2301 Fluid Mechanics Notes

Based on the 2011 autumn lectures by Prof E R Johnson

Fluid Mechanics 2301 03/10/11

Office hour; Thursday, 1pm, 805

- Magma dynamics: - dynamics of interior of the earth
  - dynamo theory
- Plasma: - sun, stars, fusion
- Blood: - biofluid dynamics
- Atmospheres & Oceans: - Meteorology, Climate Geophysical fluid dynamics (GFD)
- Air resistance: - flow past bodies - cars, planes.

How does a plane fly?

Geometry of wing cross-section $\mathcal{K}$-circulation $\mathcal{K}\rho U$
Speed - directly proportional U
Density - directly proportional ρ

How fast does a surface water wave travel?

Depth: h $[h]=L$ $[c]=LT^{-1}$
Gravity: g $[\lambda]=L$ $[g]=LT^{-2}$
Wavelength: λ

Short wave, deep water
$\lambda/h \ll 1$
$c \sim (g\lambda)^{1/2}$

max speed $c=\sqrt{g\lambda}$ different wavelen
different spe

Long waves, shallow water
$\lambda/h \gg 1$
$c \sim (gh)^{1/2}$ all waves travel at same s

Tsunami

Green's Law

speed x energy density
$cH^2 = \text{constant}$

$H^2 \propto c^{-1}$ $\quad H \propto c^{-1/2}$ $\quad H \propto h^{-1/4}$

energy a quadratic equation.

Chapter 1

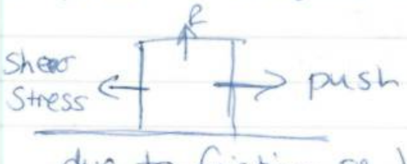
Specification and Kinematics

Continuum: a substance that we can take arbitrary small volumes of and whose properties remain the same as we do so (if $\lim_{\delta v \to 0}$ exists)

Take volume V, measure its mass m, and define mean density $\bar{\rho} = m/_V$
could take $V \supset V_1 \supset V_2 \dots$ and define the density at some point common to this sequence, $\rho = \lim_{V \to 0} m/_V$

This is a good approximation to reality provided we are interested in motions at scales large compared to the mean free path.

We will restrict attention to invicid fluids (fluids that are not viscous)
A fluid is invicid if it cannot support a shear stress.



eg. honey supports a shear stress

due to friction on bottom, opposing.

Summary

1. CONTINUUM: we can discuss infinitesimal volumes of fluid
2. INVICID: the fluid cannot support a shear stress
3. INCOMPRESSIBLE: the volume of the fluid element remains the same throughout the motion.

An element composed of the same fluid has the same mass by conservation of mass. Hence density is constant.
a) this does not mean that the density is the same everywhere

b) this is a good approximation provided speeds are small compared with the speed of sound (400 mph)
ie the mach number of the flow

$\frac{\text{typical speed}}{\text{sound speed}} = M$ is small ($1 << 1$) $M<1$ subsonic, $M>1$ supersonic

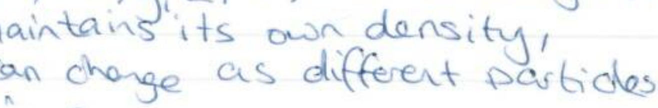
To describe the flow we have two choices
a) Lagrangian labelling - label all particles & follow their motion, ie follow particle path
strengths - conservation laws easy
drawback - simple motions can have complicated particle paths.

b) Eulerian description - set up fixed axes.


we define a vector field.
$\underline{u}(x,y,z,t)$ by defining the velocity $\underline{u}$ at time t to be the velocity of the fluid element (or fluid particle) that is at [illegible] at time t.
strengths - velocity is a vector field; we can use vector calcul[us]
drawback - conservation laws become a little more complicated.

We do the same thing for density: $\rho(x,y,z,t)$. ie. although in incompressible flow, each particle maintains its own density, the Eulerian density (at a point) can change as different particles occupy that point at different times.

Of course, in a homogeneous fluid, ρ = constant.
There are three ways of visualising or describing a motion:-

1) PARTICLE PATH: the path traced out by the fluid ~~element~~ element during a given time interval.


2) STREAKLINE/: the locus of all particles that have passe[d]
FILAMENT LINE through a given point in a given time interv[al]

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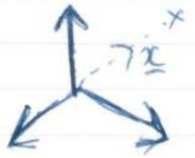
A transmits a shear stress to B

(force / unit area)

A does not transmit a shear (tangential) stress

Eulerian: $\underline{u}(\underline{x}, t)$ = velocity of particle that happens to be at $\underline{x}$ at the time t.

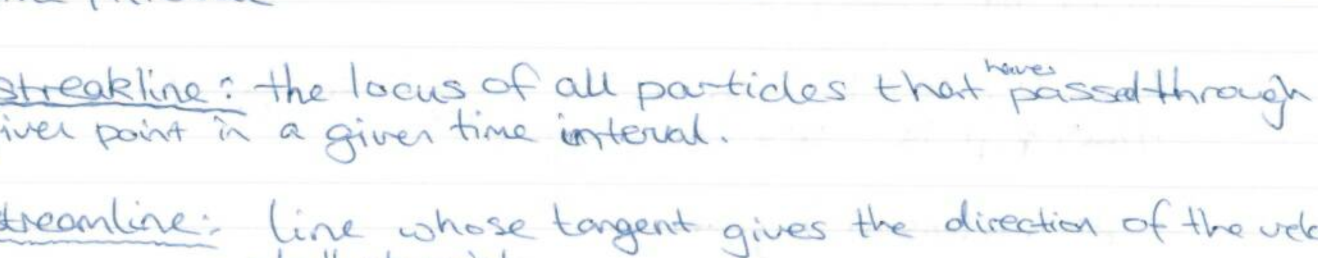


Visualise:

① particle path: path traced out by a fluid element in a gi[ven] time interval

② streakline: the locus of all particles that have passed through [a] given point in a given time interval.

③ streamline: line whose tangent gives the direction of the velo[city] at that point.



suppose we are given a velocity field $\underline{u}(\underline{r}, t)$
particle paths satisfy $\frac{d\underline{r}}{dt} = \underline{u}(\underline{r}, t)$ with $\underline{r} = r_0$ at $t = 0$

Ex: Consider the two-dimensional velocity field $\underline{u}(\underline{r}, t) = \hat{\underline{i}} - 2t\hat{e}$

2D flow field: field independent of the third direction.
i.e the same in each x-y plane.
We shall also take the velocity component in the normal directio[n] to be 0.

In Cartesians it is conventional to write $\underline{u}(x, y, z, t) = u(x, y, z, t)$
$+ v(x, y, z, t)$
$+ w(x, y, z, t)$

i.e $\underline{u} = (u, v, w)$

2D flow: $w=0$ $u=u(x,y,t)$ $v=v(x,y,t)$
flow same at each z

$\underline{u}(\underline{r},t)=\hat{\underline{i}}-2te^{-t^2}\hat{\underline{j}}$

so $\frac{d\underline{r}}{dt}=\underline{u}$

here $u=1$
$v=-2te^{-t^2}$

i.e. $\frac{dx}{dt}=u$, $\frac{dy}{dt}=v$

$\frac{dx}{dt}=1$, $\frac{dy}{dt}=-2te^{-t^2}$

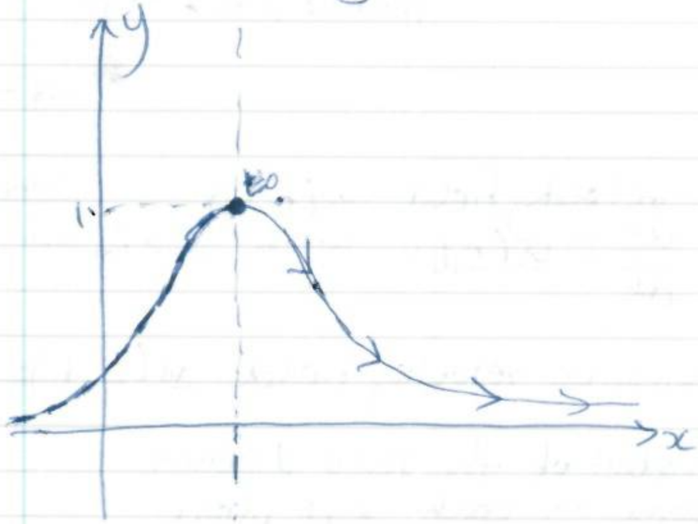
$x=t+x_0$, $y=e^{-t^2}+y_0$

What is the path traced out by the particle released from $(1,1)$ at $t=0$?

At $t=0$, $x=1$ so $x_0=1$, $y=1$ so $y_0=0$
thus p.p is $x=1+t$, $y=e^{-t^2}$

- parameterised by time t

$t=x-1$ so $y=e^{-(x-1)^2}$



What is the streakline traced out by particles released from $(1,1)$ at times $\tau<0$ when viewed at $t=0$?

particle paths: $x=t+x_0$
$y=e^{-t^2}+y_0$

At τ, particle in locus was at $(1,1)$ (when emitted)

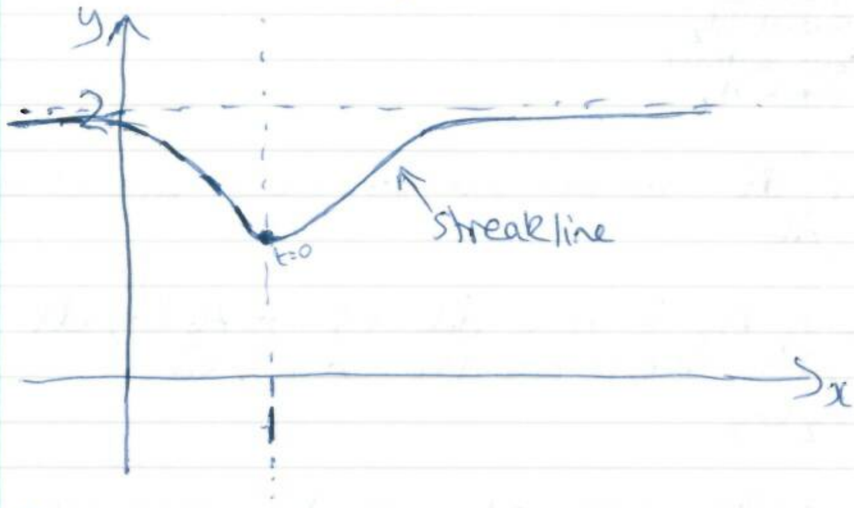
$$1 = \tau + x_0 \qquad 1 = e^{-\tau^2} + y_0$$

i.e. $x_0 = 1-\tau$, $y_0 = 1 - e^{-\tau^2}$
this particle is at (x,y) at time t where

$$x = t + 1 - \tau, \quad y = e^{-t^2} + 1 - e^{-\tau^2}$$

so at $t=0$, it is at $x = 1-\tau$, $y = 2 - e^{-\tau^2}$
- parameterised by the time of emission
sufficiently simple to eliminate τ:

$$\tau = 1 - x \quad \text{so} \quad y = 2 - e^{-(x-1)^2}$$



Streamline

parameterise $\underline{r}$ on S,
$$\frac{d\underline{r}}{ds} = \underline{u}(\underline{r}, t)$$

thus the streamlines at $t = t_0$ are given by solving $\dfrac{d\underline{r}}{ds} = \underline{u}(\underline{r}, t_0)$

Ex: for the previous velocity field, what are the streamlines at $t=0$?

$$\frac{dx}{ds} = u(x,y,0) = 1 \qquad \frac{dy}{ds} = v(x,y,0) = -2te^{-t^2} = 0$$

$y = \text{const.}$ $x = s + x_0$

At $t=0$, $\underline{u} = \hat{\underline{i}}$, i.e. tangent to ~~s~~ s'lines does give vel. field.

In a steady flow, all these are the same.

Steady: $\frac{\partial \underline{u}}{\partial t} = 0$ (does not say $\underline{u} = 0$)

2. Conservation of Mass

Suppose a incompressible fluid of constant density ρ flows through a tube of cross-sectional area A. Suppose the fluid velocity is uniform and unidirectional of size U at each cross-section.

The amount of mass between the two stations is fixed.

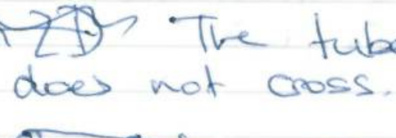
In a time interval δt, an amount of mass crosses station 1 this is $\rho A_1 U_1 \delta t$

The amount crossing A_2 in time δt is $\rho A_2 U_2 \delta t$
By conservation of mass, these are the same, so
$$A_1 U_1 = A_2 U_2$$

In terms of flux, the rate at which mass crosses A_1 is $\frac{\rho A_1 U_1 \delta t}{\delta t} = \rho A_1 U_1$

This must equal the flux across A_2, i.e. $\rho A_2 U_2$

~~Ex. In 2D~~ The tube can be any surface that fluid does not cross.



streamtube

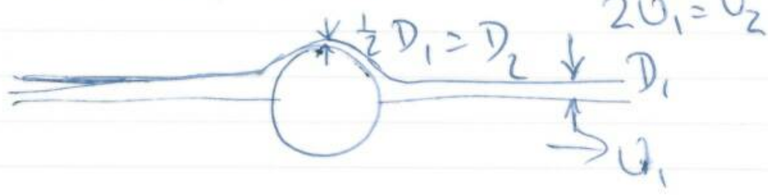
- formed by taking a closed loop of particles and drawing the streamlines emanating from them. Flow cannot cross this tube as $\underline{u}$ is tangential to the streamlines

If area halves, speed doubles.

In 2D: because the third velocity component $w = 0$, streamlines compress only in the x-y plane, so we have

$U_1 D_1 = U_2 D_2$ where D is the distance between streamlines.

i.e speed is inversely proportional to the separation of the streamlines.



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Theorem 1: If f is continuous on $[a,b]$ and $\int_c^d f = 0$ for each $(c,d) \subseteq [a,b]$.

Then $f \equiv 0$ on $[a,b]$.

Proof: suppose $\exists\ \alpha \in [a,b]$ s.t $f(\alpha) \neq 0$. WLoG w can take $f(\alpha) > 0$.
write $\delta = \frac{1}{2} f(\alpha) > 0$. Hence $\exists\ \varepsilon > 0$ s.t if $x \in (\alpha - \varepsilon, \alpha + \varepsilon)$ $|f(x) - f(\alpha)| < \delta = \frac{1}{2} f(\alpha)$

$$0 < \tfrac{1}{2} f(\alpha) < f(x) < \tfrac{3}{2} f(\alpha)$$

Thus $\int_{\alpha-\varepsilon}^{\alpha+\varepsilon} f(x)\,dx > \int_{\alpha-\varepsilon}^{\alpha+\varepsilon} \frac{1}{2} f(\alpha)\,dx = 2\varepsilon \frac{1}{2} f(\alpha)$ a
$= \varepsilon f(\alpha) > 0$

but $\int_c^d f = 0\ \forall (c,d) \subseteq [a,b]$ SO CONTRADICTION

This result extends immediately to n dimensions.

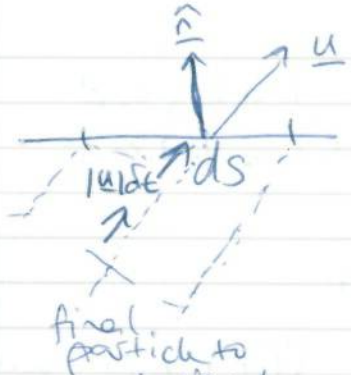
Ansatz: Suppose we wish to derive an equation $f=0$ for fluid in 3D. Let the fluid occupy a domai D in 3D. Take an <u>arbitrary</u> subdoma V of D.

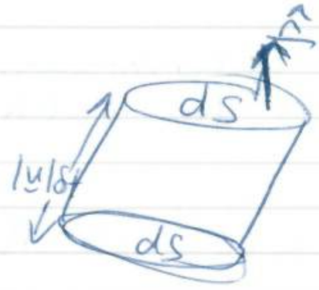
Show that $\int_V f$ vanishes. Then $f = 0$ in D
BECAUSE V IS ARBITRARY
i.e. $\int_V f = 0$ for every subdomain V of D

<u>Conservation of mass</u>

consider a fluid occupying a domain $\mathcal{D}$. Let V be an subdomain of $\mathcal{D}$, with surface S

consider a small element dS of S with outward poi unit normal $\hat{n}$. let the velocity field in D be $\underline{u}$
Then in time $\delta t << 1$, ~~an amount~~ a mass $\rho(\underline{u} \cdot \hat{\underline{n}}) dS\, \delta$ crosses dS
(take the density of the fluid to be constant, and equal t ρ everywhere).





volume = area of base × height

$$= dS\,(\underline{u}\cdot\hat{\underline{n}})\,\delta t$$

Thus the total mass passing out of V is

$$\int_S \rho(\underline{u}\cdot\hat{\underline{n}})\,dS\,\delta t = \rho\delta t\int \underline{u}\cdot\hat{\underline{n}}\,dS$$

$\rho\int \underline{u}\cdot\hat{\underline{n}}\,dS$ = outward mass flux across S

but to conserve mass in V, this must be 0

$$\int_S \underline{u}\cdot\hat{\underline{n}}\,dS = 0$$

Divergence theorem says $\int_V \underline{\nabla}\cdot\underline{u}\,dV = 0$

Thus we have

$\forall$ subregions V of D,

$$\int_V \underline{\nabla}\cdot\underline{u}\,dV = 0$$

Thus, by theorem, $\underline{\nabla}\cdot\underline{u} = 0$ in D

In 2D: if $\underline{u} = u(x,y,t)\underline{i} + v(x,y,t)\underline{j}$

then $\underline{\nabla}\cdot\underline{u} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}$

so $\underline{u} = 7x\,\hat{\underline{i}} - 5y\,\hat{\underline{j}}$ — not incompressible as $\underline{\nabla}\cdot\underline{u} = 2 \neq 0$
ie. compressible
(non-constant ρ)

i.e incompressible velocity fields are <u>not</u> arbitrary.

In 3D: $\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$

In Polars (see moodle): $\underline{u} = u_r\,\hat{\underline{r}} + u_\theta\,\hat{\underline{\theta}}$

$$\underline{\nabla}\cdot\underline{u} = \frac{1}{r}\frac{\partial}{\partial r}(r u_r) + \frac{1}{r}\frac{\partial u_\theta}{\partial \theta} \qquad \underline{\nabla} = \frac{\partial}{\partial r}\hat{\underline{r}} + \frac{1}{r}\frac{\partial}{\partial\theta}\hat{\underline{\theta}}$$

Reminder: Green's Lemma

Consider a closed region A in the plane bounded by a curve C, taken counter-clockwise.

$$\int_A \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) dA = \oint u\,dy - v\,dx$$

Thus in 2D, incompressible flow $\oint_C u\,dy - v\,dx = 0$

for any closed curve C

$d\underline{r} = dx\,\hat{\underline{i}} + dy\,\hat{\underline{j}}$ $\qquad \underline{F} = -v\hat{\underline{i}} + u\hat{\underline{j}} = \hat{\underline{k}} \wedge (u\hat{\underline{i}} + \ldots$

$= \hat{\underline{k}} \wedge \underline{u}$

$\oint_C \underline{F} \cdot d\underline{r} = 0$ for all closed curves C in $\mathcal{D}$

i.e $\underline{F}$ is a conservative vector field.

$\underline{F}$ is derivable from a potential,

i.e. $\exists \psi$ s.t $\underline{F} = \underline{\nabla}\psi$ $\qquad \hat{\underline{k}} \wedge \underline{u} = \underline{\nabla}\psi$

i.e $\underline{u} = -\hat{\underline{k}} \wedge \underline{\nabla}\psi$

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Incompressibility (constant density)
$\Rightarrow \underline{\nabla} \cdot \underline{u} = 0$

$[\Rightarrow \exists \underline{A} \text{ s.t } \underline{u} = \underline{\nabla} \wedge \underline{A}]$

In 2D: $\underline{u} = u(x,y)\,\underline{i} + v(x,y)\,\underline{j}$

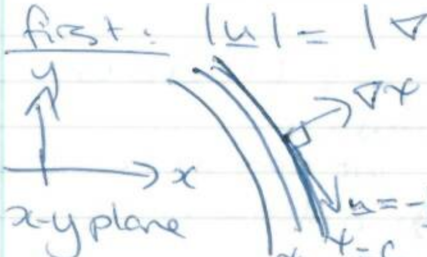
$\underline{\nabla} \cdot \underline{u} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}$

$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \Rightarrow \exists \psi \text{ s.t } \underline{u} = -\underline{k} \wedge \nabla\psi$

$= -\underline{\nabla} \wedge (\psi \underline{k})$

$\underline{u} = -\underline{k} \wedge \underline{\nabla}\psi$

first: $|\underline{u}| = |\nabla\psi|$



$c_3 > c_2 > c_1$

$\nabla\psi$ $\perp$ to level curves/isolines of ψ.

second: $\underline{u}$ and $\underline{\nabla}\psi$ $\perp$ (both in xy plane)

in fact $\underline{u}$ is $\underline{\nabla}\psi$ rotated $\pi/2$ clockwise

Finally: $\underline{u}$ is tangent to isoline $\psi = c$ for any c, i.e the isolines $\psi = c$ are streamlines.

we have shown that in an incompressible flow, $\exists$ function ψ whose isolines are streamlines.

Ex: Show that $\underline{u} = x\underline{i} - y\underline{j}$ satisfies the continuity equation, find a stream function. sketch some streamlines. (and suggest a flow.)

continuity: $\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$

here $u = x$, $\frac{\partial u}{\partial x} = 1$ $\quad v = -y$, $\frac{\partial v}{\partial y} = -1$ $\quad$ thus $\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$ as required

hence $\exists \psi$ s.t $\underline{u} = -\underline{k} \wedge \underline{\nabla}\psi$

$$\nabla\psi = \frac{\partial\psi}{\partial x}\,\underline{i} + \frac{\partial\psi}{\partial y}\,\underline{j}$$

$\underline{k} \wedge \underline{i} = \underline{j} \qquad \underline{k} \wedge \underline{j} = -\underline{i}$

so $-\underline{k} \wedge \nabla\psi = \frac{\partial\psi}{\partial y}\underline{i} - \frac{\partial\psi}{\partial x}\underline{j} = u\underline{i} + v\underline{j}$

so $u = \frac{\partial\psi}{\partial y}$, $v = -\frac{\partial\psi}{\partial x}$

for this example, $u = x$, thus $\frac{\partial\psi}{\partial y} = x$

$\psi = xy + f(x)$ $\quad$ f arbitrary function of x

$\Rightarrow \frac{\partial\psi}{\partial x} = y + f'(x)$

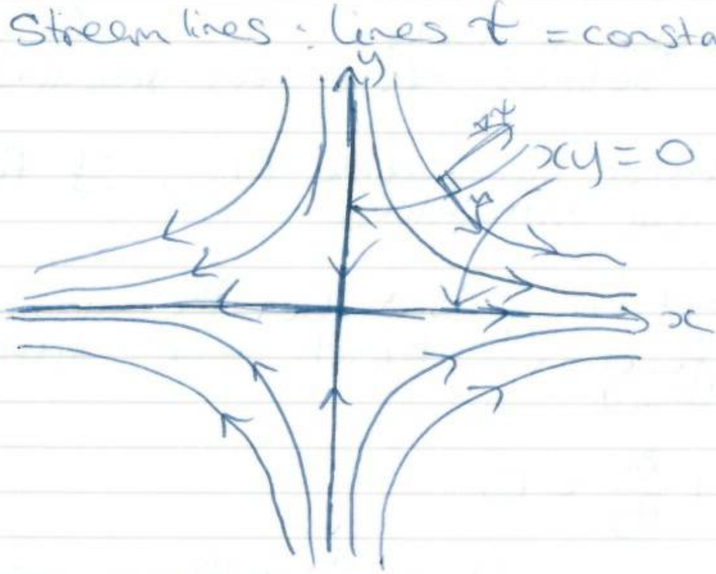
but $\frac{\partial\psi}{\partial x} = -v = y$ $\quad$ comparing gives $f'(x) = 0$
i.e f constant
w.l.o.g. we can take $f = 0$

(w.l.o.g since $\underline{u} = -\underline{k} \wedge \nabla\psi$, so adding a constant to ψ doesn't change $\underline{u}$)

ψ is unique to within an additive constant.

