2301 Fluid Mechanics Notes

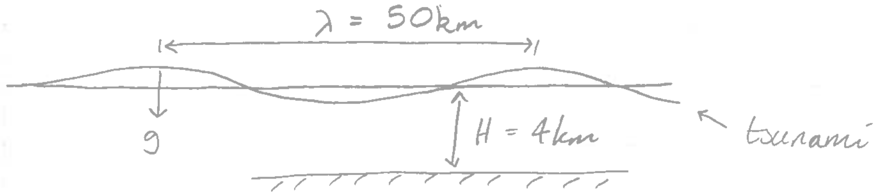
Based on the 2016 autumn lectures by Prof E R Johnson

MATH 2301

Paddison textbook is closest to course.

Problem Class 4pm (Thurs)
(Lecture this week instead)
* HW due at start of problem class
* Office hour: Mon 1pm
Ted Johnson, 805

03.10.16 Fluid Dynamics



$$[g] = LT^{-2} \qquad = 10ms^{-2}$$

$$[C] = LT^{-1}$$

$$F(\lambda/H) \qquad\qquad C = \sqrt{gH}\, F(\lambda/H)$$

Limit cases

1). $\lambda/H >> 1$ $\qquad F(\infty)$

Long waves on shallow water

Important length = H

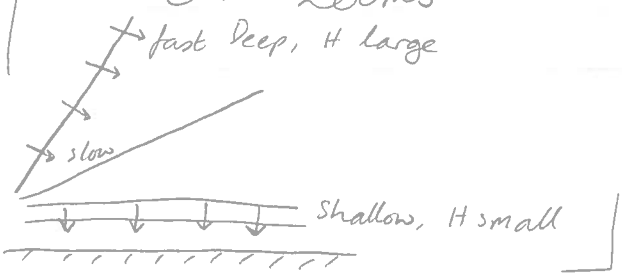
Thus C is proportional to $\sqrt{gH}$

$g = 10\,ms^{-2}$

$H = 4000m$

$gH = 40{,}000$

$C \approx 200ms^{-1} \simeq 450$ mph

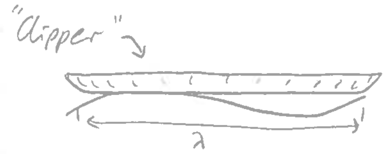


2). Short waves on deep water

$H = 4km, \; \lambda = 100m$

Important length = λ

C is proportional to $\sqrt{g\lambda}$



if ship goes faster it struggles to climb waves, so boat needs to be longer.

$C = \sqrt{gH}\, F(\lambda/H)$

$s = \lambda/H$, $s \to \infty$, $F \to$ constant (1)

$\lambda/H \to 0$, $s \to 0$, $F(s) \to \beta\sqrt{s} = \beta\sqrt{\frac{\lambda}{H}}$

$$C = \sqrt{gH}\, F(s) \qquad (\beta = 2\pi)$$
$$\longrightarrow \sqrt{gH}\,\beta\sqrt{\frac{\lambda}{H}} = \beta\sqrt{g\lambda}$$

Chapter 1

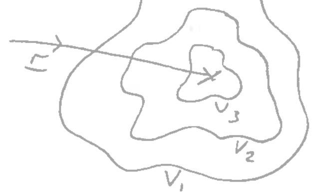
1.1 Assumptions

1.1.1 Continuum model

We suppose that our fluid (liquid, gas) is a CONTINUUM, in that we can take arbitrarily small volumes of it and its properties do not change, i.e. $\lim_{\delta V \to 0}$ makes sense.

If this is so the e.g. we can define the density at a point by

$$\rho(\underline{r}, t) = \lim_{\delta V \to 0} \left(\frac{\delta M}{\delta V} \right)$$



where δM is mass in the volume δV.

The mean free path for air at room temperature is 68nm

For a liquid it is about 0.3nm.

– very well met approximation even at biological (e.g. cell) scales

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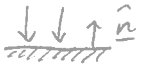
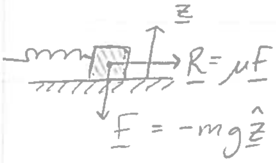
1.1.2 Inviscid

- not viscous i.e. cannot support a SHEAR STRESS



← a normal stress, the pressure-force per unit area NORMAL to a surface.

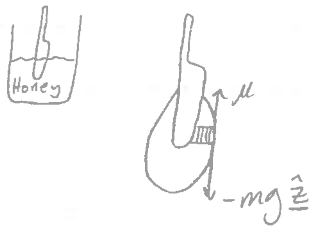


The shear stress is the force per unit area TANGENT to a surface e.g. friction on a block.

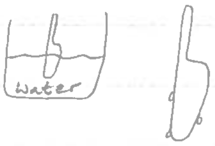
$\underline{R}$ = a shear stress

$\underline{F}$ = a normal stress

$\mu = 0$ corresponds to inviscid or frictionless



Honey is very viscous.
Held up by shear stress exerted by neighbouring fluid elements (particles)



Water is almost inviscid and so cannot hold up fluid.

$$Re = \frac{UL}{(\mu/\rho)}$$, for water $Re \gg 1$ (← Real Fluids)

1.1.3 Incompressible

- not compressible
- sound, compressional waves in the air.
- speed of sound is a fluid property.

air: $a = 340\,ms^{-1}$ (750mph)

water: $a = 1500\,ms^{-1}$

Thus compressibility is negligible for speeds, U, small compared to the speed of sound.

i.e. provided $M = \frac{U}{a} << 1$ (the Mach number)

⇒ subsonic motion.

Our theory has $M \equiv 0$. The Mach number assesses the error.

1.2 Describing fluid motion

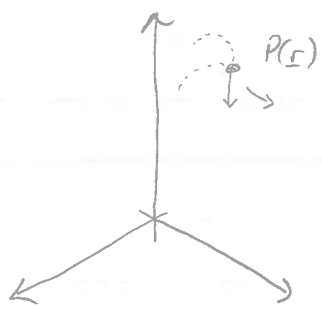
Two methods: Lagrangian or Eulerian

1.2.1 Lagrangian

Label every element by its position $\underline{r}_0$ at $t=0$ and follow it. Thus at later time this particle is at $\underline{r}(\underline{r}_0, t)$.

- Governing equations look simple.
- Spatial derivatives beome impossibly complicated.

1.2.2 Eulerian



choose some fixed axes.

Define the speed $\underline{u}(\underline{r}, t)$
as the speed of the particle that happens to be at $\underline{r}$ at time t.
(not following a particle, P is FIXED)

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Lagrange:
Look at the movement of individual particles.
When looking at a cluster of particles at a point in time, we have to trace them back to work out which ones they are - convoluted.

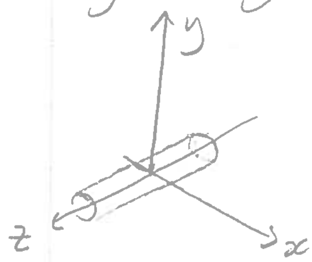
Euler:
Fix axes, don't look at individual particles but stay at a fixed point in space and look at what goes past.

Steady Flow

$$\frac{\partial \underline{u}}{\partial t} = 0$$

$\underline{u}$ does not chang in time - eg. 'a steady wind of 5mph from N'.
$\underline{u}$ can still vary with position, i.e. $\underline{u} = \underline{u}(\underline{r})$

Two-Dimensional Flow (2D flow)
e.g. a cylinder with axis O_z



In this case the flow is the same in each plane. z = constant.

i.e. $\frac{\partial \underline{u}}{\partial z} = 0$ so z is an ignorable co-ordinate.

i.e. $\underline{u} = \underline{u}(x, y, t)$.

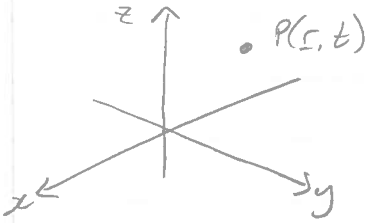
We will write the components of $\underline{u}$
as $\underline{u} = \begin{pmatrix} u \\ v \\ w \end{pmatrix} = u\hat{\underline{x}} + v\hat{\underline{y}} + w\hat{\underline{z}}$ [Note $u \neq |\underline{u}|$]

We shall also require $w = 0$.
i.e. there is no flows in the direction of the ignorable co-ordinate.

1.3 Visualising Fluid Flow

1.3.1 Particle Path

The path traced out by a given fluid particle over a given time interval.



At any point $P(\underline{r}, t)$

$$\frac{d\underline{r}}{dt} = \underline{u}(\underline{r}, t)$$

i.e. a first order ODE for $\underline{r}(t)$.

We solve this subject to the initial condition

$$\underline{r}(0) = \underline{r}_0$$

[notice this is precisely the transformation from Eulerian to Lagrangian.]

Example 1.2

Consider the flow

$\underline{u} = \hat{x} - 2te^{-t^2}\hat{y}$ (2D, unsteady, same at all x,y)

Find the path of the fluid particle that left $(1,1)$ at $t=0$.

We have $\dfrac{d\underline{r}}{dt} = \underline{u}(\underline{r}, t)$

In components $\dfrac{dx}{dt} = u$, $\dfrac{dy}{dt} = v$, $(x,y) = (1,1)$ at $t=0$.

Here $u = 1$, $y = -2te^{-t^2}$

Integrating:

$$x = t + x_0 \quad , \quad y = e^{-t^2} + y_0$$

$(x,y) = (1,1)$ at $t=0 \Rightarrow x_0 = 1$, $y_0 = 0$

so $x = t+1$, $y = e^{-t^2}$, $t > 0$

– parametric representation of the path, parameterised by time, t.

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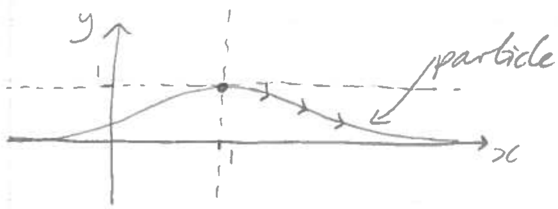
Here in fact, we can eliminate t.

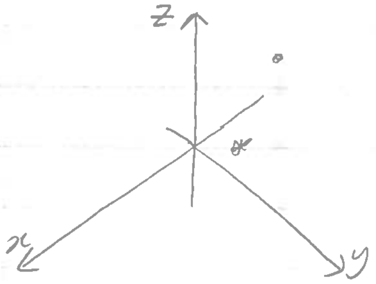
$$y = e^{-(x-1)^2}$$



1.3.2 Streakline

The locus at time t of all particles that have passed through a given point in a given time interval.

i.e. releasing dye from a fixed point.



$$\frac{d\underline{r}}{dt} = \underline{u}(\underline{r}, t)$$

subject to $\underline{r} = r_0(\tau)$, where $\tau < t$.
Solutions depend on $\underline{r}$, t and τ, plot for all $\tau < t$.

Example 1.4

Consider the same flow

$$\underline{u} = \hat{\underline{x}} - 2te^{-t^2}\hat{\underline{y}}$$

Find the streakline at $t = 0$ for particles released from $(1,1)$ at times $\tau < 0$.

As before we have

$$x = t + x_0$$
$$y = e^{-t} + y_0$$

But $(x,y) = (1,1)$ when $t = \tau$.

Thus $\begin{cases} 1 = \tau + x_0 & \text{i.e.} \quad x_0 = 1 - \tau \\ 1 = e^{-\tau^2} + y_0 & \text{i.e.} \quad y_0 = 1 - e^{-\tau^2} \end{cases}$

Thus $\begin{cases} x(t,\tau) = t + 1 - \tau \\ y(t,\tau) = e^{-t^2} + 1 - e^{-\tau^2} \end{cases}$

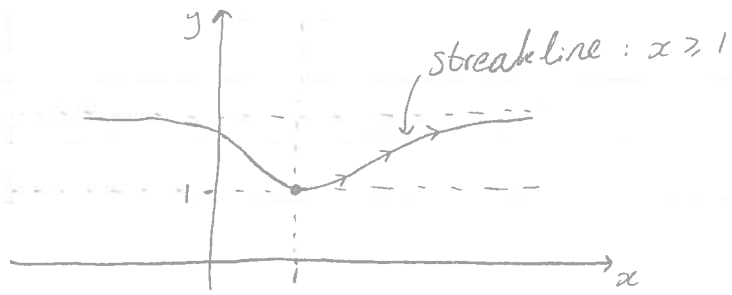
This is the streakline at time t of all particles released from $(1,1)$ at $\tau < t$.

At $t = 0$, $\begin{cases} x(\tau) = 1 - \tau \\ y(\tau) = 2 - e^{-\tau^2} \end{cases}$

The line is parameterised by τ, the release time.

Eliminating τ,

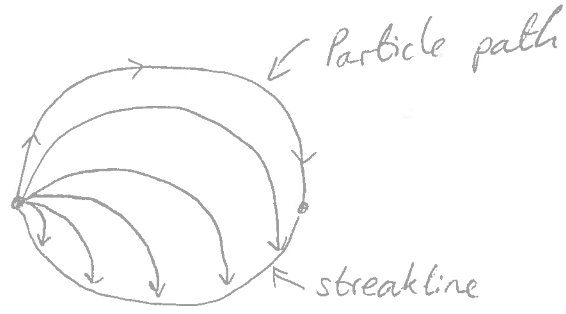
$$y = 2 - e^{-(1-x)^2}$$



Question

1). What does the smoke trail (streakline) look like at noon?

2). What is the particle path followed by the element of fluid emitted by the chimney at 6am?



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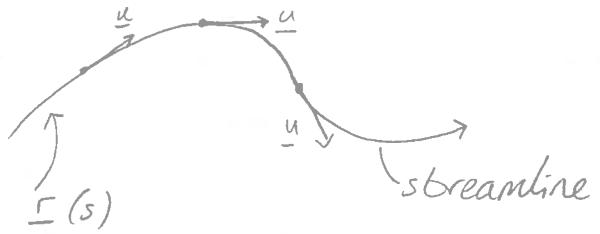
1.3.3 Streamline

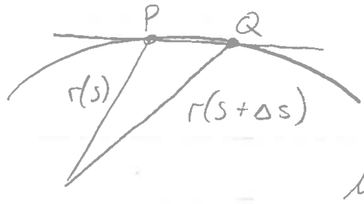
A streamline is a line whose tangent (at some fixed time t) is parallel to the velocity.



Let the streamline be the curve

$$\underline{r}(s)$$

parameterised by the variable s.



$$\frac{\vec{PQ}}{\Delta s} = \frac{\underline{r}(s+\Delta s) - r(s)}{\Delta s}$$

$$\lim_{\Delta s \to 0}\left(\frac{\vec{PQ}}{\Delta s}\right) = \frac{d\underline{r}}{ds} = \underline{u}(\underline{r}(s), t)$$

where t is fixed.

This is a first (non linear) ODE with parameter s.

To find the streamline through $\underline{r}_0$, solve

$$\frac{d\underline{r}}{ds} = \underline{u}(\underline{r}, t) \quad , \quad t \text{ fixed},$$

subject to $\underline{r}(0) = \underline{r}_0$

Example 1.6

Use the same velocity field as before

$$\underline{u} = \hat{\underline{x}} - 2te^{-t^2}\hat{\underline{y}}$$

find streamline through $(1,1)$ at $t=0$.

For streamlines, time is fixed, so put $t=0$ immediately.

At $t=0$, $\underline{u} = \hat{\underline{x}}$

Solve $\dfrac{d\underline{r}}{ds} = \hat{\underline{x}}$ i.e. $\underline{r} = s\hat{\underline{x}} + \underline{c}$

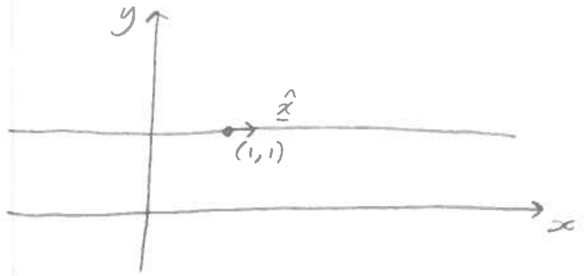
Or in components

$$\frac{dx}{ds} = 1 \quad , \quad \frac{dy}{ds} = 0$$

so $x = s + x_0$, $y = y_0$

When $s = 0$, $x = 1$ so $x_0 = 1$
When $s = 0$, $y = 1$ so $y_0 = 1$

Hence $x = s + 1$, $y = 1$ $\forall s$.



We have seen:
particle paths, streaklines, streamlines.
They are different in unsteady flows.

Exercise
Show they are the same in steady flows.
$\left[\frac{\partial u}{\partial t} = 0 \text{ , i.e. } \underline{u} = \underline{u}(r) \right]$

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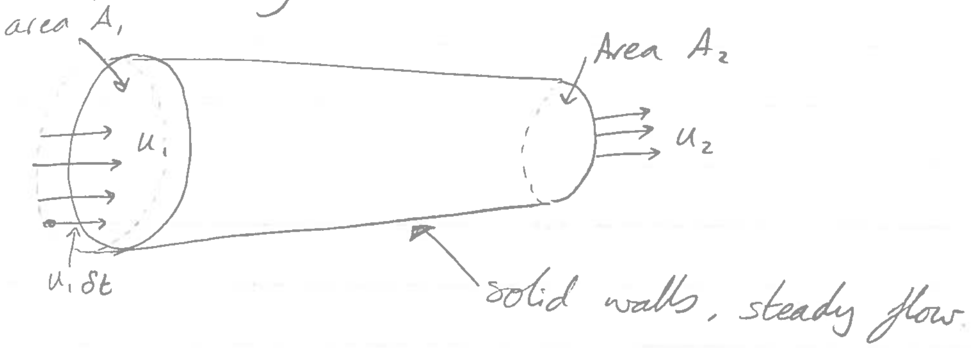
1.4 Incompressibility



Conservation of mass:
amount of mass entering pipe = amound of mass leaving.

The amount of mass entering in time δt is

$$\overset{\nearrow}{\rho}\ \underbrace{(u_1 \delta t)}_{\substack{LT^{-1}T \\ \text{length}}} A_1$$

density

The amount of fluid leaving in time δt is

$$\rho (u_2 \delta t) A_2$$

To conserve mass these balance so

$$\rho (u_1 \delta t) A_1 = \rho (u_2 \delta t) A_2$$

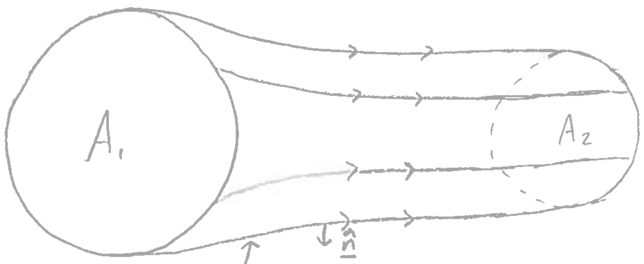
i.e. $u_1 A_1 = u_2 A_2$

i.e. $\dfrac{u_2}{u_1} = \dfrac{A_1}{A_2}$

So velocity varies inversely with area.

By dividing by δt we can say
'the rate at which mass enters' = 'the rate at which mass exits'
i.e. $\rho A_1 u_1 = \rho A_2 u_2$
The rate at which something flows = flux
i.e. mass flux in = mass flux out.
max flux in = amount of mass entering per unit time.

Streamlines give a streamtube:



$$\frac{u_2}{u_1} = \left(\frac{A_2}{A_1}\right)^{-1}$$

Streamlines – parallel to $\underline{u}$

$\underline{u} \cdot \underline{\hat{n}} = 0$

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Lemma

Let f be a continuous function of the interval $[a,b]$. For every subinterval $(c,d) \subset [a,b]$, let

$$\int_c^d f(x)\,dx = 0.$$

Then $f(x) \equiv 0$ on $[a,b]$

Proof (by contradiction)

Suppose that there exists some α in $[a,b]$ where $f(\alpha) \neq 0$.

W.l.o.g we can say $f(\alpha) > 0$.

But f is continuous at α so

$\forall\, \varepsilon > 0,\ \exists\, \delta$ s.t. if $|x-\alpha| < \delta$ then $|f(x) - f(\alpha)| < \varepsilon$.

Take $\varepsilon = \frac{1}{2} f(\alpha) > 0$.

So $\exists\, \delta$ s.t. $|x-\alpha| < \delta$, $|f(x) - f(\alpha)| < \frac{1}{2} f(\alpha)$

i.e. $\frac{1}{2} f(\alpha) < f(x) < \frac{3}{2} f(\alpha)$

Now consider

$$\int_{\alpha-\delta}^{\alpha+\delta} f(x)\,dx \geq \int_{\alpha-\delta}^{\alpha+\delta} \left[\frac{1}{2} f(\alpha)\right] dx$$

$$= 2\delta \cdot \frac{1}{2} f(\alpha) = \delta f(\alpha) > 0$$

Contradition since $\int_c^d f(x)\,dx = 0 \quad \forall\, (c,d) \subset [a,b]$

i.e. no such α exists in $[a,b]$,

i.e. $f(x) \equiv 0$ in $[a,b]$. $\square$

In 3D

Consider a domain D in 3 dimensions, and a continuous function $f(\underline{r})$.

If $\iiint_V f\, dV = 0$ for all sub intervals V,

then $f \equiv 0$ in D.

Proof

As above.

By contradiction.

i.e. assume $\exists \underline{r}_0 \in D$ s.t. $f(\underline{r}_0) \neq 0$

W.l.o.g. $f(\underline{r}_0) > 0$.

Thus by continuity, $\exists$ a ball, V_B, with radius $\delta > 0$, centred on $\underline{r}_0$ s.t.

$f(\underline{r}) > \frac{1}{2} f(\underline{r}_0) > 0$ in V_B.

Then $\iiint_{V_B} f \, dV \geqslant \iiint_{V_B} \frac{1}{2} f(\underline{r}_0) \, dV = \frac{4}{3}\pi\delta^3 \cdot \frac{1}{2} f(\underline{r}_0) > 0$

Contradiction, so $f(\underline{r}) \equiv 0$ in D.

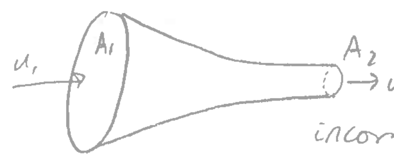
By reading week $\rightarrow$ [Advance revision: 1D version. (Leibnitz)

Differenciation under the integral sign :-

$$\frac{d}{dt}\int_{a(t)}^{b(t)} f(x,t)\, dx = \int_{a(t)}^{b(t)} \frac{df}{dt}(x,t)\, dx + b'(t) f(b(t),t) - a'(t) f(a(t),t)$$]

(Reynolds Transport Theorem)

1.4 Incompressibility



incompressible flow.

Flux. $A_1 u_1 = A_2 u_2$

1.4.1 Conservation of mass (for a fluid of fixed density)

Let our fluid domain D.

Let the fluid at any point at any time have have velocity $\underline{u}(\underline{r}, t)$.

Let the fluid be incompressible.

Take any fixed subvolume V of D, with surface S.

Then the total volume flux through

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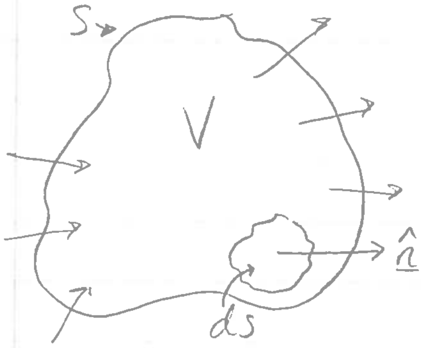
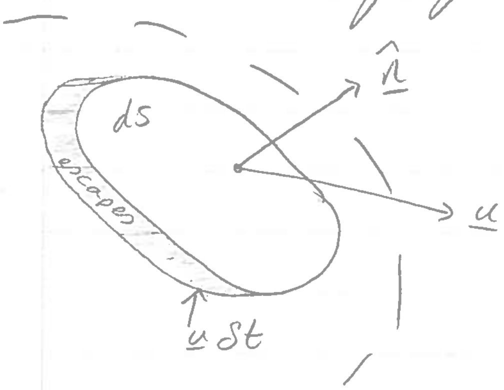
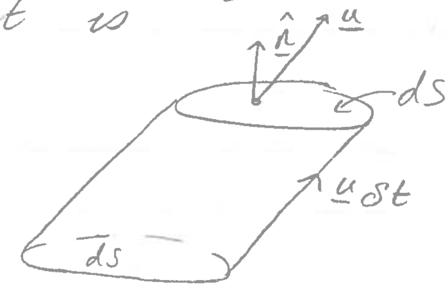
S must vanish.

Consider a surface element dS with outward normal $\hat{n}$



The amount of fluid passing through dS in time t is





Area = base × height = bl

Volume of our small cylinder is
area of base × height
= dS × height

height = component of $\underline{u}\delta t$ // to $\hat{n}$
$= (\underline{u}\cdot\hat{n})\delta t$

i.e. volume of our small cylinder is
$(\underline{u}\cdot\hat{n})dS\,dt$.

This is the amount of fluid leaving through dS in time δt, i.e. the flux [volume per unit time] through dS is
$(\underline{u}\cdot\hat{n})dS$

Thus the total flux through all of S is
$$\iint_S (\underline{u}\cdot\hat{n})\,dS$$

which vanishes by incompressibility
i.e. $\iint_S (\underline{u}\cdot\hat{\underline{n}})\,dS = 0$ in incompressible flow.

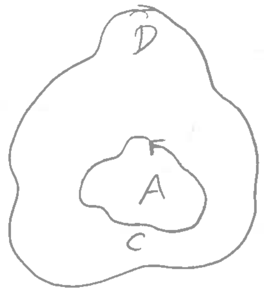
But by Gauss (or the divergence theorem)
$$\iint_S (\underline{u}\cdot\hat{\underline{n}})\,dS = \iiint_V \underline{\nabla}\cdot\underline{u}\,dV$$
Thus we have shown that for <u>all</u> V in D
$$\iiint_V \underline{\nabla}\cdot\underline{u}\,dV = 0$$
Provided $\underline{\nabla}\cdot\underline{u}$ is continuous in D (i.e. $\underline{u}$ is continuously differentiable)
then $\underline{\nabla}\cdot\underline{u} = 0$ in D
i.e. incompressible $\Rightarrow \underline{\nabla}\cdot\underline{u} = 0$. [Eqn of Continuity]
- true in 3D and 2D flow.

