2301 Fluid Mechanics Notes

Based on the 2010-2011 lectures by Prof E R Johnson

4th October 2010

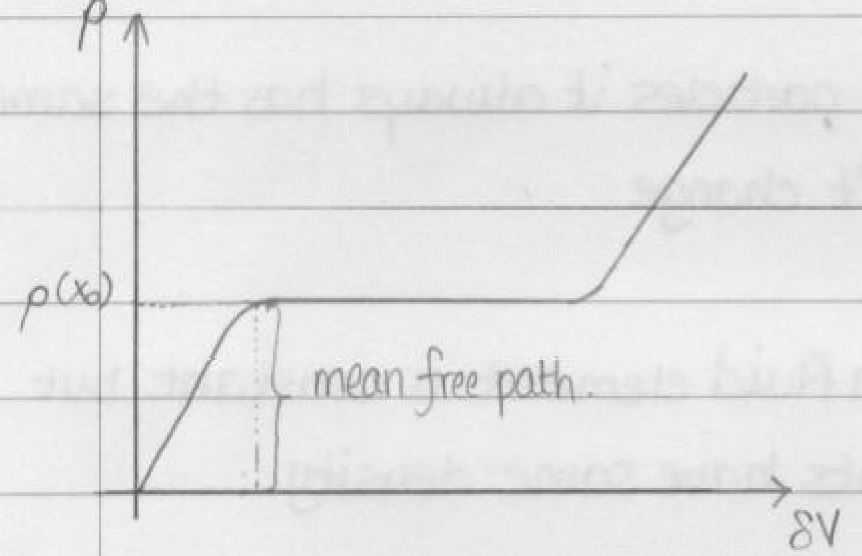
Specification + Kinematics

Continuum

A substance which we can take arbitrarily small volumes and its properties do not chance. change?

ρ = density = $\delta M / \delta V$

$\lim_{dv \to 0}$ over volumes always containing the point x_0, we call the limit $\left(\lim_{dv \to 0} \delta M / \delta V \right)$ the density at x_0.



This is an excellent approximation provided our scales are large compared to the mean free path of molecules.

In reality the limit does not make sense in the mean free path, but in maths we shall ignore the mean free path.

$\lim_{\delta V \to 0}$ makes sense. Enables us to identify properties with a point, we call this infinitesimal element, a fluid element or fluid particle.

Inviscid

Not viscous - cannot support a shear stress - tangential to surface of contact. (i.e. fluid elements slide past each other).

Viscous - honey

Inviscid - water

Incompressable

Can't be compressed - volume of fluid element does not change during the motion.

Good for air if flow speed is low compared to the speed of sound i.e. less than 600 mph in air.

Mach number = $\dfrac{\text{speed of flow / object}}{\text{speed of sound}}$.

Now there is a consequence

If fluid element always contains the same particles it always has the same mass. But incompressible flow so the volume doesn't change

∴ In incompressable flow the density of a fluid element is constant but this does not impy that all fluid elements have same density.

1.2 Two descriptions.

a) Lagrangian

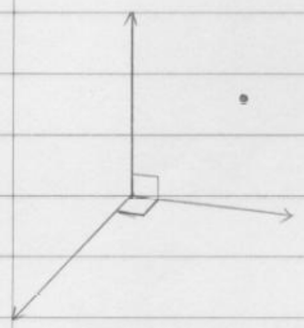
Label each particle (e.g. with initial position) and then follow each particle subsequently.

Pros : All conservation laws, newton laws apply directly : The equations are very simple

$$m\ddot{\underline{x}} = F(\underline{x})$$

Cons : Particles can follow extremely complicated paths in very simple flows.

b) Eulerian



Set up a set of fixed axes

Then for any fixed point in space can associate any fluid quantity as the value of that quantity for the particle that happens to be at that point at that time.

$\underline{u}(x,y,z,t)$: velocity of particle that happens to be at (x,y,z) at time t
$\rho(x,y,z,t)$: density " " " " " " " " " " "

Note although each particle retains its own density in incompressable flow, the density at a point could change with time - different particles at different times.

Pros : equations are standard vector calculus
Cons : equations a lot more complicated - required work to find particle paths.

1.3 Visualisation

a) Particle path : path followed by a fluid element over a given time interval.
b) Streak line : the locus formed by all points that pass through a given point in a given time interval.

c) Streamline : A line at a fixed time whose tangent at any point gives the direction of the velocity vector there.

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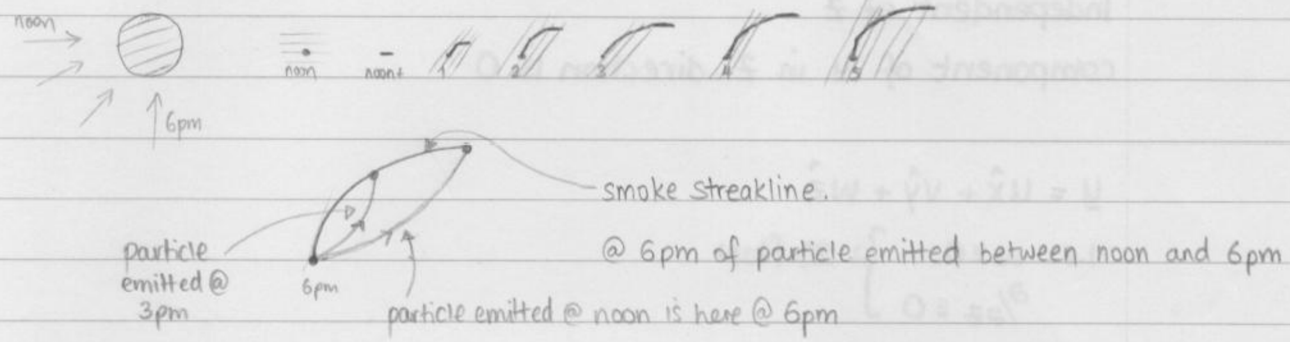


· Particle paths : paths traced out by particles in a given time interval.

Let the particle path for a given fluid element be given by $\underline{r}(t)$
Then the velocity of the particles is $\dfrac{d\underline{r}}{dt}$ at any point $\underline{r}(t)$

But by the Eulerian description $\underline{u}(\underline{r},t)$ is the velocity of the particle that happens to be at $r(\underline{t})$ at time t

i.e. $\dfrac{d\underline{r}}{dt} = \underline{u}(\underline{r},t)$

This is an o.d.e for $\underline{r}(t)$ but ~~is given~~ it is in general non-linear (+ can be very hard to solve)
To find a particle path we simply solve :

$$\frac{dr}{dt} = \underline{u}(\underline{r},t) \quad \text{given } \underline{r} = \underline{r}_0 \text{ at } t=0$$

The solution is ~~gi~~ then $\underline{r}(t)$, a curve parametrised by the time t.

Example.
Consider a particle that moves two-dimensionally with velocity field

$\underline{u}(\underline{r},t) = \hat{\underline{i}} - 2te^{-t^2}\hat{j}$ (does not vary with position).

2D

Independent of z

component of $\underline{u}$ in z-direction is 0

$$\underline{u} = u\hat{x} + v\hat{y} + w\hat{z}$$

$$\text{i.e. } \left.\begin{array}{l} w \equiv 0 \\ \partial/\partial z \equiv 0 \end{array}\right\} \text{ 2D flow}$$

Find the particle path for particle released from $(1,1)$ at $t=0$.

Ans. $\dfrac{d\underline{r}}{dt} = \underline{u}(\underline{r},t) = \hat{x} - 2te^{-t^2}\hat{y}$

with $\underline{r} = \hat{x} + \hat{y}$ at $t=0$

Or in components $\qquad (\underline{r} = x\hat{x} + y\hat{y} + z\hat{z})$

$$dx/dt = 1 \qquad\qquad dy/dt = -2te^{-t^2}$$

Thus $x = t + A \qquad\qquad y = e^{-t^2} + B$

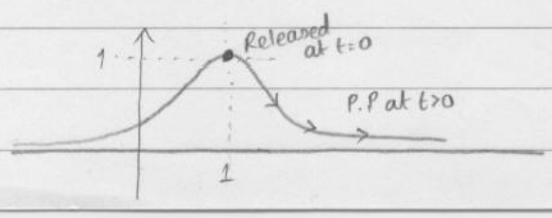
But $x = 1$ when $t = 0$ so $A = 1$

$y = 1$ when $t = 0$ so $B = 0$

i.e. $x = t + 1$ and $y = e^{-t^2}$

i.e. $(x,y) = (t+1, e^{-t^2})$ - path parametrised by time. This is sufficiently simple that we can eliminate t to get the explicit form.

$$y = e^{-(x-1)^2}$$



Example 1B.

For the above velocity field find the streakline at $t=0$ through $(1,1)$ formed by particles released from $(1,1)$ at times $t \leq 0$.

Parametrise the curve by the time of emission of the particle, τ (where $-\infty < \tau \leq 0$

i.e. at $t=\tau$, $(x,y)=(1,1)$

But $x = t + A$ and $y = e^{-t^2} + B$

i.e. $1 = \tau + A$ and $1 = e^{-\tau^2} + B$

$\Rightarrow \quad A = 1 - \tau \qquad B = 1 - e^{-\tau^2}$

so $x = t + A = t + 1 - \tau$

$y = e^{-t^2} + B = e^{-t^2} + 1 - e^{-\tau^2}$

These are the positions (x,y) at time t of particle released from $(1,1)$ at time $t = \tau$

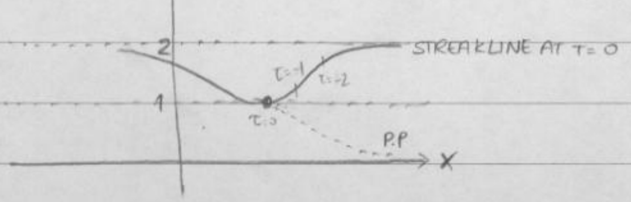
At $t=0$, these are

$$x = 1 - \tau \qquad y = 2 - e^{-\tau^2} \qquad \tau < 0$$

- curve parametrised by the time of release.

Sufficiently simple that we can eliminate τ to give the explicit curve

$$y = 2 - e^{-(1-x)^2}$$



Example 1C

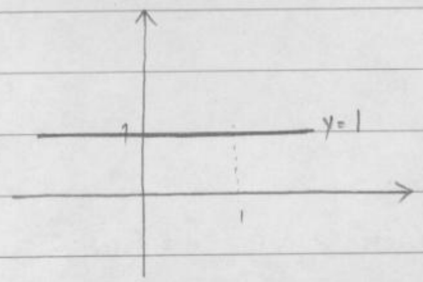
For the velocity field above, find the streamline through (1,1) at $t=0$

$$\underline{u} = \hat{x} - 2te^{-t^2}\hat{y}$$

At $t=0$, $\underline{u} = \hat{x}$

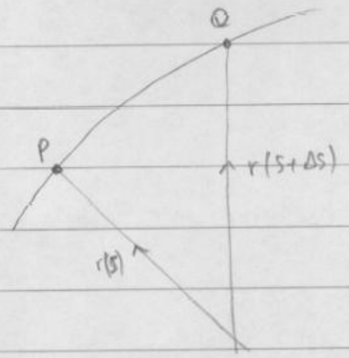
Line with direction $\hat{\underline{x}}$ passing through (1,1) is $\underline{r} = \hat{x} + \hat{y} + s\hat{x}$ $\quad -\infty < s < \infty$

i.e. $y=1$



Aside.

Suppose we have a curve $r(s)$ parametrised by s.



$$\overrightarrow{PQ} = r(s+\Delta s) - r(s)$$

direction of $\overrightarrow{PQ}$ is $\dfrac{r(s+\Delta s) - r(s)}{\Delta s}$

If the limit $\Delta s \to 0$ i.e. $Q \to P$ exists, then

$\dfrac{dr}{ds} = \lim_{\Delta s \to 0} \dfrac{r(s+\Delta s) - r(s)}{\Delta s}$ = gives the direction of the tangent to curve at P.

Thus let a streamline be parametrised by s i.e. $\underline{r}(s)$
Then the tangent of the streamline is given by dr/ds.

So at any time t_0

$$\frac{dr}{ds} = u(\underline{r}, t_0)$$

An o.d.e in s for $\underline{r}(s)$, in general nonlinear. To be solved subject to $r = r_0$ when $s=0$

Example 1c (again)

$$\frac{d\underline{r}}{ds} = u(r,0) = \hat{\underline{x}}$$

subject to $\underline{r} = \hat{x} + \hat{y}$ when $s=0$

so $\underline{r} = \hat{x}s + r_0$

But $\underline{r} = \hat{x} + \hat{y}$ @ $s=0$ so $r_0 = \hat{x} + \hat{y}$

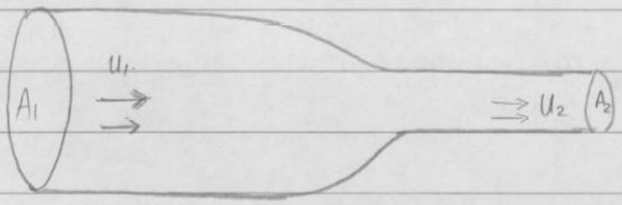
Thus $\underline{r} = s\hat{x} + \hat{x} + \hat{y}$ as before

i.e. $x = s+1$
$y = 1$

i.e. line $y=1$ for all x

Conservation of mass.

Consider fluid of constant density ρ flowing through a tube whose upstream cross-sectional area is A_1 and downstream area A_2. Suppose the fluid velocity is uniform across the tube with speed u_1 upstream and u_2 downstream.



Now in time δt a volume $(u_1 \delta t) A_1$ crosses the upstream cross-section and enters tube.

In same time interval, a volume $(u_2 \delta t) A_2$ leaves the downstream section.

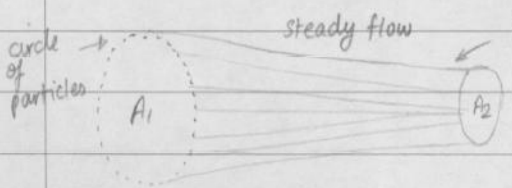
For incompressable flow, these two quantities must be the same.

$$u_1 A_1 = u_2 A_2$$

$$\frac{u_2}{u_1} = \frac{A_1}{A_2}$$

If tube halves in area, the speed doubles to conserve mass.

[Particle path, streaklines, streamlines identical in steady flow i.e. $\partial u / \partial t = 0$]



tube formed within fluid by their particle paths or streamline
- a streamtube

Then same result holds : $u_2 / u_1 = A_1 / A_2$

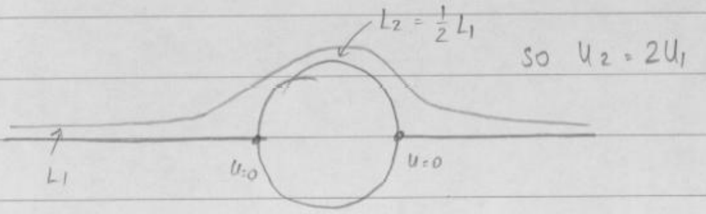
i.e. if streamtube contracts the speed of flow increases inversely as the area.

In particular in 2D, everything happens per unit width into page. Thus two streamlines seperated by distance L_1 have associated cross-~~section~~ stream area $A_1 = L_1 \times 1$ and downstream have associated area $A_2 = L_2 \times 1$.

Now,

$$\frac{U_2}{U_1} = \frac{A_1}{A_2} = \frac{L_1}{L_2} \quad \text{in 2D flow}$$

i.e. ~~stream~~ speed varies inversely as the seperation of the streamlines in 2D flow



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Lemma

If f is continuous in $[a,b]$ and $\int_c^d f$ vanishes for every $(c,d) \subset [a,b]$ then $f \equiv 0$ in $[a,b]$.

Proof

Let us have an interval $[a,b]$ and a function f s.t $\int_c^d f = 0 \;\; \forall (c,d) \subset [a,b]$

Now suppose there exists $\alpha \in [a,b]$ s.t $f(\alpha) \neq 0$

Without loss of generality we take $\cancel{\alpha > 0}$ $f(\alpha) > 0$

But f is continuous

Thus $\exists\; \delta > 0$ s.t if $|x-\alpha| < \delta$ then $|f(x) - f(\alpha)| < \frac{1}{2} f(\alpha)$

i.e. $f(x) > \frac{1}{2} f(x)$

Now consider $\displaystyle\int_{\alpha-\delta}^{\alpha+\delta} f(x)\,dx > \int_{\alpha-\delta}^{\alpha+\delta} \frac{1}{2} f(\alpha)\,dx$

$$= \frac{1}{2} f(\alpha) \int_{\alpha-\delta}^{\alpha+\delta} dx = \delta f(\alpha) > 0$$

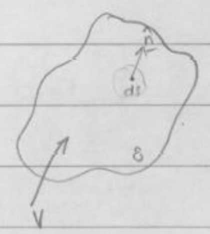
Contradiction as $\int_c^d f$ vanishes $\forall (c,d) \subset [a,b]$ so $\nexists\; f(\alpha) \neq 0$. i.e. $f \equiv 0$ in $[a,b]$.

3D version.

If we have a function f continuous in a domain D and for each subdomain V of D $\displaystyle\int_V f\,dV = 0$ then $f \equiv 0$ in D.

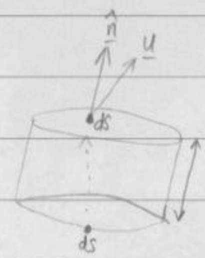
Conservation of mass (for a constant density fluid)

Consider a fluid of density ρ occupying a domain D.
Take an arbitrary subdomain V of D
Let the surface of V be S



Let the velocity field associated with the fluid be $\underline{u}(x,y,z,t)$
Consider an infinitesimal element dS of surface S with local outward normal $\hat{\underline{n}}$.

In time δt the mass of fluid passing through dS is $\rho \times$ volume



$\underline{u}\delta t$ - distance travelled by particle with velocity $\underline{u}$ in time δt.

$$\therefore \quad \rho \times \text{volume} = \rho \times \text{area of base} \times \text{height}$$
$$= \rho \, dS \, (\underline{u}\delta t \cdot \hat{\underline{n}})$$
$$= \rho \, (u \cdot \hat{\underline{n}}) \, dS \, \delta t$$

In fact we call the rate of which mass flows across dS, the - mass FLUX across dS. Here the mass flux is $\rho(u\cdot\hat{\underline{n}})\,dS$.
Thus the total mass flux out of volume V is $\int_S \rho(u\cdot\hat{n})\,dS$

$$\therefore \quad \rho \int_S (u\cdot\hat{\underline{n}})\,dS = \rho \int_V \underline{\nabla}\cdot\underline{u}\,dV.$$

this must be zero as the total mass in V is conserved.

i.e. $\int_V \underline{\nabla}\cdot\underline{u}\,dV = 0$

[Aside: for $\underline{u} = u\hat{x} + v\hat{y} + w\hat{z}$

$$\nabla = \partial/\partial x\,\hat{x} + \partial/\partial y\,\hat{y} + \partial/\partial z\,\hat{z}$$
$$\nabla\cdot u = \partial u/\partial x + \partial v/\partial y + \partial w/\partial z.$$]

Apply lemma :- Arbitrary V. Hence true $\forall\, V \subset D$. Hence $\nabla \cdot \underline{u} = 0$ in D.

i.e. for a velocity field to conserve mass in a homogeneous fluid (ρ = constant) its divergence must vanish.

In 3D $\quad \dfrac{\partial u}{\partial x} + \dfrac{\partial v}{\partial y} + \dfrac{\partial w}{\partial y} = 0$

In 2D $\quad \dfrac{\partial u}{\partial x} + \dfrac{\partial v}{\partial y} = 0$

· In 2D, conservation of mass gives : $\partial u/\partial x + \partial v/\partial y = 0$

Choose any area A bounded by a curve C

Green's theorem $\displaystyle\int_A \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) dA = \oint_C u\,dy - v\,dx$

But $\int_A$ vanishes for an incompressible fluid.

So in 2D incompressible flow, around any closed C

$$\oint_C u\,dy - v\,dx = 0$$

i.e. $\displaystyle\oint_C F \cdot dr = 0$ where $F = u\hat{y} - v\hat{x}$ $\quad d\underline{r} = dx\,\hat{x} + dy\,\hat{y}$.

i.e. F is a conservative field

i.e. $F = \nabla \Psi$ for some scalar function Ψ.

so here $\partial\Psi/\partial x = -v \qquad \partial\Psi/\partial y = u$

$\therefore \quad \underline{u} = -\hat{\underline{z}} \wedge \underline{\nabla}\Psi$

$\nabla\Psi = \dfrac{d\Psi}{dx}\hat{x} + \dfrac{d\Psi}{dy}\hat{y}$. $\quad \underline{u} = u\hat{x} + v\hat{y}$

$\hat{z} \wedge \hat{x} = \hat{y}$

$\hat{z} \wedge \hat{y} = -\hat{x}$

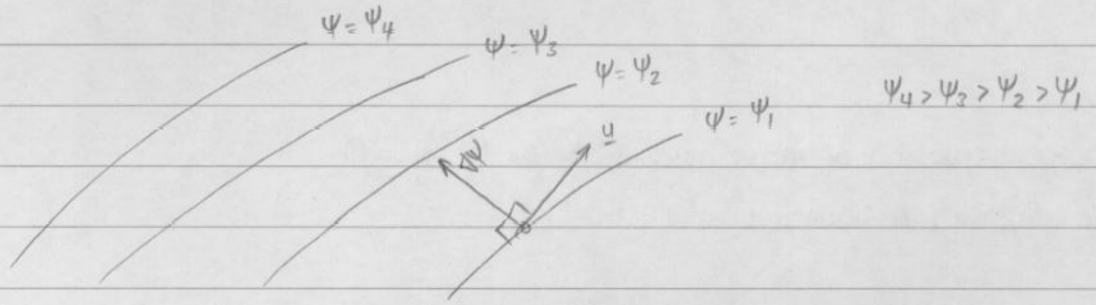
$$-\hat{z}\wedge\nabla\Psi = -{}^{\partial\Psi}/_{\partial x}\,\hat{y} + {}^{\partial\Psi}/_{\partial y}\,\hat{x}$$
$$\underline{u} = u\hat{x} + v\hat{y}.$$

We have proved:

If the flow is 2D and incompressible then $\exists\,\Psi$ s.t $\underline{u} = -\hat{z}\wedge\nabla\Psi$

(in cartesian $u = {}^{\partial\Psi}/_{\partial y}$ $v = {}^{\partial\Psi}/_{\partial x}$)

lines of constant Ψ



i.e. the lines of constant Ψ are tangent to the velocity vector at any point

i.e they are streamlines.

We call Ψ a streamfunction.

In 2D incompressible flow there exists Ψ s.t the streamline are curves $\Psi =$ const.

(it can be ~~solved by~~ found by solving ${}^{\partial\Psi}/_{\partial y} = u$, ${}^{\partial\Psi}/_{\partial x} = -v$

Example

Show that $\underline{u} = x\hat{x} - y\hat{y}$ satisfies the mass conservation (or continuity) equation.
Find a stream function and sketch the streamlines.

Continuity $\dfrac{\partial u}{\partial x} + \dfrac{\partial v}{\partial y} = 0$

Here $u = x$ $\quad {}^{\partial u}/_{\partial x} = 1$

$v = -y$ $\quad {}^{\partial v}/_{\partial y} = -1$

$\therefore \dfrac{\partial u}{\partial x} + \dfrac{\partial v}{\partial y} = 0 \qquad \Rightarrow \exists\,\Psi$ s.t $u = {}^{\partial\Psi}/_{\partial y}$ $\quad v = -{}^{\partial\Psi}/_{\partial x}$

Apply lemma :- Arbitrary V. Hence true $\forall\, V \subset D$. Hence $\nabla\cdot\underline{u} = 0$ in D.

i.e. for a velocity field to conserve mass in a homogeneous fluid (ρ = constant) its divergence must vanish.

