum_{k=0}^{\infty} \frac{s^k}{z^k}$

Thus
$$f(z) = \frac{1}{2\pi i} \int_{\gamma_2} f(s) \sum_{k=0}^{\infty} \frac{z^k}{s^{k+1}} ds + \frac{1}{2\pi i} \int_{\gamma_1} f(s) \sum_{k=0}^{\infty} \frac{s^k}{z^{k+1}} ds$$

Exchange $\int$ and Σ (not justified!)
$$f(z) = \sum_{k=0}^{\infty} z^k \frac{1}{2\pi i} \int_{\gamma_2} \frac{f(s)}{s^{k+1}} ds + \sum_{k=0}^{\infty} \frac{1}{z^{k+1}} \cdot \frac{1}{2\pi i} \int_{\gamma_1} f(s) \cdot s^k ds$$
$$= \sum_{k=-\infty}^{\infty} z^k \frac{1}{2\pi i} \int_{\gamma_2} \frac{f(s)}{s^{k+1}} ds + \sum_{k=-\infty}^{-1} z^k \frac{1}{2\pi i} \int_{\gamma_1} \frac{f(s)}{s^{k+1}} ds$$

$z^{-(k+1)}$, $m=-(k+1)$

By Cauchy Goursat for multiply connected domains,

$$\int_{\gamma_2} \frac{f(s)}{s^{k+1}} ds = \int_{\gamma_1} \frac{f(s)}{s^{k+1}} ds = \int_{S(0,r)} \frac{f(s)}{s^{k+1}} ds$$

This gives the required formula for a_k $\blacksquare$ ——— revise for test, after this point not necessary

Proof of Thm 5.1

Assume: $f \in D(z_0, R)$

Want: $f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!} (z-z_0)^n$

Use Thm 5.2. Obviously, $f \in H(D'(z_0,R))$ and $D'(z_0,R) = D_{0,R}(z_0)$. Thus $f(z)$ is represented by

$$f(z) = \sum_{k=-\infty}^{-1} a_k (z-z_0)^k + \sum_{k=0}^{\infty} a_k (z-z_0)^k$$

need to show $a_k = 0$ $\forall k \leq -1$
and $a_k = \frac{f^{(k)}(z_0)}{k!}$

write: for $m \geq 1$

$$a_{-m} = \frac{1}{2\pi i} \int_{S(z_0,R)} \frac{f(w)}{(w-z_0)^{-(m+1)}} dw = \frac{1}{2\pi i} \int_{S(z_0,R)} f(w)(w-z_0)^{m-1} dw = 0$$

by Cauchy-Goursat

closed contour, the function is a product of two analytic functions so

Write $a_k = \frac{1}{2\pi i} \int_{S(z_0,R)} \frac{f(w)}{(w-z)^{k+1}} dw \underset{\text{by Thm 4.20}}{=} \frac{f^{(k)}(z_0)}{k!}$, as required $\blacksquare$

Look at consequences

Theorem 5.3

Let f be entire, and assume $|f(z)| \leq C|z|^k$ for all $|z| \geq 1$ and some $k \in \mathbb{N}$.

example

$|z^3+1| \leq C|z|^3$, $|z| \geq 1$ with some constant $C>0$

$|e^z| \leq C|z|^k$? No, by def e^z contains all powers of z

$\sin z$ is not bounded by $|z|^k$!

cont 5.3

Then $f(z)$ is a polynomial of degree at most k.

HW 7? similar

Proof

For all z: $f(z) = \sum_{n=0}^{\infty} a_n z^n$, and the series converges absolutely, by Thm 5.1

Need to show that $a_n = 0$ for $n > k$

By Thm 5.1, 5.2

$$a_n = \frac{1}{2\pi i} \int_{S(0,r)} \frac{f(z)}{z^{n+1}} dz \quad \text{for any } r > 0$$

Let $r > 1$. Then by theorem 4.7(2)

$$|a_n| \leq \frac{C}{\cancel{2\pi}} \frac{r^k}{r^{n+1}} \cdot \cancel{2\pi r} = C r^{k-n}$$

note: $k - n < 0$ for $n > k$

as $r > 1$ is arbitrary, $a_n = 0$, $n > k$ as claimed ∎

Zeros and singularities of analytic functions

Definition 5.4

Let $f \in H(\Omega)$. Then a point $a \in \Omega$ s.t. $f(a) = 0$ is called a zero of f.

We say that a zero a is isolated if there is a number $\delta > 0$ s.t. $f(z) \neq 0$, for all $z \in D'(a, \delta)$

An isolated zero is said to have order m if in the Taylor series:

$f(z) = \sum_{k=0}^{\infty} c_k (z-a)^k$ we have

$c_0 = c_1 = \dots = c_{m-1} = 0$ and $c_m \neq 0$.

example

$(z-1)^5$ - $a = 1$ is a zero of order 5

In other words, $f(z) = \sum_{k=m}^{\infty} c_k (z-a)^k$

$= (z-a)^m \big(\underset{\neq 0}{c_m} + c_{m+1}(z-a) + c_{m+2}(z-a)^2 + \dots \big)$

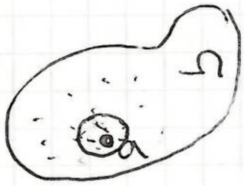
$= (z-a)^m g(z)$, where g is analytic at a and $g(a) \neq 0$

Notation: $Z(f)$ is the set of all zeros of f.

Theorem 5.5

Suppose that $f \in H(\Omega)$. Assume that $Z(f)$ has an accumulation point in Ω

Then $f(z) = 0$, $\forall z \in \Omega$



Proof

Assume that $f \not\equiv 0$ on Ω. Let $a \in \Omega$ be an accumulation point of $Z(f)$. By continuity of f, a is also a zero.

Thus for some $r > 0$, the function f can be represented in $D(a,r)$ by the series

$$f(z) = \sum_{k=0}^{\infty} c_k (z-a)^k$$

Since $f \not\equiv 0$, there is an m s.t. $c_0 = c_1 = c_2 = \dots = c_{m-1} = 0$, and $c_m \neq 0$. Hence $f(z) = (z-a)^m g(z)$, where g is analytic on $D(a,r)$ and $g(a) \neq 0$.

By continuity of g, $g(z) \neq 0$ for all $z \in D(a,\delta)$ with some $\delta > 0$.

This means that a is an isolated zero of f in $D(0,\delta)$. Thus it can not be an accumulation point of $Z(f)$. We have a contradiction, which shows that $f(z) = 0$ on $D(a,\delta)$

~~From last lecture:~~

05/12/2011

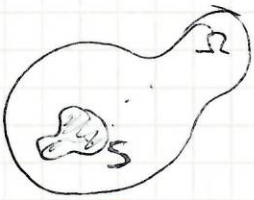
Theorem 5.5

Let $f \in H(\Omega)$. Assume that $Z(f)$ has an accumulation point in Ω. Then $f(z) = 0$ for all $z \in \Omega$

Proof of second part in lecture notes!!!

