2301 Fluid Mechanics Notes

Based on the 2011 autumn lectures by Prof E R Johnson

03/10/11

*Recommended texts;

- A First Course in Fluid Dynamics; A.R. Patterson, Cambridge.
- Ideal & incompressible fluid Dynamics; M.E. O'Neill & F. Chorlton.

*To look at;

An Album of fluid Motion; M. van Dyke, Parabolic Press.

How does a plane fly?

Speed ← Directly proportional to U.

Directly proportional to density; ρ

Geometry of wing cross-section.

↳ K; measure of the circulation; Defined later!

lift of plane; $K\rho U$!

How fast does a surface water wave travel?

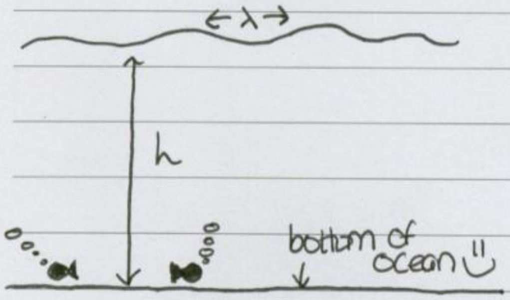
Depth of the ocean; h

Gravity (Restoring force); g

$[c] = LT^{-1}$ (dimensions of speed)

$[g] = LT^{-2}$ (dimensions of acceleration.)

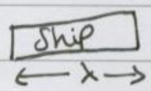
Wave length; λ $\quad [\lambda] = L = [h]$



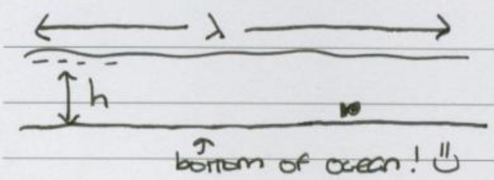
Short wave; deep water.

$\frac{\lambda}{h} << 1$

- $(g\lambda)^{1/2} \rightarrow c \sim (g\lambda)^{1/2} \rightarrow$ speed a wave travels. (different wave lengths, different speeds.)



→ max speed $= c = \sqrt{g\lambda}$



long wave - Shallow water.

$c \sim (gh)^{1/2}$

$\frac{\lambda}{h} >> 1$

Different wavelengths at same speed!

Green's Law;

- Speed x energy density $= cH^2 =$ constant. (= energy)
  ↳ This is a Quad. eqn.
- Energy $\sim H^2$
- $H^2 \sim c^{-1}$ • $H \sim c^{-1/2}$ • $c \sim h^{1/2}$ $\therefore H \propto h^{-1/4}$

Chapter 1; Specification & Kinematics!

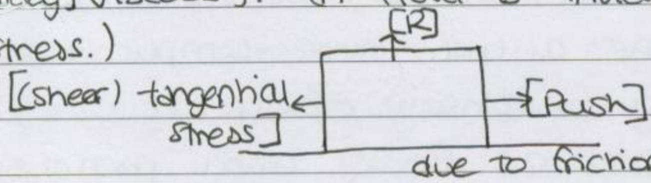
*Continuum - a substance that we can take arbitrary small volumes of and whose properties remain the same as we do so. (if $\lim_{\delta V \to 0}$ exists)

Take volume V, measure its mass m, and define its density, $\bar{\rho} = m/V$.

Could take $V > V_1 > V_2 \ldots$ and define the density at some point common to this sequence, $\rho = \lim_{V \to 0} m/V$.

This is a good approximation to reality provided that we are interested in motions at scales large compared to the mean free path. mean free path.

We will restrict attention to invicid fluids (fluids that are not [sticky] viscous). (A fluid is invicid if it cannot support a tangential (sheer) stress.)



due to friction on bottom, opposing.

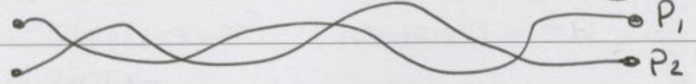
[We will not consider fluids that support a sheer stress! eg; honey!]

Summary:

1) CONTINUUM; we can discuss infintesimal volumes of fluid.
2) INVICID; the fluid cannot support a sheer stress.
3) INCOMPRESSIBLE; the volume of the fluid element remains the same throughout the motion.

An element composed of the same fluid has the same mass by conservation of mass. Hence, density is constant.

(a) This does <u>not</u> mean that the density is the same everywhere.



(b) This is a good approximation provided speeds are small compared with the speed of sound (700 mph), ie; the mach number of the flow $\frac{\text{typical speed}}{\text{sound speed}} = M$, ie; small ($\ll 1$), $M \ll 1$ Subsonic
$M > 1$ Supersonic.

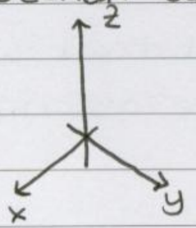
To describe the flow, we have two choices:

(a) Lagrangian labelling; label all particles & follow their motion, i.e: follow particle path.

Strength – conservation laws easy.

Drawback – simple motions can have complicated particle paths.

(b) Eulerian description – set up fixed axes.



we define a velocity field;
$\underline{u}(x, y, z, t)$ by defining the velocity $\underline{u}$ at time t to be the velocity of the fluid element (or fluid particle) that is at $\underline{x}$ at time t.

Strengths – velocity is a vector field; we can use vector calculus.

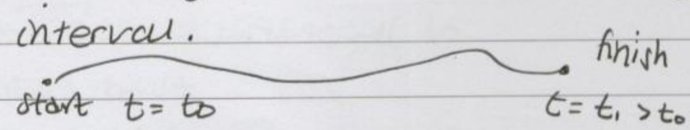
Drawback – conservation laws become a little more complicated.

we do the same thing for density: $\rho(x, y, z, t)$, i.e; although in incompressible flows, each particle maintains its own density, the Eularian density (at a point) can change as different particles occupy that point at different times.

Of course, in a homogenous fluid, ρ = constant.

There are three ways of visualising or describing a motion:-

1) PARTICLE PATH; the path traced out by the fluid element during a given time interval.



2) Streakline; the locus of all particles that have passed through a given point in a given time interval.

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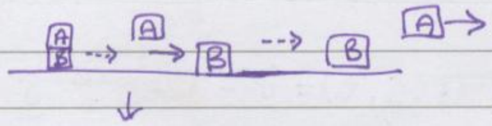
A transmits a shear stress to B.



A does not transmit shear (tangential) stress; (force/unit area)

↓
NOT VISCOUS - INVISCID

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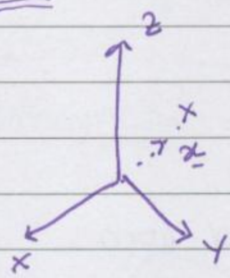
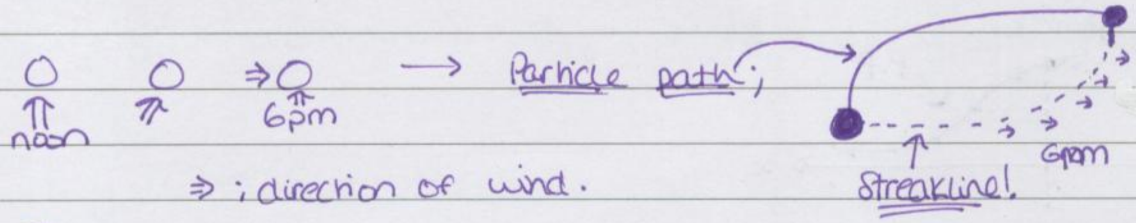


Eularian : $\underset{\sim}{u}(\underline{x}, t)$ = velocity of particle that happens to be at $\underline{x}$ at time t.

Visualise:

1) Particle path; Path traced out by a fluid element in a given time interval.
2) Streakline; The locus of all particles that have passed through a given point in a given time interval.
3) Streamline; A line whose tangent gives the direction of the velocity at that point.



⇒ ; direction of wind.

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Suppose we are given a velocity field $\underset{\sim}{u}(\underline{r}, t)$.
Particle path satisfy's $\frac{dr}{dt} = u(\underline{r}, t)$ with $\underline{r} = \underline{r_0}$ at $t=0$.

Example: Consider the two-dimensional velocity field

$$\underset{\sim}{u}(\underset{\sim}{r}, t) = \hat{\imath} - 2te^{-t^2}\hat{\jmath}$$

[2D Flow field : field independant of the third direction. i.e; the same in each x-y plane. We shall also take the velocity component in the normal direction to be zero.]

In cartesians; It is conventional to write

$$\underset{\sim}{u}(x,y,z,t) = u(x,y,z,t)\,\underline{\hat{i}} + v(x,y,z,t)\,\underline{\hat{j}} + w(x,y,z,t)\,\underline{\hat{k}}$$

i.e: $\underline{u} = (u, v, w)$.

2D flow:- $w \equiv 0$. $u = u(x,y,t)$ $v = v(x,y,t)$

Flow same at each z!

$$\underset{\sim}{u}(\underline{r}, t) = \underline{\hat{i}} - 2te^{-t^2}\underline{\hat{j}}$$

so $\frac{d\underline{r}}{dt} = \underline{u}$. i.e: $\frac{dx}{dt} = u$ $\frac{dy}{dt} = v$

Here; $u = 1$, $v = -2te^{-t^2}$.

so $\frac{dx}{dt} = 1$ and $\frac{dy}{dt} = -2te^{-t^2}$.

i.e: $x = t + x_0$ and $y = e^{-t^2} + y_0$

What is the path traced out by the particle released from (1,1) at $t=0$?

At $t=0$, $x=1$ so $x_0 = 1$.

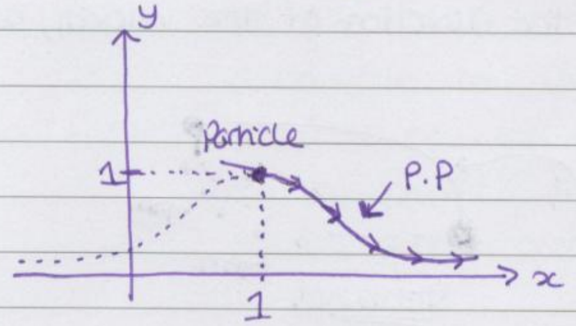
$y = 1$, so $y_0 = 0$.

Thus, the particle path is $x = 1+t$, $y = e^{-t^2}$ - Parameterised by t.

[(Parametric plot): mathematica.]

Here, $t = x-1$ so $y = e^{-(x-1)^2}$.

*P.P = Particle Path



• What is the streakline traced out at particles are released from (1,1) at times $\tau < 0$ when viewed at time $t=0$?

Particle paths: $x = t + x_0$. $y = e^{-t^2} + y_0$

At τ, Particle in laws was at (1,1): That's when it was emitted.

$1 = \tau + x_0$. $1 = e^{-\tau^2} + y_0$

i.e: $x_0 = 1 - \tau$ $y_0 = 1 - e^{-\tau^2}$

The particle is at (x,y) at time t where

$$x = t + 1 - \tau, \quad y = e^{-t^2} + 1 - e^{-\tau^2}$$

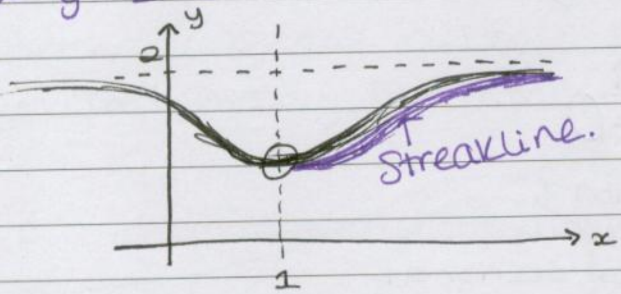
So at $t=0$, it is at $x = 1-\tau$, $y = 2-e^{-\tau^2}$.

↳ Parameterised by the time of emission.

Sufficiently simple that we can eliminate τ.

$\tau = 1-x$.

So $y = 2 - e^{-(x-1)^2}$



Emmitted at $\tau < 0$; looked at $t=0$.

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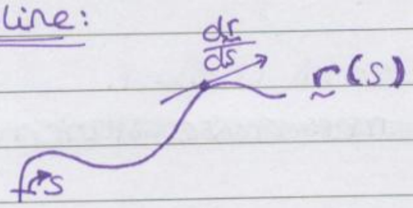
Streamline:



Parameterise $\underline{r}$ on s;

$$\frac{d\underline{r}}{ds} = \underline{u}(\underline{r}, t)$$

Thus, the streamlines at some time $t = t_0$, are given by solving

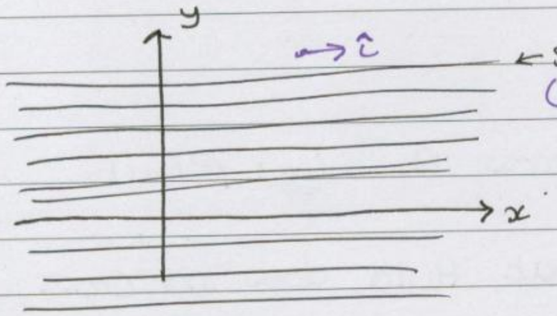
$$\frac{d\underline{r}}{ds} = \underline{u}(\underline{r}, t_0)$$

Example: for our velocity field, what are the streamlines at $t=0$?

• $\frac{dx}{ds} = u(x,y,0) = 1$, • $\frac{dy}{ds} = v(x,y,0) = -2te^{-t^2}\big|_{t=0}$

$= 0$

$\therefore y = \text{constant}$; $x = s + x_0$.



← streamlines (Horizontal)

* At $t=0$, $\underline{u} = \hat{\underline{i}}$;

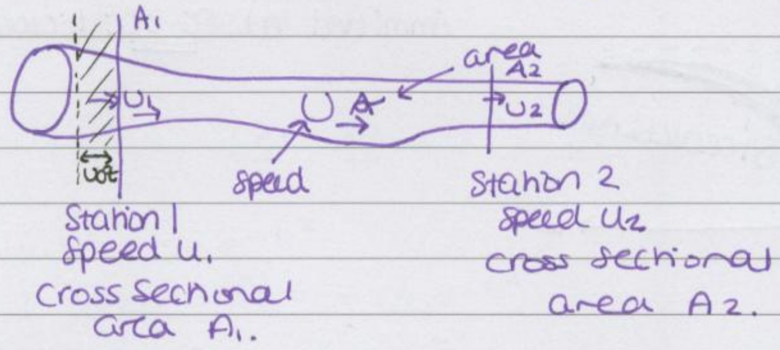
i.e: The tangent to streamline's does give the velocity field!

* In a steady flow, all these are the same.

STEADY: $\frac{d\underline{u}}{dt} = 0$ (Does NOT mean that $\underline{u} = 0$)

2) Conservation of Mass:

Suppose a fluid of constant density ρ flows through a tube of cross-sectional area A. Suppose the fluid velocity is uniform and unidirectional of size U at each cross-section.



The amount of mass between the two stations is fixed.
In a time interval, dt, an amount

$$\rho A_1 u_1 \, dt$$

of mass crosses station 1.

The amount crossing A_2 in time dt is

$$\rho A_2 u_2 \, dt.$$

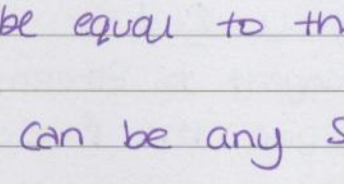
By conservation of mass, these are the same.
So $A_1 u_1 = A_2 u_2$.

* **FLUX:**
Or in terms of flux, the rate at which mass crosses A_1 is

$$\frac{\rho A_1 u_1 \, dt}{dt} = \rho A_1 U_1.$$

This must be equal to the flux across A_2. i.e: $\rho A_2 U_2$.

The tube can be any surface that fluid does not cross.



Stream tube.
↳ formed by taking a closed loop of particles and drawing the streamline eminating from them. Flow cannot cross this tube as $\underline{u}$ is tangential to streamlines.

- If area halves, speed doubles.

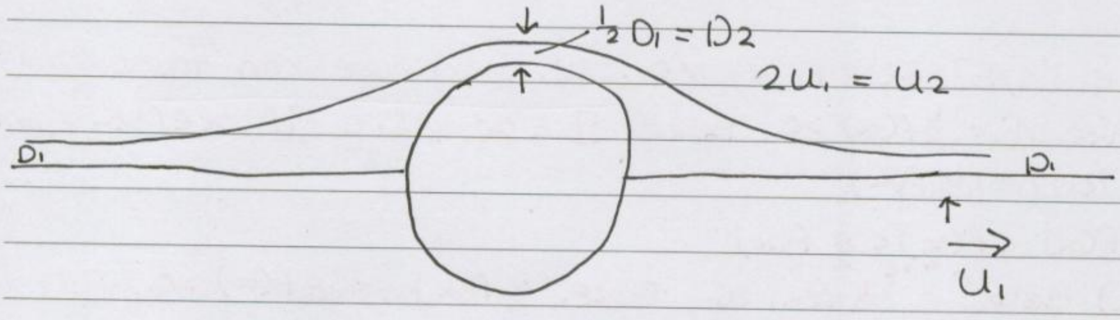
In 2D:-

Because the third velocity component doubles, $w = 0$, streamlines compress only in the x-y plane so we have

$$u_1 D_1 = u_2 D_2,$$

where D is the distance between streamlines.

i.e: Speed inversely proportional to the separation of streamlines.



Monday 9–11; G06 (Roberts)
10/10/11 Thursday 11; Chadwick

Hwk 1: due & collected Monday 17th at 9am (G06)
Hwk 2: due & collected Thursday 20th at 12 noon. (Chadwick)

Theorem 1:
If f is continuous in $[a,b]$ and $\int_c^d f = 0$ for each $(c,d) \subseteq [a,b]$, then $f \equiv 0$ on $[a,b]$.

↳ Proof:
Suppose $\exists\, \alpha \in [a,b]$ s.t $f(\alpha) \neq 0$. W.l.o.g we can take $f(\alpha) > 0$. Write $\sigma = \frac{1}{2} f(\alpha) > 0$. Hence $\exists\, \varepsilon > 0$, s.t if $x \in (\alpha - \varepsilon, \alpha + \varepsilon)$
$|f(x) - f(\alpha)| < \sigma = \frac{1}{2} f(\alpha)$,,

i.e: $0 < \frac{1}{2} f(\alpha) < f(x) < \frac{3}{2} f(x)$

Thus $\int_{\alpha-\varepsilon}^{\alpha+\varepsilon} f(x)\, dx > \int_{\alpha-\varepsilon}^{\alpha+\varepsilon} \frac{1}{2} f(\alpha)\, dx = 2\varepsilon \cdot \frac{1}{2} f(\alpha) = \varepsilon f(\alpha) > 0$.

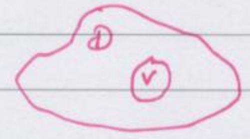
But $\int_c^d f = 0$, $\forall (c,d) \subseteq [a,b]$ # so $\nexists\, \alpha$, i.e; $f \equiv 0$ in $[a,b]$.

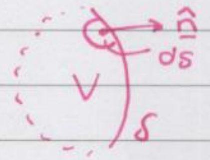
This result extends immediately to n dimensions.

Ansatz:- Suppose we ~~have~~ wish to derive an equation $f = 0$ for a fluid in 3D. Let the fluid occupy a domain $\mathcal{D}$ in 3D. Take an arbitrary subdomain V of $\mathcal{D}$. (IMPORTANT!)

Show that $\int_V f$ vanishes. Then $f \equiv 0$ in $\mathcal{D}$ because V is Arbitrary!!
i.e: $\int_V f = 0$, for every subdomain V of $\mathcal{D}$.

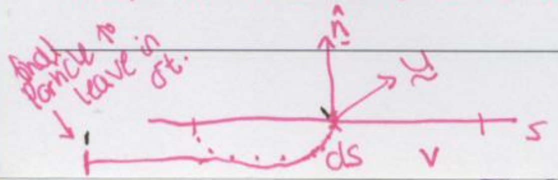
Conservation of Mass:
Consider a fluid occupying a domain $\mathcal{D}$. Let V be any subdomain of $\mathcal{D}$ with surface S.

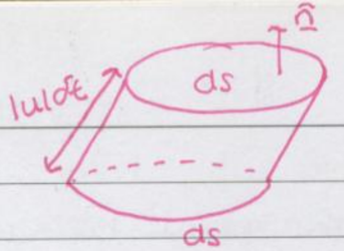




Consider a small element ds of S, with outward pointing unit normal, $\hat{n}$.
Let the velocity field in D be $\underset{\sim}{u}$. Take the density of the fluid to be constant & equal to ρ everywhere.
Then, in time $\delta t \ll 1$, a mass $\rho(\underset{\sim}{u} \cdot \hat{n})\, ds \cdot \delta t$ crosses ds.

$H = |u|\,\delta t$
↑ distance.





volume = Area of base × height = ds • $(\underline{u}\cdot\hat{\underline{n}})\,\delta t$

- component of $\underline{u}$ in direction $\hat{\underline{n}}$, i.e; height.

Thus, the total mass passing out of V, is:

$$\int_S \rho(\underline{u}\cdot\hat{\underline{n}})\,ds\,\delta t = \rho\delta t\int_S \underline{u}\cdot\hat{\underline{n}}\,ds$$

• But, to conserve mass in V, this must be 0.

i.e: $\int_S \underline{u}\cdot\hat{\underline{n}}\,ds = 0.$

[$\rho\int\underline{u}\cdot\hat{\underline{n}}\,ds$ = outward mass flux accross δS.]

Divergence Theorem says that $\int_V \underline{\nabla}\cdot\underline{u}\,dv = 0$.
Thus, we have ∀ subregions V of $\mathcal{D}$,

$$\int_V \underline{\nabla}\cdot\underline{u}\,dv = 0$$

Thus, by lemma, $\underline{\nabla}\cdot\underline{u} = 0$ in $\mathcal{D}$

In 2D: If $\underline{u} = u(x,y,t)\,\hat{\underline{i}} + v(x,y,t)\,\hat{\underline{j}}$, then

$$\underline{\nabla}\cdot\underline{u} = \frac{du}{dx} + \frac{dv}{dy}$$

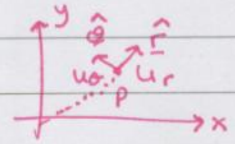
So, 1) $\underline{u} = 7x\hat{\underline{i}} - 5y\hat{\underline{j}}$ - NOT INCOMPRESSIBLE!

$$\frac{du}{dx} + \frac{dv}{dy} = 7 - 5 = 2 \neq 0.$$

i.e: COMPRESSIBLE!
(non-constant ∴ ρ)

i.e: Incompressible velocity fields are not arbitrary!

In 3D: $\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0.$



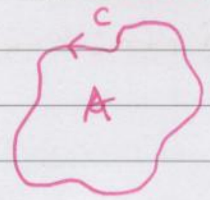
* In polars: $\underline{u} = u_r\,\hat{\underline{r}} + u_\theta\,\hat{\underline{\theta}}$

$$\underline{\nabla}\cdot\underline{u} = \frac{1}{r}\frac{\partial}{\partial r}(ru_r) + \frac{1}{r}\frac{\partial u_\theta}{\partial\theta}$$

$$\underline{\nabla} = \frac{\partial}{\partial r}\,\hat{\underline{r}} + \frac{1}{r}\frac{\partial}{\partial\theta}\,\hat{\underline{\theta}}$$

Reminder:

* Green's Lemma:



Consider a closed region A in the plane bounded by a curve C, taken counter-clockwise.

$$\int_A \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) dA = \oint u\,dy - v\,dx$$

Thus, in 2D incompressible flow, $\oint_C u\,dy - v\,dx = 0$, for any closed curve C.

$$d\underline{r} = dx\,\hat{\underline{i}} + dy\,\hat{\underline{j}}$$

$$\underset{\sim}{F} = -\underline{v}\,\hat{\underline{i}} + \underline{u}\,\underline{j} = \hat{\underline{k}} \times (u\,\hat{\underline{i}} + v\,\hat{\underline{j}}) = \hat{\underline{k}} \wedge \underline{u}$$

i.e: rotate $\underline{u}$ by 90°.

$\oint_C \underset{\sim}{F} \cdot d\underline{r} = 0 \quad \forall$ closed curves C in $\mathcal{D}$.

i.e: $\underset{\sim}{F}$ is a conservative vector field.

i.e: $\underset{\sim}{F}$ is derivable from a potential

i.e: $\exists\, \psi$ s.t $\underset{\sim}{F} = \underset{\sim}{\nabla} \psi$

i.e: $k \wedge \underline{u} = \nabla \psi$

i.e: $\underline{u} = -\underline{k} \wedge \underline{\nabla} \psi$

13/10/11

sheet 1: due 9am monday
sheet 2: due 12pm Thursday.
} Answers handed out after sheets collected.

· Incompressibility: (constant density)

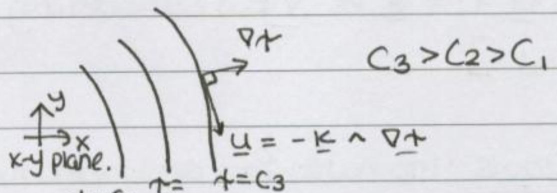
$\Rightarrow \nabla \cdot \underline{u} = 0 \qquad [\Rightarrow \exists \underline{A} \text{ s.t } \underline{u} = \nabla \wedge \underline{A}]$

In 2D: $\underline{u} = u(x,y)\hat{\underline{i}} + v(x,y)\hat{\underline{j}}$

$$\nabla \cdot \underline{u} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \Rightarrow \exists \psi \text{ s.t } \underline{u} = \hat{\underline{k}} \wedge \Delta \psi = -\nabla \wedge (\psi \hat{\underline{k}})$$

$\hat{\underline{u}} = -\hat{\underline{k}} \wedge \nabla \psi$



· First of all: $|\underline{u}| = |\nabla \psi|$

· Second of all: $\underline{u}$, $\nabla\psi$ are perpendicular.
In fact, $\underline{u}$ is $\nabla\psi$ rotated $\frac{\pi}{2}^c$ clockwise.

· Finally: $\underline{u}$ is tangent to the isoline $\psi = C$, for any C.
i.e: the lines $\psi = C$ are streamlines!

Ie: we have shown that in incompressible flow, $\exists$ a function ψ whose isolines are streamlines.

Example:

Show that $\underline{u} = x\hat{\underline{i}} - y\hat{\underline{j}}$ satisfies the continuity eq^n, find a stream – function, sketch some streamlines (+ suggest a flow).

$\rightarrow$

Continuity: $\frac{du}{dx} + \frac{dv}{dy} = 0.$ Here $u = x$, $v = -y$.

$\frac{du}{dx} = 1$, $\frac{dv}{dy} = -1$.

Thus: $\frac{du}{dx} + \frac{dv}{dy} = 0$ as required.

Hence, $\exists \psi$ s.t $\underline{u} = -\underline{k} \wedge \nabla\psi$.

$$\nabla\psi = \frac{d\psi}{dx}\hat{\underline{i}} + \frac{d\psi}{dy}\underline{j}$$

· $\underline{k} \wedge \hat{\underline{i}} = +\hat{\underline{j}}$ · $\underline{k} \wedge \hat{\underline{j}} = -\hat{\underline{i}}$

So: $-\underline{k} \wedge \nabla \psi = + \frac{d\psi}{dy}\hat{\underset{\sim}{i}} - \frac{d\psi}{dx}\hat{\underline{j}} = u\hat{\underline{i}} + v\hat{\underline{j}}$.

so $u = \frac{d\psi}{dy}$ and $v = -\frac{d\psi}{dx}$.

For this example, $u = x$. Thus; $\frac{d\psi}{dy} = x$. So $\psi = xy + f(x)$.
Where f is an arbitrary function of x.
This implies that

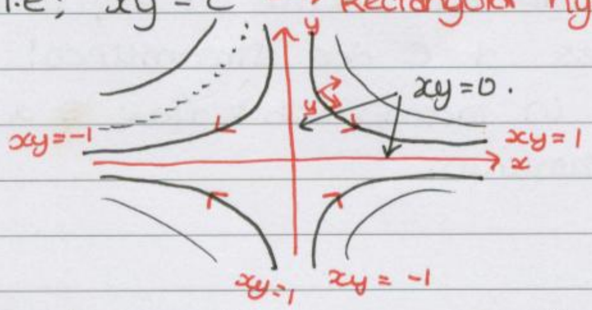
$$\frac{d\psi}{dx} = y + f'(x).$$

But $\frac{d\psi}{dx} = -v = y$.

