# 2301 Fluid Mechanics Notes (Part 1 of 2)

## Based on the 2012 autumn lectures by Prof E R Johnson

MATH2301 – Fluid Dynamics.

1 October 2012.
Prof ER JOHNSON.
Archaeology G6.

Room: 805. — office hour at thursdays 1pm.

Fluids are intrinsically non-linear.

Applications — weather pattern prediction — industrial design (e.g. for things such as cars) etc.

Homework Problem sheets due ~~at~~ Monday, 10am – submitted to Prof Vanden-Broeck.

Syllabus on Moodle, including suggestions for textbooks.

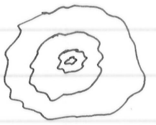
Problem sheets available on Moodle, answers subsequently available in print form (but not electronic). Answers to past papers are incomplete and will not draw full credit.

Notes will be posted online by section, after the respective topics have eventually been completed.

chapter 1.
SPECIFICATION and KINEMATICS.

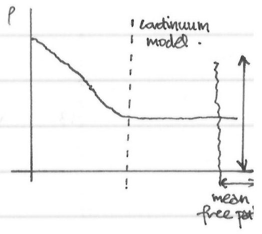
1.1 Assumptions of fluids.

1.1.1 continuum model.



Definition A continuum is something which we can take arbitrarily small amounts of.

For instance, we can define density, $\rho = \lim_{\delta V \to 0} \frac{\delta M}{\delta V}$. this is the density at the point contained in all $\delta V$.




With this continuum model, terms like "density at a point" make sense.

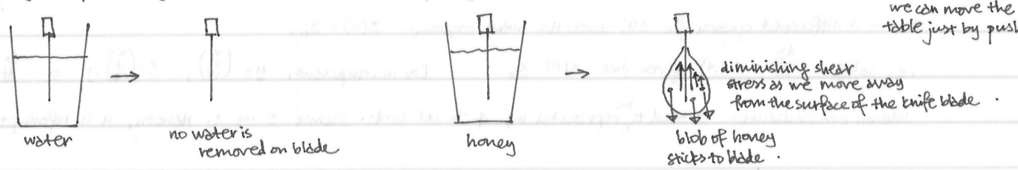
Similarly velocity, $\underline{u}$, or other quantities such as temperature, pressure.

1.1.2 Fluids are inviscid.

"inviscid": i.e. not viscous, nor "sticky". the fluid does not support a shear stress → a force tangential to the surface of contact.


we can move the paper along the table just by pushing the duster.

Imagine sticking a knife into a glass of water vs. oil (or honey), and then pulling out.




Real fluids can be viscous – but for purpose of this course we assume ideal fluids.

Air and water well approximate ideal fluids.

1.1.3 The fluid is incompressible.

"incompressible": cannot be compressed.

of course, we know that air can be compressed from daily experience. But in upper atmosphere, it is acceptable to assume incompressibility.

Compressibility is determined by the ratio of the flow, $U$, to the speed of sound, $a$.

Definition We define the Mach number, $M = \frac{U}{a}$

If $M \ll 1$, the fluid is incompressible (i.e. $U \ll a$)

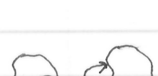
If $M \gg 1$, the fluid is compressible (i.e. $U \gg a$)

We also use the terms supersonic where $M > 1$, subsonic where $M < 1$.

Consequences: Given any labelled blob (or fluid element / fluid particle), we know that it has the same $V$ (due to incompressibility)

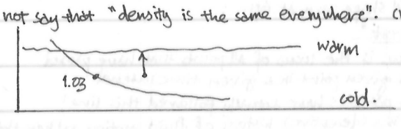
It also has the same mass, $M$, because mass cannot be created or destroyed (all particles remain the same)

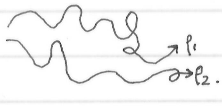
$\therefore$ it has the same $\frac{M}{V}$ i.e. same $\rho$. thus, in an incompressible fluid, fluid elements conserve ~~that~~ density.



N.B. it does not say ~~that~~ "density is the same everywhere". (we assume immiscibility).

for instance, the ocean



1.2 Describing motion.

There are two approaches to describing motion of fluids.

i) Lagrangian approach — labelling the fluid particles and following their movement (e.g. like planets).

advantage: some equations are trivial e.g. $\rho(\underline{x}_0, t) = \rho(\underline{x}_0, 0)$; hence particles originally at $\underline{x}_0$ where $t=0$ still has same density at time $t$.

this is incompressibility.

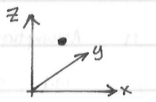
likewise, Newton's equation of motion for force per unit mass: $\frac{du}{dt} = F$.

conservation laws are trivial.

ECMWF has maps of "potential vorticity" (local spinning in atmosphere)

disadvantage: particle paths rapidly become entangled. for instance spatial derivatives are difficult.

ii) Eulerian description — taking a fixed reference frame



Eulerian grid.

We define the <u>Eulerian velocity</u>, $\underline{u}(\underline{x}, t)$ at the fixed point $\underline{x}$ and given ~~point~~ time $t$,

as the velocity of the fluid particle that happens to be at $\underline{x}$ at time $t$.

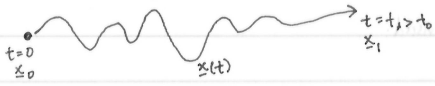
this gives us a fixed frame of reference for vector manipulations (a field is created).

advantage: we can use all standard vector apparati.

disadvantage: equations become more complicated because motion (change in particles) must be taken into account $\Rightarrow$ non-linear equations.

1.3 Visualing flow.

1.3.1
<u>Particle path:</u>




We can label a particle and see where it goes.

How do we find these? Suppose the Eulerian velocity field is $\underline{u}(\underline{x}, t)$. i.e. $\frac{d\underline{x}}{dt} = \underline{u}(\underline{x}, t)$

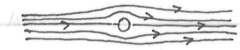
speed of particle at $\underline{x}$ (Lagrangian) — speed at $\underline{x}$ (Eulerian).

This is a differential equation for $\underline{x}(t)$ given the initial condition $\underline{x}(0) = \underline{x}_0$.

i.e. solve $\frac{d\underline{x}}{dt} = \underline{u}(\underline{x}, t)$ given that $\underline{x}(0) = \underline{x}_0$ [or in components, $\underline{u} = \begin{pmatrix} u \\ v \\ w \end{pmatrix}$, $\underline{x} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$.] $\Rightarrow$ $\frac{dx}{dt} = u$, $\frac{dy}{dt} = v$, $\frac{dz}{dt} = w$, with $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x_0 \\ y_0 \\ z_0 \end{pmatrix}$ at $t=0$.

Note on nomenclature: $\underline{u}$ used to be represented by $\underline{q}$ in old books. likewise $\underline{x}$ by $\underline{r}$. Hereon, $u$ is referring to the x-component of $\underline{u}$, not $|\underline{u}|$.

cross-section of pipe in the middle of laminar flow



this should represent the same case for all profiles in and out of paper.

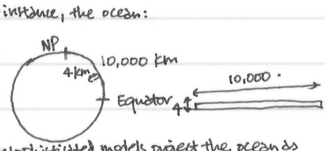
$\boxed{\text{Ex } 1}$ Consider the velocity field $\underline{u} = \hat{\underline{i}} - 2te^{-t^2}\hat{\underline{j}}$. A particle is released from $(1,1)$ at $t=0$.

Note: this is an example of a two-dimensional flow, or a <u>planar</u> flow. Hence in all planes, $z$ is constant.

<u>Soln.</u> $u=1$, $\frac{dx}{dt} = 1$, $x = t + x_0$. $v = -2te^{-t^2}$, $\frac{dy}{dt} = -2te^{-t^2}$, $y = e^{-t^2} + y_0$.

since $x_0 = 1$, $y_0 = 1$; then $x = t+1$, $y = e^{-t^2}$.

for instance, the ocean:

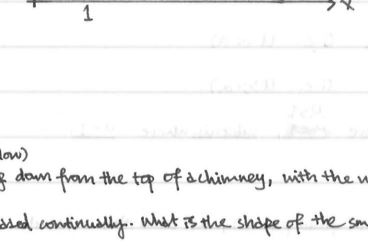



unsophisticated models project the ocean as behaving with a 2D flow.

Note: can be ploted into Mathematica/ Wolfram Alpha: ParametricPlot[{1+t, e^{-t^2}}, {t,0,10}].

For this case, we can eliminate $t$ and convert to cartesians.

$t = x-1$, so $y = e^{-(x-1)^2}$.

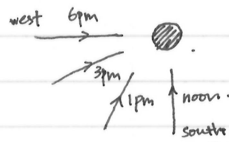


Path of particle.

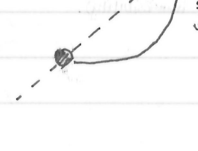
1.3.2 Streaklines.

(for unsteady flow)

Imagine looking down from the top of a chimney, with the wind changing direction throughout the day.




Smoke is released continually. What is the shape of the smoke trail seen at 6pm, and what is the path followed by the smoke blob emitted at midday?



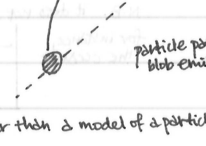
smoke trail shape seen at 6pm:

"smoke <u>streak</u>"-

<u>a streakline</u> is the locus of all points that have passed through a given point in a given time interval!

Note: no particles have actually followed this line!

it is a (deceptive) history of fluid motion rather than a model of a particle.



particle path followed by blob emitted at midday.

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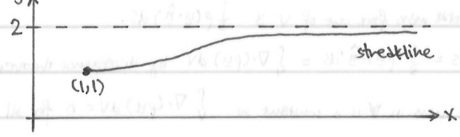
Ex 2 Consider the same flow as in Ex 1: $\underline{u} = \hat{\underline{i}} - 2te^{-t^2}\hat{\underline{j}}$. Find the streakline at $t=0$ for all particles released from $(1,1)$ at $t=\tau<0$.

Soln. As before, $x = t + x_0$, $y = e^{-t^2} + y_0$. However, initial conditions now differ.

$x=1$, $y=1$ when $t=\tau<0$. $\therefore$ $x_0 = 1-\tau$, $y_0 = 1-e^{-\tau^2}$.

$\therefore$ $x = t+1-\tau$, $y = e^{-t^2}+1-e^{-\tau^2}$. this is the position at time $t$ of particle emitted from $(1,1)$ at $t=\tau$.

For $t=0$, this gives $x = 1-\tau$, $y = 2-e^{-\tau^2}$, i.e. $y = 2-e^{-(1-x)^2}$

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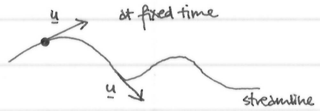
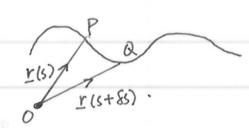
1.3.3 Streamlines.




A streamline is a line which at time $t$ is parallel to the local velocity vector.

Most mathematically useful – does not contain history.

Consider a curve $\underline{r}(s)$ parametrised by $s$. For the marked points P, Q, we have $\vec{PQ} = \underline{r}(s+\Delta s) - \underline{r}(s)$.




then $\lim_{Q\to P} \vec{PQ} = \lim_{\Delta s\to 0} \frac{\underline{r}(s+\Delta s)-\underline{r}(s)}{\Delta s} = \frac{d\underline{r}}{ds}$. thus, we know that $\frac{d\underline{r}}{ds}$ is tangential to $\underline{r}(s)$.

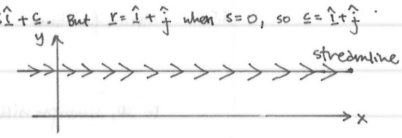
Mathematically, let our streamline be parametrised by $s$. Then $\frac{d\underline{r}}{ds} = \underline{u}(\underline{r}, t_0)$ – at fixed time.

* Note: compare this to a particle path, which has $\frac{d\underline{r}}{dt} = \underline{u}(\underline{r}, t)$

Ex 3. For the same velocity field as in Ex 2, i.e. $\underline{u} = \hat{\underline{i}} - 2te^{-t^2}\hat{\underline{j}}$. Find the streamline through $(1,1)$ at $t=0$.

Soln. We seek $\frac{d\underline{r}}{ds} = \underline{u}(\underline{r}, 0)$ solution, with $\underline{r}|_{s=0} = \hat{\underline{i}} + \hat{\underline{j}}$. Here, $\frac{d\underline{r}}{ds}\Big|_{t=0} = \hat{\underline{i}}$, i.e. $\underline{r} = s\hat{\underline{i}} + \underline{c}$. But $\underline{r} = \hat{\underline{i}} + \hat{\underline{j}}$ when $s=0$, so $\underline{c} = \hat{\underline{i}} + \hat{\underline{j}}$.

thus $\underline{r} = (s+1)\hat{\underline{i}} + \hat{\underline{j}}$, is the streamline




Plot: in 2D, $y=1$, $x=1+s$. so for $-\infty<s<\infty$, then $-\infty<x<\infty$.

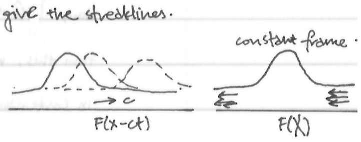
Note: In fact, $\underline{u} = \hat{\underline{i}}$ everywhere at $t=0$, so streamline through any point is straight line parallel to x-axis.

Result: these three cases are all the same in steady flow. By steady flow, we mean $\frac{\partial \underline{u}}{\partial t} = 0$ i.e. the flow does not change in time.

so, if a flow settles down in an experiment to be steady, then the streaklines are the particle paths, which in turn give the streaklines.

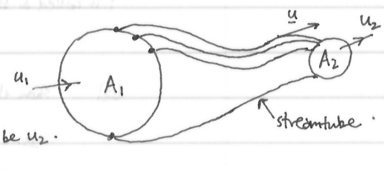
Note that fluid still moves in a steady flow! It is just that the patterns of motion remain the same.

For instance, water waves are steady. We simply consider moving into a frame translating at speed $c$ to the right




1.4 Incompressibility

We define a streamtube as a tube whose walls are streamlines. We define two terminal areas, $A_1$ and $A_2$.




There is no normal component of $\underline{u}$ across the walls of the tube i.e. no flow across tube walls.

Suppose $A_1$ is sufficiently small that the velocity across it is uniform and equal to $u_1$. Similarly, let the velocity across $A_2$ be $u_2$.

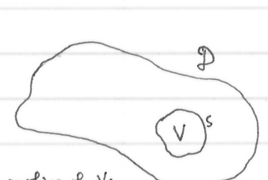
Then the amount of fluid entering the tube in time $\delta t$ is $(u_1\delta t)A_1$, because all particles within a distance $u_1\delta t$ enter.

Likewise, the amount leaving is $(u_2\delta t)A_2$. thus since mass is conserved and the fluid is incompressible, these are the same, and $\therefore$

$u_1A_1\delta t = u_2A_2\delta t$ i.e. $u_1A_1 = u_2A_2$. To conserve mass, speed varies inversely as the cross-sectional area of the streamtube.

Application The surface isobars – lines of equal surface pressure – on a weather map are very closely the streamlines for the surface flow. And so, when close together there are high winds, when far apart there are low winds.

1.4.1 Conservation of Mass for a fluid of fixed density.

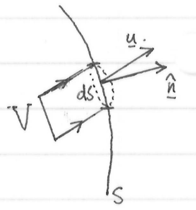



Consider a fluid of constant density $\rho$, occupying a domain $\mathcal{D}$. Take a fixed arbitrary volume V contained in $\mathcal{D}$. Let S be the surface of V.

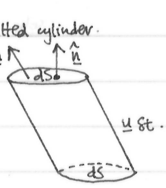
i.e. if anything applies in this V, it applies for all V in $\mathcal{D}$.

We examine a cross-section of $V$ with its surface $S$. Take $dS$ to be an area element at the surface $S$.

$\hat{\underline{n}}$ is the normal to $dS$, and the flow in the neighbourhood of $dS$ is $\underline{u}$.




The amount of fluid leaving $V$ through the element $dS$ in time $\delta t$ is given by the volume of the tilted cylinder.




This amount is $dS \times$ component of $\underline{u}\,\delta t$ in direction $\hat{\underline{n}}$ = $dS \times (\underline{u}\,\delta t)\cdot\hat{\underline{n}} = (\underline{u}\cdot\hat{\underline{n}})\, dS\, \delta t$.

We define the flux of fluid through $dS$ as the rate at which fluid leaves through $dS$, i.e. $(\underline{u}\cdot\hat{\underline{n}})\,dS$.

Here we have a volume flux. Given constant density, the mass flux is $\rho(\underline{u}\cdot\hat{\underline{n}})\,dS$.

On a general level hence, the total mass flux out of $V$ is $\int_S \rho(\underline{u}\cdot\hat{\underline{n}})\,dS$.

We know too that $\int_S \rho(\underline{u}\cdot\hat{\underline{n}})\,dS = \int_S (\rho\underline{u})\cdot\hat{\underline{n}}\,dS = \int_V \nabla\cdot(\rho\underline{u})\,dV$ by divergence theorem.

But mass is conserved. The total mass in $V$ is a constant so $\int_V \nabla\cdot(\rho\underline{u})\,dV = 0$ for all $V$ in $\mathcal{D}$.

Lemma: Let $f(x)$ be continuous on $[a,b]$. If for each $[c,d]\subset[a,b]$, $\int_c^d f(x)\,dx = 0$, then $f(x)\equiv 0$ on $[a,b]$.

Proof – Suppose the conditions for the lemma hold, but $f(x)\not\equiv 0$. i.e. $\exists\,\alpha\in[a,b]$ where $f(\alpha)\neq 0$.

WLOG, we can take $f(\alpha)>0$. Write $\varepsilon = \frac{1}{2}f(\alpha)>0$. Thus from definition of continuity, $\exists\,\delta>0$ s.t. $|x-\alpha|<\delta \Rightarrow |f(x)-f(\alpha)|<\varepsilon$.

i.e. in $(\alpha-\delta,\alpha+\delta)$, $\frac{1}{2}f(\alpha) < f(x) < \frac{3}{2}f(\alpha)$. Hence, taking integrals from $\alpha-\delta$ to $\alpha+\delta$, $\int_{\alpha-\delta}^{\alpha+\delta} f(x)\,dx > 2\delta\cdot\frac{1}{2}f(\alpha) = \delta f(\alpha) > 0$.

But that is a contradiction, and no such $\alpha$ exists. q.e.d. ($\because$ taking $c=\alpha-\delta$, $d=\alpha+\delta$, it should equal 0).

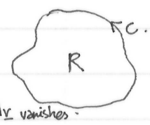
Returning to 3D, provided that $\underline{\nabla}\cdot(\rho\underline{u})$ is continuous in $\mathcal{D}$, then since $\int_V \nabla\cdot(\rho\underline{u})\,dV$ vanishes $\forall V$ in $\mathcal{D}$, then $\underline{\nabla}\cdot(\rho\underline{u})=0$ in $\mathcal{D}$.

Or, since $\rho$ is constant, $\underline{\nabla}\cdot\underline{u}=0$ in $\mathcal{D}$.

i.e. conservation of volume $\Rightarrow$ $\underline{\nabla}\cdot\underline{u}=0$ i.e. $\underline{u}$ is solenoidal, or divergence free.

## 1.5 Stream function.

In 2D incompressible flow, where $\underline{u} = u(x,y)\hat{\underline{i}} + v(x,y)\hat{\underline{j}}$, incompressibility gives $\underline{\nabla}\cdot\underline{u}=0$, i.e. $\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0$.




Introduce $\underline{F} = -v\hat{\underline{i}} + u\hat{\underline{j}}$. Then we consider an arbitrary region $R$ bounded by a curve $C$, and apply Green's theorem to $\underline{F}$.

i.e. $\oint_C \underline{F}\cdot d\underline{r} = \oint_C (-v\,dx + u\,dy) = \int_R \left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)dx\,dy = 0$. $\underline{F}$ is a vector field and the integral around every closed curve of $\oint_C \underline{F}\cdot d\underline{r}$ vanishes.

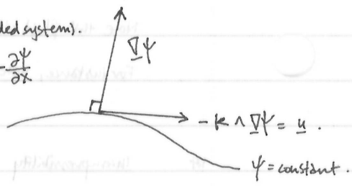
Thus, $\underline{F}$ is a conservative vector field, and $\underline{F}$ is derivable from a potential. i.e. $\exists\,\psi$ s.t. $\underline{F} = \underline{\nabla}\psi$.

Now, $-v\hat{\underline{i}} + u\hat{\underline{j}} = \underline{\nabla}\psi$. We cross this with $\hat{\underline{k}}$: $-v(\hat{\underline{k}}\wedge\hat{\underline{i}}) + u(\hat{\underline{k}}\wedge\hat{\underline{j}}) = \hat{\underline{k}}\wedge\nabla\psi \Rightarrow -v\hat{\underline{j}} - u\hat{\underline{i}} = -\underline{k}\wedge\nabla\psi$ and $\underline{u} = -\hat{\underline{k}}\wedge\underline{\nabla}\psi$.

From this, we have proven that $\boxed{\text{2D + incompressibility} \Rightarrow \exists\,\psi \text{ s.t. } \underline{u} = -\hat{\underline{k}}\wedge\underline{\nabla}\psi}$ (where $\hat{\underline{i}},\hat{\underline{j}},\hat{\underline{k}}$ form a right-handed system).