Corollary 5.6 (The unique continuation theorem)

Assume that $f, g \in H(\Omega)$. Assume also that $f(z) = g(z)$ on a set $S \subset \Omega$ which has an accumulation point in Ω. Then $f(z) = g(z)$ for all $z \in \Omega$.

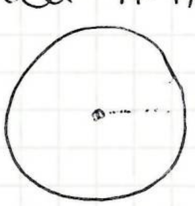


Proof

Let $h = f - g$. Use Thm 5.5 ☒

example

$\Omega = D(0,1)$. Is there a function $f \in H(\Omega)$ s.t. $f\left(\frac{1}{n}\right) = \frac{n}{n+1}$ for all $n = 1, 2, \dots$?



Let $g(z) = \frac{1}{1+z}$, so $g\left(\frac{1}{n}\right) = \frac{1}{1+\frac{1}{n}} = \frac{n}{n+1}$, for all $n \in \mathbb{N}$

Note: $g \in H(\Omega)$. Clearly, $f\left(\frac{1}{n}\right) = g\left(\frac{1}{n}\right)$ since $\frac{1}{n} \xrightarrow[n\to\infty]{} 0 \in D(0,$

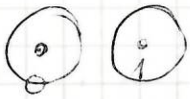
by corollary 5.6 $f(z) = g(z) = \frac{1}{1+z}$

Singularities

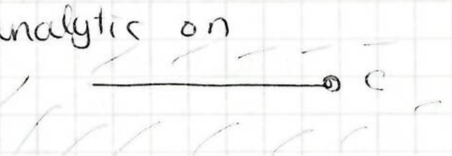
Def 5.7 We say that f has an isolated singularity at $a \in \mathbb{C}$ if f is holomorphic on $D'(a,r)$ with some $r > 0$.

examples ① $\frac{1}{z}$ has an isolated singularity at $a = 0$

② $\frac{1}{z^2(z-1)}$ isolated singularities at $a_1 = 0$ and $a_2 = 1$.



③ $\operatorname{Log} z$ $\operatorname{Log} z$ is analytic on



$\Rightarrow$ it does not have isolated singularities

Classification of singularities

as $f \in H(D'(a,r))$, by Laurent's Thm $f(z) = \underbrace{\sum_{k=-\infty}^{-1} a_k (z-a)^k}_{\text{principal part of Laurent Expansion (PP)}} + \underbrace{\sum_{k=0}^{\infty} a_k (z-a)^k}_{\text{Regular part}}$

Three types of singularities

Type ① A pole at a

If the PP of f contains finitely many terms, then we say that f has a pole at the point a. More precisely, if there is a number $M \in \mathbb{N}$ s.t $a_k = 0$ for all $k < -M$, and $a_{-M} \neq 0$, then f is said to have a pole of order M at a.

This means:

$$\text{PP of } f = \sum_{k=-M}^{-1} a_k (z-a)^k = \frac{a_{-M}}{(z-a)^M} + \frac{a_{-M+1}}{(z-a)^{M-1}} + \dots$$

$a_{-M} \neq 0$

examples

① $\frac{1}{z}$ - pole of order 1 at $a=0$, or a simple pole

② $\frac{1}{z^2(1-z)}$

$= \frac{g(z)}{z^2}$ - only contains positive powers of z $= \frac{1}{z^2}(1 + z + z^2 + z^3 + \dots) = \frac{1}{z^2} + \frac{1}{z} + 1 + z + \dots$

so Pole of order 2, we could see that from the start

Pole of order 2 at $a=0$

a simple pole at $a=1$

③ $\frac{\sin z}{z^2}$ need to write the Laurent exp since the roots at 0 cancel out

$$\frac{\sin z}{z^2} = \frac{z - \frac{z^3}{6} + \dots}{z^2} = \underbrace{\frac{1}{z}}_{PP} - \frac{1}{6} z + \dots$$

a simple pole at $a=0$

④ Type ② Essential singularity at a

If there is no number $M \in \mathbb{N}$ s.t. $a_k = 0$ for $k < -M$, then f is said to have an essential singularity at a.

example

Have to write out the series

$$g(z) = \sin\frac{1}{z} = \sum_{k=0}^{\infty} (-1)^k \frac{1}{z^{2k+1}} \frac{1}{(2k+1)!}$$

$\Rightarrow a = 0$ is an essential singularity

Type ③

$f(z) = z$, $f \in H(D'(0,1))$

a removable singularity at a

if PP of $f = 0$, then f is said to have a removable singularity at a. (In other words, if $a_k = 0$, $k \le -1$). Then f becomes analytic at a if one defines $f(a) = a_0$

example

$\frac{\sin z}{z}$ = an isolated singularity at $a=0$

$$\frac{\sin z}{z} = \frac{z - \frac{z^3}{6} + \dots}{z} = \underset{a_0}{1} - \frac{z^2}{6} + \dots$$

$PP = 0 \Rightarrow$ a removable singularity

To make it analytic at $a=0$ define $g(0) = 1$

Now, $\tilde{g}(z) = \begin{cases} \frac{\sin z}{z}, & z \ne 0 \\ 1, & z = 0 \end{cases}$ is entire

---

Mixed examples

① $h(z) = \frac{1-\cos z}{z^2} = \frac{1 - \left(1 - \frac{z^2}{2} + \frac{z^4}{24} - \dots\right)}{z^2}$

$$= \frac{1 - 1 + \frac{z^2}{2} - \frac{z^4}{24} + \dots}{z^2} = \frac{1}{2} - \frac{z^2}{24} + \dots \quad \text{removable!}$$

$\frac{1-\cos z}{z^3}$ - a simple pole at $a=0$

② $\frac{z-3}{(z-4)^3}$ - pole of order 3 at $a=4$ (no cancelation with the rooth of the numerator and denominator)

$$= \frac{z-4+1}{(z-4)^3} = \frac{1}{(z-4)^3} + \frac{1}{(z-4)^2}$$

Theorem 5.8

① The function $f \in H(D(a,R))$ has a zero of order m at a iff $\lim_{z \to a} (z-a)^{-m} f(z) = B$ with some $B \ne 0$

② The function $g \in H(D'(a,R))$ has a pole of order m at a iff $\lim_{z \to a} (z-a)^m f(z) = A$ with some $A \neq 0$

Proof of ①

Suppose f has a zero of order m. Then by defn

$$f(z) = \sum_{k=m}^{\infty} c_k (z-a)^k, \quad c_m \neq 0, \text{ i.e.}$$

$$= (z-a)^m g(z), \text{ where } g \in H(D(a,R)) \text{ and } g(a) = c_m \neq 0$$