In 3D $\dfrac{\partial u}{\partial x} + \dfrac{\partial v}{\partial y} + \dfrac{\partial w}{\partial y} = 0$

In 2D $\dfrac{\partial u}{\partial x} + \dfrac{\partial v}{\partial y} = 0$

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But $\int_A$ vanishes for an incompressible fluid.
So in 2D incompressible flow, around any closed C

$$\oint_C u\,dy - v\,dx = 0$$

i.e. $\displaystyle\oint_C F\cdot dr = 0$ where $F = u\hat{y} - v\hat{x}$ $\;d\underline{r} = dx\,\hat{x} + dy\,\hat{y}$.

i.e. F is a conservative field
i.e. $F = \nabla\Psi$ for some scalar function Ψ.

so here $\partial\Psi/\partial x = -v$ $\quad \partial\Psi/\partial y = u$

$\therefore\ \underline{u} = -\hat{\underline{z}} \wedge \underline{\nabla}\Psi$

$\nabla\Psi = \dfrac{d\Psi}{dx}\hat{x} + \dfrac{d\Psi}{dy}\hat{y}$. $\quad \underline{u} = u\hat{x} + v\hat{y}$

$\hat{z} \wedge \hat{x} = \hat{y}$
$\hat{z} \wedge \hat{y} = -\hat{x}$

i.e. $\dfrac{\partial\psi}{\partial y} = u = x$

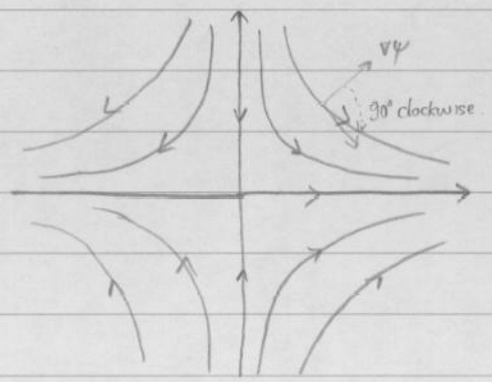
so $\psi = xy + f(x)$

$\partial\psi/\partial x = y + f'(x)$

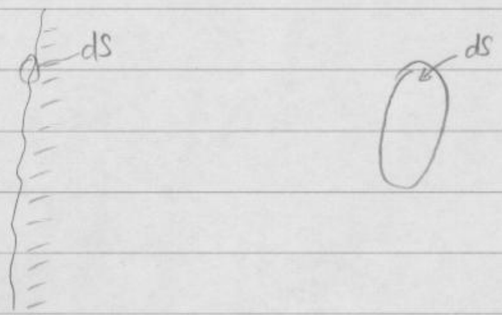
But $\partial\psi/\partial x = -v = y \quad \Rightarrow f'(x) = 0$, so f = const, taken to be zero here i.e $\psi = xy$.

Streamlines : lines ψ = const.

xy = const - rectangular hyperbolae.



Solid boundaries - impermeable.



zero flux through an element ds of an impermeable boundary.

i.e. $\rho(u \cdot \hat{n})\, ds = 0$

But $\rho \neq 0$ and $ds \neq 0$, so $u \cdot \hat{n} = 0$ at a solid boundary.

i.e. there is no velocity normal to a solid boundary

i.e. the normal velocity vanishes @ a solid boundary.

In 2D incompressible flow

$$\underline{u} = \hat{\underline{z}} \wedge \nabla\psi.$$

At a solid boundary $\underline{u} \cdot \hat{\underline{n}} = 0$

i.e. $(\hat{\underline{z}} \wedge \nabla\psi) \cdot \hat{\underline{n}} = 0$

$(z \wedge \hat{n}) \cdot \nabla\psi = 0$

$\hat{\underline{t}} \cdot \nabla\psi = 0$ where $\hat{\underline{t}}$ is unit tangent

NOT IN REVISION NOTES!!

Thus $\partial\psi/\partial s = 0$ along the boundary

i.e. ψ = constant on an impermeable boundary

Thus an impermeable boundary is a streamline and equivalently, any streamline can be taken as an impermeable boundary

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Incompressible

$\Rightarrow \underline{\nabla} \cdot \underline{u} = 0$

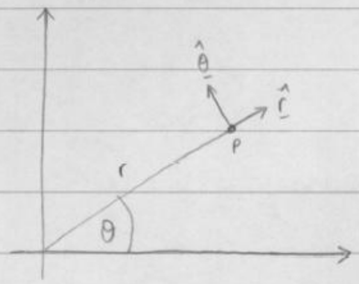
Also 2D :

$$\partial u/\partial x + \partial v/\partial y = 0$$

$\Rightarrow \exists \psi$ s.t $u = -\hat{\underline{z}} \wedge \underline{\nabla}\psi$ (streamfunction)

Natural co-ordinate system for flows with cylinders is polar coordinates (r,θ)

$x = r\cos\theta \quad y = r\sin\theta$



$$u(x,y,t) = u\hat{\underline{x}} + v\hat{\underline{y}}$$
$$= u_r\hat{\underline{r}} + u_\theta\hat{\underline{\theta}}.$$

i.e. u_r and u_θ are the polar components of the velocity vector. $\underline{u}$.

Now $\underline{u} = -\hat{\underline{z}} \wedge \underline{\nabla}\psi$

and in polars, $\underline{\nabla}\psi = \frac{\partial\psi}{\partial r}\hat{\underline{r}} + \frac{1}{r}\frac{\partial\psi}{\partial\theta}\hat{\underline{\theta}}$

$-\hat{z} \wedge \nabla\psi = -\frac{\partial\psi}{\partial r}\hat{\underline{\theta}} + \frac{1}{r}\frac{\partial\psi}{\partial\theta}\hat{\underline{r}}$

$\hat{\underline{z}} \wedge \hat{\underline{r}} = \hat{\underline{\theta}}$

$\hat{\underline{z}} \wedge \hat{\underline{\theta}} = -\hat{\underline{r}}$

$\therefore \; u_r = \frac{1}{r}\frac{\partial\psi}{\partial\theta} \qquad u_\theta = -\frac{\partial\psi}{\partial r}$

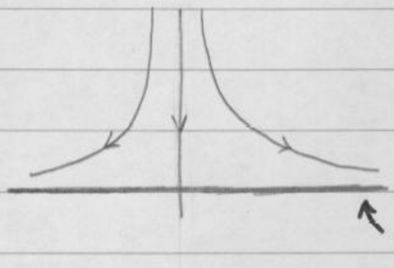
We have shown that :

$$-\hat{\underline{z}} \wedge \underline{\nabla}\psi = \frac{1}{r}\frac{\partial\psi}{\partial\theta}\,\hat{\underline{r}} - \frac{\partial\psi}{\partial r}\,\hat{\underline{\theta}}$$

But this is $\underline{u} = u_r\,\hat{\underline{r}} + u_\theta\,\hat{\underline{\theta}}$

$$\therefore \quad u_r = \frac{1}{r}\frac{\partial\psi}{\partial\theta} \qquad u_\theta = -\frac{\partial\psi}{\partial r}$$

We have show for : $\underline{u} = x\hat{x} - y\hat{y}$ $\qquad \psi = xy.$



solid boundary $\underline{u}\cdot\hat{\underline{n}} = 0$

Here $\hat{n} = \hat{y}$

$$\underline{u}\cdot\hat{\underline{n}} = \underline{u}\cdot\hat{y} = -y = 0 \text{ on } y = 0.$$

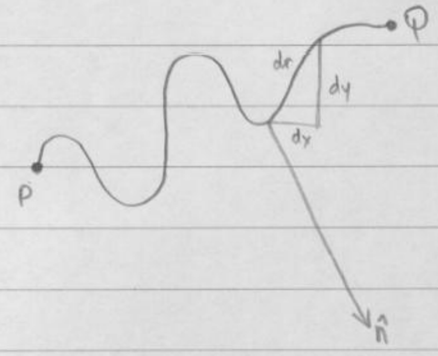
Note in this flow, at the origin, $\underline{u} = 0$. $\therefore$ The origin is the 'stagnation point'. This flow is known as stagnation point flow.

Physical Interpretation of the streamfunction, ψ.

The volume flux per unit width in a clockwise direction crossing any curve joining a point P to a point Q is given by : $\psi(Q) - \psi(P)$

Proof.

Take any pounts P, Q in the fluid and curve C joining them.



Along C

$dr = dx\,\hat{x} + dy\,\hat{y}$.

A normal to the line is given by $n = -dy\,\hat{x} + dx\,\hat{y}$

Notice $n \cdot dr = 0$ and $d\underline{r}$ is tangential

This a unit vector, in the clockwise direction is

$$\hat{n} = \frac{-dy}{\sqrt{dx^2+dy^2}}\,\hat{x} + \frac{dx}{\sqrt{dx^2+dy^2}}\,\hat{y}$$

i.e. $\hat{n} = -\frac{dy}{ds}\,\hat{x} + \frac{dx}{ds}\,\hat{y}$.

ds in the element of arc length $ds = |d\underline{r}|$.

The flux across PQ is

$$\int_P^Q (\underline{u} \cdot \underline{\hat{n}})\,ds = \int_P^Q (u\hat{x} + v\hat{y}) \cdot \left(-\frac{dy}{ds}\,\hat{x} + \frac{dx}{ds}\,\hat{y}\right) ds.$$

$$= \int_P^Q \left(\frac{\partial\psi}{\partial y}\,\hat{x} - \frac{\partial\psi}{\partial x}\,\hat{y}\right) \cdot \left(\qquad\qquad \right) ds$$

$$= \int_P^Q \left(-\frac{dy}{ds}\frac{\partial\psi}{\partial y} - \frac{dx}{ds}\frac{\partial\psi}{\partial x}\right) ds$$

18th October 2010

Summary

(1) Description

Eulearian $\underline{u}(\underline{x}, \underline{y}, \underline{z}, \underline{t})$, $\rho(x, y, z, t)$

(2) Particle paths, streamlines, streaklines.

(3) Incompressible and homogeneous (ρ = const.) flow $\Rightarrow$ $\nabla \cdot \underline{u} = 0$, $\partial u/\partial x + \partial v/\partial y + \partial w/\partial z = 0$

(4) $\nabla \cdot \underline{u} = 0$ (incompressible) plus restrict to 2D $\Rightarrow \exists \Psi$ s.t $\underline{u} = -\hat{\underline{z}} \wedge \nabla \Psi$

Ψ streamfunction $\Rightarrow$ lines Ψ = const. – streamlines

In components : Cartesian $u = \partial \Psi/\partial y \quad v = -\partial \Psi/\partial x$

Polars $u_r = 1/r \, \partial \Psi/\partial \theta \quad u_\theta = -\partial \Psi/\partial r$

(5) Boundary conditions

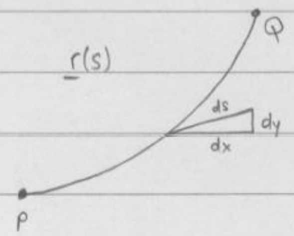
$u \cdot \hat{n} = 0$ on solid boundary

Ψ = const on solid boundary.

Vorticity

$\omega = \text{curl } \underline{u} = \underline{\nabla} \wedge \underline{u}$

Physical interpretation of the streamfunction.



Line parametrised, a function of arclength s.

$$ds = |d\underline{r}| = |dx\,\hat{x} + dy\,\hat{y}| = \sqrt{dx^2 + dy^2}$$

$\hat{n}$ = unit normal, so $\hat{n} \cdot dr = 0$

Consider $\underline{n} = dy\,\hat{\underline{x}} - dx\,\hat{y}$

Note $\underline{n} \cdot dr = dx\,dy - dx\,dy = 0$

Thus $\hat{\underline{n}} = \dfrac{dy\,\hat{x} - dx\,\hat{y}}{\sqrt{dx^2 + dy^2}}$ i.e. $\hat{n} = \dfrac{dy\,\hat{x} - dx\,\hat{y}}{ds} = \dfrac{dy}{ds}\hat{x} - \dfrac{dx}{ds}\hat{y}$

$\Rightarrow$ unit normal to curve pointing clockwise $\pi/2$

Flux across ds: $u \cdot \hat{n}\,ds$ per unit width into page

Total flux in clockwise direction across PQ is:

$$\int_P^Q (u \cdot \hat{n})\,ds = \int_P^Q \left(\frac{\partial \psi}{\partial y}\hat{x} - \frac{\partial \psi}{\partial x}\hat{y}\right) \cdot \left(\frac{dy}{ds} - \frac{dx}{ds}\right) ds$$

$$= \int_P^Q \left(\frac{\partial \psi}{\partial y}\frac{dy}{ds} + \frac{\partial \psi}{\partial x}\frac{dx}{ds}\right) ds$$

$$= \int_P^Q \left(\left.\frac{d\psi}{ds}\right|_{\text{along } PQ}\right) ds = \psi(Q) - \psi(P)$$

The streamfunction measures the volume flux (per unit length) across a line joining 2 points in the fluid.

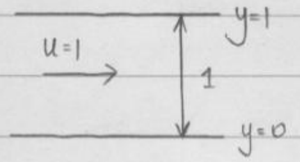
Example

$u = 1 \quad v = 0$

$\partial\psi/\partial y = 1 \quad \partial\psi/\partial x = 0$

$\therefore \ \psi = y$

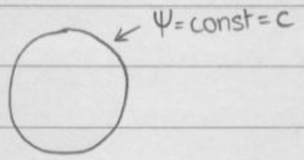
Flux (per unit width) crossing any line joining P and Q is $y(Q) - y(P)$.



$\psi(Q) - \psi(P) = 1$

$=$ seperation x speed $= 1$

Note : with 1 boundary we have

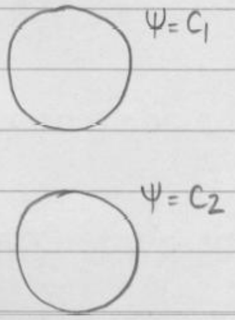


But $u = -\hat{\underline{z}} \wedge \nabla\psi$

So ψ arbitrary to within additive const.

Hence we can take $\psi = 0$ on the boundary.

2 boundaries



Take $\psi = \psi - C_1$

Thus $\psi = 0$ on one of the boundaries

But $\psi = C_2 - C_1$ on the other and this might not be zero.

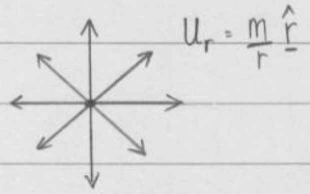
In fact $C_2 - C_1$ is the amount of flux flowing between the obstacles and in general is part of the solution to the problem.

Note : The dimensions of ψ are volume per unit time per unit width

$L^3 T^{-1} L^{-1} = L^2 T^{-1}$

Example

An isotropic line source



The streamfunction is $\psi = m\theta$.

Note $u_\theta = -\dfrac{\partial\psi}{\partial r} = 0$

$$u_r = \frac{1}{r}\frac{\partial\psi}{\partial\theta} = \frac{m}{r}$$

$$\underline{\nabla} \cdot \underline{u} = \frac{1}{r}\frac{\partial}{\partial r}(r u_r) + \frac{1}{r}\frac{\partial}{\partial\theta}(u_\theta)$$
$$= \frac{1}{r}\frac{\partial}{\partial r}(m) + \frac{1}{r}\frac{\partial 0}{\partial\theta} = 0$$

Take any curve C circuling the source at the origin. Take any P on C and Q arbitrary close to P in clockwise direction on C. Consider curve C taken in anticlockwise direction. Then the flux outwards across C is $\psi(Q) - \psi(P)$

But θ increases by 2π going around C

Thus increase in ψ; $\therefore \psi = 2\pi m + \psi_1$

Note if C does not circle origin: Then change in $\theta = 0$
Flux across C is zero.

Similarly we can shrink C as small as we like but circling origin and flux remains $2\pi m$

Hence the origin is a singularity where fluid is created at a rate $2\pi m$.

$$\underline{u} = \frac{m}{r}\,\hat{\underline{r}} \qquad (\text{sing. at } r = 0)$$

Satisfies our equations everywhere - except origin

Source of strength $2\pi m$.

Example 2.

An isotropic source of strength $2\pi m$ in a uniform stream of speed U

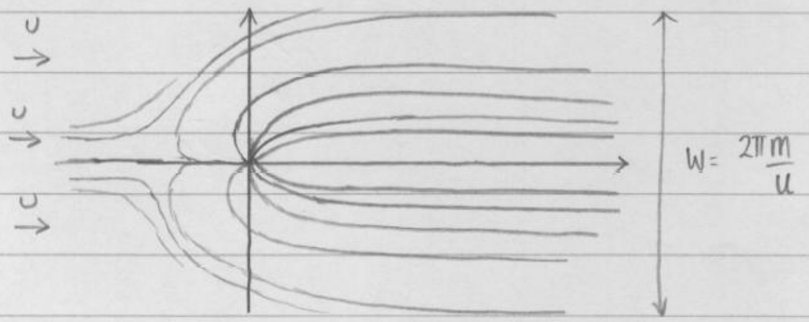
(1) Uniform stream speed U: choose x-direction in direction of stream

Then $u = U \quad v = 0$

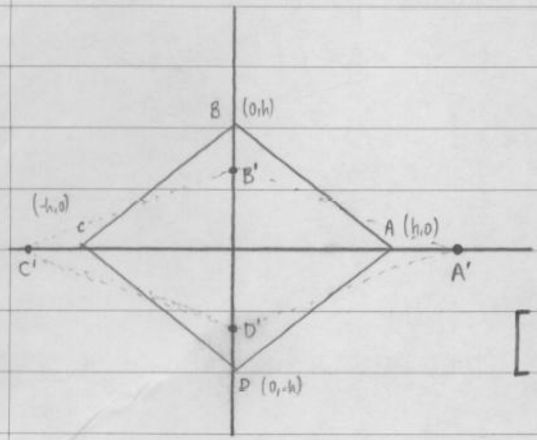
$$\frac{\partial \psi_1}{\partial y} = U \qquad \frac{\partial \psi_1}{\partial x} = 0 \qquad \text{Take } \psi_1 = Uy$$

(2) Isotropic source strength $2\pi m$ of origin $\psi_2 = m\theta$.

(3) Combine these; $\psi = \psi_1 + \psi_2 = Uy + m\theta$



21st October 2010.



Consider the square ABCD with $0<h<<1$.
Consider a time interval $0<\delta t<<1$, so that $\underline{u}$ is essentially steady

[Taylor's thm : $f(x) = f(0) + xf'(0) + R_2$
$R_2 = \frac{1}{2!}f''(\xi)h^2 \quad 0<\xi<h$]

Thus within ABCD we can write

$$u = U + \alpha x + \beta y$$
$$v = V + \gamma x + \delta y$$

with an error of order h^2.

Here $\alpha = \partial u/\partial x|_0 \quad \beta = \partial u/\partial y|_0 \quad \gamma = \partial v/\partial x|_0 \quad \delta = \partial v/\partial y|_0$

For incompressible flow: $\partial u/\partial x + \partial v/\partial y = 0$

so $\alpha + \delta = 0$

It is convenient to write

$$\theta = \tfrac{1}{2}(\beta+\gamma) \text{ and } \phi = -\tfrac{1}{2}(\beta-\gamma) = -\tfrac{1}{2}\left(\partial u/\partial y - \partial v/\partial x\right)$$
$$\theta + \phi = \beta \qquad \theta - \phi = \gamma$$

Thus

$$\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} U \\ V \end{pmatrix} + \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$

i.e.
$$\underbrace{\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} U \\ V \end{pmatrix}}_{\mathrm{I}} + \left[\alpha \underbrace{\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}}_{\mathrm{II}} + \theta \underbrace{\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}}_{\mathrm{III}} + \phi \underbrace{\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}}_{\mathrm{IV}}\right]\begin{pmatrix} x \\ y \end{pmatrix} + O(h^2)$$

translation of C.O.M

In time δt, each point ABCD moves by an amount $\begin{pmatrix} \delta x \\ \delta y \end{pmatrix} = \begin{pmatrix} u \\ v \end{pmatrix} \delta t$

Look at each term in order.

Term I : $\begin{pmatrix} \delta x \\ \delta y \end{pmatrix} = \begin{pmatrix} U \\ V \end{pmatrix} \delta t$ for every particle in ABCD. i.e. ABCD translates without change of shape, and without change of orientation

Term II : α : depends on which point you consider

Consider A : $(h,0)$

$$\begin{pmatrix} \delta x \\ \delta y \end{pmatrix}' = \alpha \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} h \\ 0 \end{pmatrix} \delta t$$

$$= \begin{pmatrix} \alpha h \delta t \\ 0 \end{pmatrix}$$

Consider C : $(-h,0)$

It moves by $\delta x = -\alpha h \delta t \qquad \delta y = 0$.

Consider B : $(0,h)$

$$\begin{pmatrix} \delta x \\ \delta y \end{pmatrix} = \alpha \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 0 \\ h \end{pmatrix} \delta t = \begin{pmatrix} 0 \\ -\alpha h \delta t \end{pmatrix}$$

Consider D : $(0,-h)$

$\delta x = 0 \qquad \delta y = \alpha h \delta t$

$\therefore$ Stretching in the x-direction and an equal and opposite contraction in the y-direction so as to conserve area : area conserving dilation.

Term III

Consider A $(h,0)$

$$\begin{pmatrix} \delta x \\ \delta y \end{pmatrix} = \theta \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} h \\ 0 \end{pmatrix} \delta t = \begin{pmatrix} 0 \\ \theta h \delta t \end{pmatrix}$$

Consider C $(-h,0)$

$\delta x = 0 \qquad \delta y = -\theta h \delta t$

Consider B $(0,h)$

$$\begin{pmatrix} \delta x \\ \delta y \end{pmatrix} = \theta \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ h \end{pmatrix} \delta t = \begin{pmatrix} \theta h \delta t \\ 0 \end{pmatrix}$$

Consider D

$\delta x = -\theta h \delta t \qquad \delta y = 0.$

Another dilation : stretching along $y=x$
contracting along $y=-x$.

Term IV

Consider A

$$\begin{pmatrix} \delta x \\ \delta y \end{pmatrix} = \phi \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} h \\ 0 \end{pmatrix} \delta t = \begin{pmatrix} 0 \\ -\phi h \delta t \end{pmatrix}$$

Consider C

$$\delta x = 0 \qquad \delta y = \phi h \delta t$$

Consider B

$$\begin{pmatrix} \delta x \\ \delta y \end{pmatrix} = \phi \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ h \end{pmatrix} \delta t = \begin{pmatrix} \phi h \delta t \\ 0 \end{pmatrix}$$

Consider D

$$\delta x = -\phi h \delta t \qquad \delta y = 0$$

Rotation about the C.O.M with angular velocity $\phi = -\frac{1}{2}\left(\partial v/\partial x - \partial u/\partial y\right)$.

$\phi > 0$ anticlockwise.

Proved

Local motion = Sum of translation of C.O.M.

Dilation

rotation about C.O.M

25th October 2010.

Local motion = sum of translation of the C.O.M
a dilation
a rotation about the C.O.M with angular velocity $\frac{1}{2}\left(\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}\right)$
$= \frac{1}{2}\underline{\hat{z}}\cdot(\nabla \wedge \underline{u})$

We call curl $\underline{u} = \underline{\nabla} \wedge \underline{u}$, the VORTICITY of the flow. (old name - rot $\underline{u}$)
The vorticity = twice the angular momentum of a fluid element about C.O.M.

In 3D; curl u has 3 non-zero components.

$$\underline{\omega} = (\xi, \eta, \zeta)$$

$$= \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \partial x & \partial y & \partial z \\ u & v & w \end{vmatrix}$$

2D flow;

$$\underline{\omega} = \begin{vmatrix} \partial x & \partial y & 0 \\ u & v & 0 \end{vmatrix}$$

$$= (\partial x v - \partial y u)\hat{\underline{z}} = (\partial v/\partial x - \partial u/\partial y)\hat{\underline{z}} = \zeta \hat{\underline{z}}.$$

Only one non-zero comp: $\perp$'r plane of motion.

In an invicid fluid no element exerts a shear stress on any other element.
If we cannot apply a shear stress we cannot apply a TORQUE, i.e. a force with a moment.
Thus we cannot change the angular momentum of a fluid element by applying a moment or torque.
Thus in 2D flow the vorticity of a fluid element never changes.
Most importantly, elements that are not spinning, can never start spinning i.e $\zeta = 0$,

for any t for any element. $\zeta \equiv 0 \ \forall t$.

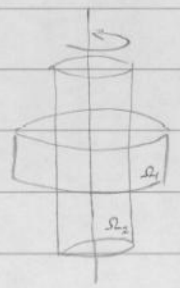
In particular for a motion started from rest, where $\underline{u} = 0$ at $t = 0$

then $\zeta = (\text{curl}\, u) \cdot \hat{z} \equiv 0$ at $t = 0$.

Hence $\zeta \equiv 0$, everywhere, for all time.

PERSISTENCE of IRROTATIONAL motion.

In 3D flow:



$\Omega_2 > \Omega_1$

Stretching a fluid element, shrinks in complementary dimensions so it spins faster. Here $\Omega_2 > \Omega_1$ so $\zeta_2 > \zeta_1$. ~~Vort~~ Vorticity can change even in inviscid flows in 3D due to vortex stretching

Vortex stretching = vorticity amplifier.

[In 2D, no motion $\perp$'r to plane $\therefore$ no stretching].