Comparing gives; $f'(x) = 0$. i.e: f is a constant.
W.L.O.G; we can take $f = 0$.
[↳ since $\underline{u} = -\underline{k} \wedge \nabla\psi$, so adding a constant to ψ does not change $\underline{u}$]

• ψ is unique to within an additive constant!
Hence the streamfunction ψ is $\psi = xy$.

Streamlines: The lines ψ = constant.
i.e; $xy = c$ → Rectangular hyperbolae!

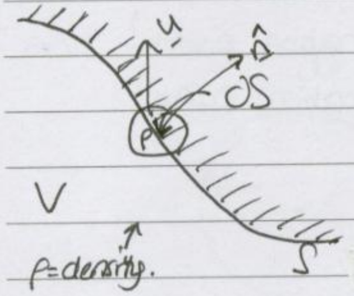


OR: $u = x$
so if $x > 0$, $u > 0$

Since $|\underline{u}| = |\nabla\psi|$, speed is directly proportional to $|\nabla\psi|$.
* or, equivalently, $|\underline{u}|$ is inversely proportional to the separation of lines of constant ψ.

↳ Known as stagnation point flow, as the origin is a stagnation point where $\underline{u} = 0$.
This flow could be 2 colliding jets of equal strength.

Flow conditions at a solid boundary:



*Solid ⇒ impermeable (no flow can go through it)

i.e: No flow through the boundary!

• The mass of fluid passing through dS in time δt is:

$$\rho(\underline{u}\cdot\hat{n})\,dS\,\delta t.$$

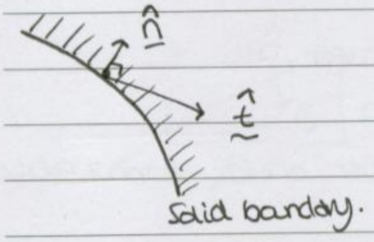
(or, there is a mass flux, $\rho(\underline{u}\cdot\hat{n})\,dS$ across dS)
(rate at which mass crossed dS)

For no mass flux, $\underline{u}\cdot\hat{n}=0$ on S.
on a solid boundary, $\underline{u}\cdot\hat{n}=0$.
i.e: velocity is tangential to the surface.

* If the fluid is also viscous (can support a sheer stress), additionally, the tangential component of $\underline{u}$ vanishes also, so $\underline{u}\equiv 0$ on a solid boundary. [→ STOKES ←] * → ASIDE

In terms of the stream-function, $\hat{n}\cdot\underline{u} = -\hat{n}\cdot\hat{\underline{k}}\wedge\underline{\nabla}\psi = -(\hat{n}\wedge\hat{\underline{k}})\cdot\underline{\nabla}\psi$

$= -\hat{\underset{\sim}{t}}\cdot\underset{\sim}{\nabla}\psi$



$\hat{\underset{\sim}{t}}$ - unit tangent to the surface.

$= -\dfrac{d\psi}{ds}$ along surface (directional derivative)

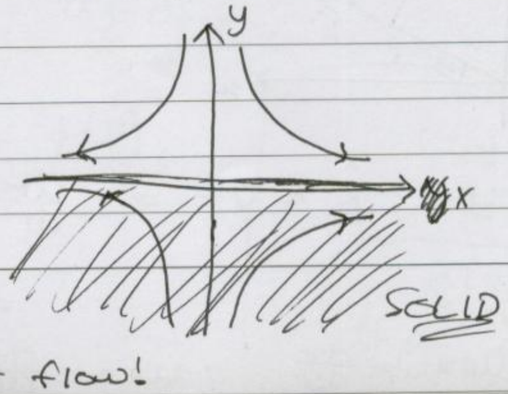
But $\underline{u}\cdot\hat{n}=0 \Rightarrow \dfrac{d\psi}{ds}=0$ along a solid boundary; i.e: ψ = constant on solid b.dary.

• Equivalently, any line ψ = constant has $\underline{u}$ tangential.
i.e: can be a solid boundary.

i.e: on a solid boundary, ψ = constant. Any line ψ = constant, can be replaced by a solid boundary without affecting a inviscid flow.

Solid boundary: $\underline{u}\cdot\hat{n}=0$ or ψ = constant.

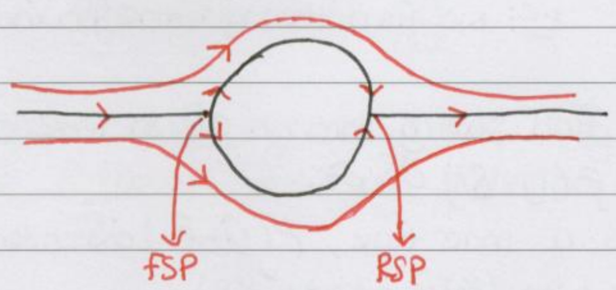
Ex: $u = x \quad v=-y \quad \psi = xy$



Replace any s'line by solid boundary, (in inviscid flow) without changing flow.
Here we obtain a jet hitting a wall. - Stagnation point flow!

e.g: Front and Rear stagnation points in uniform flow past a circular cylinder.

FSP = Front Stagnation point!
RSP = Rear Stagnation point!



Ex 2: (same Q's as in ex 1.)

Now $u = 2y$, $v = -2x$.

Thus; $\frac{d\psi}{dy} = u = 2y$; so $\psi = y^2 + f(x)$

so $\frac{d\psi}{dx} = f'(x)$

But $\frac{d\psi}{dx} = -v$, so $f'(x) = 2x$.

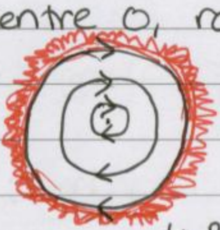
i.e: $f(x) = x^2 + c$.

W.L.O.G; take $c = 0$.

so $\psi = x^2 + y^2$.

Streamlines are lines $x^2 + y^2 = a^2$ for a, constant.

i.e: circles, centre O, radius a.



i.e: a saucepan or beaker on a turntable.

↳ Rotating as a solid body.

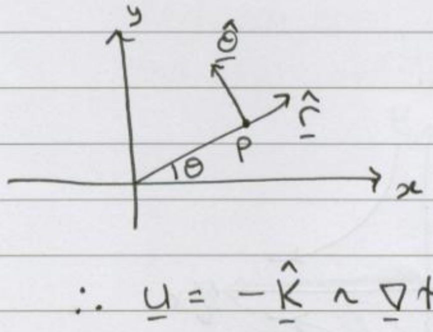
Solid body rotation; $\underline{u} = 2\hat{\underline{k}} \wedge \underline{r}$

In polar coordinates; (cylindrical polar coordinates)

$$\psi = x^2 + y^2 = r^2$$

$\underline{u} = -\hat{\underline{k}} \wedge \underline{\nabla}\psi$ ⟸ co-ordinate free.

$$\underline{\nabla}\psi = \frac{d\psi}{dr}\hat{\underline{r}} + \frac{1}{r}\frac{d\psi}{d\theta}\hat{\underline{\theta}}.$$



$$\hat{\underline{k}} \times \hat{\underline{r}} = \hat{\underline{\theta}}$$

$$\hat{\underline{k}} \wedge \hat{\underline{\theta}} = -\hat{\underline{r}}$$

$$\therefore \underline{u} = -\hat{\underline{k}} \wedge \nabla\psi = -\frac{d\psi}{dr}\hat{\underline{\theta}} + \frac{1}{r}\frac{d\psi}{d\theta}\hat{\underline{r}}$$

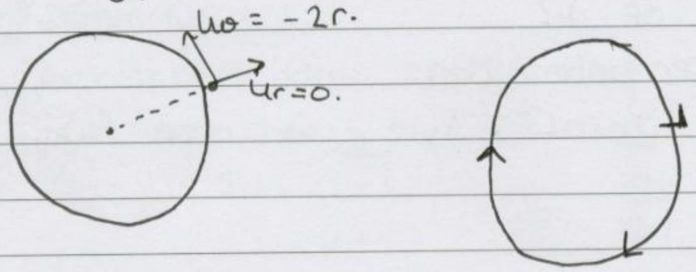
$$= u_r\hat{\underline{r}} + u_\theta\hat{\underline{\theta}}$$

• Comparing: • $u_r = \frac{1}{r}\frac{d\psi}{d\theta}$ • $u_\theta = -\frac{d\psi}{dr}$ ⇒ WORTH REMEMBERING!

we have: $\psi = r^2$ (in ar example)

Thus $u_r = \frac{1}{r}\frac{d\psi}{d\theta} = 0.$

and $u_\theta = -\frac{d\psi}{dr} = -2r.$

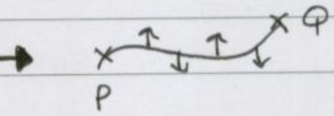


· velocity increases linearly with distance.

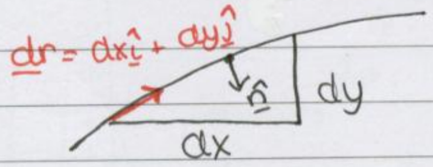
· A physical interpretation of the streamfunction:

The volume flux in a clockwise direction across any line joining a point P to a point Q in a flowfield is given by:

$$\boxed{\psi(Q) - \psi(P)} \rightarrow$$



[c.f; work done is independant of path.]



Volume flux crossing a length ds:

$$(\underline{u} \cdot \hat{\underline{n}})\, ds$$

$\cdot\ \hat{\underline{n}} \cdot \underline{dr} = 0.$

Try $\hat{\underline{n}} = dy\, \hat{\underline{i}} - dx\, \hat{\underline{j}}$

$\hookrightarrow \hat{\underline{n}} \cdot \underline{dr} = 0$

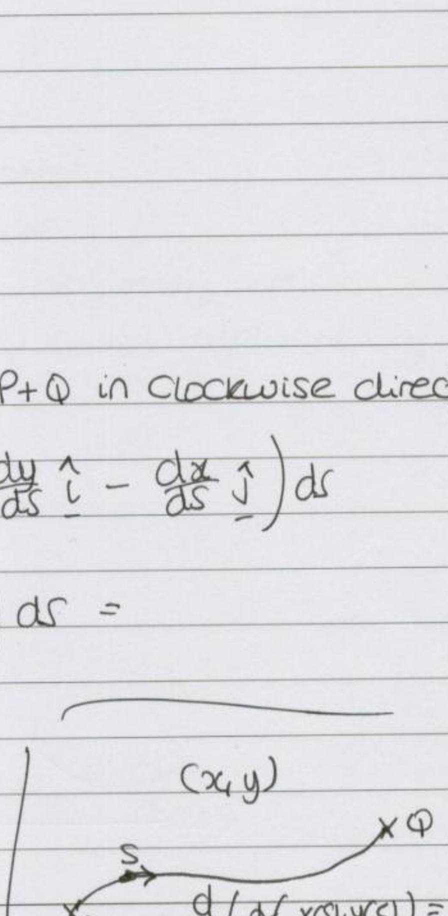
Thus, $\hat{\underline{n}} = \frac{dy\, \hat{\underline{i}} - dx\, \hat{\underline{j}}}{\sqrt{dx^2 + dy^2}} = \frac{dy}{ds}\hat{\underline{i}} - \frac{dx}{ds}\hat{\underline{j}}.$

Thus, the total flux crossing the line between P + Q in clockwise direction is

$$\int_P^Q (\underline{u} \cdot \hat{\underline{n}})\, ds = \int_P^Q \left(\frac{d\psi}{dy}\hat{\underline{i}} - \frac{d\psi}{dx}\hat{\underline{j}}\right) \cdot \left(\frac{dy}{ds}\hat{\underline{i}} - \frac{dx}{ds}\hat{\underline{j}}\right) ds$$

$$= \int_P^Q \left(\frac{d\psi}{dx}\frac{dx}{ds} + \frac{d\psi}{dy}\frac{dy}{ds}\right) ds =$$

$$\int_P^Q \frac{d\psi}{ds}\, ds = \psi(Q) - \psi(P) \,//$$

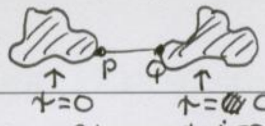


$$\frac{d}{ds}(\psi(x(s), y(s))) = \frac{d\psi}{ds}\frac{dx}{ds} + \frac{d\psi}{dy}\frac{dy}{ds} \,//$$

1 boundary

$\frac{d\psi}{dx} = -v$

$\frac{L^2T^{-1}}{L} = LT^{-1}$



• What are the dimensions of ψ?

volume / unit time per unit width.

$L^3T^{-1}L^{-1}$, i.e: L^2T^{-1}, i.e: an area flux.

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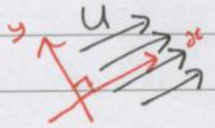
Homework hint: (Sheet 2)

vorticity $\underset{\sim}{\omega} = \underset{\sim}{\nabla} \wedge \underset{\sim}{u}$

Examples of stream functions:

1) perhaps, the simplest flow is a uniform stream.



W.L.O.G take x-axis in the direction of the flow.

Then $\underset{\sim}{u} = U$, $v = 0$ so $\frac{d\psi}{dy} = u = U$

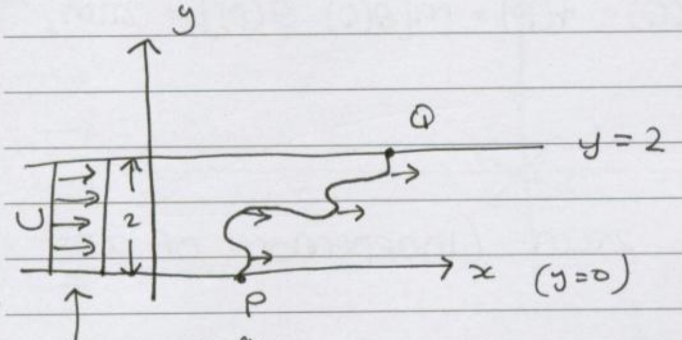
so $\psi = Uy + f(x)$

so $\frac{d\psi}{dx} = f'(x)$

But $\frac{d\psi}{dx} = -v = 0$

So $f'(x) = 0$; Hence, we can take $f = 0$

so $\psi = Uy$



velocity profile.

Flux across $x = 0$ is $2U$ i.e; length x speed.

- Flux across PQ must also be $2U$ because
  1) No fluid escapes across $y = 0$ or $y = 2$ as they are streamlines. (+ so no normal flow, i.e, could replace by solid boundary)

  OR

  2) Flux $= \psi(Q) - \psi(P) = 2U - 0 = 2U$.

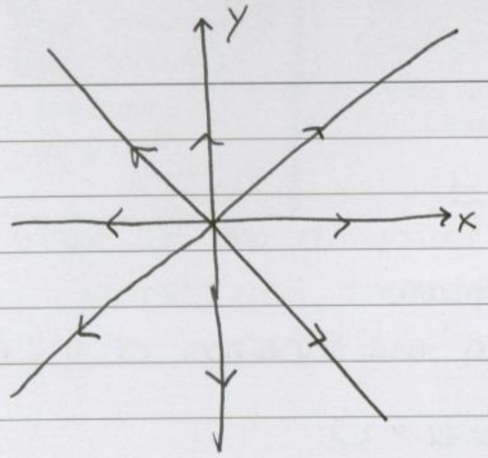
2. Isotropic Source;

This has stream function $\psi = m\theta$.

This gives $\underset{\sim}{u} = u_r \hat{\underline{r}} + u_\theta \hat{\underline{\theta}}$

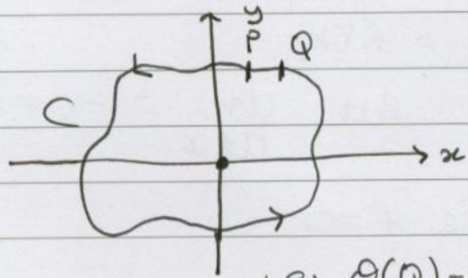
where $u_r = \frac{1}{r}\frac{d\psi}{\partial\theta}$ • $u_\theta = -\frac{d\psi}{dr}$

$\therefore u_r = \frac{m}{r}$ • $u_\theta = 0$



The velocity field is the same in all directions, i.e: independant of θ.
i.e; it is Isotropic.

· Now consider any circuit containing the origin:-

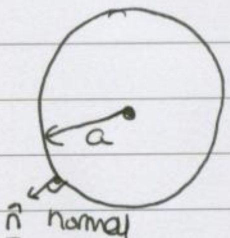


The flux across C;
Going around any closed curve containing the origin, θ increases by 2π.

i.e: $\theta(Q) - \theta(P) = 2\pi$

So $\psi(Q) - \psi(P) = m[\theta(Q) - \theta(P)] = 2\pi m$

=



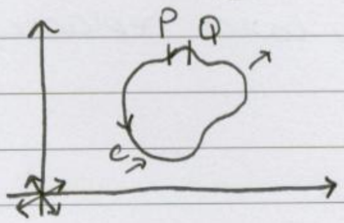
$\hat{n}$ normal velocity.

Speed $= m/a$
length $= 2\pi a$
Flux $= 2\pi a \cdot \frac{m}{a} = 2\pi m$ (Independant of a)

$\underline{u} \cdot \hat{\underline{n}} = \underline{u} \cdot \hat{\underline{r}} = u_r = m/a$

=

If curve does not circle the origin:



$\theta(P) = \theta(Q) \Rightarrow$ Flux $= 0$
I.e; no net flux accross C!

[m = rate at which fluid is created, i.e: strength of the source]

By taking successively smaller circles, we see that only at the origin is fluid created and it is created there at a flux $2\pi m$. We call $2\pi m$ the strength of the source.

i.e: a source of strength m has $\psi = m\frac{\theta}{2\pi}$

In this case (strength m) $u_r = \frac{m}{2\pi r}$, singular at the origin but well-behaved everywhere else.

$\psi = m\theta$, $2\pi m$ strength
$\frac{m\theta}{2\pi} = \psi$, m strength
$\psi = 7\theta$, 14π strength
(↳ constitent books use both.)

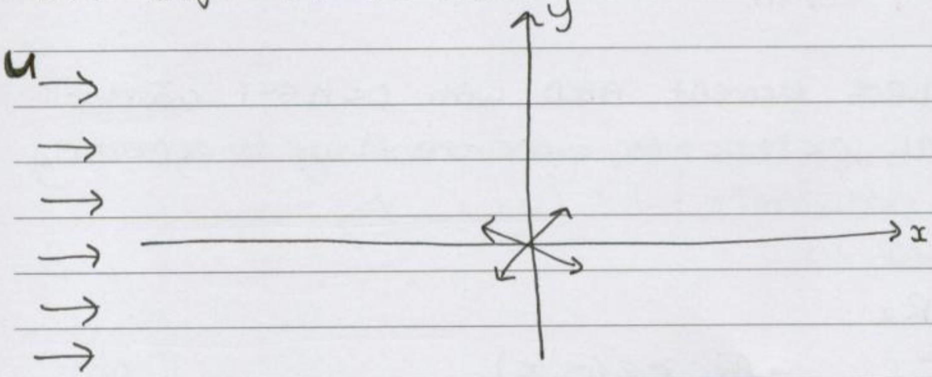
Example 3:

combine these;

i.e: An isotropic source of strength $2\pi m$ in a uniform stream of speed U.

Take x-axis in direction of stream.

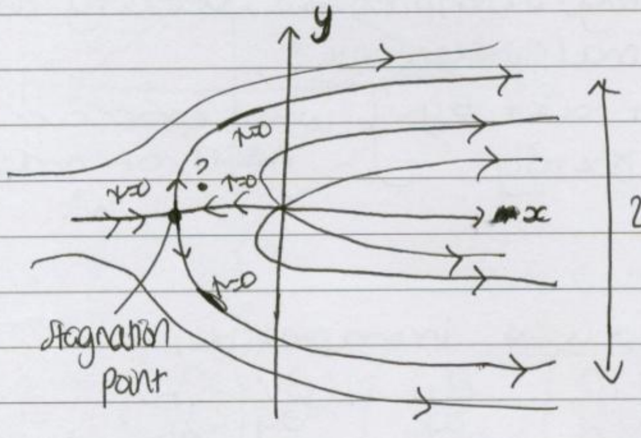
Take origin at the source.



Notice: - Source dominates ~~for~~ at origin (for r sufficiently small.)

- Stream dominates for r sufficiently large.

$u_r = m/r$.



$\psi = Uy + m\theta$

$\underset{\sim}{u} = 0$.

$u_\theta = 0 \Rightarrow \frac{d\psi}{dr} = 0$ $\quad u_r = 0$

$\frac{d\psi}{d\theta} = 0$.

Flux = width × speed

$2W \times U = 2\pi m$

$W = \frac{\pi m}{U}$

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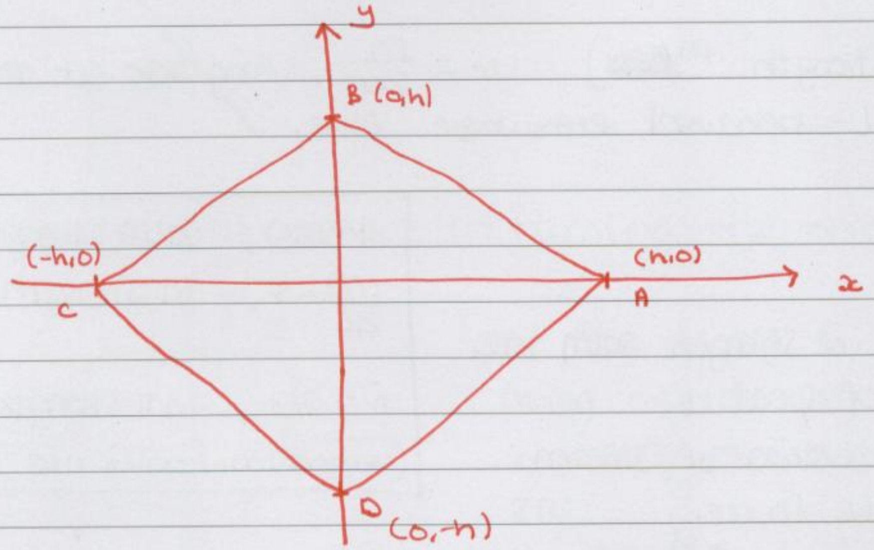
Sheet 3: flow exists! $\psi = c \to ?$

2) local motions at a point:



Consider the initially square element ABCD with $0 < h \ll 1$. consider motion in the time interval $0 < \delta t \ll 1$ so that the flow is effectively steady

• Taylor's Theorem:

$f(x) = f(0) + xf'(0) + R_2$

• $R_2 = \frac{1}{2} f''(\xi) x^2$ • for $\xi \in (0, x)$

i.e: $f(x) = a + bx$ plus error of order x^2 where $a = f(0)$, $b = f'(0)$.

What is the effect of an arbitrary, incompressible, velocity field $\underline{u}(x, y, t)$ do to our infintesimal element?

• from Taylor's theorem, (in 2d) $\begin{rcases} u = U + \alpha x + \beta y \\ v = V + \gamma x + \delta y \end{rcases}$ with error over ABCD of order h^2.

*where $U = u(0,0)$.

$\alpha = \frac{\partial u}{\partial x}(0,0)$

$\beta = \frac{\partial u}{\partial y}(0,0)$

$V = v(0,0)$

$\gamma = \frac{\partial v}{\partial x}(0,0)$

$\delta = \frac{\partial v}{\partial y}(0,0)$

• $\underline{u}$ is incompressible, so

$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$ everywhere.

In particular, $\alpha + \delta = 0$

*useful to write $\beta = \theta - \phi$

$\gamma = \theta + \phi$

Then $\theta = \frac{1}{2}(\gamma + \beta)$, $\phi = \frac{1}{2}(\gamma - \beta) = \frac{1}{2}\left(\frac{dv}{dx} - \frac{du}{dy}\right)$ //

Now
$$\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} U \\ V \end{pmatrix} + \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$

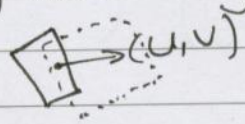
In time δt, a point $\begin{pmatrix} x \\ y \end{pmatrix}$ within ABCD moves by an amount

$$\begin{pmatrix} \delta x \\ \delta y \end{pmatrix} = \begin{pmatrix} u \\ v \end{pmatrix}\delta t = \begin{pmatrix} U \\ V \end{pmatrix}\delta t + \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}\delta t$$

i.e.:
$$\begin{pmatrix} \delta x \\ \delta y \end{pmatrix} = \underbrace{\begin{pmatrix} U \\ V \end{pmatrix}}_{I}\delta t + \left[\alpha \underbrace{\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}}_{II} + \theta \underbrace{\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}}_{III} + \phi \underbrace{\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}}_{IV}\right]\begin{pmatrix} x \\ y \end{pmatrix}\delta t$$

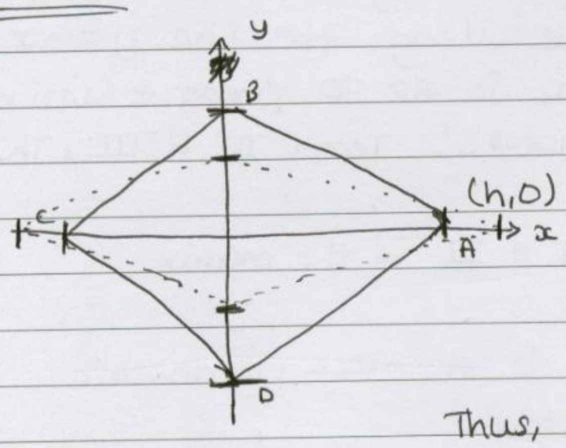
Term I:

This term simply moves every point at speed $\begin{pmatrix} u \\ v \end{pmatrix}$.



- Translation of the centre of mass at speed (U,V)

Term II:



This moves A by

$$\begin{pmatrix} \delta x \\ \delta y \end{pmatrix} = \alpha \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} h \\ 0 \end{pmatrix}\delta t$$
$$= \begin{pmatrix} \alpha h \delta t \\ 0 \end{pmatrix}$$

Thus, C moves by $\begin{pmatrix} \delta x \\ \delta y \end{pmatrix} = \begin{pmatrix} -\alpha h \delta t \\ 0 \end{pmatrix}$

B: $\begin{pmatrix} \delta x \\ \delta y \end{pmatrix} = \alpha \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 0 \\ h \end{pmatrix}\delta t = \begin{pmatrix} 0 \\ -\alpha h \delta t \end{pmatrix}$

i.e.: downwards exactly the same amount as A moves out.

D: $\begin{pmatrix} \delta x \\ \delta y \end{pmatrix} = \begin{pmatrix} 0 \\ \alpha h \delta t \end{pmatrix}$

Term II stretches the square at a rate αh in the x-direction and shrinks it at the same rate αh in the y-direction, without moving the centre of mass.

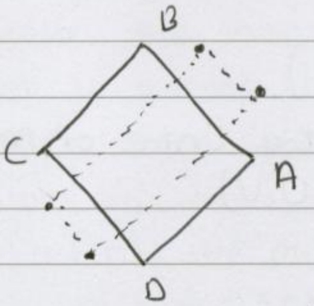
- conserving volume as expected. → a DILATION!

- A stretching in one direction.
- A shrinking in the orthogonal direction. (2D)
  - ↳ occurs at the same rate so as to conserve volume.

Term III:

At A: $\begin{pmatrix} \delta x \\ \delta y \end{pmatrix} = \begin{pmatrix} 0 \\ \theta h \delta t \end{pmatrix}$ C: $\begin{pmatrix} \delta x \\ \delta y \end{pmatrix} = \begin{pmatrix} 0 \\ -\theta h \delta t \end{pmatrix}$

B: $\begin{pmatrix} \delta x \\ \delta y \end{pmatrix} = \begin{pmatrix} \theta h \delta t \\ 0 \end{pmatrix}$



At D: $\begin{pmatrix} \delta x \\ \delta y \end{pmatrix} = \begin{pmatrix} -\theta h \delta t \\ 0 \end{pmatrix}$

I.e: another DILATION!
Stretching along the line $y=x$ and an equal & opposite shrinkage along the line $y=-x$, both at rate θh, so as to preserve volume.

- It appears that there are 2 DILATION's! Term II & III. This is not so.

The combined affect of Term II & III is the matrix

$$\begin{pmatrix} \alpha & \theta \\ \theta & -\alpha \end{pmatrix}$$ → This is a real, symmetric matrix.

