2401 Mathematical Methods 3 Notes

Based on the 2016 autumn lectures by Prof J M Vanden-Broeck

MATH 2401

Office Hour:
Friday 4pm

07-10-16 Mathematical Methods 3

3 lectures - Friday
P. class - Mon (Bowles)

Topics
* Fourier Series
* Partial differential equations
  - separation of variables
  - characteristics
* Calculus of variation

Moodle
* Notes
* Exercise sheets (due Fridays, 11·00)

1.2 Fourier Series

Any sufficiently nice function $f: [-L, L] \to R$ can be written as a Fourier Series

$$f(x) = c + \sum_{n=1}^{\infty} \left[a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right) \right] \quad (*)$$

$f(x)$ is defined for $-L < x < L$

Lemma 1.2

If $n \geq 0$ is an integer then

$$\int_{-L}^{L} \sin\left(\frac{n\pi x}{L}\right) dx = 0$$

$$\int_{-L}^{L} \cos\left(\frac{n\pi x}{L}\right) dx = 2L \, \delta_{n,0}$$

Kronecker Delta:
$$\delta_{i,j} = \begin{cases} 0 & i \neq j \\ 1 & i = j \end{cases}$$

If $m > 0$ and $n > 0$ are integers

1). $$\int_{-L}^{L} \sin\left(\frac{n\pi x}{L}\right) \sin\left(\frac{m\pi x}{L}\right) dx = L \, \delta_{m,n}$$

2). $$\int_{-L}^{L} \cos\left(\frac{n\pi x}{L}\right) \cos\left(\frac{m\pi x}{L}\right) dx = L \, \delta_{m,n}$$

3). $$\int_{-L}^{L} \cos\left(\frac{n\pi x}{L}\right) \sin\left(\frac{m\pi x}{L}\right) dx = 0$$

proof of 1).

$$\cos(A-B)-\cos(A+B) = 2\sin A\sin B$$

$$A = \frac{m\pi x}{L} \quad , \quad B = \frac{n\pi x}{L}$$

$$\sin\left(\frac{m\pi x}{L}\right)\sin\left(\frac{n\pi x}{L}\right) = \frac{1}{2}\left[\cos\left(\frac{(m-n)\pi x}{L}\right) - \cos\left(\frac{(m+n)\pi x}{L}\right)\right]$$

$$\int_{-L}^{L}\sin\left(\frac{m\pi x}{L}\right)\sin\left(\frac{n\pi x}{L}\right)dx = \frac{1}{2}\int_{-L}^{L}\cos\left(\frac{(m-n)\pi x}{L}\right)dx - \frac{1}{2}\int_{-L}^{L}\cos\left(\frac{(m+n)\pi x}{L}\right)dx$$

Calculating integrals:

$$\int_{-L}^{L}\cos\left(\frac{(m+n)\pi x}{L}\right)dx = \left[\frac{\sin\left(\frac{(m+n)\pi x}{L}\right)}{\frac{(m+n)\pi}{L}}\right]_{-L}^{L}$$

$$= \frac{1}{\pi\left(\frac{m+n}{L}\right)}\left[\sin\pi(m+n) + \sin\pi(m+n)\right] = 0$$

$m \neq n$

$$\int_{-L}^{L}\cos\left(\frac{(m-n)\pi x}{L}\right)dx = \left[\frac{\sin\left(\frac{(m-n)\pi x}{L}\right)}{\frac{(m-n)\pi}{L}}\right]_{-L}^{L}$$

$$= \frac{1}{\pi\left(\frac{m-n}{L}\right)}\left[\sin\pi(m-n) + \sin\pi(m-n)\right] = 0$$

$m = n$

$$\int_{-L}^{L}\cos\left(\frac{(m-n)\pi x}{L}\right)dx = \int_{-L}^{L}dx = 2L$$

So $$\int_{-L}^{L}\sin\left(\frac{m\pi x}{L}\right)\sin\left(\frac{n\pi x}{L}\right)dx = L\,\delta_{mn}$$