However irrotational motion is persistent in 3D also as there is no vorticity to amplify.

i.e. for motion started from rest $\underline{\nabla} \wedge \underline{u} = 0$ for all x, y, t ($+ z$ in 3D).

i.e. $\underline{u}$ is a vector field with vanishing curl.

Hence $\underline{u}$ is derivable from a scalar potential ($\underline{u}$ is conservative).

i.e. $\exists \phi$ s.t $\underline{u} = \underline{\nabla}\phi$ (in 2D/3D).

We call ϕ the velocity potential.

But our flow is incompressible so,

$$\nabla \cdot u = 0 \quad (\text{in } 2D/3D)$$

So $\nabla \cdot (\nabla \phi) = 0$ (in 2D/3D)

$$\nabla^2 \phi = 0$$

In 2D, 3D the inviscid, irrotational, fluid flow is derivable from a velocity potential satisfying Laplace's equation.

The boundary condition on a solid boundary is

$$\underline{u} \cdot \underline{\hat{n}} = 0$$

i.e. $\underline{\hat{n}} \cdot \nabla \phi = 0$

i.e. $\partial \phi / \partial n = 0$ on boundary

The normal derivative of ϕ vanishes on a solid boundary.

Example.

Find the velocity potential for a uniform stream of speed U in the x-direction.

$\underline{u} = U\underline{\hat{x}}$ i.e. $u = U$ $v = 0$

But $\underline{u} = \nabla \phi$ so $u = \partial \phi / \partial x$ $v = \partial \phi / \partial y$

so $\partial \phi / \partial x = U$ so $\phi = Ux + f(y)$

i.e. $\partial \phi / \partial y = f'(y)$ But $\partial \phi / \partial y = v = 0$ $\therefore f'(y) = 0$

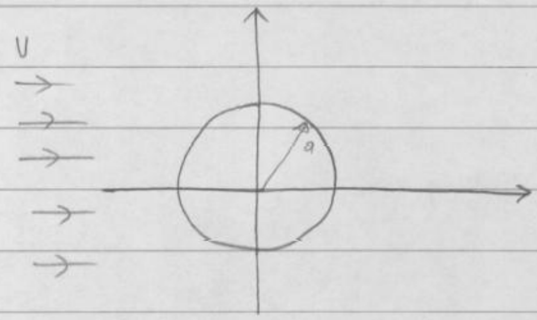
Thus take $f(y) = 0$

Then

$$\phi = Ux$$
$$\nabla \phi = U\underline{\hat{x}}.$$

Example.

Find the steady irrotational flow of a uniform stream of U in the x-direction past a circular cylinder of radius a at the origin.



Introduce polar coords.

$$x = r\cos\theta \qquad y = r\sin\theta$$

Then as $r \to \infty$, the flow becomes a uniform stream.

so

$$\phi \to Ux = Ur\cos\theta \text{ as } r \to \infty$$

The cylinder $r=a$ is solid so $\frac{\partial \phi}{\partial n} = 0$ on $r=a$.

$$\text{i.e. } \frac{\partial \phi}{\partial r} = 0 \text{ on } r=a \qquad (\underline{\hat{n}} = \underline{\hat{r}} \text{ on } r=a).$$

i.e. it remains to solve

$$\nabla^2 \phi = 0$$
$$\phi \to Ur\cos\theta \text{ as } r \to \infty$$
$$\partial\phi/\partial r = 0 \text{ on } r=a$$

or/ with stream function

$$\nabla \cdot u = 0 \text{ plus 2D } \exists\, \psi \text{ s.t } u = \partial\psi/\partial y \quad v = -\partial\psi/\partial x$$

Irrotational (+2D) : $\partial v/\partial x - \partial u/\partial y = 0$

$$-\partial^2\psi/\partial x^2 - \partial^2\psi/\partial y^2 = 0 \quad \text{i.e. } \nabla^2\psi = 0.$$

On a solid boundary $\psi = \text{const}$. If only 1 boundary as in cylinder problem, we can take $\psi = 0$ on $r = a$.

The streamfunction for a uniform ~~sp~~ stream of speed U in x-dir. $\psi = Uy$

Thus the streamfunction satisfies

$$\nabla^2 \psi = 0$$

$$\psi \to Uy = Ur\sin\theta \quad \text{as } r \to \infty$$

$$\psi = 0 \quad \text{on } r = a$$

2D irrotational, incomp. flow can use either ϕ or ψ.

ϕ but no ψ : 3D.

ψ but no ϕ : rotational, $\zeta \neq 0$

In our problems can always use both / either $u = \nabla \phi$

$u = -\hat{z} \wedge \nabla \psi$

Thus $\quad u = \partial\phi/\partial x = \partial\psi/\partial y$

$\quad v = \partial\phi/\partial y = -\partial\psi/\partial x$

These are the Cauchy-Riemann equations

$$\partial\phi/\partial x = \partial\psi/\partial y \qquad \partial\phi/\partial y = -\partial\psi/\partial x$$

Thus the function $w(z)$ where z is the complex varible $x + iy$ is a differentiable function of z when

$$w(z) = \phi(x,y) + i\,\psi(x,y)$$

Thus ϕ and ψ are the real and im parts of an analytical function $w = \phi + i\psi$.

We call $w(z)$ the complex velocity potential $\operatorname{Re} w = \phi$

$\operatorname{Im} w = \psi$.

Thus any differentiable complex function represents a 2D incompressible, irrotational flow.

e.g. 1.

$$\omega(z) = Uz$$
$$= U(x+iy)$$
$$= Ux + iUy$$

so $\phi = \mathrm{Re}\,\omega = Ux$

$\psi = \mathrm{Im}\,\omega = Uy$ i.e. uniform stream. $u = \nabla\phi = U\hat{x}$.

e.g. 2

$$\omega(z) = z^2$$
$$= (x+iy)^2 = (x^2-y^2) + i2xy.$$

$\phi = x^2-y^2$

$\psi = 2xy$

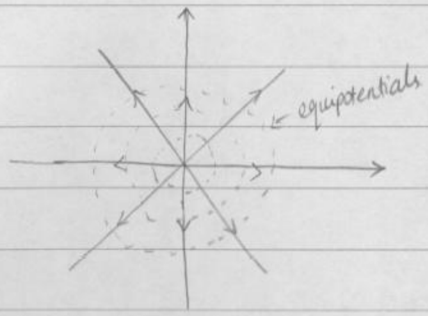
Streamlines, lines $\psi = \text{const}$ i.e. $xy = \text{const}$.

e.g 3

$$\omega(z) = m\log z$$
$$= m(\log|z| + i\arg z)$$
$$= m(\log r + i\theta) \qquad z = re^{i\theta}$$
$$= m\log r + im\theta$$

$\phi = m\log r \qquad \psi = m\theta.$

isotropic source of strength $2\pi m$.



$$u = \nabla \phi = \frac{\partial \phi}{\partial r}\,\hat{\underline{r}} + \frac{1}{r}\frac{\partial \phi}{\partial \theta}\,\hat{\underline{\theta}}.$$

$$= \frac{m}{r}\,\hat{\underline{r}}$$

$u =$

$$= -\hat{z} \wedge \nabla \psi$$

i.e. $\nabla\phi$ and $\nabla\psi$ have same magnitude $|\nabla\phi| = |\nabla\psi| = |u|$ and are $\perp r$
($\nabla\phi$ is $\nabla\psi$ rotated by $\pi/2$) (clockwise).
Thus the level curves of the real and imaginary parts of an holomorphic function are orthogonal. [except where $\nabla\phi = \nabla\psi = 0$ i.e. a stagnation point].

Now

$$u = \nabla\phi$$
$$= \partial\phi/\partial r\,\hat{r} + \frac{1}{r}\,\partial\phi/\partial\theta\,\hat{\underline{\theta}}.$$ gives $u_r = \partial\phi/\partial r$ and $u_\theta = 1/r\,\partial\phi/\partial\theta$

and $u_r = \frac{1}{r}\frac{\partial\psi}{\partial\theta}$ and $u_\theta = -\frac{\partial\psi}{\partial r}$

Thus
$$\left.\begin{aligned} \frac{\partial\phi}{\partial r} &= \frac{1}{r}\frac{\partial\psi}{\partial\theta} \\ \frac{\partial\psi}{\partial r} &= -\frac{1}{r}\frac{\partial\phi}{\partial\theta} \end{aligned}\right\}$$ C.R in polars

$$w = \phi(r,\theta) + i\psi(r,\theta)$$
$$z = re^{i\theta}.$$

Remember

$$\frac{dw}{dz} = \frac{\partial}{\partial x}(\phi + i\psi)$$
$$= \partial\phi/\partial x + i\,\partial\psi/\partial x = u - iv$$

$$u + iv = \overline{\frac{dw}{dz}}.$$

In example 1.

$$\omega = Uz$$

$$\frac{dw}{dz} = U \qquad \text{so } u = U \quad v = 0.$$

A stagnation point is a point where $u=0$ and $v=0$

i.e. $\frac{dw}{dz} = 0$.

In example 2

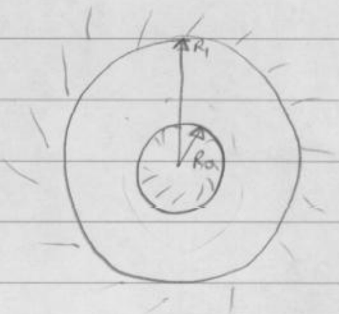
$$\omega = z^2$$

$$\frac{dw}{dz} = 2z$$

$$\frac{dw}{dz} = 0 \quad \text{at } z=0 \text{, a stagnation point.}$$

[All problems are most succinct in complex variables].

Laurent's Theorem.



If we have a function $f(z)$ analytic within an angular region $[R_0 < |z| < R_1]$ then f has the unique expansion of the form :

$$+ a_0 + a_1 z + a_2 z^2 + a_3 z^3 + \ldots\ldots$$

$$\ldots\ldots + \frac{a_{-3}}{z^3} + \frac{a_{-2}}{z^3} + \frac{a_{-1}}{z} +$$

We apply this to the complex velocity $u - iv$ (because we expect this to be bounded)
Now we must integrate to get ω:

$$\ldots + \frac{b_{-2}}{z^2} + \frac{b_{-1}}{z} + b\log z + b_0 + b_1 z + b_2 z^2 + \ldots$$

where b_i are complex constants.

Thus any ω we write down is a linear combination of $z^{\pm n}$ and $\log z$
i.e. we have proved that all solutions for ϕ and ψ are linear combinations of $\mathrm{Re}\,\omega$ and $\mathrm{Im}\,\omega$ i.e.

$$r^{\pm n}\cos n\theta,\ r^{\pm n}\sin n\theta,\ \log r,\ \theta$$

28th October 2010.

Laurent's Theorem.

Any function analytic in an annulus is simply a linear combination of the functions:

$$z^{\pm n} \quad n = 0, 1, 2, \ldots$$

$z, z^2, \ldots z^3, z^n$

Used on complex velocity $u - iv$

Integrated to get complex potential $w(z)$

All, $w(z)$ are linear combinations of $\log z, z^{\pm n}$

Or in polars:

ψ and ϕ are linear combinations of:

$$\log r,\ \theta,\ r^{\pm n}\cos n\theta,\ r^{\pm n}\sin n\theta.$$

Irrotational incomp. flow.

$\psi_1 = r^5 \cos 4\theta$ ✗

$\phi_2 = r^{-7}\cos 7\theta$ ✓

Example

Find the ideal fluid flow past a cylinder of radius a given that the flow at infinity is uniform with speed U.

Ans: Take the x-axis in the direction of flow at ∞. Let cylinder be at the origin.

Can use either Ψ or $\emptyset$, but if you have a choice use Ψ (since easy to draw s'lines).

$\emptyset$ velocity potential.

Ψ streamfunction.

$$\nabla^2\emptyset = 0 \qquad \nabla^2\Psi = 0$$

And $u = \nabla\emptyset$ or $u = -\underline{z} \wedge \underline{\nabla}\Psi$

Use Ψ.

In the far-field, $r \to \infty$, flow uniform in x-dir

i.e. $\Psi \to Uy$

On the cylinder, $r=a$, no normal flow i.e. $\Psi =$ const on $r=a$

(only one boundary so take $\Psi = 0$ on $r=a$).

Math problem:

Solve $\nabla^2\Psi = 0$ in $r > a$

with $\Psi \to Uy$ as $r \to \infty$

and $\Psi = 0$ on $r = a$.

Polar coord.

The inhomogeneous part of the problem is the far field

Here $\Psi \to Ur\sin\theta$ $r \to \infty$.

Look for a solution of the form:

$$\begin{aligned}\Psi = Ur\sin\theta &+ \cancel{a_4 r^4 \cos 4\theta} \text{ Useless} \\ &+ \cancel{a_{-1} r^{-1} \cos\theta} \text{ Useless} \\ &+ b_{-1} r^{-1} \sin\theta .\end{aligned}$$

$$\Psi = Ur\sin\theta + \frac{B}{r}\sin\theta$$

On $r=a$

$$\Psi = Ua\sin\theta + {}^{B}/_{a}\sin\theta$$

This vanishes $\forall\theta$ iff ${}^{B}/_{a} = -Ua$ i.e. $B = -Ua^2$

i.e. $\Psi = Ur\sin\theta - \frac{Ua^2}{r}\sin\theta$

$$= Ur\sin\theta\left[1 - {}^{a^2}/_{r^2}\right]$$

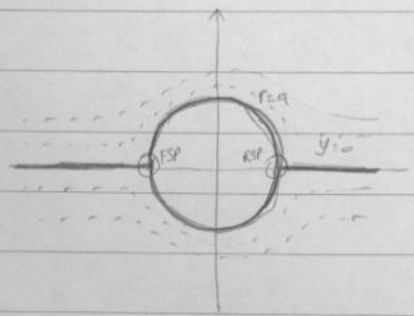
$$= Uy\left(1 - {}^{a^2}/_{r^2}\right)$$

Streamlines: $\Psi = \text{const}$

In part. $\Psi = 0$ when $Uy(1 - {}^{a^2}/_{r^2}) = 0$.

Hence $Uy = 0$ or $(1 - {}^{a^2}/_{r^2}) = 0$

i.e. $y = 0$ or $r = a$



FSP – FRONT STAG POINT

RSP – REAR STAG POINT

OR/

Use the velocity potential, ϕ

$\nabla^2\phi = 0 \quad r>a$

$\phi \to Ux$ as $r\to\infty$.

$\frac{\partial\phi}{\partial r} = 0$ on $r=a$.

$u = U\hat{x} = \nabla\phi$

Then $\phi = Ux$

On $r=a$ $\quad u\cdot\hat{\underline{n}} = 0$

i.e. $\hat{n}\cdot\nabla\phi = 0$

i.e. ${}^{\partial\phi}/_{\partial n} = 0$

${}^{\partial\phi}/_{\partial r} = 0$

Use polars.

$$\phi \to Ur\cos\theta \quad \text{as } r \to \infty$$

Try $\phi = Ur\cos\theta + \frac{A}{r}\cos\theta$

On $r = a$

$$\frac{\partial\phi}{\partial r} = U\cos\theta - \frac{A}{r^2}\cos\theta$$

Vanishes $\forall\theta$ on $r=a$ iff $U\cos\theta = A/r^2 \cos\theta$

i.e. $A = a^2 U$

Hence: $\phi = Ur\cos\theta\,(1 + a^2/r^2) = ax\,(1 + a^2/r^2)$.

1st November 2010.

Flow past cylinder.

Streamfunction $\Psi = Uy\left(1-a^2/r^2\right)$

Velocity potential $\varnothing = Ux\left(1+a^2/r^2\right)$

Complex potential $w = \varnothing + i\Psi = Ux + iUy + a^2/r^2\,(Ux - iUy)$

i.e. $w = U(x+iy) + \frac{Ua^2}{r^2}(x-iy)$

$$= Uz + \frac{Ua^2}{r^2}\bar{z}$$

$$= Uz + \frac{Ua^2\bar{z}}{z\bar{z}} = Uz + \frac{Ua^2}{z}$$ (Thanks to Laurent).

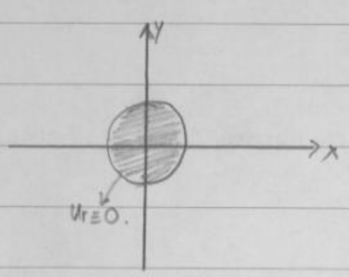
For $\Psi = Ur\sin\theta - \frac{Ua^2}{r}\sin\theta$

$$U_r = \frac{1}{r}\frac{\partial\Psi}{\partial\theta} = U\cos\theta\left(1-a^2/r^2\right)$$

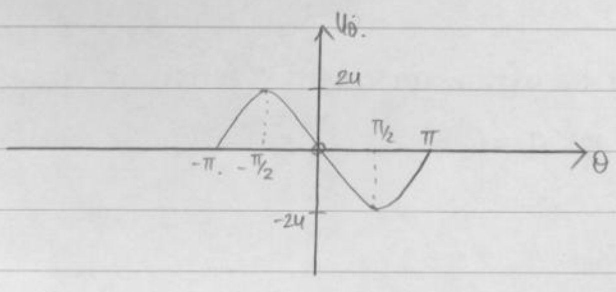
$$U_\theta = -\partial\Psi/\partial r = -U\sin\theta\left(1+a^2/r^2\right)$$

On cylinder, $|z| = a$ or $r = a$

$U_r = 0 \quad \forall\theta$ i.e. no normal velocity as expected.

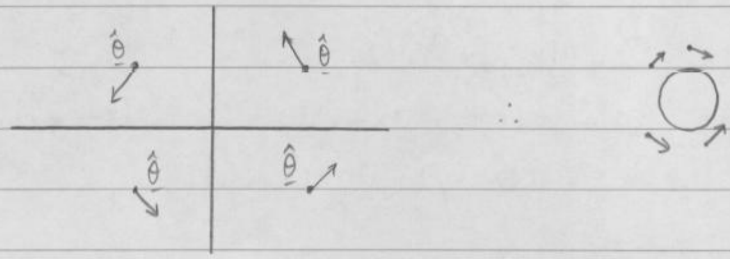


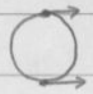
On cylinder, $U_\theta = -2U\sin\theta$



u_θ = component of u in $\hat{\underline{\theta}}$ direction.

i.e. direction of θ increasing.



Max speed:  is $\underline{u} = 2U\hat{\underline{x}}$ at $r=a$, $\theta = \pm\pi/2$

Stagnation points

When $u_\theta = 0$, $\theta = 0, -\pi$

Streamlines cut at right angles.

Streamlines in the neighbourhood of stagnation points

At stag. point $\underline{u} \equiv 0$ i.e. $\nabla\phi \equiv 0$ and $\nabla\psi \equiv 0$

so $\dfrac{dw}{dz} = u - iv = 0$

Suppose we have a stagnation point in a flow with complex potential $w(z)$.
Expand $w(z)$ as a series about this point :-

$$w(z) = a_0 + a_1 z + a_2 z^2 + a_3 z^3 + \dots$$

Since ϕ, ψ, w are only defined to within an additive constant, take $a_0 = 0$

But at stag. point $\dfrac{dw}{dz} = 0$ so $a_1 = 0$

Let the first non-zero coeff. be a_n $(n \geq 2)$

Now we can take a_n to be Real.

Consider: Az^n where $A = \alpha e^{i\phi}$. Then $Az^n = \alpha e^{i\phi} z^n$

$$\cancel{Az^n = \alpha e^{i\phi}} \quad Az^n = \alpha e^{i\phi} z^n$$
$$= \alpha e^{i\phi} (re^{i\theta})^n$$
$$= \alpha e^{i\phi} r^n e^{i\theta n}$$
$$= (\alpha^{1/n} r)^n e^{in(\theta + \phi/n)}$$

i.e. we rotate axes by an angle ϕ/n and magnify by factor $\alpha^{1/n}$.

The angle at which two streamlines cross is unchanged.

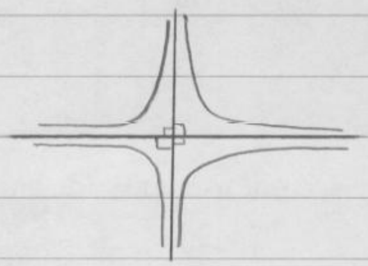
Thus w.l.o.g take A real.

Then $w \sim Az^n$

so $\psi = \operatorname{Im} w = Ar^n \sin n\theta$

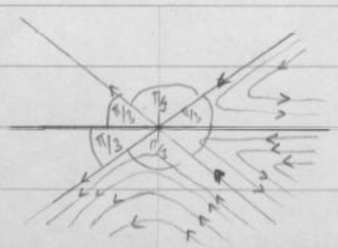
so $\psi = 0$ when $\theta = 0$, and next, with increasing θ, when $\theta = \pi/n$

e.g. $n=2$



Usual stag. point (2 streamlines crossing).

$n=3$



If n streamlines cross in irrotational flow aswell as incompressible, then they cross at π/n.

The Line Vortex.

We already have one special solution : the ~~lines~~ source of strength $2\pi m$ at the origin :

$$\psi = m\theta \qquad \phi = m\log r \qquad w = \phi + i\psi = m(\log r + i\theta) = m \log z$$

Consider the related solution obtained by 'swapping' $\phi + \psi$ i.e. make $\phi = k\theta$ and so $\psi = -k\log r$.

Check: $w = \phi + i\psi$
$= k\theta - ik\log r$
$= -ik(\log r + i\theta) = -ik\log z$.

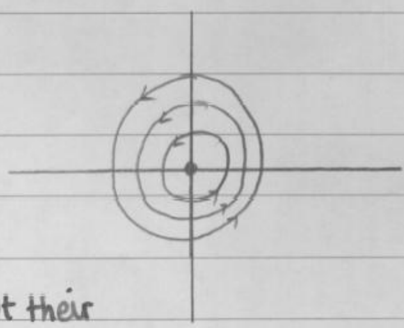
For this flow :

$$u = \nabla\phi \qquad \nabla\phi = \partial\phi/\partial r\, \hat{\underline{r}} + 1/r\, \partial\phi/\partial\theta\, \hat{\underline{\theta}}$$
$$= \frac{k}{r}\hat{\underline{\theta}}$$

i.e. always in the $\hat{\underline{\theta}}$ direction but decreasing with distance from the origin.

Streamlines $\psi = \text{const}$ i.e. $r = \text{const}$

A spinning flow - a line vortex.



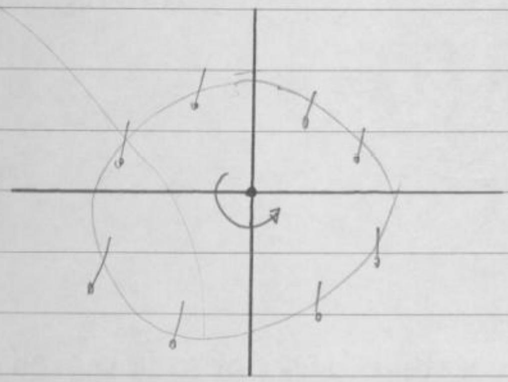
[Worry : Is this flow rotational?
No - individual elements do not rotate about their C.O.M].

$\phi = Ur\cos\theta + \frac{A}{r}\cos\theta + C\log r.$

$\nabla\phi \to U\hat{\underline{x}}$ ....(1) $\qquad \nabla(\log r) \to 0$ as $r\to\infty$.

$\underline{u} \to U\hat{\underline{x}}$ as $r\to\infty$. ....(2)

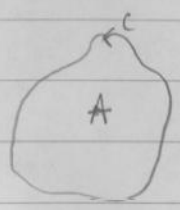
Use boundary condition (1) when surfaces or vortices.



We define the strength of a line vortex by its circulation.

$$\Gamma = \oint_c \underline{u}\cdot \underset{\sim}{dl}$$

around a given closed contour C, taken anticlockwise.



Let A be the area enclosed by C.

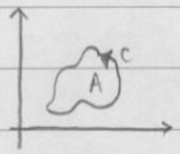
Then; $\Gamma = \oint_c u\cdot dl = \iint_A \hat{\underline{n}}\cdot(\nabla\wedge\underline{u})\,dA$

$= \iint_A \hat{\underline{z}}\cdot \zeta\hat{\underline{z}}\,dA$. in 2D flow.