It posseses two real eigenvalues.

$$\left| \begin{pmatrix} \alpha-\lambda & \theta \\ \theta & -\alpha-\lambda \end{pmatrix} \right| = 0$$

$$\therefore (\alpha-\lambda)(\alpha+\lambda) + \theta^2 = 0$$

i.e: $\alpha^2 - \lambda^2 + \theta^2 = 0$

$$\therefore \lambda^2 = \alpha^2 + \theta^2$$

Hence, we have 2 equal & opposite eigenvalues

$$\lambda = \pm\sqrt{\alpha^2+\theta^2}$$

ORTHOGONAL!

with eigenvectors $\underline{\xi}_1$ & $\underline{\xi}_2$ (say), the matrix has the form

$\begin{pmatrix} \lambda_1 & 0 \\ 0 & -\lambda_1 \end{pmatrix}$

• Precisely the form of term II. Thus expansion at rate λ_1 along ξ_1 and a contraction at rate λ_1 along the orthogonal direction $\underline{\xi}_2$.

i.e: a DILATION!

i.e: sum of 2 Dilations remains a dilation.

Next time, term IV.

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Summary:

1) $\underset{\sim}{u}(x, y, z, t)$

2) Particle paths, streaklines, streamlines

3) Incompressibility $\Rightarrow \underline{\nabla} \cdot \underline{u} = 0$ (in n-D)

4) Incompressible & 2D $\Rightarrow \underline{u} = u(x,y,t)\,\hat{i} + v(x,y,t)\,\hat{j}$

$= u_r(r,\theta,t)\,\hat{r} + u_\theta(r,\theta,t)\,\hat{\theta}$

* $\underline{\nabla} \cdot \underline{u} = 0 + 2D \Rightarrow \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \Rightarrow \exists\, \psi$ (→ scalar) s.t. $\underset{\sim}{u} = -\underline{\hat{k}} \wedge \underline{\nabla} \psi$

i.e: $u = d\psi/dy \qquad v = -\partial\psi/\partial x$

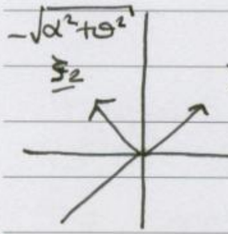
or $\quad u_r = \frac{1}{r}\frac{d\psi}{d\theta} \qquad u_\theta = -\frac{d\psi}{dr}$

5) local motion at a point

↳ Consists of a translation of the centre of mass, a dilation, and a rotation!

$$\begin{pmatrix} u \\ v \end{pmatrix} = \overset{\text{I}}{\begin{pmatrix} U \\ V \end{pmatrix}} + \left[\overset{\text{II}}{\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}} \alpha + \overset{\text{III}}{\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}} \theta + \overset{\text{IV}}{\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}} \phi \right] \begin{pmatrix} x \\ y \end{pmatrix}$$

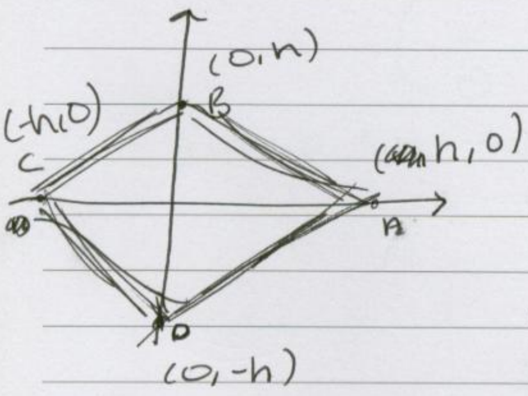
DILATION (II), DILATION (III), Rotation! (IV)



$\begin{pmatrix} \alpha & \theta \\ \theta & -\alpha \end{pmatrix}$

e'vectors ξ_1, ξ_2 ; $\xi_1 \perp \xi_2$

$$\boxed{\phi = \frac{1}{2}\left(\frac{dv}{dx} - \frac{du}{dy}\right)}$$



ϕ term:

A moves by an amount

$$\begin{pmatrix} \delta x \\ \delta y \end{pmatrix} = \begin{pmatrix} u \\ v \end{pmatrix} \delta t$$

• Term IV contributes at A:

$$\phi \begin{pmatrix} 0 \\ h \end{pmatrix} \delta t \,''$$

At B: $\begin{pmatrix} \delta x \\ \delta y \end{pmatrix} = \phi \begin{pmatrix} -h \\ 0 \end{pmatrix} \delta t$

- At A, the radial arm has length h;

Point has moved up through an angle $\phi \delta t$.

I.e: ABCD is rotating at a rate ϕ, in the anti-clockwise direction.

I.e: at a rate $\frac{1}{2}\left(\frac{dv}{dx} - \frac{du}{dy}\right)$ in the anti-clockwise direction.

Thus, we have shown that motion at a point consists of 3 and only 3 things :-

Translation of centre of mass, a dilation, and rotation about the centre-of-mass. (Notice for a solid: as above but no dilation].

Notice, $\frac{dv}{dx} - \frac{du}{dy}$ is precisely the ζ-component of $\underline{\nabla} \wedge \underline{u}$.

I.e: Curl $\underline{u}$.

It is traditional to write

$$\underset{\sim}{\omega} = \underline{\nabla} \wedge \underline{u}$$ ⇒ $\underset{\sim}{\omega}$ is the VORTICITY! of the flow.

↑ omega

I.e: The rotation of the flow.

[old name, Curl $\underline{u}$ ⇒ was rot $\underline{u}$].

- The components of $\underset{\sim}{\omega}$ are usually written

$\underset{\sim}{\omega} = \xi \hat{\underline{i}} + \eta \hat{\underline{j}} + \zeta \hat{\underline{k}}$.

(xi) (eta) (zeta)

In 2D, $\underline{u} = u(x,y,t)\,\hat{\underline{i}} + v(x,y,t)\,\hat{\underline{j}}$

$$\underline{\omega} = 0\hat{\underline{i}} + 0\hat{\underline{j}} + \zeta \hat{\underline{k}}$$

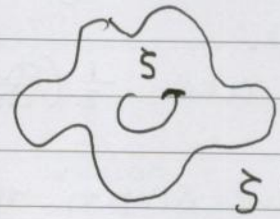
I.e: $\underline{\omega}$ is ~~[illegible]~~ solely in z-direction with $\underline{\omega} = \zeta \hat{\underline{k}}$ and

$\zeta = \frac{dv}{dx} - \frac{du}{dy}$, and it gives twice the rate of rotation of a fluid element about its C.O.M.

$[\phi = \frac{1}{2}\zeta]$

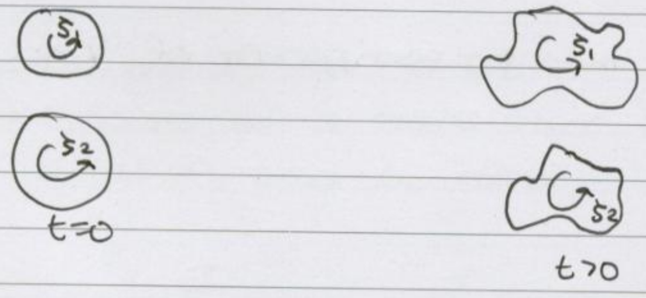
I.e: ζ is proportional to the angular momentum of a fluid element about it's C.O.M!

We can only change the rate at which a fluid element is spinning (in 2D) by applying a torque, i.e: a shear stress. But an inviscid fluid does not support a shear stress, so we cannot (in 2D) change the rate at which a fluid element spins.



$$\zeta = \frac{dv}{dx} - \frac{du}{dy}$$

i.e: a particle ~~retains~~ in a 2D inviscid fluid retains its value of ζ forever.



Consider a flow starting from rest. Then initially, $\boxed{\zeta \equiv 0}$ ($\underset{\sim}{u} \equiv 0$ at $t=0$) i.e: every particle has vorticity zero.

Hence, for all time, all particles have zero vorticity.

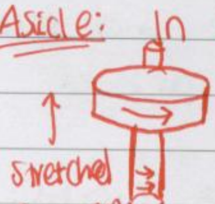
A motion where $\underline{\omega} = 0$ everywhere is called IRROTATIONAL!

e.g; any flow starting from rest is irrotational. An irrotational ~~flow~~ motion remains irrotational. $\boxed{\zeta = 0}$

(True in 3D too!) ← nothing to amplify.

↳ The persistence of irrotationality.

Aside: In 3D:



$\omega \neq 0$

$\frac{d}{\partial z} \neq 0$

e.g: Hurricane!

⇒ ζ increased

Thus, we will concentrate on irrotational flow.

Then $\underline{\nabla} \wedge \underline{u} = 0$.

Hence, $\exists \phi$ s.t $\underline{u} = \underline{\nabla} \phi$, i.e: $\underline{u}$ is derivable from a potential, the velocity potential.

(we are still in 2D or 3D) $\boxed{\underline{\nabla} \cdot \underline{u} = 0.}$

• Substituting gives $\underline{\nabla} \cdot (\underline{\nabla} \phi) = 0 \Rightarrow$ i.e; $\nabla^2 \phi = 0$

↳ Laplace's eq^n! (in 2D & 3D)

→ The governing equation for 3D incompressible, irrotational flow;

→ all we need are boundary conditions;

* on a solid boundary, $\underline{u} \cdot \hat{\underline{n}} = 0$. *

Substitute for $\underline{u}$: $\hat{\underline{n}} \cdot \underline{\nabla} \phi = 0$ on a solid boundary.

i.e: $\frac{\partial \phi}{\partial n} = 0$ on a solid boundary. → newman problem.

i.e: The normal derivative of ϕ vanishes on a solid boundary.

(The solution to Laplace's eq^n with $\frac{\partial \phi}{\partial n}$ specified on boundary, i.e: Newman problem is unique.)

Example: What is velocity potential for a uniform stream?

• Take x-axis in direction of stream.

$\underline{u} = U\hat{\underline{i}}$ so $u = U$, $v = 0$.

But $\underline{u} = \underline{\nabla} \phi = \frac{\partial \phi}{\partial x} \hat{\underline{i}} + \frac{\partial \phi}{\partial y} \hat{\underline{j}}$ so $u = \frac{\partial \phi}{\partial x}$, $v = \frac{\partial \phi}{\partial y}$

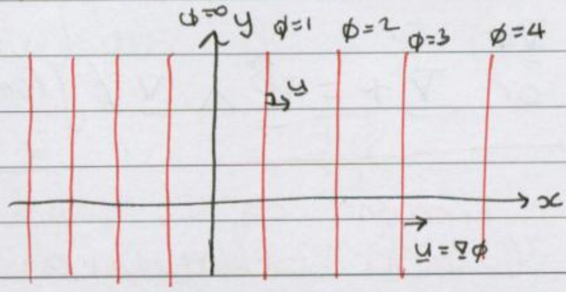
⇒ Here, $\frac{\partial \phi}{\partial x} = U$ so $\phi = Ux + f(y)$

so $\frac{\partial \phi}{\partial y} = f'(y)$, but $\frac{\partial \phi}{\partial y} = v$ and $v = 0$ so $f'(y) = 0$

⇒ f = constant.

Take $f = 0$.

$\boxed{\phi = Ux}$ → satisfies Laplace's eq^n!



equipotentials: ϕ = constant.

x = const. here.

	Good	Bad
ϕ:	3D	only irrotational
ψ:	Does not require irrotationality	only 2D

What does ψ satisfy? (in 2D irrotational flow).

2D: $\quad u = \frac{d\psi}{dy} \quad v = -\frac{d\psi}{dx}$

Irrot. 2D: $\zeta = 0, \; \frac{dv}{dx} - \frac{du}{dy} = 0.$

• Substituting $\frac{d}{dx}\left(-\frac{d\psi}{dx}\right) - \frac{\partial}{\partial y}\left(\frac{\partial \psi}{\partial y}\right) = 0$

i.e: $\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = 0$, i.e: $\nabla^2 \psi = 0$

i.e: Stream-function satisfies Laplace's eqn also.

Remember the boundary conditions on a solid boundary for ψ is ψ = constant, which can be taken as $\psi = 0$, if only one boundary.

Check: uniform stream: $\psi = Uy$ so $\nabla^2 \psi = 0$ "
Stagnation point: $\psi = xy$ so $\nabla^2 \psi = 0$ "

If flow is 2D & irrotational, then you can choose to find ϕ or ψ; whichever seems easier!
• Governing eqⁿ is same: Laplace.
• Boundary conditions different.

---

* Are ϕ and ψ related?
Yes! $\quad \underline{u} = \underline{\nabla}\phi \quad$ and $\quad \underline{u} = -\hat{\underline{k}} \wedge \underline{\nabla}\psi$ so

$$\underline{\nabla}\phi = -\hat{\underline{k}} \wedge \underline{\nabla}\psi$$
or
$$\underline{\nabla}\psi = \hat{\underline{k}} \wedge \underline{\nabla}\phi$$
} V. Famous

↓

Cauchy - Riemann eqⁿ's!
(without co-ordinates)

$u = \frac{d\phi}{dx}$ and $u = \frac{d\psi}{dy}$, so $\frac{d\phi}{dx} = \frac{d\psi}{dy}$.

$v = \frac{d\phi}{dy}$ and $v = -\frac{d\psi}{dx}$,

so $\frac{d\phi}{dy} = -\frac{d\psi}{dx}$

$\Rightarrow$ In 2D; irrotational flow

$$\frac{\partial\phi}{\partial x} = \frac{d\psi}{dy} \quad \text{and} \quad \frac{\partial\phi}{\partial y} = -\frac{\partial\psi}{\partial x}$$

– the Cauchy–Riemann eqn's.

Thus, ϕ and ψ are the real & imaginary parts of a differentiable complex function of the complex variable $z = x + iy$.

• Traditionally, this is called the COMPLEX VELOCITY POTENTIAL! and written

$$w(z) = \phi(x,y,t) + i\psi(x,y,t)$$

↑ lower case $z = x+iy$, $i = \sqrt{-1}$

* $\phi = Re(w(z))$
* $\psi = Im(w(z))$

proved; 1) Real & Imaginary parts of a complex differentiable function satisfy Laplace's eqn's.
2) constant surfaces intersect at right angles.

---

check: uniform stream:

• $\phi = Ux$ • $\psi = Uy$

$\phi + i\psi = U(x+iy) = Uz$ → a function of z alone.

So $w = Uz$ is the complex potential for a uniform stream.

Given $w(z)$, how do we get $\underline{u}$?

• consider $\frac{dw}{dz} = \frac{d}{dx}(\phi + i\psi) = \frac{d\phi}{\partial x} + i\frac{d\psi}{\partial x} = u - iv$

So: $u + iv = \overline{\frac{dw}{dz}}$ bar = conjugate.

e.g.: for $w = Uz$, $\frac{dw}{dz} = U$, $= u - iv$ so $u = U$ and $v = 0$ as expected.

e.g 2: $w = z^2$, $\frac{dw}{dz} = 2z = 2x + i2y$ so

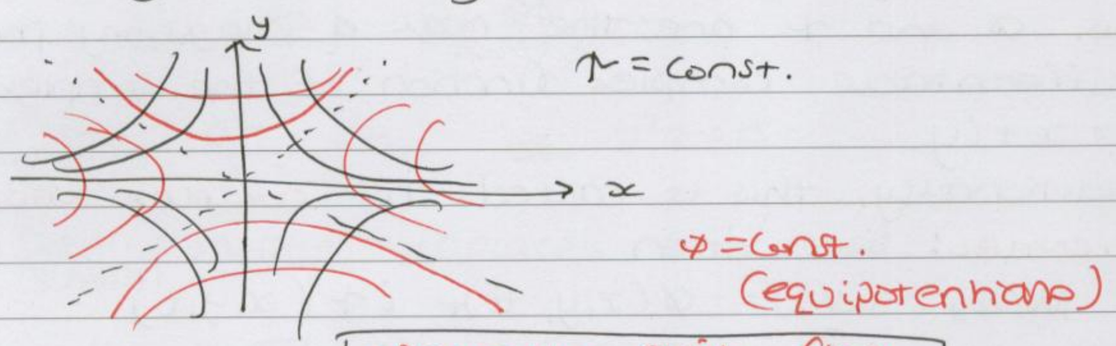
$u = 2x$, $v = -2y$.

$u=0, v=0$ only if $z=0$ (where $\frac{dw}{dz}=0$)

i.e: Stagnation point if and only if $\frac{dw}{dz}=0$,
Here, only at $z=0$.

$w = z^2$
$= (x+iy)^2 = x^2 - y^2 + 2ixy$

so $\phi = x^2 - y^2$, $\psi = 2xy$



• Stagnation point flow

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Irrotational: $\underline{\nabla} \wedge \underline{u} = 0 \Rightarrow \exists \phi$ s.t $\underline{u} = \underline{\nabla}\phi$

Incompressibility: $\underline{\nabla} \cdot \underline{u} = 0$ Plus 2D: $\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \Rightarrow \exists \psi$ s.t $\underline{u} = -\underline{k} \wedge \underline{\nabla}\psi$

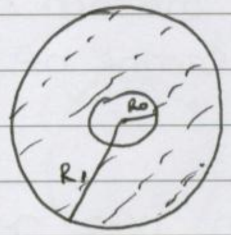
i.e: $\underline{\nabla}\phi = -\underline{k} \wedge \underline{\nabla}\psi$ – Cauchy-Riemann

$\Rightarrow \exists\, w(z)$ s.t $\frac{dw}{dz}$ exists and $w = \phi + i\psi$; $\frac{dw}{dz} = u - iv$.

Laurent Series:

A function analytic (→ holomorphic) within an annular region $R_0 < |z| < R_1$ has a unique expansion of the form

$$\dots + \frac{a_{-2}}{z^2} + \frac{a_{-1}}{z} + a_0 + a_1 z + a_2 z^2 + \dots$$



i.e: All functions analytic in the annulus are simply linear combinations $z^{\pm n}$, $n = 0, 1, 2 \dots$

Apply this to the complex velocity $u - iv$, i.e; $u - iv$ is a linear combination of $z^{\pm n}$.

Thus, $w(z)$, the complex potential, is simply a linear combination of the terms $z^{\pm n}$, $n = 0, 1, 2 \dots$, and $\log(z)$.

i.e: Our flow in any annular region (or a region that can be distorted into an anulus) or outside a single body. (Let $R_2 \to \infty$) is simply a linear combination of terms chosen from $\{z^{\pm n}, \log(z)\}$.

Note, the coefficients in the sum can be complex

In particular, in cylindrical coordinates, $z = re^{i\theta}$.

So $z^n = r^n e^{in\theta} = r^n \cos(n\theta) + i r^n \sin(n\theta)$ and

$\log(z) = \log(r) + i\theta$.

– Now $\phi = \mathrm{Re}(w)$ so ϕ must be a linear combination of terms drawn from the set.

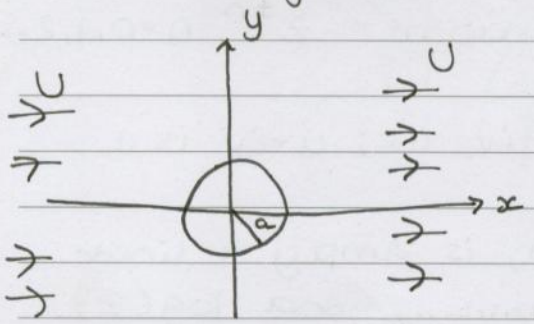
$$\{r^n \cos n\theta,\ r^n \sin n\theta,\ (n = 0, \pm 1, \pm 2, \dots),\ \log r,\ \theta\}$$

Thus, all solutions (in an annular domain) of Laplace's eq^n in polar coordinates are simply a linear combination of the terms $r^{\pm n}\cos n\theta$, $r^{\pm n}\sin n\theta$, $\log r$, θ.
Similarly, $\psi(x,y) = \text{Im}(w(z))$ is only a linear combination of terms drawn from the set

$$\{r^{\pm n}\cos n\theta,\ r^{\pm}\sin n\theta,\ (n=0,1,2,\ldots),\ \log r,\ \theta\}$$

<u>Example</u>: Find the ideal 2D flow past a cylinder of radius a given that the flow (irrotational & incompressible) at infinity is uniform with speed U.

<u>Solution</u>: Take cartesian axes with Ox in the direction of the flow at infinity and origin at centre of cylinder.



Can solve either with ϕ or ψ. Choose ψ simply so we can draw some streamlines.
Governing equations:
$\nabla^2\psi = 0$ in $r > a$.

on the cylinder ($r=a$), no flow through cylinder ($\underline{u}\cdot\hat{\underline{n}} = 0$)

$\psi =$ constant on $r=a$.

But only one body so w.l.o.g we can take $\psi = 0$ on $r=a$.

<u>As $r\to\infty$</u>: $\underline{u} \to U\hat{\underline{i}}$

<u>i.e</u>: $u \to U$, $v \to 0$

<u>i.e</u>: $\frac{\partial\psi}{\partial y} \to U$ so $\psi \to Uy + f(x)$

<u>i.e</u>: $\frac{\partial\psi}{\partial x} \to f'(x)$

But $\frac{\partial\psi}{\partial x} \to v = 0$ so $f' = 0$; w.l.o.g; $\underline{f=0}$

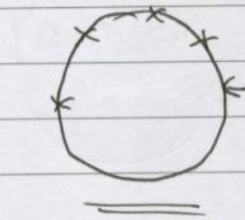
Hence, $\psi \to Uy$ as $r\to\infty$.

Summary: $\nabla^2\psi = 0$ $\quad r>a$ $\quad$ • homogeneous ($\psi=0$; soln.)
$\psi = 0$ $\quad r=a$ $\quad$ • homogeneous ($\psi=0$; soln.)
$\psi \to Uy$ $\quad r\to\infty$ $\quad$ • Inhomogeneous ($\psi=0$ NOT a soln)

• Inhomogeneous boundary conditions says
$$\psi \to U\, r\sin\theta \quad \text{as } r\to\infty$$
↑ from ar set.

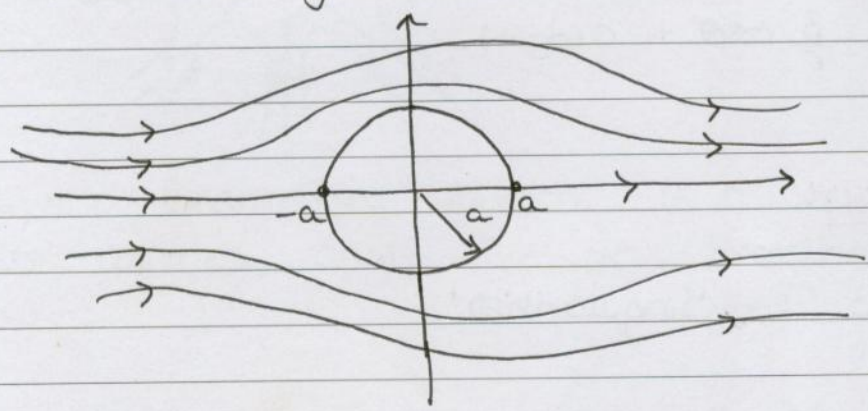
* Guess;
$$\psi = Ur\sin\theta + ~~ar^5\cos 3\theta~~ + ~~br^3\cos 3\theta~~$$
(not a soln. to Laplace.) (violates $r\to\infty$.)

$+ \frac{C}{r^3}$ ~~$\cos(3\theta)$~~ cannot balance $\sin\theta$ for every θ on $r=a$. $\quad + \frac{B}{r}\sin\theta$



on $r=a$, $\psi = Ua\sin\theta + \frac{B}{a}\sin\theta$.
This is true for all θ so $Ua + \frac{B}{a} = 0$, i.e. ⇒
$$B = -Ua^2$$

i.e: $\psi = Uy(1 - a^2/r^2)$
$\psi = 0$ when $y=0$ and $r=a$ as expected.



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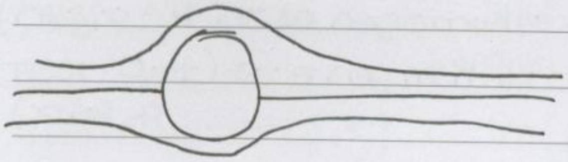


$$\psi = Uy\left(1 - a^2/r^2\right) = Uy - \frac{Ua^2y}{r^2}$$

$$= \text{Im}\left[Uz + \frac{Ua^2}{z}\right]$$

$$\frac{Ua^2y}{r^2} = \frac{Ua^2y}{|z|^2}$$

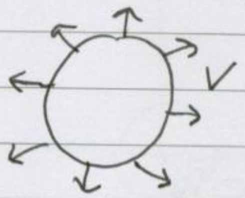
· $w(z) = Uz + Ua^2/z$, $\phi = \text{Re}(w) = Ux + Ua^2x/r^2$

$$\left[\frac{Ua^2}{z} = \frac{Ua^2\bar{z}}{|z|^2}\right]$$

$\{r^{\pm n}\cos n\theta,\ r^{\pm n}\sin n\theta,\ \log r,\ \theta\}$

* ϕ is conjugate to ψ.

$\{z^{\pm n}, \log z\}$



$\underline{u} \cdot \hat{\underline{n}} = V$ (notes have $V=0$)

$\underline{u} \to U\hat{\underline{i}}$ as $r \to \infty$

$\phi \to Ux = Ur\cos\theta$ as $r \to \infty$ (Inhomogeneous)

$\underline{u} \cdot \hat{\underline{n}} = V$ on $r = a$ (")

same for all θ.

$\therefore \phi = Ar\cos\theta + \frac{B}{r}\cos\theta + C\log(r)$.

Our Basic Solution:

1) z^{-1}, z^{-2}, z^{-3}, etc. 'Singularities'!
↑
dipole.

2) z^0 - nothing. $\frac{dw}{dz} = 0$.

3) z : $w = U$: Uniform stream, $\frac{dw}{dz} = U = u + iv$

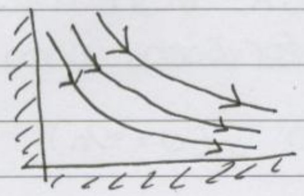
4) $w = z^2 = [re^{i\theta}]^2 = r^2\cos 2\theta + ir^2 \sin 2\theta$

so $\psi = r^2 \sin 2\theta$

so $\psi = 0$ on $\theta = 0$

and, with increasing θ, next zero when $\theta = \pi/2$.

• stagnation point flow;



$\frac{dw}{dz} = 2z = 2x + 2iy$

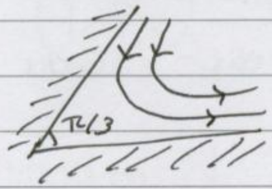
• $u = 2x$

$v = -2y$.

5) $w = z^2$

$\psi = r^3 \sin 3\theta$

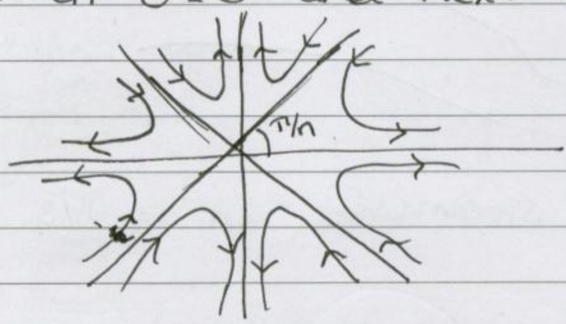
$\psi = 0$ on $\theta = 0$, next 0 when $\theta = \pi/3$.



6) $w = z^n$

$\psi = r^n \sin(n\theta)$

$\psi = 0$ on $\theta = 0$ and next at $\theta = \pi/n$



This, in fundamental solutions, if n streamlines cross, they cross at an angle π/n in irrotational, incompressible flow.

Streamlines in the neighbourhood of a stagnation point:

Suppose we have a stag. point in the flow. Move origin to that point.

In the neighbourhood of O,

$$w = a_0 + a_1 z + a_2 z^2 + a_3 z^3 + \dots$$

w.l.o.g. we can take $a_0 = 0$.

At O, $\frac{dw}{dz} = 0$. (Stag. point) $a_1 = 0$.

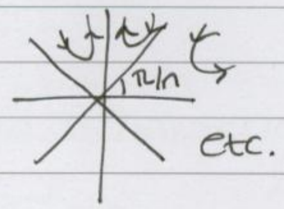
let the first non-zero term be a_n. Then $n \geq 2$.

Sufficiently close to O, $w \sim a_n z^n$ for some complex a_n.