<u>Stream flow</u>
The combination of incompressibility and 2D flow is very powerful.
2D: i.e. $\underline{u} = u(x,y,t)\hat{\underline{x}} + v(x,y,t)\hat{\underline{y}}$
Then the continuity eqn gives
$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

[conservative force fields]

(cross product, can use $\times$ or $\wedge$)

Introduce, temporarily, $\underline{u}^{\perp} = \hat{\underline{z}} \wedge \underline{u}$ ($\underline{u}$ rotated by 90°)
Show $\underline{u}^{\perp}$ is a conservative vector field.
$$\begin{aligned}\underline{u}^{\perp} &= \hat{\underline{z}} \wedge (u\hat{\underline{x}} + v\hat{\underline{y}}) \\ &= u\,\hat{\underline{z}}\wedge\hat{\underline{x}} + v\,\hat{\underline{z}}\wedge\hat{\underline{y}} \\ &= -v\hat{\underline{x}} + u\hat{\underline{y}}\end{aligned}$$



Consider fluid domain D with incompressible fluid with 2D velocity field $\underline{u}$.
Take any closed circuit C in D.

Then
$$\oint_C \underline{u}^{\perp} \cdot d\underline{r} = \oint_C (-v\hat{\underline{x}} + u\hat{\underline{y}}) \cdot (dx\,\hat{\underline{x}} + dy\,\hat{\underline{y}})$$
$$= \oint_C (-v\,dx + u\,dy)$$
$$= \int_A \left(\frac{du}{dx} + \frac{dv}{dy}\right) dx\,dy \quad [\leftarrow \text{Green's Lemma}]$$
$$= 0 \text{ by continuity}$$

Thus $\underline{u}^{\perp}$ is a conservative vector field.

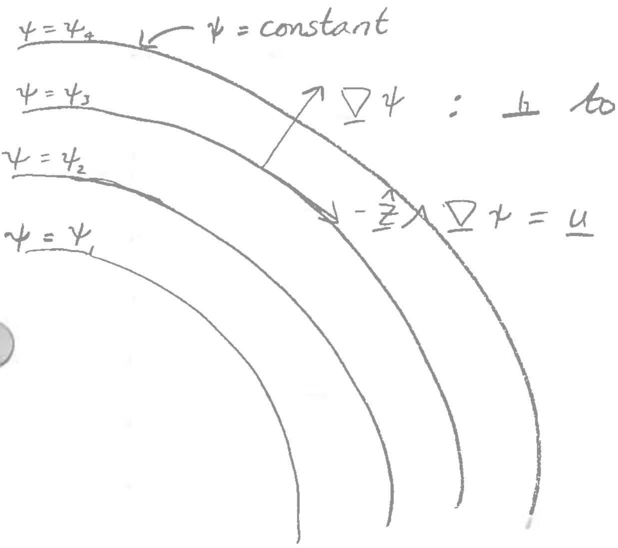
[If work done by a force around all closed contours vanishes, then the force is conservative.]

Thus there exists a function ψ s.t.
$$\underline{u}^{\perp} = \underline{\nabla}\psi$$
i.e. $\hat{\underline{z}} \wedge \underline{u} = \underline{\nabla}\psi$

i.e. $\hat{\underline{z}} \wedge (\hat{\underline{z}} \wedge \underline{u}) = \hat{\underline{z}} \wedge \underline{\nabla}\psi$

i.e. $\boxed{\exists\, \psi \text{ s.t. } \underline{u} = -\hat{\underline{z}} \wedge \underline{\nabla}\psi}$

ψ is called a STREAM FUNCTION



$\underline{\nabla}\psi$: $\perp$ to lines ψ = const., points in direction of ψ increasing.

The lines ψ = constant are tangent to the velocity field at each point: they are stream lines (s'lines)

i.e. lines $\psi(x, y, t) = \text{const}$ are streamlines.
Thus if we can find ψ we can draw the streamlines.
Notice $|\underline{u}| = |\underline{\nabla} \psi|$, so the magnitude of $\underline{\nabla}\psi$ gives the speed.

Example 1.10
Consider the 2D velocity field $\underline{u} = x\hat{\underline{x}} - y\hat{\underline{y}}$
Show that $\underline{u}$ is compressible. Hence ψ exists.
Find ψ and sketch some streamlines.

i). $u = x$, $v = -y$ so $\frac{du}{dx} + \frac{dv}{dy} = 1 - 1 = 0$

Thus flow is incompressible.
It is also 2D, thus $\exists \psi$.

ii). We know that

$\underline{u} = -\hat{\underline{z}} \wedge \underline{\nabla} \psi$

Here we are using Cartesian coordinates so

$\underline{\nabla} \psi = \frac{d\psi}{dx} \hat{\underline{x}} + \frac{d\psi}{dy} \hat{\underline{y}}$

$-\hat{\underline{z}} \wedge \underline{\nabla} \psi = -\frac{d\psi}{dx} (\hat{\underline{z}} \wedge \hat{\underline{x}}) - \frac{d\psi}{dy} (\hat{\underline{z}} \wedge \hat{\underline{y}})$

$\underline{u} = u\hat{\underline{x}} + v\hat{\underline{y}} = -\frac{d\psi}{dx} \hat{\underline{y}} + \frac{d\psi}{dy} \hat{\underline{x}}$

Hence $\boxed{u = +\frac{d\psi}{dy} \quad , \quad v = -\frac{d\psi}{dx}}$

In our example

$u = x$ so $\frac{d\psi}{dy} = x$ so $\psi = xy + f(x)$

where $f(x)$ is an arbitrary function of x
Thus $v = -\frac{d\psi}{dx} = -y - f'(x)$

But $v=-y$ so $f'(x)=0$, so f is a constant.

Notice that since

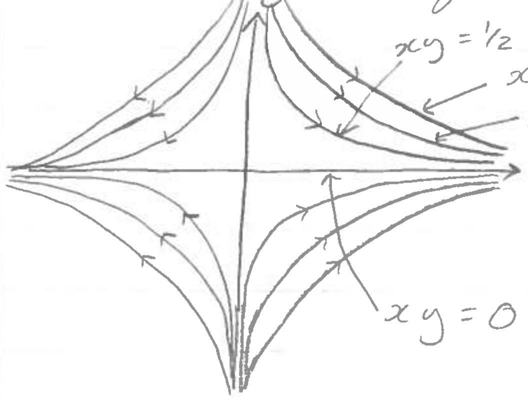
$$\underline{u} = -\hat{\underline{z}} \wedge \underline{\nabla}\psi$$

ψ is unique only to within an additive function of time.

Thus take $\psi = xy$ here.

iii). Streamlines: lines where $\psi =$ constant

ie. $xy = C$ for some C.



Either use $\nabla\psi$ or note $u = x$ so $x > 0$
$\Rightarrow u > 0$
and $x < 0 \Rightarrow u < 0$
for direction.

These are the streamlines for stagnation point flow.

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Inviscid, $Re = \frac{UL}{\nu}$ inversely proportional to viscosity

Incompressible, $\frac{\text{speed of flow}}{\text{speed of sound}} = M$, $M \ll 1$

$\Downarrow$

$\underline{\nabla} \cdot \underline{u} = 0$

Incompressible + 2D $\Rightarrow$ $\exists\, \psi$ s.t. $\underbrace{\underline{u} = -\hat{\underline{z}} \wedge \underline{\nabla} \psi}_{\text{no co-ordinates}}$

Cartesians: $\underline{\nabla}\psi = \frac{\partial \psi}{\partial x}\hat{\underline{x}} + \frac{\partial \psi}{\partial y}\hat{\underline{y}}$

so $u = \frac{\partial \psi}{\partial y}$, $v = -\frac{\partial \psi}{\partial x}$

Polar co-ordinates: $\underline{\nabla}\psi = \frac{\partial \psi}{\partial r}\hat{\underline{r}} + \frac{1}{r}\frac{\partial \psi}{\partial \theta}\hat{\underline{\theta}}$

don't need to know $\rightarrow$ $\left[\ \underline{\nabla}\psi = \sum_i \frac{1}{h_i}\frac{\partial \psi}{\partial x_i}\hat{\underline{x}}_i \right.$, h_i is the scale factor $\left.\right]$

Thus in polars

$$\begin{aligned}
\underline{u} &= -\hat{\underline{z}} \wedge \underline{\nabla}\psi \\
&= -\frac{\partial \psi}{\partial r}\hat{\underline{z}} \wedge \hat{\underline{r}} - \frac{1}{r}\frac{\partial \psi}{\partial \theta}\hat{\underline{z}} \wedge \hat{\underline{\theta}} \\
&= -\frac{\partial \psi}{\partial r}\hat{\underline{\theta}} - \frac{1}{r}\frac{\partial \psi}{\partial \theta}(-\hat{\underline{r}}) \\
&= \frac{1}{r}\frac{\partial \psi}{\partial \theta}\hat{\underline{r}} - \frac{\partial \psi}{\partial r}\hat{\underline{\theta}}
\end{aligned}$$

$\frac{1}{r}\frac{\partial\psi}{\partial\theta} = \underline{\hat{r}}\cdot\underline{u}$ which is the r-component of u, u_r, the radial component.

Similarly $\underline{\hat{\theta}}\cdot\underline{u}$ is the θ-component of u, u_θ, the azimuthal component.

$$u_r = \frac{1}{r}\frac{\partial\psi}{\partial\theta}, \quad u_\theta = -\frac{\partial\psi}{\partial r}$$

Check $\underline{\nabla}\cdot(u_r\underline{\hat{r}} + u_\theta\underline{\hat{\theta}}) = 0$.

Example 1.11

For the flow $\underline{u} = 2y\underline{\hat{x}} - 2x\underline{\hat{y}}$,

1). show that it is compressible

2). find an ψ

3). sketch some streamlines.

2). $u = \frac{\partial\psi}{\partial y} = 2y$

so $\psi = y^2 + f(x)$

so $\frac{\partial\psi}{\partial x} = f'(x)$

But $\frac{\partial\psi}{\partial x} = -v = 2x$

So $f'(x) = 2x$

i.e. $f(x) = x^2 + C$

Thus $\psi = x^2 + y^2$, taking $C = 0$.

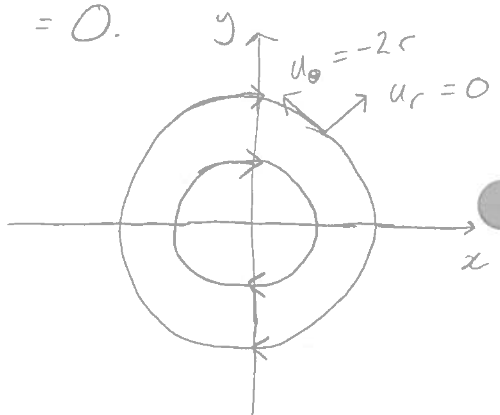
(in fact this proves 1).)

1). $\frac{\partial u}{\partial x} = 0$,

$\frac{\partial v}{\partial y} = 0$,

so $\underline{\nabla}\cdot\underline{u} = 0$

3). Streamlines: curves where ψ = constant, i.e. $x^2+y^2=a^2$



$y>0 \Rightarrow u>0$

$y<0 \Rightarrow u<0$

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Notice

$$\psi = x^2 + y^2$$
$$= (r\cos\theta)^2 + (r\sin\theta)^2$$
$$= r^2(\cos^2\theta + \sin^2\theta)$$
$$= r^2$$

$$u_r = \frac{1}{r}\frac{\partial \psi}{\partial \theta} = 0$$

$$u_\theta = -\frac{\partial \psi}{\partial r} = -2r$$ so angular speed $= -2$

Rotation in a fluid is called VORTICITY.

It is measured by curl $\underline{u}$,
which tends to be written

$$\underline{\omega} = \underline{\nabla} \wedge \underline{u}$$

Note $\underline{\nabla} \wedge \underline{u} = \begin{vmatrix} \hat{\underline{x}} & \hat{\underline{y}} & \hat{\underline{z}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ u & v & w \end{vmatrix}$

In 2D, $\left(\frac{\partial}{\partial z} \equiv 0, \; w \equiv 0\right)$

$$\underline{\nabla} \wedge \underline{u} = \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)\hat{\underline{z}}$$

vorticity is twice the local rate of rotation of a fluid

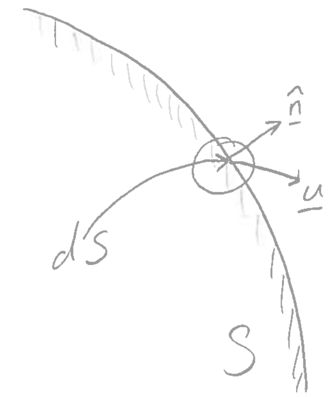
$$\underline{\omega} = \begin{pmatrix} \xi \\ \eta \\ \zeta \end{pmatrix} \quad \begin{matrix} \text{xi} \\ \text{eta} \\ \text{zeta} \end{matrix}$$

(In Ex 1.11
$u = 2y$, $v = -2x$
$\zeta = -2 - 2 = -4$
$= 2 \times$ rate of rotation)

so $\underline{\nabla} \wedge \underline{u} = \zeta \hat{\underline{z}}$ where $\zeta = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}$

1.6 Invisid flow at a Solid Boundary

i.e. an impermeable surface.



The flux of fluid through dS is

$$(\underline{u}\cdot\hat{\underline{n}})\,dS$$

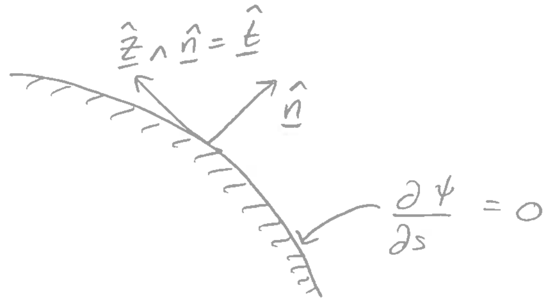
For no flow through dS we must have

$$\underline{u}\cdot\hat{\underline{n}} = 0$$

i.e. the normal component of velocity vanishes at a solid boundary.

In 2D, incompressible flow, $\exists\,\psi$ s.t. $\underline{u} = -\hat{\underline{z}}\wedge\underline{\nabla}\psi$.

Thus on a solid body



$$(\underline{z}\wedge\underline{\nabla}\psi)\cdot\hat{\underline{n}} = 0$$

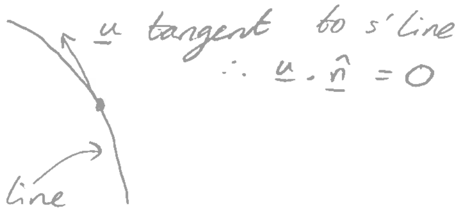
i.e. $(\underline{z}\wedge\hat{\underline{n}})\cdot\underline{\nabla}\psi = 0$

i.e. $\hat{\underline{t}}\cdot\underline{\nabla}\psi = 0$

This is the directional derivative in the direction $\hat{\underline{t}}$, i.e. we have

$$\frac{\partial\psi}{\partial s} = 0$$

in the direction $\hat{\underline{t}}$.



i.e. ψ does not change with position along the boundary, i.e. ψ continuous on boundary.

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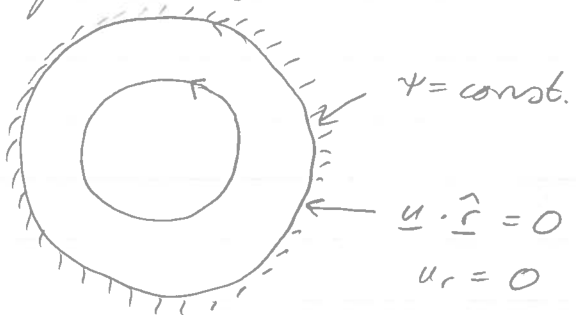
Thus any s'line can be replaced by a solid boundary without affecting the flow, and any solid boundary is a streamline.

In particular in Ex 1.11



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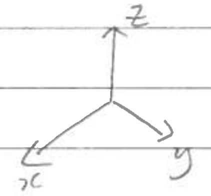
$\underline{u}(x, y, z, t)$



Incompressible $\Rightarrow$ $\underline{\nabla} \cdot \underline{u} = 0$

Also 2D:

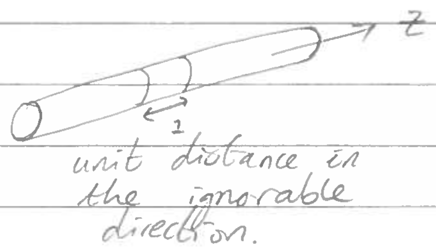
$\exists \psi$ st. $\underline{u} = -\underline{z} \wedge \underline{\nabla}\psi$

ψ is the streamfunction, ψ const. $\Rightarrow$ streamlines

§1.7 Physical Interpretation of the Streamfunction

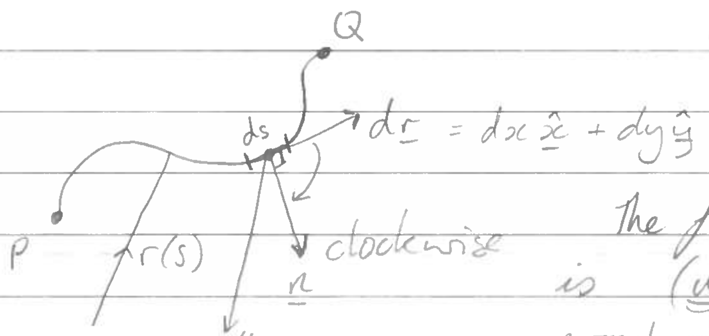
In 2D, the volume flux (per unit distance in the ignorable direction) across ANY line joining a point P to a point Q, in the clockwise direction, is given by

$$\psi(Q) - \psi(P)$$



$$\frac{\text{Volume Flux}}{\text{unit distance}} = \frac{L^3 T^{-1}}{L} = L^2 T^{-1}$$

ie. Areal Flux.



The flux of fluid across ds is $\underbrace{(\underline{u} \cdot \hat{\underline{n}})}_{\text{normal velocity}} \underbrace{ds}_{\text{length of segment.}}$

Thus for our line, the total area flux across the line (or volume flux / unit width) L joining P to Q, in the clockwise direction is

$$_L\int_P^Q (\underline{u} \cdot \hat{\underline{n}})\, ds$$

We need a vector orthogonal to $d\underline{r}$, take

$$\underline{n}_1 = -dy\,\hat{\underline{x}} + dx\,\hat{\underline{y}}.$$

Notice then $\underline{n}_1 \cdot d\underline{r} = -dx\,dy + dx\,dy = 0$

But this is in the wrong direction so take

$$\underline{n} = dy\,\hat{\underline{x}} - dx\,\hat{\underline{y}}$$

But this is not a unit vector

$$|\underline{n}|^2 = dy^2 + dx^2 = ds^2$$

ie. $|\underline{n}| = ds$

To make $\underline{n}$ a unit vector, divide by length:

$$\hat{\underline{n}} = \frac{\underline{n}}{|\underline{n}|} = \frac{dy}{ds}\hat{\underline{x}} + \frac{dx}{ds}\hat{\underline{y}}$$

For incompressible 2D flow, we have

$\underline{u} = -\hat{\underline{z}} \wedge \nabla \psi$

So $u = \dfrac{\partial \psi}{\partial y}$, $v = -\dfrac{\partial \psi}{\partial x}$

So $\underline{u} = \dfrac{\partial \psi}{\partial y}\hat{\underline{x}} - \dfrac{\partial \psi}{\partial x}\hat{\underline{y}}$ (*)

Thus the flux is

$$L\int_P^Q \left(\frac{\partial\psi}{\partial y}\hat{x} - \frac{\partial\psi}{\partial x}\hat{y}\right)\cdot\left(\frac{dy}{ds}\hat{x} - \frac{dx}{ds}\hat{y}\right) ds$$

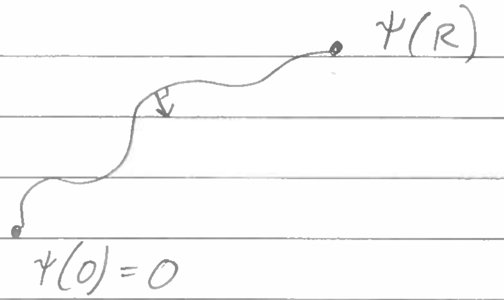
$$= L\int_P^Q \left(\frac{\partial\psi}{\partial x}\frac{dx}{ds} + \frac{\partial\psi}{\partial y}\frac{dy}{ds}\right) ds$$

(Chain rule)

$$= L\int_P^Q \frac{\partial\psi}{\partial s}\, ds$$

$$= \psi(Q) - \psi(P)$$

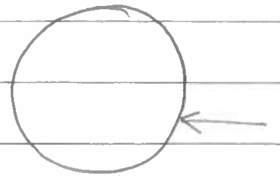
(independent of L, i.e. independent of the path).



i.e. ψ at any point R, relative to the origin, O, is simply the area flux across any line joining O to R.

Aside

@ one cylinder



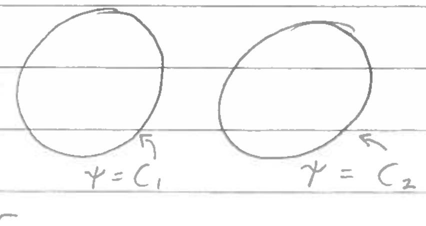
$\psi = C_1$, a constant

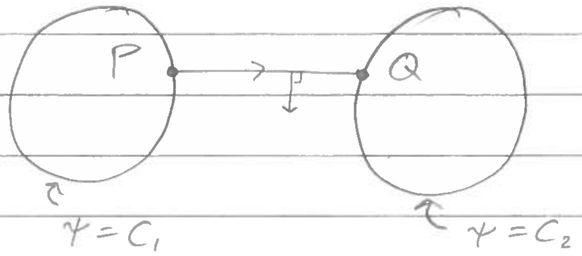
w.l.o.g we can take $C_1 = 0$

$\psi^* = \psi - C_1$.

Ⓑ two cylinders

The constants do not have to be the same, so we can only take $\psi = 0$ on one cylinder.





Flux across $\vec{PQ}$ is
$$\psi(Q) - \psi(P) = C_2 - C_1$$

$C_1 = C_2$ says no net flux between cylinders

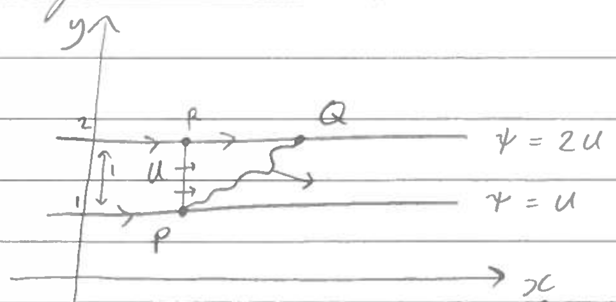
Example 1.12
Consider the velocity field
$$\underline{u} = U\,\hat{\underline{x}}$$

What is the flux between streamlines?

Streamfunction
$$\frac{\partial\psi}{\partial y} = u = U, \quad \frac{\partial\psi}{\partial x} = -v = 0$$

so $\psi = \psi(y)$
Thus $\psi'(y) = U$
i.e. $\psi = Uy$ (taking constant to be zero w.l.o.g.)

– a UNIFORM stream, a constant velocity in a fixed direction.



Flux across the line $\vec{PQ}$ (clockwise) is
$$\psi(Q) - \psi(P) = 2U - U = U$$

Flux across $\vec{PR} = U \times 1 = U$ (speed × width).

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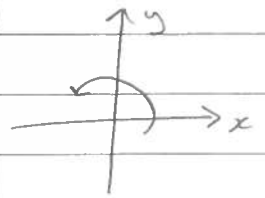
Example 1.13

Isotropic Flow

– same in all directions from the origin



$$\frac{\partial \underline{u}}{\partial \theta} = 0$$

$$\hat{\underline{\theta}} \cdot \underline{u} = 0 \quad \text{i.e.} \quad u_\theta = 0.$$

Remember $u_r = \frac{1}{r}\frac{\partial \psi}{\partial \theta}$

$$u_\theta = -\frac{\partial \psi}{\partial r}$$

But $u_\theta = 0$ so $\psi = \psi(\theta)$

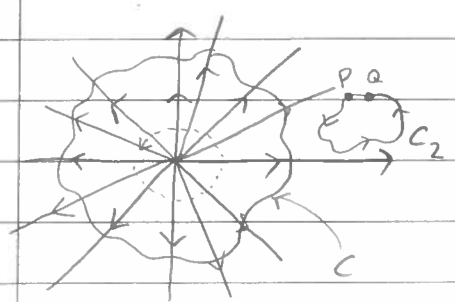
with $u_r = \frac{1}{r}\frac{\partial \psi}{\partial \theta}$

Take $\psi = m\theta$, for some constant m.

Then $u_r = \frac{m}{r} > 0$ for $m > 0$

s' lines : lines $\psi = \text{const.}$

$\Rightarrow m\theta = \text{const.} \Rightarrow \theta = \text{const.}$



Make ψ single-valued by choosing

e.g. $-\pi < \theta \leq \pi$

[Could choose $0 \leq \theta < 2\pi$]

Let C be an arbitrary closed curve circling the origin.

What is the flux across C?

$\underline{u} = u_r \hat{\underline{r}} \quad u_r > 0$

$\hat{\underline{n}} = \hat{\underline{r}} \quad \underline{u} \cdot \hat{\underline{n}} > 0.$

Consider C_2, which does not circle the origin.
Take any two points P, Q with Q slightly clockwise of P.
Then flux across PQ is $(\psi(Q) - \psi(P)) \to$ as $Q \to P$. So flux across $C_2 = 0$.

Now return to C. Then choosing Q at $\theta = \pi^-$ and P at $\theta = (-\pi)^+$ gives flux

$$\psi(Q) - \psi(P) = m\pi^- - m(-\pi)^+$$
$$= 2m\pi$$

We call this the STRENGTH of the source :- the flux of fluid across any curve circling the source

$\psi = m\theta$ is the streamfunction of an isotropic source of strength $2\pi m$.
[source strength σ, would be $\psi = \frac{\sigma}{2\pi}\theta$]
If $m < 0$ we call the flow a SINK of strength $|m|$.