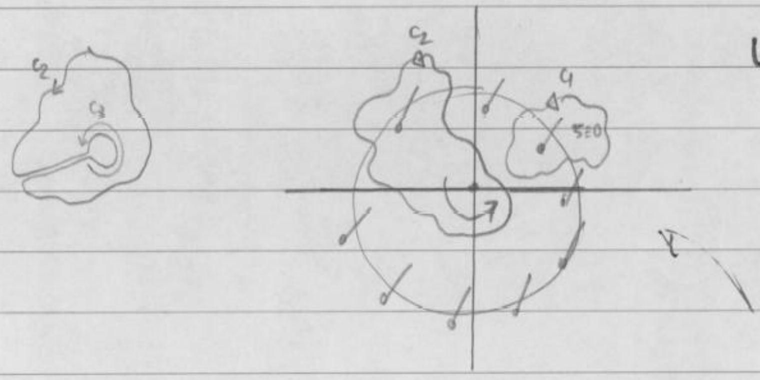
$= \iint_A \zeta\,dA$

i.e. The circulation adds up all the rotation of the individual elements enclosed.
But in irrotational flow $\zeta \equiv 0$. Hence circulation about any closed contour, ~~not~~ within

which u is differentiable, is zero.



$\Gamma = 0.$



$u = k/r\ \hat{\underline{\theta}}$ Line vortex.

The circulation about any curve that does not circle the origin, vanishes. Consider a new path consisting of old path C_2 connected to a small circle of radius ε, call this C_3, centred at the origin.

Then the area enclosed by C_2 plus C_3 contains no singularity i.e.

$$\oint_{C_1+C_3} \underline{u}\cdot d\underline{l} = 0$$

The two // lines cancel so

$$\underbrace{\oint_{C_2} u\cdot dl}_{\text{ANTICLOCKWISE}} + \underbrace{\oint_{C_3} u\cdot dl}_{\text{CLOCKWISE.}} = 0$$

$- dl = \varepsilon\, d\theta\, \hat{\underline{\theta}}$

i.e. $$\oint_{C_2} u\cdot dl = -\oint_{C_3} u\cdot dl = -\oint_0^{2\pi} \frac{k}{\varepsilon}\,\hat{\underline{\theta}}\,(\varepsilon\, d\theta(-\hat{\underline{\theta}}))$$

$$= k\oint_0^{2\pi} d\theta = 2\pi k$$

OR//

$$-\oint_{C_2}^{\text{CLOCKWISE}} u.dl = \oint_{C_3}^{\text{ANTICLOCKWISE}} u.dl = \int_0^{2\pi} \frac{k}{\varepsilon}\,\hat{\theta}\,(\varepsilon d\theta\,\hat{\theta})$$

Thus we have shown that the circulation about any contour cycling the origin is $2\pi k$
and any contour not cycling origin is zero.

c.f. line source: flux across any line cycling the origin is $2\pi m$ (for $w = m\log z$) and any not not cycling origin is zero.

Remember the only member of our set of solutions that has circulation is the line vortex: $w = -ik\log z$
The only solution that has mass flux is the line source, $w = m\log z$.

Flow past a cylinder with circulation.

This has complex potential:

$$w(z) = Uz\left(1 + a^2/r^2\right) - ik\log z.$$

This has circulation $2\pi k$ about the cylinder.

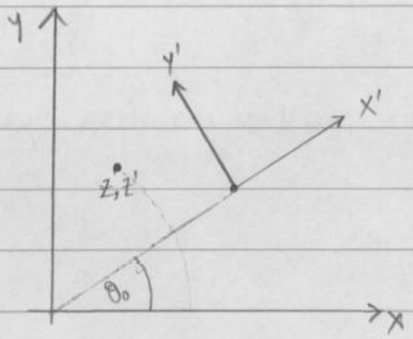
$$u - iv = dw/dz = U\left(1 - a^2/z^2\right) - i\,k/z$$

$$\rightarrow U \text{ as } z \rightarrow \infty$$

i.e. $u \rightarrow U$ $v \rightarrow 0$ i.e. uniform stream at ∞.

Can we check $U_r = 0$ on $r = a$ with complex variables?

Yes.



Consider a point $P = r_0 e^{i\theta_0}$

$$\text{Then } \; U_r - iU_\theta = \frac{dw}{dz'} = \frac{dw}{dz} \cdot \frac{dz}{dz'}$$

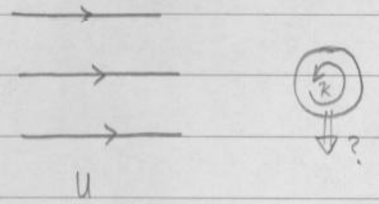
$$|z| = |z'| \; , \; \arg z = \arg z' + \theta_0 \quad \Rightarrow \quad z = z' e^{i\theta_0}$$

$$\frac{dz}{dz'} = e^{i\theta_0}$$

Polar component from complex $w(z)$

$$U_r - iU_\theta = e^{i\theta} \, dw/dz$$

4th November 2010.



Consider a cylinder of radius a, in a uniform stream of speed U (in the x-dir) with circulation k about the cylinder.

The complex velocity potential for this flow is:

$$w(z) = Uz + \frac{Ua^2}{z} - \frac{ik}{2\pi}\log z$$

$$\left[\text{or } \quad \phi = Ux\left(1+ {a^2}/{r^2}\right) + {k}/{2\pi}\,\theta \right.$$
$$\left. \text{or } \quad \psi = Uy\left(1- {a^2}/{r^2}\right) - {k}/{2\pi}\log r \right]$$

Check

$$\frac{dw}{dz} = U - \frac{Ua^2}{z^2} - \frac{ik}{2\pi z}$$

As $z \to \infty$ $\quad {dw}/{dz} = u - iv \to U \quad$ i.e. $u \to U \quad v \to 0$

On $r = a$ i.e. $|z| = a$ i.e. $z = ae^{i\theta}$ (cylinder)

Must check $u_r = 0$ on $z = ae^{i\theta}$

But $\quad u_r - iu_\theta = \dfrac{dw}{dz}e^{i\theta}$

On $z = ae^{i\theta}$,

$$u_r - iu_\theta = \left[U - \frac{Ua^2}{a^2e^{2i\theta}} - \frac{ik}{2\pi a e^{i\theta}} \right] e^{i\theta}$$

$$= Ue^{i\theta} - Ue^{-i\theta} - \frac{ik}{2\pi a}$$

$$= U(e^{i\theta} - e^{-i\theta}) - \frac{ik}{2\pi a}$$

$$= 2iU\sin\theta - \frac{ik}{2\pi a}$$

So $U_r = 0$ (no normal velocity on cyl as expected)
and $U_\theta = -2U\sin\theta + k/2\pi a$.

Where are the stagnation points?

If $k = 0$
stag. point at $\theta = 0, \pi$

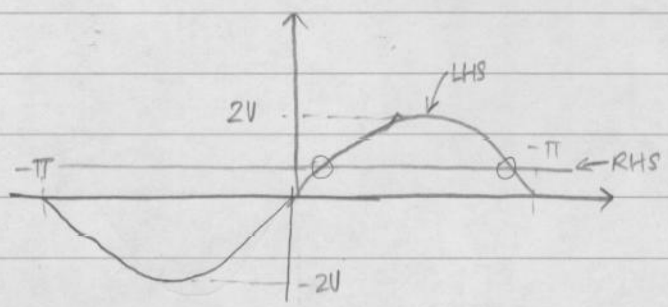
Notice $\frac{dw}{dz} = u - iv$

$$= U - \frac{Ua^2}{z^2} - \frac{ik}{2\pi z}$$

$$\frac{dw}{dz} = 0 \quad \text{when} \quad Uz^2 - Ua^2 - \frac{ikz}{2\pi} = 0$$

Thus there are two stagnation points in general.

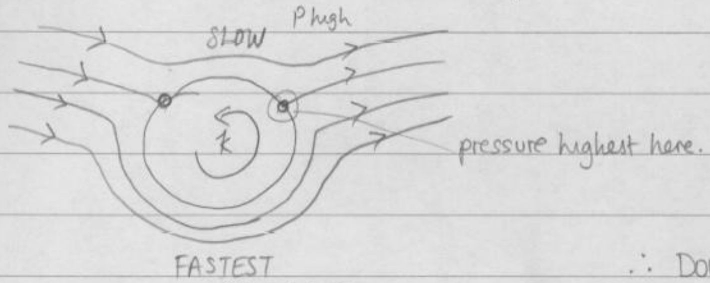
It is sufficient to find points on cyl where $U_\theta = 0$. i.e.
i.e. $2U\sin\theta = k/2\pi a$



Thus for $0 < {}^{k}/_{4\pi Ua} < 1$

there are two stagnation points at $y = a\sin\theta$

$= {}^{k}/_{4\pi U}$



$\therefore$ Downward force

Now if ${}^{k}/_{4\pi aU} = 1$

The two stagnation points merge at $y = {}^{k}/_{4\pi U} = a$

If ${}^{k}/_{4\pi aU} > 1$.

No stagnation points on the cylinder.

One stag point lies on $x = 0$ inside cylinder and other lie's outside

How do we find this S.P?

$$0 = U - \frac{Ua^2}{z^2} + \frac{ik}{2\pi z}$$

Solve this for z

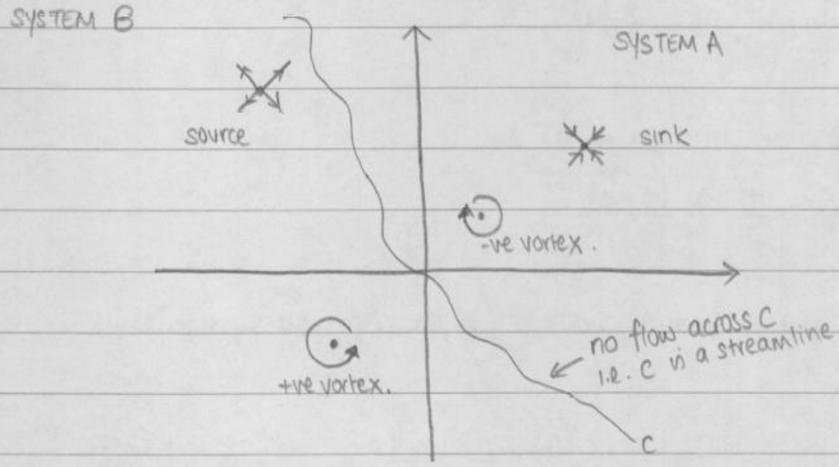
i.e. write $z = iy$

$$0 = U + {}^{Ua^2}/_{y^2} + {}^{k}/_{2\pi y} \quad \text{solve for } y.$$

15th November 2010

The method of images.

Def: If a motion of a fluid in the x-y plane is due to a distribution of singularities, (sources, sink, vortices i.e. z^{-n}) and there is a curve C drawn in that plane ~~then~~ such that there is no flow across C, then the system of singularities on one side of C is called the IMAGE of the system of singularities on the other side.

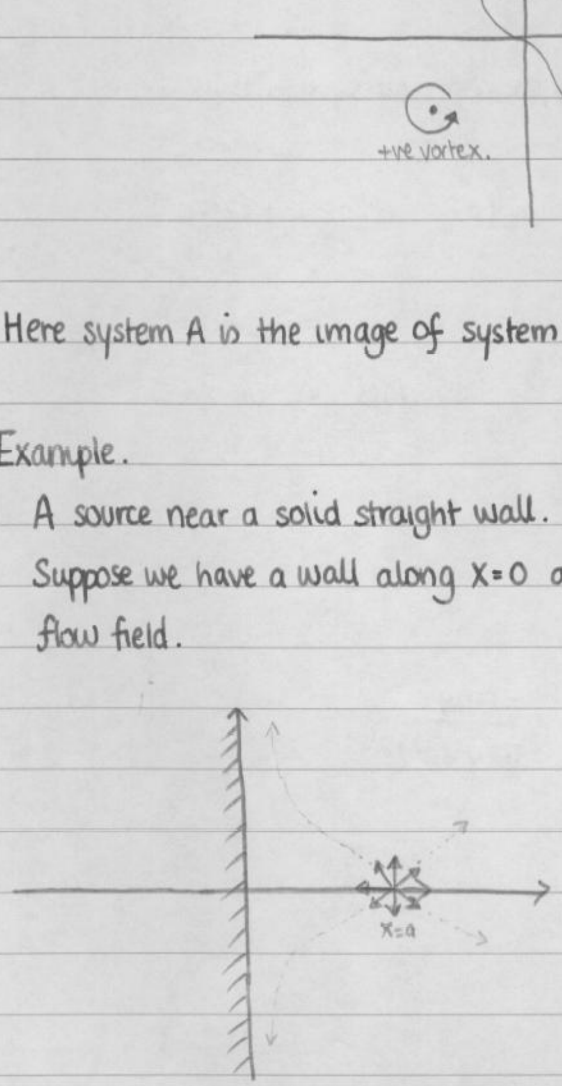


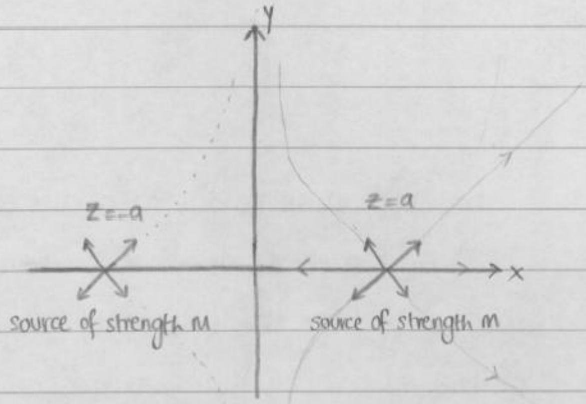
Here system A is the image of system B and system B is the image of system A.

Example.

A source near a solid straight wall.

Suppose we have a wall along $x=0$ and source of strength m at $x=a$. What is the flow field.





System A : Source of strength m at $z=a$

$$w_1 = \frac{m}{2\pi}\log(z-a)$$

System B : Source of strength m at $z=-a$

$$w_2 = \frac{m}{2\pi}\log(z+a)$$

System in whole plane, whole system, system A and system B is :

$$w = w_1 + w_2 = \frac{m}{2\pi}\log(z-a) + \frac{m}{2\pi}\log(z+a) = \frac{m}{2\pi}\log(z^2-a^2)$$

We just guessed the image. Are we right ? i.e. does $u=0$ on $x=0$

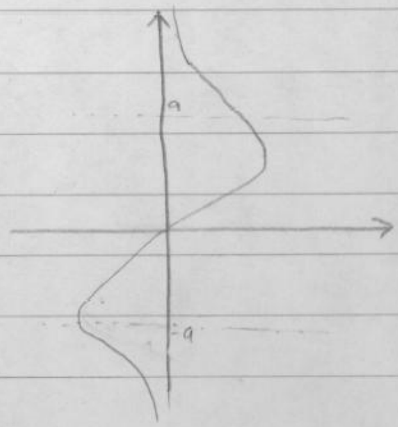
Check : $\dfrac{dw}{dz} = \dfrac{m}{2\pi}\cdot\dfrac{1}{z^2-a^2}\cdot 2z$

On $x=0$

$(z=iy)$ $\quad \dfrac{dw}{dz} = \dfrac{m}{2\pi}\dfrac{1}{-y^2-a^2}\cdot 2iy = \dfrac{-imy}{\pi(y^2+a^2)}$

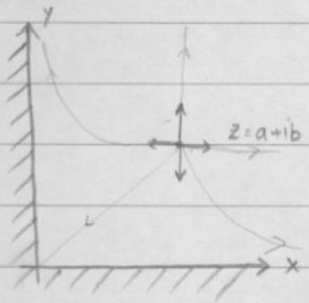
But $\dfrac{dw}{dz} = u-iv$

so $u=0$ and $v = \dfrac{my}{\pi(y^2+a^2)}$



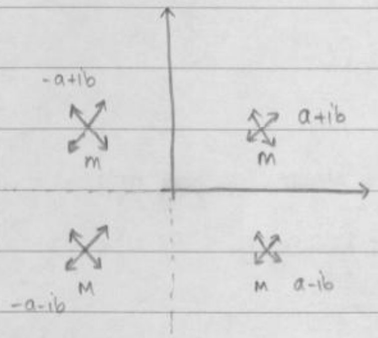
Example 2.

A source in a quater plane.



Consider a source of strength m at $z = a+ib$ in the quater plane, $x>0, y>0$.

What is the flow field.



Original system

$$w_1 = \frac{m}{2\pi} \log\left[z-(a+ib)\right]$$

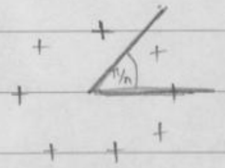
$$w_2 = \frac{m}{2\pi} \log\left[z-(-a+ib)\right] + \frac{m}{2\pi} \log\left[z-(-a-ib)\right] + \frac{m}{2\pi} \log\left[z-(a-ib)\right]$$

Full complex potential for the flow:

$$w = w_1 + w_2 = \frac{m}{2\pi} \log\left[z-(a+ib)\right] + \frac{m}{2\pi} \log\left[z-(-a+ib)\right] + \dots$$

$$\stackrel{?}{=} \frac{m}{2\pi} \log\left[z^4-(a+ib)^4\right]$$

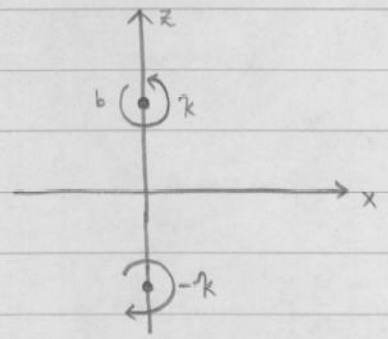
Example 3.



Source in wedge of angle π/n (n integer).

Example 4

Vortex of strength k at $z = ib$ above plane $y = 0$



Original system

$$w_1 = -\frac{ik}{2\pi} \log(z - ib)$$

Image system: Vortex at the optical image point $z = -ib$ of strength $-k$

$$w_2 = \frac{ik}{2\pi} \log(z + ib)$$

Thus the flow field is:

$$w = w_1 + w_2 = -\frac{ik}{2\pi} \log\left[\frac{z - ib}{z + ib}\right]$$

Check

$$w = \phi + i\psi \qquad \psi = \mathrm{Im}\, w$$

On $y = 0$, $z = x$

$$\psi = \mathrm{Im}\left[-\frac{ik}{2\pi} \log\left(\frac{x - ib}{x + ib}\right)\right]$$

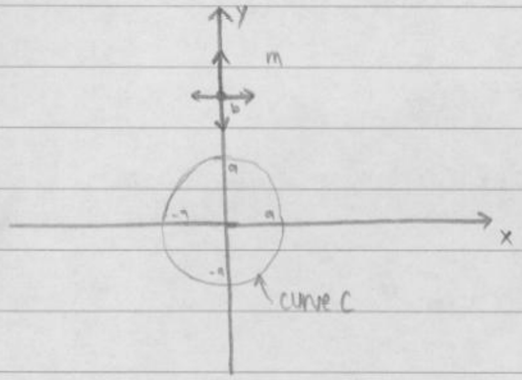
$$= -\frac{k}{2\pi} \log\left|\frac{x - ib}{x + ib}\right| = -\frac{k}{2\pi} \log 1 = 0$$

i.e. $y = 0$ is a streamline.

Example 5.

Find the flow outside a cylinder of radius a due to a source of strength m at $z = ib$.

$b > a$.



We need the image system inside C of the singularities outside C.

Original system

$w_1 = \frac{m}{2\pi} \log(z - ib)$

Image system

$w_2 =$

Optical image point is at : $z = ia^2/b$

Use the circle theorem : The imagine system in a circle $|z| = a$ of the complex potential $w_1 = f(z)$ that has no singularities in $|z| < a$ is : $w_2 = \bar{f}(a^2/z)$.

where for any complex function $g(z)$: $\bar{g}(z) = \overline{g(\bar{z})}$

Notice for

$g(z) = \ldots\ldots + a_{-1} z^{-1} + a_0 + a_1 z + \ldots\ldots$

$g(\bar{z}) = \ldots\ldots + a_{-1} \bar{z}^{-1} + a_0 + a_1 \bar{z} + \ldots\ldots$

$\overline{g(\bar{z})} = \ldots\ldots + \bar{a}_{-1} z^{-1} + \bar{a}_0 + \bar{a}_1 z + \ldots\ldots$ a power series in z, an analytic fn of z.

Proof.

Original system : $w_1 = f(z)$

Image system : $w_2 = \bar{f}(a^2/z)$

To prove : $w = w_1 + w_2$ has no flow across $|z| = a$

Notice w_1 has no singularities in $|z| < a$.

When $|z| < a$, $|a^2/z| > a$ so w_2 has no singularities in $|z| > a$

i.e. only sings in w_2 are in $|z| < a$

i.e. w_2 is a candidate for an image system

On C, i.e. $|z| = a$ i.e. $z = ae^{i\theta}$

$$\psi = \mathrm{Im}[w] = \mathrm{Im}[w_1 + w_2]$$

$$= \mathrm{Im}\left[f(ae^{i\theta}) + \bar{f}\left(a^2/ae^{i\theta}\right)\right]$$

$$= \mathrm{Im}\left[f(ae^{i\theta}) + \overline{f(ae^{i\theta})}\right] = 0$$

i.e. $\psi = 0$ on $|z| = a$ as required i.e. $r = a$ is a streamline.

Return to our example

$$w_1 = \frac{m}{2\pi}\log(z - ib) = f(z)$$

$$\bar{f}(z) = \frac{m}{2\pi}\log(z + ib)$$

$$w_2 = \bar{f}(a^2/z) = m/2\pi \log(a^2/z + ib)$$

Thus flow field

$$w = w_1 + w_2 = \frac{m}{2\pi}\log(z - ib) + \frac{m}{2\pi}\log(a^2/z + ib).$$

The image system is : $w_2 = \dfrac{m}{2\pi}\log(a^2/z + ib)$

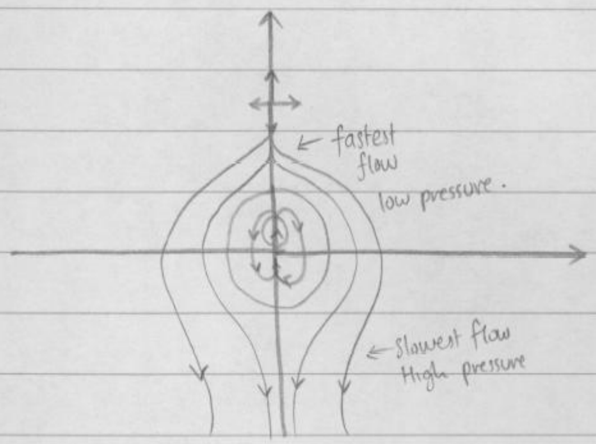
$$\Rightarrow \quad {}^{a^2}/_{z} + ib = \frac{a^2 + ibz}{z} = \frac{ib}{z} \cdot \left(z + \frac{a^2}{ib}\right)$$

$$\therefore \quad w_2 = \frac{m}{2\pi} \log(ib) - \frac{m}{2\pi} \log(z) + \frac{m}{2\pi} \log\left(z - i\,{}^{a^2}/_{b}\right)$$

nothing, no flow constant in ψ — sink of strength m of origin — source strength m at optical point.

i.e.

$$w = \frac{m}{2\pi} \log(z - ib) + \frac{m}{2\pi} \log\left(z - i\,{}^{a^2}/_{b}\right) - \frac{m}{2\pi} \log z + \frac{m}{2\pi} \log(ib)$$



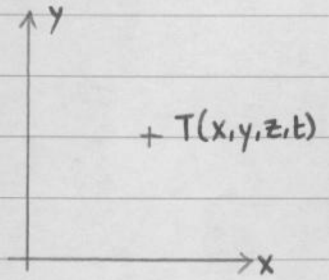
18th November 2010

Equation of motion

$$\underset{\sim}{F} = \frac{d}{dt}(m\underline{v})$$

Particle: force = rate of change of momentum.

Rate of change following a particle in a fluid.



What is the time r.o.ch of a general quantity $T(t, \underline{r})$ following a particle?

Let the particle path be $r(t)$

(Note $\partial T/\partial t$ is the time r.o.ch at a fixed point)

We use the notation $\frac{DT}{Dt}$ as time r.o.ch following a particle.

$$\frac{DT}{Dt} = \frac{d}{dt}\left[T(t, \underline{r}(t))\right]$$

$$= \frac{\partial T}{\partial t} + \frac{\partial T}{\partial x}\frac{dx}{dt} + \frac{\partial T}{\partial y}\frac{dy}{dt} + \frac{\partial T}{\partial z}\frac{dz}{dt}$$

$$= \frac{\partial T}{\partial t} + u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} + \omega\frac{\partial T}{\partial z}$$

$$= \frac{\partial T}{\partial t} + (u\hat{\underline{i}} + v\hat{\underline{j}} + \omega\hat{\underline{k}}) \cdot \left(\frac{\partial T}{\partial x}\hat{i} + \frac{\partial T}{\partial y}\hat{j} + \frac{\partial T}{\partial z}k\right)$$

$$= \frac{\partial T}{\partial t} + \underline{u} \cdot \nabla T$$

$$= \left(\frac{\partial}{\partial t} + u \cdot \nabla\right) T$$

Where the operator: $\frac{\partial}{\partial t} + \frac{u\partial}{\partial x} + \frac{v\partial}{\partial y} + \frac{\omega\partial}{\partial z}$

Examples.