Therefore,

$$(z-a)^{-m} f(z) = g(z) \xrightarrow[z \to a]{} g(a) = c_m \neq 0$$

Suppose the limit exists and $B \neq 0$. This means that

$g(z) = (z-a)^{-m} f(z)$ is bounded on $D'(a,r)$ with some $r > 0$

compute c_k, $k = 0, 1, 2, \ldots$

want: $c_0 = c_1 = \ldots = c_{m-1} = 0$ and $c_m \neq 0$

write:

$$c_k = \frac{1}{2\pi i} \int_{S(a,\rho),\ 0<\rho<r} \frac{f(z)}{(z-a)^{k+1}}\, dz$$ want a contour inside

$$= \frac{1}{2\pi i} \int_{S(a,\rho)} \frac{g(z)}{(z-a)^{k-m+1}}\, dz$$

By theorem 4.7(2)

$$|c_k| \leq \frac{1}{2\pi} \cdot \frac{C}{\rho^{k-m+1}} 2\pi\rho = C\rho^{m-k} \xrightarrow[\rho \to 0]{} 0, \ m > k$$ ρ is arbitrary

As $\rho > 0$ is arbitrary, $c_k = 0$ for all $k = 0, 1, \ldots, m-1$

This means that

$$f(z) = c_m (z-a)^m + c_{m+1}(z-a)^{m+1} + \ldots$$

and hence $\frac{f(z)}{(z-a)^m} \xrightarrow[z \to a]{} c_m = B$, as limit exists and not equal to zero

as required $\blacksquare$ ①

example

$$f(z) = \frac{1}{z^7(z-1)}$$

Note:

$$z^2 f(z) = \frac{1}{z-1} \xrightarrow[z \to 0]{} -1$$

By Thm 5.8 (2), $a=0$ is a pole of order 2
On the other hand,

$$(z-1)\,f(z) = \frac{1}{z^2} \xrightarrow[z \to 1]{} 1 \Rightarrow$$ by Thm 5.8 (2), $a=1$ is a simple pole

Corollary 5.9
The function $f \in H(D(a,R))$ has a zero of order m at a, iff $\frac{1}{f}$ has a pole of order m at a

$(z-a)^m$ – root (zero)

$(z-a)^{-m}$ –

07/12/2011

(Test: 4 questions
Test + HW = 10%) can use Cauchy residue formula and theory after Laurent expansion

① poles
② essential singularities
③ removable singularities

Residues

Def 5.10 Assume that f has an isolated singularity at a, and let

$$f(z) = \sum_{k=-\infty}^{\infty} a_k (z-a)^k$$

be its Laurent's Expansion. Then the coefficient a_{-1} is called the residue of f at the point a.
Notation: $a_{-1} = \operatorname{Res}(f,a)$

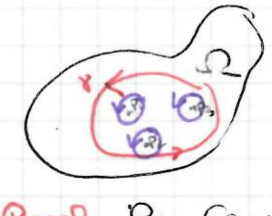
By Laurent's Theorem

$$a_{-1} = \frac{1}{2\pi i} \int_{S(a,r)} f(z)\,dz$$

Theorem 5.11 (The Cauchy's residue theorem)

Assume that f is holomorphic on Ω except for finitely many isolated singularities. Let $\gamma \subset \Omega$ be a contour st Int γ contains singularities $p_1, p_2, \ldots, p_N$.
Then

$$\int_\gamma f(z)\,dz = 2\pi i \sum_{k=1}^{N} \operatorname{Res}(f, p_k)$$



Proof. By Cauchy-Goursat for multiply connected domain (Thm 4.16)

$$\int_\gamma f(z)\,dz = \sum_{k=1}^{N} \int_{S(p_k,r)} f(z)\,dz = \sum_{k=1}^{N} 2\pi i \operatorname{Res}(f, p_k)$$

– need only to look at one by one singularity ▨

Rules for finding residues

Rule I Suppose that a is a simple pole, ie

$$f(z) = \frac{a_{-1}}{z-a} + a_0 + a_1(z-a) + \dots \qquad \text{(1 principal part)}$$

multiply:

$$(z-a)f(z) = a_{-1} + a_0(z-a) + a_1(z-a)^2 + \dots$$

Thus

$$a_{-1} = \mathrm{Res}(f,a) = \lim_{z\to a}(z-a)f(z)$$

Rule II Suppose f has a pole of order m at a:

$$f(z) = \frac{a_{-m}}{(z-a)^m} + \frac{a_{-m+1}}{(z-a)^{m-1}} + \dots + \underset{\frac{a_{-1}}{z-a}}{\hat{+}} + a_0 + a_1(z-a) + \dots$$

multiply:

$$g(z) = (z-a)^m f(z) = a_{-m} + a_{-m+1}(z-a) + \dots + a_{-1}(z-a)^{m-1} + \dots$$

Thus

$$a_{-1} = \frac{1}{(m-1)!}\frac{d^{m-1}}{dz^{m-1}} g(z)\Big|_{z=a}$$

Rule III Expand f in its Laurent series and take a_{-1}

examples

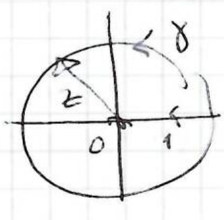
① $f(z) = \frac{\sin z}{z^4} = \frac{z - \frac{z^3}{6} + \frac{z^5}{120} - \dots}{z^4} = \frac{1}{z^3} - \frac{1}{6z} + \frac{z}{120} - $

$\Rightarrow \mathrm{Res}(f,0) = -\frac{1}{6}$

② $f(z) = \frac{1}{z^2(z-1)}$

by rule I $\mathrm{Res}(f,1) = \lim_{z\to 1}(z-1)f(z) = \lim_{z\to 1}\frac{1}{z^2} = 1$

$$\mathrm{Res}(f,0) \underset{\text{Rule II}}{=} \frac{d}{dz}\left(z^2 f(z)\right)\Big|_{z=0} = \frac{d}{dz}\frac{1}{z-1}\Big|_{z=0} = -\frac{1}{(z-1)^2}\Big|_{z=0} = -1$$



$$\int_\gamma f(z)\,dz = 2\pi i\left(\mathrm{Res}(f,1) + \mathrm{Res}(f,0)\right) = 0$$

③ $f(z) = z^2\sin\left(\frac{1}{z}\right) = z^2\left(\frac{1}{z} - \frac{1}{6z^3} + \frac{1}{120z^5} - \dots\right)$

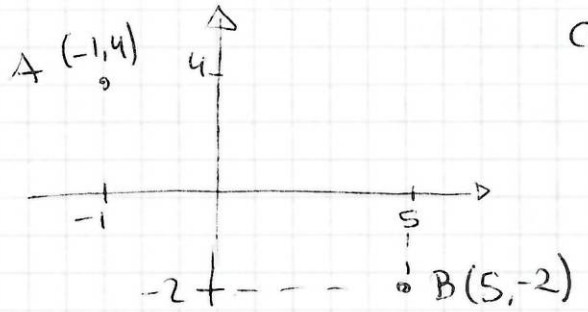
$= z - \frac{1}{6}z + \frac{1}{120z^3} - \dots$

$Res(f, 0) = -\frac{1}{6}$

Last year exam
Question: what is $Res(f, a)$ if f has a removable singularity at a?

