2301 Fluid Mechanics Notes (Part 2 of 2)

Based on the 2012 autumn problem classes by Prof J M Vanden-Broeck and Prof E R Johnson

MATH2301·P — Fluid Dynamics (Problem Class).

8 October 2012
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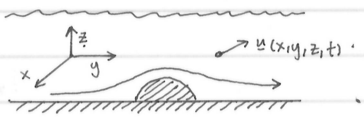
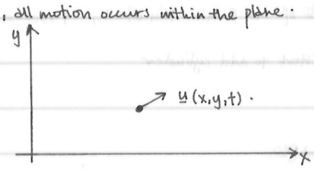
Imagine a river, with an obstacle within:



In 2D, all motion occurs within the plane.



The three key definitions (VERY IMPORTANT: state word-for-word).

- STREAMLINE — a line whose tangent is parallel to the velocity at that point.
- PARTICLE PATH — a path traced by a particle in a given time interval.
- STREAKLINE — the locus at a given time of all particles that have passed through a given point in space during a given time interval.

All three lines are the same if the flow is <u>steady</u>. $(\frac{\partial}{\partial t}(\cdots)=0)$.

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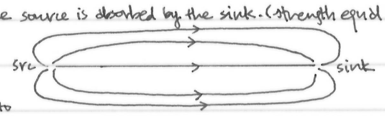
Recall that for an isotropic source of strength m, we have $(2\pi r)v_r = m$. then for this flow, we have $\phi = \frac{m}{2\pi}\ln r$, $\psi = \frac{m}{2\pi}\theta$.

In such a source, fluid is evenly emitted in all directions. Now imagine we have the opposite — i.e. fluid flows inwards into a singular point at a rate of $-m$. Then the equations for such a flow would be the same except for a sign difference. We call this a <u>sink</u>.

i.e. $(2\pi r)v_r = -m$ s.t. $\phi = -\frac{m}{2\pi}\ln r$, $\psi = -\frac{m}{2\pi}\theta$. We note also that the complex potentials are respectively $w = m\log z$ and $w = -m\log z$.

Now imagine if we put a source and a sink in close proximity. then the fluid emitted by the source is absorbed by the sink. (strength equal and opposite).

We call such a flow a <u>dipole</u> flow. Moreover, examining the streamtube, we notice that the outermost streamlines form an oval/ellipse (in 2D) with no normal flux. Hence, we refer to this system as a <u>Rankine oval</u>, with streamlines obtainable by the superposition of streamfunctions. This is able to define a closed body in a uniform flow.

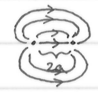


then the distance to the stagnation points up/downstream from the half-body length (intersection of semimajor/minor axes) is $\frac{l}{a} = \sqrt{\frac{m}{\pi U a}+1}$, where m is the strength of the source, U is the speed of uniform flow, and a is the half-length.

What happens when our source and sink move closer to each other? We evaluate this using complex potentials. Let the source and sink be respectively displaced from the origin by $-a$ and $+a$ respectively. As we move them closer towards being adjacent, $a \to 0$. As we get infinitesimally close, the fluid may no longer behave like a source-sink system as fluid could flow directly between, unless we presume the source/sink to have infinite strength, i.e. $m \to \infty$.

This gives us $a \to 0$, $m \to \infty$, $2am$ (or am) $\to \mu$, constant. Thus, since $W_1 = m\log(z+a)$, $W_2 = -m\log(z-a)$ then $W = m[\log(z+a) - \log(z-a)]$

And $\lim\limits_{a\to 0} w = \lim\limits_{a\to 0} m[\log(z+a) - \log(z-a)] = \lim\limits_{a\to 0}(2am)\frac{[\log(z+a)-\log(z-a)]}{2a} = \mu\frac{d}{dz}\log(z) = \frac{\mu}{z}$. In a uniform flow with speed U then, $w = Uz + \frac{\mu}{z}$.



But we know that the complex potential for flow past a stationary cylinder is the similar function $w = Uz + \frac{Ua^2}{z}$.

Hence we conclude that the flow of a dipole in uniform stream is analogous to flow past a cylinder.

Note: The term μ is known as the <u>dipole strength</u>.

29 October 2012
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<u>Techniques for Plotting ψ = const.</u> (Skills) (related to homework)

1. Draw without cylinder i.e. far-field, i.e. $r \to \infty$. e.g. $\psi = Uy(1-\frac{a^2}{r^2})$, without cylinder $\psi = Uy$.
2. Now put in cylinder with least possible distortion.
3. Make sure the streamlines hit the cylinder at the right places. i.e. at a stagnation point, where $\nabla\psi = 0$ (or $u_r = 0$ and $u_\theta = 0$ i.e. $\frac{\partial\psi}{\partial r} = 0$).

Review: If $\underline{u}$ is irrotational, then curl $\underline{u} = 0 \Rightarrow \underline{u} = \nabla\phi$. then $\nabla\phi = \frac{\partial\phi}{\partial x}\hat{i} + \frac{\partial\phi}{\partial y}\hat{j} = \frac{\partial\phi}{\partial r}\hat{r} + \frac{1}{r}\frac{\partial\phi}{\partial\theta}\hat{\theta}$.

Then if div $\underline{u} = 0$, div$(\nabla\phi) = 0 \Rightarrow \nabla^2\phi = 0$, $\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} = 0$, $\frac{\partial^2\phi}{\partial r^2} + \frac{1}{r}\frac{\partial\phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2} = 0$. (Laplace's equation).

General solutions of these equations come from terms in Laurent series: $\{r^{\pm n}\cos n\theta,\ r^{\pm n}\sin n\theta,\ \log r,\ \theta\}$.

19 November 2012.
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Some things to note when solving flow field problems:

1. the far-field must satisfy equations of motion: i.e. linear combination $\{1, \log r, \theta, r^{\pm n}\cos n\theta, r^{\pm n}\sin n\theta\}$.
2. $\underline{u} = -\underline{k}\wedge\nabla\psi$, $\underline{u} = \nabla\phi$ or $u - iv = \frac{dw}{dz}$, $u_\theta - iu_r = e^{i\theta}\frac{dw}{dz}$

3. Maximum speed is max $|\underline{u}|$, which can occur when e.g. $u<0$, $v<0$, $u_\theta<0$.

4. $y = a\sin\theta$: 2 roots of $\sin\theta$ but $|\sin\theta|>1$, 1 root $\Rightarrow$ 2 stagnation points at $\pm a\cos\theta$.

5. Draw far field – distort to add cylinder

26 November 2012
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<u>Method of (infinite) images.</u>

Setup a problem with a source placed equidistant between 2 parallel walls; we have

$$u - iv = \sum_{m=-\infty}^{+\infty} \frac{m}{z - in\pi}, \quad \text{but} \quad u - iv = \frac{dF}{dz} = m\coth z = m\frac{\cosh z}{\sinh z} \Rightarrow F = m\log\sinh z.$$

Here, infinite images are produced.