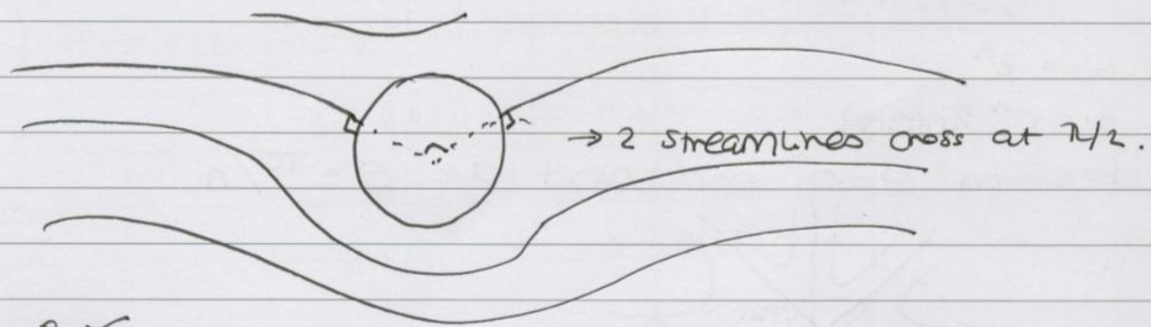
- Suppose $a_n = Ae^{i\alpha}$.

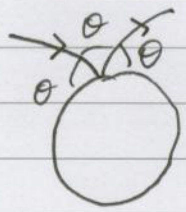
Then $w \sim Ae^{i\alpha} r^n e^{in\theta} = (A^{1/n} r)^n e^{in(\theta + \alpha/n)}$

i.e: a fundamental solution, z^n, rotated by α/n and scaled by $A^{1/n}$.



ie: exactly as before:
n streamlines must cut at π/n.





$\theta = \frac{\pi}{3}$. 3 streamlines cut at $\pi/3$.

The remaining fundamental solution is $\log(z)$.

If $w = m \log z = m(\log r + i\theta)$
$= m \log r + i m\theta$ (m real)

- so $\phi = m\log(r)$, $\psi = m\theta$

i.e: $\log r$, θ are conjugate functions. ← consequence of Cauchy's Theorem.

– isotropic source of strength $2\pi m$.

[NO OTHER FUNDAMENTAL SOLN IS A SOURCE OF FLUID]!

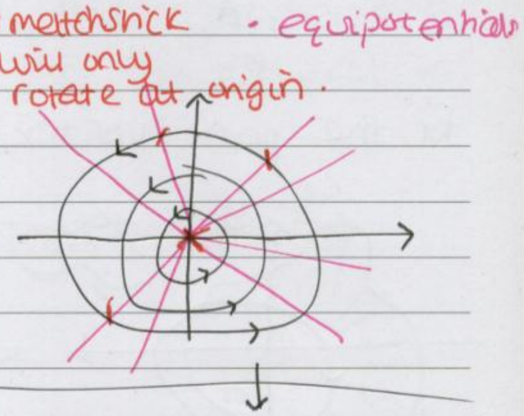
If $\omega = -iK \log(z)$ (K real)

$= -iK(\log r + i\theta)$

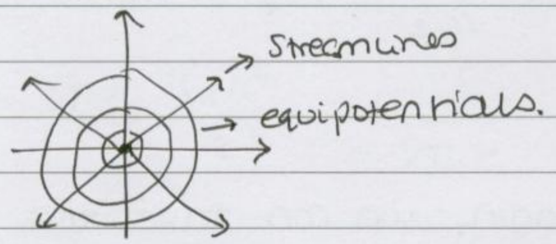
$= K\theta - iK\log(r)$

$\phi = K\theta \qquad \psi = -K\log(r)$

- Streamlines: ψ = constant.
  i.e: $\log r$ = const.
  i.e: r = const.
  i.e: circles.



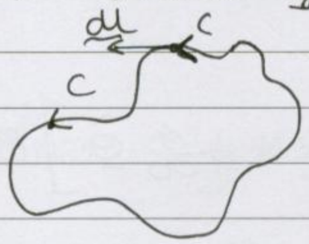
LINE VORTEX!



→ for m = real.

The strength of a line (or point) vortex.

We measure the strength of any rotational flow by its circulation about a closed contour C (say). The circulation is defined as



$$\Gamma = \oint_C \underset{\sim}{u} \cdot \underset{\sim}{dl}$$

i.e: sum of tangential velocity × distance.

(c.f work done going around a closed path)

- Notice, for an irrotational flow this is 0 for all curves C.

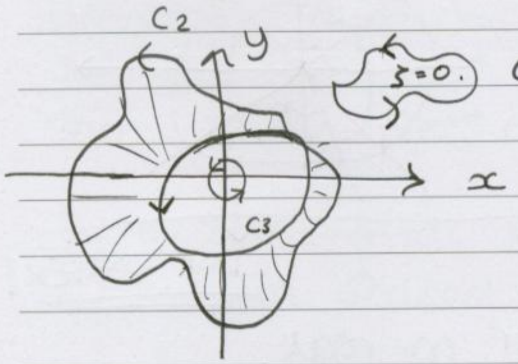
[In 3D, if $\underline{u} = \underline{\nabla}\phi$; $\int_A^B \underline{\nabla}\phi \cdot \underline{dl} = \phi(B) - \phi(A)$.
If $A = B$, this is zero.]

In 2D; $\oint_C \underline{u} \cdot \underline{dl} = \int_A (\underset{\sim}{\nabla} \wedge \underset{\sim}{u}) \cdot \hat{\underline{n}} \; dA$

But in 2D flow, $\underline{\nabla} \wedge \underline{u} = \zeta \hat{\underline{k}}$ and $\hat{\underline{n}} = \hat{\underline{k}}$

so $\Gamma = \int_A \zeta \, dA$. If $\zeta = 0$ everywhere, $\Gamma = 0$ for all C.

for the point vortex, $w = -iK \log(z)$ (Γ - capital gamma, γ)

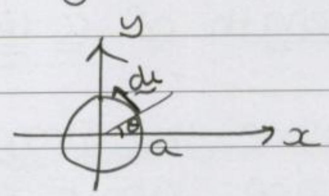


circulation around C_1 is 0.

$$\oint_{C_1} + \oint_{C_2} = 0$$

For a circuit around the origin, we can take the circuit to be a circle of radius a, w.l.o.g.

$$\Gamma = \oint_{\theta=-\pi}^{\pi} \underline{u} \cdot \underset{\sim}{dl}$$



Associated with a change $d\theta$ in θ is the vector

$$\underset{\sim}{dl} = (a \, d\theta) \, \hat{\underline{\theta}}$$

and $\underline{u} = \underline{\nabla}\phi = \underline{\nabla}(K\theta) = K \underline{\nabla} \theta = K \left[\frac{\partial}{\partial r} \hat{\underset{\sim}{r}} + \frac{1}{r}\frac{\partial}{\partial \theta} \hat{\underline{\theta}} \right] \theta$

$$= \frac{K}{r} \hat{\underline{\theta}}$$

• on $r = a$, $\underline{u} = \frac{K}{a} \hat{\underline{\theta}}$

Thus; $\Gamma = \int_{-\pi}^{\pi} \frac{K}{a} \hat{\underline{\theta}} \cdot a \, d\theta \, \hat{\underline{\theta}} = 2\pi K$

i.e: line vortex has circulation $2\pi K$.

Exercise; only fundamental soln with circulation.

Example: Consider a cylinder of radius a, in a stream of speed U, where there is circulation K about the cylinder.
Take origin at the centre of the cylinder and axis Ox in the direction of the flow at ∞.
Then:

$$w(z) = Uz + \frac{Ua^2}{z} + i\frac{K}{2\pi}\log(z).$$

Check:

$$\frac{dw}{dz} = U - \frac{Ua^2}{z^2} - \frac{iK}{2\pi z}$$ → (satisfies Laplace's eqn because a sum of fund. soln's. ϕ, ψ)

• As $z \to \infty$, $\frac{dw}{dz} \to 0$, i.e: $u \to U$ and $v \to 0$.

Thus b.c at infinity is satisfied.

Take any circuit about the cylinder. Then the circulation about C is

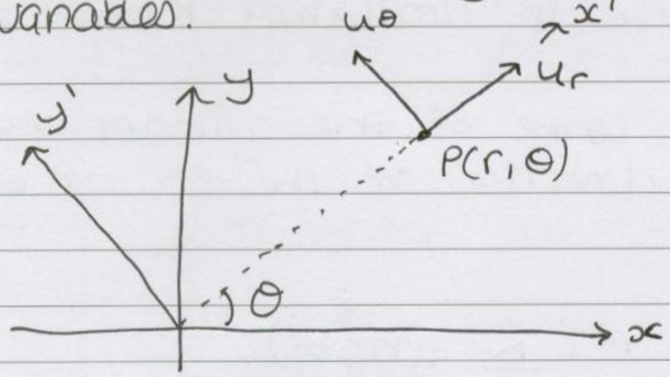
$$2\pi\left(\frac{K}{2\pi}\right) = K$$ as required,

since only line vortex has circulation.

Remains to check that $\underline{u} \cdot \hat{\underline{n}} = 0$ on the cylinder $r=a$.

i.e: $\underline{u} \cdot \hat{\underline{r}} = 0 \Rightarrow u_r = 0$. $(r=a)$

There is a nice way of doing this using complex variables.



At some point P, the cartesian components of velocity are

$$\frac{dw}{dz} = u - iv.$$

- Introduce x' and y' rotated by θ degree's anti-clockwise from x, y.

Then $\frac{dw}{dz} = u' - iv'$ (the cartesian components along dashed axis)

$= u_r - iu_\theta$

$$u_r - iu_\theta = \frac{dw}{dz'} = \frac{dw}{dz}\frac{dz}{dz'}$$

($\arg z' = \arg z - \theta$)

($\arg z = \arg z' + \theta$)

Now $z = e^{i\theta} z'$

so $\frac{dz}{dz'} = e^{i\theta}$.

Thus, $\boxed{u_r - iu_\theta = e^{i\theta}\frac{dw}{dz}}$ → v. USEFUL!

In our example; $\frac{dw}{dz} = U - \frac{Ua^2}{z^2} - i\frac{K}{2\pi z}$ //

on the cylinder $|z| = a$, i.e: $z = ae^{i\theta}$

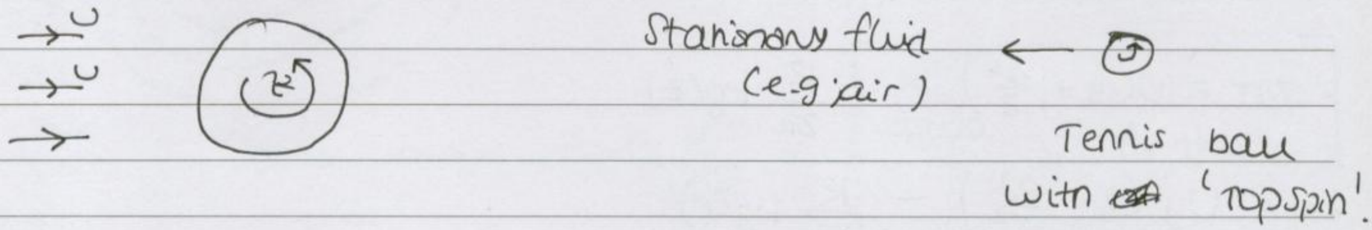
$$\frac{dw}{dz} = U - Ue^{-2i\theta} - \frac{iK}{2\pi a}e^{-i\theta}$$

$$\text{So } e^{i\theta}\frac{dw}{dz} = U\left(e^{i\theta} - e^{-i\theta}\right) - \frac{i\kappa}{2\pi a}$$

$$= 2iU\sin\theta - i\kappa/2\pi a = u_r - iu_\theta$$ //

$\therefore$ $u_r = 0$ (as required)

and $u_\theta = \frac{\kappa}{2\pi a} - 2U\sin\theta$



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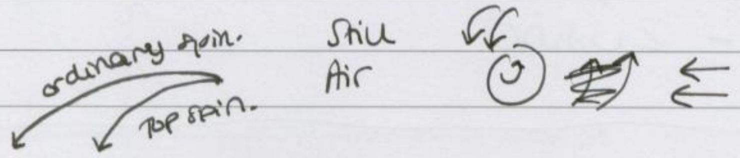
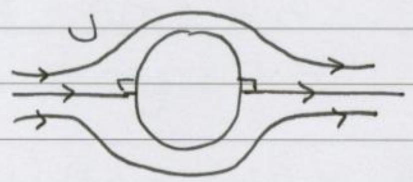
('top spin' in tennis).



- $w = U\left(z + \frac{a^2}{z}\right) - i\frac{K}{2\pi}\log(z)$
- $\psi = Uy\left(1 - \frac{a^2}{r^2}\right) - \frac{K}{2\pi}\log(r)$
- $\phi = Ux\left(1 + \frac{a^2}{r^2}\right) + \frac{K}{2\pi}\theta$

on $r=a$; $u_r = 0$
$u_\theta = \frac{K}{2\pi a} - 2U\sin\theta$

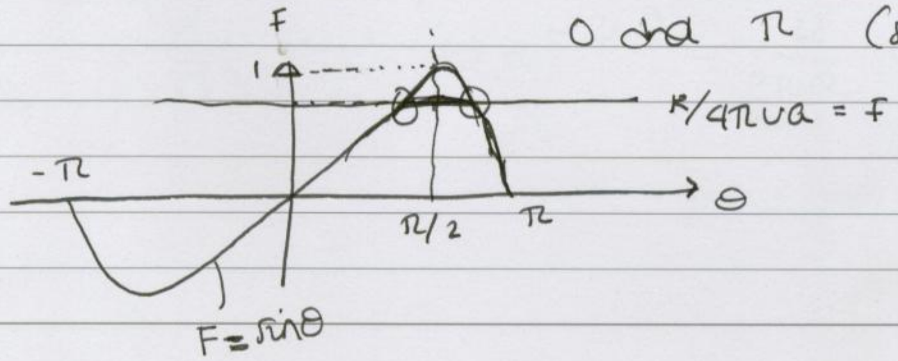
$K=0$:



At a stagnation point, $\underline{u} = 0$ or $\frac{dw}{dz} = 0$, or $u=0, v=0$
or $u_r = 0$, $u_\theta = 0$.

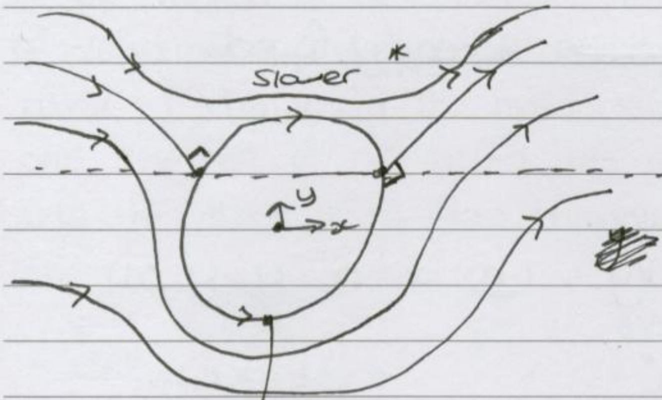
On the cylinder: $r=a$, $u_r = 0$ $\forall\theta$.
So the stagnations are where $u_\theta = 0$, i.e; $\frac{K}{2\pi a} = 2U\sin\theta$

i.e: $\sin\theta = \frac{K}{4\pi a U}$ — two roots.
for $K>0$, these are between
0 and π (symmetric at $\pi/2$)



on cylinder:

$$y = a\sin\theta = \frac{K}{4\pi U}$$



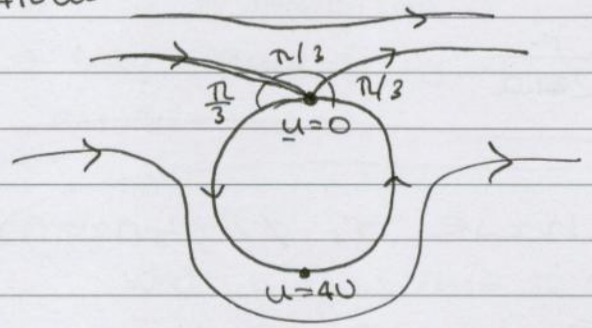
*slower ; $2U - \frac{K}{4\pi a}$

$y = \frac{K}{4\pi U}$

faster – max $2U + \frac{K}{4\pi a}$

[$\frac{K}{4\pi U a} < 1$ → only true for this]

$\frac{K}{4\pi U a} = 1$; stagnation points coincide at $y = a$.



$\frac{K}{4\pi U a} > 1$; no roots ⇒ no stagnation points on the cylinder.

Reminder:

$$w = U\left(z + \frac{a^2}{z}\right) - i\frac{K}{2\pi}\log(z).$$

$$\frac{dw}{dz} = U\left(1 - \frac{a^2}{z^2}\right) - \frac{iK}{2\pi z}$$

Stag. points: $\frac{dw}{dz} = 0$, i.e: $U\left(1 - \frac{a^2}{z^2}\right) - i\frac{K}{2\pi z} = 0$

*mult. by $\frac{z^2}{Ua^2}$

I.e: $\left(\frac{z}{a}\right)^2 - \frac{iK}{2\pi U a}\left(\frac{z}{a}\right) - 1 = 0$ → quad in $\frac{z}{a}$

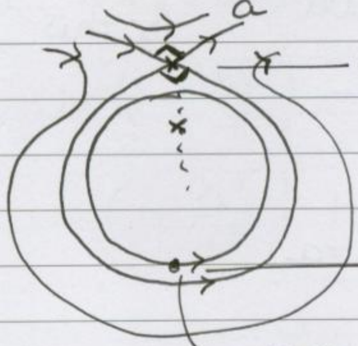
– 2 roots.

Product must be –1!

Then: $\frac{z}{a} = \frac{iK}{4\pi Ua} \pm \sqrt{1-\left(\frac{K}{4\pi Ua}\right)^2}$

- $\frac{K}{4\pi Ua} < 1$: ~~Complex conjugates~~ Already found.
  $z_1, \ -\bar{z}_1$

- $\frac{K}{4\pi Ua} > 1$: purely imaginary. i.e: $x=0$, $y=y_1(a)$
  $\frac{z}{a} = iy_1$; or $\frac{i}{y_1}$ $\qquad z = x+iy$.



slow; pressure high.

⇓ downward force. Energy!

FAST; low pressure. (K.E + pressure = C)

even faster; $2U + \frac{K}{2\pi a}$
$>4U$

---

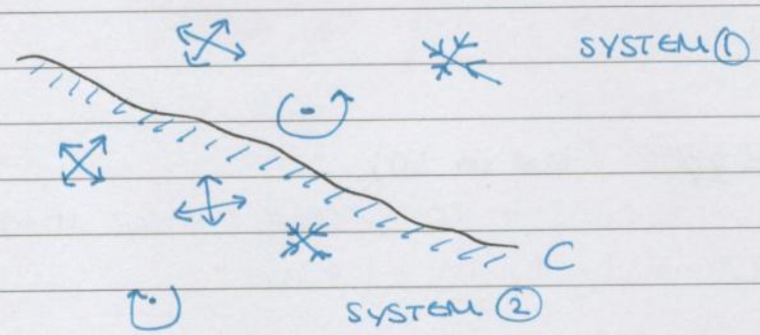
(1) Inhomogeneous; (choose r, ϕ, then:-)

(2) choose $\{ r^{\pm n}\cos n\theta,\ r^{\pm n}\sin n\theta,\ \log r,\ \theta \}$
with undetermined coefficients.

(3) BC's give coef.

The method of Images:

If the motion of a fluid in the xy-plane is due to a distribution of singularities, (e.g: sources, sinks, vortices, etc) and there is a curve C drawn in the plane, then the system of singularities on one side of C is called the image of the system on the other side IF there is no flow through C.



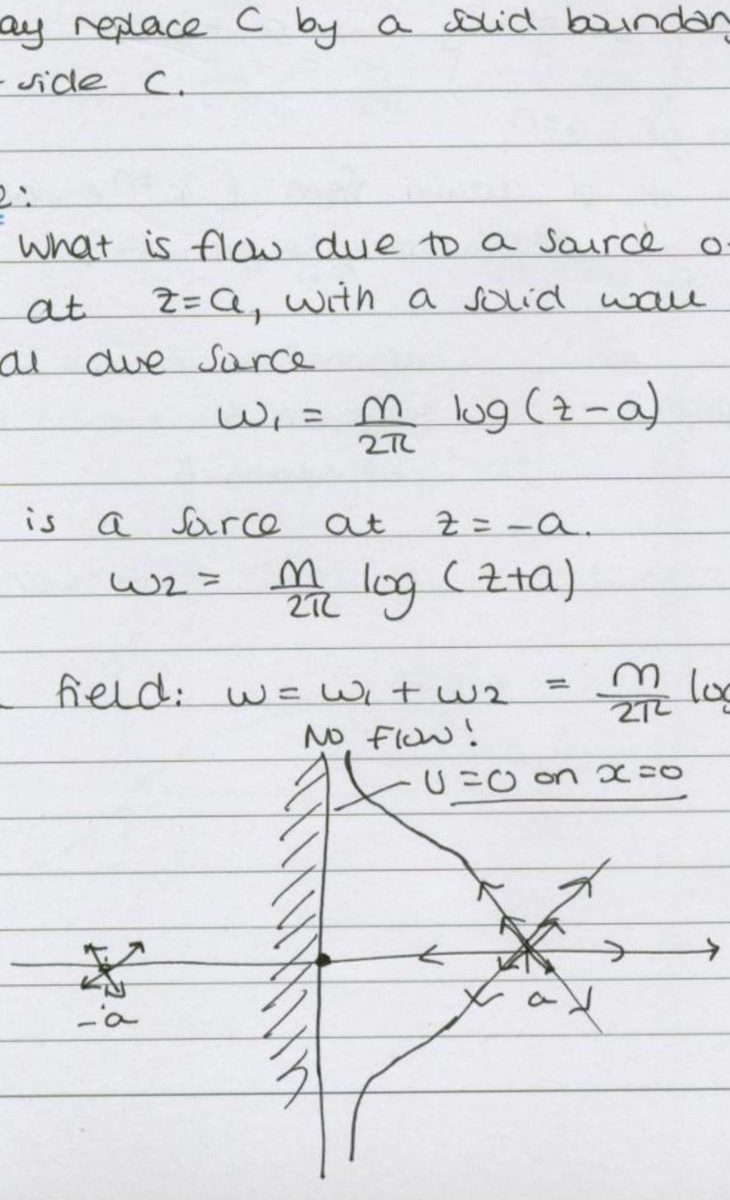
No flow across C ⇒ SYSTEM 1 image of SYSTEM 2.
⇒ C is a stream line.
⇒ May replace C by a solid boundary, without the flow outside C.

Example:
What is flow due to a source of strength m located at $z=a$, with a solid wall along $x=0$?

potential due source
$$w_1 = \frac{m}{2\pi}\log(z-a)$$

Image is a source at $z=-a$.
$$w_2 = \frac{m}{2\pi}\log(z+a)$$

Total field: $w = w_1 + w_2 = \frac{m}{2\pi}\log(z-a) + \frac{m}{2\pi}\log(z+a)$



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$\underline{u}(\underline{x},t)$: velocity field $\underline{x}$, fixed axes.

Incompressibility. $\Rightarrow \underline{\nabla}\cdot\underline{u} = 0$. ; solenoidal

plus 2D $\Rightarrow \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$.

$\Rightarrow \exists \psi$ s.t $\underline{u} = \hat{\underline{z}} \wedge \underline{\nabla}\psi$

- local motion at a point;

1) Translation of C of M.
2) Dilation
3) Rotation.

- Irrotationality persists;

$\underline{\nabla} \wedge \underline{u} = 0 \Rightarrow \exists \phi$ s.t $\underline{u} = \underline{\nabla}\phi$ (true in 3D).

* ψ : Incomp. + 2D

* ϕ : Irrot.

Irrot, Incom. + 2D: ϕ and ψ.

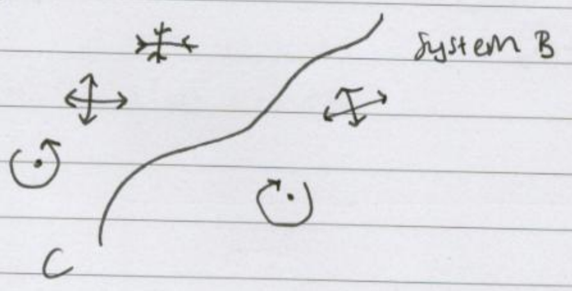
$\underline{\nabla}\phi = -\hat{\underline{z}} \wedge \underline{\nabla}\psi$ C.R

$\exists\, w(z)$ where $z = x + iy$, $w = \phi + i\psi$.

(0) Laurent: sum of $z^{\pm n}$.

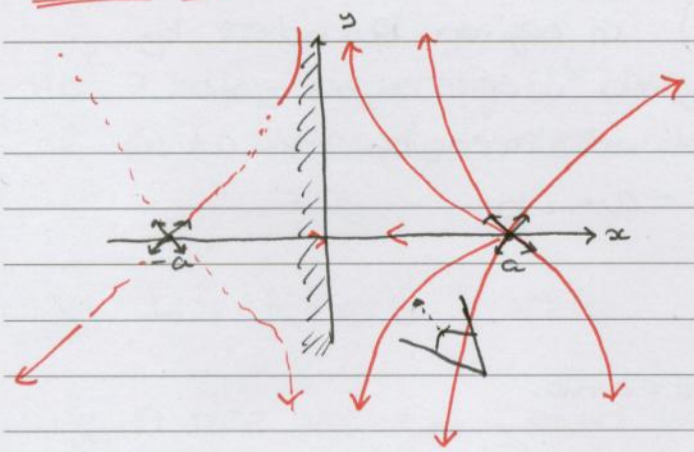
- In polars: ψ, ϕ drawn from $\{ r^{\pm n}\cos(n\theta),$ $r^{\pm}\sin(n\theta), \log r, \theta \}$

System A



If no flow across C, then system A is the IMAGE of system B.

Example (1):



- original system;
Source strength m at $x = a$;
Complex potential;

$$w_1(z) = \frac{m}{2\pi} \log(z - a)$$

- Image system :-
Source strength m at $x = -a$.
Complex potential

$$w_2(z) = \frac{m}{2\pi} \log(z + a)$$

Total system = original + image.

$$w(z) = \frac{m}{2\pi} \log(z - a) + \frac{m}{2\pi} \log(z + a)$$

- If this is correct, then $u = 0$ on $x = 0$.

$$w(z) = \frac{m}{2\pi} \log(z^2 - a^2)$$

$$\frac{dw}{dz} = \frac{m}{2\pi} \left(\frac{1}{z^2 - a^2}\right) 2z$$

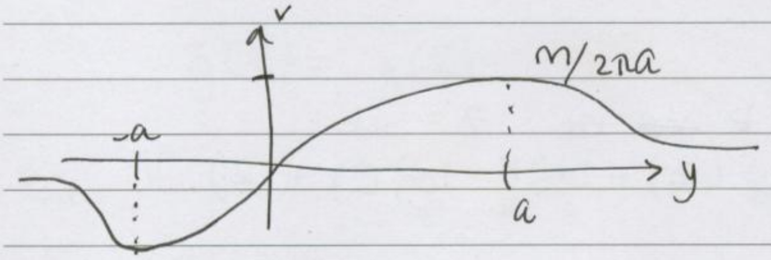
on $x = 0$; $(z = iy)$

$$u - iv = \frac{dw}{dz} = \frac{m}{2\pi} \cdot \frac{1}{-y^2 - a^2} \cdot 2iy.$$

So $u = 0$ as expected.

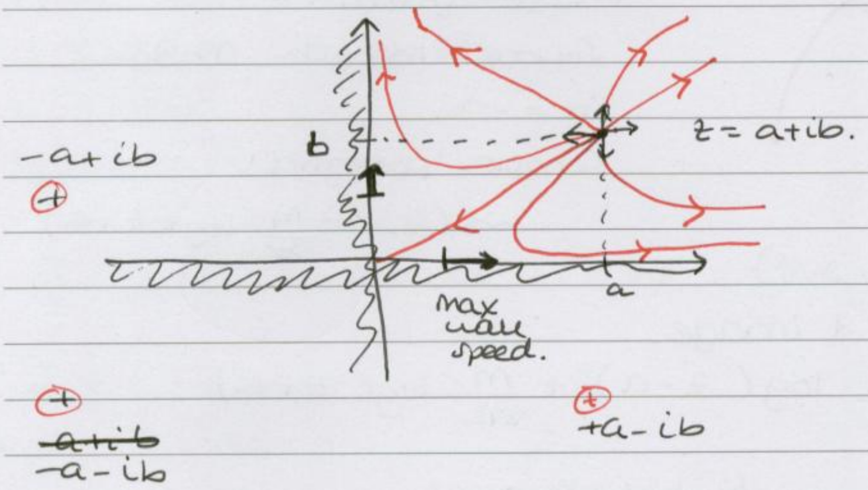
$$v = \frac{my}{\pi(y^2 + a^2)}$$

So maximum speed on wall is $V = \pm \frac{m}{2\pi a}$ when $y = \pm a$

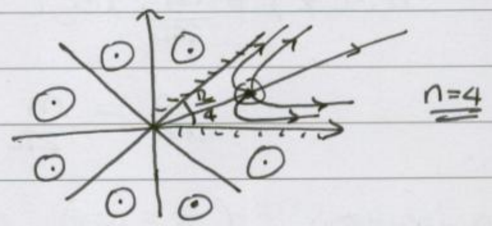


Example (2). ORIGINAL: Source of $z = a + ib$ (Strength m) in region bounded by $x = 0, y = 0$ with $x > 0, y > 0$.