Example 1.

$$T = x \qquad \frac{Dx}{Dt} = \cancel{\frac{\partial x}{\partial t}} + u\frac{\partial x}{\partial x} + v\cancel{\frac{\partial x}{\partial y}} + \omega\cancel{\frac{\partial x}{\partial z}}$$

$$= u$$

Example 2

$$T = \underline{r} \qquad \frac{D\underline{r}}{Dt} = \frac{\partial \underline{r}}{\partial t} + u\frac{\partial \underline{r}}{\partial x} + v\frac{\partial \underline{r}}{\partial y} + \omega\frac{\partial \underline{r}}{\partial z}$$

$$= \frac{\partial \underline{r}}{\partial t} + (\underline{u}\cdot\nabla)\underline{r}$$

$$= \frac{D}{Dt}(x\hat{i} + y\hat{j} + z\hat{k})$$

$$= \frac{Dx}{Dt}\hat{i} + x\frac{D\hat{i}}{Dt} + \frac{Dy}{Dt}\hat{j} + \frac{Dz}{Dt}\hat{z}$$

$$= u\hat{i} + v\hat{j} + \omega\hat{k} = \underline{u}$$ i.e. time r.o.ch of position is velocity.

Example 3

$$T = \underline{u} \qquad \frac{D\underline{u}}{Dt}$$

Time rate of change of velocity following the fluid. i.e. acceleration.

$$\frac{D\underline{u}}{Dt} = \frac{D}{Dt}(u\hat{i} + v\hat{j} + \omega\hat{k})$$

$$= \frac{Du}{Dt}\hat{i} + \frac{Dv}{Dt}\hat{j} + \frac{D\omega}{Dt}\hat{k}$$

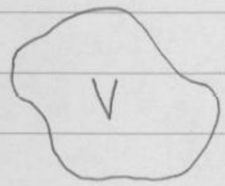
i.e. This is : $\frac{D\underline{u}}{Dt} = \left[\frac{\partial}{\partial t} + \underline{u}\cdot\underline{\nabla}\right]\underline{u}$

$$= \frac{\partial u}{\partial t} + (\underline{u}\cdot\underline{\nabla})\,\underline{u}$$

Newton laws :

$F = {}^{d}/_{dt}\,(m\underline{v})$

We wish to write down an equation that says "The time r.o.ch of the momentum of V = total force acting on V"



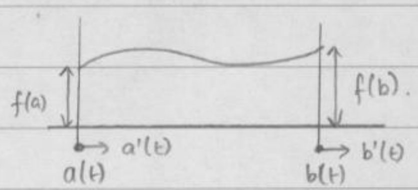
Note : we require V to be always composed of the same fluid. We need a mathematical expression for the time r.o.ch of a quantity following a volume always composed of the fluid particles.
This is given by : The Reynold's Transport Theorem (RTT)

1D : $I(t) = \int_a^b f(t,x)\,dx$

$$\frac{dI}{dt} = \int_a^b \frac{\partial f}{\partial t}(t,x)\,dx$$ provided a, b const.

If $a = a(t)$ $b = b(t)$

$$I = \int_{a(t)}^{b(t)} f(t,x)\,dx$$



then

$$\frac{dI}{dt} = \int_a^b \frac{\partial f}{\partial t}(t,x)\,dx + f(b)\,b'(t) - f(a)\,a'(t)$$

RTT is this in 3D.

Consider a quantity $\alpha(r,t)$ associated with a fluid. Consider a volume $V(t)$ always made up of the same particles. Let surface of $V(t)$ be $S(t)$. Let the velocity field be $\underline{u}(\underline{r},t)$. Consider the time-dependent integral:

$$I(t) = \int_{V(t)} \alpha(r,t)\, dV$$

What is $\frac{dI}{dt}$

$$\frac{dI}{dt} = \lim_{\delta t \to 0} \frac{I(t+\delta t) - I(t)}{\delta t}$$

$$= \lim_{\delta t \to 0} \frac{1}{\delta t} \left\{ \int_{V+\delta V} \alpha(r, t+\delta t)\, dV - \int_V \alpha(r,t)\, dV \right\}$$

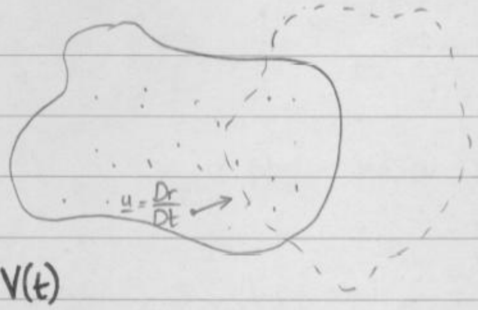
$$= \lim_{\delta t \to 0} \left\{ \int_V \alpha(r, t+\delta t)\, dV + \int_{\delta V} \alpha(r, t+\delta t)\, dV - \int_V \alpha(r,t)\, dV \right\}$$

We are going to take $\delta t \to 0$, so expand α in a Taylor series

$$\alpha(\underline{r}, t+\delta t) = \alpha(\underline{r},t) + \delta t \frac{\partial \alpha}{\partial t}(\underline{r},t) + O((\delta t)^2)$$

22nd November 2010.

RTT



$V(t+\delta t) = V + \delta V$

$V(t)$

$$I(t) = \int_{V(t)} \alpha(x,y,z,t)\,dx\,dy\,dt = \int_V \alpha \, dV$$

$$\frac{DI}{Dt} = \lim_{\delta t \to 0} \frac{I(t+\delta t) - I(t)}{\delta t}$$

$$= \lim_{\delta t \to 0} \frac{1}{\delta t} \left\{ \int_{V+\delta V} \alpha(t+\delta t)\,dV - \int_V \alpha(t)\,dV \right\}$$

Because we take limit $\delta t \to 0$, expand α in δt:

$$\cancel{\alpha(t)}\ \alpha(t+\delta t) = \alpha(t) + \partial\alpha/\partial t\,(t) \cdot \delta t + \tfrac{1}{2}\,\partial^2\alpha/\partial t^2\,(t)(\delta t)^2 + \ldots$$

Then

$$\frac{DI}{Dt} = \lim_{\delta t \to 0} \frac{1}{\delta t} \left\{ \int_{V+\delta V} \left(\alpha + \frac{\partial \alpha}{\partial t}\delta t\right) dV - \int_V \alpha \, dV \right\}$$

where all α's eval. at time t.

$$= \lim_{\delta t \to 0} \frac{1}{\delta t} \left\{ \int_V \alpha \, dV + \delta t \int_V \frac{\partial \alpha}{\partial t} dV + O(\delta t^2) + \int_{\delta V} \alpha \, dV + \delta t \int_{\delta V} \frac{\partial \alpha}{\partial t} dV + O(\delta t^2) \right.$$

$$\left. - \int_V \alpha \, dV \right\}$$

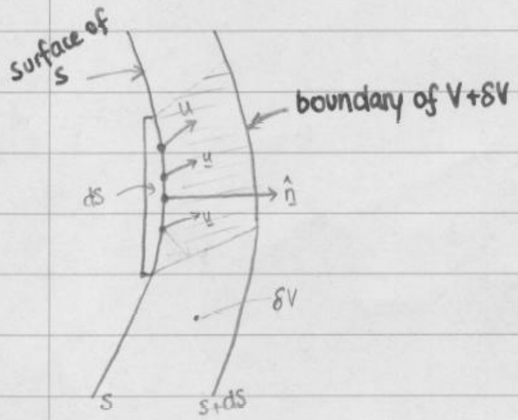
1.e

$$\frac{DT}{Dt} = \int_{V} \frac{\partial \alpha}{\partial t}\, dV + \lim_{\delta t \to 0} \int_{\delta V} \frac{1}{\delta t}\, \alpha \, dV + \lim_{\delta t \to 0} \int_{\delta V} \frac{\partial \alpha}{\partial t}\, dV$$

Now consider the final term. Suppose $\left| \partial\alpha/\partial t \right|$ in δV is bdd by K

Then $\lim_{\delta t \to 0} \left| \int_{\delta V} \partial\alpha/\partial t \, dv \right| \le \lim_{\delta t \to 0} \int_{\delta V} K \, dv = K \lim_{\delta t \to 0} |\delta V| = 0$ since δV vanishes as δt.

The last term:

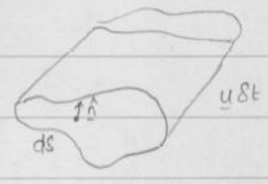


Consider a small element dS with outward normal $\hat{\underline{n}}$ of the surface S bounding the volume V. As the fluid particles lying on this element move $\underline{u}\delta t$ in time δt, they map out an element.

$$dV = (\underline{u} \cdot \hat{\underline{n}})\, \delta t \, dS$$

of the volume δV

[the volume element is a cylinder



so its volume

$= \text{area of base} \times \text{height}$
$= dS \, (\underline{u} \cdot \hat{\underline{n}})\, \delta t$]

This is simply the flux of volume through dS in time δt.

Thus

$$\lim_{\delta t \to 0} \frac{1}{\delta t} \int_{\delta V} \alpha \, dV$$

$$= \lim_{\delta t \to 0} \frac{1}{\delta t} \int_S \underbrace{(u.\underline{\hat{n}})\,\delta t\, dS}_{dV} . \, \alpha$$

$$= \int_S \alpha \, (\underline{u} \cdot \underline{\hat{n}}) \, dS$$

i.e. the flux of α through the surface S.

i.e.

flux of α out of V

$$\frac{D}{Dt} \int_V \alpha \, dV = \int_V \frac{\partial \alpha}{\partial t} \, dV + \int_S \alpha \, (u.\underline{\hat{n}}) \, dS \qquad \text{(RTT1)}$$

r.o.ch of α in V following V

local r.o.ch inside V

Notice $\int_S (\alpha \underline{u}) . \hat{n} \, dS = \int_V \underline{\nabla} \cdot (\alpha \underline{u}) \, dV$ by Gauss.

So we have RRT2

$$\frac{D}{Dt} \int_V \alpha \, dV = \int_V \left[\frac{\partial \alpha}{\partial t} + \underline{\nabla} \cdot (\alpha \underline{u}) \right] dV$$

Now $\underline{\nabla} . (\alpha \underline{u}) = \alpha \underline{\nabla} \cdot \underline{u} + u \cdot \underline{\nabla} \alpha$ (so $\partial \alpha / \partial t + \nabla \cdot (\alpha \underline{u}) = \underbrace{\partial \alpha / \partial t + u \cdot \nabla \alpha}_{} + \alpha \nabla \cdot \underline{u}$)

$$= \frac{D\alpha}{Dt} + \alpha \underline{\nabla} \cdot \underline{u}$$

so we have RTT3 :

$$\frac{D}{Dt} \int_V \alpha \, dV = \int_V \left[\frac{D\alpha}{Dt} + \alpha \underline{\nabla} \cdot \underline{u} \right] dV$$

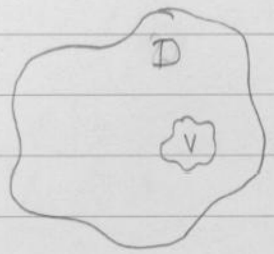
r.o.ch following V

sum of r.o.ch of particle in V

non-zero for compressible flow

Example

Conservation of Mass (for possible compressible fluid).



Consider a fluid of density $\rho(x,y,z,t)$ that occupies a region D. Take an arbitrary subregion V of D.
Follow the particles composing this region forward in time.
Use the RTT2 on the quantity $\rho(x,y,z,t)$.

Then

$$\underbrace{\frac{D}{Dt}\int_V \rho \, dV}_{\text{Total mass of particles comprising of V.}} = \int_V \left[\partial\rho/\partial t + \underline{\nabla}\circ(\rho\underline{u}) \right] dV$$

But mass is conserved so its r.o.ch following same particle is zero.

i.e. we have shown for $\underline{\underline{\text{all}}}$ subregions (since V is arbitrary) V of D.

$$\int_V \left[\frac{\partial\rho}{\partial t} + \nabla\cdot(\rho\underline{u}) \right] dV = 0$$

The only way this can be true is if the integrand vanishes everywhere i.e.

$$\frac{\partial\rho}{\partial t} + \underline{\nabla}\circ(\rho\underline{u}) = 0 \text{ in } D. \qquad \text{(conservation of mass).}$$

This is :

$$\frac{\partial\rho}{\partial t} + (u\circ\nabla)\rho + \rho\nabla\circ\underline{u} = 0$$

i.e. $$\frac{D\rho}{Dt} + \rho(\nabla\circ\underline{u}) = 0$$

If flow is incompressible particles do not change their volume. They don't change mass.

So particles in incomp. flow maintain their density ρ.

i.e. $\frac{D\rho}{Dt} = 0$ so our equation gives $\nabla \cdot \underline{u} = 0$.

[notice this is more general than our previous result as it does not require $\rho = \text{const.}$].

RTT4:

Consider: $$\frac{D}{Dt}\int_V (\rho f)\, dV = \int_V \left[\frac{\partial}{\partial t}(\rho f) + \underline{\nabla} \cdot (\rho f \underline{u})\right] dV \qquad \alpha = \rho f \text{ using RTT2.}$$

$$= \int_V \rho \frac{\partial f}{\partial t} + f\frac{\partial \rho}{\partial t} + f\,\nabla \cdot (\rho \underline{u}) + \rho \underline{u} \cdot \nabla f \quad dV$$

$$= \int_V f\Big(\overbrace{\cancel{\tfrac{\partial \rho}{\partial t} + \nabla \cdot (\rho \underline{u})}}^{\text{C.O.M}}\Big) + \rho\left(\frac{\partial f}{\partial t} + u \cdot \nabla f\right) dV$$

$$= \int_V \rho \frac{Df}{Dt}\, dV = \int_V \frac{Df}{Dt}\, \underline{\rho\, dV}$$ ← mass of small element.

i.e.

$$\frac{D}{Dt}\int_V f \rho\, dV = \int_V \frac{Df}{Dt}\rho\, dV \qquad \text{RTT4.}$$

Example : Newton (Euler)

Consider a fluid of density $\rho(x,y,z,t)$ with velocity field $\underline{u}(x,y,z,t)$ occupying a domain D. Take an ARBITRARY subregion V of D with surface S. The total momentum of - all particles comprising V is :

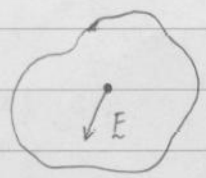
$$\underset{\sim}{m} = \int_V \underline{u} \rho \, dV$$

By RTT4 :

$$\frac{Dm}{Dt} = \frac{D}{Dt}\int_V \underline{u} \rho dV = \int_V \frac{D\underline{u}}{Dt} \rho \, dV$$

Newton : R.o.ch of momentum = total force acting on particles comprising V.

Let there be an external force $\underline{F}$ per unit mass acting on each fluid particle.



e.g.

gravity $F = -g\hat{z}$ for $\hat{z}$ vertically upwards.

electric field, magnetic field

Inviscid : no tangential force on surface. But there is a normal force, the pressure, (force per unit area),

$-p\hat{\underline{n}} dS$ on an element dS of the surface of V.

Total force acting on V is

$$\int_V \underset{\sim}{F} \rho \, dV + \int_S -p\underline{\hat{n}} \, dS = \int_V \underset{\sim}{F} \rho \, dV - \int_V \underline{\nabla} p \, dV$$

$$= \int_V (\rho \underline{F} - \nabla p) \, dV$$

Using :- $\int_V \nabla G = \int_S G \underline{\hat{n}} \, dS$: vector div. thm.

R.o.ch = Force acting

$$\int_V \frac{D\underline{u}}{Dt} \rho \, dV = \int_V (-\nabla p + \rho \underset{\sim}{F}) \, dV$$

i.e. $\displaystyle\int_V \left(\rho \frac{D\underline{u}}{Dt} + \underline{\nabla} p - \rho \underset{\sim}{F}\right) dV = 0$

But V is arbitrary so this is true for all V.

Hence integrand must vanish everywhere in D

i.e. $\rho \frac{Du}{Dt} + \nabla p - \rho F = 0$ in D.

i.e. $\displaystyle\frac{D\underline{u}}{Dt} = -\frac{1}{\rho} \underline{\nabla} p + \underset{\sim}{F}$ Euler's equations.

Momentum eqn. for the fluid.

Accel. pressure gradient external forces.

Together with conservation of mass :

$$\partial\rho/\partial t + \nabla\cdot(\rho\underline{u}) = 0$$

we have 4 scalar equations (or 1 vector + 1 scalar) in 5 unknowns $\underline{u}, p, \rho$

(1) Gas dynamics : $p = f(\rho)$

(2) Here in GFD : Incompressible $\rightsquigarrow$ cons. of mass splits : (1). $\dfrac{D\rho}{Dt} = 0$

(2). $\underline{\nabla}\cdot\underline{u} = 0$

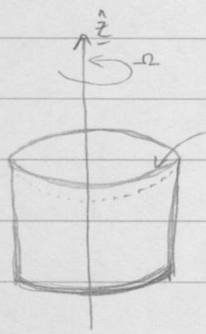
(3) Or even more simply, take ρ=const.

ρ= const. $\qquad \dfrac{D\underline{u}}{Dt} = -\dfrac{1}{\rho}\underline{\nabla}p + F$

$\underline{\nabla}\cdot\underline{u} = 0 \qquad$ 4 equations in 4 unknowns.

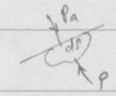
Example

Find the shape of the free surface of a partially filled cylinder in solid body rotation.



boundary condition on the surface is : " the pressure in the fluid at the surface must balance the constant atmospheric pressure, p_a "

i.e. $p = p_a$, const on surface.



The fluid is in solid body rotation

$$\underline{u} = \underline{\Omega} \wedge \underline{r} \quad \text{with} \quad \underline{\Omega} = \Omega \hat{\underline{z}}$$

$$\underline{u} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 0 & 0 & \Omega \\ x & y & z \end{vmatrix} = -y\Omega\hat{\underline{i}} + x\Omega\hat{\underline{j}}.$$

i.e. $u = -\Omega y$
$v = \Omega x$

Check

$$\underline{\nabla} \cdot \underline{u} = \partial u/\partial x + \partial u/\partial y + \partial u/\partial z$$
$$= 0 + 0 + 0 = 0 \quad \checkmark$$

Euler: $\dfrac{D}{Dt} = \dfrac{\partial}{\partial t} + u \cdot \underline{\nabla}$

$$= \cancel{\frac{\partial}{\partial t}} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + w\frac{\partial}{\partial z}$$

$$= -\Omega y\, \partial/\partial x + \Omega x\, \partial/\partial y$$

Euler $\therefore$ $\dfrac{Du}{Dt} = -\dfrac{1}{\rho}\dfrac{\partial p}{\partial x}$ $\quad \dfrac{Dv}{Dt} = -\dfrac{1}{\rho}\dfrac{\partial p}{\partial y}$ $\quad \dfrac{Dw}{Dt} = -\dfrac{1}{\rho}\dfrac{\partial p}{\partial z} - g$ $\qquad F = -g\hat{\underline{z}}$

$$-\Omega y \overset{-\Omega y}{\partial u/\partial x} + \Omega x \overset{-\Omega y}{\partial u/\partial y} = -1/\rho \, \partial p/\partial x$$

$$-\Omega y \overset{\Omega x}{\partial v/\partial x} + \Omega x \overset{\Omega x}{\partial v/\partial y} = -1/\rho \, \partial p/\partial y$$

$$0 = -1/\rho \, \partial p/\partial z - g$$

25th November 2010.

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + w\frac{\partial}{\partial z}$$

$$\frac{d}{dt}\int_{V(t)} \alpha \, dV = \int_V \frac{\partial \alpha}{\partial t}\, dV + \int_S \alpha(u \cdot \hat{n})\, dS \qquad \text{RTT1.}$$

$$= \int_V \frac{\partial \alpha}{\partial t} + \nabla \cdot (\alpha \underline{u})\, dV \qquad \text{RTT2.}$$

$$= \int_V \frac{D\alpha}{Dt} + \alpha(\underline{\nabla} \cdot \underline{u})\, dV \qquad \text{RTT3.}$$

Mass : $\frac{\partial \rho}{\partial t} + \underline{\nabla}(\rho \underline{u}) = 0$

cons. of mass + RTT2 $\Rightarrow$ $\frac{D}{Dt}\int_V \alpha \rho dV = \int_V \frac{D\alpha}{Dt}\rho dV$ RTT4

momentum + RTT4 : Euler: $\frac{Du}{Dt} = \frac{1}{\rho}\nabla p + \underset{\sim}{F}$ $\quad \underset{\sim}{F}$ external force per unit mass.

In this course just take $\rho = \text{const}$.

Then : $\nabla \cdot \underline{u} = 0$

and $\frac{D\underline{u}}{Dt} = -\frac{1}{\rho}\nabla p + \underline{F}$

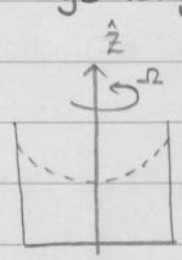
Example

$\underline{u} = \underline{\Omega} \wedge \underline{r}$

$$\cancel{\partial \underline{u}/\partial t} + (u \cdot \nabla)\underline{u} = -\frac{1}{\rho}\nabla p + F = -\frac{1}{\rho}\nabla p - g\hat{z} \qquad F = -g\hat{z} \text{ for gravity.}$$

Steady
= 0

$$(-\Omega y\, \partial/\partial x + \Omega x\, \partial/\partial y)\, \underline{u} = -\frac{1}{\rho}\nabla p - g\hat{z}.$$



In components:

$$\hat{x} : (-\Omega y\, \partial/\partial x + \Omega x\, \partial/\partial y)(-\Omega y) = -\frac{1}{\rho}\frac{\partial p}{\partial x}$$

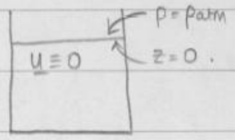
$$\hat{y} : (-\Omega y\, \partial/\partial x + \Omega x\, \partial/\partial y)(\Omega x) = -\frac{1}{\rho}\frac{\partial p}{\partial y}$$

$$\hat{z} : (\qquad\qquad)\, 0 = -\frac{1}{\rho}\frac{\partial p}{\partial z} - g$$

Integrate to get $p(x,y,z)$.
Put $p = p_{atm}$ to get shape of surface $P(x,y,z) = p_{atm}$.

Example 2.
$\underline{u} \equiv \underline{0}$
Hydrostatic equilibrium.

$$\cancel{\frac{D\underline{u}}{Dt}} = -\frac{1}{\rho}\nabla p - g\hat{z}$$



Take $z=0$ at the surface, where $p = p_{atm}$, the constant atmospheric pressure and measure z increasing upwards so $\hat{\underline{z}}$ is vertically upwards and gravitational acceleration is $\underline{F} = -g\hat{z}$

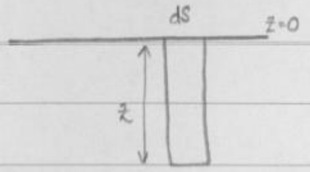
Then $\underline{\nabla} p = -\rho g \hat{\underline{z}}$

i.e. $p = -\rho g \hat{z} + \text{const}$

But $p = p_{atm}$ at $z=0$, so $p_H = p_a - \rho g z$.

Here p_H is hydrostatic pressure.

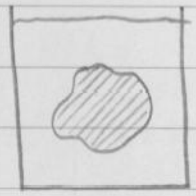
$p_H = p_a - \rho g z$ (i.e. pressure = weight (per unit area) of water above you).



Weight of column $= g\rho z \, dS$

Atm pressure $= 14 \text{ lbs}/(\text{in})^2$
$= 1$ bar
$= 1000$ millibar.

Example 3.
A submerged body.



What force is experienced by a body of volume V surface S submerged in a fluid at rest at density ρ.

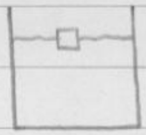
$$p = p_H$$
$$= p_a - \rho g z$$

Total force on body $= \displaystyle\int_S (-p\underline{\hat{n}})\, dS = -\int_V (\nabla p)\, dV = -\int_V \nabla(p_a - \rho g z)\, dV$

$$= -\int_V -\rho g \underline{\hat{z}}\, dV = \cancel{\rho} g \hat{z} \int_V \rho\, dV = g\hat{z} \times \text{mass of water displaced}$$

= weight water displaced acting upwards. ARCHIMEDES.