Removable singularity $\Leftrightarrow$ PP of $f = 0$. In particular $a_{-1} = 0 \Rightarrow Res(f, a) = 0$

Problem class $\boxed{PS\ 7}$



Cannot compare A & B

Qu 6

$$\frac{1}{z-b} \qquad |z| < |b| \qquad \left|\frac{z}{b}\right| < 1$$

$$= -\frac{1}{b}\frac{1}{1-\frac{z}{b}} = \left(-\sum_{n=0}^{\infty}\left(\frac{z}{b}\right)^n\right)\frac{1}{b} = -\sum_{n=0}^{\infty}\frac{z^n}{b^{n+1}}$$

(ii) $$\frac{1}{z-b} = \sum_{n=-\infty}^{-1} b^{-k-1} z^k \qquad |z| > |b|$$

(iii) $$\frac{1}{z-b} = \sum_{n=-\infty}^{-1} \underbrace{(b-a)}_{c}{}^{-n-1} \underbrace{(z-a)}_{w}{}^{n}$$ then (iii) is the same as part (ii)

$|z-a| > |b-a|$
$|w| > |c|$

Qu 4

$$\sin z = \sin(z_0 + (z - z_0)) \qquad \text{at } z_0 \ \sum_{n=0}^{\infty} a_n (z-z_0)^n$$

$$= \sin z_0 \cdot \underbrace{\cos(z-z_0)} + \underbrace{\sin(z-z_0)} \cdot \cos z_0$$

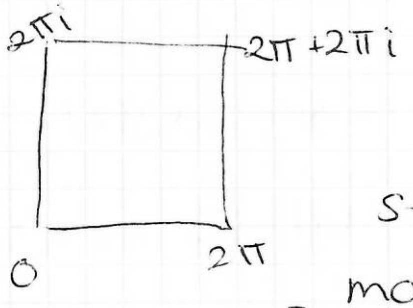
$$= \sin z_0 \sum_{n=0}^{\infty}\frac{(-1)^n (z-z_0)^{2n}}{(2n)!} + \left(\sum_{n=0}^{\infty}\frac{(-1)^n(z-z_0)^{2n+1}}{(2n+1)!}\right)\cdot \cos z_0$$

Qu 2

$f(z) = f(z+2\pi) = f(z+2\pi i)$ $\quad$ f-entire = f-holomorphic on $\mathbb{C}$

Need to show $f = \text{const}$

Notice that $f(z)$ is 2π-periodical so, it's enough to consider it on S.



f - entire $\Rightarrow f$ - holomorphic on S

f - diff $\forall z \in S \Rightarrow f$ - cont on S

S - compact set $\Rightarrow f$ - bounded on S $\quad |f(z)| \le M$

remember modulus or you can't compare it $\rightarrow \max_{z \in S} |f(z)| \le M \quad , M > 0$

periodicity implies that $f(z)$ is bounded $\forall z \in \mathbb{C}$

f - entire & bounded $\Rightarrow f = \text{const}$ (by Liouville's Thm)

Qu 8

$f \in H(D'(a,R))$, $R>0$ $\qquad |f(z)| \le \frac{M}{|z-a|^\alpha} = \frac{M}{r^\alpha}$ since boundary ; $0 \le \alpha < 1$, $M \ge 0$

need to prove $a_{-n} = 0 \quad \forall n \in \mathbb{N} = \{1, 2, \dots, n, \dots\}$

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z-a)^n$$

$$a_n = \frac{1}{2\pi i} \int_\gamma \frac{f(z)}{(z-a)^{n+1}} dz \qquad \text{consider } S(a,r)$$

$$= \frac{1}{2\pi i} \int_{\substack{S(a,r) \\ r<R}} \frac{f(z)}{(z-a)^{n+1}} dz$$

$$\Rightarrow |a_n| = \left| \frac{1}{2\pi i} \int_{S(a,r)} \frac{f(z)}{(z-a)^{n+1}} dz \right| \underset{\text{M-L result}}{\le}$$

$$\frac{1}{2\pi} \int_{|z-a|=r} \left| \frac{f(z)}{(z-a)^{n+1}} \right| |dz| \qquad \text{on the } S(a,r) \quad r = |z-a|$$

$$= \frac{1}{2\pi} \max_{|z-a|=r} |f(z)| \cdot r^{+n+1} \cdot 2\pi r$$

$$\int_{S(a,r)} |dz| = 2\pi r \qquad \le r^{n-\alpha} \xrightarrow{r \to 0} 0$$

we can make r sufficiently small

~~PS 8~~

Qu 2

$\{a_n\} \subset \mathbb{C}$ (1) $\sum |a_n|$ converges

(2) $\sum \frac{a_n}{k^n} = 0 \quad \forall k \in \mathbb{N}$

show $a_n = 0 \quad \forall n$

consider $f(z) = \sum a_n z^n$ - analytic

Note that $f(\frac{1}{k}) \stackrel{(?)}{=} 0$ $\quad z_k = \frac{1}{k}$ - set of zeros of $f(z)$, $k \in \mathbb{N}$

$\{\frac{1}{k}\}$ - zeros of $f(z)$ $\quad Z(f) = \{\frac{1}{k}, k \in \mathbb{N}\}$

0 - accumulation point for $Z(f)$

$f(z) = 0$ because of unique cont. thm
$\Leftrightarrow$
$\forall n \; a_n = 0$

$f(\frac{1}{n}) = \begin{cases} 0, & n\text{-odd} \\ \frac{1}{n}, & n\text{-even} \end{cases}$

assume $f(z)$ exists

$z_0 = 0 \quad Z(f) = \{\frac{1}{2}, \frac{1}{4}, \quad \frac{1}{2k}, \dots\}$

$\llcorner$ accumulation point for $Z(f)$

$\Rightarrow f(z) = 0$ by UCTh
contradiction

b) only positive answer is b TRUE

a) apply def of derivative $\quad$ lim doesn't exist, not holomorphic

Q4) f-entire $\quad f(z) = f(\frac{1}{z}) \;\; \forall z \in \mathbb{C} \setminus \{0\}$
need $f(z) = $ const

f - entire $\rightarrow f \in H(D(0,1))$

$\rightarrow f$ - continuous on $\bar{D}(0,1)$

$\rightarrow f$ - bounded on $\bar{D}(0,1)$

(*) $\exists M > 0$ st $\forall z \in D(0,1) \quad |f(z)| < M$

$\forall w \notin \bar{D}(0,1) \quad \exists z \in D(0,1)$ st $f(w) = f(z)$

(*) $|f(w)| \le M \quad \forall w \notin \bar{D}(0,1) \quad f$-entire and bounded

