2301 Fluid Mechanics Notes
Based on the autumn 2011 lectures
by Prof E R Johnson.

To the times that we suffered for the sake of understanding mathematics and to which we shall be rewarded for it.

[illegible].

3/10/11

What about?
Magma Dynamics :- dynamics of the interior of Earth.
- dynamo theory.

Plasma : - Sun, star and fusion.

Blood : - Biofluid dynamics.

Atmostphere + Oceans
- Meteorolgy
- Climate.

Eg:

"Air resistance"
- flow past bodies
- Cars
- Planes

Geophysical Fluid Dynamics
"GFD".

Books
A first course in Fluid Dynamics
A. R Patterson, Cambridge.

Ideal and Incompressiable Fluid Dynamics.

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$\underline{u}(x, y, z, t)$ <u>Clay Institute</u>.

$$\frac{\partial \underline{u}}{\partial t} + (\underline{u} \cdot \underline{\nabla})\underline{u} = -\underline{\nabla} p + \nu \nabla^2 \underline{u}$$

$$\underline{\nabla} \cdot \underline{u} = 0 \qquad \text{(Navier Stokes)}.$$

<u>Either</u>

$\exists \underline{u}(x, y, z, 0)$ somooth, bdd.

But $|\underline{u}| \to \infty$ at $t = t_0 > 0$.

<u>Or</u>

No such $\underline{u}(x, y, z, 0)$ exists.

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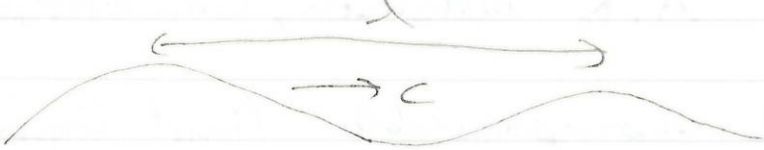
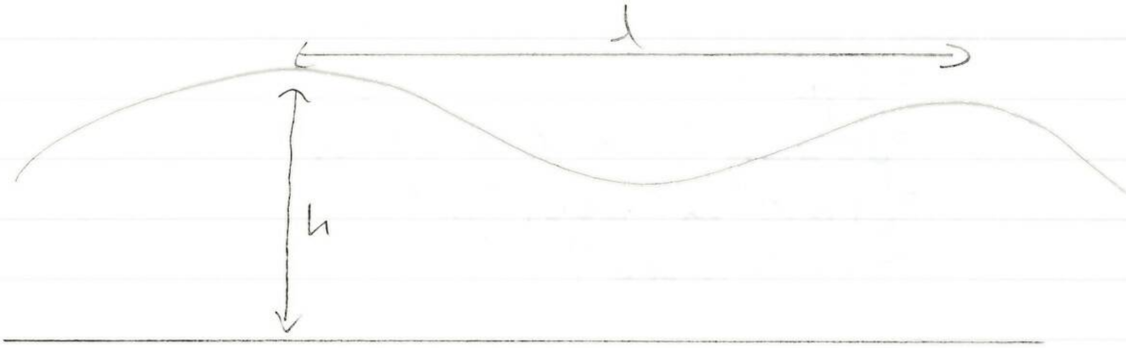
How does a plane fly?
Geometry of wind cross section.
Speed ← directly poportional to $\underline{u}$.
Directly porportional density → It is the product of $K_J \rho_A V$.

— / —

How fast does a surface water travel?
Depth : h
Gravity : g
Wavelength : λ.



$$[c] = LT^{-1}$$
$$[g] = LT^{-2}$$



Short waves, deep waters:

$\lambda / h << 1$

$c \sim (gh)^{1/2}$ → speed a wave travels (different lengths, different speeds).

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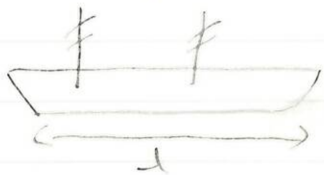

Max speed $c = \sqrt{g\lambda}$ ← make them longer.

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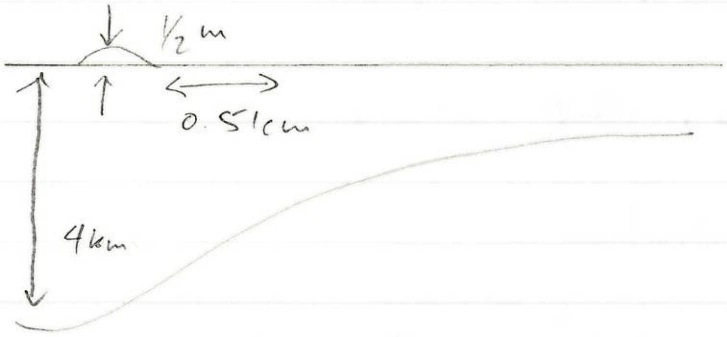
Long waves on shallow water:

$\lambda / h >> 1$

$c \sim (gh)^{1/2}$ All waves travel at same speed.

$$c = \sqrt{gh}$$
$$= [10\,\mathrm{m\,sec^{-2}} \times 4 \times 10^3\,\mathrm{m}]^{1/2}$$
$$= [4 \times 10^4\,\mathrm{m^2\,sec^{-2}}]^{1/2}$$
$$= 200\,\mathrm{m\,sec^{-1}}.$$



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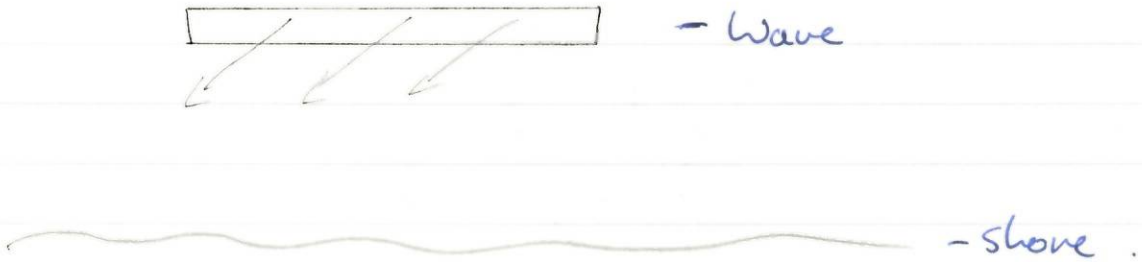

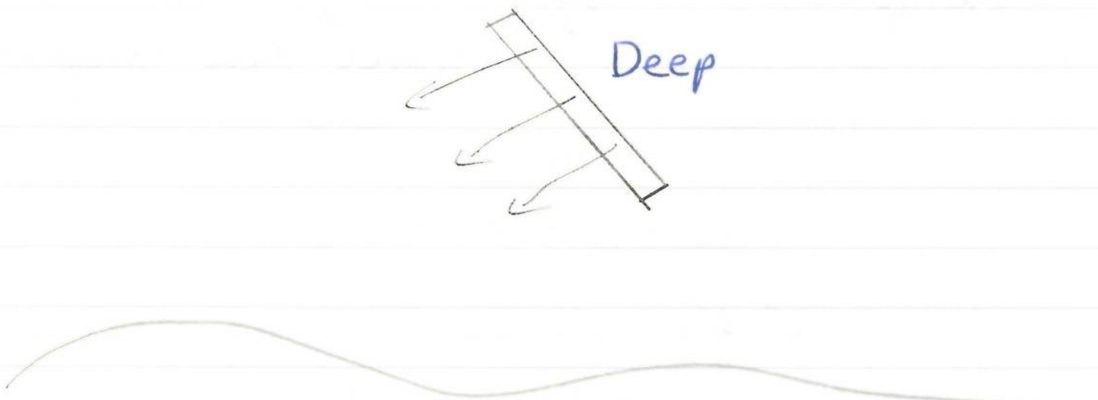

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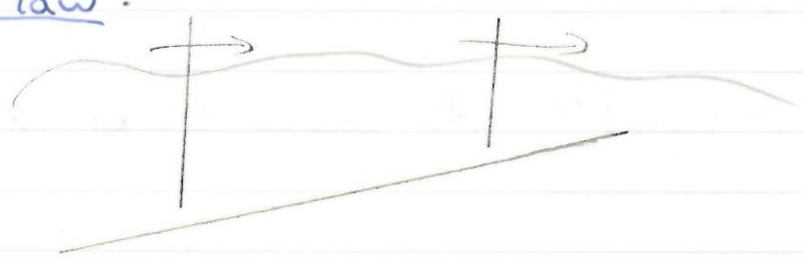
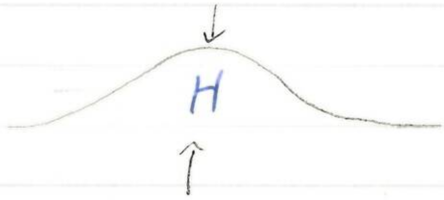
<u>Green's law</u>:



Speed × Energy Density
cH^2



Energy a quadratic equation:

$$cH^2 = \text{Constant}.$$

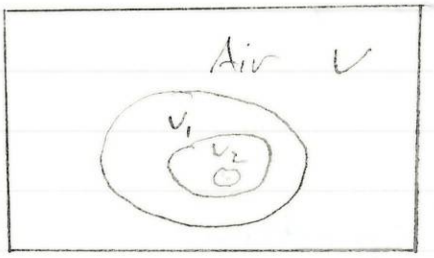
$H^2 \propto c^{-1}$, $H \propto c^{-1/2}$, $c \propto h^{1/2}$, $H \propto h^{-1/4}$.

4km → 0.4m
h decreases so H increase by 10.

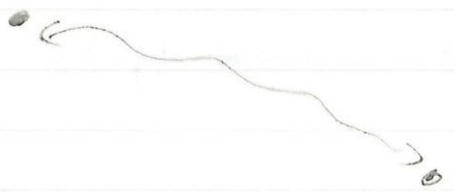
<u>Chapter 1</u>
<u>Specification and Kinematics</u>
Continuum - a substance that we take arbitrary small **volumes** of and whose properties remain the same.
 ↳ We do so ($\lim_{\delta v \to 0}$ exist).

Take a volume V, measure "its" mass m, and define its density $\bar{\rho} = m/v$.

Could take $V \supset V_1 \supset V_2 \ldots$ and defines the density at some point common to this sequence: $\rho = \lim_{v \to 0} (m/v)$.

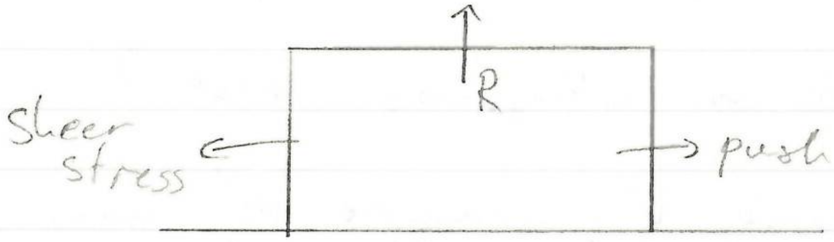


This is a good approximation to reality provided we are interested in motions at scales large compared to the mean free path.



mean free path.

We will restrict attention to invicid fluids (fluids that are not viscous). A fluids are invicid if it cannot support a shear stress.



Due to friction on bottom, opposing.

<u>Summary</u>

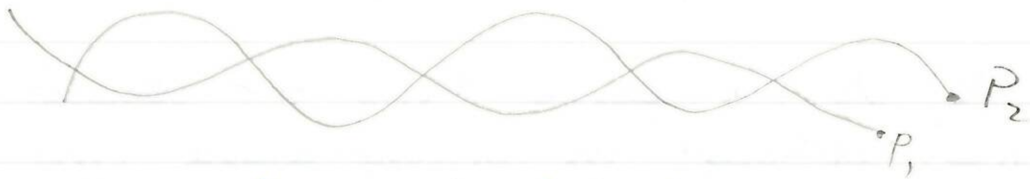
1) Continuum: We can dissus inifinitesimal volume of fluid.

2) Invicid : The fluid cannot support a shear stress.

3) Incompessible : The volume of the fluid element remains the same throughout the motion.

A element composed of the same fluid has the same. **Hence density is constant**.

a) This does <u>not</u> mean that the density is the same everywhere.



b) This is a good approximatation provided speeds are small compare with the speed of sound (700 mph) i.e the mach number of the flow; (typical speed ÷ sound speed = m) is small (<< 1), m < 1 supsonic, m > 1 supersonic.

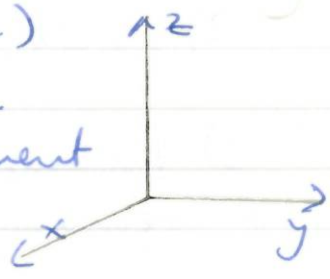
To describle the flow we have two choices.

a) <u>Lagragian labelling</u> - Label all particles + follow their motions i.e follow particle path.

Strength - Conservation lows easy.

Drawback - Simple motion can have complicated paths.

b) <u>Eulerican description</u> - set up fixed axies, we define a velocity field $\underline{u}(x, y, z, t)$ by defining the velocity $\underline{u}$ at time t to be the velocity of the fluid element (or fluid particle) that is at $\underline{x}$ at time t.



Strengths - velocity is a vector field; we can use vector calculus.

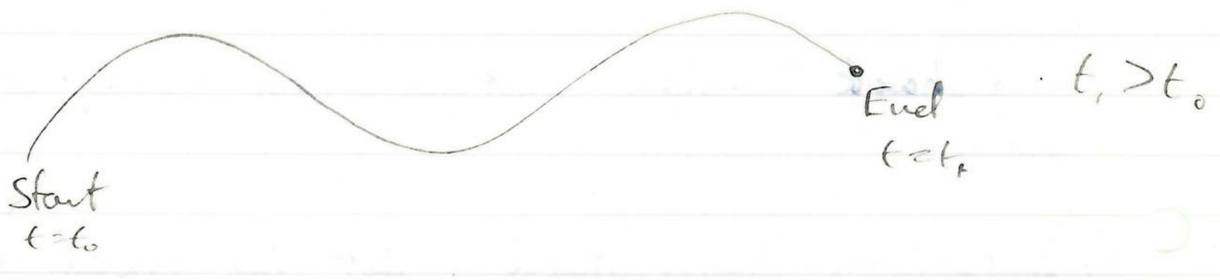
Drawback - conservation laws become a little more complicated.

We do the same thing for density $\rho(x, y, z, t)$, i.e. although in incompressable flow, each particle maintains its own density, the Eulerian density (at a point) can change as different particle occupy that point at different times.

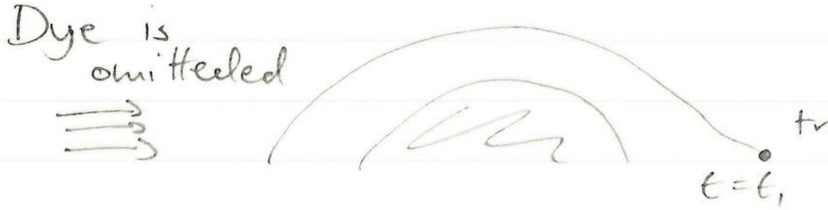
Of course, in homogeneous fluid, ρ = constant.

there are three **ways of increasing** or decreasing a motion.

1) Particle path: the path traced out by the fluid element during a given time interval.



2) Streakline / Filament line : the locus of all particles, that have passed through a given point in a given time interval.



Take snapshot at t_1

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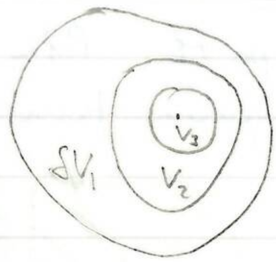
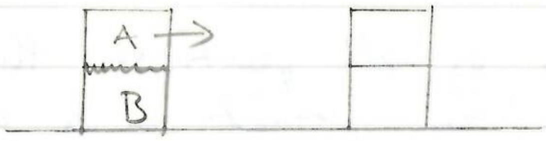
$\lim_{\delta V \to 0}$ ~~fluid~~, Continuum,

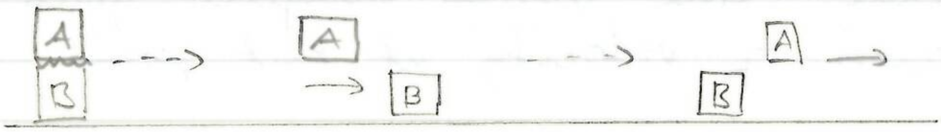


$$\lim_{\delta V \to 0} \left(\frac{\Delta M}{\Delta V} \right) = \rho .$$

—/—

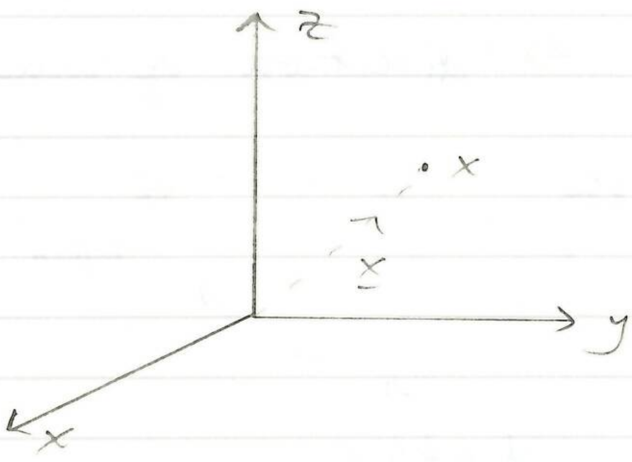


A transmit a **shear** stess to B.



A does not transmit shear (tangential) stress:
(force unit area).

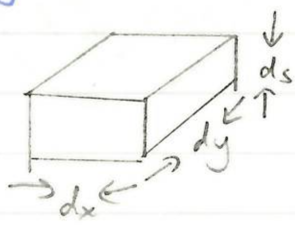
NOT VISCOUS - INVISCI



Eulerian: $\underline{u}(x,t)$ = velocity of particle that happens to be at $\underline{x}$ at time t.

Visualise :

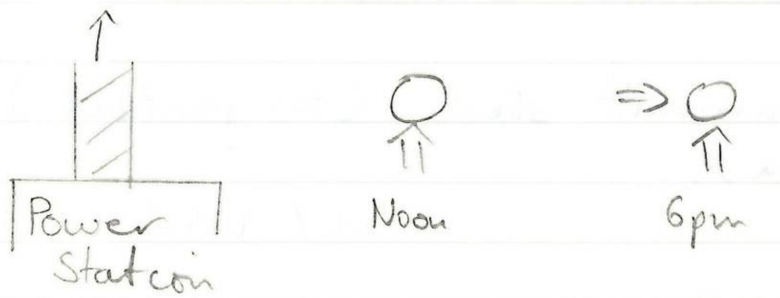
1) Particle path : Path traced out by a **fluid** element in a given time interval.



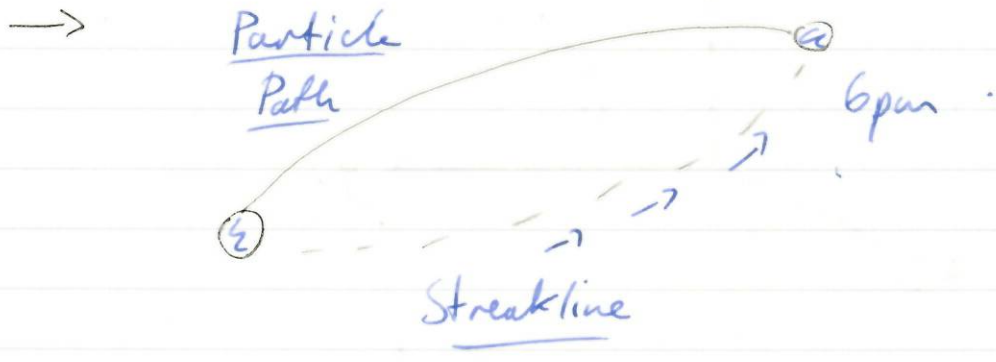
Infinitesmal fluid element.

2) Streakline : The locus of all particle that have passed through a given point in a given time interval

3) Streamline : Line whose tangent gives the direction of velocity at that point.



$\Rightarrow$: direction of wind.



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Suppose we are given a velocity field $\underline{u}(r,t)$

Particle Path satisfy

$$\frac{d\underline{r}}{dt} = u(\underline{r},t) \quad \text{with} \quad \underline{r} = \underline{r_0} \quad \text{at } t=0.$$

Example:
Consider the two-dimensonal velocity field.

$$\underline{u}(\underline{r},t) = \underline{i} + -2te^{-t^2}\underline{j}$$

—/—

2D flow field : field indepenant of the thriol directcon i.e the same in each x y plane.

We shall also take the velocity components in the normal directcon to be zero.

In cartesian it is conventional to write:

$$\underline{u}(x,y,z,t) = u(x,y,z,t)\underline{\hat{i}} + v(x,y,z,t)\underline{\hat{j}} + w(x,y,z,t)\underline{\hat{k}}$$

i.e $\underline{u} = (u,v,w)$.

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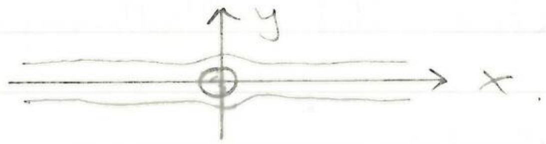
2D flow $- \; w \equiv 0$, $u = u(x,y,t)$, $v = v(x,y,t$

Flow is the same at each z:



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Now: $\underline{u}(\underline{r}, t) = \hat{\underline{i}} - 2te^{-t^2}\hat{\underline{j}}$

So: $\frac{d\underline{r}}{du} = \underline{u}(\underline{r}, t)$.

i.e $\frac{dx}{dt} = u$, $\frac{dy}{dt} = v$.

Here: $u = 1$
$v = -2te^{-t^2}$.

So: $\frac{dx}{dt} = 1$ and $\frac{dy}{dt} = -2te^{-t^2}$.

i.e $x = t + x_0$ and $y = e^{-t^2} + y_0$.

What is the path traced out by the particle released from $(1, 1)$ at $t = 0$?

At $t = 0$, $x = 1$ so $x_0 = 1$, $y = 1$ so $y_0 = 0$

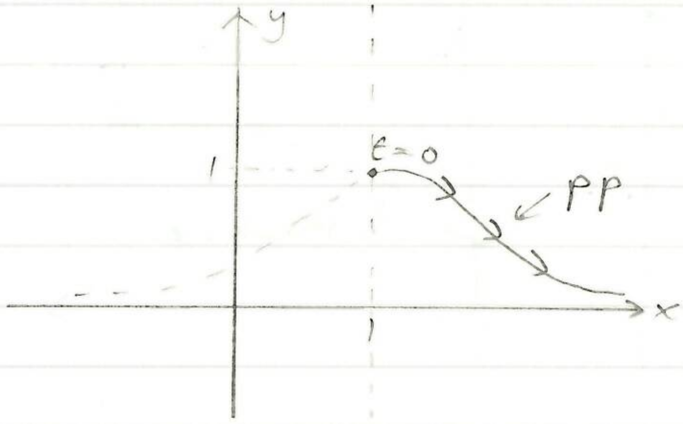
Thus p.p (particle path) is:

$$x = 1 + t$$
$$y = e^{-t^2}.$$

Parameterised by time t.

[Use Parametric Plot: Mathematica]

Here $t = x - 1$ so $y = e^{-(x-1)^2}$



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What is the streakline traced out particles relased from $(1,1)$ at time $\tau < 0$ when viewed at time $t = 0$?

Particle Path: $x = t + x_0$.
$y = e^{-t^2} + y_0$.

At τ, particle in locus was at $(1,1)$. [That's when it was emitted].

$$1 = \tau + x_0, \quad 1 = e^{-\tau^2} + y_0.$$

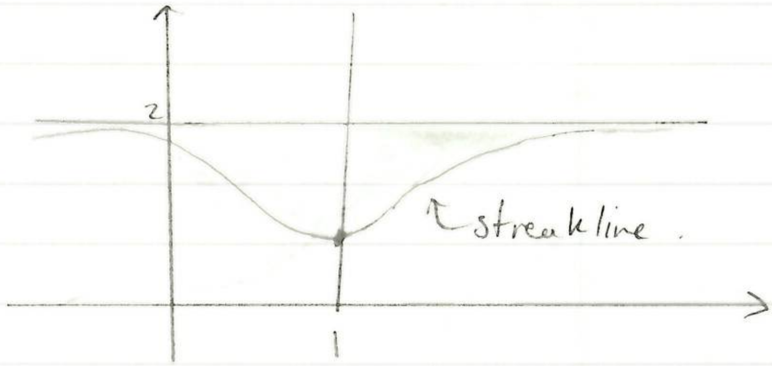
i.e $x_0 = 1 - \tau$, $y_0 = 1 - e^{-\tau^2}$

The particle is at (x, y) at time t when

$$x = t + 1 - \tau, \quad y = e^{-t^2} + 1 - e^{-\tau^2}$$

So at $t = 0$ it is at $x = 1 - \tau$, $y = 2 - e^{-\tau^2}$

Parameterised by time of emission. Sufficiently simple that we can eliminate τ.



Emitted at $\tau < 0$, look at $t = 0$

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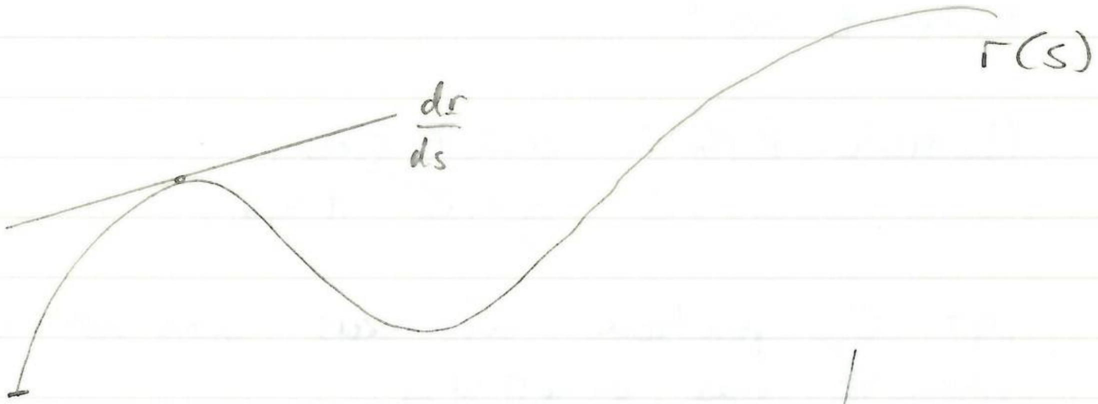
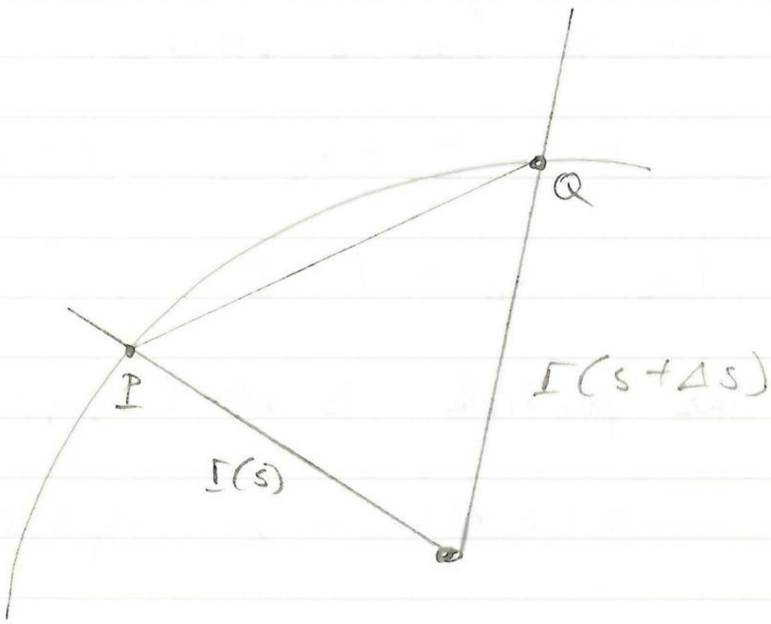
<u>Streamline</u>.





$$\lim_{\substack{P\to 0\\ \Delta s\to 0}} \frac{r(s+\Delta s) - r(s)}{\Delta s} = \frac{dr}{ds}.$$

Parameterise $\underline{r}$ on s.

$$\frac{d\underline{r}}{ds} = \underline{u}(\underline{r}, t).$$

This the streamline at $t = t_0$, are given by solving

$$\frac{d\underline{r}}{ds} = u(\underline{r}, t_0).$$

Example: For our velocity field, what are the streamline at $t=0$?

$$\frac{dx}{ds} = u(x, y, 0) = 1,$$

$$\frac{dy}{ds} = v(x, y, 0) = -2te^{-t^2}\Big|_{t=0} = 0.$$

$\therefore\ y = \text{const}$

$x = s + x_0.$

* At $t=0$, $\underline{u} = \underline{\hat{c}}$; i.e: The tangent to streamline does give the velocity field!

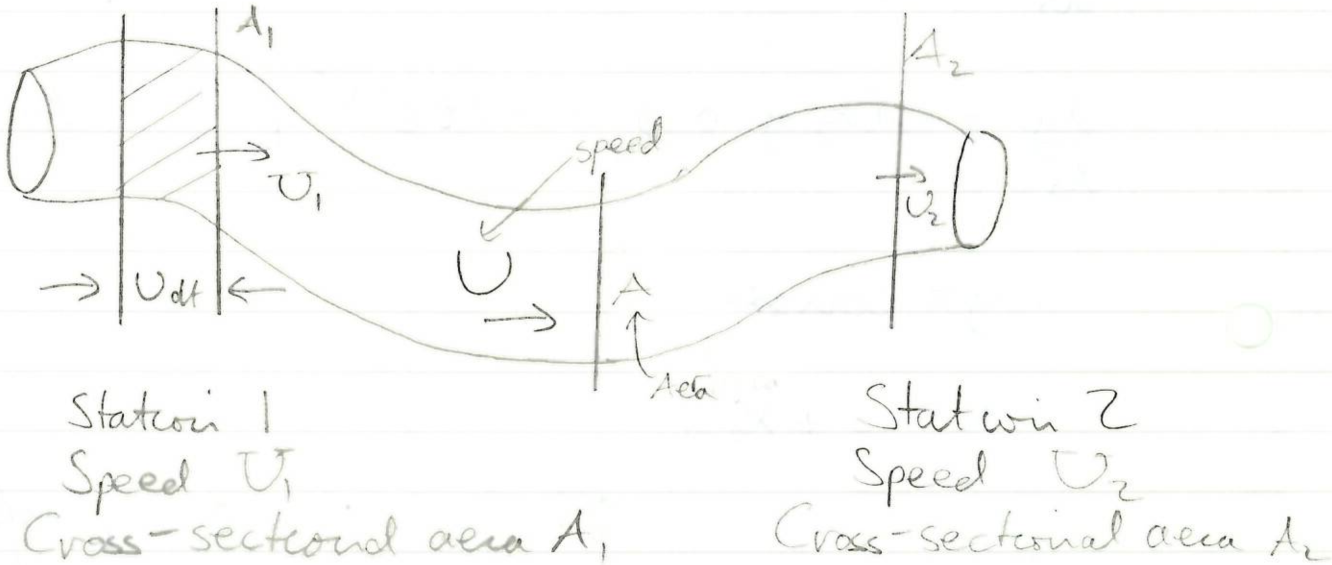
— / —

In a steady flow, all these are the same.

Steady: $\dfrac{\partial \underline{u}}{\partial t} = 0$. (Does not say $\underline{u} = 0$)

2. Conservation of Mass

Suppose a fluid of constant density ρ flows through a tube of cross-sectional area A. Suppose the fluid velocity is uniform and unidirectional of size U at each cross-section:



The amount of mass between the two stations is fixed.

In a time interval dt an amount

$$\rho A_1 U_1 \, dt.$$

of mass crosses station 1.

The amount crossing A_2 in time dt is:

$$\rho A_2 U_2 \, dt .$$

By conversation of mass, these are the same so $A_1 U_1 = A_2 U_2$.

Flux

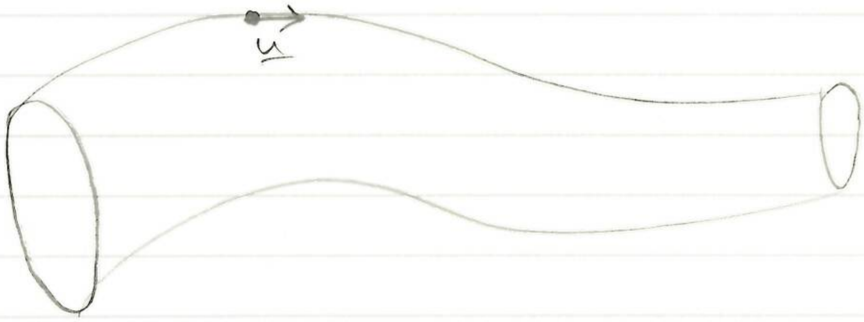
Or in terms of flux, the rate at which mass crosses A_1 is:

$$\frac{\rho A_1 U_1 \, dt}{dt} = \rho A_1 U_1$$

This must equal the flux across A_2 i.e

$$\rho A_2 U_2 .$$

The tube can be any surface that fluid does not cross



Stream tube is formed by taking a closed loop of particles and drawing the streamline emminating from them. Flow cannot cross the tube as $\underline{u}$ tangental to s'lines.

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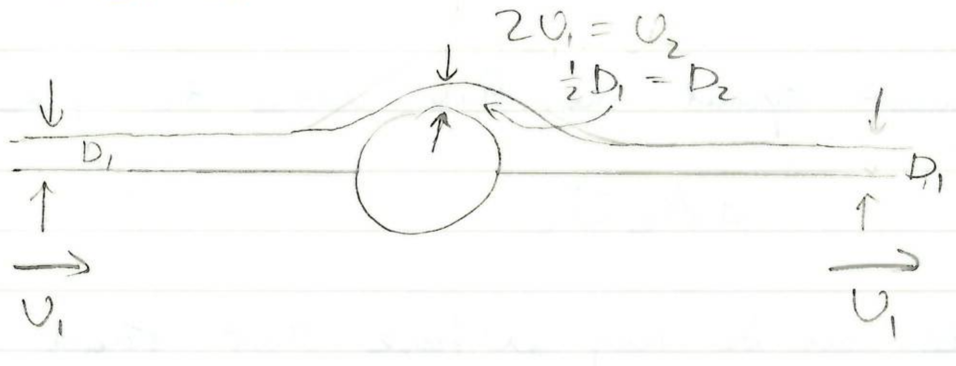
If area halves, speed doubles.

In 2D: - Because thrid velocity component, $w=0$ streamlines compress only in xy plane so we have:

$$U_1 D_1 = U_2 D_2$$

where D is the distance between s'lines i.e. speed inversely portportional to the separation of the s'lines.



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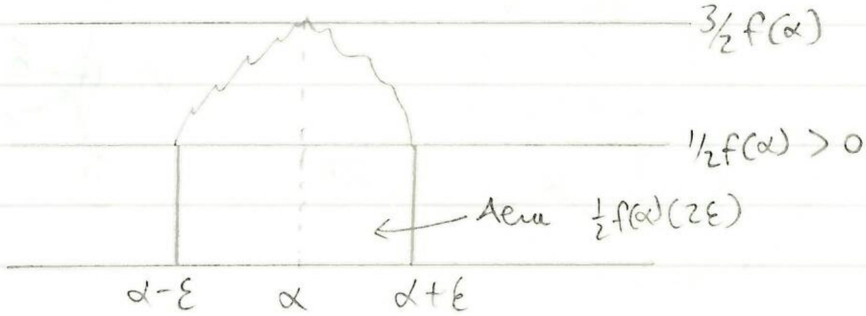
Theorem 1: If f is continuous in $[a,b]$ and $\int_c^d f = 0$ for each $(c,d) \subset [a,b]$ then $f = 0$ on $[a,b]$.

Proof: Suppose $\exists \alpha \in [a,b]$ st $f(\alpha) \neq 0$ w.l.o.g we can take $f(\alpha) > 0$.

Write $\delta = \frac{1}{2} f(\alpha) > 0$. Hence $\exists\, \varepsilon > 0$ st if $x \in (\alpha - \varepsilon, \alpha + \varepsilon)$ then $|f(x) - f(\alpha)| < \delta = \frac{1}{2} f(\alpha)$ i.e: $0 < \frac{1}{2} f(\alpha) < f(x) < \frac{3}{2} f(\alpha)$.



Thus $\int_{\alpha-\varepsilon}^{\alpha+\varepsilon} f(x)\,dx > \int_{\alpha-\varepsilon}^{\alpha+\varepsilon} \frac{1}{2} f(\alpha)\,dx$

$$\begin{aligned} RHS &= 2\varepsilon \tfrac{1}{2} f(\alpha) \\ &= \varepsilon f(\alpha) > 0. \end{aligned}$$

But $\int_c^d f = 0 \ \forall (c,d) \subset [a,b]$ so contradiction so $\nexists \alpha$ i.e $f \equiv 0$ in $[a,b]$.

This result extends immediately to n dimension

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Ansatz: Suppose we wish to derive an equation $f = 0$ for a fluid in 3D.

Let the fluid occupy a domain D in 3D.

Take an arbitrary subdomain V of D
↳ Terribly imporant

Show that $\int_V f$ vanishes. Then $f \equiv 0$ in D because V is abitrary i.e $\int_V f = 0$ for every subdomain V of D.

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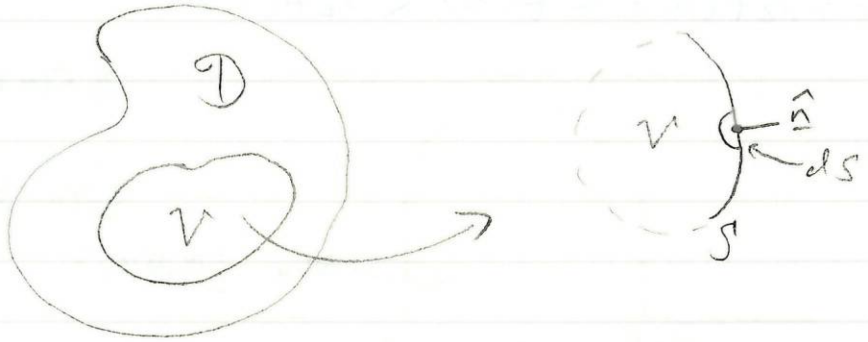
Conservation of Mass

Consider a fluid occupying a domain D with surface S.

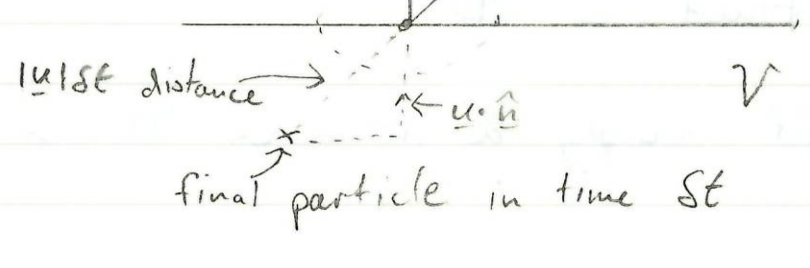


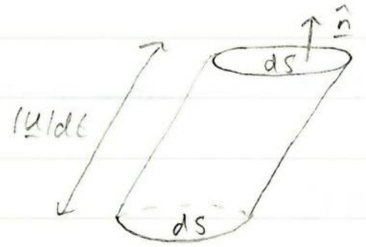
Consider a small element dS of S with outward pointing unit normal $\hat{\underline{n}}$.

Let the velocity field in D be $\underline{u}$

Take the density of the **fluid** to be constant and equal to ρ everywhere.

Then in time $\delta t << 1$, an amount mass $\rho(\underline{u} \cdot \hat{\underline{n}})\, dS\, \delta t$ crosses dS.





Volume = Area of Base × Height.
$= dS' \cdot (\underline{u} \cdot \hat{\underline{n}}) \delta t$.

Note that : $(\underline{u} \cdot \hat{\underline{n}}) \delta t$ - component of $\underline{u} \delta t$ in the direction $\hat{\underline{n}}$ i.e hieght.

The total mass passing out of V is

$$\int_S \rho (\underline{u} \cdot \hat{n}) \, dS' \delta t = \rho \delta t \int_{S'} \underline{u} \cdot \hat{\underline{n}} \, dS'.$$

$\hookrightarrow \rho \delta t \int_{S'} \underline{u} \cdot \hat{\underline{n}} \, dS' =$ Outward mass flux across dS'.

