2401 Mathematical Methods 3 Notes

Based on the 2012 autumn lectures by Dr J Evans

MATH2401 — Mathematical Methods 3.

3 October 2012
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Dr. Jonny EVANS. Homework distributed and collected on Fridays. Deadline 5pm.

General Logistics — 10% coursework, 90% examinations. Homework problems up on course page.

Notes will be provided — comprehensive electronic versions.

Resources up at http://www.jde27.co.uk >> Methods
> Schedule.
> Notes
> Videos
> Problems — answers on Moodle.

Feedback can be provided via j.d.evans@ucl.ac.uk

Methods 3 — study of 18th century mathematics: Lagrange, Laplace, Fourier, etc. Building upon Newton-Leibniz theories.

Outline of syllabus: partial differentiation and beyond. In two parts.

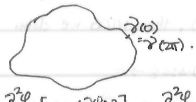
PART 1. Some problems in physics and geometry → differential equations (ODEs and PDEs)

PART 2. Techniques to solve PDEs.

TASTER TO THE COURSE.

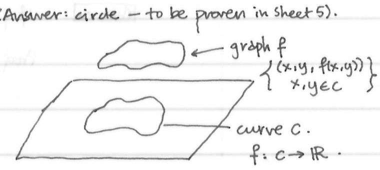
Problem: Prove that a straight line is the shortest path between P and Q in $\mathbb{R}^2$. (see video)

Problem: (Isoperimetric problem). Find a shape in $\mathbb{R}^2$ which maximises area amongst all shapes with the same perimeter. (Answer: circle — to be proven in sheet 5).



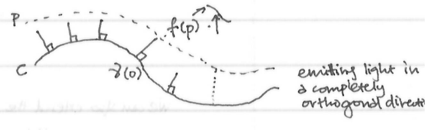
$\gamma : \mathbb{R} \to \mathbb{R}^2$. $\gamma(t) = (\gamma_1(t), \gamma_2(t))$

$\Rightarrow \ddot{\gamma}_1 = \lambda \dot{\gamma}_1, \quad \ddot{\gamma}_2 = \lambda \dot{\gamma}_2$



Solve. $\frac{\partial^2 \varphi}{\partial x^2}\left[1+\left(\frac{\partial \varphi}{\partial y}\right)^2\right] + \frac{\partial^2 \varphi}{\partial y^2}\left[1+\left(\frac{\partial \varphi}{\partial x}\right)^2\right] = 2\frac{\partial \varphi}{\partial x}\cdot\frac{\partial \varphi}{\partial y}\cdot\frac{\partial^2 \varphi}{\partial x \partial y}$.

Take some function $\varphi : B \to \mathbb{R}$, $\varphi|_C = f$



Solving equations such as $\frac{\partial f}{\partial x} + 2\frac{\partial f}{\partial y} = 7$. to obtain $f(x,y)$ using the method of characteristics.

EIKONAL EQUATION (from geometric optics): $\left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2 = 1$.

Introduce parametrisation of path — $\gamma(s) : [0,t] \to \mathbb{R}^2$. $\frac{dt}{ds} = 1$. $f(t) = \int_0^t ds = t$. But what if a point can be reached from multiple origins along a curve? (e.g. conca

Problem — characteristic lines can cross, creating singularities in the solution set.

Consider an ellipse — numerous singularities are created. $\frac{\partial \varphi}{\partial t} = k\frac{\partial^2 \varphi}{\partial x^2}$, $\varphi : [0,1] \times \mathbb{R} \to \mathbb{R}$. (Maxwell's equations). $\frac{1}{c^2}\frac{\partial^2 \varphi}{\partial t^2} = \frac{\partial^2 \varphi}{\partial x^2}$ predicts existence of EM waves.

We will then move on to consider second-order differential equations. only three specific ones — heat eqn, wave eqn, Laplace's eqn. — $\frac{\partial^2 \varphi}{\partial x^2} + \frac{\partial^2 \varphi}{\partial y^2} = 0$.

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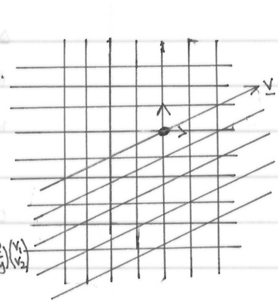
CALCULUS IN SEVERAL VARIABLES.

Consider functions $f : \mathbb{R}^2 \to \mathbb{R}$.



Definition: The partial derivatives of f are $\frac{d}{dt}\Big|_{t=0} f(x_0+t, y_0) = \frac{\partial f}{\partial x}(x_0,y_0) = \lim_{h\to 0}\frac{f(x_0+h,y_0)-f(x_0,y_0)}{h}$. Likewise, $\frac{\partial f}{\partial y} = \frac{d}{dt}\Big|_{t=0} f(x_0, y_0+t)$.

Note: Other conventional notation for $\frac{\partial f}{\partial x}$ include $\partial_x f$, f_x.

There is no reason why we should be compelled to pick the Cartesian axes as our basis.

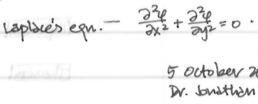
We can pick any vector in $\mathbb{R}^2$, say $\underline{v} = (v_1, v_2)$. Then we consider $f(x_0+v_1, y_0+v_2)$.

Definition: The directional derivative of f in direction $\underline{v}$ at $p = (x_0, y_0)$ is $v_p(f) = \frac{d}{dt}\Big|_{t=0} f(x_0+tv_1, y_0+tv_2)$. Also denoted $d_pf(v)$, or $\left(\frac{\partial f}{\partial x}\ \frac{\partial f}{\partial y}\right)\begin{pmatrix}v_1\\ v_2\end{pmatrix}$

EX: Let $f(x,y) = x^2+y^2$. At the point $p = (1,2)$, find the directional derivative in the direction $\underline{v} = (3,5)$.

Soln. $f(1+3t, 2+5t) = (1+3t)^2 + (2+5t)^2 = 5 + 26t + 34t^2$. $v_p(f) = \frac{d}{dt}\Big|_{t=0} f = 26 + 68(0) = 26$.

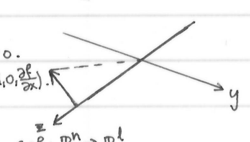
Theorem: Suppose $f : \mathbb{R}^2 \to \mathbb{R}$ and that $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ exist everywhere and are continuous. Then $\forall \underline{v} = (v_1, v_2)$,
$v_p(f) = v_1\frac{\partial f}{\partial x}(p) + v_2\frac{\partial f}{\partial y}(p)$



In the calculus of one variable, we know that $f(x_0+\varepsilon) = f(x_0) + \varepsilon f'(x_0) + \underbrace{\varepsilon\eta(\varepsilon)}_{\text{correction term}}$. $\eta(\varepsilon) \to 0$ as $\varepsilon \to 0$.

The tangent line is close to the graph f.

Analogously for two variables, $f(x_0+v_1, y_0+v_2) = f(x_0,y_0) + v_1\frac{\partial f}{\partial x}(x_0,y_0) + v_2\frac{\partial f}{\partial y}(x_0,y_0) + |v|\,\eta(|v|)$, where $\eta \to 0$ as $|v| \to 0$.



Theorem (Chain Rule)

Suppose $f : \mathbb{R}^n \to \mathbb{R}^m$, $g : \mathbb{R}^m \to \mathbb{R}^l$. We can think of them as $\begin{pmatrix} f_1(x_1,\dots,x_n) \\ \vdots \\ f_m(x_1,\dots,x_n)\end{pmatrix}$ and $\begin{pmatrix} g_1(y_1,\dots,y_m) \\ \vdots \\ g_l(y_1,\dots,y_m)\end{pmatrix}$. Compose to get $g \circ f : \mathbb{R}^n \to \mathbb{R}^l$.

Let $df = \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & & \vdots \\ \frac{\partial f_m}{\partial x_1} & \cdots & \frac{\partial f_m}{\partial x_n}\end{pmatrix}$, $dg = \begin{pmatrix} \frac{\partial g_1}{\partial y_1} & \cdots & \frac{\partial g_1}{\partial y_m} \\ \vdots & & \vdots \\ \frac{\partial g_l}{\partial y_1} & \cdots & \frac{\partial g_l}{\partial y_m}\end{pmatrix}$. Then $dg \cdot df = d(g\circ f)$.

[Ex] Let $f: (x,y) \longmapsto (u(x,y), v(x,y))$, $\mathbb{R}^2 \to \mathbb{R}^2$. $g: \mathbb{R}^2 \to \mathbb{R}$. Then $g \circ f: \mathbb{R}^2 \to \mathbb{R}$.

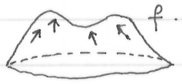
We note that $dg = \begin{pmatrix} \frac{\partial g}{\partial u} & \frac{\partial g}{\partial v} \end{pmatrix}$, $df = \begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{pmatrix}$. Then $d(g \circ f) = dg \cdot df$. *Note: we cannot combine $\frac{\partial g}{\partial u}\frac{\partial u}{\partial x}$ to get $\frac{\partial g}{\partial x}$!

Note: df is known as the Jacobian matrix.

Another perspective: we consider the gradient of $f: \mathbb{R}^2 \to \mathbb{R}$. Then $\nabla f = \begin{pmatrix} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \end{pmatrix} = (df)^T$.

[Lemma] ∇f points in the direction of greatest increase of f and has $|\nabla f|$ equal to the directional derivative in that direction (provided gradient is non-zero).

Proof – $\underline{v} = \frac{\nabla f}{|\nabla f|}$. f changes if we move in direction $\underline{w}$: $\underline{w}_p(f) = d_p f(w) = \begin{pmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \end{pmatrix}\begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = (\nabla f) \cdot w = |\nabla f||\underline{w}| \cos\theta$.



this value is maximised when $\cos\theta = 1 \Rightarrow \theta = 0 \Rightarrow$ greatest increase of f is when it is in ∇f direction. // q.e.d.

$\underline{v}(f) = \nabla f \cdot \frac{\nabla f}{|\nabla f|} = \frac{|\nabla f|^2}{|\nabla f|} = |\nabla f|$ // q.e.d.

Critical Points.

[Definition] A critical point of $f: \mathbb{R}^2 \to \mathbb{R}$ is a point $p \in \mathbb{R}^2$ where $\nabla f(p) = 0 \Leftrightarrow d_p f = 0 \Leftrightarrow v_p(f) = 0 \quad \forall v$.

[Definition] A local maximum of $f: \mathbb{R}^2 \to \mathbb{R}$ is a point $p \in \mathbb{R}^2$ s.t. $\exists$ neighbourhood U of p s.t. $f(x) \leq f(p) \quad \forall x \in U$. (replace inequality for minimum)

[Theorem] If p is a local maximum/minimum of $f: \mathbb{R}^2 \to \mathbb{R}$, then p is critical i.e. $\frac{\partial f}{\partial x} = \frac{\partial f}{\partial y} = 0$ ($\because$ all directional derivatives are linear combinations of these).

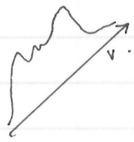
Proof – suppose that f has a local maximum at p, then assume that $\nabla f(p) \neq 0$. i.e. if $p = (x_0, y_0)$, then WLOG we claim $\frac{\partial f}{\partial x} \neq 0$ (or $\frac{\partial f}{\partial y}$).

we know that taking Taylor series, $f(x_0 + \varepsilon, y_0) = f(x_0, y_0) + \varepsilon \frac{\partial f}{\partial x} + \varepsilon \eta(\varepsilon)$, where $\eta(\varepsilon) \to 0$ as $\varepsilon \to 0$

so for all small ε, $\eta(\varepsilon) < \frac{1}{2}\frac{\partial f}{\partial x}$. then $f(x_0 + \varepsilon, y_0) = f(x_0, y_0) + \varepsilon\left(\frac{\partial f}{\partial x} + \eta\right) \geq f(x_0, y_0) + \varepsilon \cdot \frac{1}{2}\frac{\partial f}{\partial x} > f(x_0, y_0)$.

this means $f(x_0, y_0)$ is not a maximum at $P \Rightarrow$ contradiction. Hence $\nabla f(p) = 0$ // q.e.d.

We can also extend the theory of directional derivatives to second derivatives. Let $\underline{v}$ be a vector $\underline{v} = (v_1, v_2)$.



Then $v^2(f) = \frac{d^2}{ds^2}\Big|_{s=0} f(x_0 + sv_1, y_0 + sv_2)$.

Recall that $v(f) = v_1 \frac{\partial f}{\partial x} + v_2 \frac{\partial f}{\partial y}$. then $v^2(f) = \left(v_1 \frac{\partial}{\partial x} + v_2 \frac{\partial}{\partial y}\right)^2 f = v_1^2 \frac{\partial^2 f}{\partial x^2} + v_2^2 \frac{\partial^2 f}{\partial y^2} + v_1 v_2 \frac{\partial^2 f}{\partial x \partial y} + v_2 v_1 \frac{\partial^2 f}{\partial y \partial x} = \begin{pmatrix} v_1 & v_2 \end{pmatrix} \begin{pmatrix} \frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial x \partial y} \\ \frac{\partial^2 f}{\partial y \partial x} & \frac{\partial^2 f}{\partial y^2} \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$.

The matrix $\begin{pmatrix} \partial^2 f/\partial x^2 & \partial^2 f/\partial x \partial y \\ \partial^2 f/\partial y \partial x & \partial^2 f/\partial y^2 \end{pmatrix}$ is called the Hessian of f, denoted $\text{Hess}_p(f)$.

[Definition] If all derivatives of f up to n^{th} order exist and are continuous, then we say f is $\mathcal{C}^n$.

[Theorem] Assume $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}$ exist and are continuous, and all second derivatives are continuous. Then $\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}$.

The Hessian is symmetric if f is $\mathcal{C}^2$.

Second Derivative Test: Suppose $f: \mathbb{R}^2 \to \mathbb{R}$ ($\mathcal{C}^2$), and suppose $p \in \mathbb{R}^2$ is a critical point of f.

Then if both eigenvalues of $\text{Hess}_p(f)$ are both negative, then p is a local maximum. Likewise if both positive, then p is a local minimum.

Note: We know these eigenvalues exist and are real because $\text{Hess}_p(f)$ is symmetric, and symmetric matrices are diagonalisable by orthogonal matrices.

[U s.t. $U^T = U^{-1}$ denote orthogonal matrices]. Then if $A^T = A$, $\exists$ U orthogonal st. $U^{-1}AU = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$, $\lambda_1 \lambda_2 \in \mathbb{R}$.

One trick we can use to test this without calculating eigenvalues by using determinants. $\det(U^{-1}AU) = \det\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \Rightarrow \det U^{-1} \det A \det U = \lambda_1 \lambda_2 \Rightarrow \det A = \lambda_1 \lambda_2$

Then the product of eigenvalues is the determinant of $\text{Hess}_p(f)$. Hence, if $\det \text{Hess}_p(f) > 0$, we have an extremum. If $\det \text{Hess}_p(f) < 0$, we have a saddle point.

If $\text{Tr}\,\text{Hess}_p(f) > 0$, we have a minimum $(\lambda_1, \lambda_2 > 0)$; and if $\text{Tr}\,\text{Hess}_p(f) < 0$, we have a maximum $(\lambda_1, \lambda_2 < 0)$.

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Recap: we have seen from above that $v_p(f) = v_1 \frac{\partial f}{\partial x}(p) + v_2 \frac{\partial f}{\partial y}(p)$ at point p. Then $v_p(f) = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}\begin{pmatrix} \partial f/\partial x \\ \partial f/\partial y \end{pmatrix} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \nabla f = \underline{v} \cdot \nabla f$.

Then, we introduce the total derivative $d_p f = \begin{pmatrix} \frac{\partial f}{\partial x}(p) & \frac{\partial f}{\partial y}(p) \end{pmatrix}$.

The motivation of this comes from the chain rule, which tells us that the total derivative of a composition of functions is a matrix product of total derivatives.

→ Now, we return to the second derivative test and attempt to prove it. We first need another theorem.

[Theorem] (Taylor's theorem for two variables).

If f is $\mathcal{C}^2$, then for all vectors $\underline{v}$ with small magnitude, $f(p + \underline{v}) = f(p) + d_p f(\underline{v}) + \frac{1}{2}\underline{v}^T \text{Hess}_p(f)\underline{v} + \text{error}(\underline{v})$,

where the error term goes to 0 faster than $|v|^2$, i.e. $\frac{|\text{error}(v)|}{|v|^2} \to 0$ as $\underline{v} \to 0$.

Note: The linear term could be written $v_p(f)$ or $\nabla f(p)$.

Proof – omitted; should be covered in analysis.

Theorem (Second Derivative Test) — see pg 2.

Proof — By Taylor's theorem, $f(p+\underline{v}) = f(p) + \frac{1}{2}\underline{v}^T \text{Hess}_p(f)\underline{v} + \text{error}(\underline{v})$ $\because$ $d_pf = 0$ as p is critical and all directional derivatives vanish.

Since $\text{Hess}_p(f)$ is symmetric, we can diagonalise it by an orthogonal matrix U s.t. $U^{-1}\text{Hess}_p(f)U = D = \begin{pmatrix}\lambda_1 & 0\\ 0 & \lambda_2\end{pmatrix}$.

Then since λ_1, λ_2 are the eigenvalues, the corresponding eigenvectors are $\underline{v}_1 = U\begin{pmatrix}1\\0\end{pmatrix}$, $\underline{v}_2 = U\begin{pmatrix}0\\1\end{pmatrix}$.

From this, $f(p + \overbrace{a\underline{v}_1 + b\underline{v}_2}^{\text{lin comb.}}) = f(p) + \frac{1}{2}(a\underline{v}_1 + b\underline{v}_2)^T \text{Hess}_p(f)(a\underline{v}_1 + b\underline{v}_2) + \text{error}(a\underline{v}_1 + b\underline{v}_2)$.

We concentrate on the second term: $\text{Hess}_p(f) = U\begin{pmatrix}\lambda_1 & 0\\ 0 & \lambda_2\end{pmatrix}U^{-1}$. Then $\underline{v}_1 = U\begin{pmatrix}1\\0\end{pmatrix}$, $U^{-1}\underline{v} = \begin{pmatrix}1\\0\end{pmatrix} \Rightarrow [U^{-1}\underline{v}]^T = (1\ 0) = \underline{v}_1^T U$ ($\because U^{-1} = U^T$, orthogonal)

Likewise, $\underline{v}_2 = U\begin{pmatrix}0\\1\end{pmatrix} \Rightarrow \underline{v}_2^T U = (0\ 1)$. Then $\frac{1}{2}(a\underline{v}_1 + b\underline{v}_2)^T\text{Hess}_p(f)(a\underline{v}_1 + b\underline{v}_2) = \frac{1}{2}(a\underline{v}_1^T + b\underline{v}_2^T)U\begin{pmatrix}\lambda_1 & 0\\ 0 & \lambda_2\end{pmatrix}U^{-1}(a\underline{v}_1 + b\underline{v}_2)$.

$= \frac{1}{2}(a\ b)\begin{pmatrix}\lambda_1 & 0\\ 0 & \lambda_2\end{pmatrix}\begin{pmatrix}a\\b\end{pmatrix} = \frac{1}{2}a^2\lambda_1 + \frac{1}{2}b^2\lambda_2$.

$\therefore f(p + a\underline{v}_1 + b\underline{v}_2) = f(p) + \frac{1}{2}a^2\lambda_1 + \frac{1}{2}b^2\lambda_2 + \text{error}(a\underline{v}_1 + b\underline{v}_2)$. Now if λ_1 and λ_2 are both positive (or negative), this is the second term. When $a\underline{v}_1 + b\underline{v}_2$ is very small (i.e. $a^2+b^2 \to 0$), this term dominates the error, and we see that $f(p + a\underline{v}_1 + b\underline{v}_2) > f(p)$ [respectively $< f(p)$]

$\Rightarrow p$ is a local minimum (or maximum).

Ex Find the nature of the critical point of $f(x,y) = x^2 - y^2$.

Soln. Critical point is $(x,y) = (0,0)$. Here, $\text{Hess}_p(f) = \begin{pmatrix}2 & 0\\ 0 & -2\end{pmatrix}$, so eigenvalues are of opposite sign $\Rightarrow$ saddle point.

What happens if one of the eigenvalues is 0? We do not have enough information to understand the function, which vanishes to 1st and 2nd order.

Definition A critical point is <u>nondegenerate</u> if all eigenvalues of the Hessian are non-zero. (also called a <u>Morse critical point</u>)

There are three kinds of non-degenerate critical points for a 2-D function:

- $\lambda_1, \lambda_2 > 0$ e.g. $f(x,y) = x^2 + y^2$
- $\lambda_1, \lambda_2 < 0$ e.g. $f(x,y) = -x^2 - y^2$
- WLOG $\lambda_1 > 0$, $\lambda_2 < 0$ e.g. $f(x,y) = x^2 - y^2$

For an n-dimensional function $f(x_1, \dots, x_n) = x_1^2 + \dots + x_p^2 - x_{p+1}^2 - \dots - x_n^2$, there are p posive and $n-p$ negative eigenvalues.

Lemma If $\text{Tr}(A) = a+d$ is positive, then $\lambda_1 + \lambda_2$ is positive too. (More strictly, $a + d = \lambda_1 + \lambda_2$).

Proof — We use the fact that $\text{Tr}(ABC) = \text{Tr}(BCA) = \text{Tr}(CAB)^*$, to be proved later. (cyclicity of trace).

Then $\text{Tr}(A) = \text{Tr}[U\begin{pmatrix}\lambda_1 & 0\\ 0 & \lambda_2\end{pmatrix}U^{-1}] = \text{Tr}[UU^{-1}\begin{pmatrix}\lambda_1 & 0\\ 0 & \lambda_2\end{pmatrix}] = \text{Tr}\begin{pmatrix}\lambda_1 & 0\\ 0 & \lambda_2\end{pmatrix} = \lambda_1 + \lambda_2$ q.e.d.

*Proof — (cyclicity of Trace).

We write $A = \begin{pmatrix}a_{11} & \cdots & a_{1n}\\ \vdots & \ddots & \vdots\\ a_{n1} & \cdots & a_{nn}\end{pmatrix} \Rightarrow \text{Tr}(A) = \sum_{i=1}^n a_{ii}$. Now if $D = AB$, then $D_{ik} = \sum_{j=1}^n a_{ij}b_{jk}$.

Similarly if $E = ABC$, then $E_{il} = \sum_{j=1}^n\sum_{k=1}^n a_{ij}b_{jk}c_{kl}$. Finally, $\text{Tr}(ABC) = \text{Tr}(E) = \sum_{i=1}^n e_{ii} = \sum_{i=1}^n\sum_{j=1}^n\sum_{k=1}^n a_{ij}b_{jk}c_{ki}$

But then^, $a_{ij}b_{jk}c_{ki} = b_{jk}c_{ki}a_{ij} = c_{ki}a_{ij}b_{jk}$ because they are numbers and commute [^ use Einstein summation convention to eliminate $\sum_{i=1}\sum_{j=1}\sum_{k=1}$].