Example 1.14
An isotropic source in a uniform stream.
Uniform stream: $\psi_1 = Uy$
Isotropic source of strength $2\pi m$: $\psi_2 = m\theta$
Isotropic source in a uniform stream:

$$\psi = \psi_1 + \psi_2$$
$$= Uy + m\theta$$

Streamline: lines ψ = constant.
[Try plotting $\psi = y + \tan^{-1}(y/x)$]

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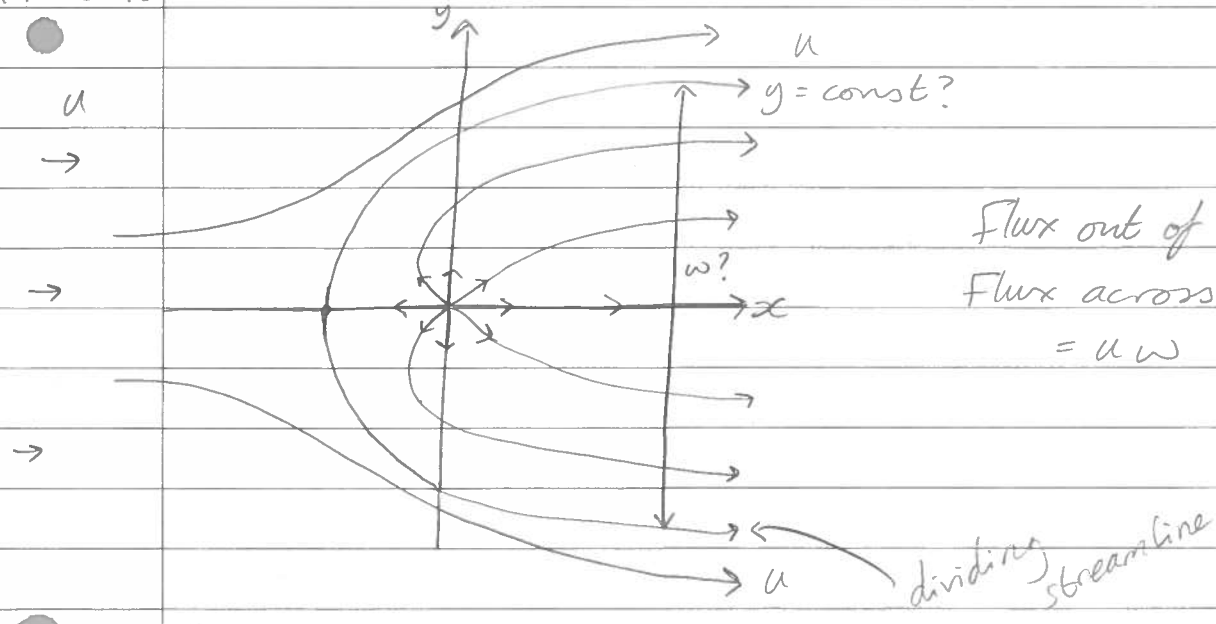


Flux out of source $2\pi m$

Flux across vertical line $= u w$

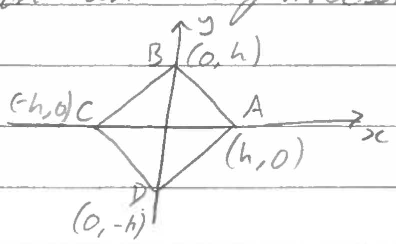
Source $u_r = \frac{m}{r}$

$\to \infty$ as $r \to 0$

$\to 0$ as $r \to \infty$

§1.8 Local motion of a fluid element.

What does an arbitrary incompressible 2D velocity field $\underline{u}(x, y, t)$ do to an arbitrarily small (typical dimension $h \ll 1$) square fluid element in an infinitessimal time δt



Taylor series for small x,

$$f(x) = f(0) + x f'(0) + \frac{1}{2!} x^2 f''(0) + \dots + R_n$$

In 2D for small (x, y),

$$f(x,y) = f(0,0) + x \frac{\partial f}{\partial x}(0,0) + y \frac{\partial f}{\partial y}(0,0) + (x^2, xy, y^2) + \dots + R_n$$

Thus for our $\underline{u} = u\underline{\hat{x}} + v\underline{\hat{y}}$

$u = U + \alpha x + \beta y$
$v = V + \gamma x + \delta y$

where $U = u(0,0)$, $V = v(0,0)$,
$\alpha = \frac{\partial u}{\partial x}(0,0)$, $\beta = \frac{\partial u}{\partial y}(0,0)$, $\gamma = \frac{\partial v}{\partial x}(0,0)$, $\delta = \frac{\partial v}{\partial y}(0,0)$.

Here we make an error of order h^2.

Now the flow is incompressible so
$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$
ie. $\alpha + \delta = 0$
ie $\delta = -\alpha$

Thus in matrix form
$$\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} U \\ V \end{pmatrix} + \begin{pmatrix} \alpha & \beta \\ \gamma & -\alpha \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$

For any $\underline{A}$ we have
$$\underline{A} = \underbrace{\tfrac{1}{2}(\underline{A} + \underline{A}^T)}_{\text{symmetric}} + \underbrace{\tfrac{1}{2}(\underline{A} - \underline{A}^T)}_{\text{anti (skew)-symmetric}}$$
$B^T = B$ (symmetric)
$B^T = -B$ (anti-symmetric)

Thus we can write
$$\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} U \\ V \end{pmatrix} + \left[\alpha\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + \theta\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + \phi\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\right]\begin{pmatrix} x \\ y \end{pmatrix}$$
Here $\theta = \frac{1}{2}(\beta + \gamma)$, $\phi = \frac{1}{2}(\beta - \gamma) = \frac{1}{2}\left(\frac{\partial u}{\partial y} - \frac{\partial v}{\partial x}\right)$

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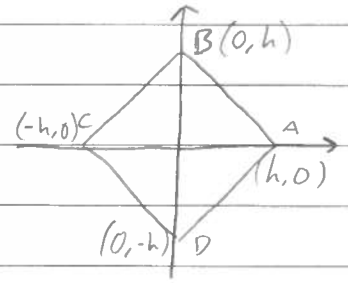
Incompressibility $\Rightarrow \underline{\nabla} \cdot \underline{u} = 0$ (Eqn of cont.)

Plus 2D $\Rightarrow \exists \psi$

Local motion at a point



$\underline{u}(\underline{x}, t)$

$0 < h \ll 1$

$$\begin{pmatrix} u \\ v \end{pmatrix} = \underbrace{\begin{pmatrix} U \\ V \end{pmatrix}}_{\mathrm{I}} + \left[\alpha \underbrace{\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}}_{\mathrm{II}} + \theta \underbrace{\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}}_{\mathrm{III}} + \phi \underbrace{\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}}_{\mathrm{IV}} \right] \begin{pmatrix} x \\ y \end{pmatrix} + O(x^2, y^2, xy)$$

where $|x| \ll 1$, $|y| \ll 1$, ie. order h^2

In general for $0 < \delta t \ll 1$ then any point in ABCD moves an amount

$$(\delta x, \delta y) = \underline{u}(\underline{x}, 0)\, \delta t + O(\delta t^2)$$

Term I (on its own)

ie. all points in ABCD move an amount

$(\delta x, \delta y) = (U, V)\delta t = (U\delta t, V\delta t)$

since $\begin{pmatrix} U \\ V \end{pmatrix}$ same for every point.

— This is a translation (no change of shape or orientation).

Term II

$$\begin{pmatrix} u \\ v \end{pmatrix} = \alpha \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

- a function of position
- has different effects on different points.

Moves a point by amount

$$\begin{pmatrix} u \\ v \end{pmatrix} \delta t = \alpha \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \delta t$$

At A, $(h, 0)$:

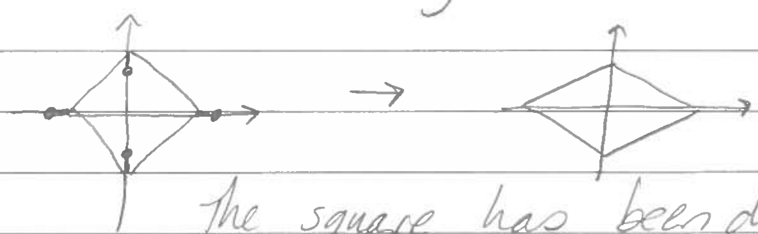
$$\begin{pmatrix} \delta x \\ \delta y \end{pmatrix} = \alpha \begin{pmatrix} h \\ 0 \end{pmatrix} \delta t \quad , \quad \begin{cases} \delta x = \alpha h \delta t \\ \delta y = 0 \end{cases}$$

At C, $(-h, 0)$: $\begin{cases} \delta x = -\alpha h \delta t \\ \delta y = 0 \end{cases}$

At B, $(0, h)$:

$$\begin{pmatrix} \delta x \\ \delta y \end{pmatrix} = \alpha \begin{pmatrix} 0 \\ -h \end{pmatrix} \delta t \quad , \quad \begin{cases} \delta x = 0 \\ \delta y = -\alpha h \delta t \end{cases}$$

At D, $(0, -h)$: $\begin{cases} \delta x = 0 \\ \delta y = \alpha h \delta t \end{cases}$



The square has been distorted to a rhombus (stretched along x-axis and shrunk by precisely the same amount along the y-axis). Thus conserving area (as it must by conservation of volume) to order h^2

- a dilation

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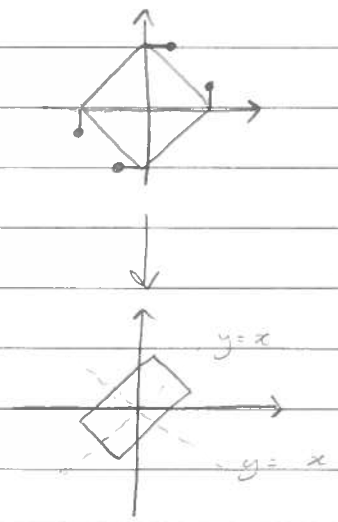
Term III

$$\begin{pmatrix} \delta x \\ \delta y \end{pmatrix} = \theta \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \delta t$$

At A, $(h, 0)$:

$$\begin{pmatrix} \delta x \\ \delta y \end{pmatrix} = \begin{pmatrix} 0 \\ \theta h \delta t \end{pmatrix}$$



At C, $(-h, 0)$:

$$\begin{pmatrix} \delta x \\ \delta y \end{pmatrix} = \begin{pmatrix} 0 \\ -\theta h \delta t \end{pmatrix}$$

At B, $(0, h)$:

$$\begin{pmatrix} \delta x \\ \delta y \end{pmatrix} = \begin{pmatrix} \theta h \delta t \\ 0 \end{pmatrix}$$

Another Dilation

At D, $(0, -h)$:

$$\begin{pmatrix} \delta x \\ \delta y \end{pmatrix} = \begin{pmatrix} -\theta h \delta t \\ 0 \end{pmatrix}$$

- Stretching along $y = x$ and an equal and opposite shrinking along the orthogonal direction $y - x$ so as to conserve area to order h^2.

Note: the sum of two dilations is itself a dilation.

Put terms II and III together to give

$$\begin{pmatrix} \alpha & \theta \\ \theta & -\alpha \end{pmatrix}$$ Real, symmetric matrix.
$\Rightarrow$ eigenvalues real, orthogonal eigenvectors.

This has eigenvalues λ st.

$$\begin{vmatrix} \alpha - \lambda & \theta \\ \theta & -\alpha - \lambda \end{vmatrix} = 0$$

$$-(\alpha-\lambda)(\alpha+\lambda)-\theta^2=0$$

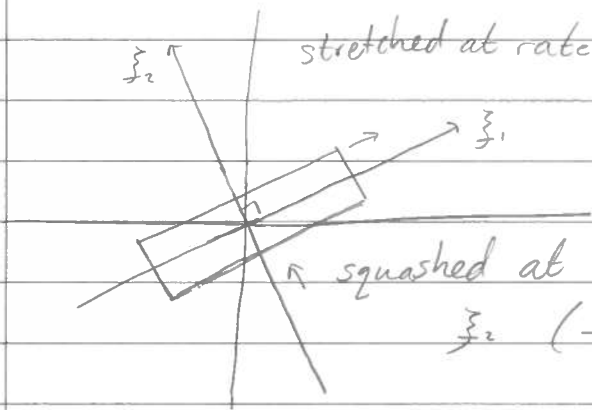
i.e. $\lambda^2=\alpha^2+\theta^2$

i.e. $\lambda_{1,2}=\pm\sqrt{\alpha^2+\theta^2}$

- real eigenvalues (as expected) but also equal and opposite.

- distinct $\therefore$ corresponding eigenvectors, ξ_1, ξ_2 are orthogonal, i.e. $\xi_1\cdot\xi_2=0$

Expressed relative to the orthogonal basis $\{\xi_1,\xi_2\}$ the matrix is diagonal with eigenvalues on the diagonal $\begin{pmatrix}\lambda_1 & 0\\ 0 & \lambda_2\end{pmatrix}=\begin{pmatrix}\sqrt{\alpha^2+\theta^2} & 0\\ 0 & -\sqrt{\alpha^2+\theta^2}\end{pmatrix}$



stretched at rate $\lambda_1=\sqrt{\theta^2+\alpha^2}$ along ξ_1

squashed at rate $\lambda_2=-\sqrt{\theta^2+\alpha^2}$ along ξ_2 ($\perp$ to ξ_1)

Term IV

$$\begin{pmatrix}\delta x\\ \delta y\end{pmatrix}=\phi\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix}\delta t$$

$A,(h,0)$: $\begin{pmatrix}\delta x\\ \delta y\end{pmatrix}=\begin{pmatrix}0\\ -\phi h\,\delta t\end{pmatrix}$

$C,(-h,0)$: $\begin{pmatrix}\delta x\\ \delta y\end{pmatrix}=\begin{pmatrix}0\\ \phi h\,\delta t\end{pmatrix}$

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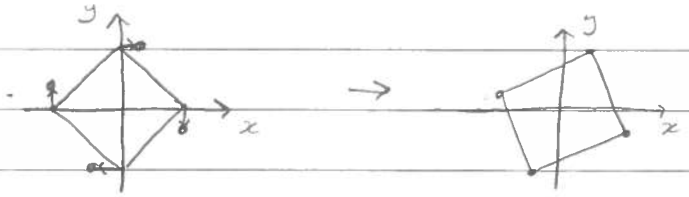
$B, (0, h): \begin{pmatrix} \delta x \\ \delta y \end{pmatrix} = \begin{pmatrix} \phi h \delta t \\ 0 \end{pmatrix}$

$D, (0, -h): \begin{pmatrix} \delta x \\ \delta y \end{pmatrix} = \begin{pmatrix} -\phi h \delta t \\ 0 \end{pmatrix}$



[arm of length h, moves distance $-\phi h \delta t$ so $|\text{angle}| = \phi \delta t$]

- a rotation

Rotating clockwise at a rate of increase of angle ϕ.

The angular velocity of our square is ϕ in the clockwise direction.

ϕ gives the rate at which our body is spinning clockwise about its centre of mass.

$$\phi = \frac{1}{2}\left(\frac{\partial u}{\partial y} - \frac{\partial v}{\partial x}\right)$$

$$= \left[\frac{-1}{2}\, \hat{\underline{z}} \cdot (\underline{\nabla} \wedge \underline{u})\right]$$

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Incompressible $M << 1$

$\Downarrow$

$\underline{\nabla} \cdot \underline{u} = 0$ (3D & 2D)

Add: flux is 2D
then $\exists \psi$ s.t. $\underline{u} = -\hat{\underline{z}} \wedge \underline{\nabla}\psi$

streamlines: ψ constant [a solid body is a streamline]

Just shown that the local motion at a point consists of
1). Translation of centre of mass of a fluid element
2). Dilation of the element (conserving area in 2D, or volume in 3D, in incompressible flow).
3). Rotation about the centre of mass at rate ϕ clockwise, $\phi = \frac{1}{2}\left(\frac{\partial u}{\partial y} - \frac{\partial v}{\partial x}\right)$

Traditionally we take +ve anticlockwise, ie. rate of rotation about centre of mass (anticlockwise) is $\frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)$.

Add in that flow is inviscid (cannot supports a sheer stress) and we get a second restriction on $\underline{u}$. This is sufficient to determine $\underline{u}$ uniquely (given boundary conditions).

Remember we define vorticity as
$\underline{\omega} = \underline{\nabla} \wedge \underline{u}$
and for 2D flow [$\underline{u} = u(x,y,t)\hat{\underline{x}} + v(x,y,t)\underline{y} + 0$]
$\underline{\omega} = \zeta \hat{\underline{z}}$
where $\zeta = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}$ = twice the local rate of rotation of a fluid element about its Centre of Mass
(also true in 3D)

ζ a scalar , $\underline{\omega} = \zeta \hat{\underline{z}}$ a vector

Recall:

Angular momentum:

$$\underline{J} = \underline{r} \wedge m \underline{v}$$

$$\frac{d\underline{J}}{dt} = \underbrace{\overbrace{\frac{d\underline{r}}{dt}}^{\underline{u}} \wedge m\underline{v}}_{=0} + \underline{r} \wedge \frac{d}{dt}(m\underline{v})$$

Newton: $\underline{F} = \frac{d}{dt}(m\underline{v})$

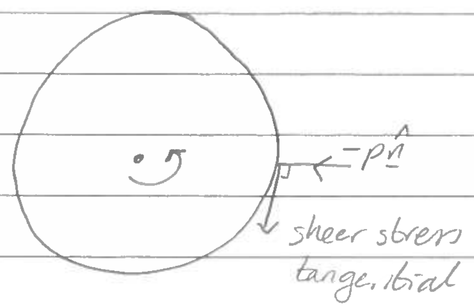
so $\frac{d\underline{J}}{dt} = \underline{r} \wedge \underline{F}$

central force $\underline{F} = f(r)\hat{\underline{r}}$

so $\frac{d\underline{J}}{dt} = (\underline{r} \wedge \hat{\underline{r}}) f(r) = 0$

Conservation of angular momentum.

Now consider a circular fluid element:



p: pressure
(force per unit area $\perp$ to surface, inwards $\Rightarrow -\hat{\underline{n}}$)

(zero in an inviscid fluid)

Angular momentum is conserved as for the circle (or sphere in 3D), pressure is a normal and therefore

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a circular force (i.e. body spins at a constant rate)
Thus in 2D a fluid element retains its value of ζ. [Immediately extends to arbitrary 2D elements]
Important subclass: Consider flow started from rest:

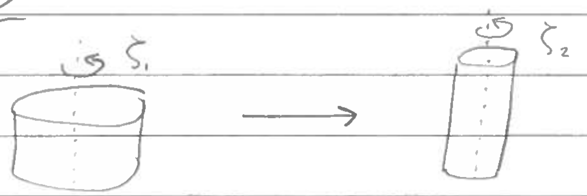
$\underline{u} = 0$ at $t = 0$, so $\underline{\nabla} \wedge \underline{u} = 0$ at $t = 0$.

Then in 2D, ζ remains zero for all t.

$\zeta \equiv 0$: irrotational flow.

Thus in an inviscid fluid, irrotational flow is persistent – "the persistance of irrotational flow."

In 3D



Ballerina effect
(Ice-skater effect)

$\zeta_2 > \zeta_1$

Stretching a column amplifies vorticity.
Squashing shrinks vorticity.
(only in 3D, not 2D since $\frac{\partial}{\partial x} \equiv 0$)

So vorticity is NOT conserved by elements in 3D.
However because it is an amplifier, we have persistance of irrotationality in 3D also.

Thus we will consider (almost always) irrotational flow.
i.e. $\underline{\nabla} \wedge \underline{u} = 0$ (in 2D and 3D)
– the second restriction on $\underline{u}$.

Since curl $\underline{u}$ vanishes $\exists\, \phi$ s.t.

$\underline{u} = \underline{\nabla} \phi$ (did not need 2D).

ϕ is called the velocity potential.

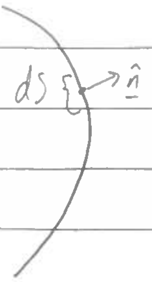
Then for incompressible flow, $\underline{\nabla} \cdot \underline{u} = 0$

so $\underline{\nabla} \cdot (\underline{\nabla} \phi) = 0$

ie. $\nabla^2 \phi = 0$, ie. Laplace's equation

[Cartesians: $\underline{\nabla} \phi = \frac{\partial \phi}{\partial x} \hat{\underline{x}} + \frac{\partial \phi}{\partial y} \hat{\underline{y}} + \frac{\partial \phi}{\partial z} \hat{\underline{z}}$

$$\underline{\nabla} \cdot (\underline{\nabla} \phi) = \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2}$$]



On a solid boundary $\underline{u} \cdot \hat{\underline{n}} = 0$

Flux through dS is $(\underline{u} \cdot \hat{\underline{n}})\, dS$.

Vanishes $\forall\, dS$ at a solid boundary so

$\underline{u} \cdot \hat{\underline{n}} = 0$ on a solid boundary

Substitute for ϕ,

$\hat{\underline{n}} \cdot \underline{\nabla} \phi = 0$ on a solid boundary

Rate of change of ϕ in $\hat{\underline{n}}$ direction is zero

ie. $\frac{\partial \phi}{\partial n} = 0$

– the normal derivative of ϕ vanishes on a solid boundary.

– complete problem for ϕ.

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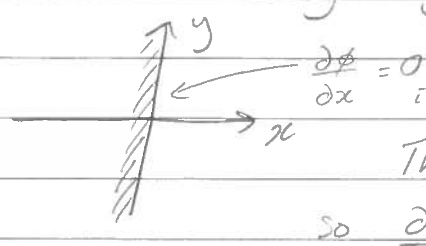
Irrotational $(\underline{\nabla} \wedge \underline{u} = 0) \Rightarrow \exists \phi$ s.t. $\underline{u} = \underline{\nabla}\phi$

plus incompressible $(\underline{\nabla} \cdot \underline{u} = 0) \Rightarrow \nabla^2 \phi = 0$

Boundary condition on solid boundary is $\dfrac{\partial \phi}{\partial n} = 0$

Examples

① solid boundary: y-axis (ie line $x = 0$ or plane $x = 0$ in 3D)



Then $\hat{\underline{n}} = \hat{\underline{x}}$

so $\dfrac{\partial \phi}{\partial n} = \dfrac{\partial \phi}{\partial x}$

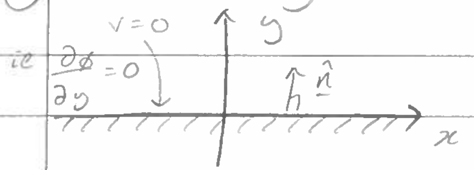
In cartesians,

$$\underline{\nabla}\phi = \frac{\partial \phi}{\partial x}\hat{\underline{x}} + \frac{\partial \phi}{\partial y}\hat{\underline{y}} + \frac{\partial \phi}{\partial z}\hat{\underline{z}}$$

and $\underline{u} = u\hat{\underline{x}} + v\hat{\underline{y}} + w\hat{\underline{z}}$

so $u = \dfrac{\partial \phi}{\partial x}$, $v = \dfrac{\partial \phi}{\partial y}$, $w = \dfrac{\partial \phi}{\partial z}$

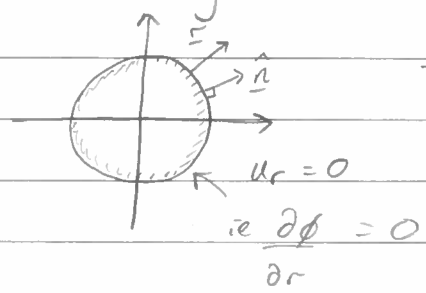
② Solid body: x-axis (plane $y = 0$ in 3D)



$\hat{\underline{n}} = \hat{\underline{y}}$

ie $\dfrac{\partial \phi}{\partial n} = \dfrac{\partial \phi}{\partial y}$

③ Solid body: circle radius a : $x^2 + y^2 = a^2$



$\hat{\underline{n}} = \hat{\underline{r}}$

$\dfrac{\partial \phi}{\partial n} = \dfrac{\partial \phi}{\partial r}$

streamfunction:

$$\underline{\nabla} \cdot \underline{u} = 0 \quad \text{(incompressible)}$$

Plus 2D: $\exists \psi$ s.t. $\underline{u} = -\hat{\underline{z}} \wedge \underline{\nabla} \psi$

Now add in irrotational, we have

$$\underline{\nabla} \wedge (\hat{\underline{z}} \wedge \nabla \psi) = 0$$

In Cartesians:

$$\underline{u} = \frac{\partial \psi}{\partial y} \hat{\underline{x}} - \frac{\partial \psi}{\partial x} \hat{\underline{y}} \qquad u = \frac{\partial \psi}{\partial y}, \; v = -\frac{\partial \psi}{\partial x}$$

$$\zeta = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = -\frac{\partial^2 \psi}{\partial x^2} - \frac{\partial^2 \psi}{\partial y^2}$$

$$= -\nabla^2 \psi$$

Vorticity is minus Laplacian of ψ

Irrotational flow: $\zeta \equiv 0$ everywhere for all time, i.e. $\nabla^2 \psi = 0$

Boundary condition on ψ, ψ = constant on a solid boundary or $\psi = 0$ if there is a single boundary.

To exist: ϕ : irrotational (if also incompressible $\nabla^2 \phi = 0$ in 3D)
ψ : incompressible + 2D.

They both exist for incompressible, irrotational, 2D flow, which we will concentrate on. Thus, in general, we will have both ϕ and ψ.
We can solve any problem using either.

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When they both exist, are they related?

Yes: $\underline{u} = \underline{\nabla}\phi$
and $\underline{u} = -\hat{\underline{z}} \wedge \underline{\nabla}\psi$

so $\boxed{\underline{\nabla}\phi = -\hat{\underline{z}} \wedge \underline{\nabla}\psi}$

i.e.

$$\begin{cases} u = \dfrac{\partial\psi}{\partial y} & \text{but } u = \dfrac{\partial\phi}{\partial x} \\ v = -\dfrac{\partial\psi}{\partial x} & \text{but } v = \dfrac{\partial\phi}{\partial y} \end{cases}$$

Thus $\boxed{\begin{aligned} \frac{\partial\phi}{\partial x} &= \frac{\partial\psi}{\partial y} \\ \frac{\partial\phi}{\partial y} &= -\frac{\partial\psi}{\partial x} \end{aligned}}$ $\swarrow$ Cauchy-Riemann equations (MATH 2101)

Thus ϕ is the real part and ψ the imaginary part of a holomorphic (i.e. differentiable) complex function $w(z)$ (say) of the complex variable $z = x + iy$.

We call $w(z)$: the complex velocity potential.
i.e. $w(z,t) = \phi(x,y,t) + i\,\psi(x,y,t)$

Note: we have shown that both the real and imaginary parts of a holomorphic function satisfy Laplace's equation.

$\Rightarrow \quad -\hat{\underline{z}} \wedge \underline{\nabla}\psi = \underline{\nabla}\phi \longleftrightarrow$ Cauchy Riemann equation coordinate free!

Equally any holomorphic function gives an incompressible, irrotational, 2D flow.

Examples

(1) $w = C$, a constant.