IMAGE SYSTEM: 3 Sources of Strength m at $z = \pm a - ib, \; -a + ib$.



Example (3): walls at $\frac{\pi}{n}$;



Example (4): Vortex of Strength K at $z = ib$, above a plane $y = 0$.

Complex potential $w_1(z) = -\frac{iK}{2\pi} \log(z - ib)$

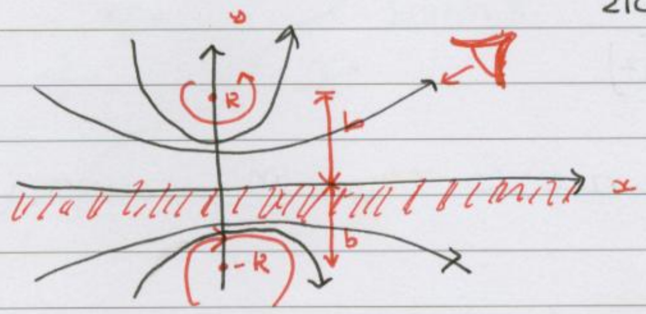


Image: vortex of Strength $-K$ ~~and~~ at $z = -ib$

Complex potential $w_2(z) = +\frac{iK}{2\pi} \log(z + ib)$

- Total System = original + image $= -\frac{iK}{2\pi} \log(z - ib) + \frac{iK}{2\pi} \log(z + ib)$

Check $v = 0$ on $y = 0$ as expected.

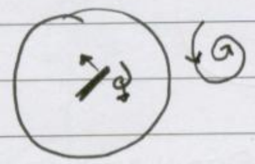
velocity field: $\frac{dw}{dz} = -\frac{iK}{2\pi} \cdot \frac{1}{z-ib} + \frac{iK}{2\pi(z+ib)}$

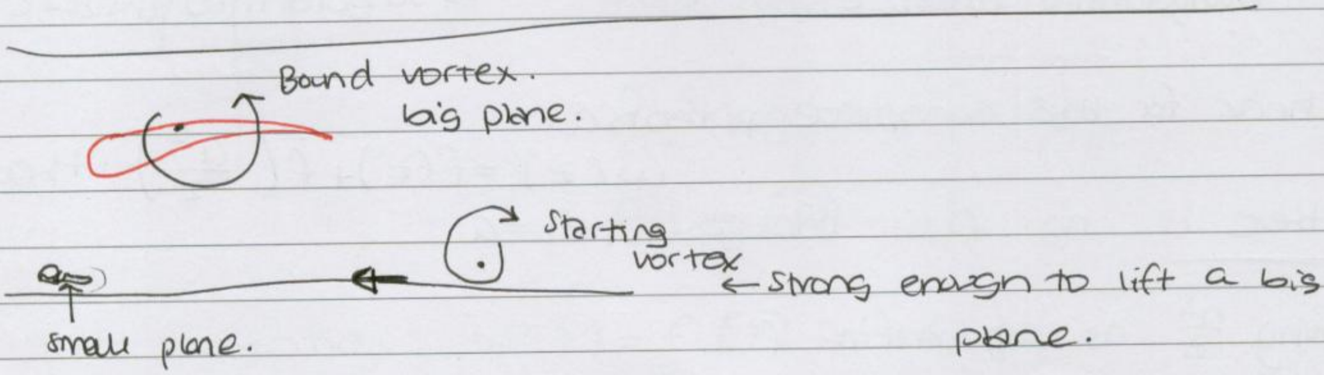
At $z = ib$, neglecting 1st term, which is just the spinning of an isolated vortex about its centre,

$$\frac{dw}{dz} = \frac{iK}{4\pi i b}$$

i.e: $u = K/4\pi b$, $v = 0$.

i.e: A free vortex would be driven along parallel to the plane $x=0$, by it's image in the plane.



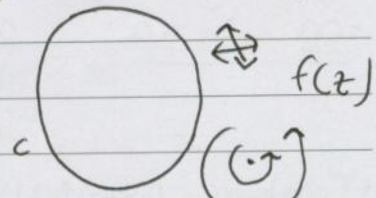


• Can we only do planar boundaries? NO!

Circle Theorem:

The image system in the circle $|z| = a$ of the complex potential $w(z) = f(z)$ where $f(z)$ has no singularities inside the circle. (i.e: original system all on one side of line), $|z| < a$, is $\bar{f}\left(\frac{a^2}{z}\right)$ where for any analytic function $g(z)$,

$$\bar{g}(z) = \overline{g(\bar{z})}$$



e.g: If $g(z) = +\frac{a_{-2}}{z^2} + \frac{a_{-1}}{z} + a_0 + a_1 z + a_2 z^2 + \ldots$

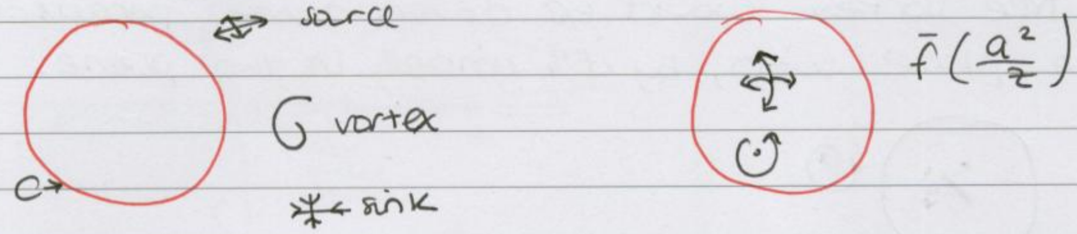
$g(\bar{z}) = a_{-2}\bar{z}^{-2} + a_{-1}\bar{z}^{-1} + a_0 + a_1\bar{z} + a_2\bar{z}^2 + \ldots$

↳ still an analytic function of z.

so $\bar{f}\left(\frac{a^2}{z}\right)$ is an analytic function of $\frac{a^2}{z}$.
Now f has no singularities in $|z| < a$.

So $f(a^2/z)$ has no singularities in $|z| > a$, since if $|z| > a$, then $\frac{|z|}{a^2} > \frac{1}{a}$ so $\frac{a^2}{|z|} < a$.

Similarly, for $\bar{f}(a^2/z)$;



↳ singularities outside C ↳ singularities inside C.

Check for the complete potential $w(z) = f(z) + \bar{f}\left(\frac{a^2}{z}\right)$ that there is no flow through $|z| = a$.

Why $\frac{a^2}{z}$ as argument of $\bar{f}$?

On the circle $|z| = a$, i.e: $z\bar{z} = a^2$; i.e: $\frac{a^2}{z} = \bar{z}$,

i.e; $\frac{a^2}{z}$ is analytic (except $z=0$) but equal to $\bar{z}$ on C.

The general problem is "find an analytic function of z (possibly singularities) which equals $\bar{z}$ on some curve C."*

- on $|z| = a$; $\frac{a^2}{z} = \bar{z}$ so $\bar{f}\left(\frac{a^2}{z}\right) = \bar{f}(\bar{z}) = \overline{f(\bar{\bar{z}})} = \overline{f(z)}$ on C.

- Combine potentials; $w(z) = f(z) + \bar{f}\left(\frac{a^2}{z}\right)$ on C;

$$w(z) = f(z) + \overline{f(z)}$$
$$= 2\,\mathrm{Re}\{f(z)\}$$

So $\mathrm{Im}\{w(z)\} = 0$ on $|z| = a$.

i.e; $\psi = 0$ on $|z| = a$.

* The Schwarz funct. for C.

i.e: Circle is a streamline as required.

i.e: no flow across $|z| = a$.

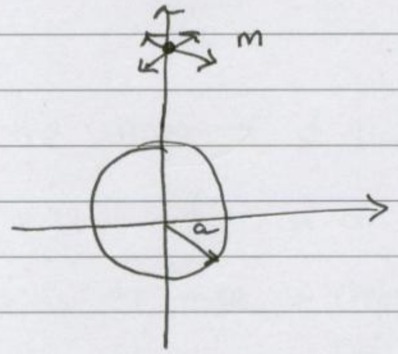
i.e; $\bar{f}(a^2/z)$ is the image of $f(z)$ in $|z| = a$.

Find the schwartz function $h(z)$ for C.

On C, $h(z) = \bar{z}$, so image of $f(z)$ is $\bar{f}(h(z))$ //

Example:

Find the image system & the total complex potential for a source of strength m at $z = ib$ outside the cylinder $|z| = a$, where $b > a$.



Original system; $f_1(z) = \frac{m}{2\pi}\log(z - ib)$

Image system; $f_2(z) = \bar{f_1}(a^2/z)$

$$= \frac{m}{2\pi}\overline{\log\left(\overline{\left(\frac{a^2}{z}\right)} - ib\right)}$$

$$= \frac{m}{2\pi}\log\left(\frac{a^2}{z} + ib\right)$$

· Total potential; $w(z) = f_1(z) + f_2(z)$

$$= \frac{m}{2\pi}\log(z - ib) + \frac{m}{2\pi}\log\left(\frac{a^2}{z} + ib\right)$$

↳ what is this?

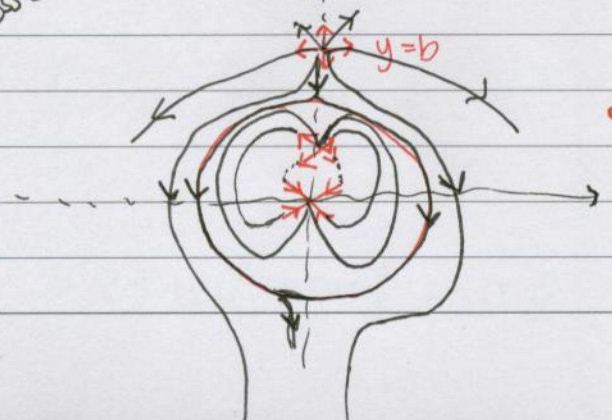
* Image system; $\frac{m}{2\pi}\log\left(\frac{a^2}{z} + ib\right) = -\frac{m}{2\pi}\log(z) + \frac{m}{2\pi}\log(a^2 + ibz)$

$$= -\frac{m}{2\pi}\log(z) + \frac{m}{2\pi}\log(ib) + \frac{m}{2\pi}\log\left(z - \frac{ia^2}{b}\right)$$

↑ sink strength m at origin. ↑ no surprise - no flow across C.

↑ arbitrary constant - no effect.

↑ Source strength m at $z = \frac{ia^2}{b}$, i.e: $x = 0$, $y = a^2/b$ // → optical image point.

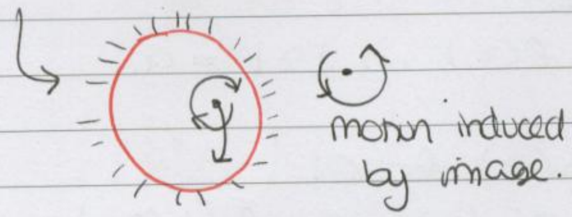


• $\frac{a^2}{b} < a$

• $(b > a)$
↑
Guaranteed

Example (2):

Vortex in a Coffee Cup:



Equations of Motion:

* $\underset{\sim}{F} = mg$

* $\underset{\sim}{F} = \frac{d}{dt}(m\underline{v})$

↑ rate of change of momentum particle following the particle.

• We thus need to define rate of change following a particle for a fluid.

* Suppose we have some field, known for all time t, and positions $\underset{\sim}{r}$, $\phi(t, \underset{\sim}{x})$. Now suppose we follow some particle whose path is given by

$$\frac{d\underset{\sim}{r}}{dt} = \underline{u}$$

Then the values of ϕ along the particle path are $\phi(t, \underset{\sim}{r}(t))$ where $\frac{d\underset{\sim}{r}}{dt} = \underline{u}$, a function of t alone!

What is the rate of change of ϕ along this path?

$$\frac{D\phi}{Dt} = \left.\frac{d\phi}{dt}\right|_{\underset{\sim}{x} = \underset{\sim}{r}(t)} = \frac{\partial\phi}{\partial t} + \frac{\partial\phi}{\partial x}\frac{dx}{dt} + \frac{\partial\phi}{\partial y}\frac{dy}{dt} + \frac{\partial\phi}{\partial z}\frac{dz}{dt}$$

↑ Chain Rule!

i.e.: $$\frac{D\phi}{Dt} = \frac{\partial\phi}{\partial t} + u\frac{\partial\phi}{\partial x} + v\frac{\partial\phi}{\partial y} + w\frac{\partial\phi}{\partial z}$$

$$= \frac{\partial\phi}{\partial t} + (u\hat{\underline{i}} + v\hat{\underline{j}} + w\hat{\underline{k}}) \cdot \left(\frac{\partial\phi}{\partial x}\hat{\underline{i}} + \frac{\partial\phi}{\partial y}\hat{\underline{j}} + \frac{\partial\phi}{\partial z}\hat{\underline{k}}\right)$$

$$= \frac{\partial\phi}{\partial t} + \underline{u}\cdot\nabla\phi = \left(\frac{\partial}{\partial t} + \underline{u}\cdot\underline{\nabla}\right)\phi$$

↑ speed at which move through gradient!

CONVECTIVE OR ADVECTIVE derivative

Ex ①: $\phi = x$; $$\frac{Dx}{Dt} = \frac{\partial x}{\partial t} + u\frac{\partial x}{\partial x} + v\frac{\partial x}{\partial y} + w\frac{\partial x}{\partial z} = u$$

Ex ②: $\phi = \underset{\sim}{r}$; $$\frac{D\underset{\sim}{r}}{Dt} = \frac{\partial}{\partial t}(x\hat{\underline{i}} + y\hat{\underline{j}} + z\hat{\underline{k}}) + u\frac{\partial}{\partial x}(\dots) + v\frac{\partial}{\partial y}(\dots) + w\frac{\partial}{\partial z}(\dots)$$

$$= 0 + u\hat{\underline{i}} + v\hat{\underline{j}} + w\hat{\underline{k}} = \underline{u}$$

Ex ③: $\phi = \underline{u}$; $\frac{D\underline{u}}{Dt} = \frac{\partial \underline{u}}{\partial t} + (\underset{\sim}{u} \cdot \underline{\nabla})\underset{\sim}{u}$

$$\frac{d}{dt}\left(\int_{a(t)}^{b(t)} f(t,x)\,dx\right) = \int_a^b \frac{df}{dt}\,dx + f(t,b)\,b'(t) - f(t,a)\,a'(t)$$

↓

Leibnitz

17/11/11

Reynold's Transport Theorem; RTT

Eventually we want to apply Newton's laws to a fluid.

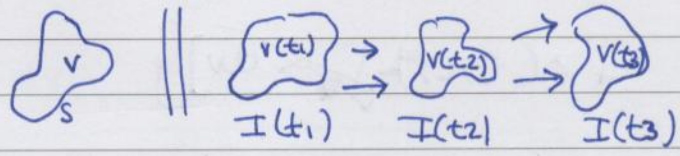
i.e; $\frac{d}{dt}$ (momentum) = force.

Consider a quantity $\alpha(\underline{r}, t)$ associated with a fluid. Let the fluid occupy a domain $\mathcal{D}$ and ~~tet~~ have the specified velocity fluid $\underline{u}(\underline{r}, t)$.

Consider a subvolume V contained in $\mathcal{D}$ with surface S.

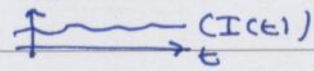
We take V to consist always of the same fluid elements or particles.

Thus V moves. i.e; $V = V(t)$.



we define $I(t) = \int_{V(t)} \alpha(\underline{r}, t)\, d\underline{r}$ //

i.e: the total amount of α in V at any time ;



Reynolds; what is the rate of change of I, $\frac{dI}{dt}$?

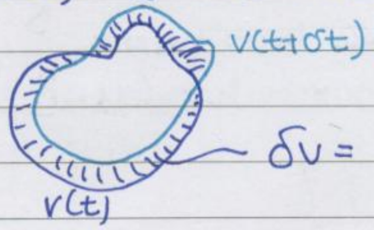
$$\frac{dI}{dt} \text{ or } \frac{DI}{Dt} \ (*)$$

↳ emphasises that we are following particle.

$$(*) = \lim_{\delta t \to 0} \left(\frac{I(t+\delta t) - I(t)}{\delta t} \right) //$$

· Here, $I(t+\delta t) = \int_{V(t+\delta t)} \alpha(\underline{r}, t+\delta t)\, d\underline{r}$

Here, $V(t+\delta t)$ is the volume position at an interval δt after t:



$\delta V = V(t+\delta t) - V(t)$ //

and by Taylor's theorem: $\alpha(\underline{r}, t+\delta t) = \alpha(\underline{r}, t) + \delta t \frac{\partial \alpha}{\partial t}(\underline{r}, t) + \frac{1}{2}(\delta t)^2 \frac{\partial^2 \alpha}{\partial t^2}(\underline{r}, \tau)$ where τ lies in $(0, \delta t)$ //

$$\text{So } I(t+\delta t) = \int_{V+\delta V} \left[\alpha(\underline{r}, t) + \delta t \frac{\partial \alpha}{\partial t}(\underline{r}, t) \right] d\underline{r} + \frac{1}{2}(\delta t)^2 \int_{V+\delta V} \frac{\partial^2 \alpha}{\partial t^2}(\underline{r}, \tau)\, d\underline{r}$$

$$= \int_{V+\delta V} \alpha \, dV + \delta t \int_{V+\delta V} \frac{\partial \alpha}{\partial t} dV + O((\delta t)^2)$$

argument is $(\underline{r}, t)$

• Now $\frac{DI}{Dt} = \lim_{\delta t \to 0} \left\{ \frac{1}{\delta t} \left[I(t+\delta t) - I(t) \right] \right\}$

$$= \lim_{\delta t \to 0} \left\{ \frac{1}{\delta t} \left[\cancel{\int_V \alpha \, dV} + \int_{\delta V} \alpha \, dV + \delta t \int_V \frac{\partial \alpha}{\partial t} dV + \delta t \int_{\delta V} \frac{\partial \alpha}{\partial t} dV + O((\delta t)^2) - \cancel{\int_V \alpha \, dV} \right] \right\}$$

$$\therefore \frac{DI}{Dt} = \lim_{\delta t \to 0} \left\{ \frac{1}{\delta t} \int_{\delta V} \alpha \, dV + \int_V \frac{\partial \alpha}{\partial t} dV + \underline{\int_{\delta V} \frac{\partial \alpha}{\partial t} dV} + O(\delta t) \right\}$$

• The underlined term is

$$\int_{\delta V} \frac{\partial \alpha}{\partial t} dV \leq \left| \int_{\delta V} \frac{\partial \alpha}{\partial t} dV \right| \leq \int_{\delta V} \left| \frac{\partial \alpha}{\partial t} \right| dV \leq \underbrace{\int_{\delta V} A \, dV}_{(\cdot)}$$

$\cdot \; A = \max_V \left| \frac{\partial \alpha}{\partial t} \right| \Rightarrow (\cdot) = A \int_{\delta V} dV = A |\delta V| \to 0$ as $\delta t \to 0$

Thus $\frac{DI}{Dt} = \int_V \frac{\partial \alpha}{\partial t} dV + \lim_{\delta t \to 0} \frac{1}{\delta t} \int_{\delta V} \alpha \, dV$

particles making up dS have moved a distance $\underline{u}\,\delta t$ in time δt.

They sweep out a volume ; Area base times height

$dS \times h = (\underline{u} \cdot \hat{\underline{n}}) \delta t$

i.e: $dV \sim (\underline{u} \cdot \hat{\underline{n}}) \delta t \, dS$

Thus, $\int_{\delta V} \alpha \, dV = \int_S \alpha (\underline{u} \cdot \hat{\underline{n}}) \delta t \, dS$

$$= \delta t \int_S \alpha (\underline{u} \cdot \hat{\underline{n}}) \, dS$$

• Thus, $\boxed{\frac{DI}{Dt} = \int_V \frac{\partial \alpha}{\partial t} dV + \int_S \alpha (\underline{u} \cdot \hat{\underline{n}}) \, dS}$ RTT I

↑ 1st form of the Reynolds Transport Theorem!

RTT 1: $\frac{D}{Dt}\left(\int_V \alpha \, dV\right) = \int_V \frac{\partial \alpha}{\partial V} \, dV + \int_S \alpha (\underline{u} \cdot \hat{\underline{n}}) \, dS$

local R.O.Ch. — flux of α through bound. of V.

↳ The 3D version of Leibnitz rule for differentiating under the integral sign.

[1D Leibnitz; $\frac{d}{dt}\int_{a(t)}^{b(t)} \alpha(t,x)\,dx = \int_a^b \frac{\partial \alpha}{\partial t} dx + b'(t)\alpha(t,b) - a'(t)\alpha(t,a)$]

But divergence theorem says that for any vector $\underset{\sim}{F}$,

$$\int_S \underset{\sim}{F} \cdot \hat{\underline{n}} \, dS = \int_V (\underline{\nabla} \cdot \underset{\sim}{F}) \, dV$$

Putting $\underset{\sim}{F} = \alpha \underline{u}$; $\boxed{\frac{D}{Dt}\left(\int_V \alpha \, dV\right) = \int_V \left[\frac{\partial \alpha}{\partial t} + \underline{\nabla} \cdot (\alpha \underline{u})\right] dV}$ RRT 2

• Now; $\underline{\nabla} \cdot (\alpha \underline{u}) = (\underline{u} \cdot \underline{\nabla})\alpha + \alpha \underline{\nabla} \cdot \underline{u}$

So; $\frac{\partial \alpha}{\partial t} + \underline{\nabla} \cdot (\alpha \underline{u}) = \frac{\partial \alpha}{\partial t} + (\underline{u} \cdot \underline{\nabla})\alpha + \alpha \underline{\nabla} \cdot \underline{u} = \frac{D\alpha}{Dt} + \alpha \underline{\nabla} \cdot \underline{u}$

• Thus; $\boxed{\frac{D}{Dt}\left(\int_V \alpha \, dV\right) = \int_V \left[\frac{D\alpha}{Dt} + \alpha \underline{\nabla} \cdot \underline{u}\right] dV}$ RTT 3

Example: Take $\alpha = \rho$ density.

Then $M = \int_V \rho \, dV$ is the mass of particles making up the volume V.

Then, by RTT2, $\frac{DM}{Dt} = \frac{D}{Dt}\int_V \rho \, dV = \int_V \left(\frac{\partial \rho}{\partial t} + \underline{\nabla} \cdot (\rho \underline{u})\right) dV$

But mass is ~~constant~~ conserved so $\frac{DM}{Dt} = 0$.

i.e: for any V in $\mathcal{D}$;

$$\int_V \left[\frac{\partial \rho}{\partial t} + \underline{\nabla} \cdot (\rho \underline{u})\right] dV = 0.$$

By our lemma, this implies

$$\frac{\partial \rho}{\partial t} + \underline{\nabla} \cdot (\rho \underline{u}) = 0 \text{ everywhere in } \mathcal{D}.$$

↳ Conservation of mass for a compressible fluid!

↳ Contains $\underline{\nabla} \cdot \underline{u} = 0$ when ρ is constant.

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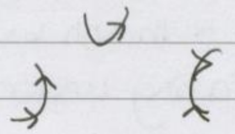
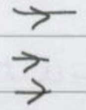


Straining $r^2\cos 2\theta$



uniform stream $r\sin\theta$.

RTT: 3D version of Leibnitz.

RTT 1: $\frac{D}{Dt}\int_V \alpha \, dv = \int_V \frac{\partial \alpha}{\partial t} \, dv + \int_S \alpha \underline{u} \cdot \underline{\hat{n}} \, ds$

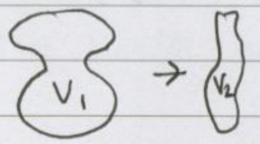
RTT 2: $\frac{D}{Dt}\int_V \alpha \, dv = \int_V \left[\frac{\partial \alpha}{\partial t} + \underline{\nabla} \cdot (\alpha \underline{u}) \right] dV$

RTT 3: $\frac{D}{Dt}\int_V \alpha \, dv = \int_V \left(\frac{D\alpha}{Dt} + \alpha \, \underline{\nabla} \cdot \underline{u} \right) dV$

- Any scalar; $\alpha(\underline{x}, t)$

Example: Conservation of mass ($\alpha = \rho$; density).
Consider a fluid of variable density $\rho(\underline{x}, t)$, that occupies a domain $\mathcal{D}$. Let V be any subdomain of $\mathcal{D}$. [i.e; V must be arbitrary.]. Consider the mass (total mass) of all the particles comprising V, i.e:

$$M = \int_V \rho \, dv.$$



The rate of change of mass M, staying with the same particles must be zero. (Conservation of mass.)

i.e: $\frac{DM}{Dt} = 0$

But by RTT 2; $\frac{D}{Dt}\int_V \rho \, dv = \int_V \left[\frac{\partial \rho}{\partial t} + \underline{\nabla} \cdot (\rho \underline{u}) \right] dv.$

So $\int_V \left[\frac{\partial \rho}{\partial t} + \underline{\nabla} \cdot (\rho \underline{u}) \right] dv = 0.$

But V is arbitrary, so this is true for all V. Hence, by our theorem:

$\frac{\partial \rho}{\partial t} + \underline{\nabla} \cdot (\rho \underline{u}) = 0$ everywhere in $\mathcal{D}$!

This can also be written as $\frac{D\rho}{Dt} + \rho \underline{\nabla} \cdot \underline{u} = 0$. } True whether compressible or not!

Notice, if the flow is incompressible, fluid elements cannot be squashed, i.e; they preserve their volume. But they preserve their mass. Hence, they preserve their density. i.e: $\frac{D\rho}{Dt} = 0$

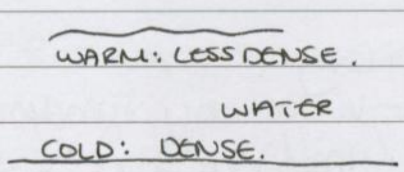
• By conservation of mass;

$$\rho \underline{\nabla} \cdot \underline{u} = 0$$

$$\text{i.e; } \underline{\nabla} \cdot \underline{u} = 0 \quad \text{(as before)}$$

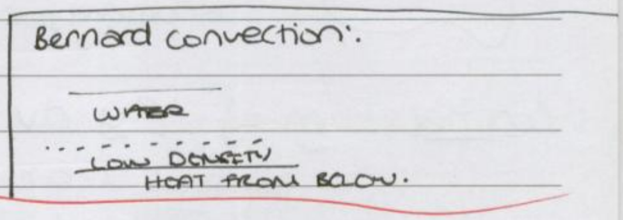
[Notice; This does not require all particles to have the same density.

e.g: oceans



Incompressible: not constant density.
Fluid particles retain density.
c.f colour.

e.g: Antarctic Bottom Water!