In Cartesians: $\nabla\psi = \frac{\partial\psi}{\partial x}\hat{\underline{i}} + \frac{\partial\psi}{\partial y}\hat{\underline{j}} \Rightarrow \underline{k}\wedge\nabla\psi = \frac{\partial\psi}{\partial x}\hat{\underline{j}} - \frac{\partial\psi}{\partial y}\hat{\underline{i}} = -u\hat{\underline{i}} - v\hat{\underline{j}} = \underline{u}$. Thus, $u = \frac{\partial\psi}{\partial y}$ and $v = -\frac{\partial\psi}{\partial x}$

$$\left[\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y} = \frac{\partial^2\psi}{\partial x\partial y} - \frac{\partial^2\psi}{\partial x\partial y} = 0\right].$$




The lines $\psi$ = constant are streamlines, because their tangents are parallel to the velocity vector.

$\psi$ is called a stream function.

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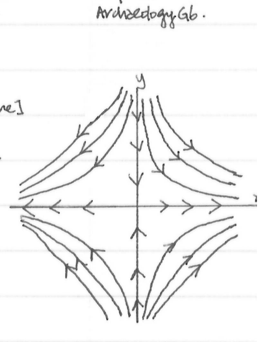
Ex: Show that velocity field $\underline{u} = x\hat{\underline{i}} - y\hat{\underline{j}}$ is an incompressible flow (steady) $\left(\frac{\partial\underline{u}}{\partial t}=0\right)$, and that it is 2D so $\exists\,\psi$. Find $\psi$.

Soln. $\underline{\nabla}\cdot\underline{u} = \frac{\partial u}{\partial x}+\frac{\partial v}{\partial y} = 1-1=0 \Rightarrow$ it is an incompressible flow. q.e.d. ($u=x$, $v=-y$).

Hence, $\exists\,\psi$. $\frac{\partial\psi}{\partial y}=u$, $\frac{\partial\psi}{\partial x}=-v$, but $u=x$ so $\frac{\partial\psi}{\partial y}=x$, and so $\psi = xy + f(x)$ [$f(x)$ is an arbitrary function of $x$ alone]




This means that $\frac{\partial\psi}{\partial x} = y + f'(x)$. But $\frac{\partial\psi}{\partial x} = -v = y$, and thus $f'(x)=0 \Rightarrow f(x) = \text{const}$. Thus, $\psi = xy + \text{constant}$.

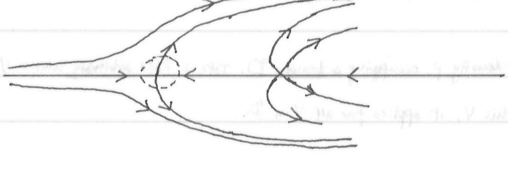
Note: the value of this added constant is arbitrary and irrelevant, as only derivatives of $\psi$ appear in our definition of $\psi$.

Here, we take const $=0$. The streamlines are the lines $\psi = \text{const}$. i.e. const $= xy \Rightarrow xy = A$ (say).

If $A=0$, $xy=0$ so $x=0$ or $y=0$. When $y>0$, $v<0$; when $y<0$, $v>0$. When $x>0$, $u>0$; when $x<0$, $u<0$.

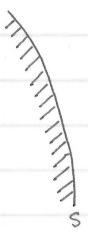
Then try $A=1$, we get $xy=1$. We continue in this manner.

Notice that at origin $(0,0)$, $\underline{u}=\underline{0} \Rightarrow$ flow is at rest. We call this a stagnation point. The flow is termed stagnation point flow.



( ) denotes a stagnation point.

1.6 Inviscid flow at a solid boundary.




A solid boundary is also called an impermeable boundary, i.e. there is no flow through the surface S.

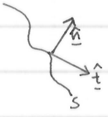
the flux through an element $dS$ is $(\underline{u}\cdot\hat{\underline{n}})\,dS$. the only way that can vanish for all $dS$ on S is if $\underline{u}\cdot\hat{\underline{n}}=0$ on S.

i.e. the normal component of the velocity vanishes on a solid surface.

Aside. In a real fluid, even with infinitesimally small viscosity, if you are sufficiently close to the boundary, the roughness of the boundary dominates.


sufficiently close to a boundary, a real fluid also has no tangential velocity i.e. $\underline{u}\cdot\hat{\underline{t}}=0$. combining with $\underline{u}\cdot\hat{\underline{n}}=0$. This gives $\underline{u}=0$ (stokes)

But we can write this in 2D, in terms of $\psi$. Our boundary condition is $\underline{u}\cdot\hat{\underline{n}}=0$ on S. then $\underline{u}=-\underline{k}\wedge\nabla\psi$

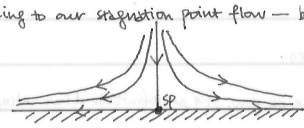



Hence, $\hat{\underline{n}}\cdot(\hat{\underline{k}}\wedge\nabla\psi)=0$ i.e. $(\underline{n}\wedge\underline{k})\cdot\nabla\psi=0$ i.e. $\hat{\underline{t}}\cdot\nabla\psi=0$. Take s as the distance along the boundary.

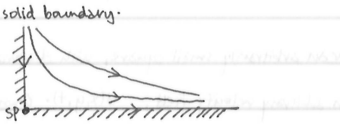
Then the directional derivative vanishes, giving $\frac{\partial\psi}{\partial s}=0$ on curve S in 2D, i.e. $\psi=$ constant on S.

We have thus proven that if C is solid, then $\psi=$ const. on C. conversely, if $\psi=$ const. on some curve, then we can replace that curve by a solid boundary without altering flow.

Returning to our stagnation point flow — but now replace any streamline by a solid boundary.

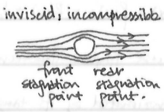



flow in a half-plane towards a solid planar wall




inviscid, incompressible 2D flow in a quarter plane.

this phenomenon can be seen in a cylinder placed across a planar flow.




11 October 2012
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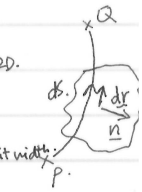
Roberts G06.

$\psi$ is a quantity with a physical meaning, and we term it as the streamfunction.

1.7 Physical meaning of $\psi$.

Given any two point in the fluid, say P and Q, the flux in the clockwise direction across any line joining P and Q is $\psi(Q)-\psi(P)$.

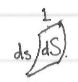



$\psi$ is the volume flux per unit distance perpendicular to the plane of motion: dimensionality of $L^2T^{-1}/L \Rightarrow L^2T$, i.e. it is an area flux in 2D.

Take an area element $dS$ as shown — then the volume flux across the element of area $dS$ is $(\underline{u}\cdot\hat{\underline{n}})dS$.

Or if $dS$ was length $ds$ along PQ and width 1 into the page, then the volume flux per unit width i.e. the area flux is $(\underline{u}\cdot\hat{\underline{n}})\,ds$ per unit width.

Tangent to PQ is $d\underline{r}=dx\,\hat{\underline{i}}+dy\,\hat{\underline{j}}$. thus a normal to this is $\underline{n}=dy\,\hat{\underline{i}}-dx\,\hat{\underline{j}}$. then $|\underline{n}|=\sqrt{dx^2+dy^2}=ds$, and so $\hat{\underline{n}}=\frac{dy}{ds}\hat{\underline{i}}-\frac{dx}{ds}\hat{\underline{j}}$.



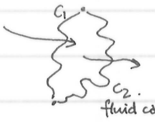

$dS = ds\times 1$

This vector $\hat{\underline{n}}$ is the unit normal to PQ in clockwise direction.

The total flux per unit width across PQ is

$$\int_P^Q(\underline{u}\cdot\hat{\underline{n}})\,ds=\int_P^Q\left(\frac{\partial\psi}{\partial y}\hat{\underline{i}}-\frac{\partial\psi}{\partial x}\hat{\underline{j}}\right)\cdot\left(\frac{dy}{ds}\hat{\underline{i}}-\frac{dx}{ds}\hat{\underline{j}}\right)ds=\int_P^Q\frac{\partial\psi}{\partial y}\frac{dy}{ds}+\frac{\partial\psi}{\partial x}\frac{dx}{ds}\,ds$$

$$=\int_P^Q\frac{d}{ds}\psi(x(s),y(s))\,ds=\int_P^Q\frac{d\psi}{ds}ds=\psi(Q)-\psi(P).$$



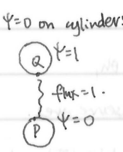

fluid cannot pile up

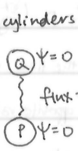
$\Rightarrow$ flux across any $C_1$ is the same as that across $C_2$

Aside: in fact, $\psi=0$ on cylinders. flow about two cylinders?

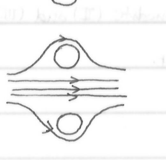
$\otimes\,\psi=\text{const}_1$

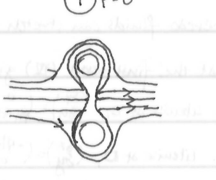
$\otimes\,\psi=\text{const}_2$. we can remove one constant, but the other remains




i.e.

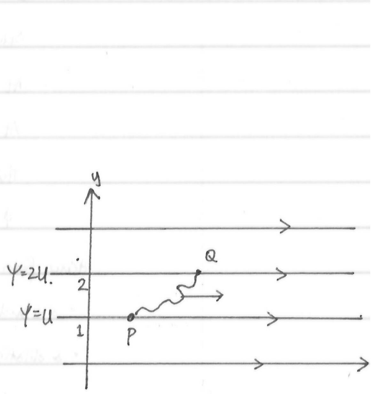




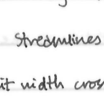
Ex uniform flow, $\underline{u}=U\hat{\underline{i}}$, so $\frac{\partial\psi}{\partial y}=U$, $\frac{\partial\psi}{\partial x}=0$.

soln. thus take $\psi=Uy$. Streamlines are lines $y=$const.

Then flux per unit width crossing PQ is $U$.




Ex (An isotropic source).



the natural coordinate system is cylindrical polar coordinates.

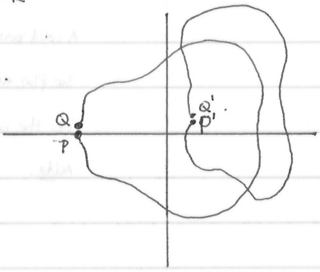
soln. so note that $\underline{u}=-\hat{\underline{k}}\wedge\nabla\psi$ where $\nabla\psi=\frac{\partial\psi}{\partial r}\hat{\underline{r}}+\frac{1}{r}\frac{\partial\psi}{\partial\theta}\hat{\underline{\theta}}$ and $\underline{u}=u_r\hat{\underline{r}}+u_\theta\hat{\underline{\theta}}$.

But $\underline{u}=-\frac{\partial\psi}{\partial r}\hat{\underline{\theta}}+\frac{1}{r}\frac{\partial\psi}{\partial\theta}\hat{\underline{r}}$; so by comparison $\boxed{u_r=\frac{1}{r}\frac{\partial\psi}{\partial\theta},\quad u_\theta=-\frac{\partial\psi}{\partial r}}$

The streamfunction for an isotropic source is $\psi = m\theta$ where $m$ is a constant, measuring the strength of the source.

$u_\theta = -\frac{\partial\psi}{\partial r} = 0$, $u_r = \frac{1}{r}\frac{\partial\psi}{\partial\theta} = \frac{m}{r}$. *note the decay of the radial velocity over time.

Note: In a 3D isotropic source, with components $\hat{R}$, $\hat{\theta}$ and $\hat{\phi}$, then $u_\theta = 0$, $u_\phi = 0$, $u_R = \frac{K}{R^2}$.




We now generalise for an arbitrary curve. Let $-\pi \le \theta < \pi$. Fluid clockwise across PQ is $\psi(Q) - \psi(P) = m\pi - (-m\pi) = 2m\pi$.

But for the path P'Q', $\psi(Q') - \psi(P') = 0 \Rightarrow$ no fluid is generated within any closed curve that does not cycle origin.

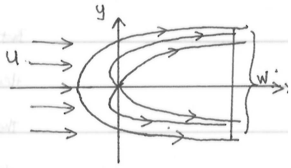
Thus all fluid comes from origin.

Ex Consider an isotropic source in a uniform stream. Find the distance between the line boundaries downstream.

Soln. $\psi_1 = Uy$, $\psi_2 = m\theta$. $\psi_3 = Uy + m\theta$, then streamlines are $\psi_3 = $ const.

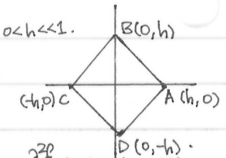
All fluid between the lines comes from origin and all this crosses PQ. But for downstream, this is $UW$

$\Rightarrow W = \frac{2\pi m}{U}$ //




1.8 Local motion of a fluid element.

Consider an arbitrarily small square, with diagonals $2h$ and vertices aligned with axes.




Take an arbitrary velocity field, $\underline{u}(x,y,t)$. Consider an infinitesimal fluid element in 2D flow, over an infinitesimal time interval $\delta t$.

Using Taylor's theorem, $f(x) = f(0) + x f'(0) + \frac{1}{2}x^2 f''(0) + O(x^3)$. Take $\delta t$ sufficiently small that we can regard $\underline{u}$ as steady.

For our small element, we use Taylor's theorem in 2D, which gives $f(x,y) = f(0,0) + x\frac{\partial f}{\partial x}(0,0) + y\frac{\partial f}{\partial y}(0,0) + \frac{1}{2}x^2\frac{\partial^2 f}{\partial x^2}(0,0) + xy\frac{\partial^2 f}{\partial x \partial y}(0,0) + \frac{1}{2}y^2\frac{\partial^2 f}{\partial y^2}(0,0) + \cdots$

Any function $f$ can be approximated by its tangent plane for sufficiently small distances: i.e. write in components — $u = U + \alpha x + \beta y$, $v = V + \gamma x + \delta y$.

Here, $\alpha = \frac{\partial u}{\partial x}(0,0)$, $\beta = \frac{\partial u}{\partial y}(0,0)$, $\gamma = \frac{\partial v}{\partial x}(0,0)$, $\delta = \frac{\partial v}{\partial y}(0,0)$. and $U = u(0,0)$, $V = v(0,0)$. i.e. $\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} U \\ V \end{pmatrix} + \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$.

We know that the flow is incompressible, i.e. $\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$, i.e. $\alpha + \delta = 0$.

Then $\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} U \\ V \end{pmatrix} + \begin{pmatrix} \alpha & \beta \\ \gamma & -\alpha \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$. Recall the decomposition $A = \underbrace{\tfrac{1}{2}(A + A^T)}_{\text{symmetric}} + \underbrace{\tfrac{1}{2}(A - A^T)}_{\text{anti-symmetric}}$. Then, $\begin{pmatrix} u \\ v \end{pmatrix} = \overset{(\mathrm{I})}{\begin{pmatrix} U \\ V \end{pmatrix}} + \left[\overset{(\mathrm{II})}{\alpha\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}} + \overset{(\mathrm{III})}{\theta\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}} + \overset{(\mathrm{IV})}{\phi\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}}\right]\begin{pmatrix} x \\ y \end{pmatrix}$.

where $\theta = \frac{1}{2}(\beta + \gamma)$ and $\phi = \frac{1}{2}(\beta - \gamma) = \frac{1}{2}\left(\frac{\partial u}{\partial y} - \frac{\partial v}{\partial x}\right)$.

Now, we work out what each of these terms does to the square

(I): Translates the square at a speed of $(U,V) = U\hat{\underline{i}} + V\hat{\underline{j}}$. — translation of the centre of mass.

(II): A point $\begin{pmatrix} x \\ y \end{pmatrix}$ moves to $\begin{pmatrix} x + u\delta t \\ y + v\delta t \end{pmatrix}$ in time $\delta t$, i.e. it moves by an amount $\begin{pmatrix} u\,\delta t \\ v\,\delta t \end{pmatrix} = \begin{pmatrix} \delta x \\ \delta y \end{pmatrix}$

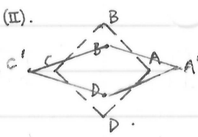
Here, particles move by $\alpha\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}\delta t$. At point A, $x = h, y = 0$. So A moves by $\begin{pmatrix} \delta x \\ \delta y \end{pmatrix} = \begin{pmatrix} \alpha h \delta t \\ 0 \end{pmatrix}$. Likewise, C moves by $\begin{pmatrix} -\alpha h \delta t \\ 0 \end{pmatrix}$.

For B, $\begin{pmatrix} \delta x \\ \delta y \end{pmatrix} = \begin{pmatrix} 0 \\ -h\alpha\delta t \end{pmatrix}$, and likewise for D, $\begin{pmatrix} \delta x \\ \delta y \end{pmatrix} = \begin{pmatrix} 0 \\ \alpha h \delta t \end{pmatrix}$.

i.e. the square is stretched along one axis and squashed along the orthogonal axis by equal and opposite amounts.

— dilation on orthogonal axes; preserves area (to order $h$).

(II).




(III): For A, $\begin{matrix} x=h \\ y=0 \end{matrix}$, $\begin{pmatrix} \delta x \\ \delta y \end{pmatrix} = \theta\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} h \\ 0 \end{pmatrix}\delta t = \begin{pmatrix} 0 \\ \theta h \delta t \end{pmatrix}$. By symmetry, for C, $\begin{pmatrix} \delta x \\ \delta y \end{pmatrix} = \begin{pmatrix} 0 \\ -\theta h \delta t \end{pmatrix}$.

For B, $\begin{pmatrix} \delta x \\ \delta y \end{pmatrix} = \theta\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 \\ h \end{pmatrix}\delta t = \begin{pmatrix} \theta h \delta t \\ 0 \end{pmatrix}$. By symmetry, for D, $\begin{pmatrix} \delta x \\ \delta y \end{pmatrix} = \begin{pmatrix} -\theta h \delta t \\ 0 \end{pmatrix}$.

This represents another dilation, involving a stretching along the line $y = x$ at a rate of $\theta h$,

and shrinking along the orthogonal line $y = -x$ at an equal and opposite rate, so as to conserve area.

15 October 2012.
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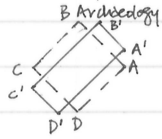
B Archaeology G06.

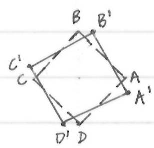



(IV): So far, solids have only been able to do (I), whereas fluids can stretch to accomodate (II) and (III).

Solids can also rotate, so we need to establish that the final term (IV) gives that.

At A, $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} h \\ 0 \end{pmatrix}$, $\begin{pmatrix} \delta x \\ \delta y \end{pmatrix} = \phi\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} h \\ 0 \end{pmatrix}\delta t = \begin{pmatrix} 0 \\ -\phi h \delta t \end{pmatrix}$. Likewise at C, $\begin{pmatrix} \delta x \\ \delta y \end{pmatrix} = \begin{pmatrix} 0 \\ \phi h \delta t \end{pmatrix}$.

At B, $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ h \end{pmatrix}$, $\begin{pmatrix} \delta x \\ \delta y \end{pmatrix} = \phi\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} 0 \\ h \end{pmatrix}\delta t = \begin{pmatrix} \phi h \delta t \\ 0 \end{pmatrix}$. Likewise at D, $\begin{pmatrix} \delta x \\ \delta y \end{pmatrix} = \begin{pmatrix} -\phi h \delta t \\ 0 \end{pmatrix}$.




This represents a rotation about the centre of mass, clockwise by an amount $\phi h\,\delta t$ in time $\delta t$, at angular speed $\phi$.

$\phi = \frac{1}{2}\left(\frac{\partial u}{\partial y} - \frac{\partial v}{\partial x}\right)$. Hence, the element is rotating in anti-clockwise direction at rate $\frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)$.

Hence from these, we have shown that the local motion consists of:

- a translation of the centre of mass;
- a dilation, and
- a rotation about the centre of mass at a rate $\frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)$ in the anti-clockwise direction.

2301-006.

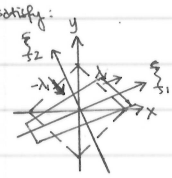
To understand the dilation, we use linear algebra to prove that the combination of two dilations in another dilation.

combine dilations (II) and (III) to get $\alpha\begin{pmatrix}1&0\\0&-1\end{pmatrix}+\theta\begin{pmatrix}0&1\\1&0\end{pmatrix}=\begin{pmatrix}\alpha&\theta\\\theta&-\alpha\end{pmatrix}$. This has eigenvalues $\lambda_{1,2}$ and corresponding eigenvectors $\underline{\xi}_{1,2}$ ($\because$ symmetric), which satisfy:

$(A-\lambda_{1,2}I)\underline{\xi}_{1,2}=0$. Then $|A-\lambda I|=\begin{vmatrix}\alpha-\lambda&\theta\\\theta&-\alpha-\lambda\end{vmatrix}=-(\alpha-\lambda)(\alpha+\lambda)-\theta^2=\lambda^2-(\alpha^2+\theta^2)$. This gives two roots, $\lambda_1=\sqrt{\theta^2+\alpha^2}$, $\lambda_2=-\sqrt{\theta^2+\alpha^2}$.