Notice $\frac{\partial w}{\partial z} = \frac{\partial w}{\partial x}$

$= \frac{\partial \phi}{\partial x} + i\frac{\partial \psi}{\partial x}$

$= u - iv$ $\quad u = \mathrm{Re}\left(\frac{\partial w}{\partial z}\right)$

$\quad v = -\mathrm{Im}\left(\frac{\partial w}{\partial z}\right)$

Here $\frac{\partial w}{\partial z} = 0$ i.e. $u = 0$, $v = 0$ i.e. no flow.

(2) $w = Uz$ $\quad U$ real

Then $\frac{\partial w}{\partial z} = U$

so $u = U$, $v = 0$

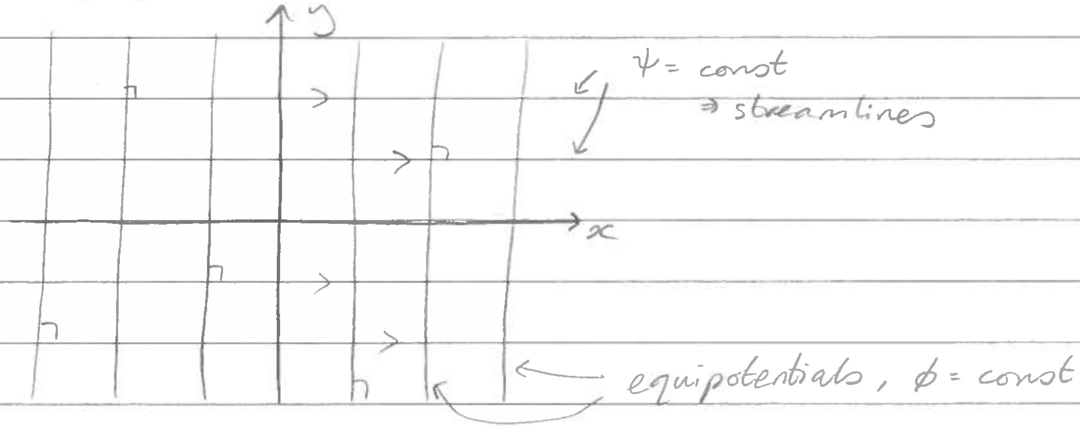
i.e. a uniform stream in the x direction

or: $w = Uz = Ux + iUy$

so $\phi = Ux$, $\psi = Uy$

Streamlines: Lines Uy = constant i.e. y = constant

Equipotentials: Lines Ux = constant i.e. x = constant.



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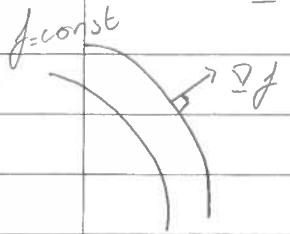
$$-\hat{\underline{z}} \wedge \underline{\nabla}\psi = \underline{\nabla}\phi$$

$\Rightarrow \underline{\nabla}\phi \perp$ to $\underline{\nabla}\psi$ (except when both vanish)

i.e. except when $\underline{u} = 0$

i.e. stagnation points.



Thus we have proved that the isolines of th real and imaginary parts of a holomorphic function intersect at right angles except at points where the derivative vanishes.

Examples

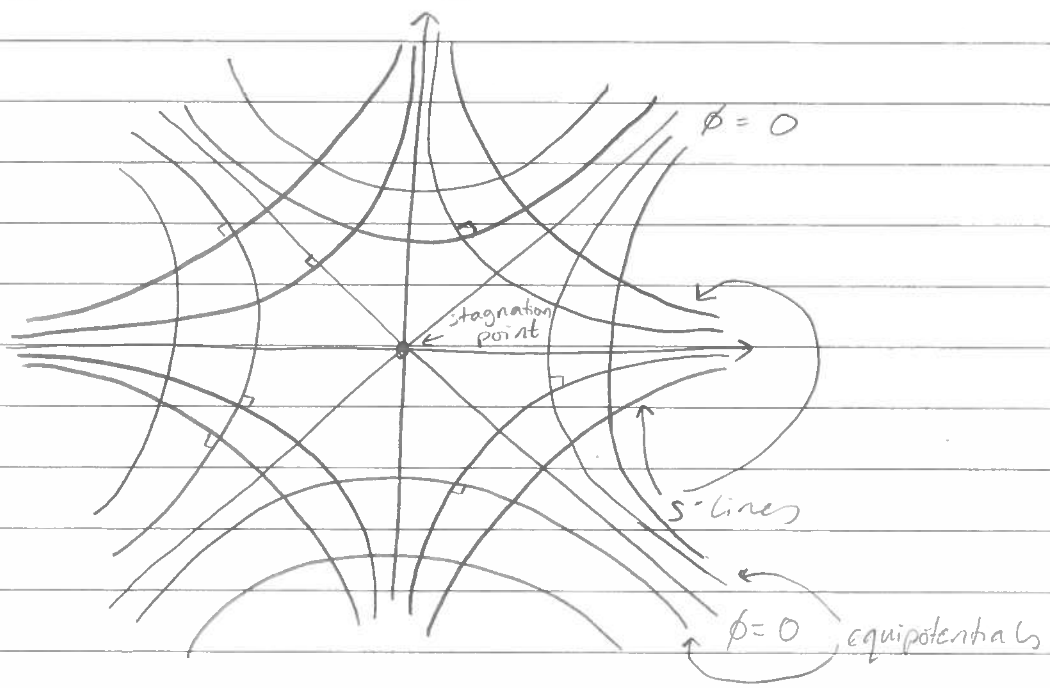
(3) $w = z^2$

$$= (x+iy)^2 = (x^2-y^2) + i(2xy)$$

so $\phi = x^2 - y^2$, $\psi = 2xy$

streamlines: $xy = $ const.

equipotentials: $x^2 - y^2 = $ const.



easiest way to find stagnation points: look at $\frac{\partial w}{\partial z}$

(4) $w = z^3$

Simpler to introduce polar coordinates

$$z = re^{i\theta}$$
$$= r(\cos\theta + i\sin\theta)$$

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Incompressible $\nabla\cdot\underline{u} = 0$

& 2D $\Rightarrow$ s'fn, ie. $\exists\,\psi$ st. $\underline{u} = -\hat{\underline{z}}\wedge\underline{\nabla}\psi$

(plus irrot.) $\Rightarrow \nabla^2\psi = 0$

Irrotational $\underline{\nabla}\wedge\underline{u} = 0$

$\Rightarrow \exists\,\phi$ s.t. $\underline{u} = \underline{\nabla}\phi$ (velocity potential)

plus incomp. $\underline{\nabla}\cdot(\underline{\nabla}\phi) = 0 \quad \Rightarrow \nabla^2\phi = 0$

& 2D : $\nabla^2\phi = 0$

2D, incomp, irrot

$\nabla^2\psi = 0$

$\nabla^2\phi = 0$

$\underline{\nabla}\phi = -\hat{\underline{z}}\wedge\underline{\nabla}\psi$ Cauchy-Riemann equations.

$\exists\, w(z)$, w holomorphic function

complex $z = x+iy = re^{i\theta}$

$w=$ const : $\dfrac{\partial w}{\partial z} = 0$: no flow

$w = Uz$: $\dfrac{\partial w}{\partial z} = U = u+iv$: uniform stream, x-direction.

$w = z^2$: $\dfrac{\partial w}{\partial z} = 2z$: stagnation point flow.

$w = z^3$ introduce polars, $z = re^{i\theta}$

$z^3 = r^3 e^{3i\theta}$

$= r^3(\cos 3\theta + i\sin 3\theta)$

$w = \phi + i\psi$ so $\phi = r^3\cos 3\theta$

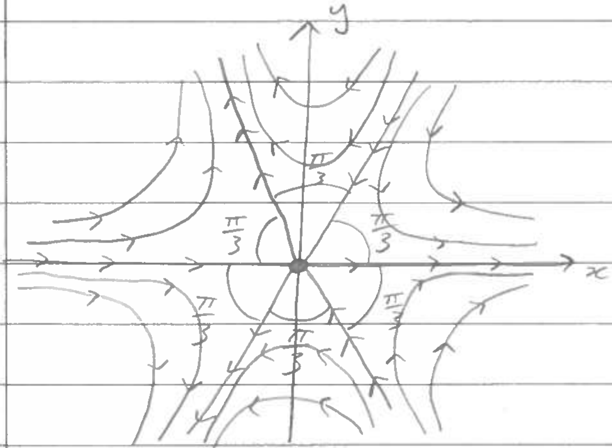
$\psi = r^3\sin 3\theta$

s'lines : $r^3\sin 3\theta = $ const.

Consider first : $\psi = 0$ then $r^3\sin 3\theta = 0$

Then $\sin 3\theta = 0$ if $r \neq 0$

so $\theta = \pm\frac{\pi}{3}, \pm\frac{2\pi}{3}, \pm\frac{3\pi}{3}, 0$

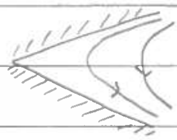


$u - iv = \frac{\partial w}{\partial z} = 3z^2$

On $y=0$, $u - iv = 3x^2$

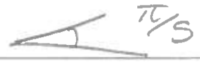
i.e. $u = 0$, $u \geq 0$.

stagnation point flow.

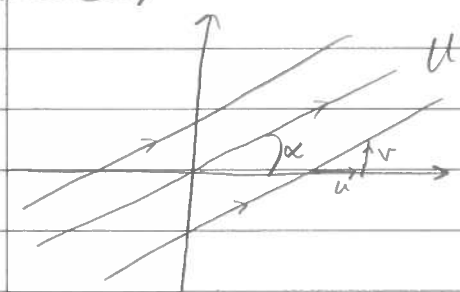
equivalent to  flow in a wedge of angle $\frac{\pi}{3}$

Rotate image by $\pi/6$ to see equipotentials due to sine and cosine periodicity.

$w = z^4$ 

$w = z^5$ 

Example 5



Uniform flow at angle α to the x-axis.

Thus $u + iv = Ue^{i\alpha}$ ← rate of change of position

So $u - iv = Ue^{-i\alpha}$ (conjugate)

← holomorphic function

Thus $w = Ue^{-i\alpha} z$.

This is in fact general,

$w_z = F(ze^{-i\alpha})$

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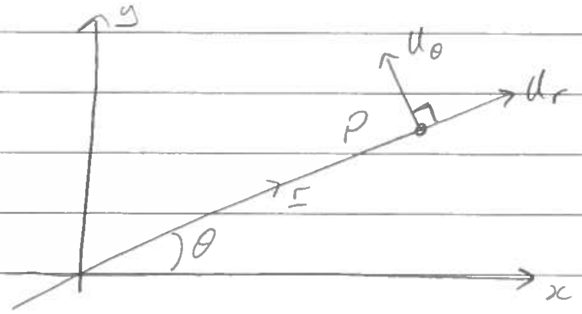
is the same flow as

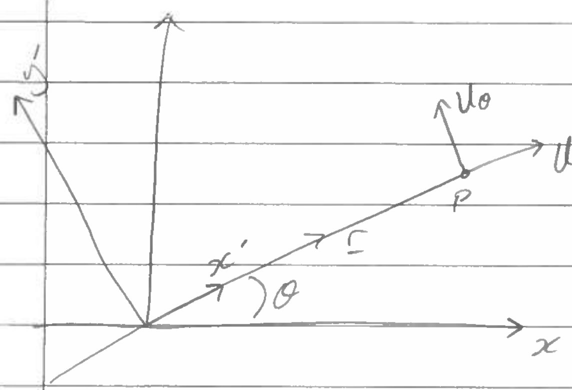
$$w_1 = F(z)$$

but rotated by α in the anti-clockwise direction.

The polar velocity components (u_r, u_θ) form the complex potential.



Notice if we introduce Cartesian axes z' - z-axis rotated by θ



$$u' - iv' = \frac{dw}{dz'}$$

$$u_r - iu_\theta = \frac{dw}{dz} \Big/ \frac{dz'}{dz}$$

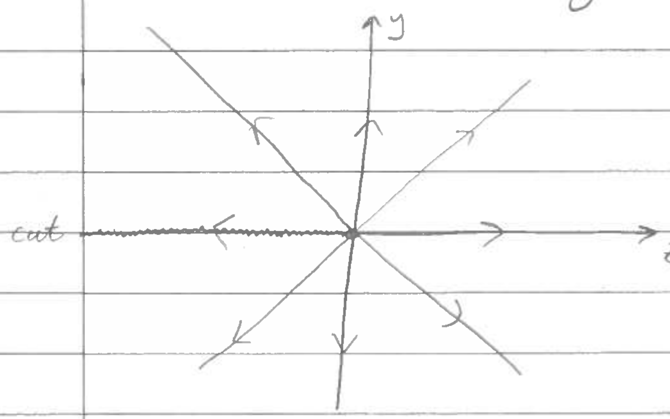
$$z' = ze^{-i\theta}$$

$$\frac{dz'}{dz} = e^{-i\theta}$$

So $u_r - iu_\theta = e^{i\theta} \dfrac{dw}{dz}$

Example

Consider $w = m\log z$



$$\begin{aligned} w &= m\log z \\ &= m\log(re^{i\theta}) \\ &= m\log r + m\log e^{i\theta} \\ &= m\log r + im\theta \end{aligned}$$

$$\Rightarrow \begin{cases} \phi = m\log r \\ \psi = m\theta \end{cases}$$

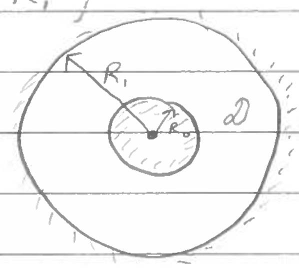
$\frac{dw}{dz} = \frac{m}{z}$ defined everywhere except $z=0$ where it is singular.

Then $u_r - iu_\theta = e^{i\theta} \cdot \frac{m}{re^{i\theta}} = \frac{m}{r}$

So $\begin{cases} u_r = \frac{m}{r} \\ u_\theta = 0 \end{cases}$ — An isotropic source of strength $2\pi m$.

Laurent Series

A function that is holomorphic within an annular (ring) region ($R_0 < |z| < R_1$) has a unique expansion of the form



$$f(z) = \sum_{n=-\infty}^{+\infty} a_n z^n$$

$$= \dots + a_{-2}z^{-2} + a_{-1}z^{-1} + a_0 + a_1 z + a_2 z^2 + \dots$$

where the coefficients a_n are usually complex.

We apply this to $u - iv$ so

$$u - iv = \sum_{n=-\infty}^{+\infty} a_n z^n$$

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Integrating gives

$$w = \ldots + b_{-2} z^{-2} + b_{-1} z^{-1} + b_0 \log z + b_{00} + b_1 z + b_2 z^2 + \ldots$$

Thus all answers for incompressible, 2D, irrotational flow are of this form.

Real and imaginary parts of w satisfy Laplace's eqn, i.e. they are drawn from the set
$\{1, r^{\pm n}\cos n\theta, r^{\pm n}\sin n\theta, \log r, \theta\}$

- thus all solutions are linear combinations of these functions.

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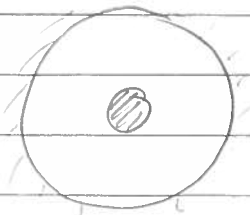
Laurent:



Any function (holomorphic) in an annulus can be expressed as a linear combination of powers of z, i.e. functions drawn from the set $\{z^{\pm n}, n = 0, 1, 2, \dots\}$ (roots can be complex)

Apply to $u + iv$.

Thus w can be expressed as a linear combination of functions drawn from the set $\{\log z, z^{\pm n}, n = 0, 1, 2, 3 \dots\}$

The stream function and velocity potential, ψ and ϕ, are $Im(w)$ and $Re(w)$ respectively and so are linear combinations of $\{\log r, \theta, r^{\pm n}\cos(n\theta), r^{\pm n}\sin(n\theta), n = 0, 1, 2, \dots\}$

We are guaranteed that all solns of Laplace's eqn in annulus are real linear combinations of functions from this set.

Example

Uniform flow at speed U about a cylinder of radius a.

1). stream function ψ

2). velocity potential ϕ

3). complex potential $w = \phi + i\psi$

1). stream function.

(i) governing equation: irrotational, incompressible, 2D:

Laplace's eqn:

$$\nabla^2\psi = 0$$ ← homogeneous, $\psi = 0$ is a soln

(ii) boundary condition on cylinder $(r=a)$

$\underline{u}\cdot\hat{\underline{n}} = 0$ ie. ψ = constant.

Only one body so w.l.o.g. take $\psi = 0$ on $r=a$.

↖ homogeneous $\psi = 0$ is soln

(iii) far-field boundary condition

as $r \to \infty$, flow becomes uniform, speed U.

Take this to be x-direction.

Then $\underline{u} \to U\hat{x}$

ie. $\left.\begin{array}{l} u \to U \\ v \to 0 \end{array}\right\}$ as $r \to \infty$

Note this says $\left\{\begin{array}{l} u = \dfrac{\partial\psi}{\partial y} = U \\ v = -\dfrac{\partial\psi}{\partial x} = 0 \end{array}\right.$ as $r \to \infty$

Thus as $r \to \infty$, $\psi = f(y)$

and $\psi_y = f'(y) = U$

So $\psi = Uy$ as $r \to \infty$

In polars this is $\psi \to Ur\sin\theta$ as $r \to \infty$

↖ inhomogeneous, $\psi = 0$ not a soln

Thus try $\psi = Ur\sin\theta + Br^{-1}\sin\theta$

↖ must cancel $\forall\theta$ on $r=a$.

This satisfies the far-fied and Laplace.

Chose B so it satisfies conditions on cylinder

ie. $0 = Ua\sin\theta + \dfrac{B}{a}\sin\theta \quad \forall\theta$.

For this to be true

$$Ua + \frac{B}{a} = 0 \Rightarrow B = -Ua^2$$

$$\begin{aligned} \text{So } \psi &= Ur\sin\theta - \frac{Ua^2}{r}\sin\theta \\ &= Ur\sin\theta\left[1 - a^2/r^2\right] \\ &= Uy\left(1 - a^2/r^2\right) \end{aligned}$$

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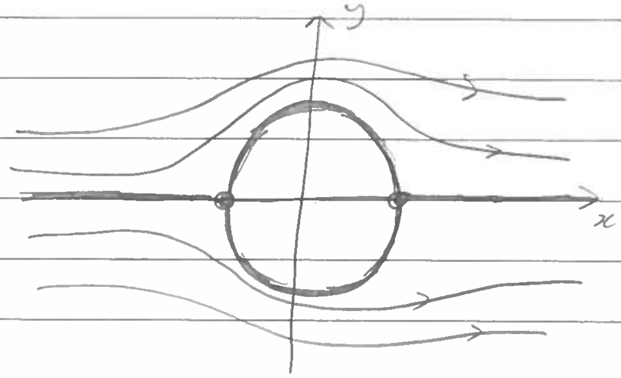
Note as $r \to \infty$, $\psi \to Uy$,
on $r = a$, $\psi = 0$.

Draw s'lines:
Easiest is $\psi = 0$

$$Uy(1 - a^2/r^2) = 0$$

i.e. $y = 0$ or $r = a$



2). Velocity potential

(i) Governing eqn: Invisid, irrotational, incompressible

$\underline{u} = \nabla \phi$

$\nabla \cdot \underline{u} = 0$ i.e. $\boxed{\nabla^2 \phi = 0}$ (Laplace)

homogeneous, $\psi = 0$ is soln

(ii) Boundary condition on $r = a$

$\underline{u} \cdot \hat{\underline{n}} = 0$

$\hat{\underline{n}} \cdot \nabla \phi = 0$

i.e. $\dfrac{\partial \phi}{\partial n} = 0$ on $r = a$

i.e. $\boxed{\dfrac{\partial \phi}{\partial r} = 0 \text{ on } r = a}$ homogeneous, $\psi = 0$ is soln

(iii) Far-field

$\underline{u} \to U\hat{\underline{x}}$

i.e. $u \to U$, $v \to 0$ as $r \to \infty$

But $u = \dfrac{\partial \phi}{\partial x}$, $v = \dfrac{\partial \phi}{\partial y}$

So as $r \to \infty$, $\phi = g(x)$

$$\frac{\partial \phi}{\partial x} = g'(x) = u = U$$

So $\phi \to Ux$ as $r \to \infty$

In polars this is $\phi \to Ur\cos\theta$ as $r \to \infty$

← inhomogeneous

Thus try

$$\phi = Ur\cos\theta + Ar^{-1}\cos\theta$$

(satisfies Laplace, satisfies far-field as extra term vanishes as $r \to \infty$ $\forall A$).

Remains to satisfy b.c. on $r = a$.

Thus on $r = a$,

$$\frac{\partial \phi}{\partial r} = 0 \quad \text{for all } \theta$$

But $\frac{\partial \phi}{\partial r} = U\cos\theta - \frac{A}{r^2}\cos\theta$ at any r

$$= \cos\theta\left[U - \frac{A}{a^2}\right] \quad \text{on } r = a$$

For this to $= 0$ we need $A = a^2 U$.

Thus $\phi = Ur\cos\theta + \frac{Ua^2}{r}\cos\theta$

$$= Ur\cos\theta\left[1 + \frac{a^2}{r^2}\right]$$

$$= Ux\left(1 + a^2/r^2\right)$$

3). Complex potential

$$w = \phi + i\psi$$
$$= Ux(1 + a^2/r^2) + iUy(1 - a^2/r^2)$$
$$= U(x + iy) + \frac{Ua^2}{r^2}(x - iy)$$
$$= Uz + \frac{Ua^2}{\bar{z}}$$

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$$= U\left(z + \frac{a^2}{z}\right)$$

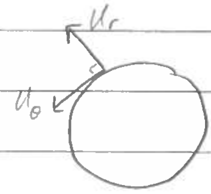
— holomorphic in $|z| > a$

$$\left[\ \frac{1}{z} = \frac{1}{r} e^{-i\theta} = \frac{1}{r}[\cos\theta - i\sin\theta] = \frac{1}{r^2}(x - iy)\ \right]$$

Properties of the solution

We have

$$\psi = U_r \sin\theta \left(1 - \frac{a^2}{r^2}\right)$$



Now $U_r = \dfrac{1}{r} \dfrac{\partial\psi}{\partial\theta}$

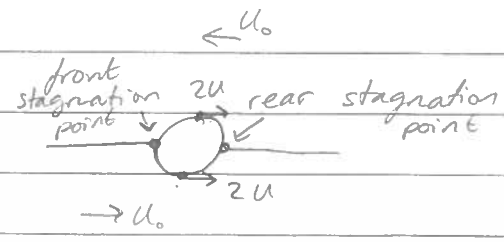
$$U_\theta = -\frac{\partial\psi}{\partial r}$$

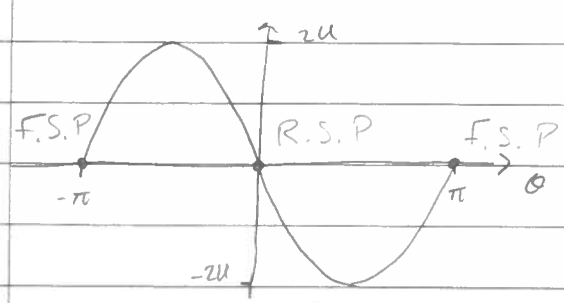
Thus $U_r = U\cos\theta(1 - a^2/r^2)$

Note $U_r = 0$ on $r = a$ (as constructed)

$$U_\theta = -U\sin\theta - \frac{Ua^2}{r^2}\sin\theta$$

on $r = a$: $U_\theta = -2U\sin\theta$





$U_\theta = 0$ at $\theta = 0, \pm\pi$

R.S.P F.S.P

Example
Cylinder in a strain-field.

Consider a cylinder of radius a in the velocity field $\left.\begin{array}{l} u - ky \to 0 \\ v - kx \to 0 \end{array}\right\}$ as $r \to \infty$

far-field streamfunction

$$\frac{\partial \psi}{\partial x} = -v = -kx$$

$$\frac{\partial \psi}{\partial y} = u = ky$$

Thus $\psi = -\frac{1}{2}kx^2 + f(y)$

so $\psi_y = f'(y) = ky$

So $f(y) = \frac{1}{2}ky^2$

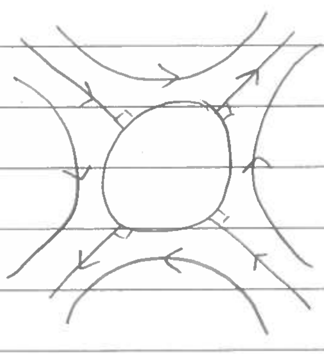
This gives $\psi \to \frac{1}{2}k(y^2 - x^2)$ as $r \to \infty$

$$= \frac{1}{2}k(r^2\sin^2\theta - r^2\cos^2\theta)$$

$$= -\frac{1}{2}kr^2\cos 2\theta$$

[We will also need $\frac{c}{r^2}\cos 2\theta$]

Full soln. in notes



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§2.3.1 Streamlines at a stagnation point

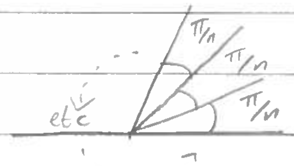
Suppose we have a flow with a stagnation point.
w.l.o.g. take this point to be $z=0$.
Also the complex velocity potential can be written
$$w(z) = a_0 + a_1 z + a_2 z^2 + a_3 z^3 + \ldots$$

w.l.o.g. can take $a_0 = 0$ (constant does not affect the flow).

But $\dfrac{dw}{dz} = 0$ at $z=0$ (a stagnation point), so $a_1 = 0$.

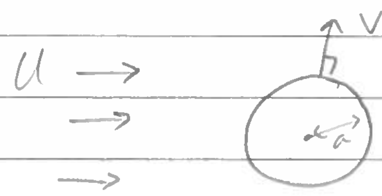
Let the first non-zero term be a_n (so $n \geq 2$).
Thus sufficiently close to $z=0$, $w \approx a_n z^n$
so $a_n = \rho e^{i\alpha}$.