$\Rightarrow$ by L Thm $\quad f = $ const

12/12/2011

Trigonometric integrals

looking at:

$$\int_0^{2\pi} f(\cos\theta, \sin\theta)\, d\theta$$

example:

$$I = \int_0^{2\pi} \frac{1}{5 - 4\cos\theta}\, d\theta$$

Define: $z = e^{i\theta}$

$dz = ie^{i\theta}d\theta \qquad d\theta = iz\,d\theta \qquad \Rightarrow d\theta = \frac{dz}{iz}$

Thus

$$\int_0^{2\pi} f(\cos\theta, \sin\theta)\, d\theta = \int_{S(0,1)} f\left(\tfrac{1}{2}\left(z+\tfrac{1}{z}\right), \tfrac{1}{2i}\left(z-\tfrac{1}{z}\right)\right) \frac{dz}{iz}$$

$$\cos\theta = \frac{1}{2}\left(z + \frac{1}{z}\right)$$

$$\sin\theta = \frac{1}{2i}\left(z - \frac{1}{z}\right)$$

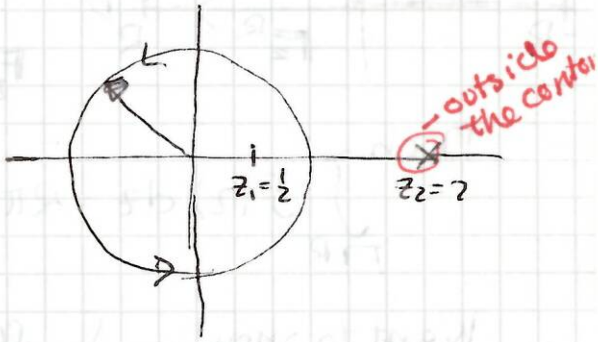
---

example (cont.)

$$I = \int_{S(0,1)} \frac{1}{5 - 4\frac{1}{2}\left(z+\frac{1}{z}\right)} \frac{dz}{iz}$$

$$= -i\int_{S(0,1)} \frac{1}{5z - 2z^2 - 2}\, dz = \int_{S(0,1)} \frac{i}{2z^2 - 5z + 2}$$

$$= \int_{S(0,1)} \frac{i}{(2z-1)(z-2)}\, dz$$



Two singularities: $z_1 = \frac{1}{2}$, $z_2 = 2$

Therefore:

$$I = 2\pi i \left(\mathrm{Res}\left(g, \tfrac{1}{2}\right) + \cancel{\mathrm{Res}(g, 2)}\right)$$

$$g(z) = \frac{i}{(2z-1)(z-2)}$$ $\{\frac{1}{2}, 2\}$ are simple poles

simple pole — $$\mathrm{Res}\left(g, \tfrac{1}{2}\right) \underset{\text{rule 1}}{=} \lim_{z \to \frac{1}{2}} \left(z - \tfrac{1}{2}\right) \frac{i}{(2z-1)(z-2)} =$$

$$= \frac{i}{2} \lim_{z \to \frac{1}{2}} \frac{1}{z-2} = -\frac{i}{3}$$

$$\mathrm{Res}(g, 2) = \lim_{z \to 2} (z-2) \frac{i}{(2z-1)(z-2)} = \frac{i}{3}$$

, is outside the contour, so we shouldn't consider it !!

$$\mathrm{Res}\left(g, \tfrac{1}{2}\right) + \mathrm{Res}(g, 2) = 0$$

$$\Rightarrow \cancel{I = 0}$$ something went wrong!

$$I = 2\pi i \left(-\frac{i}{3}\right) = \frac{2\pi}{3}$$

$$I = \int_0^{2\pi} \frac{1}{(5-4\cos\theta)^2}\, d\theta$$

$$I = \int_{S(0,1)} \frac{i}{(2z-1)^2(z-2)^2}\, dz$$ poles of order 2

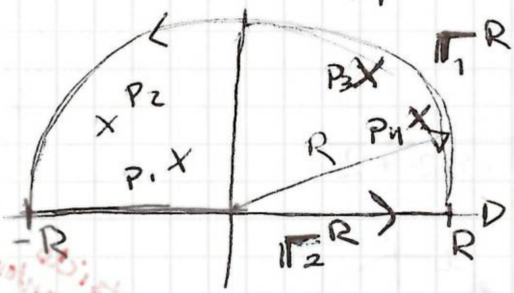
Do it!

Improper real integrals

Integrals of the type

$$\int_{-\infty}^{\infty} f(t)\, dt = I$$

Assume: f has finitely many isolated singularities in the upper half-plane $P_1, P_2, \dots, P_N$



Represent: $I = \lim_{R\to\infty} \int_{-R}^{R} f(t)\, dt$

$$\Gamma_1^R = \{z \in \mathbb{C} : \operatorname{Im} z \geq 0,\ |z| = R\}$$

$$\Gamma_2^R = \{z \in \mathbb{C} : \operatorname{Im} z = 0 \quad |z| \leq R\}$$

$$\Gamma^R = \Gamma_1^R \cup \Gamma_2^R$$

Then

$$\int_{\Gamma R} f(z)\, dz = 2\pi i \sum_{k\geq 1}^{N} \operatorname{Res}(f, P_k)$$

Want to show: $\int_{\Gamma_1^R} f(z)\, dz \to 0$ as $R \to \infty$, so

$$I = \lim_{R\to\infty} \int_{\Gamma R} f(z)\, dz = 2\pi i \sum_{k=1}^{N} \operatorname{Res}(f, P_k)$$

Lemma 5.12 Suppose that f is continuous in the upper half-plane, and that ~~max~~ $\leq |f(z)|$

$$\max_{z\in\Gamma_1^R} |f(z)| \leq \frac{C}{R^{\alpha}}, \quad \alpha > 1$$

Then $\int_{\Gamma_1 R} f(z)\, dz \to 0,\ R \to \infty$

important that $\alpha > 1$!