Since $\underline{\xi}_1$ and $\underline{\xi}_2$ correspond to distinct eigenvalues of a real symmetric matrix, and so are orthogonal. Taking orthogonal axes $\xi_1$ and $\xi_2$,

our matrix becomes $\begin{pmatrix}\lambda_1&0\\0&\lambda_2\end{pmatrix}=\lambda_1\begin{pmatrix}1&0\\0&-1\end{pmatrix}$. This is a similar form to term (II), but with different axes from Cartesians.




Then, we try to interpret our rotational term. First, we recall our equation of continuity, $\nabla\cdot\underline{u}=0$.

$\frac{1}{2}\left(\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}\right)$.

In 2D, the only way the rate of spinning of a fluid element can be altered is by exerting a shear stress on the element.

Thus, in a 2D inviscid fluid, each element preserves its angular momentum about its centre of mass (i.e. its rate of spinning about its centre of mass).

We call twice this rate of spinning the <u>vorticity</u> of the fluid element. In 3D, one obtains vorticity, $\underline{\omega}=\text{curl }\underline{u}=\nabla\wedge\underline{u}$.

The 2D case is just a special case of this, where $\frac{\partial u}{\partial z}=0$, $w=0$ so $\underline{\omega}=\zeta\hat{\underline{z}}$, $\zeta=\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}$.

Thus, each particle in a 2D inviscid flow conserves its value of $\zeta$. In general, these values can differ.

However, if $\zeta=0$ for all elements at some time, then $\zeta=0$ for all elements for all time $\Rightarrow$ $\nabla\wedge\underline{u}=0$ (true in 3D as well) $\rightarrow$ i.e. irrotational flow.

We call this property the <u>persistence of irrotationality</u>.

18 October 2012
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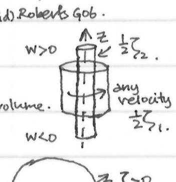
Recall that the rate of change of angular momentum = moment of force about centre of mass = 0 for a normal force (no shear stress as fluid is inviscid) Roberts G06.

We know that in 2D, every particle retains its value of $\zeta$. Why only in 2D? In 3D, imagine a rotating cylinder with any velocity $\frac{1}{2}\zeta$.

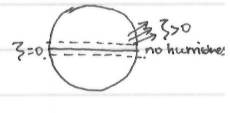
Along the z-axis, the cylinder can have $w>0$ above ~~xy axes~~ and $w<0$ below. Then it can dilate (stretch vertically), becoming thinner to conserve volume.

Moreover, to conserve angular momentum it must spin faster, i.e. $\zeta_1<\zeta_2$.




Aside: Hurricanes form where water is sufficiently hot. But why do we not find hurricanes at the equator?

Hurricanes amplify the local vertical component of the Earth's rotation. At the equator, there is no local component of vertical vorticity to amplify through vertical stretching.




Unlike the retention of $\zeta=0$, the persistence of irrotationality is conserved in 3D.

Thus even in 3D, for irrotational flow, $\nabla\wedge\underline{u}=0$ for all $t$ $\Rightarrow$ $\exists\,\phi$ s.t. $\underline{u}=\nabla\phi$. We call $\phi$ the <u>velocity potential</u>.

For incompressible flow, $\nabla\cdot\underline{u}=0$ so $\nabla\cdot(\nabla\phi)=0\Rightarrow\boxed{\nabla^2\phi=0}$ i.e. satisfies <u>Laplacian equation</u>.

$\phi$ exists for irrotational flow in 3D, and if it is also incompressible, $\nabla^2\phi=0$.

We have already had the streamfunction $\psi$. For incompressible and 2D flow, we have $\underline{u}=-\hat{\underline{k}}\wedge\nabla\psi$ (does not have to be irrotational).

$\underline{u}=-\nabla\wedge(\psi\hat{\underline{k}})$. If $\underline{u}$ is also irrotational, then $\nabla\wedge\underline{u}=0\Rightarrow\nabla\wedge(-\nabla\wedge\psi\hat{\underline{k}})=-\nabla^2\psi\hat{\underline{k}}=0$.

In cartesians, $u=\frac{\partial\psi}{\partial y}$, $v=-\frac{\partial\psi}{\partial x}$, $\zeta=\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}=-\frac{\partial^2\psi}{\partial x^2}-\frac{\partial^2\psi}{\partial y^2}=-\nabla^2\psi$. Thus if $\zeta=0$ everywhere, $\nabla^2\psi=0$ everywhere.

Consider the case where both sets of conditions apply — 2D, incompressible, irrotational flow — we have both $\phi$ and $\psi$.

$\underline{u}=\nabla\phi$ and $\underline{u}=-\underline{k}\wedge\nabla\psi$; so $\boxed{\nabla\phi=-\underline{k}\wedge\nabla\psi}$. In cartesians, $\underline{u}=\nabla\phi$, $u=\frac{\partial\phi}{\partial x}$, $v=\frac{\partial\phi}{\partial y}$. But $u=\frac{\partial\psi}{\partial y}$, $v=-\frac{\partial\psi}{\partial x}$ $\because$ $\underline{u}=-\underline{k}\wedge\nabla\psi$.

Thus, $\frac{\partial\phi}{\partial x}=\frac{\partial\psi}{\partial y}$ and $\frac{\partial\phi}{\partial y}=-\frac{\partial\psi}{\partial x}$ $\Rightarrow$ which are the <u>Cauchy-Riemann equations</u>. Thus $\phi(x,y,t)$ and $\psi(x,y,t)$ are conjugate harmonics $\Rightarrow$

We have that $\phi$ and $\psi$ are respectively the real and imaginary parts of a holomorphic complex function of the complex variable $z=x+iy$.

Thus we write $w(z,t)=\phi(x,y,t)+i\psi(x,y,t)$ with differentiable $w$.

We can also derive the Cauchy-Riemann equations in polar form — $\nabla\phi=\frac{\partial\phi}{\partial r}\hat{\underline{r}}+\frac{1}{r}\frac{\partial\phi}{\partial\theta}\hat{\underline{\theta}}$ $\Rightarrow$ $\frac{\partial\phi}{\partial r}=\frac{1}{r}\frac{\partial\psi}{\partial\theta}$, $\frac{1}{r}\frac{\partial\phi}{\partial\theta}=-\frac{\partial\psi}{\partial r}$; then $w(z)$ is a differentiable function of $z=re^{i\theta}$

Remember, for example, that in a velocity vector field, $u=\frac{\partial\phi}{\partial x}=\frac{\partial\psi}{\partial y}$; $v=\frac{\partial\phi}{\partial y}=-\frac{\partial\psi}{\partial x}$. Now $\frac{dw}{dz}=\frac{\partial w}{\partial x}=\frac{\partial}{\partial x}(\phi+i\psi)=\frac{\partial\phi}{\partial x}+i\frac{\partial\psi}{\partial x}=u-iv$

This gives us a formula for $w$, the <u>complex velocity potential</u> which satisfies the important result, $\boxed{\underset{\text{(any dir.)}}{\frac{dw}{dz}}=u-iv}$

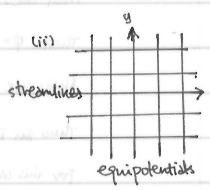
**EX** Find the flow for (i) $w=C$, constant; (ii) $w=Uz$, $U$ constant; (iii) $w=Az^2$; and sketch it if any.

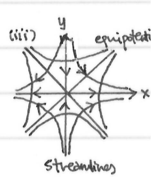
Soln. (i) $w=C$, $\frac{dw}{dz}=0$ i.e. $u=v=0\Rightarrow$ no flow.

(ii) $\frac{dw}{dz}=U$. So $u=U$ and $v=0\Rightarrow$ a uniform stream in x-direction.

Here $w=Uz=Ux+i(Uy)$, so $\phi=Ux$, $\psi=Uy$.

(iii) $w=Az^2=A(x^2-y^2)+2iAxy$, so $\phi=A(x^2-y^2)$, $\psi=2Axy$. $\Rightarrow$ stagnation point flow.



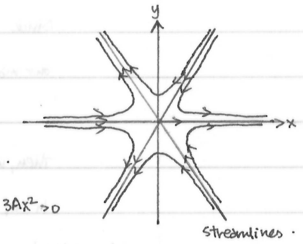



When are the streamlines and equipotentials orthogonal?

We know that $\underline{u} = \underline{\nabla}\phi$ and $\underline{u} = -\hat{\underline{k}} \wedge \underline{\nabla}\psi$, which means that $\underline{\nabla}\phi = -\hat{\underline{k}} \wedge \underline{\nabla}\psi$ i.e. $\underline{\nabla}\phi$ is perpendicular to $\underline{\nabla}\psi$ except where both equal zero.

Thus the level curves of $\phi$ and $\psi$ cut at right angles except where $\underline{\nabla}\phi = \underline{\nabla}\psi = 0$, i.e. $\underline{u} = 0$, a stagnation point.

$\boxed{\text{Ex}}$ Find the flow for $w = Az^3$, and sketch it.



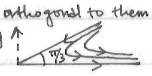
Streamlines.

Soln. We switch into polars, then $w = Ar^3 e^{3i\theta} = Ar^3(\cos 3\theta + i \sin 3\theta)$ s.t. $\phi = Ar^3 \cos 3\theta$, $\psi = Ar^3 \sin 3\theta$.

(Notice that power of $r$ and multiple in $\theta$ are the same integer). To plot, we draw $\psi = 0 \Rightarrow Ar^3 \sin 3\theta = 0$.

i.e. $\sin 3\theta = 0$, i.e. $\theta = 0, \pm\frac{\pi}{3}, \pm\frac{2\pi}{3}, \pi$. If $\psi$ = small const, streamlines are very close $\Rightarrow$ we just bend them.

Then equipotentials $\phi = k$ are just orthogonal to them. We then find directions: on $y=0$, $w = Ax^3$, $\frac{\partial w}{\partial x} = 3Ax^2 > 0$

In one sector, the flow for the example above gives 

and we see this by replacing any streamline by a solid boundary.

As such, we have established that any differentiable complex function gives a 2D, incompressible and inviscid flow, that is also irrotational.

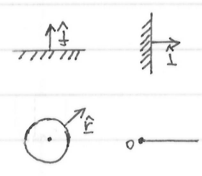
We generally call such a flow an <u>ideal flow</u>.

Then given any 2D ideal fluid flow problem, we have three choices of approach:

1) the streamfunction, $\psi$, satisfies $\nabla^2\psi = 0$. On a <u>single</u> solid boundary can take $\psi = 0$, plus a boundary condition at infinity.

2) the velocity potential, $\phi$, satisfies $\nabla^2\phi = 0$. On a solid boundary, $\underline{u} \cdot \hat{\underline{n}} = 0$ i.e. $\hat{\underline{n}} \cdot \underline{\nabla}\phi = 0$ i.e. normal component of $\underline{\nabla}\phi$ vanishes, $\frac{\partial \phi}{\partial n} = 0$.

For instance, if the boundary is $y=0$, condition is $\hat{\underline{j}} \cdot \underline{\nabla}\phi = 0$ i.e. $\frac{\partial \phi}{\partial y} = 0$; if the boundary is $x=0$, condition is $\hat{\underline{i}} \cdot \underline{\nabla}\phi = 0$ i.e. $\frac{\partial \phi}{\partial x} = 0$.

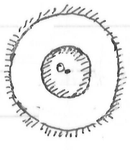



If the boundary is a circle $r=a$, condition is $\frac{x\hat{\underline{i}} + y\hat{\underline{j}}}{\sqrt{x^2+y^2}} \cdot \underline{\nabla}\phi = 0$, $\hat{\underline{r}} \cdot \underline{\nabla}\phi = 0$ i.e. $\frac{\partial \phi}{\partial r} = 0$.

If the boundary is a half-plane $x>0$, $y=0$. We can use Cartesians, $\frac{\partial \phi}{\partial y} = 0$ where $y=0$; or polars, $\frac{\partial \phi}{\partial \theta} = 0$ where $\theta = 0$.

3) the complex velocity potential, $w(z)$ is holomorphic in the flow domain. Boundary condition: No flow through the boundary.

Perhaps the simplest condition is $\psi$ = const, i.e. Im $w$ = const.




To solve these, we use the <u>Laurent series</u>. A function that is analytic (holomorphic) within an annular region $R_0 < |z| < R_1$

has an expansion of the form $f(z) = \dots + a_{-2}z^{-2} + a_{-1}z^{-1} + a_0 + a_1 z + a_2 z^2 + \dots = \sum_{n=-1}^{\infty} a_n z^n + \sum_{n=0}^{\infty} a_n z^n$.

(i.e. infinite number of negative poles + power series) — to be proved in Analysis.

We require that $\frac{dw}{dz} = u - iv$ is holomorphic in the flow domain. Thus all 2D, incompressible, irrotational flows within an annular region are of the form:

$u - iv = \sum_{n=-1}^{\infty} a_n z^n + \sum_{n=0}^{\infty} a_n z^n$. Hence it then remains to find the values of $a_n$.

Note that while the flow in the Laurent series has a singularity in the origin, the fact that a solid body is present and covers the origin will not affect flow.

(i.e. we must allow for the possibility of singularities (i.e. non-differentiable terms such as $\frac{1}{z}$) inside the body — because it is not part of flow field).

Note: of course, if there is no solid body, then there will be no singularities $\Rightarrow a_{-n} = 0$, $n = 1, 2, 3, \dots$

As long as solid body is at origin, we need not use circles, we can use any arbitrary curves.

Take a typical term: $z^n = r^n(\cos n\theta + i \sin n\theta) \Rightarrow \phi = r^n \cos n\theta$, $\psi = r^n \sin n\theta$ satisfy Laplacian equation: which in polar form is $\psi_{rr} + \frac{1}{r}\psi_r + \frac{1}{r^2}\psi_{\theta\theta} = 0$.

$z^{-n} = r^{-n}(\cos(-n\theta) + i \sin(-n\theta)) \Rightarrow$ real and imaginary parts are $r^{-n}\sin(-n\theta)$, $r^{-n}\cos(-n\theta)$: i.e. $-r^{-n}\sin(n\theta)$, $r^{-n}\cos(n\theta)$ which satisfy Laplacian eqn.

(− not necessary)

Thus any linear combination of functions drawn from the set $\{n=0,\dots : r^n \cos n\theta,\ r^n \sin n\theta,\ r^{-n} \cos n\theta,\ r^{-n} \sin n\theta\}$ satisfies Laplacian equation.

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In the 3D case, we do not have ideal fluid flow as defined above, so we are forced to use conditions $\underline{u} = \nabla\phi$, $\nabla^2\phi = 0$.

Whereas in 2D, we also can have $\underline{u} = -\underline{k} \wedge \underline{\nabla}\psi$ and $\exists\, w = \phi + i\psi$, which is a complex differentiable function.

We have shown that an arbitrary velocity field in an annulus can be expressed as a linear combination of $z^n$, $n = 0, \pm 1, \pm 2, \dots$

In polars, these are $1$, $r^{\pm n}\cos n\theta$, $r^{\pm n}\sin n\theta$ for $\psi, \phi$. To get from $u - iv = \frac{dw}{dz}$ back to $w$, integrate w.r.t. $z$.

Then $w$ is a linear combination of $\{\log z, z^n, n = 0, \pm 1, \pm 2, \dots\}$. Notice we include $\log z$, which comes from integrating $z^{-1}$.

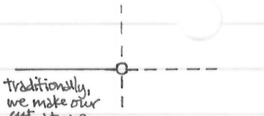
What does $\log z$ represent? $w = \phi + i\psi$. Consider $w = m \log z = m \log(re^{i\theta}) = m[\log r + \log e^{i\theta}] = m[\log r + i\theta]$ (i.e. $\log z = \log|z| + i \arg z$).

Then $\phi = m \log r$, $\psi = m\theta$. This is an isotropic source of strength $2\pi m$.

Note: we can make the cut at any half-line beginning at the origin (by convention, $\theta = \pi$).

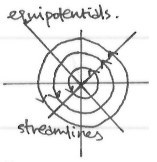
Thus an isotropic source of strength $2\pi m$ at $z = z_0$ is thus given by $w = m \log(z - z_0)$.

For this case, $\phi = m \log|z - z_0|$, $\psi = m \arg(z - z_0)$.



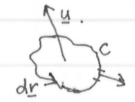
Traditionally, we make our cut at -ve real axis.
This gives principal value, Arg $z$, which is discontinuous across cut.

When we form linear combinations of $\log z$, $z^n$; $n = 0, \pm 1, \pm 2, \dots$ there is no reason why the constants should be real.




Consider $w = -ik \log z = -ik[\log r + i\theta] = k\theta - ik\log r$. Here, $\phi = k\theta$, $\psi = -k\log r$. $\Rightarrow$ equipotentials and streamlines swap around.

This is a <u>point vortex</u>, centred on the origin: this is going "around" — is it still irrotational?




Remember, for flux across the curve $C$, we have flux across curve $C = \oint_C (\underline{u}\cdot\hat{\underline{n}})\, dl$:

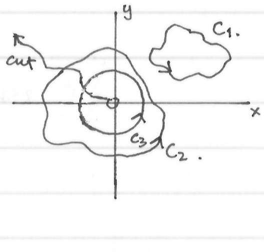
This is zero unless there is a source within. If there is a source within, then flux = strength.

We do the same for our "rotation". We define <u>circulation</u> about a curve $C$ as $\Gamma = \oint_C \underline{u}\cdot d\underline{r}$ (depends only on tangential component of $\underline{u}$.

Now if $\underline{u}$ is irrotational, $\underline{\nabla}\wedge\underline{u} = 0$ by definition. Then $\forall$ curves $C$, $\oint_C \underline{u}\cdot d\underline{r} = 0$ i.e. in irrotational flow, circulation about closed curve is zero.

Notice, this is true is general, but like our definition of flux over a curve, it requires differentiability of $w$. So it applies immediately to $z^{\pm n}$, provided we are in the annulus. For a sum of $z^{\pm n}$, flux across $C$ and circulation about $C$ are both zero.

Thus, the $\log z$ term is the only one potentially able to generate mass flux.



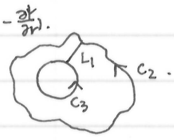

Let us calculate circulation due to $\underline{w} = -ik\log z$.

- For $C_1$, it is differentiable, so $\underline{\nabla}\wedge\underline{u} = \underline{0}$, $\oint_{C_1} = 0 \Rightarrow$ circulation about $C_1$ is $0$.
- For $C_2$, the arbitrary closed curve surrounding the origin, $\Gamma = \oint_{C_2} \underline{u}\cdot d\underline{r}$. But $\underline{u}$ is not differentiable at $0$.

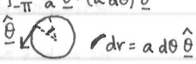
  There exists a circle say of radius $a$, inside $C_2$. Then if we draw a connecting line segment $L_1$,

  We get $-\oint_{C_3} \underline{u}\cdot d\underline{r} + \int_{C_2} \underline{u}\cdot d\underline{r} + \int_{L_1} \underline{u}\cdot d\underline{r} - \int_{L_1} \underline{u}\cdot d\underline{r} = 0$.

  Now we have $\phi = k\theta$, $\psi = -k\log r$. Then $\underline{u} = \nabla\phi$, $u_r = \frac{\partial\phi}{\partial r}$, $u_\theta = \frac{1}{r}\frac{\partial\phi}{\partial\theta}$. Then $u_r = 0$, $u_\theta = \frac{k}{r}$. (or $u_r = \frac{1}{r}\frac{\partial\psi}{\partial\theta}$, $u_\theta = -\frac{\partial\psi}{\partial r}$).




  On a circle of radius $a$, $u_\theta = \frac{k}{a}$, $u_r = 0$ i.e. $\underline{u} = \frac{k}{a}\hat{\underline{\theta}}$. Then around $C_3$, $\Gamma = \oint_{C_3} \underline{u}\cdot d\underline{r} = \int_{-\pi}^{\pi} \frac{k}{a}\hat{\underline{\theta}}\cdot(a\, d\theta)\,\hat{\underline{\theta}}$




  Then $\Gamma = k\int_{-\pi}^{\pi} |\hat{\underline{\theta}}|^2 d\theta = k\int_{-\pi}^{\pi} d\theta = 2\pi k$, independent of $a$, we can shrink radius $a$ to $\varepsilon > 0$.

  $\Rightarrow$ curve is irrotational everywhere, except at origin.

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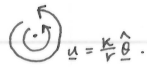
Recall that $w = m\log z \Rightarrow \phi = m\log r$, $\psi = m\theta$. This is a line source of strength $2\pi m$ at origin.




Also, $w = -ik\log z \Rightarrow \phi = k\theta$, $\psi = -k\log r$. This is a line vortex of strength (= circulation) $2\pi k$ at the origin.

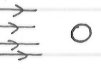
Thus, we will draw $w$ from linear combinations of the functions $\{\log z, z^{\pm n}\}$. So, we draw $\phi$ or $\psi$ from the real and imaginary parts i.e. from the set $\{\log r, \theta, r^n\cos n\theta, r^n\sin n\theta, r^{-n}\cos n\theta, r^{-n}\sin n\theta\}$ — there are no others (by Laurent series theory).

Consider an irrotational line vortex: $\underline{u} = \frac{k}{r}\hat{\underline{\theta}}$  Why is it irrotational? Consider an infinitesimal fluid element (e.g. a matchstick).

Then it moves in a circle — but it does not rotate about centre of mass, i.e. $\zeta = 0$. (Contrast this with solid body rotation, $\underline{u} = \underline{\Omega}\wedge\underline{r}$, $u = \Omega r\hat{\theta}$).