We only took cyclic permutations, so relative positions of the indices are the same, we relabel $i \to j \to k \to i$, and this yields $\text{Tr}(ABC) = \text{Tr}(BCA) = \text{Tr}(CAB)$ q.e.d.

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Ex Let $F(x,y) = x^2 + \sin(x+y)$. Find critical points and describe their nature.

Soln. At a critical point, $\frac{\partial F}{\partial x} = 2x + \cos(x+y) = 0$, $\frac{\partial F}{\partial y} = \cos(x+y) = 0 \Rightarrow x = 0$, $y = (n+\frac{1}{2})\pi$, $\forall n \in \mathbb{Z}$.

Find $\text{Hess}_p(F) = \begin{pmatrix}\partial^2F/\partial x^2 & \partial^2F/\partial x\partial y\\ \partial^2F/\partial y\partial x & \partial^2F/\partial y^2\end{pmatrix} = \begin{pmatrix}2-\sin(x+y) & -\sin(x+y)\\ -\sin(x+y) & -\sin(x+y)\end{pmatrix} = \begin{pmatrix}2-(-1)^n & (-1)^{n+1}\\ (-1)^{n+1} & (-1)^{n+1}\end{pmatrix}$.

$\det \text{Hess}_p(F) = (-1)^{n+1}\cdot 2$. $\det\text{Hess}_p(F) < 0$ if n is even, > 0 if n is odd.

For n is even, we have a saddle point. For n is odd, $\text{Tr Hess}_p(F) = 2 - (-1)^n + (-1)^{n+1} = 2 - 2(-1)^n = 4 > 0 \Rightarrow$ we have minima.

With more variables, in higher dimensions, ∇F has more components: i.e. $\nabla F = \begin{pmatrix}\partial F/\partial x_1\\ \vdots \\ \partial F/\partial x_n\end{pmatrix}$. Then $\text{Hess}_p(F) = \begin{pmatrix}\partial^2F/\partial x_1^2 & \partial^2F/\partial x_1\partial x_2 & \cdots & \partial^2F/\partial x_1\partial x_n\\ & \partial^2F/\partial x_2^2 & \cdots & \\ & \vdots & & \vdots\end{pmatrix}$.

Ex Help Armin the Axolotl. He finds himself at the origin, surrounded by a cloud of toxic waste with density $10 - \frac{3}{2}(x^2+z^2) - 2xy - 4xz - 2yz$. Which way should he swim?

Soln. Let the density of the toxic waste be $f(x,y,z) = 10 - \frac{3}{2}(x^2+z^2) - 2xy - 4xz - 2yz$. Then $\frac{\partial f}{\partial x} = -3x - 2y - 4z$, $\frac{\partial f}{\partial y} = -2x - 2z$, $\frac{\partial f}{\partial z} = -3z - 4x - 2y$.

Since Armin is at origin, $(x,y,z) = (0,0,0) \Rightarrow \frac{\partial f}{\partial x} = \frac{\partial f}{\partial y} = \frac{\partial f}{\partial z} = 0 \Rightarrow$ critical point. So we consider the Hessian; where $x = x_1$, $y = x_2$, $z = x_3$.

$\text{Hess}_p(f) = \begin{pmatrix}\partial^2F/\partial x_1^2 & \partial^2F/\partial x_1\partial x_2 & \partial^2F/\partial x_1\partial x_3\\ \partial^2F/\partial x_2\partial x_1 & \partial^2F/\partial x_2^2 & \partial^2F/\partial x_2\partial x_3\\ \partial^2F/\partial x_3\partial x_1 & \partial^2F/\partial x_3\partial x_2 & \partial^2F/\partial x_3^2\end{pmatrix} = \begin{pmatrix}-3 & -2 & -4\\ -2 & 0 & -2\\ -4 & -2 & -3\end{pmatrix}$. We then find eigenvalues: $A\underline{v} = t\underline{v} \Rightarrow \det(tI - A)\underline{v} = \underline{0} \Rightarrow |tI - A| = 0 \therefore \left|\begin{pmatrix}t+3 & 2 & 4\\ 2 & t & 2\\ 4 & 2 & t+3\end{pmatrix}\right| = 0$.

$\det\begin{pmatrix}t+3 & 2 & 4\\ 2 & t & 2\\ 4 & 2 & t+3\end{pmatrix} = \det\begin{pmatrix}t+3 & 2 & 4\\ 2 & t & 2\\ 0 & 2-2t & t-1\end{pmatrix} = (t-1)\det\begin{pmatrix}t+3 & 2 & 4\\ 2 & t & 2\\ 0 & -2 & 1\end{pmatrix} = (t-1)\left[(t+3)\begin{vmatrix}t & 2\\ -2 & 1\end{vmatrix} - 2\begin{vmatrix}2 & 4\\ -2 & 1\end{vmatrix}\right] = (t-1)[(t+3)(t+4) - 2(10)] = (t-1)^2(t+8)$. $\lambda_{1,2} = 1$ and $\lambda_3 = -8$.

Recall by Taylor's theorem, $f(p+\underline{v}) = f(p) + \frac{1}{2}\underline{v}^T\text{Hess}_p(f)\underline{v} + \eta(\underline{v})$. Let $\underline{v} = a\underline{v}_1 + b\underline{v}_2 + c\underline{v}_3$, then from above, $f(p+\underline{v}) = f(p) + \frac{1}{2}a^2 + \frac{1}{2}b^2 + \frac{1}{2}(-8)c^2$. Objective is to minimise $f(p+$

Take $a=0, b=0$, maximise c^2. Then find $\underline{v}_3$. $\lambda_3 I - A = \begin{pmatrix}-5 & 2 & 4\\ 2 & -8 & 2\\ 4 & 2 & -5\end{pmatrix} \leftrightarrow \begin{pmatrix}-1 & 4 & -1\\ 1 & -4 & 1\\ 0 & 18 & -9\end{pmatrix} \leftrightarrow \begin{pmatrix}1 & -4 & 1\\ 0 & 2 & -1\\ 0 & 0 & 0\end{pmatrix} \leftrightarrow \begin{pmatrix}1 & 0 & -1\\ 0 & 2 & -1\\ 0 & 0 & 0\end{pmatrix}$. Let $z = \alpha$, $\begin{pmatrix}x\\y\\z\end{pmatrix} = \begin{pmatrix}1\\1/2\\1\end{pmatrix}\alpha$, $\underline{v}_3 = \begin{pmatrix}2\\1\\2\end{pmatrix}$.

2401-003.

CONSTRAINED OPTIMISATION.

Imagine we want to maximise the equation $F(x,y)$, subject to the constraint that $G(x,y)=0$.

For instance, consider $x^2-y^2=F(x,y)$ s.t. $x=0$ [i.e. $G(x,y)=0$]. Then $F|_{G=0}(y) = -y^2 \Rightarrow$ max at $y=0$.

Ex. Let $F(x,y)=x$, $G(x,y)=x^2+y^2-1$, $\{G=0\}$.

Soln. F has no critical points $\because \partial F/\partial x = 1$, but $F|_{(G=0)}$ has two critical points. We then use the Lagrangian Multiplier... (see later cont'd *).

<u>Algorithm to solve such problems.</u>

→ Introduce a new parameter $\lambda \in \mathbb{R}$ (<u>Lagrange multiplier</u>)

→ consider $H(x,y,\lambda) = F(x,y) - \lambda G(x,y)$

→ Find critical points of H: $\frac{\partial H}{\partial x} = \frac{\partial H}{\partial y} = \frac{\partial H}{\partial \lambda} = 0$. Remark: $\frac{\partial H}{\partial \lambda} = -G(x,y)$. At these points, $(x,y,\lambda) = (x_0, y_0, \lambda_0)$.

→ Ignore λ_0 to get (x_0, y_0); then

(x_0, y_0) is a critical point of $F|_{(G=0)}$.

Ex. Let $F(x,y)=x^2-y^2$, $G(x,y)=x$. Find critical points where $G(x,y)=x=0$.

Soln. Consider $H(x,y,\lambda) = (x^2-y^2) - \lambda x$. $\{\frac{\partial H}{\partial x} = 2x - \lambda,\ \frac{\partial H}{\partial y} = -2y,\ \frac{\partial H}{\partial \lambda} = -x\}$. ⇒ since all $=0$, $x=0$, $y=0$, $\lambda=0$.

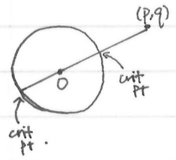
critical point of H is $(0,0,0) = (x,y,\lambda)$; so critical point of $F|_{G=0} = (0,0)$ //.

Ex* (cont'd). $F(x,y)=x$, $G(x,y)=x^2+y^2-1$.

Soln. consider $H(x,y,\lambda) = x - \lambda(x^2+y^2-1) = x + 1 - \lambda x^2 - \lambda y^2 \Rightarrow \{\frac{\partial H}{\partial x} = 1 - 2\lambda x,\ \frac{\partial H}{\partial y} = -2\lambda y,\ \frac{\partial H}{\partial \lambda} = -x^2-y^2+1\}$. since all $=0$,

$\Rightarrow \lambda = 0,\ y=0,\ x=\pm 1$. ⇒ critical points of $F|_{G=0} = (\pm 1, 0)$ //.

Ex. Find that the point $\frac{(p,q)}{\sqrt{p^2+q^2}}$ is the closest point on the unit circle centred at the origin, to the point (p,q) outside the circle.

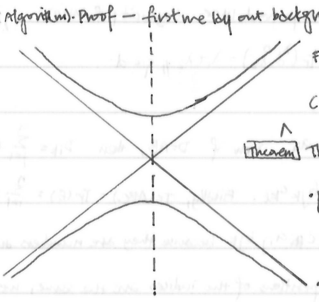


Soln. $F(x,y) = (x-p)^2 + (y-q)^2$, $G(x,y) = x^2+y^2-1$. Then $H(x,y,\lambda) = (x-p)^2+(y-q)^2 - \lambda x^2 - \lambda y^2 + \lambda$.

$\frac{\partial H}{\partial x} = 2(x-p) - 2\lambda x = 0$ ①, $\frac{\partial H}{\partial y} = 2(y-q) - 2\lambda y = 0$ ②, $\frac{\partial H}{\partial \lambda} = -x^2-y^2+1 = 0$ ③ ⇒ rearranging, ①: $x = \frac{p}{1-\lambda}$, $y = \frac{q}{1-\lambda}$.

substitute into ③: $\frac{p^2}{(1-\lambda)^2} + \frac{q^2}{(1-\lambda)^2} = 1 \Rightarrow p^2+q^2 = (1-\lambda)^2 \Rightarrow 1-\lambda = \pm\sqrt{p^2+q^2}$. ⇒ critical points are $\pm\frac{(p,q)}{\sqrt{p^2+q^2}}$ //

(Lagrange Multiplier Algorithm). Proof — first we lay out background, proof follows on pg 5.

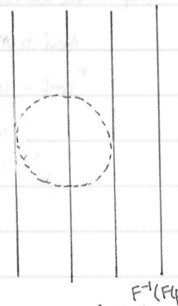


$F = x^2 - y^2$.

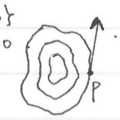
$C = \{(x,y) : G(x,y) = 0\}$.

Theorem^ Then the critical points of $F|_C$ come in two flavours.

- $p \in \mathbb{R}^2$ s.t. $d_pF = 0$ and $p \in C$. (p is a crit pt of F)
- $p \in \mathbb{R}^2$ s.t. $p \in C$ and the level set (contour) of F through p is tangent to C.

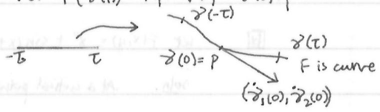


Lemma. Suppose that $F: \mathbb{R}^2 \to \mathbb{R}$, $p \in \mathbb{R}^2$ is such that $d_pF \neq 0$. Then the tangent vectors to the <u>contour of F through p</u> are $\{\underline{v} : \underline{v} \cdot \underline{\nabla F} = 0\}$ ($d_pF(\underline{v}) = 0$)



Proof — we parametrise our contour, $\gamma: (-\tau, \tau) \to \mathbb{R}^2$ s.t. $\gamma(t) \in F^{-1}(F(p))\ \forall t$. i.e. $F(\gamma(t)) = F(p)\ \forall t$, $\gamma(0) = p$.

Tangent vector at p to γ is $\frac{d\gamma}{dt}(0) = (\dot\gamma_1(0), \dot\gamma_2(0))$



Then $F \circ \gamma: (-\tau, \tau) \to \mathbb{R}$ is constant, i.e. $F \circ \gamma = F(p)$. Then $\frac{d}{dt}[F \circ \gamma(t)] = 0$

By chain rule, $\frac{\partial F}{\partial x}\frac{d\gamma_1}{dt} + \frac{\partial F}{\partial y}\frac{d\gamma_2}{dt} = \underline{\nabla F} \cdot \underline{\dot\gamma} = 0 \Rightarrow \nabla F$ and $\underline{\dot\gamma}$ are orthogonal

⇒ tangent vectors to contours of F are orthogonal to gradient of F // q.e.d.

Back to theorem^. Proof — C is a level set of C. ⇒ the tangents to C are the vectors $\perp$ to ∇G.

similarly, contours have tangents $\underline{v}$ s.t. $\underline{v} \perp \underline{\nabla F}$. Now take a parametrisation of C, $\delta: (-S, S) \to C$.

Let $\delta(0) = p$, $\dot\delta(0)$ is tangent to C at p. Then consider $\frac{d}{dt}\big|_{t=0}(F \circ \delta)$, in order to find the critical points of $F|_C$.

But $F \circ \delta$ is $F|_C : C \to \mathbb{R}$. We need to find points where $0 = \frac{d}{dt}\big|_{t=0}(F \circ \delta) = \underline{\nabla F} \cdot \underline{\dot\delta}$.

case 1: $\nabla F(p) = 0 \Rightarrow \nabla F \cdot \dot\delta = 0 \Rightarrow p$ is a critical point of $F \circ \delta = F|_C$.

case 2: $\nabla F(p) \neq 0$. Then lemma applies, and $\dot\delta$ is tangent to contours of $F \Rightarrow \underline{\nabla F} \cdot \underline{\dot\delta} = 0 \Rightarrow p$ is critical too // q.e.d.

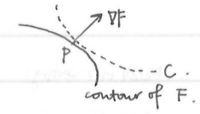
Remark: In our lemma, which identified tangents to contours of F, we assigned $\nabla F \neq 0$, because we picked $\gamma: (-\tau, \tau) \to$ contour.

The existence of this unique parametrisation is based upon the <u>implicit function theorem</u>,

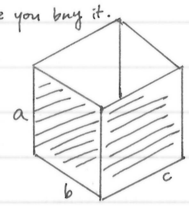
which assumes that $\nabla F \neq 0$.

Proof of algorithm – From our critical point equation, $H = F - \lambda G \Rightarrow \nabla H = \nabla F - \lambda \nabla G = 0 \iff \nabla F = \lambda \nabla G$.

$\frac{\partial H}{\partial \lambda} = -G = 0 \iff (x,y) \in C$, q.e.d.



Ex You have enough money to buy $4m^2$ of cardboard and make it into a lidless box. You can fold it into a desired dimension before you buy it. How should it be cut to maximise volume?



Soln. $F(a,b,c) = abc$, $G(a,b,c) = -4 + 2ab + 2ac + bc = 0$

$H(a,b,c,\lambda) = F - \lambda G = abc - \lambda(2ab + 2ac + bc - 4)$. $0 = \frac{\partial H}{\partial b} = ac - \lambda(2a+c)$, $0 = \frac{\partial H}{\partial c} = ab - \lambda(2a+b)$.

$\Rightarrow 2\lambda a = ac - c\lambda = (a-\lambda)c$. Likewise, $ab - b\lambda = (a-\lambda)b \Rightarrow b = c$.

Then $H(a,b,\lambda) = ab^2 - \lambda(4ab + b^2 - 4) \Rightarrow \frac{\partial H}{\partial a} = b^2 - 4\lambda b = 0$ and $\frac{\partial H}{\partial b} = 2ab - 4\lambda a - 2\lambda b = 0 \Rightarrow b = 4\lambda = c$, $a = \frac{4-b^2}{4b} = \frac{1-4\lambda^2}{4\lambda}$

Since $4ab + b^2 = 4$, $4\left(\frac{1-4\lambda^2}{4\lambda}\right)(4\lambda) + 16\lambda^2 = 4 \Rightarrow (1-4\lambda^2) + 4\lambda^2 = 1 \Rightarrow 12\lambda^2 = 1$.

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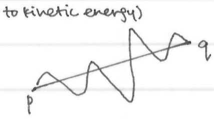
Theorem A straight line is the shortest path between two points in the plane.

Proof – consider the path $\gamma: [0,1] \to \mathbb{R}$. Then the position along any path at time t is $(\gamma_1(t), \gamma_2(t))$.

Let $L(\gamma) = \int_0^1 |\dot\gamma|\, dt = \int_0^1 \sqrt{\dot\gamma_1^2 + \dot\gamma_2^2}\, dt$. This is a complicated expression, so instead we aim to minimise action, $A(\gamma) = \int_0^1 (\dot\gamma_1^2 + \dot\gamma_2^2)\, dt$. (similar to kinetic energy)

Proposition A straight line minimises action over all paths from p to q.



Proof – Let γ be the straight line. Suppose δ is another path, $\varepsilon(t) = \delta(t) - \gamma(t)$.

$A(\delta) = \int_0^1 |\dot\delta|^2 dt = \int_0^1 |\dot\gamma + \dot\varepsilon|^2 dt = \int_0^1 (\dot\gamma + \dot\varepsilon)(\dot\gamma + \dot\varepsilon)\, dt = \int_0^1 |\dot\gamma|^2 + 2\dot\gamma\cdot\dot\varepsilon + |\dot\varepsilon|^2 dt$

$= A(\gamma) + 2\int_0^1 \dot\gamma\cdot\dot\varepsilon\, dt + (\text{other positive term } \int_0^1 |\dot\varepsilon|^2 dt) \geq A(\gamma) + 2\int_0^1 \dot\gamma\cdot\dot\varepsilon\, dt = A(\gamma) + 2\dot\gamma\cdot\int_0^1 \dot\varepsilon\, dt$ ($\because \gamma$ is a straight line, $\dot\gamma$ is const.)

$= A(\gamma) + 2\dot\gamma\int \frac{d\varepsilon}{dt} dt = A(\gamma) + 2\dot\gamma\cdot[\varepsilon]_0^1 = A(\gamma) + 2\dot\gamma\cdot[0-0] = A(\gamma)$ $\because$ δ and γ do not differ at $t=0$ and $t=1$.

$\therefore$ For any path δ, $A(\delta) \geq A(\gamma)$, with equality $\iff \int_0^1 |\dot\varepsilon|^2 dt = 0$ i.e. $\dot\varepsilon = 0$, $\varepsilon = 0$, $\delta = \gamma$.

Hence, γ is the shortest path, q.e.d.

We know that the straight line minimises action, but we need further to show that it minimises L: so we must relate $A(\delta)$ and $L(\delta)$.

$A(\delta) = \int_0^1 \dot\delta_1^2 + \dot\delta_2^2\, dt$, $L(\delta) = \int_0^1 \sqrt{\dot\delta_1^2 + \dot\delta_2^2}\, dt$. We replace δ by α, which is the path followed by a particle moving along δ with constant speed.

This moves at constant speed $L(\delta)$ per unit time. Then $L(\alpha) = L(\delta)\cdot 1 = L(\delta)$. And $A(\alpha) = \int |\dot\alpha|^2 dt = \int L(\delta)^2 dt = L(\delta)^2 = L(\alpha)^2$.

But we showed that the straight line γ minimises action. So $L(\gamma)^2 = A(\gamma) \leq A(\alpha) = L(\delta)^2$. γ has constant speed $\therefore A(\gamma) = L(\gamma)^2$

$\Rightarrow \gamma$ minimises L too, q.e.d.

FUNCTIONALS.

In our earlier proof, we had the following

- a space of functions X [paths]
- a functional (functional space of functions) $L: X \to \mathbb{R}$ [length]
- a supposed critical point γ of L (extremum) [straight line].

Then we work with an equation, such as $L(\gamma + \underset{\text{variation}}{\underline{\varepsilon}}) = L(\gamma) + dL(\varepsilon) + \text{higher order terms}$, $A(\gamma + \varepsilon) = A(\gamma) + 2\int \dot\gamma\cdot\dot\varepsilon + \text{higher order terms}$.

Ex Consider $X = \{\phi: [0,1] \to \mathbb{R}$ s.t. $\phi(0) = a, \phi(1) = b\}$, functional $L = \int_0^1 \sqrt{1 + \dot\phi^2}\, dt$. Find extremum of L.

Soln. $L(\phi + \varepsilon) = \int_0^1 \sqrt{1 + (\dot\phi + \dot\varepsilon)^2}\, dt = \int_0^1 \sqrt{(1 + \dot\phi^2) + 2\dot\phi\dot\varepsilon + \dot\varepsilon^2}\, dt = \int_0^1 \sqrt{1 + \dot\phi^2} + \frac{2\dot\phi\dot\varepsilon}{2\sqrt{1+\dot\phi^2}} + \text{higher order } dt$ (by Taylor series expansion $\sqrt{A + x} = \sqrt{A} + \frac{x}{2\sqrt{A}} + \dots$)

$= L(\phi) + \int_0^1 \frac{\dot\phi\dot\varepsilon}{\sqrt{1+\dot\phi^2}} dt + \text{higher order terms}$. We claim that $\int_0^1 \frac{\dot\phi\dot\varepsilon}{\sqrt{1+\dot\phi^2}}$ is merely the directional derivative in direction of ε. $\because$ ε is a direction in function space.

$\overset{\text{IBP}}{=} L(\phi) + \left[\frac{\dot\phi\varepsilon}{\sqrt{1+\dot\phi^2}}\right]_0^1 - \int_0^1 \varepsilon \frac{d}{dt}\left(\frac{\dot\phi}{\sqrt{1+\dot\phi^2}}\right) dt = L(\phi) + 0 - \int_0^1 \varepsilon \frac{d}{dt}\left(\frac{\dot\phi}{\sqrt{1+\dot\phi^2}}\right) dt$ $\because \varepsilon(0) = \varepsilon(1) = 0$.

The directional derivative of L in the ε direction is $-\int \varepsilon \frac{d}{dt}\left(\frac{\dot\phi}{\sqrt{1+\dot\phi^2}}\right) dt \dots$ (to be continued)

Theorem (Fundamental Theorem of Calculus of Variations)

If $y: [a,b] \to \mathbb{R}$ has property that $\int_a^b y\varepsilon\, dt = 0 \;\forall \varepsilon(t)$, then $y = 0$. Proof – see next page.

Corollary If ϕ is a critical point of L, then $\frac{d}{dt}\left(\frac{\dot\phi}{\sqrt{1+\dot\phi^2}}\right) = 0$.