Hence the streamfunction ψ is
$$\psi = xy$$

Streamlines: lines ψ = constant. i.e $xy = c$



since $|\underline{u}| = |\nabla\psi|$ speed is directly proportional to $|\underline{\nabla}\psi|$ or equivalently $|\underline{u}|$ is inversely proportional to the separation of lines of constant ψ

This is stagnation point flow, as the origin is a stagnation point where $\underline{u} = 0$. This flow could be 2 colliding jets of equal strength.

<u>Flow conditions at a solid boundary</u>

solid
- impermeable.
no flow through boundary

the mass of fluid passing through dS in time δt is $\rho\,(\underline{u}\cdot\hat{\underline{n}})\,dS\,\delta t$

or, there is a mass flux, $\rho\,(\underline{u}\cdot\hat{\underline{n}})\,dS$ across dS
(rate at which mass crosses dS)

For no mass flux, $\underline{u}\cdot\hat{\underline{n}} = 0$ on S

On a solid boundary, $\underline{u}\cdot\hat{\underline{n}} = 0$

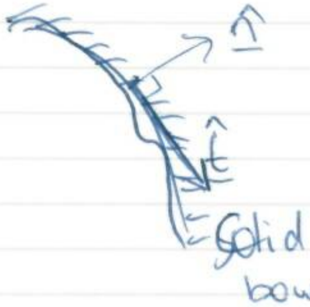
i.e velocity tangential to surface

If the fluid is also viscous, additionally, the tangential component of $\underline{u}$ vanishes also, so $\underline{u} = 0$ on a solid boundary → Stokes ← (Real fluids)

In terms of the streamfunction,

$$\hat{\underline{n}}\cdot\underline{u} = -\hat{\underline{n}}\cdot(\underline{k}\wedge\nabla\psi)$$

$$= -(\hat{\underline{n}}\wedge\underline{k})\cdot\nabla\psi$$



$$= -\hat{\underline{t}}\cdot\nabla\psi \qquad \hat{\underline{t}} \text{ unit tangent to surface}$$

$$= -\frac{\partial\psi}{\partial s} \quad \text{along surface (directional derivat}$$

but $\underline{u}\cdot\hat{\underline{n}} = 0$ so $\dfrac{\partial\psi}{\partial s} = 0$ along a solid boundar

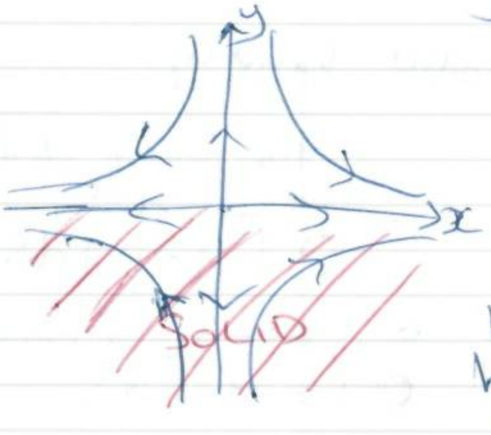
i.e ψ = constant on solid boundary

equivalently any line ψ = constant has $\underline{u}$ tangential i.e can be a solid boundary.

i.e. on a solid boundary, ψ = constant.
Any line ψ = constant can be replaced by solid boundary without affecting an inviscid flow

solid boundary: $\underline{u} \cdot \hat{n} = 0$ or $\psi = \text{constant}$.

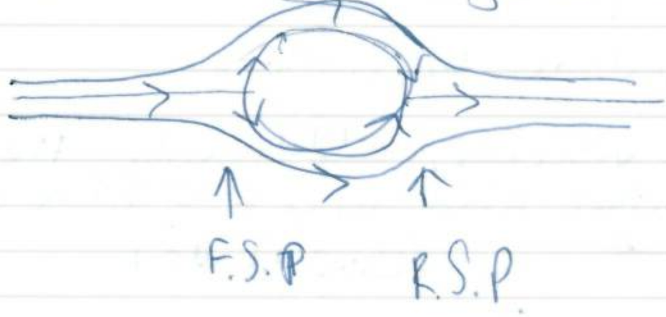
Ex $\quad u = x \quad v = -y \quad \psi = xy$



replace ANY streamline by solid boundary (in inviscid flow) without changing flow

here we obtain a jet hitting a wall
- stagnation point flow.

eg. Front and rear stagnation points in uniform flow past a circular cylinder.



Ex 2 $\quad$ same question as in ex 1.
now ~~...~~ $u = 2y, v = -2x$

$$\frac{\partial \psi}{\partial y} = u = 2y$$

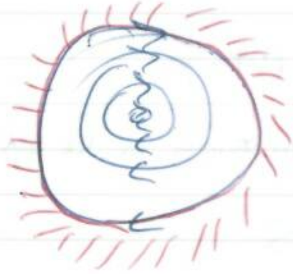
$$\psi = y^2 + f(x)$$
$$\frac{\partial \psi}{\partial x} = f'(x)$$

but $\frac{\partial \psi}{\partial x} = -v$ so $f'(x) = 2x$
i.e $f(x) = x^2 + C$

W.l.o.g. take $c = 0$

so $\psi = x^2 + y^2$

Streamlines are lines $x^2 + y^2 = a^2$ for a, constant.
i.e circles centre O radius a



i.e saucepan / beaker on turntable

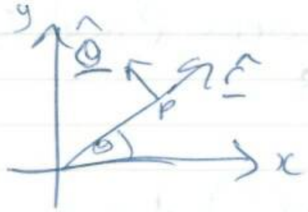
rotating as a solid body.
solid body rotation $\underline{u} = 2\hat{\underline{k}} \wedge \underline{r}$

$\psi = x^2 + y^2 = r^2$

using cylindrical polar coordinates:

$\underline{u} = -\hat{\underline{k}} \wedge \underline{\nabla}\psi$ coordinate free

$$\underline{\nabla}\psi = \frac{\partial\psi}{\partial r}\hat{\underline{r}} + \frac{1}{r}\frac{\partial\psi}{\partial\theta}\hat{\underline{\theta}}$$



$$\hat{\underline{k}} \wedge \hat{\underline{r}} = \hat{\underline{\theta}}$$

$$\hat{\underline{k}} \wedge \hat{\underline{\theta}} = -\hat{\underline{r}}$$

$$\underline{u} = -\hat{\underline{k}} \wedge \underline{\nabla}\psi$$

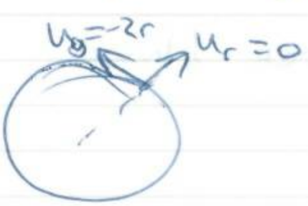
$$\underline{u} = -\frac{\partial\psi}{\partial r}\hat{\underline{\theta}} + \frac{1}{r}\frac{\partial\psi}{\partial\theta}\hat{\underline{r}}$$

$$= u_r\hat{\underline{r}} + u_\theta\hat{\underline{\theta}}$$

comparing $\quad u_r = \dfrac{1}{r}\dfrac{\partial\psi}{\partial\theta} \quad u_\theta = -\dfrac{\partial\psi}{\partial r}$] worth remembering

we have $\psi = r^2$ in our example

thus $u_r = \frac{1}{r}\frac{\partial\psi}{\partial\theta} = 0, \quad u_\theta = -\frac{\partial\psi}{\partial r} = -2r$



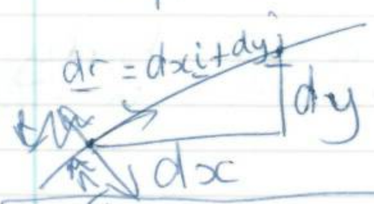
velocity increases linearly with distance

A physical interpretation of the streamfunction

The volume flux in a clockwise direction across any line joining a point P to a point Q in a flow field is given by

$\psi(Q) - \psi(P)$

[c.f. Work done is independent of path]



Volume flux crossing a length ds

$(\underline{u} \cdot \hat{\underline{n}})\, ds$

$$\boxed{\begin{array}{l} \text{want } \hat{\underline{n}} \cdot \underline{dr} = 0 \\ \text{try } \underline{n} = dy\,\underline{i} - dx\,\underline{j} \\ \text{so } \underline{n} \cdot \underline{dr} = 0 \end{array}}$$

thus $\hat{\underline{n}} = \dfrac{dy\,\underline{i} - dy\,\underline{i}}{\sqrt{dx^2 + dy^2}}$

$$\hat{\underline{n}} = \frac{dy}{ds}\underline{i} - \frac{dx}{ds}\underline{j}$$

thus the total flux crossing line between P and Q in clockwise direction is $\int_P^Q (\underline{u} \cdot \hat{\underline{n}})\, ds$

$$= \int_P^Q \left(\frac{\partial \psi}{\partial y}\underline{i} - \frac{\partial \psi}{\partial x}\underline{j} \right) \cdot \left(\frac{dy}{ds}\underline{i} - \frac{dx}{ds}\underline{j} \right) ds$$

$$= \int_P^Q \left(\frac{\partial \psi}{\partial x}\frac{dx}{ds} + \frac{\partial \psi}{\partial y}\frac{dy}{ds} \right) ds$$

$$= \int_P^Q \frac{\partial \psi}{\partial s}\, ds = \psi(Q) - \psi(P)$$

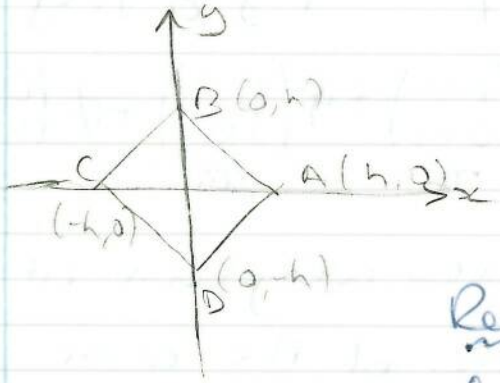
What are the dimensions of ψ?

volume / unit ~~time~~ per unit width

$L^3 T^{-1} L^{-1}$ i.e $L^2 T^{-1}$ area flux

20/11/10

2. Local motion at a point



Consider the initially square element ABCD with $0 < h \ll 1$.
Consider motion in the time interval $0 < \delta t \ll 1$ so that the flow is effectively steady.

Reminder: Taylor's Theorem

$$f(x) = f(0) + x f'(0) + R_2$$
$$R_2 = \tfrac{1}{2} f''(\xi) x^2 \quad \text{for } \xi \in (0,x)$$

i.e $f(x) = a + bx$ plus error of order x^2 where
$a = f(0)$
$b = f'(0)$

What is the effect of an arbitrary incompressible velocity field $\underline{u}(x,y,t)$ do to our infinitesimal element?

From Taylor's theorem (in 2D)

$$u = U + \alpha x + \beta y \quad , \quad v = V + \gamma x + \delta y$$ with error after ABCD of order h^2

where $U = u(0,0)$, $\alpha = \frac{\partial u}{\partial x}(0,0)$

$\beta = \frac{\partial u}{\partial y}(0,0)$, $V = v(0,0)$

$\gamma = \frac{\partial v}{\partial x}(0,0)$ $\quad \delta = \frac{\partial v}{\partial y}(0,0)$

now $\underline{u}$ is incompressible, so $\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$ everywhere

In particular, $\alpha + \delta = 0$
useful to write $\beta = \theta - \phi$
$\gamma = \theta + \phi$

Then $\theta = \frac{1}{2}(\gamma + \beta)$
$\phi = \frac{1}{2}(\gamma - \beta) = \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)$

now $\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} U \\ V \end{pmatrix} + \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$

In time δt, a point $\begin{pmatrix} x \\ y \end{pmatrix}$ within ABCD moves by an amount

$$\begin{pmatrix}\delta x\\ \delta y\end{pmatrix} = \begin{pmatrix}u\\ v\end{pmatrix}\delta t = \begin{pmatrix}U\\ V\end{pmatrix}\delta t + \begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix}\delta t$$

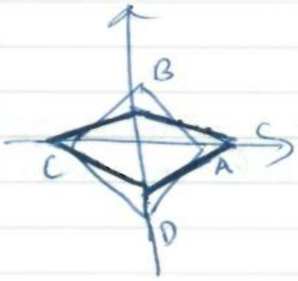
i.e. $$\begin{pmatrix}\delta x\\ \delta y\end{pmatrix} = \underset{\mathrm{I}}{\begin{pmatrix}U\\ V\end{pmatrix}}\delta t + \left[\alpha\underset{\mathrm{II}}{\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}} + \theta\underset{\mathrm{III}}{\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}} + \phi\underset{\mathrm{IV}}{\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}}\right]\begin{pmatrix}x\\ y\end{pmatrix}\delta t$$

<u>term I:</u> this term simply moves every point at speed $\begin{pmatrix}U\\ V\end{pmatrix}$

– translation of the centre of mass (at speed U, V)

<u>term II:</u> this moves A by $\begin{pmatrix}\delta x\\ \delta y\end{pmatrix} = \alpha\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\begin{pmatrix}h\\ 0\end{pmatrix}\delta t$

$$= \begin{pmatrix}\alpha h \delta t\\ 0\end{pmatrix}$$



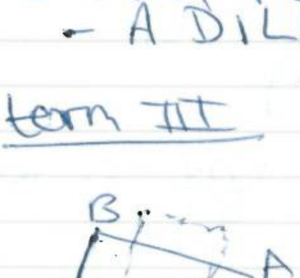
thus C moves by $\begin{pmatrix}\delta x\\ \delta y\end{pmatrix} = \begin{pmatrix}-\alpha h \delta t\\ 0\end{pmatrix}$

At B $\begin{pmatrix}\delta x\\ \delta y\end{pmatrix} = \alpha\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\begin{pmatrix}0\\ h\end{pmatrix}\delta t = \begin{pmatrix}0\\ -\alpha h \delta t\end{pmatrix}$

At D $\begin{pmatrix}\delta x\\ \delta y\end{pmatrix} = \begin{pmatrix}0\\ \alpha h \delta t\end{pmatrix}$

thus term II stretches the square at a rate αh in the x-direction and shrinks at the same rate αh in the y-direction without moving the centre of mass, conserving volume as expected
– A DILATION

<u>term III</u> At A $\begin{pmatrix}\delta x\\ \delta y\end{pmatrix} = \begin{pmatrix}0\\ \theta h \delta t\end{pmatrix}$



C $\begin{pmatrix}\delta x\\ \delta y\end{pmatrix} = \begin{pmatrix}0\\ -\theta h \delta t\end{pmatrix}$

B $\begin{pmatrix}\delta x\\ \delta y\end{pmatrix} = \begin{pmatrix}\theta h \delta t\\ 0\end{pmatrix}$

D $\begin{pmatrix}\delta x\\ \delta y\end{pmatrix} = \begin{pmatrix}-\theta h \delta t\\ 0\end{pmatrix}$

Another DILATION. Stretching along line $y = x$ and an equal & opposite shrinkage along the line $y = -x$, both at rate θh, so as to preserve volume

It appears that there are two dilations, term II and III
This is not so.

The combined effect of II and III is the matrix

$\begin{pmatrix} \alpha & \theta \\ \theta & -\alpha \end{pmatrix}$ This is a real, symmetric matrix.
It possesses 2 real eigenvalues

$$\begin{vmatrix} \alpha-\lambda & \theta \\ \theta & -\alpha-\lambda \end{vmatrix} = 0$$

$$-(\alpha-\lambda)(\alpha+\lambda) - \theta^2 = 0$$
$$-\alpha^2 + \lambda^2 - \theta^2 = 0$$
$$\lambda^2 = \alpha^2 + \theta^2$$

hence we have 2 equal and opposite eigenvalues
$\lambda = \pm\sqrt{\alpha^2+\theta^2}$

with eigenvectors ξ_1 and ξ_2 (say) ORTHOGONAL
In the basis $\{\xi_1, \xi_2\}$ the matrix has form

$\begin{pmatrix} \lambda_1 & 0 \\ 0 & -\lambda_1 \end{pmatrix}$ precisely the form of term II

Thus expansion at rate λ_1 along ξ_1 and contraction
at rate λ_1 along the orthogonal direction ξ_2
i.e a DILATION

2301 24/10/11

Summary:

1. $\underline{u}(x, y, z, t)$
2. particle paths, streaklines, streamlines
3. incompressibility $\Rightarrow \underline{\nabla} \cdot \underline{u} = 0$ (in n-D)
4. incompressible + 2D $\Rightarrow \underline{u} = u(x,y,t)\hat{\underline{i}} + v(x,y,t)\hat{\underline{j}} = u_r(r,\theta,t)\hat{\underline{r}} + u_\theta(r,\theta,t)\hat{\underline{\theta}}$

$\nabla \cdot \underline{u} = 0$ + 2D $\Rightarrow \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \Rightarrow \exists \underset{\text{SCALAR}}{\psi}$ s.t. $\underline{u} = -\hat{\underline{k}} \wedge \nabla \psi$

i.e. $u = \frac{\partial \psi}{\partial y}$ $\quad v = -\frac{\partial \psi}{\partial x}$ $\quad$ or $\quad u_r = \frac{1}{r}\frac{\partial \psi}{\partial \theta}$ $\quad u_\theta = -\frac{\partial \psi}{\partial r}$

5. local motion at a point
- consists of a translation of centre of mass, a dilation and ...

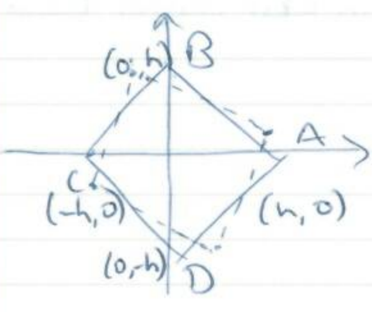
previously we had:

$$\begin{pmatrix} u \\ v \end{pmatrix} = \underbrace{\begin{pmatrix} U \\ V \end{pmatrix}}_{\mathrm{I}} + \left[\underbrace{\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}}_{\substack{\mathrm{II} \\ \text{dilation}}} \alpha + \underbrace{\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}}_{\substack{\mathrm{III} \\ \text{dilation}}} \theta + \underbrace{\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}}_{\substack{\mathrm{IV} \\ ?}} \phi \right] \begin{pmatrix} x \\ y \end{pmatrix} \qquad \phi = \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)$$

$\begin{pmatrix} \alpha & \theta \\ \theta & -\alpha \end{pmatrix}$

eigenvectors

$\underline{\xi}_1, \underline{\xi}_2$, $\underline{\xi}_1 \perp \underline{\xi}_2$



Term IV contributes:

at A $\phi\begin{pmatrix} 0 \\ h \end{pmatrix}\delta t$

A moves by an amount $\begin{pmatrix} \delta x \\ \delta y \end{pmatrix} = \begin{pmatrix} u \\ v \end{pmatrix}\delta t$

At B $\begin{pmatrix} \delta x \\ \delta y \end{pmatrix} = \phi\begin{pmatrix} -h \\ 0 \end{pmatrix}\delta t$

At A the radial arm has length h. Point has moved up distance $\phi h \delta t$
Hence moved through an angle $\phi\,\delta t$. i.e. ABCD is rotating at a rate of ϕ in the anticlockwise direction, i.e at a rate $\frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)$ in the anticlockwise direction.

Thus we have shown that the motion at a point consists of 3 and only 3 things:
translation of C. of. M, a dilation, and rotation about the C. of. M.

[notice a solid is as above, ~~bgg~~ but no dilation].

note that $\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}$ is precisely the z component of $\underline{\nabla} \wedge \underline{u}$, i.e. curl $\underline{u}$

It is traditional to write $\underline{\omega} = \underline{\nabla} \wedge \underline{u}$

$\underline{\omega}$ is the VORTICITY of the flow, i.e the rotation of the flow.

[old name curl $\underline{u}$ was rot $\underline{u}$]

The components of $\underline{\omega}$ are usually written $\underline{\omega} = \xi \hat{\underline{i}} + \eta \hat{\underline{j}} + \zeta \hat{\underline{k}}$

xi eta zeta

In 2D, $\underline{u} = u(x,y,t)\hat{\underline{i}} + v(x,y,t)\hat{\underline{j}}$

$\underline{\omega} = 0\hat{\underline{i}} + 0\hat{\underline{j}} + \zeta \hat{\underline{k}}$

i.e $\underline{\omega}$ is solely in the z direction with $\underline{\omega} = \zeta \hat{\underline{k}}$

and $\zeta = \dfrac{\partial v}{\partial x} - \dfrac{\partial u}{\partial y}$

and it gives twice the rate of rotation of a fluid element about the C of M.

$[\,\phi = \tfrac{1}{2}\zeta\,]$

i.e. ζ is proportional to the angular momentum of a fluid about its centre of mass.

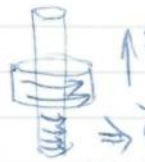
We can only change the rate at which a fluid element is spinning in 2D by applying a torque, i.e a shear stress.

But an inviscid fluid does not support a shear stress, so we cannot change the rate at which spins (in 2D), i.e a particle in a 2D inviscid fluid retains its value of ζ forever.

Consider a flow started from rest. Then initially $\zeta \equiv 0$ ($\underline{u} \equiv 0$ at $t=0$). i.e every particle has vorticity zero. Hence for all time, all particles have zero vorticity.

A motion where $\underline{\omega} = 0$ everywhere is called IRROTATIONAL. For example, any flow started from rest is irrotational. An irrotational motion remains irrotational - the persistence of irrotationality. (true in 3D also, as there is nothing to amplify).

Aside: In 3D



stretched column $\omega \neq 0$

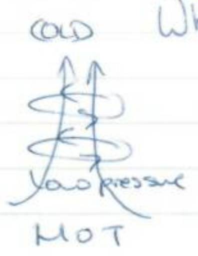
$\Rightarrow \zeta$ increased $\dfrac{\partial}{\partial z} \neq 0$

eg hurricane

Amplifier

Hurricanes: Why in tropics, not equator? ← no rotation to amplify
Why not at poles, northern oceans?
Why spin?

$\boxed{\text{Isaac Held}}$



HEAT ENGINE

amplifier

no heat to drive it

We will concentrate on IRROTATIONAL flow. Then $\underline{\nabla} \wedge \underline{u} = 0$
Hence $\exists\, \phi$ s.t. $\underline{u} = \underline{\nabla} \phi$ i.e $\underline{u}$ is derivable from a potential, the velocity potential.
(we are still in 2D or 3D)

In incompressible flow, in 2D or 3D, $\underline{\nabla} \cdot \underline{u} = 0$

substituting gives $\underline{\nabla} \cdot (\underline{\nabla} \phi) = 0$ i.e. $\nabla^2 \phi = 0$
Laplace's equation in 2D and 3D
- The governing equation for 3D incompressible, irrotational flow, all we need are boundary conditions.