The only point where it spins is the origin, where the flow is not irrotational. Here, this is not irrotational — $\underline{\nabla}\wedge\underline{u} = \zeta\hat{\underline{z}} = 2\underline{\Omega}$.

<u>Uniform flow past a cylinder</u>




Imagine a cylinder with a stream $U$ passing over it. It is a 2D motion if we consider the cross-section — the cylinder has radius $a$, and the stream at large distance has $\underline{u}\to U\hat{\underline{i}}$ as $r\to\infty$. We can approach this three ways.

1) Streamfunction, $\psi$: then $\nabla^2\psi = 0$ where $r > a$. Boundary condition — at $r = a$, $\psi$ = constant. This is the only one solid boundary, so WLOG take $\psi = 0$ on $r = a$.

   At $\infty$, $u\to U$ $v\to 0$ i.e. $\frac{\partial\psi}{\partial y}\to U$, $\frac{\partial\psi}{\partial x}\to 0$ so $\psi = Uy$. Thus $\nabla^2\psi = 0$ subject to $\psi = 0$ on $r = a$, $\psi\to Uy$ at $r\to\infty$.

   Consider the inhomogeneous equation first: $r\to\infty$ gives $\psi\to Ur\sin\theta$, $r\to\infty$. (i.e. $\psi - Ur\sin\theta\to 0$)

   Recall that $\{1, \log r, \theta, r^{\pm n}\cos n\theta, r^{\pm n}\sin n\theta\}$ is a basis for $\psi, \phi$, so $Ur\sin\theta$ is in the set since flow is irrotational.

   We guess $\psi = \underbrace{Ur\sin\theta}_{\text{we know this.}}$ + something. We cannot add terms like $Ar^7\sin 6\theta$, because it is not in set. Also, terms like $Br^3\sin 3\theta$ will not lead to $\psi = 0$ at $r = a$

   The only term that can balance it is a $\sin\theta$ term, and the only 'other' $\sin\theta$ term available is $r^{-1}\sin\theta \Rightarrow \psi = Ur\sin\theta + \frac{c}{r}\sin\theta$.

   where $r = a$, $\psi = 0$ i.e. $0 = (Ua + \frac{c}{a})\sin\theta\ \forall\theta \Rightarrow -Ua = \frac{c}{a} \Rightarrow c = Ua^2 \Rightarrow \psi = Ur\sin\theta - \frac{Ua^2}{r}\sin\theta = Ur\sin\theta(1 - \frac{a^2}{r^2}) = Uy(1 - \frac{a^2}{r^2})$

2) Velocity potential, $\phi$: then $\underline{\nabla^2\phi = 0} \Rightarrow$ on $r = a$, $\frac{\partial\phi}{\partial n} = 0$ i.e. $\underline{\frac{\partial\phi}{\partial r} = 0}$. As $r\to\infty$, $\underline{u}\to U\hat{\underline{i}}$, $\underset{(\phi_x)}{u}\to U$ and $\underset{(\phi_y)}{v}\to 0 \Rightarrow \phi\to Ux = \underline{Ur\cos\theta}$

   Thus $\phi\to Ur\cos\theta$ as $r\to\infty$, $\phi = Ur\cos\theta + A\frac{1}{r}\cos\theta$ by drawing element from Laurent series set: needs to satisfy $\frac{\partial\phi}{\partial r} = 0$ on $r = a$

   $\phi_r(a) = U\cos\theta - \frac{A}{r^2}\cos\theta = 0 \Rightarrow (U - \frac{A}{a^2}) = 0 \Rightarrow A = a^2U$, so $\phi = U(r + \frac{a^2}{r})\cos\theta = Ux + \frac{a^2U}{r}\cos\theta = Ux(1 + \frac{a^2}{r^2})$.

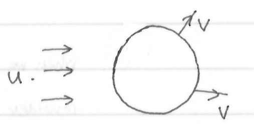
3) Use $w$, the complex potential (generally difficult to guess, unless pre-intuition given). Recall that $\psi = Ur\sin\theta - Ua^2\frac{\sin\theta}{r}$, and $\psi = \operatorname{Im} w$.

   Hence $w = Uz + \frac{Ua^2}{z}$ (note sign: because $\frac{1}{z} = \frac{1}{r}e^{-i\theta} \Rightarrow \operatorname{Im}\frac{1}{z} = -\frac{\sin\theta}{r}$). Then $\phi = \operatorname{Re} w = Ur\cos\theta + \frac{Ua^2}{r}\cos\theta$.

   (This means that we can transition from one to the other using the complex potential).

Consider the problem on sheet 4. There, we have $\underline{u}\to U\hat{\underline{i}}$ as $r\to\infty$, $\underline{u}\to Ux$ as $r\to\infty$ ($\because u\to U, v\to 0$).

We have boundary condition $\underline{u}\cdot\underline{\hat{n}}=V$ on $r=a$ $\Rightarrow$ $\frac{\partial\phi}{\partial n}=V$ on $r=a$ i.e. $\frac{\partial\phi}{\partial r}=V$ on $r=a$. Then we know:




$\nabla^2\phi=0$, $\phi\to Ur\cos\theta$ as $r\to\infty$, $\frac{\partial\phi}{\partial r}=V$ on $r=a$. Then we have a linear combination of terms in Laurent series –

$\phi = Ur\cos\theta + \frac{A}{r}\cos\theta + B\log r$. TBC: Sheet 4 Q1.

**Ex** Consider a cylinder in a flow that, as $r\to\infty$, has also the flow $u=-ky$, $v=-kx$. Show that flow is irrotational and find $\psi$.

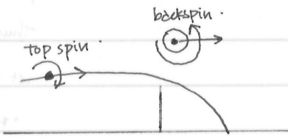
soln. $v_x - u_y = 0$ and $v_y + u_x = 0 \Rightarrow$ satisfies CRE. Then $\frac{\partial\psi}{\partial y}=-ky \Rightarrow \psi = -\frac{1}{2}ky^2 + f(x) \Rightarrow \frac{\partial\psi}{\partial x}=f'(x)=kx \Rightarrow f=\frac{1}{2}kx^2$.

Thus as $r\to\infty$, $\psi\to -\frac{1}{2}ky^2+\frac{1}{2}kx^2 = \frac{1}{2}k(x^2-y^2)=\frac{1}{2}k(r^2\cos^2\theta - r^2\sin^2\theta)=\frac{1}{2}kr^2(\cos^2\theta-\sin^2\theta)=\frac{1}{2}kr^2\cos 2\theta$.

Define $\psi=0$ on $r=a$, so try $\psi=\frac{1}{2}kr^2\cos 2\theta + \frac{A}{r^2}\cos 2\theta$. $\Rightarrow$ $0=\frac{1}{2}ka^2\cos 2\theta + \frac{A}{a^2}\cos 2\theta \Rightarrow \frac{ka^2}{2}-\frac{A}{a^2}=0 \Rightarrow A=\frac{ka^4}{2}$.

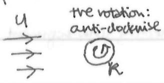
Then, $\psi = \cos 2\theta\left(\frac{1}{2}kr^2+\frac{ka^4}{2r^2}\right) = \frac{k}{2}\left(r^2+\frac{a^4}{r^2}\right)\cos\theta_{//}$.

## 2.3.2 A cylinder with circulation




Consider a cylinder of radius $a$ in a stream that at infinity in uniform with speed $\underline{u}=U\hat{\underline{i}}$. Let there be a circulation $K$ about the cylinder.

Frame of ball:




Rather than have fluid move over our cylinder, we can consider the cylinder moving left with speed $U$ in stationary air. i.e. this models a ball travelling to left at speed $U$, into stationary air, with 'topspin'.

Due to Laurent series, we try finding $\psi$ from set $\{1, \theta, \log r, r^{\pm n}\sin n\theta, r^{\pm n}\cos n\theta\}$. Then we try: $\psi = \underbrace{U\left(r-\frac{a^2}{r}\right)\sin\theta}_{\text{no circ.}} - \underbrace{\frac{K}{2\pi}\log\left(\frac{r}{a}\right)}_{\text{vortex} \div \text{circ. } K}$.

We know this because $\log r$ is the only case that has circulation. Likewise, $\phi = U\left(r+\frac{a^2}{r}\right)\cos\theta + \frac{K}{2\pi}\theta$

* Recall. source (strength $m$) $\phi=\frac{m}{2\pi}\log r$, $\psi=\frac{m}{2\pi}\theta$; vortex (strength $K$) $\phi=\frac{K}{2\pi}\theta$, $\psi=-\frac{K}{2\pi}\log r$

Let us check our solution: $\psi = Ur\sin\theta - \frac{Ua^2}{r}\sin\theta - \frac{K}{2\pi}\log\left(\frac{r}{a}\right)$. Then $\nabla^2\psi=0$ (✓, since drawn from Laurent set), $\psi=0$ as $r=a$ (✓), circulation is $K$ (✓)

But what about as $r\to\infty$? $\psi = Uy\left(1-\frac{a^2}{r^2}\right)-\frac{K}{2\pi}\log\left(\frac{r}{a}\right)$, then $\psi\to Uy - \frac{K}{2\pi}\log\left(\frac{r}{a}\right)$, which seems problematic. But what we really want is $\underline{\nabla}(\psi - Uy) = \underline{\nabla}\left(-\frac{K}{2\pi}\log\left(\frac{r}{a}\right)\right) = -\frac{K}{2\pi r}\hat{\underline{r}} \to 0$ as $r\to\infty$; i.e. (✓ – velocity field is correct at $\infty$).

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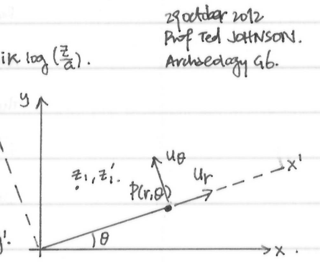
Archaeology G6.

Consider a cylinder with circulation $\psi = Uy\left(1-\frac{a^2}{r^2}\right)-K\log\left(\frac{r}{a}\right)$ – or, with complex velocity potential $w$: $w=Uz\left(1+\frac{a^2}{z^2}\right)-iK\log\left(\frac{z}{a}\right)$.

Check this – Now, $u-iv=\frac{dw}{dz}=U-\frac{a^2U}{z^2}-\frac{iK}{z}$, so as $z\to\infty$, $u-iv\to U$ i.e. $u\to U$, $v\to 0$ as required.




We would like to show that $u_r=0$ on $r=a$. We want to get $u_r$ ($u$ in the $r$ direction) from $w$ (or $\frac{dw}{dz}$).

Now, $u_r - iu_\theta = \frac{dw}{dz'}$ (i.e. cartesian velocity components relative to axes $Ox'y'$). We know that $\frac{dw}{dz'}=\frac{dw}{dz}\cdot\frac{dz}{dz'}$.

What is the significance of $\frac{dz}{dz'}$? Let $Q$ be any point in the plane with complex coordinates $z_1$ wrt $Oxy$, $z_1'$ w.r.t. $Ox'y'$.

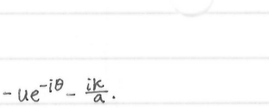
Then $|z_1|=|z_1'|$, $\arg z_1 = \theta + \arg z_1'$ i.e. $z_1 = |z_1|e^{i\arg z_1} = |z_1'|e^{i\arg z_1' + i\theta} = z_1' e^{i\theta}$, so $\frac{dz}{dz'}=e^{i\theta}$

Thus we have $\boxed{u_r - iu_\theta = e^{i\theta}\frac{dw}{dz}}$, and in our problem, $u_r - iu_\theta = e^{i\theta}\left[U-\frac{a^2U}{z^2}-\frac{iK}{z}\right]$.

On the cylinder, $r=a$, i.e. $|z|=a$, $z=ae^{i\theta}$. Thus on $z=ae^{i\theta}$, $u_r - iu_\theta = e^{i\theta}\left[U-\frac{a^2U}{a^2e^{2i\theta}}-\frac{iK}{ae^{i\theta}}\right] = Ue^{i\theta}-Ue^{-i\theta}-\frac{iK}{a}$.

$\therefore$ $u_r - iu_\theta = 2U\sinh(i\theta)-\frac{iK}{a} = 2iU\sin\theta - i\frac{K}{a}$. Hence comparing components, $u_r=0$ as required, $u_\theta=\frac{K}{a}-2U\sin\theta$

We convince ourselves that the circulation is $2\pi$: circulation $=\int_{-\pi}^{\pi} u_\theta\hat{\underline{\theta}}\cdot a\hat{\underline{\theta}}\,d\theta = \int_{-\pi}^{\pi}\left(\frac{K}{a}-2U\sin\theta\right)a\,d\theta = 2\pi K_{//}$ as expected.



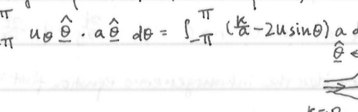

Then, we perform stagnation point analysis:

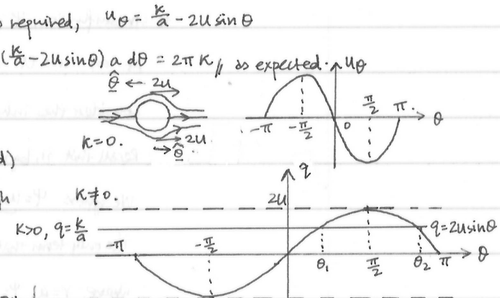
- $K=0$, no circulation. $u_r=0$, $u_\theta=-2U\sin\theta$, so $u_\theta=0$ where $\theta=0$ or $\pi$. (as expected)
- $K>0$, positive circulation. $u_r=0$, $u_\theta=-2U\sin\theta+\frac{K}{a}=0 \Rightarrow 2U\sin\theta=\frac{K}{a}$.

  stagnation points are roots $\theta_1, \theta_2$ provided $\frac{K}{a}<2U$; where $\theta_2 = -\theta_1$

  the energy of fluid / unit mass $=\frac{1}{2}|\underline{u}|^2+p$ conserved.

  Note: This is only where $K<2Ua$, such that there are




Two real roots to the solution; if this does not hold (i.e. circulation too large, we have other cases)

1 November 2012
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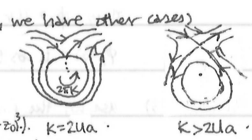
- $K=2Ua$. then we have a single stagnation point, at the top as $\theta=\frac{\pi}{2}$.

  At that point, there is a double zero of $\frac{dw}{dz}$ (i.e. triple zero of $w$) $\Rightarrow$ streamlines intersect at $\frac{\pi}{3}$ (similar to flow at $w=(z-z_0)^3$).




- $K>2Ua$ (cylinder spins faster). Then there are no solutions for $\theta$ on $r=a$. Now $u-iv=\frac{dw}{dz}$

  $u-iv=\frac{dw}{dz}=U\left(1-\frac{a^2}{z^2}\right)-\frac{iK}{z}$. Stagnation points occur where $\frac{dw}{dz}=0$ $\Rightarrow$ $U(z^2-a^2)-iKz=0 \Rightarrow \underbrace{z^2-\frac{iK}{U}z-a^2=0}_{\text{(quadratic – 2 roots)}} \Rightarrow \left(\frac{z}{a}\right)^2-\frac{iK}{aU}\left(\frac{z}{a}\right)-1=0$.

  i.e. $\left(\frac{z}{a}\right)=\frac{iK}{2aU}\pm\sqrt{\left(\frac{iK}{2Ua}\right)^2+1}$. Check our formula: $K=0$, $\frac{z}{a}=\pm 1$, $0<K<2Ua \Rightarrow \frac{z}{a}=\frac{iK}{2aU}\pm\sqrt{1-\left(\frac{K}{2Ua}\right)^2} \Rightarrow \left|\frac{z}{a}\right|=1$, lie on unit circle, same $y$, opposite $x$.

2301-010.

If $K=2Ua$, $\frac{z}{a}=i \Rightarrow z=ia$ as before. Finally, if $K>2Ua$, $\left(\frac{z}{a}\right)=\frac{iK}{2Ua}\pm i\sqrt{\left(\frac{K}{2Ua}\right)^2-1} = i\left[\frac{K}{2Ua}\pm\sqrt{\left(\frac{K}{2Ua}\right)^2-1}\right] \Rightarrow$ purely imaginary. Hence, these points lie on the line $x=0$. We know that since $\left(\frac{z}{a}\right)^2-\frac{iK}{aU}\left(\frac{z}{a}\right)-1=0$, product of roots is $-1$, so if one root has $|y|>1$, the other has $|y|<1$.
i.e the y-values are reciprocals of each other. (see diagram at base of pg 10). There is a free stagnation point.

For the cylinders, top speed is $\frac{K}{a}+2U$, bottom speed is $\frac{K}{a}-2U$. So as KE per unit volume is $\frac{1}{2}\rho u^2+P$,
then the pressure difference is proportional to difference of squares of $u$, i.e. $\rho\left[\left(\frac{K}{a}+2U\right)^2-\left(\frac{K}{a}-2U\right)^2\right]=\rho\frac{2K}{a}\cdot 4U=\frac{8UK\rho}{a}$.

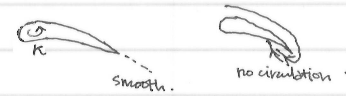
Thus the force per unit width of cylinder scales as $\underbrace{\rho}_{\text{air density}}\underbrace{U}_{\text{cylinder speed}}\underbrace{K}_{\text{circular spin}}$.

Application: For aircraft, lift is achieved by

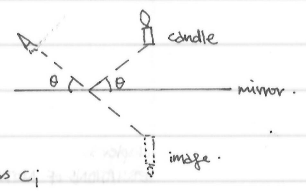
allowing air to leave smoothly off the edge of the wing — hence circulation is necessary. ($K$)

High altitude airports are rare due to insufficient lift generated ($\rho$ low) by low air density.

Faster planes (high $U$) require shorter wings (per unit width force is greater).

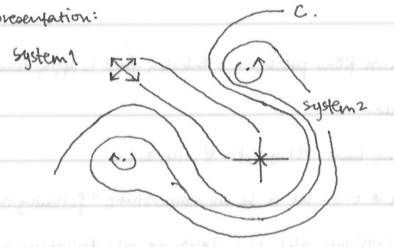



## 24 Method of images.

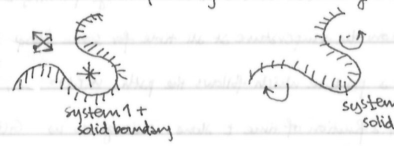



Imagine a candle in front of a mirror, then an image is produced. This motivates our understanding for fluids:
If the motion of a fluid in the plane is due to a distribution of singularities (i.e. $(z-z_0)^{-n}$, $n\geq 1$, $\log(z-z_0)$:
such as line vortices, line sources, line sinks, dipoles), and there exists a curve $C$ drawn in the plane with no flow across $C$;
then the system of singularities on $C$ is the image of the system of singularities on the other side.

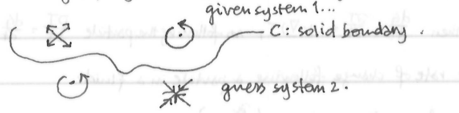
Diagramatic representation:




If there is no flow across $C$, then system 1 and system 2 are mutual images.
But then $C$ can be replaced by a solid boundary without changing the flow on either side.




This gives a method for solving problems with solid boundaries. Given a set of singularities (call them system 1) and a solid boundary (call that $C$),
then we can guess the image of system 1 in $C$, and that guess we call system 2. (guess system 2).

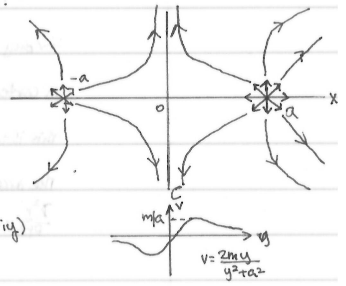



The solution is system 1 PLUS system 2.
(system 2 incorporates information about our boundary $C$).

$\boxed{\text{Ex}}$ Suppose there is a source of strength $2\pi m$ at $z=a\in\mathbb{R}$, and a solid wall along $x=0$. What is the flow field?

Soln. Let our solid wall be $C$, system 1 be on right side of $C$. Then system 1 gives
$w_1=m\log(z-a)$. Guess system 2 — using $C$ as a mirror, we have a source at $-a$ with
strength $2\pi m$. $\Rightarrow$ $w_2=m\log(z+a)$. Then total system, in its complete form, is
$w=w_1+w_2= m\log(z+a)+m\log(z-a)=m\log(z^2-a^2)$.

Verify: Prove that $u=0$ on $x=0$. $u-iv=\frac{dw}{dz}=\frac{2mz}{z^2-a^2}$. On $x=0$, $u-iv=\frac{2imy}{-a^2-y^2}$ $(\because z=x+iy)$
so $u=0$, $(v)$ $v=\frac{2my}{a^2+y^2}$, and $v_{max}=\frac{m}{a}$.





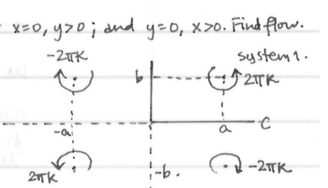


$\boxed{\text{Ex}}$ As above, but we now have vortex of strength (circulation) $2\pi K$. What is the flow field.