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Recall our setting: $X=\{\phi:[a,b]\to\mathbb{R}$ s.t. $\phi(a)=A, \phi(b)=B\}$ is a function space, $F:X\to\mathbb{R}$ is a functional s.t. $F(\phi)=\int_a^b L(t,\phi(t),\dot\phi(t))\,dt$,

where $L(p,q,r)$ is a function of three variables. We aim to find critical points of F on X, which are called extrema,

i.e. we seek $\phi\in X$ such that $d_\phi F(\varepsilon)=0$ (all directional derivative vanish) $\forall \varepsilon\in\{g:[a,b]\to\mathbb{R}$ s.t. $g(a)=0, g(b)=0\}$, because $\phi+g\in X$. Then ε is called a variation

To make sense of the term $d_\phi F(\varepsilon)$, we use Taylor's theorem – taking F at ϕ and perturbing it by ε, then $F(\phi+\varepsilon)=F(\phi)+d_\phi F(\varepsilon)$ + higher order terms. ($d_\phi F(\varepsilon)$: directional derivative in ε-direction.)

$F(\phi)=\int_a^b L(t,\phi(t),\dot\phi(t))\,dt$, $F(\phi+\varepsilon)=\int_a^b L(t,\phi(t)+\varepsilon(t),\dot\phi(t)+\dot\varepsilon(t))\,dt$. We use $L(p,q+\alpha,r+\beta)=L(p,q,r)+\alpha\frac{\partial L}{\partial q}(p,q,r)+\beta\frac{\partial L}{\partial r}(p,q,r)+\eta$.

Then $F(\phi+\varepsilon)=\int_a^b L(t,\phi(t),\dot\phi(t))+\int_a^b \varepsilon(t)\frac{\partial L}{\partial q}(t,\phi(t),\dot\phi(t))+\int_a^b\dot\varepsilon(t)\frac{\partial L}{\partial r}(t,\phi(t),\dot\phi(t))\,dt+\eta$, where q,r are second and third variables

the final integral is evaluated by parts: $\int_a^b\dot\varepsilon(t)\frac{\partial L}{\partial r}(t,\phi(t),\dot\phi(t))=\Big[\varepsilon(t)\frac{\partial L}{\partial r}(t,\phi(t),\dot\phi(t))\Big]_a^b-\int_a^b\varepsilon(t)\frac{d}{dt}\Big[\frac{\partial L}{\partial r}(t,\phi(t),\dot\phi(t))\Big]dt=-\int_a^b\varepsilon(t)\frac{d}{dt}\Big(\frac{\partial L}{\partial r}(t,\phi(t),\dot\phi(t))\Big)dt$. (The boundary term vanishes at both a and b.)

L is a function of p,q,r, thus so is $\frac{\partial L}{\partial t}$. Since $p=t$, $q=\phi(t)$, $r=\dot\phi(t)$, then the term $\frac{d}{dt}\frac{\partial L}{\partial r}$ makes sense as it depends only on t.

$\Rightarrow F(\phi+\varepsilon)=F(\phi)+\int_a^b\varepsilon(t)\Big(\frac{\partial L}{\partial q}-\frac{d}{dt}\frac{\partial L}{\partial r}\Big)dt+\eta \Rightarrow \phi$ is an extremum of F iff $\int\varepsilon(t)\Big(\frac{\partial L}{\partial q}-\frac{d}{dt}\big(\frac{\partial L}{\partial r}\big)\Big)dt=0\ \forall\varepsilon$.

We claim that this implies $\frac{\partial L}{\partial q}-\frac{d}{dt}\big(\frac{\partial L}{\partial r}\big)=0$, which is a form of the Euler-Lagrange equation*.

Theorem (Euler-Lagrange Equation)

$$\frac{\partial L}{\partial\phi}-\frac{d}{dt}\Big(\frac{\partial L}{\partial\dot\phi}\Big)=0 \iff \phi \text{ is an extremum of } F.$$

Ex Let $L=\sqrt{1+r^2}$. Show that all extrema are straight lines.

Soln. $F(\phi)=\int_a^b\sqrt{1+\dot\phi^2}\,dt$ is the arc length of graph ϕ. $\frac{\partial L}{\partial\phi}=0$ (no explicit ϕ dependence), $\frac{\partial L}{\partial\dot\phi}=\frac{2\dot\phi}{2\sqrt{1+\dot\phi^2}}=\frac{\dot\phi}{\sqrt{1+\dot\phi^2}}$. By Euler-Lagrange equation,

$\frac{\partial L}{\partial\phi}-\frac{d}{dt}\big(\frac{\partial L}{\partial\dot\phi}\big)=0\Rightarrow\frac{d}{dt}\big(\frac{\partial L}{\partial\dot\phi}\big)=0\Rightarrow\frac{\partial L}{\partial\dot\phi}=C\Rightarrow\frac{\dot\phi}{\sqrt{1+\dot\phi^2}}=C\Rightarrow\frac{\dot\phi^2}{1+\dot\phi^2}=C\Rightarrow\dot\phi=D$ const., so $\phi(t)=Dt+E\Rightarrow$ straight line.

We now prove the Fundamental theorem of Calculus of Variations, which helps us make the claim marked * above. Statement of theorem is found at bottom of page 5.

Proof – (Fundamental Theorem of Calculus of Variations)

Suppose $\exists x\in[a,b]$ s.t. $y(x)>0\Rightarrow y>0$ on interval $(x-\delta,x+\delta)$ by inertia principle.

Then $\varepsilon(t)=\begin{cases}\exp\big(\frac{-1}{\delta^2-(t-x)^2}\big) & \text{for } t\in(x-\delta,x+\delta)\\ 0 & \text{otherwise}\end{cases}$ is ≥0 everywhere, and positive in $(x-\delta,x+\delta)\Rightarrow\int y\varepsilon\,dt>0$.

But we realise that this means $\exists\varepsilon(t)$ s.t. $\int_a^b y(t)\varepsilon(t)\neq0$. $\Rightarrow$ contradiction. Repeat with $y(x)<0$ case to get another contradiction.

Thus by elimination, $y=0$ // q.e.d.

The Euler-Lagrange equations can be simplified in some instances:

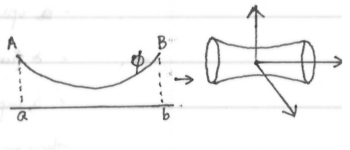
Theorem (Beltrami Identity).

Let $L(p,q,r)$ be a function in three variables. Then suppose $\frac{\partial L}{\partial p}=0$, ϕ satisfies Euler-Lagrange equation $\Rightarrow L-\dot\phi\frac{\partial L}{\partial\dot\phi}=$ constant.

Proof – Differentiate throughout w.r.t t, then $\frac{d}{dt}\Big(L(t,\phi(t),\dot\phi(t))-\dot\phi(t)\frac{\partial L}{\partial\dot\phi}(t,\phi(t),\dot\phi(t))\Big)=\frac{\partial L}{\partial p}\frac{dt}{dt}+\frac{\partial L}{\partial q}\frac{d\phi}{dt}+\frac{\partial L}{\partial r}\frac{d\dot\phi}{dt}-\big(\frac{d}{dt}\dot\phi\big)\frac{\partial L}{\partial\dot\phi}-\dot\phi\frac{d}{dt}\big(\frac{\partial L}{\partial\dot\phi}\big)$

This gives $(0)(1)+\frac{\partial L}{\partial\phi}\dot\phi+\frac{\partial L}{\partial\dot\phi}\ddot\phi-\dot\phi\frac{d}{dt}\big(\frac{\partial L}{\partial\dot\phi}\big)-\ddot\phi\frac{\partial L}{\partial\dot\phi}=\frac{\partial L}{\partial\phi}\dot\phi-\dot\phi\frac{d}{dt}\big(\frac{\partial L}{\partial\dot\phi}\big)=\dot\phi\big(\frac{\partial L}{\partial\phi}-\frac{d}{dt}\frac{\partial L}{\partial\dot\phi}\big)=\dot\phi(0)=0$ // q.e.d.

Catenoids.



Take a twice-differentiable function $\phi:[a,b]\to\mathbb{R}$, $\phi(a)=A$, $\phi(b)=B$ and $\phi>0$. We rotate it to form a surface of revolution.

We know that its surface area is given by Area $=2\pi\int_a^b\phi(t)\sqrt{1+\dot\phi(t)^2}\,dt=F(\phi)$ [see video online].

Question: What function ϕ minimises the surface area of its surface of revolution?

To answer this, we need to solve the Euler-Lagrange equations. Since $L(p,q,r)=\phi\sqrt{1+\dot\phi^2}=q\sqrt{1+r^2}$, L has no explicit p-dependence i.e. $\frac{\partial L}{\partial p}=0$

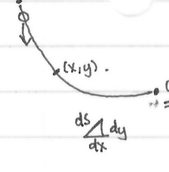
$\Rightarrow$ we only need to solve Beltrami identity, i.e. $L-\dot\phi\frac{\partial L}{\partial\dot\phi}=c$. We know $L=\phi\sqrt{1+\dot\phi^2}$, so $\frac{\partial L}{\partial\dot\phi}=\phi\frac{\dot\phi}{\sqrt{1+\dot\phi^2}}\Rightarrow\phi\sqrt{1+\dot\phi^2}-\dot\phi\frac{\phi\dot\phi}{\sqrt{1+\dot\phi^2}}=c$

$\Rightarrow\phi\Big(\sqrt{1+\dot\phi^2}-\frac{\dot\phi^2}{\sqrt{1+\dot\phi^2}}\Big)=\phi\Big(\frac{1+\dot\phi^2-\dot\phi^2}{\sqrt{1+\dot\phi^2}}\Big)=\frac{\phi}{\sqrt{1+\dot\phi^2}}=c\Rightarrow\frac{\phi^2}{1+\dot\phi^2}=c^2\Rightarrow\dot\phi=\sqrt{\frac{\phi^2}{c^2}-1}$; $\dot\phi\,dt=d\phi$

$\Rightarrow\int\frac{\dot\phi}{\sqrt{(\phi/c)^2-1}}\,dt=\int dt\Rightarrow\int\frac{d\phi}{\sqrt{(\phi/c)^2-1}}=t+A\Rightarrow c\log2+c\cosh^{-1}\frac{\phi}{c}=t+A\Rightarrow\frac{\phi}{c}=\cosh\frac{t-D}{c}\Rightarrow\phi=c\cosh\frac{t-D}{c}$.

This minimises area of its surface of revolution – graph is called catenary, surface is called catenoid.

Brachistochrone Problem.



We pick fixed points P and Q, connecting them by a frictionless curve. A heavy bead falls under gravity along the curve.

Question: What shape of wire minimises the time taken by the bead falling from p to q? Let P be origin, Q be (x_0,y_0); $x>0$, $y<0$.

Consider energy of bead $=\frac{1}{2}mv^2+mgy$. At P, this means $E=\frac{1}{2}m(0)^2+mg(0)=0$. $\therefore$ By conservation of energy, $\frac{1}{2}mv^2+mgy=0$.

Then $v=\sqrt{-2gy}$. Then time taken is given by bead is $T=\int_p^q\frac{\text{arc length}}{\text{velocity}}=\int_p^q\frac{ds}{v}=\int_p^q\frac{ds}{\sqrt{-2gy}}=\int_p^q\frac{\sqrt{1+(y')^2}}{\sqrt{-2gy}}\,dx$.

We aim to find y s.t. T is minimised, where $T(y) = \int \frac{\sqrt{1+(y')^2}}{\sqrt{-2gy}}\,dx$. Then $L(p,q,r) = \frac{\sqrt{1+r^2}}{\sqrt{-2gq}}$, which is p (i.e. x) independent

$\therefore$ Beltrami identity holds $\Rightarrow L - y'\frac{\partial L}{\partial y'} = c$. Here, $\frac{\partial L}{\partial y'} = \frac{y'}{\sqrt{-2gy}\sqrt{1+(y')^2}}$, so $C = \frac{\sqrt{1+(y')^2}}{\sqrt{-2gy}} - y'\frac{y'}{\sqrt{-2gy}\sqrt{1+(y')^2}} = \frac{1+(y')^2-(y')^2}{\sqrt{-2gy}\sqrt{1+(y')^2}} = \frac{1}{\sqrt{-2gy}\sqrt{1+(y')^2}}$

$\Rightarrow 2gc^2 = -\frac{1}{y(1+(y')^2)} = A$, so $Ay(1+(y')^2) = -1 \Rightarrow y' = \sqrt{\frac{1}{-Ay}-1} \Rightarrow \frac{dy}{dx} = \sqrt{-\frac{1}{Ay}-1} \Rightarrow \frac{dx}{dy} = \frac{1}{\sqrt{-\frac{1}{Ay}-1}}$ [strictly speaking illegal, we assume that graph is bijective]

Then $\frac{dx}{dy} = \sqrt{\frac{-y}{A^{-1}+y}} \Rightarrow \int dx = \int dy \sqrt{\frac{-y}{A^{-1}+y}} \Rightarrow x + \text{const}$, and $x = A^{-1}\sin^{-1}\sqrt{-yA} - \sqrt{-y}\sqrt{A^{-1}+y}$, is the shortest path.

The curve $x = A^{-1}\sin^{-1}\sqrt{-yA} - \sqrt{-y}\sqrt{A^{-1}+y}$ is called the bachistochrone curve.

Extension: this curve also obeys the tautochrone property, in which a bead dropped from anywhere along the curve takes the same time to reach Q.

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In general, given a function space $X = \{f : [a,b] \to \mathbb{R}$ s.t. $\phi(a) = A, \phi(b) = B\}$, $F: X \to \mathbb{R}$ is a functional $F(\phi) = \int L(t, \phi(t), \dot{\phi}(t))\,dt$.

The function L is a function of three variables, and we call it the Lagrangian density.

The Euler-Lagrange theorem states that if ϕ is an extremum of F, then it satisfies $\frac{\partial L}{\partial \phi} = \frac{d}{dt}\frac{\partial L}{\partial \dot{\phi}}$.

We can extend this to other cases, including vector-valued ϕ, constrained problems, functions $\phi(x,y)$ of two variables.

①: Vector-valued ϕ: Here $\phi : [a,b] \to \mathbb{R}^n$ i.e. $(\phi_1(t), \phi_2(t), \ldots, \phi_n(t))$. Then $F(\phi) = \int L(t, \phi_1(t), \dot{\phi}_1(t), \phi_2(t), \dot{\phi}_2(t), \ldots, \phi_n(t), \dot{\phi}_n(t))\,dt$.

L is a function of $2n+1$ variables, and we get a system of Euler-Lagrange equations: $\left\{\frac{\partial L}{\partial \phi_1} = \frac{d}{dt}\frac{\partial L}{\partial \dot{\phi}_1}, \frac{\partial L}{\partial \phi_2} = \frac{d}{dt}\frac{\partial L}{\partial \dot{\phi}_2}, \ldots, \frac{\partial L}{\partial \phi_n} = \frac{d}{dt}\frac{\partial L}{\partial \dot{\phi}_n}\right\}$.

See Exercise on Sheet 4.

Ex Recall that for paths in plane, $\gamma(t) = (\gamma_1(t), \gamma_2(t))$, $F(\gamma) = \int(\dot{\gamma}_1^2 + \dot{\gamma}_2^2)\,dt$.

Soln. We get two Euler-Lagrange equations: $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\gamma}_1}\right) = \frac{\partial L}{\partial \gamma_1} = \frac{d}{dt} 2\dot{\gamma}_1 = 0 \Rightarrow \ddot{\gamma}_1 = 0$. Likewise, $\ddot{\gamma}_2 = 0$. Then $\gamma_1 = At + B$, $\gamma_2 = Ct + D$.

Thus, γ is a straight line.

②: Constrained problems: We seek min/max $F(\phi)$ over X subject to the constraint $\int_a^b M(t, \phi(t), \dot{\phi}(t))\,dt = K$ constant. For instance we want to maximise area under graph $\int_a^b \phi\,dt$, given fixed arc length $\int_a^b \sqrt{1+\dot{\phi}^2}\,dt$.

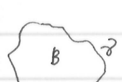
We use the finite dimension algorithm.

- Introduce $\lambda \in \mathbb{R}$.
- Consider $H(\phi) = F(\phi) - \lambda G(\phi)$, where $G(\phi) = \int M(t, \phi, \dot{\phi})\,dt - K$. Then $H(\phi) = \int_a^b (L - \lambda M - \frac{K}{b-a})\,dt$.
- Find critical points of H. Varying $\lambda \Rightarrow G(\phi) = 0$, varying $\phi \Rightarrow$ new equation $\frac{\partial}{\partial \phi}(L - \lambda M) = \frac{d}{dt}\left(\frac{\partial}{\partial \dot{\phi}}(L - \lambda M)\right)$

Ex (Isoparametric problem) — Maximising area with fixed perimeter.

If $\exists$ a closed curve in $\mathbb{R}^2$ with length 2π which maximises area it bounds amongst all curves with length 2π, then show that it is a circle.

Soln. Consider $\gamma : \mathbb{R} \to \mathbb{R}^2$ that is 2π-periodic, i.e. $\gamma(t+2\pi) = \gamma(t)$. Area $= \iint_B dx\,dy = \oint_\gamma x\,dy$ (Green's theorem)



We want to maximise $\oint_\gamma x\,dy$ subject to $\int_0^{2\pi} |\dot{\gamma}|\,dt = 2\pi$. Apply the finite dimension algorithm, we should find critical points of modified functional $H(\gamma) = \int x\,dy - \int \lambda(\sqrt{\dot{x}^2+\dot{y}^2} - 1)\,dt = \int x\dot{y}\,dt - \int \lambda(\sqrt{\dot{x}^2+\dot{y}^2} - 1)\,dt$. $(x = \gamma_1, y = \gamma_2)$.

$H(\gamma) = \int_0^{2\pi} \underbrace{[x\dot{y} - \lambda(\sqrt{\dot{x}^2+\dot{y}^2} - 1)]}_{L(t,x,\dot{x},y,\dot{y})}\,dt$. We then write the Euler-Lagrange equations: $\frac{\partial L}{\partial x} = \frac{d}{dt}\frac{\partial L}{\partial \dot{x}}$, $\frac{\partial L}{\partial y} = \frac{d}{dt}\frac{\partial L}{\partial \dot{y}}$

Manipulating these, we then get $\dot{y} = \frac{d}{dt}\left(\frac{-\lambda \dot{x}}{\sqrt{\dot{x}^2+\dot{y}^2}}\right)$, $0 = \frac{d}{dt}\left(x - \frac{\lambda \dot{y}}{\sqrt{\dot{x}^2+\dot{y}^2}}\right)$.

If we have a maximum $\gamma(t) = (x(t), y(t))$, then we can find another γ with same area, same length, but moving with constant speed 1. $1 = |\dot{\gamma}| = \sqrt{\dot{x}^2+\dot{y}^2}$. It's still a maximum, and therefore solves Euler-Lagrange equations. Then $\dot{y} = -\frac{d}{dt}(\lambda\dot{x})$, $0 = \frac{d}{dt}(x - \lambda\dot{y})$

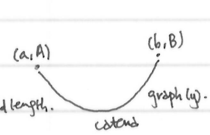
$\Rightarrow \dot{y} = -\lambda\ddot{x}$, $x = \lambda\dot{y} \Rightarrow \ddot{y} = -\lambda\dddot{x}$, $\ddot{y} = \frac{\dot{x}}{\lambda} \Rightarrow -\lambda\dddot{x} = \frac{\dot{x}}{\lambda} \Rightarrow \{\dddot{x} = -\frac{1}{\lambda^2}\dot{x}, \dddot{y} = -\frac{1}{\lambda^2}\dot{y}\}$ which gives SHM.

Then $\dot{y} = A\sin(\frac{t}{\lambda}) + B\cos(\frac{t}{\lambda})$, $\dot{x} = -A\cos(\frac{t}{\lambda}) + B\sin(\frac{t}{\lambda}) \Rightarrow x =$ circular function $+ C_1$, $y =$ circular function $+ C_2$.

This parametrises a circle $\Rightarrow$ max area is bounded by circle.

Note: In sheet 5, Q1, we actually prove the existence of a maximising circle.

Ex (Catenary Problem)



A chain hangs freely between (a,A) and (b,B) in a way designed to minimise total potential energy (naturally), but with fixed length.

Describe this shape mathematically.

Soln. $F(y) = \int \rho g y\,ds = \int \rho g y \sqrt{1+(y')^2}\,dx$. (total PE) $\quad G(y) = \underbrace{\int_a^b \sqrt{1+(y')^2}\,dx}_{\text{arclength}} - \underset{\text{constant}}{K}$. Having set up the problem — solution is left as Ex in sheet 5.

$ds = \sqrt{dx^2+dy^2} = \sqrt{1+(y')^2}\,dx$. with uniform density ρ, then mass of infinitesimal chunk is $\rho\,ds$. And gravitational potential is mgy.

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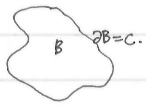
Ex Let $F(y) = \int (y')^2 - xy\,dx$, $G = \int_0^1 y - 1\,dx$. conditions: $\int_0^1 y\,dx = 1$, $y(0)=0$, $y'(0)=0$.

Soln. By Euler-Lagrange equation, let $(y')^2 - xy - \lambda(y-1) = S(y)$. Find critical points: $\frac{\partial S}{\partial y} = \frac{d}{dx}\left(\frac{\partial S}{\partial y'}\right) \Rightarrow -x-\lambda = \frac{d}{dx}(2y') = 2y''$. Solve $y'' = -\frac{1}{2}(x+\lambda)$

$y'' = -\frac{1}{2}(x+\lambda) \Rightarrow y' = -\frac{1}{4}x^2 - \frac{1}{2}\lambda x + \mu$, $y'(0) = \mu = 0$, $y = -\frac{1}{12}x^3 - \frac{\lambda}{4}x^2 + \nu$, $y(0) = \nu \Rightarrow y = -\frac{1}{12}x^3 - \frac{\lambda}{4}x^2$. Then to find λ, use $\int_0^1 y\,dx = 1$ //

(3): Functions of 2 variables: Let $B \subset \mathbb{R}^2$ be a domain, $\partial B = C$ be the boundary. $X = \{\phi : B \to \mathbb{R},\ \phi|_C = \phi_0 : C \to \mathbb{R} \text{ fixed}\}$



$F(\phi) = \iint_B L(x, y, \phi(x,y), \phi_x(x,y), \phi_y(x,y))\,dx\,dy$.

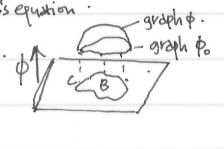
Theorem The extrema (critical points) of $F: X \to \mathbb{R}$ satisfy a partial differential equation $\frac{\partial L}{\partial \phi} = \frac{\partial}{\partial x}\left(\frac{\partial L}{\partial \phi_x}\right) + \frac{\partial}{\partial y}\left(\frac{\partial L}{\partial \phi_y}\right)$. Proof – see Problem set 5, Q3.

Ex (Dirichlet energy).

$F(\phi) = \iint_B (\phi_x^2 + \phi_y^2)\,dx\,dy$ i.e. $F(\phi) = \iint_B |\nabla\phi|^2\,dx\,dy$ i.e. the total gradient of ϕ, the <u>Dirichlet energy</u>.