On a solid boundary $\underline{u} \cdot \hat{\underline{n}} = 0$. Substitute for $\underline{u}$: $\hat{\underline{n}} \cdot \underline{\nabla} \phi = 0$ on solid boundary
i.e $\frac{\partial \phi}{\partial n} = 0$, the normal derivative of ϕ vanishes on a solid boundary.

(The solution to Laplace's equation with $\frac{\partial \phi}{\partial n}$ specified on boundary, i.e Neuman problem, is unique) 2D or 3D

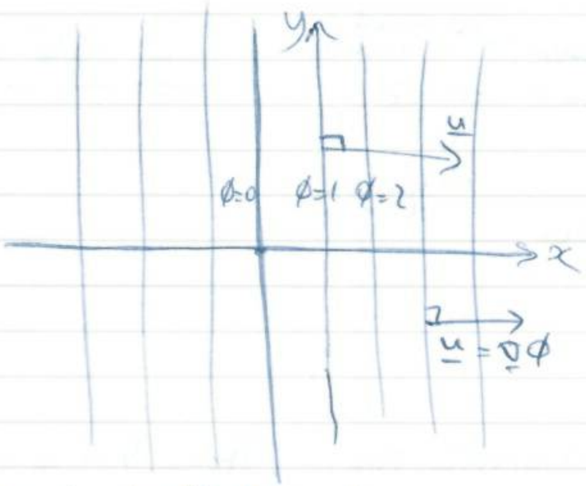
Example: What is velocity potential for a uniform stream?
Take x-axis in direction of stream.
$\underline{u} = U\hat{\underline{i}}$ so $u = U$, $v = 0$

but $\underline{u} = \underline{\nabla} \phi = \frac{\partial \phi}{\partial x} \hat{\underline{i}} + \frac{\partial \phi}{\partial y} \hat{\underline{j}}$ so $u = \frac{\partial \phi}{\partial x}$, $v = \frac{\partial \phi}{\partial y}$

here $\frac{\partial \phi}{\partial x} = U$ so $\phi = Ux + f(y)$

so $\frac{\partial \phi}{\partial y} = f'(y)$ but $\frac{\partial \phi}{\partial y} = v$ and $v = 0$ so $f'(y) = 0$
so f constant
take $f = 0$

$\phi = Ux$
(notice satisfies Laplace's eqn)



	good	bad
ϕ	3D	Only irrotational
ψ	does not require irrotationality	only 2D

equipotentials, ϕ = constant
x = constant here.

What does ψ satisfy in 2D, irrotational flow?

2D: $u = \dfrac{\partial \psi}{\partial y}$ $\quad v = -\dfrac{\partial \psi}{\partial x}$

irrotational: $\zeta = 0$ $\quad \dfrac{\partial v}{\partial x} - \dfrac{\partial u}{\partial y} = 0$
2D

substituting: $\dfrac{\partial}{\partial x}\left(-\dfrac{\partial \psi}{\partial x}\right) - \dfrac{\partial}{\partial y}\left(\dfrac{\partial \psi}{\partial y}\right) = 0$

i.e $\dfrac{\partial^2 \psi}{\partial x^2} + \dfrac{\partial^2 \psi}{\partial y^2} = 0$

i.e. $\nabla^2 \psi = 0$

so the streamfunction satisfies Laplace's equation as well.
Remember that on a solid boundary for ψ ~~is~~ ψ = constant, which can be taken as $\psi = 0$ if only one boundary.

check: uniform stream $\psi = Uy$ so $\nabla^2 \psi = 0$

stagnation point $\psi = xy$ so $\nabla^2 \psi = 0$

If the flow is 2D and irrotational then you ~~too~~ can choose to find ϕ or ψ - whichever seems easier.
the governing equation is the same: Laplace
but boundary conditions are different.

Are ϕ and ψ related?

Yes

$\underline{u} = \underline{\nabla}\phi$ and $\underline{u} = -\hat{\underline{k}} \wedge \underline{\nabla}\psi$

so $\underline{\nabla}\phi = -\hat{\underline{k}} \wedge \underline{\nabla}\psi$
or $\underline{\nabla}\psi = \hat{\underline{k}} \wedge \nabla\phi$] Cauchy Rieman equations.

$u = \frac{\partial \phi}{\partial x}$, $u = \frac{\partial \psi}{\partial y}$ so $\frac{\partial \phi}{\partial x} = \frac{\partial \psi}{\partial y}$

$v = \frac{\partial \phi}{\partial y}$ and $v = -\frac{\partial \psi}{\partial x}$ so $\frac{\partial \phi}{\partial y} = -\frac{\partial \psi}{\partial x}$

In 2D irrotational flow

$\frac{\partial \phi}{\partial x} = \frac{\partial \psi}{\partial y}$ and $\frac{\partial \phi}{\partial y} = -\frac{\partial \psi}{\partial x}$ the Cauchy-Riemann equations.

thus ϕ and ψ are the real and imaginary parts of a differentiable complex function of the complex ~~functi~~ variable $z = x + iy$.

traditionally this is called the <u>complex velocity potential</u> and written

$w(z) = \phi(x,y,t) + i\psi(x,y,t)$ $\quad z = x+iy$

$\phi = \mathrm{Re}\{w(z)\}$, $\psi = \mathrm{Im}\{w(z)\}$

Proved: 1) Real and imaginary parts of complex differentiable function satisfy Laplace's equation

2) Constant surfaces intersect at right angles.

<u>check</u> uniform stream

$\phi = Ux$
$\psi = Uy$

$\phi + i\psi = U(x+iy) = Uz$ (a function of z alone)

so $w = Uz$ is the complex potential for a uniform stream.

Given $w(z)$ how do we get $\underline{u}$?

consider $\frac{dw}{dz} = \frac{\partial}{\partial x}(\phi + i\psi) = \frac{\partial \phi}{\partial x} + i\frac{\partial \psi}{\partial x} = u - iv$

so $u + iv = \overline{\frac{dw}{dz}}$ (conjugate of $\frac{dw}{dz}$)

eg. 1 for $w = Uz$

$$\frac{dw}{dz} = U$$
$$= u - iv$$

So $u = U$ and $v = 0$ as expected.

2 $w = z^2$

$$\frac{dw}{dz} = 2z = 2x + 2iy$$

$u = 2x, \ v = -2y$

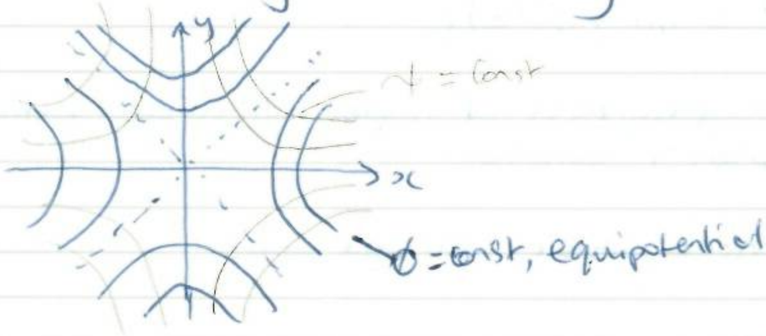
$u = 0$ and $v = 0$ only if $z = 0$ (where $\frac{dw}{dz} = 0$)

i.e stagnation point iff $\frac{dw}{dz} = 0$

here only at $z = 0$

$w = z^2 = (x+iy)^2 = x^2 - y^2 + 2ixy$

so $\phi = x^2 - y^2$, $\psi = 2xy$



Irrotational: $\nabla \wedge \underline{u} = 0 \Rightarrow \exists \phi$ s.t $\underline{u} = \nabla\phi$

incompressibility : $\underline{\nabla} \cdot \underline{u} = 0$

plus 2D : $\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$

$\Rightarrow \exists \psi$ s.t $\underline{u} = -k \wedge \underline{\nabla}\psi$

i.e $\underline{\nabla}\phi = -k \wedge \underline{\nabla}\psi$ Cauchy-Riemann

$\Rightarrow \exists\, \omega(z)$ s.t $\frac{d\omega}{dz}$ exists and $\omega = \phi + i\psi$

$\frac{d\omega}{dz} = u - iv$

Laurent Series

A function analytic within an annular region $R_0 < |z| < R_1$

has a unique expension of the form

$\dots + \frac{a_2}{z^2} + \frac{a_1}{z} + a_0 + a_1 z + a_2 z^2 + \dots$

i.e all functions analytic in the annulus are simply linear combinations of

$z^{\pm n}$, $n = 0, 1, 2, 3, 4, \dots$

Apply this to the complex velocity $u - iv$
i.e $u - iv$ is a linear comb. of $z^{\pm n}$

Thus $\omega(z)$, the complex potential, is simply a linear comb. of the terms

$z^{\pm n}$, $n = 0, 1, 2, 3 \dots$ and $\log z$

i.e our flow in any annular region (or a region that can be distorted into an annulus) or outside a single body (let $R_1 \to \infty$) is simply a ~~sum~~ linear comb. of terms chosen from

$\{ z^{\pm n}, \log z \}$

note : coefficients in sum can be complex

In particular, in cylindrical coordinates, $z = re^{i\theta}$

so $z^n = r^n e^{in\theta} = r^n \cos n\theta + i r^n \sin n\theta$

$\log z = \log r + i\theta$

now $\phi = \mathrm{Re}\, \omega$ so ϕ must be a linear comb. of terms drawn from the set

$$\{ r^n \cos n\theta, r^n \sin n\theta, (n = 0, \pm 1, \pm 2, \ldots), \log r, \theta \}$$

thus all solutions (in an annular domain) of Laplace's equation in polar coordinates are simply a linear comb. of the terms

$r^{\pm n} \cos n\theta$, $r^{\pm n} \sin n\theta$, $\log r$, θ

similarly, $\psi(x,y) = \mathrm{Im}\, \omega(z)$ is only a linear comb. of terms drawn from the set

$$\{ r^{\pm n} \cos n\theta, r^{\pm n} \sin n\theta, (n = 0, 1, 2, 3, \ldots), \log r, \theta \}$$

Example:

Find the ideal 2D flow past a cylinder of radius a (irrot. and incomp.) given that the flow at infinity is uniform with speed U

Solution: take ~~Cartesian~~ axes with Ox in the direction of flow at infinity and origin at centre of cylinder

can solve either with ϕ or ψ

Choose ψ simply so we can draw some streamlines.

Governing equation $\nabla^2 \psi = 0$ in $r > a$

On the cylinder $(r = a)$, no flow through cylinder $(\underline{u} \cdot \hat{\underline{n}} = 0)$

ψ = constant on $r=a$

but only one body so w.l.o.g we can take $\psi=0$ on $r=a$

As $r\to\infty$: $\underline{u}\to U\underline{i}$

i.e $u\to U$ $v\to 0$

$\frac{\partial\psi}{\partial y}\to U$ so $\psi\to Uy+f(x)$

$\frac{\partial\psi}{\partial x}\to f'(x)$

but $\frac{\partial\psi}{\partial x}\to v=0$ so $f'=0$ w.l.o.g $f=0$

hence $\psi\to Uy$ as $r\to\infty$

Summary:
$$\nabla^2\psi=0 \quad r>a$$
$$\psi=0 \quad r=a$$
$$\psi\to Uy \quad r\to\infty$$

homogeneous ($\psi=0$ is a solution)

inhomogeneous ($\psi=0$ not a solution)

Inhomogeneous boundary condition
$\psi\to Ur\sin\theta$ as $r\to\infty$

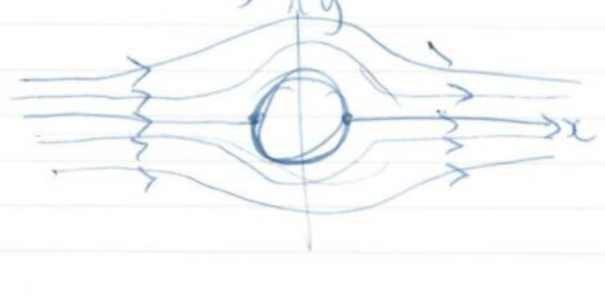
Guess $\psi = Ur\sin\theta + \cancel{ar\cos3\theta} + \cancel{br^3\cos3\theta} + \frac{c}{r^3}\cos3\theta$

violates $r\to\infty$

$+\frac{b}{r}\sin\theta$

ie $\psi=Uy\left(1-\frac{a^2}{r^2}\right)$

$\psi=0$ when $y=0$, $r=a$ as expected



17th Octobe

vorticity: $\underline{\omega} = \underline{\nabla} \wedge \underline{u}$ (curl of the velocity)

examples of streamfunctions

1) perhaps the simplest flow is a uniform stream.
W.l.o.g take x axis in the direction of the flo

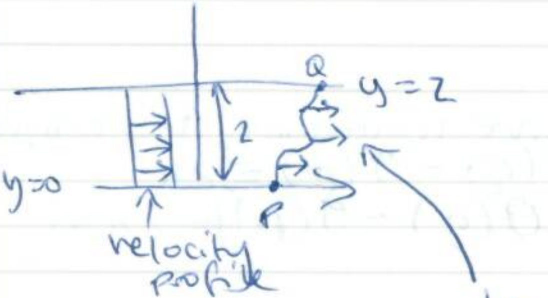
then $u = U$, $v = 0$ $\quad \frac{d\psi}{dy} = u = U$

so $\psi = Uy + f(x)$

$\frac{d\psi}{dx} = f'(x)$

but $\frac{d\psi}{dx} = -v = 0$

so $f' = 0$ hence we can take $f = 0 \Rightarrow \psi = Uy$



Flux across $x = 0$ is $2U$ i.e. length x speed.

How much fluid crosses PQ?

(1) flux across PQ must also be $2U$ because no fluid escapes through $y = 0$ or $y = 2$ as they are streamlines and so flow is tangential to them, i.e. we can replace them by a solid boundary

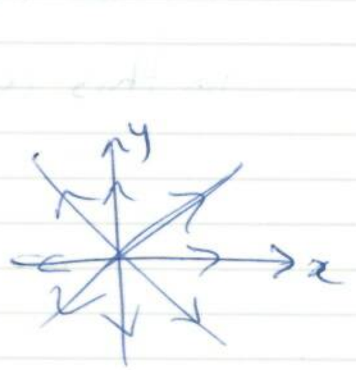
OR

(2) the flux $= \psi(Q) - \psi(P) = 2U - 0 = 2U$

2) Another important flow is the isotropic source.
This has streamfunction $\psi = m\theta$

This gives $\underline{u} = U_r \hat{\underline{r}} + U_\theta \hat{\underline{\theta}}$

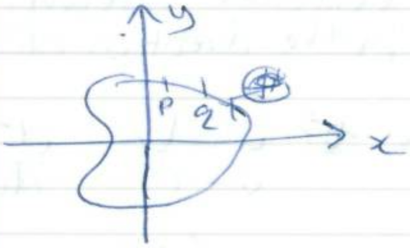
where $u_r = \frac{1}{r}\frac{d\psi}{d\theta} \quad u_\theta = -\frac{d\psi}{dr}$



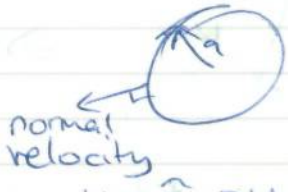
hence $u_r = \frac{m}{r}$, $u_\theta = 0$

velocity field is the same in all directions, i.e independent of θ i.e. it is isotropic

now consider any circuit containing the origin.



the flux across C is



$\underline{u} \cdot \hat{\underline{n}} = \underline{u} \cdot \hat{\underline{r}} = u_r$

speed $= \frac{m}{a}$

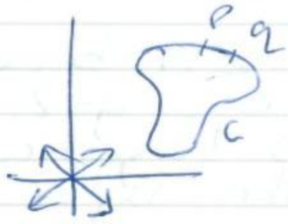
length $= 2\pi a$

flux $= 2\pi a \frac{m}{a} = 2\pi m$

Going around any closed curve containing the origin θ increases by 2π i.e $\theta(q) - \theta(p) = 2\pi$
so $\psi(q) - \psi(p) = m[\theta(q) - \theta(p)] = 2\pi m$

(surround)

It curve does not cycle origin.



$\theta(P) = \theta(Q)$

so flux $= 0$ i.e no net flux across C

by making successively smaller circles we see that only the origin is fluid created and it is created there at flux $2\pi m$.
We call $2\pi m$ the <u>strength</u> of the source

(two statements are equivalent)

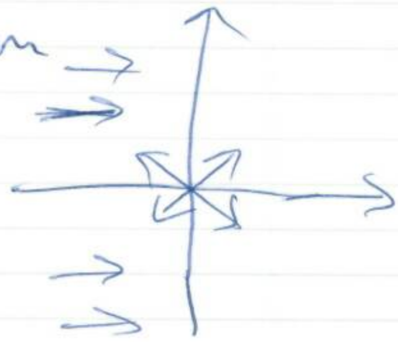
i.e a source of strength then $\psi = \frac{m\theta}{2\pi}$

in this case strength $u_r = \frac{m}{2\pi r}$, singular at the origin but well behaved everywhere else

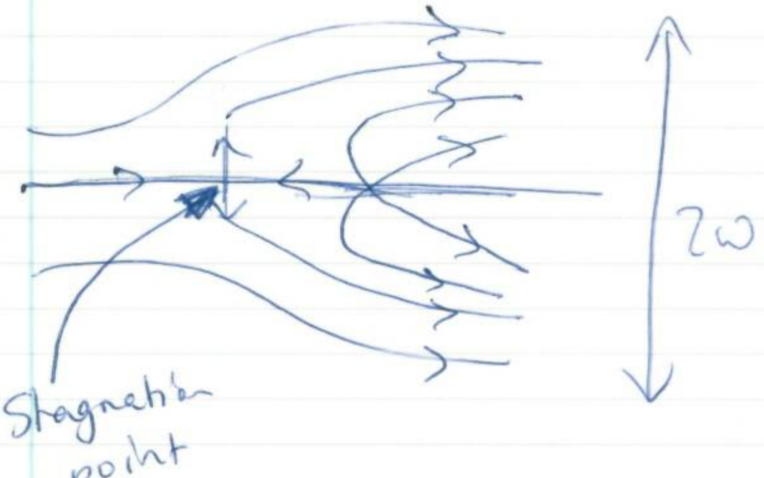
3) Combine these

An isotropic source of strength $2\pi m$ in a uniform stream of speed U

take x axis in direction of stream
take origin at the source



Notice source dominates for r sufficiently small (near origin)
stream dominates if r is sufficiently large as $U_r = \frac{u}{r}$



$$\psi = U_y + m\theta$$

working out properties:

$$\psi = U_y + m\theta$$
$$\underline{u} = 0 \;,\; U_r = 0$$
$$U_\theta = 0 \Rightarrow \frac{d\psi}{dr} = 0 \quad \frac{d\psi}{d\theta} = 0$$

flux = width × speed
$= 2$[illegible] $\times$ [illegible] $= 2\pi m$

so $\omega = \frac{\pi m}{U}$

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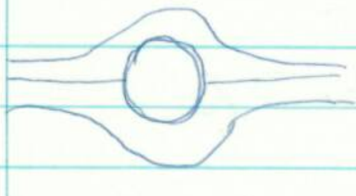


$$\psi = Uy\left(1-\frac{a^2}{r^2}\right)$$
$$= Uy - \frac{Ua^2y}{r^2}$$
$$= \operatorname{Im}\left[Uz - \frac{Ua^2}{z}\right]$$

$$\frac{Ua^2y}{r^2} = \frac{Ua^2y}{|z|^2}$$

$$w(z) = Uz + \frac{Ua^2}{z}$$

$$\phi = \operatorname{Re} w$$
$$= Ux + \frac{Ua^2x}{r^2}$$

ϕ is conjugate to ψ

h/w

$\underline{u}\cdot\hat{\underline{n}} = V$ (notes have $V=0$)

$\underline{u} \to U\underline{i}$ as $r\to\infty$

$\phi \to Ux = Ur\cos\theta$ as $r\to\infty$ inhomogeneous

$\underbrace{\underline{u}\cdot\hat{\underline{n}} = V}_{\text{same for all }\theta}$ on $r=a$ inhomogeneous.

$$\phi = Ar\cos\theta + \frac{B}{r}\cos\theta + C\log r \perp$$

<u>Our basic solutions:</u>

1) z^{-1}, z^{-2}, z^{-3} etc 'singularities'
$\uparrow$ dipole

2) z^0 - nothing $\frac{dw}{dz} = 0$

3) z: $w = Uz$ uniform stream $\frac{dw}{dz} = U = u - iv$

4) $w = z^2 = (re^{i\theta})^2 = r^2\cos 2\theta + ir^2\sin 2\theta$

so $\psi = r^2\sin 2\theta$ so $\psi = 0$ on $\theta = 0$

and with increasing θ, next zero when $\theta = \pi/2$

stag. point flow

$$\frac{dw}{dz} = 2z = 2x + 2iy$$

$u = 2x$
$v = -2y$

5) $w = z^3$

$\psi = r^3 \sin 3\theta$

$\psi = 0$ on $\theta = 0$, next zero when $\theta = \pi/3$

6. $w = z^n$

$\psi = r^n \sin n\theta$

$\psi = 0$ on $\theta = 0$ and next at $\theta = \pi/n$

Thus in fundamental solutions, if n streamlines cross, they cross at an angle π/n in irrotational, incompressible flow.

<u>Streamlines in the neighbourhood of a stagnation point.</u>

suppose we have a stag. point in the flow. Move origin to that point

In the neighbourhood of O,

$w = a_0 + a_1 z + a_2 z^2 + a_3 z^3 + \dots$

w.l.o.g we can take $a_0 = 0$

at O, $\frac{dw}{dz} = 0$ (stag. point) thus $a_1 = 0$

let the first non-zero term be a_n. Then $n \geq 2$.

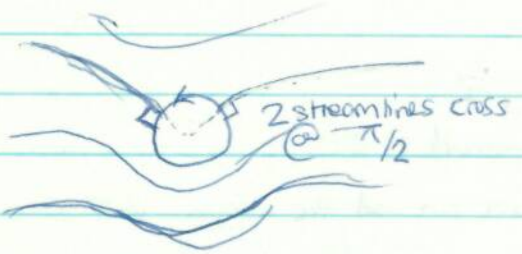
sufficiently close to O, $w \sim a_n z^n$ for some complex a_n

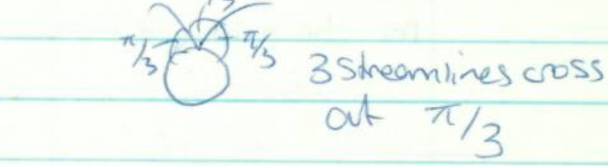
suppose $a_n = Ae^{i\alpha}$ then $w \sim Ae^{i\alpha} r^n e^{in\theta}$

$$= (A^{1/n} r)^n e^{in(\theta + \frac{\alpha}{n})}$$

ie. a fundamental solution z^n, rotated by α/n and scaled by $A^{1/n}$

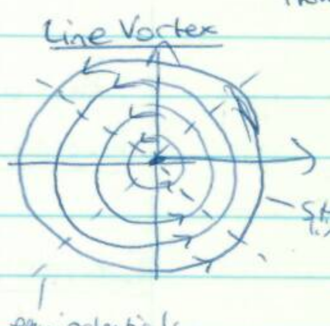
i.e exactly as before: n streamlines must cut at π/n





the remaining fundamental solution is $\log z$.
if $w = m\log z = m(\log r + i\theta) = m\log r + im\theta$ (m real)
so $\phi = m\log r$, $\psi = m\theta$ i.e $\log r$, θ are conjugate f
– isotropic source of strength $2\pi m$

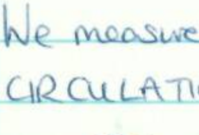
Exercise: no other fundamental solution is a source of fluid. (consequence of Cauchy Thm)



If $w = -iK\log z$ (K real)
$= -iK(\log r + i\theta) = K\theta - iK\log r$
so $\phi = K\theta$, $\psi = -K\log r$
(streamlines and equipotentials are swapped from $w = m\log z$).
irrotational everywhere but the origin (singularity).