Soln. System 1: $w_1=-iK\log(z-a)$; its image in $C:\{x=0\}$ gives our guess for system 2.
$w_2=iK\log(z+a)$. Complete flow is then $w=w_1+w_2=-iK\log(z-a)+iK\log(z+a)=iK\log\left(\frac{z+a}{z-a}\right)$.

$\boxed{\text{Ex}}$ We have a vortex of strength (circulation) $2\pi K$ at $z=a+ib$, $a,b>0$ (in first quadrant). There are solid walls at $x=0, y>0$; and $y=0, x>0$. Find flow.

Soln. $w_1=-iK\log[z-(a+ib)]$ gives system 1. Then, system 2 are the three reflected vortices
$w_2=iK\log[z-(-a+ib)]+iK\log[z-(a-ib)]-iK\log[z-(-a-ib)]$. Then our complete system
is $w=w_1+w_2$.

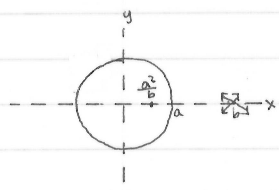


Ex Consider solid boundaries separated by a degree $\frac{\pi}{n}$, $n \in \mathbb{N}$. How many images are produced in the flow? What if $n \notin \mathbb{Z}$.

Soln. For $\theta = \frac{\pi}{n}$, $2n-1$ images are produced. If $n \notin \mathbb{Z}$,

Ex Consider a source of strength $2\pi m$ at $z=b$ outside a cylinder of radius $a$ $[0<a<b]$. Show that the image is given by

Soln. We use the circle theorem. (see below). Solution after.




Theorem (Circle theorem)

The image system in the circle $|z|=a$ of the complex potential $W_1(z)=f(z)$ where $f(z)$ has no singularities inside the circle $|z| \le a$ (so it can be system 1) is $W_2(z) = \bar{f}\left(\frac{a^2}{z}\right)$, where for any function $g(z)$, $\bar{g}(z) = \overline{g(\bar{z})}$.

Proof – NTP: there is no flow across $C$, i.e. $|z|=a$. The complex velocity potential is $W = W_1 + W_2 = f(z) + \bar{f}\left(\frac{a^2}{z}\right)$.

On $|z|=a$, $z\bar{z} = a^2$, so $\frac{a^2}{z} = \bar{z}$. Thus on $|z|=a$, $\bar{f}\left(\frac{a^2}{z}\right) = \bar{f}(\bar{z}) = \overline{f(\bar{\bar{z}})} = \overline{f(z)}$, so $W = f(z) + \overline{f(z)} = 2\,\mathrm{Re}\, f(z) \in \mathbb{R} \Rightarrow$

$\mathrm{Im}\, W = \Psi = 0$ i.e. $\Psi = 0$ on $C \Rightarrow$ on the circle, we have a streamline $\Rightarrow$ no net flux. q.e.d.

Soln (cont'd) – By the circle theorem, $W_1 = m\log(z-b) \Rightarrow W_2 = \bar{f}\left(\frac{a^2}{z}\right) = \overline{m\log\left(\overline{\frac{a^2}{z}} - b\right)} = m\log\left(\frac{a^2}{z} - b\right) = m\log\left(\frac{1}{z}\right) + m\log(b) + m\log\left(\frac{a^2}{b} - z\right)$

so $W_2 = \underbrace{-m\log(z)}_{\text{sink of strength } 2\pi m \text{ at origin}} + \underbrace{m\log(-b)}_{\text{constant, part of } \phi} + \underbrace{m\log\left(z - \frac{a^2}{b}\right)}_{\text{source of strength } 2\pi m \text{ at optical point } a^2/b}$.

Chapter 3.

EQUATIONS OF MOTION.

12 Novem 2
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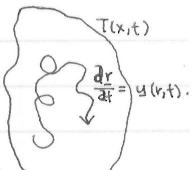
Archaeology 406.

We know that Force on a particle = its acceleration × its mass. We must follow particles in a Eulerian field to apply Newton's law.




We introduce notation $\frac{D}{Dt}$ to mean the rate of change following a particle.

Suppose we know the temperature at all time for some flow i.e. we have $T(x,y,z,t)$ $\forall\, x,y,z,t$.

Now consider a particle which follows the path $\underline{u}(r,t) = \frac{\partial \underline{r}}{\partial t}$, $\underline{r} = \underline{r}_0$ at $t=0$. Let $G$ be the temperature "following the particle".

Then $G = G(t)$, a function of time $t$ alone. This gives us $G(t) = T(\underbrace{x(t), y(t), z(t)}_{\text{position of our particle at time } t \text{ along its path.}}, t)$. Thus, we take derivatives with respect to $t$ ∴ by chain rule,

$\frac{dG}{dt} = \frac{\partial T}{\partial x}\frac{dx}{dt} + \frac{\partial T}{\partial y}\frac{dy}{dt} + \frac{\partial T}{\partial z}\frac{dz}{dt} + \frac{\partial T}{\partial t}$. But on our path, we have

$\frac{dx}{dt} = u$, $\frac{dy}{dt} = v$, $\frac{dz}{dt} = w$ i.e. $\frac{dG}{dt} = \frac{\partial T}{\partial t} + u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} + w\frac{\partial T}{\partial z}$. Then since $\underline{\nabla} T = \begin{pmatrix} T_x \\ T_y \\ T_z \end{pmatrix}$, $\underline{u} = \begin{pmatrix} u \\ v \\ w \end{pmatrix}$, then $\frac{dG}{dt} = \frac{\partial T}{\partial t} + \underline{u}\cdot\underline{\nabla} T$

We introduce the operator $(\underline{u}\cdot\nabla) = \left(u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + w\frac{\partial}{\partial z}\right)$, then $\frac{dG}{dt} = \frac{\partial T}{\partial t} + (\underline{u}\cdot\underline{\nabla})T$, so following the particle $\frac{DT}{Dt} = \frac{dG}{dt}$ s.t. $\boxed{\frac{DT}{Dt} = \frac{\partial T}{\partial t} + (\underline{u}\cdot\underline{\nabla})T}$

This is an advective, convective derivative – giving the rate of change following a particle in a fluid.

Notice that it is possible (and likely) that $\frac{DT}{Dt} \neq 0$ even if the flow is steady $\left(\frac{\partial \underline{u}}{\partial t} = 0\right)$

For any $Q(x,y,z,t)$, $\frac{DQ}{Dt} = \frac{\partial Q}{\partial t} + (\underline{u}\cdot\underline{\nabla})Q = Q_t + uQ_x + vQ_y + wQ_z$. For instance, if $Q = x$, $\frac{Dx}{Dt} = 0 + u + 0 + 0 = u$, similarly $\frac{Dy}{Dt} = v$, $\frac{Dz}{Dt} = w$.

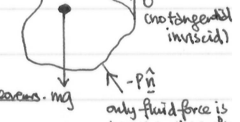
In Cartesians, $\frac{D\underline{r}}{Dt} = \frac{D}{Dt}(x\hat{\underline{i}} + y\hat{\underline{j}} + z\hat{\underline{k}}) = \frac{Dx}{Dt}\hat{\underline{i}} + x\frac{D\underline{i}}{Dt} + \frac{Dy}{Dt}\hat{\underline{j}} + y\frac{D\underline{j}}{Dt} + \frac{Dz}{Dt}\hat{\underline{k}} + z\frac{D\underline{k}}{Dt} = u\underline{i} + v\underline{j} + w\underline{k} = \underline{u}$.

This is true for all coordinate systems, however it requires care in calculation for Non-Cartesian systems: $\frac{D\hat{\underline{r}}}{Dt} \neq 0$ so $(\underline{u}\cdot\underline{\nabla})\hat{\underline{r}} \neq 0$ since $\underline{\nabla}\hat{\underline{r}} \neq 0$.

The acceleration following a particle, $\frac{D^2\underline{r}}{Dt^2}$, is given by:

$\frac{D^2\underline{r}}{Dt^2} = \frac{D\underline{u}}{Dt} = \frac{D}{Dt}(u\underline{i} + v\underline{j} + w\underline{k}) = \left(\frac{Du}{Dt}\right)\underline{i} + \left(\frac{Dv}{Dt}\right)\underline{j} + \left(\frac{Dw}{Dt}\right)\underline{k} = \left[\frac{\partial u}{\partial t} + (\underline{u}\cdot\underline{\nabla})u\right]\underline{i} + \left[\frac{\partial v}{\partial t} + (\underline{u}\cdot\underline{\nabla})v\right]\underline{j} + \left[\frac{\partial w}{\partial t} + (\underline{u}\cdot\underline{\nabla})w\right]\underline{k} = \left(\frac{\partial}{\partial t} + \underline{u}\cdot\underline{\nabla}\right)\underline{u}$, i.e. $\frac{D^2\underline{r}}{Dt^2} = \frac{D\underline{u}}{Dt} = \frac{\partial \underline{u}}{\partial t} + (\underline{u}\cdot\underline{\nabla})\underline{u}$.

Newton's law gives us $\underline{F} = m\underline{a}$. Imagine a fluid element, and consider the forces acting on it.




only fluid force is normal to the surface. It is pressure (force per unit area acting inwards).

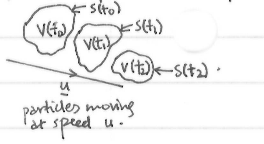
We must balance rate of change of the momentum of a moving blob of fluid, with the forces acting on it.

Thus, we need to know the positive rate of change of anything following a fluid blob (e.g. temp. / heat). This is given by the Reynold Transport Theorems.

## Reynold Transport Theorems.

Consider a quantity $\alpha(\underline{r},t)$ defined throughout a fluid domain $D$. Take any subvolume $V$ of $D$ with surface $S$, s.t. $V$ is always composed of the same fluid particles.

Let the velocity associated with the motion be $\underline{u}(x,t)$. Consider the quantity: $\int_{V(t)} \alpha(\underline{x},t)\, dV = I(t)$.




particles moving at speed u.

This is purely a function of time alone (as integrated over $V$). Then,

$\frac{dI}{dt} = \lim_{\delta t \to 0} \frac{I(t+\delta t) - I(t)}{\delta t}$.

15 November 2012
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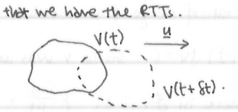
Recall that we have the RTTs.




After time $\delta t$, the same particles have moved to a new point. We have some quantity (temp/mass/momentum etc.) $\alpha(\underline{r},t)$.

We write $I(t) = \int_{V(t)} \alpha(\underline{r},t)\, dV$. $I$ is a function of $t$ alone when we follow a given set of particles.

(Taylor's thm)

Now $\frac{dI}{dt} = \lim_{\delta t\to 0} \frac{I(t+\delta t)-I(t)}{\delta t} = \lim_{\delta t\to 0} \frac{1}{\delta t}\left[\int_{V+\delta V} \alpha(\underline{r},t+\delta t)dV - \int_V \alpha(\underline{r},t)\,dV\right]$. But we are going to let $\delta t\to 0$, so expand $\alpha(\underline{r},t+\delta t) = \alpha(\underline{r},t) + \delta t\frac{\partial\alpha}{\partial t}(\underline{r},t) + \frac{1}{2}(\delta t)^2\frac{\partial^2\alpha}{\partial t^2}(\underline{r},t) + \dots$

Notice that all arguments of $\alpha$ and its time derivatives are $(\underline{r},t)$, so we drop the $(\underline{r},t)$ from our notation, implicitly: $\alpha(\underline{r},t+\delta t) = \alpha + \delta t\frac{\partial\alpha}{\partial t} + \frac{1}{2}(\delta t)^2\frac{\partial^2\alpha}{\partial t^2}$.

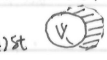
Then $I(t+\delta t) = \int_{V+\delta V}\left(\alpha + \delta t\frac{\partial\alpha}{\partial t} + \frac{1}{2}(\delta t)^2\frac{\partial^2\alpha}{\partial t^2}+\dots\right)dV = \int_V \alpha + \delta t\frac{\partial\alpha}{\partial t} + \frac{1}{2}(\delta t)^2\frac{\partial^2\alpha}{\partial t^2}dV + \int_{\delta V}\alpha + \delta t\frac{\partial\alpha}{\partial t} + \frac{1}{2}(\delta t)^2\frac{\partial^2\alpha}{\partial t^2}dV = \int_V\alpha\, dV + \delta t\int_V\frac{\partial\alpha}{\partial t}dV + \frac{1}{2}(\delta t)^2\int_V\frac{\partial^2\alpha}{\partial t^2}dV + \dots$

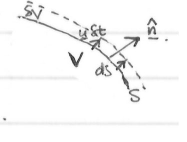
Thus, $\frac{dI}{dt} = \lim_{\delta t\to 0}\frac{1}{\delta t}\Big[\int_V\alpha\, dV + \delta t\int_V\frac{\partial\alpha}{\partial t}dV + \frac{1}{2}(\delta t)^2\int_V\frac{\partial^2\alpha}{\partial t^2}dV + \int_{\delta V}\alpha\, dV + \delta t\int_{\delta V}\frac{\partial\alpha}{\partial t}dV + \frac{1}{2}(\delta t)^2\int_{\delta V}\frac{\partial^2\alpha}{\partial t^2}dV + \dots - \underbrace{\int_V\alpha\, dV}_{I(t)}\Big] + \int_{\delta V}\alpha\, dV + \delta t\int_{\delta V}\frac{\partial\alpha}{\partial t}dV + \dots$

$= \lim_{\delta t\to 0}\left[\int_V\frac{\partial\alpha}{\partial t}dV + \frac{1}{2}\delta t\int_V\frac{\partial^2\alpha}{\partial t^2}dV + \dots + \frac{1}{\delta t}\int_{\delta V}\alpha\, dV + \int_{\delta V}\frac{\partial\alpha}{\partial t}dV + \dots\right] = \int_V\frac{\partial\alpha}{\partial t}dV + 0 + \lim_{\delta t\to 0}\frac{1}{\delta t}\int_{\delta V}\alpha\, dV + 0$

$\because$ for final term, $\left|\lim_{\delta t\to 0}\int_{\delta V}\frac{\partial\alpha}{\partial t}dV\right| = \lim_{\delta t\to 0}\left|\int_{\delta V}\frac{\partial\alpha}{\partial t}dV\right| \le \lim_{\delta t\to 0}\int_{\delta V}\left|\frac{\partial\alpha}{\partial t}\right|dV \le \lim_{\delta t\to 0}\int_{\delta V}M\, dV$ (where $M = \max_V\left|\frac{\partial\alpha}{\partial t}\right|$) $= M\lim_{\delta t\to 0}\int_{\delta t}dV = M\lim_{\delta t\to 0}|\delta V| \to 0$.

i.e. we have shown $\frac{dI}{dt} = \frac{d}{dt}\int_V\alpha\, dV = \int_V\frac{\partial\alpha}{\partial t}dV + \lim_{\delta t\to 0}\frac{1}{\delta t}\int_{\delta V}\alpha\, dV$.




Take a small element on surface $dS$. In time $\delta t$, it moves outwards by $\underline{u}\,\delta t$. Then $dV$ = area of base × height = $dS \times (\underline{u}\cdot\hat{\underline{n}})\delta t$

So $\frac{dI}{dt} = \int_V\frac{\partial\alpha}{\partial t}dV + \lim_{\delta t\to 0}\frac{1}{\delta t}\int_S\alpha(\underline{u}\cdot\hat{\underline{n}})\,\delta t\, dS = \int_V\frac{\partial\alpha}{\partial t}dV + \int_S\alpha\,\underline{u}\cdot\hat{\underline{n}}\, dS$. This gives RTT1: <u>Reynold's Transport Theorems</u>, version 1.

$\boxed{\frac{dI}{dt} = \int_V\frac{\partial\alpha}{\partial t}dV + \int_S\alpha\,\underline{u}\cdot\hat{\underline{n}}\, dS}$, or in words: rate of change following a blob = local rate of change over blob + flux of $\alpha$ through boundary of V.

We can make this formulation more useful by using the divergence theorem: $\int_S\underline{v}\cdot\hat{\underline{n}}\, dS = \int_V\underline{\nabla}\cdot\underline{v}\, dV$.

Then, $\frac{d}{dt}\int\alpha\, dV = \int\frac{\partial\alpha}{\partial t}dV + \int_V\underline{\nabla}\cdot(\alpha\underline{u})\, dV \Rightarrow \boxed{\frac{dI}{dt} = \int_V\left[\frac{\partial\alpha}{\partial t} + \underline{\nabla}\cdot(\alpha\underline{u})\right]dV}$. This gives RTT2.

But $\underline{\nabla}\cdot(\alpha\underline{u}) = \alpha\underline{\nabla}\cdot\underline{u} + \underline{u}\cdot\underline{\nabla}\alpha$, hence $\frac{\partial\alpha}{\partial t} + \underline{\nabla}\cdot(\alpha\underline{u}) = \frac{\partial\alpha}{\partial t} + \underline{u}\cdot\underline{\nabla}\alpha + \alpha\underline{\nabla}\cdot\underline{u} = \frac{D\alpha}{Dt} + \alpha\underline{\nabla}\cdot\underline{u}$. So we have RTT3: $\boxed{\frac{dI}{dt} = \int_V\left(\frac{D\alpha}{Dt} + \alpha\underline{\nabla}\cdot\underline{u}\right)dV}$.

Examine RTT3 — in an incompressible fluid, $\underline{\nabla}\cdot\underline{u} = 0$, then $\frac{d}{dt}\int_V\alpha\, dV = \int_V\frac{D\alpha}{Dt}$ indeed.

$\boxed{\text{Ex}}$ Given a fluid element with volume V. The mass of V is $M = \int_V\rho\, dV$. Find the rate of change of mass.

soln. Following the same particles, $\frac{dM}{dt} = \frac{d}{dt}\int_V\rho\, dV \overset{RTT2}{=} \int_V\left[\frac{\partial\rho}{\partial t} + \underline{\nabla}\cdot(\rho\underline{u})\right]dV$. where $\rho = \rho(\underline{r},t)$

We present a more careful argument to the previous example — let a fluid occupy a domain $\mathcal{D}$ and have density $\rho(\underline{r},t)$, and velocity field $\underline{u}(\underline{r},t)$. Take any subregion V of $\mathcal{D}$

then $\frac{dM}{dt} = \frac{d}{dt}\int_V\rho\, dV = \int_V\left[\frac{\partial\rho}{\partial t} + \underline{\nabla}\cdot(\rho\underline{u})\right]dV$. But for the same particles, $\frac{dM}{dt} = 0$ by conservation of mass. i.e. we have show that

<u>for all</u> V in $\mathcal{D}$, $\int_V\left[\frac{\partial\rho}{\partial t} + \underline{\nabla}\cdot(\rho\underline{u})\right]dV = 0$. By our lemma, $\boxed{\frac{\partial\rho}{\partial t} + \underline{\nabla}\cdot(\rho\underline{u}) = 0}$ everywhere in $\mathcal{D}$. [This is the equation of <u>conservation of mass</u> without requiring <u>incompressibility</u>]

It applies for things such as sound waves.

How does this tie in with our incompressibility criterion? Incompressibility implies that fluid elements retain their volume (cannot be squashed). But they also retain their mass $\Rightarrow$ it has the same density throughout its motion. Hence, following a particle, $\frac{D\rho}{Dt} = 0$, which is our equation of incompressibility.

(i.e. each particle maintains its respective density, not that all particles in blobs have the same density).

But $\frac{\partial\rho}{\partial t} + \nabla\cdot(\rho\underline{u}) = 0 \Rightarrow \frac{\partial\rho}{\partial t} + \underline{u}\cdot\underline{\nabla}\rho + \rho\underline{\nabla}\cdot\underline{u} = 0 \Rightarrow \frac{D\rho}{Dt} + \rho\underline{\nabla}\cdot\underline{u} = 0$ (mass conservation for any fluid). So in an incompressible fluid, $\underline{\nabla}\cdot\underline{u} = 0$ as before.

We can also use the conservation of mass property to obtain RTT4: let $\underline{v} = \rho\underline{u}$, $f\underline{v} = f\rho\underline{u}$.

Take any quantity $f$ and write $\alpha = \rho f$. Then apply RTT2: $\frac{d}{dt}\int_V\rho f\, dV = \int_V\left[\frac{\partial}{\partial t}(\rho f) + \underline{\nabla}\cdot(\rho f\underline{u})\right]dV = \int_V\left[\rho\frac{\partial f}{\partial t} + f\frac{\partial\rho}{\partial t} + f\underline{\nabla}\cdot(\rho\underline{u}) + \rho\underline{u}\cdot\underline{\nabla}f\right]dV$

then our integral becomes $\int_V\left\{f\left[\frac{\partial\rho}{\partial t} + \underline{\nabla}\cdot(\rho\underline{u})\right] + \rho\left[\frac{\partial f}{\partial t} + (\underline{u}\cdot\underline{\nabla})f\right]\right\}dV$. By conservation of mass, $\frac{\partial\rho}{\partial t} + \underline{\nabla}\cdot(\rho\underline{u}) = 0$ everywhere. Also, $\frac{\partial f}{\partial t} + (\underline{u}\cdot\underline{\nabla})f = \frac{Df}{Dt}$.