RTT4: For a fluid of density $\rho(\underline{x},t)$ consider any quantity $f(\underline{x},t)$.
Put $\alpha = \rho f$ in RTT3:

$$\frac{D}{Dt}\int_V f \rho \, dv = \int_V \left[\frac{\partial}{\partial t}(f\rho) + \underline{\nabla} \cdot (f\rho\underline{u}) \right] dv$$

$$* \; [\ldots] = \rho \frac{\partial f}{\partial t} + f \frac{\partial \rho}{\partial t} + f\rho \, \underline{\nabla} \cdot \underline{u} + \underline{u} \cdot \underline{\nabla}(f\rho)$$

$$= f\left(\frac{\partial \rho}{\partial t} + \rho \underline{\nabla} \cdot \underline{u} \right) + \rho \frac{\partial f}{\partial t} + f \underline{u} \cdot \underline{\nabla}\rho + \rho \underline{u} \cdot \underline{\nabla} f$$

$$= f\left[\frac{\partial \rho}{\partial t} + \underline{\nabla} \cdot (\rho \underline{u}) \right] + \rho \left[\frac{\partial f}{\partial t} + (\underline{u} \cdot \underline{\nabla}) f \right]$$

↓ 0 by C.O.M (Conservation...)

$$= \rho \frac{Df}{Dt}$$

i.e: RTT4:

$$\frac{D}{Dt}\int_V \rho f \, dV = \int_V \rho \frac{Df}{Dt} \, dV$$

or

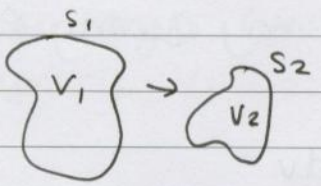
$$\frac{D}{Dt}\int_V f \rho \, dV = \int_V \frac{Df}{Dt} \rho \, dV$$

[i.e: $\frac{D}{Dt}\int_V f \, dM = \int_V \frac{Df}{Dt} \, dM$ i.e; integrals WRT mass.]

Example (2): Force = R.o.change of Momentum.
i.e: "$\alpha = \rho \underset{\sim}{u}$"

consider a fluid of ~~constant~~ density $\rho(\underset{\sim}{x},t)$ occupying a domain $\mathcal{D}$. Let V be any subdomain, with surface S, of $\mathcal{D}$. (i.e; Important that V is arbitrary for our theorem.)

Consider: $\underset{\sim}{m} = \int_V \rho \underset{\sim}{u} \, dV$, the total momentum of the fluid particles making up V.

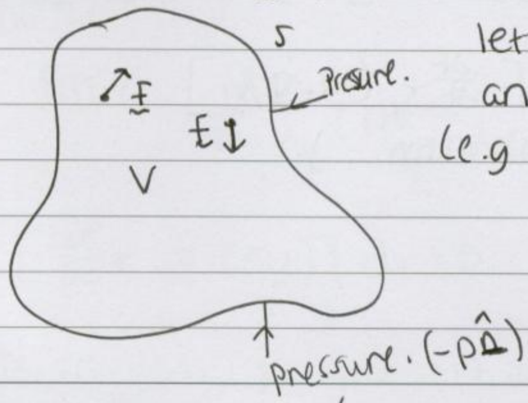


Then following these particles, by RTT4;

$$\frac{D\underset{\sim}{m}}{Dt} = \frac{D}{Dt}\int_V \rho \underset{\sim}{u} \, dV = \int_V \rho \frac{D\underset{\sim}{u}}{Dt} \, dV$$

↑ acceleration.

By Newton, this must ~~exceed~~ equal the total external force acting on the particle comprising V. (The internal forces sum to zero).



let each particle be subject to an external force $\underset{\sim}{F}$ per unit mass.
(e.g gravity, $\underset{\sim}{F} = -g\hat{\underset{\sim}{z}}$
(or magnetic force.
or electric force.)

↳ normal stress. $= p(\underset{\sim}{x},t)$ inwards.
i.e; $-p\hat{n}$.

That is all for an inviscid fluid as elements cannot exert a shear (i.e: tangential stress). [extra term in a viscous fluid.].

· Thus, the total force on all particles comprising V is

$$\int_V \rho \underline{F}\, dv + \int_S (-p)\, \hat{\underline{n}}\, dS$$

$$= \int_V \rho \underline{F}\, dv + \int_V (-\underline{\nabla} p)\, dV$$

by vector form of divergence thm.

$$\equiv \int (-\underline{\nabla} p + \rho \underline{F})\, dV$$

* R.o.cn of momentum = force acting.

$$\int_V \rho \frac{D\underline{u}}{Dt}\, dV = \int_V (-\underline{\nabla} p + \rho \underline{F})\, dV$$

$$\Rightarrow \int_V \left(\rho \frac{D\underline{u}}{Dt} + \underline{\nabla} p - \rho \underline{F}\right) dV = 0.$$

But V arbitrary, so this is true for all V. ~~Then~~ Hence, by our theorem,

$$\rho \frac{D\underline{u}}{Dt} + \underline{\nabla} p - \rho \underline{F} = 0 \quad \text{everywhere in } \mathcal{D}.$$

This is Euler's eqⁿ for an inviscid fluid,

$$\rho \frac{D\underline{u}}{Dt} = -\underline{\nabla} p + \rho \underline{F}$$

mass x acceleration = force.

Equations of motion for a (possibly compressible) fluid:
Density $\rho(\underline{x},t)$, Pressure $P(\underline{x},t)$, velocity $\underline{u}(x,t)$.

Mass: $\frac{\partial \rho}{\partial t} + \underline{\nabla}\cdot(\rho\underline{u}) = 0$.

Euler: $\rho \frac{Du}{Dt} = -\underline{\nabla}P + \rho\underline{F}$

• 2 unknown scalar fields & one unknown vector field. ($\underline{u}$).
we have 1 scalar & 1 vector equation.
Missing a scalar eq^n.
Geophysical fluid Dynamics: Atmospheres + oceans.

Incompressibility: $\frac{D\rho}{Dt} = 0$.

Mass: $\underline{\nabla}\cdot\underline{u} = 0$

Euler: $\rho\frac{D\underline{u}}{Dt} = -\underline{\nabla}P + \rho\underline{F}$.

Gas Dynamics: (cosmology)

$\rho = f(P)$.

For an ideal gas; $P = \rho^{\alpha}$, for some α. - second scalar eq^n.
• We will continue by taking the density to be constant.
i.e: all particles have the same density.
Then:
mass = Incompressibility = 'continuity'
Euler.

$\underline{\nabla}\cdot\underline{u} = 0$.

$\rho\frac{D\underline{u}}{Dt} = -\underline{\nabla}P + \rho\underline{F}$

1 scalar unknown: P
1 vector unknown: $\underline{u}$

1 scalar eq^n.
1 vector eq^n.

ρ given - constant.

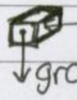
Next: examples.
• Open channel flow - Hydraulics.
• Surface water waves.
} need gravity.

Example: Find the free surface shape for a cylindrical container partially filled with fluid of constant density ρ in a solid body rotation with angular speed Ω about a vertical axis.

Ans: let the flow have settled to a steady state. Then $\frac{\partial}{\partial t} = 0$.

continuity: $\underset{\sim}{\nabla} \cdot \underline{u} = 0$.

Euler: $\rho \frac{D\underline{u}}{Dt} = -\underline{\nabla} p + \rho \underset{\sim}{f}$.



$\underset{\sim}{f}$ force / unit mass (dimensions: acceleration)

$= -g\hat{\underline{z}}$.

$$\frac{D\underline{u}}{Dt} = \frac{\partial \underline{u}}{\partial t} + (\underline{u} \cdot \underline{\nabla})\underline{u}$$

- Euler becomes: $(\underline{u} \cdot \underline{\nabla})\underline{u} = -\underline{\nabla} p - \rho g \hat{\underline{z}}$

we are told that the fluid is in solid body rotation, i.e:

$$\underline{u} = \underline{\Omega} \wedge \underline{r}$$

$$\underset{\sim}{u} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 0 & 0 & \Omega \\ x & y & z \end{vmatrix} = \underline{i}(-y\Omega) + \underline{j}(x\Omega)$$

i.e: $u = -\Omega y, \quad v = \Omega x$.

$$\underline{u} \cdot \underline{\nabla} = u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + w\frac{\partial}{\partial z} = -\Omega y\frac{\partial}{\partial x} + \Omega x\frac{\partial}{\partial y}$$

So $(\underline{u} \cdot \underline{\nabla})u = \left(-\Omega y\frac{\partial}{\partial x} + \Omega x\frac{\partial}{\partial y}\right)(-\Omega y) = -\Omega^2 x$

$(\underline{u} \cdot \underline{\nabla})v = (\quad " \quad)(\Omega x) = -\Omega^2 y$

Euler (In components):

- x-momentum → $-\rho\Omega^2 x = -\dfrac{\partial p}{\partial x}$ ①
- y-momentum → $-\rho\Omega^2 y = -\dfrac{\partial p}{\partial y}$ ②
- z-momentum → $0 = -\dfrac{\partial p}{\partial z} - \rho g$ ③

$(\underset{\sim}{F} = -\rho g \hat{\underset{\sim}{z}})$

* Integrate ①:

$$p = \tfrac{1}{2}\rho\Omega^2 x^2 + f(y, z)$$

* Differentiate wrt y: $p_y = f_y$

c.f: $f_y = \rho\Omega^2 y$

i.e: $f = \tfrac{1}{2}\rho\Omega^2 y^2 + h(z)$.

$$\therefore\ p = \tfrac{1}{2}\rho\Omega^2(x^2+y^2) + h(z)$$

Diff WRT z:

$$p_z = h'(z)$$

c.f with ③: $h'(z) = -\rho g$

i.e: $h = -\rho g z + C$ (C = constant).

i.e: $p = \tfrac{1}{2}\rho\Omega^2(x^2+y^2) - \rho g z + C$.

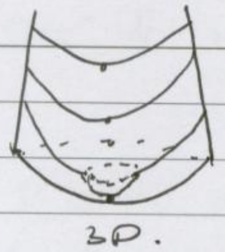
Isobars = lines of constant pressure.
Isobaric surface = surface of constant pressure.

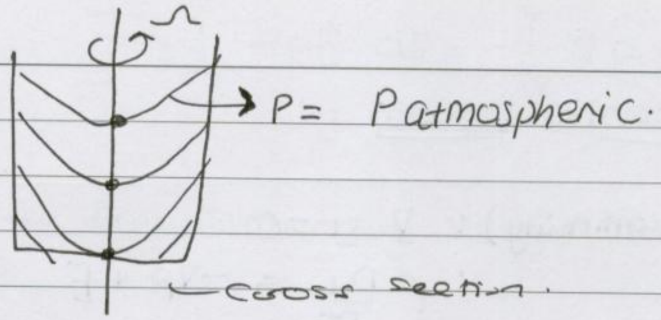
p = constant. e.g: $p = A$, constant.

$$\rho g z = \tfrac{1}{2}\rho\Omega^2(x^2+y^2) + C - A$$

i.e: $z = \tfrac{1}{2}\dfrac{\Omega^2}{g}(x^2+y^2) + \underbrace{\dfrac{C-A}{\rho g}}_{z_0}$

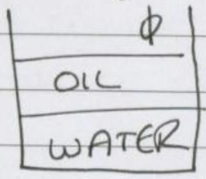
$\therefore\ z - z_0 = \tfrac{1}{2}\dfrac{\Omega^2}{g}(x^2+y^2)$ → a paraboloid with origin $(0, 0, z_0)$





Archimedes?

Example 2: Consider a submerged body of volume V with surface S. Then the force on the body is upwards and equal to the weight of water displaced.



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Hydrostatic Pressure:

Cty (continuity) : $\underline{\nabla} \cdot \underline{u} = 0$

Euler : $\rho \frac{D\underline{u}}{Dt} = -\nabla p + \underline{F}$

• When gravity is the only ~~external~~ ~~function~~ external force;

$$\underset{\sim}{F} = -g\hat{\underline{z}}$$

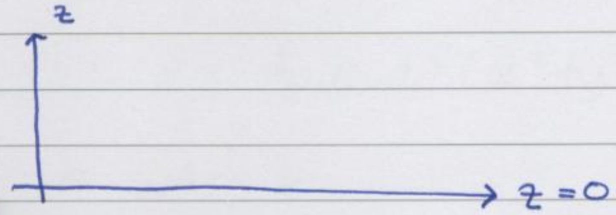
* If the fluid is at rest, $\underline{u} = 0$.

$$0 = -\nabla p - \rho g \hat{\underline{z}}$$

i.e: $\frac{\partial p}{\partial x} = 0, \; \frac{\partial p}{\partial y} = 0, \; \frac{\partial p}{\partial z} = -\rho g$

$\Downarrow$ $p = p(y,z)$ so $\Downarrow$ $p = p(z)$

i.e: $p = -\rho g z$ + constant.

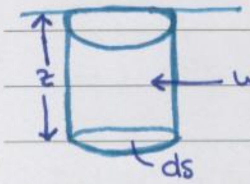


: surface where pressure is atmospheric, i.e: $p = p_a$.

Then $p = p_a$ when $z = 0$.

So $p = p_a - \rho g z$.

↳ We call this Hydrostatic Pressure!

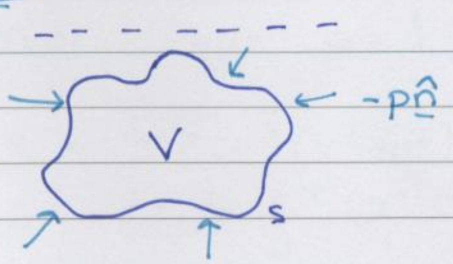


← weight of water above ds is $(z\,ds) \times \rho \times g$.

Force per unit area $= \rho g z$

i.e: Hydrostatic pressure supports water above.

Example:



Consider a submerged body occupying a volume V with surface S, immersed in a fluid of density ρ.

The force on the body is $\underset{\sim}{F} = \int_S -p\underline{\hat{n}}\, dS = -\int_V \underline{\nabla} p \, dV$
(by divergence Theorem).

Here, P is the pressure in the fluid surrounding V. But fluid at rest, so $p = p_H = p_a - \rho g z$.
so $\underline{\nabla} p = \underline{\nabla} p_H = -\rho g \underline{\hat{z}}$

density of fluid.

$$\text{so } \underset{\sim}{F} = -\int_V (-\rho g \underline{\hat{z}})\, dV = \rho g \underline{\hat{z}} \int_V dV = \rho V g \underline{\hat{z}}$$

* not mass of body!!

* ρV = mass of fluid displaced.
  $\rho V g$: weight " " "
  $\rho V g \underline{\hat{z}}$: upward force equal to the weight of fluid displaced (ARCHIMEDES).

For a moving fluid, it is often convenient to split the pressure into hydrostatic and the rest, called dynamic pressure.
i.e: write $p = p_H + p_d$

Then the Euler equations under gravity become

$$\rho \frac{D\underline{u}}{Dt} = -\underline{\nabla} p - \rho g \underline{\hat{z}} = -\underline{\nabla} p_H - \underline{\nabla} p_d - \rho g \underline{\hat{z}}$$

$$= -(-\rho g \underline{\hat{z}}) - \underline{\nabla} p_d - \rho g \underline{\hat{z}} = -\underline{\nabla} p_d$$

i.e: we can ignore gravity in the equations of motion, provided we measure pressure as the deviation from Hydrostatic. This is not useful when a free surface is present since the bc there $p = p_a$ is on the total pressure $p = p_H + p_d$

Bernoulli's Equation:

We have the identity $(\underset{\sim}{u} \cdot \underline{\nabla})\underset{\sim}{u} = \underline{\nabla}\left(\frac{1}{2}\underline{u}^2\right) + \underset{\sim}{\omega} \wedge \underset{\sim}{u}$

where $\underset{\sim}{\omega} = \underline{\nabla} \wedge \underline{u}$ is the vorticity.

· Thus: $\frac{D\underline{u}}{Dt} = \frac{\partial \underline{u}}{\partial t} + (\underline{u} \cdot \underline{\nabla})\underline{u} = \frac{\partial \underline{u}}{\partial t} + \underline{\nabla}\left(\frac{1}{2}\underline{u}^2\right) + \underline{\omega} \wedge \underline{u}$

* Now Euler is $\rho \frac{D\underline{u}}{Dt} = -\underline{\nabla} p + \rho \underline{F}$

Now consider only external forces derivable from a potential,

i.e: $\underset{\sim}{F} = -\underline{\nabla} V_e$

i.e: for gravity; $V_e = gz$.

i.e: F is conservative.

Then: $\rho\left[\frac{\partial \underline{u}}{\partial t} + \underline{\omega} \wedge \underline{u}\right] = -\underline{\nabla} p - \rho \underline{\nabla}\left(\frac{1}{2}\underline{u}^2\right) - \rho \underline{\nabla} V_e$

$= -\underline{\nabla} H.$

where $H = p + \frac{1}{2}\rho \underline{u}^2 + \rho V_e$

In steady flow, $\frac{\partial \underline{u}}{\partial t} = 0$, so $\rho\, \underline{\omega} \wedge \underline{u} = -\underline{\nabla} H$.

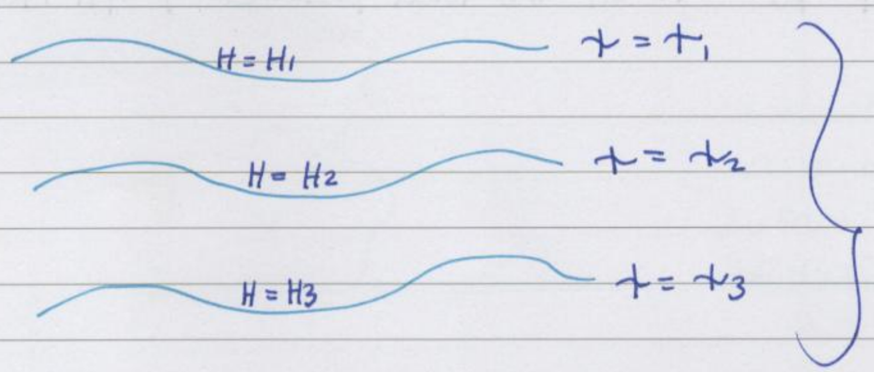
Dot with u: $0 = (\underline{u} \cdot \underline{\nabla}) H$

i.e: $\frac{DH}{Dt} = 0$, i.e: H is constant following a particle.

But flow is steady: particle paths are streamlines.

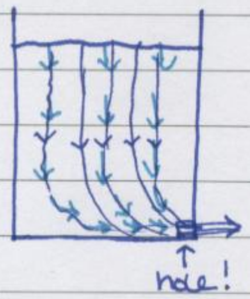
* Thus, H is constant on streamlines!

BERNOULLI'S THEOREM!



H can have different values on different streamlines.

Example:



**Important: The surface is connected to the exit by streamlines.

Assume the hole is sufficiently small so that flow is steady!

Hence, Bernoulli applies.

on any streamline;

$H =$ constant (not necessarily the same)

- Here, $H = P + \frac{1}{2}\rho \underline{u}^2 + \rho V_e$.

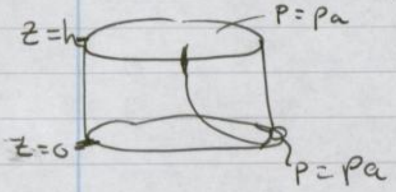
The external potential is $V_e = gz$ //

(with zero taken at level of hole).

Fluids:

Bernoulli examples:

Example 1: draining cylinder.



Streamline connecting surface to exit.
∴ can apply Bernoulli on this streamline.
i.e: $p+\frac{1}{2}\rho u^2 + \rho V_e$ is constant on streamline.
Hence, same at top & bottom.

Now $V_e = gz$ ($z=0$ at level of exit)
pressure is atmospheric at top & bottom.

At top: $p+\frac{1}{2}\rho \underline{u}^2 + \rho V_e = p_a + \rho U^2 + \rho g h$ where $u=U$ at surface.

At bottom: $p+\frac{1}{2}\rho \underline{u}^2 + \rho V_e = p_a + \frac{1}{2}\rho V^2 + 0$ where $u=V$ at exit.

Thus $p_a + \frac{1}{2}\rho U^2 + \rho g h = p_a + \frac{1}{2}\rho V^2$

i.e: $V^2 = U^2 + 2gh$

Then the mass flux at top is $\rho U A$
" " " " bottom is $\rho V a$.
These are the same so $UA = Va$

Thus $V^2 = \left(\frac{Va}{A}\right)^2 + 2gh$ i.e: $V^2\left[1-\left(\frac{a}{A}\right)^2\right] = 2gh$

If hole is small $\frac{a}{A} \ll 1$ so $\left(\frac{a}{A}\right)^2 \lll 1$

Then $V^2 \simeq 2gh$
i.e $V = \sqrt{2gh}$

i.e: exactly as for a free-falling particle under gravity.

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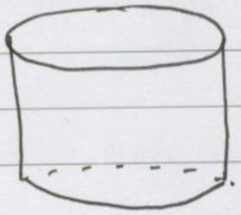
Bernoulli examples:

Example 1: Draining cylinder.



Now $V_e = gz$ ($z = 0$ at level of exit)

pressure is atmosphere at top and bottom.

At top: $p + \frac{1}{2}\underset{\sim}{u}^2 + \rho V_e = p_a + \frac{1}{2}\rho U^2 + \rho g h$

where $u = U$ at surface.

~~Those~~ Thus $p_a + \frac{1}{2}\rho U^2 + \rho g h = p_a + \frac{1}{2}\rho v^2$

i.e: $v^2 = U + 2gh$.

The mass flux is conserved. Let surface area at the top be A and the surface area at the bottom be a.

Then the mass flux at top is $\rho U A$.

" " " " bottom is $\rho v a$.

These are the same so $UA = va$.

Thus, $v^2 = \left(\frac{va}{A}\right)^2 + 2gh$.

i.e: $v^2\left[1 - \left(\frac{a}{A}\right)^2\right] = 2gh$.

If hole is small, $a/A \ll 1$. So $\left(\frac{a}{A}\right)^2 <<<< 1$.

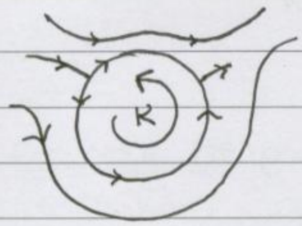
Then $v^2 \simeq 2gh$

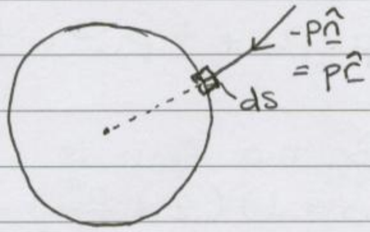
i.e $v = \sqrt{2gh}$,

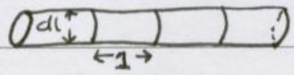
i.e: exactly as for a free ~~standing~~ falling particle under gravity.

Example 2: Spinning cylinder.

Consider a cylinder of radius a in a stream, uniform at infinity with speed U in the x-direction. Let the cylinder be spinning so that the circulation about the cylinder is K.



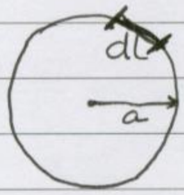




- we will find the force per unit length on the cylinder. Our unit of area is $ds = dl \times 1$.

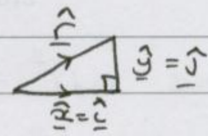
Thus, total force (per unit length) =

$$\int -p\hat{n}\, ds = -\int p\, \hat{r}\, dl \rightarrow \text{per unit length.}$$



And $dl = a\, d\theta$

$$\underline{F} = -\int_{\theta=-\pi}^{\pi} p\hat{r}\, a\, d\theta$$



$$\therefore \hat{r} = \cos\theta\, \hat{i} + \sin\theta\, \hat{j}$$

$$\equiv -a\int_{-\pi}^{\pi} p(\cos\theta\, \hat{i} + \sin\theta\, \hat{j})\, d\theta$$

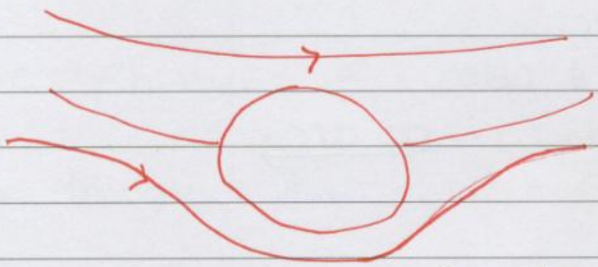
$$= \mathcal{D}\hat{i} + \mathcal{L}\hat{j}.$$

- where the drag, $\mathcal{D}$ is

$$\mathcal{D} = \int_{-\pi}^{\pi} -a\,(p\cos\theta)\, d\theta$$

and the lift $\mathcal{L}$ is

$$\mathcal{L} = -a\int_{-\pi}^{\pi} p\sin\theta\, d\theta$$



Now all streamlines originate upstream, ~~(~~ ~~)~~ & we are taking flow to be steady. So use Bernoulli.

Steady iff $\frac{d\underline{u}}{dt} = 0$.

$p + \frac{1}{2}\rho u^2 =$ constant on streamlines in the absence of external forces.

At infinity, $p = p_\infty$, constant

$$\underset{\sim}{u} = U\hat{\underset{\sim}{i}}.$$

So $p + \frac{1}{2}\rho \underline{u}^2 = p_\infty + \frac{1}{2}\rho U^2$.

Anywhere in the flow, $p = p_\infty + \frac{1}{2}\rho U^2 - \frac{1}{2}\rho |\underline{u}|^2$ //

The complex potential for the flow is

$$w = U\left(z + \frac{a^2}{z}\right) - \frac{iK}{2\pi}\log(z)$$

· so $\frac{dw}{dz} = u - iv = U\left(1 - \frac{a^2}{z^2}\right) - \frac{iK}{2\pi z}$

so $u_r - iu_\theta = e^{i\theta}\frac{dw}{dz} = 2iU\sin\theta - \frac{iK}{2\pi a}$ on $z = ae^{i\theta}$ //

i.e: $u_r = 0$ as expected. } on cylinder.
$u_\theta = \frac{K}{2\pi a} - 2U\sin\theta$

Thus $|\underline{u}|^2 = \frac{K^2}{4\pi^2 a^2} - \frac{2UK}{\pi a}\sin\theta + 4U^2\sin^2\theta$.

Now $\mathcal{D} = -a\int_{-\pi}^{\pi}\cos\theta \cdot p \, d\theta$ //

$$* \int_{-\pi}^{\pi}\cos\theta \cdot C \, d\theta = 0$$

$$* \int_{-\pi}^{\pi}\cos\theta \cdot \sin\theta \, d\theta = 0$$

$$* \int_{-\pi}^{\pi}\cos\theta\sin^2\theta \, d\theta = 0$$

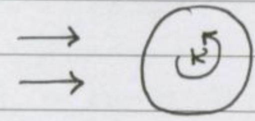
· Thus $\mathcal{D} = 0$, i.e: no drag.
or velocity symmetric before & after.
So pressure the same before & after.
∴ no drag.

Now $\mathcal{L} = -a\int_{-\pi}^{\pi}\sin\theta \cdot P \, d\theta$

$\int_{-\pi}^{\pi}\sin\theta \, C \, d\theta = 0$ // ; $\int_{-\pi}^{\pi}\sin^3\theta \, d\theta = 0$ // (odd)

$$\int_{-\pi}^{\pi}\sin^2\theta \, d\theta = \pi$$ //

$\therefore \mathcal{L} = -a\int_{-\pi}^{\pi} \left(-\frac{1}{2}\rho\right)\left(-2U\frac{K}{\pi a}\right) \sin^2\theta \, d\theta = -\rho U K$

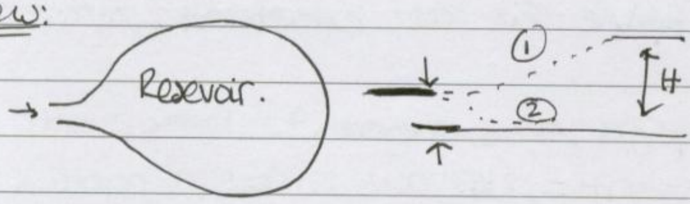


i.e: downward force: $\rho U K$. (Independent of a)
(per unit length)

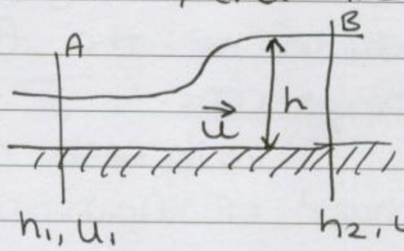
$u^2 = |\underline{u}|^2 = \underline{u} \cdot \underline{u}$

Example 3: Open channel flow. (Flow down (along) a channel that is open to the air.)
e.g: River, aqueduct.

Plan view:



Initially, let us consider a channel of constant width b, and horizontal floor. Let any changes in the flow be slow in the flow direction. Let the local depth be h, and the local speed be U downstream.



By conservation of mass, in steady flow, mass flux across station A must equal mass flux across station B.

i.e: $\rho h_1 b u_1 = \rho h_2 b u_2$
i.e: $h_1 u_1 = h_2 u_2$

or throughout flow; $uh = Q$, constant.