Soln. Euler-Lagrange equation is $\frac{\partial L}{\partial \phi} = \frac{\partial}{\partial x}\left(\frac{\partial L}{\partial \phi_x}\right) + \frac{\partial}{\partial y}\left(\frac{\partial L}{\partial \phi_y}\right) \Rightarrow 0 = \frac{\partial}{\partial x}(2\phi_x) + \frac{\partial}{\partial y}(2\phi_y) = 2\phi_{xx} + 2\phi_{yy} = 2\Delta\phi$ // by Laplace's equation.

Ex $\phi: B \to \mathbb{R}$; $\phi|_C = \phi_0 : C \to \mathbb{R}$, $C = \partial B$. By the diagram on the right, the surface area of the graph ϕ is $\iint_B \sqrt{1 + \phi_x^2 + \phi_y^2}\,dx\,dy$.



Theorem If ϕ minimises the surface area of its graph amongst all ϕ with $\phi|_C = \phi_0$, then we have the relation

$$\frac{\partial^2\phi}{\partial x^2}\left(1 + \left(\frac{\partial\phi}{\partial y}\right)^2\right) + \frac{\partial^2\phi}{\partial y^2}\left(1 + \left(\frac{\partial\phi}{\partial x}\right)^2\right) = 2\frac{\partial\phi}{\partial x}\frac{\partial\phi}{\partial y}\frac{\partial^2\phi}{\partial x\partial y}.$$

Proof – see Problem set 5, Q4

PARTIAL DIFFERENTIAL EQUATIONS.

PDEs are differential equations with partial derivatives evaluated at the same point, i.e. $\frac{\partial\phi}{\partial x}(x,y) + \frac{\partial\phi}{\partial y}(x-1, y-1) = 0$ is not a PDE, it is a <u>difference equation</u>.

It must also have finitely many derivatives, i.e. $\phi_x + \phi_{xx} + \phi_{xxx} + \cdots = 0$ is not a PDE.

A PDE is an equation $E(x, y, \phi, \phi_x, \phi_y, \phi_{xx}, \phi_{yy}, \phi_{xy}, \ldots) = E(\phi) = 0$. where ϕ, ϕ_x, etc are evaluated at point (x,y).
(ϕ, ϕ_x, ϕ_y: has some dependence on derivative terms; $\ldots$: finite list.)

Examples of PDEs: Burgers's equation — $\partial_t\phi + \phi\partial_x\phi = 0$ models an inviscid fluid, $(\partial_x\phi)^2 + (\partial_y\phi)^2 = 1$ is the eikonal equation, $\phi_t = \phi_{xx}$ is the heat equation.

Definition The order of a PDE is the highest order of derivatives taken in an equation

	order
Burgers's eqn.	1
eikonal eqn.	2
heat eqn.	2.

Our course generally deals with first and second order PDEs.

Definition A linear PDE $E(\phi) = 0$ is a PDE with the property that $E(\lambda_1\phi_1 + \lambda_2\phi_2) = \lambda_1 E(\phi_1) + \lambda_2 E(\phi_2)$ $\lambda_1, \lambda_2 \in \mathbb{R}$.

i.e. linear combinations of solutions are solutions.

Note: Not many PDEs are linear. Linear equations are sums of terms, where each term has the form $A(x,y)\,\partial_1\partial_2\cdots\partial_k\phi$ (combination of partials).

of our three examples above, only the heat equation is linear

Definition A linear PDE is called <u>constant coefficient linear</u> if A is constant $\forall$ terms.

Definition An inhomogeneous linear equation is of the form $E(\phi) = F$ where E is linear and F only depends on x, y (e.g. $\partial_x\phi + \partial_y\phi = 7x^3$).

Definition A <u>quasi-linear</u> equation of order n, is where in any term, an n^{th} derivative only occurs linearly.

For instance, consider the eikonal equation $(\partial_x\phi)^2 + (\partial_y\phi)^2 = 1$, it is not quasilinear $\because$ $\partial_x\phi, \partial_y\phi$ occur non-linearly. However, $\partial_t\phi + \phi\partial_x\phi = 0$ is quasilinear because $\partial_x\phi$ is linear. i.e. if you treat everything as constant except the highest derivative terms, then it is linear (in this case inhomogenous).

Furthermore, $\phi_x\phi_{yy} + \phi_{xx}\phi_y = 7$ is quasi-linear, $\partial_{xxx}\phi + \partial_{xx}\phi\,\partial_{yy} - (\partial_{xy}\phi)^2 = 0$ is quasilinear.

"Classification" of 2^{nd} order quasilinear equations in 2 variables. A general such equation is of the form $A\frac{\partial^2\phi}{\partial x^2} + B\frac{\partial^2\phi}{\partial x\partial y} + C\frac{\partial^2\phi}{\partial y^2} = R$, (*) where A, B, C, R are all expressions dependent on $x, y, \phi, \phi_x, \phi_y$.

Definition The discriminant, $\Delta = B^2 - 4AC$. This is a function of $x, y, \phi, \phi_x, \phi_y$.

Consider the equation (*). If $\Delta > 0$, it is hyperbolic; if $\Delta = 0$, it is parabolic; if $\Delta < 0$, it is elliptic. Of course, this is only a partial guideline.

For instance consider $x\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} = 0$, then its character changes over different regions.

If A, B, C, are all constant, then $\Delta \in \mathbb{R}$, so equation falls into one of these categories. For instance,

- $\phi_{xx} + \phi_{yy} = 0$, $\Delta = -4 \Rightarrow$ elliptic,
- $\phi_{xx} = \frac{1}{c}\phi_{tt} \Rightarrow \Delta = \frac{4}{c^2} \Rightarrow$ hyperbolic,
- $\phi_t = \phi_{xx}$ (ϕ_t = R) $\Rightarrow \Delta = 0$ ($\because$ there is no ϕ_{tt} term).

Solutions to the quasilinear elliptic equation share many properties with solutions of $\Delta\phi = 0$.

We can check that $\phi_{xx}(1 + \phi_y^2) + \phi_{yy}(1 + \phi_x^2) = 2\phi_x\phi_y\phi_{xy}$ is elliptic.

2401-008

Solution to constant coefficient 2^{nd} order hyperbolic equation.

Let it be homogeneous: then $A\phi_{xx} + B\phi_{xy} + C\phi_{yy} = 0$; $A, B, C \in \mathbb{R}$. Then $\Delta = B^2 - 4AC > 0$ (hyperbolic).

Recall the quadratic equation, $At^2 + Bt + C = 0$. This has distinct real roots $t_\pm = \frac{-B \pm \sqrt{\Delta}}{2A}$. We use a trick to change to light cone coordinates: $u = y + xt_+$, $v = y + xt_-$

These are LI, so they set up a coordinate system. Then $x = \frac{u-v}{t_+ - t_-}$, $y = \frac{t_+ v - t_- u}{t_+ - t_-}$. Then $\frac{\partial}{\partial u} = \frac{\partial x}{\partial u}\frac{\partial}{\partial x} + \frac{\partial y}{\partial u}\frac{\partial}{\partial y} = \frac{1}{t_+ - t_-}\left(\frac{\partial}{\partial x} - t_-\frac{\partial}{\partial y}\right)$, and

$\frac{\partial}{\partial v} = \frac{\partial x}{\partial v}\frac{\partial}{\partial x} + \frac{\partial y}{\partial v}\frac{\partial}{\partial y} = -\frac{1}{t_+ - t_-}\left(\frac{\partial}{\partial x} - t_+\frac{\partial}{\partial y}\right)$. Thus, $At^2 + Bt + C = A(t - t_+)(t - t_-)$. Then $\frac{\partial}{\partial u}\frac{\partial}{\partial v}\phi = -\frac{1}{(t_+ - t_-)^2}\left(\frac{\partial}{\partial x} - t_-\frac{\partial}{\partial y}\right)\left(\frac{\partial}{\partial x} - t_+\frac{\partial}{\partial y}\right)\phi$

expanding, $\frac{\partial}{\partial u}\frac{\partial}{\partial v}\phi = -\frac{1}{A(t_+ - t_-)}\left(A\frac{\partial^2}{\partial x^2} + B\frac{\partial^2}{\partial x \partial y} + C\frac{\partial^2}{\partial y^2}\right)\phi = 0 \Rightarrow \frac{\partial^2 \phi}{\partial u \partial v} = 0$, $\phi = F(u) + G(v)$ where F and U are arbitrary functions.

Theorem (restatement) If ϕ solves $A\phi_{xx} + B\phi_{xy} + C\phi_{yy} = 0$, then $\phi(x,y) = F(y + xt_+) + G(y + xt_-)$ where $t_\pm$ are roots of $At^2 + Bt + C = 0$, and F, G are arbitrary.

Proof - $At^2 + Bt + C = A(t - t_+)(t - t_-)$, where $t_\pm$ are real and distinct roots $\because \Delta > 0$. Then set $u = y + t_+ x$, $v = y + t_- x$.

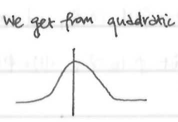
By manipulation, we can re-express our operators as $\frac{\partial}{\partial u} = \frac{1}{t_+ - t_-}\left(\frac{\partial}{\partial x} - t_-\frac{\partial}{\partial y}\right)$, $\frac{\partial}{\partial v} = -\frac{1}{t_+ - t_-}\left(\frac{\partial}{\partial x} - t_+\frac{\partial}{\partial y}\right)$.

Then, multiplying together, $\frac{\partial}{\partial u}\frac{\partial}{\partial v}\phi = -\frac{1}{(t_+ - t_-)^2}\left(\frac{\partial}{\partial x} - t_-\frac{\partial}{\partial y}\right)\left(\frac{\partial}{\partial x} - t_+\frac{\partial}{\partial y}\right)\phi = -\frac{1}{(t_+ - t_-)^2}\left[\frac{\partial^2\phi}{\partial x^2} - (t_- + t_+)\frac{\partial^2\phi}{\partial x \partial y} + t_+ t_- \frac{\partial^2\phi}{\partial y^2}\right]$

$= \frac{1}{A}(At^2 + Bt + C)$. Thus $\frac{\partial^2\phi}{\partial u \partial v} = 0$, $\phi = F(u) + G(v)$. (but at least one of F or G is C^1).

Ex Solve the wave equation $\frac{\partial^2\phi}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2\phi}{\partial y^2} \Rightarrow c^2\frac{\partial^2\phi}{\partial x^2} - \frac{\partial^2\phi}{\partial y^2} = 0$.

Soln. We get from quadratic equation, $t^2 - c^2 = 0 \Rightarrow t_+ = c$, $t_- = -c \Rightarrow \phi(x,y) = F(x+cy) + G(x-cy)$ [Here time = y].
(or for clarity, $t_\pm = \frac{0 \pm \sqrt{0 + 4c^2}}{2c^2}$).
wave moving backwards / forwards in x-direction.



As y (time) increases, x+cy increases (for each x),
F is being translated to the right, G is being translated to the left.

For example, let $F(x) = \frac{1}{2}x$, $G(x) = \frac{1}{2}x$ is a solution to wave equation. Then $\phi(x,y) = x = \frac{1}{2}(x+cy) + \frac{1}{2}(x-cy) = F(x+cy) + G(x+cy)$.

For all time, $\phi(x,y) = x \Rightarrow$ this is a standing solution.

Another solution — take $F(x) = x$, $G(x) = 0$. Corresponding solution is $\phi(x,y) = F(x+cy) + G(x-cy) = x + cy$ (time-dependent).

In both examples, $\phi(x,0) = x$. For a unique solution, we have to specify more initial conditions; such as $\partial_y\phi(x,0)$.

For first, $\partial_y\phi(x,0) = 0$; for second $\partial_y\phi(x,0) = c$.

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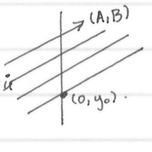
First order inhomogeneous linear PDEs.

These take the form $A\phi_x + B\phi_y + C\phi = D(x,y)$, where A, B, C, D are all functions of x, y.

Case 1: A, B, C all constants. Then observe $A\frac{\partial\phi}{\partial x} + B\frac{\partial\phi}{\partial y}$ is the directional derivative of ϕ in the (A, B) direction (assume $A \neq 0$).



i.e. lines $(x(t), y(t)) = (tA, y_0 + tB)$. Define $u(t) = \phi(tA, y_0 + tB)$. Then directional derivative $= \frac{d}{dt}\Big|_{t=T} u(t)$, $A\frac{\partial\phi}{\partial x} + B\frac{\partial\phi}{\partial y} = \dot{u}$

$\Rightarrow \dot{u} = D(x(t), y(t)) - u(t)C = D(tA, y_0 + tB) - Cu(t)$. This is an ODE for u.

Key Idea: Turn 1^{st} order partial differential operator into ordinary derivative. (for homogeneous eqn, $D = 0 \Rightarrow \dot{u} = -Cu \Rightarrow \frac{\dot{u}}{u} = -C \Rightarrow u = Le^{-ct}$.

Hence $u(t) = Le^{-ct} \Leftrightarrow \phi(x,y) = Le^{-\frac{cx}{A}}$ for homogeneous case. $x = tA$, $y = y_0 + tB \Rightarrow t = \frac{x}{A}$, $y_0 = y - \frac{xB}{A}$, so $u(t) = \phi(x,y)$ when these two conditions hold.)

i.e. General solution: $\phi(x,y) = L\left(y - \frac{xB}{A}\right)e^{-\frac{cx}{A}}$.

e.g.
If on $x = 0$, we specify $\phi(0,y) = y$, then $L(y) = y$. We can use initial conditions to determine "constants of integration".

Ex Solve $A\phi_x + B\phi_y + C\phi = x$.

Soln. $\dot{u} = D(tA, y_0 + tB) - Cu(t) = tA - Cu(t) \Rightarrow \dot{u} + Cu(t) = tA$ Multiply throughout by IF: $e^{ct} \Rightarrow \dot{u}e^{ct} + Cue^{ct} = tAe^{ct}$.

$\frac{d}{dt}(ue^{ct}) = tAe^{ct} \Rightarrow ue^{ct} = \int tAe^{ct}\,dt = \frac{A}{c}e^{ct}\left(t - \frac{1}{c}\right) + K(y_0). \Rightarrow u = \frac{A}{c}\left(t - \frac{1}{c}\right) + K(y_0)e^{-ct}$, $t = \frac{x}{A}$, $y_0 = y - \frac{xB}{A}$.

$\phi(x,y) = \frac{A}{c}\left(\frac{x}{A} - \frac{1}{c}\right) + e^{-\frac{cx}{A}}K\left(y - \frac{xB}{A}\right)$ //.

Algorithm:

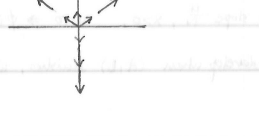
→ Identify vector (A, B) and observe that $A\phi_x + B\phi_y$ is the directional derivative of ϕ in (A, B) direction.

→ Restrict to the lines $\dot{u} = D - Cu$, $u = \phi$ is restricted to line with slope (A, B), $\dot{u} + Cu = e^{-Ct}\frac{d}{dt}(ue^{ct})$.

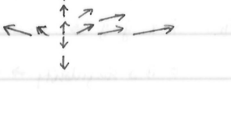
General solution for all D: $\frac{d}{dt}(ue^{ct}) = De^{ct} \Rightarrow \phi(x,y) = \underbrace{e^{-\frac{cx}{A}}\int_0^{x/A} e^{ct}D\left(tA, y - \frac{xB}{A} + tB\right)dt}_{PI} + \underbrace{K\left(y - \frac{xB}{A}\right)e^{-\frac{cx}{A}}}_{CF}$.

Case 2: A, B, C are all functions of (x, y). This is still linear, but vector field is no longer uniformly constant. For instance, we could have:

① $A = x$, $B = y$



② $A = 2x$, $B = 1$.



etc.

Definition If $(A(x,y), B(x,y))$ is a vector field, then an <u>integral curve</u> is a path $\gamma(t) = (x(t), y(t))$ s.t. $\dot\gamma(t) = (A(\gamma(t)), B(\gamma(t)))$.

i.e. $\dot\gamma(t) = (\dot x(t), \dot y(t))$, and $\dot x(t) = A(x(t), y(t))$, $\dot y(t) = B(x(t), y(t))$ which is a coupled system of ODEs.

e.g. $A(x,y) = x$, $B(x,y) = y \Rightarrow \dot x = x, \dot y = y \Rightarrow x = Ae^t, y = Be^t \Rightarrow$ parametrise a line through 0 with slope $\frac{B}{A}$.

$A = 2x$, $B = 1$. Need to solve $\dot x = 2x, \dot y = 1 \Rightarrow x = Ke^{2t}, y = t + L$.

We restrict ϕ to integral curves and solve ODEs along them. If $\gamma(t) = (x(t), y(t))$ is an integral curve of $(A(x,y), B(x,y))$, define $u(t) = \phi(x(t), y(t))$.

Then, we apply chain rule: $\frac{du}{dt} = \frac{\partial\phi}{\partial x}\frac{dx}{dt} + \frac{\partial\phi}{\partial y}\frac{dy}{dt} = A\frac{\partial\phi}{\partial x} + B\frac{\partial\phi}{\partial y}$ ($\because \dot x = A(x,y), \dot y = B(x,y)$) $= D - Cu = D(x(t), y(t)) - C(x(t), y(t))\,u(t)$ ← this is an ODE.

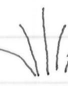
Remark: To solve this PDE, we need to solve three ODEs: $\dot x = A(x,y)$, $\dot y = B(x,y)$, $\dot u = D(x(t), y(t)) - C(x(t), y(t))\,u(t)$.

Ex Solve $2x\phi_x + \phi_y = 0$.

Soln $A = 2x = \dot x$, $B = 1 = \dot y \Rightarrow$ $x(t) = Ke^{2t}$, $y(t) = t + L$. We re-parametrise and define $s = t + L$, then $x(s) = \boxed{Ke^{-2L}}\,e^{2s}$, $y(s) = s$. $Ke^{-2L} = M$, new const. (i.e. $\frac{dy}{ds}$)

Then our integral curves are $x(s) = Me^{2s}$, $y(s) = s$. Then we want to solve $2x\phi_x + \phi_y = 0 \Rightarrow \dot u = 0$, where $u = \phi(x(s), y(s)) = \phi(Me^{2s}, s)$.

$\Rightarrow \phi(Me^{2s}, s)$ does not depend on s. $\Rightarrow \phi$ is an arbitrary function of $M = \frac{x}{e^{2y}}$.



so $\phi(x,y) = f\left(\frac{x}{e^{2y}}\right)$, an arbitrary function of xe^{-2y}.

Remark: A simpler example would be $\frac{\partial\phi}{\partial x} = 0 \Rightarrow \phi$ is an arbitrary function of y.

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Recall that given $A\phi_x + B\phi_y + C\phi = D$, where A, B, C, D are functions of x, y. We define the vector field (A,B), and seek integral curves $(\dot x(t), \dot y(t)) = (A,B)$.

These curves are called "characteristics". Given a characteristic $\gamma(t) = (x(t), y(t))$, we restrict ϕ to γ, $u(t) = \phi(\gamma(t))$. By chain rule,

$\frac{d\phi}{dt} = \frac{\partial\phi}{\partial x}\dot x + \frac{\partial\phi}{\partial y}\dot y = A\phi_x + B\phi_y = D - C\phi$, $\dot u = D - Cu$, ODE for u.

Ex Solve $\phi_x + 2x\phi_y = 1$.

Soln. $A = 1$, $B = 2x \Rightarrow$ characteristic curves satisfy $\dot x = 1$, $\dot y = 2x \Rightarrow x = t$, $y = t^2 + K$ (WLOG, reparametrise to get rid of constant)

i.e. $y = x^2 + K$ (parabola). The PDE reduces to $\dot u = 1 \Rightarrow u = t + L$ (constant depends on choice of curve (i.e. K)). $K = y - x^2$, $t = x$.

$\therefore \phi(x,y) = u(t) = x + L(y - x^2)$

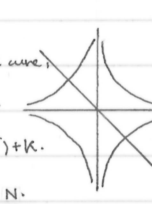
Check: $\frac{\partial\phi}{\partial x} = 1 - 2xL'$ (chain rule) $\frac{\partial\phi}{\partial y} = L' \Rightarrow \phi_x + 2x\phi_y = 1 - 2xL' + 2xL' = 1$ (verified).

To fix our arbitrary function, we require an initial condition. For instance, given our prior example, use $\phi(0,y) = \sin y$.

then $\phi(0,y) = 0 + L(y - 0^2) = L(y) = \sin y$. So the initial condition determines L. Hence $\phi(x,y) = x + \sin(y - x^2)$.

Ex Solve $y\phi_x + x\phi_y + xy\phi = 0$. Also solve using the initial condition $\phi(s, 1-s) = \tan s$.

Soln. $\dot x = y, \dot y = x \Rightarrow \ddot x = x, \ddot y = y$, so $x = Me^t + Ne^{-t}$, $y = Me^t - Ne^{-t}$. Then, $x^2 - y^2 = 4MN$ is the characteristic curve, which is a hyperbola. We reparametrise t, WLOG $M: -1, 0, 1$. Restrict to area of plane where $M > 0$, WLOG, $M = 1$.



$\dot u = -xyu = -(e^t + Ne^{-t})(e^t - Ne^{-t})\,u = -(e^{2t} - N^2e^{-2t})u \Rightarrow \frac{\dot u}{u} = -(e^{2t} - N^2e^{-2t}) \Rightarrow \ln u = -\frac{1}{2}(e^{2t} + N^2e^{-2t}) + K$.

Note that $x^2 + y^2 = e^{2t} + N^2e^{-2t} + e^{2t} + N^2e^{-2t} = 2(e^{2t} + N^2e^{-2t})$. Hence, $\ln u = -\frac{1}{4}(x^2 + y^2) + K$. K depends on N.

$N = \frac{x^2 - y^2}{4} \Rightarrow \phi(x,y) = e^{-\frac{1}{4}(x^2+y^2)}e^{K\left(\frac{x^2-y^2}{4}\right)} = L(x^2 - y^2)\,e^{-\frac{1}{4}(x^2+y^2)}$.

$\phi(x,y) = \phi(s, 1-s) = L(s^2 - (1-s)^2)\,e^{-\frac{1}{4}(s^2 + (1-s)^2)} = L(2s-1)\,e^{-\frac{1}{4}(2s^2 - 2s + 1)} = \tan s \Rightarrow L(2s-1) = \tan s \cdot e^{\frac{1}{4}(2s^2 - 2s + 1)}$.

Let $w = 2s - 1$, $s = \frac{1}{2}(w+1) \Rightarrow L(w) = \tan\frac{1}{2}(w+1)\,e^{\frac{1}{4}(\dots)}$. Let $w = x^2 - y^2$, substitute and solve.

Ex Solve $x^2\phi_x + x\phi_y + \phi = 1$, $\phi(x,0) = x$.