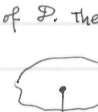
Hence, $\boxed{\frac{d}{dt}\int_V\rho f\, dV = \int_V\rho\frac{Df}{Dt}dV}$. This is RTT4. Note: this is also written $\frac{d}{dt}\int_V f\rho dV = \int_V\frac{Df}{Dt}\rho dV \Rightarrow \frac{d}{dt}\int_V f\, dM = \int_V\frac{Df}{Dt}dM$.

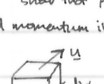
This is applicable because $dM$, the unit of measure, is invariant under $\frac{D}{Dt}$, unlike $dV \Rightarrow$ hence it only commutes for this case.

$\boxed{\text{Ex}}$ With the standard setup, take $\underline{\alpha} = \rho\underline{u}$ to be the momentum per unit volume. If there is a pressure / unit area of $p$ and external force $\underline{F}$ per unit volume, show that $\rho\frac{D\underline{u}}{Dt} = -\nabla p + \rho\underline{F}$

soln. Let a fluid of density $\rho(\underline{r},t)$ and velocity $\underline{u}(\underline{r},t)$ occupy a domain $\mathcal{D}$. Take any subvolume V of $\mathcal{D}$. Then the total momentum in V is

$\underline{m} = \int_V\rho\underline{u}\, dV$. By RTT4, $\frac{d\underline{m}}{dt} = \frac{d}{dt}\int_V\rho\underline{u}\, dV = \int_V\rho\frac{D\underline{u}}{Dt}dV$.



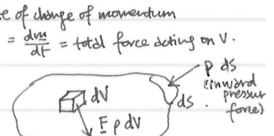

Density $\rho(\underline{r},t)$.

By Newton's law, $F = \frac{d}{dt}(m\underline{u}) = \frac{d\underline{m}}{dt}$. Let there be an arbitrary force per unit mass $\underline{F}$ acting on the fluid (e.g. $\underline{F} = -g\hat{\underline{z}}$ for $\underline{z}$ upwards, magnetic field like in sun, electric field in plasma).

rate of change of momentum $= \frac{d\underline{m}}{dt}$ = total force acting on V.

In an inviscid fluid, the only internal force is the pressure, $p$. (no tangential stress, only a normal force per unit area).

The total vector force on blob V is $\int_S(\underbrace{-\hat{\underline{n}})p}_{\text{inward pressure}}\, dS + \int_V\rho\underline{F}\, dV$. We know that the vector form of the divergence theorem gives $\int p\hat{\underline{n}}\, dS = \int\underline{\nabla}p\, dV$ for any $p$.



Thus the total force acting on V is $\int_V (-\nabla p + \rho F)\, dV$. This must equal rate of change of momentum.

i.e. $\int_V \rho \frac{Du}{Dt}\, dV = \int_V (-\nabla p + \rho \underline{F})\, dV \Rightarrow \int_V (\rho \frac{Du}{Dt} + \nabla p - \rho \underline{F})\, dV = 0$. Since V was arbitrary, this integral vanishes for all $V \in \mathcal{D}$. So this implies $\rho \frac{Du}{Dt} + \nabla p - \rho \underline{F} = 0$ everywhere in $\mathcal{D}$. i.e. $\rho \frac{Du}{Dt} = -\nabla p + \rho F$ // q.e.d.

Our manipulations give us the relation (Euler equation) for an inviscid fluid: $\boxed{\rho \frac{Du}{Dt} = -\nabla p + \rho \underline{F}}$ ← external forces. ← pressure gradient. ↑ rate of change of momentum

i.e. mass × acceleration / unit volume = force / unit volume.

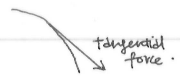
This gives us the natural implication that large accelerations occur when pressure gradients are large ⇒ isobars are close.

Hence we have mass: $\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \underline{u}) = 0$ and momentum: $\rho \frac{Du}{Dt} = -\nabla p + \rho \underline{F}$ form our governing equations.

Check — our unknowns include: $\rho(\underline{r}, t)$, $p(\underline{r}, t)$ and $\underline{u}(\underline{r}, t)$; i.e. 2 scalar unknowns + 1 vector unknown.

But we only have one vector & one scalar equation. We need another scalar equation: depends on the field of interest.

Aside: real fluids have tangential force at surface




- Gas dynamics — equation of state $p = p(\rho)$
- Geophysical Fluid Dynamics — incompressibility, then $\frac{D\rho}{Dt} = 0$, $\nabla \cdot \underline{u} = 0$ and Euler $\rho \frac{Du}{Dt} = -\nabla p + \rho \underline{F}$.

To address this underdetermination, we take (for this year) $\rho$ = constant everywhere (not unknown, variable). This gives us a <u>homogeneous, barotropic flow</u>. $\frac{D\rho}{Dt} \equiv 0$. We are left with [ $\nabla \cdot \underline{u} = 0$ (incompressibility) and $\frac{Du}{Dt} = -\frac{1}{\rho} \nabla p + \underline{F}$ (Euler) ] as our governing equations → 1 scalar unknown P, 1 vector unknown $\underline{u}$.

19 November 2012
Prof ER Johnson
Archaeology G6.

Our fundamental equations of fluids are

- $\nabla \cdot \underline{u} = 0$ (continuity)
- $\frac{Du}{Dt} = -\frac{1}{\rho} \nabla p + \underline{F}$ (Euler equation of momentum — i.e. Newton's equation for fluids).

These give a vector unknown $\underline{u}(x, y, z, t)$ and one scalar unknown $p(x, y, z, t)$ ⇒ closed system.

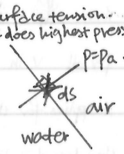
i.e. the first equation can determine the pressure, through compatibility with the second.

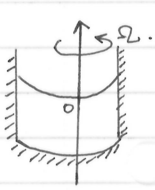
**Ex** Find the pressure fields associated with the flows:

(1) Solid body rotation, $\underline{u} = \underline{\Omega} \wedge \underline{r}$, where $\underline{\Omega} = \Omega \underline{k}$. Let the top surface be free (i.e. open to the atmosphere). Ignore surface tension. Where does highest pressure occur?

outside $p = p_a$ (atmospheric) constant
surface balances.
inside

Soln. Consider at our boundary, pressure outside is $p = p_a$. By Newton's 3rd law, in the water, pressure $= p_a$ as the region $dS$ of infinitismal thickness would have force $p_a\, ds$ inwards and $p\, ds$ but zero mass ⇒ infinite acceleration if $p_a\, ds \neq p\, ds$.




Thus, on the surface, we have boundary condition $p = p_a$ (constant)

We have $\underline{u} = \underline{\Omega} \wedge \underline{r} = \det \begin{pmatrix} \underline{i} & \underline{j} & \underline{k} \\ 0 & 0 & \Omega \\ x & y & z \end{pmatrix} = -\Omega y \hat{\underline{i}} + \Omega x \hat{\underline{j}}$. Check our continuity equation, $\nabla \cdot \underline{u} = 0$.

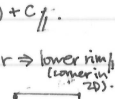
$\nabla \cdot \underline{u} = \frac{\partial}{\partial x}(-\Omega y) + \frac{\partial}{\partial y}(\Omega x) + \frac{\partial}{\partial z}(0) = 0$ (verified). Then use Euler equation: $\frac{D}{Dt} = \frac{\partial}{\partial t} + \nabla \cdot \underline{u} = \frac{\partial}{\partial t} + u \frac{\partial}{\partial x} + v \frac{\partial}{\partial y} + w \frac{\partial}{\partial z} = -\Omega y \frac{\partial}{\partial x} + \Omega x \frac{\partial}{\partial y}$

letting $\underline{F}$ be the external force (i.e. gravity) $\underline{F} = -g \hat{\underline{k}}$. Then by components of Euler equation: $[-\Omega y \frac{\partial}{\partial x} + \Omega x \frac{\partial}{\partial y}](-\Omega y) = -\frac{1}{\rho} \frac{\partial p}{\partial x}$ (since $\frac{Du}{Dt} = -\frac{1}{\rho} \frac{\partial p}{\partial x}$) $\Rightarrow -\Omega^2 x = -\frac{1}{\rho} \frac{\partial p}{\partial x} \Rightarrow \frac{\partial p}{\partial x} = \rho \Omega^2 x$ i.e. $p = \frac{1}{2} \rho \Omega^2 x^2 + f(y, z)$. For y-component: $\frac{Dv}{Dt} = -\frac{1}{\rho} \frac{\partial p}{\partial y} + F_y$, so $[-\Omega y \frac{\partial}{\partial x} + \Omega x \frac{\partial}{\partial y}](\Omega x) = -\frac{1}{\rho} \frac{\partial p}{\partial y} \Rightarrow -\Omega^2 y = -\frac{1}{\rho} \frac{\partial p}{\partial x} \Rightarrow \frac{\partial p}{\partial y} = \rho \Omega^2 y$ (we knew this since prob is rotationally symmetric). For z-component: $\frac{Dw}{Dt} = -\frac{1}{\rho} \frac{\partial p}{\partial z} - g$, but $w = 0$, so $\frac{\partial p}{\partial z} = -\rho g$.

But $p = \frac{1}{2} \rho \Omega^2 x^2 + f(y, z)$, so $\frac{\partial p}{\partial z} = \frac{\partial f}{\partial z} = -\rho g$, so $f(y, z) = -\rho g z + h(y)$. $\Rightarrow h(y) = \frac{1}{2} \rho \Omega^2 y^2 + C \Rightarrow p = -\rho g z + \frac{1}{2} \rho \Omega^2 (x^2 + y^2) + C$ //.

Surfaces $p$ = const $\Rightarrow z = \frac{\Omega^2}{g} r^2 + D$, which is a paraboloid. From our equation, we want to maximise $p$ ⇒ minimise $z$, max $r$ ⇒ lower rim (corner in 2D).

Comment: if we are given $\phi_x = \frac{x}{x^2 + y^2}$, $\phi_y = \frac{-y}{x^2 + y^2}$. Solve by integrating one and differentiating, rather than integrate and combine.




(2) (Archimedes' Principle): "A submerged body experiences an upward force (buoyancy) equal to weight of fluid displaced.

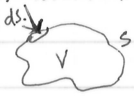
We can generalise it to any stratified fluid. The fluid is at rest i.e. $u \equiv 0$, i.e. it is static. We call the associated pressure field the <u>hydrostatic pressure</u>. Find

Soln. Since $\underline{u} \equiv 0$, we immediately notice trivially that $\nabla \cdot \underline{u} = 0$. Euler equation gives $\frac{Du}{Dt} = -\frac{1}{\rho} \nabla p + \underline{F} = -\frac{1}{\rho} \nabla p - g \hat{z}$. But $\underline{u} \equiv 0$, so.

$\nabla p = -\rho g \hat{z}$ (i.e. $\frac{\partial p}{\partial x} = 0$, $\frac{\partial p}{\partial y} = 0$, $\frac{\partial p}{\partial z} = -\rho g$). i.e. $p = -\rho g z + C$. Set $z = 0$ at surface, $p = p_a$. Then $p = p_a - \rho g z$ //.

This is the pressure field that balances gravity in a fluid at rest, i.e. $\boxed{p_H = p_a - \rho g z}$

Implications: we see that hydrostatic pressure increases linearly with depth ⇒ atmospheric pressure at ocean is 1 bar, 2 bar at 10m, 4 bar at 30m etc. (about the limit for divers to attain safely.




Torricelli established that the atmosphere can support a 10m column of water.

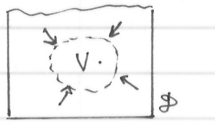
Suppose we have an arbitrary body occupying a volume V, fully submerged in a fluid at rest. Then the fluid force is $-p_H \hat{n}\, ds$ on an element

2301-014

in $S$, an inward force. Total force on $V$ is $\int_S -p\underline{n}\,dS \overset{(\text{div thm})}{=} -\int_V \underline{\nabla} p_H\,dV = -\int_V (-\rho g)\hat{\underline{z}}\,dV = \rho g \hat{\underline{z}} \int_V dV = \rho g \hat{\underline{z}}\,\underset{(\text{volume})}{|V|} = mg\hat{\underline{z}} = W\hat{\underline{z}}$.

Hence, this creates an upwards force ($\hat{\underline{z}}$ direction) equal to the weight of the water ($\rho$ associated to water) displaced.

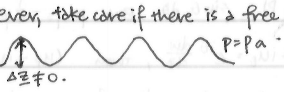
In fluid at rest, pressure balances gravity. For any $V$ in $\mathcal{D}$, the total fluid forces must balance total external forces (i.e. weight). i.e. fluid supports the weight of water occupying $V$.




Now replace $V$ by a solid body. Outside $V$, nothing has changed, so fluid exerts same forces on $V$ i.e. buoyancy = weight of water displaced.

Challenge: What are the forces acting on a body exhibiting solid body rotation.

In many situations, where gravity is the only external force, it is useful to measure pressure by its deviation from hydrostatic pressure i.e. to write $p(x,y,z,t) = p_H(z) + p_D(x,y,z,t)$.
Here, $p_D$ is the dynamic pressure. Then Euler equation becomes $\frac{D\underline{u}}{Dt} = -\frac{1}{\rho}\nabla p - g\hat{\underline{z}} = -\frac{1}{\rho}\underline{\nabla}(p_H + p_D) - g\hat{\underline{z}} = -\frac{1}{\rho}\underline{\nabla} p_H - \frac{1}{\rho}\underline{\nabla} p_D - g\hat{\underline{z}} = g\hat{\underline{z}} - \frac{1}{\rho}\underline{\nabla} p_D - g\hat{\underline{z}}$
i.e. $\frac{D\underline{u}}{Dt} = -\frac{1}{\rho}\underline{\nabla} p_D$ i.e. if we use $p_D$ as the pressure, gravity disappears from the Euler equation. Governing system becomes $\{\underline{\nabla}\cdot\underline{u} = 0,\ \frac{D\underline{u}}{Dt} = -\frac{1}{\rho}\underline{\nabla} p_D\}$. However, take care if there is a free surface because the pressure there is $p = p_a$ i.e. $p_D + p_H = p_a$
e.g. in water waves.




There is a much easier way of introducing pressure which, when relevant, is much simpler: Bernoulli equations.

Bernoulli equations: These arise from vector identity $(\underline{u}\cdot\underline{\nabla})\underline{u} = \nabla(\frac{1}{2}\underline{u}^2) + \underline{\omega}\wedge\underline{u}$ where $\underline{\omega} = \underline{\nabla}\wedge\underline{u}$ (vorticity).
Then $\frac{D\underline{u}}{Dt} = \frac{\partial \underline{u}}{\partial t} + (\underline{u}\cdot\underline{\nabla})\underline{u} = \frac{\partial \underline{u}}{\partial t} + \underline{\nabla}(\frac{1}{2}\underline{u}^2) + \underline{\omega}\wedge\underline{u}$, giving the Euler equations $\frac{\partial \underline{u}}{\partial t} + \underline{\omega}\wedge\underline{u} = -\nabla(\frac{1}{2}\underline{u}^2) - \frac{1}{\rho}\underline{\nabla} p + \underline{F}$.
For conservative external forces, $\underline{F} = -\underline{\nabla} V_e$ i.e. derivable from external potential $V_e$, then $\frac{\partial \underline{u}}{\partial t} + \underline{\omega}\wedge\underline{u} = -\frac{1}{\rho}\underline{\nabla}(p + \frac{1}{2}\rho\underline{u}^2 + \rho V_e)$.
Outline: if flow is steady and $\underline{\omega}\wedge\underline{u} = 0$, $p + \frac{1}{2}\rho\underline{u}^2 + \rho V_e$ = pressure + KE + PE = constant.

22 November 2012
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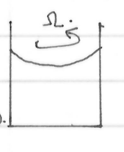
We know that $\underline{\nabla}\cdot\underline{u} = 0$, $\underline{\nabla}\wedge\underline{u} = 0$ (persistence of irrotationality), $\exists\,\phi$ s.t. $\underline{u} = \underline{\nabla}\phi$, $\nabla^2\phi = 0$ (even in 3D).
In 2D, $\exists\,\psi$ s.t. $\underline{u} = -\hat{\underline{k}}\wedge\underline{\nabla}\psi$, $\nabla^2\psi = 0$. If we are interested in where particles go, this is fine. However, if we are interested in forces (lift, drag etc.) or if there is a free surface ($p = p_a$, constant there), we need pressure i.e. forces. Then for constant density, $\underline{\nabla}\cdot\underline{u} = 0$, $\frac{D\underline{u}}{Dt} = -\frac{1}{\rho}\underline{\nabla} p + \underline{F}$.

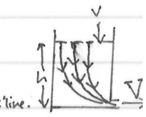



Without any assumptions this becomes $\frac{\partial \underline{u}}{\partial t} + \underline{\omega}\wedge\underline{u} = \frac{1}{\rho}\underline{\nabla} p - \underline{\nabla}(\frac{1}{2}\underline{u}^2) - \underline{\nabla} V_e$ (where $\underline{u}^2 = \underline{u}\cdot\underline{u}$) $= \frac{1}{\rho}\underline{\nabla} H$.
$\Rightarrow H = p + \frac{1}{2}\rho\underline{u}^2 + \rho V_e$: $H$ is an energy/unit volume. We expect that $H$ is 'conserved' in some sense (no viscosity i.e. no dissipation).
In particular, if the flow is steady, $\frac{\partial \underline{u}}{\partial t} = 0$. Dot the equation with $\underline{u}$, then $\underline{u}\cdot\underline{\nabla} H = 0$ (note: we do not require $\omega = 0$, true for rotational flows as well).
i.e. $\underline{\nabla} H$ is perpendicular to $\underline{u}$, but $\underline{u}$ is parallel to the streamlines i.e. $\underline{\nabla} H$ is perpendicular to streamlines $\Rightarrow$ surfaces of constant $H$ lie along the streamlines i.e. $H$ is constant on streamlines (still true in 3D). This is Bernoulli's theorem: $p + \frac{1}{2}\rho\underline{u}^2 + \rho V_e$ is a constant along streamlines (in a constant density fluid). This does not say $p + \frac{1}{2}\rho\underline{u}^2 + \rho V_e$ = constant, because there can be a different constant on each streamline.
e.g. in previous example, in rotating solid body, streamlines are concentric circles with different values of $H$.

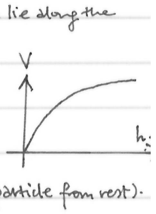
Ex. Picture a container with surface area $A$, filled to depth $h$ with fluid of density $\rho$. The fluid is incompressible. A hole (an aperture of area $\delta\cdot A$ at the base), $0 < \delta \ll 1$, is punched at the bottom of the container. How fast does the fluid exit?

Note: this is analogous to the case where air escapes a space capsule through a hole (except air is compressible, so it will take longer than this lower bound for time due to the escape airway being "choked").

Soln. We will suppose that the hole is sufficiently small, that the flow at any instant is approximately steady. There are streamlines joining the top surface to the aperture. On these streamlines, we can apply Bernoulli theorem: i.e. $p + \frac{1}{2}\rho\underline{u}^2 + \rho V_e$ = const. along s'line.

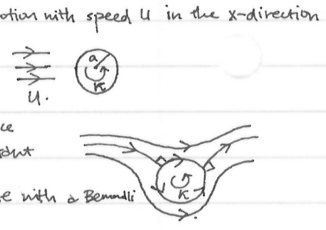



In particular, it is the same on the free surface as at the aperture, and in this case, the same for each streamline joining the free surface to the hole (because every streamline starts with the same value of $H$). On the free surface, we have $p + \frac{1}{2}\rho\underline{u}^2 + \rho V_e = p_a + \frac{1}{2}\rho v^2 + \rho g h$ ($v$ — downward speed at surface.)
At surface of height $h$, the gravitational potential per unit mass is $gh$. (i.e. $\frac{mgh}{m}$). At the aperture,
At base, aperture is opened to atmosphere, so $p = p_a$. Then $p + \frac{1}{2}\rho\underline{u}^2 + \rho V_e = p_a + \frac{1}{2}\rho V^2 + 0$. Since surface and aperture lie along the same streamline, we see that $p_a + \frac{1}{2}\rho v^2 + \rho g h = p_a + \frac{1}{2}\rho V^2 + 0 \Rightarrow V^2 = v^2 + 2gh$.
This behaves exactly like a particle in uniform gravity (because pressures cancel). Thus, final KE = initial KE + decrease in PE
But this is an incompressible fluid, so downward flux at surface = outward flux at hole. i.e. $vA = V\cdot\delta\cdot A \Rightarrow v = \delta V$.




i.e. $(1-\delta^2)V^2 = 2gh \Rightarrow$ hence if $\delta \ll 1$, to order $\delta^2$, $V^2 = 2gh$ i.e. $V = \sqrt{2gh}$ (precisely the speed of releasing a particle from rest).

[E] The spinning cylinder. Consider a cylinder of radius $a$ in a fluid of constant density $\rho$, which at $\infty$ is in uniform motion with speed $U$ in the x-direction (WLOG). Let the circulation on the cylinder be $K$ (WLOG, $K>0$). Analyse, finding drag and lift.