But to conserve mass in V, this must be zero, i.e $\int_{S'} \underline{u} \cdot \hat{\underline{n}} \, dS' = 0$.

Divergence theorem says that $\int_V \underline{\nabla} \cdot \underline{u} \, dV = 0$

Thus we have, $\forall$ subregions V of $\mathcal{D}$, $\int_V \nabla \cdot \underline{u} \, dV = 0$

Thus by theorem 1 $\underline{\nabla} \cdot \underline{u} = 0$ in $\mathcal{D}$.

— / —

In 2D : If $\underline{u} = u(x, y, t) \hat{\underline{i}} + v(x, y, t) \hat{\underline{j}}$ then

$$\underline{\nabla} \cdot \underline{u} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}$$

— / —

So, 1) $\underline{u} = 7x\,\hat{\underline{i}} - 5y\,\hat{\underline{j}}$. — Not incompressible!

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} = 7 - 5 = 2 \neq 0.$$

i.e compressible (non-constant ρ)

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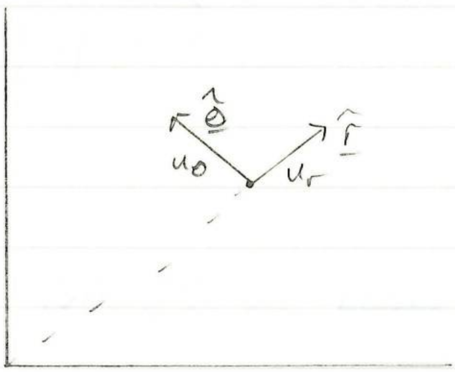
In 3D : $\dfrac{\partial u}{\partial x} + \dfrac{\partial u}{\partial y} + \dfrac{\partial u}{\partial z} = 0.$



$$\underline{\nabla} \cdot \underline{u} = \frac{1}{r}\frac{\partial}{\partial r}(r u_r) + \frac{1}{r}\frac{\partial u_\theta}{\partial \theta}$$

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Reminder : Green's lemma :

Consider a closed region A in the plane bounded by a curve C, taken counter-clockwise :

$$\int_A \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) dA = \oint u\,\partial y - v\,dx$$

line integral.

where $\underline{u} = u\hat{\underline{i}} + v\hat{\underline{j}}$.

Thus in 2D, incompressible flow.

$$\oint_C u\,dy - v\,dx = 0.$$

for any closed curve C.

—/—

Note : $d\underline{r} = dx\hat{\underline{i}} + dy\hat{\underline{j}}$
Note : $\underline{u} = u\hat{\underline{i}} + v\hat{\underline{j}}$.

Consider the vector field:

$$\underset{\sim}{F} = -v\hat{\underline{i}} + u\hat{\underline{j}}$$

$$= \hat{\underline{k}} \wedge (u\hat{\underline{i}} + v\hat{\underline{j}})$$

$$= \hat{\underline{k}} \wedge \underline{u}.$$

i.e rotates $\underline{u}$ by $90°$.

Hence: $\oint \underline{F} \cdot d\underline{r} = 0$ for all closed vector field.

i.e $\underline{F}$ is derivable from a potential.

i.e $\exists \psi$ st $\underset{\sim}{F} = \underset{\sim}{\nabla}\psi$ i.e $\hat{k} \wedge \underline{u} = \nabla\psi$

i.e $\underline{u} = -k \wedge \underline{\nabla}\psi$

—/—

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Incompressibility (constant density)

$\Rightarrow \underline{\nabla} \cdot \underline{u} = 0 \quad [\Rightarrow \exists \underline{A} \text{ st } \underline{u} = \underline{\nabla} \wedge \underline{A}]$

—/—

In 2D: $\underline{u} = u(x,y)\,\hat{\underline{i}} + v(x,y)\,\hat{\underline{j}}$

$$\underline{\nabla} \cdot \underline{u} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}$$

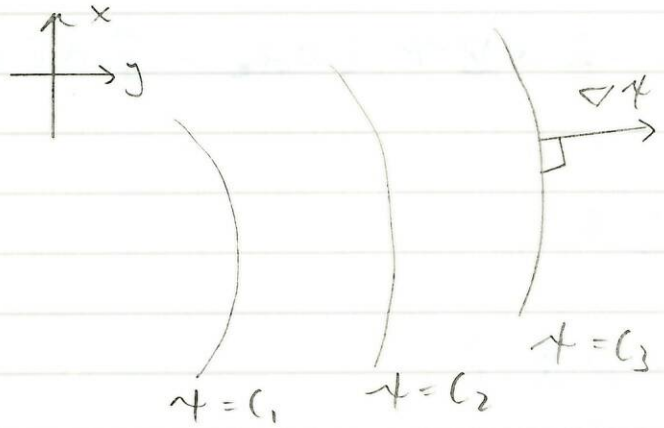
If $\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \Rightarrow \exists \psi$ st $\underline{u} = -\hat{\underline{k}} \wedge \underline{\nabla}\psi$
$= \underline{\nabla}\psi \wedge \hat{\underline{k}}$.

—/—

First: $|\underline{u}| = |\underline{\nabla}\psi|$

90 degrees to ψ

$\nabla\psi$ is $\perp$'r to level curves or isolines of ψ.



$c_3 > c_2 > c_1$

Second: $\underline{u}$, $\underline{\nabla}\psi$ are $\perp$'r (both x, y plane). In fact $\underline{u}$ is $\underline{\nabla}\psi$ rotated $\pi/2$ clockwise.

Finally: $\underline{u}$ is tangent to isoline $\psi = C$ for any C, i.e. isolines $\psi = c$ are streamlines, i.e. we have shown that incompressible flow $\exists$ a function ψ whose isoline are s'lines

—/—

Example: Show that $\underline{u} = x\hat{\underline{i}} - y\hat{\underline{j}}$ satisfies the continuity equation, find a streamline, sketch some s'lines (and suggest a flow).

Continuity equation: $\dfrac{\partial u}{\partial x} + \dfrac{\partial v}{\partial y} = 0$

Here $u = x$, $v = -y$.

$$\Rightarrow \frac{\partial u}{\partial x} = 1, \quad \frac{\partial v}{\partial y} = -1$$

Thus $\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$ as required.

Hence $\exists \psi$ st $\underline{u} = -\hat{\underline{k}} \wedge \underline{\nabla}\psi$ (Note: $\hat{\underline{k}} \wedge \hat{\underline{i}} = \hat{\underline{j}}$, $\hat{\underline{k}} \wedge \hat{\underline{j}} = \hat{\underline{i}}$).

So: $\underline{u} = -\underline{k} \wedge \underset{\sim}{\nabla}\psi$

$$= \frac{\partial \psi}{\partial y}\hat{\underline{i}} - \frac{\partial \psi}{\partial x}\hat{\underline{j}}.$$

$$= u\hat{\underline{i}} + v\hat{\underline{j}}.$$

$$\Rightarrow \quad u = \frac{\partial \psi}{\partial y} \quad \text{and} \quad v = -\frac{\partial \psi}{\partial x}.$$

For this example $u = x$, thus $\frac{\partial \psi}{\partial y} = x$.

So $\psi = xy + f(x)$ where f is an arbitrary function of x.

This implies : $\frac{\partial \psi}{\partial x} = y + f'(x)$.

But $\frac{\partial \psi}{\partial x} = -v = f'(x)$

Comparing gives $f'(x) = 0$ i.e f is constant
w.l.o.g we can take the function $f = 0$.

—/—

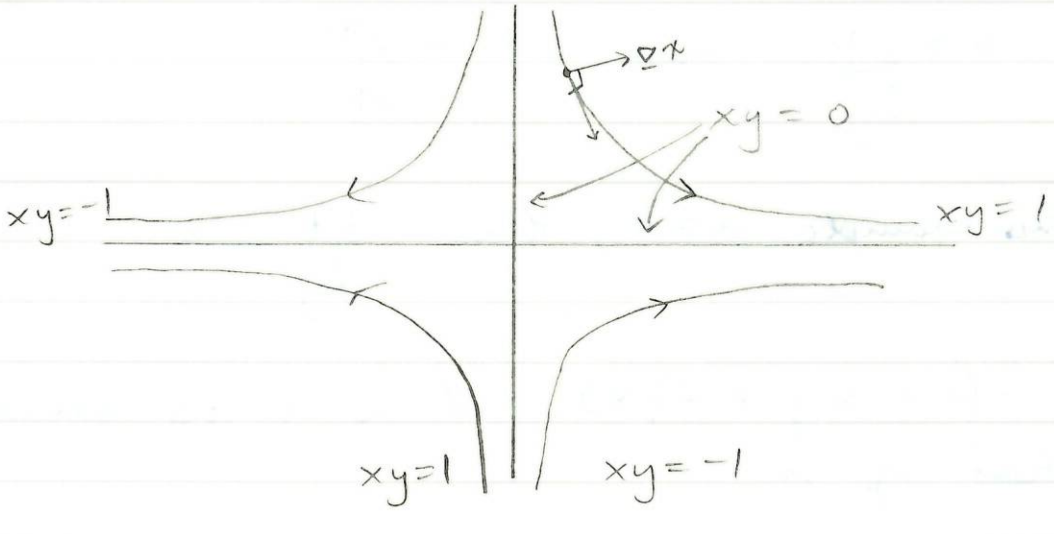
[w.l.o.g since $\underline{u} = -\underline{k} \wedge \underline{\nabla}\psi$, so adding a constant to ψ does not change $\underline{u}$]

—/—

ψ is unique to within an additive constant, hence the streamfunction ψ is

$$\psi = xy.$$

Streamlines; the lines ψ = constant i.e $xy = c$. (Rectangular hyperbola)



or $u = x$, So if $x > 0$, $u > 0$.

—/—

Since $|\underline{u}| = |\underline{\nabla} \chi|$, speed is directly porportenial to $|\underline{\nabla} \chi|$ or equivalently $|\underline{u}|$ is inversely porportional to the separation of lines of χ.

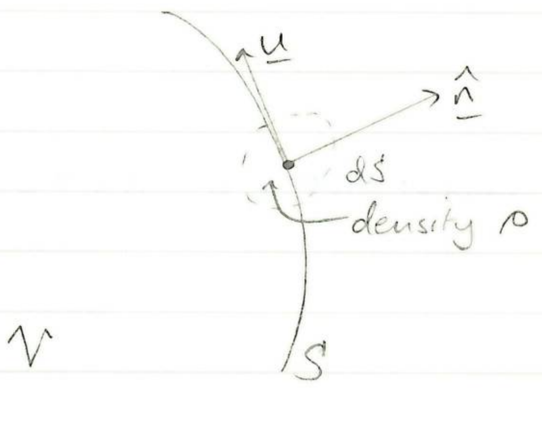
—/—

There is stagnation point flow as the origin is a stagnation point where $\underline{u} = 0$.

this flow could be 2 colliding jets of equal strength can be an example of this flow.

—/—

Flow condition at a solid boundary



Solid $\Rightarrow$ impermeable i.e no flow through the boundary.

The mass of fluid passing through dS in time δt is:

$$\rho(\underline{u}\cdot\hat{\underline{n}})\, dS\,\delta t.$$

or there is a mass flux (rate at which mass crosses dS).

For no mass, $\underline{u}\cdot\hat{\underline{n}} = 0$ on S. On a solid boundary: $\underline{u}\cdot\hat{\underline{n}} = 0$ i.e velocity is tangential to surface.

— / —

able to support shear stess.

[If the fluid is also viscous, additionally the tangental component to $\underline{u}$ vanishes also so $\underline{u} = 0$ on a solid boundary.]

→ Stokes ← (Real fuilds)

Aside for 3rd year course

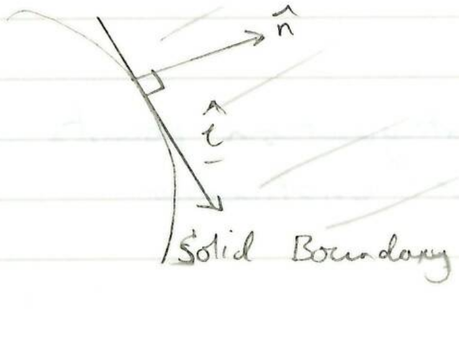
— / —

In terms of the s'fn:

$$\hat{\underline{n}}\cdot\underline{u} = -\hat{\underline{n}}\cdot(\hat{\underline{k}}\wedge\nabla\psi)$$

$$= -(\hat{\underline{n}}\wedge\hat{\underline{k}})\cdot\underline{\nabla}\psi$$

$$= -\frac{\partial\psi}{\partial s}$$ along the surface (direction derivative)



$\hat{t}$ - unit tangent surface

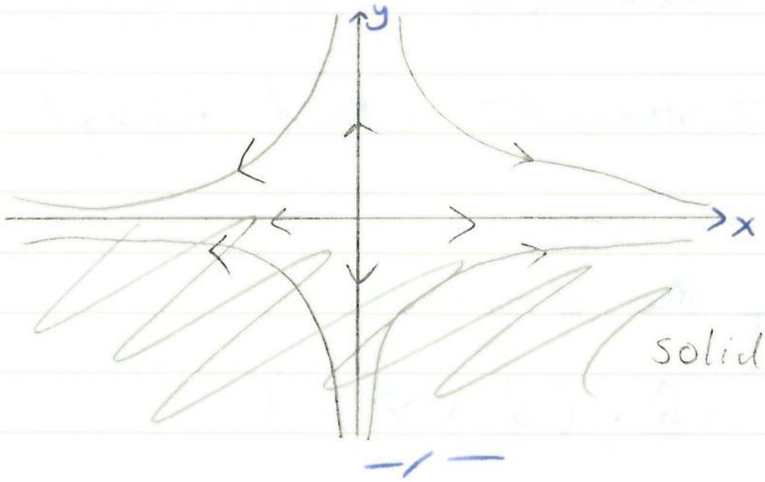
But $\underline{u} \cdot \hat{\underline{n}} = 0$, so $\frac{\partial \psi}{\partial s} = 0$ along a solid boundary

i.e ψ = constant on solid bdry.

Equivalently any line ψ constant has $\underline{u}$ tangential i.e can be a solid bdry i.e on a solid boundary, ψ = const. Any line ψ = constant can be replaced by solid bdry without affecting an invicid flow.

Solid bdry: $\underline{u} \cdot \hat{\underline{n}} = 0$ or ψ = constant.

Ex: $u = x$, $v = -y$, $\psi = xy$ (w/o - without)



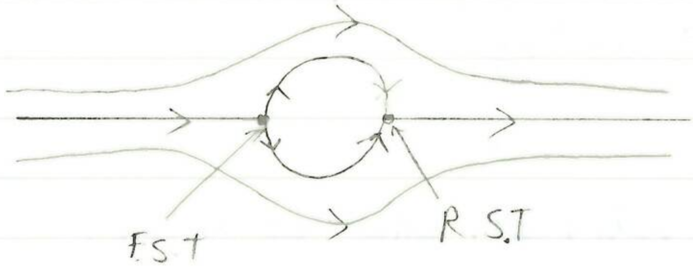
— / —

Replace any s'line by solid boundary (in inviscid flow) w/o changing flow.

— / —

Here we obtain a jet hitting a wall. - stagnation point flow.

e.g: Front and rear stagnation points in uniform flow past a circular cylinder:



F.S.T - Front stagnation point.
R.S.T - Rear stagnation point.

— / —

Ex 2: (Same question in ex. 1.)

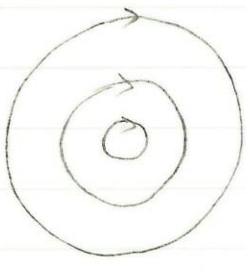
Now $u = 2y$, $v = 2x$

Thus $\frac{\partial \psi}{\partial y} = u = 2y$.

So $\psi = y^2 + f(x)$

So $\frac{\partial \psi}{\partial x} = f'(x)$. $\Rightarrow \psi = x^2 + y^2$

i.e circles centre O radius a:



i.e saucepan or beaker on a turntable.

Rotating as a solid body; $\underline{u} = -2\underline{\hat{k}} \wedge \underline{r}$, where

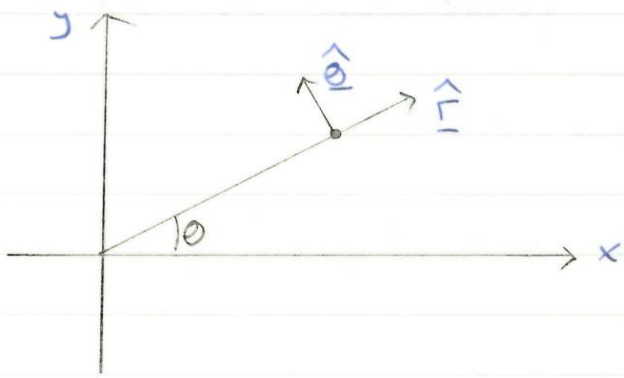
$\psi = r^2, \quad r = \sqrt{x^2+y^2}.$

— / —

Redo in cylindrical polar co-ords.

$\underline{u} = -k \wedge \underline{\nabla} \psi \Leftarrow$ co-ordinates free.

$$\underline{\nabla}\psi = \frac{\partial \psi}{\partial r}\hat{\underline{r}} + \frac{1}{r}\frac{\partial \psi}{\partial \theta}\hat{\underline{\theta}}.$$



$$\hat{\underline{k}} \wedge \hat{\underline{r}} = \hat{\underline{\theta}}$$

$$\hat{\underline{k}} \wedge \hat{\underline{\theta}} = -\hat{\underline{r}}$$

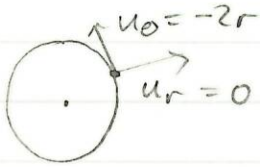
$$\underline{u} = -\hat{\underline{k}} \wedge \underline{\nabla}\psi$$

$$= -\frac{\partial \psi}{\partial r}\hat{\underline{\theta}} + \frac{1}{r}\frac{\partial \psi}{\partial \theta}\hat{\underline{r}}$$

Comparing:

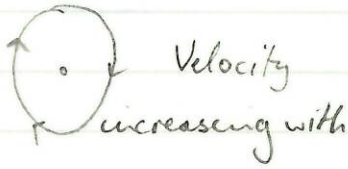
$$\underbrace{u_r = \frac{1}{r}\frac{\partial \psi}{\partial \theta}, \qquad u_\theta = -\frac{\partial \psi}{\partial r}}_{\text{Worth remembering}}$$

We have $\psi = r^2$ in our example.



Thus, $u_r = \dfrac{1}{r}\dfrac{\partial \psi}{\partial \theta} = 0$.

and $u_\theta = -\dfrac{\partial \psi}{\partial r} = -2r$.



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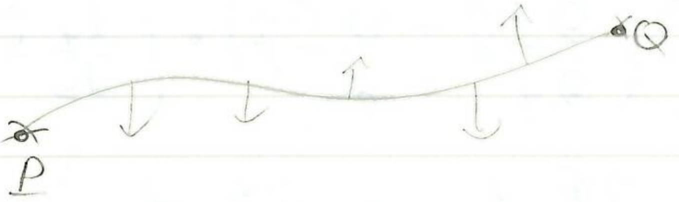
<u>A phyisical interpertation of the streamfunction</u>

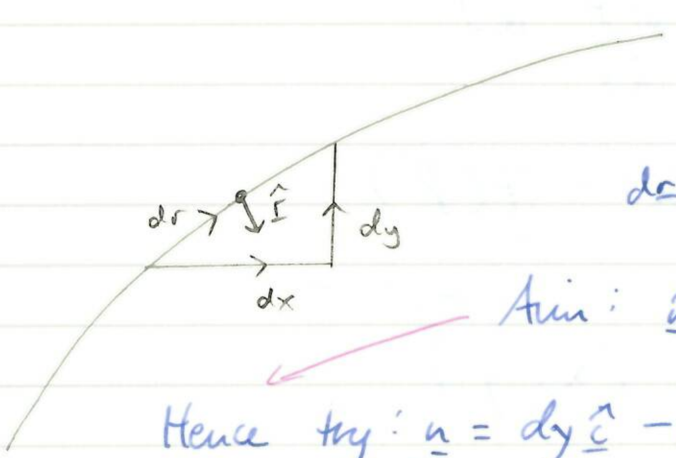
The volume flux in a clockwise directoin across any line joining a point P to a point Q in a flow field is given by:

$$\psi(Q) - \psi(P)$$



c.f – compare with

[c.f work done is indepenant of path]



$d\underline{r} = dx\,\hat{\underline{i}} + dy\,\hat{\underline{j}}$

Aim: $\hat{\underline{n}} \cdot d\underline{r} = 0$

Hence try: $\underline{n} = dy\,\hat{\underline{i}} - dx\,\hat{\underline{j}}$ so $\underline{n} \cdot d\underline{r} = 0$

Normal vector but not unit normal vector.

Volume flux crossing a length dS.

$$\int_P^Q (\underline{u} \cdot \underline{\hat{n}}) \, dS.$$

Thus : $$\underline{\hat{n}} = \frac{dy\,\underline{\hat{\imath}} - dx\,\underline{\hat{\jmath}}}{\sqrt{dx^2 + dy^2}}$$

$$= \frac{dy}{ds}\underline{\hat{\imath}} - \frac{dx}{ds}\underline{\hat{\jmath}}$$

Thus total flux crossing line P and Q in clockwise direction is:

$$\int_P^Q (\underline{u} \cdot \underline{\hat{n}}) \, dS$$

$$= \int_P^Q \left(\frac{\partial \psi}{\partial y}\underline{\hat{\imath}} - \frac{\partial \psi}{\partial x}\underline{\hat{\jmath}} \right) \cdot \left(\frac{dy}{ds}\underline{\hat{\imath}} - \frac{dx}{ds}\underline{\hat{\jmath}} \right) dS.$$

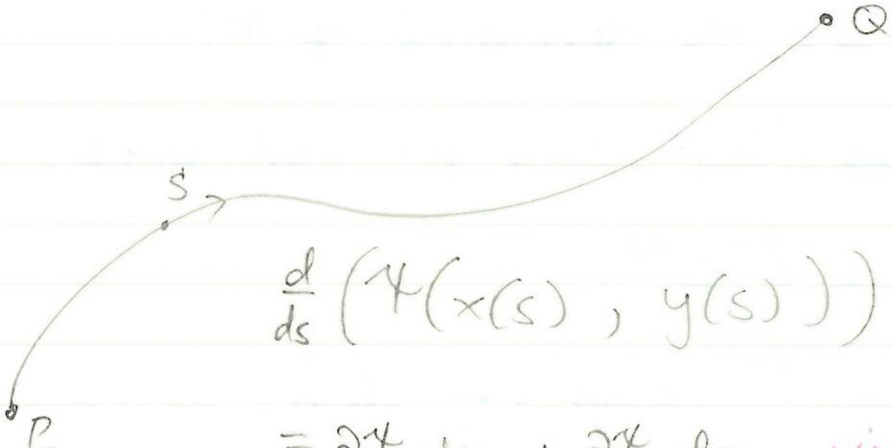
$$= \int_P^Q \left(\frac{\partial \psi}{\partial x}\frac{dx}{ds} + \frac{\partial \psi}{\partial y}\frac{dy}{ds} \right) dS$$

$$= \int_P^Q \frac{d\psi}{ds} \, dS$$

$$= \psi(Q) - \psi(P).$$

—/—

<u>Aside</u> : On (x, y).

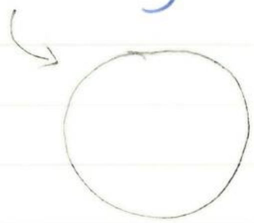


$$\frac{d}{ds}\left(\psi(x(s),\, y(s))\right)$$

$$= \frac{\partial \psi}{\partial x}\frac{dx}{ds} + \frac{\partial \psi}{\partial y}\frac{dy}{ds}$$ via chain rule.

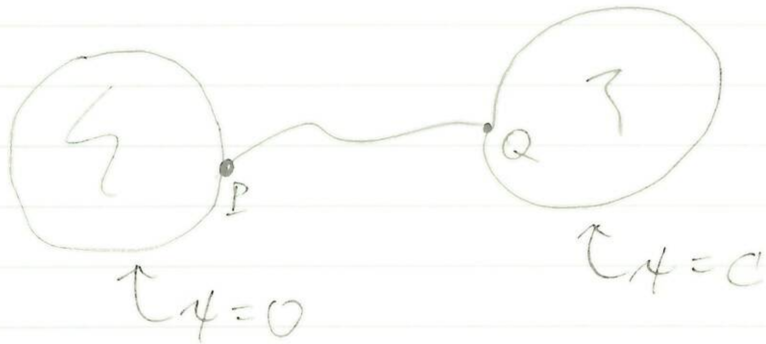
— / —

On 1 boundary:

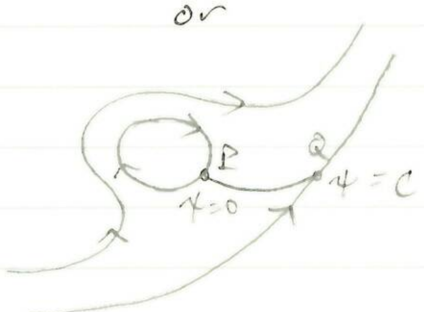


$\psi = 0$
w.l.o.g

On 2 boundaries:



or



— / —

What are the dimension of τ?

Volume / Unit time per unit width.

$$L^3 T^{-1} L^{-1}$$

i.e $L^2 T^{-1}$ i.e an area flux.

— 1 —

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— / —

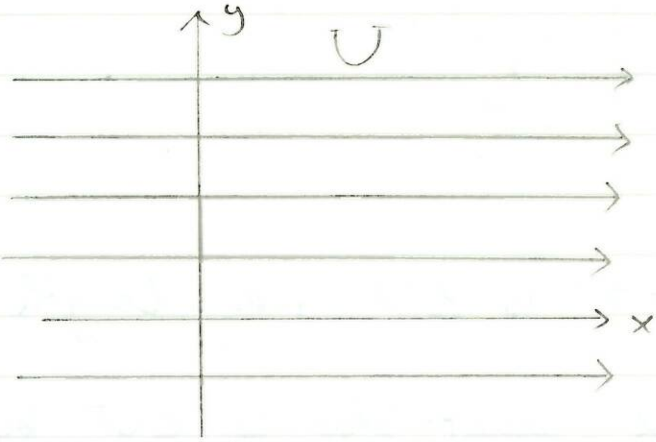
Vorticity : $\underline{\omega} = \underline{\nabla} \wedge \underline{u}$.

— / —

Example of streamfunction

1. Perhaps the simplest flow is a uniform stream.

W.l.o.g take the x-axis in the direction of the flow



Then $u = U$, $v = 0$, so

$$\frac{\partial \psi}{\partial y} = u = U$$

So $\psi = Uy + f(x)$

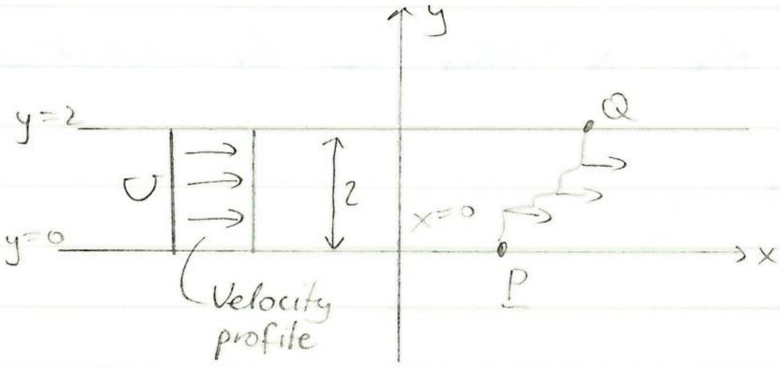
So : $\dfrac{\partial \psi}{\partial x} = f'(x)$

But $\dfrac{\partial \psi}{\partial x} = -v = 0$

So $f'(x) = 0$.

Hence we can take $f = 0$.

So $\psi = Uy$.



Flux across $x=0$ is $2U$ i.e Length × Speed.

Flux across PQ must also be $2U$ because:

1) No fluid escapes across $y=0$ as they are s'lines (and so no normal flow i.e. could replace by solid bdry)

OR

2) Flux $= \psi(Q) - \psi(P) = 2U - 0 = 2U$.

2. Another example of stream function:

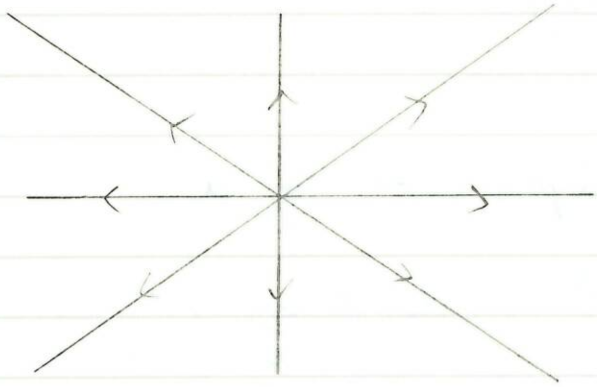
Isotropic source: - this has stream-function:

$$\psi = m\theta .$$

This gives: $\underline{u} = u_r \hat{\underline{r}} + u_\theta \hat{\underline{\theta}}$ where

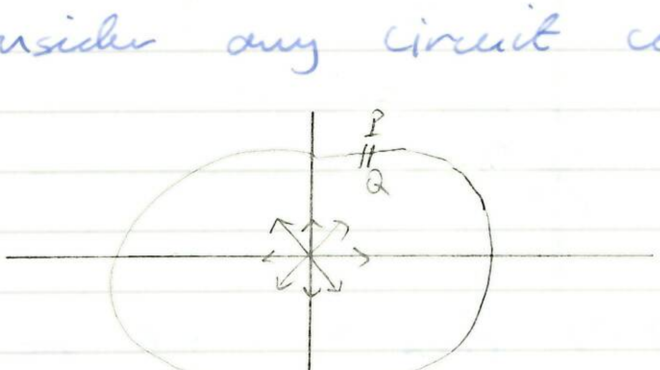
$$u_r = \frac{1}{r}\frac{\partial \psi}{\partial \theta} , \quad u_\theta = -\frac{\partial \psi}{\partial r}$$

Hence: $u_r = \frac{m}{r}$, $u_\theta = 0$.



The velocity field is the same in all direction i.e independant of θ i.e it is isotropic.

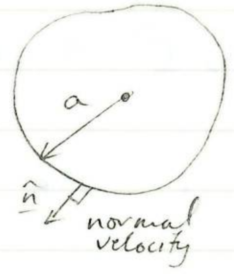
Now consider any circuit containing the origin:



The flux of C is going around any closed curve containing the origin O increases by 2π. So $\psi(Q) - \psi(P) = m(\theta(Q) - \theta(P))$
$= 2\pi m$.

—/—

If



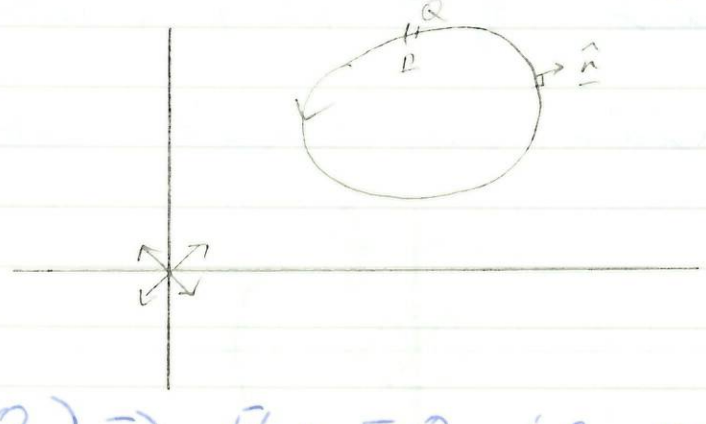
Speed $= m/a$
Length $= 2\pi a$
Flux $= 2\pi a(m/a)$
$= 2\pi m$.

The flux is indepenant of a.

$$\underline{u} \cdot \underline{\hat{n}} = \underline{u} \cdot \underline{\hat{r}} = u_r = \frac{m}{a}.$$

—/—

If curve does not circle the origin:



$\theta(P) = \theta(Q) \Rightarrow$ Flux $= 0$ i.e no net flux across C.

[m = rate at which fluid is created i.e. strength of the source].

—/—

By taking successive smaller circles we see that only at the orgin fluid is created and it is created there at a volume flux $2\pi m$. We call $2\pi m$ the strength of the source i.e a source of strength m has $\psi = \frac{m\theta}{2\pi}$.

In this case the strength is m.

—/—

If $U_r = m/2\pi r$ singular at the orgin but well behaved everywhere else.

If $\psi = m\theta$, strength $= 2\pi m$
If $\psi = \frac{m}{2\pi}\theta$, strength $= 2\pi m/2\pi$
If $= \psi = 7\theta$, strength $= 14m$

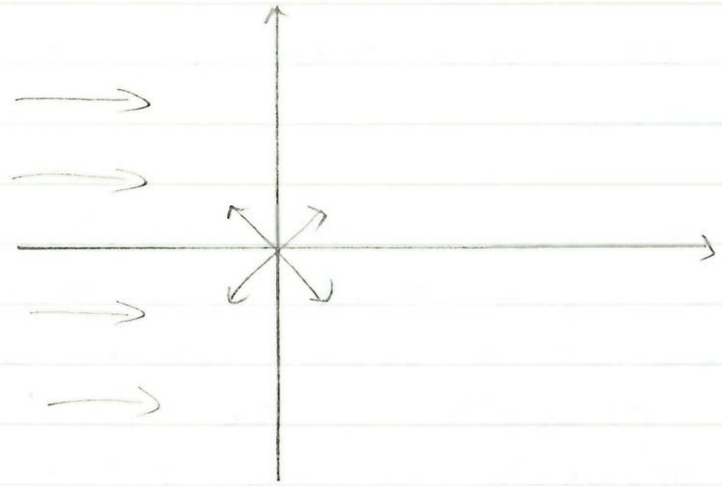
—/—

3) Example of another streamfunction :

Combine these: i.e an isotropic source of strength $2\pi m$ in a uniform stream of speed U.

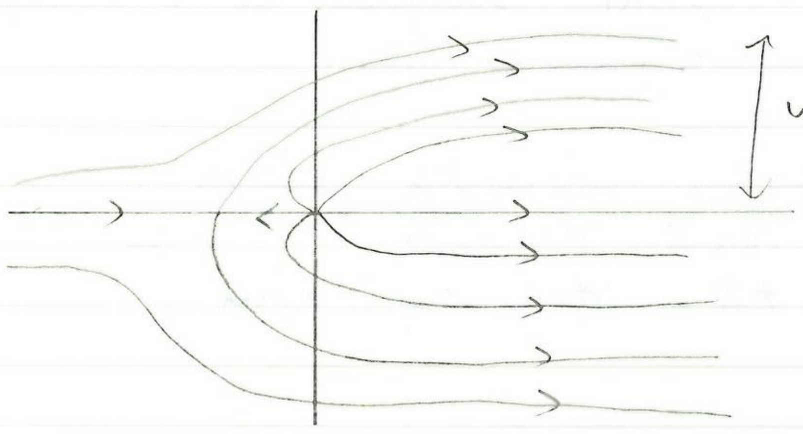
Take x-axis in direction of stream

Take orgin at the source.



Notice:

- source dominates at orgin (for r sufficiently small).
- Stream dominates if r sufficently Large, $u_r = \frac{m}{r}$



$$\psi = Uy + m\theta.$$

–/–

Stagnation point: If $\underline{u} = 0 \Rightarrow u_\theta = 0$
$\Rightarrow \frac{\partial \psi}{\partial r} = 0$ and $u_r = 0 \Rightarrow \frac{\partial \psi}{\partial \theta} = 0.$

–/–

Flux = Width × Speed = $2w \times U = 2\pi m$
So $w = \pi m / U$

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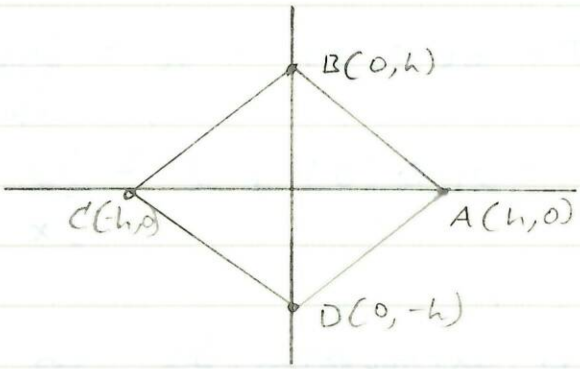
2. Local motion at a point



Consider the initially square element ABCD with $0 < h << 1$.

Consider motion in the time interval $0 < \delta t << 1$ so that the flow is effectively steady.

—/—

Reminder: Taylor's thm:

$$f(x) = f(0) + x f'(0) + R_2$$

$$R_2 = \frac{1}{2} f''(\xi) x^2 \quad \text{for} \quad \xi \in (0, x)$$

i.e $f(x) = a + bx$ plus an error of order x^2 where $a = f(0)$, $b = f'(0)$

—/—

What is the effect of an arbitrary, incompressible velocity feild $\underline{u}(x, y, t)$ do to our infinitemental element?

From Taylor's theorem (in 2D)

$$u = U + \alpha x + \beta y$$
$$v = V + \gamma x + \delta y$$
} with error over ABCD of order h^2.

where: $U = u(0,0)$, $\alpha = \frac{\partial u}{\partial x}(0,0)$, $\beta = \frac{\partial u}{\partial y}(0,0)$

where: $V = v(0,0)$, $\gamma = \frac{\partial v}{\partial x}(0,0)$, $\delta = \frac{\partial v}{\partial y}(0,0)$

Now $\underline{u}$ is incompressible, so $\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$.
everywhere

comes from $\underline{\nabla} \cdot \underline{u} = 0$

In particular $\alpha + \delta = 0$.

Useful to write:

$$\beta = \theta - \phi$$
$$\gamma = \theta + \phi.$$

Then $\theta = \frac{1}{2}(\gamma + \beta)$, $\phi = \frac{1}{2}(\gamma - \beta)$

$$\Rightarrow \quad \theta = \frac{1}{2}\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right), \quad \phi = \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)$$

Now:

$$\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} U \\ V \end{pmatrix} + \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$

In time δt, a point $\binom{x}{y}$ within ABCD moves by an amount:

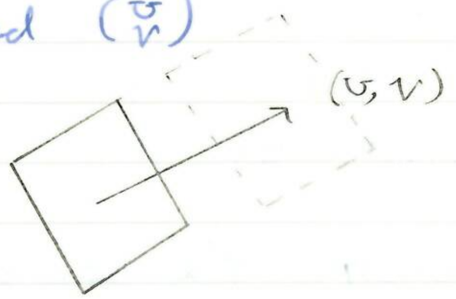
$$\begin{pmatrix} \delta x \\ \delta y \end{pmatrix} = \begin{pmatrix} u \\ v \end{pmatrix} \delta t$$

$$= \begin{pmatrix} U \\ V \end{pmatrix} \delta t + \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \delta t .$$

i.e :

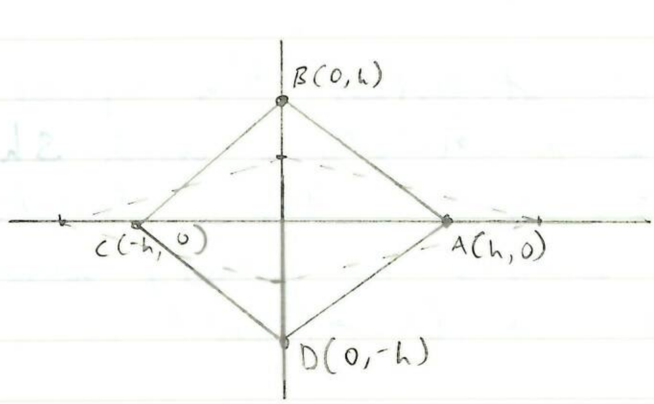
$$\begin{pmatrix} \delta x \\ \delta y \end{pmatrix} = \begin{pmatrix} U \\ V \end{pmatrix}^{(I)} \delta t + \left[\alpha \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}^{(II)} + \beta \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}^{(III)} + \phi \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}^{(IV)} \right] \begin{pmatrix} x \\ y \end{pmatrix} \delta t .$$

Term I : This term simply moves every point at speed $\begin{pmatrix} U \\ V \end{pmatrix}$



Translation of the centre of mass (at speed U, V).