The strength of a line (or point) vortex.

We measure the strength of ANY ROTATIONAL flow by its CIRCULATION about a closed contour C (say). The circulation is defined as



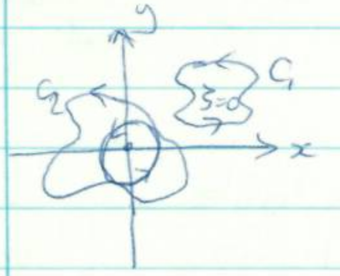
$\Gamma = \oint_C \underline{u} \cdot d\underline{l}$ i.e sum of tangential velocity × distance
Γ (gamma capital) (c.f work done going around a closed path)

notice for irrotational flow this is 0 for all curves C
In 3D, if $\underline{u} = \underline{\nabla}\phi$ $\int_A^B \underline{\nabla}\phi \cdot d\underline{l} = \phi(B) - \phi(A)$. If A=B this is 0

In 2D $\oint_C \underline{u} \cdot d\underline{l} = \int_A (\underline{\nabla} \wedge \underline{u}) \cdot \hat{\underline{n}} \, dA$ but in 2D flow $\underline{\nabla} \wedge \underline{u} = \zeta$ and $\hat{\underline{n}} = \hat{\underline{k}}$

so $\Gamma = \int_A \zeta \, dA$ If $\zeta = 0$ everywhere $\Gamma = 0$ for all C]

For the point vortex, $w = -iK\log z$



circulation around C_1 is 0

For a circuit around the origin, we can take the circuit to be a circle of radius a, w.l.o.g

$$\Gamma = \oint_{\theta=-\pi}^{\pi} \underline{u}\cdot d\underline{l}$$

Associated with a change $d\theta$ in θ is the vector $d\underline{l} = (a\,d\theta)\hat{\underline{\theta}}$

and $\underline{u} = \nabla\phi = \nabla(K\theta) = K\nabla\theta = K\left[\frac{\partial}{\partial r}\hat{\underline{r}} + \frac{1}{r}\frac{\partial}{\partial\theta}\hat{\underline{\theta}}\right]\theta = \frac{K}{r}\hat{\underline{\theta}}$

On $r=a$ $\underline{u} = \frac{K}{a}\hat{\underline{\theta}}$

thus $\Gamma = \int_{-\pi}^{\pi} \frac{K}{a}\hat{\underline{\theta}} \cdot a\,d\theta\,\hat{\underline{\theta}} = 2\pi K$

the vortex has circulation $2\pi K$.

<u>Exercise:</u> only fundamental solution with circulation.

<u>Example:</u> Consider a cylinder of radius a, in a ~~direction~~ stream of speed U where there is circulation K around the cylinder.

Take origin at the centre of the cylinder and axis Ox in the direction of the flow at infinity. Then

$$w(z) = Uz + \frac{Ua^2}{z} + i\frac{K}{2\pi}\log z$$

<u>check:</u>

$$\frac{dw}{dz} = U - \frac{Ua^2}{z^2} - \frac{iK}{2\pi z} \qquad \text{as } z\to\infty,\ \frac{dw}{dz}\to U \text{ ie } \begin{matrix} u\to U \\ v\to 0\end{matrix}$$

thus b.c. at infinity is satisfied.

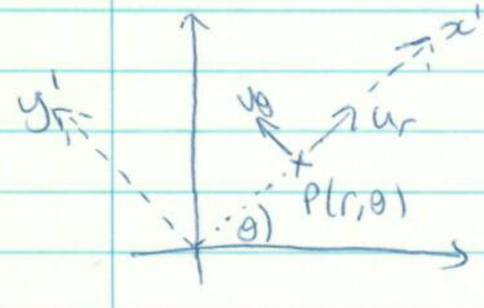
Take any circuit about the cylinder. Then the circulation about C is $2\pi\left(\frac{K}{2\pi}\right) = K$ as required, since only line vortex has circulation

Satisfies Laplace's equation because a sum of fundamental solutions).

Remains to check $\underline{u}\cdot\hat{\underline{n}} = 0$ on the cylinder $r=a$ i.e $\underline{u}\cdot\hat{\underline{r}} = 0$ ie. $u_r = 0$ on $r=a$

There is a nice way to do this using complex variables

There is a nice way of doing this using complex variables.



At some point P the cartesian components of velocity

$\frac{dw}{dz} = u - iv.$

Introduce x' and y' rotated by θ degrees anticlockwise from x, y. Then

$\frac{dw}{dz'} = u' - iv'$ (the Cartesian components along desired axes)

$= u_r - iu_\theta$

$u_r - iu_\theta = \frac{dw}{dz'} = \frac{dw}{dz}\frac{dz}{dz'}$ now $z = e^{i\theta}z'$ ($\arg z' = \arg z - \theta$, $\arg z = \arg z' + \theta$)

Thus $\boxed{u_r - iu_\theta = e^{i\theta}\frac{dw}{dz}}$ ⇐ very useful

In our example, $\frac{dw}{dz} = U - \frac{Ua^2}{z^2} - \frac{iK}{2\pi z}$

on the cylinder, $|z| = a$, i.e $z = ae^{i\theta}$

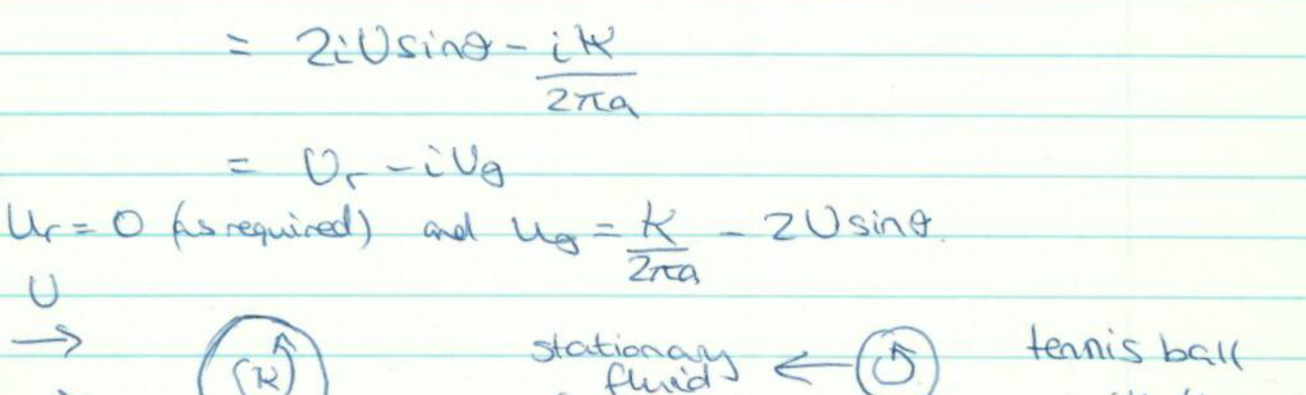
$\frac{dw}{dz} = U - Ue^{-2i\theta} \frac{-iK}{2\pi a} e^{-i\theta}$

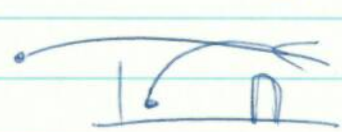
so $e^{i\theta}\frac{dw}{dz} = U(e^{i\theta} - e^{-i\theta}) - \frac{iK}{2\pi a}$

$= 2iU\sin\theta - \frac{iK}{2\pi a}$

$= U_r - iU_\theta$

$U_r = 0$ (as required) and $U_\theta = \frac{K}{2\pi a} - 2U\sin\theta$.





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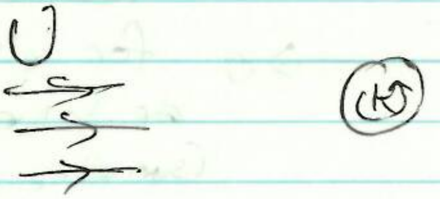
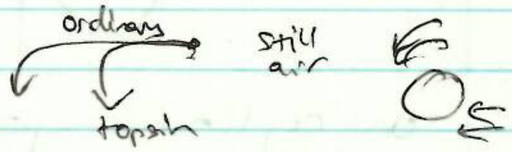


'top-spin' in tennis



$$w = U\left(z + \frac{a^2}{z}\right) - \frac{i\kappa}{2\pi}\log z$$

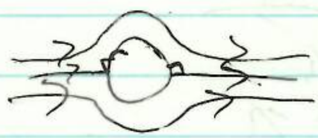
$$\psi = Uy\left(1 - \frac{a^2}{r^2}\right) - \frac{\kappa}{2\pi}\log r$$

$$\phi = Ux\left(1 + \frac{a^2}{r^2}\right) + \frac{\kappa}{2\pi}\theta$$

On $r = a$, $u_r = 0$

$$u_\theta = \frac{\kappa}{2\pi a} - 2U\sin\theta$$

$\kappa = 0$

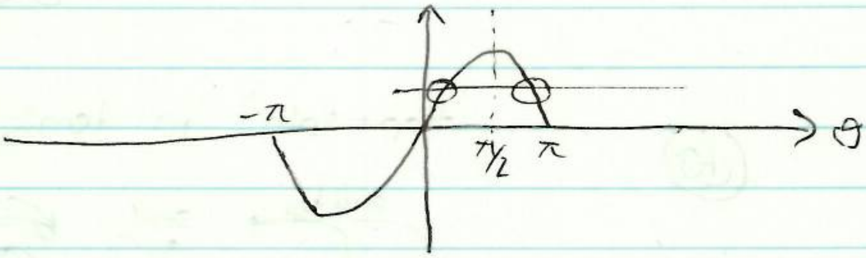


At a stagnation point, $\underline{u} = 0$ or $\frac{dw}{dz} = 0$
or $u = 0, v = 0$ or $u_r = 0, u_\theta = 0$

On the cylinder $r = a$, $u_r = 0 \ \forall \theta$
so stagnation pts where $u_\theta = 0$

$$\frac{\kappa}{2\pi a} = 2U\sin\theta$$

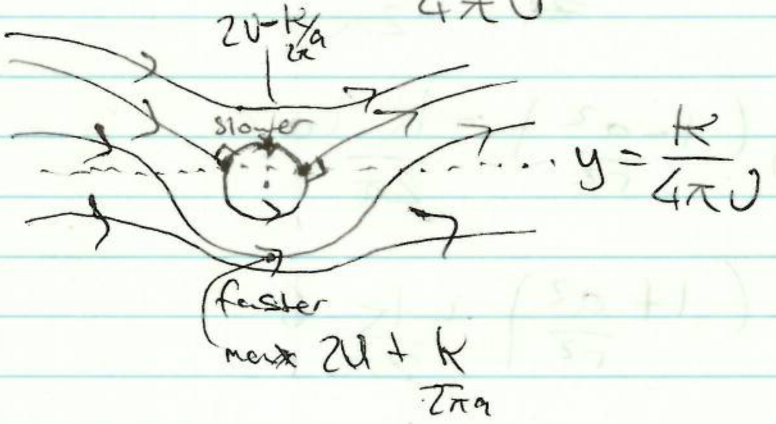
$$\sin\theta = \frac{\kappa}{4\pi a U}$$



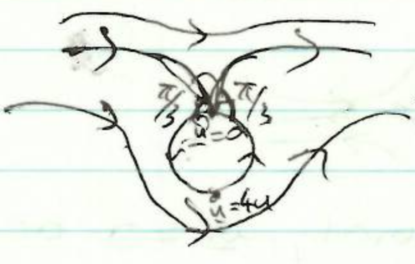
two roots for $x > 0$ these are between 0 and π (symmetric at $\pi/2$)

on cylinder $y = a\sin\theta$
$$= \frac{K}{4\pi U}$$

$$\frac{K}{4\pi U a} < 1$$



$\frac{K}{4\pi U a} = 1$ stag. pts coincide at $y = a$



$\frac{K}{4\pi U a} > 1$ no ~~s~~ roots $\Rightarrow$ no stagnation points on the cylinder

Remember $w = U\left(z + \frac{a^2}{z}\right) - \frac{iK}{2\pi}\log z$

$$\frac{dw}{dz} = U\left(1 - \frac{a^2}{z^2}\right) - \frac{iK}{2\pi z}$$

stag. pts. $\frac{dw}{dz} = 0$

$$U\left(1 - \frac{a^2}{z^2}\right) - \frac{iK}{2\pi} = 0$$

$$\left(\frac{z}{a}\right)^2 - \frac{iK}{2\pi Ua}\left(\frac{z}{a}\right) - 1 = 0 \qquad \left(\text{mult. by } \frac{z^2}{Ua^2}\right)$$

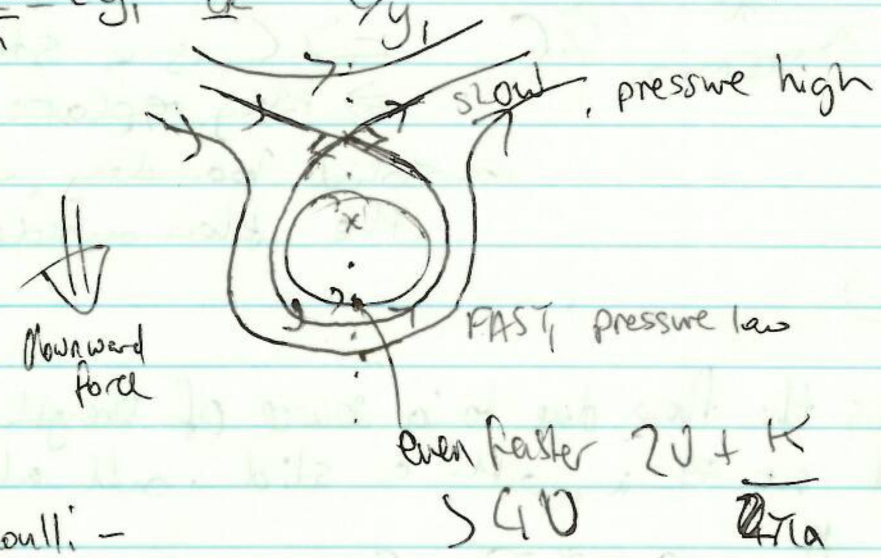
quadratic in $\frac{z}{a}$. 2 roots (product $= -1$)

$$\frac{z}{a} = \frac{iK}{4\pi Ua} \pm \sqrt{1 - \left(\frac{K}{4\pi Ua}\right)^2}$$

$\frac{K}{4\pi Ua} < 1$ ~~complex conjugate~~ (already found) $(z_1, -\bar{z}_1)$

$\frac{K}{4\pi Ua} > 1$, purely imag. roots.
i.e $x = 0$, $y = ay_1$

$\frac{z}{a} = iy_1$ or i/y_1



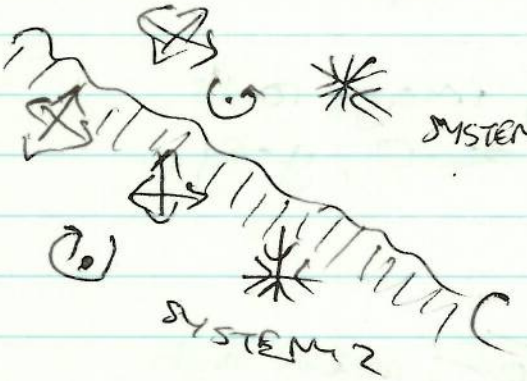
Bernoulli -
KE + pressure = const.

Choose ψ, ϕ then
① In homog.
② choose $\{r^{\pm n}\cos n\theta, r^{\pm n}\sin n\theta, \log r, \theta\}$
with undetermined coeff.
③ BCs give coeff.

The method of images

If the motion of a fluid in the xy plane is due to a distribution of singularities (eg. sources, sinks, vortices etc) and there is a curve C drawn in the plane then the system of singularities on on side of C is called the IMAGE of the system on the other side.

If there is no flow through C

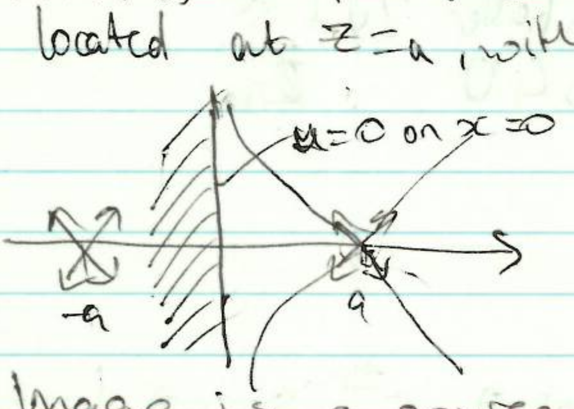


no flow across C
$\Rightarrow$ SYSTEM 1 image of SYSTEM 2

$\Rightarrow$ C is a streamline
$\Rightarrow$ May replace C by a solid boundary, without the flow outside C

Example:

What is the flow due to a source of strength m located at $z=a$, with a solid wall along $x=0$?



Potential due source

$$w_1 = \frac{m}{2\pi} \log(z-a)$$

Image is a source at $z=-a$

$$w_2 = \frac{m}{2\pi} \log(z+a$$

total field. $w = w_1 + w_2 = \frac{m}{2\pi} \log(z-a) + \frac{m}{2\pi} \log(z+a)$

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$\underline{u}(\underline{x}, t)$ velocity field, $\underline{x}$ fixed axes.

incompressibility $\Rightarrow$ $\underline{\nabla} \cdot \underline{u} = 0$ SOLENOIDAL

plus 2D $\Rightarrow$ $\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$

$\Rightarrow \exists \psi$ s.t $\underline{u} = - \hat{\underline{z}} \wedge \underline{\nabla} \psi$

local motion at a point

1) translation of C of M
2) Dilation
3) rotation

Irrotationality persists.

$\underline{\nabla} \wedge \underline{u} = 0 \Rightarrow \exists \phi$ s.t $\underline{u} = \underline{\nabla}$

(true in 3D)

ψ: incomp and 2D

ϕ: irrot.

Irrot, incomp. + 2D : ϕ and ψ $\underline{\nabla} \phi = - \hat{\underline{z}} \wedge \underline{\nabla} \psi$

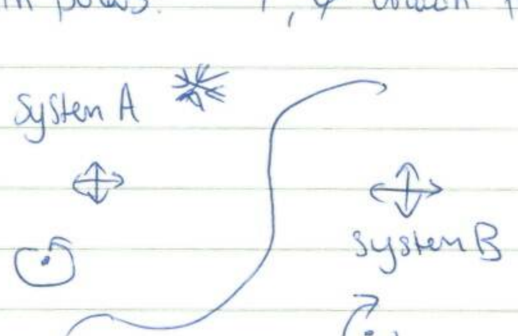
Laurent

$\exists\ \omega(z)$ where $z = x + iy$

$\omega = \phi + i\psi$

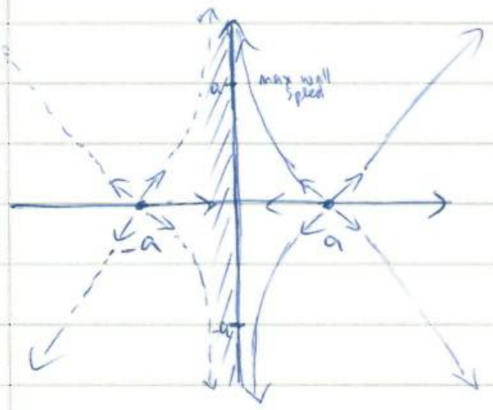
Sum of $z^{\pm n}$

In polars: ψ, ϕ drawn from $\{ r^{\pm n} \cos n\theta, r^{\pm n} \sin n\theta, \log r, \theta$



If no flow across C, then system A is the IMAGE of system B.

Example 1:



original system:

Source strength m at $x=a$

Complex potential $w_1(z) = \frac{m}{2\pi}\log(z-a)$

image system:

Source strength m at $x=-a$

complex potential $w_2(z) = \frac{m}{2\pi}\log(z+a)$

total system = original + image.

$$w(z) = \frac{m}{2\pi}\log(z-a) + \frac{m}{2\pi}\log(z+a)$$

if this is correct then $u=0$ on $x=0$

$$w(z) = \frac{m}{2\pi}\log(z^2-a^2) \Rightarrow \frac{dw}{dz} = \frac{m}{2\pi}\frac{2z}{z^2-a^2}$$

on $x=0$, $(z=iy)$

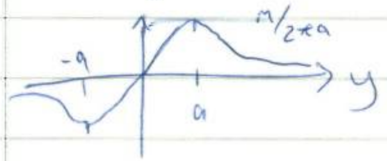
$$u - iv = \frac{dw}{dz}$$

$$= \frac{m}{2\pi}\left(\frac{2iy}{-y^2-a^2}\right)$$

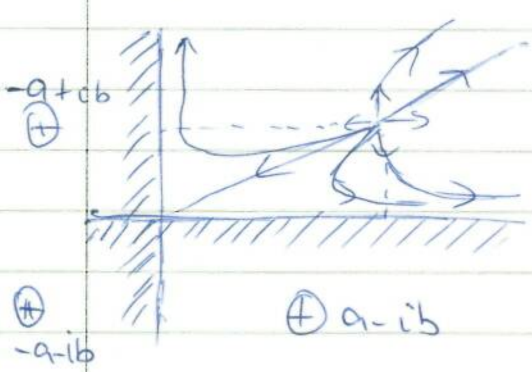
so $u=0$ (as expected)

$$v = \frac{my}{\pi(y^2+a^2)}$$

so maximum speed on wall is $v = \pm\frac{m}{2\pi a}$ when $y = \pm a$



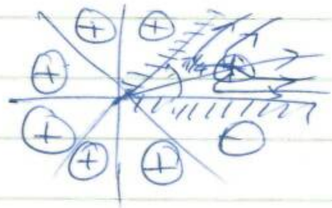
Example 2:



Original: source at $z=a+ib$ (strength m)
in region bdd by $x=0$, $y=0$
with $x>0$, $y>0$

Image system: 3 sources of strength m
at $z=\pm a-ib$, $-a+ib$

Example 3: walls at π/n



eg $n=4$

Example 4: vortex of strength K
at $z=ib$, above a plane $y=0$

complex potential $w_1(z) = -\frac{iK}{2\pi}\log(z-ib)$

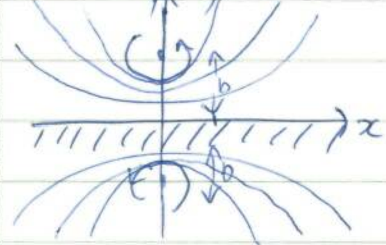


image: vortex of strength $-K$ at $z=-ib$

complex potential $w_2(z) = +\frac{iK}{2\pi}\log(z+ib)$

total system = original + image
$= -\frac{iK}{2\pi}\log(z-ib) + \frac{iK}{2\pi}\log(z+ib)$

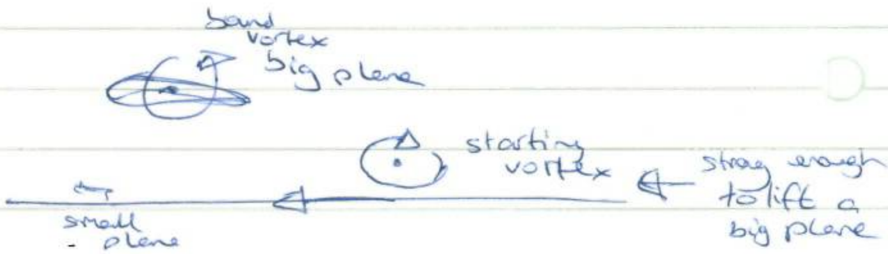
Check: $v=0$ on $y=0$
as expected.

velocity field

$$\frac{dw}{dz} = \frac{-iK}{2\pi}\left(\frac{1}{z-ib}\right) + \frac{iK}{2\pi(z+ib)}$$

At $z=ib$, neglecting first term, which is just the spinning of an isolated vortex about its centre, $\frac{dw}{dz} = \frac{iK}{4\pi ib}$

i.e $u = \frac{K}{4\pi b}$, $v=0$ i.e a free vortex would be driven along parallel to the plane $x=0$ by its image in the plane.



can we only do plane boundaries? <u>No</u>

<u>Circle Theorem</u>: The image system in the circle $|z|=a$ of the complex potential $w(z)=f(z)$ where $f(z)$ has no singularities inside the circle (i.e original system all on one side of line), $|z|<a$ is
$$\bar{f}\left(\frac{a^2}{z}\right),$$ where for any analytic function $g(z)$, $\bar{g}(z)=\overline{g(\bar{z})}$

eg. if $g(z)=\dots+\frac{a_{-2}}{z^2}+\frac{a_{-1}}{z}+a_0+a_1z+a_2z^2+\dots$

$$g(\bar{z})=\dots+a_{-2}\bar{z}^{-2}+a_{-1}\bar{z}^{-1}+a_0+a_1\bar{z}+a_2\bar{z}^2+\dots$$

$$\bar{g}(z)=\dots+\bar{a}_{-2}z^{-2}+\bar{a}_{-1}z^{-1}+\bar{a}_0+\bar{a}_1z+\bar{a}_2z+\dots$$

still an analytic function of z.