Soln. $A = x^2$, $B = x$, so characteristic curves satisfy $\dot x = x^2$, $\dot y = x \Rightarrow \dot x x^{-2} = 1 \Rightarrow -x^{-1} = t$ (reparametrise to remove const. of integration.) $x = -\frac{1}{t}$, $y = -\ln|t| + L$. Then, $y = -\ln|t| + L$

Then $y = L - \ln|\frac{1}{x}|$ then $\dot u = 1 - \phi = 1 - u \Rightarrow \dot u + u = 1 \Rightarrow e^t\dot u + e^t u = e^t \Rightarrow \frac{d}{dt}(e^t u) = e^t \Rightarrow e^t u = e^t + M \Rightarrow u = 1 + Me^{-t}$.

$t = -\frac{1}{x}$, so $u = 1 + Me^{\frac{1}{x}}$. M is a constant that depends on $L = y + \ln|\frac{1}{x}|$. $= y + \ln(\frac{1}{|x|}) = y - \ln|x|$; $\phi(x,y) = 1 + M(y - \ln|x|)\,e^{\frac{1}{x}}$.

Then imposing initial condition, $\phi(x,0) = x = 1 + M(-\ln|x|)e^{\frac{1}{x}} \Rightarrow M(-\ln|x|) = (x-1)\,e^{-1/x}$. Let $w = -\ln|x|$, $x = e^{-w}$.

Thus, $M(w) = (e^{-w} - 1)\cdot e^{-e^{w}}$

Ex Solve $x\phi_x + y\phi_y = 0$; if $\phi(x,y) = x$ $\forall x \in D(0,1)$ and evaluate it at the origin.

Soln. $\dot x = x, \dot y = y \Rightarrow x = Ne^t$, $y = Me^t \Rightarrow$ lines through 0 with slope $\frac{M}{N}$, $x > 0$. $\dot u = 0 \Rightarrow \phi$ is constant along these lines.

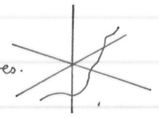
0 is a singularity $\Rightarrow$ not well defined. Note: singularities can develop when (A,B) vanishes, like at 0 in this example.

2401-010.

__Quasilinear case__ (first order)

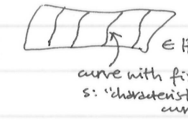
These take the form $A(x,y,\phi)\frac{\partial\phi}{\partial x} + B(x,y,\phi)\frac{\partial\phi}{\partial y} + C(x,y,\phi) = 0$. Then we impose the equations for a vector field, $\dot{x} = A(x,y,\phi)$, $\dot{y} = B(x,y,\phi)$. We also have $\dot{\phi} = -C(x,y,\phi)$ (note the -ve sign!) Rename ϕ as z, then this gives us a vector field in $\mathbb{R}^3 = \{(x,y,z)\}$, which is our characteristic vector fields.

Hence, we seek characteristic curves $(x(t), y(t), z(t))$ s.t. $\dot{x} = A$, $\dot{y} = B$, $\dot{z} = -C$.



We really want a solution for $\phi(=z)$ i.e. a surface $z = \phi(x,y)$. Hence we need a parameter s and a 1-parameter family of characteristic curves.

i.e. $(s,t) \mapsto (x_s(t), y_s(t), z_s(t))$. This is a map from $\mathbb{R}^2 \to \mathbb{R}^3$, which specifies a curve dependent on t, for fixed s.



__Theorem__ ($\Rightarrow$) Given any 1-parameter family of characteristic curves, their union is the graph of a solution (a surface in $\mathbb{R}^3$ – and when this surface is a graph, it's the graph of a solution to the equation). i.e. it is a graph if projection to (xy) plane is injective.

($\Leftarrow$) All solutions arise this way.

Proof – ($\Rightarrow$) Suppose $(x_s(t), y_s(t), z_s(t))$ is a 1-parameter family of characteristic curves (at least locally)

$\bigcup_{s,t}(x_s(t), y_s(t), z_s(t)) = \text{graph}(\phi) = \{(x,y,\phi(x,y)) : (x,y) \in \mathbb{R}^2\}$. Restrict ϕ to $(x_s(t), y_s(t)) \subseteq \mathbb{R}^2$ for each s.

$u(t) = \phi(x_s(t), y_s(t))$. Then by the chain rule, $\dot{u} = \frac{dx}{dt}\frac{\partial\phi}{\partial x} + \frac{dy}{dt}\frac{\partial\phi}{\partial y} = A\frac{\partial\phi}{\partial x} + B\frac{\partial\phi}{\partial y} = -C = \dot{z} \Rightarrow u = z = \phi(x,y) \Rightarrow \phi$ solves the PDE // q.e.d.

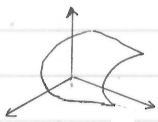
($\Leftarrow$). Conversely, suppose that $\phi(x,y)$ is a solution to PDE, then define vector field $\begin{pmatrix} V \\ W \end{pmatrix} = \begin{pmatrix} A(x,y,\phi(x,y)) \\ B(x,y,\phi(x,y)) \end{pmatrix}$, and find its integral curves. $(x(t), y(t))$.

i.e. $\dot{x} = A(x,y,\phi(x,y))$, $\dot{y} = B(x,y,\phi(x,y))$. Define $z(t) = \phi(x(t), y(t))$. We can check:

$\dot{z} = \dot{x}\frac{\partial\phi}{\partial x} + \dot{y}\frac{\partial\phi}{\partial y} = A\frac{\partial\phi}{\partial x} + B\frac{\partial\phi}{\partial y} = -C$ $\because \phi$ is a solution $\Rightarrow$ $(x(t), y(t), z(t))$ solves this equation, and is a characteristic curve.

The main problem with this ~~theorem~~ is the problem of __caustics__: the issue that $\bigcup_{s,t}(x_s(t), y_s(t), z_s(t))$ is not usually a graph.

For instance, we see that the solution can "foldover itself" at some points, which are called caustics.



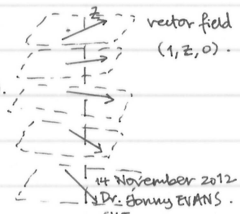
__Ex__ (Burgers equation). Solve $\frac{\partial\phi}{\partial x} + \phi\frac{\partial\phi}{\partial y} = 0$, given $\phi(0,y) = y$; and find any singularities.

Soln. Let $\phi = z$. Then $\dot{x} = 1$, $\dot{y} = z$, $\dot{z} = 0$ $\Rightarrow$ $z = M$, $x = t$, $y = Mt + N$. (const of integration vanishes by reparametrising WLOG.)

$\because t = 0$, $y = Mt + N = N$, $M = z = y = N$.

From initial condition, when $x = 0$, $t = 0$ $\Rightarrow$ $\phi(0,y) = z = y \Rightarrow N = M$. Then solution surface is $(M,t) \mapsto (t, Mt+M, M)$. (this is to help pick out a 1-parameter family).

Then $M \overset{z=}{=} \frac{y}{t+1} = \frac{y}{x+1} = \phi(x,y)$. // This solution has a singularity at $x = 1$ //



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We continue with 1st order quasilinear equations. To refresh our memory, consider the example $x\phi_x + \phi_y = y$. Then $\binom{x}{1}$ is our characteristic vector field.

We seek solutions to $\begin{cases} \dot{x} = x \\ \dot{y} = 1 \end{cases}$. Having found this, restrict ϕ to this curve $u(t) = \phi(x(t), y(t))$. By chain rule, $\dot{u} = \dot{x}\frac{\partial\phi}{\partial x} + \dot{y}\frac{\partial\phi}{\partial y} = x\phi_x + \phi_y$, so using original eqn, $\dot{u} = y(t)$

$\dot{y} = 1 \Rightarrow y(t) = t + M = t$ (WLOG), $\dot{x} = x \Rightarrow x = Ne^t$, so $\binom{x(t)}{y(t)} = \binom{Ne^t}{t}$. $\dot{u} = y(t) = t \Rightarrow u = \frac{t^2}{2} + c$, where c depends on $N = \frac{x}{e^t} = \frac{x}{e^y}$, so $\phi(x,y) = \frac{y^2}{2} + c\left(\frac{x}{e^y}\right)$ //.

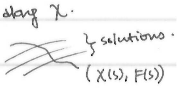
__Generalised Quasilinear case__.

These take the form $A(x,y,\phi)\frac{\partial\phi}{\partial x} + B(x,y,\phi)\frac{\partial\phi}{\partial y} + C(x,y,\phi) = 0$ ⊛. Let $\phi = z$, then as before, $\dot{x} = A(x,y,z)$, $\dot{y} = B(x,y,z)$, $\dot{z} = -C(x,y,z)$.

Return to our example of the Burgers equation. We used an initial condition to pick out our 1-parameter family $\Rightarrow$ addressing the __Cauchy problem__.

Cauchy problem: Given a curve $\chi : \overset{(s)}{\mathbb{R}} \to \mathbb{R}^2$ and a function $F : \mathbb{R} \to \mathbb{R}$, find a solution ϕ to PDE ⊛ s.t. $\phi \circ \chi = F$ i.e. ϕ agrees with F along χ.

e.g. $\chi(s) = (0,s)$, $F(s) = s$. Then Cauchy problem asks for $\phi(0,s) = s$ i.e. $\phi(0,y) = y$. This is an initial condition for ⊛.



In general, given Cauchy data χ, F, we get $\mathbb{R} \to \mathbb{R}^3$, $s \mapsto (\chi(s), F(s))$. Then solution to ⊛ is the union of characteristic curves passing through $(\chi(s), F(s))$.

__Ex__ Solve Burgers equation $\phi_x + \phi\phi_y = 0$, but now with initial condition $\phi(0,s) = s^2$.

Soln. $x = t$, $y = Et + F$, $z = E$. $\phi(0,s) = s^2 \Rightarrow x = t = 0$, $y = Et + F = s$, $z = E = s^2$, so $s^2 = E$ and $s = F$. Then the curves we require are

$x_s(t) = t$, $y_s(t) = s^2t + s$, $z_s(t) = s^2$. Then $(s,t) \mapsto (t, s^2t + s, s^2)$. This is the solution surface, which in this case does not form a graph.

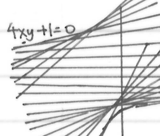
Then $z = s^2 = \frac{y-s}{t} = \frac{y-s}{x} \Rightarrow s^2x + s - y = 0 \Rightarrow s = \frac{-1 \pm \sqrt{1+4xy}}{2x}$. Since $z = s^2$, $z = \left(\frac{-1 \pm \sqrt{1+4xy}}{2x}\right)^2$.

__Definition__ If $(x_s(t), y_s(t), z_s(t))$ is a particular characteristic curve (i.e. s fixed), then $(x_s(t), y_s(t)) \subseteq \mathbb{R}^2$ is a __characteristic projection__.

We generally run into problems where characteristic projections cross.

Question: where do solution surfaces fold over themselves? Recall our earlier example, $(s,t) \mapsto (t, s^2t + s, s^2)$. $s^2t + s = y$ and $t = x \Rightarrow s^2x + s - y = 0$.

This is quadratic, and we solved earlier to get $z = \left(\frac{1 \pm \sqrt{1+4xy}}{2x}\right)^2$. So something goes wrong along $4xy + 1 = 0$. Where $4xy + 1 < 0$, z does not make sense. Thus $4xy + 1$ is the boundary of our characteristic projection.



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Last time we saw that

1. $(s,t) \mapsto (t, st+s, s)$
2. $(s,t) \mapsto (t, s^2t+s, s^2)$

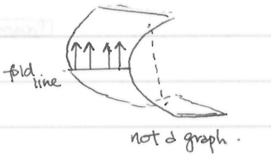
} are parametrisations for solution surfaces to $\phi_x + \phi\phi_y = 0$ with Cauchy data $\phi(0,s) = \{^s_{s^2}$ respectively.

In other words, 1. $\{(t, st+s, s) : t,s \in \mathbb{R}\} = \{(x,y,\phi(x,y)) : x,y \in \mathbb{R}\}$ for some solution ϕ. Likewise, 2. $\{(t, s^2t+s) : t,s \in \mathbb{R}\} = \{(x,y,\phi(x,y)) : x,y \in \mathbb{R}\}$.

i.e. $(t, st+s, s) = (x, y, \phi(x,y))$ — if we can express the third coordinate in terms of the other two, we get ϕ, e.g. $s = \frac{st+s}{t+1} = \frac{y}{x+1}$, $\phi(x,y) = \frac{y}{x+1}$

In other example, we get $\phi(x,y) = \left(\frac{-1 \pm \sqrt{1+4xy}}{2x}\right)^2$. We want to see where solutions develop singularities or go multivalued just from the parametrisation. (i.e. not a graph).

Consider the solution on the right, which is not a graph as it folds over itself. We can see that this is because it has a *fold line*.



Definition A *fold line* of a solution surface is the set of points in $\mathbb{R}^3$ where the surface has a vertical tangency.

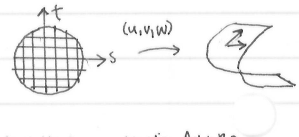
The image of the fold line under the projection to (x,y)-coordinates is called the *caustic*.

Lemma A parametrised surface $(s,t) \mapsto (u(s,t), v(s,t), w(s,t))$ has a vertical tangency if $\det\begin{pmatrix} \partial u/\partial s & \partial u/\partial t \\ \partial v/\partial s & \partial v/\partial t \end{pmatrix} = 0$. [but not the other way around].

e.g. Consider the mapping $(s,t) \mapsto (t, st+s, s)$. Recall that this is the graph of $\phi(x,y) = \frac{y}{x+1}$, so we already know that trouble occurs at $x=-1$.

using our lemma, $\det\begin{pmatrix} \partial u/\partial s = 0 & \partial u/\partial t = 1 \\ \partial v/\partial s = t+1 & \partial v/\partial t = s \end{pmatrix} = -(t+1)$. Where $\det J = 0$, $t=-1$. Since $u=t$, $x=-1$.

Proof — consider the ball in $\mathbb{R}^3$, mapped by (u,v,w) to a curved surface.



A tangent vector t,s in domain is transformed into tangent vectors $\begin{pmatrix} \partial u/\partial t \\ \partial v/\partial t \\ \partial w/\partial t \end{pmatrix}, \begin{pmatrix} \partial u/\partial s \\ \partial v/\partial s \\ \partial w/\partial s \end{pmatrix}$.

$\alpha \mapsto \left(\frac{\partial u}{\partial s}, \frac{\partial v}{\partial s}\right)$, $\beta \mapsto \left(\frac{\partial u}{\partial t}, \frac{\partial v}{\partial x}\right)$ for a projection of solution onto (x,y)-plane. So if a tangent vector is a linear combination $A\alpha + B\beta$,

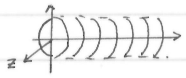
a linear combination of $A\begin{pmatrix} \partial u/\partial s \\ \partial v/\partial s \end{pmatrix} + B\begin{pmatrix} \partial u/\partial t \\ \partial v/\partial t \end{pmatrix} = 0 \Rightarrow \begin{pmatrix} \partial u/\partial s \\ \partial v/\partial s \end{pmatrix}$ and $\begin{pmatrix} \partial u/\partial t \\ \partial v/\partial t \end{pmatrix}$ are linearly dependent $\Rightarrow \det J = 0$ // q.e.d.

Ex Find the caustics of mapping $(s,t) \mapsto (t, s^2t+s, s^2)$.

Soln. then $\det\begin{pmatrix} \partial u/\partial s & \partial u/\partial t \\ \partial v/\partial s & \partial v/\partial t \end{pmatrix} = \det\begin{pmatrix} 0 & 1 \\ 2st+1 & s^2 \end{pmatrix} = -2st-1 = 0 \Rightarrow s = -\frac{1}{2t}$. Hence, we have a caustic at $(u(-\frac{1}{2t},t), v(-\frac{1}{2t},t), w(-\frac{1}{2t},t))$.

substituting, we get $(t, \frac{t}{4t^2} - \frac{1}{2t}, \frac{1}{4t^2}) \Rightarrow \frac{1}{4t} - \frac{1}{2t} = -\frac{1}{4t} = y \Rightarrow xy = -\frac{1}{4} \Rightarrow 4xy+1 = 0$ //.

Ex Consider the function $-\sin\phi \frac{\partial\phi}{\partial x} + \cos\phi \frac{\partial\phi}{\partial y} = 1$, initial condition $\phi(s,0) = 0$. (i.e. $\chi(s) = (s,0)$ ← Cauchy hypersurface, $F(s) = 0$). Solve and find caustics.



Soln. characteristic vector field: $\dot{x} = -\sin z$, $\dot{y} = \cos z$, $\dot{z} = 1$. $\Rightarrow$ $z = t$ (reparametrise to remove const.) $\Rightarrow$ $\dot{x} = -\sin t$, $\dot{y} = \cos t$

$x = \cos t + M$, $y = \sin t + N$. $\Rightarrow$ characteristic curves $(x(t), y(t), z(t)) = (\cos t + M, \sin t + N, t)$ are helices.

From initial condition: $\phi(s,0) = 0$. We aim to get M and N in terms of s. $\cos t + M = s$, $\sin t + N = 0$, $t = 0 \Rightarrow N = 0, M = s-1$.

$\Rightarrow$ solution surface is parametrised by $(s,t) \mapsto (s-1+\cos t, \sin t, t)$? $\phi = \arcsin y$. Note that $y = \sin t$, which is always bounded. So we could plausibly

guess that the caustic is $y = \pm 1$. $\det\begin{pmatrix} \partial u/\partial s & \partial u/\partial t \\ \partial v/\partial s & \partial v/\partial t \end{pmatrix} = \det\begin{pmatrix} 1 & -\sin t \\ 0 & \cos t \end{pmatrix} = \cos t = 0 \Rightarrow \cos^2 t = 0 \Rightarrow \sin^2 t = 1 \Rightarrow y^2 = 1 \Rightarrow y = \pm 1$ //.

Ex Solve $y\phi^2 \frac{\partial\phi}{\partial x} - x\frac{\partial\phi}{\partial y} = 0$; with initial condition $\phi(s,0) = s$.

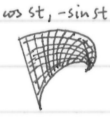
Soln. $\dot{x} = yz^2$, $\dot{y} = -x$, $\dot{z} = 0 \Rightarrow z = k$, $\dot{x} = k^2 y$, $\dot{y} = -x \Rightarrow \ddot{x} = k^2\dot{y} = -k^2 x$; $\ddot{y} = -\dot{x} = -k^2 y$. $\Rightarrow$ $x = M\cos kt + N\sin kt$, $y = \frac{\dot{x}}{k^2} = \frac{-Mk\sin kt + Nk\cos kt}{k^2}$.

$\therefore$ $y = -\frac{M}{k}\sin kt + \frac{N}{k}\cos kt$. WLOG, reparametrising allows us to assume $N = 0 \Rightarrow x = M\cos kt$, $y = -\frac{M}{k}\sin kt$; $z = k$. are characteristic curves

The characteristic projections $(x,y) = (M\cos kt, -\frac{M}{k}\sin kt)$ are ellipses: $\left(\frac{x}{M}\right)^2 + \left(\frac{ky}{M}\right)^2 = 1 \Rightarrow x^2 + (ky)^2 = M^2$. From initial conditions:

$M\cos kt = s$, $\frac{M}{k}\sin kt = 0$, $s = k$, $\sin kt = 0 \Rightarrow t = 0$, $M\cos(k\cdot 0) = M = s = k$. Hence our surface of solution is $(s,t) \mapsto (s\cos st, -\sin st, s)$.

then $\phi(x,y) = s$. $\left(\frac{x}{s}\right)^2 + y^2 = 1 \Rightarrow s^2 = \frac{x^2}{1-y^2} \Rightarrow \phi = \frac{x}{\sqrt{1-y^2}}$ //. Compute caustics: $\det\begin{pmatrix} \partial u/\partial s & \partial u/\partial t \\ \partial v/\partial s & \partial v/\partial t \end{pmatrix} = 0$.

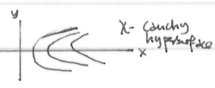


$\Rightarrow \det\begin{pmatrix} \cos(st) - st\sin(st) & -s^2\sin(st) \\ -t\cos(st) & -s\cos(st) \end{pmatrix} = -s\cos^2(st) + s^2 t\sin(st)\cos(st) - s^2 t\sin(st)\cos(st) = -s\cos^2(st) = 0$.

$\Rightarrow s = 0$ or $st = \frac{\pi}{2} + n\pi$, $n \in \mathbb{Z}$. $s=0$ does not arise from vertical tangencies, however. Along $s=0$, $(0,t) \mapsto (0,0,0) \Rightarrow$ problem along surface.

For $st = (n+\frac{1}{2})\pi$. $\Rightarrow x = s\cos[(n+\frac{1}{2})\pi] = s(0) = 0$, $y = -\sin[(n+\frac{1}{2})\pi] = (-1)^{n+1} \Rightarrow$ caustic is pair of points $(x,y) = (0,1), (0,-1)$ //.

Ex Solve $\phi\frac{\partial\phi}{\partial x} + \frac{\partial\phi}{\partial y} = y^2$; with initial condition $\phi(\cos s, \sin s) = s$, for $s \in (0, \frac{\pi}{2})$.



$\dot{x} = z$, $\dot{y} = 1$, $\dot{z} = y^2 \Rightarrow y = t$, $\dot{z} = y^2 = t^2 \Rightarrow z = \frac{1}{3}t^3 + M \Rightarrow x = \frac{1}{12}t^4 + Mt + N$. Thus for initial condition, on Cauchy hypersurface,

$\frac{t^4}{12} + Mt + N = \cos s$, $t = \sin s$, $\frac{t^3}{3} + M = s$. $s = \frac{t^3}{3} + M = \frac{\sin^3 s}{3} + M \Rightarrow M = s - \frac{\sin^3 s}{3}$, $\frac{\sin^4 s}{12} + (s - \frac{\sin^3 s}{3})\sin s + N = \cos s$.

$\Rightarrow N = \cos s - \frac{\sin^4 s}{12} + s\sin s - \frac{\sin^4 s}{3} = \cos s - s\sin s - \frac{\sin^4 s}{4}$. Then $(s,t) \mapsto \left(\frac{t^4}{12} + (s - \frac{\sin^3 s}{3})t + \cos s + \frac{\sin^4 s}{4} - s\sin s, t, \frac{t^3}{3} + s - \frac{\sin^3 s}{3}\right)$.

Note: General algorithm is as follows. • Write down characteristic vector field • solve for char curves • impose initial condition • compute constants for characteristic curves in terms of s.

• obtain parametrisation of surface • compute ϕ (find caustics).

This gives us a general notion of approach.

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2^{nd} order linear PDEs with constant coefficients.

Examples of such equations include the <u>heat equation</u>: $\phi(x,t)$ = temperature distribution on an interval $x \in [0,L]$, t = time $\in [0,\infty)$. Then $\frac{\partial \phi}{\partial t} = \frac{\partial^2 \phi}{\partial x^2}$.