Term II :



This moves A by:

$$\begin{pmatrix} \delta x \\ \delta y \end{pmatrix} = \alpha \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} h \\ 0 \end{pmatrix} \delta t .$$

$$= \begin{pmatrix} \alpha h \delta t \\ 0 \end{pmatrix}$$

Thus C moves by:

$$\begin{pmatrix} \delta x \\ \delta y \end{pmatrix} = \begin{pmatrix} -\alpha h \delta t \\ 0 \end{pmatrix}$$

At B: $\begin{pmatrix} \delta x \\ \delta y \end{pmatrix} = \alpha \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 0 \\ h \end{pmatrix} \delta t .$

$$= \begin{pmatrix} 0 \\ -\alpha h \delta t \end{pmatrix}$$

i.e. downwards exactly the same amount as A moves out.

At D: $\begin{pmatrix} \delta x \\ \delta y \end{pmatrix} = \begin{pmatrix} 0 \\ \alpha h \delta t \end{pmatrix}$

Thus Term II stretches the square at a rate αh in the x-direction and **shrinks** it at the same rate αh in the y-direction, without moving the centre of mass, conserving volume as expected – a DILATION!

Note: A streching in on direction and shrinking in the orthogonal dirn (in 2D) at the same rate so as to conserve volume.

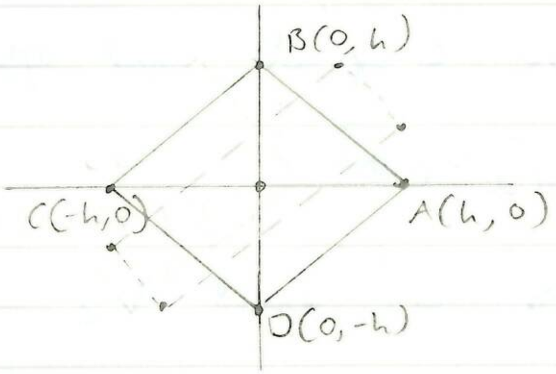
Term III :

At A, $\begin{pmatrix} \delta x \\ \delta y \end{pmatrix} = \begin{pmatrix} 0 \\ \theta h \delta t \end{pmatrix}$

At C, $\begin{pmatrix} \delta x \\ \delta y \end{pmatrix} = \begin{pmatrix} 0 \\ -\theta h \delta t \end{pmatrix}$

At B, $\begin{pmatrix} \delta x \\ \delta y \end{pmatrix} = \begin{pmatrix} \theta h \delta t \\ 0 \end{pmatrix}$

At D, $\begin{pmatrix} \delta x \\ \delta y \end{pmatrix} = \begin{pmatrix} -\theta h \delta t \\ 0 \end{pmatrix}$



i.e another DILATION: streching along line $y = -x$, both a rate θh, so as to preserve volume.

—/—

It appears that there are 2 DILATIONS: Term II and Term III. This is not so.

The combine effect of term II + III is the martix:

$$\begin{pmatrix} \alpha & \theta \\ \theta & -\alpha \end{pmatrix}$$

This is a real symmetric martix:

It possesses 2 real eigenvalues:

$$\begin{vmatrix} \alpha - \lambda & \theta \\ \theta & -\alpha - \lambda \end{vmatrix} = 0.$$

$$\Rightarrow \quad -(\alpha - \lambda)(\alpha - \lambda) - \theta^2 = 0$$

$$\Rightarrow \quad -\alpha^2 + \lambda - \theta^2 = 0$$

$$\Rightarrow \quad \lambda^2 = \alpha^2 + \theta^2 .$$

Hence we have 2 equal and oppsite eigenvalues

$$\lambda = \pm \sqrt{\alpha^2 + \theta^2}$$

ORTHOGONAL.

with e'vectors ξ_1 and ξ_2 (say)

In the basis ξ_1, ξ_2 the martix has the form

$$\begin{pmatrix} \lambda_1 & 0 \\ 0 & -\lambda_1 \end{pmatrix}$$

Precisely the form of term III.

Thus expansion at rate λ_1 along ξ_1 and and a contraction at rate λ_1 along the orthogonal ξ_2 i.e a DILATION

i.e sum of 2 DILATION

—/—

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A Summary of what is done so far:

1) $\underline{u}(x, y, z, t)$

2) Particles Path, Streaklines, Streamlines

3) Incompressibility $\Rightarrow \underline{\nabla} \cdot \underline{u} = 0$ (in n-D)

4) Incompressibility + 2D $\Rightarrow$ $\underline{u} = u(x,y,t)\hat{\underline{i}} + v(x,y,t)\hat{\underline{j}} = u_r(r,\theta,t)\hat{\underline{r}} + u_\theta(r,\theta,t)\hat{\underline{\theta}}$.

$$\nabla \cdot u = 0 \text{ and in 2D} \Rightarrow \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

scalar

$$\Rightarrow \exists \psi \text{ st } \underline{u} = -\underline{k} \wedge \underline{\nabla} \psi \text{ i.e}$$

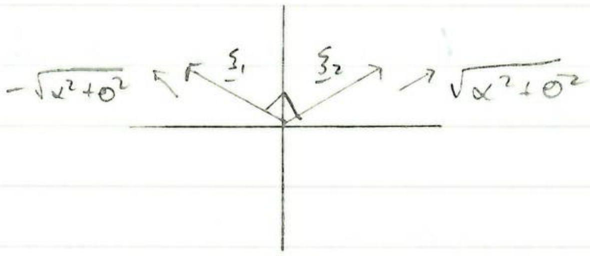
$$u = \frac{\partial \psi}{\partial y}, \quad v = -\frac{\partial \psi}{\partial x}$$

$$\text{or} \quad u_r = \frac{1}{r}\frac{\partial \psi}{\partial \theta}, \quad u_\theta = -\frac{\partial \psi}{\partial r}$$

5) Local motion at a point - consists of a translation of centre of mass, a dilation, and ...

$$\begin{pmatrix} u \\ v \end{pmatrix} = \overset{(I)}{\begin{pmatrix} U \\ V \end{pmatrix}} + \Bigg[\underbrace{\overset{(II)}{\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}} \alpha + \overset{(III)}{\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}} \theta}_{(*)} + \overset{IV}{\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}} \phi \Bigg] \begin{pmatrix} x \\ y \end{pmatrix}$$

Considering $(*) \Rightarrow \begin{pmatrix} \alpha & \theta \\ \theta & -\alpha \end{pmatrix}$, e'vectors: $\underline{\xi}_1, \underline{\xi}_2, \underline{\xi}_1 \perp \underline{\xi}_2$



<u>Term IV</u>:

A moves by an amount:

$$\begin{pmatrix} \delta x \\ \delta y \end{pmatrix} = \begin{pmatrix} u \\ v \end{pmatrix} \delta t$$

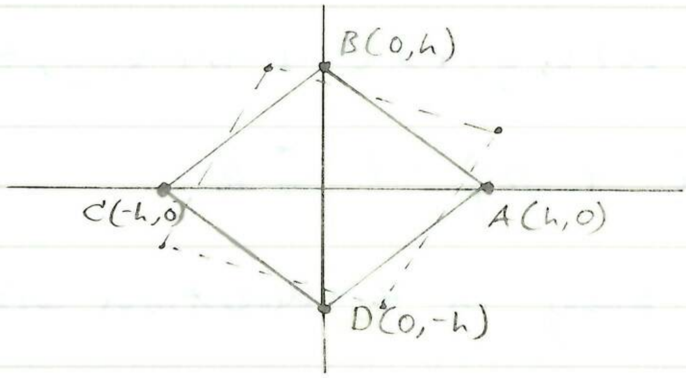
At B:

$$\begin{pmatrix} \delta x \\ \delta y \end{pmatrix} = \phi \begin{pmatrix} -h \\ 0 \end{pmatrix} \delta t .$$

Term IV contribute of A:

$$\phi \begin{pmatrix} 0 \\ h \end{pmatrix} \delta t$$

At A the vertical arm have length h.
Point moved up distance $\phi h \delta t$
Hence moved through an angle $\phi \delta t$
i.e ABCD is rotating at a rate ϕ in the anti-clockwise direction.



i.e at a rate $\frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)$ in the anti-clockwise direction.

—/—

Thus we have shown that motion at a point consist of 3 and only 3 things:
• Translation of C of M, a dilation, and a rotation about the C of M. [Notice for a solid: as above but no dilation].

—/—

Notice: $\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}$ is precisely the z-component of $\underline{\nabla} \wedge \underline{u}$ i.e curl $\underline{u}$.

It is traditional to write

$$\underline{\omega} = \underline{\nabla} \wedge \underline{u}$$

Omega.

$\underline{\omega}$ is the vorticity of the flow i.e rotation of the flow. [old name curl $\underline{u}$ was rot $\underline{u}$]

The component of $\underline{\omega}$ usally written:

$$\underline{\omega} = \underset{(xi)}{\xi}\,\underline{\hat{i}} + \underset{(eta)}{\eta}\,\underline{\hat{j}} + \underset{(zeta)}{\zeta}\,\underline{\hat{k}}$$

In 2D, $\underline{u} = u(x, y, t)\hat{i} + v(x, y, t)\hat{j}$.

$$\underline{\omega} = 0\hat{i} + 0\hat{j} + \zeta\hat{k}$$

i.e $\underline{\omega}$ is solely in z-direction with

$$\underline{\omega} = \zeta\hat{k}. \quad \text{and} \quad z = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}.$$

and it gives twice the rate of rotation of a fluid element about its C of M [$\Phi = \frac{1}{2}\zeta$] i.e ζ is portportial to the angular momentum of a fluid element its C of M.

—/—

We can only change the rate at which a fluid element is spinning (in 2D) by applying a torque i.e shear stress

But an invised fluid does not support a shear stress, so we cannot (in 2D) change the rate at which a fluid element spins, i.e a particle in 2D invised fluid retains its values of ζ forever.

At $t=0$ At $t>0$

—/—

Consider a flow started from rest. Then initially $\zeta = 0$ ($\underline{u} = 0$ at $t = 0$) i.e every particle has vorticity zero.

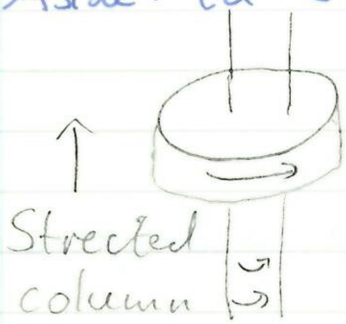
Hence for all time, all particle have zero vorticity.

A number where $\underline{\omega} = 0$ everywhere is called IRROTATIONAL, e.g any flow started from rest is irrotational. — the persistence of irrotationality ($\zeta \equiv 0$).

True in 3D - nothing to amplify.

— / —

Aside: In 3D: $\underline{\omega} \neq 0$, $\frac{\partial}{\partial z} \neq 0$.

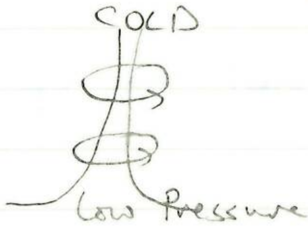


$\Rightarrow \zeta$ increased

e.g: Hurricane.
Amplifier.

Hurricane: Why in tropics not equator?
Why not at poles, northen oceans?
Why spin?

Isaac Held



Heat Engine.

— / —

Thus we will concentrate on IRROTIONAL flow.

Then $\underline{\nabla} \wedge \underline{u} = 0$.

Hence $\exists \phi$ st $\underline{u} = \underline{\nabla}\phi$ i.e. $\underline{u}$ is derivable from a potential, the velocity potential. (we are still in 2D or 3D).

In incompressible flow in 2D or 3D, $\underline{\nabla} \cdot \underline{u} = 0$.

Substituting gives:

$$\underline{\nabla} \cdot (\underline{\nabla}\phi) = 0$$

$$\text{i.e } \nabla^2 \phi = 0.$$

Laplance's equation 2D and 3D.
- The governing equation for 3D incompressible, irrotational flow.
- All we need are boundary conditions.

On a solid boundary $\underline{u} \cdot \underline{\hat{n}} = 0$.

Substitue for $\underline{u} = \nabla\phi$, $\underline{\hat{n}} \cdot \underline{\nabla}\phi = 0$ on a solid body i.e $\frac{\partial \phi}{\partial n} = 0$ on a solid body.

i.e the normal derivatives of ϕ vanishes on a solid body.

(The solution to Laplaces' eqn with $\frac{\partial \phi}{\partial n}$ specified in boundary i.e Newman problem, is unique).

—/—

<u>Example</u>: What is velocity potential for a uniform stream?

Take x-axis in direction of stream.

$\underline{u} = U\hat{\underline{i}}$ so $u = U$, $v = 0$.

But $\underline{u} = \underline{\nabla}\phi = \frac{\partial \phi}{\partial x}\hat{\underline{i}} + \frac{\partial \phi}{\partial y}\hat{j}$

So $u = \frac{\partial \phi}{\partial x}$, $v = \frac{\partial \phi}{\partial y}$.

Here $\frac{\partial \phi}{\partial x} = U$.

So $\phi = Ux + f(y)$

So $\frac{\partial \phi}{\partial y} = f'(y)$.

But $\frac{\partial \phi}{\partial y} = v$ and $v = 0$.

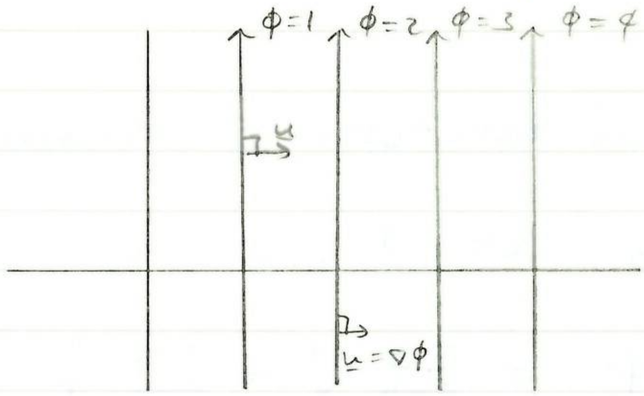
So $f'(y) = 0$

So f is constant.

Take $f = 0$.

$$\phi = Ux.$$

(Notice: satisifies Laplace's equ).



equipotentials, ϕ = constant
x = const here.

—/—

	Good	Bad
ϕ:	3D	Only irrotational
ψ:	Does not require irrotationality	Only 2D.

—/—

What does ψ satisfy? (in 2D, irrotational flow).

2D: $u = \dfrac{\partial \psi}{\partial y}$, $v = -\dfrac{\partial \psi}{\partial x}$

Irrot 2D : $\zeta = 0$; $\quad \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = 0$

Substituting :

$$\frac{\partial}{\partial x}\left(-\frac{\partial \psi}{\partial x}\right) - \frac{\partial}{\partial y}\left(\frac{\partial \psi}{\partial y}\right) = 0 .$$

i.e $\frac{\partial^2 \psi}{\partial^2 x} + \frac{\partial^2 \psi}{\partial y^2} = 0$

i.e $\nabla^2 \psi = 0$.

i.e s'fn satisfies Laplace's eqn also :

Remember the b.c. on a solid boundary for ψ is ψ = constant which is taken as $\psi = 0$ if only one boundary.

—/—

Check :

Uniform stream : $\psi = Uy$ so $\nabla^2 \psi = 0$.

Stagnation point : $\psi = xy$ so $\nabla^2 \psi = 0$.

—/—

If flow is 2D and irrotational, then you can choose to find ϕ or ψ .

- whichever seem easier
- governing eqn is the same : Laplance .
- boundary conditions different .

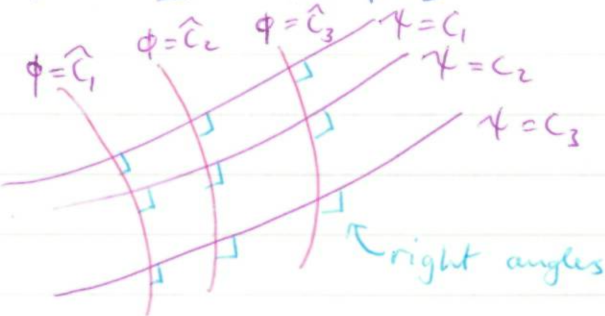
Are ϕ and ψ related ?

Yes! of course .

$\underline{u} = \underline{\nabla}\phi$ and $\underline{u} = -\underline{\hat{k}} \wedge \underline{\nabla}\psi$.

$$\left.\begin{array}{ll} \text{So} & \underline{\nabla}\phi = -\underline{\hat{k}} \wedge \underline{\nabla}\psi \\ \text{or} & \underline{\nabla}\psi = \underline{\hat{k}} \wedge \underline{\nabla}\phi \end{array}\right\}\text{very famous.}$$

} Cauchy-Riemann eq^{n}s! without co-ordinates



Cauchy - Riemann equation :

$$u = \frac{\partial \phi}{\partial x} \quad \text{and} \quad u = \frac{\partial \psi}{\partial y}$$

$$\text{so} \quad \frac{\partial \phi}{\partial x} = \frac{\partial \psi}{\partial y}.$$

$$v = \frac{\partial \phi}{\partial y} \quad \text{and} \quad v = -\frac{\partial \psi}{\partial x}$$

$$\text{so} \quad \frac{\partial \phi}{\partial y} = -\frac{\partial \psi}{\partial x}.$$

In 2D, irrotational flow :

$$\frac{\partial \phi}{\partial x} = \frac{\partial \psi}{\partial y} \quad \text{and} \quad \frac{\partial \phi}{\partial y} = -\frac{\partial \psi}{\partial x}$$

the Cauchy - Riemann equations.

In polars:

$$\underline{u} = \underline{\nabla}\phi = \frac{\partial \phi}{\partial r}\underline{\hat{r}} + \frac{1}{r}\frac{\partial \phi}{\partial \theta}\underline{\hat{\theta}}$$

thus: $\frac{\partial \phi}{\partial r} = \frac{1}{r}\frac{\partial \psi}{\partial \theta}$, $\frac{1}{r}\frac{\partial \phi}{\partial \theta} = -\frac{\partial \psi}{\partial r}$

thus ϕ and ψ are the real and imaginary parts of a differentiable complex function of the complex variable $z = x + iy$.

Traditionally this is called the COMPLEX VELOCITY POTENTIAL and written

$$w(z) = \phi(x,y,t) + i\,\psi(x,y,t).$$

lower case $\quad \sqrt{-1}$

where: $z = x + iy$.

$$\phi = Re\{w(z)\}$$

$$\psi = Im\{w(z)\}$$

Proved:

1) Real + Imag parts of a complex differentiable function, satisfy Laplace's equ.

2) Constant sufaces intersect at right angles.

—/—

Check : Uniform stream :

$$\phi = Ux$$
$$\psi = Uy .$$

$\phi + i\psi = U(x+iy) = Uz$ (a function of z alone).

So $w = Uz$ is the complex potential for a uniform stream.

— / —

Given $w(z)$ how do we get $\underline{u}$?

Consider :

$$\frac{dw}{dz} = \frac{\partial}{\partial x}(\phi + i\psi) = \frac{\partial \phi}{\partial x} + i\frac{\partial \psi}{\partial x}$$

$$= u - iv .$$

So $u + iv = \overline{\frac{\partial w}{\partial z}}$ → bar = conjugate.

— / —

Eg :

1) for $w(z) = Uz$

$$\Rightarrow \frac{dw}{dz} = U$$
$$= u - iv .$$

So $u = U$ and $v = 0$ as expected.

2) $w(z) = z^2$

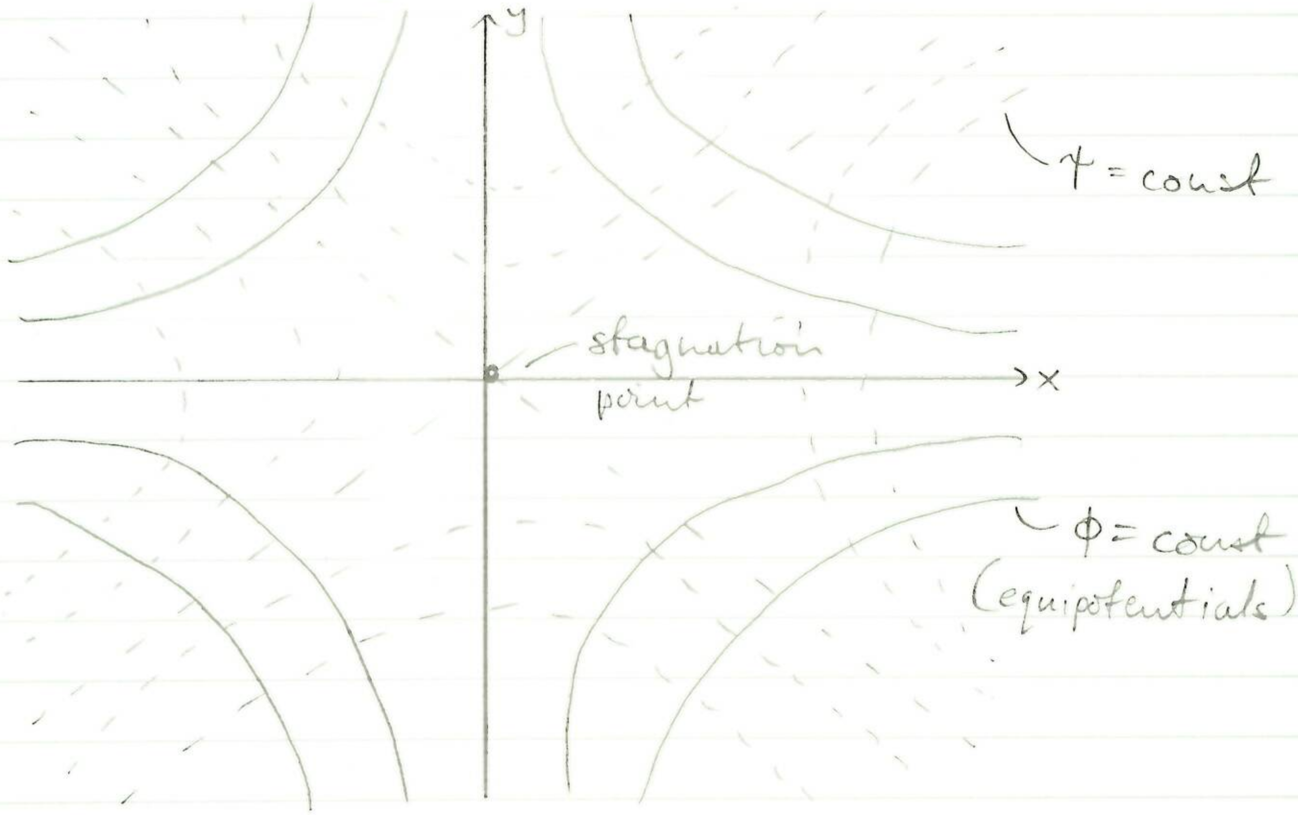
$$\frac{dw}{dz} = 2z = 2x + i2y.$$

So $u = 2x$, $v = -2y$

$u = 0$, $v = 0$ only if $z = 0$ (where $\frac{dw}{dz} = 0$)
i.e a stagnation point if and only if $\frac{dw}{dz} = 0$
Here only at $z = 0$.

$$w = z^2 = (x+iy)^2 = x^2 - y^2 + 2ixy$$

So $\phi = x^2 - y^2$, $\psi = 2xy$.



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Summary:

Irrotational: $\underline{\nabla} \wedge \underline{u} = 0 \Rightarrow \exists\ \underline{u} = \underline{\nabla}\phi$

Incompressibility: $\underline{\nabla} \cdot \underline{u} = 0$ plus 2D:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0.$$

i.e $\underline{\nabla}\phi = -\underline{k} \wedge \underline{\nabla}\psi$ (Cauchy-Riemann)

$\Rightarrow \exists\ w(z)$ st $\frac{dw}{dz}$ exist and $w = \phi + i\psi$.

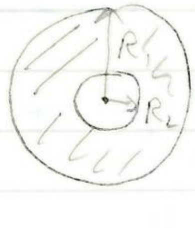
$$\frac{dw}{dz} = u - iv.$$

—/—

Laurent Series

A function analytic or holomorphic within an annular region: $R_0 < |z| < R_1$ has a unique expansion of the form:

$$\dots + \frac{a_2}{z^2} + \frac{a_1}{z} + a_0 + a_1 z^1 + a_2 z^2 + \dots$$



i.e all functions analytic in the annulus are simply linear combination of $z^{\pm n}$, $n = 0, 1, 2, 3, 4, \dots$

Apply this to the complex velocity $u - iv$ i.e $u - iv$ is a linear combination of $z^{\pm n}$

These $w(z)$, the complex potential is simply a linear combination of the term: $z^{\pm n}$ $n = 0, 1, 2, \ldots$ and $\mathrm{Log}\, z$.

i.e our flow in any annular region (or a region that can be distorted into an annulus, or outside a single body (let $R_1 \to \infty$) is simply a linear combination of the terms choosen from:

$$\{ z^{\pm n}, \mathrm{Log}\, z \}$$

Note that the coeffienants in the sum can be complex.

In particular, in cylindrical coordinates $z = re^{i\theta}$.

So $z^n = r^n e^{i\theta n}$
$= r^n \cos(n\theta) + i \sin(n\theta)$

And $\mathrm{Log}\, z = \underline{\mathrm{Log}\, r} + i\theta$.

Now $\phi = \mathrm{Re}\, w$ so ϕ must be a linear combination of the terms drawn from the set:

$\{r^n\cos(n\theta), r^n\sin(n\theta), (n = 0, \pm 1, \pm 2, \dots),$
$\log r, \theta\}$.

Thus all solution (in an annular domain) of Laplace's eqn in polar co-ords are simply a linear combination of the terms; $r^{\pm n}\cos(n\theta)$, $r^{\pm}\sin(n\theta)$, $\log r$, θ.

—/—

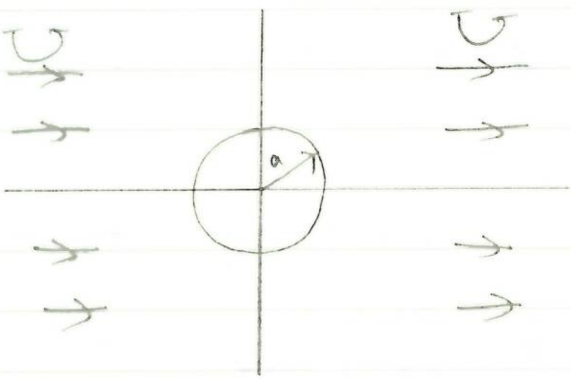
Similarly $\Psi(x, y) = \mathrm{Im}(z)$ is only a linear combination of terms drawn from the set:

$\{r^n\cos(n\theta), r^n\sin(n\theta), (n = 0, \pm 1, \pm 2, \dots),$
$\log r, \theta\}$.

—/—

Example: Find the ideal (irrot + incomp) 2D flow past a cylinder of radius a given that the flow at infinity is uniform with speed U.

Solution: Take Cartesian axes with Ox in direction of flow at infinity and origin of centre of cylinder.



Can slove either with Φ or Ψ.

Choose ψ simply so we can draw some s'lines

Governing equations: $\nabla^2 \psi = 0$ in $r > a$.

On the cylinder $r = a$, there is no flow through cylinder, recall that $(\underline{u} \cdot \hat{\underline{n}} = 0)$ on boundary.

Therefore take: $\psi = \text{const}$ at $r = a$.

But there is only one boundary so make $\psi = 0$ on $r = a$.

As $r \to \infty$ $\quad \underline{u} \to U \hat{\underline{i}}$.

i.e $u \to U$, $v \to 0$.

i.e $\dfrac{\partial \psi}{\partial y} \to U$ so $\psi \to Uy + f(x)$

i.e $\dfrac{\partial \psi}{\partial x} \to f'(x)$.

But $\dfrac{\partial \psi}{\partial x} \to v = 0$

So $f' = 0$ w.l.o.g $f = 0$.

Hence $\psi \to Uy$ as $r \to \infty$.

Summary:

$\nabla^2 \psi = 0$ $\quad r > a$ homogenous ($\psi = 0$ is a solution).

$\psi = 0$ $\quad r = a$ homogenous ($\psi = 0$ is a solution.

$\psi \to Uy$ $\quad r \to \infty$ inhomogenous ($\psi = 0$ is not a solution.

· Inhomogenous b.c. says:

$$\psi \to U r \sin\theta \quad \text{as } r \to \infty.$$

from our set

Guess:

(n=+1) from our set

Not sol^n to Laplace

Violates $r \to \infty$

$$\psi = U r \sin\theta + a r^5 \cancel{\cos}(3\theta) + b r^3 \cancel{\cos}(3\theta)$$

$$+ \frac{c}{r^3} \cancel{\cos}(3\theta) + \frac{B}{r} \sin\theta.$$

cannot balance $\sin\theta$ for even θ on $r = a$

(n=-1) from our set.

$\leftarrow \psi = 0$ on $r = a$ $\forall \theta$.

So on $r = a$

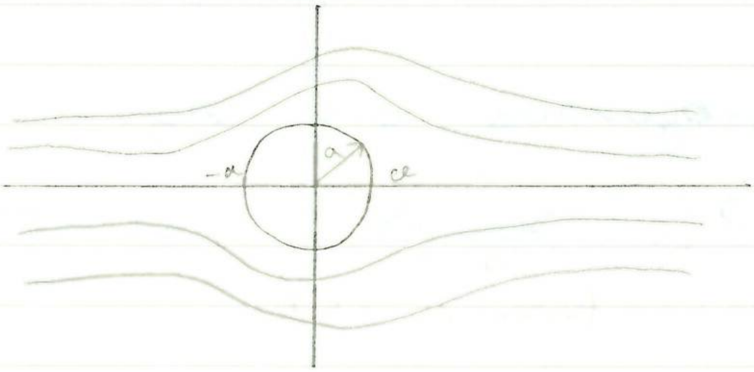
$$\psi = U a \sin\theta + \frac{B}{a} \sin\theta.$$

This is true for all θ so:

$$Ua + \frac{B}{a} = 0 \quad \text{i.e} \quad B = -Ua^2.$$

$$\text{i.e} \quad \psi = Uy + \left(1 - \frac{a^2}{r^2}\right)$$

$\psi = 0$ when $y = 0$ and $r = a$ as expected



Hence:

$$\psi = Ur\sin\theta - \frac{Ua^2}{r}\sin\theta.$$

$$\psi = Ur\sin\theta\left[1 - \frac{a^2}{r^2}\right]$$

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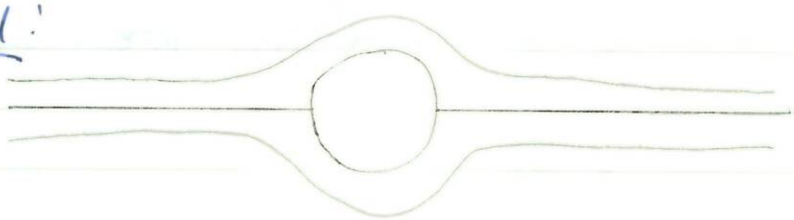
Recall:



$$\psi = Uy\left(1 - \frac{a^2}{r^2}\right)$$

$$= Uy - \frac{Ua^2y}{r^2}$$

Note that:

$$\frac{Ua^2y}{r^2} = \frac{Ua^2y\bar{z}}{|z|^2}$$

$$w(z) = Uz + \frac{U}{z}a^2$$

|| $w(z)$ comes from the set: $\{z^{\pm n}, \operatorname{Log} z\}$

So $\psi = \operatorname{Im}\left[Uz + \frac{Ua^2}{z}\right]$

So $\phi = \operatorname{Re}\{w(z)\}$

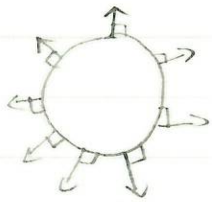
$$= Ux + \frac{Ua^2x}{r^2}$$

|| ψ, ϕ comes from the set: $\{r^{\pm n}\cos(n\theta), r^{\pm n}\sin(n\theta), \log r, \theta\}$.

ϕ is conjugate to ψ.

—/—

Suppose that fluid is injected out of the body and the same conditions previously.



$\underline{u} \cdot \hat{\underline{n}} = V$ (Notes previous had $V = 0$)

$\underline{u} \to U\hat{\underline{i}}$ as $r \to \infty$
$\phi \to Ux = U\cos\theta$ as $r \to \infty$ (Inhomogenous)

$\underbrace{\underline{u} \cdot \hat{\underline{n}} = V}_{\text{Same for all } \theta}$ on $r = a$ (Inhomgenous)

So $\phi = Ar\cos\theta + \frac{B}{r}\cos\theta + C\log(r)$

—/—

Our basic solution:

1) z^1, z^2, z^3 etc "singularities"
$\uparrow$ Dipole.

2) z^0 – Nothing ; $\frac{dw}{dz} = 0$

3) z : $w(z) = Uz$; uniform stream: $\frac{dw}{dz} = U = u - iv$.

4) $w(z) = z^2 = [re^{i\theta}]^2$

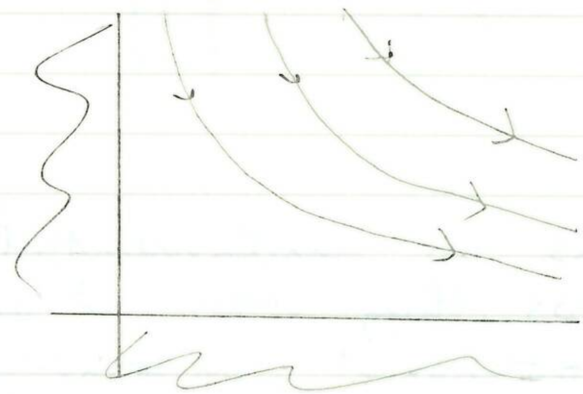
$= r^2\cos 2\theta + ir^2 \sin 2\theta.$

So $\psi = r^2 \sin 2\theta.$

So $\psi = 0$ on $\theta = 0$.

and with increasing θ, next zero when $\theta = \frac{\pi}{2}$
hence:

Stagnation
point flow:



$\frac{dw}{dz} = 2z = 2x + 2iy$

$\Rightarrow u = 2x, \ v = -2y.$

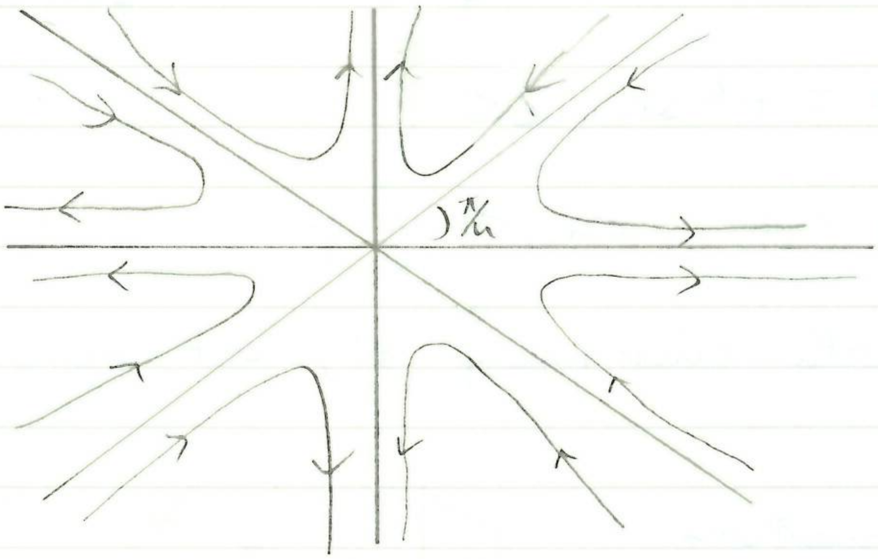
5) $w(z) = z^3$

$\psi = r^3 \sin(3\theta)$

$\psi = 0$ on $\theta = 0$, next zero when
$\theta = \pi/3$.

6) $w(z) = z^4$

At $\psi = 0$ on $\theta = \frac{m\pi}{n}$, at $m = 0, \pm 1, \ldots, \pm n$.



Thus in fundamental soln's, if n streamlines cross, they cross at an angle π/n in irrotational, incompressible flow.

<u>Streamlines in the neighbourhood of a stagnation point</u>:

Suppose we have a stag pt. in the flow. Move origin to that point.

In the neighbourhood of O.

$$w(z) = a_0 + a_1 z^1 + a_2 z^2 + a_3 z^3 + \ldots$$

W.l.o.g we can take $a_0 = 0$.

At O (origin):

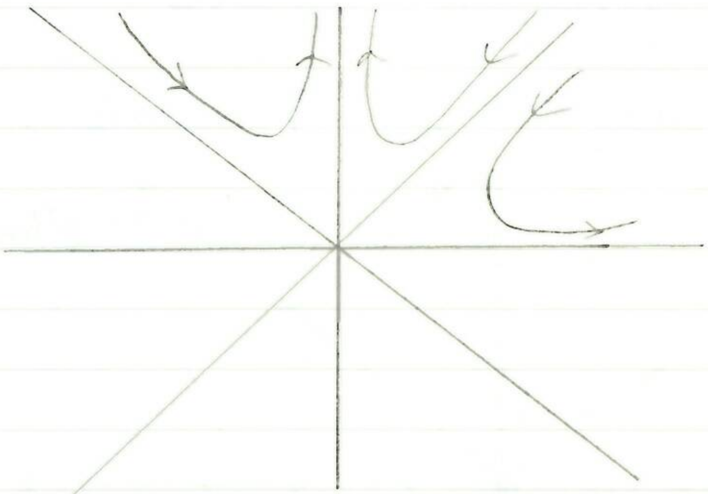
$\frac{dw}{dz} = 0$ (stag pt) thus $a_1 = 0$.

Let the first non-zero term be a_n. Thus $n \geq 2$ sufficiently to O, $w \sim a_n z^n$ for some complex a_n.

Suppose $a_n = Ae^{i\alpha}$

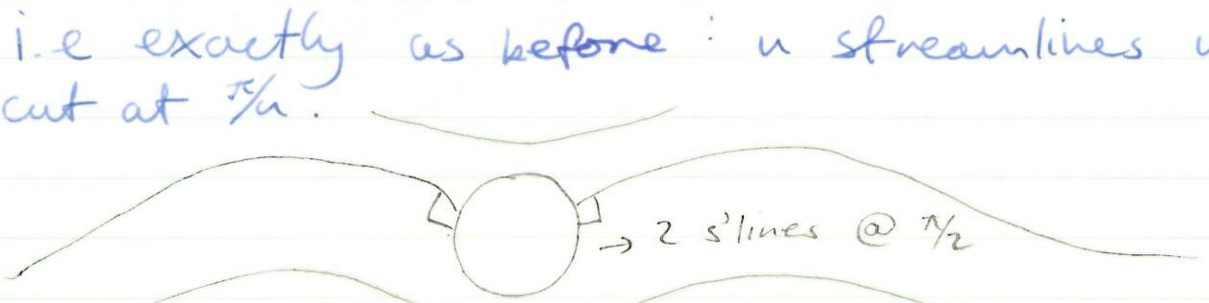
Then $w \sim Ae^{i\alpha} r^n e^{i\theta}$.
$= (A^{1/n} r)^n e^{in(\theta + \alpha/n)}$

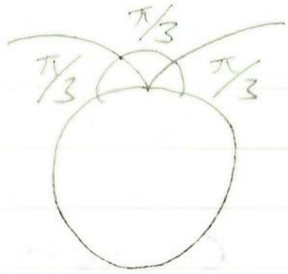
i.e a fundermental soln, z^n rotated by α/n and scaled $A^{1/n}$



etc :

i.e exactly as before : n streamlines must cut at π/n.





3 s'lines cut at $\pi/3$.