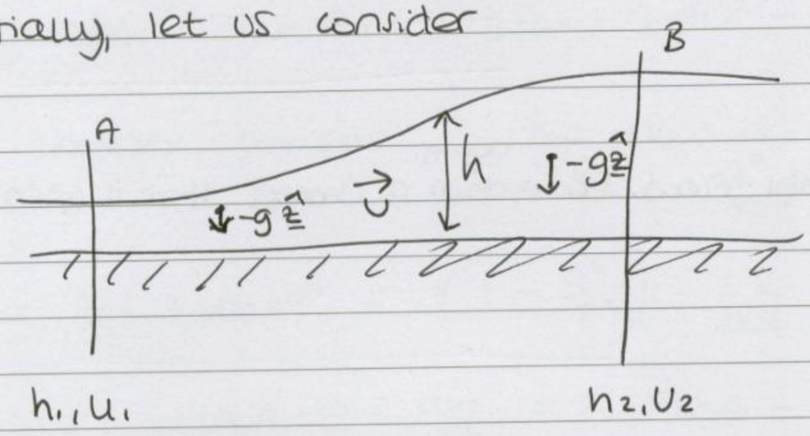
i.e: $Q = uh$, is constant of the motion.
[volume flux per unit width]
i.e: dimensions $L^2 T^{-1}$

provided the flow is smooth, a particle on the surface stays there.

i.e: surface is a streamline.

Hence, we can apply Bernoulli there.

Initially, let us consider



only force acting on the fluid is gravity.

Bernoulli (on the surface, a streamline)

$$p + \frac{1}{2}\rho|\underline{u}|^2 + \rho V_e \text{ is a constant.}$$

Here, $p = p_a + \frac{1}{2}\rho U^2 + \rho g h = \text{constant}$.

i.e: $\frac{1}{2}u^2 + gh = \text{constant} = gH$.

where H is a second constant of the motion.

Dimensions of H are length.

H is the depth the fluid would occupy were it to come to rest.

i.e: $h \to H$ if $u \to 0$.

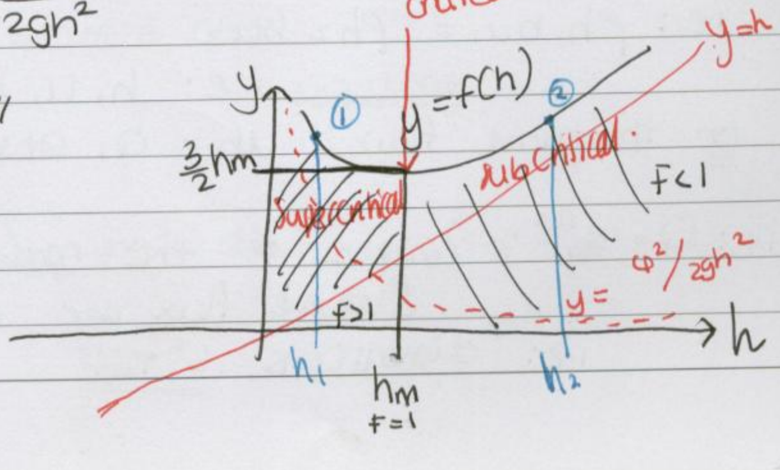
H: 'head' of flow.

Thus we have $uh = Q$, $\frac{1}{2}u^2 + gh = gH$.

Eliminate u: $u = Q/h$

so $\frac{Q^2}{2gh^2} + h = H$.

let $f(h) = \frac{Q^2}{2gh^2} + h$



This graph has a single minimum for $h>0$, when $f'(h)=0$.

i.e: $-\frac{2Q^2}{2gh^3}+1=0$ i.e: $h = h_m = (Q^2/g)^{1/3}$

* $f(h_m) = h_m + \frac{h_m^3}{2h_m^2} = \frac{3}{2}h_m$ //

At $h=h_m$; $h_m^3 = \frac{Q^2}{g} = (h_m u_m)^2/g = h_m^2 u_m^2/g$

$$\Rightarrow \frac{u_m^2}{g h_m} = 1$$

We define the Froude number $F = u/\sqrt{gh}$ at any point in the flow.

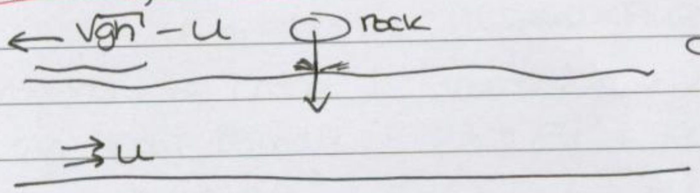
Then $F=1$ when $h=h_m$.

If $h>h_m$, then $u<u_m$ so $F<1$.

If $h<h_m$, then $u>u_m$ so $F>1$.

Fact: The speed of long waves on shallow water is $\sqrt{gh}$.
(shown in water waves)

So if $F<1$; flow is slower than waves. (subcritical)

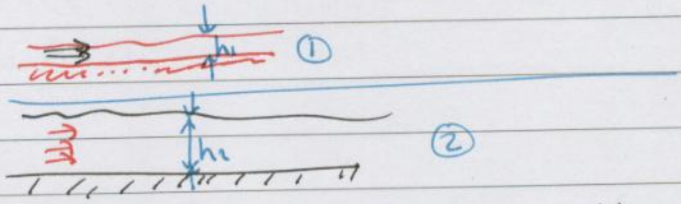


drop a rock!

- waves can travel upstream.

If $F>1$, flow is faster than waves.
Information cannot travel upstream.
(supercritical)

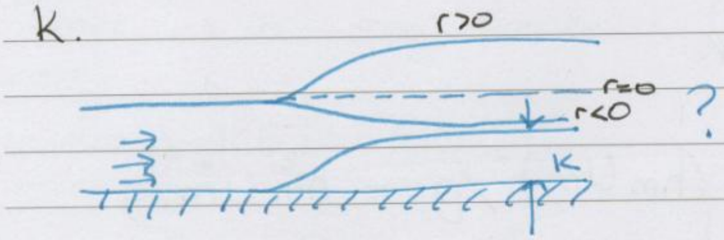
- Supercritical: shallow & fast.
- Subcritical: deep & slow.

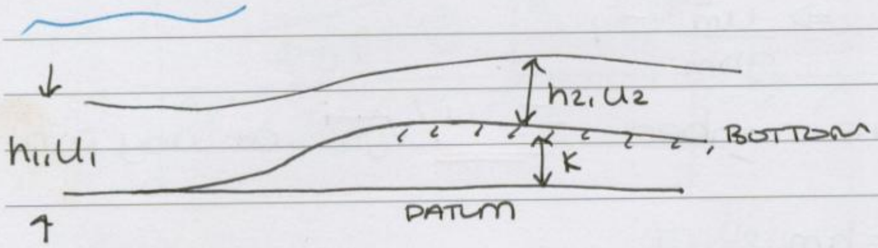


* These are two flows with the same Q and the same H, but different h.

Example:

Now suppose the channel remains of constant width but the floor of the channel rises smoothly by an amount K.





Depth h2.

upstream surface height h_1

Downstream surface height $h_2 + K$.

Rise in surface.

$$r = h_2 + K - h_1$$

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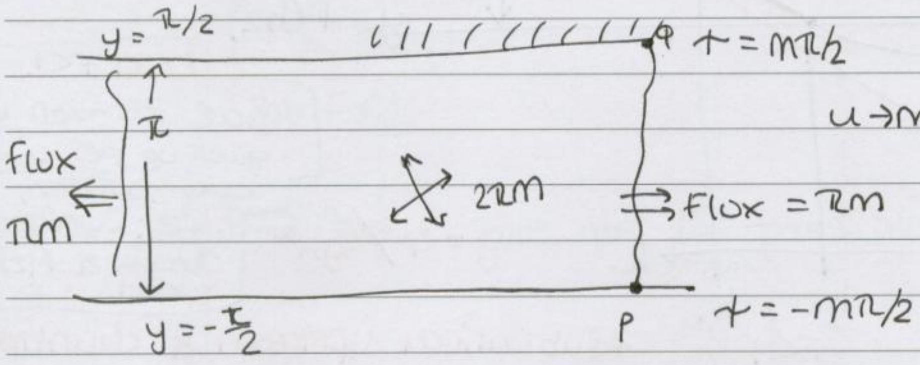
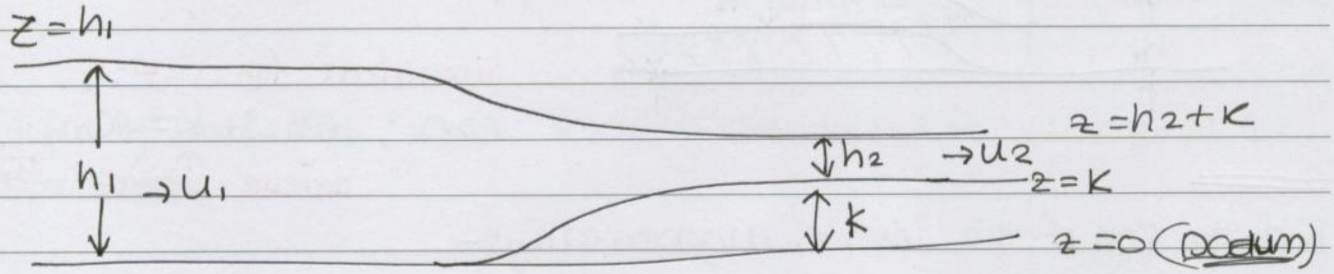


$u \to m$ as $x \to \infty$.

Sheet (6)

$u \to -m$ as $x \to -\infty$.



rise ~~...~~; r = new − old. = $h + K - h_1$

Mass flux constant

$$u_1 h_1 = u_2 h_2 \quad \text{(speed x fluid depth)}$$

Now provided the change is smooth, the surface is a streamline. Thus, we can apply Bernoulli :− $p + \frac{1}{2}\rho \underline{u}^2 + \rho g z$ = constant on surface.

upstream: $p + \frac{1}{2}\rho \underline{u}^2 + \rho g z = p_a + \frac{1}{2}\rho u_1^2 + \rho g h_1$

Downstream: $p + \frac{1}{2}\rho u^2 + \rho g z = p_a + \frac{1}{2}\rho u_2^2 + \rho g \,(\underline{h_2 + K})$

↳ NOT THE DEPTH! Height of surface.

=

Thus $\frac{1}{2}\rho u_1^2 + \rho g h_1 = \frac{1}{2}\rho u_2^2 + \rho g (h_2 + K)$

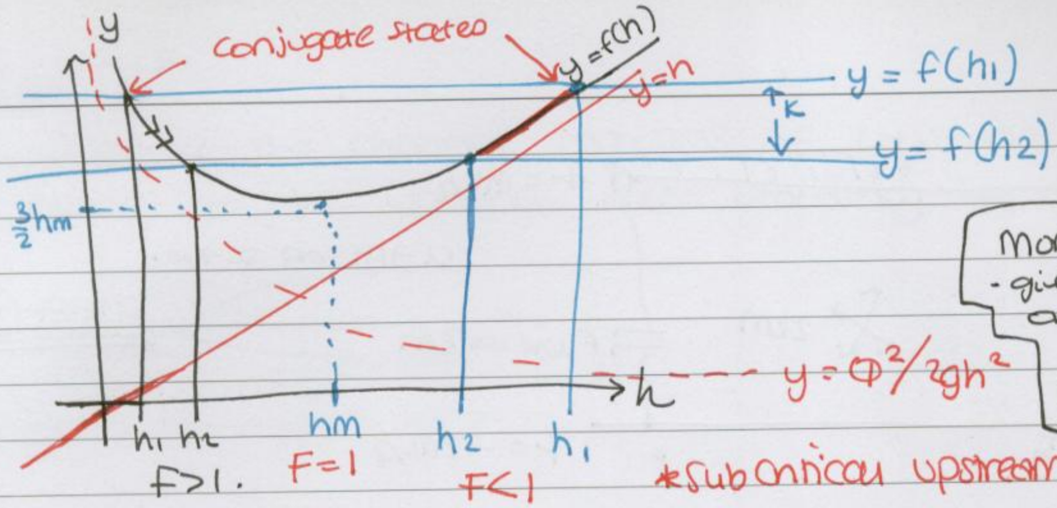
i.e: $\frac{u_1^2}{2g} + h_1 = \frac{u_2^2}{2g} + h_2 + K$

$$\& \; u_1 h_1 = u_2 h_2 = Q \quad \text{(volume flux/ unit width)}$$

i.e: $\frac{Q^2}{2gh_1^2} + h_1 = \frac{Q^2}{2gh_2^2} + h_2 + K$

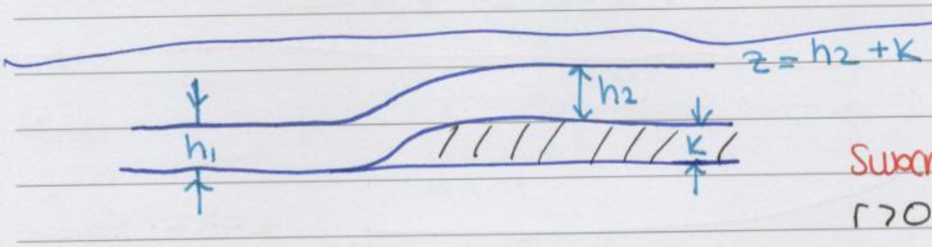
i.e: $f(h_1) = f(h_2 + K)$

where $f(h) = \frac{Q^2}{2gh} + h$ //



sub F<1,
More PE than KE
- gives up PE to get over barrier,
i.e: Surface drops & flow speeds up.

*subcritical upstream & downstream.



Subcritical flow F>1.

$r>0$; $f(h_2) + K = f(h_1)$ ↳ FIXED!

decrease   increase

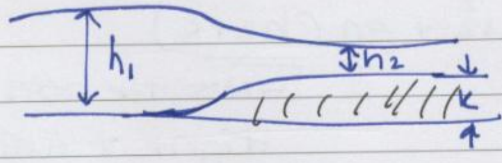
Find the sign of the rise, r, algebraically:-

$$r = h_2 + K - h_1$$

$$= \frac{Q^2}{2g}\left(\frac{1}{h_1^2} - \frac{1}{h_2^2}\right) = \frac{Q^2}{2gh_1^2h_2^2}\left(h_2^2 - h_1^2\right)$$

Thus: $r>0$ when $h_2 > h_1$ and $r<0$ when $h_2 < h_1$.

$$\therefore \frac{Q^2}{2gh_1^2} + h_1 = \frac{Q^2}{2gh_2^2} + h_2 + K$$

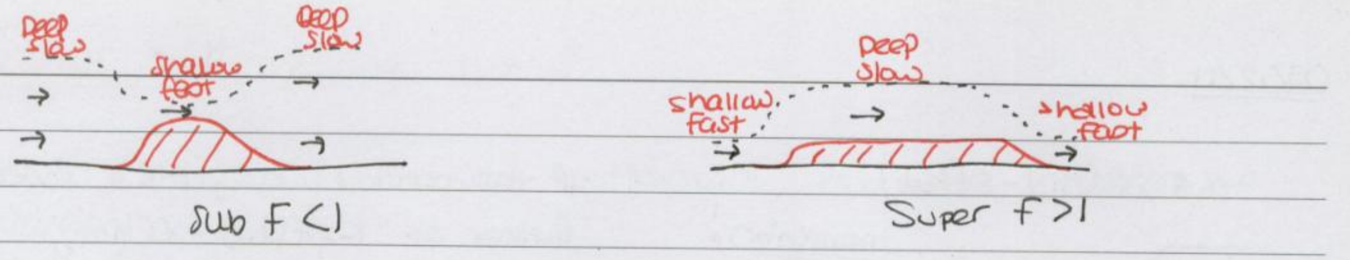


F<1 subcritical.

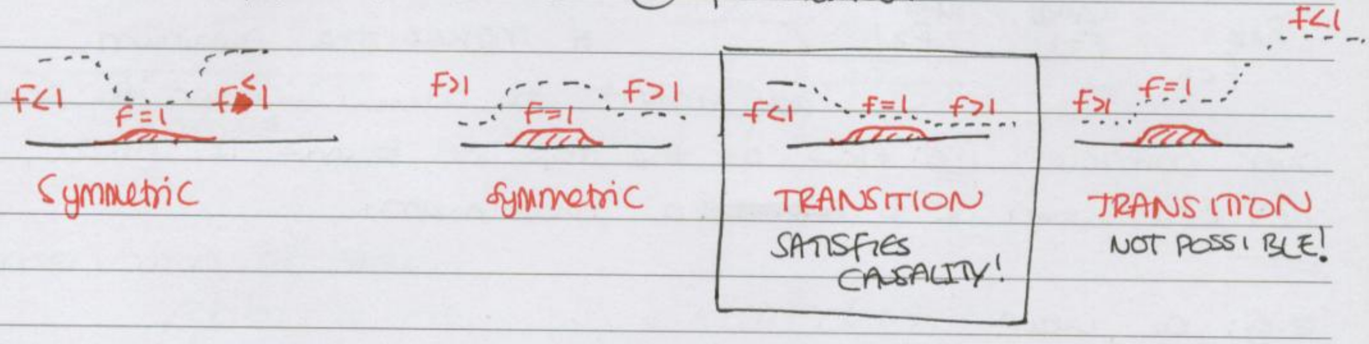
$$F = \frac{u}{\sqrt{gh}} \quad \therefore F^2 = u^2/gh = \frac{\frac{1}{2}\rho u^2}{\frac{1}{2}\rho g h} = \frac{\text{kinetic energy}}{\text{potential energy}}$$

(u → flow speed; $\sqrt{gh}$ → wave speed.)

More KE and PE to get over barrier gives up some KE to get PE.



If $K = f(h_1) - f(h_m)$; there are ④ possibilities:-



Causality shows that smooth transitions are always from subcritical to supercritical.

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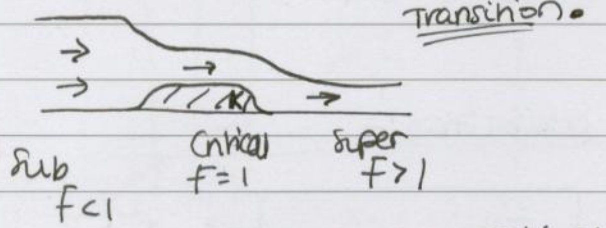
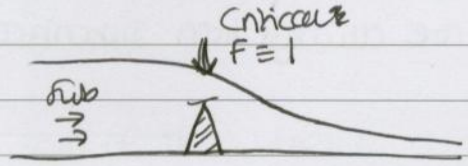
$K > f(h_1) - f(h_m)$



• If the obstacle height K is increased further so $K > f(h_1) - f(h_m)$, then the upstream flow banks up, deepens, flux decreases & makes the minimum adjustment to allow water to pass over obstacle; i.e: flow at the top of bump is critical, i.e: $F=1$ when K is ~~critical~~ a maximum.

e.g: a weir.

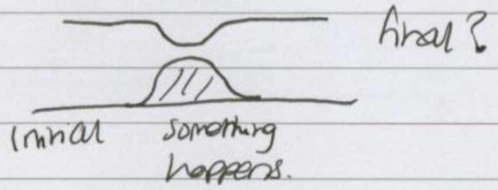


Notice, if you know the depth ~~of~~ at at weir you know the flux without having to measure speed.

$$\frac{Q^2}{g} = h_m^3$$

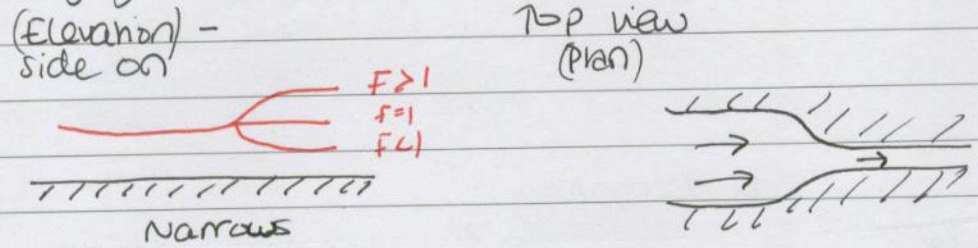
$$\therefore Q = (g h_m^3)^{1/2}, \quad u_m = \sqrt{g h_m} \quad \text{since } F = 1.$$

Remember, one solution may be $h_2 = h_1$.



Example 4: Converging channel

Consider flow through a flat-bottomed, horizontal channel of varying width, b.



① Conservation of mass:

$$\rho h b u = \rho Q, \quad h = \text{fluid depth.}$$

$\Rightarrow Q = hbu$ is the constant volume flux.

② provided the surface remains smooth, the surface is a streamline so we can apply Bernoulli there.

$\therefore p + \frac{1}{2}\rho u^2 + \rho g z = C$, z = height of surface

i.e: $p_a + \frac{1}{2}\rho u^2 + \rho g h = \text{const.}$: $p = p_a$; constant atmospheric pressure on surface.

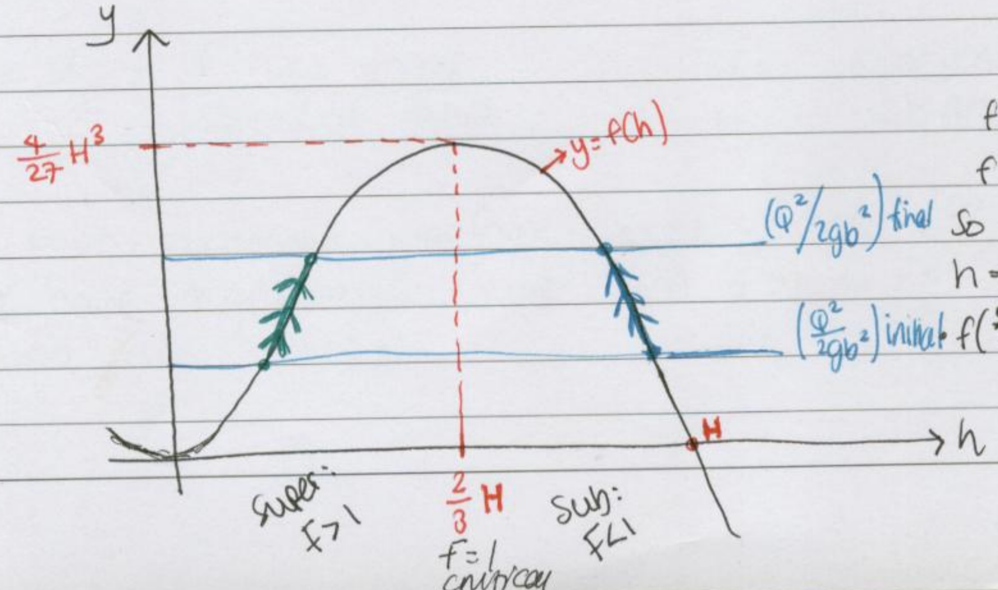
i.e: $\frac{u^2}{2g} + h = H$, constant.

Eliminating u:

$$\frac{Q^2}{2gh^2b^2} + h = H.$$

$$* \; (H-h)h^2 = \frac{Q^2}{2gb^2}$$

Write: $f(h) = (H-h)h^2$



$f(h) = h^2H - h^3$

$f'(h) = 2hH - 3h^2$

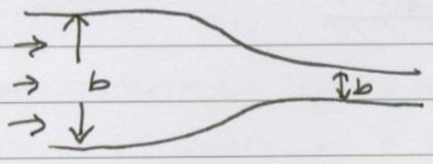
so $f'(h) = 0$ if $h = 0$, or $h = \frac{2}{3}H$

$f(\frac{2}{3}H) = \left(\frac{4}{9} - \frac{8}{27}\right)H^3$

At $h = \frac{2}{3}H$; $\frac{u^2}{2g} = H - h = \frac{1}{3}H$

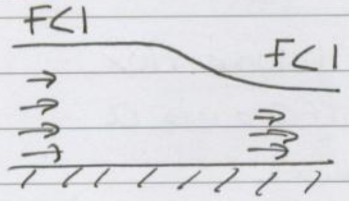
So $\frac{u^2}{gh} = \frac{u^2}{g(\frac{2}{3}H)} = \frac{\frac{2}{3}H}{\frac{2}{3}H} = 1$ i.e. $F = 1$ when $h = \frac{2}{3}H$.

*Plan view:



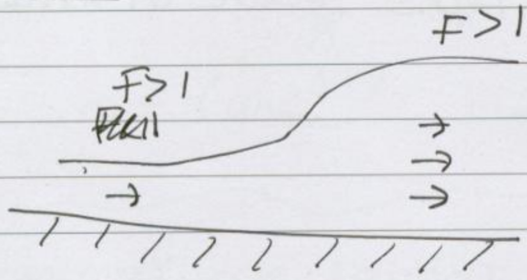
b decreasing.

$\frac{Q^2}{2gb^2}$ Increasing.



Flows ~~more~~ more towards critical at a constriction.

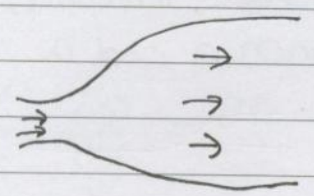
sub $F<1$, $h > \frac{2}{3}H$.



Super $F>1$, $h < \frac{2}{3}H$

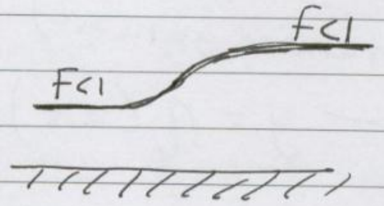
Example 5: Expanding channel.

① SUB: $F<1$, $h>\frac{2}{3}H$



Plan.

Elevation:

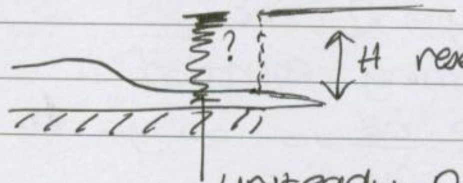


· River flowing into a reservoir.
Then h increases to H as $b\to\infty$.

i.e: $\frac{Q^2}{gb^2}=0$, $u=0$. (stagnant)

River smoothly enters stagnant reservoir.

② super $F>1$ $h<\frac{2}{3}H$; fast, shallow.



unsteady, non-smooth jump.

Here, $h\downarrow 0$ as $b\to\infty$.
River cannot smoothly join reservoir.

$F=\frac{u}{\sqrt{gh}}=\frac{\text{flow speed}}{\text{wave speed}}$. $F>1$ supercritical.

$M=\frac{u}{a}=\frac{\text{Flow speed}}{\text{speed of sound}}$. $M>1$ supersonic, $M<1$ subsonic

Sonic boom: spontaneous jump from $M>1$ to $M<1$. (as this gives out energy, - sound).
Only have hydraulic jump from $F>1$ to $F<1$ as this gives out energy.

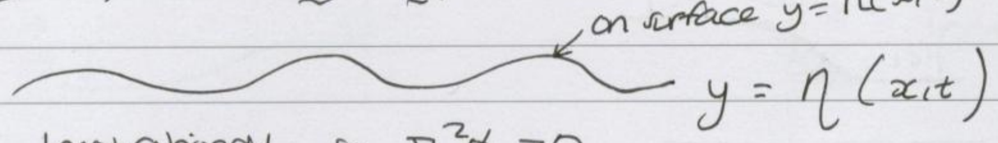
chapter . . . ?

Water waves:

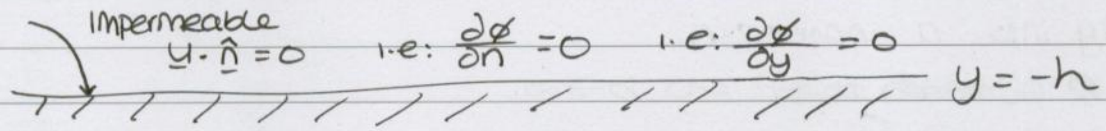
↳ free surface gravity waves.

- we will take the flow to be 2D, irrotational, inviscid, incompressible. Thus we have a streamfunction and a velocity potential and a complex potential.

i.e: $\exists \phi$ s.t $\underset{\sim}{u} = \underset{\sim}{\nabla} \phi$



Irrotational, so $\nabla^2 \phi = 0$.



(1) $P = P_e$, constant.
(2) ?

1) we have $\exists \phi$ s.t $\underline{u} = \nabla \phi$.
Let the unknown free-surface be $y = \eta(x,t)$.
Then, in the fluid, $-h < y < \eta$, governing equation is $\nabla^2 \phi = 0$. on lower boundary $v = 0$ so $\frac{\partial \phi}{\partial y} = 0$ on $y = -h$.

we need two bc's on the surface (because η is unknown).
The two bc's are the Kinematic & Dynamic conditions.
- Dynamic (forces): $p = p_a$ on $y = \eta$.
- Kinematic : particle on the surface remains on the surface.