We will need boundary conditions both in x and t (conditions in t are often called initial conditions. $\phi(x,0)$ = temp distribution at $t=0$, for instance we impose the ① <u>Dirichlet boundary conditions</u>: $\phi(0,t) = S_0$, $\phi(L,t) = S_1$. or we can set the ② <u>von Neumann boundary conditions</u>: $\frac{\partial \phi}{\partial x}(0,t) = \frac{\partial \phi}{\partial x}(L,t) = 0$.

Physically, Dirichlet conditions fix the temperature at the boundary, whereas von Neumann conditions insulate the boundary. Heat flows down a temperature gradient, so $\frac{\partial \phi}{\partial x}(0,t) = 0$ means that no heat flows out of $x=0$.

The strategy for solving the heat equation was outlined by Fourier in 1822 (analytic theory):

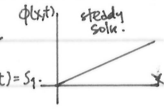
Step 1 – Find many special solutions ϕ_n, and try to fit them to boundary conditions but not the initial conditions.

Step 2 – Use linearity of heat equation to take linear combinations: $\phi = \sum A_n \phi_n$, $A_n \in \mathbb{R}$. Since heat equation is linear, ϕ is still a solution (assuming $\sum A_n \phi_n$ converges uniformly). We try to pick A_n s.t. ϕ satisfies initial conditions $\phi(x,0)$.

We look for solutions of the separable form $\phi(x,t) = X(x)T(t)$ (we call these separated solutions). Assuming this form of solution, equation becomes $\frac{\partial \phi}{\partial t} = \frac{\partial^2 \phi}{\partial x^2} \Rightarrow X(x)T'(t) = X''(x)T(t)$ —— ⊛.

Divide ⊛ by $XT \Rightarrow \frac{X\frac{dT}{dt}}{XT} = \frac{\frac{d^2X}{dx^2}T}{XT} \Rightarrow \frac{1}{T}\frac{dT}{dt} = \frac{1}{X}\frac{d^2X}{dx^2}$. LHS only depends on t, RHS only depends on $x \Rightarrow$ equality implies no dependence on x or t, so $\frac{dT/dt}{T} = \lambda = \frac{d^2X/dx^2}{X}$. We call λ the "constant of separation". Then we solve $\frac{T'}{T} = \lambda$ and $\frac{X''}{X} = \lambda$ separately. Hence, $T(t) = T = e^{\lambda t}$ (ignore constant), and $X'' - \lambda X = 0 \Rightarrow$ solutions for $X(x)$ depend on sign of λ. $X(x) = \begin{cases} A\cos(x\sqrt{-\lambda}) + B\sin(x\sqrt{-\lambda}) & \lambda<0 \\ Ax+B & \lambda=0 \\ A\cosh(x\sqrt{\lambda}) + B\sinh(x\sqrt{\lambda}) & \lambda>0 \end{cases} \Rightarrow$ we know what ϕ is.

$\phi(x,t) = X(x)T(t) = X(x)e^{\lambda t}$ where X is as defined above. Let's think about these solutions further. Where $\lambda=0$, $\phi(x,t) = Ax+B$. This is called the <u>steady solution</u>; which has no t-dependence. Where $\lambda>0$, $T = e^{\lambda t}$ grows exponentially in time. We will see shortly that this soln can be ignored. Finally, for $\lambda<0$, we have exponential decay. Recall that our Dirichlet boundary conditions are $\phi(0,t) = S_0$, $\phi(L,t) = S_1$.



Then the straight line $\frac{S_1 - S_0}{L}x + S_0 = \phi_0(t)$ is a solution satisfying the Dirichlet conditions. Suppose I have $\phi(x,t)$ that solves the heat equation and $\phi(x,0) = F(x)$ specified. Then $\theta(x,t) = \phi(x,t) - \phi_0(x,t)$ is also a solution of the heat equation. Then $\theta(x,0) = F(x) - \frac{S_1 - S_0}{L}x - S_0$. But $\theta(0,t) = \phi(0,t) - \phi_0(0,t) = 0$; and $\theta(L,t) = 0$, by the same logic. $\Rightarrow$ WLOG, $S_0 = S_1 = 0$. Suppose that $X(0) = 0$ and $X(L) = 0$. $\Leftrightarrow$ $\phi(0,t) = \phi(L,t) = 0$. Then if $\lambda>0$, $X = A\cosh(x\sqrt{\lambda}) + B\cosh(x\sqrt{\lambda}) \Rightarrow X(0) = A = 0$, $X(L) = B\sinh(L\sqrt{\lambda}) = 0 \Rightarrow B = 0$. i.e. only such solutions which satisfy $\lambda>0$ are where $A=B=0 \Rightarrow$ only has constant solution, reduces. When $\lambda>0$, there are no separated solutions that work. However, we try $X(0)=0$, $X(L)=0$ for $\lambda<0$, i.e. $X = A\cos(x\sqrt{-\lambda}) + B\cos(x\sqrt{-\lambda})$. $\Rightarrow X(0) = A\cos(0) = A = 0$, $X(L) = B\sin(L\sqrt{-\lambda}) = 0$. We can require $L\sqrt{-\lambda} = n\pi$ for some $n \in \mathbb{Z}$. $\Rightarrow$ this gives a discrete set of solutions $L\sqrt{-\lambda} = n\pi \Rightarrow \lambda_n = -\frac{n^2\pi^2}{L^2}$. Then solutions are $X_n(x) = B_n \sin\left(\frac{n\pi x}{L}\right)$, while $T_n(t) = e^{\lambda_n t} = e^{-\frac{n^2\pi^2}{L^2}t}$. Thus, $\phi_n(x,t) = B_n e^{-\frac{n^2\pi^2}{L^2}t}\sin\left(\frac{n\pi x}{L}\right)$. Then, we fit this to our initial condition. Set $\phi(x,t) = \sum_{n=1}^{\infty} A_n \phi_n$ for some $A_n \in \mathbb{R}$, and solve $\phi(x,0) = F(x) = \sum A_n \sin\left(\frac{n\pi x}{L}\right)$, and if we set $A_n = \frac{1}{L}\int_{-L}^{L} \tilde{F}(x)\sin\frac{n\pi x}{L}\,dx$ (half-range) we get a solution.

(Note: Here, we extend odd function F to $\tilde{F}$, such that $\tilde{F}(x) = \begin{cases} F(x) & x>0 \\ -F(x) & x<0 \end{cases}$.) $\Rightarrow \phi(x,t) = \sum A_n e^{-\frac{n^2\pi^2}{L^2}t}\sin\left(\frac{n\pi x}{L}\right)$. Letting S_0, S_1 vary again, $\phi(x,t) = \left[\sum A_n e^{-\frac{n^2\pi^2}{L^2}t}\sin\left(\frac{n\pi x}{L}\right)\right] + \frac{S_1 - S_0}{L}x + S_0$ solves $F(x) = \phi(x,0)$, $\phi(0,t) = S_0$, $\phi(L,t) = S_1$. When n is large, the Fourier mode $\sin\frac{n\pi x}{L}$ has a lot of gradient, but is suppressed by $e^{-\frac{n^2\pi^2}{L^2}t}$, because heat flows more quickly to rectify the gradient.

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In our example, we take $F(x) = \cos x + \frac{2\pi}{x} - 1$ to an odd function on $[-\pi,\pi]$. $\tilde{F}(x) = \begin{cases} \cos x + 2x/\pi - 1 & x \in [0,\pi] \\ -\cos x + 2x/\pi + 1 & x \in [-\pi, 0] \end{cases}$.

Then $A_n = \frac{1}{\pi}\int_{-\pi}^{\pi} \tilde{F}(x)\sin nx\,dx$. Perform this integral: $\phi(x,t) = \theta(x,t) + \phi_0(x,t) = 1 - \frac{2x}{\pi} + \sum A_n e^{-n^2 t}\sin(nx)$.

– comments: Higher Fourier modes (large n) decay faster in time than lower modes – because heat flows down a gradient.

When $t \to 0$, all $e^{-n^2 t} \to 0$, solution $\phi(x,t) \to 1 - \frac{2x}{\pi}$ tends to steady temperature distribution.

Laplace's Equation:

Steady temperature distribution on interval are just linear functions $Ax+B$. With a solution in 2D (i.e. coordinates x,y), $\frac{\partial \phi}{\partial t} = \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = \Delta \phi$

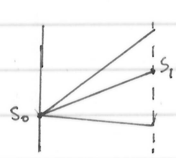
We call $\Delta\phi$ the Laplacian. A steady temperature distribution has no t-dependence $\Rightarrow \frac{\partial \phi}{\partial t} = 0$, $\Rightarrow \Delta\phi = 0$ i.e. $\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0$.

Also, we have seen that $\Delta\phi = 0$ is the Euler-Lagrange equation for $\phi \mapsto \iint \left[\left(\frac{\partial \phi}{\partial x}\right)^2 + \left(\frac{\partial \phi}{\partial y}\right)^2\right] dx\,dy$, which is the Dirichlet energy measuring total gradient of temperature distribution.

<u>Definition</u> If $\Delta\phi = 0$, we say that ϕ is a steady temperature distribution, or ϕ is harmonic.

Harmonic functions on $[a,b]$ are just linear $Ax+B$, but we use it for intuition:

→ Linear function depends heavily on S_0, S_1, and if you change either, the whole function changes.



→ Mean Value Property of Harmonic Functions: If F is linear, then $F\left(\frac{x+y}{2}\right) = \frac{1}{2}\left(F(x) + F(y)\right)$.

Theorem (Mean Value Principle, 2D).
If $F: U \to \mathbb{R}$ is harmonic, $U \subseteq \mathbb{R}^2$ is a domain, then $F(x,y) = \dfrac{\int_0^{2\pi} F(x+r\cos\theta,\ y+r\sin\theta)\, r\, d\theta}{\int_0^{2\pi} r\, d\theta}$ i.e. average of F over a small circle centred at x.

Proof – omitted.

Aim: Uniqueness of solutions to $\Delta\phi = 0$ with given boundary data.

Corollary (Maximum Principle)

If $F: U \to \mathbb{R}$ and $\Delta F = 0$, then F takes on its maximum and minimum along ∂U, and if F achieves its maximum in the interior of U, then F is constant.

Proof – suppose for contradiction that F achieves its maximum in interior and is not constant at $x \in U$.

$\Rightarrow \exists y \in U$ st. $F(x) \neq F(y)$ i.e. $F(x) > F(y)$. This is a sequence of discs $D(z_k, r_k)$. Then $z_{k+1} \in \partial D(z_k, r_k)$. (i.e. lies on boundary).

By Mean Value Principle, $\overset{max}{F(x)} = \dfrac{\int F(x+r_0 e^{i\theta})}{\int r_0\, d\theta} \leq F(x)\dfrac{\int r\, d\theta}{\int r\, d\theta}$. Inequality comes from fact that on $\partial(r_0, d_0)$, $F(x+r_0 e^{i\theta}) \leq F(x)$ $\because$ x is max.

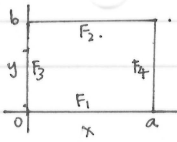
$F(z_1) = F(x)$ is maximal. Equality holds $\iff F(x+r_0 e^{i\theta}) = F(x)\ \forall\theta \Rightarrow F(x+r_0 e^{i\theta}) \equiv F(x)$. Iterate this argument for y until we get to y, thus $F(x) = F(y)$. contradiction, q.e.d.

Corollary (Uniqueness of solutions to Laplace's Equation)

If $\phi_1, \phi_2: U \to \mathbb{R}$ satisfy $\Delta\phi_1 = \Delta\phi_2 = 0$, and if $\phi_1|_{\partial U} = \phi_2|_{\partial U}$, then $\phi_1 = \phi_2$.

Proof – consider $\theta = \phi_1 - \phi_2$. By linearity of Laplace's equation, $\Delta\theta = 0$. But $\theta|_{\partial U} = \phi_1|_{\partial U} - \phi_2|_{\partial U} = 0 \Rightarrow$ by maximum principle, the maximum of θ and minimum of θ are achieved on $\partial U \Rightarrow \max\theta = 0 = \min\theta \Rightarrow \theta = 0 \Rightarrow \phi_1 = \phi_2$, q.e.d.

Laplace: Separation of Variables.



• Boundary conditions – $\Delta\phi = 0$ on $U = [0,a]\times[0,b]$. We impose Dirichlet conditions: $\phi(x,0) = F_1(x)$, $\phi(x,b) = F_2(x)$, $\phi(0,y) = F_3(y)$, $\phi(a,y) = F_4(y)$.

similar to heat equation, we can have von Neumann boundary conditions: $\frac{\partial\phi}{\partial y}(x,0) = \frac{\partial\phi}{\partial x}(0,y) = \frac{\partial\phi}{\partial y}(x,b) = \frac{\partial\phi}{\partial y}(a,y) = 0$.

(With Dirichlet conditions)
Separated solutions: $\phi(x,y) = X(x)Y(y)$, so $\Delta\phi = 0 \Rightarrow X''Y + XY'' = 0$ [here both $\frac{d}{dx}$ and $\frac{d}{dy}$ are denoted by primes].

$\therefore \frac{X''Y}{XY} + \frac{XY''}{XY} = 0 \Rightarrow \frac{X''}{X} + \frac{Y''}{Y} = 0 \Rightarrow \frac{X''}{X} = -\frac{Y''}{Y} = \lambda \Rightarrow X'' = \lambda X,\ Y'' = -\lambda Y$. We have three cases for solution, based on values of λ:

$\lambda < 0$; $X = A\cos(x\sqrt{-\lambda}) + B\sin x\sqrt{-\lambda}$, $Y = C\cosh(y\sqrt{-\lambda}) + D\sinh(y\sqrt{-\lambda})$. If $\lambda = 0$, $X = Ax + B$, $Y = Cy + D$.

$\lambda > 0$, $X = A\cosh(x\sqrt{\lambda}) + B\sinh(x\sqrt{\lambda})$, $Y = C\cos(y\sqrt{\lambda}) + D\sin(y\sqrt{\lambda})$.

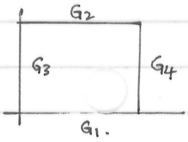
As before, start with $\lambda = 0$. This helps us to fit ϕ to the boundary data at the corners of U. Then Dirichlet conditions give

• $S_{0,0} = \phi(0,0) = F_1(0) = F_3(0)$ • $S_{0,1} = \phi(0,b) = F_3(b) = F_2(0)$ • $S_{1,0} =$ • $S_{1,1} =$

We want $\phi_0(x,y) = (Ax+B)(Cy+D)$ st. ϕ agrees with ϕ_0 at corners of U. $\phi_0(0,0) = (0+B)(0+D) = BD = S_{0,0} = \phi(0,0)$, and analogous equations.

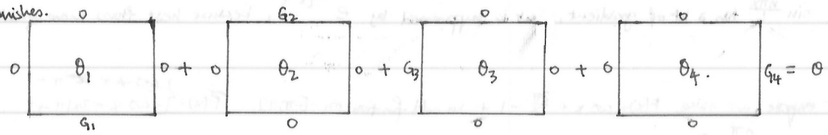
Finally, we get $\phi_0(x,y) = S_{0,0} + \frac{S_{1,0} - S_{0,0}}{a}x + \frac{S_{0,1} - S_{0,0}}{b}y + xy\frac{S_{1,1} - S_{1,0} - S_{0,1} + S_{0,0}}{ab}$. Now define $\theta(x,y) = \phi(x,y) - \phi_0(x,y)$. $\Delta\theta = 0$ by linearity.

But our boundary conditions have changed: let $G_1 = F_1 - S_{0,0} - \frac{S_{1,0} - S_{0,0}}{a}x$, $G_3 = F_3 - S_{0,0} - \frac{S_{0,1} - S_{0,0}}{b}y$, etc…



We can check to verify that $G_1(0) = 0$, $G_1(a) = 0$, …… i.e. θ vanishes at corners of U. i.e. WLOG, $S_{0,0} = S_{1,0} = S_{0,1} = S_{1,1} = 0$.

We split the problem of finding θ into four separate problems, and then add them up to get θ. We can do this because at corners, θ vanishes.



still solution to Laplace by linearity.

i.e. for instance, θ_1 is harmonic and has these boundary conditions. Solve for θ_1: $\theta_1(x,0) = 0 = \theta_1(x,b)$, $\theta_1(0,y) = G_1$, $\theta_1(a,y) = 0$.

So we want separated solutions $\Psi(x,y) = X(x)Y(y)$, such that $X(0) = X(a) = 0$, because then we will satisfy $\Psi(0,y) = \Psi(a,y) = 0$. and $X(0) = X(a) = 0$.

Then $X'' = \lambda X$, $X = A\cos(x\sqrt{-\lambda}) + B\sin(x\sqrt{-\lambda})$ for $\lambda < 0$ or $X = A\cosh(x\sqrt{\lambda}) + B\sinh(x\sqrt{\lambda})$ for $\lambda > 0$. Try $\lambda > 0$: then $X(0) = A = 0$,

$X(a) = B\sinh(a\sqrt{\lambda}) = 0 \Rightarrow B = 0 \Rightarrow$ no solutions, so we need $\lambda < 0$. $X(0) = 0 \Rightarrow A + B\sin 0 = 0 \Rightarrow A = 0$. $X(a) = B\sin(a\sqrt{-\lambda}) = 0$, we want B non-zero, thus $\sin(a\sqrt{-\lambda}) = 0 \Rightarrow a\sqrt{-\lambda} = n\pi$ for some $n \in \mathbb{Z}$, hence $\lambda = -\frac{n^2\pi^2}{a^2}$. So we have $\theta_1(x,y) = \sum_{n=1}^{\infty} \underbrace{\left(C_n \sinh\frac{n\pi y}{a} + D_n\cosh\frac{n\pi y}{a}\right)}_{Y(y)} \sin\frac{n\pi x}{a}$.

We generate solutions like these by taking linear combinations of infinitely many separate solutions. Fit $\theta(x,y)$ to $\theta_1(x,0) = G_1(x)$, $\theta_1(x,b) = 0$.

$\theta_1(x,0) = G_1(x) \Rightarrow \sum_{n=1}^{\infty}\left(C_n\sinh\frac{n\pi 0}{a} + D_n\right)\sin\frac{n\pi x}{a} = G_1(x) \Rightarrow$ If $\tilde{G}_1(x) = \begin{cases} G_1(x) & x \in [0,a] \\ -G_1(-x) & x \in [-a,0] \end{cases}$ then. And, $D_n = \frac{1}{a}\int_{-a}^{a}\tilde{G}_1(x)\sin\frac{n\pi x}{a}\,dx$, by

Fourier analysis. We still need C_n. $\theta_1(x,b) = 0 \Rightarrow \sum\left(C_n\sinh\frac{n\pi b}{a} + D_n\cosh\frac{n\pi b}{a}\right)\sin\frac{n\pi x}{a} = 0 \Rightarrow C_n\sinh\frac{n\pi b}{a} + D_n\cosh\frac{n\pi b}{a} = 0 \Rightarrow C_n = -D_n\coth\frac{n\pi b}{a}$.

$\Rightarrow \theta_1(x,y) = \sum D_n\left(-\coth\frac{n\pi b}{a}\sinh\frac{n\pi y}{a} + \cosh\frac{n\pi y}{a}\right)\sin\frac{n\pi x}{a} = \sum\frac{\left(\cosh\frac{n\pi y}{a}\sinh\frac{n\pi b}{a} - \cosh\frac{n\pi b}{a}\sinh\frac{n\pi y}{a}\right)}{\sinh\left(\frac{n\pi b}{a}\right)}D_n \cdot \sin\frac{n\pi x}{a}$ and

$= \sum D_n\underbrace{\frac{\sinh\left(\frac{n\pi}{a}(b-y)\right)}{\sinh\frac{n\pi b}{a}}}\sin\frac{n\pi x}{a}$. We consider analogously $\theta_2(x,y)$. θ_2: $0 = \sum D_n\sin\frac{n\pi x}{a}$ $\Rightarrow D_n = 0$, while also we have

$G_2(x) = \sum\left(C_n\sinh\frac{n\pi b}{a} + D_n\cosh\frac{n\pi b}{a}\right)\sin\frac{n\pi x}{a} = \sum C_n\sinh\frac{n\pi b}{a}\sin\frac{n\pi x}{a} \Rightarrow C_n\sinh\frac{n\pi b}{a}$ are the Fourier coefficients of $\tilde{G}_2$. We perform θ_3, θ_4 analogously.

This eventually gives us a solution for θ.

2401-014.

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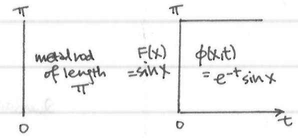
At this juncture, we pause to take stock of what we have done for the heat and Laplace equation, in a review. For this lecture, we simplify $\left.\begin{matrix}L\\a\\b\end{matrix}\right\} = \pi$.

Heat equation (review)

Consider a metal rod of length π. At time 0, $\phi = \sin x$ is an initial condition. Then $\phi(x,0) = \sin x$ at $t=0$. For this case,



$\phi = e^{-t}\sin x$ is a solution, because $\frac{\partial\phi}{\partial t} = -e^{-t}\sin x$, $\frac{\partial^2\phi}{\partial t^2} = -e^{-t}\sin x$. Likewise, if $\phi(x,0) = \sin nx$, $\phi(x,t) = e^{-n^2t}\sin(nx)$.

These are particular separated solutions to the heat equation. And if we take $\phi(x,t) = \sum A_n e^{-n^2t}\sin(nx)$ will be to a solution to initial condition $\phi(x,0) = \sum A_n \sin(nx)$ $\Rightarrow$ If we can express our initial condition $\phi(x,t)$ as a Fourier series $\sum A_n \sin(nx)$.

Then, it evolves over time as $\sum A_n e^{-n^2t}\sin(nx)$. For a broader example, let $\phi(x,0) = \cos x$, and set $\phi(0,t) = 1$ while $\phi(\pi,t) = -1$.

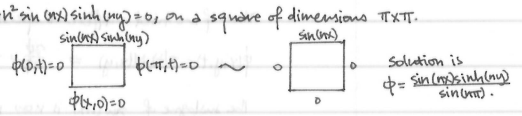
Reduce this to the previous problem. We define $\phi_0 = 1 - \frac{2x}{\pi}$. Consider $\theta = \phi - \phi_0$. Then $\theta(x,0) = \phi(x,0) - \phi_0(x,0) = \cos x - 1 + \frac{2x}{\pi} = \sum_{n=1}^{\infty}\frac{2}{\pi}\frac{1+(-1)^n}{n(n^2-1)}\sin(nx)$ by Fourier expansion.

$\Rightarrow \theta(x,t) = \sum_{n=1}^{\infty}\frac{2}{\pi}\frac{1+(-1)^n}{n(n^2-1)}e^{-n^2t}\sin(nx)$, and our solutions are $\phi(x,t) = \phi_0 + \theta = 1 - \frac{2x}{\pi} + \sum_{n=1}^{\infty}\frac{2}{\pi}\frac{1+(-1)^n}{n(n^2-1)}e^{-n^2t}\sin(nx)$ //.