07-10-16

Integrating (*): (we want to find c)

$$\int_{-L}^{L} F(x)\,dx = \int_{-L}^{L} c\,dx + \sum_{n=1}^{\infty}\left[a_n \underbrace{\int_{-L}^{L}\cos\left(\frac{n\pi x}{L}\right)dx}_{0} + b_n \underbrace{\int_{-L}^{L}\sin\left(\frac{n\pi x}{L}\right)dx}_{0}\right]$$

$$= c\int_{-L}^{L} dx = c\,2L$$

so $c = \frac{1}{2L}\int_{-L}^{L} F(x)\,dx$

Try to find a_2:
Multiply fourier series by $\cos\left(\frac{2\pi x}{L}\right)$ and integrate:

$$\int_{-L}^{L} F(x)\cos\left(\frac{2\pi x}{L}\right)dx = \cancel{c\int_{-L}^{L}\cos\left(\frac{2\pi x}{L}\right)dx} + \cancel{a_1\int_{-L}^{L}\cos\left(\frac{\pi x}{L}\right)\cos\left(\frac{2\pi x}{L}\right)dx}$$

$$+ a_2\int_{-L}^{L}\cos^2\left(\frac{2\pi x}{L}\right)dx + \cancel{a_3\int_{-L}^{L}\cos\left(\frac{3\pi x}{L}\right)\cos\left(\frac{2\pi x}{L}\right)dx + \dots}$$

$$+ \cancel{b_1\int_{-L}^{L}\sin\left(\frac{\pi x}{L}\right)\cos\left(\frac{2\pi x}{L}\right)dx} + \cancel{b_2\int_{-L}^{L}\sin\left(\frac{2\pi x}{L}\right)\cos\left(\frac{2\pi x}{L}\right)dx}$$

$$+ \cancel{b_3\int_{-L}^{L}\sin\left(\frac{3\pi x}{L}\right)\cos\left(\frac{2\pi x}{L}\right)dx + \dots}$$

$$= a_2\int_{-L}^{L}\cos^2\left(\frac{2\pi x}{L}\right) = a_2 L$$

so $a_2 = \frac{1}{L}\int_{-L}^{L} F(x)\cos\left(\frac{2\pi x}{L}\right)dx.$

How to find a_m:

Multiply (*) by $\cos\left(\frac{m\pi x}{L}\right)$ and integrate:

$$\int_{-L}^{L} F(x)\cos\left(\frac{m\pi x}{L}\right) dx = c\int_{-L}^{L}\cos\left(\frac{m\pi x}{L}\right)dx + \sum_{n=1}^{\infty}\left[a_n\int_{-L}^{L}\cos\left(\frac{n\pi x}{L}\right)\cos\left(\frac{m\pi x}{L}\right)dx\right.$$
$$\left. + \, b_n\int_{-L}^{L}\sin\left(\frac{n\pi x}{L}\right)\cos\left(\frac{m\pi x}{L}\right)dx\right]$$
$$= \sum_{n=1}^{\infty} a_n\int_{-L}^{L}\cos\left(\frac{n\pi x}{L}\right)\cos\left(\frac{m\pi x}{L}\right)dx$$
$$= \sum_{n=1}^{\infty} a_n L\,\delta_{m,n}$$
$$= a_m L$$

So $a_m = \frac{1}{L}\int_{-L}^{L} F(x)\cos\left(\frac{m\pi x}{L}\right)dx.$ $\quad (m \geq 1)$

Finding b_m:
We can use the same method with $\sin\left(\frac{m\pi x}{L}\right)$
we get:

$$\int_{-L}^{L} F(x)\sin\left(\frac{m\pi x}{L}\right)dx = \sum_{n=1}^{\infty} b_n\int_{-L}^{L}\sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{m\pi x}{L}\right)dx$$
$$= \sum_{n=1}^{\infty} b_n L\,\delta_{m,n}$$
$$= b_m L$$

So $b_m = \frac{1}{L}\int_{-L}^{L} F(x)\sin\left(\frac{m\pi x}{L}\right)dx$

MATH 2401

07-10-16

To conclude:

$$F(x) = c + \sum_{n=1}^{\infty}\left[a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right)\right]$$