Kinematic: i.e: on the surface, $y = \eta(x,t)$ $\forall x,t$. ($y = \eta$)

i.e: $y - \eta(x,t) = 0$ $\forall x,t$.

Following a particle on surface,

$$\frac{D}{Dt}\left(y - \eta(x,t)\right) = 0 \quad \left(\text{on } y = \eta \ \forall x,t\right)$$

i.e: $v - \frac{D\eta}{Dt} = 0$ on $y = \eta$

i.e: $v = \frac{\partial \eta}{\partial t} + u \frac{\partial \eta}{\partial x}$ on $y = \eta(x,t)$

i.e: $\frac{\partial \phi}{\partial y} = \frac{\partial \eta}{\partial t} + \frac{\partial \phi}{\partial x}\frac{\partial \eta}{\partial x}$ on $y = \eta(x,t)$

↳ Kinematic B.C!

To deal with the dynamic condition on pressure we would like to use Bernoulli. But the flow must be STEADY!
i.e $\frac{\partial}{\partial t} \equiv 0$ for the form of Bernoulli up to now. We need a new Bernoulli for unsteady flow.

Remember, $\frac{D\underline{u}}{Dt} = -\frac{1}{\rho}\underline{\nabla} p + \underline{f}$

i.e: $\frac{\partial \underline{u}}{\partial t} + (\underline{u}\cdot\underline{\nabla})\underline{u} = -\frac{1}{\rho}\underline{\nabla} p - \underline{\nabla} V_e$

$\underline{f} = -\underline{\nabla} V_e$ – a conservative force.

$$\frac{\partial \underline{u}}{\partial t} + \underline{\nabla}\left(\tfrac{1}{2}\underline{u}^2\right) + \underline{\omega}\wedge\underline{u} = -\frac{1}{\rho}\nabla p - \underline{\nabla} V_e$$

[Last time: steady, dot with $\underline{u}$ to get rid of $\underline{\omega}\wedge\underline{u}$.]

(This time: use fact that irrotational, $\underline{u} = \underline{\nabla}\phi$ and $\underline{\omega} = 0$)

Thus $\frac{\partial}{\partial t}\underline{\nabla}\phi + \underline{\nabla}\left(\tfrac{1}{2}\underline{u}^2\right) = -\frac{1}{\rho}\underline{\nabla} p - \underline{\nabla} V_e$

i.e:

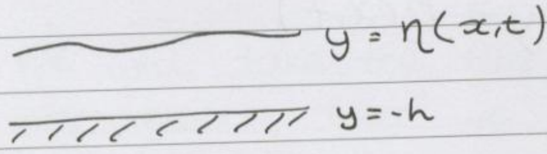
$$\underline{\nabla}\left[\frac{\partial \phi}{\partial t} + \frac{1}{2}|\underline{\nabla}\phi|^2 + \frac{1}{\rho}p + V_e\right] = 0$$

i.e: $\rho\left(\frac{\partial \phi}{\partial t} + \frac{1}{2}|\underline{\nabla}\phi|^2\right) + p + \rho V_e = G(t)$

↳ new Bernoulli!

08/12/11

Water waves:



- Equation: $\nabla^2 \phi = 0$ $\quad [\underline{u} = \nabla\phi, \ \underline{\nabla} \cdot \underline{u} = 0]$
- lower bc: $\frac{\partial \phi}{\partial y} = 0, \ y = -h$
- upper bc's: Kinematic – $v = \frac{D\eta}{Dt}$ on $y = \eta$

i.e: $\frac{\partial \phi}{\partial y} = \frac{\partial \eta}{\partial t} + \frac{\partial \phi}{\partial x}\frac{\partial \eta}{\partial x}$ on $y = \eta$

Dynamic – $p = p_a$ on $y = \eta$.

Bernoulli: (time-dep, irrotational)

$$\rho \frac{\partial \phi}{\partial t} + \frac{1}{2}\rho |\underline{\nabla}\phi|^2 + p + \rho V_e = F(t)$$

The restoring force is gravity: so $V_e = gy$

Thus $\rho \frac{\partial \phi}{\partial t} + \frac{1}{2}\rho |\underline{\nabla}\phi|^2 + p + \rho g y = F(t)$

- $F(t)$ can be absorbed into ϕ.

Redefine:

$$\hat{\phi} = \phi - \frac{1}{\rho}\int^t F(\tau)\, d\tau$$

then $\underline{\nabla}\hat{\phi} = \nabla\phi = \underline{u}$ and $\rho\frac{\partial \hat{\phi}}{\partial t} = \rho\frac{\partial \phi}{\partial t} - F(t)$.

Thus, W.L.O.G, we can take $F \equiv 0$, (since if $F \neq 0$, we can redefine ϕ as above.]

Hence everywhere in the flow, $\rho\frac{\partial \phi}{\partial t} \ \frac{1}{2}\rho |\nabla\phi|^2 + \rho g y + p = 0$

↑ UNSTEADY BERNOULLI!

---

on surface, $y = \eta$ and $p = p_a$, (constant)

Thus, $\rho \frac{\partial \phi}{\partial t} + \frac{1}{2}\rho |\underline{\nabla}\phi|^2 + \rho g \eta = -p_a$ (constant).

By above argument, can absorb p_a (constant) into ϕ so we have the Dynamic condition

$$\frac{\partial \phi}{\partial t} + \frac{1}{2}|\underline{\nabla}\phi|^2 + g\eta = 0 \text{ on } y = \eta \text{ ; Dynamic b.C.}$$

• Equation: Laplace: $\nabla^2 \phi = 0$

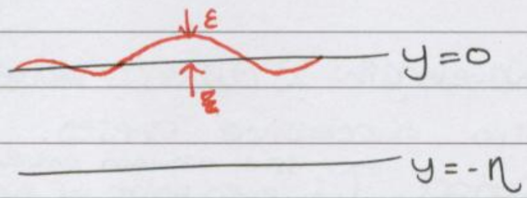
Lower b.c: $\frac{\partial \phi}{\partial y} = 0$ on $y = -h$

Kinematic: $\frac{\partial \phi}{\partial y} = \frac{\partial \eta}{\partial t} + \frac{\partial \phi}{\partial x}\frac{\partial \eta}{\partial x}$ on $y = \eta$.

Dynamic: $\frac{\partial \phi}{\partial t} + \frac{1}{2}|\underline{\nabla}\phi|^2 + g\eta = 0$ on $y = \eta$.

$\eta(x,t)$ unknown; Full, non-linear, surface water problem.

To make progress, we 'linearise', i.e: we consider waves of infinitesimal amplitude, $0 < \varepsilon \ll 1$,
i.e: We take $\eta(x,t)$ to be of order ε.



we expect velocities and so ϕ to be order ε also.

• Kinematic bc: $\frac{\partial \phi}{\partial y} = \frac{\partial \eta}{\partial t} + \frac{\partial \phi}{\partial x}\frac{\partial \eta}{\partial x}$

$\varepsilon \quad \varepsilon \quad \varepsilon \cdot \varepsilon$

i.e; $\varepsilon : \varepsilon : \varepsilon^2$

or ; $1 : 1 : \varepsilon$

Thus, in limit $\varepsilon \to 0$, the final term dissappears.
we have $\frac{\partial \phi}{\partial y} = \frac{\partial \eta}{\partial t}$ on $y = \eta$ (Linear)

(with error of order ε)

• Notice, for any function $f(y)$,

$$f(\varepsilon) = f(0) + \varepsilon f'(0) + \tfrac{1}{2}\varepsilon^2 f''(0) + \dots$$
$$= f(0) \text{ (with error of order } \varepsilon.)$$

Thus, can move bc from $y = \eta$ (of order ε) to $y = 0$ with error of order ε. Thus we have

$$\frac{\partial \phi}{\partial y} = \frac{\partial \eta}{\partial t}, \text{ on } y = \eta$$ (Now linear, on known surface)

Dynamic bc: $\frac{\partial \phi}{\partial t} + \frac{1}{2}|\underline{\nabla}\phi|^2 + g\eta = 0$ on $y=\eta$ $\downarrow y=0$

$\quad\varepsilon \qquad\quad \varepsilon^2 \qquad\quad \varepsilon$

linearised bc: $\frac{\partial \phi}{\partial t} + g\eta = 0$ on $y=0$

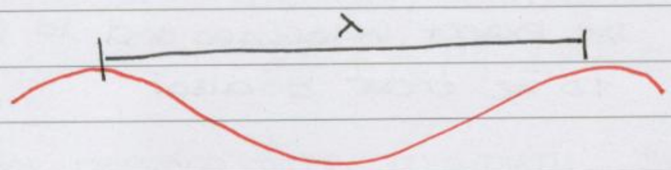
Linear on known surface.

Summary: Linear water waves!

Equation: $\nabla^2\phi = 0$ (already linear)

lower bc: $\frac{\partial \phi}{\partial y} = 0$ on $y = -h$ (already linear, already on known surface).

upper bc's:
$$\left.\begin{aligned} \frac{\partial \phi}{\partial y} &= \frac{\partial \eta}{\partial t} \\ \frac{\partial \phi}{\partial t} + g\eta &= 0 \end{aligned}\right\} \text{ on } y=0.$$



- wavelength λ, distance between two successive crests.
- period τ; time between arrival at a given point of successive crests.
- speed c, at which crests advance

$[c = \lambda/\tau]$

Any periodic function can be expressed as a sum (within reason) of sines & cosines. Thus it is sufficient ~~enough~~ to consider

$$\eta = A\sin\left[\frac{2\pi}{\lambda}(x - ct)\right]$$

– wave with amplitude A, wavelength λ, speed c to the right, also $\tau = \lambda/c$.

fluids lectures

Water waves

1. Irrotational, inviscid, incompressible => New Bernoulli => Fully non-linear eqn set.
2. Infinitesimal waves, linearised => linear wave equations.

Gov. Eqn: $\nabla^2\phi = 0$

lower b.c: $\frac{\partial\phi}{\partial y} = 0$ on $y=-h$

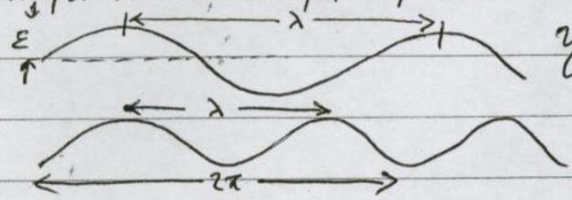
upper b.c: kinematic: $\frac{\partial\phi}{\partial y} = \frac{\partial\eta}{\partial t}$ on $y=0$

dynamic: $\frac{\partial\phi}{\partial t} + g\eta = 0$ on $y=0$

linear
fixed domain
$0 \geq y \geq -h$

linear constant coefficients: all solutions are sums or integrals of sinusoids.

look for solutions of the form



$\eta(x,t) = \varepsilon \sin\left[\frac{2\pi}{\lambda}(x-ct)\right]$

$= \varepsilon \sin[k(x-ct)]$

$= \varepsilon \sin[kx-\omega t]$

ε amplitude
λ wavelength
c speed (phase speed)

Period $\tau = \lambda/c$

wave number: $k = \frac{2\pi}{\lambda}$

no of waves in distance 2π

frequency: $\omega = kc = \frac{2\pi}{\tau}$

$c = \frac{\omega}{k}$

We wish to find ϕ

On the surface $\frac{\partial\phi}{\partial t} = -g\eta$

$= -\varepsilon g \sin[kx-\omega t]$

ω - omega

Thus ϕ behaves like $-\frac{\varepsilon g}{\omega}\cos[kx-\omega t]$ on $y=0$

$\nabla^2\phi = 0$

$\frac{\partial\phi}{\partial y} = 0$ $y=-h$

$\frac{\partial\phi}{\partial y} = \frac{\partial\eta}{\partial t}$ on $y=0$

$\frac{\partial\phi}{\partial t} + g\eta = 0$ on $y=0$

Equivalently $\frac{\partial\phi}{\partial y} = \frac{\partial\eta}{\partial t}$

$= -\varepsilon\omega\cos[kx-\omega t]$ on $y=0$

Both of these say the (x,t) behaviour of $\phi(x,y,t)$ is like $\cos(kx-\omega t)$.

Look for a solution

$\phi(x,y,t) = Y(y)\cos(kx-\omega t)$

stick $-\varepsilon\omega$ in front since it's const

$\phi(x,y,t) = -\varepsilon\omega\, Y(y)\cos(kx-\omega t)$

Then $\frac{\partial\phi}{\partial y} = -\varepsilon\omega Y'(y)\cos(kx-\omega t)$

But on $y=0$ $\frac{\partial\phi}{\partial y} = \frac{\partial\eta}{\partial t} = -\varepsilon\omega\cos(kx-\omega t)$

Thus we require $Y'(0)=1$

Similarly for $\frac{\partial \phi}{\partial y}=0$ on $y=-h$ for all x,t satisfies $\frac{\partial \phi}{\partial y}=0$ on $y=-h$

$$Y'(-h)=0$$

The governing equation is

$$\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2}=0$$

$$\phi(x,y,t)=-\varepsilon\omega Y(y)\cos(kx-\omega t)$$

$$\frac{\partial^2\phi}{\partial x^2}=+\varepsilon\omega k^2 Y(y)\cos(kx-\omega t)$$

$$\frac{\partial^2\phi}{\partial y^2}=-\varepsilon\omega Y''(y)\cos(kx-\omega t)$$

adding

$$0=Y''-k^2Y$$

Thus we have

$$Y''-k^2Y=0, \quad -h\le y\le 0$$ (notice no x's, t's, ω's, $\cos$'s, $\sin$'s)

$$Y'(0)=1, \quad Y'(-h)=0$$ (justifies form of ϕ assumed)

one form of complementary function is

$$Y(y)=Ce^{ky}+De^{-ky}$$

or $Y(y)=E\cosh ky+F\sinh ky$

But best is

$$Y(y)=A\cosh[k(y+h)]+B\sinh[k(y+h)]$$

This gives

$$Y'(y)=Ak\sinh[k(y+h)]+Bk\cosh[k(y+h)]$$

we require

$$Y'(-h)=0 \text{ so } B=0$$

It remains to require

$$Y'(0)=1 \qquad Ak\sinh kh=1$$

Thus $Y(y)=\dfrac{\cosh[k(y+h)]}{k\sinh kh}$

This gives $\phi(x,y,t)=-\varepsilon\omega\dfrac{\cosh[k(y+h)]}{k\sinh kh}\cos[kx-\omega t]$

$$=-\varepsilon c\frac{\cosh[k(y+h)]}{\sinh kh}\cos[k(x-ct)]$$

We still have a condition to satisfy.

$$\frac{\partial\phi}{\partial t}+g\eta=0 \text{ on } y=0$$

here

$$-\varepsilon c\coth kh.(+\omega\sin(kx-\omega t))+\varepsilon g\sin(kx-\omega t)=0 \ \forall x,t$$

(2)

since true $\forall x, t$, divide by $\sin(kx-\omega t)$.

$$-\omega c \coth kh + g = 0$$

ie $-\omega^2 \coth kh + gk = 0 \quad c = \omega/k$

ie $\omega^2 = gk \tanh kh$

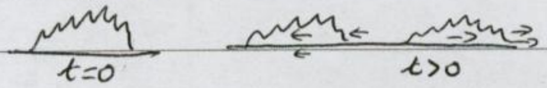
ie k and ω aren't independent

Thus $c^2 = \frac{g}{k}\tanh kh$

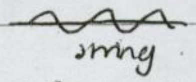
$$= \frac{g\lambda}{2\pi}\tanh\left(\frac{2\pi h}{\lambda}\right)$$

So waves of different wavelengths travels at different speeds.

i.e waves disperse



"the speed" of the waves, NOT DISPERSIVE



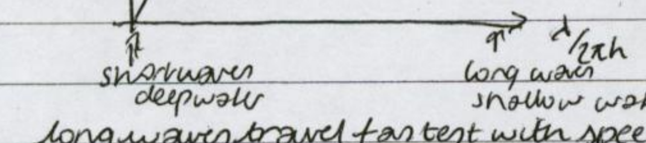
$u_{tt} = c^2 u_{xx}$

nothing to do with λ $\quad c^2 = \frac{T}{\rho}$

soundwaves all travel at 'speed of sound'
NON DISPERSIVE

$$\frac{c^2}{gh} = \frac{\tanh kh}{kh} = \frac{\tanh\left(\frac{2\pi h}{\lambda}\right)}{\left(\frac{2\pi h}{\lambda}\right)}$$

electromagnetic radiation
- speed 'c' - unique (for given radius)
NON DISPERSIVE



(eqn plus tanh)

$\theta = \lambda/2\pi h$

$$\frac{c^2}{gh} = \theta \tanh\left(\frac{1}{\theta}\right)$$

$\theta >> 1 \quad \frac{c^2}{gh} \sim \theta \frac{1}{\theta} = 1$

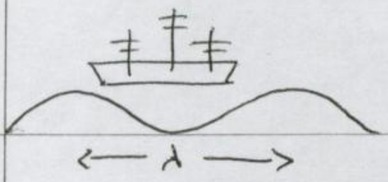
$\theta << 1 \quad \frac{c^2}{gh} \sim \theta$

long waves travel fastest with speed $c = \sqrt{gh}$

long waves are non-dispersive: all have speed $\sqrt{gh}$.
on shallow water

Short waves are dispersive on deep water $\quad \frac{\lambda}{2\pi h} << 1 \quad \frac{c^2}{gh} \to \frac{\lambda}{2\pi h}$ i.e $c \to \left(\frac{g\lambda}{2\pi}\right)^{1/2}$

$= (2\pi)^{-1/2}\sqrt{g\lambda}$

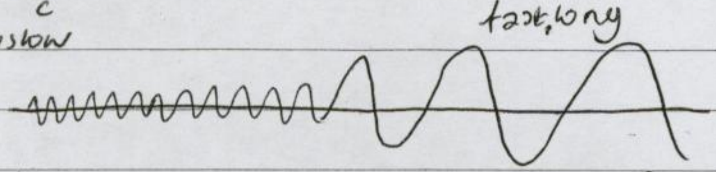


ship speed on deep water proportional to square root of length

Notice in a flow with $u > c$ i.e $u > \sqrt{gh}$ all waves swept downstream ie supercritical

$\left(F = \frac{u}{\sqrt{gh}}\right)$ compare Mach no. $M = \frac{u}{a}$, a speed sound)

$= \frac{u}{c}$



<u>Particle paths in a water wave</u>

$\frac{dx}{dt} = u(x,y,t)$

But the amplitude of the motion is of order ε. i.e particles only move an amount ε. So write

$\phi \ldots = \varepsilon$

$\eta = \varepsilon$

function + derivatives products → ε^2 justifying linearisation

$$x = x_0 + \varepsilon X$$

$$y = y_0 + \varepsilon Y$$

Then $\frac{dx}{dt} = \varepsilon \frac{dX}{dt}$

$= u(x,y,t)$

$= \frac{\partial \phi}{\partial x}(x_0 + \varepsilon X, y_0 + \varepsilon Y, t)$

$= \frac{\partial \phi}{\partial x}(x_0, y_0, t) + \varepsilon X \frac{\partial^2 \phi}{\partial x^2} + \varepsilon Y \frac{\partial^2 \phi}{\partial y^2} + \ldots$ cancel higher-order terms

$\varepsilon \qquad \varepsilon^2 \qquad \varepsilon^2$

Thus to order ε^2,

$$\varepsilon \frac{dX}{dt} = \frac{\partial \phi}{\partial x}(x_0, y_0, t) \qquad \varepsilon \frac{dY}{dt} = \frac{\partial \phi}{\partial y}(x_0, y_0, t)$$

$$\phi = -\varepsilon c \frac{\cosh[k(y+h)]}{\sinh kh} \cos(kx - \omega t)$$

$$\left.\frac{\partial \phi}{\partial x}\right|_{x_0, y_0} = +\varepsilon k c \frac{\cosh[k(y_0+h)]}{\sinh kh} \sin(kx_0 - \omega t)$$

change to x_0, y_0 from x, y

$$\left.\frac{\partial \phi}{\partial y}\right|_{x_0, y_0} = -\varepsilon k c \frac{\sinh[k(y_0+h)]}{\sinh kh} \cos(kx_0 - \omega t)$$

↑ indep of x, y, i.e X, Y

(F3)

Thus $\frac{dX}{dt} = \frac{\omega \cosh[k(y_0+h)]}{\sinh kh} \sin(kx_0 - \omega t)$

so $X = \frac{\cosh[k(y_0+h)]}{\sinh kh} \cos(kx_0 - \omega t)$ (absorb constant into x_0)

and $Y = \frac{\sinh[k(y_0+h)]}{\sinh kh} \sin(kx_0 - \omega t)$

$\alpha = \frac{\cosh[k(y_0+h)]}{\sinh kh}$ $\beta = \frac{\sinh[k(y_0+h)]}{\sinh kh}$

$\frac{X}{\alpha} = \cos(kx_0 - \omega t)$

$\frac{Y}{\beta} = \sin(kx_0 - \omega t)$

so $\left(\frac{X}{\alpha}\right)^2 + \left(\frac{Y}{\beta}\right)^2 = 1$

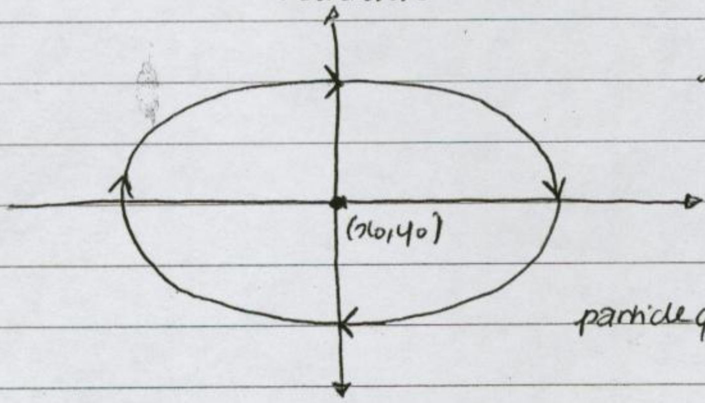
Ellipses with vertical semi-axis

$\beta = \frac{\sinh[k(y_0+h)]}{\sinh kh}$

horizontal semi-axis

$\alpha = \frac{\cosh[k(y_0+h)]}{\sinh kh}$

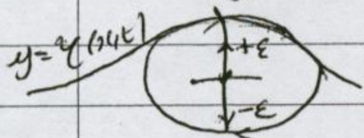
$\alpha > \beta$ so major axis is horizontal



when Y is most +ve, (largest)
$\frac{\partial \phi}{\partial x}$ is the largest
i.e u is largest

particle goes around clockwise

At top, $y_0 = 0$, $\alpha = \coth kh$, $\beta = 1$, so $-1 \leq Y \leq 1$



$y = y_0 + \varepsilon Y$

- exactly the amplitude of wave $\eta = Y$ here

- particles move forward at the crest + backwards at the trough

At bottom $y_0 = -h$, $\alpha = \operatorname{cosech} kh$, $\beta = 0$, $Y = 0$

$2\varepsilon \operatorname{cosech} kh$

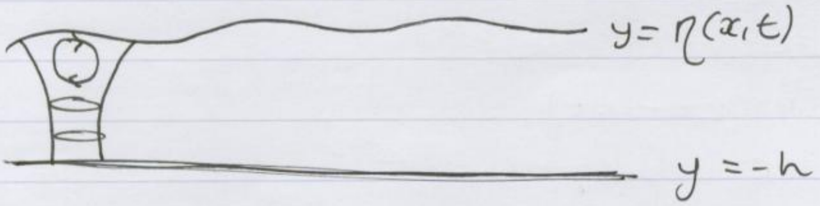
~~...~~ $y = h$

$h \to \infty$, use $\phi = Ae^{ky}$

much easier

$\frac{d}{2\pi h}$

Fluids: 15/12/11



$c=\frac{1}{\sqrt{2\pi}}\sqrt{g\lambda}$ short waves, deep water, $\frac{\lambda}{2\pi h} << 1$

$c=1\cdot\sqrt{gh}$ long waves, shallow water. $\frac{\lambda}{2\pi h} >> 1$.

Infinitely deep water; $h\to\infty \Rightarrow$ all waves short.

Take $\eta = \varepsilon\sin(kx-\omega t)$

$\phi = -\varepsilon\omega\cos(kx-\omega t)\,Y(y)$

Kinematic: $Y'(0)=1$ as before.

lower b.c; $Y\to 0$ as $y\to -\infty$

Equation; $Y''-k^2Y=0$

C.F: $Y = Ce^{kY} + De^{-kY}$

Bounded as $y\to-\infty \Rightarrow D=0$.

But $Y'(0)=1$ so $C=1/k$

and $\phi = -\frac{\varepsilon\omega}{k}\cos(kx-\omega t)e^{ky} = \underbrace{-\varepsilon c\cos(kx-\omega t)e^{ky}}_{\text{infinite depth velocity potential}}$ $(h\to\infty)$

Substitute in dynamic condition and find $c^2=g/k$, $c=g\lambda/2\pi$,

$c=\frac{1}{\sqrt{2\pi}}\sqrt{g\lambda}$ as expected. $\to \frac{\lambda}{2\pi h}\to 0$

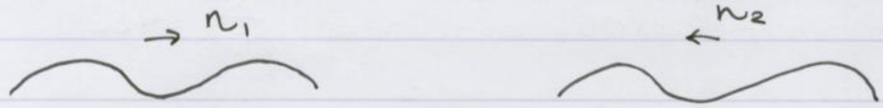
Kinematic: $\frac{\partial\phi}{\partial y}=\frac{\partial\eta}{\partial t}$
Dynamic: $\frac{\partial\phi}{\partial t}+g\eta=0$

* Particle paths are circles!

Radius; e^{ky_0};

Reflected waves:

$$\eta_1(x,t) = \varepsilon \sin(kx-\omega t)$$
$$\phi_1(x,y,t) = -\varepsilon c\cos(kx-\omega t)e^{ky}$$
$$\eta_2(x,t) = \varepsilon \sin(kx+\omega t)$$
$$\phi_2(x,t) = +\varepsilon c\cos(kx+\omega t)e^{ky}$$



same k, same ω, opposite direction.

Add them:

$$\eta = \eta_1 + \eta_2 = \varepsilon[\sin(kx-\omega t) + \sin(\omega t + kx)]$$

and

$$\phi = \phi_1 + \phi_2 = \varepsilon c e^{ky}[\cos(\omega t + kx) - \cos(kx - \omega t)]$$

$$= -2\varepsilon c e^{ky}\sin(kx)\sin(\omega t)$$

• Particle paths:

$$\frac{dx}{dt} = u = \frac{\partial\phi}{\partial x} = -2\varepsilon c \cdot k\cos(kx) \cdot e^{ky}\sin(\omega t)$$

$$\cdot \frac{dy}{dt} = v = \frac{\partial\phi}{\partial y} = -2\varepsilon c\, k e^{ky}\sin kx \cdot \sin\omega t$$

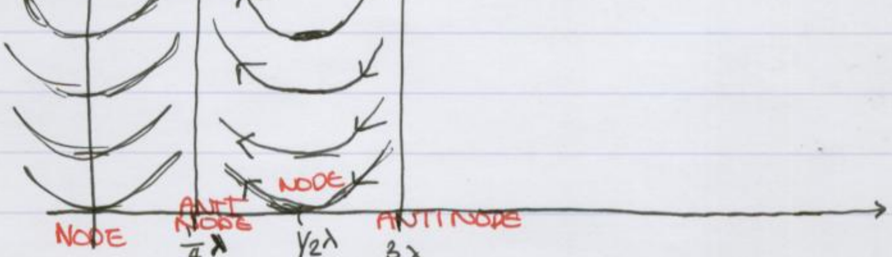
Now $\frac{\partial y}{\partial x} = \frac{dy/dt}{dx/dt} = \frac{\sin kx}{\cos kx} = \underline{\underline{\tan(kx)}}$

$$\left(\text{so } y = \frac{1}{k}\ln(\sec(kx))\right)$$

To lowest order;

$$\frac{dy}{dx} = \tan(kx_0)$$

lines of slope $\tan(kx_0)$



$kx_0 = \pi/2$
$x = \pi/2k$
$2\pi/\lambda\, x_0 = \pi/2 \Rightarrow x_0 = \frac{1}{4}\lambda$