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The remaining fundermental solutcoin is

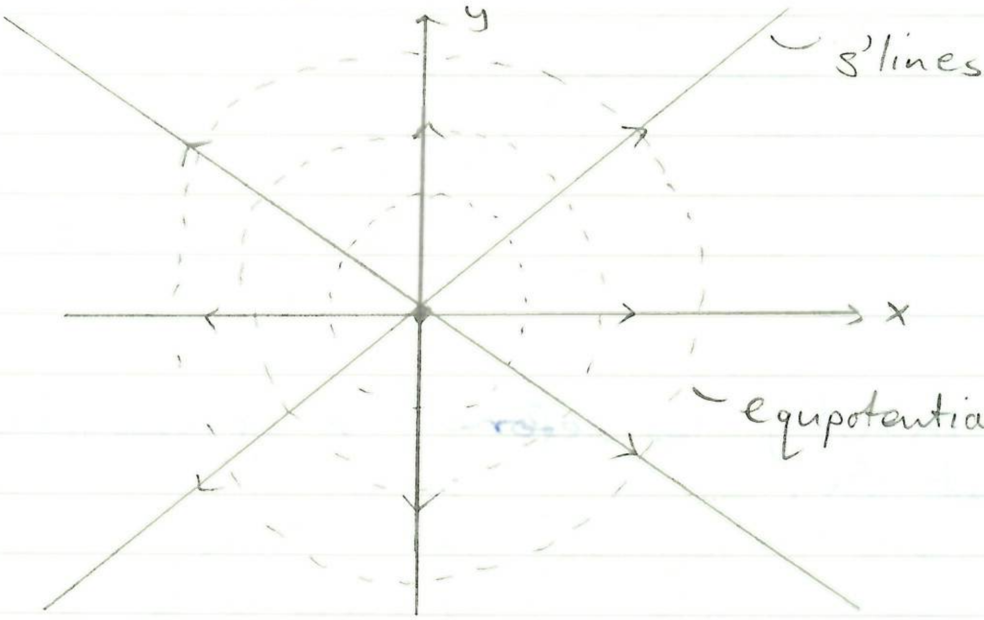
$$\log z.$$

If $w(z) = m \log z$ (m real)

$$= m(\log r + i\theta)$$

$$= m \log r + im\theta.$$

So $\phi = m \log r$, $\psi = m\theta$.



↳ for m real

i.e $\log r$, θ are conjugate functcoins - isotropic source strength $2\pi m$.

— / —

<u>Exercise</u>: No other fundamental sol^n is a source of fluid (consequence of Cauchy's thm).

If $w(z) = -iK\log z$ (K real)

$$= -iK(\log r + i\theta)$$

$$= K\theta - iK\log r.$$

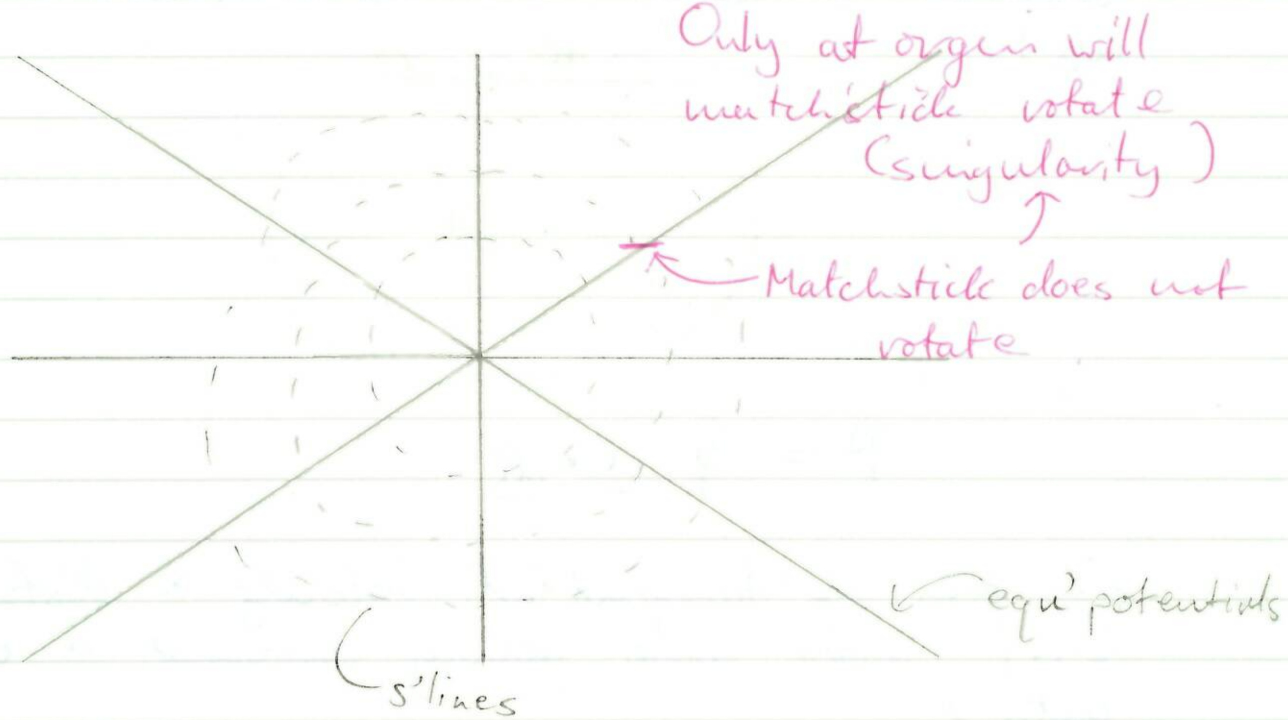
$$\phi = K\theta, \quad \psi = -K\log r.$$

S'lines: $\psi = $ const

i.e $\log r = $ const

i.e $r = $ const

i.e circles!



LINE VORTEX!

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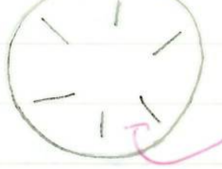
Aside:

$\underline{\omega} = \text{curl } \underline{u}$

$$= \xi \hat{\underline{i}} + \eta \hat{\underline{j}} + \zeta \hat{\underline{k}} .$$

How to let a solid body rotate $\zeta = 2\Omega$ everywhere.



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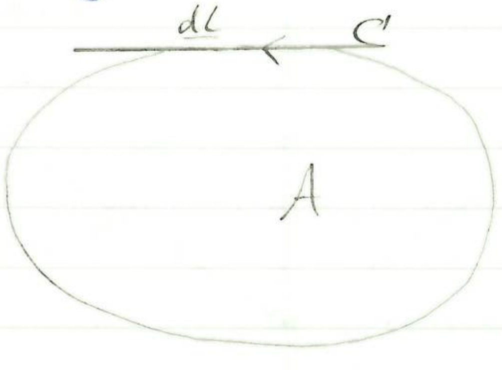
The strength of a line (or point) vortex

We measure the strength of ANY ROTATIONAL flow by its CIRCULATION about a closed contour C (say). The circulation is defined as:



$$\Gamma = \oint \underline{u} \cdot \underline{dl}$$

i.e sum of tangential velocity × distance. (c.f work done going around a closed path).

Notice for irrotational flow this is zero for all curves C.

[In 3D, if $\underline{u} = \underline{\nabla}\phi$; $\int_A^B \underline{\nabla}\phi \cdot \underline{dl} = \phi(A)-\phi($
if $A = B$ this is zero].

In 2D, $\oint_c \underline{u}\cdot\underline{dl} = \int_A (\underline{\nabla}\wedge\underline{u})\cdot\underline{\hat{n}}\, dA$.

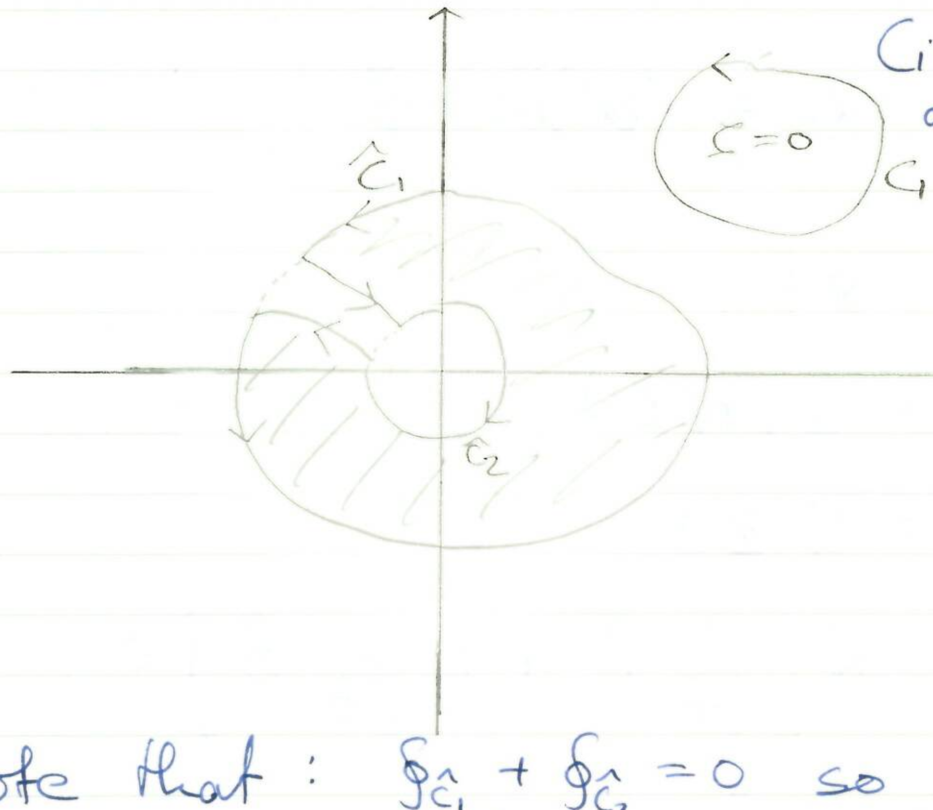
Both in 2D flow, $\underline{\nabla}\wedge\underline{u} = \zeta\underline{\hat{k}}$ and $\underline{\hat{n}} = \underline{\hat{k}}$
so:

$$\Gamma = \int_A \zeta\, dA.$$

If $\zeta = 0$ everywhere $\Gamma = 0$ for all C.

For the point vortex:

$$w(z) = -iK\log z.$$

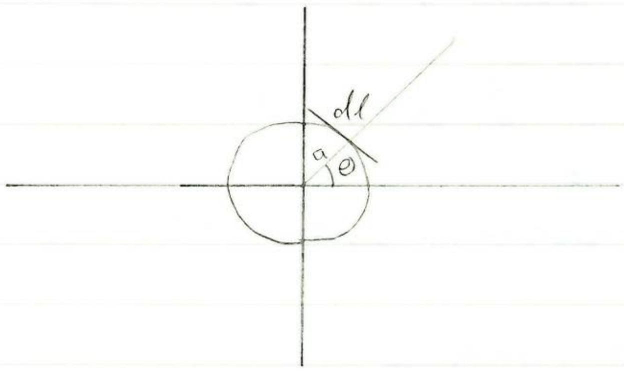


Circulation around C_1 is 0

Note that: $\oint_{\hat{C}_1} + \oint_{\hat{C}_2} = 0$ so $\oint_{\hat{C}_1} = -\oint_{\hat{C}_2}$

For a circuit around the origin, we can take the circuit to be a circle of radius a w.l.o.g.

$$\Gamma = \oint_{\theta=-\pi}^{\pi} \underline{u} \cdot \underline{dl}$$



Associated with a change $d\theta$ in θ is the vector:

$$\underline{dl} = (a \, d\theta) \, \underline{\hat{\theta}}$$

and

$$\underline{u} = \underline{\nabla} \phi$$

$$= \underline{\nabla} (K \theta)$$

$$= K \underline{\nabla} \theta$$

$$= K \left[\frac{\partial}{\partial r} \underline{\hat{r}} + \frac{1}{r} \frac{\partial}{\partial \theta} \underline{\hat{\theta}} \right] \theta$$

$$= \frac{K}{r} \underline{\hat{\theta}} .$$

On $r = a$, $\underline{u} = \frac{K}{a}\underline{\hat{\theta}}$.

thus:

$$\Gamma = \int_{-\pi}^{\pi} \frac{K}{a}\,\underline{\hat{\theta}} \cdot a\,d\theta\,\underline{\hat{\theta}}$$

$$= 2\pi K.$$

i.e line vortex has circulation $2\pi K$

—/—

<u>Excercise</u>: Only fundermental soln with circulation.

—/—

<u>Example</u>: Consider a cylinder of radius a, in a stream of speed U where there is circulation K about the cylinder.

Take origin at the centre of the cylinder and axis Ox in the direction of the flow at infinity. Then:

$$w(z) = Uz + \frac{Ua}{z} - \frac{iK}{2\pi}\log z$$

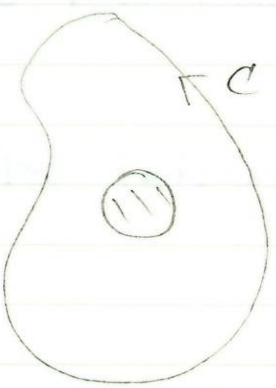
(satifies Laplace's eq^n because a sum of fundermental sol^n ψ, ϕ).

Check:

$$\frac{dw}{dz} = U - \frac{Ua^2}{z} - \frac{iK}{2\pi z}$$

As $z \to \infty$, $\frac{dw}{dz} \to U$ i.e. $u \to 0$ and $v \to 0$. Thus b.c. (boundary conditions) at infinity is satisfy.

Take any circuit about the cylinder:



Then the circulate about C is:

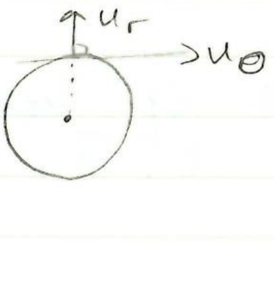
$$2\pi\left(\frac{K}{2\pi}\right) = K.$$

as required, since only line vortex has circulation.

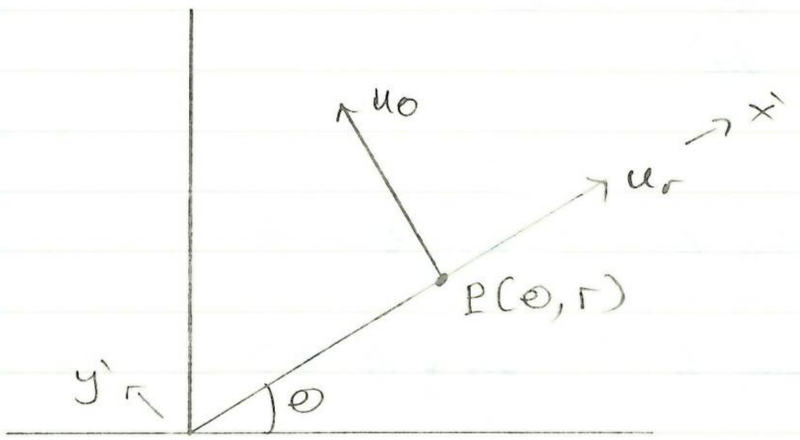
Remains to check at $\underline{u} \cdot \underline{\hat{n}} = 0$ on the cylinder $r = a$.

i.e $\underline{u} \cdot \underline{\hat{r}} = 0$

i.e $u_\theta = 0$ on $r = a$.



Here is a nice way of doing this using complex variable:



At some point P the Cartesian components of velocity are:

$$\frac{dw}{dz} = u - iv.$$

Introduce x' and y' rotated by θ degree anti-clockwise from x, y.

The $\dfrac{dw}{dz'} = u' - iv'$ (the cartesian component along the dashed axis).

$$= u_r - iu_\theta .$$

So $u_r - iu_\theta = \dfrac{dw}{dz'} = \dfrac{dw}{dz}\dfrac{dz}{dz'}$

$$= e^{i\theta}\frac{dw}{dz}$$

Now: $z = e^{i\theta} z'$

so $\dfrac{dz}{dz'} = e^{i\theta}$

—/—

$\arg z' = \arg z - \theta$

$\arg z = \arg z' + \theta$

—/—

Thus: $u_r - i u_\theta = e^{i\theta} \dfrac{dw}{dz}$ ← Very useful.

In our example:

$$\frac{dw}{dz} = -\frac{Ua^2}{z^2} - \frac{iK}{2\pi z}.$$

On the cylinder, $|z| = a$ i.e. $z = ae^{i\theta}$

$$\frac{dw}{dz} = U - Ue^{-2i\theta} - \frac{i e^{-i\theta}}{2\pi z}$$

so $$e^{i\theta}\frac{dw}{dz} = Ue^{i\theta} - Ue^{-i\theta} - \frac{iK}{2\pi a}$$

$$= 2iU\sin\theta - \frac{iK}{2\pi a}$$

$$= u_r - iu_\theta.$$

Thus $u_r = 0$ (as required)

and $u_\theta = \frac{K}{2\pi a} - 2U\sin\theta$.

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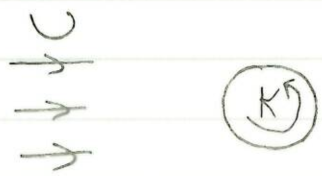
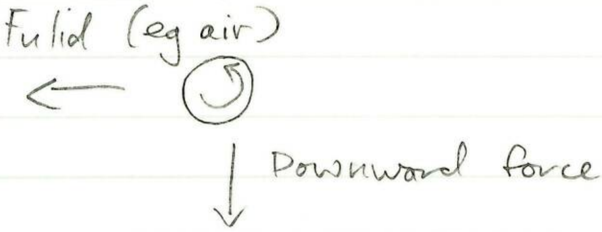
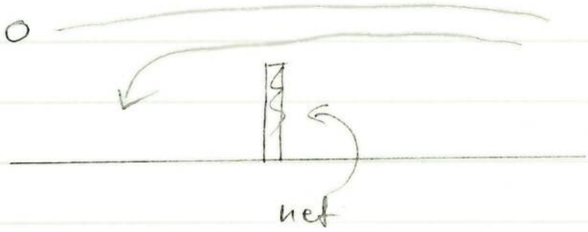


Tenis ball with "top spin"



So



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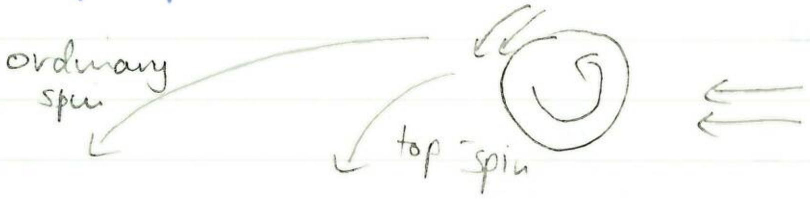
Eg "Top spin" tennis:



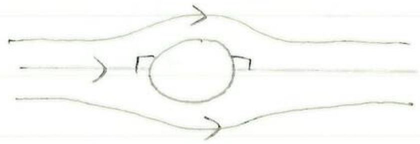
$$w(z) = U\left(z + \frac{a^2}{z}\right) - \frac{iK}{2\pi} \operatorname{Log} z.$$

$$\psi = Uy\left(1 - \frac{a^2}{r^2}\right) - \frac{K}{2\pi} \log r$$

$$\phi = Ux\left(1 + \frac{a^2}{r^2}\right) + \frac{K}{2\pi} \theta.$$

On $r = a$, $u_r = 0.$, $u_\theta = \frac{K}{2\pi a} - 2U \sin\theta.$

When $K = 0$



At a stagnation point:

$\underline{u} = 0$ or $\frac{dw}{dz} = 0$

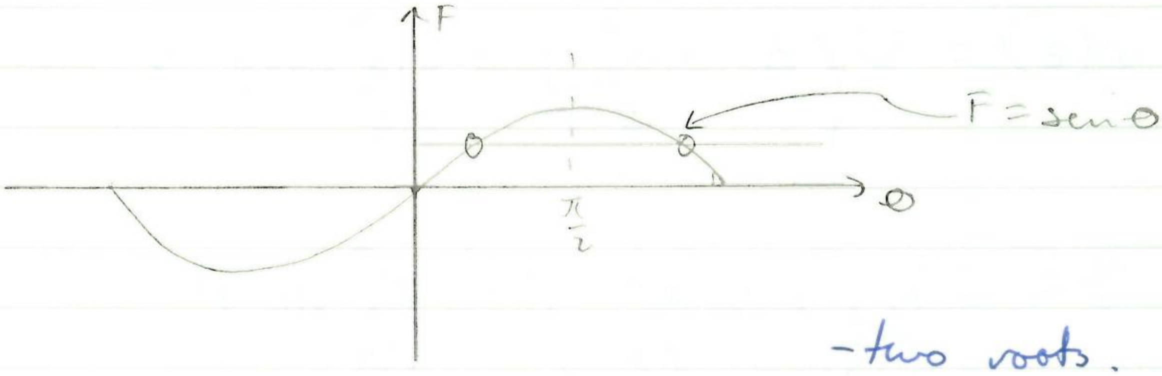
or $u = 0$, $v = 0$

or $u_r = 0$, $u_\theta = 0$.

On the cylinder; $r=a$, $u_r=0$ for all θ
so the stag point are where $u_\theta=0$

i.e $\dfrac{K}{2\pi a} = 2U\sin\theta$.

i.e $\sin\theta = \dfrac{K}{4\pi a U}$



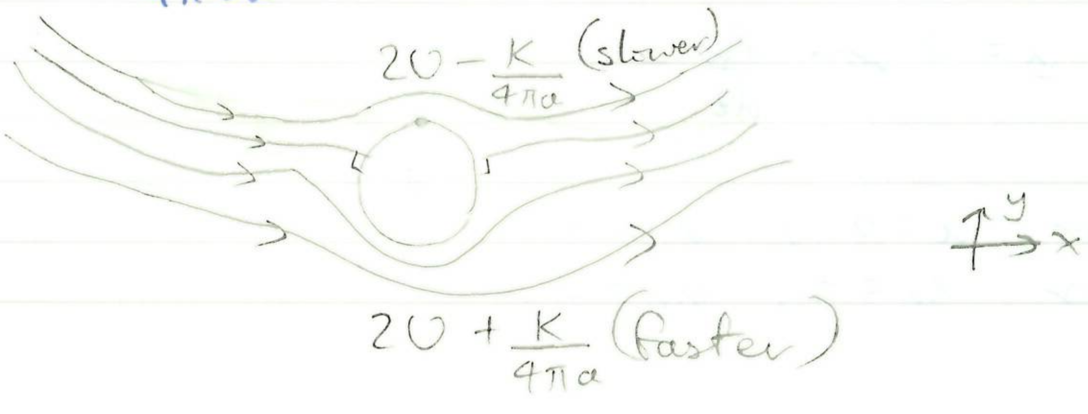
- two roots.

For $x>0$, there are between 0 and π (symmetric at $\pi/2$).

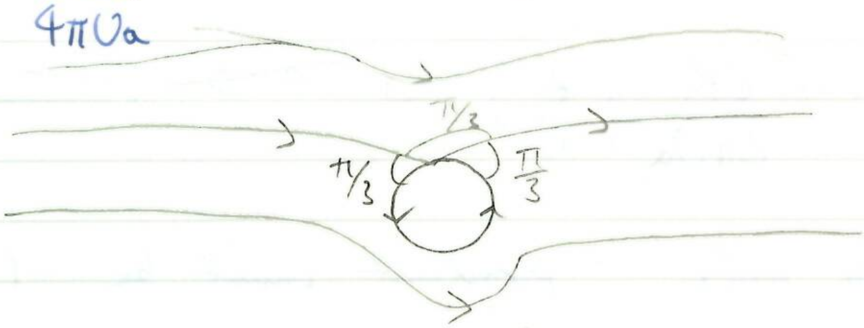
On cylinder:

$$y = a\sin\theta = \frac{K}{4\pi U}$$

where $\dfrac{K}{4\pi U a} < 1$



If $\frac{K}{4\pi U a} = 1$



Stagnation points coincide at $y = a$.

If $\frac{K}{4\pi a U} > 1$, No roots $\Rightarrow$ no stagnation point on the cylinder

Remember

$$w(z) = U\left(z + \frac{a^2}{z}\right) - \frac{iK}{2\pi} \log z$$

$$\frac{dw}{dz} = U\left(1 - \frac{a^2}{z^2}\right) - \frac{iK}{2\pi z}$$

Stag points:

$$\frac{dw}{dz} = 0$$

i.e $$U\left(1 - \frac{a^2}{z^2}\right) - \frac{iK}{2\pi z} = 0.$$

Multiply by $\frac{z^2}{Ua^2}$

Quadratic in z/a, i.e:

$$\left(\frac{z}{a}\right)^2 - \frac{iK}{2\pi U a}\left(\frac{z}{a}\right) - 1 = 0 .$$

So 2 ~~roots~~ roots - product must be -1.

Then:

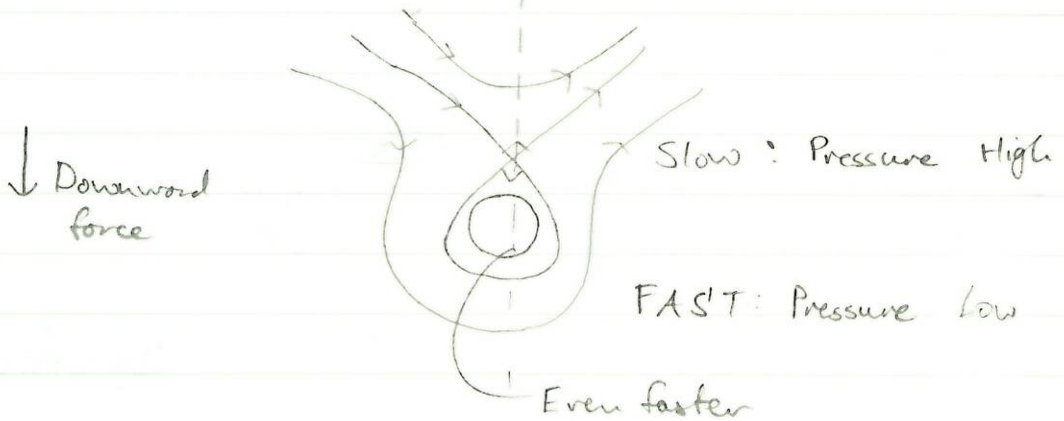
$$\frac{z}{a} = \frac{iK}{4\pi U a} \pm \sqrt{1 - \left(\frac{K}{4\pi U a}\right)^2}$$

If $\frac{K}{4\pi a U} < 1$: Already found $(z_1, -\bar{z}_1)$

If $\frac{K}{4\pi a U} > 1$: Purely imaginary roots
i.e $x = 0$., $y = y_1(a)$

$\Rightarrow \frac{z}{a} = iy$, $y = y_1$, $z = x + iy$

or $\frac{i}{y_1}$



$$2U + \frac{K}{2\pi a} > 4U$$

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Summary: choose ψ, ϕ then:

1) Inhomogenous.

2) Choose $\{r^{\pm n}\cos(n\theta),\ r^{\pm n}\sin(n\theta),\ \log r,\ \theta\}$ with undetermined coeffiant.

3) BC's give coefficients.

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<u>The method of images</u>

If the motion of a fluid in the xy plane is due to a distribution of singularties (eg.: sources, sinks, vortices etc) and there is curve C in the plane then the system of singularites on one side of C is called the Image of the systems on the other side. If there is no flow through C:

No flow across C

$\Rightarrow$ System 1 image of system 2

$\Rightarrow$ C is a streamline.

$\Rightarrow$ May replace C by a solid boundary, without the flow outside C.

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Example: What is flow due to a source of strength m located at $z = a$, with a source wall along $x = 0$?

Potential due source:

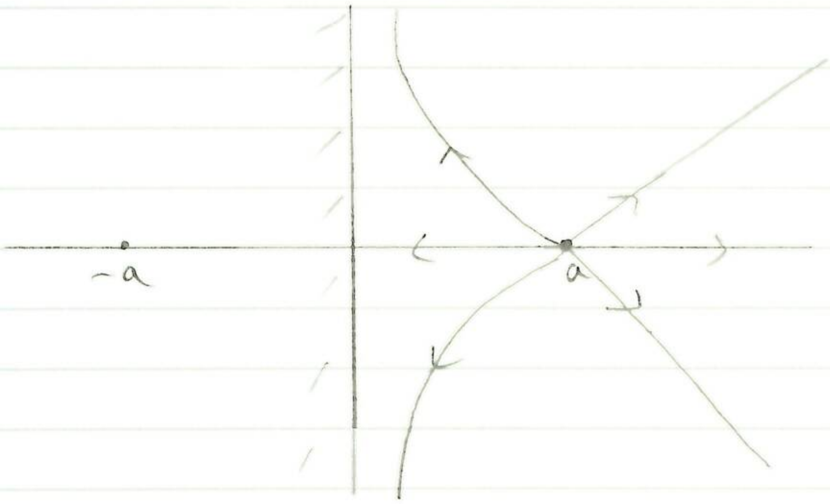
$$w_1(z) = \frac{m}{2\pi} \log (z-a).$$

Image is a source at $z = -a$.

$$w_2(z) = \frac{m}{2\pi} \log(z+a).$$

So $w(z) = w_1(z) + w_2(z)$

$$= \frac{m}{2\pi}\log(z-a) + \frac{m}{2\pi}\log(z+a).$$



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$\underline{u}(x,t)$, Velocity field, $\underline{x}$ fixed axes.

Incompressibility $\Rightarrow$ $\underline{\nabla}\cdot\underline{u} = 0$.

Local motion at a point
1) Translation of C of M.
2) Dilation.
3) Rotation.

Irrotationality persist:

$\underline{\nabla} \wedge \underline{u} = 0 \Rightarrow \exists\, \phi$ st $\underline{u} = \underline{\nabla}\phi$. (True in 3D)

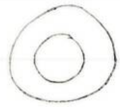
ψ: Incompressible and 2D
ϕ: Irrotational.

Irrotational, Incompressible and 2D: ϕ and ψ then

$$\underline{\nabla}\phi = -\hat{z} \wedge \nabla \psi.$$

$\exists\, w(z)$ where $z = x + iy$, $w(z) = \phi + i\psi$.

<u>Laurent</u>: Sum of $z^{\pm n}$.



In polars:
ψ, ϕ drawn from $\{ r^{\pm n}\cos n\theta, r^{\pm n}\sin n\theta, \log r, \theta \}$.

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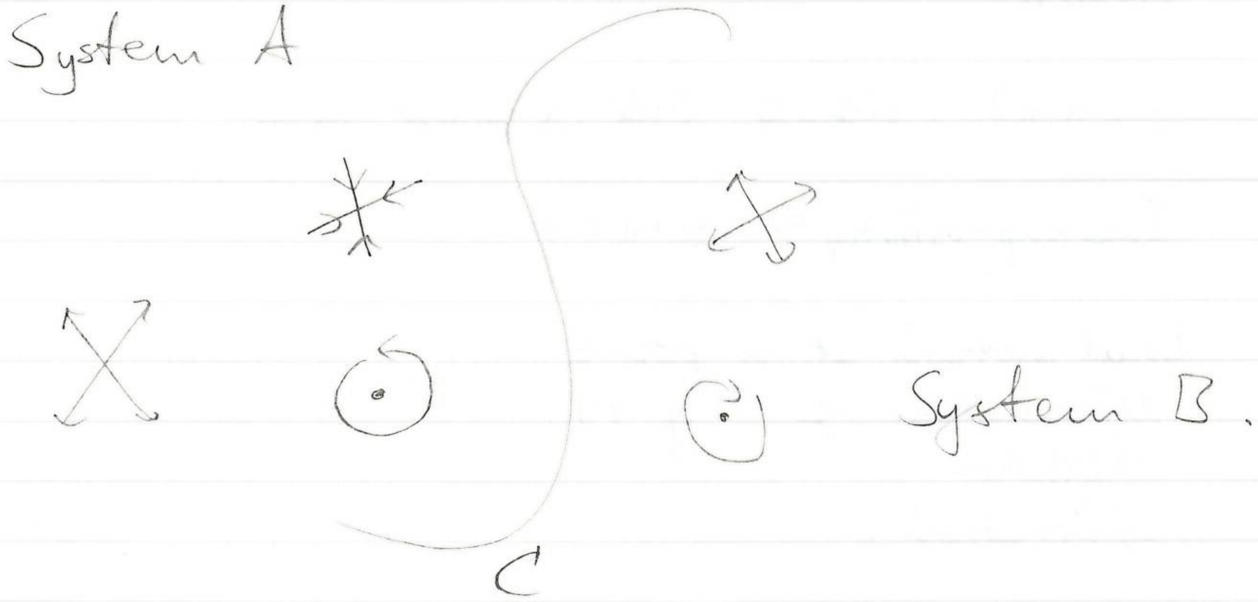
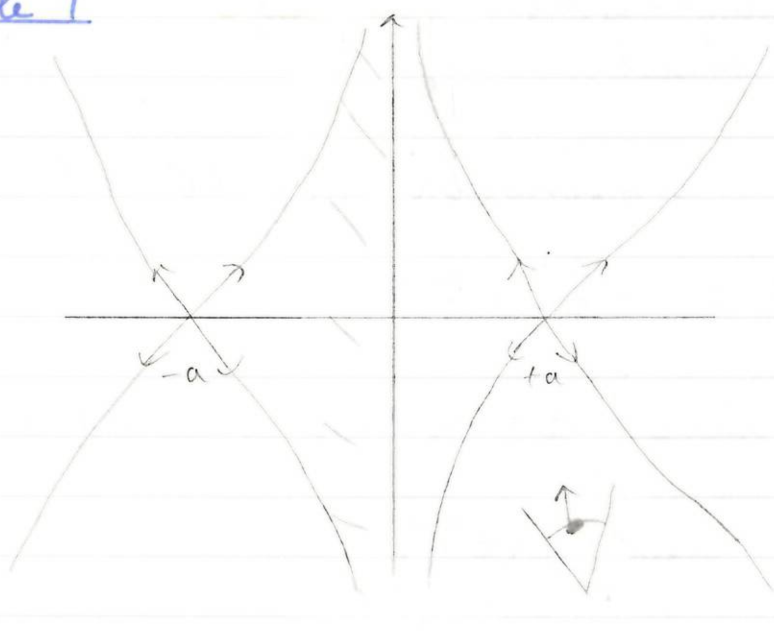


If no flow across C, then system A is the <u>image</u> of system B.

<u>Example 1</u>



- Original System:

Source strength m at $x = a$

Complex potential

$$w_1(z) = \frac{m}{2\pi} \operatorname{Log}(z - a)$$

• Image system :
Source strength m at $x = -a$.
Complex potential

$$w_2(z) = \frac{m}{2\pi} \operatorname{Log}(z+a).$$

Total system = Orginal + Image.

$$w(z) = \frac{m}{2\pi} \operatorname{Log}(z-a) + \frac{m}{2\pi} \operatorname{Log}(z+a).$$

If this is correct then $u = 0$ on $x = 0$.

$$w(z) = \frac{m}{2\pi} \operatorname{Log}(z^2 - a^2)$$

$$\frac{dw}{dz} = \frac{m}{2\pi} \left(\frac{1}{z^2 - a^2} \right) 2z$$

On $x = 0$ or $z = iy$.

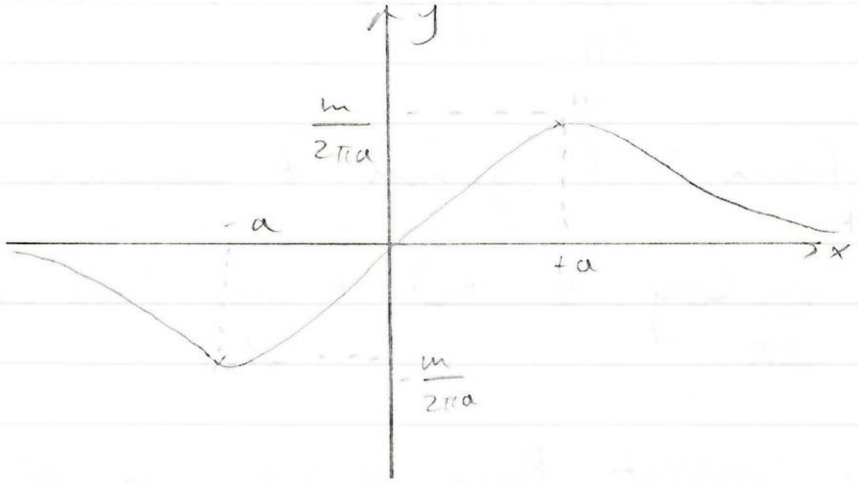
$$u - iv = \frac{dw}{dz}$$

$$= \frac{m}{2\pi} \left(\frac{1}{-y^2 - a^2} \right) 2iy.$$

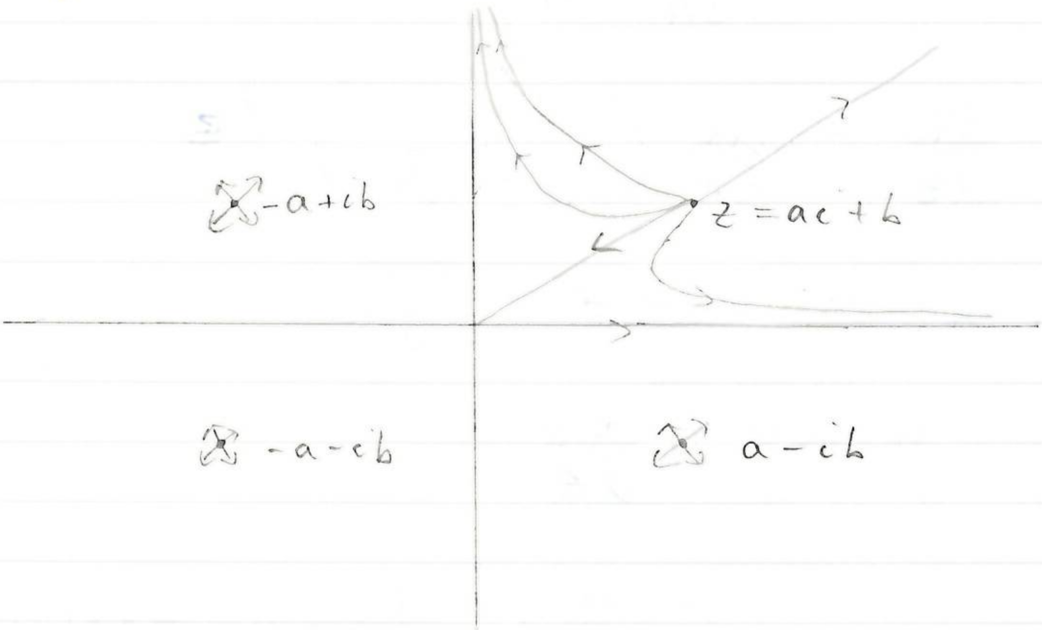
So $u = 0$ (as expected) and $v = \dfrac{my}{\pi(y^2+a^2)}$

So maximum speed on wall is

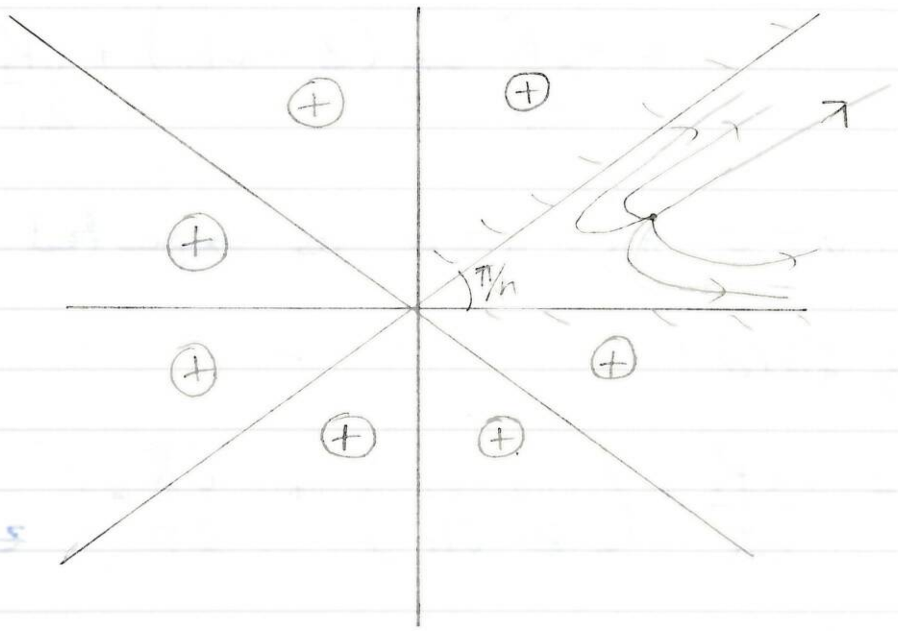
$$V = \pm \frac{m}{2\pi a} \quad \text{when} \quad y = \pm a .$$



Example 2



Example 3: Wall at π/n.