All a_n does is magnify and rotate the s'lines.
So consider $a_n = 1$,
$\Rightarrow w = z^n$



So $\psi = r^n \sin(n\theta)$
$= 0$ when $\theta = \pi/n$

H/w hint



i) $\nabla^2 \phi = 0$ homo

ii) $\phi \to Ux$ as $r \to \infty$ inhomo

iii) $U_r = \dfrac{\partial \phi}{\partial r} = V$ on $r = a$ inhomogeneous

Could solve $\begin{cases} \text{Prob A: } U_r = 0 \text{ on } r=a \text{ (notes)} \\ \text{Prob B: } \phi \to 0 \text{ at } \infty \quad \dfrac{\partial\phi}{\partial r} = V \text{ on } r=a \end{cases}$ then A + B

§2.3.2 Uniform flow (at speed U) about a cylinder of radius a with circulation κ

Remember we discuss a point (or line) vortex where
$\psi = -\kappa \log r$ [source $w = m\log z$
$= m\log r + im\theta = \phi + i\psi$]

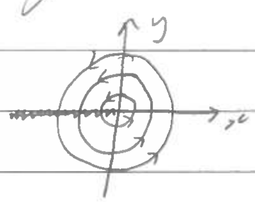
[Source strength $2\pi m$
$\psi = m\theta = Im[m\log z]$
$\phi = Re[m\log z] = m\log r$]

Consider $\psi = -\kappa \log r$
$= Im[-i\kappa \log r]$
$= Im[-i\kappa \log z]$

i.e. The corresponding w is
$w = -i\kappa \log z$
So $\psi = -\kappa \log r$ and $\phi = Re\{w\}$
$= (-i\kappa)(i\theta) = \kappa\theta$

Streamlines: lines ψ = constant i.e. r = const,
$\underline{u} = \underline{\nabla}\phi$
$= \frac{\kappa}{r}\underline{\hat{\theta}}$

This flow is a vortex (in 2D: point vortex,
in 3D: line vortex)



Note w is holomorphic in the cut plane.

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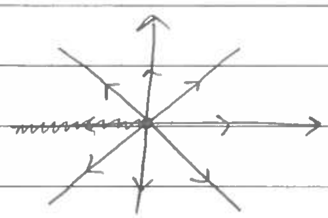
These are 2 important, fundamental solutions.

Source (strength $2\pi m$)

$w = m \log z$

$\phi = m \log r$

$\psi = m\theta$



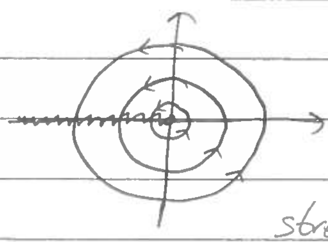
the strength is the flux of fluid across any circuit containing O.

Vortex (strength $2\pi\kappa$)

$w = -i\kappa \log z$

$\phi = \kappa\theta$

$\psi = -\kappa \log r$



strength

- any circuit around origin.

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Line source: $w = \frac{m}{2\pi} \log z$ m real, strength.

$\phi = \frac{m}{2\pi} \log r \qquad \psi = \frac{m}{2\pi} \theta$

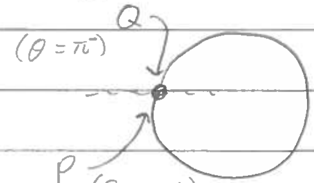
Strength (in 2D) flux / unit width

$\oint \underline{u} \cdot \hat{n} \, dl$

for any curve enclosing the source.

Obtain m.

$\underline{u} = -\hat{\underline{z}} \wedge \nabla \psi$

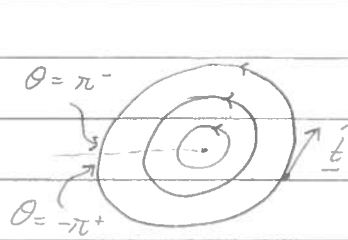


$\psi(Q) - \psi(P) = \frac{m}{2\pi}(\pi - (-\pi)) = m$

Line vortex

$w = \frac{-i\kappa}{2\pi} \log z$ κ real, strength

$\phi = \frac{\kappa}{2\pi} \theta \qquad \psi = -\frac{\kappa}{2\pi} \log r$



Strength: circulation

capital gamma

$\Gamma = \oint (\underline{u} \cdot \hat{\underline{t}}) \, dl = \oint \underline{u} \cdot d\underline{r}$

for any curve enclosing the vortex.

But for irrotational flow, $\underline{u} = \underline{\nabla} \phi$

so $\Gamma = \oint \underline{\nabla} \phi \cdot d\underline{r} = \phi(Q) - \phi(P) = \kappa$

No rotation anywhere except for at the origin.

$\{\theta,\ \log r,\ r^{\pm n}\cos n\theta,\ r^{\pm n}\sin n\theta\}$

not single valued without a jump

single valued fn of θ

$-\frac{i\kappa}{2\pi}\log z$: Only flow with circulation has $\phi = \frac{\kappa\theta}{2\pi}$ $\left[\psi = -\frac{\kappa}{2\pi}\log r\right]$

$\frac{m}{2\pi}\log z$: Only flow with mass flux has $\psi = \frac{m}{2\pi}\theta$ $\left[\phi = \frac{m}{2\pi}\log r\right]$

Example

Find the flow past a cylinder of radius a in a uniform stream of speed U where the cylinder is spinning so that the circulation about the cylinder is κ.

'top spin'

With no spin ($\kappa = 0$) the complex velocity potential is $w_1 = U\left(z + \frac{a^2}{z}\right)$

The one and only flow with circulation κ is

$w_2 = \frac{-i\kappa}{2\pi}\log z$, our line vortex.

Thus try

$$w = w_1 + w_2$$
$$= U\left(z + \frac{a^2}{z}\right) - \frac{i\kappa}{2\pi}\log z$$

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Check: $\dfrac{dw}{dz} = U\left(1 - \dfrac{a^2}{z^2}\right) - \dfrac{i\kappa}{2\pi z}$

$\longrightarrow \underset{= u + iv}{U}$ as $|z| \to \infty$

so $u \to U$ and $v \to 0$ as $|z| \to \infty$

ie. uniform flow at infinity as required.

Now $U_r - iU_\theta = e^{i\theta}\dfrac{dw}{dz}$ [on cylinder where $|z| = a$, $z = ae^{i\theta}$]

$$= e^{i\theta}\left[U\left(1 - \frac{a^2}{a^2 e^{2i\theta}}\right) - \frac{i\kappa}{2\pi a e^{i\theta}}\right]$$

$$= U\left(e^{i\theta} - e^{i\theta}\right) - \frac{i\kappa}{2\pi a}$$

$$= 2U i \sin\theta - \frac{i\kappa}{2\pi a}$$

Thus $U_r = 0$, $U_\theta = -2U\sin\theta + \dfrac{\kappa}{2\pi a}$

expected as no flow through $|z| = a$.

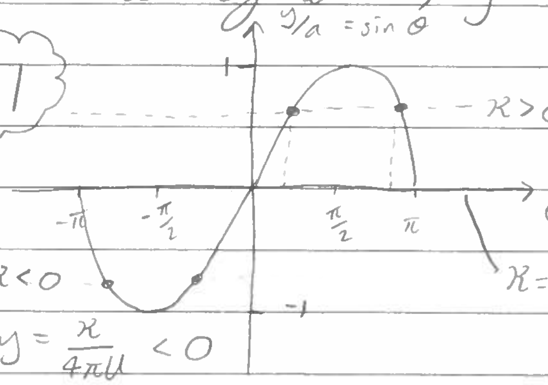
The stagnation points on the cylinder occur when $U_\theta = 0$ ie.

$$2U\sin\theta = \frac{\kappa}{2\pi a}$$

ie. $\sin\theta = \kappa / 4\pi U a$ ← non dimensional.

On the cylinder, $y = a\sin\theta$, so $\sin\theta = \dfrac{y}{a}$

$\boxed{\left|\dfrac{\kappa}{4\pi U a}\right| < 1}$



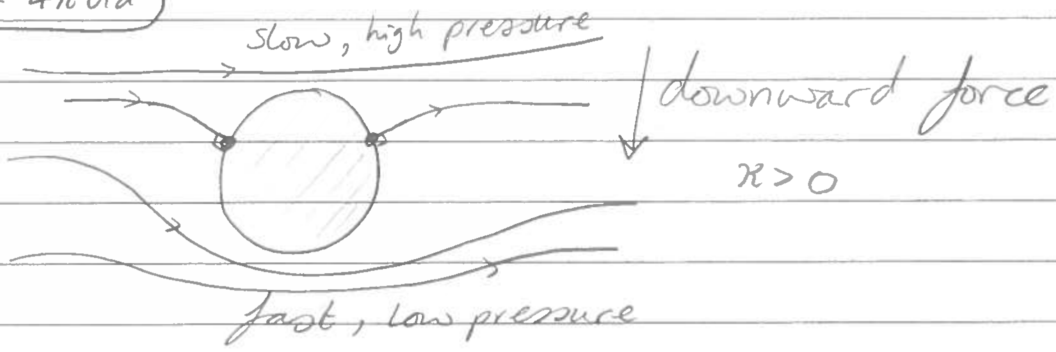
$\kappa > 0$: 2 roots between 0 and π, $y = \dfrac{\kappa}{4\pi U}$

$\kappa < 0$

$\kappa = 0$ → stagnation points are at $0, \pm\pi$ (on $y = 0$)

2 roots in $(-\pi, 0)$, $y = \dfrac{\kappa}{4\pi U} < 0$

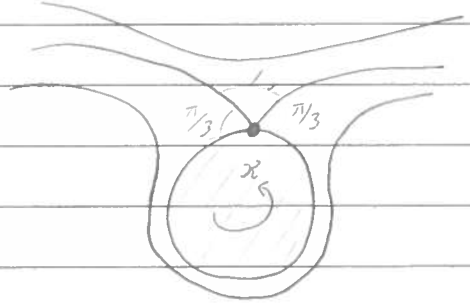
$\kappa < 4\pi U a$



Energy = K.E. + P.E. + pressure

large K.E. $\Rightarrow$ small pressure (to keep E constant

If $\kappa = 4\pi U a$, we get one repeated root $y = a$ $(x = 0)$. Stagnation points "collide" at $z = ia$

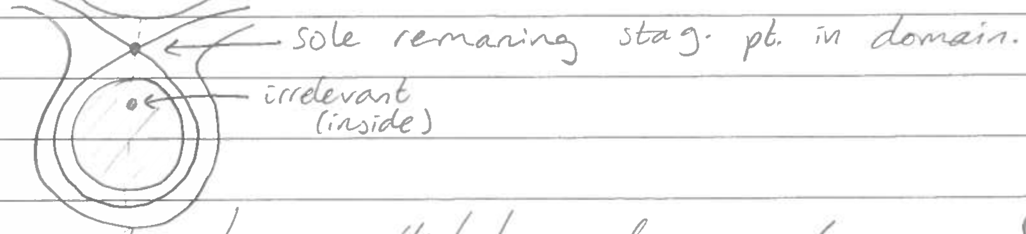


If $\kappa > 4\pi U a$ U_θ on cylinder never vanishes
$\therefore$ no stagnation points on cylinder.

$$\frac{dw}{dz} = U\left(1 - \frac{a^2}{z^2}\right) - \frac{i\kappa}{2\pi z} = 0$$

— always has 2 roots

Thus solve for z.



Stagnation points have collided and moved off the cylinder.

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Incompressible $\underline{\nabla} \cdot \underline{u} = 0$
Also 2D, $\exists \psi$ st. $\underline{u} = -\underline{\hat{z}} \wedge \underline{\nabla} \psi$ (stream function)

Inviscid : no sheer stress
particles preserved their velocity in 2D
In 2D & 3D irrotational motion persists.
Irrotational $\underline{\nabla} \wedge \underline{u} = 0$
$\Rightarrow \exists \phi$ st. $\underline{u} = \underline{\nabla} \phi$ (velocity potential)

2D, incomp, irrot : both ϕ, ψ related by Cauchy - Riemann equations.
$\exists\, \omega(z)$, $z = x + iy$ (differentiable)
$\omega = \phi + i\psi$
$\nabla^2 \phi = 0$, $\nabla^2 \psi = 0$
Solutions are linear combinations from the set
$\{ \log r, \theta, r^{\pm} \sin(n\theta), r^{\pm n} \cos(n\theta) \}$

2.4 Method of Images

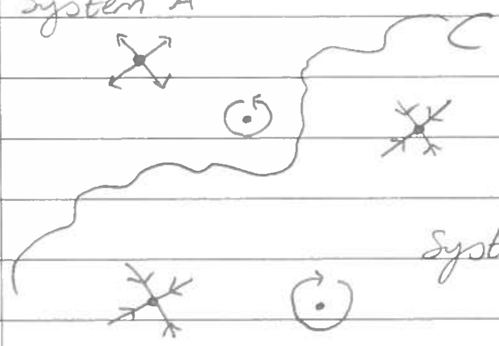
If the motion in the complex plane (actually works in 3D also) is due to a distribution of singularities

[eg. source at z_0 : $\frac{m}{2\pi} \log(z - z_0)$ (strength m)

vortex at z_1 : $\frac{-i\kappa}{2\pi} \log(z - z_1)$

higher order singularity: $(z - z_2)^{-n}$ $n \geq 1$]

and there exists a curve C in the plane with no flow across it (ie. C is a s'line) then the system of singularities on one side of C is called the IMAGE of the system on the other side.



System B is the image of system A.

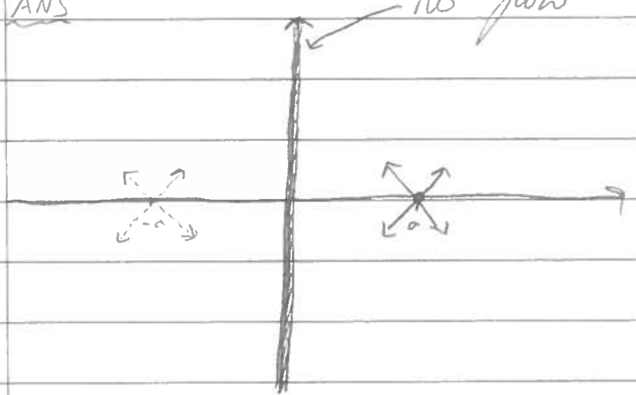
System A is the image of system B.

__Example__

Suppose we have a source of strength m at $z = a$ and a solid wall lying along $x = 0$. Find the flow field and the maximum velocity on the wall.

__ANS__

no flow across wall : make this curve C



System A: source, strength m, at $z = a$,
complex potential $\omega_1 = \frac{m}{2\pi} \log(z-a)$

System B: the image of system A in C is a source of strength m at $z = -a$,
complex potential $\omega_2 = \frac{m}{2\pi} \log(z+a)$

The flow field requires __BOTH__ contributions to satisfy the no flow condition, so it is $\omega = \omega_1 + \omega_2$

$$\Rightarrow \quad \omega = \frac{m}{2\pi} \log(z-a) + \frac{m}{2\pi} \log(z+a)$$

$$= \frac{m}{2\pi} \log\left[(z-a)(z+a)\right] = \frac{m}{2\pi} \log(z^2 - a^2)$$

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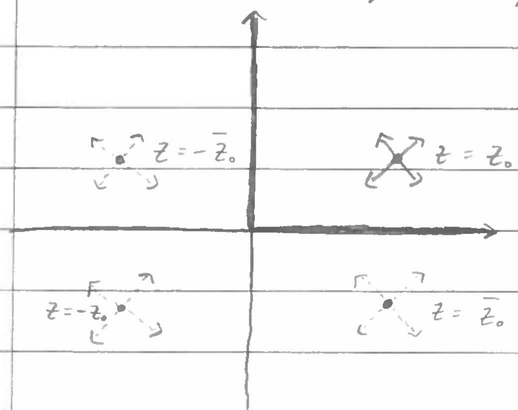
Now $u - iv = \frac{dw}{dz} = \frac{m}{2\pi} \cdot \frac{2z}{z^2 - a^2}$

On $z = iy$, $u - iv = \frac{m}{\pi} \cdot \frac{iy}{-(y^2 + a^2)}$

So $u = 0$ (as expected),
and $v = \frac{my}{\pi(y^2 + a^2)}$

This has maximum magnitude at $y = \pm a$ where
$v = \frac{\pm m}{2\pi a}$

Example 2
A source of strength m lies at $z = z_0$ in the first quadrant with walls $x = 0$, $y > 0$, and $y = 0$, $x > 0$.
What is the image system?
Find the complex potential for the flow.



System A : source at z_0
$w_1 = \frac{m}{2\pi} \log(z - z_0)$

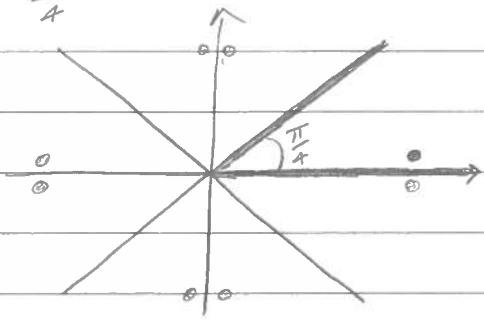
C drawn.

System B : 3 sources at $-z_0$, $\pm \bar{z}_0$
$w_2 = \frac{m}{2\pi} \log\left[(z + z_0)(z - \bar{z}_0)(z + \bar{z}_0)\right]$

Complex potential for both:
$w = w_1 + w_2 = \frac{m}{2\pi} \log\left[(z^2 - z_0^2)(z^2 - \bar{z}_0^2)\right]$

Example 3

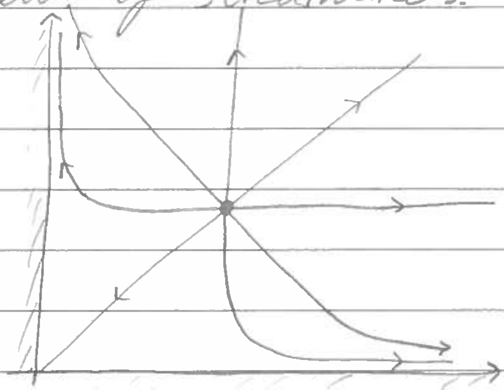
$\frac{\pi}{4}$ walls



Example 4

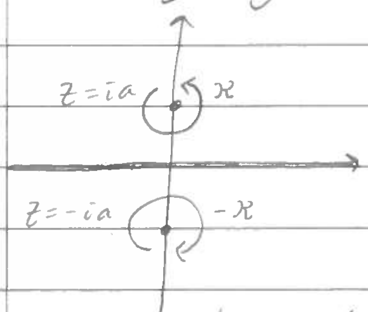
Walls at angle α (α must by $\frac{\pi}{n}$)

Sketch of streamlines.



Example 5

Find the complex potential due to a line vortex of strength κ at $z = ia$ above the planar boundary $y=0$.



System A: $w_1 = \frac{-i\kappa}{2\pi} \log(z - ia)$

System B: line vortex of strength $-\kappa$ at $z = -ia$, $w_2 = \frac{i\kappa}{2\pi} \log(z + ia)$

So complex potential: $w = w_1 + w_2 = \frac{-i\kappa}{2\pi} \log\left[\frac{z - ia}{z + ia}\right]$

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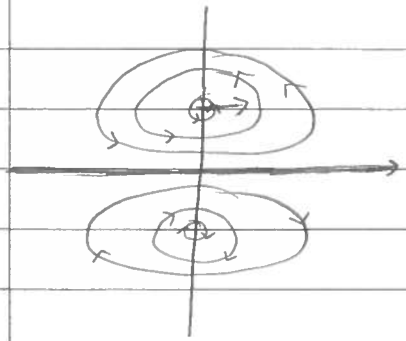
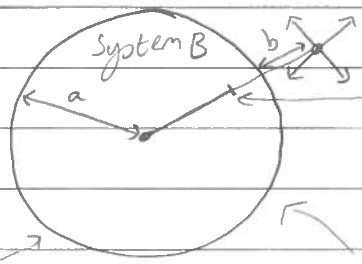


the image vortex has a velocity component at the real vortex

$v = 0, \quad u = \kappa / 4\pi a)$

2.5 Circle Theorem



optical range pt a^2/b.

System A : $f(z)$

C

We have $z\bar{z} = a^2$, $|z| = a$, on C, $\frac{a^2}{z} = \bar{z}$.

The image system in the circle $|z| = a$ of the complex potential $w_1(z) = f(z)$ where f has no singularities inside the circle (so f can be system A) is $w_2(z) = \bar{f}(\bar{z})$ where, for any function $g(z)$, $\bar{g}(z) = \overline{g(\bar{z})}$

Proof : Take z such that $|z| > a$

Then $|\bar{z}| > a$

So $\left|\frac{a^2}{\bar{z}}\right| < a$

? So $f\left(\frac{a^2}{\bar{z}}\right)$ is non singular (inside C)

Thus w_2 has no singularities outside C.

(on C) So $w(z) = f(z) + \bar{f}(\bar{z}) = f(z) + \overline{f(\bar{\bar{z}})}$

$= f(z) + \overline{f(z)} = 2\,\mathrm{Re}(f(z))$

$\Rightarrow w(z) = 2\,\mathrm{Re}(f(z)) \leftarrow$ real

$= \phi + i\psi$

So $\psi = 0$ (streamline)

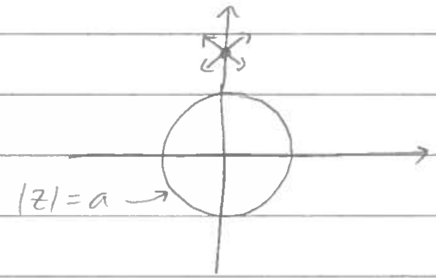
So C is a streamline

i.e. no flow across C

i.e. system B is the image of system A.

Example

What is the complex potential for a source of strength m at $z = ib$ outside the cylinder $|z| = a$ (where $a < b$)



System A: $w_1 = \frac{m}{2\pi}\log(z - ib)$

so $f(z) = \frac{m}{2\pi}\log(z - ib)$

$$\bar{f}(z) = \overline{f(\bar{z})}$$

i.e. $\bar{f}(z) = \overline{\frac{m}{2\pi}\log(\bar{z} - ib)}$

$$= \frac{m}{2\pi}\log\left(\overline{\bar{z} - ib}\right)$$

$$= \frac{m}{2\pi}\log(z + ib)$$

So $w_2(z) = \bar{f}\left(\frac{a^2}{z}\right) = \frac{m}{2\pi}\log\left(\frac{a^2}{z} + ib\right)$

Thus $w = w_1 + w_2$

$$= \frac{m}{2\pi}\log(z - ib) + \frac{m}{2\pi}\log\left(\frac{a^2}{z} + ib\right)$$

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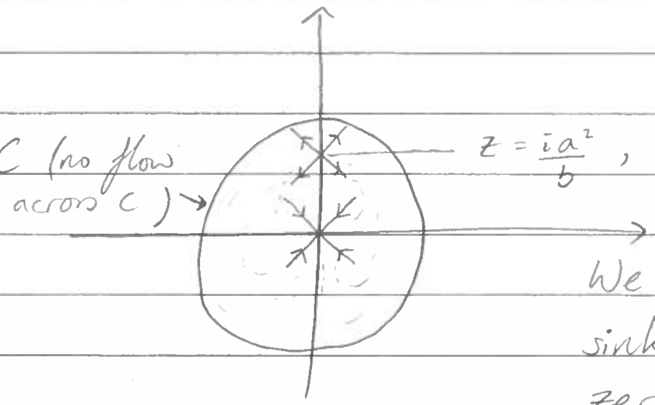
Let's examine the image more carefully:-

$$\log\left(\frac{a^2}{z} + ib\right) = \log\left[\frac{1}{z}\left(a^2 + ibz\right)\right] = \log\left[\frac{1}{z}\cdot ib\left(z + \frac{a^2}{ib}\right)\right]$$

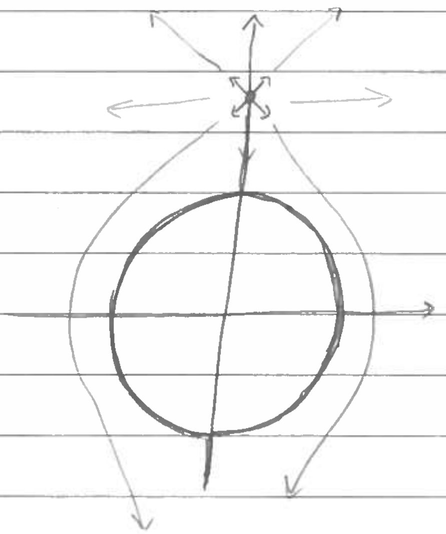
Thus $$\omega_z = \underbrace{-\frac{m}{2\pi}\log z}_{\text{sink of strength } m \text{ at origin}} + \underbrace{\frac{m}{2\pi}\log(ib)}_{\text{constant}} + \frac{m}{2\pi}\log\left(z - \frac{ia^2}{b}\right)$$

source of strength m at $z = \frac{ia^2}{b}$

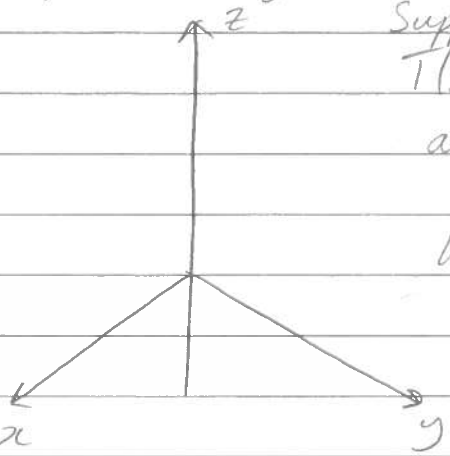


$z = \frac{ia^2}{b}$, the optical image point.

We require the total sum of sinks plus sources to be zero inside closed C.



Chapter 3: Dynamics



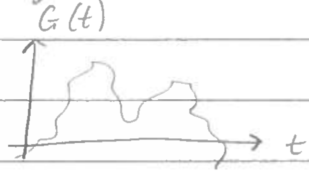
Suppose we know $T(x, y, z, t)$, the temperature at every point at every time.

What is the temperature of a fluid particle?

The particle follows a particle path: $\underline{r}(t) = (x(t), y(t), z(t))$.

At time t it has temperature $G(t) = T(x(t), y(t), z(t), t)$

The rate of change of the temperature of the particle is



$$\frac{\partial G}{\partial t} = \frac{\partial T}{\partial x}\frac{dx}{dt} + \frac{\partial T}{\partial y}\frac{dy}{dt} + \frac{\partial T}{\partial z}\frac{dz}{dt}$$

$$= \frac{\partial T}{\partial t} + u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} + w\frac{\partial T}{\partial z}$$

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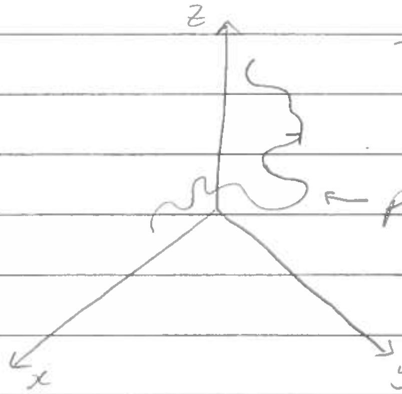
Forces: dynamics

Newton's Law

Force = time rate of change of momentum.