Proof By theorem 4.7 (2)

$$\left| \int_{\Gamma_1 R} f(z)\, dz \right| \leq \frac{C}{R^{\alpha}} \pi R = C\pi R^{1-\alpha} \to 0,\ R \to \infty$$

example

$$I = \int_{-\infty}^{\infty} \frac{x^2}{x^4+5x^2+4}\, dx$$ define, $f(z) = \frac{z^2}{z^4+5z^2+4}$

$$= \frac{z^2}{(z^2+1)(z^2+4)} = \frac{z^2}{(z-i)(z+i)(z+2i)(z-2i)}$$

Two simple poles: $z_1 = i$, $z_2 = 2i$, $\operatorname{Im} z_1 > 0$, $\operatorname{Im} z_2 > 0$ (we only look at upper half-plane)

will appear in the exam

Suppose that f is continuous in $\Pi_+ \setminus D(0, R_0)$ with some $R_0 > 0$, and that

$$\max_{z \in \Gamma_1^R} |f(z)| \leq \frac{C}{R^{\alpha}}; \quad R \geq 1, \ \alpha > 1$$

Therefore, by Cauchy Residue Theorem

Need to write: "by Cauchy Residue Theorem"

$$\boxed{\int_{\Gamma R} f(z)\, dz = 2\pi i \left(\operatorname{Res}(f, i) + \operatorname{Res}(f, 2i)\right)}$$

By Rule 1, $\operatorname{Res}(f, i) = \lim_{z \to i} (z - i) f(z) =$

Need to state lemma 5.12 and use it

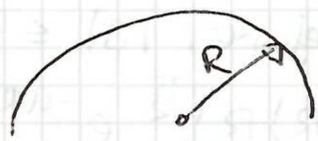
$$= \lim_{z \to i} \frac{z^2}{(z+i)(z^2+4)} = \frac{-1}{2i \cdot 3} = \frac{i}{6}$$

$$\operatorname{Res}(f, 2i) = \lim_{z \to 2i} (z - 2i) f(z) = \lim_{z \to 2i} \frac{z^2}{(z^2+1)(z+2i)}$$

$$= \frac{-4}{(-3)(4i)} = -\frac{i}{3}$$

Thus $\displaystyle\int_{\Gamma R} f(z)\, dz = 2\pi i \left(-\frac{i}{3} + \frac{i}{6}\right) = \frac{\pi}{3}$

Estimate f on Γ_1^R:



$$\left| \frac{z^2}{(z^2+1)(z^2+4)} \right| = \frac{|z|^2}{|z^2+1|\,|z^2+4|}$$

want $R \to \infty$ so can ignore 1&4

$|z| = R$

$$\leq \frac{R^2}{(|z|^2-1)(|z|^2-4)} = \frac{R^2}{(R^2-1)(R^2-4)} \leq \frac{CR^2}{R^2 R^2}$$

$$= \frac{C}{R^2}, \text{ with some constant } C > 0$$

Now use lemma 5.12 $\to$ $\boxed{\int_{\Gamma_1^R} f(z)\, dz \to 0, \ R \to \infty}$

Put everything together:

$$I = \lim_{R \to \infty} \int_{\Gamma_1^R} f(z)\, dz = 2\pi i \left(\operatorname{Res}(f, i) + \operatorname{Res}(f, 2i)\right) = \frac{\pi}{3}$$

$$I = \int_{-\infty}^{\infty} \frac{x^2}{(x^4+5x^2+4)^2}\,dx$$

same, but pole of order two, so need to find $\mathrm{Res}(f, \ )$ in an other way) (Rule 2)

$$\max_{z\in\Gamma_1^R} |f(z)| \le \frac{C}{R^{\varkappa}}, \quad \varkappa > 1$$

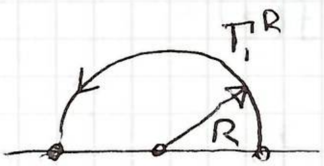
· Integrals containing exponentials

$$J = \int_{-\infty}^{\infty} e^{iaz} f(z)\,dz$$

Lemma 5.13 (Jordan's lemma)

Suppose that f is continuous in $\Pi_+ \setminus D(0,R_0)$ with some $R_0>0$, and that :

$$M(R) = \max_{z\in\Gamma_1^R} |f(z)| \to 0, \ R\to\infty$$

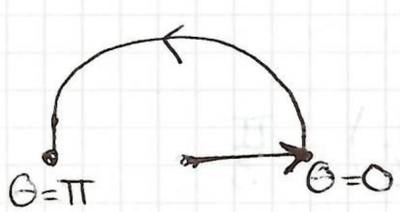


If $a>0$, then

$$\int_{\Gamma_1^R} e^{iaz} f(z)\,dz \to 0, \ R\to\infty$$

Proof

Let $z = Re^{i\theta}$, $\theta\in[0,\pi]$. Then

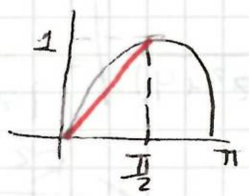


$$J = \int_0^{\pi} e^{iaRe^{i\theta}} f(Re^{i\theta})\, iRe^{i\theta}\,d\theta$$

$$= iR\int_0^{\pi} e^{-aR\sin\theta} e^{iaR\cos\theta} f(Re^{i\theta}) \cdot e^{i\theta}$$

Therefore, $|J| \le R\int_0^{\pi} e^{-aR\sin\theta}|f(Re^{i\theta})|\,d\theta \le M(R) \cdot R\int_0^{\pi} e^{-aR\sin\theta}$

$= 2M(R)R\int_0^{\frac{\pi}{2}} e^{-aR\sin\theta}\,d\theta$



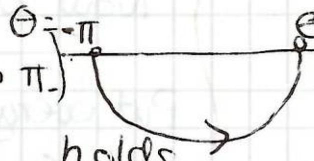
Observe: $\sin\theta \ge \frac{2}{\pi}\theta$, so

$$|J| \le 2RM(R)\int_0^{\pi/2} e^{-a\frac{2}{\pi}R\theta}\,d\theta \qquad \frac{2}{\pi}\theta$$

$$\le 2RM(R)\int_0^{\infty} e^{-\frac{2a}{\pi}R\theta}\,d\theta = \frac{2RM(R)\pi}{2aR} = \frac{\pi}{a}M(R) \xrightarrow[R\to\infty]{} 0$$

as required ∎

(if $a<0$ we should change the path to Π_-)



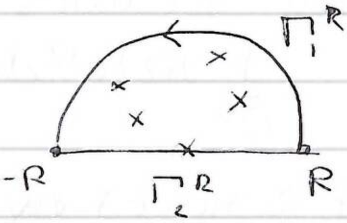
Remark If $a<0$, then lemma 5.13 still holds if one replaces Γ_1^R by the pat $\Gamma_1^R = \{z = Re^{i\theta}, \theta\in[-\pi,0]\}$

14/12/2011

$\int_{-\infty}^{\infty} f(x)\,dx \qquad \int_{-\infty}^{\infty} e^{iax} f(x)\,dx$