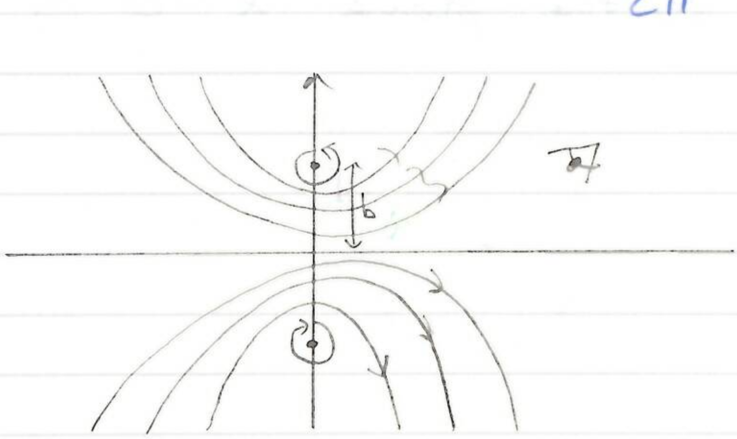
e.g $n=4$.

Example 4: Vortex of strength K_v at $z=ib$. above the plane $y=0$.

Complex potential $w_1(z) = \frac{-iK_v}{2\pi} \text{Log}(z-ib)$

Image: vortex of strength $-K_v$ at $z=-ib$

Complex potential $w_2(z) = \frac{+iK_v}{2\pi} \text{Log}(z+ib)$.



Total system = original + image

$$= \frac{-iK}{2\pi} \log(z - ib) + \frac{iK}{2\pi} \log(z + ib)$$

Check $v = 0$ on $y = 0$ as expected.

Velocity field:

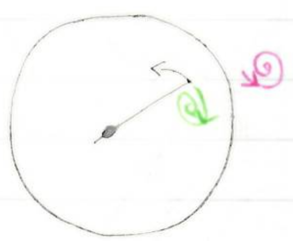
$$\frac{dw}{dz} = \frac{-iK}{2\pi} \cdot \left(\frac{1}{z - ib} \right) + \frac{iK}{2\pi} \cdot \left(\frac{1}{z + ib} \right)$$

At $z = ib$, neglecting first term, which is just the spinning of an isolated vortex about its centre.

$$\frac{dw}{dz} = \frac{iK}{4\pi i b}.$$

i.e $u = \frac{K}{4\pi b}$, $v = 0$.

i.e a free vortex would be driven along parallel to the plane $x = 0$ by its image in the plane:



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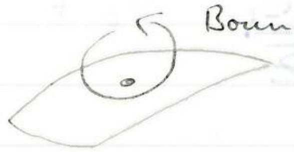

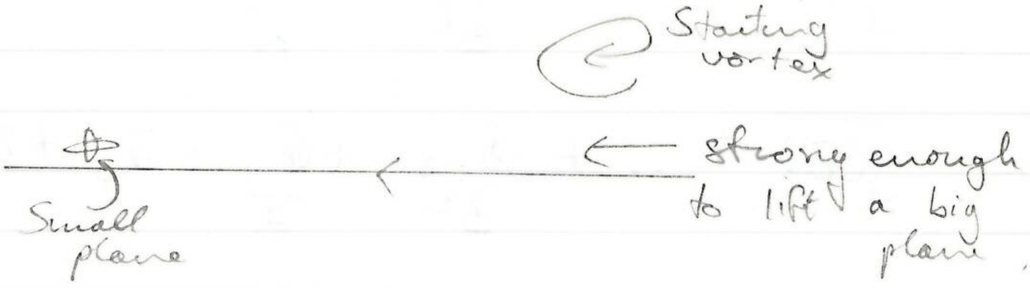

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Circle theorem:

The image system in the circle $|z| = a$ of the complex potential $w(z) = f(z)$ where $f(z)$ has no singularities inside the circle (i.e original system all on one side of line) $|z| < a$, is $\bar{f}\left(\frac{a^2}{z}\right)$ where for any analytic function $g(z)$, $\bar{g}(z) = \overline{g(\bar{z})}$

eg. if

$$g(z) = \ldots + \frac{a_{-2}}{z^2} + \frac{a_{-1}}{z} + a_0 + a_1 z + a_2 z^2 + \ldots$$

$$g(\bar{z}) = \dots + \frac{a_{-2}}{\bar{z}^2} + \frac{a_{-1}}{\bar{z}} + a_0 + a_1\bar{z} + a_2\bar{z}^2 + \dots$$

$$\bar{g}(z) = \dots + \frac{\bar{a}_{-2}}{z^2} + \frac{\bar{a}_{-1}}{z} + \bar{a}_0 + \bar{a}_1 z + \bar{a}_2 z^2 + \dots$$

$$\overline{g(\bar{z})} = \dots + \frac{\bar{a}_{-2}}{z^2} + \frac{\bar{a}_{-1}}{z} + \bar{a}_0 + \bar{a}_1 z + \bar{a}_2 z^2 + \dots$$

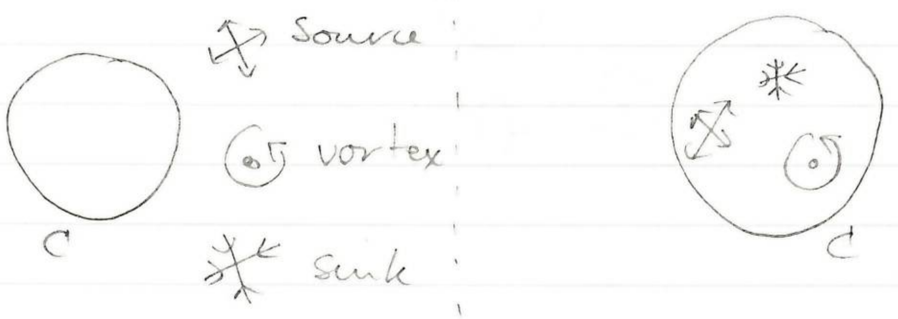
$$= \bar{g}(z)$$

Still an analytic function at $\frac{a^2}{z}$

Now f has no singularities in $|z| < a$ so $f(a^2/z)$ has no singularities in $|z| > a$, since

$$|z| > a \Rightarrow \frac{|z|}{a^2} > \frac{1}{a} \quad \text{so} \quad \frac{a^2}{|z|} < a.$$

Similarly for $\bar{f}(a^2/z)$



Singularities ouside C : Singularities inside C

Check for the complete potential:

$$w(z) = f(z) + \bar{f}\left(\frac{a^2}{z}\right)$$

that there is no flow through $|z| = a$.

Why a^2/z as argument of $\bar{f}$?

On the circle $|z| = a$, i.e. $z\bar{z} = a^2$ i.e $a^2/z = \bar{z}$
i.e a^2/z is analytic (except $z = 0$) but equal to $\bar{z}$ on C.
- this is the Schwarz function for C.

On $|z| = a$, $a^2/z = \bar{z}$.

so $\bar{f}(a^2/z) = \bar{f}(\bar{z})$

$= \overline{f(\bar{\bar{z}})}$

$= \overline{f(z)}$ on C.

Combine potentials

$$w(z) = f(z) + \bar{f}(a^2/z)$$

on C.

Note $w(z) = f(z) + \overline{f(z)}$

$= \mathrm{Re}\{w(z)\}$ ie $\psi = 0$ on $|z| =$

So

$$\operatorname{Im}\{w(z)\} = 0$$

on $|z| = a$.

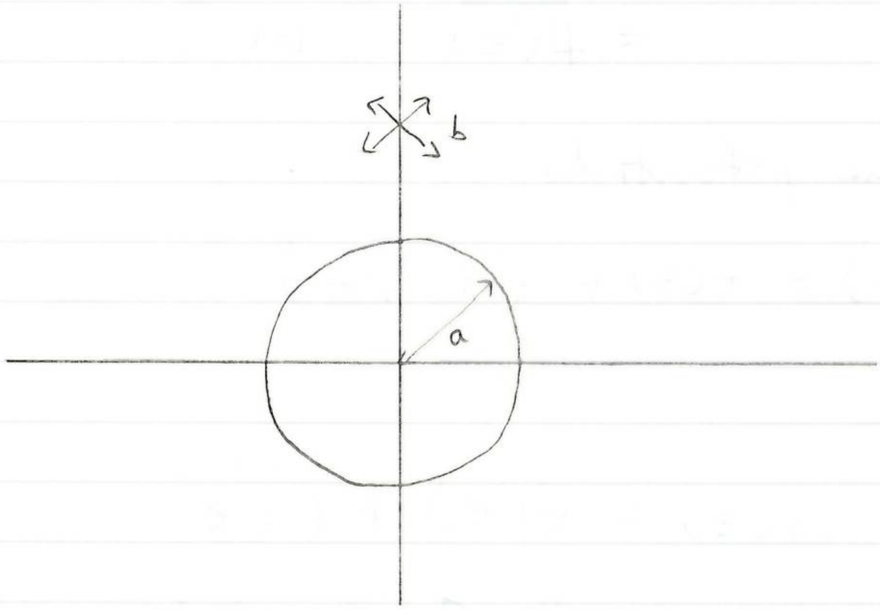
i.e circle is a streamline as required i.e no flow across $|z| = a$.

i.e $\bar{f}\left(\frac{a^2}{z}\right)$ is image of $f(z)$ in $|z| = a$.

Find Schwarz function $h(z)$ for C.

On C $h(z) = \bar{z}$. So image of $f(z)$ is $\bar{f}(h(z))$.

<u>Example 1:</u>
Find the image system + the total complex potential for a source of strength m at $z = a + ib$ outside the cylinder $|z| = a$ where $b > a$.



Original System:

$$f_1(z) = \frac{m}{2\pi} \operatorname{Log}(z - ib).$$

Image system:

$$f_2(z) = \overline{f_1}\left(\frac{a^2}{z}\right)$$

$$= \frac{m}{2\pi} \overline{\operatorname{Log}\left(\left(\overline{\frac{a^2}{z}}\right) - ib\right)}$$

$$= \frac{m}{2\pi} \operatorname{Log}\left(\frac{a^2}{z} + ib\right)$$

Total potential.

$$w(z) = f_1(z) + f_2(z)$$

$$= \frac{m}{2\pi} \operatorname{Log}\left(z - ib\right) + \frac{m}{2\pi} \operatorname{Log}\left(\frac{a^2}{z} + ib\right)$$

What is this.

$$\text{Image system} = \frac{m}{2\pi} \operatorname{Log}\left(\frac{a^2}{z} + ib\right)$$

$$= -\frac{m}{2\pi} \operatorname{Log}(z) + \frac{m}{2\pi} \operatorname{Log}\left(a^2 + ibz\right)$$

$$= \frac{-m}{2\pi} \operatorname{Log}(z) + \frac{m}{2\pi} \operatorname{Log}(ib) + \frac{m}{2\pi} \operatorname{Log}\left[z - \frac{ia^2}{b}\right]$$

Sink strength m at origin

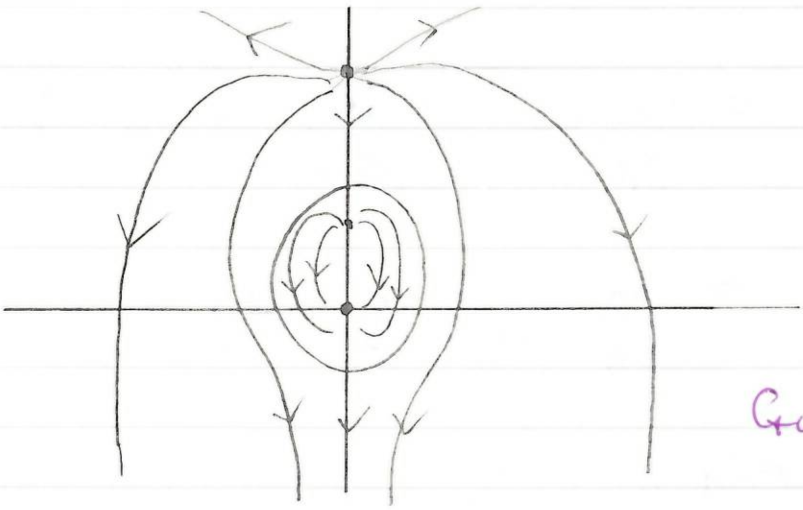
(no surprise)

Arbitrary constant ⇒ no effect

(no flow across C)

Source strength m at $z = \frac{ia^2}{b}$ i.e $x = 0$, $y = a^2/b$, $b > a$

Opitical image point

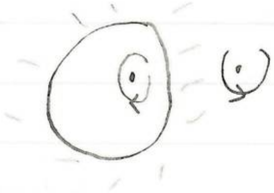


$\frac{a^2}{b} < a$

$(b > a)$

Guaranteed.

Example 2: Vortex in coffee cup.



motion induced by image.

Equation of Motion

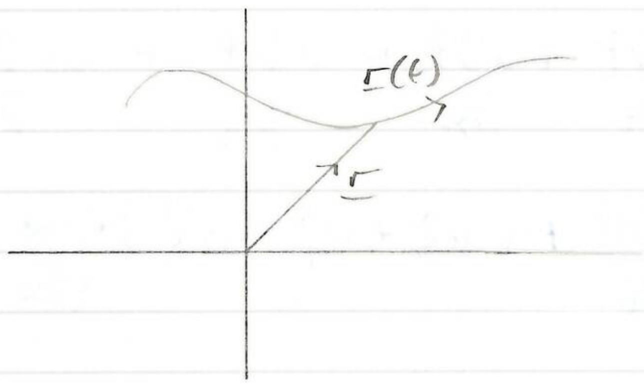
$$\underline{F} = m\underline{a}$$

$$= \frac{d}{dt}(m\underline{v})$$

Rate of change of mom. of particle, following the particle.

We thus need to define rate of change following a particle for a fields.

Suppose we have some field, known for all time t, and postion $\underline{r}$, $\phi(t, \underline{r})$.



Now suppose we follow some particle whose path is given by:

$$\frac{d\underline{r}}{dt} = \underline{u}$$

Then the values of ϕ along the particle path are:

$\phi(t, \underline{r}(t))$ when $\frac{d\underline{r}}{dt} = \underline{u}$

a function of t, alone.

What is the rate of change of ϕ along this path

$$\frac{D\phi}{Dt} = \left.\frac{d\phi}{dt}\right|_{x=\underline{r}(t)}$$

$$\ldots = \frac{\partial \phi}{\partial t} + \frac{\partial \phi}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial \phi}{\partial y}\frac{\partial y}{\partial t} + \frac{\partial \phi}{\partial z}\frac{\partial z}{\partial t}.$$

((Chain rule)

i.e $$\frac{D\phi}{Dt} = \frac{\partial \phi}{\partial t} + u\frac{\partial \phi}{\partial x} + v\frac{\partial \phi}{\partial y} + w\frac{\partial \phi}{\partial z}$$

$$= \frac{\partial \phi}{\partial t} + (u\underline{\hat{i}} + v\underline{\hat{j}} + w\underline{\hat{k}})\cdot\left(\frac{\partial \phi}{\partial x}\hat{i} + \frac{\partial \phi}{\partial y}\hat{j} + \frac{\partial \phi}{\partial z}\underline{\hat{k}}\right)$$

$$= \frac{\partial \phi}{\partial t} + \underline{u}\cdot\underline{\nabla}\phi$$

Speed at which move through gradient.

$$= \left(\frac{\partial}{\partial t} + \underline{u}\cdot\underline{\nabla}\right)\phi.$$

Convective or advective derivative.

Example:

Ex. 1. $\phi = x$.

$$\frac{Dx}{Dt} = \cancel{\frac{\partial x}{\partial t}} + u\frac{\partial x}{\partial x} + v\cancel{\frac{\partial x}{\partial y}} + w\cancel{\frac{\partial x}{\partial z}}$$

$$= u.$$

Ex. 2: $\phi = \underline{r}$

$$\frac{D\underline{r}}{Dt} = \frac{\partial}{\partial t}(x\hat{\underline{i}} + y\hat{\underline{j}} + z\hat{\underline{k}})$$

$$+ u\frac{\partial}{\partial x}(x\hat{\underline{i}} + y\hat{\underline{j}} + z\hat{\underline{k}})$$

$$+ v\frac{\partial}{\partial y}(x\hat{\underline{i}} + y\hat{\underline{j}} + z\hat{\underline{k}})$$

$$+ w\frac{\partial}{\partial z}(x\hat{\underline{i}} + y\hat{\underline{j}} + z\hat{\underline{k}})$$

$$= 0 + u\hat{\underline{i}} + v\hat{\underline{j}} + z\hat{\underline{k}}$$

$$= \underline{u}$$

Ex 3: $\underline{\phi} = \underline{u}$;

$$\frac{D\underline{u}}{Dt} = \frac{\partial \underline{u}}{\partial t} + (\underline{u} \cdot \underline{\nabla}) \underline{u} .$$

—/—

$$\frac{d}{dt} \int_{a(t)}^{b(t)} f(t,x)\, dx$$

$$= \int_a^b \frac{df}{dt} dx + f(t,b)\, b'(t) - f(t,a)\, a'(t)$$

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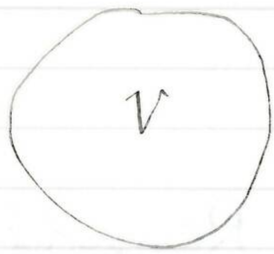
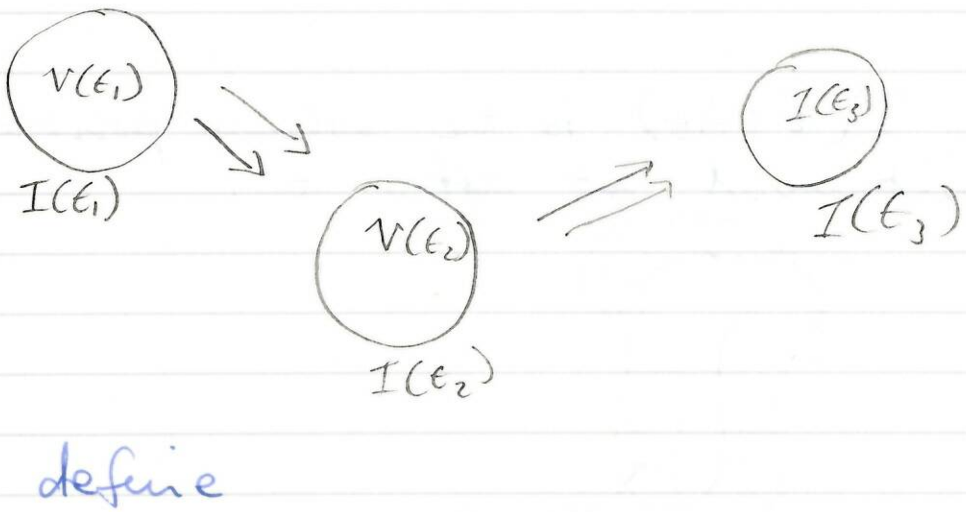
Reynolds transport theorem (RTT)

Why? Eventually we want to apply Newton's Law to a fluid, i.e



$$\frac{d}{dt}(\text{mom'm}) = \text{force}.$$

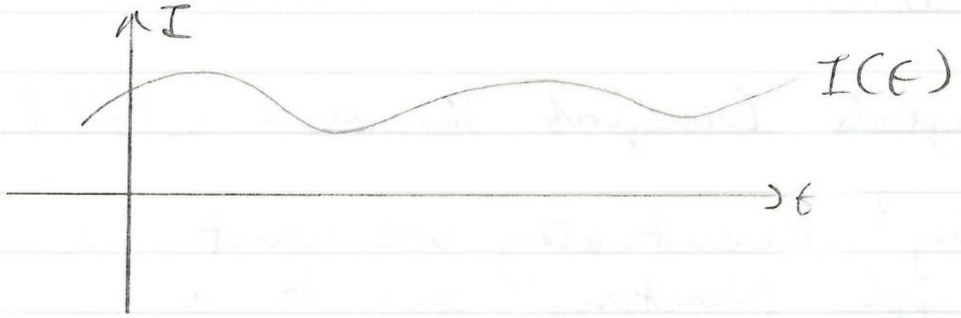
Consider a quantity $\alpha(\underline{r}, t)$ associated with a fluid. Let the fluid occupy a domain $\mathcal{D}$ and have specified velocity field $\underline{u}(\underline{r}, t)$. Consider a sub-volume V contained in $\mathcal{D}$ with surface S. We take V to consist always of the same fluid element or particles. Thus V moves i.e $V = V(t)$.



We define

$$I(t) = \int_{V(t)} \alpha(\underline{r}, t)\, d\underline{r}.$$

i.e the total amount of α in V at any time.



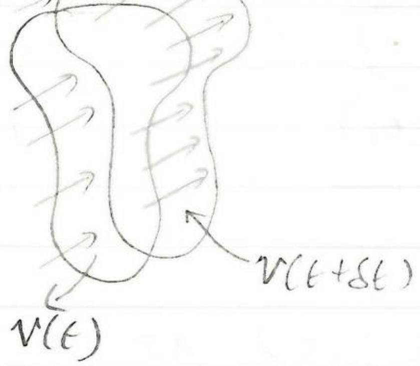
Reynold's : What is $\frac{dI}{dt}$?

$$\frac{dI}{dt} \text{ or } \frac{DI}{Dt} = \lim_{\delta t \to 0} \frac{I(t+\delta t) + I(t)}{\delta t} .$$

Emphasises that
we are following particles.

Here $I(t+\delta t) = \int_{V(t+\delta t)} \alpha(\underline{x}, t+\delta t) \, d\underline{x}$.

Here $V(t+\delta t)$ is the volume position at an interval δt after t .



and by Taylor's Theorem :

$$\alpha(\underline{r}, t+\delta t) = \alpha(\underline{r}, t) + \delta t \frac{\partial \alpha}{\partial t}(\underline{r}, t)$$

$$+ \frac{1}{2}(\delta t)^2 \frac{\partial^2 \alpha}{\partial t^2}(\underline{r}, \tau)$$

where τ lies in $(0, \delta t)$

So:

$$I(t+\delta t) = \int_{V+\delta V} \left[\alpha(\underline{r}, t) + \delta t \frac{\partial \alpha}{\partial t}(\underline{r}, t)\right] d\underline{r}$$

$$+ \frac{1}{2}(\delta t)^2 \int_{V+\delta V} \frac{\partial^2 \alpha}{\partial t^2}(\underline{r}, \tau)\, d\underline{r}.$$

$$= \underbrace{\int_{V+\delta V} \alpha \, dV + \delta t \int_{V+\delta V} \frac{\partial \alpha}{\partial t} \, dV}_{\text{Arguement is } (\underline{r}, t)} + O((\delta t)^2)$$

Now:

$$\frac{DI}{Dt} = \lim_{\delta t \to 0} \left\{ \frac{1}{\delta t} \left[I(t+\delta t) - I(t) \right] \right\}$$

$$\ldots = \lim_{\delta t \to 0} \left\{ \frac{1}{\delta t} \left[\cancel{\int_V \alpha \, dV} + \int_{\delta V} \alpha \, dV + \delta t \int_V \frac{d\alpha}{dt} \, dV \right.\right.$$

$$\left.\left. + O((\delta t)^2) - \cancel{\int_V \alpha \, dV} \right] \right\}$$

$$\Rightarrow \frac{DI}{Dt} = \lim_{\delta t \to 0} \left\{ \frac{1}{\delta t} \int_{\delta V} \alpha \, dV + \int_V \frac{\partial \alpha}{\partial t} \, dV \right.$$

$$\left. + \underbrace{\int_{\delta V} \frac{\partial \alpha}{\partial t} \, dV} + O(\delta t) \right\}.$$

The underlined term is:

$$\int_{\delta V} \frac{\partial \alpha}{\partial t} \, dV \leq \left| \int_{\delta V} \frac{\partial \alpha}{\partial t} \, dV \right|$$

$$\leq \int_{\delta V} \left| \frac{\partial \alpha}{\partial t} \right| dV$$

$$\leq \int_{\delta V} A \, dV.$$

where $A = \max_V \left(\left| \frac{\delta \alpha}{\delta t} \right| \right)$

$$= A \int_{\delta V} dV.$$

$$= A |\delta V| \longrightarrow 0$$

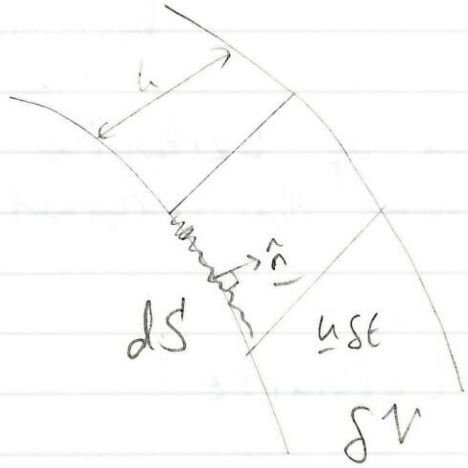
As $\delta t \to 0$.

Thus: $\frac{DI}{Dt} = \int_V \frac{\partial \alpha}{\partial t} \, dV + \lim_{\delta t \to 0} \frac{1}{\delta t} \int_{\delta V} \alpha \, dV.$

Particles making up dS' have moved a distance $\underline{u}\delta t$ in time δt.

They sweep out a volume = area base times height i.e

$$dV = dS \times h = (\underline{u} \cdot \hat{\underline{n}})\delta t.$$



i.e $dV \sim (\underline{u} \cdot \hat{\underline{n}})\delta t \, dS'$.

Thus is $\int_{dV} \alpha \, dV = \int_{S'} \alpha(\underline{u} \cdot \hat{\underline{n}}) \delta t \, dS'$.

$$= \delta t \int_{S'} \alpha(\underline{u} \cdot \hat{\underline{n}}) \, dS'.$$

Thus:

$$\underbrace{\frac{DI}{Dt} = \int_V \frac{\partial \alpha}{\partial t} dV + \int_S \alpha(\underline{u} \cdot \hat{\underline{n}}) \, dS}$$

Reynolds Transport Theorem (1)
or RTT (1)

The first form of RTT I:

$$\frac{D}{Dt}\left(\int_V \alpha \, dV\right) = \int_V \frac{\partial \alpha}{\partial t} \, dV + \int_S \alpha (\underline{u} \cdot \hat{n}) \, dS.$$

Local r.o.c.h (rate of change)

flux of α through boundary of V.

(The 3D version of Leibnitz rule for differentiating under the integral sign.)

—/—

Recall the 1D Leibnitz:

$$\frac{d}{dt}\left(\int_{a(t)}^{b(t)} \alpha(x,t) \, dx\right) = \int_a^b \frac{\partial \alpha}{\partial t} \, dx + b'(t)\alpha(t,b)$$

$$- a'(t)\alpha(t,a).$$

—/—

But the divergence theorem says that for any vector $\underline{F}$:

$$\int_S \underline{F} \cdot \hat{\underline{n}} \, dS = \int_V (\underline{\nabla} \cdot \underline{F}) \, dV.$$

Putting $\underline{F} = \alpha \underline{u}$.

Putting $\underline{F} = \alpha \underline{u}$.

$$\frac{D}{Dt}\left(\int_V \alpha \, dV\right) = \int_V \left[\frac{\partial \alpha}{\partial t} + \underline{\nabla} \cdot (\alpha \underline{u})\right] dV.$$

RRT 2

Now:

$$\underline{\nabla} \cdot (\alpha \underline{u}) = (\underline{u} \cdot \underline{\nabla})\alpha + \alpha \underline{\nabla} \cdot \underline{u}.$$

So:

$$\frac{\partial \alpha}{\partial t} + \underline{\nabla} \cdot (\alpha \underline{u}) = \frac{\partial \alpha}{\partial t} + (\underline{u} \cdot \underline{\nabla})\alpha + \alpha (\underline{\nabla} \cdot \underline{u})$$

$$= \frac{D\alpha}{Dt} + \alpha (\underline{\nabla} \cdot \underline{u}).$$

Thus:

$$\frac{D}{Dt}\left(\int_V \alpha \, dV\right) = \int_V \left(\frac{D\alpha}{Dt} + \alpha (\underline{\nabla} \cdot \underline{u})\right) dV.$$

RRT 3

Example: Take $\alpha = \rho$, density:

Then $M = \int_V \rho \, dV$

is the mass of particles, making up the volume V

Then by RTT2:

$$\frac{DM}{Dt} = \frac{D}{Dt} \int_V \rho \, dV.$$

$$= \int_V \left(\frac{\partial \rho}{\partial t} + \underline{\nabla} \cdot (\rho \underline{u}) \right) dV.$$

But mass is conserved so $\frac{DM}{Dt} = 0$.

i.e. for any V in $\mathcal{D}$:

$$\int_V \left[\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \underline{u}) \right] dV = 0 .$$

By the lemma, this implies:

$$\frac{\partial \rho}{\partial t} + \underline{\nabla}(\rho \underline{u}) = 0$$

everywhere in $\mathcal{D}$.

× Conservation of mass for a compressible

× Contains $\underline{\nabla} \cdot \underline{u} = 0$ when ρ is constant.

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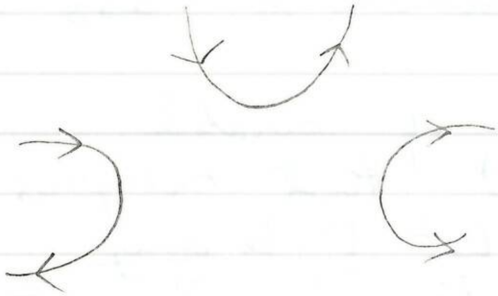


streamline
$r^2 \cos 2\theta$.

Uniform Stream
$r \sin\theta$.

—/—

RTT: 3D version of Leibnitz:

RTT1: $\frac{D}{Dt}\int_{\mathcal{V}} \alpha \, dt = \int_{\mathcal{V}} \frac{\partial \alpha}{\partial t} \, d\mathcal{V} + \int_{S} \alpha \, \underline{u} \cdot \hat{\underline{n}} \, dS$

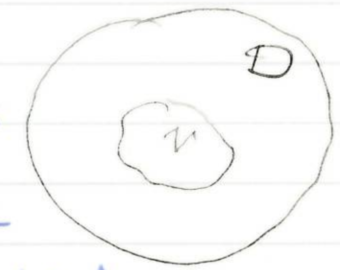
RTT2: $\frac{D}{Dt}\int_{\mathcal{V}} \alpha \, d\mathcal{V} = \int_{\mathcal{V}} \left[\frac{\partial \alpha}{\partial t} + \underline{\nabla} \cdot (\alpha \underline{u}) \right] d\mathcal{V}$.

RTT3: $\frac{D}{Dt}\int_{\mathcal{V}} \alpha \, d\mathcal{V} = \int_{\mathcal{V}} \left[\frac{D\alpha}{Dt} + \alpha(\nabla \cdot u) \right] d\mathcal{V}$.

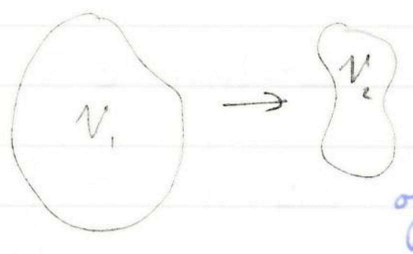
Any scalar $\alpha(\underline{x}, t)$.

Example: Conservation of Mass ($\alpha = \rho$ density)

Consider a fluid of variable density $\rho(\underline{x}, t)$ that occupies a domain $\mathcal{D}$. Let V be any subdomain of $\mathcal{D}$. [i.e V must be ARBITRARY]. Consider the total mass of all the particles comprising V i.e



$$M = \int_V \rho \, dV.$$



The rate of change of mass M, staying with the same particles must be zero (conservation of mass) i.e

$$\frac{DM}{Dt} = 0.$$

But by RTT 2:

$$\frac{D}{Dt}\int_V \rho \, dV = \int_V \left[\frac{\partial \rho}{\partial t} + \nabla\cdot(\rho \underline{u})\right] \partial V.$$

So $$\int_V \left[\frac{\partial \rho}{\partial t} + \underline{\nabla}\cdot(\rho \underline{u})\right] dV = 0.$$

But V is ARBITRARY, so this is TRUE FOR ALL V.

Hence by our theorem:

$$\frac{\partial \rho}{\partial t} + \underline{\nabla} \cdot (\rho \underline{u}) = 0$$

everywhere in $\mathcal{D}$.

This can be written:

$$\frac{D\rho}{Dt} + \rho \underline{\nabla} \cdot \underline{u} = 0 .$$

TRUE whether COMPRESSIBLE or INCOMPRESSIBLE.

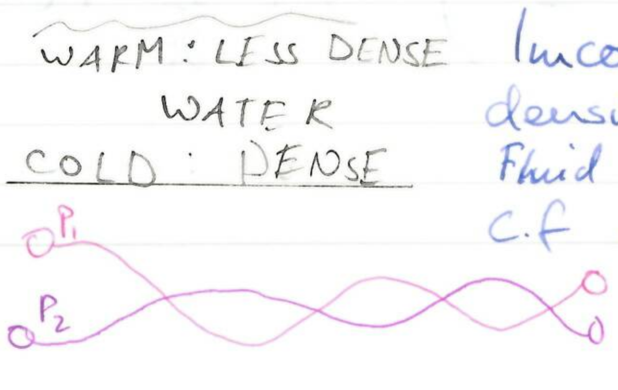
Notice, if the flow is <u>incompressiable</u>, fluid elements cannot be squashed i.e they preserve their volume. But they preserve their mass

$$\rho \underline{\nabla} \cdot \underline{u} = 0$$

$$\text{i.e} \quad \underline{\nabla} \cdot \underline{u} = 0 \quad (\text{As below}).$$

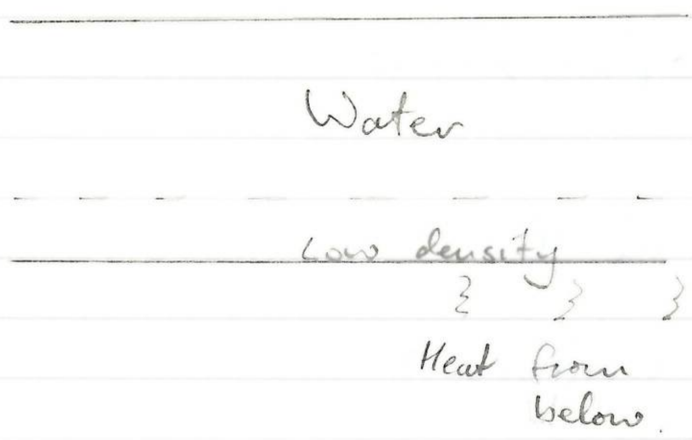
[Notice: This does not require all particle to have the same density].

Eg:



Incompressible not constant density.
Fluid particles retain density c.f colour.

BERNARD CONVECTION:



RTT4: For a fluid of density $\rho(\underline{x},t)$ considered any quantity $f(\underline{x},t)$

Put $\alpha = \rho f$ in RTT3:

$$\frac{D}{Dt}\int_V f\rho \, dV = \int_V \left[\frac{\partial}{\partial t}(f\rho) + \nabla\cdot(f\rho\underline{u})\right] dV$$

$$[\ldots] = \rho\frac{\partial f}{\partial t} + f\frac{\partial \rho}{\partial t} + f\rho\,\underline{\nabla}\cdot\underline{u} + \underline{u}\cdot\nabla(f\rho)$$

$$= f\left(\frac{\partial\rho}{\partial t} + \rho\,\underline{\nabla}\cdot\underline{u}\right) + \rho\frac{\partial f}{\partial t} + f\;\underline{\nabla}\cdot u\rho$$

$$+\rho\,\underline{u}\cdot\underline{\nabla} f.$$

$$= f\left(\frac{\partial\rho}{\partial t} + \underline{\nabla}\cdot(\rho\underline{u})\right) + \rho\left[\frac{\partial f}{\partial t} + (\underline{u}\cdot\underline{\nabla})f\right]$$

$$= \rho\frac{Df}{Dt}$$

i.e RTT4:

$$\frac{D}{Dt}\int_V \rho f \, dV = \int_V \rho \frac{Df}{Dt} \, dV$$

or

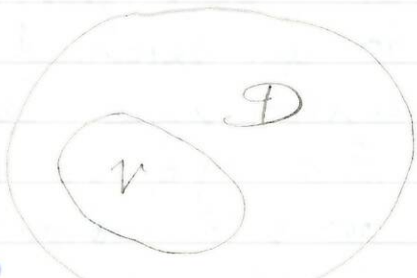
$$\frac{D}{Dt}\int_V f\rho \, dV = \int_V \frac{Df}{Dt} \rho \, dV.$$

i.e:

$$\frac{D}{Dt}\int_V f \, dM = \int \frac{Df}{Dt} \, dM.$$

i.e integrating w.r.t mass.

Example 2: Force = R.o.c.h of momentum
i.e $\alpha = \rho \underline{u}$.



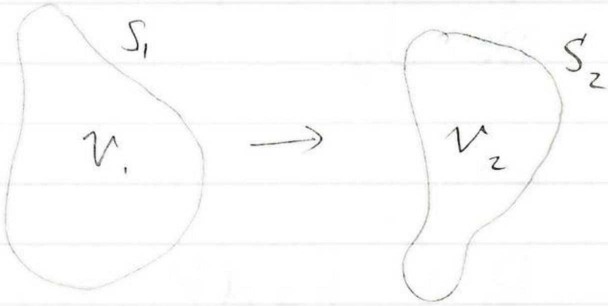
Consider a fluid of density $\rho(\underline{x}, t)$ occupying a domain $\mathcal{D}$. Let V be any subdomain, with surface S of D (i.e. important that V is ARBITRARY for our theorem).

Consider

$$\underline{m} = \int_V \rho \underline{u} \, dV.$$

the total momentum of the fluid particle making up V.

The following these particles, by RTT4,

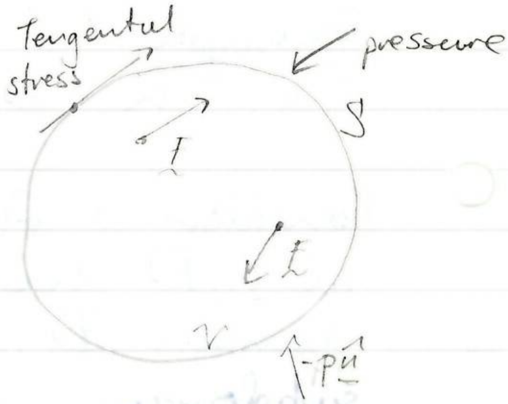


$$\frac{D\underline{m}}{Dt} = \frac{D}{Dt}\int_V \rho \underline{u} \, dV$$

$$= \int_V \rho \frac{D\underline{u}}{Dt} \, dV.$$

Acceleration.

By Newton this must equal the total external force acting on the particle comprising V. (The internal forces sum to zero).



Let each particle be subject to an external force $\underline{F}$ per unit mass. (e.g gravity, $\underline{F} = -g\hat{\underline{z}}$ or magnetic force or or a "fictitious" force e.g Coriolis)

Pressure: normal stress inwards $\underline{p}(\underline{x}, t) = p\hat{\underline{n}}$

That is ALL for INVISCID fluid as elements cannot exert a shear force (i.e. tangential stress) - there would be an extra term in viscious fluid.

Thus, the total force on all particles comprising

V is

$$\int_V \rho E \, dV + \int_S (-p)\,\underline{\hat{n}} \, dS.$$

$$= \int_V \rho \underline{F} \, dV + \int_V (-\nabla p) \, dV$$

by vector form of divergence theorem.

$$= \int_V (-\nabla p + \rho \underline{F}) \, dV$$

R.o.ch mom'm = force density.

$$\int_V \rho \frac{D\underline{u}}{Dt} \, dV = \int_V (-\underline{\nabla} p + \rho \underline{F}) \, dV.$$

i.e $$\int_V \left(\rho \frac{D\underline{u}}{Dt} + \underline{\nabla} p - \rho \underline{F} \right) dV = 0.$$

But V is arbitrary, so this is true for all V. Hence by our thm:

$$-\rho \frac{D\underline{u}}{Dt} + \underline{\nabla} p - \rho \underline{F} = 0,$$

Everywhere in $\mathcal{D}$.

This is Euler's Equation for an inviscid fluid!

$$\rho \frac{D\underline{u}}{Dt} = -\nabla p + \rho \underline{\nabla} \underline{F}.$$

Mass $\times$ ACCELERATION = FORCE.

Note that:

ρ - rho - density

p - "p" - pressure.

Equations of motion for a (possibly compressible) fluid.

Density $\rho(\underline{x}, t)$, pressure $p(\underline{x}, t)$, velocity $\underline{u}(\underline{x}, t)$

Mass: $\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \underline{u}) = 0.$

Euler: $\rho \frac{D\underline{u}}{Dt} = -\nabla p + \rho \underline{F}.$

Note: 2 unknown scalar fields (ρ, p) fields and one unknown vector fields $(\underline{u})$.