Soln. Complex potential: $w(z) = U(z + \frac{a^2}{z}) - \frac{iK}{2\pi}\log(\frac{z}{a})$, $u_\theta - iu_r = e^{i\theta}\frac{dw}{dz}$. (Almost) every streamline starts upstream, where $H = p + \frac{1}{2}\rho \underline{u}^2 = p_\infty + \frac{1}{2}\rho U^2$ (ignoring gravity $\rightarrow$ we use dynamic pressure $\because$ no free surface, where $p_\infty$ is some constant upstream pressure). Here the only effect of gravity is a normal buoyancy force. This is a second example with a Bernoulli constant existing. On all streamlines, $H = p_\infty + \frac{1}{2}\rho U^2$ i.e. everywhere in the flow (since all streamlines come from upstream for value of $K$ sketched). $H = p_\infty + \frac{1}{2}\rho U^2 = p + \frac{1}{2}\rho \underline{u}^2$ i.e. everywhere pressure is given by $p = p_\infty + \frac{1}{2}\rho(U^2 - \underline{u}^2)$.

At a stagnation point $\underline{u} \equiv 0$ so $p = p_s = p_\infty + \frac{1}{2}\rho U^2$ is the stagnation pressure. $p$ has a minimum when $|\underline{u}|$ max.

Thus $p = p_s - \frac{1}{2}\rho \underline{u}^2$. Per unit distance into page, the force on the cylinder is $\underline{F} = \oint -(p\hat{\underline{n}})\,dl$ on $r = a = -\oint p\hat{\underline{n}}\,dl = ?$

We call the component in the direction of the flow at $\infty$ the drag, and the upward force orthogonal to it the lift. i.e. $\underline{F} = \mathcal{D}\hat{\underline{i}} + \mathcal{L}\hat{\underline{j}}$.

Then $\hat{\underline{r}} = \hat{\underline{n}} = (\cos\theta)\hat{\underline{i}} + (\sin\theta)\hat{\underline{j}}$. Thus, $\underline{F} = -\int_{-\pi}^{\pi} p|_{r=a}[\cos\theta\,\hat{\underline{i}} + \sin\theta\,\hat{\underline{j}}]\,a\,d\theta = \mathcal{D}\hat{\underline{i}} + \mathcal{L}\hat{\underline{j}}$. Hence, we have $\mathcal{D} = -a\int_{-\pi}^{\pi} p|_{r=a}\cos\theta\,d\theta$, $\mathcal{L} = -a\int_{-\pi}^{\pi} p|_{r=a}\sin\theta\,d\theta$. But $p = p_s - \frac{1}{2}\rho\underline{u}^2$. On $|\underline{r}| = a$, $\underline{u} = u_r\hat{\underline{r}} + u_\theta\hat{\underline{\theta}} = u_\theta\hat{\underline{\theta}}$, $|\underline{u}|^2 = u_\theta^2$.

$\therefore p|_{r=a} = p_s - \frac{1}{2}\rho u_\theta^2$. Since $u_r - iu_\theta = e^{i\theta}\frac{dw}{dz}$, then $(u_r - iu_\theta)_{r=a} = (e^{i\theta}\frac{dw}{dz})_{r=a} = -i(\frac{K}{2\pi a} - 2U\sin\theta)$. Hence, as expected, $u_r = 0$, $u_\theta = \frac{K}{2\pi a} - 2U\sin\theta$. Thus $p|_{r=a} = p_s - \frac{1}{2}\rho u_\theta^2 = p_s - \frac{1}{2}\rho(\frac{K}{2\pi a} - 2U\sin\theta)^2 = \frac{1}{2}\rho(\frac{K^2}{4\pi^2 a^2} - \frac{2KU}{\pi a}\sin\theta + 4U^2\sin^2\theta)$.

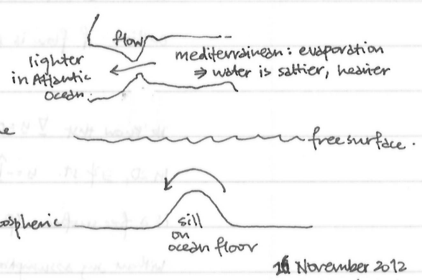
Hence $\mathcal{D} = -a\int_{-\pi}^{\pi} p|_{r=a}\cos\theta\,d\theta$. Recall that, from Fourier theory, $\{1, \cos n\theta, \sin n\theta\}$ are an orthogonal set, $\int_{-\pi}^{\pi} f_i f_j\,d\theta = 0$ if $i \neq j$.

Thus, since $\sin^2\theta = \frac{1}{2}(1 - \cos 2\theta)$, $\mathcal{D} = 0$. $\mathcal{L} = -a\int_{-\pi}^{\pi} p|_{r=a}\sin\theta\,d\theta = -a\int_{-\pi}^{\pi}(-\frac{1}{2}\rho)(-\frac{2KU}{\pi a})\sin^2\theta\,d\theta = -a\int_{-\pi}^{\pi}\frac{\rho KU}{\pi a}\sin^2\theta\,d\theta$.

$\therefore \mathcal{L} = -\int_{-\pi}^{\pi}\frac{\rho KU}{\pi}\sin^2\theta\,d\theta = -\rho KU$ (independent of $a$, downwards, proportional to density, circulation and speed).

A further, important application of Bernoulli's theorem is Open Channel Flow.

Consider the straits of Gibraltar. It is hemmed in by both sides, with water flowing through a narrow channel. How does flow occur? It moves down a pressure gradient — formed by heavier water in the Mediterranean flowing into the Atlantic.




Alternatively, consider a bump (sill) on the ocean floor, with a free surface over it and open to atmospheric pressure. Is the surrounding water light enough to pass over it?

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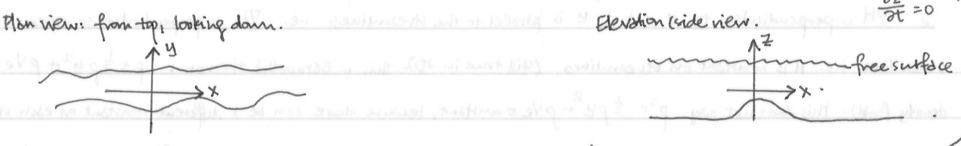
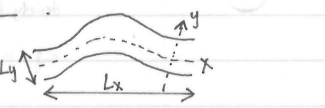
Open channel flows are slowly-varying long wave equations: consider either the plan view or the elevation. We assume flow is steady. $\frac{\partial \underline{u}}{\partial t} = 0$

Plan view: from top, looking down. Elevation (side view).




Slowly-varying assumption: everything varies slowly in x — distance along channel.

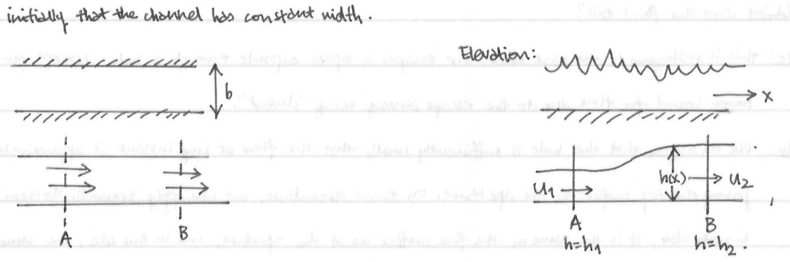
i.e. in diagram on right, $\frac{L_y}{L_x} \ll 1$.




Suppose initially, to simplify case, that bottom is flat and horizontal (i.e. $z=0$) and take the zero of our external gravitational potential to lie at $z=0$.

We will also suppose initially that the channel has constant width.

Plan view: Elevation:




$\frac{\partial h}{\partial t} = 0$.

We suppose that the variations are so slow that flow remains uniform across the channel $\underline{u} = u(x)\,\hat{\underline{i}}$. Let the slowly varying depth be $h(x)$.

The amount of fluid passing A = the amount passing B (otherwise mass between A and B not steady).

Flux at A $= \rho u_1 h_1 b$, flux at B $= \rho u_2 h_2 b$, so $u_1 h_1 = u_2 h_2$, but these are arbitrary points, so $uh = \text{const}$, say $Q$, throughout the flow.

We require one more relation between $u$ and $h$, to give us a system of two equations with two unknowns.

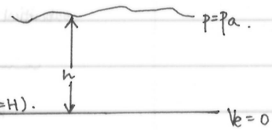
We consider first smooth variations. Thus the surface is smooth. A particle on the surface stays there $\Rightarrow$ surface is a streamline. Then the flow is steady, we have a streamline, thus we can apply Bernoulli equation at the surface. (not generally so, we need to assume smoothness – c.f. Korteweg de Vries equation, inverse scattering theory)

2301-016

Along the surface, $p = \frac{1}{2}\rho u^2 + \rho V_e = \text{const}$. Then $V_e = gh$, and we have $p_a + \frac{1}{2}\rho u^2 + \rho g h = \text{const}$.

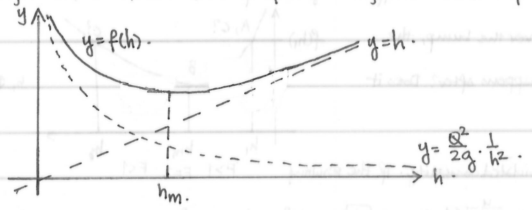
Rearranging, $h + \frac{1}{2g}u^2 = \text{const}, H$. We call $H$ the head of flow, a measure of the internal energy of fluid.

Note: this is so called because if fluid is released against a wall, it will attain height $H$ against the wall (i.e. $u=0$, $h=H$).




The whole system becomes: (mass) $uh = Q$, (energy) $h + \frac{1}{2g}u^2 = H$. To solve, we eliminate $u$, so $u = \frac{Q}{h}$.

then $h + \frac{Q^2}{2gh^2} = H$, or $f(h) = H$, where $f(h) = h + \frac{Q^2}{2gh^2}$ is called the specific energy. We plot a graph showing $y = f(h)$ as $h$ varies.



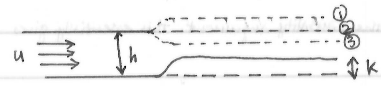

The function has a single minimum:

$f'(h) = 1 - \frac{Q^2}{gh^3}$, so $f'(h) = 0$ where $h^3 = \frac{Q^2}{g} \Rightarrow h = \left(\frac{Q^2}{g}\right)^{1/3} = h_m$.

Then at the point which minimises specific energy, $h_m$,

$f(h_m) = h_m + \frac{Q^2}{2gh_m^2} = h_m\left[1 + \frac{Q^2}{2gh_m^3}\right] = h_m\left[1 + \frac{1}{2}\frac{Q^2}{g}\cdot\frac{g}{Q^2}\right] = \frac{3}{2}h_m$.

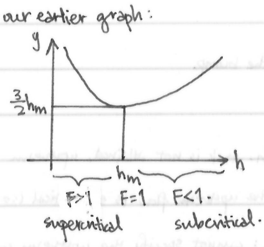
Then consider a constant width channel, where the floor rises by $k$. Water approaches with a height of $h$. What happens to the flow?




Does the water ① go up, ② remain the same, or ③ go down?

In a sense, all answers are correct, and it depends on circumstances.

Consider our earlier graph:




At $h_m$, $\frac{Q^2}{gh^3} = 1 \Rightarrow \frac{u^2h^2}{gh^3} = 1$, $\frac{u^2}{gh} = 1$. We introduce the Froude number, $F = \frac{u}{\sqrt{gh}}$.

At $h_m$, $F=1$; when $h > h_m$, $u < u_m$ since $uh$ is constant, so $F<1$. Also, when $h<h_m$, $u>u_m$.

Note that $F = \frac{\text{flow speed}}{\text{long wave speed}}$. Where

- $F=1$, flow speed = wave speed, flow is critical
- $F>1$, flow speed > wave speed, flow is supercritical
- $F<1$, flow speed < wave speed, flow is subcritical.

Note: the Mach number is a specific case for sound, where analogously, $M = \frac{\text{flow speed}}{\text{sound speed}}$.

29 November 2012
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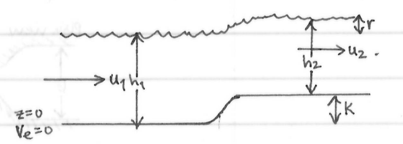
Our earlier assumptions called for channel flow of • constant width and • flat bottom.

Now we relax the lower boundary condition: let the floor of our channel rise by an amount $k$.

On our diagram, let $r$ be the rise in the water level be $r = (k + h_2) - h_1$. Is $r=0$, $r<0$ or $r>0$?

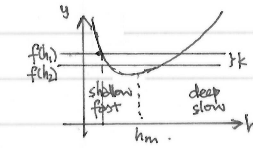



By conservation of mass, $u_1h_1 = u_2h_2$. The ① flow is steady and the ② surface is smooth (and is a streamline).

And so, we can use Bernoulli's equation on surface: note that for potential here, we use $z$, the height of the surface above the zero of $V_e$. (NOT $h$!)

Then $p + \frac{1}{2}\rho u^2 + \rho g z = \text{const}$. i.e. $p_a + \frac{1}{2}\rho u_1^2 + \rho g h_1 = p_a + \frac{1}{2}\rho u_2^2 + \rho g(k + h_2)$. $\Rightarrow$ $\frac{u_1^2}{2g} + h_1 = \frac{u_2^2}{2g} + h_2 + k$. But from earlier, $f(h) = h + \frac{Q^2}{2gh^2} = h + \frac{u^2}{2g}$

Hence, $f(h_1) = f(h_2) + k$. We consider cases where $k$ is not large i.e. $k < f(h_1) - f(h_m)$.

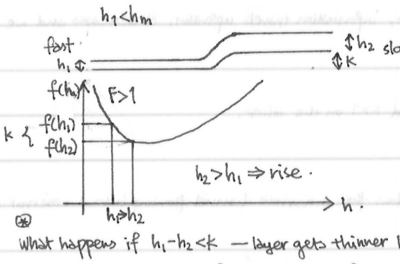


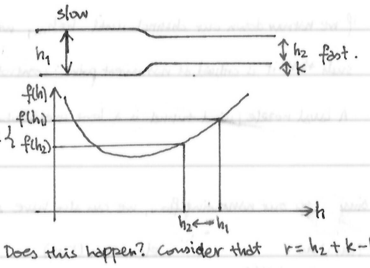

- If $h_1 < h_m$, then upstream, Froude number $F>1$, supercritical. Then $f(h_2) = f(h_1) - k \Rightarrow f(h_2) < f(h_1)$, so $h_2 > h_1$.

  The flow gets deeper and slows down. i.e. surface rises. KE is converted to PE. Flow remains supercritical.

- If $h_1 > h_m$, then upstream, Froude number $F<1$, subcritical. Then $f(h_2) = f(h_1) - k \Rightarrow f(h_2) < f(h_1)$, so $h_2 < h_1$.

  The flow gets shallower and speeds up i.e. surface falls if $h_1 - h_2 > k$. ⊛ (regardless, layer gets thinner). PE is converted to KE.



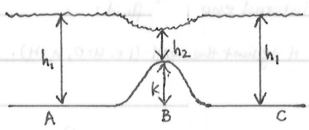

⊛ What happens if $h_1 - h_2 < k$ — layer gets thinner but surface rises/remains flat. Does this happen? Consider that $r = h_2 + k - h_1$. Since $h_1 + \frac{Q^2}{2gh_1^2} = h_2 + \frac{Q^2}{2gh_2^2} + k$.

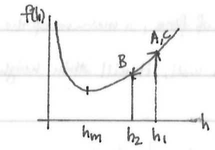
Hence, $r = h_2 + k - h_1 = \frac{Q^2}{2g}\left(\frac{1}{h_1^2} - \frac{1}{h_2^2}\right) = \frac{Q^2}{2gh_1^2h_2^2}(h_2^2 - h_1^2)$. Therefore $r<0$ when $h_2 < h_1$, i.e. the surface always falls, regardless. i.e. $h_1 - h_2 > k$ in all cases.

Alternative argument: How do we show that $h_1 - h_2 \not\leq k$? Use a graphical argument: if $h_1 - h_2 \leq k$, $k>0$, then $\frac{k}{h_1 - h_2} \geq 1 \Rightarrow \frac{f(h_1) - f(h_2)}{h_1 - h_2} \geq 1$. Hence, the gradient on the segment $(h_2, f(h_2))$ to $(h_1, f(h_1))$ is $\geq 1$. But for $h > h_m$, $f(h) \to h$ from above, so $f(h)$ is less steep than $h$, and $\frac{f(h_1) - f(h_2)}{h_1 - h_2} \leq h' = 1$, so we need not worry as such cases never exist.

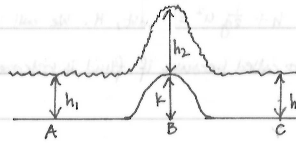
Let us consider a more complex system — flow over a bump.

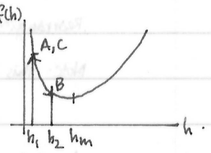
Subcritical flow over a bump:



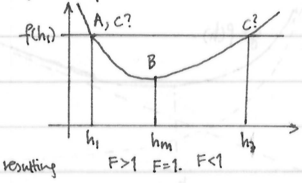

Supercritical flow over a bump:

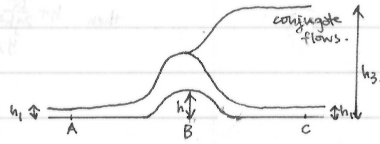



We know that at $h=h_m$, $f(h)=\frac{3}{2}h_m$. Let us consider a flow with $F>1$ approaching a bump.

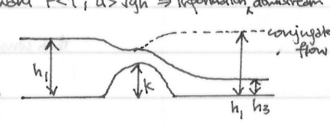
Initially, at point A, height is $h_1$ and $f(h)=f(h_1)$. Then over the bump, the thickness of the layer increases, surface rises. But what happens after? Does it increase to $h_3$ or return to $h_1$?



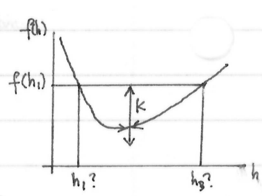



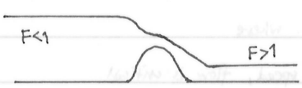

Answer: It always returns to $h_1$, as the other solution violates causality. If the resulting solution had been $h_3$, after the bump $F<1 \Rightarrow \frac{u}{\sqrt{gh}}<1 \Rightarrow u<\sqrt{gh} \Rightarrow$ information can flow upstream. Instead we want $F<1$, $u>\sqrt{gh} \Rightarrow$ information swept downstream.

Likewise, if we consider that initially $F<1$, water at $h_1$. Then over the bump, the water level dips. What happens after? The water level does not return to $h_1$, but rather drops to $h_3$ — again by the causality argument. This essentially gives us a <u>transition from subcritical to supercritical</u> flow.




If $k>f(h_1)-\frac{3}{2}h_m$, then no smooth solution joins our given upstream flow to the flow over the bump. There must be a transition at the bump (subcritical to supercritical flow):



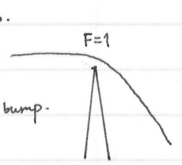




This gives us a point ($h_1$) on our graph. As k is not attained, upstream conditions change as they can, as the upstream flow is subcritical (so information can travel upstream).

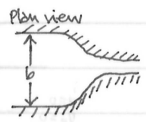
A little more PE is needed to get over a higher bump. We will require a <u>controlled flow</u>: cannot specify the upstream conditions.

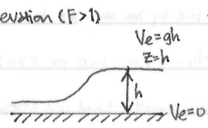
Consider the weir at Marlowe: the height of it is so great that the flow is always critical, thus there is always a transition.

Then this is an example of a controlled flow — unless it is resubmerged by large water flow, then we get subcritical flow over bump.




Now to a different example: we keep the bottom horizontal but consider variable width b.



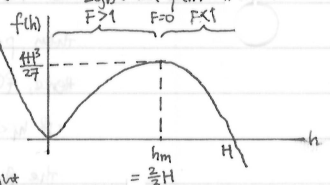

Work using conservation of volume flux, which is given by area × speed, at any station A.

i.e. $bhu = \text{const} = Q$. Then Bernoulli's equation yields (since flow is steady, surface is smooth ⇒ streamline)

$$p+\tfrac{1}{2}\rho u^2+\rho V_e=\text{const} \Rightarrow p_a+\tfrac{1}{2}\rho u^2+\rho g h=\text{const.} \Rightarrow \frac{u^2}{2g}+h=H$$

Eliminate u by $u=\frac{Q}{bh}$, then $\frac{Q^2}{2gb^2h^2}+h=H$, then $h^2(H-h)=\frac{Q^2}{2gb^2}$. Let $f(h)=h^2(H-h)$

then $f(h)=\frac{Q^2}{2gb}$. f is a parabola near $h=0$, simple 0 at $h=H$, leading coefficient negative.




f(h) has a maximum at $h_m$ where $f'(h_m)=0$. $f'(h)=2h(H-h)-h^2$. For $h\neq 0$, $2H-2h-h=0$, i.e. $h=\frac{2}{3}H$, $f(h_m)=\frac{4}{27}H^3$

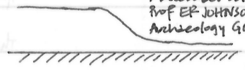
Suppose we have a slowly-narrowing channel with flat bottom. Let flow be subcritical upstream.

Since $b_2<b_1$, then since $f(h)=\frac{Q^2}{2gb^2}$, $f(h_2)>f(h_1) \Rightarrow$ for $F<1$, $h_2<h_1$ and the surface falls: as in diagram to right (→)

The system gives up PE for KE. We can also consider the case where the flow is supercritical upstream.