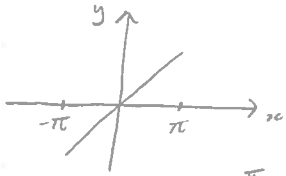
$$c = \frac{1}{2L}\int_{-L}^{L} F(x)\,dx$$

$$a_n = \frac{1}{L}\int_{-L}^{L} F(x)\cos\left(\frac{n\pi x}{L}\right)dx$$

$$b_n = \frac{1}{L}\int_{-L}^{L} F(x)\sin\left(\frac{n\pi x}{L}\right)dx$$

<u>Example</u>

$F(x) = x$ on the interval $[-\pi, \pi]$. (so $L = \pi$)



$$c = \frac{1}{2\pi}\int_{-\pi}^{\pi} x\,dx = \frac{1}{2\pi}\left[\frac{x^2}{2}\right]_{-\pi}^{\pi} = \frac{1}{2\pi}\left[\frac{\pi^2}{2} - \frac{\pi^2}{2}\right] = 0$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} x\cos nx\,dx = \frac{1}{\pi}\left[\cancel{\left[\frac{x\sin nx}{n}\right]_{-\pi}^{\pi}} - \int_{-\pi}^{\pi}\frac{\sin nx}{n}\,dx\right]$$ by parts.

$$= -\frac{1}{n^2\pi}\left[\cos nx\right]_{-\pi}^{\pi} = 0 \quad \Rightarrow \quad a_n = 0 \ \forall n.$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} x\sin nx\,dx = \frac{1}{\pi}\left[\left[-\frac{x\cos nx}{n}\right]_{-\pi}^{\pi} + \cancel{\frac{1}{n}\int_{-\pi}^{\pi}\cos nx\,dx}\right]$$ by parts

$$= \frac{1}{n\pi}\left[-\pi\cos n\pi - \pi\cos(-n\pi)\right]$$

$$= -\frac{2\cos n\pi}{n} = -\frac{2}{n}(-1)^n = \frac{2}{n}(-1)^{n+1}$$

So $c = 0$, $a_n = 0$, $b_n = \frac{2}{n}(-1)^{n+1}$

$$x = \sum_{n=1}^{\infty} \frac{2}{n}(-1)^{n+1} \sin(nx)$$

$$= 2\left[\sin x - \frac{\sin 2x}{2} + \frac{\sin 3x}{3} - \frac{\sin 4x}{4} + \dots\right]$$

If F is an odd function $(F(-x) = -F(x))$
then $c = 0$, $a_n = 0$ $\forall n$.

If F is an even function $(F(-x) = F(x))$
then $b_n = 0$ $\forall n$.

Lemma 1.5

$$F(x) = c + \sum_{n=1}^{\infty}\left[a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right)\right]$$

If $F(x)$ is even then

$$b_n = 0$$

$$a_n = \frac{2}{L}\int_0^L F(x)\cos\left(\frac{n\pi x}{L}\right)dx$$

$$c = \frac{1}{L}\int_0^L F(x)\,dx$$

If $F(x)$ is odd then

$$a_n = 0$$

$$c = 0$$

$$b_n = \frac{2}{L}\int_0^L F(x)\sin\left(\frac{n\pi x}{L}\right)dx$$

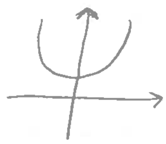
MATH 2401

07-10-16

In general

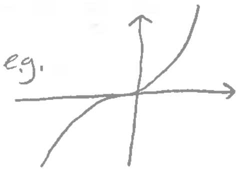
1). If $g(x)$ is even then

$$\int_{-L}^{L} g(x)\,dx = 2\int_{0}^{L} g(x)\,dx$$

eg. 

2). If $g(x)$ is odd then

$$\int_{-L}^{L} g(x)\,dx = 0$$

eg. 

Proof of 1).

$$\int_{-L}^{L} g(x)\,dx = \underbrace{\int_{-L}^{0} g(x)\,dx}_{\text{let } u=-x} + \int_{0}^{L} g(x)\,dx$$

so

$$\int_{-L}^{L} g(x)\,dx = -\int_{-L}^{0} g(-u)\,du + \int_{0}^{L} g(x)\,dx$$

$$= \int_{0}^{L} g(u)\,du + \int_{0}^{L} g(x)\,dx \quad \text{note } g \text{ even.}$$

$$= 2\int_{0}^{L} g(x)\,dx$$

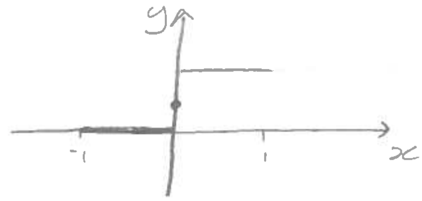
Examples:

even: x^2, $\cos x$, x^4, $|x|$, ...