We have 1 scalar + 1 vector equation

This is missing a scalar equation.

—/—

Geophysical Fluid Dynamics :- Atmosphere + Ocean

Incompressibility : $\frac{D\rho}{Dt} = 0$

Mass : $\underline{\nabla} \cdot \underline{u} = 0$.

Euler : $\rho \frac{D\underline{u}}{Dt} = -\nabla p + \rho \underline{F}$.

Gas Dynamics (Cosmology)

$$p = f(\rho.)$$

For an ideal gas

$$p = \rho^{\alpha}$$

for some α — second scalar equation.

—/—

We will continue by taking the density to be constant, all particles have same density

Then

Mass = Incompressibility = "Continuity".

~~For~~ Euler:

1 scalar unknown : p, $\nabla \cdot \underline{u} = 0$.
1 vector unknown : $\underline{u}$, $\rho \frac{D\underline{u}}{Dt} = -\nabla p + \rho \underline{F}$.

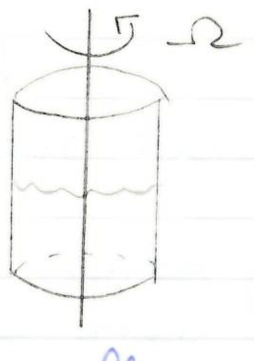
1 scalar equation
1 vector equation.

—/—

Next examples using this:

Open channel flow: hydrodics
Surface water waves
$\Big\}$ Need gravity

<u>Example</u>: Find the free surface shape for a cylinderical container partially filled with fluid of contsant density ρ in a solid body rotation with angular speed Ω about a vertical axis.

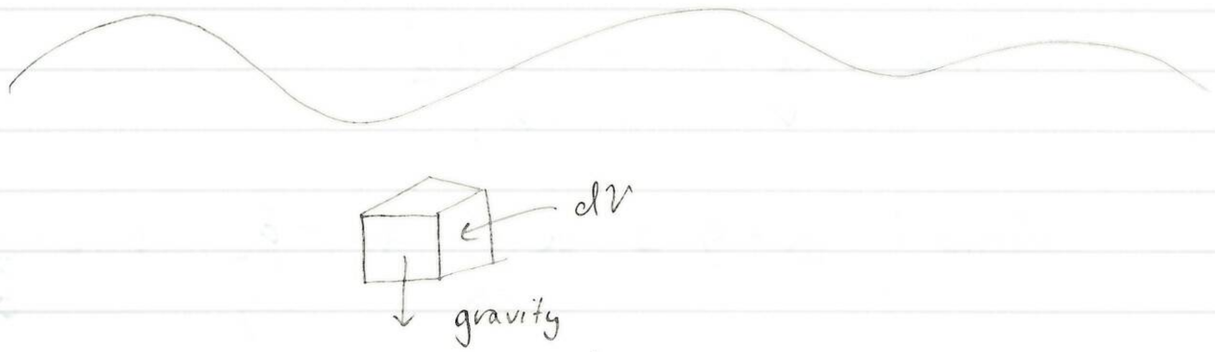


<u>Ans</u>: Let the flow have settled to a steady state, then:

$$\frac{\partial}{\partial t} \equiv 0.$$

Continuity : $\underline{\nabla} \cdot \underline{u} = 0$.

Euler : $\rho \frac{D\underline{u}}{Dt} = -\underline{\nabla} p + \rho \underline{F}$.



$\underline{F}$ force / unit mass ← dimension Acceleration

$$\text{So } \underline{F} = -g\hat{\underline{z}}.$$

$$\frac{D\underline{u}}{Dt} = \frac{\partial \underline{u}}{\partial t} + (\underline{u} \cdot \underline{\nabla})\underline{u}.$$

So Euler becomes :

$$(\underline{u} \cdot \underline{\nabla})\underline{u} = -\underline{\nabla} p - \rho g \hat{\underline{z}}.$$

important.

since $\underline{F} = -\rho g \hat{\underline{z}}$

We are told that the fluids is in solid body rotation i.e :

$$\underline{u} = \underline{\Omega} \wedge \underline{r}.$$

$$\Rightarrow \quad \underline{u} = \begin{vmatrix} \hat{\underline{i}} & \hat{\underline{j}} & \hat{\underline{k}} \\ 0 & 0 & \Omega \\ x & y & z \end{vmatrix}$$

$$= -\Omega y \hat{\underline{i}} + \Omega x \hat{\underline{j}}.$$

i.e $u = -\Omega y \hat{\underline{i}}$
$v = \Omega x \hat{\underline{j}}$.

Now: $\underline{u} \cdot \underline{\nabla} = u \frac{\partial}{\partial x} + v \frac{\partial}{\partial y} + w \frac{\partial}{\partial z}$

So:

$$(\underline{u} \cdot \underline{\nabla}) u = \left(-\Omega y \frac{\partial}{\partial x} + \Omega x \frac{\partial}{\partial y} \right) (-\Omega y)$$

$$= -\Omega^2 x.$$

$$(\underline{u} \cdot \underline{\nabla}) v = \left(-\Omega y \frac{\partial}{\partial x} + \Omega x \frac{\partial}{\partial y} \right) (\Omega x)$$

$$= -\Omega^2 y.$$

Eulers: (in components)

x-mom'm: $-\rho \Omega^2 x = -\frac{\partial p}{\partial x}$ (1)

y-mom'm: $-\rho \Omega^2 y = -\frac{\partial p}{\partial y}$ (2)

z-mom'm: $0 = -\frac{\partial p}{\partial z} - \rho g$. (3)

Integrate (1)

$$p = \frac{1}{2}\rho\Omega^2 x^2 + f(y,x)$$

Diff. w.r.t y:

$$p_y = f_y(y,x).$$

compare with (2): $f_y = \rho\Omega^2 y$

i.e $f = \frac{1}{2}\rho\Omega^2 y^2 + h(z)$

i.e $p = \frac{1}{2}\rho\Omega^2(x^2+y^2) + h(z)$

Diff. w.r.t z:

$$p_z = h'(z)$$

compare with (3)

$$h'(z) = -\rho g$$

i.e $h(z) = -\rho g z + C$

where C is a constant

i.e. $p = \frac{1}{2}\rho\Omega^2(x^2+y^2) + \rho g z + C$.

— / —

isobars = lines of constant pressure.

isobaric surface = surface of constant pressure

— / —

p = constant , $p = A$, constant A.

Take:

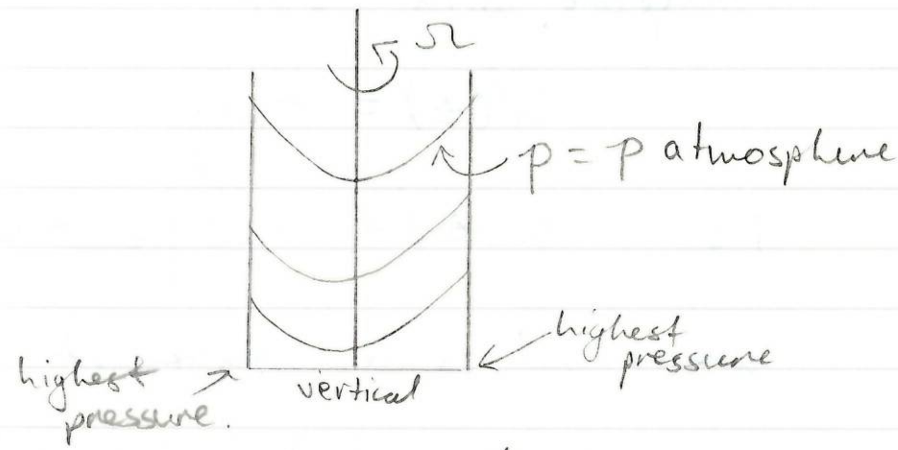
$$\rho g z = \frac{1}{2}\rho \Omega^2 (x^2+y^2) + C - A .$$

i.e. $$z = \frac{1}{2}\frac{\Omega^2}{g}(x^2+y^2) + \frac{C-A}{\rho g}$$

Let $\frac{C-A}{\rho g} = z_0$ = constant

$$z - z_0 = \frac{1}{2}\frac{\Omega^2}{g}(x^2+y^2).$$

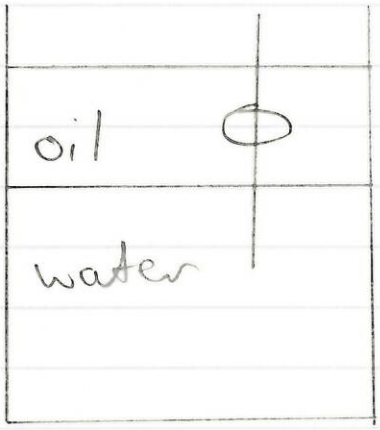
– a paraboloid with $(0, 0, z_0)$



cross-section

Archimedes ?

Example 2 : Consider a submerged body of volume V with surface S. Then the force on the body is upwards and equal to the weight of water displaced.



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Hydrostatic Pressure

Cty (Continuity) : $\underline{\nabla} \cdot \underline{u} = 0$.

Euler : $\rho \frac{D\underline{u}}{Dt} = -\nabla p + \rho \underline{F}$.

When gravity is the only external force :

$$\underline{F} = -g\hat{\underline{z}}.$$

If the fluid is at rest :

$$\underline{u} = 0, \quad 0 = -\underline{\nabla} p - g\hat{\underline{z}}.$$

i.e $\frac{\partial p}{\partial x} = 0$, $\frac{\partial p}{\partial y} = 0$, $\frac{\partial p}{\partial z} = -\rho g$

$\Downarrow$ $\Downarrow$

i.e $p = p(y, z)$, $p = p(z)$

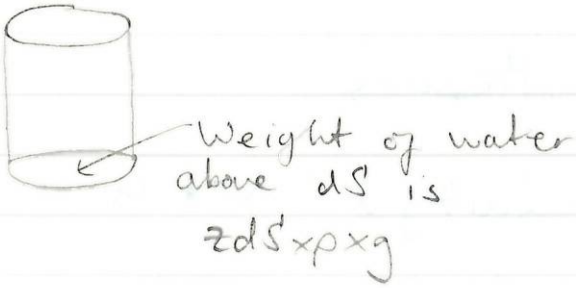
i.e $p = -\rho g z + \text{const}$.

$z = 0$: surface where pressure is atmospheric i.e $p = p_a$

Then $p = p_a$ when $z = 0$

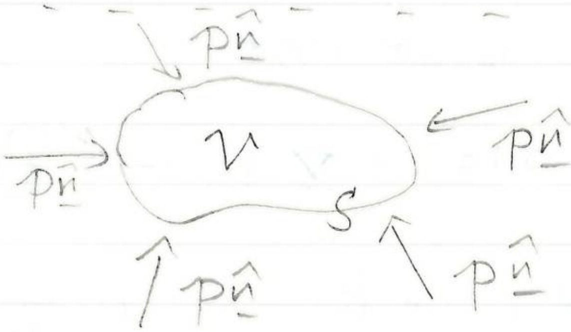
$$p = p_a - \rho g z.$$

we call this HYDROSTATIC pressure.



i.e Hydrostatic pressure supports above.

Example: Consider a submerged body occupying a volume V with surface S, immersed in a fluid of density ρ.



The force on the body is

$$\underline{F} = \int_S p\hat{\underline{n}}\, dS$$

$$= \int_V \underline{\nabla} p \, dV$$

by the divergence theorem.

Here p is the pressure in the fluid surrounding V. But fluid at rest, so:

$$p = p_H = p_a - \rho g \hat{\underline{z}}.$$

So

$$\underline{\nabla} p = \underline{\nabla} p_H = -\rho g \hat{\underline{z}}$$

So

$$\underline{F} = -\int_V (-\rho g \hat{\underline{z}})\, dV.$$

$$= \rho g \hat{\underline{z}} \int_V dV$$

$$= \underbrace{\overset{\text{Density of fluid}}{\rho} V}_{\text{Not mass of body}} g \hat{\underline{z}}$$

ρV = mass of fluid

$\rho V g$ = weight

$\rho V g \hat{\underline{z}}$ = an upward force equal to the weight of the fluid displaced

ARCHEDLES

—/—

For a moving fluid, it is often convenient to split the pressure into hydrostatic and the rest, called dynamic

i.e. $p = p_H + p_d$.

Then the Euler equation under gravity become

$$\rho \frac{D\underline{u}}{Dt} = -\nabla p - \rho g \hat{\underline{z}}$$
$$= -\nabla p_H - \nabla p_d - \rho g \hat{\underline{z}}$$
$$= -(\rho g \hat{\underline{z}}) - \nabla p_d - \rho g \hat{\underline{z}}$$
$$= -\underline{\nabla} p_d .$$

i.e. we can ignore gravity in the equation of motion, provided we measure pressure as the deviation from hydrostatic.

This is not useful when a free surface is pressed since the b.c. there

$$p = p_a .$$

is on the total pressure $p = p_H + p_d$.

<u>Bernoulli's Equation</u>
We have the identity:

$$(\underline{u}\cdot\underline{\nabla})\underline{u} = \underline{\nabla}(\tfrac{1}{2}\underline{u}^2) + \underline{\omega} \wedge \underline{u} .$$

where $\underline{\omega} = \underline{\nabla} \wedge \underline{u}$ is the vorticity.

Thus $\frac{D\underline{u}}{Dt} = \frac{\partial \underline{u}}{\partial t} + (\underline{u}\cdot\underline{\nabla})\underline{u}$.

$$\ldots = \frac{\partial \underline{u}}{\partial t} + \underline{\nabla}\left(\tfrac{1}{2}u^2\right) + \underline{\omega} \wedge \underline{u} .$$

Now Euler is:

$$\rho \frac{D\underline{u}}{Dt} = -\nabla p + \rho \underline{F} .$$

Now consider only external forces derivable from a potential i.e:

$$\underline{F} = -\underline{\nabla} \hat{V_e} .$$

i.e for gravity.

$$\hat{V_e} = gz$$

i.e $\underline{F}$ is conservative.

Then:

$$\rho\left[\frac{\partial \underline{u}}{\partial t} + \underline{\omega} \wedge \underline{u}\right] = -\nabla p_a - \rho \underline{\nabla}\left(\tfrac{1}{2}\underline{u}^2\right) - \rho \underline{\nabla} \hat{V_e} .$$

$$= -\underline{\nabla} H .$$

where

$$H = p = + \tfrac{1}{2}\rho \underline{u}^2 + \rho \hat{V_e}$$

In steady flow;

$$\frac{\partial \underline{u}}{\partial t} = 0$$

So $\rho(\underline{\omega} \wedge \underline{u}) = -\underline{\nabla} H$.

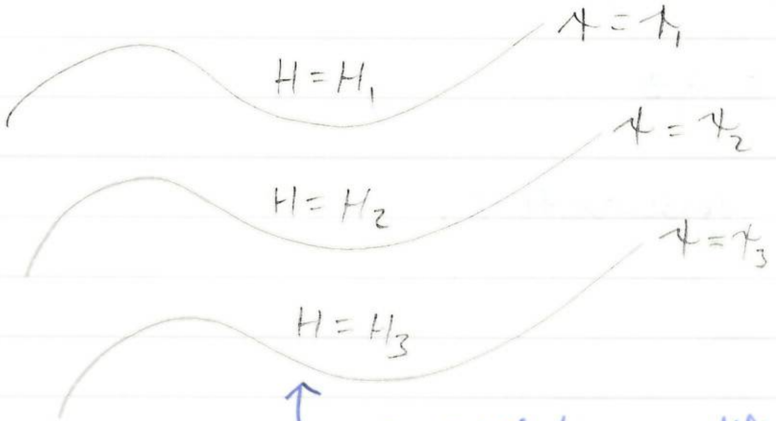
Dot with $\underline{u}$:

$$0 = (u \cdot \underline{\nabla}) H.$$

i.e $\frac{D\underline{u}}{Dt} = 0$

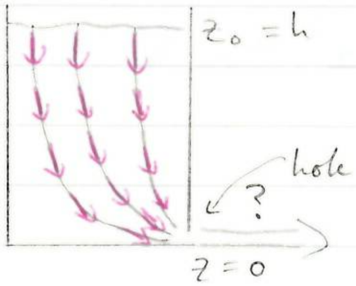
i.e Particle paths are streamlines.

Thus H is constant on STREAMLINES, — BERNOLLI's THEOREM.



H CAN have different values on different streamlines. H sometimes; incorrectly called Bernolli's constant.

Example:



Imporant : The surface is connected to the exit by STREAMLINES

Assume the hold is sufficiently small that the flow is steady.

Hence Bernolli's applies on any streamlines:

H = Constant (Not nesscesscally same constant)

Hence :

$$H = p + \frac{1}{2}\rho u^2 + \rho V_e$$

The external potential is $V_e = gz$ (with level of hole).

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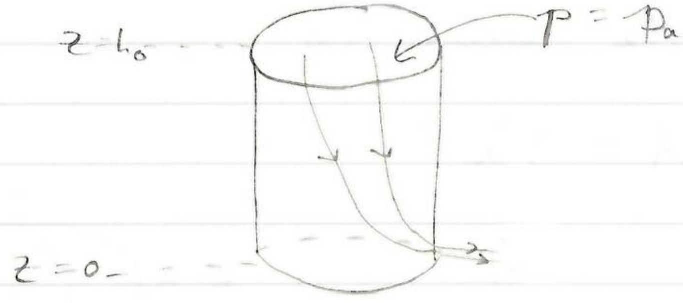
Bernoulli Examples

Example 1: Draining cylinder



S'line connecting surface to exit.

$\therefore$ Can apply Bernoulli on this s'line.

i.e $p + \frac{1}{2}\rho \underline{u}^2 + \rho V_e$ is constant on streamline

Hence same at top and bottom.

Now: $V_e = gz$ ($z = 0$ at level of exit)

Pressure is atmospheric at top and bottom

At top: $p + \frac{1}{2}\rho \underline{u}^2 + \rho V_e = p_a + \frac{1}{2}\rho U^2 + \rho g h$.
where $u = U$ at surface.

At bottom: $p + \frac{1}{2}\rho u^2 + \rho V_e = p_a + \frac{1}{2}\rho V^2 + 0$.
where $u = V$ at exit

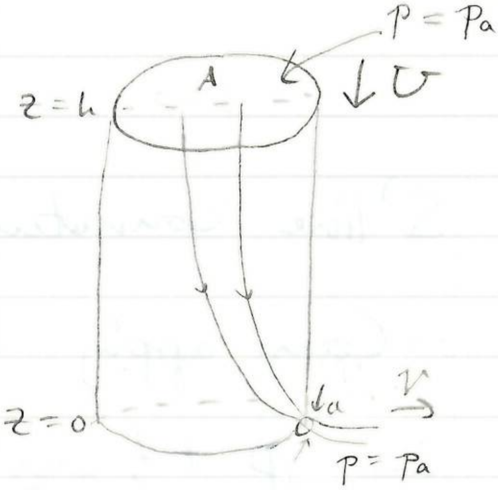
Thus $p_a + \frac{1}{2}\rho U^2 + \rho g h = p_a + \frac{1}{2}\rho V^2$

i.e $V^2 = U^2 + 2gh$.

The mass flux at top is $\rho U A$.
The mass flux at bottom is $\rho V a$.

These are the same so:

$$UA = Va.$$



Thus

$$V^2 = \left(\frac{Va}{A}\right)^2 + 2gh.$$

$$\Rightarrow V^2\left[1 - \left(\frac{a}{A}\right)^2\right] = 2gh$$

If hole is small $\frac{a}{A} \ll 1$

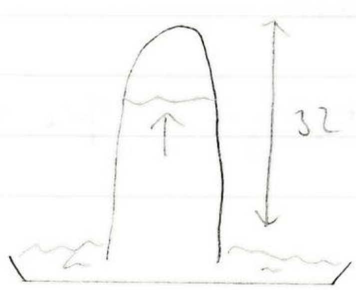
so $\left(\frac{a}{A}\right)^2 \lll 1$

Then $V^2 \simeq 2gh$

i.e $V = \sqrt{2gh}$

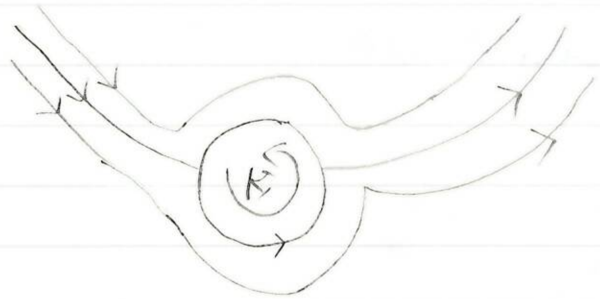
i.e exactly as for a free-falling particle under gravity: $\frac{1}{2}\sqrt{2gh} \lesssim V \leq \sqrt{2gh}$

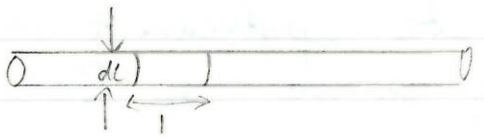
Torcelli:

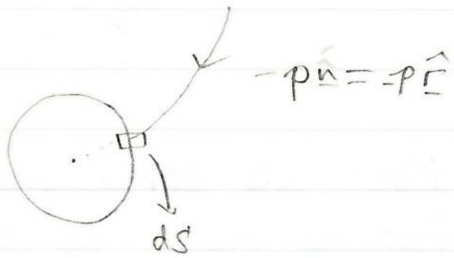


Example 2: Spinning Cylinder

Consider a cylinder of radius a in a stream, uniform at in infinity with speed U in the x-direction. Let the cylinder be spinning so that the circulation about the cylinder is K:

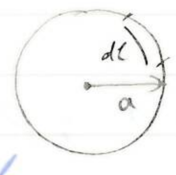






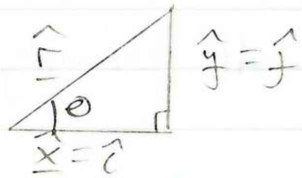
We will find the force per unit length on the cylinder. Our unit of aera is $dS = dl \cdot x \cdot 1$. Thus total force (per unit length):

$$= \int -p\,\underline{\hat{n}}\,dS$$



$$= -\int p\,\underline{\hat{r}}\,dl \quad \text{per unit length.}$$

And $dl = a\,d\theta$



$\underline{\hat{r}} = \cos\theta\,\underline{\hat{i}} + \sin\theta\,\underline{\hat{j}}$.

$$\underline{F} = -\int_{\theta=-\pi}^{\pi} p\,\underline{\hat{r}}\,a\,d\theta$$

$$= -a\int_{-\pi}^{\pi} p(\cos\theta\,\underline{\hat{i}} + \sin\theta\,\underline{\hat{j}})\,d\theta$$

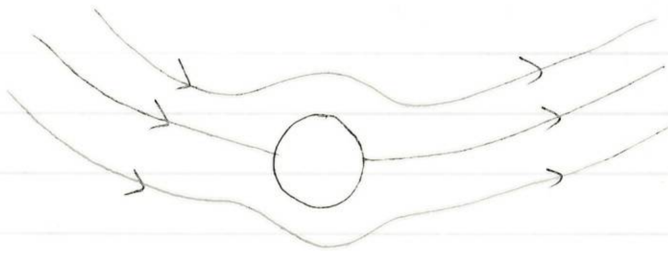
$$= \mathcal{D}\,\underline{\hat{i}} + \mathcal{L}\,\underline{\hat{j}}.$$

Where the drag, $\mathcal{D}$ is

$$\mathcal{D} = -a \int_{-\pi}^{\pi} p\cos\theta \, d\theta$$

and the lift $\mathcal{L}$ is

$$\mathcal{L} = -a \int_{-\pi}^{\pi} p\sin\theta \, d\theta .$$



Now all streamline orginate upstream + we are taking flow to be steady so we use Bernolli:

—/—

Note: $\frac{\partial \underline{u}}{\partial t} = 0 \iff$ steady

—/—

$p + \frac{1}{2}\rho \underline{u}^2 =$ constant on streamline in the absence of external force.

At infinity, $p = p_\infty$, constant

$$\underline{u} = U\hat{\underline{i}} .$$

So $p + \frac{1}{2}\rho \underline{u}^2 = p_\infty + \frac{1}{2}\rho U^2$

Anywhere in the flow, $p = p_0 + \frac{1}{2}\rho U^2 - \frac{1}{2}\rho |\underline{u}|^2$

The complex potential for the flow is:

$$w(z) = U\left(z + \frac{a^2}{z}\right) - \frac{iK}{2\pi} \operatorname{Log} z.$$

So $\frac{dw}{dz} = u + iv$

$$= U\left(1 - \frac{a^2}{z^2}\right) - \frac{iK}{2\pi z}$$

So $$u_r - iu_\theta = e^{i\theta}\frac{dw}{dz}$$

$$= 2iU\sin\theta - \frac{iK}{2\pi a} \quad \text{on } z = ae^{i\theta}.$$

i.e $\left.\begin{array}{l} u_r = 0 \quad \text{as expected} \\ u_\theta = \frac{K}{2\pi a} - 2U\sin\theta \end{array}\right\}$ On cylinder

Thus:

$$|\underline{u}|^2 = \frac{K^2}{4\pi} - \frac{2UK}{\pi a}\sin\theta + 4U^2\sin^2\theta.$$

Now: $$D = -a\int_{\theta=-\pi}^{\theta=\pi} \cos\theta \, p \, d\theta$$

Note that:
- $\int_{-\pi}^{\pi} \cos\theta \, C \, d\theta = 0$
- $\int_{-\pi}^{\pi} \cos\theta\sin\theta \, d\theta = 0$
- $\int_{-\pi}^{\pi} \cos\theta\sin^2\theta \, d\theta = 0.$

Thus $\mathcal{D} = 0$ i.e no drag.

<u>Or</u>: Velocity symmetric before + after
so pressure same before + after. $\therefore$ no drag.

Anywhere in the flow:

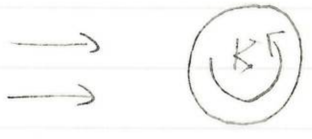
$$p = p_\infty + \frac{1}{2}\rho U^2 - \frac{1}{2}\rho|\underline{u}|^2 .$$

Now: $\mathcal{L} = -a\int_{-\pi}^{\pi} \sin\theta \; p \; d\theta .$

Note : $\times \int_{-\pi}^{\pi} \sin\theta \; C \; d\theta = 0$
$\times \int_{-\pi}^{\pi} \sin^3\theta \; d\theta = 0 .$
$\times \int_{-\pi}^{\pi} \sin^2\theta \; d\theta = \pi$

$$\mathcal{L} = -a\int_{-\pi}^{\pi} \left(-\frac{1}{2}\rho\right)\left(-\frac{2UK}{\pi a}\right)\sin^2\theta \quad d\theta$$

$$= -\rho U K .$$



i.e. downward force

$\rho U K$ (indep of a)

(per unit length).

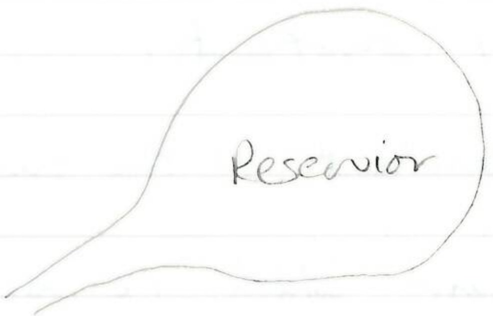
—/—

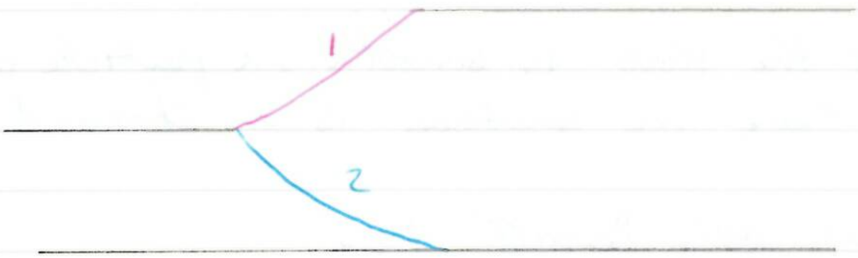
Aside $\underline{u}^2 = |\underline{u}|^2 = \underline{u}\cdot\underline{u}$

—/—

Example 3: Open channel flow – flow (down along) a channel that open to the air. e.g river, aqueduct.

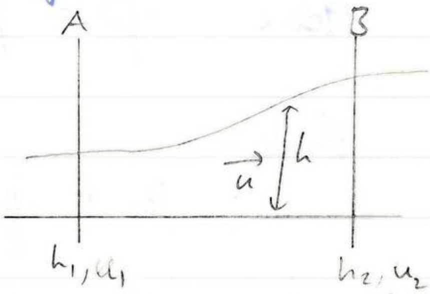
1) Rises
2) Stays same
3) Falls.





1) Get deeper, join smoothly
2) Gets shallower

Initially, let us consider a channel of constant width b, and horiztal floor.



Let any changes in the flow be slow in the flow direction.

Let the local depth be h and local speed u direction.

By conversatcoin of mass, in steady flow, mass flux across statcoin A must equal mass flux across statcoin B.

i.e $\rho h_1 b u_1 = \rho h_2 b u_2$

i.e $h_1 u_1 = h_2 u_2$.

<u>OR</u>; throughout the flow

$$uh = Q \text{ , constant.}$$

i.e $Q = uh$, is constant of the motion.

[volume flux per unit width i.e dimension : $L^2 T^{-1}$]

Provided the flow is smooth, a particle on the surface stays there i.e surface is a streamline

Hence we apply Bernoulli there:

Bernoulli (on a surface, a streamline):

$$p + \tfrac{1}{2}\rho|\underline{u}|^2 + \rho V_e \text{ is a constant.}$$

Here: $p = p_a$, $\underline{u} = u\hat{\underline{i}}$, $V_e = gz$.

Thus:

$$p_a + \frac{1}{2}\rho u^2 + \rho g h = \text{const}.$$

i.e $\frac{1}{2}u^2 + gh = \text{constant} = gH$.

where H is a second constant of the motion.

Dimensions of H are length.

H is the depth the fluid would occupy were it to come to rest i.e $h \to H$ if $u \to 0$.

H: "head of the flow".

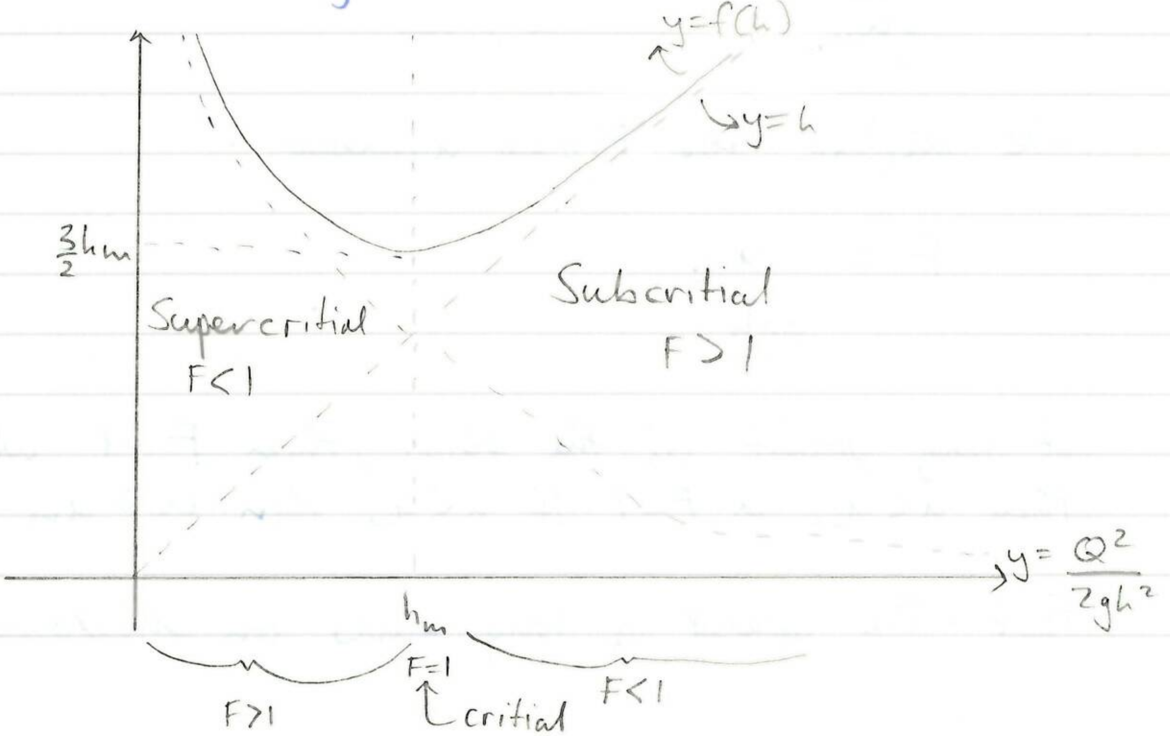
Thus we have

$$uh = Q$$
$$\tfrac{1}{2}u^2 + gh = gH.$$

Eliminate u, $u = Q/h$

so $$\frac{Q}{2gh^2} + h = H.$$

Let $f(h) = Q^2/2gh^2 + h$.



This graph has a single minimum for $h>0$ where $f(h)=0$

i.e $\quad -\frac{2Q^2}{2gh^3}+1=0$.

i.e $\quad h=h_m=\left(\frac{Q^2}{g}\right)^{1/3}$

$$f(h_m)=h_m+\frac{h_m^3}{2h_m^2}=\frac{3}{2}h_m.$$

—/—

At $h=h_m$

$$h_m^3=Q^2/g$$

$$=\frac{h_m^2u_m^2}{g}$$

i.e $\quad \frac{u_m^2}{gh_m}=1$.

—/—

We define the Froude number:

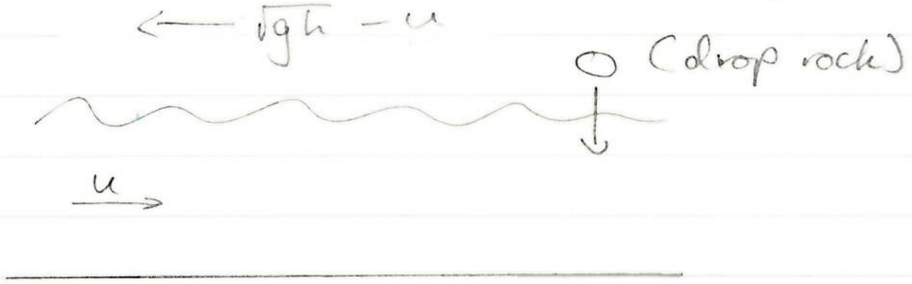
$$F=\frac{u}{\sqrt{gh}}$$

at any point in the flow, then $F=1$ when $h=h_m$, then $u<u_m$ so $F<1$. If $h<h_m$ then $u>u_m$ so $F>1$

Fact: The speed of long waves on shallow water is

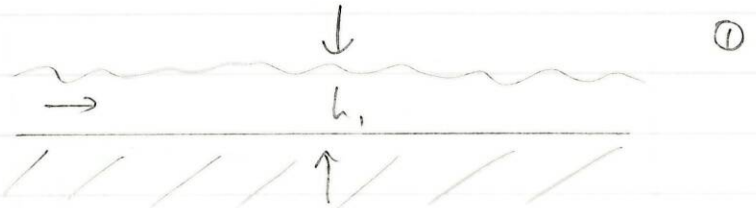
$\sqrt{gh}$ (shown in water waves).

So if $F<1$, flow is slower than waves, waves can travel upstream: (Subcritical)

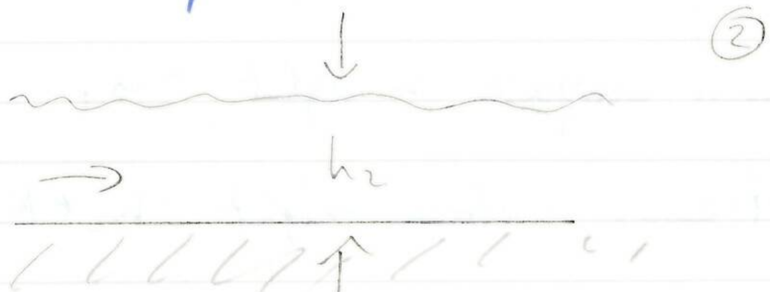


If $F>1$, flow is faster than waves, information cannot travel upstream. (Superscritical)

Supercritial: Shallow and fast:

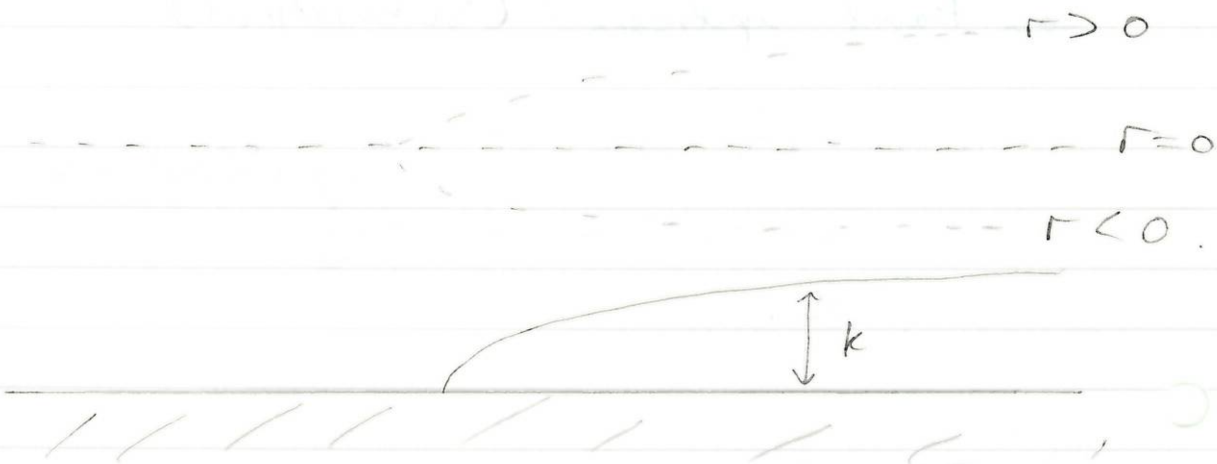


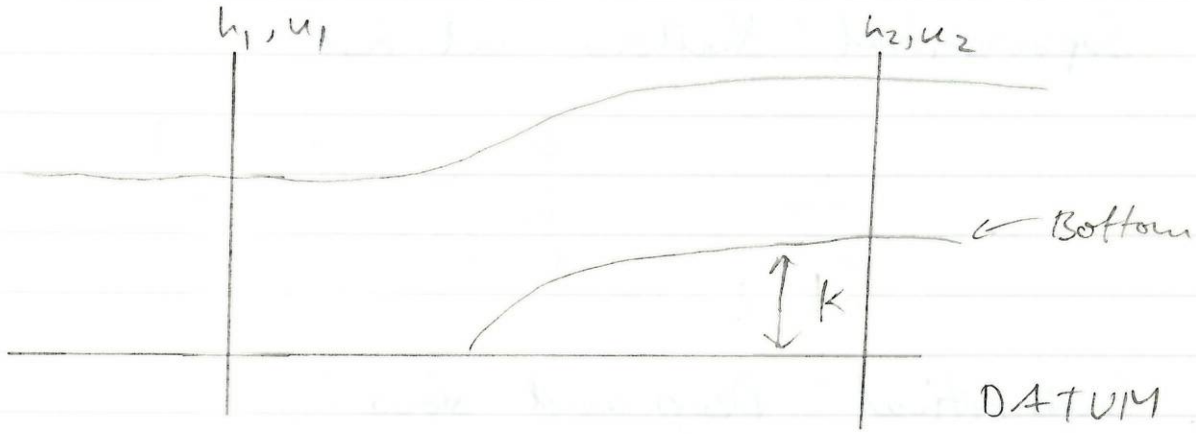
Subcritical: Deep and slow:



These are two flows with same Q or same H but different h.

Example. Now suppose the channel remains constant width but the floor of the channel rises smoothly by an amount K.





Upstream surface height $- h_1$

Downstream surface height $- h_2 + k$

Rise in surface $r = (h_2 + k) - h_1$.