Euler

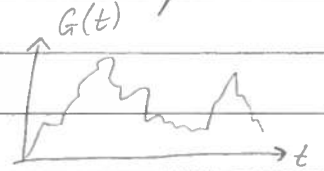
$*\quad \underline{u}(\underline{x},t)\quad ,\ T(\underline{x},t)$

← particle moves along a particle path

$$\frac{d\underline{x}}{dt} = \underline{u}$$

$$G(t) = T(\underline{x}(t),t)$$

On a particle path, $G(t) = T(\underline{x}(t),t)$



$$G'(t) = \frac{dG}{dt}$$

use chain rule

$$= \frac{\partial T}{\partial x}\frac{dx}{dt} + \frac{\partial T}{\partial y}\frac{dy}{dt} + \frac{\partial T}{\partial z}\frac{dz}{dt} + \frac{\partial T}{\partial t}$$

$$= \frac{\partial T}{\partial t} + u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} + w\frac{\partial T}{\partial z}$$

$$\underline{\nabla} T = \frac{\partial T}{\partial x}\underline{\hat{x}} + \frac{\partial T}{\partial y}\underline{\hat{y}} + \frac{\partial T}{\partial z}\underline{\hat{z}}$$

$$\underline{u} = u\underline{\hat{x}} + v\underline{\hat{y}} + w\underline{\hat{z}}$$

So $G'(t) = \dfrac{\partial T}{\partial t} + \underline{u}\cdot\underline{\nabla} T$

$$= \left(\frac{\partial}{\partial t} + \underline{u}\cdot\underline{\nabla}\right)T$$

$$= \left(\frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + w\frac{\partial}{\partial z}\right)T$$

$= \dfrac{DT}{Dt}$ time rate of change of T following the fluid, i.e. following a particle path.

This is called the material derivative / advective derivative / convective derivative.

Example 1

$\frac{D\underline{r}}{Dt}$ — rate of change of position following a particle.

$$= \left(\frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + w\frac{\partial}{\partial z}\right)(x\hat{\underline{x}} + y\hat{\underline{y}} + z\hat{\underline{z}})$$

$$= 0 + u\hat{\underline{x}} + v\hat{\underline{y}} + w\hat{\underline{z}}$$

note $\hat{\underline{x}}, \hat{\underline{y}}, \hat{\underline{z}}$ are constant unit vectors so differentiating them gives 0.

$= \underline{u}$ (velocity, as expected)

Example 2 (acceleration)

$$\frac{D\underline{u}}{Dt} = \frac{D}{Dt}(u\hat{\underline{x}} + v\hat{\underline{y}} + w\hat{\underline{z}})$$

$$= \frac{Du}{Dt}\hat{\underline{x}} + \frac{Dv}{Dt}\hat{\underline{y}} + \frac{Dw}{Dt}\hat{\underline{z}}$$

where $\frac{Du}{Dt} = \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}$

3.2 - Reynold's Transport Theorem

Leibniz: $I(t) = \int_{\alpha(t)}^{\beta(t)} f(x,t)\,dt$

1D →

$$\frac{dI}{dt} = \lim_{\delta t \to 0}\left[\frac{1}{\delta t}\left(I(t+\delta t) - I(t)\right)\right]$$

$$= \int_{\alpha(t)}^{\beta(t)} \frac{\partial f}{\partial t}(x,t)\,dt - f(\alpha(t),t)\alpha'(t) + f(\beta(t),t)\beta'(t)$$

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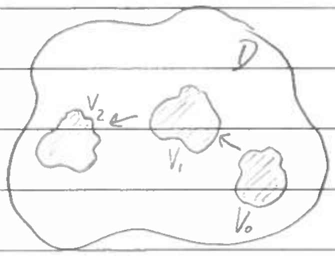
Suppose we have a fluid domain D, with velocity field $\underline{u}(\underline{x}, t)$ defined in D and some quantity $\alpha(\underline{x}, t)$ defined in D.

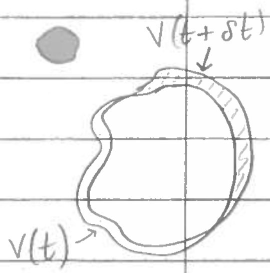
Take any volume $V(t)$, a subvolume of D, always composed of the same particles.



Let $I(t) = \int_{V(t)} \alpha(\underline{x}, t)\, dV$

Reynolds: This is a simple function of time.
What is it's rate of change?

$$\frac{dI}{dt} = \lim_{\delta t \to 0}\left[\frac{I(t+\delta t) - I(t)}{\delta t}\right]$$



$$= \lim_{\delta t \to 0}\left[\frac{\int_{V(t+\delta t)} \alpha(\underline{x}, t+\delta t)\, dV - \int_{V(t)} \alpha(\underline{x}, t)\, dV}{\delta t}\right]$$

Label the difference $V(t+\delta t) - V(t)$ by δV.

$0 < \delta t \ll 1$.

Expand $\alpha(\underline{x}, t+\delta t) = \alpha(\underline{x}, t) + \frac{\partial \alpha}{\partial t}(\underline{x}, t)\delta t + \frac{1}{2}(\delta t)^2 \frac{\partial^2 \alpha}{\partial t^2} + \ldots$

Then $I(t+\delta t) = \int_{V+\delta V}\left[\alpha + \frac{\partial \alpha}{\partial t}\delta t + O(\delta t)^2\right] dV$ taylor series

where $O(\delta t)^2$ means "behaves like $(\delta t)^2$."

So $I(t+\delta t) = \int_V \alpha \, dV + \delta t \int_V \frac{\partial \alpha}{\partial t} dV + \int_{\delta V} \alpha \, dV$

$$+ \delta t \int_{\delta V} \frac{\partial \alpha}{\partial t} dV + O((\delta t)^2)$$

Now $I(t) = \int_V \alpha \, dV$

Thus $\frac{dI}{dt} = \lim_{\delta t \to 0} \frac{1}{\delta t} \left[\delta t \int_V \frac{\partial \alpha}{\partial t} dV + \int_{\delta V} \alpha \, dV + \delta t \int_{\delta V} \frac{\partial \alpha}{\partial t} dV \right.$

$$\left. + O((\delta t)^2) \right]$$

$$= \int_V \frac{\partial \alpha}{\partial t} dV + \lim_{\delta t \to 0} \left[\frac{1}{\delta t} \int_{\delta V} \alpha \, dV \right]$$

$$\cancel{+ \lim_{\delta t \to 0} \left[\int_{\delta V} \frac{\partial \alpha}{\partial t} dV \right]}$$

Now $\left| \lim_{\delta t \to 0} \int_{\delta V} \frac{\partial \alpha}{\partial t} dV \right| = \lim_{\delta t \to 0} \left| \int_{\delta V} \frac{\partial \alpha}{\partial t} dV \right|$

$$\leq \lim_{\delta t \to 0} \int_{\delta V} \left| \frac{\partial \alpha}{\partial t} \right| dV$$

For bounded $\frac{\partial \alpha}{\partial t}$, i.e. $\left| \frac{\partial \alpha}{\partial t} \right| < M$ for some M then

this limit is $\leq \lim_{\delta t \to 0} \int_{\delta V} M \, dV$

$$= \lim_{\delta t \to 0} M \cdot \delta V = 0$$

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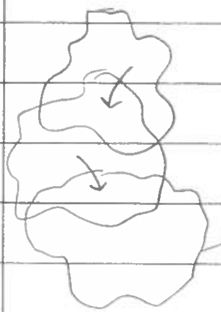
Newton - Euler



Time rate of change of momentum = Force acting

p: pressure : force per unit area normal to surface, no tangential stress: inviscid

$\alpha(\underline{r}, t)$ $$\frac{D\alpha}{Dt} = \frac{\partial \alpha}{\partial t} + (\underline{u}\cdot\underline{\nabla})\alpha$$

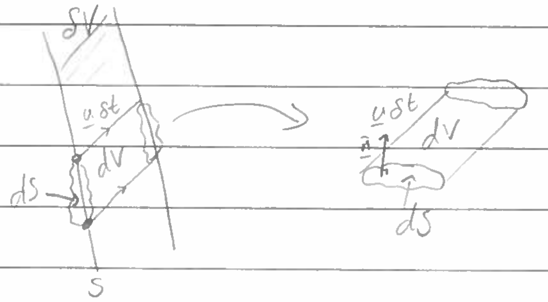
Take any volume V consisting of the same fluid (elements) and follow it.

$$I_v(t) = \int_{V(t)} \alpha \, dV$$

$$\frac{dI_v(t)}{dt} = ?$$

From last time we have

$$\frac{dI}{dt} = \int_V \frac{\partial \alpha}{\partial t} \, dt + \lim_{\delta t \to 0}\left[\frac{1}{\delta t}\int_{\delta V} \alpha \, dV\right]$$



volume = area of base × height

so $dV = dS\left[(\underline{u}\,\delta t)\cdot\hat{\underline{n}}\right]$

So the final term is $\displaystyle\lim_{\delta t \to 0}\left[\frac{1}{\delta t}\int_S \alpha(\underline{u}\cdot\hat{\underline{n}})\,\delta t \, dS\right]$

$$= \int_S \alpha\,\underline{u}\cdot\hat{\underline{n}}\,dS$$

Thus $$\frac{dI_v}{dt} = \int_V \frac{\partial \alpha}{\partial t}\,dV + \int_S \alpha\,\underline{u}\cdot\hat{\underline{n}}\,dS$$

ie. $$\boxed{\frac{D}{Dt}\left[\int_V \alpha \, dV\right] = \underbrace{\int_V \frac{\partial \alpha}{\partial t} \, dV}_{\text{local rate of change inside } V} + \underbrace{\int_S \alpha \underline{u} \cdot \underline{\hat{n}} \, dS}_{\text{flux of } \alpha \text{ through } S}}$$

[Reynolds Transport Theorem 1] RTT1

Divergence theorem:

$$\int_V \underline{\nabla} \cdot \underline{v} \, dV = \int_S \underline{v} \cdot \underline{\hat{n}} \, dS$$

$v = \alpha \underline{u}$: $$\boxed{\frac{D}{Dt}\int_V \alpha \, dV = \int_V \left[\frac{\partial \alpha}{\partial t} + \underline{\nabla} \cdot (\alpha \underline{u})\right] dV}$$

[Reynolds Transport theorem 2] RTT2

$$\underline{\nabla} \cdot (\alpha \underline{u}) = \alpha \underline{\nabla} \cdot \underline{u} + \underline{u} \cdot \underline{\nabla} \alpha$$

Thus $$\frac{\partial \alpha}{\partial t} + \underline{\nabla} \cdot (\alpha \underline{u}) = \frac{\partial \alpha}{\partial t} + \underline{u} \cdot \underline{\nabla} \alpha + \alpha \underline{\nabla} \cdot \underline{u}$$

$$= \frac{D\alpha}{Dt} + \alpha \underline{\nabla} \cdot \underline{u}$$

Thus $$\boxed{\frac{D}{Dt}\int_V \alpha \, dV = \int_V \left(\frac{D\alpha}{Dt} + \alpha \underline{\nabla} \cdot \underline{u}\right) dV}$$

[Reynolds Transport Theorem 3] RTT3

This holds even for compressible fluids.

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§3-2-1 RTT4

note this is fairly common in exams. ✓

Example

Take $\alpha = \rho$, the fluid density.
Consider a fluid of density $\rho(\underline{r}, t)$ occupying a domain D with velocity field $\underline{u}(\underline{r}, t)$. Take ANY subvolume V of D, consisting always of the same fluid (elements).
Consider

$$M(t) = \int_{V(t)} \rho(\underline{r}, t)\, dV$$

This is the mass of the fluid comprising V.
By conservation of mass $\frac{dM}{dt}$ or $\frac{DM}{Dt} = 0$.

– not creating or destroying any mass.

Apply RTT2:

$$\frac{D}{Dt}\int_V \rho(\underline{r}, t)\, dV = \int_V \left[\frac{\partial \rho}{\partial t} + \underline{\nabla}\cdot(\rho \underline{u})\right] dV$$

$$= 0 \text{ by conservation of mass.}$$

But V is arbitrary. Thus this is true for all V in D.
By our lemma, the integral can vanish for all V only if the integrand is identically zero everywhere in D.

i.e. $\frac{\partial \rho}{\partial t} + \underline{\nabla}\cdot(\rho \underline{u}) = 0$ in D

– conservation of mass for a compressible fluid.

Example

Now consider

$$\alpha = \rho f$$

where f is any scalar fn of position and time, $f(\underline{r},t)$

Then RTT2 is

$$\frac{D}{Dt}\int_V \rho f \, dV = \int_V \left[\frac{\partial}{\partial t}(\rho f) + \underline{\nabla}\cdot(\rho f \underline{u})\right] dV$$

i.e. $\dfrac{\partial \rho}{\partial t} + \underline{\nabla}\cdot(\rho \underline{u}) = 0$ in D [CM1]

– conservation of mass for a compressible fluid.

As in RTT3 this can also be written

$$\frac{D\rho}{Dt} + \rho \underline{\nabla}\cdot\underline{u} = 0 \quad [CM2]$$

Integrand is

$$\frac{\partial \rho}{\partial t}\cdot f + \rho\frac{\partial f}{\partial t} + \rho\underline{u}\cdot\underline{\nabla} f + f\,\underline{\nabla}\cdot(\rho\underline{u})$$

$$= f\left[\frac{\partial \rho}{\partial t} + \underline{\nabla}\cdot(\rho\underline{u})\right] + \rho\left[\frac{\partial f}{\partial t} + \underline{u}\cdot\underline{\nabla} f\right]$$

$$= \underset{\substack{\uparrow \\ \text{conservation of mass}}}{f\cdot 0} + \rho\frac{Df}{Dt}$$

i.e. we have shown

$$\frac{D}{Dt}\int_V f\rho \, dV = \int_V \frac{Df}{Dt}\rho \, dV \qquad RTT4$$

i.e. $\dfrac{D}{Dt}$ and $\displaystyle\int_V$ do commute provided element of integration is $dM = \rho\, dV$ (which is fixed by conservation of mass).

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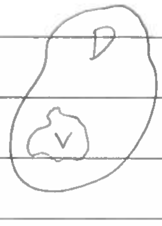
§3.2.2 Newton

force = time rate of change of momentum.

Suppose we have a fluid of density $\rho(\underline{r},t)$ (can be compressible) occupying a domain D and with velocity field $\underline{u}(\underline{r},t)$ [Eulerian description].

Take an ARBITRARY subvolume V of D, always composed of the same fluid (elements).

Consider $\quad \underline{m} = \int_V \rho \underline{u} \, dV \quad$: momentum of fluid (elements) comprising V



dM mass of element dV

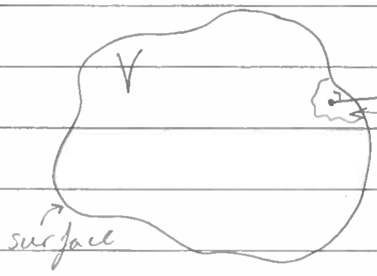
$\underline{u}$ = velocity of element dV

$$d\underline{m} = \rho \underline{u} \, dV$$
$$= \underline{u} \rho \, dV = \underline{u} \, dM \leftarrow \text{momentum of } dV$$

By RTT4: $\quad \frac{D}{Dt} \int_V \rho \underline{u} \, dV = \int_V \rho \frac{D\underline{u}}{Dt} \, dV$

— time rate of change of momentum of fluid (elements) comprising V.

By Newton this equals the net force acting on the fluid (elements) comprising V.



element dS of the surface S, with normal $\hat{n}$.

There is only a normal stress (inviscid) — the pressure

Pressure force is $-p\hat{n} \, dS$

The total pressure force on V is $\int_S -p\hat{n} \, dS$.

$$\int_S -p\underline{\hat{n}}\, dS = \int_V -\underline{\nabla} p \, dV$$ by divergence theorem.

$\left[\begin{array}{l}\text{Arbitrary } \underline{a} \\ \text{Dot one side, get dot of other side} \\ \text{Take } \underline{a} = \underline{\hat{x}}, \underline{\hat{y}}, \underline{\hat{z}}\end{array}\right]$ ← Hw hint

Also allow for an arbitrary external force per unit mass $\underline{F}$.

eg. gravity $\underline{F} = \dfrac{-mg\underline{\hat{z}}}{m}$

$= -g\underline{\hat{z}}$

$\underline{\hat{z}}$ $\underline{\hat{x}}$ $\underline{\hat{y}}$

This exerts a total force
$$\int_V (-g\underline{\hat{z}})\, dV = \int_V \underline{F}\, dV$$

Thus the total force on the fluid (elements) comprising V is $\int_V -\underline{\nabla} p \, dV + \int_V \rho \underline{F}\, dV$

pressure external (gravity electric force)

Thus Newton: $\displaystyle\int_V \rho \frac{Du}{Dt}\, dV = \int_V (-\nabla p + \rho \underline{F})\, dV$

ie. $$\int_V \left[\rho \frac{Du}{Dt} + \underline{\nabla} p - \rho \underline{F}\right] dV = 0$$

But V arbitrary so integral vanishes for all V in D so integrand identically zero in D.
ie. $\rho \dfrac{Du}{Dt} + \underline{\nabla} p - \rho \underline{F} = 0$ in D

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This is Newton's Law for a fluid, derived by Euler : Euler equations.

Traditional to write this as

$$\frac{D\underline{u}}{Dt} = -\frac{1}{\rho}\underline{\nabla} p + \underline{F} \qquad \text{eqn of motion}$$

accel = pressure gradient force + $\frac{\text{external force}}{\text{unit mass}}$

Already had $\frac{\partial \rho}{\partial t} + \underline{\nabla}\cdot(\rho \underline{u}) = 0$.

These are not complete.

Unknowns: $p, \rho, \underline{u}$ 5 scalars

Equations: 3+1 ; 4 equations

<u>Gas Dynamics</u>

<u>Lighthill</u> Gas Law : $p = f(\rho)$ for some function f (e.g. $PV = nRT$)

mach $\gtrsim 1$ 5 eqns in 5 unknowns.

OR/

If soundwaves unimportant (low mach no. mach << 1)

(Incompressible)

We have

$$\frac{D\rho}{Dt} + \rho \underline{\nabla}\cdot\underline{u} = 0$$

Incompressible: volume does not change, but mass does not change, so density does not change

ie, the density of a fluid element does not change, following the element

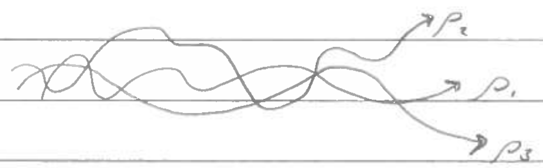
i.e. $\frac{D\rho}{Dt} = 0$ incompressibility

Hence $\underline{\nabla} \cdot \underline{u} = 0$.

5 eqns in 5 unknowns (conservation of mass split into 2).

Notice this is more general than saying ρ = constant, since different fluid elements can have different densities.

eg.



- extremely imortant in geophysical flow

This year: homogeneous flow

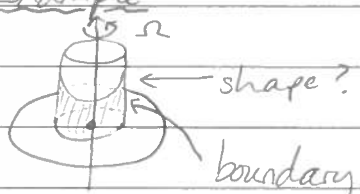
i.e. $\rho \equiv$ constant

Then $\frac{D\rho}{Dt} = 0$ identically satisfied.

We have $\frac{D\underline{u}}{Dt} = -\frac{1}{\rho}\underline{\nabla} p + \underline{F}$ 4 eqns, 4 unknowns.

$\underline{\nabla} \cdot \underline{u} = 0$ closed system.

<u>Example</u>



$\underline{F} = -g\,\hat{\underline{z}}$

boundary condition: pressure in fluid
= pressure in atmosphere.

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Example

The free-surface of a fluid in solid body rotation under gravity is a paraboloid (Liquid Mirror Telescope).

Solution

The velocity field is $\underline{u} = \underline{\Omega} \wedge \underline{r}$
with $\underline{\Omega} = \Omega \hat{\underline{z}}$, vertical.
The force/unit mass is gravity, $\underline{F} = -g\hat{\underline{z}}$.

$$\underline{u} = \begin{vmatrix} \hat{\underline{x}} & \hat{\underline{y}} & \hat{\underline{z}} \\ 0 & 0 & \Omega \\ x & y & z \end{vmatrix} = -y\Omega\hat{\underline{x}} + x\Omega\hat{\underline{y}}$$

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + w\frac{\partial}{\partial z}$$

$$= 0 - y\Omega\frac{\partial}{\partial x} + x\Omega\frac{\partial}{\partial y} + 0$$

Eqn of motion

$$\frac{D\underline{u}}{Dt} = -\frac{1}{\rho}\underline{\nabla}p + \underline{F}$$

In components:

x-component $\quad \frac{Du}{Dt} = -\frac{1}{\rho}\frac{\partial p}{\partial x}$

y-component $\quad \frac{Dv}{Dt} = -\frac{1}{\rho}\frac{\partial p}{\partial y}$

z-component $\quad \frac{Dw}{Dt} = -\frac{1}{\rho}\frac{\partial p}{\partial z} - g$

x-momentum

$$\left(-y\Omega\frac{\partial}{\partial x} + x\Omega\frac{\partial}{\partial y}\right)(-y\Omega) = -\frac{1}{\rho}\frac{\partial p}{\partial x}$$

$$x\Omega(-\Omega) = -\frac{1}{\rho}\frac{\partial p}{\partial x} \quad \text{ie. } \frac{\partial p}{\partial x} = \rho\Omega^2 x \quad (1)$$

y-momentum

$$\left(-y\Omega\frac{\partial}{\partial x} + x\Omega\frac{\partial}{\partial y}\right)(x\Omega) = \frac{-1}{\rho}\frac{\partial p}{\partial y}$$

$$-y\Omega^2 = \frac{-1}{\rho}\frac{\partial p}{\partial y} \quad \text{i.e.} \quad \frac{\partial p}{\partial y} = \rho\Omega^2 y \quad (2)$$

z-momentum

$(w=0)$

$$\frac{Dw}{Dt} = 0$$

$$\text{So } 0 = \frac{-1}{\rho}\frac{\partial p}{\partial z} - g \quad \text{i.e.} \quad \frac{\partial p}{\partial z} = -\rho g \quad (3)$$

Integrate (1) w.r.t. x: $p = \frac{1}{2}\rho\Omega^2 x^2 + f(y,z)$

$$\text{So } \frac{\partial p}{\partial y} = 0 + \frac{\partial f}{\partial y} = \rho\Omega^2 y \quad \text{by } (2)$$

Integrate w.r.t. y:

$$f(y,z) = \frac{1}{2}\rho\Omega^2 y^2 + h(z)$$

$$\text{So } p = \frac{1}{2}\rho\Omega^2(x^2+y^2) + h(z)$$

$$\text{i.e. } \frac{\partial p}{\partial z} = h'(z) = -\rho g \quad \text{by } (3)$$

Integrate w.r.t. z:

$$h(z) = -\rho g z + C$$

$$\text{So } p = \frac{1}{2}\rho\Omega^2(x^2+y^2) - \rho g z + C$$

which is the pressure field associated with the motion. Thus the pressure is constant on the isobaric surfaces

$$\frac{1}{2}\rho\Omega^2(x^2+y^2) - \rho g z = \text{const.}$$

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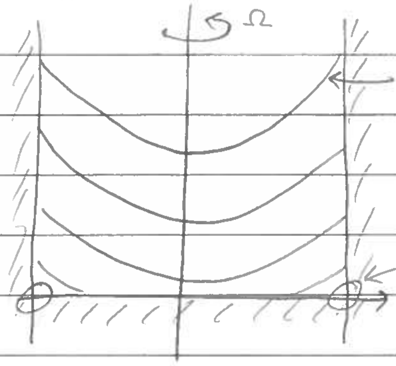
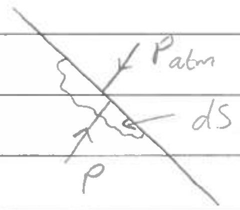
i.e. surfaces

$$z = \frac{1}{2}\frac{\Omega^2}{g}(x^2+y^2) + \text{const}$$

i.e. paraboloids



The surface boundary condition is that p (pressure in water) equals p_{atm} (atmospheric pressure)



Surface element dS.
Inward force, $p_{atm}\,dS$
Outward force, $p\,dS$
Net force outwards $(p - p_{atm})\,dS$

The mass of dS is zero as it has only area, no volume.

The only way Accel = force / mass is finite here is if Force = 0, i.e. $p = p_{atm}$.

[By the same argument the sheer stress is continuous at a fluid surface, e.g. atmosphere - ocean interface]

§3.3 Hydrostatic Pressure

When the flow is at rest

$$\underline{u} = 0$$

$$\text{so } \frac{D\underline{u}}{Dt} = 0$$

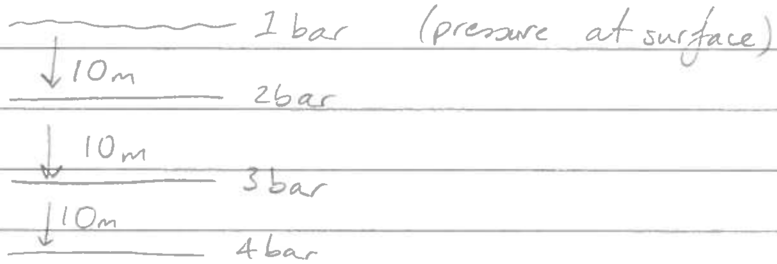
So $-\frac{1}{\rho}\nabla p - g\hat{\underline{z}} = 0$ under gravity alone.

i.e. $\dfrac{\partial p}{\partial x} = 0, \quad \dfrac{\partial p}{\partial y} = 0, \quad \dfrac{\partial p}{\partial z} = -\rho g$

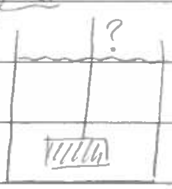
Thus $p = -\rho g z + \text{const}$

Put origin at the free surface
so $p = p_{atm}$ when $z = 0$.

Then $\boxed{p_{hyd} = p_{atm} - \rho g z}$



Example



Archimedes Principle?
Consider a body immersed totally in a fluid.