$\bar{f}\left(\frac{a^2}{z}\right)$ is an analytic function of $\frac{a^2}{z}$

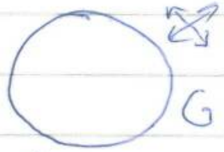
now f has no singularities in $|z|<a$

~~$\frac{|z|}{a^2}<\frac{1}{a}$ i.e $\left|\frac{a^2}{z}\right|>a$~~

so $f\left(\frac{a^2}{z}\right)$ has no singularities in $|z|>a$

since if $|z|>a$, then $\frac{|z|}{a^2}>\frac{1}{a}$, so $\frac{a^2}{|z|}<a$

similarly for $\bar{f}\left(\frac{a^2}{z}\right)$



$f(z)$
sings outside C

$\bar{f}\left(\frac{a^2}{z}\right)$
sings inside C

Check for complete potential: $w(z) = f(z) + \bar{f}\left(\frac{a^2}{z}\right)$ that there is no flow through $|z| = a$

why $\frac{a^2}{z}$ as argument of $\bar{f}$?

on the circle $|z| = a$, i.e $z|z| = a^2$, i.e. $\frac{a^2}{z} = \bar{z}$
i.e $\frac{a^2}{z}$ is analytic (except $z=0$) but equal to $\bar{z}$ on C.

The general problem is 'find an analytic function of z (possibly with singularities) which equals $\bar{z}$ on some curve C. - this is the Schwarz function for C.

On $|z| = a$, $\frac{a^2}{z} = \bar{z}$ so $\bar{f}\left(\frac{a^2}{z}\right) = \bar{f}(\bar{z})$
$= \overline{f(\bar{\bar{z}})} = \overline{f(z)}$ on C.

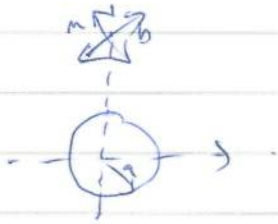
combine potentials
$$w(z) = f(z) + \bar{f}\left(\frac{a^2}{z}\right)$$

on C, $w(z) = f(z) + \overline{f(z)} = 2\,\mathrm{Re}\{f(z)\}$
So $\mathrm{Im}\{w(s)\} = 0$ on $|z| = a$ i.e $\psi = 0$ on $|z| =$
i.e circle is a streamline, as required (no flow across $|z| = a$).

so $\bar{f}\left(\frac{a^2}{z}\right)$ is image of $f(z)$ in $|z| = a$

find Schwarz function $h(z)$ for C
$h(z) = \bar{z}$
so image of $f(z)$ is $\bar{f}(h(z))$

Example: Find the image system and the total complex potential for a ~~sour~~ source of strength m at $z = ib$ outside the cylinder $|z| = a$, where $b > a$



Original system $\quad f_1(z) = \frac{m}{2\pi} \log(z - ib)$

$$f_2(z) = \overline{f_1\left(\frac{a^2}{\bar{z}}\right)}$$
$$= \frac{m}{2\pi} \log\left(\overline{\left(\frac{a^2}{\bar{z}}\right)} - ib\right)$$
$$= \frac{m}{2\pi} \log\left(\frac{a^2}{z} + ib\right)$$

Total potential:

$$w(z) = f_1(z) + f_2(z)$$

$$w(z) = \frac{m}{2\pi} \log(z - ib) + \underbrace{\frac{m}{2\pi} \log\left(\frac{a^2}{z} + ib\right)}_{\text{what is this?}}$$

$$\text{Image system} = \frac{m}{2\pi} \log\left(\frac{a^2}{z} + ib\right) = -\frac{m}{2\pi} \log z + \frac{m}{2\pi} \log(a^2 + ibz)$$

$$= -\frac{m}{2\pi} \log z + \frac{m}{2\pi} \log(ib) + \frac{m}{2\pi} \log\left(z - \frac{ia^2}{b}\right)$$

sink, strength m at origin

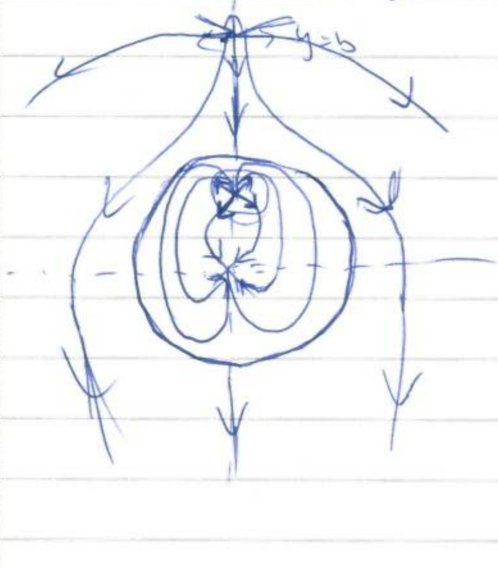
arbitrary constant - no effect

source strength m at $z = \frac{ia^2}{b}$

optical image pt

ie $x = 0 \quad y = \frac{a^2}{b}$

$b > a$

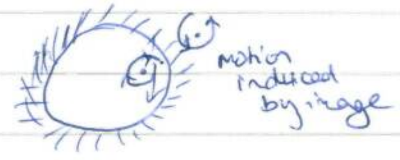


$\frac{a^2}{b} < a \quad (b > a)$

guaranteed.

no surprise as no flow across C.

Example 2 Vortex in coffee cup.



Equations of Motion

$\underline{F} = m\underline{a}$ $\qquad \underline{F} = \dfrac{d}{dt}(m\underline{v})$

rate of change of momentum of particle, following the particle.

* We need to define rate of change following a particle for a fluid.

Suppose we have some field, known for all time t, and positions $\underline{r}$ $\phi(t, \underline{r})$.

Now suppose we follow some particle whose path is given by $\dfrac{d\underline{r}}{dt} = \underline{u}$

then the values of ϕ along the particle path are $\phi(t, \underline{r}(t))$ where $\dfrac{d\underline{r}}{dt} = \underline{u}$, a function of t alone.

What is the rate of change of ϕ along this path

$$\frac{D\phi}{Dt} = \left.\frac{d\phi}{dt}\right|_{\underline{x}=\underline{r}(t)} = \frac{\partial \phi}{\partial t} + \frac{\partial \phi}{\partial x}\frac{dx}{dt} + \frac{\partial \phi}{\partial y}\frac{dy}{dt} + \frac{\partial \phi}{\partial z}\frac{dz}{dt}$$

i.e $\dfrac{D\phi}{Dt} = \dfrac{\partial \phi}{\partial t} + u\dfrac{\partial \phi}{\partial x} + v\dfrac{\partial \phi}{\partial y} + w\dfrac{\partial \phi}{\partial z}$ Chain rule. $\left(\dfrac{d\underline{r}}{dt} = \underline{u}\right)$

$$= \frac{\partial \phi}{\partial t} + (u\hat{\underline{i}} + v\hat{\underline{j}} + w\hat{\underline{k}})\left(\frac{\partial \phi}{\partial x}\hat{\underline{i}} + \frac{\partial \phi}{\partial y}\hat{\underline{j}} + \frac{\partial \phi}{\partial z}\hat{\underline{k}}\right)$$

$$\frac{D\phi}{Dt} = \frac{\partial \phi}{\partial t} + \underline{u} \cdot \nabla \phi$$

speed at which move through gradient.

$$\frac{D\phi}{Dt} = \left(\frac{\partial}{\partial t} + \underline{u} \cdot \nabla\right) \phi$$

CONVECTIVE or ADVECTIVE derivative

Examples

Ex 1 $\phi = x$ $\quad \frac{Dx}{Dt} = \frac{\partial x}{\partial t} + u\frac{\partial x}{\partial x} + v\frac{\partial x}{\partial y} + w\frac{\partial x}{\partial z}$

$= u$

Ex 2: $\phi = \underline{r}$

$$\frac{D\underline{r}}{Dt} = \frac{\partial}{\partial t}(x\underline{\hat{i}} + y\underline{\hat{j}} + z\underline{\hat{k}}) + u\frac{d}{dx}(\quad) + v\frac{d}{dy}(\quad) + w\frac{d}{dz}(\quad)$$

$$= 0 + u\underline{\hat{i}} + v\underline{\hat{j}} + w\underline{\hat{k}}. = \underline{u}$$

Ex 3: $\phi = u$: $\quad \frac{D\underline{u}}{Dt} = \frac{\partial \underline{u}}{\partial t} + (u \cdot \underline{\nabla})\,\underline{u}$

$$\frac{d}{dt}\int_{a(t)}^{b(t)} f(t,x)\,dx = \int_a^b \frac{\partial f}{\partial t}\,dx + f(t,b)\,b'(t) - f(t,a)\,a'(t)$$

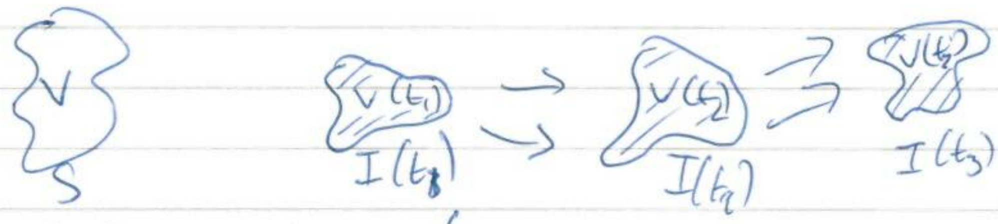
Liebnitz

2301 17/11/11

Reynolds' Transport Theorem: (RTT)

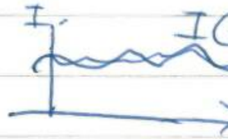
Why: eventually we want to apply Newton's laws to a fluid. i.e $\frac{d}{dt}$ (Momentum) = force.

Consider a quantity $\alpha(\underline{r}, t)$ associated with a fluid. Let the fluid occupy a domain D and have the specified velocity field $\underline{u}(\underline{r}, t)$.
Consider a subvolume V contained in D with a surface S. We take V to consist always of the same fluid elements or particles. Thus V moves.
i.e. $V = V(t)$



We define $I(t) = \int_{V(t)} \alpha(\underline{r}, t)\, d\underline{r}$

ie the total amount of α in V at any time.



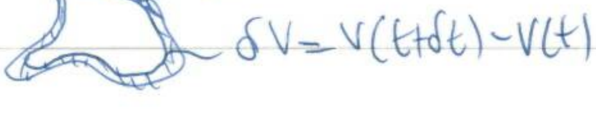
Reynolds: What is $\frac{dI}{dt}$?

$$\frac{dI}{dt} \text{ or } \frac{DI}{Dt} = \lim_{\delta t \to 0} \frac{I(t+\delta t) - I(t)}{\delta t}$$

emphasises we are following particles?

here $I(t+\delta t) = \int \alpha(\underline{r}, t+\delta t)\, d\underline{r}$

here $V(t+\delta t)$ is the volume position at an interval δt after t



~~and~~ by Taylor's theorem, ~~$\alpha(\underline{r}, t+\delta t) =$~~

$$\alpha(\underline{r}, t+\delta t) = \alpha(\underline{r}, t) + \delta t \frac{\partial \alpha}{\partial t}(\underline{r}, t) + \frac{1}{2}(\delta t)^2 \frac{\partial^2 \alpha}{\partial t^2}(\underline{r}, t)$$

$$\text{so } I(t+\delta t) = \int_{V+\delta V} \left[\alpha(r,t) + \delta t \frac{\partial \alpha}{\partial t}(r,t)\right] d\underline{r}$$

$$+ \frac{1}{2}(\delta t)^2 \int_{V+\delta V} \frac{\partial^2 \alpha}{\partial t^2}(\underline{r}, \tau)\, d\underline{r}$$

$$= \underbrace{\int_{V+\delta V} \alpha\, dV + \delta t \int_{V+\delta V} \frac{\partial \alpha}{\partial t}\, dV}_{\text{argument is } (\underline{r}, t)} + O((\delta t)^2)$$

$$\text{now } \frac{DI}{Dt} = \lim_{\delta t \to 0} \left\{ \frac{1}{\delta t} \left[I(t+\delta t) - I(t) \right] \right\}$$

$$= \lim_{\delta t \to 0} \left\{ \frac{1}{\delta t} \left[\int_V \alpha\, dV + \int_{\delta V} \alpha\, dV + \delta t \int_V \frac{\partial \alpha}{\partial t}\, dV + \delta t \int_{\delta V} \frac{\partial \alpha}{\partial t}\, dV + O((\delta t)^2) - \int_V \alpha\, dV \right] \right\}$$

$$\frac{DI}{Dt} = \lim_{\delta t \to 0} \left\{ \frac{1}{\delta t} \int_{\delta V} \alpha\, dV + \int_V \frac{\partial \alpha}{\partial t}\, dV + \underline{\int \frac{\partial \alpha}{\partial t}\, dV} + O(\delta t) \right\}$$

the underlined term is

$$\int_{\delta V} \frac{\partial \alpha}{\partial t}\, dV \leq \left| \int_{\delta V} \frac{\partial \alpha}{\partial t}\, dV \right|$$

$$\leq \int_{\delta V} \left| \frac{\partial \alpha}{\partial t} \right| dV$$

$$\leq \int_{\delta V} A\, dV \qquad A = \max \left| \frac{\partial \alpha}{\partial t} \right|$$

$$= A\int_{\delta V} dV = A|\delta V| \to 0 \text{ at } \delta t \to 0$$

$$\text{Thus } \frac{DI}{Dt} = \int_V \frac{\partial \alpha}{\partial t}\, dV + \lim_{\delta t \to 0} \frac{1}{\delta t}\int_{\delta V} \alpha\, dV$$

Particles making up dS have moved a distance $\underline{u}\,\delta t$ in time δt

they sweep out a volume Area base $\times$ height dS $h = (\underline{u}\cdot\hat{\underline{n}})$

i.e $dV \sim (\underline{u}\cdot\hat{\underline{n}})\,\delta t\, dS$

$$\text{Thus } \int_{dV} \alpha\, dV = \int_S \alpha\, \underline{u}\cdot\hat{\underline{n}}\, dS$$

$$\text{Thus } \frac{DI}{Dt} = \int_V \frac{d\alpha}{dt}\, dV + \int_S \alpha\, (\underline{u}\cdot\hat{\underline{n}})\, dS$$

First form of RTT,

$$\text{RTT1: } \frac{D}{Dt}\left(\int_V \alpha\, dV\right) = \underbrace{\int_V \frac{\partial \alpha}{\partial t}\, dV}_{\text{local r.o.ch.}} + \int_S \alpha(\underline{u}\cdot\hat{\underline{n}})\, dS$$

flux of α through boundary of V

(The 3D version of Leibnitz rule for diff. under integral sign

$$\frac{d}{dt}\int_{a(t)}^{b(t)} \alpha(t,x)\, dx = \int_a^b \frac{\partial \alpha}{\partial t}\, dx + b'(t)\alpha(t,b) - a'(t)\alpha(t,a)$$

- 1D Leibnitz

but the divergence theorem says that for any vector $\underline{F}$

$$\int_S \underline{F} \cdot \hat{\underline{n}} \, dS = \int_V (\underline{\nabla} \cdot \underline{F}) \, dV$$

Putting $\underline{F} = \alpha \underline{u}$,

$$\frac{D}{Dt}\left(\int_V \alpha \, dV\right) = \int_V \left[\frac{\partial \alpha}{\partial t} + \underline{\nabla} \cdot (\alpha \underline{u})\right] dV$$ RRT2

now $\underline{\nabla} \cdot (\alpha \underline{u}) = (\underline{u} \cdot \underline{\nabla})\alpha + \alpha \underline{\nabla} \cdot \underline{u}$

so $$\frac{\partial \alpha}{\partial t} + \underline{\nabla} \cdot (\alpha \underline{u}) = \frac{\partial \alpha}{\partial t} + (\underline{u} \cdot \underline{\nabla})\alpha + \alpha \underline{\nabla} \cdot \underline{u}$$

$$= \frac{D\alpha}{Dt} + \alpha \underline{\nabla} \cdot \underline{u}$$

Thus $$\frac{D}{Dt}\left(\int_V \alpha \, dV\right) = \int_V \left[\frac{D\alpha}{Dt} + \alpha \underline{\nabla} \cdot \underline{u}\right] dV$$

RTT3.

Example: take $\alpha = \rho$, density.
then $M = \int_V \rho \, dV$ is the mass of particles making up the volume

then by RTT2, $$\frac{DM}{Dt} = \frac{D}{Dt}\int_V \rho \, dV$$

$$= \int_V \left(\frac{\partial \rho}{\partial t} + \underline{\nabla} \cdot (\rho \underline{u})\right) dV$$

but mass is conserved so $\frac{DM}{Dt} = 0$

i.e for any V in D

$$\int_V \left[\frac{\partial \rho}{\partial t} + \underline{\nabla} \cdot (\rho \underline{u})\right] dV = 0$$

by our lemma, this implies

$$\frac{\partial \rho}{\partial t} + \underline{\nabla} \cdot (\rho \underline{u}) = 0 \quad \text{everywhere in } D$$

- conservation of mass for a compressible fluid.
- contains $\underline{\nabla} \cdot \underline{u} = 0$ when ρ is constant.

2301 21/11/11

RTT: 3D version of Leibnitz

RTT1: $\frac{D}{Dt}\int_V \alpha \, dV = \int_V \frac{\partial \alpha}{\partial t} dV + \int_S \alpha \underline{u}\cdot\hat{\underline{n}} \, dS$

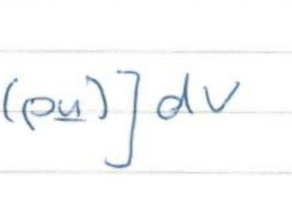
RTT2: $\frac{D}{Dt}\int_V \alpha \, dV = \int_V \left[\frac{\partial \alpha}{\partial t} + \underline{\nabla}\cdot(\alpha \underline{u})\right] dV$

RTT3: $\frac{D}{Dt}\int_V \alpha \, dV = \int_V \left(\frac{D\alpha}{Dt} + \alpha \underline{\nabla}\cdot\underline{u}\right) dV$

any scalar $\alpha(\underline{x}, t)$

Example: Conservation of mass. ($\alpha = \rho$, density)

consider a fluid of variable density, $\rho(\underline{x}, t)$ that occupies a domain D. let V be any subdomain of D. [i.e V must be arbitrary]. Consider the total mass of all the particles comprising V, i.e
$M = \int_V \rho \, dV$



The rate of change of mass M, staying with the same particles, must be 0 (conservation of mass). ie. $\frac{DM}{Dt} = 0$

but by RTT2, $\frac{D}{Dt}\int_V \rho \, dV = \int_V \left[\frac{\partial \rho}{\partial t} + \underline{\nabla}\cdot(\rho \underline{u})\right] dV$

so $\int_V \left[\frac{\partial \rho}{\partial t} + \underline{\nabla}\cdot(\rho \underline{u})\right] dV = 0$

but V is arbitrary, so this is true for all V.
hence by our theorem,

$$\frac{\partial \rho}{\partial t} + \underline{\nabla} \cdot (\rho \underline{u}) = 0 \quad \text{everywhere in } D$$

this can also be written $\frac{D\rho}{Dt} + \rho \underline{\nabla} \cdot \underline{u} = 0$

· notice, if the flow is incompressible, fluid elements cannot be squashed, i.e they preserve their volume. But they preserve their mass, hence they preserve their density, i.e $\frac{D\rho}{Dt} = 0$

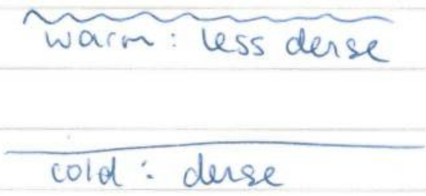
By conservation of mass,

$$\rho \underline{\nabla} \cdot \underline{u} = 0$$

$$\text{i.e } \underline{\nabla} \cdot \underline{u} = 0 \quad \text{(as before)}$$

[notice this does not require all particles to have the same density]

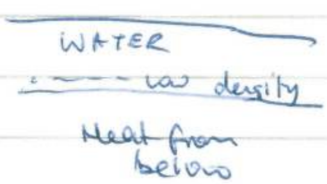
eg. ocean



incompressible
not constant density
fluid particles retain density

eg. Antarctic Bottom Water.