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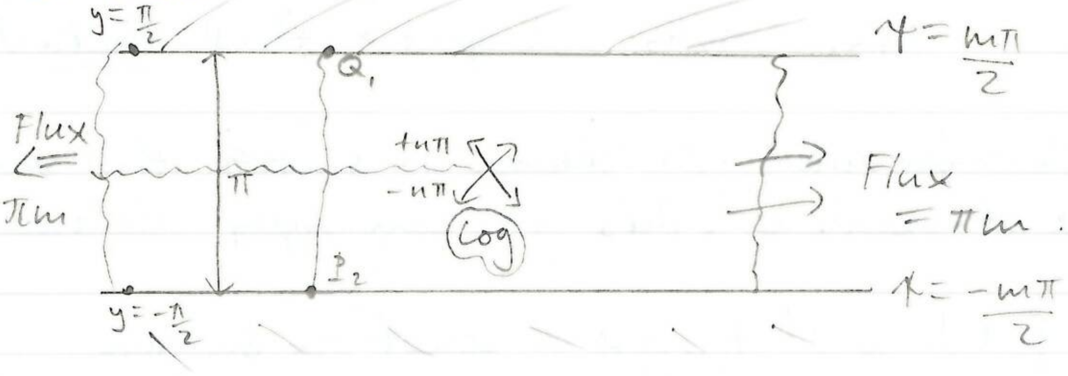
—/—

From homework:



$u \to m$, as $x \to \infty$.

$$\left(\frac{\pi}{2}m - \pi m\right) + \left(-m\pi + \frac{\pi}{2}m\right)$$

$$= -\pi m.$$

Note also $u \to -m$ as $x \to -\infty$

—/—

rise r = new − old
$= h_2 + k - h_1$

Mass flux constant

$$u_1 h_1 = u_2 h_2 \quad (\text{speed} \times \text{fluid depth})$$

Now provided the change is smooth the surface is a streamline. This we can apply Bernoulli

$$p + \frac{1}{2}\rho \underline{u}^2 + \rho g z = \text{const on surface}$$

Upstream:

$$p + \frac{1}{2}\rho \underline{u}^2 + \rho g z = p_a + \frac{1}{2}\rho u_1^2 + \rho g h_1$$

Downstream:

$$p + \frac{1}{2}\rho \underline{u}^2 + \rho g z = p_a + \frac{1}{2}\rho u_2^2 + \rho g (\underline{h_2 + k})$$

Not the depth
height of surface

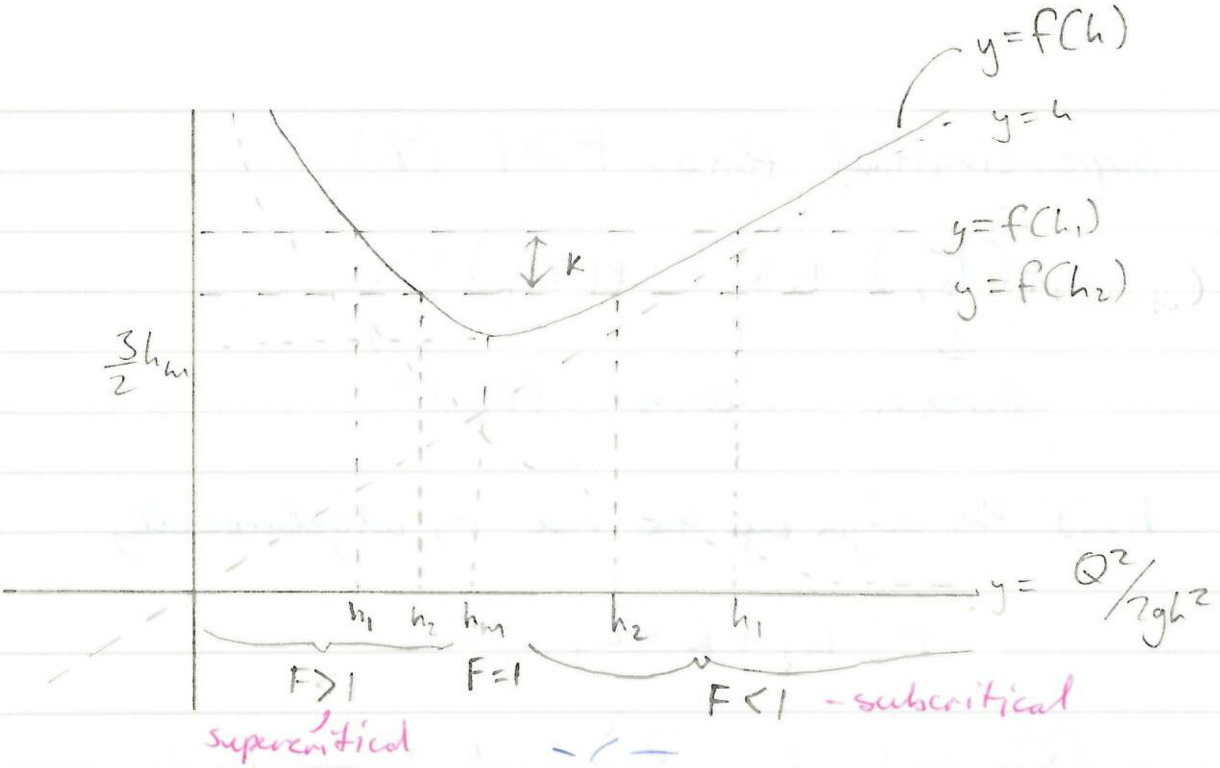
Thus: $\frac{1}{2}\rho u_1^2 + \rho g h_1 = \frac{1}{2}\rho u_2^2 + \rho g (h_2 + k)$.

i.e $\frac{u_1^2}{2g} + h_1 = \frac{u_2^2}{2g} + h_2 + k$.

i.e $\frac{Q^2}{2gh_1^2} + h_1 = \frac{Q^2}{2gh_2^2} + h_2 + k$.

$$u_1 h_1 = u_2 h_2 = Q \quad (\text{volume flux / unit width})$$

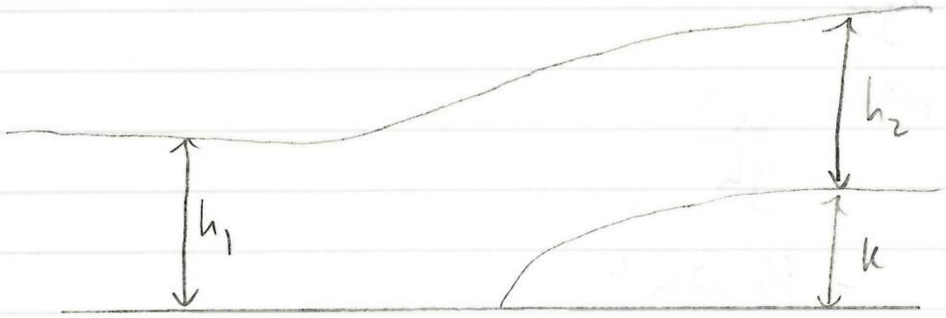
i.e $f(h_1) = f(h_2) + k$, where $f(h) = \frac{Q^2}{2gh^2} + h$.



— / —

$$f(h_1)-k \quad < f(h_m) \quad ?$$

Subcritical flow $F<1$.
More PE than KE gives up PE to get over barrier
i.e surface, drops + flows speed up.



Supercritical flow: $F > 1$ (*)

$$\underset{\text{decreases}}{\underset{\downarrow}{f(h_2)}} + \underset{\text{increases}}{\underset{\uparrow}{k}} = \underset{\text{fixed}}{f(h_1)}$$

Find the sign of the rise r, algebraically

$$r = h_2 + k - h_1$$

$$\frac{Q^2}{2gh_1^2} + h_1 = \frac{Q^2}{2gh_2^2} + h_2 + k$$

$$r = \frac{Q^2}{2g}\left[\frac{1}{h_1^2} - \frac{1}{h_2^2}\right]$$

$$= \frac{Q^2}{2gh_1^2h_2^2}\left(h_2^2 - h_1^2\right)$$

Thus $r > 0$ when $h_2 > h_1$ and $r < 0$ when $h_2 < h_1$

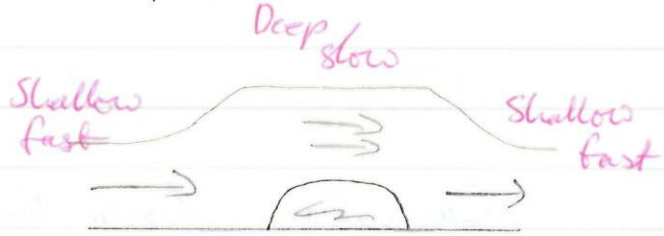
—/—

$$F = \frac{u}{\sqrt{gh}}$$

$$\Rightarrow \quad F^2 = \frac{u^2}{gh}$$

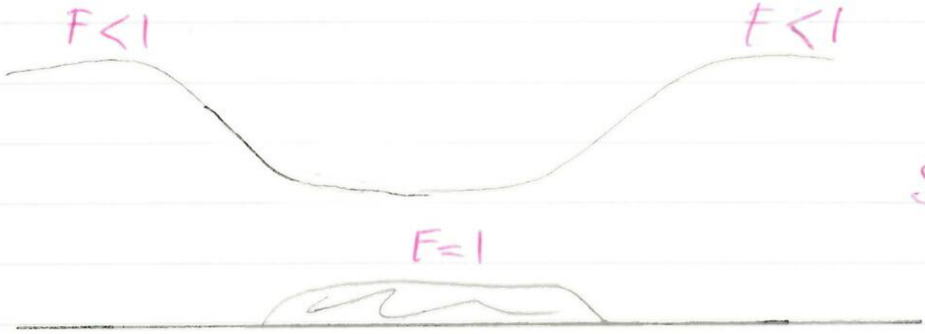
$$= \frac{\frac{1}{2}\rho u^2}{\frac{1}{2}\rho gh}.$$

(*) More KE than PE. To get over barrier gives up some KE to get PE.

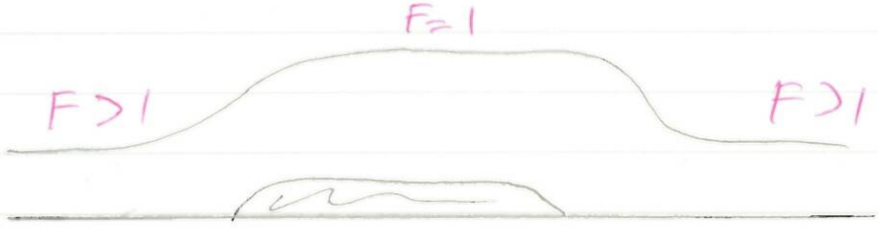


Sup. $F<1$ Super $F>1$.

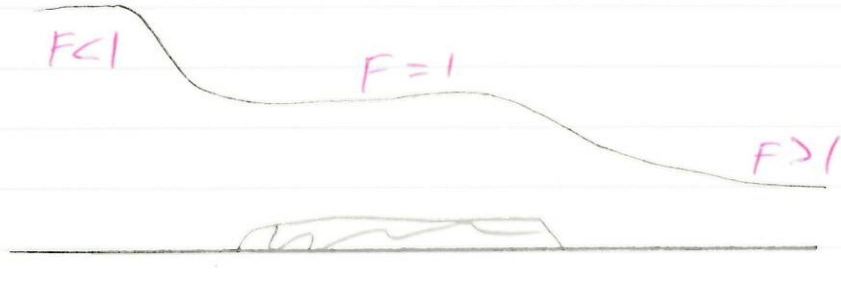
If $k = f(h_1) - f(h_m)$ there are four 4 possilites.



Symmetric



Symmetric.



Transition (Satisfies causality)

Causality shows that smooth transitoin are always from sub-ciritical to super-critical.

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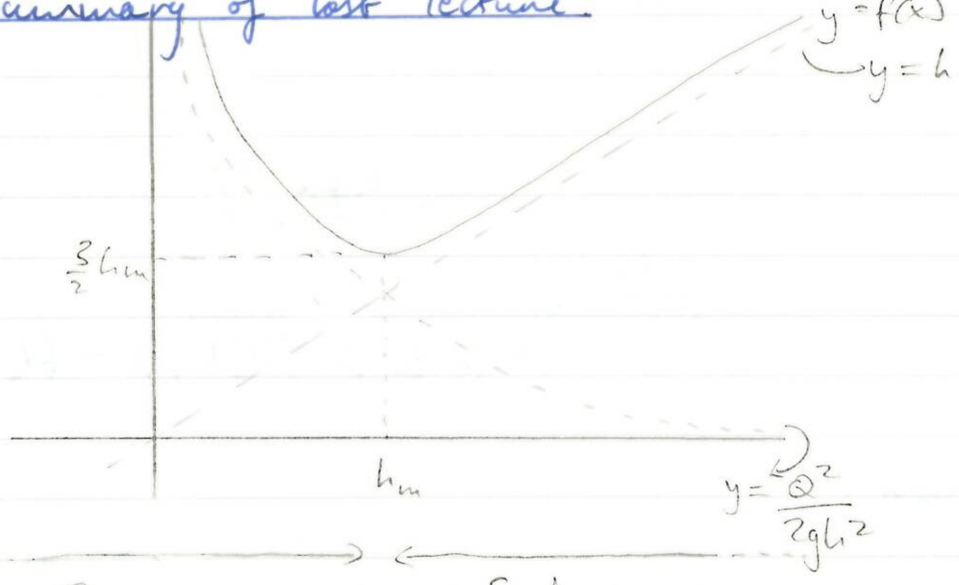
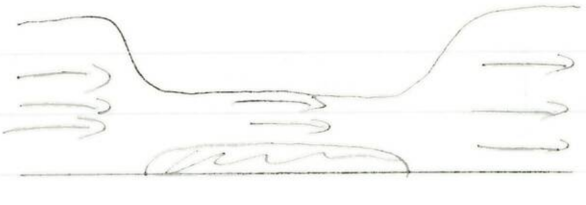
Summary of last lecture



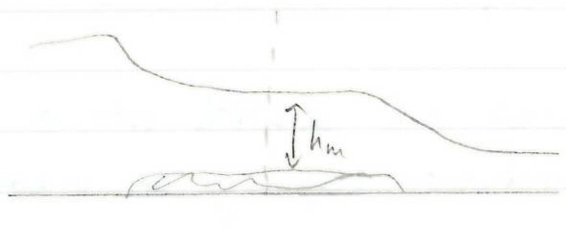
Super
$F > 1$
$h < h_m$

Sub
$F < 1$
$h > h_m$

$F > 1$
Super
Shallow, fast.

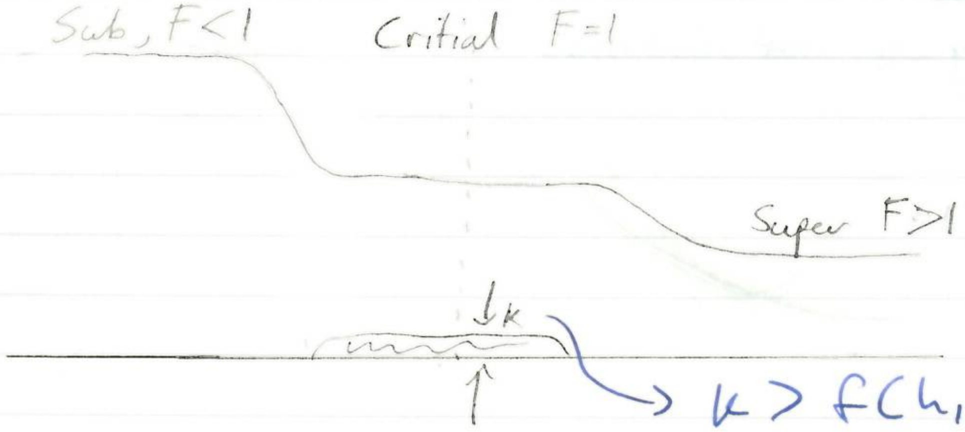


$F < 1$
Sub
Deep, slow.



Super
$F = 1$ at maximum height
Transition.

—/—

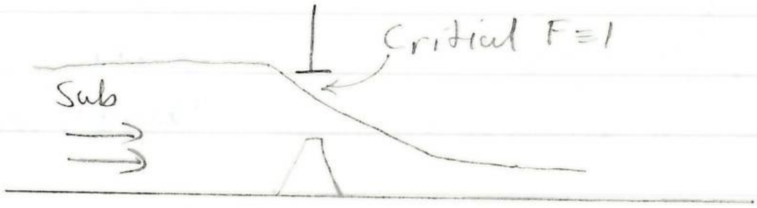


$\rightarrow k > f(h_1) - f(h_m)$

Transitcoin!

If the obstacle height k is increased further so $k > f(h_1) - f(h_m)$ then the upstream flow breaks up, deepens, flux decreases and makes the minimum adjustment to allow water to pass over obstacle i.e flow at the top of bump is critial i.e $F=1$ when k is a maximum.

Eg: a weir:



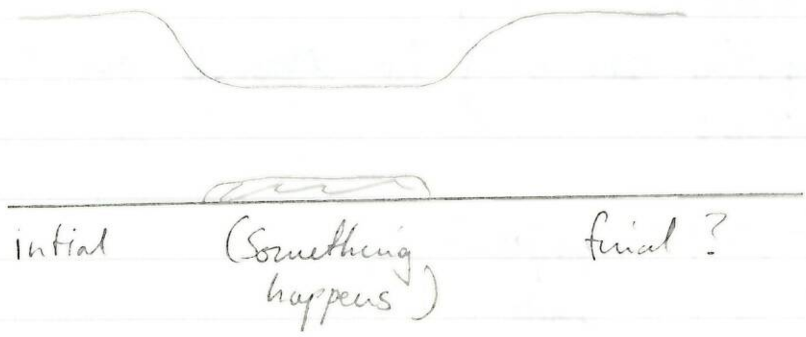
Notice if you know the depth at a weir you know the flux without having to measure speed:

$$\frac{Q^2}{g} = h_m^3$$

$$\Rightarrow Q = (g h_m^3)^{1/2} \qquad \text{so } u_m = \sqrt{g h_m} \text{ since } F = 1.$$

—/—

— / —

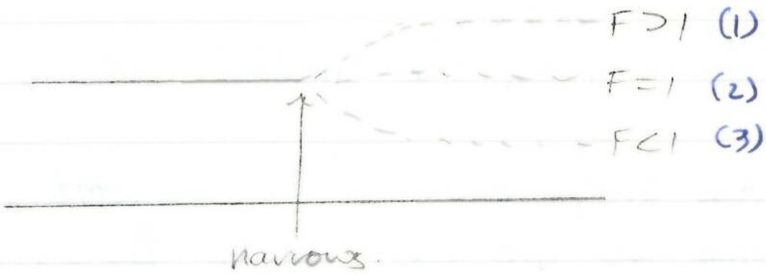


Remember one solution may be $h_2 = h$.

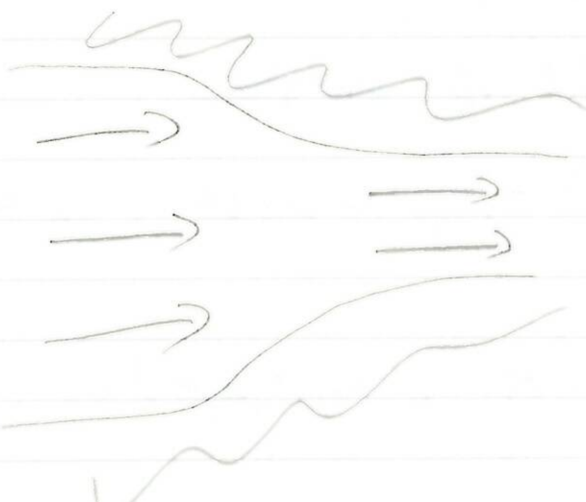
— / —

<u>Example</u>: Converging channel
Consider flow through a flat - bottom, horizontal channel of varying width, b.

Side on: (Elevation)



Top view: (Plan view).



(1) Deepens
(2) Stay the same
(3) Shallow

$\left.\right\}$ It depends on F.

If $F > 1$ super
$F < 1$ sub.

$$\frac{u}{\sqrt{gh}} = \frac{\frac{1}{2}\rho u^2}{\frac{1}{2}\rho g h} = \frac{KE}{PE}.$$

① Conservation of mass:

$$\rho h b u = \rho Q$$

so $Q = hbu$ is the cirtical volume flux.

② Provided the surface remains smooth, the surface is streamline so we can apply Bernoulli there:

$$p + \frac{1}{2}\rho u^2 + \rho z g = \text{const}.$$

z - height of surface.

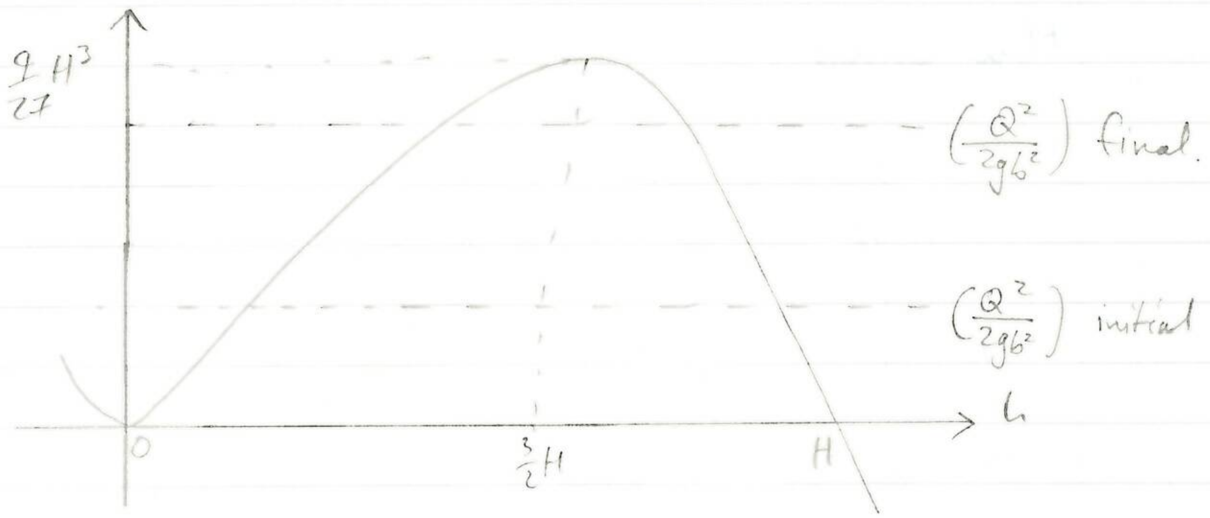
i.e $p_a + \frac{1}{2}\rho u^2 + \rho g h = \text{const}$, $p = p_a$ constant atmosphere pressure on surface.

i.e $\frac{u^2}{2g} + h = H$ (constant).

Eliminating u, $\frac{Q^2}{2gh^2b^2} + h = H$.

i.e $(H-h)h^2 = \dfrac{Q^2}{2gb^2}$

Write $(H-h)h^2 = f(h)$.



Information to draw the curve:

$$f(h) = h^2H - h^3$$
$$f'(h) = 2hH - 3h^2$$

so $f'(h) = 0$ if $h = 0$ or $h = \dfrac{3}{2}H$

At $h = 3/2 H$

$$f\left(\tfrac{2}{3}H\right) = \left(\frac{4}{9} - \frac{8}{27}\right)H^3$$

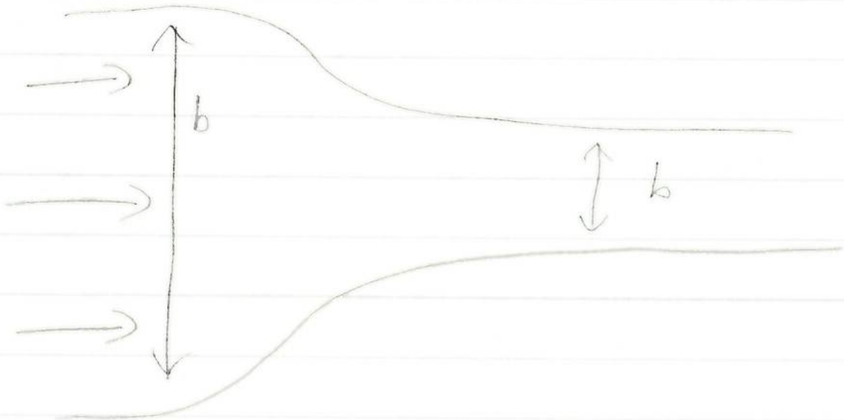
So at $h = 2/3 H$.

$$\frac{u^2}{2g} = H - h = \frac{1}{3}H.$$

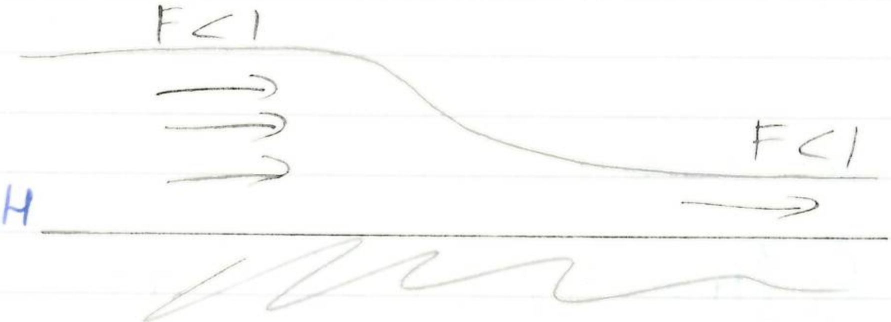
So: $\frac{u^2}{gh} = \frac{u^2}{g\left(\frac{2}{3}H\right)} = \frac{2/3\,H}{2/3\,H} = 1$

i.e $F=1$ when $h = \frac{3}{2}H$.

Plan view:



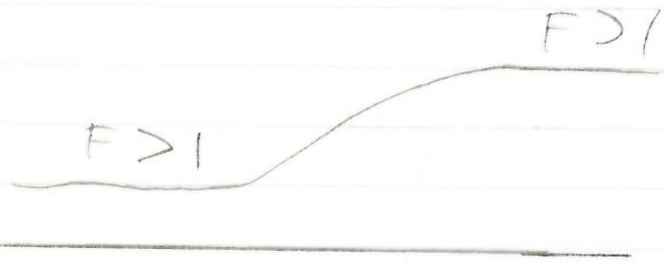
b decreasing $\Rightarrow \frac{Q^2}{2gb^2}$ increasing.



Sub
$F<1$
$h > \frac{2}{3}H$

Flows moves towards critical at a constriction.

Aside:

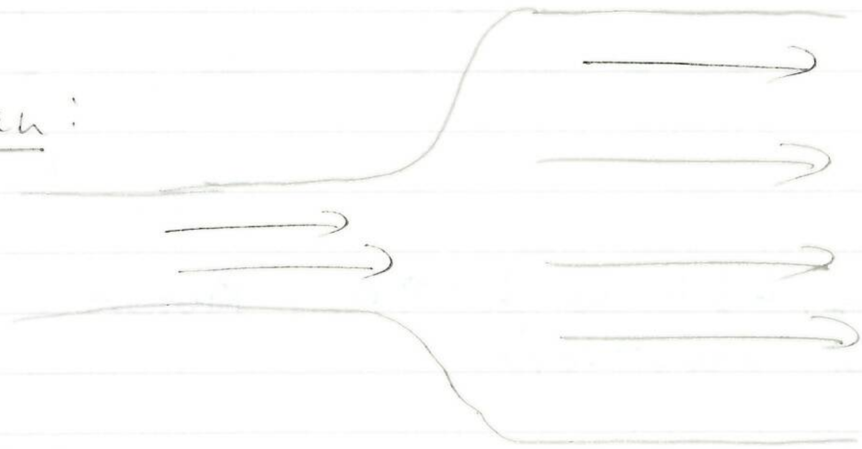


Super
$F>1$
$h < \frac{2}{3}H$

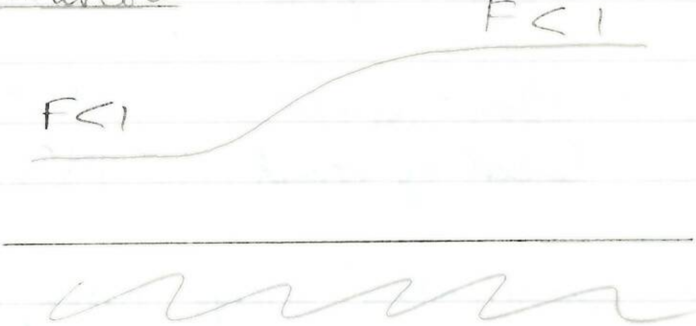
Example 5 : Expanding channel,

① SUB $F<1$, $h > \frac{3}{2}H$.

Plan:



Elevation:

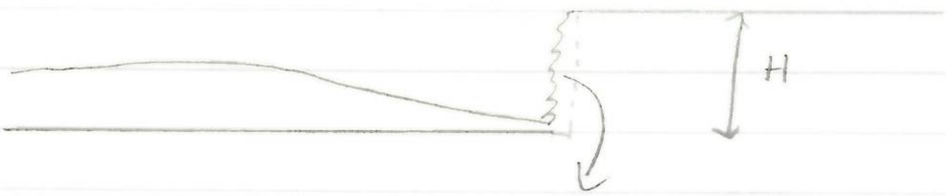


River flowing into reservoir:
Then $h \uparrow H$ as $b \to \infty$

i.e $\frac{Q^2}{2g^2} = 0$, $u = 0$ (stagnant)

River smoothly enters stagnant reservoir.

② : SUPER $F > 1$, $h < \frac{3}{2}h$ fast shallow.



Unsteady Non-smoth jump.

Here $h \downarrow 0$ as $b \to 0$. Rivier cannot smoothly join reservoir.

$$F = \frac{u}{\sqrt{gh}} = \frac{\text{Flow speed}}{\text{Wave speed}}$$, $F > 1$ supercritial

$$M = \frac{u}{a} = \frac{\text{Flow speed}}{\text{Speed of sound}}$$ $M > 1$ Supersonic
$M < 1$ Subsonic

Sonic boom: Spontanous jump from $M > 1$ to $M < 1$ (as this gives out energy - sound)

Only have hydraulic jump from $F > 1$ to $F < 1$ as this gives out energy.

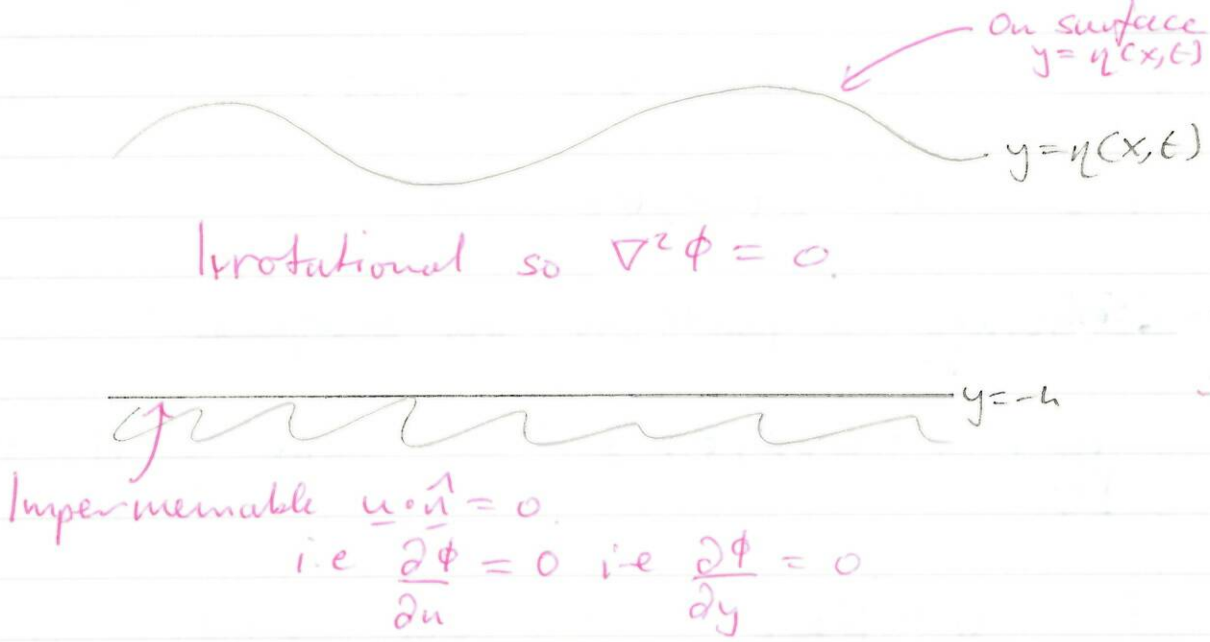
—/—

Water waves

- Free surface gravity waves.
- Water will take the flow to be 2D, irrotational, inviscid, incompressible.

Thus we have a streamfunction and a velocity potential and complex potential.

i.e $\exists \phi$ st $\underline{u} = \underline{\nabla} \phi$



1) We have $\exists \phi$ st $\underline{u} = \underline{\nabla} \phi$.

Let the unknown, free surface by $y = \eta(x,t)$.

Then in the fluid, $-h < y < \eta$, governing equation is $\nabla^2 \phi = 0$.

On lower boundary $v = 0$ on $y = -h$.

We need the b.c's on the surface (because η is unknown). The two b.c's are the KINEMATIC and DYNAMIC condition.

DYNAMICS (forces): $p = p_a$ on $y = \eta$

KINEMATICS: Particles on the surface remains on the surface.

Kinematics: i.e: on the surface ($y = \eta$)

$$y = \eta(x,t) \quad \forall x, t.$$

i.e $\quad y - \eta(x,t) = 0.$

Following a particle on surface:

$$\frac{D}{Dt}\left(y - \eta(x,t)\right) = 0 \quad \text{on } y = \eta \quad \forall x, t.$$

i.e $\quad v - \frac{D\eta}{Dt} = 0 \quad$ on $y = \eta$

i.e $\quad v = \frac{\partial \eta}{\partial t} + u \frac{\partial \eta}{\partial x} \quad$ on $y = \eta(x,t)$.

i.e $\quad \frac{\partial \phi}{\partial y} = \frac{\partial \eta}{\partial t} + \frac{\partial \phi}{\partial x}\frac{\partial \eta}{\partial x} \quad$ on $y = \eta(x,t)$

- kinematics b.c.

To deal with the dynamic conditions on pressure we would like to use Bernoulli. But the flow must be STEDY i.e $\partial/\partial t \equiv 0$ for the form of Bernoulli up to now. We need a new Bernoulli for unsteady flow.

Remember:

$$\frac{D\underline{u}}{Dt} = -\frac{1}{\rho}\underline{\nabla} p + \underline{F}$$

i.e $$\frac{\partial \underline{u}}{\partial t} + (\underline{u}\cdot\nabla)\underline{u} = -\frac{1}{\rho}\underline{\nabla} p - \underline{\nabla} V_e.$$

Note: $F = -\underline{\nabla} V_e$ (a conservative force)

i.e $$\frac{\partial \underline{u}}{\partial t} + \underline{\nabla}\left(\tfrac{1}{2}\underline{u}^2\right) + (\underline{\omega}\wedge\underline{u}) = -\frac{1}{\rho}\underline{\nabla} p - \underline{\nabla} V_e$$

[last time: steady, dotted with $\underline{u}$ to get rid of $\underline{\omega}\wedge\underline{u}$].

This time: Use the fact that irrotiational, $\underline{u} = \underline{\nabla}\phi$ and $\underline{\omega} = 0$.

Thus:

$$\frac{\partial}{\partial t}\underline{\nabla}\phi + \underline{\nabla}\left(\tfrac{1}{2}\underline{u}^2\right) = -\frac{1}{\rho}\nabla p - \underline{\nabla} V_e.$$

i.e $$\underline{\nabla}\left[\frac{\partial\phi}{\partial t} + \frac{1}{2}|\underline{\nabla}\phi|^2 + \frac{1}{\rho}p + V_e\right] = 0.$$

i.e $\rho\left(\frac{\partial\phi}{\partial t} + \frac{1}{2}|\nabla\phi|^2\right) + p + \rho V_e = G(t)$.

New Benoulli !

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Water waves:

Equation: $\nabla^2 \phi = 0$ $[\underline{u} = \underline{\nabla}\phi, \ \underline{\nabla}\cdot\underline{u} = 0]$.

Lower B.C.: $\dfrac{\partial \phi}{\partial y} = 0$, $y=-h$.

Upper B.C: Kinematics - $v = \dfrac{D\eta}{Dt}$ on $y=\eta$

i.e $\dfrac{\partial \phi}{\partial y} = \dfrac{\partial \eta}{\partial t} + \dfrac{\partial \phi}{\partial x}\dfrac{\partial \eta}{\partial x}$ on $y=\eta$.

Dynamic: $p = p_a$ on $y=\eta$.

Bernolli (time - dep, irrotational).

$$\rho \frac{\partial \phi}{\partial t} + \frac{1}{2}\rho |\underline{\nabla}\phi|^2 + p + \rho V_e = F(t).$$

The restoring force is gravity so $V_e = gy$

Thus: $\rho \dfrac{\partial \phi}{\partial t} + \dfrac{1}{2}\rho|\underline{\nabla}\phi|^2 + p + \rho g y = F(t)$.

$F(t)$ can be absorbed into ϕ.

Redefine $\hat{\phi} = \phi - \rho \int^t F(\tau)\, d\tau$.

then $\underline{\nabla} \hat{\phi} = \underline{\nabla} \phi = \underline{u}$.

and $\rho \dfrac{\partial \hat{\phi}}{\partial t} = \rho \dfrac{\partial \phi}{\partial t} - F(t)$.

Thus w.l.o.g we can take $F \equiv 0$, [Since if $F \not\equiv 0$ we can redefine ϕ as above].

Hence everywhere in the flow:

$$\rho \frac{\partial \phi}{\partial t} + \frac{1}{2}\rho |\underline{\nabla} \phi|^2 + \rho g y + p = 0.$$

UNSTEADY BERONOULLI

On surface, $y = \eta$ and $p = p_a$ (constant)

Thus: $\rho \dfrac{\partial \phi}{\partial t} + \dfrac{1}{2}\rho |\underline{\nabla} \phi|^2 + \rho g \eta = p_a$. (constant)

By the above argument, can absorb p_a (constant) into ϕ so we have the dynamic conditions:

$$\frac{\partial \phi}{\partial t} + \frac{1}{2} |\underline{\nabla} \phi|^2 + g\eta = 0 \quad \text{on } y = \eta.$$

Dynamic B.C.

Equation Laplace: $\nabla^2 \phi = 0$.

Lower B.C: $\dfrac{\partial \phi}{\partial y} = 0$ on $y = -h$.

Upper BC

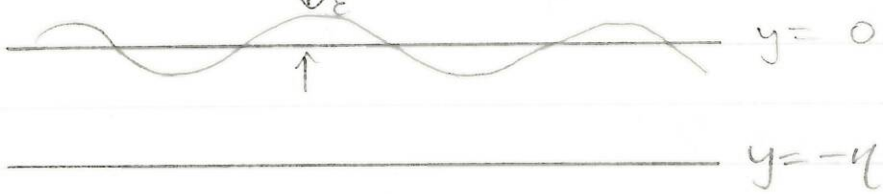
- Kinematic: $\dfrac{\partial \phi}{\partial y} = \dfrac{\partial \eta}{\partial t} + \dfrac{\partial \phi}{\partial x}\dfrac{\partial \eta}{\partial x}$. on $y = \eta$

- Dynamic: $\dfrac{\partial \phi}{\partial t} + \dfrac{1}{2}|\nabla \phi|^2 + g\eta = 0$ on $y = \eta$

$\eta(x,t)$ is unknown, Full, Non-linear, Surface water pattern.

To make progess we "linearise", i.e we consider waves of infitesimal amplitude, $0 < \varepsilon << 1$.

i.e we take $\eta(x,t)$ to be of order ε.



We expect velocities and so ϕ to be ε also.

Kinematics B.C: $\dfrac{\partial \phi}{\partial y} = \dfrac{\partial \eta}{\partial t} + \dfrac{\partial \phi}{\partial x}\dfrac{\partial \eta}{\partial x}$

$\varepsilon \qquad \varepsilon \cdot \varepsilon$

i.e $\varepsilon, \qquad \varepsilon : \varepsilon^2$

or $1, \qquad 1 : \varepsilon$

Thus, in limit $\varepsilon \to 0$ the fenial term disappear.

We have $\dfrac{\partial \phi}{\partial y} = \dfrac{\partial \eta}{\partial t}$ on $y = \eta$ (linear).
(with error of order ε)

Notice for any function $f(y)$.

$$f(\varepsilon) = f(0) + \varepsilon f'(0) + \frac{1}{2} \varepsilon f''(0) + \dots = f(0)$$
(with error of order ε).