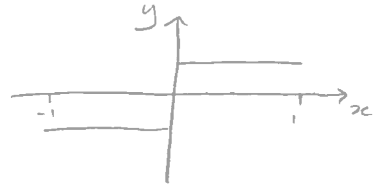
odd: x, $\sin x$, ...

neither: $x + x^2$, e^x, ...

$$F(x) = \begin{cases} 0 & -1 \le x < 0 \\ \frac{1}{2} & x = 0 \\ 1 & 0 < x \le 1 \end{cases}$$



$$G(x) = f(x) - \frac{1}{2} = \begin{cases} -\frac{1}{2} & -1 \le x < 0 \\ 0 & x = 0 \\ \frac{1}{2} & 0 < x \le 1 \end{cases}$$



$G(x)$ is odd! $(L = 1)$

$\Rightarrow a_n = 0, \quad c = 0$

$$b_n = 2\int_0^1 G(x)\sin(n\pi x)\,dx$$

when $0 < x \le 1$, $G(x) = \frac{1}{2}$

so
$$b_n = \int_0^1 \sin(n\pi x)\,dx$$
$$= \left[\frac{-\cos(n\pi x)}{n\pi}\right]_0^1$$
$$= \frac{-\cos n\pi + 1}{n\pi} = \frac{-(-1)^n + 1}{n\pi}$$
$$= \frac{(-1)^{n+1} + 1}{n\pi} = \begin{cases} 0 & n \text{ even} \\ \frac{2}{n\pi} & n \text{ odd} \end{cases}$$

So
$$G(x) = \sum_{n=1}^{\infty} b_n \sin(n\pi x)$$
$$= \frac{2}{\pi}\sin \pi x + \frac{2}{3\pi}\sin 3\pi x + \dots$$

$$F(x) = G(x) + \frac{1}{2} = \frac{1}{2} + \frac{2}{\pi}\sin \pi x + \frac{2}{3\pi}\sin 3\pi x + \dots$$

07-10-16

$f(x) = x^2 \quad -\pi < x < \pi \quad , \quad L = \pi$

even

$\Rightarrow b_n = 0$

$$a_n = \frac{2}{\pi}\int_0^{\pi} x^2 \cos nx \, dx = \ldots = \frac{4(-1)^n}{n^2}$$ (notes p.14)

(calculate as exercise)

$$c = \frac{1}{\pi}\int_0^{\pi} x^2 \, dx = \frac{1}{\pi}\left[\frac{x^3}{3}\right]_0^{\pi} = \frac{\pi^2}{3}$$

so $x^2 = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \cos nx$.

Half-range Fourier series

$f(x)$ defined for $0 < x < L$

We want to represent $f(x)$ as a series of sin (half range sine series).

Extend $f(x)$ for $-L < x < 0$ so that $f(x)$ is an odd function: $f_{odd}(x) = \begin{cases} f(x) & x > 0 \\ -f(-x) & x < 0 \end{cases}$

$$f_{odd}(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right)$$ [$f_{odd}(x) = f(x)$ for $0 < x < L$]

$$b_n = \frac{2}{L}\int_0^L f(x) \sin\left(\frac{n\pi x}{L}\right) dx$$

To represent $f(x)$ as a series of cos (half range cosine series), we extend $f(x)$ for $-L < x < 0$ so that $f(x)$ is an even function

$$f_{even}(x) = \begin{cases} F(x) & , x>0 \\ F(-x) & , x<0 \end{cases}$$

$$f_{even}(x) = c + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right)$$

$$a_n = \frac{2}{L}\int_0^L F(x)\cos\left(\frac{n\pi x}{L}\right)dx$$

$$c = \frac{1}{L}\int_0^L f(x)dx$$

Half range sine series of $F(x) = x(\pi - x)$ $\quad 0 \leq x \leq \pi$

$L = \pi$

$$f(x) = \sum_{n=1}^{\infty} b_n \sin(nx)$$

$$b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin(nx)dx$$

Half range cosine series:

$$f(x) = c + \sum_{n=1}^{\infty} a_n \cos nx$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\, dx$$

$$c = \frac{1}{\pi}\int_0^{\pi} F(x)dx$$

MATH 2401

07-10-16

Parseval's Theorem

$$\frac{1}{L}\int_{-L}^{L} f^2(x)\,dx = 2c^2 + \sum_{n=1}^{\infty}(a_n^2 + b_n^2)$$

Proof

$$\frac{1}{L}\int_{-L}^{L} f(x)\,f(x)\,dx$$

$$= \frac{1}{L}\int_{-L}^{L} f(x)\left[c + \sum_{n=1}^{\infty} a_n\cos\left(\frac{n\pi x}{L}\right) + b\sin\left(\frac{n\pi x}{L}\right)\right]dx$$

$$= \frac{c}{L}\int_{-L}^{L} f(x)\,dx + \sum_{n=1}^{\infty}\left[\frac{a_n}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)dx + \frac{b_n}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)dx\right]$$

$$= 2c^2 + \sum_{n=1}^{\infty}(a_n^2 + b_n^2)$$

$\square$

Recall $f(x) = x$, $-\pi < x < \pi$

$c = 0$, $a_n = 0$, $b_n = \frac{2(-1)^{n+1}}{n}$, $L = \pi$

$$f(x) = \sum_{n=1}^{\infty} \frac{2(-1)^{m+1}}{m}\sin(mx)$$

$$\frac{1}{\pi}\int_{-\pi}^{\pi} x^2\,dx = \sum_{n=1}^{\infty}\frac{4}{n^2}$$

$$\Rightarrow \frac{2\pi^2}{3} = 4\sum_{n=1}^{\infty}\frac{1}{n^2}$$

$$\Rightarrow \frac{\pi^2}{6} = \sum_{n=1}^{\infty}\frac{1}{n^2}$$

14-10-16

<u>Fourier Series</u>

$$F(x) = c + \sum_{n=1}^{\infty} \left(a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right) \right)$$

defined for
$-L < x < L$

$c = ?$ $\quad a_n = ?$ $\quad b_n = ?$

<u>Convergence</u>

Partial Sums $F_N(x) = c + \sum_{n=1}^{N} \left(a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right) \right)$

$F_N(x) \to F(x)$ as $N \to \infty$.

We require $F^2(x)$ is integrable,
i.e. $\int_{-L}^{L} F^2(x)\,dx$ exists and is finite.

$$\int_{-L}^{L} (F_N(x) - F(x))^2\,dx \to 0 \text{ as } N \to \infty.$$

<u>Periodic Functions</u>

$$\cos\left(\frac{n\pi}{L}(x+2L)\right) = \cos\left(\frac{n\pi x}{L} + 2n\pi\right) = \cos\left(\frac{n\pi x}{L}\right)$$

$$\sin\left(\frac{n\pi}{L}(x+2L)\right) = \sin\left(\frac{n\pi x}{L} + 2n\pi\right) = \sin\left(\frac{n\pi x}{L}\right)$$

Chapter 2 Second order partial differential equations (PDEs)

Heat equation:

$$\frac{\partial \phi}{\partial t} = K \frac{\partial^2 \phi}{\partial x^2} \quad , \quad \phi(x,t) \ , \ \text{hyperbolic}$$

Wave equation:

$$\frac{1}{c^2} \frac{\partial^2 \phi}{\partial t^2} = \frac{\partial^2 \phi}{\partial x^2} \quad , \quad \phi(x,t) \ , \ \text{parabolic}$$

Laplace equation:

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0 \quad , \quad \phi(x,y) \ , \ \text{elliptic}$$

[Contrast with ordinary differential equations (ODE)
e.g. $\frac{d^2 y}{dx^2} + y = 0$]

Method of seperation of variables

We will use the heat equation with $K=1$,
so $\frac{\partial \phi}{\partial t} = \frac{\partial^2 \phi}{\partial x^2}$