Bernard convection



RTT4: For a fluid of density $\rho(\underline{x}, t)$ consider any quantity $f(\underline{x}, t)$. Put $\alpha = \rho f$ in RTT3.

$$\frac{D}{Dt}\int_V f\rho \, dV = \int_V \left[\frac{\partial}{\partial t}(f\rho) + \underline{\nabla}\cdot(f\rho\underline{u})\right] dV$$

$$\begin{aligned}[\ ] &= \rho\frac{\partial f}{\partial t} + f\frac{\partial \rho}{\partial t} + f\rho\underline{\nabla}\cdot\underline{u} + \underline{u}\cdot\underline{\nabla}(f\rho) \\ &= f\left(\frac{\partial \rho}{\partial t} + \rho\underline{\nabla}\cdot\underline{u}\right) + \rho\frac{\partial f}{\partial t} + f\underline{u}\cdot\underline{\nabla}\rho + \rho\underline{u}\cdot\underline{\nabla}f \\ &= f\underbrace{\left[\frac{\partial \rho}{\partial t} + \underline{\nabla}\cdot(\rho\underline{u})\right]}_{\text{zero by conserv. of mass}} + \rho\left[\frac{\partial f}{\partial t} + (\underline{u}\cdot\underline{\nabla})f\right] \\ &= \rho\frac{Df}{Dt}\end{aligned}$$

ie. RTT4: $\displaystyle \frac{D}{Dt}\int_V \rho f \, dV = \int_V \rho\frac{Df}{Dt} \, dV$

or $\displaystyle \frac{D}{Dt}\int_V f\rho \, dV = \int_V \frac{Df}{Dt}\rho \, dV$

[i.e $\displaystyle \frac{D}{Dt}\int_V f \, dM = \int_V \frac{Df}{Dt} \, dM$ ie integrating w.r.t mass]

<u>Example 2</u>: Force = R.o.c.h. of momentum.
rate of change
i.e "set $\alpha = \rho\underline{u}$"

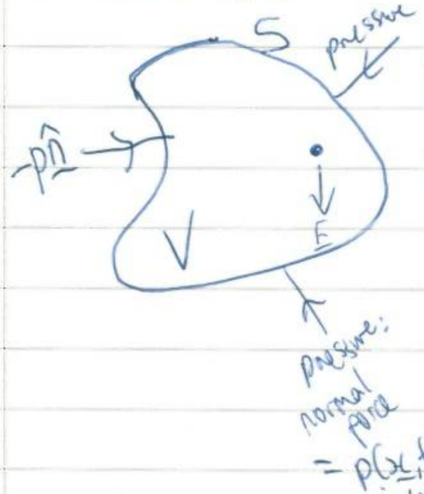
consider a fluid of density $\rho(x,t)$, occupying a domain D.
let V be any subdomain, with surface S, of D
(i.e important V is arbitrary, for our theorem)

Consider $\underset{\sim}{M} = \int_V \rho\underline{u} \, dV$, the total momentum of the fluid particles making up V

Then, following these particles, by RTT4,

$$\frac{D\underline{m}}{Dt} = \frac{D}{Dt}\int_V \rho \underline{u}\, dV = \int_V \rho \underbrace{\frac{D\underline{u}}{Dt}}_{\text{acceleration}} dV$$

by Newton, this must equal the total force (external) acting on the particles comprising V (the internal forces sum to 0)



let each particle be subject to an external force $\underline{F}$ per unit mass. (eg. gravity, $\underline{F} = -g\hat{\underline{z}}$ or magnetic force, or electric force) or a 'fictitious' force, eg Coriolis.

That is all for an inviscid fluid as elements cannot exert a shear (ie tangential) stress. [extra term in a viscous fluid].

Thus the total force on all particles comprising V is

$$\int_V \rho \underline{F}\, dV + \int_S (-p)\hat{\underline{n}}\, dS$$

$$= \int_V \rho \underline{F}\, dV + \int_V (-\underline{\nabla} p)\, dV \quad \text{by vector form of divergence theorem.}$$

$$= \int_V (-\nabla p + \rho \underline{f})\, dV$$

R.o.ch momentum = force acting.

$$\int_V \rho \frac{D\underline{u}}{Dt}\, dV = \int_V (-\nabla p + \rho \underline{f})\, dV$$

i.e. $\int_V \left(\rho \frac{D\underline{u}}{Dt} + \underline{\nabla} p - \rho \underline{F}\right) dV = 0$

but V is arbitrary, so this is true for all V, hence by our theorem,

$$\rho \frac{D\underline{u}}{Dt} + \underline{\nabla} p - \rho \underline{F} = 0 \quad \text{everywhere in } D$$

This is Euler's equation for an inviscid fluid,

$$\rho \frac{D\underline{u}}{Dt} = -\underline{\nabla} p + \rho \underline{F}$$

mass x acceleration = force.

equations of motion for a (possibly compressible) fluid. density $\rho(\underline{x},t)$, pressure $p(x,t)$, velocity $\underline{u}(x,t)$

mass: $\frac{\partial \rho}{\partial t} + \underline{\nabla}\cdot(\rho \underline{u}) = 0$

Euler: $\rho \frac{D\underline{u}}{Dt} = -\underline{\nabla} p + \rho \underline{F}$

2 unknown scalar fields (ρ, p) and one unknown vector field $(\underline{u})$. We have 1 scalar and 1 vector equation. Missing a scalar equa

Geophysical fluid dynamics: Atmospheres and Oceans.

Incompressibility: $\frac{D\rho}{Dt} = 0$ mass: $\underline{\nabla}\cdot\underline{u} = 0$

Euler: $\rho \frac{D\underline{u}}{Dt} = -\nabla p + \rho \underline{F}$

Gas Dynamics (Cosmology)

$~~\rho~~$ $\rho = f(p)$

for an ideal gas $p = \rho^{\alpha}$ for some α - second scalar eqn.

we will continue by taking the density to be constant, i.e all particles have the same density. Then

mass $\simeq$ incompressibility = 'continuity' : $\underline{\nabla} \cdot \underline{u} = 0$

Euler : $\rho \dfrac{D\underline{u}}{Dt} = -\underline{\nabla} p + \rho \underline{F}$

1 scalar unknown: P , 1 vector unknown: $\underline{u}$, 1 scalar eqn
1 vector eqn. ρ given constant.

next: examples using this

open channel flow: hydraulics (need gravity)
surface water waves (need gravity).

<u>Example</u>: Find the free surface shape for a cylindrical container partially filled with fluid of constant density ρ in solid body rotation with angular speed Ω about a vertical axis

let the flow have settled to a steady state. Then
$$\frac{\partial}{\partial t} \equiv 0$$

dV
gravity
$\underline{F} = -g\hat{\underline{z}}$
force per unit mass

Continuity: $\underline{\nabla} \cdot \underline{u} = 0$
Euler: $\rho \dfrac{D\underline{u}}{Dt} = -\underline{\nabla} p + \rho \underline{F}$

$$\frac{D\underline{u}}{Dt} = \frac{\partial \underline{u}}{dt} + (\underline{u} \cdot \underline{\nabla})\,\underline{u}$$

Euler becomes: $\rho(\underline{u} \cdot \underline{\nabla})\,\underline{u} = -\underline{\nabla} p - \rho g \hat{\underline{z}}$

we are told that the fluid is in solid body rotation, i.e
$$\underline{u} = \underline{\Omega} \wedge \underline{r}$$

$$\underline{u} = \begin{vmatrix} \hat{\underline{i}} & \hat{\underline{j}} & \hat{\underline{k}} \\ 0 & 0 & \Omega \\ x & y & z \end{vmatrix} = -\Omega y \hat{\underline{i}} + \Omega x \hat{\underline{j}}$$ ~~ke~~ $u = -\Omega y$, $v = \Omega x$

$$\underline{u} \cdot \underline{\nabla} = u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + w\frac{\partial}{\partial z} = -\Omega y\frac{\partial}{\partial x} + \Omega x\frac{\partial}{\partial y}$$

so $(\underline{u} \cdot \underline{\nabla})u = \left(-\Omega y\frac{\partial}{\partial x} + \Omega x\frac{\partial}{\partial y}\right)(-\Omega y) = -\Omega^2 x$

$(\underline{u} \cdot \underline{\nabla})v = \left(-\Omega y\frac{\partial}{\partial x} + \Omega x\frac{\partial}{\partial y}\right)(\Omega x) = -\Omega^2 y$

Euler (in components):

x-momentum: $-\rho\Omega^2 x = -\dfrac{\partial p}{\partial x}$ ①

y-momentum: $-\rho\Omega^2 y = -\dfrac{\partial p}{\partial y}$ ②

z-momentum: $0 = -\dfrac{\partial p}{\partial z} - \rho g$ ③

integrate ①: $p = \frac{1}{2}\rho\Omega^2 x^2 + f(y,z)$

diff w.r.t y: $p_y = f_y$ compare with ②, $f_y = \rho\Omega^2 y$

i.e $f = \frac{1}{2}\rho\Omega^2 y^2 + h(z$

i.e $p = \frac{1}{2}\rho\Omega^2(x^2+y^2)$

diff w.r.t z: $p_z = h_z$ compare with ③ $h_z = -\rho g$

i.e ~~gz~~ $h = -\rho g z + \text{const}$

ie. $p = \frac{1}{2}\rho\Omega^2(x^2+y^2) - \rho g z + C$.

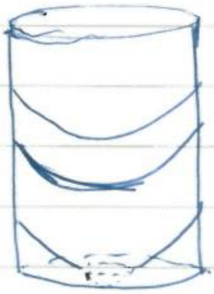
isobars = lines of constant pressure, isobaric surface = surface of constant pre

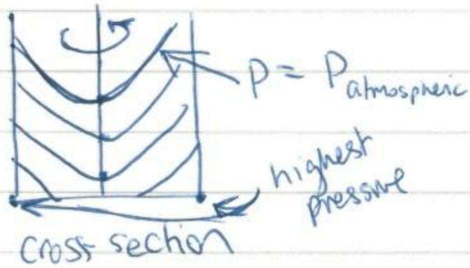
$p = \text{const}$. eg. $p = A$, const.

$$\rho g z = \frac{1}{2}\rho\Omega^2(x^2+y^2) + C - A$$

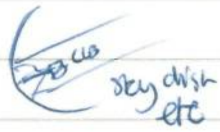
i.e $z = \dfrac{1}{2}\dfrac{\Omega^2}{g}(x^2+y^2) + \dfrac{C-A}{\rho g}$ z_0

$$z - z_0 = \frac{1}{2}\frac{\Omega^2}{g}(x^2+y^2)$$ – paraboloids with origin $(0, 0, z_0)$



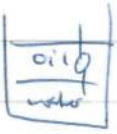


Peter Rhines
U of W. Seattle.



Archimedes?

Example 2: Consider a submerged body of volume V with surface S. Then the force on the body is upwards and equal to the weight of water displaced.



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Hydrostatic pressure.

Cty: $\nabla \cdot \underline{u} = 0$

Euler: $\rho \frac{D\underline{u}}{Dt} = -\underline{\nabla}p + \rho\underline{f}$

when gravity is the only external force $\underline{f} = -g\hat{\underline{z}}$

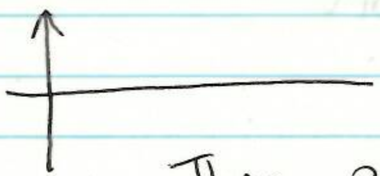
If the fluid is at rest, $\underline{u} = 0$

$$0 = -\underline{\nabla}p + \rho g \hat{\underline{z}}$$

i.e. $\frac{\partial p}{\partial x} = 0$, $\frac{\partial p}{\partial y} = 0$, $\frac{\partial p}{\partial z} = -\rho g$

i.e. $p = p(y,z)$ so $p = p(z)$

i.e. $p = -\rho g z + \text{const}$



$z=0$: surface where pressure is atmospheric i.e $p = p_a$

Then $p = p_a - \rho g z$

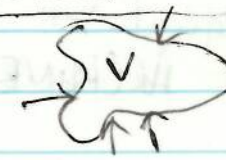
we call this HYDROSTATIC PRESSURE

weight of water above dS is
$(z\,dS)\rho g$

force per unit area $= \rho g z$

ie hydrostatic pressure supports water above

Example



$-p\hat{\underline{n}}$ consider a submerged body occupying a volume V with surface S, immersed in a fluid of density ρ

The force on the body is

$$\mathcal{F} = \int_S -p\hat{\underline{n}}\, dS$$

$$= -\int_V \underline{\nabla} p \, dV \quad \text{by divergence theorem}$$

here p is the pressure in the fluid surrounding V. But fluid at rest, so $p = p_H = p_a - \rho g z$

so $\underline{\nabla} p = \underline{\nabla} p_H = -\rho g \hat{\underline{z}}$

so $$\mathcal{F} = -\int_V (-\rho g \hat{\underline{z}})\, dV$$

$$= \rho g \hat{\underline{z}} \int_V dV$$

$$= \underline{\rho V}\, g \hat{z}$$

density of fluid

not mass of body

ρV: mass of fluid displaced
$\rho V g$: weight of fluid displaced
$\rho V g \hat{\underline{z}}$: an upward force equal to the weight of fluid displaced - ARCHIMEDES.

For a moving fluid, it is often convenient to split the pressure into hydrostatic and the rest, called dynamic pressure, i.e write $p = p_H + p_d$

Then the Euler equations under gravity become

$$\rho \frac{D\underline{u}}{Dt} = -\underline{\nabla} p - \rho g \hat{\underline{z}}$$

$$= -\nabla p_H - \underline{\nabla} p_d - \rho g \hat{\underline{z}}$$

$$= -(-\rho g \hat{\underline{z}}) - \underline{\nabla} p_d - \rho g \hat{\underline{z}}$$

$$= -\underline{\nabla}\, p_d$$

i.e we can ignore gravity in the equation of motion, provided we measure pressure as the deviation from hydrostatic.

This is not useful when a free surface is present since the bc there $p = p_a$ is on the initial pressure $p = p_H + p_d$

Bernoulli's Equation

we have the identity

$$(\underline{u} \cdot \underline{\nabla})\underline{u} = \underline{\nabla}\left(\tfrac{1}{2}\underline{u}^2\right) + \underline{\omega} \wedge \underline{u}$$

where $\omega = \underline{\nabla} \wedge \underline{u}$ is the vorticity

Thus $$\frac{D\underline{u}}{Dt} = \frac{\partial \underline{u}}{\partial t} + (\underline{u} \cdot \underline{\nabla})\underline{u} = \frac{\partial \underline{u}}{\partial t} + \underline{\nabla}\left(\tfrac{1}{2}\underline{u}^2\right) + \underline{\omega} \wedge \underline{u}$$

now Euler is $$\rho \frac{D\underline{u}}{Dt} = -\underline{\nabla} p + \rho \underline{F}$$

now consider only external forces derivable from a potential i.e $\underline{F} = -\underline{\nabla} V_e$

i.e for gravity $V_e = gz$

i.e. $\underline{f}$ is conservative

Then $\rho\left[\frac{\partial \underline{u}}{\partial t} + \underline{\omega} \wedge \underline{u}\right] = -\underline{\nabla} p - \rho \underline{\nabla}\left(\tfrac{1}{2}\underline{u}^2\right) - \rho \underline{\nabla} V_e$

$$= -\underline{\nabla} H \quad (*)$$

where $H = p + \frac{1}{2}\rho \underline{u}^2 + \rho V_e$

In steady flow, $\frac{\partial \underline{u}}{\partial t} = 0$ so

$$\rho\, \underline{\omega} \wedge \underline{u} = -\underline{\nabla} H$$

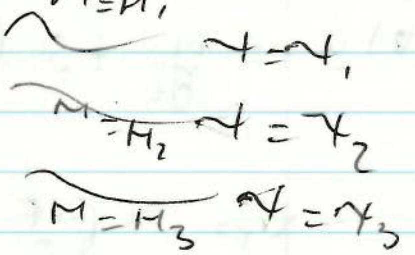
dot with $\underline{u}$: $\quad 0 = (\underline{u} \cdot \underline{\nabla}) H$

ie. $\frac{DH}{Dt} = 0$ i.e H is constant following a particle.

But flow is steady ∴ ~~PPS~~ particle paths are straight lines

Thus H is constant on streamlines

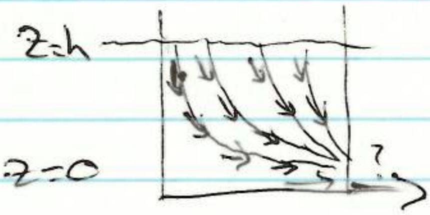
Bernoulli's theorem



H can have different values on different s. lines

H sometimes, incorrectly called Bernoulli's constant

Example:



Important: the surface is connected to ~~the~~ the exit by streamlines

Assume, hole is sufficiently small that the flow is steady. ~~Hence~~

Hence Bernoulli applies,

on any streamline

$$H = \text{const} \quad (\text{not necessarily same const})$$

here $$H = p + \frac{1}{2}\rho \underline{u}^2 + \rho V_e$$

the external potential is $V_e = gz$
(with zero taken at level of hole)

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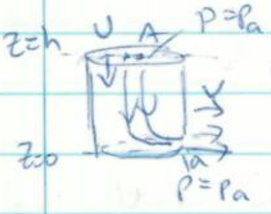
Bernoulli Examples.

Example 1: Draining cylinders.



streamline connecting surface to exit.
∴ can apply Bernoulli on this streamline.
i.e $p + \frac{1}{2}\rho u^2 + \rho V_e$ is constant on streamline. Hence same at top and bottom.

Now $V_e = gz$ ($z = 0$ at level of exit)
pressure is atmospheric at top and bottom.
At top $p + \frac{1}{2}\rho \underline{u}^2 + \rho V_e = p_a + \frac{1}{2}\rho U^2 + \rho g h$

where $u = U$ at surface

At bottom $p + \frac{1}{2}\rho \underline{u}^2 + \rho V_e = p_a + \frac{1}{2}\rho v^2 + 0$

where $u = v$ at exit.

Thus

$$p_a + \frac{1}{2}\rho U^2 + \rho g h = p_a + \frac{1}{2}\rho v^2$$

$$\text{ie. } v^2 = U^2 + 2gh$$

The mass flux is conserved. Let surface area at the top be A and the surface area at the bottom be a.
Then the mass flux at top is $\rho U A$
bottom is $\rho v a$
these are the same, so $UA = va$

Thus $v^2 = \left(\frac{va}{A}\right)^2 + 2gh$

$$v^2 \left(1 - \left(\frac{a}{A}\right)^2\right) = 2gh$$

If hole is small, $\frac{a}{A} \ll 1$ so $\left(\frac{a}{A}\right)^2 <<<< 1$

Then $v^2 \simeq 2gh$

i.e. $v \simeq \sqrt{2gh}$

i.e exactly as for a free-falling particle under gravity

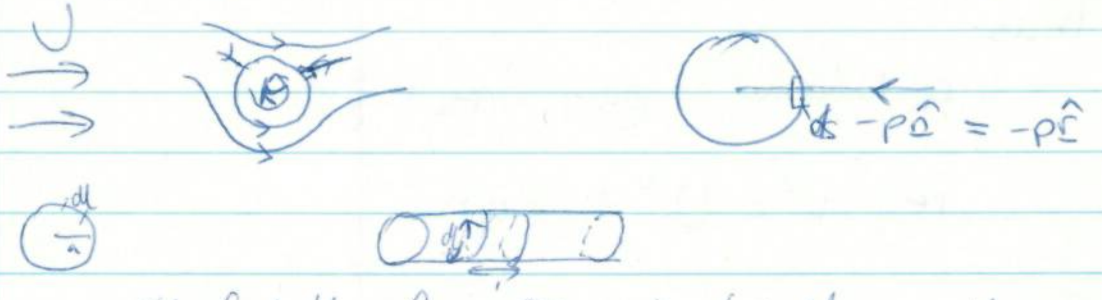
TORRICELLI

exp

$\frac{1}{2}\sqrt{2gh} \leq v \leq \sqrt{2gh}$

Example 2: Spinning cylinder

Consider a cylinder of radius a in a stream, uniform at infinity with speed U in the x-direction. Let the cylinder be spinning so that the circulation about the cylinder is K.

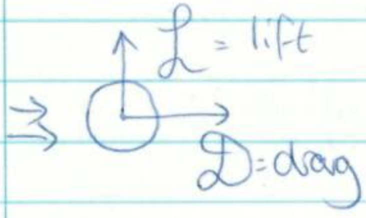


we will find the force per unit length on the cylinder.
Our unit of area is $dS = dl \times 1$
Thus total force (per unit length) $= \int -p\hat{n}\, dS = -\int p\hat{r}\, dl$ per unit length

and $dl = a\,d\theta$

$$\underset{\sim}{F} = -\int_{\theta=-\pi}^{\pi} p\,\hat{r}\, a\, d\theta$$

$$= -a\int_{-\pi}^{\pi} p\,(\cos\theta\, \underline{i} + \sin\theta\, \underline{j})\, d\theta$$



$$\underset{\sim}{\mathcal{F}} = \mathcal{D}\,\underline{\hat{i}} + \mathcal{L}\,\underline{\hat{j}}$$

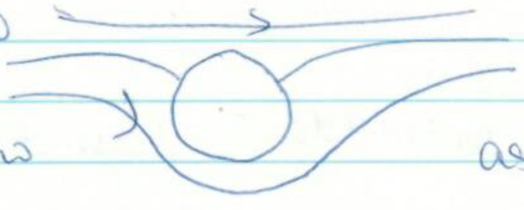
where the drag, $\mathcal{D}$ is

$$\mathcal{D} = -a\int_{-\pi}^{\pi} p\cos\theta \, d\theta$$

and the Lift, $\mathcal{L}\underset{\sim}{\hat{j}}$ is

$$\mathcal{L} = -a\int_{-\pi}^{\pi} p\sin\theta \, d\theta$$

Now



flow

all streamlines originate upstream, and we take the as steady. So use Bernoulli.

$p + \frac{1}{2}\rho \underline{u}^2 =$ const on streamlines in the absence of external forces.

$\boxed{\frac{\partial \underline{u}}{\partial t} = 0 \text{ STEADY}}$

At infinity $p = p_\infty$, constant,
$U = U$

~~so $p_1 > p_1$~~

So $p + \frac{1}{2}\rho \underline{u}^2 = p_\infty + \frac{1}{2}\rho U^2$

anywhere in the flow, $p = p_\infty + \frac{1}{2}\rho U^2 - \frac{1}{2}\rho|\underline{u}|^2$

The complex potential for the flow is ~~$w = U(z+a$~~

$$w = U\left(z + \frac{a^2}{z}\right) - \frac{i\kappa}{2\pi}\log z$$

$$\frac{dw}{dz} = u - iv = U\left(1 - \frac{a^2}{z^2}\right) - \frac{i\kappa}{2\pi} z$$

so $u_r - iu_\theta = e^{i\theta}\frac{dw}{dz} = 2iU\sin\theta - \frac{i\kappa}{2\pi a}$ on $z = ae^{i\theta}$

i.e $u_r = 0$ as expected
$$\left.\begin{aligned} u_r &= 0 \\ u_\theta &= \frac{K}{2\pi a} - 2U\sin\theta \end{aligned}\right] \text{ on cylinder}$$

Thus $$|\underline{u}|^2 = \frac{K^2}{4\pi^2 a^2} - \frac{2UK}{\pi a}\sin\theta + 4U^2\sin^2\theta$$

now $$\mathcal{D} = -a\int_{-\pi}^{\pi} p\cos\theta \, d\theta$$

$$\int_{-\pi}^{\pi} \cos\theta \, d\theta = 0 \quad \int_{-\pi}^{\pi} \cos\theta\sin\theta \, d\theta = 0 \quad \int_{-\pi}^{\pi} \cos\theta\sin^2\theta \, d\theta = 0$$

Thus $\mathcal{D} = 0$ i.e no drag

Or velocity is symmetric fore and aft.
so pressure same fore and aft.
$\therefore$ no drag.