$\Gamma_1^R = \{ z \in \mathbb{C} : z = Re^{i\theta}, \theta \in [0, \pi] \}$

Important:



$$\int_{\Gamma_1^R} f(z)\,dz \to 0, \ R \to \infty$$

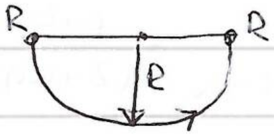
Lemma 5.13 (Jordan's lemma)

$$M(R) = \max_{z \in \Gamma_1^R} |f(z)| \to 0, \ R \to \infty$$

If $a > 0$, then

$$\int_{\Gamma_1^R} e^{iaz} f(z)\,dz \to 0 \text{ as } R \to \infty$$

if $a < 0$, look at



the lower half-plane

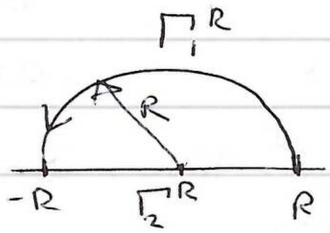
example

$$I_1 = \int_{-\infty}^{\infty} \frac{x \sin x}{(x^2+a^2)^2}\,dx, \qquad I_2 = \int_{-\infty}^{\infty} \frac{x e^{ix}}{(x^2+a^2)^2}\,dx, \quad a > 0$$

$f(z) = \dfrac{z}{(z^2+a^2)^2}$. Then $I_2 = \int_{-\infty}^{\infty} f(x)\, e^{ix}\,dx$

and $I_1 = \operatorname{Im} I_2$

Rewrite:



$$I_2 = \lim_{R \to \infty} \int_{-R}^{R} f(x) e^{ix}\,dx$$

Need to do: ① find singularities of f in the upper-half plane, and evaluate the residues.

② Show that $\int_{\Gamma_1^R} f(z) e^{iz}\,dz \to 0, \ R \to \infty$

Remember to check conditions

② $a=1>0$, and $\max_{|z|=R}|f(z)| = \max_{|z|=R} \dfrac{|z|}{|z^2+a^2|^2} \le \dfrac{R}{(R^2-a^2)^2}$

$\le CR^{-3} \to 0,\ R\to\infty$

By Jordan's lemma

$$\int_{\Gamma_1^R} f(z)e^{iz}\,dz \to 0,\ R\to\infty$$

① Residues of f: two singular points: $+ia, -ia$

only $p=ia$ is in the upper half-plane. Write:

$f(z) = \dfrac{z}{(z+ia)^2(z-ia)^2}$ $\qquad p=ia$ is a pole of order 2

Thus by Rule II:

$$\mathrm{Res}(fe^{iz}, p) = \frac{d}{dz}\Big[(z-ia)^2 f(z)e^{iz}\Big]\Big|_{z=ia}$$

$$= \frac{d}{dz}\left(\frac{ze^{iz}}{(z+ia)^2}\right)\Big|_{z=ia} = \left[\frac{e^{iz}}{(z+ia)^2} - 2\frac{ze^{iz}}{(z+ia)^3} + \frac{ize^{iz}}{(z+ia)^2}\right]_{z=ia}$$

$$= -\frac{e^{-a}}{4a^2} + \frac{2\cancel{i}\cancel{a}e^{-a}}{8\cancel{i}a^{\cancel{3}2}} + \frac{ae^{-a}}{4a^2}$$

$$= \frac{e^{-a}}{4a^2}\left[\cancel{-1}+\cancel{1}+a\right] = \frac{e^{-a}}{4a}$$

Therefore

$$I_2 = \lim_{R\to\infty}\int_{-R}^{R} f(x)e^{ix}\,dx = 2\pi i\,\mathrm{Res}(fe^{iz}, ia)$$

$$= 2\pi i\frac{e^{-a}}{4a} = \underline{\frac{\pi i}{2a}e^{-a}}$$

Also, $I_1 = \dfrac{\pi}{2a}e^{-a}$

$\int_{S(0,1)} \frac{e^{iz}}{z^2}dz$ Need to use Cauchy integral formula for higher derivatives

The indentation trick

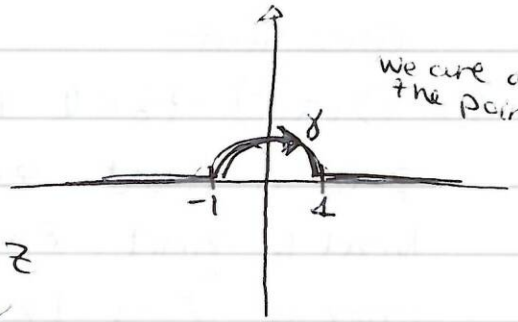
$I = \int_{-\infty}^{\infty} \dfrac{\sin x}{x}\,dx$ $\qquad \int_0^1 \frac{1}{x}dx$ - doesn't exist

Try: $I = \mathrm{Im}\int_{-\infty}^{\infty}\dfrac{e^{ix}}{x}\,dx$? $\qquad$ (won't work as $e^{ix}\to 1,\ x\to 0$)

Let $f(z) = \frac{\sin z}{z}$. Know: f is entire.

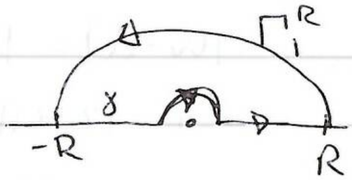
Therefore

$$I = \int_\gamma f(z)\,dz$$



We are avoiding the point 0

$$= \underbrace{\frac{1}{2i}\int_\gamma \frac{e^{iz}}{z}\,dz}_{I_1} - \underbrace{\frac{1}{2i}\int_\gamma \frac{e^{-iz}}{z}\,dz}_{I_2}$$



$I_1 = ?$

Need to show:

$$\int_{\Gamma_1^R} \frac{e^{iz}}{z}\,dz \to 0,\ R \to \infty$$

$a=1$ $f(z)$ decays

By Jordan's lemma, it is the case

$\frac{e^{iz}}{z}$ has no singularities inside the contour,

and hence $\int_{-\infty}^{\infty} \frac{e^{iz}}{z}\,dz = \lim_{R\to\infty} \int \frac{e^{iz}}{z}\,dz = 0$

$$I_2 = \frac{1}{2i}\int_\gamma \frac{e^{-iz}}{z}\,dz = ?$$

$a=-1$, so use lower half-plane

By Jordan's lemma:

$$\int_{\tilde{\Gamma}_1^R} \frac{e^{-iz}}{z}\,dz \to 0,\ R\to\infty$$

one singularity

$$\mathrm{Res}\left(\frac{e^{-iz}}{z}, 0\right) = \lim_{z\to 0} z\,\frac{e^{-iz}}{z} = 1$$

Thus $\int_{-\gamma} \frac{e^{-iz}}{z}\,dz = \lim_{R\to\infty} \int \frac{e^{-iz}}{z}\,dz = 2\pi i$

Therefore $I_2 = \frac{1}{2i}\int_\gamma \frac{e^{-iz}}{z}\,dz = -\frac{1}{2i}\int_{-\gamma} \frac{e^{-iz}}{z}\,dz = -\frac{2\pi i}{2i} = -\pi$

$I_1 = 0$

Thus $I = I_1 - I_2 = \pi$

<u>Problem class</u>

Class test?