$\phi(x,t) = X(x)T(t)$ (assume this)

$$\frac{\partial \phi}{\partial t} = X(x)\frac{dT}{dt} = X(x)T'(t)$$

$$\frac{\partial^2 \phi}{\partial x^2} = \frac{d^2 X}{dx^2} T(t) = X''(x)T(t)$$

So $\frac{\partial \phi}{\partial t} = \frac{\partial^2 \phi}{\partial x^2}$ becomes $X(x)T'(t) = X''(x)T(t)$

14-10-16

So $\underbrace{\frac{T'(t)}{T(t)}}_{\text{does not depend on } x} = \underbrace{\frac{X''(x)}{X(x)}}_{\text{does not depend on } t}$,

$$\Rightarrow \frac{T'(t)}{T(t)} = \frac{X''(x)}{X(x)} = \text{constant} = -\lambda$$

So $\frac{T'(t)}{T(t)} = -\lambda$, $\frac{X''(x)}{X(x)} = -\lambda$

So $T'(t) + \lambda T(t) = 0$, $X''(x) + \lambda X(x) = 0$

$T'(t) + \lambda T(t) = 0 \Rightarrow T(t) = \tilde{A} e^{-\lambda t}$ (can choose $\tilde{A} = 1$ here)

$$X''(x) + \lambda X(x) = 0$$

$$\Rightarrow \begin{cases} \lambda > 0, \text{ so } \lambda = p^2 & \text{so } X''(x) + p^2 X(x) = 0 \\ \lambda = 0 & \text{so } X''(x) = 0 \\ \lambda < 0, \text{ so } \lambda = -p^2 & \text{so } X''(x) - p^2 X(x) = 0 \end{cases}$$

$$\Rightarrow \begin{cases} X(x) = A\cos(px) + B\sin(px) \\ X(x) = Ax + B \\ X(x) = A\cosh(px) + B\sinh(px) \left[= Ce^{px} + De^{-px}\right] \end{cases}$$

Note: $\cosh(px) = \frac{e^{px} + e^{-px}}{2}$, $\sinh(px) = \frac{e^{px} - e^{-px}}{2}$

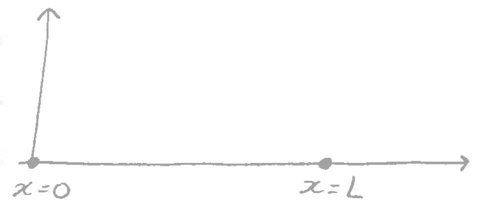
$\cosh(px) + \sinh(px) = e^{px}$, $\cosh(px) - \sinh(px) = e^{-px}$

$$\begin{aligned} X(x) &= Ce^{px} + De^{-px} \\ &= C[\cosh(px) + \sinh(px)] + D[\cosh(px) - \sin(px)] \\ &= (C+D)\cosh(px) + (C-D)\sinh(px) \end{aligned}$$

Boundary conditions

Dirichlet boundary conditions : $\phi(0,t) = M \quad \phi(L,t) = N$

Neuman boundary conditions: $\frac{\partial \phi}{\partial x}(0,t) = 0 \quad \frac{\partial \phi}{\partial x}(L,t) = 0$



distribution of temperature in a rod.

Initial condition

$\phi(x,0) = F(x)$ given

Summary

$$\frac{\partial \phi}{\partial t} = \frac{\partial^2 \phi}{\partial x^2}$$

Dirichlet: $\phi(0,t) = M \quad , \quad \phi(L,t) = N$

Initial: $\phi(x,0) = F(x)$

Define

$$\theta(x,t) = \phi(x,t) - \phi_0(x,t)$$

where $\phi_0(x,t) = M + \frac{N-M}{L}x$.