Since $b_2<b_1$ then $f(h_2)>f(h_1)$. For $F>1$, $h_2>h_1$ and the surface rises: as in diagram above. (↑).

Elevation (F<1)



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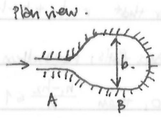
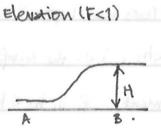
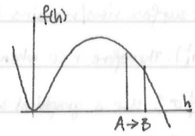
If we narrow down our channel until $h=h_m$, we get $F=1$. If we narrow it more, information travels upstream, Q decreases and we move to a different graph such that it is critical at narrowest point: controlled flow.

A Laval nozzle / wind tunnel is a long streamtube such that $F<1$ on one side and $F>1$ on the other.

Similar to our narrowing flow, we can also have a divergent flow, where fluid flows from a narrow channel flowing into a reservoir at rest, $u=0$.

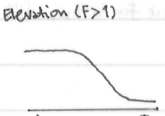


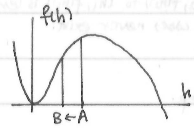




For $F<1$, b is increasing $\Rightarrow f(h)=\frac{Q^2}{2gb^2}$ decreasing, h increasing

As $b\to\infty$, $\frac{Q^2}{2gb^2}\to 0$ and $h\to H$.




For $F>1$, b is increasing $\Rightarrow f(h)=\frac{Q^2}{2gb^2}$ decreasing, h decreasing.

As $b\to\infty$, $\frac{Q^2}{2gb^2}\to 0$.

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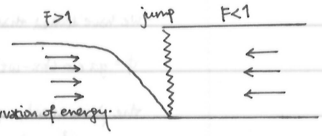
One implication of this is that no fast rivers ever enter a reservoir.

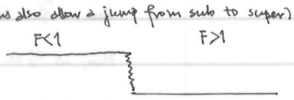
There cannot be a smooth transition with smoothly altered upstream conditions as the oncoming flow is supercritical ($F>1$), and characteristics all point downstream, hence no information travels upstream by causality. Thus a non-smooth jump occurs – we call this a hydraulic jump.




In our diagram, the jump occurs from $F>1$ to $F<1$.

It is not smooth $\Rightarrow$ no streamlines along surface to connect upstream and downstream $\Rightarrow$ no Bernoulli equation $\Rightarrow$ no conservation of energy.

Mass is conserved and momentum is conserved – these two equations are sufficient to determine $h_2$ given $h_1$. (Note: equations also allow a jump from sub to super)




Here, energy flux at B is greater than energy flux at A, thus cannot occur spontaneously. For super to sub, energy is given out in the form of waves/noise etc. (e.g. with planes – sonic boom).

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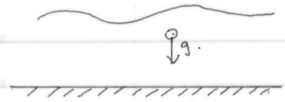
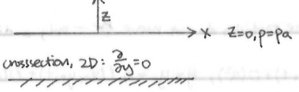
## 6. SURFACE WATER WAVES.

How fast do waves travel? If the wave moves under a restoring force of $g$, prove that $c=\sqrt{gh}$ if fluid is shallow.




We will consider only infinitesimal waves. Suppose that the basic flow is absolutely at rest with flat, horizontal surface, and thus hydrostatic pressure $p=p_a-\rho g z$. Now consider small waves on the surface, so $p=p_a-\rho g z+\rho\phi$, where $\phi$ is some small quantity (s.t. $\phi^2$ is negligible).




Then our momentum equations are $\frac{\partial \underline{u}}{\partial t}+(\underline{u}\cdot\underline{\nabla})\underline{u}=-\frac{1}{\rho}\underline{\nabla}p-g\hat{\underline{z}}$, i.e. in components:

- x-component is $\frac{\partial u}{\partial t}+(\underline{u}\cdot\underline{\nabla})u=-\frac{1}{\rho}\rho\frac{\partial\phi}{\partial x}$. The $(\underline{u}\cdot\underline{\nabla})\underline{u}$ term is quadratic in small qty, everything else is linear $\Rightarrow \frac{\partial u}{\partial t}=-\frac{\partial\phi}{\partial x}$
- z-component is $\frac{\partial w}{\partial t}+(\underline{u}\cdot\underline{\nabla})w=-\frac{1}{\rho}(-\rho g+\rho\frac{\partial\phi}{\partial z})-g=-\frac{\partial\phi}{\partial z}$.

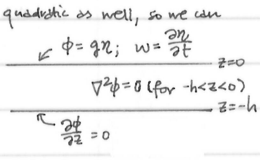
So our governing equations are Euler: $\frac{\partial u}{\partial t}=-\frac{\partial\phi}{\partial x}$, $\frac{\partial w}{\partial t}=-\frac{\partial\phi}{\partial z}$. Continuity: $\frac{\partial u}{\partial x}+\frac{\partial w}{\partial z}=0$, so $\frac{\partial}{\partial t}(\frac{\partial u}{\partial x}+\frac{\partial w}{\partial z})=0$ i.e. $\frac{\partial}{\partial x}(\frac{\partial u}{\partial t})+\frac{\partial}{\partial z}(\frac{\partial w}{\partial t})=0$, and $\frac{\partial}{\partial x}(-\frac{\partial\phi}{\partial x})+\frac{\partial}{\partial z}(-\frac{\partial\phi}{\partial z})=-\frac{\partial^2\phi}{\partial x^2}-\frac{\partial^2\phi}{\partial z^2}=0$ i.e. $\nabla^2\phi=0 \Rightarrow$ our problem is governed by Laplace's equation.

If there is a lower boundary, then $w=0$ there $\forall t$, so $\frac{\partial\phi}{\partial z}=0$ on $z=-h$.


Let the surface at any time be given by $z=\eta(x,t)$. Then the pressure in the water at the surface must balance atmospheric pressure i.e. when $z=\eta$, $p=p_a$.

$p=p_a-\rho g z+\rho\phi \Rightarrow p_a=p_a-\rho g\eta+\rho\phi$ on $z=\eta$, thus $\phi=g\eta$ on $z=\eta$. Thus, since $\eta$ is small, we can evaluate our boundary condition on $z=0$ (instead of $z=\eta$) and we make an error of order $\eta\frac{\partial\phi}{\partial z}$ in our equation. But $\eta\frac{\partial\phi}{\partial z}$ is quadratic as well, so we can well neglect it $\Rightarrow \phi=g\eta$ on $z=0$. We need one more relation between $\phi$ and $\eta$.


When we claimed that "$z=\eta$ is a surface", we meant that a particle on the surface defines the surface for all time:




i.e. a particle on the surface stays on the surface. i.e. Following a particle, $z=\eta(x,t)$ is always true on $z=\eta$; i.e. we have $\frac{D}{Dt}(z-\eta(x,t))=0$ on $z=\eta$ i.e. $w=\frac{D\eta}{Dt}$ on $z=\eta$. $\Rightarrow w=\frac{D\eta}{Dt}=\frac{\partial\eta}{\partial t}+\underbrace{(\underline{u}\cdot\underline{\nabla})\eta}_{\text{quadratic, neglect}}=\frac{\partial\eta}{\partial t}$ on $z=\eta \Rightarrow w=\frac{\partial\eta}{\partial t}$ on $z=0$ (neglecting terms again).

This completes the setup/derivation of our problem.

Initially let us suppose that the flow is infinitely deep: $h\to\infty$. Then $\nabla^2\phi=0$, $\phi=g\eta$ on $z=0$, $w=\frac{\partial\eta}{\partial t}$ on $z=0$ i.e. $\frac{\partial\phi}{\partial z}=-\frac{\partial^2\eta}{\partial t^2}$ on $z=0$; $\frac{\partial\phi}{\partial z}\to 0$ as $z\to-\infty$

This problem is linear, so all solutions are superpositions of sinusoids (equivalent to a Fourier series for periodic flow, or a Fourier transform for non-periodic flow).

i.e. sufficient to consider $\eta(x,t)=a\cos(kx-\omega t)$. Each of these variables has a name: $a$ is amplitude, $\omega$ is frequency (in radians), so temporal period $\tau=\frac{2\pi}{\omega}$

$k$ is wavenumber, and spatial period (or wavelength) is $\lambda=\frac{2\pi}{k}$. $k$ is called a wavenumber as it is the number of waves in a distance $2\pi$.

Then on $z=0$, $\phi=ag\cos(kx-\omega t)$ and $\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2}=0$. Try $\phi=ag\cos(kx-\omega t)Z(z)$ with $Z(0)=1$, $\phi_{xx}=-ak^2\cos(kx-\omega t)Z(z)$, and hence, in Laplace's:

$-ak^2g\cos(kx-\omega t)Z+ag\cos(kx-\omega t)Z''=0$, so $Z''-k^2Z=0$. $Z(0)=1$, $Z'\to 0$ as $z\to-\infty$. Thus, general solution is $Z=Ae^{kz}+Be^{-kz}$ as $k^2>0$, and $Z'\to 0$ as $z\to-\infty$, so $B=0$, and $Z(0)=1 \Rightarrow A=1$ i.e. $Z(z)=e^{kz}$. Hence $\phi=ag\cos(kx-\omega t)e^{kz}$ //.

The wave travels to the right with speed $c=\frac{\omega}{k}$, decaying with an e-folding scale of $\frac{1}{k}=\frac{\lambda}{2\pi}$. (distance over which the quantity falls by a factor $\frac{1}{e}$).

But we have not satisfied $\phi_z=-\eta_{tt}$ on $z=0$ i.e. $g\phi_z=-\phi_{tt}$ on $z=0$ i.e. $\phi_{tt}+g\phi_z=0$ on $z=0$. $\Rightarrow -ag\omega^2\cos(\ )e^{k0}+gkag\cos(\ )e^{k0}=0$.

so $-\omega^2+gk=0$, $\omega^2=gk$. Hence $\phi=ag\cos(kx-\omega t)e^{kt}$ is a solution to our problem iff $\omega^2=gk \Rightarrow \omega=\sqrt{gk} \Rightarrow c=\frac{\omega}{k}=\sqrt{\frac{g}{k}}=\sqrt{\frac{g\lambda}{2\pi}}$.

i.e. a wave of wavelength $\lambda$ travels at speed (over infinite depth) $c=\sqrt{\frac{g\lambda}{2\pi}}$.

Put the lower boundary back in: i.e. $\frac{\partial\phi}{\partial z}=0$ on $z=-h$. As above, $\eta=a\cos(kx-\omega t)$, so $\phi=ag\cos(kx-\omega t)Z(z)$. Cf:

$Z(z)=A\cosh[k(z+h)]+B\sinh[k(z+h)] \Rightarrow Z'(z)=Ak\sinh[\ ]+Bk\cosh[\ ] \Rightarrow Z'(-h)=Bk$ so $B=0$.

Now $Z(0)=A\cosh(kh)$, so $A=\frac{1}{\cosh(kh)}$, giving $\phi=ag\cos(kx-\omega t)\cosh[k(z+h)]/\cosh(kh)$. (As $h\to 0$, open channel flow?). But as before, we have to satisfy:

$\phi_{tt}+g\phi_z=0$ on $z=0 \Rightarrow -\omega^2ag\cos(\ )\cosh(kh)/\cosh(kh)+gag\cos(\ )\cdot k\sinh(kh)/\cosh(kh) \Rightarrow \omega^2=gk\tanh(kh)$.

2301-019

10 December 2012
Prof Jean-Marc VANDEN-BROEC[K]
Arch[illegible]

Two snippets of videos on previous material were screened in class.

13 December 2012
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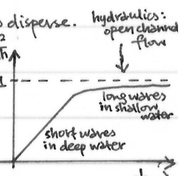
We have derived that surface displacement, $\eta$, is modelled by $\eta = a\cos(kx-\omega t)$, $0<a<<1$. Then we have

$\phi = ga\cos(kx-\omega t)\dfrac{\cosh k(z+h)}{\cosh kh}$ iff $\omega^2 = gk\tanh kh$. This is a dispersion relation, as different wavelengths travel at different speeds i.e. waves disperse.

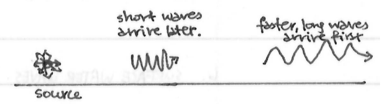
The phase speed, given by $c$, appears in the equation $\cos(kx-\omega t) = \cos[k(x-ct)]$. Then, $c^2 = \dfrac{\omega^2}{k^2} = \dfrac{g}{k}\tanh kh = \dfrac{g\lambda}{2\pi}\tanh\left(2\pi\dfrac{h}{\lambda}\right)$ $\because k = \dfrac{2\pi}{\lambda}$.

Thus, $\dfrac{c^2}{gh} = \dfrac{\lambda}{2\pi h}\tanh\dfrac{2\pi h}{\lambda}$. We plot a graph of $\dfrac{c^2}{gh}$ against $\dfrac{1}{\theta} = \dfrac{2\pi h}{\lambda}$. Observe that as $\dfrac{1}{\theta} >> 1$, $\theta << 1$, $\tanh\theta \approx \theta$, so $\dfrac{c^2}{gh} = \dfrac{1}{\theta}\tanh\theta \approx 1$.

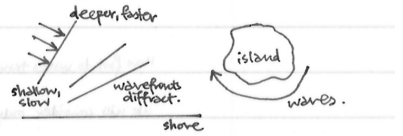
Also, as $\dfrac{1}{\theta} << 1$, $\theta >> 1$, $\tanh\theta \approx 1$, so $\dfrac{c^2}{gh} \approx \dfrac{1}{\theta} = \dfrac{\lambda}{2\pi h}$ (linear relation). Hence, this gives us the relation shown in graph:




Thus, long waves travel fastest with maximum speed $\sqrt{gh}$ (exactly as in the definition of the Froude number $F = \dfrac{u}{\sqrt{gh}}$: e.g. $F>1$, flow faster than any wave: no wave can go upstream.) Wavespeed depends on wavelength.




Aside: this phenomenon allows for predictions of tsunamis based on observation of wave flow. Also, it explains why around some islands the waves travelled in a direction parallel to the shore, because of diffraction: waves travel slower closer to the shore: tsunamis are long-waves that are non-dispersive, so in sufficiently shallow water, a limit is achieved and the long waves behave like normal waves, slower in shallow water.




What are the particle paths? We have $\dfrac{dx}{dt} = u(x,z,t)$ and $\dfrac{dy}{dt} = v(x,z,t)$. The amplitudes are small, so the particles do not move far — only a distance of order $a$ in the Taylor series expansions: $0<a<<1$. We write the position of the particle as $(x,y)$, where $x = x_0 + aX(x_0,z_0,t) + O(a^2)$, $z = z_0 + aZ(x_0,z_0,t) + O(a^2)$.


Then $a\dfrac{dX}{dt} + O(a^2) = u(x_0,y_0,t) + O(a)\left[\dfrac{\partial u}{\partial x}, \dfrac{\partial u}{\partial z}\right]$. But $\dfrac{\partial u}{\partial t} = -\dfrac{\partial \phi}{\partial x} = gak\sin(kx-\omega t)\dfrac{\cosh[\,]}{\cosh kh}$ i.e. $u = \dfrac{gak}{\omega}\cos(kx-\omega t)\dfrac{\cosh[\,]}{\cosh kh}$

Note that $a\dfrac{dX}{dt} + O(a^2) = \overbrace{u(x_0,z_0,t)}^{O(a),\text{ from above}} + O(a)\overbrace{\left[\dfrac{\partial u}{\partial x}, \dfrac{\partial u}{\partial z}\right]}^{O(a)}$ $\Rightarrow$ to order $a$, $a\dfrac{dX}{dt} = \dfrac{gak}{\omega}\cos(kx_0-\omega t)\dfrac{\cosh[k(z_0+h)]}{\cosh kh}$. We cancel $a$, and integrate to get

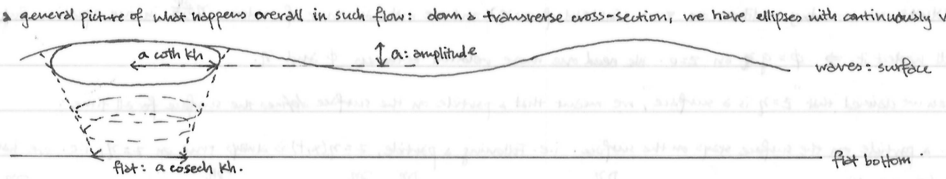
$X = -\dfrac{gk}{\omega^2}\sin(kx_0-\omega t)\dfrac{\cosh[k(z_0+h)]}{\cosh kh}$. And since $\omega^2 = gk\tanh kh$, $X = \alpha\sin(kx_0-\omega t)$ where $\alpha = \dfrac{\cosh[k(z_0+h)]}{\sinh kh}$.

Similarly, $Z = \beta\cos(kx_0-\omega t)$ where $\beta = \dfrac{\sinh[k(z_0+h)]}{\sinh kh}$. Thus, we get $\dfrac{X^2}{\alpha^2} + \dfrac{Y^2}{\beta^2} = 1$, which is an ellipse with horizontal semi-axis $\alpha$ and vertical semi-axis $\beta$.

We check our result. On $z_0 = 0$, i.e. the surface; $\alpha = \coth kh$, $\beta = 1$ and $Z = \cos(kx_0-\omega t)$. Then particles move in circular contours as $h\to\infty$


If $\alpha=\beta=1$ as $h\to-\infty$, we also get circles. We can perform a final check at the bottom, $z_0 = -h$. $\alpha = \dfrac{1}{\sinh kh}$, $\beta = 0$ so $Z = 0$ (makes sense). Particles oscillate horizontally with amplitude cosech kh

This gives us a general picture of what happens overall in such flow: down a transverse cross-section, we have ellipses with continuously varying aspect ratios.



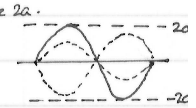

Finally, we consider the case of reflected waves:

A wave of amplitude $a$ moving to the right with speed $c$ has $\eta_1 = a\cos[k(x-ct)]$ and $\phi_1 = ag\cos[k(x-ct)]\dfrac{\cosh[k(z+h)]}{\cosh kh}$.

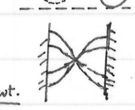
A reflected wave of the same frequency (and so the same wavelength) has $\eta_2 = a\cos[k(x+ct)]$, so $\phi_2 = ag\cos[k(x+ct)]\dfrac{\cosh[k(z+h)]}{\cosh kh}$

The combined wave is $\eta = \eta_1 + \eta_2 = a\cos[k(x-ct)] + a\cos[k(x+ct)] = 2a\cos kx\cos kct = 2a\cos kx\cos\omega t$, which is a standing wave of amplitude $2a$.

For this wave, $\phi = \phi_1 + \phi_2 = ag\dfrac{\cosh[k(z+h)]}{\cosh kh}\{\cos[k(x-ct)] + \cos[k(x+ct)]\} = 2ag\dfrac{\cosh[\,]}{\cosh kh}\cos kx\cos\omega t$.




For a solid wall at some $x$, $u=0$ i.e. $\dfrac{\partial\phi}{\partial x} = 0$ i.e. $kx = n\pi$ i.e. $x = \dfrac{n\pi}{k}$. We have $\dfrac{\partial\phi}{\partial x} = 0$ at each of the crests/troughs $\Rightarrow \dfrac{\partial\eta}{\partial x} = 0$ too.



Hence, we can put in solid walls at each of these nodes to treat each component as an isolated wave unit:

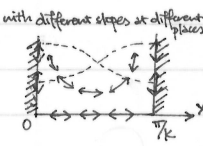
For the particle paths, $\dfrac{dx}{dt} = u(x,z,t)$, $\dfrac{dz}{dt} = w(x,z,t)$. Now, $\dfrac{\partial u}{\partial t} = -\dfrac{\partial\phi}{\partial x}$ here, so $\dfrac{\partial u}{\partial t} = -2ag\dfrac{\cosh[\,]}{\cosh kh}(-k\sin kx\cos\omega t) \Rightarrow u = \dfrac{2agk}{\omega}\dfrac{\cosh[\,]}{\cosh kh}\sin kx\,\underline{\sin\omega t}$.

And also, $\dfrac{\partial w}{\partial t} = -\dfrac{\partial\phi}{\partial z} \Rightarrow \dfrac{\partial w}{\partial t} = -2agk\dfrac{\sinh[k(z+h)]}{\cosh kh}\cos kx\cos\omega t \Rightarrow w = -\dfrac{2agk}{\omega}\dfrac{\sinh[\,]}{\cosh kh}\cos kx\,\underline{\sin\omega t}$. We observe that both $u$ and $w$ have the same time dependence,

so we can just take ratios: $\dfrac{dz}{dx} = \dfrac{dz/dt}{dx/dt} = \dfrac{w}{u} = \tanh[k(z+h)]\cot kx$. We can solve this as it stands. However, we are interested only in small waves, $0<a<<1$. We write

$x = x_0 + aX + O(a^2)$, $z = z_0 + aZ + O(a^2)$. Then $\dfrac{dZ}{dX} = \tanh[k(z_0+h)]\cot kx_0$, a constant in time (just a function of position) $\Rightarrow$ i.e. straight lines with different slopes at different places

So slopes are infinite when $\cot kx_0 = \infty$ i.e. $\sin kx_0 = 0 \Rightarrow x_0 = \dfrac{n\pi}{k}$, which are our solid walls. When $z_0 = -h$, at bottom, $\dfrac{dZ}{dX} = 0$.




END OF SYLLABUS.

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