This can move BC from $y = \eta$ (of order ε) to $y = 0$ with error of order ε. Thus we have:

$$\frac{\partial \phi}{\partial y} = \frac{\partial \eta}{\partial t} \quad \text{on} \quad y = 0.$$
(Now linear on known surface)

Dynamic BC:

$$\underbrace{\frac{\partial \phi}{\partial t}}_{\varepsilon} + \underbrace{\frac{1}{2} |\nabla \phi|^2}_{\varepsilon^2} + \underbrace{g\eta}_{\varepsilon} = 0 \quad \text{on} \quad y = \eta$$
$\downarrow$
$y = 0$.

Linearised BC:

$$\frac{\partial \phi}{\partial t} + g\eta = 0 \quad \text{on} \quad y = 0.$$

(Linear, on known surface)

—/—

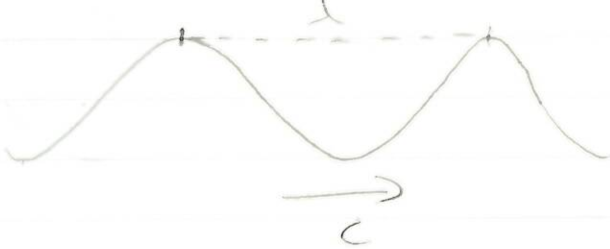
Summary: Linear water waves.

Equation : $\nabla^2 \phi = 0$ (already linear).

Lower BC: $\dfrac{\partial \phi}{\partial y} = 0$ on $y = -h$ (already linear, already on known surface).

Upper BC:
$$\left.\begin{aligned} \frac{\partial \phi}{\partial y} &= \frac{\partial \eta}{\partial t} \\ \frac{\partial \phi}{\partial t} + g\eta &= 0. \end{aligned}\right\} \text{ On } y = 0.$$

—/—



- Wavelength λ; distance between two successive creasts.
- Period τ; time between arrival at a given point of successive creasts.
- Speed c; at which the creast advance.

$$\left[c = \lambda / \tau \right]$$

—/—

Any (within reason) periodic function can be expressed a sum of sines and cosines. Thus it is sufficient to consider:

$$\eta = A\sin\left[\frac{2\pi}{\lambda}\left[x - ct\right]\right]$$

- waves with amplitude A, wavelength λ, speed c, to the right (as period $\tau = \lambda/c$).

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Water Waves

Summary:

1) Irrotational, Inviscid, Incompressible $\Rightarrow$ New Bernoulli $\Rightarrow$ Fully non-linear equation set.

2) Infinitesimal waves, linearised $\Rightarrow$ linear wave equation

Gov. Eqn : $\nabla^2 \phi = 0$.

Lower BC: $\dfrac{\partial \phi}{\partial y} = 0$ on $y = -h$.

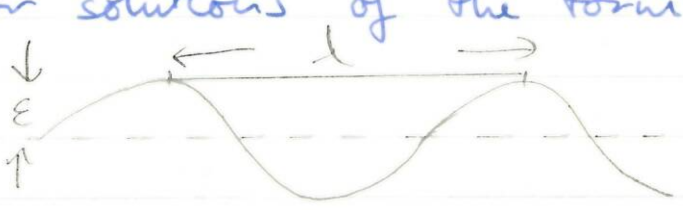
Upper BC:

- Kinematics : $\dfrac{\partial \phi}{\partial y} = \dfrac{\partial \eta}{\partial t}$ on $y = 0$.
- Dynamic : $\dfrac{\partial \phi}{\partial t} + g\eta = 0$ on $y = 0$

linear fixed domain $0 \geq y \geq -h$.

Linear, constant coefficient : all solution are sums or integrals of sinusoids.

Look for solutions of the form:



$$\eta(x,t) = \varepsilon \sin\left[\frac{2\pi}{\lambda}(x - ct)\right] = \varepsilon \sin\left[k(x - ct)\right]$$

$\ldots = \varepsilon \sin[kx - \omega t]$

where:

ε - Amplitude.

λ - Wavelength

c - Speed (phase speed).

Period : $\tau = \lambda / c$.

Wavenumber : $k = 2\pi/\lambda$, no of waves in distance 2π.

Frequency : $\omega = kc = 2\pi/\tau$, $c = \omega/k$.

—/—

We wish to find ϕ.

On the surface $\frac{\partial \phi}{\partial t} = -g\eta$

$$= -\varepsilon g \sin[kx - \omega t]$$

Thus ϕ behaves like $\frac{-\varepsilon g}{\omega} \cos\left[kx - \omega t\right]$ on $y = 0$.

—/—

$\nabla^2 \phi = 0$ ①

$\frac{\partial \phi}{\partial y} = 0$, $y = -h$. ②

$\frac{\partial \phi}{\partial y} = \frac{\partial \eta}{\partial t}$ on $y = 0$. ③

$\frac{\partial \phi}{\partial t} + g\eta = 0$ on $y = 0$ ④

Equivalently: $\frac{\partial \phi}{\partial y} = \frac{\partial \eta}{\partial t}$

$$= -\varepsilon \omega \cos\left[kx - \omega t\right] \quad \text{on } y=0.$$

Both of these say that (x,t) behaviour of $\phi(x,y,t)$ is like $\cos(kx - \omega t)$.

Look for a solution

$$\phi(x,y,t) = -\varepsilon \omega\, Y(y) \cos(kx - \omega t)$$

Then $\frac{\partial \phi}{\partial y} = -\varepsilon \omega\, Y'(y) \cos(kx - \omega t)$.

But on $y=0$, $\frac{\partial \phi}{\partial y} = \frac{\partial \eta}{\partial t} = -\varepsilon \omega \cos(kx - \omega t)$.

Thus require: $Y'(0) = 1$ satisfies (2).

Similarly for $\frac{\partial \phi}{\partial y} = 0$ on $y = -h$, for all x, t.

$$Y'(-h) = 0 \quad \text{satisfies (3)}$$

The governing eqn is

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0.$$

$$\frac{\partial^2 \varphi}{\partial x^2} = +\varepsilon \omega k^2 Y(y) \cos(kx - \omega t).$$

$$\frac{\partial^2 \varphi}{\partial y^2} = -\varepsilon \omega k^2 Y''(y) \cos(kx - \omega t).$$

Adding:

$$0 = Y'' - k^2 Y$$

Thus we have:

(α) $Y'' - k^2 Y = 0.$
(β) $Y'(0) = 1$
(γ) $Y'(-h) = 0$

(notice no x's, t's, ω's cos's, sin's)
(justifies form of ϕ assumed).

One form of Complementary Function is:

$$Y(y) = Ce^{ky} + De^{-ky}$$
or $Y'(y) = E\cosh ky + F\sinh ky.$

But best is:

$$Y(y) = A\cosh[k(y+h)] + B\sinh[k(y+h)].$$

(satisfies (α))

This gives.

$$Y'(y) = Ak\sinh[k(y+h)] + Bk\cosh[k(y+h)].$$

We require:

$$Y'(-h) = 0, \quad \text{so} \quad B = 0$$

It remains to require

$$Y'(0) = 1 \quad \text{so} \quad Ak\sinh kh = 0.$$

Thus $$Y(y) = \frac{\cosh[k(y+h)]}{k\sin(kh)}.$$

(Solves (α), (β), (γ))

Thus gives:

$$\phi(x,y,t) = -\frac{\varepsilon\omega\cosh[k(y+h)]}{k\sinh kh}\cos[kx-ct]$$

$$= -\frac{\varepsilon c\cosh[k(y+h)]}{\sinh kh}\cos[k(x-ct)]$$

We still have a condition to satisfy:-

$$\frac{\partial\phi}{\partial t} + g\eta = 0. \quad \text{on} \quad y=0.$$

Here:

$$-\varepsilon c\coth(kh)\left[\omega\sin(kx-\omega t)\right] + \varepsilon g\sin\left[kx-\omega t\right] = 0 \ \forall x,$$

Since true $\forall x, t$, divide by $\sin(kx - \omega t)$.

$$-\omega c \coth(kh) + g = 0.$$

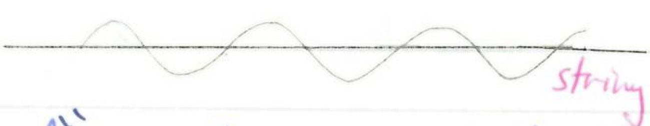
i.e $-\omega^2 \coth(kh) + gk = 0$. $\quad c = \omega/k$.

i.e $\omega^2 = gk \tanh(kh)$

i.e k and ω are <u>NOT</u> independant.

Thus
$$c^2 = \frac{g}{k}\tanh(kh)$$
$$= \frac{g\lambda}{2\pi}\tanh\left[\frac{2\pi h}{\lambda}\right]$$

So waves of different wavelength travel at different speed i.e wave disperse.



$u_{tt} = c^2 u_{xx}$

"the speed" of the waves NOT DISPERSIVE.

Nothing to do with λ $\quad c^2 = \frac{T}{\rho}$.

Sound waves ALL travel at "speed of sound" DISPERSIVE.

Electromagnetic radition : - speed "c" - unquie (for given medium) NON - DISPERSIVE.

$$\frac{c^2}{gh} = \frac{\tanh(kh)}{kh} = \frac{\tanh\left(\frac{2\pi h}{\lambda}\right)}{(2\pi h/\lambda)}$$

Information about the curve in order to draw it.

$$\theta = \lambda / 2\pi h .$$

$$\frac{c^2}{gh} = \theta \tan\left(\frac{1}{\theta}\right)$$

for $\theta >> 1$, $\frac{c^2}{2g} \sim \theta \cdot \frac{1}{\theta} = 1.$

for $\theta << 1$, $\frac{c^2}{gh} \sim \theta.$

—/—

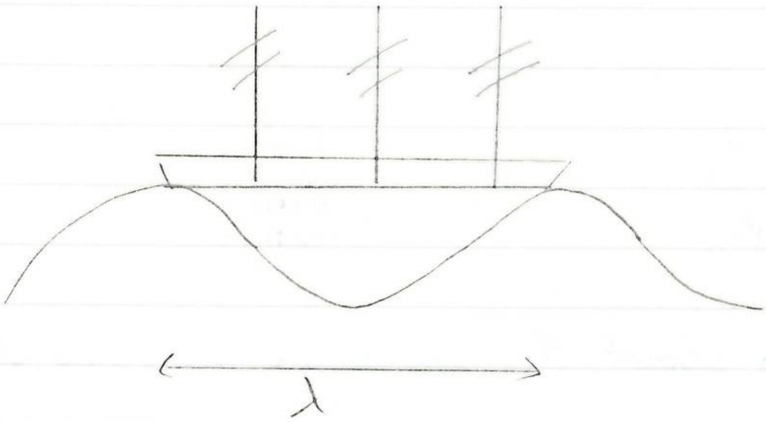
Long waves travel faster with speed $c = \sqrt{gh}$

Long waves on shallow water are non-dispersive : all have speed $\sqrt{gh}$.

Short waves on deep water are disperive on deep water

$\frac{\lambda}{2\pi h} << 1$, $\frac{c^2}{gh} \to \frac{\lambda}{2\pi h}$ i.e $c \to \left(\frac{gh}{2\pi}\right)^{\frac{1}{2}} = (2\pi)^{-\frac{1}{2}} \sqrt{g\lambda}$.

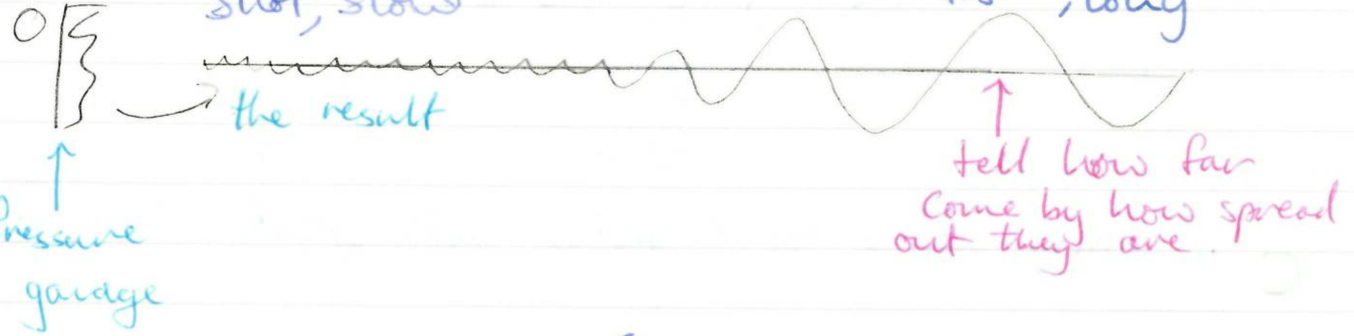
—/—



Ship speed water portportional to square root of length.

Notice in flow with $u > c$ i.e $u > \sqrt{gh}$ all waves swept downstream i.e supercritical. ($F = u/\sqrt{gh} = u/c$)
(c.f Mach no. $M = u/a$ where a speed sound)

—/—



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Particle Paths in a water wave

$$\frac{dx}{dt} = u(x, y, t).$$

But the amplitude of the motion is of order ε i.e particles only move an amount ε.

So write:

$$x = x_0 + \varepsilon X$$
$$y = y_0 + \varepsilon Y.$$

Then $\dfrac{dx}{dt} = \varepsilon \dfrac{dX}{dt}$

$$= u(x, y, t)$$

$$= \frac{\partial \phi}{\partial x}(x_0 + \varepsilon X, y_0 + \varepsilon Y, t).$$

$$= \underset{\varepsilon}{\frac{\partial \phi}{\partial x}(x_0, y_0, t)} + \underset{\varepsilon^2}{\cancel{\varepsilon X \frac{\partial^2 \phi}{\partial x^2}}}\,{}_{\varepsilon} + \underset{\varepsilon^2}{\cancel{\varepsilon Y \frac{\partial^2 \phi}{\partial x \partial y}}}\,{}_{\varepsilon}$$

Thus to order ε^2

$$\varepsilon \frac{dX}{dt} = \frac{\partial \phi}{\partial x}(x_0, y_0, t)$$
$$\varepsilon \frac{dY}{dt} = \frac{\partial \phi}{\partial y}(x_0, y_0, t).$$

$$\phi = -\varepsilon c \frac{\cosh[k(y+h)]}{\sinh kh} \cos[kx - \omega t]$$

So:

$$\left.\frac{\partial \phi}{\partial x}\right|_{x_0, y_0} = +\varepsilon k c \frac{\cosh[k(y_0+h)]}{\sinh kh} \sin(kx_0 - \omega t)$$

$$\left.\frac{\partial \phi}{\partial y}\right|_{x_0, y_0} = -\varepsilon k c \frac{\sinh[k(y_0+h)]}{\sinh kh} \cos(kx_0 + \omega t)$$

indep of x, y i.e X, Y.

Thus $\frac{dX}{dt} = \omega \frac{\cosh[k(y_0+h)]}{\sinh kh} \sin[kx_0 - \omega t]$

So: $X = \frac{\cosh[k(y_0+h)]}{\sinh kh} \cos[kx_0 - \omega t]$

(absorbs constant into x_0)

and $Y = \frac{\sinh[k(y_0+h)]}{\sinh kh} \sin(kx_0 - \omega t)$

$$\alpha = \frac{\cosh[k(y_0+h)]}{\sinh kh}, \quad \beta = \frac{\sinh[k(y_0+h)]}{\sinh kh}.$$

Hence: $\frac{X}{\alpha} = \cos(kx_0 - \omega t)$, $\frac{Y}{\beta} = \sin(k_0 - \omega t)$.

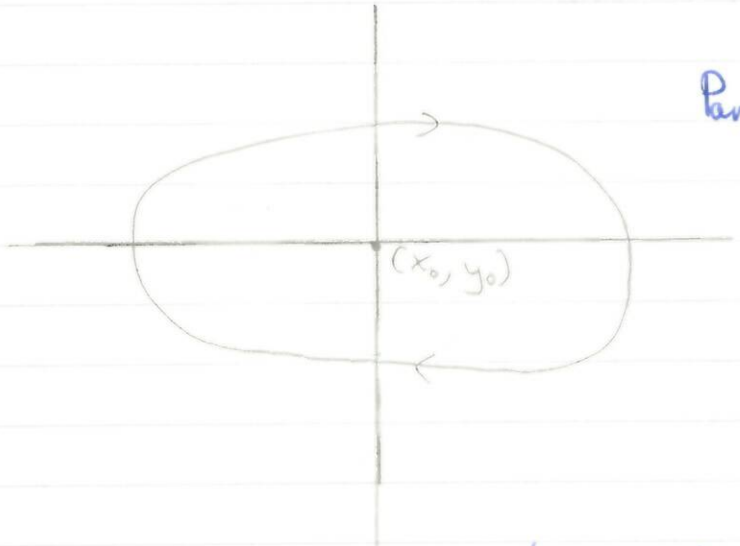
So $\left(\frac{X}{\alpha}\right)^2 + \left(\frac{Y}{\beta}\right)^2 = 1$, $\alpha > \beta$ so major axis is horizontal.

Ellipises with vertical semi-axes.

$$\beta = \frac{\sinh[k(y_0+h)]}{\sinh(kh)}.$$

Also horizontal - axes.

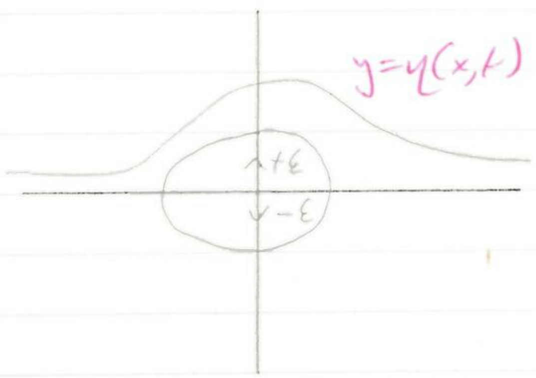
$$\alpha = \frac{\cosh[k(y_0+h)]}{\sinh(kh)}.$$



Particle goes around clockwise

When Y is largest, $\frac{\partial \phi}{\partial x}$ is the largest i.e u is largest.

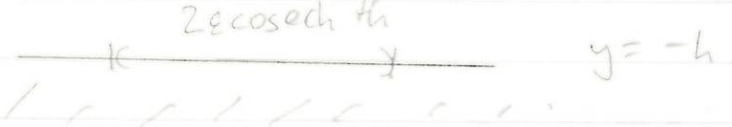
At top, $y_0 = 0$, $\alpha = \coth kh$, $\beta = 1$ so $-1 \le Y \le 1$.



$y = y_0 + \varepsilon Y$.

- exactly the amplitude of the wave $\eta = Y$ here
- particle move toward at the creast + backward at the trough.

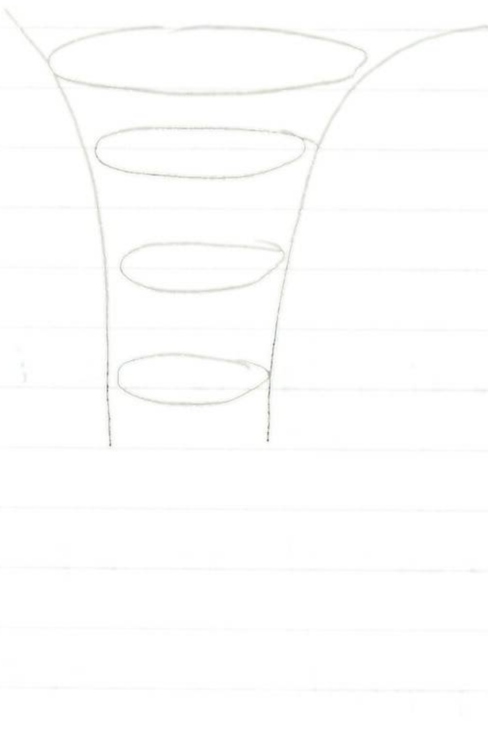
At bottom $y_0 = -h$, $\alpha = -\operatorname{cosech} kh$, $\beta = 0$, $Y \equiv 0$.



$h \to \infty$, Use $\phi = Ae^{-ky}$

Much easier

$\frac{\lambda}{2\pi h}$



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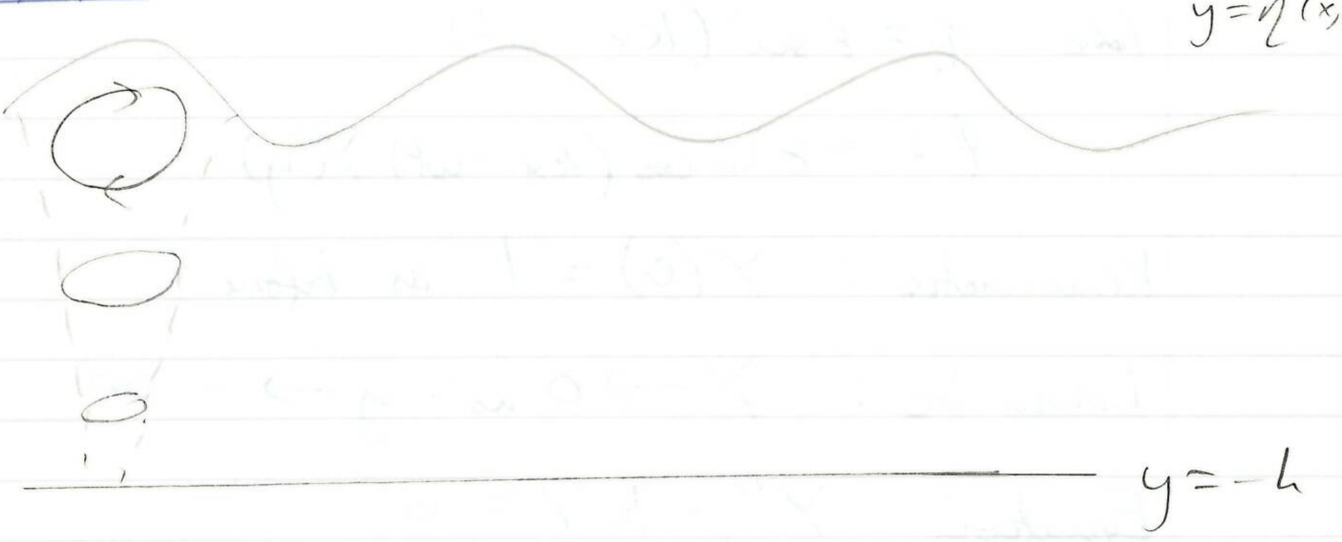
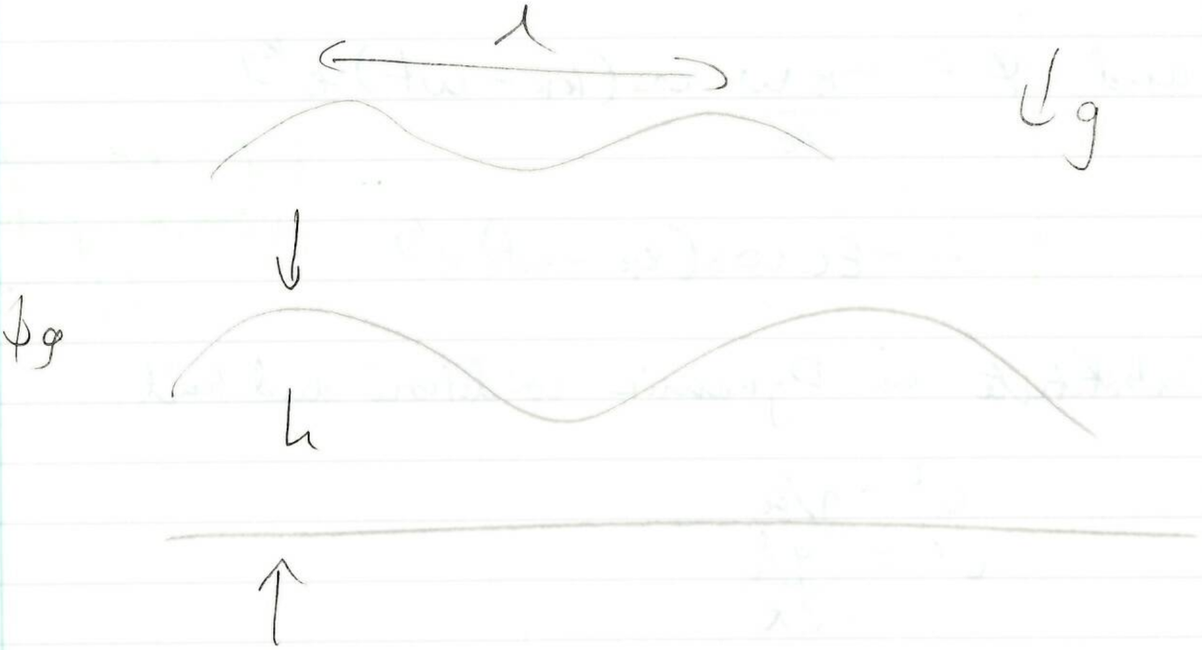


$c = \frac{1}{\sqrt{2\pi}}\sqrt{g\lambda}$ short waves. deep water $\frac{\lambda}{2\pi h} \ll 1$

$c = 1\sqrt{gh}$ long waves shallow waves $\frac{\lambda}{2\pi h} \gg 1$



Infinitely deep water $h \to \infty$
$\Rightarrow$ all waves short.

Take $\eta = \varepsilon \sin(kx - \omega t)$

$\phi = -\varepsilon \omega \cos(kx - \omega t)\, Y(y)$.

Kinematic : $Y'(0) = 1$ as before.

Lower bc : $Y \to 0$ as $y \to -\infty$

Equation $Y'' - kY = 0$

CF $Y = Ce^{+ky} + De^{-ky}$.

Bdd as $y \to -\infty \Rightarrow D = 0$.

bounded

But $Y'(0) = 1$

so $C = 1/k$

and $\boxed{\phi = -\dfrac{\varepsilon\omega}{k}\cos(kx - \omega t)\, e^{ky} = -\varepsilon c \cos(kx - \omega t)\, e^{ky}}$

Infinite depth velocity potential $(h \to \infty)$

Substitute in Dynamic condition and find

$c^2 = g/k$

$c^2 = \dfrac{g\lambda}{2\pi}$.

$\boxed{c = \dfrac{1}{\sqrt{2\pi}}\sqrt{g\lambda}}$ as expected.

$\dfrac{\lambda}{2\pi h} \to 0$.

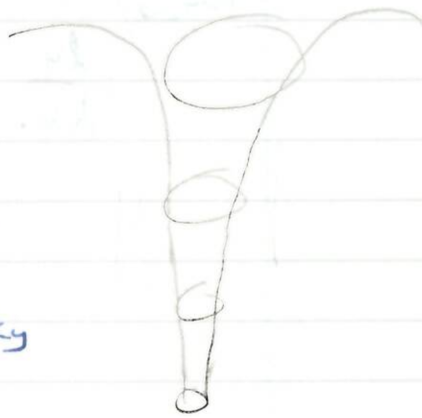
Kinematics $\frac{\partial \phi}{\partial y} = \frac{\partial \eta}{\partial t}$

Dynamic

$$\frac{\partial \phi}{\partial t} + g\eta = 0.$$

Particle path are Circles

Radius e^{ky_0}



Reflected Waves:

$\eta_1(x,t) = \varepsilon \sin(kx - \omega t)$

$\phi_1(x,y,t) = -\varepsilon C \cos(kx - \omega t) e^{ky}$

$\eta_2(x,t) = \varepsilon \sin(kx - \omega t)$

$\phi_2(x,t) = +\varepsilon C \cos(kx - \omega t) e^{ky}$

Add them:

$$\eta = \eta_1 + \eta_2 = \varepsilon\left[\sin(kx - \omega t) + \sin(kx + \omega t)\right]$$

and

$$\phi = \phi_1 + \phi_2 = \varepsilon C e^{ky}\left[\cos(kx - \omega t) - \cos(kx - \omega t)\right]$$

$= -2\varepsilon c e^{ky} \sin kx \sin \omega t$

Particle path

$$\frac{dx}{dt} = u = \frac{\partial \phi}{\partial x}.$$

$$= -2\varepsilon c k \cos kx \, e^{ky} \sin \omega t.$$

$$\frac{dy}{dt} = v = \frac{\partial \phi}{\partial y}$$

$$= -2\varepsilon c k e^{ky} \sin kx \sin \omega t$$

Now:

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$$

$$= \frac{\sin kx}{\cos kx}$$

$$= \tan kx.$$

$$\left[\text{so} \quad y = \frac{1}{k} \ln(\sec kx) \right]$$

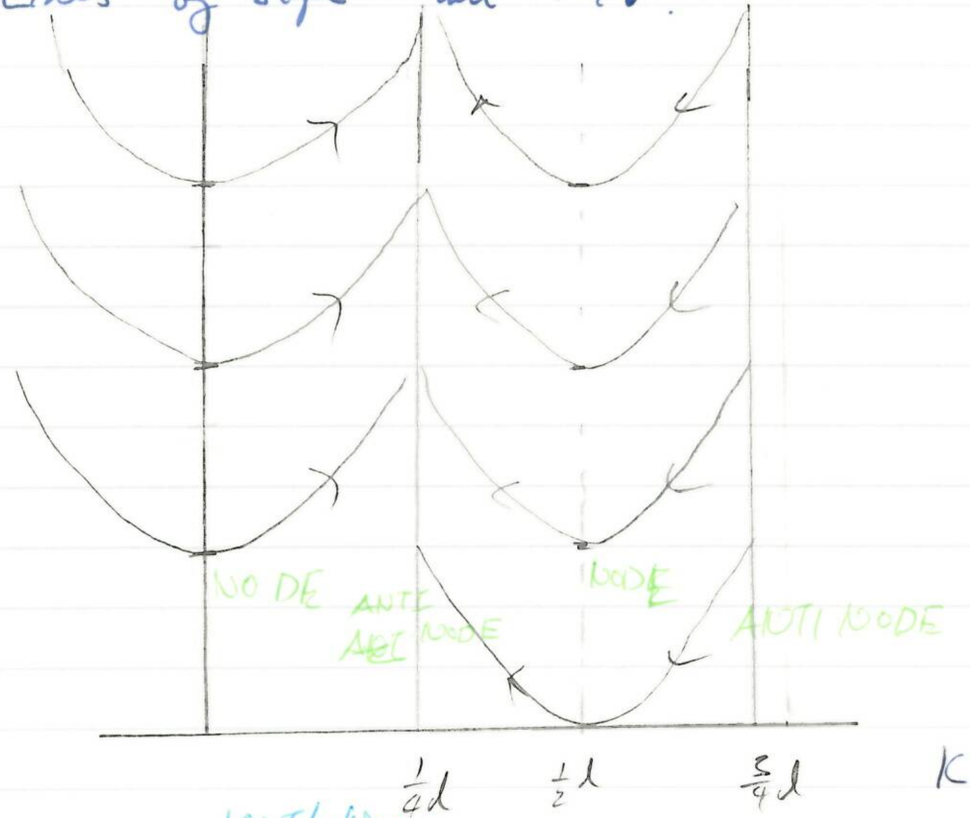
$x = x_0 + \varepsilon X$

$y = y_0 + \varepsilon Y.$

To lowest order

$$\frac{dy}{dx} = \tan kx_0 + O(\varepsilon)$$

Lines of slope tan kx_0.



$kx_0 = \pi/2$

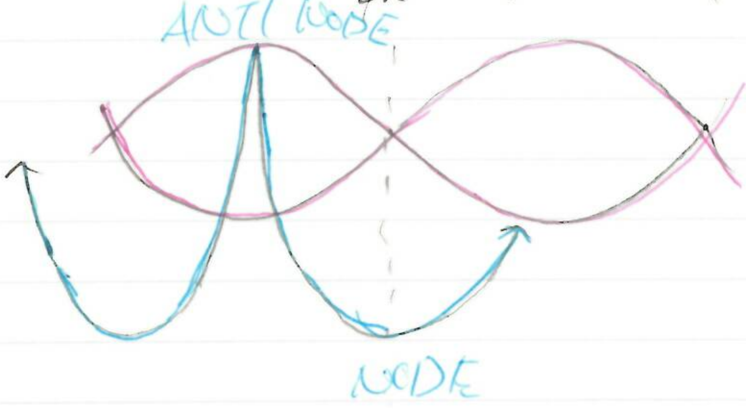
$x_0 = \frac{\pi}{2k}$

—/—

$\frac{2\pi}{\lambda} x_0 = \frac{\pi}{2}$

$\Rightarrow x_0 = \frac{1}{4}\lambda$

—/—



2.5 Hydraulic jump ($q \equiv u$ here)

Consider the jump as shown in a channel of width d and horizontal bed. The suffixes 1 and 2 refer to conditions before and after the jump, and we consider the fluid bounded by A_1B_1 and A_2B_2. In time δt, the fluid has moved to the region bounded by $A_1'B_1'$ and $A_2'B_2'$. Then at the two locations we have the following quantities.

Height	h_1	h_2
Mean velocity	q_1	q_2
Pressure	$\rho g(h_1 - z)$	$\rho g(h_2 - z)$
Thickness'	$q_1\delta t$	$q_2\delta t$
Mass	$m_1 = \rho d h_1 q_1 \delta t$	$m_2 = \rho d h_2 q_2 \delta t$

Conservation of mass shows that $m_1 = m_2$ and the flow rate Q is the same at 1 and 2

Flow rate	$dh_1q_1 = Q$	$dh_2q_2 = Q$
Momentum	m_1q_1	m_2q_2
Force in flow direction	$F_1 = \int_0^{h_1} \rho g d(h_1 - z)dz = \frac{1}{2}\rho g d h_1^2$	$F_2 = -\int_0^{h_2} \rho g d(h_2 - z)dz = -\frac{1}{2}\rho g d h_2^2$

Force equals rate of change of momentum gives $F_1 - F_2 = (m_2q_2 - m_1q_1)/\delta t$ or $\frac{1}{2}\rho g d(h_1^2 - h_2^2) = \rho Q(q_2 - q_1) = \frac{\rho Q^2}{d}(\frac{1}{h_2} - \frac{1}{h_1})$. Hence either $h_2 - h_1 = 0$, in which case the flow is continuous and there is no jump, or

$$h_1h_2(h_1 + h_2) = \frac{2Q^2}{gd^2}.$$

For given upstream conditions this equation determines h_2 and hence $q_2 = Q/dh_2$.

Kinetic energy	$\frac{1}{2}m_1q_1^2$	$\frac{1}{2}m_2q_2^2$
Work done by force	$F_1q_1\delta t$	$F_2q_2\delta t$

If $D\delta t$ is the amount of knetic energy lost in time δt, conservation of energy gives

$$(F_1q_1 - F_2q_2)\delta t = \frac{1}{2}m_2q_2^2 - \frac{1}{2}m_1q_1^2 + D\delta t,$$

from which it follows that

$$D = \frac{1}{2}\rho g d(h_1^2 q_1 - h_2^2 q_2) + \frac{1}{2}\rho Q(q_1^2 - q_2^2) = \frac{1}{2}\rho g Q(h_1 - h_2) + \frac{\rho Q^3}{2d^2}(\frac{1}{h_1^2} - \frac{1}{h_2^2})$$

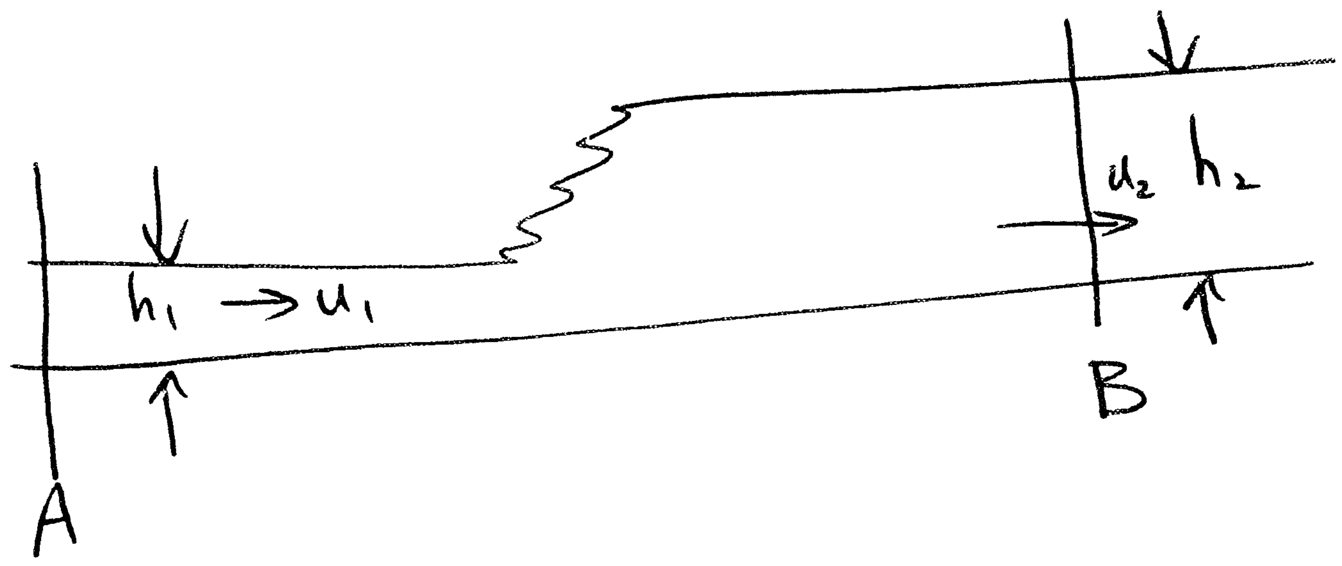
or

$$D = \frac{1}{2}\rho g Q\frac{(h_2 - h_1)^3}{2h_1h_2}.$$

Since $D \geq 0$, $h_2 \geq h_1$. In a hydraulic jump, the level of the water rises and the speed falls. For the flow upstream,

$$\frac{q_1^2}{gh_1} - 1 = \frac{Q^2}{gd^2h_1^3} - 1 = \frac{h_1h_2(h_1 + h_2)}{2h_1^2} - 1 = \frac{(h_2 - h_1)(h_2 + 2h_1)}{2h_1^2},$$

Energy



From mass & momentum conservation we have shown that

$$h_1 h_2 (h_1 + h_2) = \frac{2Q^2}{g d^2} \quad (1)$$

Work done at A = force × distance, and distance = $u_1 \delta t$ at all heights

Force = pressure, integrated over area = $\rho g (h_1 - z)$ integrated over area (taking $p_a = 0$).

Hence force = $\frac{1}{2} \rho g h_1^2 d$

Thus work done at A = $\frac{1}{2} \rho g h_1^2 u_1 d \delta t = \frac{1}{2} \rho g Q \delta t . h_1$

Similarly, work done at B = $- \frac{1}{2} \rho g Q \delta t . h_2$ (since displacement in opposite dirn. to force).

Thus net work done on fluid = $\frac{1}{2} \rho g Q \delta t (h_1 - h_2)$.

KE in at A = $\frac{1}{2} \rho u_1^2$ / unit volume.

∴ Total KE in at A = $\frac{1}{2} \rho u_1^2 Q \delta t$.

PE = $\rho g z$ / unit volume

∴ PE in at A = $\frac{1}{2} \rho g h_1^2 d u_1 \delta t = \frac{1}{2} \rho g h_1 Q \delta t$ (integrating from 0 to h_1)

∴ Total energy in at A = $\frac{1}{2} \rho Q \delta t \; (u_1^2 + g h_1)$

Similarly, total energy out at B = $\frac{1}{2} \rho Q \delta t \; (u_2^2 + g h_2)$

Energy lost = Work done + Energy In − Energy Out

$$= (\rho Q \delta t) \left[g (h_1 - h_2) + \tfrac{1}{2} (u_1^2 - u_2^2) \right]$$

But $u_1^2 = Q^2 / h_1^2 d^2 = \frac{g h_2}{2 h_1} (h_1 + h_2)$ by (1)

and $u_2^2 = \frac{g h_1}{2 h_2} (h_1 + h_2)$

Thus lost energy = $\frac{\rho g Q \delta t}{4 h_1 h_2} \left[(h_1 + h_2) h_2^2 - (h_1 + h_2) h_1^2 + 4 h_1 h_2 (h_1 - h_2) \right]$

$$= \frac{\rho g Q \delta t}{4 h_1 h_2} (h_2 - h_1)^3$$

For energy to be lost, $h_2 > h_1$ i.e. an upward jump.