(1) $D(a,r) = \{z \in \mathbb{C} : |z-a| < r\}$ $r>0$ - open set

pick a point $z \in D(a,r)$

Need to find $\varepsilon > 0$ st $\forall w \in D(z,\varepsilon) \to w \in D(a,r)$

Choose $\varepsilon = r - |z-a| > 0$

fix $w \in D(z,\varepsilon)$ and consider

2 points $|w-a| = |w-z+z-a| \le |w-z| + |z-a|$

$= r - |z-a| + |z-a| = r$

(can also write $0 < \varepsilon < (r - |z-a|) > 0$)

(2)

$\sum_{n=2}^{\infty} \frac{z^{5n}}{\ln n}$ apply root test

remember modulus $\sqrt[5]{\left|\frac{z^{5n}}{\ln n}\right|} = |z| < 1$ converges $\sqrt[n]{n} \xrightarrow[n\to\infty]{} 1$

$R = 1$

$\sum_{k=1}^{\infty} k!\,(z+1)^k$ Ratio test: (Stirling's formula)

2 points $\frac{(k+1)!\,|z+1|^{k+1}}{k!\,|z+1|^k} = (k+1)\,|z+1| \xrightarrow[k\to\infty]{} \infty$

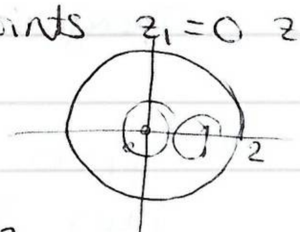
conv only at $z_0 = -1$ $\underline{\underline{R=0}}$

$\sum_{l=1}^{\infty} l^2 e^l z^l$ root test

$\lim_{l\to\infty} \sqrt[l]{l^2 e^l z^l} = |z|\,e \cdot \underbrace{\lim_{l\to\infty} \sqrt[l]{l^2}}_{=1} = |z|\,e < 1$

$R = \frac{1}{e}$

(3) $\int_{S(0,2)} \frac{e^{iz}}{z^2(1-z)}\,dz$ two singular points $z_1 = 0$ $z_2 = 1$



3 points $= \overset{①}{\int_{S(0,\frac{1}{10})}} \frac{e^{iz}\,dz}{z^2(1-z)} + \overset{②}{\int_{S(1,\frac{1}{10})}} \frac{e^{iz}}{z^2(1-z)}\,dz$

$\underbrace{\int_{S(1,\frac{1}{10})} \frac{e^{iz}}{z^2(1-z)}\,dz}_{= -\int \frac{e^{iz}}{z^2(z-1)}\,dz}$

CIF: ① $\int \frac{e^{iz}}{z^2(z-1)}\,dz = -2\pi i e^{i}$

② Cauchy's integral formula for higher derivatives

$$\int_{S(0,\frac{1}{10})} \frac{\frac{e^{iz}}{1-z}}{z^2}\,dz$$

$$f^{(n)}(z_0) = \frac{n!}{2\pi i}\int_{\gamma} \frac{f(z)}{(z-z_0)^{n+1}}\,dz$$

our case $n+1 = 2 \rightarrow n = 1$

$$f'(0) = \frac{1}{2\pi i}\int_{S(0,\frac{1}{10})} \frac{f(z)}{(z-0)^2}\,dz$$

$$\left(\frac{e^{iz}}{1-z}\right)' = \frac{ie^{iz}(1-z)+e^{iz}}{(1-z)^2}$$

$$\left(\frac{e^{iz}}{1-z}\right)'\Bigg|_{z=0} = \frac{i+1}{1} = i+1$$

$$\int_{S(0,\frac{1}{10})} \frac{\frac{e^{iz}}{1-z}}{z^2}\,dz = 2\pi i\,(i+1)$$

④ $g(z) = \frac{z}{(1+z)(1-z)} = \frac{z}{1-z^2}$

3 points

(a) $|z|<1$

$|z|^2<1$

$$g(z) = \frac{z}{1-z^2} = z\sum_{n=0}^{\infty} z^{2n} = \sum_{n=0}^{\infty} z^{2n+1}$$

(b) $|z-1|>2 \Rightarrow \frac{2}{|z-1|}<1$

$$g(z) = \underbrace{\frac{1}{2}\frac{1}{1-z}} - \frac{1}{2}\frac{1}{1+z} =$$

$$= -\frac{1}{2}\frac{1}{z-1} - \frac{1}{2}\frac{1}{2+z-1} = -\frac{1}{2}\frac{1}{z-1} - \frac{1}{2}\frac{1}{z-1}\cdot\frac{1}{\frac{2}{z-1}+1}$$

$$= -\frac{1}{2}\frac{1}{z-1} - \frac{1}{2}\frac{1}{z-1}\sum_{n=0}^{\infty}\frac{(-1)^n 2^n}{(z-1)^n}$$

$$= -\frac{1}{2}\frac{1}{z-1} + \frac{1}{2}\frac{1}{z-1}\sum_{n=0}^{\infty}\frac{(-1)^{n+1}2^n}{(z-1)^n}$$

Problem sheet 9

Qu 3

(a) $f(z) = \dfrac{z^2+z-1}{z^2(z-1)}$

singularities $z_1 = 0$ - pole of ord 2

$z_2 = 1$ - simple pole

$$\lim_{z \to z_0} f(z) = \begin{cases} A & A \neq \pm\infty \text{ - removable singularities} \\ \infty & \text{- pole} \\ \text{does not exist} & \text{- essential} \end{cases}$$

$$\operatorname{Res}(f(z), z_1) \overset{\text{Rule II}}{=} \lim_{z \to z_1} \frac{d}{dz}\left[(z-z_1)^2 f(z)\right]$$

$$\operatorname{Res}(f(z), z_2) \overset{\text{Rule I}}{=} \lim_{z \to z_2} (z-z_2) f(z) =$$

$$\frac{1}{\sin z + \cos z} = \frac{\sqrt{2}/2}{\underbrace{\frac{\sqrt{2}}{2}}_{\sin\frac{\pi}{4}} \sin z + (\cos z)\underbrace{\frac{\sqrt{2}}{2}}_{\cos\frac{\pi}{4}}} = \frac{\frac{\sqrt{2}}{2}}{\cos\left(\frac{\pi}{4} - z\right)}$$

$\frac{1}{g(z)} \leftarrow$ roots of $\frac{1}{g(z)} \Leftrightarrow$ poles of $g(z)$

$\cos\left(\frac{\pi}{4} - z\right) = 0 \qquad -\frac{\pi}{4} + z_n = \frac{\pi}{2} + \pi n$

$z_n = \frac{3\pi}{4} + \pi n, \; n \in \mathbb{Z}$ - simple poles

$$\operatorname{Res}(g(z), z_k) = \lim_{z \to z_k} (z - z_k) \cdot f(z)$$