Then
$$\begin{cases} \frac{\partial \theta}{\partial t} = \frac{\partial^2 \theta}{\partial x^2} \\ \theta(0,t) = 0 \quad , \quad \theta(L,t) = 0 \\ \theta(x,0) = F(x) - \phi_0(x,0) \end{cases}$$

MATH 2401

14-10-16

$$\frac{\partial \theta}{\partial t} = \frac{\partial \phi}{\partial t} - \cancel{\frac{\partial \phi_0}{\partial t}} = \frac{\partial \phi}{\partial t}$$

$$\frac{\partial^2 \theta}{\partial x^2} = \frac{\partial^2 \phi}{\partial x^2} - \cancel{\frac{\partial^2 \phi_0}{\partial x^2}} = \frac{\partial^2 \phi}{\partial x^2}$$

$$\theta(0,t) = \underbrace{\phi(0,t)}_{M} - \underbrace{\phi_0(0,t)}_{M} = 0$$

$$\theta(L,t) = \underbrace{\phi(L,t)}_{N} - \underbrace{\phi_0(L,t)}_{N} = 0$$

$$\theta(x,0) = \phi(x,0) - \phi_0(x,0)$$
$$= F(x) - \phi_0(x,0)$$

$$\frac{\partial \phi}{\partial t} = \frac{\partial^2 \phi}{\partial x^2}$$

$\phi(0,t) = 0 \qquad \phi(L,t) = 0$

$\phi(x,0) = F(x)$

$\phi(x,t) = X(x)T(t)$

$\phi(0,t) = 0 \Rightarrow X(0)T(t) = 0 \Rightarrow X(0) = 0$

$\phi(L,t) = 0 \Rightarrow X(L)T(t) = 0 \Rightarrow X(L) = 0$

<u>Lemma 2.5</u>

A separated solution $X(x)T(t)$ of the heat equation satisfying $X(0) = X(L) = 0$ has the form $B\sin\left(\frac{n\pi x}{L}\right)e^{\left(\frac{-n^2\pi^2 t}{L^2}\right)}$

<u>Proof</u>

(i) $\lambda = -p^2$

$X(x) = A\cosh px + B\sinh px$

$X(0) = A = 0$

$X(L) = B\underbrace{\sinh(pL)}_{\neq 0 \text{ when } pL \neq 0} = 0 \Rightarrow B = 0$

$X(x) = 0$ (trivial solution).

(ii) $\lambda = 0$

$X(x) = Ax + B$

$X(0) = B = 0$

$X(L) = AL = 0 \rightarrow A = 0$

$X(x) = 0$ (trivial).

(iii) $\lambda = p^2$

$X(x) = A\cos(px) + B\sin(px)$

$X(0) = A = 0$ so $X(x) = B\sin(px)$

$X(L) = B\sin(pL) = 0$

$B = 0$ (trivial) or $\sin(pL) = 0$

$\Rightarrow pL = n\pi \quad , n = 1, 2, 3, \dots$

So $\lambda = \frac{n^2\pi^2}{L^2}$

$T(t) = e^{-\lambda t} = e^{-\frac{n^2\pi^2}{L^2}t}$

$X(x) = B\sin(px) = B\sin\left(\frac{n\pi}{L}x\right)$

$\square$

<u>Lemma 2.6</u>

A seperated solution $X(x)T(t)$ of the heat equation satisfying $X'(0) = 0$ $X'(L) = 0$ has the form

$$A\cos\left(\frac{n\pi x}{L}\right)e^{-\frac{n^2\pi^2}{L^2}t}$$

(proof as exercise).

<u>Neumann</u>

$\frac{\partial\phi}{\partial x}(0, t) = 0 \qquad \frac{\partial\phi}{\partial x}(L, t) = 0$

$X'(0)T(t) = 0 \qquad X'(L)T(t) = 0$

$X'(0) = 0 \qquad X'(L) = 0$

(i) $\lambda = -p^2$ (trivial)

(ii) $\lambda = 0$ (trivial)

(iii) $\lambda = p^2$

$$X(x) = A\cos(px) + B\sin(px)$$
$$X'(x) = -Ap\sin(px) + Bp\cos(px)$$
$$X'(0) = Bp = 0 \Rightarrow B = 0$$
$$\Rightarrow X'(L) = -Ap\sin(pL) = 0$$
$$pL = n\pi$$
$$X(x) = A\cos\frac{n\pi x}{L}$$

Cont. of lemma 2.5

Superposition of the seperated solutions

$$\phi(x,t) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{L}\right) e^{\left(\frac{-n^2\pi^2}{L^2}t\right)}$$