Laplace equation (review)

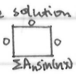
$\phi(x,y) = \sin(nx)\sinh(ny)$ is a general solution to our equation $\Delta\phi = \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} = -n^2\sin(nx)\sinh(ny) + n^2\sin(nx)\sinh(ny) = 0$, on a square of dimensions $\pi\times\pi$.

These ϕ satisfy $\phi(0,y) = \phi(\pi,y) = 0$, $\phi(x,0) = 0$; which settles the boundary conditions on three sides

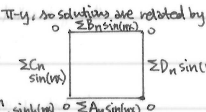


Then on the final sides (i.e. the top side), $\phi(x,\pi) = \sin(nx)\sinh(n\pi)$

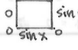
Hence, if $\phi(x,\pi) = \sum A_n \sin(nx)$ (Fourier expansion of boundary data), the solution is $\phi = \sum A_n \sin(nx)\frac{\sinh(ny)}{\sinh(n\pi)}$ //.

Now, assume we set the lower (rather than upper) boundary as $\sum A_n\sin(nx)$, i.e. flip to get  These 2 pictures are related by reflection $y \mapsto \pi - y$, so solutions are related by same transformation: $\phi = \sum A_n \sin(nx)\frac{\sinh n(\pi-y)}{\sinh(n\pi)}$.

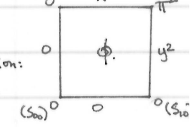
General case: corners are 0; but boundaries are non-zero, but instead series of sines: as shown in the diagram on the right

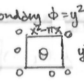


Solution is formed by linear combinations: $\phi(x,y) = \sum A_n\sin(nx)\frac{\sinh n(\pi-y)}{\sinh(n\pi)} + \sum B_n \sin(nx)\frac{\sinh(ny)}{\sinh(n\pi)} + \sum C_n\sin(ny)\frac{\sinh n(\pi-x)}{\sinh(n\pi)} + \sum D_n \sin(ny)\frac{\sinh(nx)}{\sinh(n\pi)}$.

For example, consider the case with . Then solution is $\phi = \sin x\frac{\sinh(\pi-y)}{\sinh\pi} + \sin y\frac{\sinh x}{\sinh\pi}$ //.

Even more generally, what happens if ϕ does not vanish at the corner? On the top boundary set $\phi = x^2$, right boundary $\phi = y^2$, with 0 on other boundaries. Suppose ϕ is a solution:

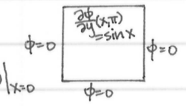


Let $\phi_0 = xy$ (comes from formula on last lecture), then indeed $\phi_0(\pi,\pi) = \pi^2$. Define $\theta = \phi - \phi_0$, then θ is a solution to . Corners all are set to 0.

Take Fourier expansions: $x^2 - \pi x = -\frac{4}{\pi}\sum_{k=1}^{\infty}\frac{[1+(-1)^k]}{k^3}\sin kx$, $y^2 - \pi y = -\frac{4}{\pi}\sum_{k=1}^{\infty}\frac{[1+(-1)^k]}{k^3}\sin ky$. Hence, we solve for θ then ϕ:

$\theta = -\frac{4}{\pi}\sum\frac{[1+(-1)^k]}{k^3}\sin(kx)\frac{\sinh(ky)}{\sinh(k\pi)} - \frac{4}{\pi}\sum\frac{[1+(-1)^k]}{k^3}\sin(ky)\frac{\sinh(kx)}{\sinh(k\pi)} = -\frac{4}{\pi}\sum_{k=1}^{\infty}\frac{1+(-1)^k}{k^3}\frac{\sin kx\sinh ky + \sinh kx \sin ky}{\sinh k\pi}$, so $\phi = xy - \frac{4}{\pi}\sum_{k=1}^{\infty}\frac{1+(-1)^k}{k^3}\frac{\sin kx \sinh ky + \sinh kx\sin ky}{\sinh k\pi}$

We can also set von Neumann boundary conditions.



Here, we consider a case with mixed boundary conditions as on right — these boundary conditions are consistent i.e. $\frac{\partial}{\partial y}[\phi(0,y)]\big|_{y=\pi} = 0 = \sin 0 = \frac{\partial\phi}{\partial y}(x,\pi)\big|_{x=0}$

Separate variables: $\phi = X(x)Y(y) \Rightarrow X''Y + XY'' = 0 \Rightarrow \frac{X''}{X} = -\frac{Y''}{Y} = 0$

x boundary conditions are still Dirichlet: $X(0) = 0 = X(\pi) \Rightarrow B_n = 0$, $\sqrt{-\lambda} = n$, $n\in\mathbb{Z} \Rightarrow X = A\sin(nx)$. Then $Y'' = -\lambda Y = n^2Y$, $Y = A_n\sinh(ny) + B_n\cosh(ny)$.

Then $Y(0) = 0 \Rightarrow B_n = 0$, $Y'(\pi) = \sin x$. Since $Y'(y) = A_n\cosh(ny)$, $Y'(\pi) = A_n\cosh(n\pi) = \sin x$. How is this possible?

Now we take $\phi(x,y) = \sum A_n \sin(nx)\sinh(ny)$. Hence, $\frac{\partial\phi}{\partial y}(x,\pi) = \sum A_n n\cosh(n\pi)\sin(nx) = \sin x \Rightarrow A_n = 0$ when $n>1$ and $A_1\cosh\pi = 1 \Rightarrow A_1 = \operatorname{sech}\pi$.

$\therefore$ solution is $\phi(x,y) = \sin x\sinh y\operatorname{sech}\pi = \frac{\sin x\sinh y}{\cosh\pi}$ // q.e.d.

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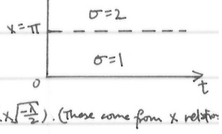
Ex Consider the heat equation with constant of thermal diffusivity σ. It has initial conditions $\phi(0,t) = \phi(2\pi,t) = 0$, with $\sigma(x) = \begin{cases}1 & x\in[0,\pi)\\ 2 & x\in(\pi,2\pi]\end{cases}$. Find continuous, separated solutions.

Soln. We want to have $\phi(x,t) = \begin{cases}X_1(x)T_1(t), & x\in[0,\pi)\\ X_2(x)T_2(t) & x\in(\pi,2\pi]\end{cases}$. Separating variables gives $XT' = \sigma X''T \Rightarrow X_1T_1' = X_1''T_1$, $X_2T_2' = 2X_2''T_2$



Thus, $\frac{X_1''}{X_1} = \frac{T_1'}{T_1} = \lambda$, $2\frac{X_2''}{X_2} = \frac{T_2'}{T_2} = \mu$. These equations give $T_1 = e^{\lambda t}$, $T_2 = e^{\mu t}$. If ϕ is to be continuous at $x=\pi$, then we have

$X_1(\pi)T_1(t) = X_2(\pi)T_2(t) \Rightarrow X_1(\pi)e^{\lambda t} = X_2(\pi)e^{\mu t}$. Unless $\lambda = \mu$, we have a linear relation between $e^{\lambda t}$ and $e^{\mu t}$, which is impossible.

Hence $\lambda = \mu$. As usual, ignore $\lambda \geq 0$, and concentrate on $X_1(x) = A_1\sin(x\sqrt{-\lambda}) + B_1\cos(x\sqrt{-\lambda})$, $X_2(x) = A_2\sin(x\sqrt{-\frac{\lambda}{2}}) + B_2\cos(x\sqrt{-\frac{\lambda}{2}})$. (These come from x relation

We now have to solve for $A_1, B_1, A_2, B_2, \lambda$. Using boundary conditions, $\phi(0,t) = 0 \Rightarrow X_1(0) = 0$, $X_1(x) = A_1\sin(x\sqrt{-\lambda}) + B_1\cos(x\sqrt{-\lambda})$. $X_1(0) = B_1 = 0$.

Also, $\phi(2\pi,t) = 0 \Rightarrow X_2(2\pi) = 0$, and $X_2(2\pi) = A_2\sin(2\pi\sqrt{-\frac{\lambda}{2}}) + B_2\cos(2\pi\sqrt{-\frac{\lambda}{2}}) = 0 \Rightarrow B_2 = -A_2\tan(2\pi\sqrt{-\frac{\lambda}{2}})$. ① To fix A_1 and λ, consider $\phi(\pi,t)$.

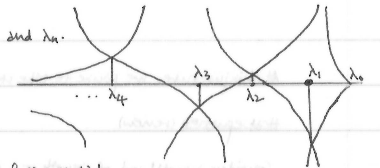
This can be written in two ways: $X_1(\pi)T_1(t)$ and $X_2(\pi)T_2(t)$. Since $T_1(t)$ and $T_2(t)$ are equal, we only have to consider x-components: For ϕ to be continuous at $x=\pi$, we need $X_1(\pi) = X_2(\pi)$. i.e. $A_1\sin(\pi\sqrt{-\lambda}) + B_1\cos(\pi\sqrt{-\lambda}) = A_2\sin(\pi\sqrt{-\frac{\lambda}{2}}) + B_2\cos(\pi\sqrt{-\frac{\lambda}{2}})$. ② We have to make a second assumption to solve for last constant: that $\frac{\partial\phi}{\partial x}$ is continuous at $x=\pi \Rightarrow X_1'(\pi) = \sqrt{-\lambda}A_1\cos(\pi\sqrt{-\lambda}) - \sqrt{-\lambda}B_1\sin(\pi\sqrt{-\lambda}) = X_2'(\pi) = \sqrt{-\frac{\lambda}{2}}A_2\cos(\pi\sqrt{-\frac{\lambda}{2}}) - \sqrt{-\frac{\lambda}{2}}B_2\sin(\pi\sqrt{-\frac{\lambda}{2}})$ ③

Using ① and $B_1 = 0$, ② becomes: $A_1\sin(\pi\sqrt{-\lambda}) = A_2\left(\sin(\pi\sqrt{-\frac{\lambda}{2}}) - \tan(2\pi\sqrt{-\frac{\lambda}{2}})\cos(\pi\sqrt{-\frac{\lambda}{2}})\right)$ ④ and ③ becomes: $\sqrt{-\lambda}A_1\cos(\pi\sqrt{-\lambda}) = \sqrt{-\frac{\lambda}{2}}A_2\left(\cos(\pi\sqrt{-\frac{\lambda}{2}}) + \tan(2\pi\sqrt{-\frac{\lambda}{2}})\sin(\pi\sqrt{-\frac{\lambda}{2}})\right)$ ⑤

Divide ④/⑤: $\frac{1}{\sqrt{-\lambda}}\tan(\pi\sqrt{-\lambda}) = \frac{1}{\sqrt{\lambda/2}}\frac{\sin(\pi\sqrt{-\frac{\lambda}{2}}) - \tan(2\pi\sqrt{-\frac{\lambda}{2}})\cos(\pi\sqrt{-\frac{\lambda}{2}})}{\cos(\pi\sqrt{-\lambda/2}) - \tan(2\pi\sqrt{-\lambda/2})\sin(\pi\sqrt{-\lambda/2})} = \frac{1}{\sqrt{\lambda/2}}\frac{\tan(\pi\sqrt{-\lambda/2}) - \tan(2\pi\sqrt{-\lambda/2})}{1 + \tan(2\pi\sqrt{-\lambda/2})\tan(\pi\sqrt{-\lambda/2})} = \frac{1}{\sqrt{-\lambda/2}}\tan\left((\pi-2\pi)\sqrt{-\frac{\lambda}{2}}\right) = -\frac{1}{\sqrt{-\lambda/2}}\tan(\pi\sqrt{-\frac{\lambda}{2}})$

Therefore, our solution set involves all λ where $\tan(\pi\sqrt{-\lambda}) = -\sqrt{2}\tan(\pi\sqrt{-\frac{\lambda}{2}})$. This has no closed form solution, as seen on graph overleaf.

It has a discrete set of solutions λ_n. We could substitute back and find A_2, B_2 in terms of A_1 and λ_n.



$\Rightarrow \phi_n$ has a discrete set of separated solutions

Note: this may not seem particularly useful, but often we just want to find values of λ_n.

e.g. For quantum particle "trapped" in a potential well, λ_n represents the possible energies of the particle.

Question: Why does $\sin \frac{n\pi x}{L}$ keep appearing? And if we looked at a different equation, what functions would crop up?

Define vector space Y over $\mathbb{R}$, with $Y = \{\phi : [-\pi, \pi] \to \mathbb{R} : \int_{-\pi}^{\pi} \phi^2 \, dx < \infty\}$ i.e. set of square integrable functions. Define the subspace $X \leq Y$, where $X = \{\phi \in Y : \phi(0) = \phi(\pi) = 0,\ \phi \text{ is odd}\}$. Take the linear map $\frac{d^2}{dx^2} : Y \to Y$, and restrict it to $\frac{d^2}{dx^2} : X \to Y$. Then we look for eigenvalues and eigenfunctions, i.e. we seek $\phi \in X$ s.t. $\frac{d^2\phi}{dx^2} = \lambda\phi$ for some λ. Since $\phi \in X \Rightarrow \phi$ is odd, and $\phi(\pi) = 0$, the solutions of $\frac{d^2\phi}{dx^2} = \lambda\phi$ are $\sin(nx)$, $\lambda = n^2 \Rightarrow$ eigenvalues of $\frac{d^2}{dx^2}$ are $-n^2$, and the corresponding eigenfunctions are $\sin(nx)$. Our next step in Fourier theory is to prove orthogonality, i.e. $\frac{1}{\pi}\int_{-\pi}^{\pi} \sin nx \sin mx = \begin{cases} 0 & m \neq n \\ 1 & m = n \end{cases}$ — ⊛. In our definition of Y, we required $\int_{-\pi}^{\pi} \phi^2 \, dx < \infty$. We can use this to define a dot product on Y, $\langle \phi, \phi \rangle = \frac{1}{\pi}\int_{-\pi}^{\pi} \phi \cdot \phi \, dx$. From ⊛, $\sin(nx)$ and $\sin(mx)$ are orthogonal in Y.

Now, if we tried a different operator, what would happen? For example, find $\phi(x,y,t)$ for modified heat/Laplace equation $\frac{\partial\phi}{\partial t} = \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2}$. We separate variables: $\phi(x,y,t) = T(t)M(x,y) \Rightarrow \frac{\partial\phi}{\partial t} = T'M = \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} = (\Delta M)T \Rightarrow \frac{T'}{T} = \frac{\Delta M}{M} = \lambda$, const. $\Rightarrow T = e^{\lambda t}$, $\Delta M = \lambda M$ i.e. Δ is a linear map between function spaces, and the analogue of $\sin(nx)$ is now the set of eigenfunctions of Δ (Helmholtz equation).

We need to be able to expand "arbitrary" functions as sums of eigenfunctions of Δ. This works in quite a lot of generality. For ordinary differential operators, this theory is called Sturm-Liouville Theory.

We look at an alternative proof for $\int_{-\pi}^{\pi} \sin nx \sin mx \, dx = 0$ if $m \neq n$. Consider $\int_{-\pi}^{\pi} \sin nx (\sin mx)'' \, dx \overset{IBP}{=} -\int_{-\pi}^{\pi} \frac{d}{dx}(\sin nx) \frac{d}{dx}(\sin mx) \, dx \overset{IBP}{=} \int (-n^2) \sin mx \sin nx \, dx$.
Hence, since $\int_{-\pi}^{\pi} (-m^2) \sin(mx) \sin(nx) \, dx$, if $m \neq n$, $\int_{-\pi}^{\pi} (n^2 - m^2) \sin nx \sin mx \, dx = 0 \Rightarrow \int_{-\pi}^{\pi} \sin(nx) \sin(mx) \, dx = 0 \because n^2 - m^2 \neq 0$.

In our new language, this means that $\langle \sin nx, \frac{d^2}{dx^2} \sin mx \rangle = \langle \frac{d^2}{dx^2} \sin nx, \sin mx \rangle$ i.e. if $v = \sin nx$, $w = \sin mx$, $A = \frac{d^2}{dx^2}$, then $v \cdot Aw = (Av) \cdot w$. i.e. $(v_1 \cdots)(A)\begin{pmatrix} w_1 \\ \vdots \end{pmatrix} = (v_1 \cdots)\left[(A^T)\begin{pmatrix} w_1 \\ \vdots \end{pmatrix}\right]$, and A is its own adjoint (symmetric), $A = A^T$ is a symmetric $\infty \times \infty$ matrix.

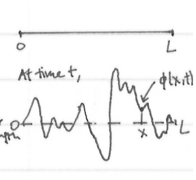
Theorem Let M be a symmetric matrix and v, w be λ, μ eigenvectors $(\lambda \neq \mu)$ i.e. $Mv = \lambda v$, $Mw = \mu w$ (finite dimensional). Then $v \cdot w = 0$

Proof – $v \cdot (Mw) = (Mv) \cdot w = \mu v \cdot w \Rightarrow (\lambda - \mu) v \cdot w = 0 \Rightarrow v \cdot w = 0$ unless $\lambda = \mu$, q.e.d.
$v \cdot (Mw) = \lambda(v \cdot w) \because Mw = \lambda w$

<u>Wave Equation</u>

The equation is given by $\frac{1}{c^2}\frac{\partial^2\phi}{\partial t^2} = \frac{\partial^2\phi}{\partial x^2}$, with c constant, $\phi(x,t)$ is a wavefunction.

Derivation: Let there be a string of length L, which is displaced from initial rest at time t_0. At time t, we get a graph of $\phi(x,t)$, where $\phi(x,t)$ is the displacement of string at time t above x. We claim that approximately (approximation), when ϕ and its derivatives are small, ϕ satisfies the wave equation. We first claim that the potential energy of the string obeys Hooke's law (by experiment), the potential energy of the string (i.e. lengthening of string) is approximately $\tau\int_0^L \left(\sqrt{1 + (\frac{\partial\phi}{\partial x})^2} - 1\right) dx$ (arclength; length of unperturbed string), where τ is a constant denoting tension per unit length.



Kinetic energy is $\frac{1}{2}\rho\int_0^L (\frac{\partial\phi}{\partial t})^2 \, dx$, where ρ is the mass by unit length and $\partial\phi/\partial t$ is the velocity of vertical displacement.

By Lagrange's formulation of mechanics, we know that a path/configuration will minimise the quantity KE−PE, i.e. ϕ will minimise
$\int_0^L \left[\frac{1}{2}\rho(\frac{\partial\phi}{\partial t})^2 - \tau\left[\sqrt{1 + (\partial\phi/\partial x)^2} - 1\right]\right] dx \overset{\text{Taylor expansion}}{=} \int_0^L \frac{1}{2}\rho(\frac{\partial\phi}{\partial t})^2 - \tau(1 - 1 + \frac{1}{2}(\frac{\partial\phi}{\partial x})^2 + \dots) \, dx = \int_0^L \frac{1}{2}\rho(\frac{\partial\phi}{\partial t})^2 - \frac{1}{2}\tau(\frac{\partial\phi}{\partial x})^2 \, dx$. We then use the expression $L = \frac{1}{2}\rho(\frac{\partial\phi}{\partial t})^2 - \frac{1}{2}\tau(\frac{\partial\phi}{\partial x})^2$ as the functional in the 2-variable Euler-Lagrange equation: $\frac{\partial L}{\partial\phi} = \frac{\partial}{\partial t}(\frac{\partial L}{\partial\phi_t}) + \frac{\partial}{\partial x}(\frac{\partial L}{\partial\phi_x})$, which gives us overall,
$0 = \rho\frac{\partial}{\partial t}(\frac{\partial\phi}{\partial t}) - \tau\frac{\partial}{\partial x}(\frac{\partial\phi}{\partial x}) = \rho\frac{\partial^2\phi}{\partial t^2} - \tau\frac{\partial^2\phi}{\partial x^2} \Rightarrow \frac{\rho}{\tau}\frac{\partial^2\phi}{\partial t^2} = \frac{\partial^2\phi}{\partial x^2}$, which gives us the wave equation if $c = \sqrt{\frac{\tau}{\rho}}$.

Among the major classes of solutions are <u>d'Alembert's solution</u> using $\frac{\partial^2}{\partial u \partial v} = 0$, which we will revisit, and <u>Fourier's solution</u> which we will now examine.

In Fourier's solution: We separate variables, $\phi(x,t) = X(x)T(t) \Rightarrow \frac{\partial^2\phi}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2\phi}{\partial t^2}$ i.e. $X''T = \frac{1}{c^2}XT''$. Dividing by XT, $\frac{X''}{X} = \frac{T''}{c^2 T} = \lambda$ const. Then this gives us $X'' = \lambda X$, $T'' = c^2\lambda T$. Now, to solve it, we impose some boundary conditions: fix our ends such that $\phi(0,t) = 0$, $\phi(L,t) = 0$

If $\lambda = 0$, then $X'' = 0$, $X = Ax + B \Rightarrow X(0) = B = 0$, $X(L) = AL = 0 \Rightarrow A = 0 \Rightarrow X = 0$ (trivial solution). If $\lambda > 0$, $X'' = \lambda X \Rightarrow X = A\sinh(x\sqrt{\lambda}) + B\cosh(x\sqrt{\lambda})$. Then $X(0) = B = 0$, so $X(x) = A\sinh(x\sqrt{\lambda})$ and $X(L) = 0 = A\sinh(L\sqrt{\lambda}) \Rightarrow A = 0$, again we have only the trivial solution. Thus, we need only consider $\lambda < 0$, so write $\lambda = -p^2$. $\Rightarrow X = A\sin px + B\cos px$, $T'' = \lambda c^2 T = -p^2c^2T \Rightarrow T = C\sin(pct) + D\cos(pct)$ i.e. solution is symmetric/oscillatory in time.
This is a key property of the wave equation.

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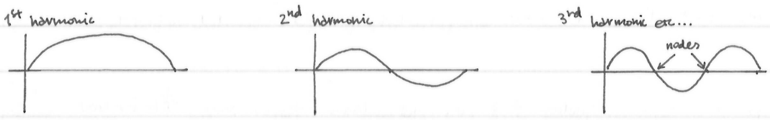
We have shown earlier that our solution is $X(x) = A\sin(px) + B\cos(px)$, $T(t) = C\sin(pct) + D\cos(pct)$. Then from $X(0) = X(L) = 0$, we get $B=0$,

$A\sin pL = 0 \Rightarrow pL = n\pi,\ n\in\mathbb{Z},\ p = \frac{n\pi}{L}$. Then $\phi_n(x,t) = [C_n \sin\frac{n\pi ct}{L} + D_n\cos\frac{n\pi ct}{L}](\sin\frac{n\pi x}{L})$, which is called the n^{th} harmonic.

At $t=0$, we have:



Recall that for heat equation, the n^{th} Fourier mode decays like $e^{-n^2 t}$. Then the period of oscillation is inversely proportional to n.

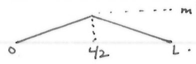
We impose initial conditions on our wave equation: $\phi(x,0) = F(x)$, $\frac{\partial\phi}{\partial t}(x,0) = G(x)$. Then our solutions are of form $\phi = \sum_{n=1}^{\infty}(C_n\sin\frac{n\pi ct}{L} + D_n\cos\frac{n\pi ct}{L})(\sin\frac{n\pi x}{L})$.