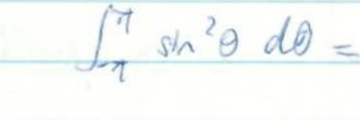
Now $$\mathcal{L} = -a\int_{-\pi}^{\pi} p\sin\theta \, d\theta$$

$$\int_{-\pi}^{\pi} \sin\theta \, d\theta = 0 \quad \int_{-\pi}^{\pi} \sin^3\theta \, d\theta = 0 \quad \text{(integrand odd)}$$

$$\int_{-\pi}^{\pi} \sin^2\theta \, d\theta = \pi$$

$$\mathcal{L} = -a\int_{-\pi}^{\pi} \left(-\tfrac{1}{2}\rho\right)\left(\frac{-2UK}{\pi a}\sin\theta\right)\sin\theta \, d\theta$$

$$= -\rho U K$$



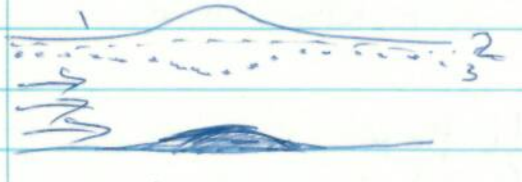
'top spin'

i.e. downward force $\rho U K$ (indep. of a)
(per unit length)

note: $\underline{u}^2 = |\underline{u}|^2 = \underline{u}\cdot\underline{u}$

Example 3: Open Channel flow

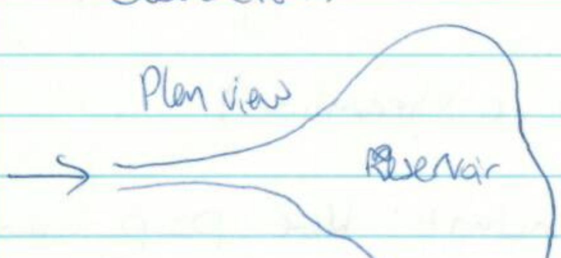
- flow {along / down} a channel that is open to the air.
eg. river, aqueduct,

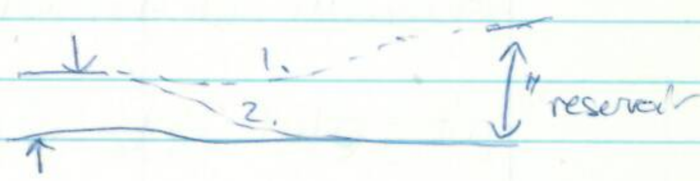


1. rises
2. stays same
3. falls.

?

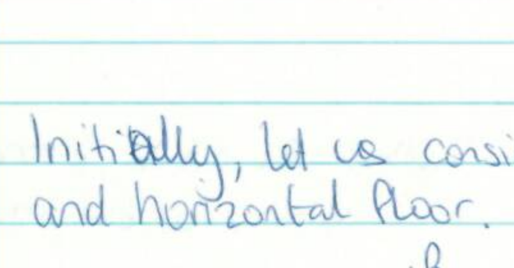
elevation.





1. gets deeper, joins smoothly.
2. gets shallower.

Initially, let us consider a channel of constant width b and horizontal floor.



Let any changes in the flow be slow in the flow direction. Let the local depth be h and the local ~~velocity~~ speed be u downstream.

By conservation of mass, in steady flow, mass flux across A must equal mass flux across B. ie.

$$\rho h_1 b u_1 = \rho h_2 b u_2$$

ie $h_1 u_1 = h_2 u_2$

OR throughout the flow, $uh = Q$, constant.
i.e $Q = uh$ is a constant of the motion
(volume flux per unit width)
ie dimensions $L^2 T^{-1}$

~~notice a particle~~

Provided the flow is smooth, a particle on the surface stays there, ie. the surface is a streamline.

Hence we can apply Bernoulli there.
only force acting on the fluid is gravity.

Bernoulli: (on the surface, a streamline)

$p + \frac{1}{2}\rho|\underline{u}|^2 + \rho V_e$ is a constant. Here $p = p_a$ $\underline{u} = u\underline{i}$ $V_e = gz$

$$P_a + \frac{1}{2}\rho u^2 + \rho gh = \text{const.}$$

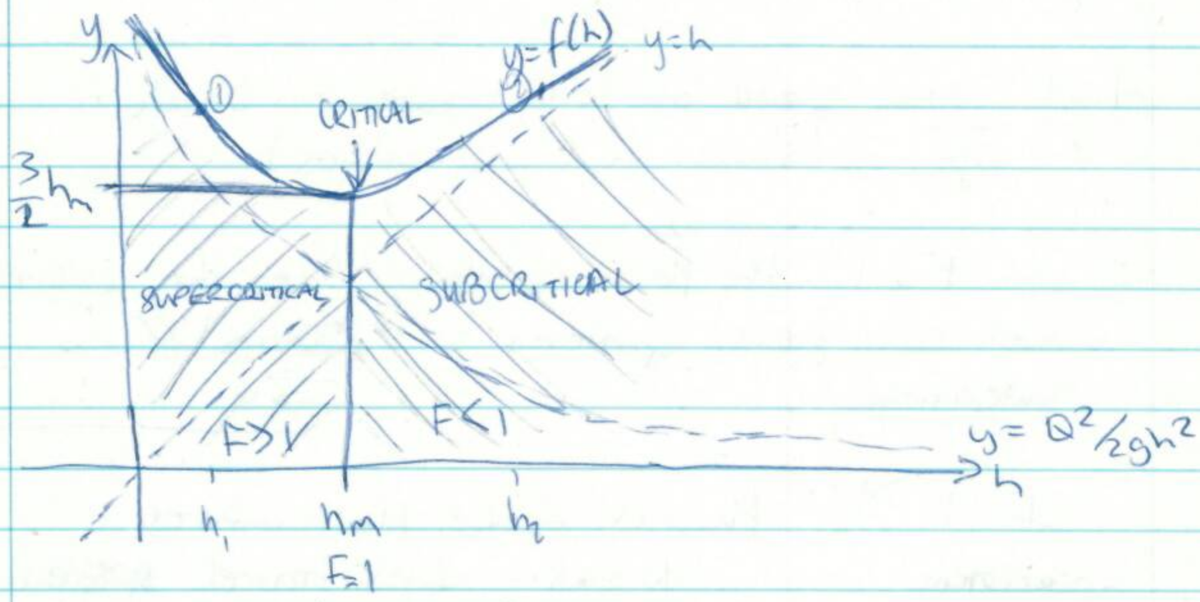
i.e $\frac{1}{2}u^2 + gh = \text{const.}$
$= gH$ where H is a second constant of the motion.

Dimensions of H are length.
H is the depth the fluid would occupy, were it to come to rest
i.e $h \to H$ if $u \to 0$
H: the 'head' of the flow.

Thus we have $uh = Q$
$$\frac{1}{2}u^2 + gh = gH$$

eliminate u, $u = Q/h$ so $\frac{Q^2}{2gh^2} + h = H$

let $f(h) = \dfrac{Q^2}{2gh^2} + h$



this graph has a single minimum for $h>0$ when $f'(h)=0$

$$\frac{-2Q^2}{2gh^3} + 1 = 0$$

i.e. $h = h_m = \left(\dfrac{Q^2}{g}\right)^{1/3}$

$$f(h_m) = h_m + \frac{Q^2}{2gh_m^2} = h_m + \frac{h_m^3}{2h_m^2} = \frac{3}{2}h_m$$

at $h = h_m$, $h_m^3 = \dfrac{Q^2}{g} = h_m^2 u_m^2 / g$

i.e. $\dfrac{u_m^2}{gh_m} = 1$

we define the Froude number $F = \dfrac{u}{\sqrt{gh}}$ at any point in the flow.

Then $F = 1$ when $h = h_m$

If $h > h_m$, then $u < u_m$ so $F < 1$

If $h < h_m$, then $u > u_m$, so $F > 1$

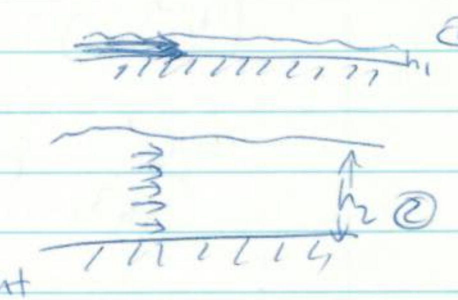
Fact: the speed of long waves on shallow water is $\sqrt{gh}$. (shown in waterwaves).

So if $F < 1$ the flow is slower than the waves. waves can travel upstream. SUBCRITICAL

if $F > 1$ flow is faster then waves information cannot travel upstream. SUPERCRITICAL

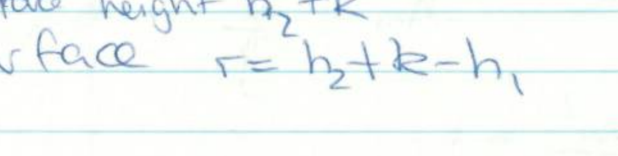
SUPERCRITICAL: shallow + fast

SUBCRITICAL: deep + slow



These are two flows with the same Q and same H but different h.

Example:



now suppose the channel remains of constant width but the floor of the channel rises smoothly by k.

upstream surface height h_1,

downstream surface height $h_2 + k$

rise in surface $r = h_2 + k - h_1$

$r > 0$
$r = 0$?
$r < 0$.

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ex cont.

Mass flux constant,

$u_1 h_1 = u_2 h_2$ (speed * fluid depth)

now provided the change is smooth the surface is a streamline thus we can apply Bernoulli:

$p + \frac{1}{2}\rho \underline{u}^2 + \rho g z = \text{const on surface}$

upstream: $p + \frac{1}{2}\rho \underline{u}^2 + \rho g z = p_a + \frac{1}{2}\rho u_1^2 + \rho g h$
downstream: $p + \frac{1}{2}\rho \underline{u}^2 + \rho g z = p_a + \frac{1}{2}\rho u_2^2 + \rho g (h_2 +$

not depth of surface

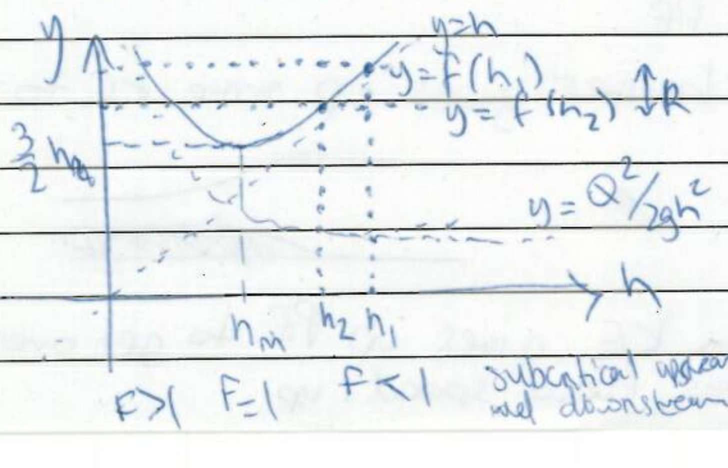
Thus

$$\frac{1}{2}\rho u_1^2 + \rho g h_1 = \frac{1}{2}\rho u_2^2 + \rho g (h_2 + k)$$

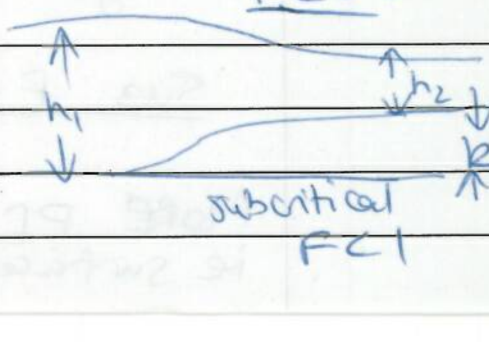
i.e $$\frac{u_1^2}{2g} + h_1 = \frac{u_2^2}{2g} + h_2 + k$$

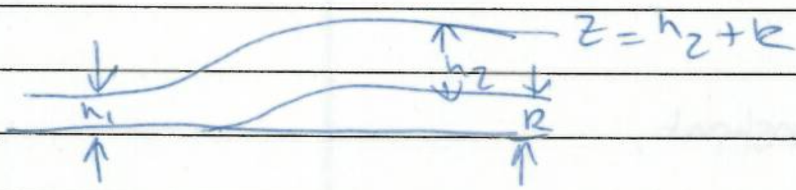
i.e $$\frac{Q^2}{2gh_1^2} + h_1 = \frac{Q^2}{2gh_2^2} + h_2 + k$$

i.e $$f(h_1) = f(h_2) + k$$



F<0?





supercritical flow $F > 1$

$\underline{r > 0}$

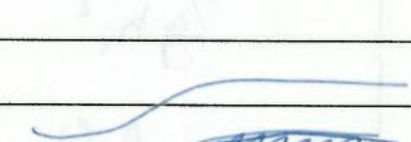
$$\underbrace{f(h_2)}_{\text{decrease}} + \underbrace{k}_{\text{increase}} = \underbrace{f(h_1)}_{\text{fixed}}$$

Find the sign of the rise, r
algebraically:

$$r = h_2 + k - h_1$$
$$= \frac{Q^2}{2g}\left(\frac{1}{h_1^2} - \frac{1}{h_2^2}\right)$$
$$= \frac{Q^2}{2g h_1^2 h_2^2}\left(h_2^2 - h_1^2\right)$$

$$\frac{Q^2}{2g h_1^2} + h_1 = \frac{Q^2}{2g h_2^2} + h_2 + k$$

SUPER $F > 1$ $r > 0$

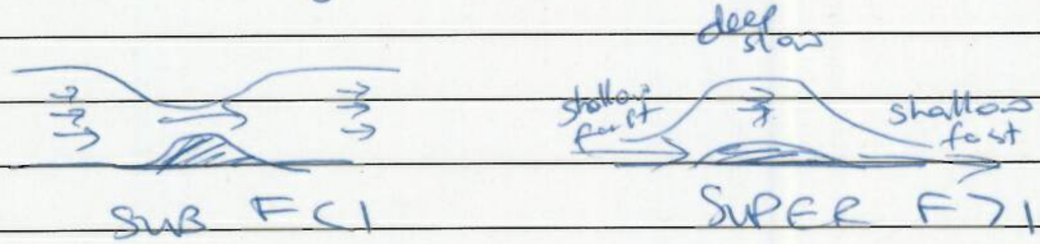


more KE than PE
To get over barrier gives up some KE to get PE

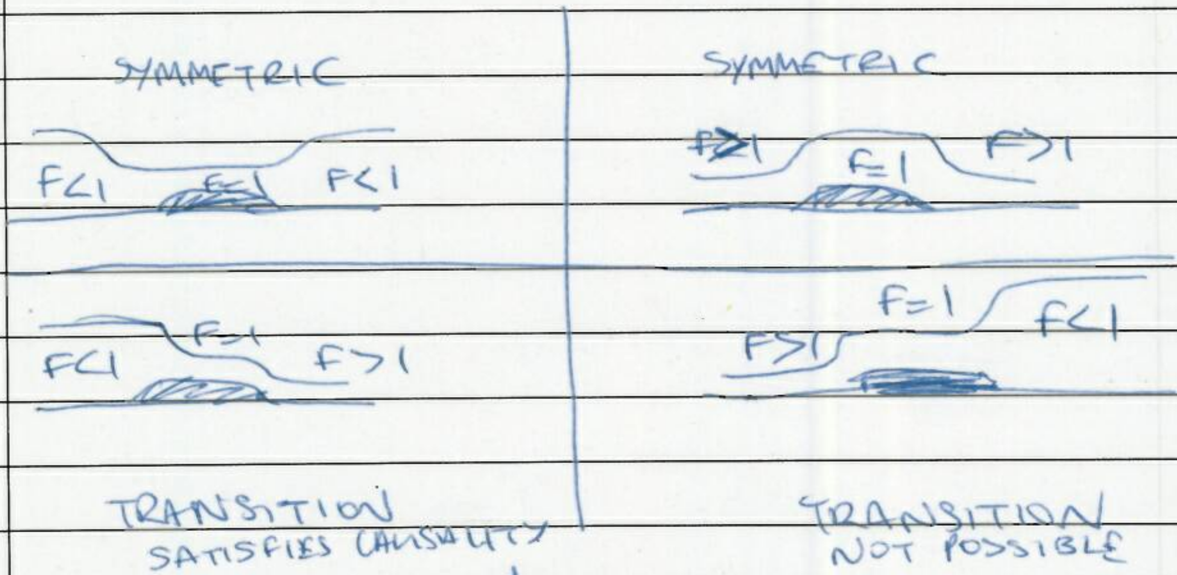
SUB $F < 1$ $\cancel{r}$

MORE PE than KE, gives up PE to get over barrier
ie surface drops, flow speeds up

~~Find the sign of~~



If $D = f(h_1) - f(h_m)$ there are 4 possibilities.



$$F = \frac{U}{\sqrt{gh}}$$

flow speed

wave speed

$$F^2 = \frac{u^2}{gh} = \frac{\frac{1}{2}\rho u^2}{\frac{1}{2}\rho g h} = \frac{KE}{PE}$$

causality shows that smooth transitions are alwa
from subcritical to supercritical

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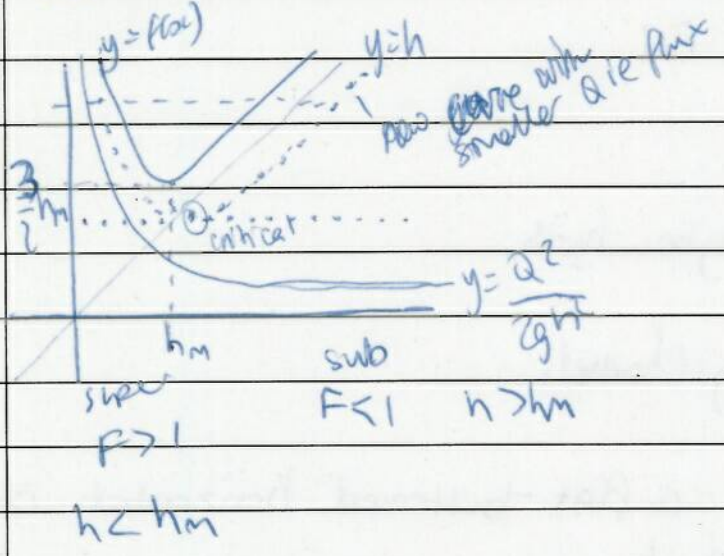
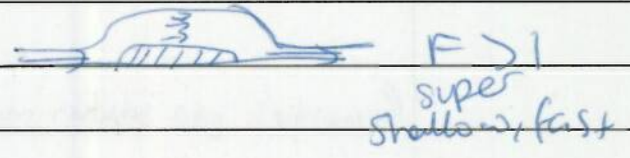
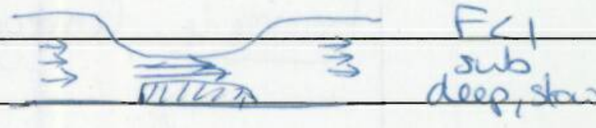
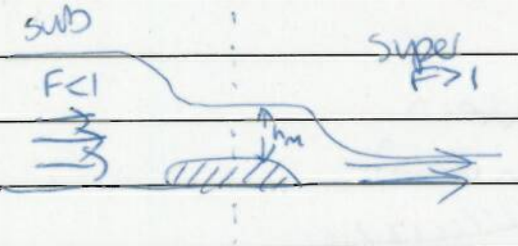
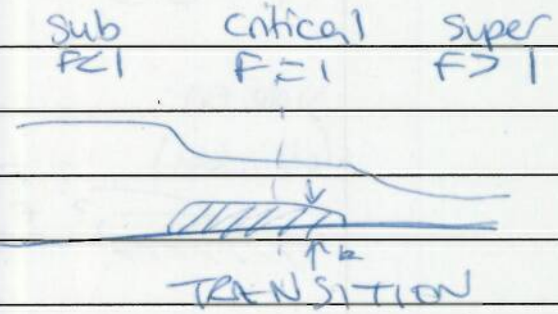










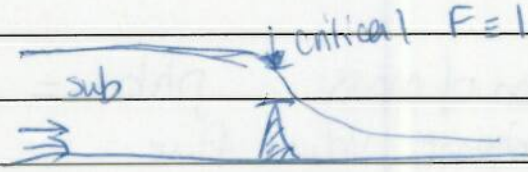
F=1 at max height

TRANSITION

$k > f(h_1) - f(h_m)$

If the obstacle height k is increased further so $k > f(h_1) - f(h_m)$ then the upstream flow banks up, deepens, flux decreases and makes the minimum adjustment to allow water to pass over obstacle ie flow at the top of bump is critical ie. F=1 when k is a maximum.

eg. a weir.



notice if you know the depth at a weir, you know the flux without having to measure speed. $\frac{Q^2}{g} = h_m^3$

$Q = (gh_m^3)^{1/2}$

$u_m = \sqrt{g h_m}$ since $F=1$

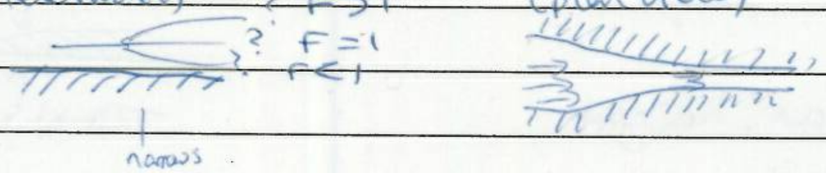
Remember one solution may be $h_2 = h_1$

Example 4 - Converging Channel

consider flow through a flat-bottomed horizontal channel of varying width, b.

side on:
(elevation)

top view
(plan view)



deepens / stays same / shallows ?