The force on the body is

$-p\hat{\underline{n}}$ on each element dS
(force per unit area)

$$\int_S (-p\hat{\underline{n}})\, dS = -\int_V \underline{\nabla} p \, dV$$

$$= -\int_V -\rho g \hat{\underline{z}} \, dV \qquad (p = p_{hyd})$$

fluid density

$$= \rho g \hat{\underline{z}} \int_V dV = \rho g \,(\text{body volume})\, \hat{\underline{z}} \quad (\text{upwards})$$

= weight of the fluid displaced

So the weight of the body is reduced by an amount equal to the weight of the water displaced.

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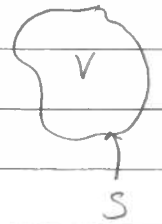

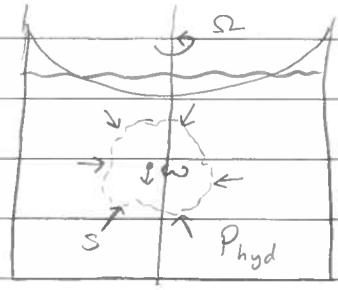

Euler's eqn

$$\frac{D\underline{u}}{Dt} = -\frac{1}{\rho}\underline{\nabla}p + \underline{F} \qquad \text{take } \underline{F} = -g\hat{\underline{z}}$$

hydrostatic pressure (p_h)

Thus we can write $p = p_{hyd} + p_d$ ← dynamic pressure

($p_d = p - p_{hyd}$)

Note $\frac{1}{\rho}\underline{\nabla}p_h = -g\hat{\underline{z}}$

So $\frac{1}{\rho}\underline{\nabla}p = \frac{1}{\rho}\underline{\nabla}p_h + \frac{1}{\rho}\underline{\nabla}p_d$

Thus $$\frac{D\underline{u}}{Dt} = -\frac{1}{\rho}\underline{\nabla}p_h - \frac{1}{\rho}\underline{\nabla}p_d - g\hat{\underline{z}}$$

$$= -\frac{1}{\rho}\underline{\nabla}p_d$$

i.e. only p_d accelerates the fluid.
p_h simply balances gravity, i.e. weight of water

i.e. in many problems we can treat p_h and p_d separately so compute dynamic forces and then simply add in buoyancy.

The only time you cannot do this is when there is a free surface, because the B.C. there is

$p = p_{atm}$, i.e. $p_d + p_h = p_{atm}$

§3.4 ## Bernoulli equation

We have
$$(\underline{u}\cdot\underline{\nabla})\underline{u} = \underline{\nabla}\left(\tfrac{1}{2}\underline{u}^2\right) + \underline{\omega}\wedge\underline{u}$$

Thus Euler is
$$\frac{\partial \underline{u}}{\partial t} + \underline{\nabla}\left(\tfrac{1}{2}\underline{u}^2\right) + \underline{\omega}\wedge\underline{u} = -\frac{1}{\rho}\underline{\nabla}p \underbrace{-\underline{\nabla}V_e}_{\underline{F}}$$
when $\underline{F}$ is a conservative force so there exists a potential V_e (e: external) s.t. $\underline{F} = -\underline{\nabla}V_e$

e.g. gravity $\underline{F} = -g\hat{\underline{z}}$

so $V_e = gz$ where $V_e = 0$ when $z = 0$

i.e. $z=0$ is the DATUM for the potential V_e.

i.e. $$\frac{\partial \underline{u}}{\partial t} + \underline{\omega}\wedge\underline{u} = -\frac{1}{\rho}\underline{\nabla}H$$
where $H = p + \frac{1}{2}\rho\underline{u}^2 + \rho V_e$

taking density, ρ, to be a constant.

(where subsequently we will see that H is related to the pressure head).

If the flow is STEADY, $\frac{\partial \underline{u}}{\partial t} = 0$ (still have $\underline{u} = \underline{u}(\underline{x})$)

then dotting with $\underline{u}$ gives
$$\underbrace{\rho\underline{u}\cdot(\underline{\omega}\wedge\underline{u})}_{=0} = -\underline{u}\cdot\underline{\nabla}H \qquad \left(\frac{\partial H}{\partial t} = 0\right)$$

$$\text{i.e. } \underline{u}\cdot\underline{\nabla}H = 0, \quad \text{i.e. } \frac{DH}{Dt} = 0$$

i.e. H is constant on particle paths.

But flow is steady so p.p's are s'lines.

Thus H is constant along streamlines (in steady flow of constant density).

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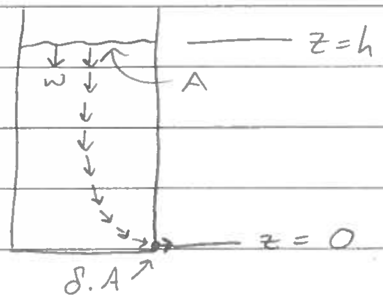
i.e. $p + \frac{1}{2}\rho u^2 + \rho V_e$ is constant along s'lines
(Bernoulli's Theorem)
– Can be different values on different streamlines.

<u>Example</u> (Torricelli)
Consider a vessel open to the air with free surface of instantaneous area A and depth h. Suppose a small hole of area $\delta . A$ is punctured at the bottom of the vessel, $0 < \delta << 1$, so that the flow is approximately steady, with surface falling at speed u and fluid exiting at speed U. Find U when $0 < \delta << 1$.



The external force is gravity
$\underline{F} = -g\hat{\underline{z}}$
$V_e = gz$
Choose the DATUM at exit level.

Thus water : constant density
small hole : approx steady flow
line tangential to velocity field links surface to exit.
i.e. there is a s'line joining surface to exit
(notice no particle has actually made that journey)

Thus use Bernoulli on this s'line :-
$p + \frac{1}{2}\rho \underline{u}^2 + \rho V_e$ is a constant along this s'line.

In particular value at top = value at bottom.
Both places open to air, so $p = p_{atm}$.
Hence $p_{atm} + \frac{1}{2}\rho \underline{u}^2 + \rho g \underset{\uparrow z=h}{h} = p_{atm} + \frac{1}{2}\rho U^2 + 0$

The other relation between u and U is conservation of mass :-

Mass flux at top = Mass flux at bottom (flow steady)

$$\rho A u = \rho \delta . A U$$

so $u = \delta . U$

Hence we have $\frac{1}{2}\rho \delta^2 U^2 + \rho g h = \frac{1}{2}\rho U^2$

i.e. $U^2(1-\delta^2) = 2gh$

i.e. $U = \sqrt{2gh}$ to order δ^2

— same as freely falling particle under gravity.

Bernoulli : conservation of energy

pressure energy + K.E. + P.E. is conserved on s'line.

Example

Force on a spinning cylinder

Consider a cylinder of radius a in a uniform stream of speed U in the $\hat{\underline{x}}$ direction and spinning that the circulation about the cylinder is $\varkappa$.

What is the force on the cylinder?

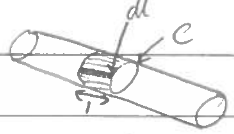
The complex velocity potential for this flow is

$$w(z) = U\left(z + \frac{a^2}{z}\right) - \frac{i\varkappa}{2\pi}\log z$$

$\xrightarrow{U}$ 

'top spin.'

We find the force per unit length in the ignorable coordinate.



This is

$$\underline{F} = \oint_C -p\hat{\underline{n}}\, 1 \cdot dl$$

element of area $\boxed{ds}\updownarrow dl$, $\overleftrightarrow{1}$

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We can divide $\underline{\mathcal{F}}$ into components:—

Drag, $\mathcal{D}$, which is the component of $\underline{\mathcal{F}}$ in the direction of the flow

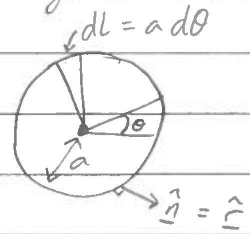
$$\mathcal{D} = \underline{\hat{x}} \cdot \underline{\mathcal{F}}$$

and the lift, $\mathcal{L}$, which is the component of $\underline{\mathcal{F}}$ perpendicular to the flow

$$\mathcal{L} = \underline{\hat{y}} \cdot \underline{\mathcal{F}}$$

Thus

$$\mathcal{D} = -\oint_C p\,\underline{\hat{x}} \cdot \underline{\hat{r}}\, dl \qquad \text{(drag / unit length)}$$



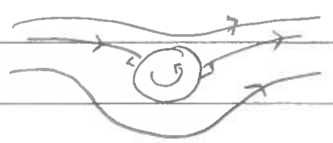
$$\underline{\hat{r}} = \cos\theta\, \underline{\hat{x}} + \sin\theta\, \underline{\hat{y}}$$

i.e. $$\mathcal{D} = -\int_{-\pi}^{\pi} p\cos\theta \cdot a\, d\theta.$$

Similarly

$$\mathcal{L} = -a\int_{-\pi}^{\pi} p\sin\theta\, d\theta$$

The flow is steady, the density is constant and we have streamlines originating from $x = -\infty$



On any of these s'lines we can use Bernoulli $p + \frac{1}{2}\rho \underline{u}^2 = \text{const}$, ignoring gravity which contributes only a buoyancy force (can add in later if important).

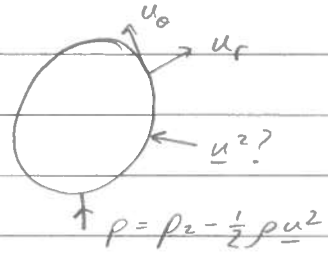
All s'lines originate upstream where $p = p_0$ (say) and $\underline{u} = U\underline{\hat{x}}$, so $\underline{u}^2 = U^2$

ie. we have $p + \frac{1}{2}\rho\underline{u}^2 = p_0 + \frac{1}{2}\rho U^2$ everywhere.

$$= p_s \quad \text{(a constant)}$$

where p_s is the stagnation point pressure ($\underline{u} = 0$)

$p = p_s - \frac{1}{2}\rho \underline{u}^2$: p_s max pressure.



On the cylinder

$$\underline{u}^2 = u_r^2 + u_\theta^2$$
$$= u_\theta^2 \quad (u_r = 0)$$

Using $u_r - iu_\theta = e^{i\theta}\frac{dw}{dz}$

we obtained $u_\theta = -2U\sin\theta + \kappa/2\pi a$ on $r = a$

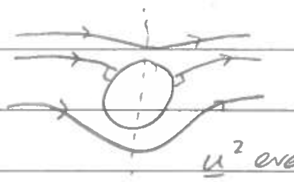
Thus $p = p_s + \frac{1}{2}\rho\left[-2U\sin\theta + \frac{\kappa}{2\pi a}\right]^2$

$$= p_s - \frac{1}{2}\rho\left[\frac{\kappa^2}{4\pi^2 a^2} - \frac{2U\kappa\sin\theta}{\pi a} + 4U^2\sin^2\theta\right]$$

Now $D = -a\int_{-\pi}^{\pi} p\cos\theta\, d\theta$

But $[1, \cos\theta, \sin\theta, \cos 2\theta, \sin 2\theta, \cos 3\theta, \sin 3\theta, \ldots]$ forms an orgonal set on $\int_{-\pi}^{\pi}$ ie. $\int_{-\pi}^{\pi} f_1 f_2 \, d\theta$ if $f_1 \neq f_2$.

Thus $D = 0$, ie. no drag.



Flow symmetric (before and after)

$\underline{u}^2$ even in x, p even in x.

Now for $L = -a\int_{-\pi}^{\pi} p\sin\theta\, d\theta$

$\int_{-\pi}^{\pi}\sin^2\theta\, d\theta = \frac{1}{2}\cdot 2\pi$

$$= -a\left(\frac{-1}{2}\rho\right)\left(-\frac{2U\kappa}{\pi a}\right)\pi$$

$$= -\rho U\kappa$$

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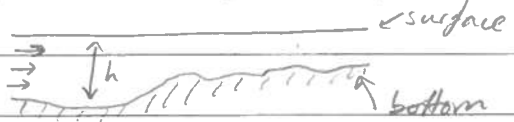
i.e. there is a DOWNWARD force of magnitude $\rho U \varkappa$

Lift has magnitude $\rho U \varkappa$ per unit width
- proportional to speed : slow planes $\Rightarrow$ long wings, fast planes $\Rightarrow$ short wings.
- proportional to $\varkappa$: the more spin, the more a ball floats
- proportional to ρ : eg. plane loads in deserts

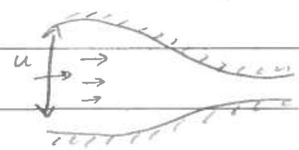
§3.5 Open channel flow
- a third example of Bernoulli

We consider water open to the air flowing down a channel whose width b may vary and whose base may rise and fall.

ELEVATION (side view)



PLAN (top view)



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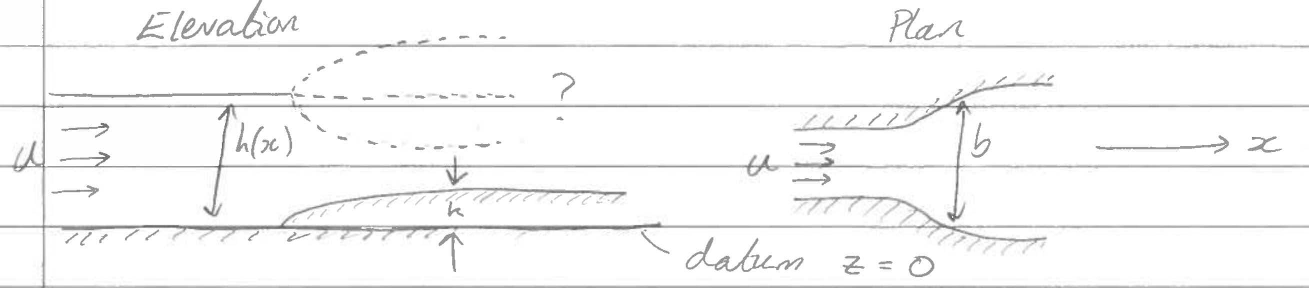
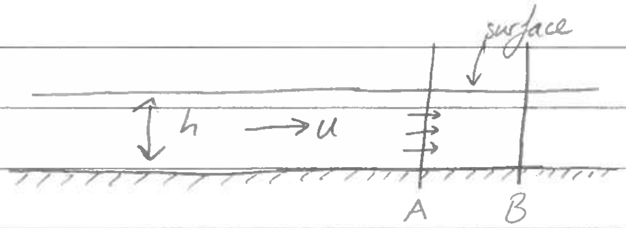


Assume variations along the channel are slow.
Assume therefore that the flow is independent of depth and uniform across the channel.
Assume that the flow is steady.
Thus the flow is a function of x alone.
Also $v=0$ and $w=0$. Thus only velocity component is $u(x)$.
The only other variable is the depth $h(x)$.

First consider a constant width channel with flat horizontal bottom. (Constant density ρ)



The flux of fluid across any station A is
speed $\times$ area $= uhb$ (b is width of river)
The flux across station B is the same,
ie. uhb is a constant of the motion.
Here b is a constant width, so
$Q = uh$ is a constant of the motion
(volume flux / unit width).

1). Constant density, ρ
2). Steady flow
3). The flow is smooth & a particle on the surface stays there, ie. the surface is a s'line.

$\therefore$ We can use Bernoulli.

i.e. on surface

$p + \frac{1}{2}\rho u^2 + \rho V_e = \text{constant}$

External force is gravity

$\therefore V_e = gz$

where z = height above datum

Hence $p_a + \frac{1}{2}\rho u^2 + \rho g z = \text{const.}$ $\quad p = p_a$ on $z = h$

Thus $\frac{1}{2}u^2 + gh = \text{const}$

i.e. $H = h + \dfrac{u^2}{2g}$ is constant

H = pressure head (max depth attained when $u=0$)

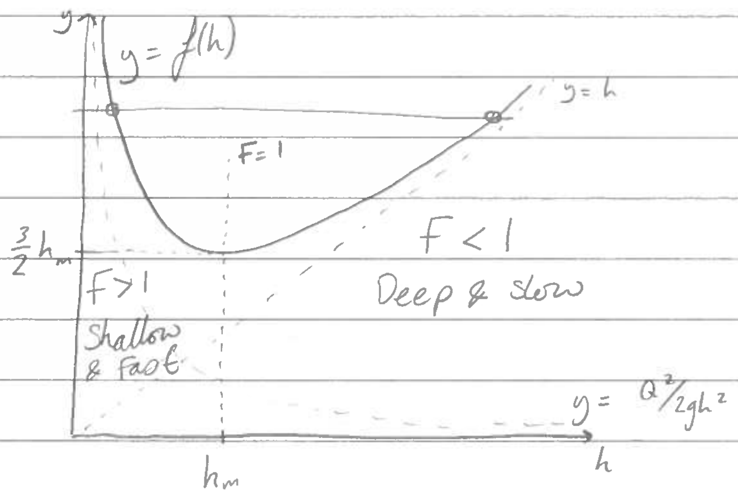
– a second constant of the motion.

We have our 2 constants of our motion

$Q = uh$

$H = h + \dfrac{u^2}{2g}$

So $H = h + \dfrac{Q^2}{2gh^2} = f(h)$



This has a unique minimum (for $h>0$) where $f'(h_m) = 0$.

So $1 - \dfrac{Q^2}{gh_m^3} = 0 \Rightarrow h_m = \left(\dfrac{Q^2}{g}\right)^{1/3}$

and $f(h_m) = h_m\left[1 + \dfrac{Q^2}{2gh_m^3}\right] = \frac{3}{2}h_m$

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Note $Q^2 = gh_m^3$

So $u_m^2 h_m^2 = g h_m^3$

$\Rightarrow \dfrac{u_m^2}{g h_m} = 1$

Froude number : $F = \dfrac{u}{\sqrt{gh}}$

with $F = 1$ when $h = h_m$

CRITICAL (SONIC)

SUBCRITICAL (SUBSONIC)

If $h > h_m$, $u < m$. So $F < 1$. (note $Q = uh = \text{const.}$)

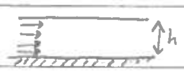
If $h < h_m$, $u > m$. So $F > 1$.

SUPERCRITICAL (SUPERSONIC)

$F = \dfrac{u}{\sqrt{gh}}$

$F^2 = \dfrac{u^2}{gh} \sim \dfrac{K.E.}{P.E.}$

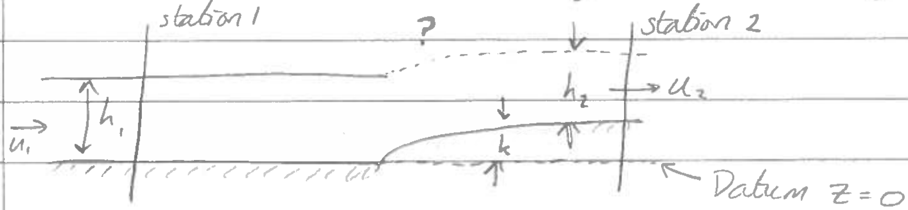
$F > 1 \rightarrow$ large K.E., small P.E. (fast) 

$F < 1 \rightarrow$ small K.E., large P.E. (slow) 

$F = \dfrac{u}{\sqrt{gh}} = \dfrac{(\text{flow speed})}{(\text{long surface wave speed})}$

Example Rising Floor

Consider a channel of constant width whose depth far upstream is h_1 where the flow speed is u_1. Let the floor of the channel slowly rise by an amount k. Does the surface rise or fall?



The flow is steady, so by conservation of mass, flux per unit width is the same at 1 and 2.

i.e. $u_1 h_1 = u_2 h_2 = Q$ (say)

In exam need to check conditions for Bernoulli

Flow is steady (✓), density is constant (✓).
The flow is smooth, particle on surface stays there.
So surface is a streamline
∴ use Bernoulli

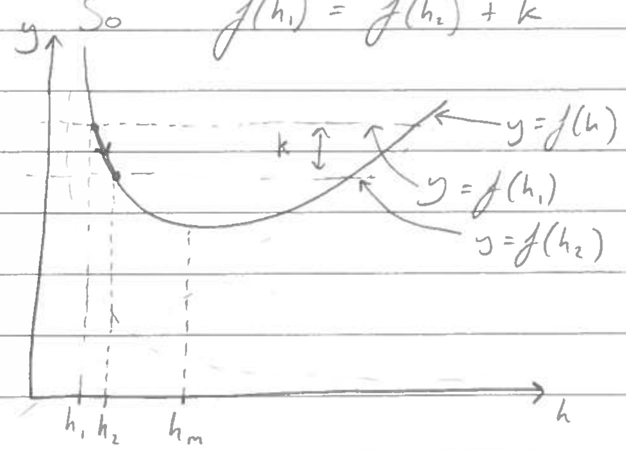
$$p + \frac{1}{2}\rho u^2 + \rho g z = \text{const.} \qquad z = \text{height above datum.}$$

$$p_a + \frac{1}{2}\rho u_1^2 + \rho g h_1 = p_a + \frac{1}{2}\rho u_2^2 + \rho g(h_2 + k) \qquad p = p_a \text{ on surface}$$

i.e. $$h_1 + \frac{u_1^2}{2g} = h_2 + \frac{u_2^2}{2g} + k$$

$$h_1 + \frac{Q^2}{2gh_1^2} = h_2 + \frac{Q^2}{2gh_2^2} + k$$

So $$f(h_1) = f(h_2) + k$$



suppose oncoming flow is SUPERCRITICAL, i.e. $h_1 < h_m$

As k increases from zero, h increases from h_1 to h_2, i.e. depth increases.

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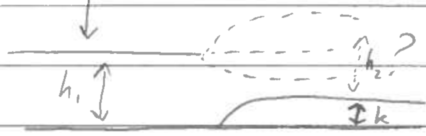
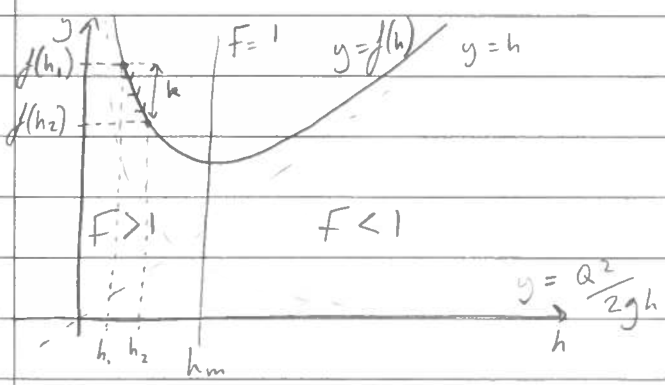
Bernoulli : $\begin{cases} \text{steady } \checkmark \\ \text{homogeneous } \checkmark \\ \text{smooth - s'line } \checkmark \end{cases}$

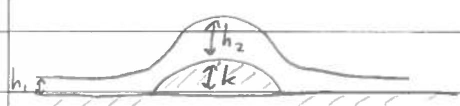


← Datum ($z=0$) - at the bottom upstream.

$$\frac{Q^2}{2gh_1^2} + h_1 = \frac{Q^2}{2gh_2^2} + \underbrace{h_2 + k}_{\text{height above datum downstream.}}$$

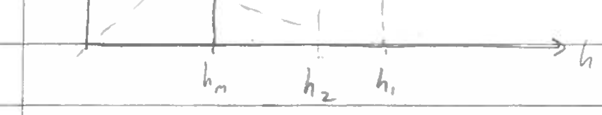
So $f(h_1) = f(h_2) + k$



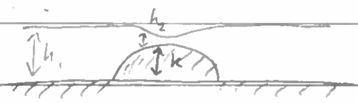


fast, shallow flow, $F>1$

symmetric, supercritical flow over a bump.

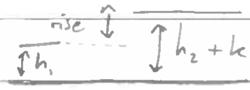


Oncoming flow subcritical, $F<1$, i.e. deep & slow
$h_2 < h_1 \Rightarrow$ flow becomes shallower.



$$h_1 - h_2 - k = \frac{Q^2}{2g}\left(\frac{1}{h_2^2} - \frac{1}{h_1^2}\right) > 0$$

$h_2 + k - h_1$ = rise in surface

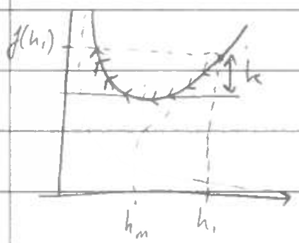


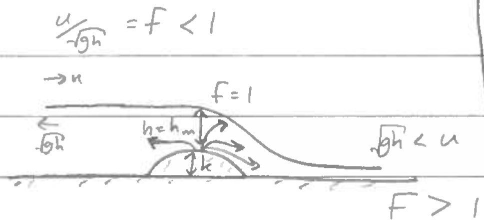
$r = h_2 + k - h_1$:
$<0 \Rightarrow F<1$
$>0 \Rightarrow F>1$

$F<1$: - large P.E., small K.E.
- gets over bump by converting some P.E. to K.E.

$F>1$: - high K.E., low P.E.
- converts some K.E. to P.E. to get over the bump.

Critical flow: $k = f(h_1) - f(h_m)$

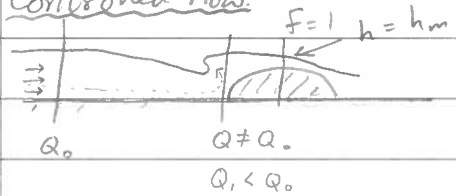


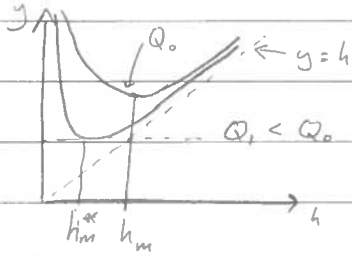


1st order pde
$u_t + K(u)u_x = L(x)$
on characteristic
$\frac{dx}{dt} = K(u)$
$\frac{du}{dt} = L(x)$

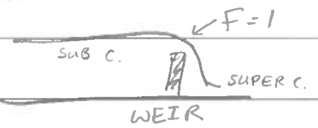
At the highest point of obstacle $F=1$, $h=h_m$, and flow transitions smoothly from subcritical upstream to supercritical downstream. CAUSALITY.

<u>Controlled Flow.</u>

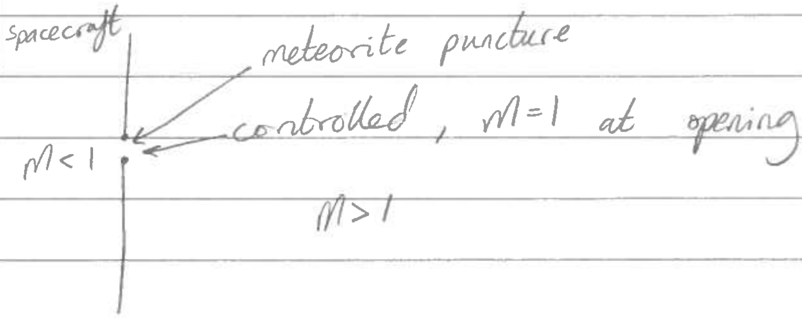




The flow is critical at the obstacle again but Q is reduced to reduce h_m^* i.e. allow a bigger k.