Without maths.
The fluid is at rest
The fluid would also be at rest with the fluid particles in the same position were the body to be replaced by water. Hence forces around the surface must precisely cancel the weight of the water.



Since hydrostatic pressure is large it can be useful to eliminate it

Write: $p = p_H + p_d$

then $\nabla p = \nabla p_H + \nabla p_d$

$$= -\rho g \hat{\underline{z}} + \nabla p_d$$

Euler: $\dfrac{D\underline{u}}{Dt} = -\dfrac{1}{\rho} \underline{\nabla} p - g\hat{\underline{z}}$

$$= -\frac{1}{\rho}\left[-\rho g \hat{\underline{z}} + \nabla p_d\right] - g\hat{\underline{z}}$$

$$= -\frac{1}{\rho} \nabla p_d$$

i.e. When density is const. then gravity does not accelerate the flow, it is simply accomodated by hydrostatic pressure.

29th November 2010.

Governing equations. (ρ = constant)

Euler equations

$$\partial \underline{u}/\partial t + (\underline{u}\cdot\nabla)\underline{u} = -1/\rho \, \nabla p + \underset{\sim}{F}$$

Continuity

$$\underline{\nabla}\cdot\underline{u} = 0$$

[Have solved for p in solid body rotation]

Bernoulli equation

Observe $(\underline{u}\cdot\underline{\nabla})\underline{u} = \underline{\omega}\wedge\underline{u} + \nabla(\frac{1}{2}u^2)$ $\qquad \underline{\omega} = \underline{\nabla}\wedge\underline{u}$

If, also the external force is derived from a potential

i.e.

$$\underset{\sim}{F} = -\underline{\nabla} V_e$$

e.g If $\underset{\sim}{F} = -g\hat{z}$

then $V_e = +gz$

(others : centrifugal force, electric field, magnetic etc)

Then Euler becomes :

$$\partial \underline{u}/\partial t + \underline{\omega}\wedge\underline{u} + \nabla(\tfrac{1}{2}u^2) = -\frac{1}{\rho}\underline{\nabla}p - \underline{\nabla}V_e$$

$$\Rightarrow \partial \underline{u}/\partial t + \underline{\omega}\wedge\underline{u} = -\left\{\frac{1}{\rho}\underline{\nabla}p + \underline{\nabla}(\tfrac{1}{2}u^2) + \underline{\nabla}V_e\right\} = -\frac{1}{\rho}\nabla H$$

Where $H = p + \frac{1}{2}\rho u^2 + \rho V_e$

- Steady flow, $\partial u / \partial t = 0$

If we dot with $\underline{u}$ we get;

$$0 = u \cdot (\underline{\omega} \wedge \underline{u}) = -\frac{1}{\rho}(u \cdot \nabla H)$$

Thus in steady flow:

$$\underline{u} \cdot \underline{\nabla} H = 0$$

Steady, so $\partial H / \partial t = 0$, giving:

$$\frac{DH}{Dt} = \partial H / \partial t + \underline{u} \cdot \underline{\nabla} H$$

$$= 0$$

i.e. H is constant on particle paths.

But steady, so particle paths are streamlines.

Thus H is constant along streamlines.

Bernoulli's theorem;

In steady flow (incompressible, constant density) with conservative external forces,

$$H = p + \frac{1}{2}\rho \underline{u}^2 + \rho V_e$$

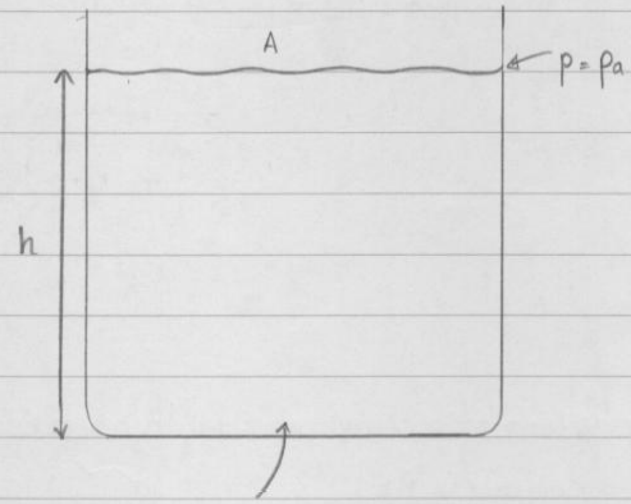
is constant along streamlines.

This is nothing more than conservation of energy :-

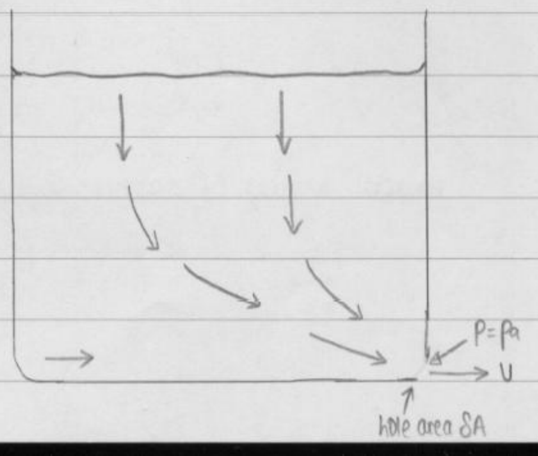
$$\underbrace{P}_{\text{Pressure energy}} + \underbrace{\tfrac{1}{2}\rho \underline{u}^2}_{\text{K.E per unit Vol.}} + \underbrace{\rho V_e}_{\text{P.E per unit Vol.}}$$

Example

A large deep container with surface area A open to atmosphere and depth h is punctured at the bottom by a hole of size $\delta . A$ where $\delta \ll 1$.
How fast does the fluid flow out ?



$p = p_H = p_a + \rho g h > p_a$



Let the exist velocity be U.

Mass flux leaving $= U\delta A$

Let surface fall at at speed u

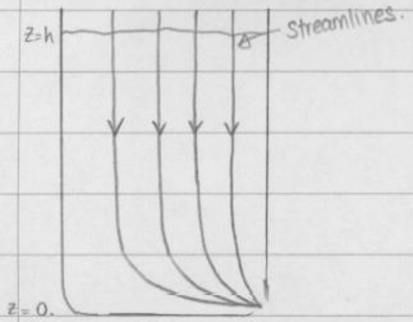
Then flux across any horizontal line is uA

Thus $uA = \delta U A$

i.e. $u = \delta U$ i.e. $u \ll U$

No particle path joins top to bottom,

But streamlines do join the surface to the exit.



We can apply Bernoulli on any of these streamlines.

i.e. $p + \frac{1}{2}\rho \underline{u}^2 + \rho g z = \text{const}$

$$= \begin{cases} p_a + \frac{1}{2}\rho u^2 + \rho g h & \text{at top} \\ p_a + \frac{1}{2}\rho U^2 + 0 & \text{at bottom} \end{cases}$$

Thus $\frac{1}{2}\rho U^2 = \frac{1}{2}\rho u^2 + \rho g h$

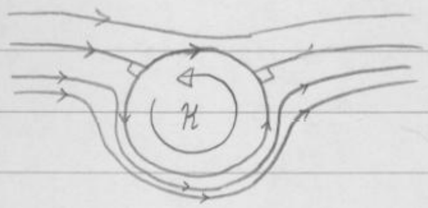
Pressure cancels in this problem to give us precisely acceleration of particle under gravity. [increase in KE = decrease in P.E]

i.e. $U^2(1 - (u/U)) = 2gh$

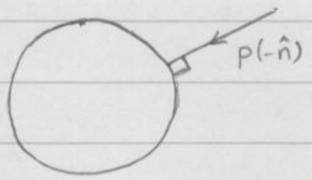
$$U^2(1 - \delta^2) = 2gh$$

For $\delta \ll 1$ $U \simeq \sqrt{2gh}$ exactly speed of particle falling under gravity.

Example 2.



Complex potential : $w(z) = U\left(z + a^2/z\right) - \frac{i\kappa}{2\pi}\log z$

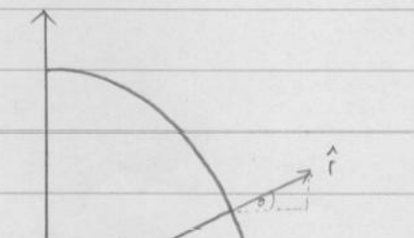


Consider a cylinder of radius a in a stream which at large distances has speed U in the x-direction. Let the cylinder be spinning at a rate such that the circulation about the cylinder is κ

The pressure force on the cylinder is ;

$$\int_S -p\hat{\underline{n}}\, dS = \oint_C -p\hat{n}\, dl \quad \text{per unit distance } \perp_r \text{ to board.}$$

$$= \int_{\theta=-\pi}^{\pi} -p\hat{\underline{n}}\, a\,d\theta \ / \text{ unit width.}$$



$|\hat{\underline{r}}| = 1$

$\hat{\underline{n}} = \hat{\underline{r}} = \cos\theta\,\hat{\imath} + \sin\theta\,\hat{\jmath}.$

i.e. total force (per unit width) on cylinder is :

$$\left(-a\int_{-\pi}^{\pi} p\cos\theta \, d\theta\right)\hat{\imath} + \left(-a\int_{-\pi}^{\pi} p\sin\theta \, d\theta\right)\hat{\jmath}$$

$$= \mathcal{D}\hat{\imath} + \mathcal{L}\hat{\jmath}.$$

Here $\mathcal{L}$ is the lift (per unit width)
and $\mathcal{D}$ is the drag (— " —)

Now we can obtain p on the cylinder (where $\dot{u}$ is a function of θ) using Bernoulli.

For upstream, $x \to -\infty$

$\underline{u} \to U\hat{\imath}$ and $p \to p_0$, const.

Thus on all streamlines,

$$p + \frac{1}{2}\rho \underline{u}^2 = p_0 + \frac{1}{2}\rho U^2$$

(since all streamlines start at ∞ where conditions are uniform)
i.e. can call this the Bernoulli constant.

Hence everywhere:

$$p = \left(p_0 + \frac{1}{2}\rho U^2\right) - \frac{1}{2}\rho \underline{u}^2.$$

Thus on the cylinder:

$$\mathcal{D} = \frac{1}{2}a\rho \int_{-\pi}^{\pi} \underline{u}^2 \cos\theta \, d\theta \quad \text{on } r=a$$

$$\mathcal{L} = \frac{1}{2}a\rho \int_{-\pi}^{\pi} \underline{u}^2 \sin\theta \, d\theta \quad \text{on } r=a$$

Since $\int_{-\pi}^{\pi} \cos\theta \, d\theta = 0$

$\int_{-\pi}^{\pi} \sin\theta \, d\theta = 0$

$$w(z) = U\left(z + a^2/z\right) - \frac{ik}{2\pi}\log z$$

$$\frac{dw}{dz} = U\left(1 - a^2/z^2\right) - \frac{ik}{2\pi z}$$

$$U_r - iU_\theta = e^{i\theta}\frac{dw}{dz} = e^{i\theta}\left[U(1-e^{-2i\theta}) - \frac{ik}{2\pi a}e^{-i\theta}\right] \qquad z = ae^{i\theta}$$

$$= U(e^{i\theta} - e^{-i\theta}) - \frac{ik}{2\pi a}$$

$$= 2iU\sin\theta - \frac{ik}{2\pi a}$$

Thus; $U_r = 0$ $\qquad \frac{dw}{dz} = u - iv$

$$U_\theta = -2U\sin\theta + \frac{k}{2\pi a}$$

so $\underline{u}^2 = U_r^2 + U_\theta^2$

$$= 4U^2\sin^2\theta - \frac{2Uk}{\pi a} + \frac{k^2}{4\pi^2 a^2}$$

$$\therefore \int_{-\pi}^{\pi} \sin^2\theta\cos\theta = 0 \qquad \int_{-\pi}^{\pi} \cos\theta \, d\theta = 0$$

$$\int_{-\pi}^{\pi} \sin\theta\cos\theta = 0$$

$$\therefore \mathfrak{D} \equiv 0$$

Integrals appearing in $\mathcal{L}$:

$$\int_{-\pi}^{\pi} \sin^3\theta \, d\theta = 0$$

$$\int_{-\pi}^{\pi} \sin\theta \, d\theta = 0$$

$$\int_{-\pi}^{\pi} \sin^2\theta \, d\theta = \pi$$

Thus $\mathcal{L} = \frac{1}{2}\rho a \cdot \pi \cdot (-{}^{2U\kappa}/_{\pi a})$

$$= -\rho U \kappa$$

Force per unit width on a spinning cylinder is directly proportional to

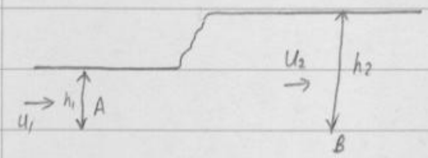
(1) the speed of oncoming flow.
(2) the fluid density.
(3) the circulation about cylinder.

Open Channel Flow

Fully non-linear flow.
Open = open to air.
application of Bernoulli.

Consider flow with a free surface along a channel whose geometry varies only slowly ~~distance~~ with distance along the channel. Initially assume that the channel has constant width.

Suppose that the fluid speed is independent of depth and position across the channel and so is simply a scalar, u.
Let the local fluid depth be h.



Consider two stations A and B, with $(u,h) = (u_1,h_1)$ and (u_2,h_2) respectively

Then conservation of mass gives :

$$\text{Flux A} = \text{Flux B}$$
$$u_1h_1 = u_2h_2$$

For this flow $uh = Q = \text{const.}$

If the fluid surface is smooth between A and B, particles on surface stay on surface so surface is a streamline so we can apply Bernoulli there.

$$p + \tfrac{1}{2}\rho u^2 + \rho g h = \text{const.}$$

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1

Open Channel Flow

Constant width

Speed independent of depth, constant across channel,
ie, single variable u.

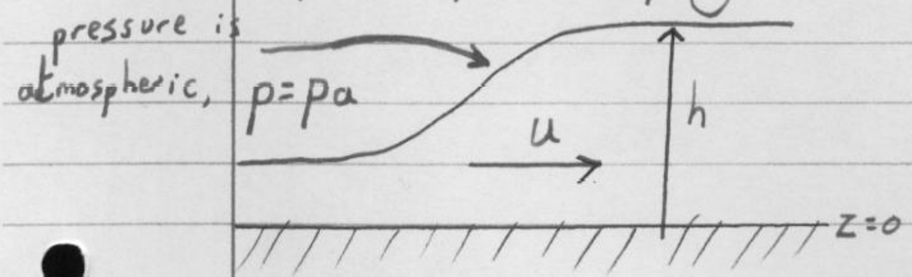
Conservation of Mass, $uh = Q$

(h depth when speed u)

Q constant Volume Flux per Unit Width.

If surface remains smooth, then the surface is a Streamline, thus we can apply Bernoulli:

$$p + \frac{1}{2}\rho u^2 + \rho g z = \text{Const.}$$



Take the datum to be the base of the channel where $z=0$. So surface is $z=h$.

Thus on Surface (because Surface is Streamline).

$$p_a + \frac{1}{2}\rho u^2 + \rho g h = \text{Const.}$$

$$= p_a + \rho g H \quad \text{(say)}$$

Here $\frac{1}{2g}u^2 + h = H$, a Constant

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2

H is the depth the flow would take, where it is to be brought to rest.

H is known as the 'pressure head'.

Two Constants of motion, Q, h.

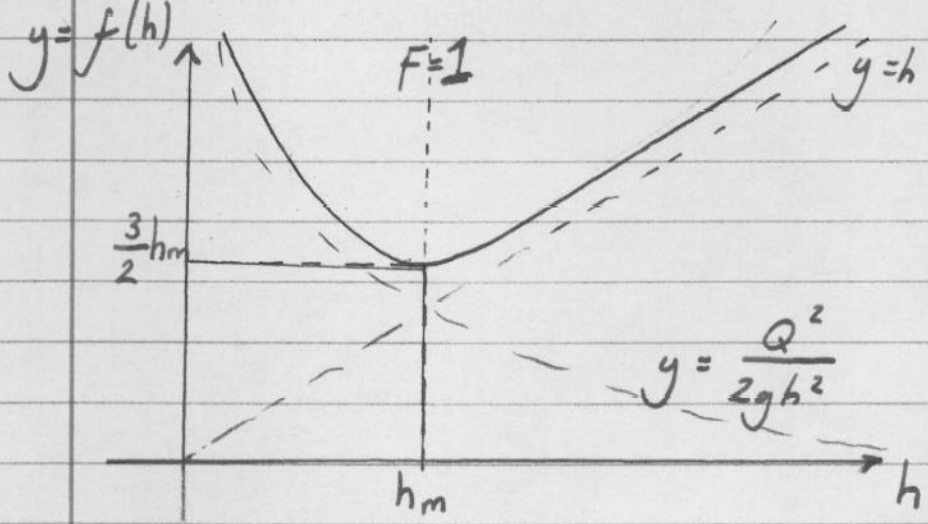
$$uh = Q$$

$$\frac{1}{2g}u^2 + h = H$$

Eliminate u : $u = Q/h$

$$\frac{1}{2g}\frac{Q^2}{h^2} + h = H$$

Consider $f(h) = h + \frac{Q^2}{2gh^2}$ ($h>0$ fluid depth)



$$f'(h) = 1 - \frac{2Q^2}{2gh^3}$$

This has a Single Zero in $h>0$ when $h^3 = \frac{Q^2}{g}$

$$\Rightarrow h = \left(\frac{Q^2}{g}\right)^{1/3} = h_m \quad \text{(say)}$$

$$H(h_m) = h_m + \left(\frac{Q^2}{2gh_m^3}\right) h_m$$

$$= h_m + \frac{1}{2} h_m$$

$$= \frac{3}{2} h_m$$

$$u = \frac{Q}{h}$$

$$u_m = \frac{Q}{h_m} = Q \times g^{1/3} Q^{-2/3}$$

$$= Q^{1/3} g^{1/3}$$

Eliminate Q between h_m and u_m.

$$h_m = \frac{Q^{2/3}}{g^{1/3}} = \frac{u_m^2 g^{-2/3}}{g^{1/3}} = \frac{u_m^2}{g}$$

ie, $u_m^2 = g h_m$

Introduce the Parameter

$F = \dfrac{u}{\sqrt{gh}}$, the 'Froude Number',

Then at h_m, $F = 1$

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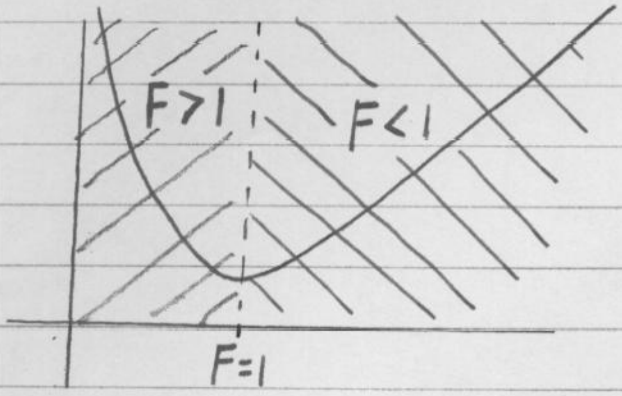
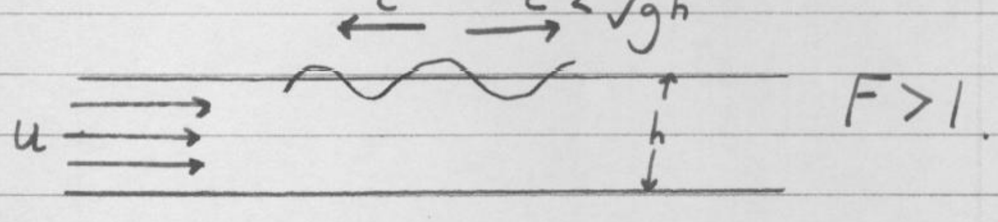
4



- Then we have two types of flow: those where $F<1$ and those where $F>1$.
- The fastest infinitesimal wave on fluid of depth h has speed $\sqrt{gh}$ (to be proved in water waves, last topic).
- $F>1$: Speed of flow greater than the fastest wave.

  All information of a Disturbance is swept downstream.

  Called 'Super Critical Flow.'



- $F<1$: Information can Propagate Upstream.

  The flow is Subcritical.

Precise analogy with Mach Number.

$M>1$: Supersonic, speed flow > speed of sound.

$M<1$: Subsonic speed of flow < speed of sound.

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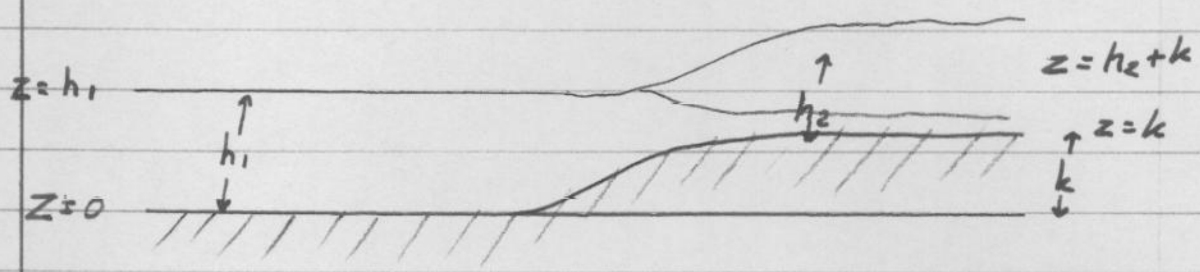
5

Example: Consider a Channel whose floor slowly (with distance down channel) rises by an amount k.

Does the fluid surface rise or fall?



Define the Rise r of the surface

$$r = h_2 + k - h_1$$

Where h_2 is the flow DEPTH and h_1 is the Initial flow DEPTH.

Conservation of Mass,

$$u_1 h_1 = u_2 h_2$$

Bernoulli on Surface:

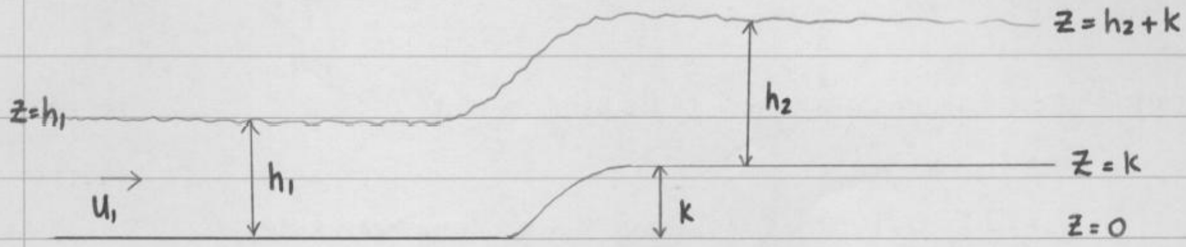
$$p_a + \tfrac{1}{2}\rho u_1^2 + \rho g h_1 = p_a + \tfrac{1}{2}\rho u_2^2 + \rho g (h_2 + k)$$

$$\Rightarrow \frac{u_1^2}{2g} + h_1 = \frac{u_2^2}{2g} + h_2 + k$$

$$\Rightarrow \frac{Q^2}{2gh_1^2} + h_1 = \frac{Q^2}{2gh_2^2} + h_2 + k$$

$\Rightarrow H(h_1) = H(h_2) + k$ where $H(h) = \dfrac{Q^2}{2gh^2} + h$, as before

6th December 2010.



Old depth : h_1

New depth : h_2

Old height free surface : h_1

new " " " : $h_2 + k$

$u_1 h_1 = Q = u_2 h_2$ conservation of mass

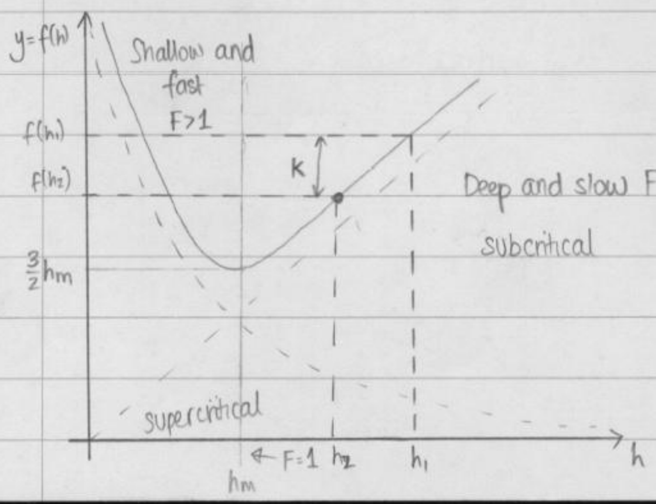
Can use Bernoulli because surface is a streamline.

$$P + \frac{1}{2}\rho u^2 + \rho g \underbrace{z}_{\text{free surface}} = \text{const}$$

$$P_a + \frac{1}{2}\rho u_1^2 + \rho g h_1 = P_a + \frac{1}{2}\rho u_2^2 + \rho g (h_2 + k)$$

$$\frac{u_1^2}{2g} + h_1 = \frac{u_2^2}{2g} + h_2 + k$$

$$f(h_1) = f(h_2) + k \qquad \therefore\ f(h) = \frac{Q^2}{2gh^2} + h$$



$f(h_2)$ is less than $f(h_1)$ by an amount k.