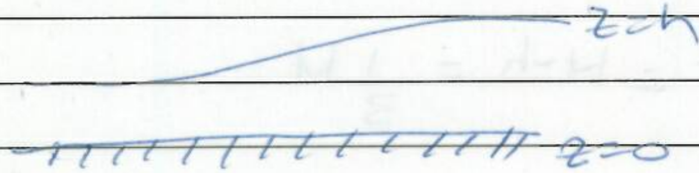
It depends on F

$F > 1$ super
$F < 1$ sub

$$\frac{U}{\sqrt{gh}} \quad \frac{\frac{1}{2}\rho u^2}{\frac{1}{2}\rho g h} \quad \frac{PE}{KE}$$

① Conservation of mass $\rho h b u = \rho Q$ so $Q = hbu$ is the constant volume flux

② Provided the surface remains smooth, the surface is a streamline, so apply Bernoulli there.



h fluid depth

$$p + \frac{1}{2}\rho u^2 + \rho g z = \text{const}$$

z height of surface

i.e. $p_a + \frac{1}{2}\rho u^2 + \rho g h = \text{const}$

$p = p_a$ constant atmosph pressure on surfa

$$\frac{u^2}{2g} + h = H, \text{ constant}$$

eliminate u, $$\frac{Q^2}{2gh^2b^2} + h = H$$

i.e. $$(H-h)h^2 = \frac{Q^2}{2gb^2}$$

$f(h) = h^2H - h^3$

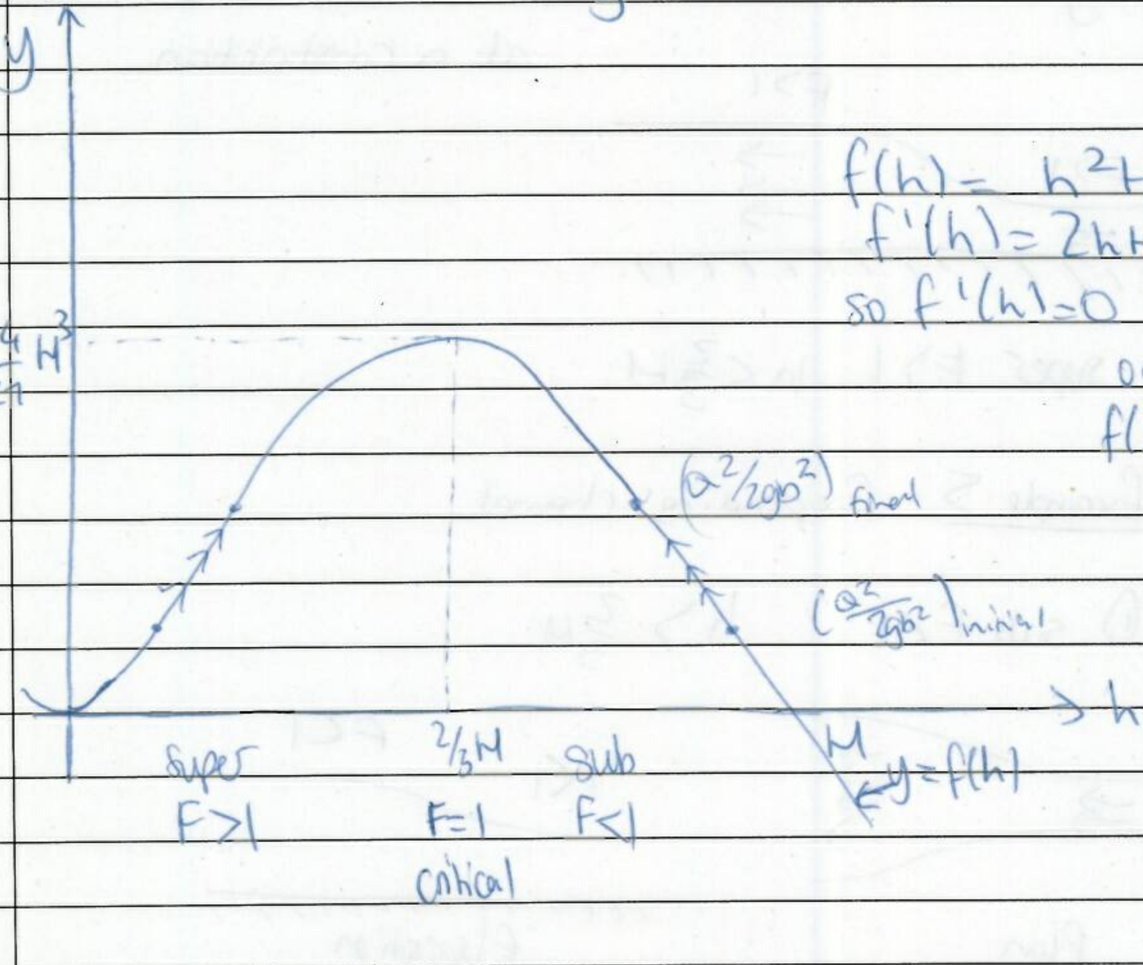
$f'(h) = 2hH - 3h^2$

so $f'(h) = 0$ if $h = 0$

or $h = \frac{2}{3}H$

$f(\frac{2}{3}H) = \left(\frac{4}{9} - \frac{8}{27}\right)$

$=$

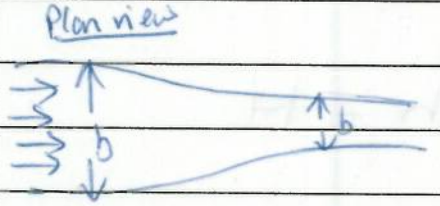


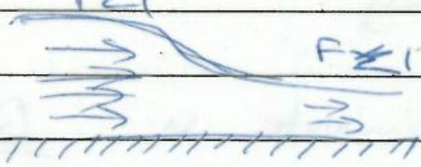
At $h = \frac{2}{3}H$, $\frac{u^2}{2g} = H - h = \frac{1}{3}H$

so $\frac{u^2}{gh} = \frac{u^2}{g(\frac{2}{3}H)} = \frac{\frac{2}{3}H}{\frac{2}{3}H} = 1$

i.e $F = 1$ when ~~at~~ $h = \frac{2}{3}H$

Plan view



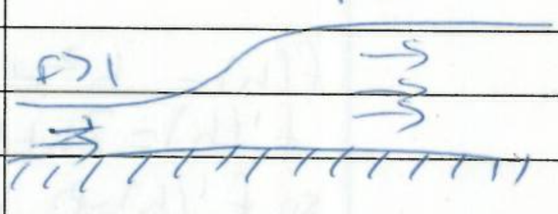


b decreasing

sub $F<1$ $h > \frac{2}{3}H$

$\frac{Q^2}{2gb^2}$ increasing.

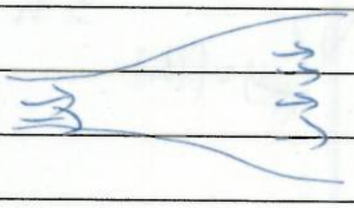
Flows move towards critical at a constriction.

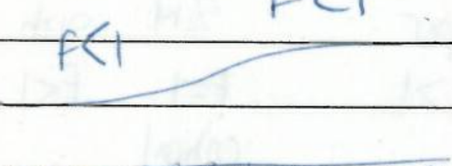


super $F>1$ $h < \frac{2}{3}H$

Example 5 Expanding channel.

① sub $F<1$ $h > \frac{2}{3}H$





Plan

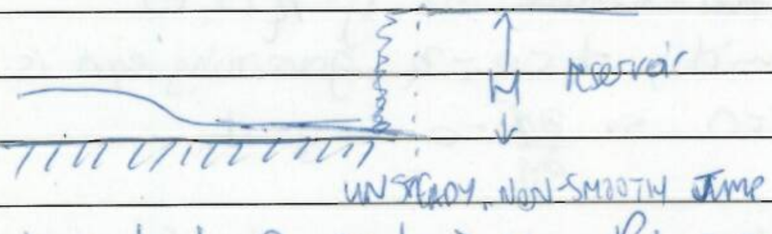
Elevation

River flows into reservoir. Then $h \uparrow H$ (increases to) as $b \to a$

i.e $\frac{Q^2}{gh^2} = 0$ $u = 0$ stagnant

River smoothly enters stagnant reservoir.

(2) super $F > 1$ $h < \frac{2}{3}H$ fast, shallow



here $h \downarrow 0$ (decreases to) as $b \to \infty$. River cannot smoothly join reservoir.

$$F = \frac{u}{\sqrt{gh}} = \frac{\text{flow speed}}{\text{wave speed}}$$ $F > 1$ super critical

$$M = \frac{u}{a} = \frac{\text{flow speed}}{\text{speed of sound}}$$ $M > 1$ supersonic, $M < 1$ subsonic

Sonic boom: spontaneous jump from $M > 1$ to $M < 1$

Only have hydraulic jump from $F > 1$ to $F < 1$ as this gives out energy.

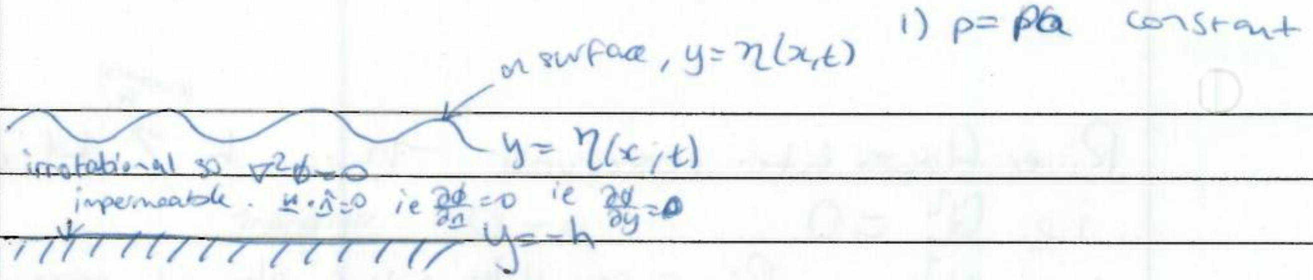
Water waves

Free surface gravity waves.

- We will take the flow to be 2D irrotational, inviscid, incompressible

Thus we have a stream function and a velocity potential and complex potential

i.e $\exists\, \phi$ st $\underline{u} = \underline{\nabla}\phi$



1) $\exists\ \phi$ s.t. $u = \nabla\phi$

let the unknown ~~free~~-surface be $y = \eta(x,t)$.

Then in the fluid, $-h < y < \eta$, governing eqn is $\nabla^2\phi = 0$

on lower bdd $v = 0$ so $\frac{\partial\phi}{\partial y} = 0$ on $y = -h$

we need 2 bdd. conditions on surface (since η unknown)

The two bcs are the KINEMATIC and DYNAMIC condition

DYNAMIC (forces): $p = p_a$ on $y = \eta$

KINEMATIC : particle on the surface remains on the surface.

i.e on the surface $y = \eta(x,t)$ $\forall x,t$

i.e $y - \eta(x,t) = 0$ $\forall x,t$

following a particle on surface, $\frac{D}{Dt}(y - \eta(x,t)) = 0$ on $y = \eta$ $\forall x,t$.

i.e $v - \frac{D\eta}{Dt} = 0$ on $y = \eta$

i.e. $v = \frac{\partial\eta}{\partial t} + u\frac{\partial\eta}{\partial x}$ on $y = \eta(x,t)$.

i.e $\frac{\partial\phi}{\partial y} = \frac{\partial\eta}{\partial t} + \frac{\partial\phi}{\partial x}\frac{\partial\eta}{\partial x}$ on $y = \eta(x,t)$

To deal with the dynamic condition on pressure we would like to use Bernoulli. But the flow must be STEADY ie $\frac{\partial}{\partial t} = 0$ for the form of Bernoulli up to now. We need a new

Bernoulli for unsteady flow

Remember $\dfrac{D\underline{u}}{Dt} = -\dfrac{1}{\rho}\underline{\nabla}p + \underline{F}$

i.e $\dfrac{\partial \underline{u}}{\partial t} + (\underline{u}\cdot\underline{\nabla})\underline{u} = -\dfrac{1}{\rho}\nabla p - \underline{\nabla} V_e$ $\quad \underline{F} = -\underline{\nabla} V_e$ a conservative force

$$\frac{\partial \underline{u}}{\partial t} + \underline{\nabla}\left(\tfrac{1}{2}\underline{u}^2\right) + \underline{\omega}\wedge\underline{u} = -\frac{1}{\rho}\underline{\nabla}p - \underline{\nabla}V_e$$

[last time: steady, dot with $\underline{u}$ to get rid of $\underline{\omega}\wedge\underline{u}$]
This time: use fact that irrotational, $\underline{u} = \underline{\nabla}\phi$ and $\underline{\omega} = \underline{0}$

Thus $\dfrac{\partial}{\partial t}\nabla\phi + \underline{\nabla}\left(\tfrac{1}{2}\underline{u}^2\right) = -\dfrac{1}{\rho}\underline{\nabla}p - \underline{\nabla}V_e$

ie $\underline{\nabla}\left[\dfrac{\partial\phi}{\partial t} + \dfrac{1}{2}|\underline{\nabla}\phi|^2 + \dfrac{1}{\rho}p + V_e\right] = 0$

ie $\rho\left(\dfrac{\partial\phi}{\partial t} + \dfrac{1}{2}|\underline{\nabla}\phi|^2\right) + \cancel{\tfrac{1}{\rho}}\,p + \rho V_e = G(t)$

new Bernoulli

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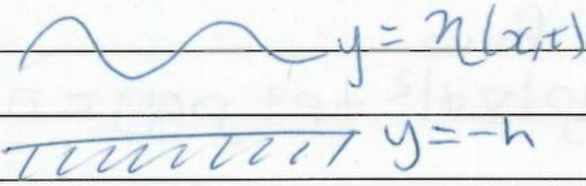
Water waves



equation: $\nabla^2 \phi = 0$ $\quad [\underline{u} = \underline{\nabla}\phi,\ \nabla \cdot \underline{u} = 0]$

lower bc: $\dfrac{\partial \phi}{\partial y} = 0 \quad y = -h$

upper bc kinematic $\quad v = \dfrac{D\eta}{Dt}$ on $y = \eta$

i.e $\dfrac{\partial \phi}{\partial y} = \dfrac{\partial \eta}{\partial t} + \dfrac{\partial \phi}{\partial x}\dfrac{\partial \eta}{\partial x}$ on $y = \eta$

dynamic $\quad p = p_a$ on $y = \eta$

Bernoulli (time deps, irrotational)

$$\rho \frac{\partial \phi}{\partial t} + \frac{1}{2}\rho |\nabla \phi|^2 + p + \rho V_e = F(t)$$

the restoring force is gravity so $V_e = gy$

Thus $\rho \dfrac{\partial \phi}{\partial t} + \frac{1}{2}\rho |\underline{\nabla} \phi|^2 + p + \rho g y = F(t)$

F(t) can be absorbed into ϕ

redefine $\hat{\Phi} = \phi - \frac{1}{\rho}\int^t F(\tau)\, d\tau$

then $\underline{\nabla}\hat{\phi} = \underline{\nabla}\phi$ and $\rho \dfrac{\partial \hat{\phi}}{\partial t} = \rho \dfrac{\partial \phi}{\partial t} - F(t)$

thus w.l.o.g we can take $F=0$
(since if $F\neq 0$ we can redefine ϕ as above).

hence everywhere in the flow

$$\rho\frac{\partial\phi}{\partial t} + \frac{1}{2}\rho|\nabla\phi|^2 + p + \rho g y = 0$$

UNSTEADY BERNOULLI

On surface, $y=\eta$ and $p=p_a$ (const)

thus $\rho\frac{\partial\phi}{\partial t} + \rho|\nabla\phi|^2 + \rho g\eta = -p_a$ (const)

by above argument, can absorb p_a (const) into ϕ
So we have the dynamic condition

$$\rho\frac{\partial\phi}{\partial t} + \frac{1}{2}\rho|\nabla\phi|^2 + \rho g\eta = 0$$

$$\frac{\partial\phi}{\partial t} + \frac{1}{2}|\nabla\phi|^2 + g\eta = 0$$

Equation: Laplace: $\nabla^2\phi = 0$

lower b.c: $\frac{\partial\phi}{\partial y} = 0$ on $y=-h$

upper b.c: kinematic: $\frac{\partial\phi}{\partial y} = \frac{\partial\eta}{\partial t} + \frac{\partial\phi}{\partial x}\frac{\partial\eta}{\partial x}$ on $y=\eta$

dynamic: $\frac{\partial\phi}{\partial t} + \frac{1}{2}|\nabla\phi|^2 + g\eta = 0$ on $y=\eta$

Full, nonlinear, surface water wave problem
$\eta(x,t)$ unknown

To make progress we 'linearise', i.e consider waves of infinitesimal amplitude, $0 < \varepsilon \ll 1$
i.e we take $\eta(x, t)$ to be of order ε

we expect velocities and so ϕ to be order ε also

$y = \eta$

Kinematic b.c: $\dfrac{\partial \phi}{\partial y} = \dfrac{\partial \eta}{\partial t} + \dfrac{\partial \phi}{\partial x}\dfrac{\partial \eta}{\partial x}$

$$\varepsilon : \varepsilon : \varepsilon^2$$
$$1 : 1 : \varepsilon$$

thus in limit $\varepsilon \to 0$, the final term disappears.
we have $\dfrac{\partial \phi}{\partial y} = \dfrac{\partial \eta}{\partial t}$ on $y = \eta$ (linear)
(with error of order ε)

notice for any function $f(y)$, $f(\varepsilon) = f(0) + \varepsilon f'(0) + \frac{1}{2}\varepsilon^2 f''(0) + \dots$
$= f(0)$ with error order ε

Thus can move b.c from $y = \eta$ (of order ε) to $y = 0$ with error of order ε - thus we have $\dfrac{\partial \phi}{\partial y} = \dfrac{\partial \eta}{\partial t}$ on $y = 0$

(now linear, on known surface)

dynamic bc. $\dfrac{\partial \phi}{\partial t} + \cancel{\frac{1}{2}|\nabla \phi|^2} + g\eta = 0$ on $y = \eta \to y = 0$

$$\varepsilon \qquad \varepsilon^2 \qquad \varepsilon$$

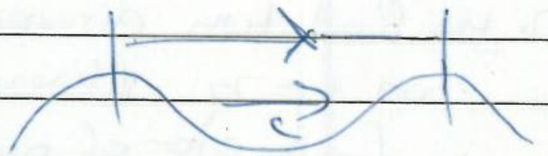
linearised b.c $\dfrac{\partial \phi}{\partial t} + g\eta = 0$ on $y = 0$

Summary: linear water waves

equation: $\nabla^2\phi = 0$ — already linear

lower b.c $\frac{\partial \phi}{\partial y} = 0$ on $y = -h$ — already linear + on known surface

upper b.c $$\left.\begin{aligned} \frac{\partial \phi}{\partial y} &= \frac{\partial \eta}{\partial t} \\ \frac{\partial \phi}{\partial t} + g\eta &= 0 \end{aligned}\right\} \text{ on } y = 0$$



- wavelength λ, distance between two successive crests.
- period τ, time between arrival at a given point of successive crests.
- speed c, at which crests advance

$$\boxed{c = \frac{\lambda}{\tau}}$$

Any periodic function can be expressed as a sum of sines and cosines.
Thus it is sufficient to consider $\eta = A\sin\left(\frac{2\pi}{\lambda}(x - ct)\right)$

- wave with amplitude A, wavelength λ, speed c to the right and period $\tau = \frac{\lambda}{c}$

2.5 Hydraulic jump ($q \equiv u$ here)

Consider the jump as shown in a channel of width d and horizontal bed. The suffixes 1 and 2 refer to conditions before and after the jump, and we consider the fluid bounded by A_1B_1 and A_2B_2. In time δt, the fluid has moved to the region bounded by $A_1'B_1'$ and $A_2'B_2'$. Then at the two locations we have the following quantities.

Height	h_1	h_2
Mean velocity	q_1	q_2
Pressure	$\rho g(h_1 - z)$	$\rho g(h_2 - z)$
Thickness	$q_1\delta t$	$q_2\delta t$
Mass	$m_1 = \rho d h_1 q_1 \delta t$	$m_2 = \rho d h_2 q_2 \delta t$

Conservation of mass shows that $m_1 = m_2$ and the flow rate Q is the same at 1 and 2

Flow rate	$dh_1q_1 = Q$	$dh_2q_2 = Q$
Momentum	m_1q_1	m_2q_2
Force in flow direction	$F_1 = \int_0^{h_1} \rho g d(h_1 - z)dz = \frac{1}{2}\rho g d h_1^2$	$F_2 = -\int_0^{h_2} \rho g d(h_2 - z)dz = -\frac{1}{2}\rho g d h_2^2$

Force equals rate of change of momentum gives $F_1 - F_2 = (m_2q_2 - m_1q_1)/\delta t$ or $\frac{1}{2}\rho g d(h_1^2 - h_2^2) = \rho Q(q_2 - q_1) = \frac{\rho Q^2}{d}(\frac{1}{h_2} - \frac{1}{h_1})$. Hence either $h_2 - h_1 = 0$, in which case the flow is continuous and there is no jump, or

$$h_1h_2(h_1 + h_2) = \frac{2Q^2}{gd^2}.$$

For given upstream conditions this equation determines h_2 and hence $q_2 = Q/dh_2$.

Kinetic energy	$\frac{1}{2}m_1q_1^2$	$\frac{1}{2}m_2q_2^2$
Work done by force	$F_1q_1\delta t$	$F_2q_2\delta t$

If $D\delta t$ is the amount of knetic energy lost in time δt, conservation of energy gives

$$(F_1q_1 - F_2q_2)\delta t = \tfrac{1}{2}m_2q_2^2 - \tfrac{1}{2}m_1q_1^2 + D\delta t,$$

from which it follows that

$$D = \tfrac{1}{2}\rho g d(h_1^2q_1 - h_2^2q_2) + \tfrac{1}{2}\rho Q(q_1^2 - q_2^2) = \tfrac{1}{2}\rho g Q(h_1 - h_2) + \frac{\rho Q^3}{2d^2}(\frac{1}{h_1^2} - \frac{1}{h_2^2})$$

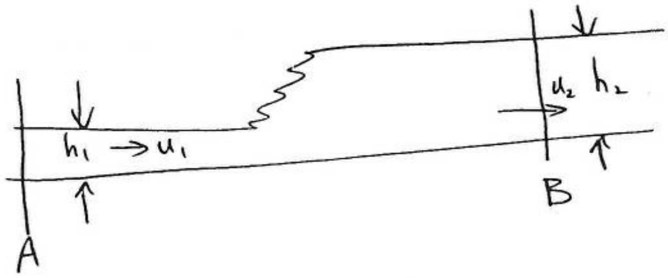
or

$$D = \tfrac{1}{2}\rho g Q\frac{(h_2 - h_1)^3}{2h_1h_2}.$$

Since $D \geq 0$, $h_2 \geq h_1$. In a hydraulic jump, the level of the water rises and the speed falls. For the flow upstream,

$$\frac{q_1^2}{gh_1} - 1 = \frac{Q^2}{gd^2h_1^3} - 1 = \frac{h_1h_2(h_1 + h_2)}{2h_1^2} - 1 = \frac{(h_2 - h_1)(h_2 + 2h_1)}{2h_1^2},$$

Energy



From mass & momentum conservation we have shown that

$$h_1 h_2 (h_1 + h_2) = \frac{2Q^2}{g d^2} \quad (1)$$

Work done at A = force × distance, and distance = $u_1 \delta t$ at all heights

Force = pressure, integrated over area = $\rho g (h_1 - z)$ integrated over area (taking $p_a = 0$).

Hence force = $\frac{1}{2} \rho g h_1^2 d$

Thus work done at A = $\frac{1}{2} \rho g h_1^2 u_1 d \delta t = \frac{1}{2} \rho g Q \delta t . h_1$

Similarly, work done at B = $-\frac{1}{2} \rho g Q \delta t . h_2$ (since displacement in opposite dirn. to force).

Thus net work done on fluid = $\frac{1}{2} \rho g Q \delta t (h_1 - h_2)$.

KE in at A = $\frac{1}{2} \rho u_1^2$ / unit volume.

∴ Total KE in at A = $\frac{1}{2} \rho u_1^2 Q \delta t$.

PE = $\rho g z$ / unit volume

∴ PE in at A = $\frac{1}{2} \rho g h_1^2 d u_1 \delta t = \frac{1}{2} \rho g h_1 Q \delta t$ (integrate from 0 to h_1)

∴ Total energy in at A = $\frac{1}{2} \rho Q \delta t \; (u_1^2 + g h_1)$

Similarly, total energy out at B = $\frac{1}{2} \rho Q \delta t \; (u_2^2 + g h_2)$

Energy lost = Work done + Energy In − Energy Out

$$= (\rho Q \delta t) \left[g (h_1 - h_2) + \tfrac{1}{2} (u_1^2 - u_2^2) \right]$$

But $u_1^2 = Q^2 / h_1^2 d^2 = \frac{g h_2}{2 h_1} (h_1 + h_2)$ by (1)

and $u_2^2 = \frac{g h_1}{2 h_2} (h_1 + h_2)$

Thus lost energy = $\frac{\rho g Q \delta t}{4 h_1 h_2} \left[(h_1 + h_2) h_2^2 - (h_1 + h_2) h_1^2 + 4 h_1 h_2 (h_1 - h_2) \right]$

$$= \frac{\rho g Q \delta t}{4 h_1 h_2} (h_2 - h_1)^3$$

For energy to be lost, $h_2 > h_1$ i.e. an upward jump.