$F=1 \Rightarrow u_m = \sqrt{gh_m}$ so knowing $h_m \Rightarrow$ know u_m at weir.
$\therefore$ know flux.



Froude $= \frac{\text{flow speed}}{\text{wave speed}}$

Air $= \frac{\text{flow speed}}{\text{speed of sound}} = M$

$=$ Mach number

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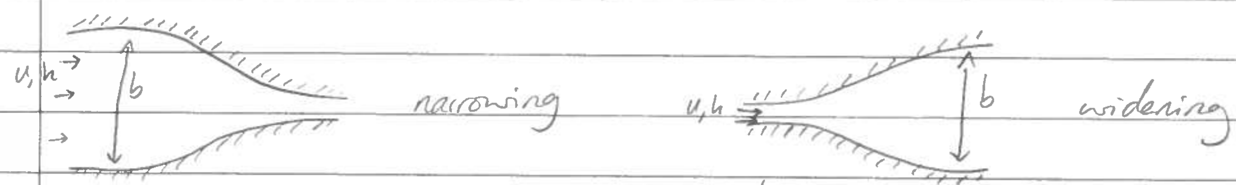
Thus given density and pressure the speed is determined exactly like flow over a weir.

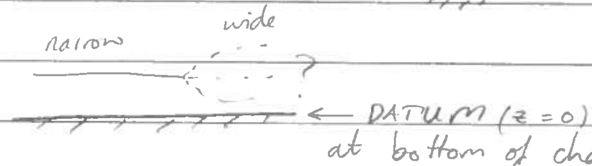
A narrowing / widening channel

Consider a flat-bottomed channel of local width b.

Plan:



Elevation:



Bernoulli?

1). steady ✓

2). homogeneous ✓

3). surface is a particle path & smooth ∴ s'line ✓

∴ Bernoulli on surface.

$$p + \frac{1}{2}\rho u^2 + \rho V_e = \text{const}$$

i.e. $p_a + \frac{1}{2}\rho u^2 + \rho g z = \text{const}$ $p = p_a$ on surface, p_a const.

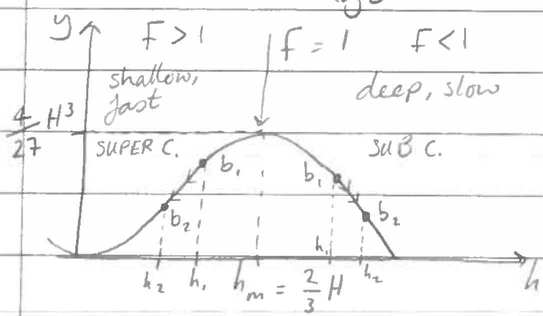
i.e. $\frac{1}{2}u^2 + gh = \text{const}$ (sinz $z = h$ everywhere this time)

i.e. $\dfrac{u^2}{2g} + h = H$, const

i.e. $\dfrac{Q^2}{2gh^2b^2} + h = H$

i.e. $(H - h)h^2 = \dfrac{Q^2}{2gb^2}$

i.e. $f(h) = \dfrac{Q^2}{2gb^2}$ where $f(h) = h^2(H - h)$ (cubic).



$f'(h) = 2hH - 3h^2 = 0$ at $h = 0$

and $2H = 3h$

i.e. $h = \frac{2}{3}H$

$$f\left(\tfrac{2}{3}H\right) = \frac{4}{27}H^3$$

At $h = h_m = \frac{2}{3}H$ we have

$$\frac{u^2}{2g} + \frac{2}{3}H = H$$

i.e. $$\frac{u^2}{2g} = \frac{1}{3}H$$

So $$\frac{u^2}{2gh} = \frac{\frac{1}{3}H}{\frac{2}{3}H} = \frac{1}{2}$$

$$\Rightarrow \frac{u^2}{gh} = 1 \quad \text{i.e.} \quad F_m = \frac{u_m}{\sqrt{gh_m}} = 1$$

Oncoming flow subcritical, $F < 1$

widening

b increasing

$$f(h) = h^2(H-h) = \frac{Q^2}{2gb^2}, \quad \text{RHS decreases}$$

So { subcritical widening ⇒ deepening
{ supercritical widening ⇒ shallowing

and { subcritical narrowing ⇒ shallowing
{ supercritical narrowing ⇒ deepening

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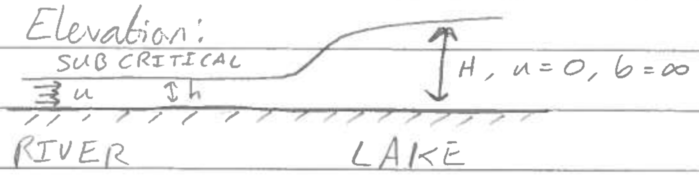
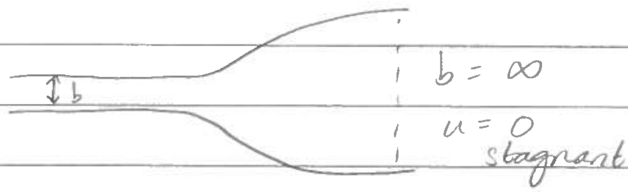
Elevation:

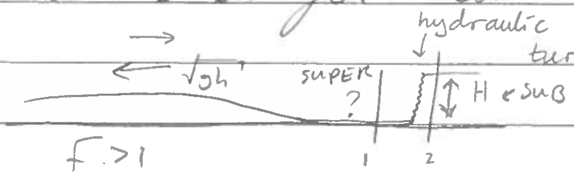


Plan:



$u = 0$ so $h = H$

makes sense for subcritical flow (as above)



turbulent, unsteady, no Bernoulli

$h \to 0$ as $b \to \infty$

for $F > 1$

$\sqrt{gh} < u$ - no information propagates upstream
- no change of upstream condition
- not a control.

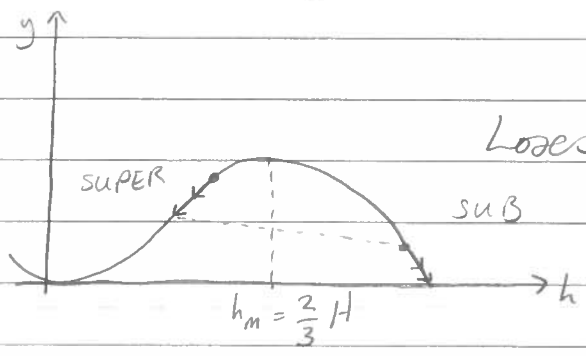
Between stations 1 & 2 still have conservation of mass.

We don't have Bernoulli and so need one more relation between the stations.

Because the jump is narrow ignore bottom friction.

Thus use Newton conservation of momentum.

momentum flux in at 1, plus force acting (pressure)
- momentum flux out at 2



Loses energy

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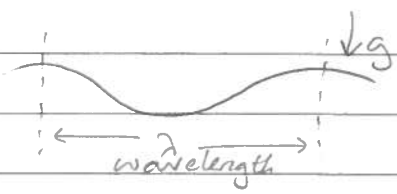
Chapter 4 - Surface Water Waves

$F = \dfrac{u}{\sqrt{gh}}$ ← flow speed

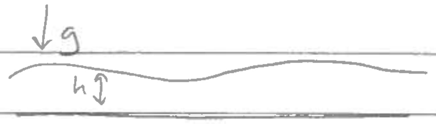
long wave speed



$g: LT^{-2}$, $\lambda : L$

$C = a(gh)^{1/2} \quad : LT^{-1}$

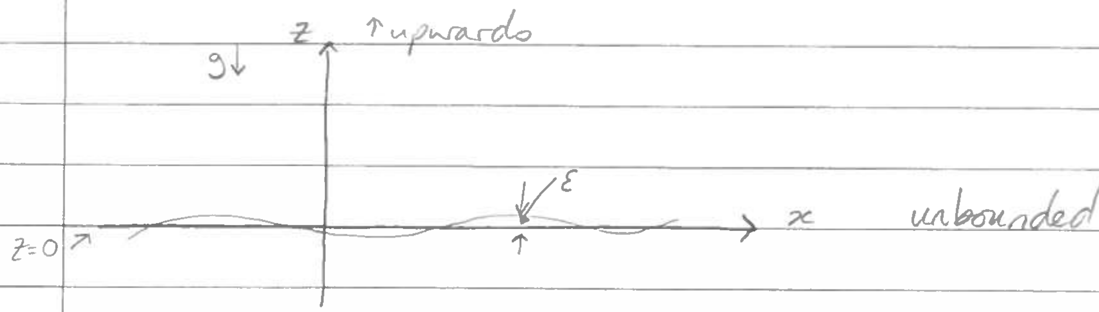
$a = \text{constant}$



long waves on shallow water (eg. channel flow)

$g: LT^{-2}$, $h: L$

$C = a(gh)^{1/2} : L$, $a = \text{constant}$.



Undisturbed fluid has surface $z=0$

Take the flow to be independent of y, $\dfrac{d}{dy} \equiv 0$

Then $p = p_a - \rho g z$ (hydrostatic) when flow at rest.

Now suppose the surface is perturbed by an amount ε when $0 < \varepsilon << 1$.

Let the new surface be given by

$z = \eta(x,t)$, where η is of order ε.

The fluid velocities u and w will be of the same order ε, and they will decay away from the surface.

Let the pressure under the wave be given by

$p = p_a - \rho g z + \rho \phi$

where ϕ is a quantity of order ε.

Notice ϕ is <u>not</u> the velocity potential (but it is close to being it).

The euler equations are:

x-momentum:

$$\frac{\partial \overset{\varepsilon}{u}}{\partial t} + \overset{\varepsilon}{u}\frac{\partial \overset{\varepsilon}{u}}{\partial x} + \overset{\varepsilon}{w}\frac{\partial \overset{\varepsilon}{u}}{\partial z} = -\frac{1}{\rho}\frac{\partial \overset{\varepsilon}{p}}{\partial x}$$

$$= -\frac{\partial \phi}{\partial x}$$

$\varepsilon + \varepsilon^2 + \varepsilon^2 = \varepsilon$

$\Rightarrow 1 + \varepsilon + \varepsilon = 1$, let $\varepsilon \to 0$.

The linearised x-momentum eqn is simply

$$\frac{\partial u}{\partial t} = -\frac{\partial \phi}{\partial x} \qquad (1)$$

[Remember with velocity potential, $\underline{u} = \underline{\nabla}\Phi$, $u = \frac{\partial \Phi}{\partial x}$

so $\phi = -\frac{\partial \Phi}{\partial t}$]

The linearised z momentum eqn is

$$\frac{\partial w}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial z} - \rho g \qquad [\underline{F} = -g\hat{\underline{z}}]$$

$$= \rho g - \frac{\partial \phi}{\partial z} - \rho g$$

$$\Rightarrow \frac{\partial w}{\partial z} = -\frac{\partial \phi}{\partial z} \qquad (2)$$

Governing eqn: cty

$$\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} = 0, \text{ so } \frac{\partial}{\partial t}\left(\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z}\right) = 0 \qquad (3)$$

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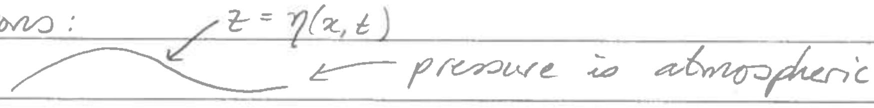
Putting (1), (2) into (3):

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0$$

ie. $\nabla^2 \phi = 0$

Boundary conditions:



On surface (unknown) need 2 conditions

$p = p_a$ on $z = \eta$

ie. $p_a - \rho g \eta + \rho \phi = p_a$

ie. $\phi = g\eta$ (on $z = \eta$)

Consider an function $f(z)$

$$f(z) = f(0) + z f'(0) + \frac{1}{2} z^2 f''(0) + \dots$$

We can replace $f(z)$ by $f(0)$ with error of order z.

Thus we can apply our boundary condition on $z = 0$ with error of order η, ie. of order ε.
But the functions are already $O(\varepsilon)$, so error is $O(\varepsilon^2)$ just like the omitted non-linear terms.

Thus we have $\phi = g\eta$ on $z = 0$.

On the surface, for any particle $z = \eta(x,t)$ on $z = \eta$.
A particle on the surface remains there.
Following a particle $z - \eta(x,t) = 0$ on $z = \eta$ $\forall t$.
ie. $\frac{D}{Dt}(z - \eta(x,t)) = 0$ on $z = \eta$ $\forall t$.

i.e. $\underset{\varepsilon}{w} - \left(\underset{\varepsilon}{\frac{\partial \eta}{\partial t}} + \underset{\varepsilon^2}{u \frac{\partial \eta}{\partial x}} \right) = 0$ on $z = \eta$

i.e. $w = \frac{\partial \eta}{\partial t}$ on $z = \eta$ for $0 < \varepsilon << 1$

With same error, move to $z = 0$

$w = \frac{\partial \eta}{\partial z}$ on $z = 0$

$z = 0 \begin{cases} \phi = g\eta \\ w = -\frac{\partial \eta}{\partial t} \end{cases}$

$\nabla^2 \phi = 0$

as $z \to \infty$, disturbance vanishes, i.e. $\phi \to 0$.

Look for wavelike solutions of the form

$\eta(x,t) = a\cos(kx - wt)$ (where a is of order ε)

$= a\cos\underbrace{[k(x-ct)]}_{\text{phase}}$

a = amplitude

w = frequency (radian frequency) PERIOD $= \frac{2\pi}{w} = \tau$

k = wave number WAVELENGTH $= \lambda = 2\pi/k$

= number of wavelengths so $k = 2\pi/\lambda$

in distance 2π, large $k \Rightarrow$ short waves

$c = w/k$ = PHASE speed, wave propagates to the right at speed c.

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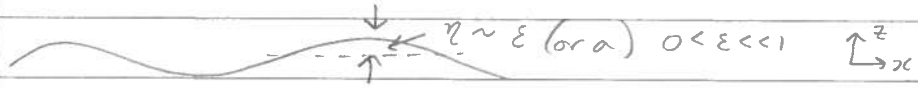

$p = p_a - \rho g z + \rho \phi$, $\phi \sim \varepsilon$ (or a)

Governing eqn: $\nabla^2 \phi = 0$ In 2D: $\dfrac{\partial^2 \phi}{\partial x^2} + \dfrac{\partial^2 \phi}{\partial z^2} = 0$

Surface (linearised to $z=0$) $\phi = g z$ on $z = 0$ DYNAMIC (forces)

$w = \dfrac{\partial \eta}{\partial t}$ on $z = 0$ KINEMATIC (from motion)

Lower b.c. $\phi \to 0$ as $z \to \infty$

Sufficient to consider $\eta(x,t) = a\cos(kx - \omega t)$

$= a\cos(k(x - ct))$, $c = \omega/k$

$\Rightarrow$ sinusoid travelling to right with speed c.

On $z = 0$, $\phi = g\eta = ag\cos(kx - \omega t)$

Try $\phi = ag\cos(kx - \omega t)\, Z(z)$

Then $Z(0) = 1$

and $Z \to 0$ as $z \to -\infty$

Laplace: $\underbrace{-agk^2\cos(kx-\omega t)Z(z)}_{\phi_{xx}} + \underbrace{ag\cos(kx-\omega t)Z''(z)}_{\phi_{zz}} = \underset{=0}{0}$

For η not identically zero ($a \neq 0$),

$$\begin{cases} Z'' - k^2 Z = 0 \\ Z(0) = 1 \\ Z \to 0 \text{ as } z \to -\infty \end{cases}$$

[w.l.o.g. $k > 0$]

C.F. $Z = Ae^{kz} + Be^{-kz}$

For $Z \to 0$ as $z \to -\infty$, $B = 0$.

Thus $Z = Ae^{kz}$

But $Z(0) = 1$ so $A = 1$.

i.e. $Z(z) = e^{kz}$

Thus $\phi(x,z,t) = ag\cos(kx-\omega t)e^{kz}$.

Now $\frac{\partial w}{\partial t} = -\frac{\partial \phi}{\partial z}$

so $\frac{\partial w}{\partial t} = -agk\cos(kx-\omega t)e^{kz}$

so $w = \frac{agk}{\omega}\sin(kx-\omega t)$ on $z=0$

$\frac{\partial \eta}{\partial t} = a\omega\sin(kx-\omega t)$ as $\eta = a\cos(kx-\omega t)$

$\frac{\partial \eta}{\partial t} = w$, so $\omega = \frac{gk}{\omega}$

ie. $\omega^2 = gk \longleftarrow$ Dispersion relation

$c^2 = \frac{\omega^2}{k^2} = \frac{g}{k} = \frac{g\lambda}{2\pi}$

so $c = (2\pi)^{-\frac{1}{2}}\sqrt{g\lambda}$

Waves of different wavelengths travel at different speeds (unlike sound, light or EM radiation (in vaccum)).
(long displacement hulls travel fastest)

λ (m)	c (m/s)	$\tau = 2\pi/\omega$ (s)
100	12.5	8
1	1.25	0.8
0.01	0.125	0.08

$u, w, \phi \sim e^{kz}$

So if we are 1 wavelength below the surface ie. $z = -\lambda$, decay is given by

$e^{-k\lambda} = e^{-2\pi} = 0.002$

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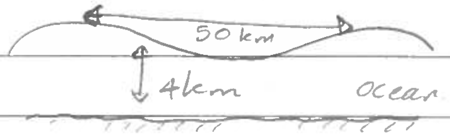
Thus only if the depth is less than about a wavelength does the bottom influence the motion.

Tsunami - bottom is important



Modify theory to include finite depth.
ie. rigid boundary at $z=-h$.
Everything is the same, except replace the lower b.c. by 'no normal flow' at $z=-h$

ie. $w=0$ at $z=-h$

ie. $\dfrac{\partial\phi}{\partial z}=0$ at $z=-h$

As before, $\eta = a\cos(kx-\omega t)$

$\phi = g\eta$ on $z=0$

Try $\phi = ag\cos(kx-\omega t)\,Z(z)$

Laplace eqn. again so

$$\begin{cases} Z''-k^2Z=0 & \text{as before} \\ Z(0)=1 & \\ \text{Now } Z'(-h)=0 & \end{cases}$$

C.F, $Z(z)= A\sinh k(z+h) + B\cosh k(z+h)$

Now $Z'(-h) = Ak\cosh(0) = Ak$

But this vanishes so $A=0$.

But $Z(0)-1$ so $B\cosh kh = 1$

Hence $Z(z) = \dfrac{\cosh k(z+h)}{\cosh kh}$

Hence $\phi = ag\cos(kx-\omega t)\cos k(z+h)/\cosh kh$

Remains to do kinematic condition at $z=0$, as before.

$$\frac{\partial w}{\partial t} = -\frac{\partial \phi}{\partial z}$$

$$= -agk \cos(kx-\omega t) \sinh k(z+h) \cosh kh$$

$$= \frac{agk}{\omega} \sin(kx-\omega t) \tanh kh \qquad \text{on } z=0$$

But $\frac{\partial \eta}{\partial t} = \omega a \sin(kx-\omega t)$

$w = \frac{\partial \eta}{\partial t}$ on $z=0$

So $\frac{gk}{\omega} \tanh kh = \omega$

$\Rightarrow \omega^2 = gk \tanh kh$ freq. - wavenumber relation

$\Rightarrow$ dispersion relation extended to finite depth.

$$\frac{\omega^2}{k^2} = \frac{g}{k} \tanh kh$$

i.e. $\frac{c^2}{gh} = \frac{\tanh(kh)}{kh}$

$$= \frac{\tanh(2\pi h/\lambda)}{(2\pi h/\lambda)}$$

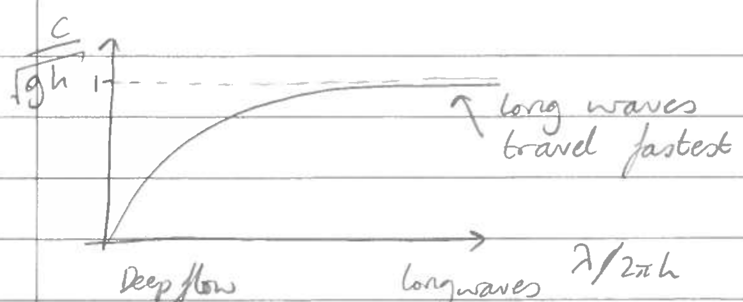
$\to 1$ if $2\pi h/\lambda \to 0$ (shallow flow)

then $c = \sqrt{gh}$ as used in open channel flow

$x \to \infty$, $\frac{h}{\lambda} \to \infty$ (infinitely deep)

$\frac{c^2}{gh} = \frac{\lambda}{2\pi h}$ i.e. $c^2 = \left(\frac{1}{2\pi}\right) \lambda g$ as befor

Depth	Max speed
1 m	3.1 m/s
1 cm	31 cm/s



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4.2 – Particle Paths

We have

$$\frac{\partial u}{\partial t} = -\frac{\partial \phi}{\partial x} = \frac{agk}{\omega\cosh kh}\cosh k(z+h)\cos(kx-\omega t)$$

$$\frac{\partial w}{\partial t} = -\frac{\partial \phi}{\partial z} = \frac{agk}{\omega\cosh kh}\sinh k(z+h)\sin(kx-\omega t)$$

For a p.p. $\frac{dx}{dt} = u \quad \frac{dz}{dt} = w$

notice η, u, w are all $a, \varepsilon << 1$ so $\Delta x \sim \varepsilon$, $\Delta z \sim \varepsilon$.

Write $x = x_0 + \xi$ where ξ, ζ are small $(\sim \varepsilon)$

$z = z_0 + \zeta$

Then $\frac{d\xi}{dt} = \frac{agk}{\omega\cosh kh}\cosh k(z_0+h)\cos(kx_0-\omega t)$

Where we have replaced x by x_0 and z by z_0 with error of order ε in the function and so ε^2 in u (because of the factor a) i.e. no further error.

Integrate w.r.t. t:

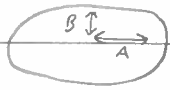
$$\xi = A\sin(kx_0-\omega t)$$

where $A = \frac{agk}{\omega^2\cosh kh}\cosh k(z_0+h)$

Similarly $\zeta = B\cos(kx_0-\omega t)$, $B = \frac{agk}{\omega^2\cosh kh}\sinh k(z_0+h)$

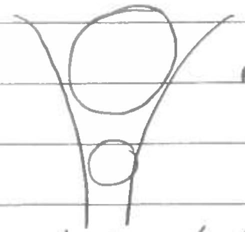
Thus $\left(\frac{\xi}{A}\right)^2 + \left(\frac{\zeta}{B}\right)^2 = 1 \qquad A > B.$

This is an ellipse with semi-major axis A and semi-minor axis B.



At the bottom $z_0 = -h$, $B = 0$

As $h \to \infty$, $A \to B$ i.e. p.p. become circles.



radius decreases exponentially with depth.

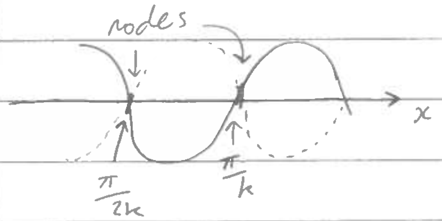
or from the infinite depth solution.

4.3 - Standing waves

Consider two co-existing waves of same amplitude, same wavelength, same period, but travelling in opposite directions.

So $\eta = a\cos(kx - \omega t) + a\cos(kx + \omega t)$

$= 2a\cos\omega t \cos kx$



We already have the solution for the first term in η, $\phi_1 = ag\frac{\cosh(k(z+h))}{\cosh kh}\cos(kx - \omega t)$

Thus solution for second term is $(\omega \to -\omega)$

$$\phi_2 = \frac{ag\cosh(k(z+h))}{\cosh kh}\cos(kx + \omega t)$$

Adding (as problem linear)

$$\phi = \frac{2ag\cosh k(z+h)}{\cosh kh}\cos\omega t\cos kx$$

Thus $u = \frac{agk}{\omega\cosh kh}\cosh k(z+h)\sin\omega t\sin kx$

$w = \frac{agk}{\omega\cosh kh}\sinh k(z+h)\sin\omega t\cos kx$

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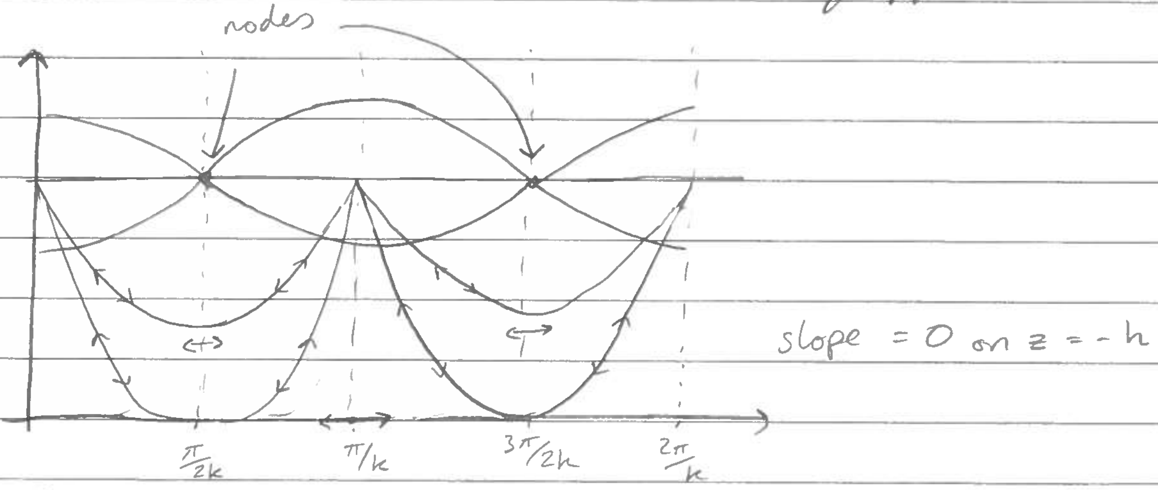
Now for p.p.'s $\frac{dx}{dt} = u$, $\frac{dz}{dt} = w$

Thus $\frac{dz}{dx} = \frac{w}{u} = \tanh k(z+h) \cot kx$

slope of p.p.'s



Replace any antinode by a solid wall (check $u=0$ there)