Thus $h_2 < h_1$. Flow gets shallower.

Rise; r = new surface height - old surface height

$$= h_2 + k - h_1$$

$$= \frac{u_1^2}{2g} - \frac{u_2^2}{2g} \qquad \text{Taking } h_1 + \frac{u_2^2}{2g} \text{ from both sides.}$$

$$= \frac{Q}{2g}\left(\frac{1}{h_1^2} + \frac{1}{h_2^2}\right)$$

Now $h_2 < h_1 \Rightarrow \frac{1}{h_2^2} > \frac{1}{h_1^2}$

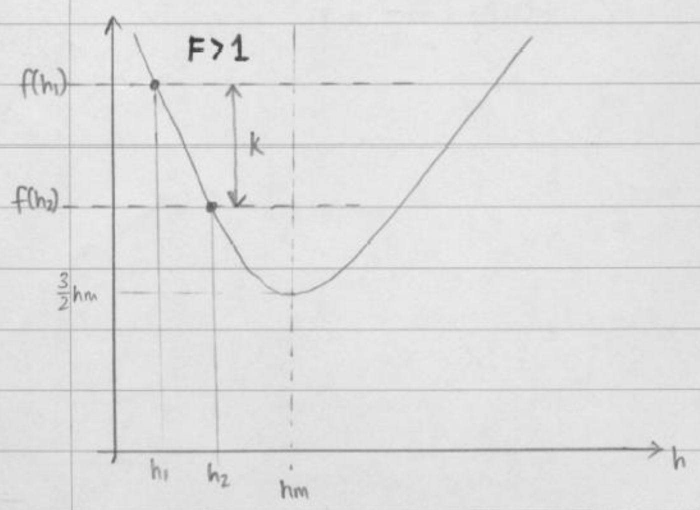
so $r < 0$

i.e. surface falls, i.e. drop in depth is greater than rise in base.

$F < 1$

SURFACE FALLS. If the upstream flow is subcritical.

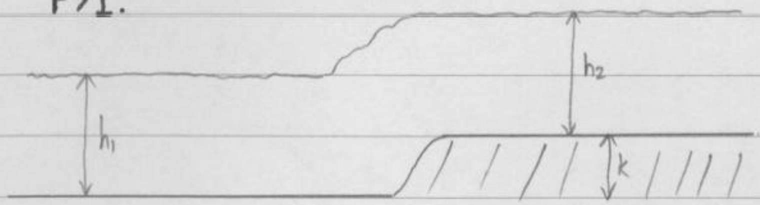
$h_2 + k < h_1$



$h_2 > h_1$. Flow gets deeper

$\therefore$ surface rises

F>1.



$h_2 + k > h_1$

Surface rises if the upstream flow is supercritical, i.e. has $F > 1$.

We conclude:

Both flows move towards critical as the floor rises.

$$F = \frac{u}{\sqrt{gh}} \qquad F^2 = \frac{u^2}{gh} = \frac{\rho u^2}{\rho g h} = \frac{K.E}{P.E}$$

SUPERCRITICAL; $F > 1$

Flow has more K.E than P.E (fast and shallow)

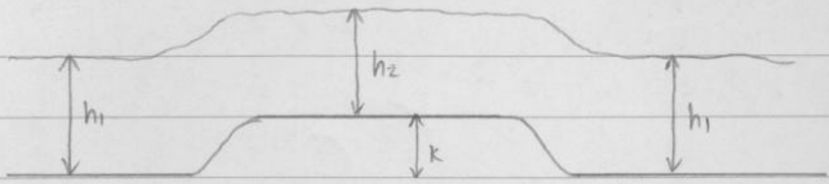
To get over bump flow converts some K.E to P.E (raising surface) to get over.

SUBCRITICAL; $F < 1$.

Flow has more P.E than K.E (deep and slow)

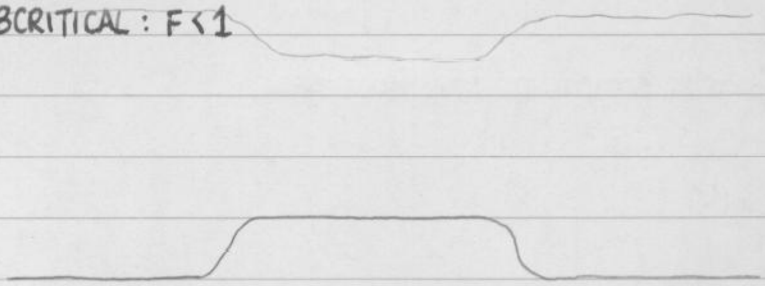
To get over bump it gives up P.E (ie surface falls) to get K.E.

SUPERCRITICAL: $F>1$



$h_2+k>h_1$ E.g. Fast, shallow river over rock.

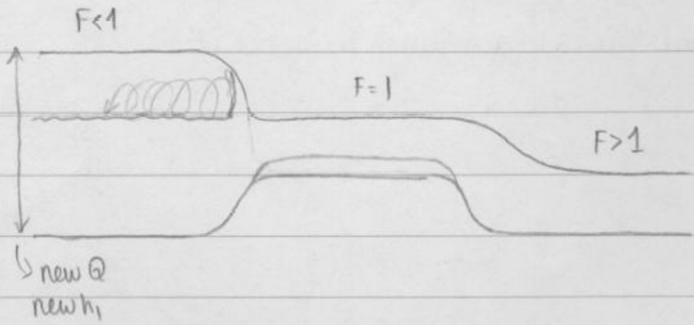
SUBCRITICAL: $F<1$



Deep and slow river

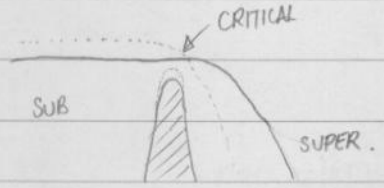
- Note that if k is sufficiently large i.e. $k+\frac{3}{2}h_m > f(h_1)$, then we have moved below the minimum of $f(h)$. Hence there is no smooth solution.

What happens?



Transition from SUB to SUPER. at a lower value of Q. Bump has changed flow. Bump determines Q. $\therefore$ Upstream is subcritical (info. travels upstream).

Weirs:



Raise Weir

Waves change upstream boundary conditions.

Flow deepens and slows to remain critical at weir

Now $h_m = \left(\frac{Q^2}{g}\right)^{1/3}$

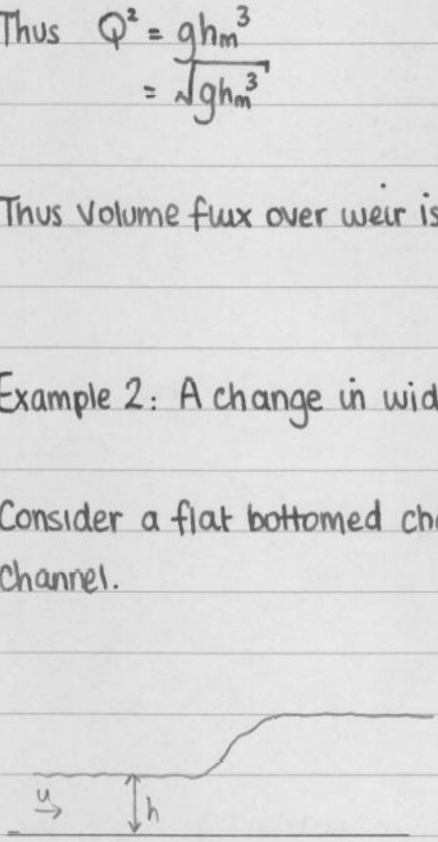
Thus $Q^2 = g h_m^3$

$= \sqrt{g h_m^3}$

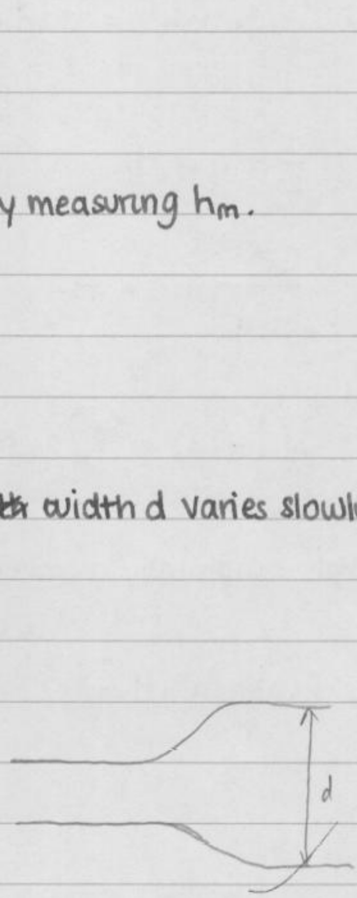
Thus volume flux over weir is known simply by measuring h_m.

Example 2: A change in width.

Consider a flat bottomed channel, whose ~~depth~~ width d varies slowly along the channel.



Side view



Top view

speed u

depth h

width d

Consider mass flux :

$$Q = uhd\,, \quad \text{a constant.}$$

Surface smooth. ∴ streamline ∴ Bernoulli applies here.

elevation of surface.

$$P + \frac{1}{2}\rho u^2 + \rho g h = \text{const.}$$

$$P_a + \frac{1}{2}\rho u^2 + \rho g h = \rho g H + P_a$$

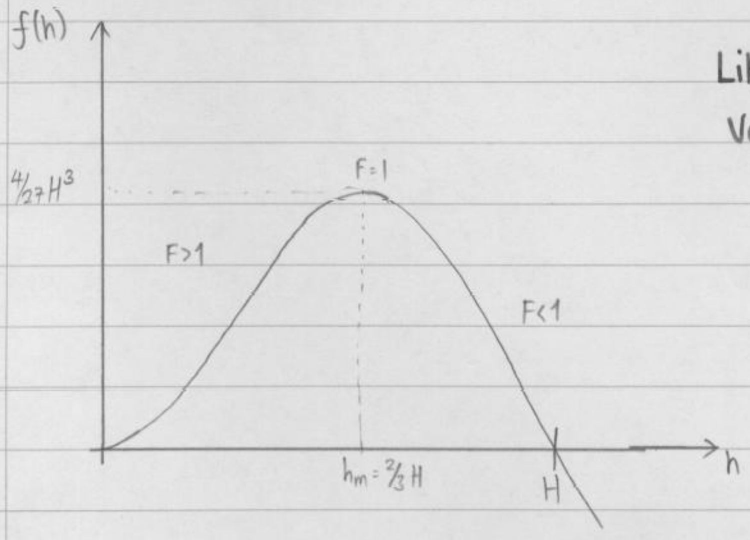
i.e. $$\frac{u^2}{2g} + h = H.$$

$$\frac{Q^2}{2gh^2d^2} + h = H$$

$$h^2(H-h) = Q^2/2gd^2.$$

Solve this graphically as in example 1.

Call this $f(h) = h^2(H-h)$



Like h^2 near $h \simeq 0$

Vanishes when $h = H$.

It has a single max. where $f'(h_m) = 0$

i.e. $2h_m H - 3h_m^2 = 0 \qquad \Rightarrow h_m = \frac{2}{3}H$

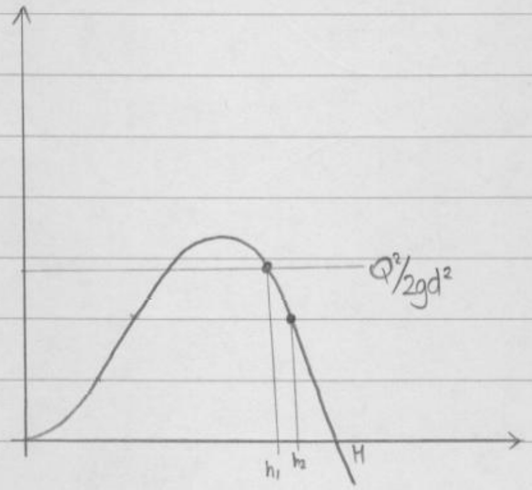
$f(h_m) = \frac{4}{27} H^3$

when $h_m = \frac{2}{3}H \Rightarrow \frac{U_m^2}{2g} + h_m = H$

$$\frac{U_m^2}{2g} = \frac{1}{3}H$$

$$U_m^2 = \frac{2}{3}gH$$

so $\frac{U_m^2}{gh_m} = F^2 = \frac{\frac{2}{3}gH}{\frac{2}{3}gH} = 1$



F<1

Channel widens : d increases

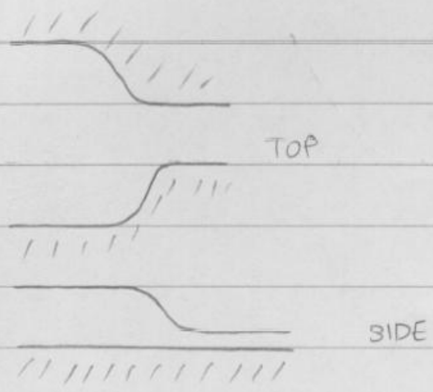
RHS decreases

$\therefore$ depth increases.

Channel decreases : d decreases

RHS increases

$\therefore$ depth decreases.

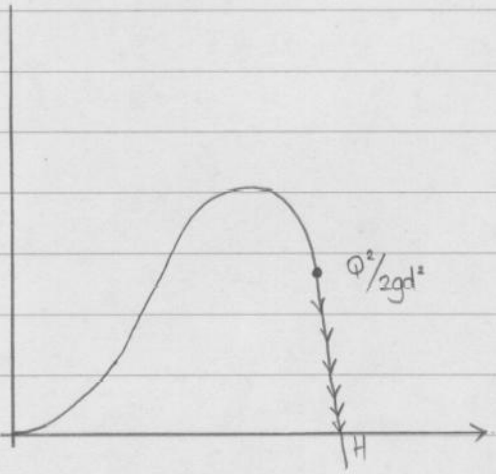


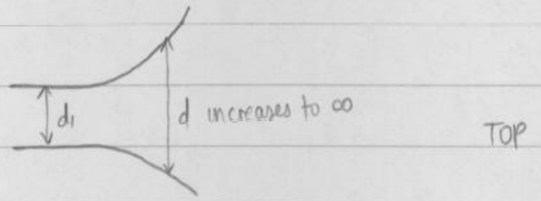
Important application: Consider a flow down a flat-bottomed channel that slowly widens into an infinitely wide reservoir (at rest)

In reservoir: $\cancel{p_a} + \cancel{\tfrac{1}{2}\rho u^2} + \rho g h = \cancel{p_a} + \rho g H$

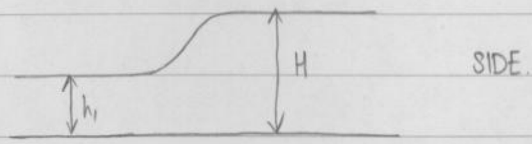
$$h = H$$

i.e. reservoir has depth H.

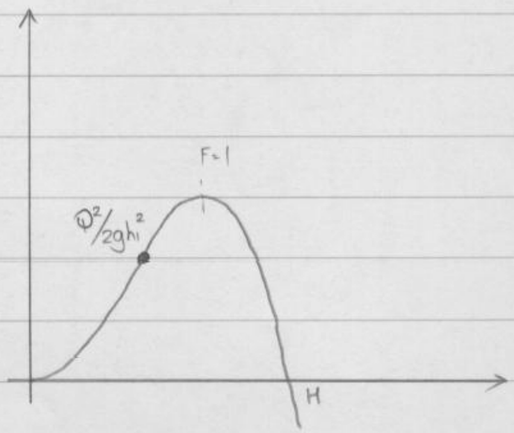




$\frac{Q^2}{2gd^2}$ decreases to zero



But, if the upstream flow is supercritical, $F > 1$.



$Q^2/2gd^2$ decreases smoothly to zero as $d \to \infty$.

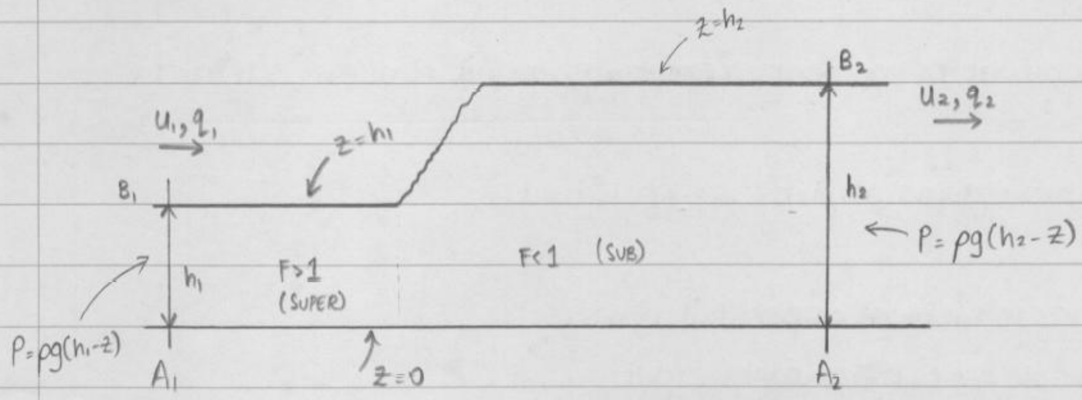
Thus $h \to 0$.

i.e. fluid dries out.

i.e. Cannot join smoothly to a stagnant reservoir.

9th December 2010.

Hydraulic Jump.



Take surface to be zero.

Zero on surface, increasing linearly with depth. i.e. hydrostatic

Constant width channel, d, fixed.

Conservation of mass flux : $Q = U_1 h_1 = U_2 h_2$

Rate of change of downstream momentum = Force in downstream direction [Newton]

At $A_1 B_1$ pressure applies a force downstream

$$\int_0^{h_1} \rho g (h_1 - z)\, dz = \frac{1}{2}\rho g h_1^2 \text{ per unit width}$$

At $A_2 B_2$ pressure acts upstream on the fluid volume $A_1 B_1 A_2 B_2$. It gives a downstream force : $-\frac{1}{2}\rho g h_2^2$ per unit width.

$\therefore$ Total downstream force on region is :

$$\frac{1}{2}\rho g h_1^2 d - \frac{1}{2}\rho g h_2^2 d$$

$= $ r.o.ch of momentum in $A_1 B_1 B_2 A_2$.

Fluid enters the region with volume flux $u_1 h_1 d$
This fluid has momentum ρu_1 per unit volume

Thus the amount of momentum entering per unit time is $\rho u_1^2 h_1 d$

The amount leaving at A_2B_2 is: $\rho u_2^2 h_2 d$.

The increase in momentum per unit time is :
amount we get out - amount we put in
$\rho u_2^2 h_2 d - \rho u_1^2 h_1 d$

Balancing force with rate of increase of fluid momentum

$$\Rightarrow \frac{1}{2}\rho g h_1^2 d - \frac{1}{2}\rho g h_2^2 d = \rho u_2^2 h_2 d - \rho u_1^2 h_1 d$$

$$\frac{1}{2}\rho g d \,(h_1^2 - h_2^2) = \rho \frac{Q^2}{d}\left(\frac{1}{h_2} - \frac{1}{h_1}\right)$$ where $Q = uhd$, $u^2 = \dfrac{Q^2}{h^2 d^2}$

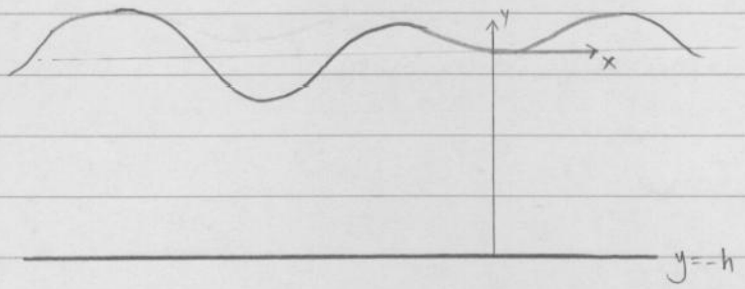
One possibility in these problem is often 'nothing happens'.
i.e. $h_1 = h_2$ (always be aware of this as it reduces a hard cubic to an easy quadratic, with perhaps only a single +ve root)
e.g. channel fat-thin-fat
channel deep-shallow-deep.

Similarly here there could be no jump.
Then $h_1 = h_2$
Note we have $\frac{1}{2}\rho g d\, h_1 h_2\,(h_1 - h_2)(h_1 + h_2) = \rho \frac{Q^2}{d}(h_1 - h_2)$

Either $h_1 = h_2$ nothing happen
or $\frac{1}{2}\rho g d\, h_1 h_2 (h_1 + h_2) = \frac{\rho Q^2}{d}$

$$\Rightarrow h_1 h_2 (h_1 + h_2) = \frac{2Q^2}{g d^2}.$$

Surface water waves.



Take waves to be 2D with y vertical. Take the equulibrium surface level as $y=0$ and the bottom boundary at $y=-h$

Take fluid to be incompressible and inviscid and irrotational.

Then we have a velocity potential $\underline{u} = \nabla\phi$

But $\nabla\cdot\underline{u} = 0$ so $\nabla\cdot(\nabla\phi) = 0$ i.e. $\nabla^2\phi = 0$

Lower boundary conditions:

$\underline{u}\cdot\hat{\underline{n}} = 0$ on $y=-h$

i.e. $\frac{\partial\phi}{\partial n} = 0$ on $y=-h$

$$\frac{\partial\phi}{\partial y} = 0 \text{ on } y=-h$$

Let the surface be given by:

$$y = \eta(x,t)$$

Then we have, so far,

$$\nabla^2\phi = 0 \qquad -h<y<\eta$$

$$\frac{\partial\phi}{\partial y} = 0 \qquad y=-h$$

On the surface $y = \eta(x,t)$ we have 2 b.c's

kinematic ~~s~~ b.c

'A particle on the surface stays on the surface'

For a particle on surface :

$$y = \eta(x,t) \ \forall t \qquad \text{on } y = \eta$$

Following a particle

$$\frac{Dy}{Dt} = \frac{D\eta}{Dt} \qquad \text{on } y=\eta$$

$$\text{i.e. } v = \frac{\partial \eta}{\partial t} + u\frac{\partial \eta}{dx} \qquad \text{on } y = \eta$$

$$\text{or} \quad \frac{\partial \phi}{\partial y} = \frac{\partial \eta}{\partial t} + \frac{\partial \phi}{\partial x}\cdot\frac{\partial \eta}{\partial x} \qquad \text{on } y = \eta$$

The second b.c on the surface brings in gravity (the restoring force for these waves) We use Bernoulli (but modified)

Remember, $$\frac{D\underline{u}}{Dt} = -\frac{1}{\rho}\underline{\nabla}p + \underline{F}$$

$$= -\frac{1}{\rho}\underline{\nabla}p - g\underline{\hat{z}}$$

$$\frac{D\underline{u}}{Dt} = \frac{\partial u}{\partial t} + \underline{u}\cdot\underline{\nabla}\underline{u}$$

$$= \frac{\partial u}{\partial t} + \omega \wedge u + \nabla\left(\tfrac{1}{2}u^2\right)$$

[before we said $\partial u/\partial t \equiv 0$ and dotted with $\underline{u} \Rightarrow$ traditional Bernoulli]

Here $\omega = 0$ (irrotational flow) and $\underline{u} = \nabla\phi$ so $\partial u/\partial t = \nabla(\partial\phi/\partial t)$

i.e. $\nabla\left(\frac{p}{\rho} + \frac{1}{2}u^2 + gy - \frac{\partial\phi}{\partial t}\right) = 0$

Thus $\frac{p}{\rho} + \frac{1}{2}u^2 + gy - \frac{\partial\phi}{\partial t}$ is a function of time alone.