Where $t=0$, $\phi(x,0) = F(x) = \sum_{n=1}^{\infty} D_n \sin\frac{n\pi x}{L}$, and D_n is the n^{th} Fourier coefficient of F. Remember, we are thinking of F as an odd function on $[-L,L]$, where $\tilde{F}(x) = \begin{cases} F(x), & x>0 \\ -F(x), & x<0\end{cases}$. Then $\frac{1}{L}\int_{-L}^{L}\tilde{F}(x)\sin\frac{n\pi x}{L}dx = \frac{2}{L}\int_0^L F(x)\sin\frac{n\pi x}{L}dx = D_n$. Likewise, $\frac{\partial\phi}{\partial t} = \sum_{n=1}^{\infty}(C_n\frac{n\pi c}{L}\cos\frac{n\pi ct}{L} - D_n\frac{n\pi c}{L}\sin\frac{n\pi ct}{L})(\sin\frac{n\pi x}{L})$, and where $t=0$, $\sum_{n=1}^{\infty} C_n\frac{n\pi c}{L}\sin\frac{n\pi x}{L} = G(x)$ s.t. $C_n = \frac{L}{n\pi c}\cdot n^{th}$ Fourier coefficient of G.

Ex Take $L=\pi$, $F(x) = \sin x$, $G(x) = 0$.

Soln. $D_1 = 1$, $D_n = 0$, $n\neq 1$. Also $C_n = 0$. Thus, solution is $\phi(x,t) = \cos(ct)\sin x$ //.

Ex Take $F(x) = \begin{cases} mx & x\in[0,L/2] \\ \frac{mL}{2} - mx & x\in[L/2, L]\end{cases}$, $G(x) = 0$.



Soln. $C_n = 0$. Then $F(x) = \sum_{n=1}^{\infty}\frac{2mL}{n\pi}(\frac{2}{n\pi}\sin\frac{n\pi}{2} + \frac{\cos n\pi - \cos\frac{n\pi}{2}}{2})\sin\frac{n\pi x}{L}$. by Fourier expansion.

Then $\phi(x,t) = \sum_{n=1}^{\infty}\frac{2mL}{n\pi}(\frac{2}{n\pi}\sin\frac{n\pi}{2} + \frac{\cos n\pi - \cos\frac{n\pi}{2}}{2})\sin\frac{n\pi x}{L}\cos\frac{n\pi ct}{L}$ //

Note: We know how each Fourier mode evolves, so if we know Fourier expansion of initial condition, we know solution.

Ex Take $F(x) = 0$, $G(x) = \sin x$, $L = \pi$.

Soln. $D_n = 0$, $G(x) = \sin x \Rightarrow C_1 = 1\cdot\frac{1}{1c} = \frac{1}{c}$, $C_n = 0$ where $n\neq 1$. Thus, $\phi(x,t) = \frac{1}{c}\sin ct\sin x$ //.

Ex Solve wave equation on $[0,L]$ with Neumann boundary conditions at $x=0$ and Dirichlet at $x=L$: $\phi(L,t) = 0$, $\frac{\partial\phi}{\partial x}(0,t) = 0$.

Soln. Look for separated solutions: $\phi = XT \Rightarrow X''T = \frac{1}{c^2}XT'' \Rightarrow X'' = \lambda X$, $T'' = c^2\lambda T$. In terms of x and t, our initial conditions are – $X(L) = 0$, $X'(0) = 0$.

If $\lambda = 0$: $X = Ax + B \Rightarrow A = B = 0$, so no nontrivial solutions. Likewise for $\lambda>0$, no nontrivial separated solutions. Only left with case $\lambda<0$. Let $\lambda = -p^2$ s.t. $\sqrt{-\lambda} = p$.

$X(x) = A\sin px + B\cos px$. $X'(0) = 0 \Rightarrow pA\cos 0 - pB\sin 0 = 0 \Rightarrow A = 0$. $X(L) = 0 \Rightarrow B\cos pL = 0$. Then $pL = (n+\frac{1}{2})\pi$, $n\in\mathbb{Z}$.

Let $B=1$.

$\Rightarrow X = \underline{\underline{B}}\cos(n+\frac{1}{2})\pi\frac{x}{L}$. Likewise, $T = C_n\frac{\sin(n+\frac{1}{2})\pi ct}{L} + D_n\frac{\cos(n+\frac{1}{2})\pi ct}{L}$. Set $\phi(x,0) = F(x)$, $\frac{\partial\phi}{\partial x}(x,0) = G(x)$. Then we have:

$\phi(x,t) = \sum(C_n\sin\frac{(n+\frac{1}{2})\pi ct}{L} + D_n\cos\frac{(n+\frac{1}{2})\pi ct}{L})\cos\frac{(n+1/2)\pi x}{L}$. Then $\phi(x,0) = \sum D_n\cos\frac{(n+\frac{1}{2})\pi ct}{L}\cos\frac{(n+\frac{1}{2})\pi x}{L} = \sum D_n\cos\frac{(n+\frac{1}{2})\pi x}{L}$.

Recall that $\int_{-L}^{L}\cos\frac{(n+\frac{1}{2})\pi x}{L}\cos\frac{(m+\frac{1}{2})\pi x}{L}dx = 0$ if $m\neq n$. [Proof of claim: $\int\cos\frac{(n+\frac{1}{2})\pi x}{L}\frac{d^2}{dx^2}(\cos\frac{(m+\frac{1}{2})\pi x}{L})dx$. Note that $\cos\frac{(m+\frac{1}{2})\pi x}{L}$ is $[\frac{(m+\frac{1}{2})\pi}{L}]^2$ eigenfunction for $\frac{d^2}{dx^2}$.

$= -\frac{(m+\frac{1}{2})^2\pi^2}{L^2}\int\cos\frac{(n+\frac{1}{2})\pi x}{L}\cos\frac{(m+\frac{1}{2})\pi x}{L}dx \overset{\text{(doing the other way)}}{=} -\frac{(n+\frac{1}{2})^2\pi^2}{L^2}\int\cos\frac{(n+\frac{1}{2})\pi x}{L}\cos\frac{(m+\frac{1}{2})\pi x}{L}dx \Rightarrow \frac{(m+\frac{1}{2})^2\pi^2}{L^2} = \frac{(n+\frac{1}{2})^2\pi^2}{L^2}$ if integral is non-zero $\Rightarrow m=n$.]

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D'Alembert's solution to the wave equation

We recall that the wave equation is actually a hyperbolic 2nd order PDE: of form $A\frac{\partial^2}{\partial x^2} + B\frac{\partial^2}{\partial x\partial y} + C\frac{\partial^2}{\partial y^2} = \text{something}\times\frac{\partial}{\partial u}\frac{\partial}{\partial v}$ for suitable coordinates u and v.

$B^2 - 4AC > 0 \Leftrightarrow$ hyperbolicity. When we change variables, $\frac{\partial}{\partial x} = \frac{\partial u}{\partial x}\frac{\partial}{\partial u} + \frac{\partial v}{\partial x}\frac{\partial}{\partial v}$, $\frac{\partial}{\partial y} = \frac{\partial u}{\partial y}\frac{\partial}{\partial u} + \frac{\partial v}{\partial y}\frac{\partial}{\partial v}$, so

$A(u_x^2\frac{\partial^2}{\partial u^2} + v_x^2\frac{\partial^2}{\partial v^2} + 2u_xv_x\frac{\partial^2}{\partial u\partial v}) + B(u_xu_y\frac{\partial^2}{\partial u^2} + (u_xv_y + v_xu_y)\frac{\partial^2}{\partial u\partial v} + v_xv_y\frac{\partial^2}{\partial v^2}) + C(u_y^2\frac{\partial^2}{\partial u^2} + v_y^2\frac{\partial^2}{\partial v^2} + 2u_yv_y\frac{\partial^2}{\partial u\partial v})$.

$\Rightarrow$ To get an operator proportional to $\frac{\partial^2}{\partial u\partial v}$, we need ① $Au_x^2 + Bu_xu_y + Cu_y^2 = 0$, ② $Av_x^2 + Bv_xv_y + Cv_y^2 = 0$, then we will be left with:

$(2Au_xv_x + (u_xv_y + v_xu_y)B + 2Cu_yv_y)\frac{\partial^2}{\partial u\partial v}$, ①: $A(\frac{u_x}{u_y})^2 + B(\frac{u_x}{u_y}) + C = 0 \Rightarrow \frac{u_x}{u_y}$ solves $As^2 + Bs + C = 0$, likewise $\frac{v_x}{v_y}$, then $S_\pm = \frac{-B\pm\sqrt{B^2-4AC}}{2A}$.

So take $S_+ = \frac{u_x}{u_y}$, $S_- = \frac{v_x}{v_y}$. So if we define $u = xS_+ + y$, $u_x = S_+$, $u_y = 1 \Rightarrow \frac{u_x}{u_y} = S_+$. $v = xS_- + y$ correspondingly.

In coordinates, $u = xS_+ + y$, $v = xS_- + y$, ⊛: $A\frac{\partial^2}{\partial x^2} + B\frac{\partial^2}{\partial x\partial y} + C\frac{\partial^2}{\partial y^2}$ becomes something $\times\frac{\partial^2}{\partial u\partial v}$. To find coefficients, note that

$2Au_xv_x + B(u_xv_y + u_yv_x) + 2Cu_yv_y = 2AS_+S_- + B(S_+ + S_-) + 2C$. But remember that $As^2 + Bs + C = A(s-S_+)(s-S_-) = As^2 - As(S_+ + S_-) + AS_+S_-$,

$B = -A(S_+ + S_-)$, $C = S_+S_-A$. This coefficient is $2C - \frac{B^2}{A} + 2C = -\frac{1}{A}(B^2 - 4AC)$. Thus, in these coordinates, operator becomes $-\frac{1}{A}(B^2-4AC)\frac{\partial^2}{\partial u\partial v}$.

In the context of the wave equation, $\frac{\partial^2\phi}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2\phi}{\partial t^2} = 0$. $s^2 - \frac{1}{c^2} = 0$, $S_\pm = \pm\frac{1}{c}$, our algorithm gives $\tilde{u} = \frac{x}{c} + t$, $\tilde{v} = -\frac{x}{c} + t$. By convention though, we will take $u = x + ct$, $v = x - ct$. Thus, $x = \frac{1}{2}(u+v)$, $t = \frac{1}{2c}(u-v)$. $\frac{\partial}{\partial u} = \frac{\partial x}{\partial u}\frac{\partial}{\partial x} + \frac{\partial t}{\partial u}\frac{\partial}{\partial t} = \frac{1}{2}\frac{\partial}{\partial x} + \frac{1}{2c}\frac{\partial}{\partial t}$, $\frac{\partial}{\partial v} = \frac{1}{2}\frac{\partial}{\partial x} - \frac{1}{2c}\frac{\partial}{\partial t} \Rightarrow \frac{\partial}{\partial u}\frac{\partial}{\partial v} = \frac{1}{4}(\frac{\partial^2}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2})$.

So ϕ solves the wave equation $\Leftrightarrow \frac{\partial^2\phi}{\partial u\partial v} = 0 \Rightarrow \phi = F(u) + G(v)$. Hence, solutions are $\phi = F(x+ct) + G(x-ct)$.

How does this solution $\phi = F(x+ct) + G(x-ct)$ relate to the separated solutions we already found? e.g. $\phi(x,t) = \sin x \sin ct$

We know that $\phi(x,t) = \sin x \sin ct = \frac{\cos(x-ct) - \cos(x+ct)}{2} = F(x+ct) + G(x-ct)$ where $F = \frac{1}{2}\cos$, $G = -\frac{1}{2}\cos$.

In general, the Fourier style solutions are good for solving Dirichlet and Neumann problems on a bounded string.

On the other hand, d'Alembert solutions are particularly good for unbounded (infinitely long) strings.

General problem: solve wave equation on $\mathbb{R}$ with initial conditions $\phi(x,0) = M(x)$, $\frac{\partial\phi}{\partial t}(x,0) = N(x)$; specified functions. Remember that we have $\phi(x,t) = F(x+ct) + G(x-ct) \Rightarrow \phi(x,0) = F(x) + G(x) = M(x)$ — ①. Also, $\frac{\partial\phi}{\partial t}(x,0) = cF'(x) - cG'(x) = N(x)$. —② We differentiate ①: $F'(x) + G'(x) = M'(x)$.

$\therefore F'(x) = \frac{1}{2}\left[M'(x) + \frac{N(x)}{c}\right] \Rightarrow F(x) = \frac{1}{2}\left[M(x) + \frac{1}{c}\int_0^x N(\xi)\,d\xi\right]$, $G(x) = M(x) - F(x) = \frac{1}{2}\left[M(x) - \frac{1}{c}\int_0^x N(\xi)\,d\xi\right]$.

Theorem (d'Alembert's solution)

The solution to the wave equation with $\phi(x,0) = M(x)$, $\frac{\partial\phi}{\partial t}(x,0) = N(x)$ is $\frac{1}{2}\left(M(x+ct) + \frac{1}{c}\int_0^{x+ct} N(\xi)\,d\xi\right) + \frac{1}{2}\left(M(x-ct) - \frac{1}{c}\int_0^{x-ct} N(\xi)\,d\xi\right) = \phi(x,t)$.

Or in another expression: $\phi(x,t) = \frac{1}{2}\left(M(x+ct) + M(x-ct) + \frac{1}{c}\int_{x-ct}^{x+ct} N(\xi)\,d\xi\right)$.

Ex $\phi(x,0) = \sin x$, $\frac{\partial\phi}{\partial t}(x,0) = \cos x$.

Soln. our equations are $F(x) + G(x) = \sin x$, $F'(x) - G'(x) = \frac{\cos x}{c}$, so $F'(x) + G'(x) = \cos x \Rightarrow 2F' = \cos(1 + \frac{1}{c}) \Rightarrow F = \frac{1}{2}(1+c)\sin x$.

$G = \sin x - F(x) = \sin x - \frac{1}{2}(1+\frac{1}{c})\sin x = \frac{1}{2}(1 - \frac{1}{c})\sin x \Rightarrow \phi(x,t) = \frac{1}{2}(1+\frac{1}{c})\sin(x+ct) + \frac{1}{2}(1-\frac{1}{c})\sin(x-ct)$.

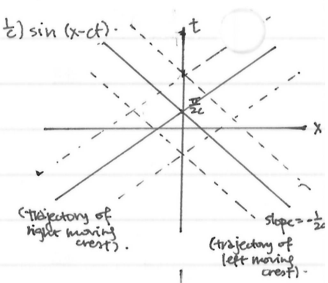
To interpret our results, we can draw spacetime diagrams: $\phi(x,t) = \frac{1}{2}(1+\frac{1}{c})\sin(x+ct) + \frac{1}{2}(1-\frac{1}{c})\sin(x-ct)$

has crests at $x + ct = 2\pi n + \frac{1}{2}\pi$; $t = -\frac{1}{c}x + \frac{2\pi n}{c} + \frac{\pi}{2c}$ is a line in spacetime. we call this left-moving. ($\sin(x-ct)$: right-moving.)

These lines in spacetime diagram form our new coordinates.

They are called by several names in the wave equation:

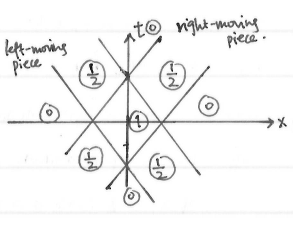
• characteristics • null lines • light-like rays

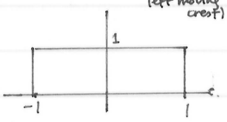


Ex $M(x) = \phi(x,0) = \begin{cases} 1 & x \in [-1,1] \\ 0 & x \notin [-1,1] \end{cases}$. $N(x) = 0 = \frac{\partial\phi}{\partial t}(x,0)$.

Soln. D'Alembert solution gives $\phi(x,t) = \frac{1}{2}(M(x+ct) + M(x-ct))$.

We plot the solutions in spacetime accordingly: and see that it is split into regions valued at $0, \frac{1}{2}, 1$.





More complicated hyperbolic equations

Ex consider $\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial x\partial y} - 2\frac{\partial^2\phi}{\partial y^2} = e^x$. solve using coordinate transform. Plug into solutions the initial conditions $\phi(x,0) = x^2$, $\frac{\partial\phi}{\partial y}(x,0) = x^3$.

Soln. Solve $s^2 + s - 2 = (s+2)(s-1)$. Then $v = -2x + y$, $u = x + y \Rightarrow x = \frac{u-v}{3}$, $y = \frac{2u+v}{2}$. The operator under this change of coordinates becomes $-\frac{B^2-4AC}{A}\frac{\partial^2}{\partial u\partial v} = -9\frac{\partial^2}{\partial u\partial v}$. Then RHS $= e^x = e^{\frac{u-v}{3}} \Rightarrow \frac{\partial^2\phi}{\partial u\partial v} = -\frac{1}{9}e^{\frac{u-v}{3}} \Rightarrow \phi = e^{\frac{u-v}{3}} + F(u) + G(v) = e^x + F(x+y) + G(y-2x)$.

① $\phi(x,0) = e^x + F(x) + G(-2x) = x^2$, ② $\frac{\partial\phi}{\partial y} = F'(x) + G'(-2x) = x^3$ ① $\Rightarrow e^x + F'(x) - 2G'(-2x) = 2x$. Then $3F' = 2(x^3 + x) - e^x$ s.t. $F' = \frac{2}{3}(x^3+x) - \frac{e^x}{3}$

Then $F(x) = \frac{1}{6}x^4 + \frac{x^2}{3} - \frac{e^x}{3}$. $\Rightarrow G(-2x) = x^2 - F(x) - e^x = \frac{2x^2}{3} - \frac{1}{6}x^4 - \frac{2e^x}{3}$. sub $z = -2x$, then $x = -\frac{z}{2}$ s.t. $G(z) = \frac{z^2}{6} - \frac{z^4}{96} - \frac{2}{3}e^{-\frac{z}{2}}$.

$\therefore \phi(x,y) = F(x+y) + G(y-2x) + e^x = \frac{1}{6}(x+y)^4 + \frac{1}{3}(x+y)^2 - \frac{1}{3}e^{x+y} + \frac{1}{6}(y-2x)^2 - \frac{1}{96}(y-2x)^4 - \frac{2}{3}e^{\frac{2x-y}{2}} + e^x$ //

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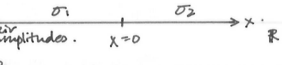
We now consider an example with discontinuities, using d'Alembert solution.

Ex consider $c(x)^2\frac{\partial^2\phi}{\partial x^2} = \frac{\partial^2\phi}{\partial t^2}$, with $c(x) = \begin{cases} \sigma_1 & x<0 \\ \sigma_2 & x\geq 0 \end{cases}$. At $x=0$, specify $G_1 = A\sin(kx+\psi)$ and $F_2 = 0$. calculate G_2, F_1 and their amplitudes.



Soln. As before, we will have to impose interface condition: $\phi(x,t)$ and $\frac{\partial\phi}{\partial x}(x,t)$ are continuous at $x=0$.

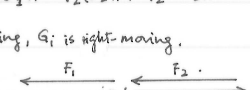
Imagine that $c(x)$ changes sharply but smoothly. $\therefore$ we guess that ϕ is smooth; this is making ϕ as smooth as it can be.

We know, from d'Alembert's method, that $\phi(x,t) \begin{cases} \phi_1(x,t) & x<0 \\ \phi_2(x,t) & x>0 \end{cases}$, $\phi_1(x,t) = F_1(x+\sigma_1 t) + G_1(x-\sigma_1 t)$, $\phi_2(x,t) = F_2(x+\sigma_2 t) + G_2(x-\sigma_2 t)$.

Our conditions imply that $F_1(\sigma_1 t) + G_1(-\sigma_1 t) = F_2(\sigma_2 t) + G_2(-\sigma_2 t)$ —① and $\frac{\partial\phi_1}{\partial t}(0,t) = \frac{\partial\phi_2}{\partial t}(0,t) \Rightarrow F_1'(\sigma_1 t) + G_1'(-\sigma_1 t) = F_2'(\sigma_2 t) + G_2'(-\sigma_2 t)$. —②

$\frac{d}{dt}$①: $\sigma_1 F_1'(\sigma_1 t) - \sigma_1 G_1'(\sigma_1 t) = \sigma_2 F_2'(\sigma_2 t) - \sigma_2 G_2'(\sigma_2 t)$. Aside: Why do we specify G_1 and F_2? F_i is left-moving, G_i is right-moving.

Here F_1 is the reflected wave, G_2 is the transmitted wave. How much light is reflected and transmitted?



$G_1 = A\sin(kx+\psi)$ is a plane wave, A is amplitude, k is wavenumber and ψ represents a phase shift.

① becomes: $F_1(\sigma_1 t) + A\sin(-k\sigma_1 t + \psi) = 0 + G_2(-\sigma_2 t) \Rightarrow \sigma_1 F_1'(\sigma_1 t) - \sigma_1 kA\cos(-k\sigma_1 t + \psi) = 0 - \sigma_2 G_2'(-\sigma_2 t)$. Also, equation ② is:

$F_1'(\sigma_1 t) + kA\cos(-k\sigma_1 + \psi) = 0 + G_2'(-\sigma_2 t)$. To isolate F_1', take $\sigma_2\times$② $+ \frac{d}{dt}$①: $(\sigma_1+\sigma_2)F_1'(\sigma_1 t) + (\sigma_2 - \sigma_1)kA\cos(\psi - k\sigma_1 t) = 0$.

Then $F_1'(\sigma_1 t) = \frac{\sigma_1 - \sigma_2}{\sigma_1+\sigma_2}kA\cos(\psi - k\sigma_1 t)$. Substitute $z = \sigma_1 t$, $F_1'(z) = \frac{\sigma_1-\sigma_2}{\sigma_1+\sigma_2}kA\cos(\psi - kz) \Rightarrow F_1(z) = \frac{-\sigma_1+\sigma_2}{\sigma_1+\sigma_2}A\sin(\psi - kz)$.

$G_2(-\sigma_2 t) = F_1(\sigma_1 t) + A\sin(\psi - k\sigma_1 t) \Rightarrow G_2(u) = F_1(-\frac{\sigma_1}{\sigma_2}u) + A\sin(\psi + k\frac{\sigma_1}{\sigma_2}u) = \left(\frac{\sigma_2-\sigma_1}{\sigma_2+\sigma_1}+1\right)A\sin(\psi + k\frac{\sigma_1}{\sigma_2}u)$. Amp of $F_1 = A\frac{\sigma_2-\sigma_1}{\sigma_2+\sigma_1}$ // Amp of $G_2 = A\frac{2\sigma_2}{\sigma_1+\sigma_2}$ //

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For example, if $\sigma_1 = \sigma_2$, then amp $G_2 = A$ i.e. all light transmitted. If $\frac{\sigma_1}{\sigma_2} \to \infty$, amp $G_2 = \frac{2}{\sigma_1/\sigma_2 + 1} A \to 0$ i.e.
if light goes between media (from fast into very slow), then most of it is reflected.

END OF SYLLABUS.