Suppose this equalled $b(t)$

Redefine ϕ as $\hat{\phi} - \int^t b(t')\,dt$

$$\frac{\partial\phi}{\partial t} = \frac{\partial\hat{\phi}}{\partial t} - b(t) \qquad \text{But } \underset{\sim}{u} = \nabla\phi = \nabla\hat{\phi}$$

Thus w.l.o.g

$$\frac{p}{\rho} + \frac{1}{2}u^2 + gz - \frac{\partial\phi}{\partial t} = 0$$

This is true everywhere. In particular, on the surface $y = \eta$

$$\frac{p_a}{\rho} + \frac{1}{2}|\nabla\phi|^2 + g\eta - \frac{\partial\phi}{\partial t} = 0 \qquad \text{on } y = \eta$$

Similarly absorb $\frac{p_a}{\rho}$ into $\frac{\partial\phi}{\partial t}$

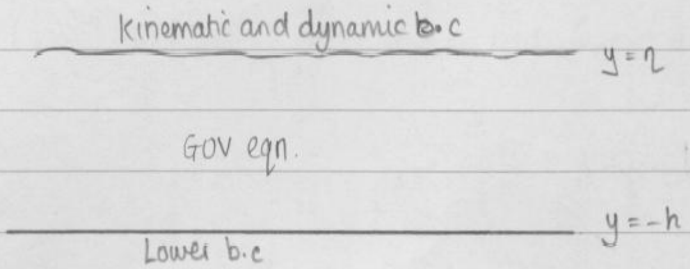
Summary.

Gov. eqn. $\nabla^2\phi = 0 \qquad -h < y < \eta$

Lower b.c $\frac{\partial\phi}{\partial y} = 0 \qquad y = -h$

Surface b.c

$$\left.\begin{aligned} &\frac{\partial\eta}{\partial t} + \frac{\partial\phi}{\partial x}\cdot\frac{\partial\eta}{\partial x} = \frac{\partial\phi}{\partial y} \quad \text{kinematics} \\ &\frac{\partial\phi}{\partial t} = \frac{1}{2}|\nabla\phi|^2 + g\eta \end{aligned}\right\} \quad y = \eta$$



The full problem is almost intractable analytically. Thus we confine attension to infitesimal waves

We consider waves where η (and so ϕ is of order ε where $\varepsilon \ll 1$

$$\frac{\partial \eta}{\partial t} + \frac{\partial \phi}{\partial x}\frac{\partial \eta}{\partial x} = \frac{\partial \phi}{\partial y}$$

$$\varepsilon : \varepsilon \ \varepsilon : \varepsilon$$

$$1 : \varepsilon : 1$$

For infitesimal waves, the middle term is negligible.

i.e. $\frac{\partial \eta}{\partial t} = \frac{\partial \phi}{\partial y}$ on $y = \eta$.

Similarly for infitesimal waves the dynamic bs becomes

$$\frac{\partial \phi}{\partial t} = g\eta \quad \text{on } y = \eta$$

$$f(\varepsilon) = f(0) + \varepsilon f'(0) + \tfrac{1}{2}\varepsilon^2 f''(0)$$

thus with error order ε, we can replace $f(\varepsilon)$ by $f(0)$

i.e. we can place the surface b.c's on $y = 0$

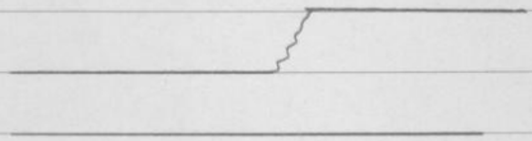
Linear summary

Gov. eqn. $\nabla^2 \phi = 0$ $\quad -h \le y \le 0$

Lower. b.c $\partial \phi / \partial y = 0$ $\quad y = -h$

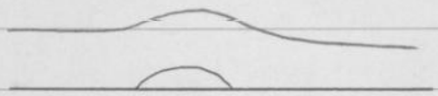
Surface b.c
$$\left.\begin{aligned} \frac{\partial \eta}{\partial t} &= \frac{\partial \phi}{\partial y} \\ \frac{\partial \phi}{\partial t} &= g\eta \end{aligned}\right\} \quad \text{on } y = 0$$

13th December 2010.



Mass

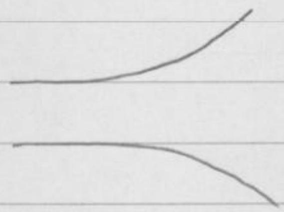
Momentum



Mass

Bernoulli (because surface is a streamline)

Momentum is not conserved.



Gov.

Lower b.c

Dynamic b.c → unsteady Bernoulli ($\underline{u} = \nabla \phi$)

Kinematic b.c → particle on surface stay $\frac{D}{Dt}(y-\eta) = 0$ on $y = \eta$

Transfer b.c from $y = \eta$

to $y = 0$ if $y \sim \varepsilon$

Linear water waves.

Gov. eqn $\nabla^2 \phi = 0$ $-h < y < 0$

Lower b.c $\partial\phi/\partial y = 0$ $y = -h$

Upper b.c

$$\left.\begin{array}{ll} \text{Dynamic} & \partial\phi/\partial t + g\eta = 0 \\ \text{Kinematic} & \partial\eta/\partial t = \partial\phi/\partial y \end{array}\right\} y = 0$$

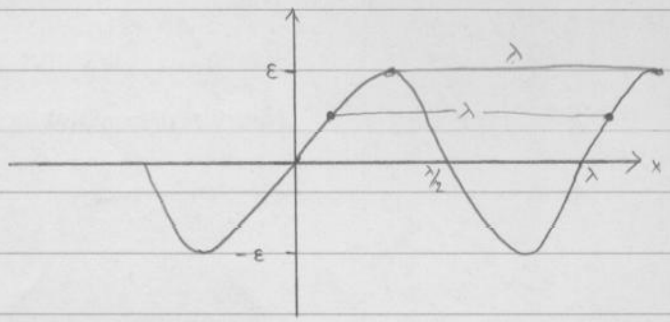
It is sufficient to consider sinusoidal solutions (because every solution can be expressed as an integral of sinusoids)

Consider the surface elevation

$$\eta(x,t) = \varepsilon \sin\left[\tfrac{2\pi}{\lambda}(x-ct)\right]$$

Notice this of the form $F(x-ct)$. (a solution propagating to the right with speed c)

At $t=0$



i.e. η is a sinusoid with wavelength λ (the distance between successive peaks). We call $\frac{2\pi}{\lambda}(x-ct)$ the phase of the wave i.e. the argument of sine. Hence λ is the distance between any two points of the same phase.

The wave has radian frequency $2\pi c/\lambda = \omega$.
So the period $\tau = 2\pi/\omega = \lambda/c$
[It takes time λ/c for the next peak to arrive at speed c].
It is convenient to the wavenumber - the number of wavelengths in distance 2π.
i.e.

$$k = 2\pi/\lambda \qquad [k] = L^{-1} \qquad [\lambda] = L$$

Then $\eta = \varepsilon \sin(kx - \omega t)$ $\qquad \lambda = 2\pi/k \qquad \tau = 2\pi/\omega$

Now we solve: we know $\partial\phi/\partial y = \partial\eta/\partial t$ on $y=0$ $\forall x,t$

$$= -\varepsilon\omega\cos(kx-\omega t). \quad \forall x,t$$

Try $\phi(x,y,t) = -\varepsilon\omega Y(y)\cos(kx-\omega t)$ as the solution. $\partial\phi/\partial y = Y' . [-\varepsilon\omega\cos(kx - ..)]$

Then this b.c requires $Y'(y)=1$ at $y=0$

$\phi(x,y,t) = -\varepsilon\omega\cos(kx-\omega t) . Y(y)$

Kinematic : $Y'(0)=1$

Lower b.c : $\partial\phi/\partial y = 1$ at $y=-h$

$Y'(-h)=0$

$\partial\phi/\partial y = -\varepsilon\omega\cos(kx-\omega t)Y'(y) = 0$

$\forall x,t$ iff $Y'(-h)=0$.

Governing equation.

$$\phi_{xx} + \phi_{yy} = 0$$

$\phi_{xx} = \varepsilon k^2 \cos(kx-\omega t)Y(y)$

$\phi_{yy} = -\varepsilon\omega\cos(kx-\omega t)Y''(y)$

Thus, adding : $0 = -\varepsilon\omega\cos(kx-\omega t)(Y''-k^2Y) \quad \forall x,t, \quad 0>y>-h$

Thus $Y''-k^2Y=0 \qquad 0>y>-h$

So far : $\phi = -\varepsilon\omega\cos(kx-\omega t)Y(y)$

where

Gov. $Y''-k^2Y=0 \qquad -h<y<0$

L. B.C $Y'(-h)=0$

K. B.C $Y'(0)=1$

Solutions to gov. eqn : e^{ky}, e^{-ky}, $\sinh ky$, $\cosh ky$, $\sinh k(y+h)$, $\cosh k(y+h)$.

Try $Y(y) = A\cosh k(y+h)$.

This satisfy : $Y'' - k^2 Y = 0$

$Y'(-h) = 0$

Now need ; $Y'(0) = 1$

i.e. $\left[A\, k \sinh k(y+h)\right]_{y=0} = 1$

i.e. $A = 1/k\sinh kh$

Thus $Y(y) = \dfrac{\cosh k(y+h)}{k \sinh kh}$

Thus $\phi(x,y,t) = -\varepsilon\omega \cos(kx-\omega t)\dfrac{\cosh k(y+h)}{k\sinh kh}$

What about the surface dynamic b.c. ϕ = velocity potential

$\underline{u} = \nabla\phi$.

On $y=0$

$\partial\phi/\partial t = -\varepsilon\omega^2 \sin(kx-\omega t)\cosh kh / k\sinh kh$

$-g\eta = -\varepsilon g \sin(kx-\omega t)$

These are the same $\forall x,t$ thus

$$\omega^2 = gk \tanh kh$$

$[\omega^2] = T^{-2}$ $[gk] = LT^{-2}L^{-1} = T^{-2}$ $[kh] = L^{-1}L$

The dynamic b.c has provided a relationship between the frequency of a wave and its wavelength.

$c^2k^2 = gk \tanh kh$ $\omega = ck$

$c^2 = g/k \tanh kh$ $k = 2\pi/\lambda$.

i.e. $c^2 = \frac{g\lambda}{2\pi} \tanh \frac{2\pi h}{\lambda}$

- the speed of a wave depends on its wavelength.

NOT waves on a string
NOT sound
NOT light (in a vaccuum) } speed light is const.
NOT radio waves

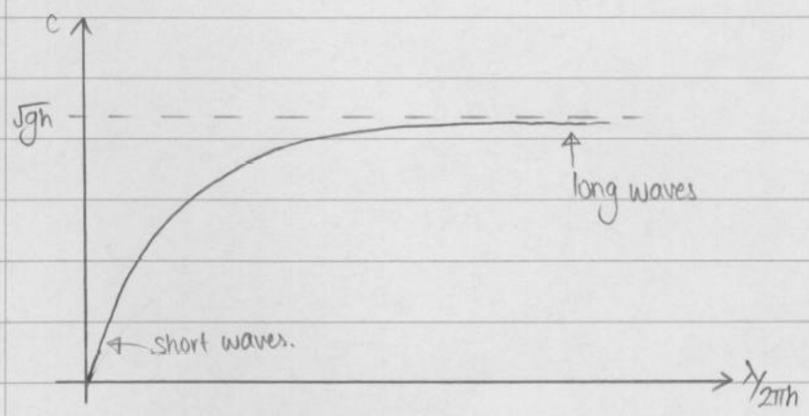
Suppose $h \gg \lambda$ i.e. very deep flow

$\tanh \alpha \to 1$ as $\alpha \to \infty$

$c^2 \to g\lambda/2\pi$ i.e. $c \to \sqrt{g\lambda/2\pi}$ as $h/\lambda \to \infty$.

i.e. larger a ship, the larger the waves it generates the larger the λ, the faster it goes.

i.e. speed of ship is proportional to the square root of length.



Now consider $h \ll \lambda$ i.e. very shallow flow (i.e. as in open channel flow where $\lambda \to \infty$, h fixed so $\lambda/h \to \infty$)

But $\tanh \alpha \to \alpha$ as $\alpha \to 0$.

As $h/\lambda \to 0$

$$c^2 \to \frac{g\lambda}{2\pi} \cdot \frac{2\pi h}{\lambda} = gh$$

i.e. $c \to \sqrt{gh}$, the speed of the longest wave
(long waves are not dispersive).

In fixed $\overset{\nearrow \text{depth}}{\text{flow}}$, long waves travel fastest.

Precisely as in our def. of the Frude number $F = u/\sqrt{gh}$ = flow speed/speed of fast waves

When $F > 1$, no infitesimal wave is fast enough to travel upstream : all (infitesimal) disturbances are swept downstream.

The relationship between speed and wavelength of a wave is called a dispersion relation.

Particle Paths.

$$\frac{dx}{dt} = u(x,y,t) = \frac{\partial\phi}{\partial x}(x,y,t)$$

$$\frac{dy}{dt} = v(x,y,t) = \frac{\partial\phi}{\partial y}(x,y,t)$$

$$\phi = -\frac{\varepsilon\omega\cos(kx-\omega t)\cosh k(y+h)}{k\sinh kh}$$

Then $\frac{dx}{dt} = \varepsilon k\omega\sin(kx-\omega t)\cosh k(y+h)/k\sinh kh$

$$\frac{dy}{dt} = -\varepsilon k\omega\cos(kx-\omega t)\sinh k(y+h)/k\sinh kh.$$

Write $x = x_0 + \varepsilon X(t)$.

$y = y_0 + \varepsilon Y(t)$. (not previous Y).

$\therefore\ \varepsilon\frac{dX}{dt} = \varepsilon k\omega\sin(kx_0-\omega t)\cosh(y_0+h)/k\sinh kh$

(replacing x by x_0, y by y_0 makes an error of order ε^2 on RHS and is neglected in the limit $\varepsilon\to 0$).

i.e. $\frac{dX}{dt} = k\omega\sin(kx_0-\omega t)\,.\,\frac{k\omega\cosh(y_0+h)}{k\sinh(kh)}$

$X = \frac{k\cosh(y_0+h)}{k\sinh(kh)}\,.\,\cos(kx_0-\omega t)\ [+\text{ const}]$ ← absorb into x_0.

$$\varepsilon\, dY/dt = -\varepsilon k\omega\cos(kx_0-\omega t)\sinh k(Y_0+h)/k\sinh kh.$$

RHS a function of time.

$$Y = -\cancel{k\omega}\ \frac{\sinh k(y_0+h)}{\sinh kh}\ \sin(kx_0-\omega t).$$

$$x = x_0 + \varepsilon X$$
$$y = y_0 + \varepsilon Y.$$

$$(x-x_0) = \varepsilon X$$
$$(y-y_0) = \varepsilon Y \qquad \Rightarrow \quad (X-X_0)^2$$

But $X = a\cos(kx_0-\omega t)$ $\qquad a = \cosh k(y_0+h)/\sinh kh.$

$\qquad Y = b\sin(kx_0-\omega t)$ $\qquad b = \sinh k(y_0+h)/\sinh kh$

$$\frac{X^2}{a^2}+\frac{Y^2}{b^2} = 1.$$

$$\frac{(x-x_0)^2}{a^2}+\frac{(y-y_0)^2}{b^2} = \varepsilon^2$$

Now $a>b$

i.e. ellipse with semi-major axis b (horizontal) and semi-minor axis a (vertical).

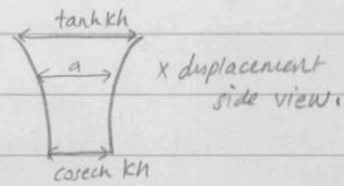
When $y_0=-h$, $b=0$ no y-displacement. particles move back + forth along bottom.

When $y_0=0$, $b=1$ $\qquad Y = \sin(kx_0-\omega t)$

$\qquad y = \varepsilon\sin(kx_0-\omega t) = \eta$ (as expected).

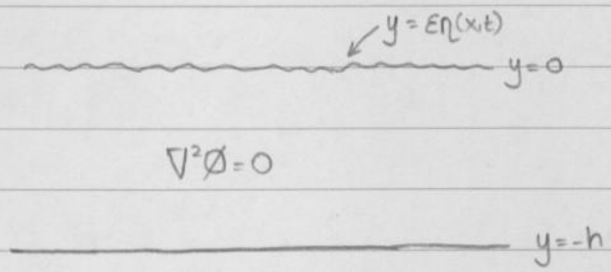
Since cosh is monotonically increasing; as $y_0\to -h$, a decreases ~~for~~ from tanh kh at y_0 to cosech at $y=-h$.

As $h\to\infty$. infinitely deep. $a\to b$ and particle paths are circles of radius εe^{-2th} (check).



15th December 2010.

Water waves.



$\eta(x,t) = \varepsilon \sin(kx-\omega t)$

$\varnothing(x,y,z,t) = -\varepsilon\omega\cos(kx-\omega t)\cosh[k(y+h)]/k\sinh kh$

$$\frac{c^2}{gh} = \frac{\left(\frac{\omega}{k}\right)^2}{gh} = \frac{\lambda}{2\pi h}\tanh\left(\frac{2\pi h}{\lambda}\right)$$

$c^2 = \frac{g}{k}\tanh(kh)$

$\omega^2 = gk\tanh(kh)$.

$\frac{c^2}{gh} = \frac{1}{kh}\tanh(kh)$.

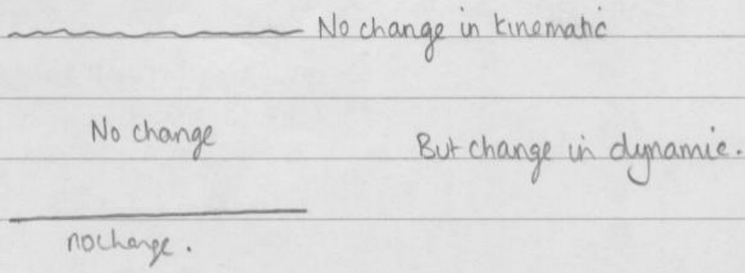
$\frac{\left(\frac{\omega^2}{k^2}\right)}{gh} = \frac{g}{k}\tanh(kh)$.

Waves in string with tension T.

$\underset{\text{curvature.}}{C^2U_{xx}} = U_{tt}$ $\quad C = \sqrt{T/\rho}$

σ - surface tension.
- tries to shorten surface.
- increases pressure in fluid
- proportional to surface tension α to curvature.

$$\boxed{P = P_a + (-\eta_{xx})\frac{\sigma}{\rho}}$$



Dynamic was :

$$\rho \phi_t + \rho g z + p = \text{const.}$$

On surface $y = \eta(x,t)$

$$\rho \phi_t + \rho g z + P_a - \sigma \eta_{xx} = \text{const.}$$

i.e. we have dynamic condition

$$\phi_t + g\eta - \frac{\sigma}{\rho}\eta_{xx} = 0 \qquad (P_a \text{ absorbed into velocity potential}).$$

Get ϕ as before
Substitute in

$$\phi_t + \varepsilon g \sin(kx - \omega t) - \frac{\sigma}{\rho}(-k^2)\varepsilon \sin(kx - \omega t) = 0$$

i.e. $\phi_t + \varepsilon \sin(kx - \omega t)\left[g + \frac{\sigma k^2}{\rho}\right] = 0$

As before but g has becomes

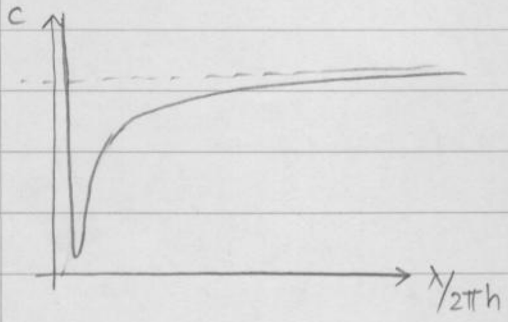
$$g + \frac{\sigma k^2}{\rho} = g\left[1 + \frac{\sigma k^2}{\rho g}\right]$$

Thus $\frac{c^2}{gh} = \cancel{\tanh}\ \frac{\tanh kh}{kh}\left[1 + \frac{\sigma k^2}{\rho g}\right]$

$$\frac{c^2}{gh} = \frac{\lambda}{2\pi h}\tanh\frac{2\pi h}{\lambda}\left[1 + \underline{\frac{\sigma}{\rho g h^2}\left(\frac{2\pi h}{\lambda}\right)^2}\right]$$

↖ only important for small λ.

surface tension only affects short waves.



Reflected waves.

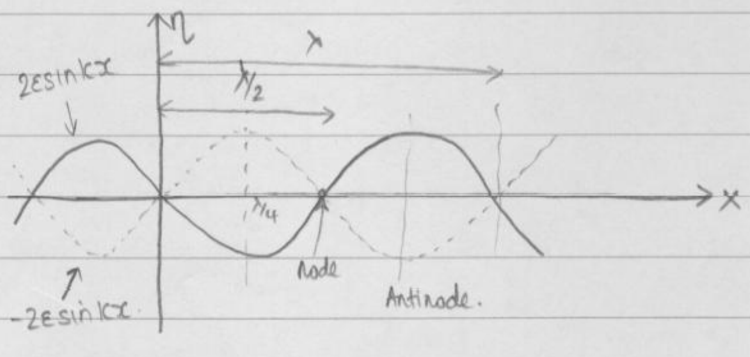
$\eta_1 = \varepsilon \sin(kx - \omega t)$

$\eta_2 = \varepsilon \sin(kx + \omega t)$ ← same wave, opp. direction.

Total surface displacement is

$$\eta = \eta_1 + \eta_2 = 2\varepsilon \sin kx \cos \omega t$$

a standing wave.



Solve for η_1: get

$\phi_1 = -\varepsilon\omega \cos(kx - \omega t) \cosh[k(y+h)] / k\sinh kh$

For η_2 change ω to $-\omega$

$\phi_2 = \varepsilon\omega \cos(kx + \omega t) \cosh[k(y+h)] / k\sinh kh.$

Total ϕ is : $\phi_1 + \phi_2$

i.e. $\phi = -2\varepsilon\omega \sin kx \sin \omega t \cosh[k(y+h)] / k\sinh kh$.

$$\frac{dx}{dt} = \frac{\partial \phi}{\partial x} = u = -2\varepsilon\omega \cos kx \sin \omega t \cosh[k(y+h)]/\sinh kh$$

$$\frac{dy}{dt} = v = \frac{\partial \phi}{\partial y} = -2\varepsilon\omega \sin kx \sin \omega t \sinh[k(y+h)]/\sinh kh.$$

Need to eliminate time to get particle paths

$$\cancel{\frac{dx}{dy}} \quad \frac{dy}{dx} = \frac{dy}{dt}\cdot\frac{dt}{dx} = \tan kx \tanh(k(y+h)).$$

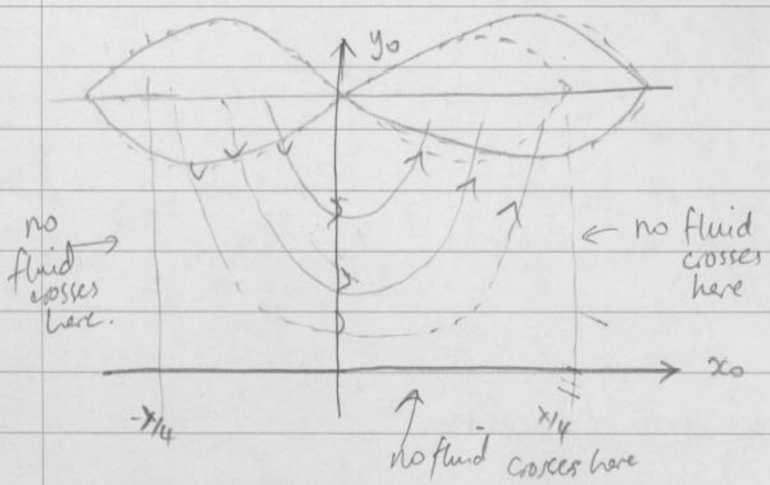
Can integrate w.o approx to get

$$\cos kx \sinh[k(y+h)] = \text{const}$$

However we can use our previous linearisation

$$x = x_0 + \varepsilon X(x, y, t)$$
$$y = y_0 + \varepsilon Y(x, y, t)$$

Then $\frac{dY}{dX} = \tan kx_0 \tanh(k(y_0+h))$ (want to find gradient to be ∞ i.e. $kx_0 = \pi/2$).



$kx_0 = \pi/2$

$\frac{2\pi}{\lambda} x_0 = \frac{\pi}{2}$

$x_0 = \lambda/4$.

Thus we can replace $x = -\lambda/4$ and $x = \lambda/4$ by solid boundaries

Lowest mode standing wave in a container

B.C is in fact $\eta_x = 0$ in solid boundaries

i.e. surface is flat

sides are antinodes (opp. of string where ends are nodes.