Check
$$\frac{\partial\phi}{\partial t} - \frac{\partial^2\phi}{\partial x^2} = \sum_{n=1}^{\infty} B_n\left[\frac{\partial}{\partial t}\left(\sin\left(\frac{n\pi x}{L}\right)e^{\left(\frac{-n^2\pi^2 t}{L^2}\right)}\right) - \frac{\partial^2}{\partial x^2}\left(\sin\left(\frac{n\pi x}{L}\right)e^{\left(\frac{-n^2\pi^2}{L^2}\right)t}\right)\right] = 0$$

$B_n = ?$

Initial condition $\phi(x,0) = F(x)$

$$\sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{L}\right) = F(x)$$ Fourier series

$F(x)$ gives $0 < x < L$

$$\frac{\partial \phi}{\partial t} = \frac{\partial^2 \phi}{\partial x^2} \qquad 0 < x < \pi$$

$$\phi(0,t) = 0 \qquad \phi(\pi, t) = -\pi^2 \qquad \text{(Dirichlet) (BC)}$$

$$\phi(x,0) = -x^2 \qquad \text{(IC)}$$

$$\phi_0(x,t) = M + \frac{N-M}{\pi} x \quad = -\pi x$$

$$M = 0 \qquad N = -\pi^2$$

$$\phi_0(x,t) = \alpha x + \beta$$
$$\phi_0(0,t) = 0 \qquad \phi_0(L,t) = -\pi^2$$
$$\alpha L = -\pi^2, \quad \alpha = -\frac{\pi^2}{L} = -\pi$$
$$L = \pi$$

$$\theta = \phi - \phi_0 = \phi + \pi x$$
$$\frac{\partial \theta}{\partial t} = \frac{\partial^2 \theta}{\partial x^2}$$
$$\theta(0,t) = 0 \qquad \theta(\pi, t) = 0$$
$$\theta(x,0) = \phi(x,0) + \pi x$$
$$= -x^2 + \pi x = x(\pi - x)$$

$$\theta(x,t) = \sum_{n=1}^{\infty} B_n \sin(nx) e^{-n^2 t}$$

$$x(\pi - x) = \sum_{n=1}^{\infty} B_n \sin(nx)$$

$$B_n = \frac{4}{n^3 \pi} \left[(-1)^{n+1} + 1 \right]$$

$$\left[e^{-n^2 t} \to 0 \text{ as } t \to \infty \right]$$

$$\phi = \theta - \pi x$$
$$= \sum_{n=1}^{\infty} \frac{4}{n^3 \pi} \left[(-1)^{n+1} + 1 \right] \sin x \, e^{-n^2 t} \underbrace{- \pi x}_{\phi_0}$$

$$\phi \to \phi_0 = -\pi x \text{ as } t \to \infty$$

14-10-16

Heat equation

$$\frac{\partial \phi}{\partial t} = \frac{\partial^2 \phi}{\partial x^2} \qquad 0 < x < L$$

Dirichlet BC $\quad \phi(0,t) = 0 \;, \quad \phi(L,t) = 0$

IC $\quad \phi(x,0) = F(x)$

$$\phi(x,t) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{L}\right) e^{\left(-\frac{n^2\pi^2}{L^2} t\right)}$$

$$\sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{L}\right) = F(x)$$

(fourier)

Neumann BC

$$\frac{\partial \phi}{\partial x}(0,t) = 0 \qquad \frac{\partial \phi}{\partial x}(L,t) = 0$$

$$\phi(x,t) = \sum_{n=0}^{\infty} B_n \cos\left(\frac{n\pi x}{L}\right) e^{\left(-\frac{n^2\pi^2}{L^2} t\right)}$$

$$\left\{\begin{array}{l} \dfrac{\partial \phi}{\partial t} = \dfrac{\partial^2 \phi}{\partial x^2} \\ \dfrac{\partial \phi}{\partial x}(0,t) = 0 \qquad \dfrac{\partial \phi}{\partial x}(\pi,t) = 0 \end{array}\right. \qquad L = \pi \;, \text{ so } 0 < x < \pi$$

$$\phi(x,0) = \sin^2 x$$

$$\phi(x,t) = \sum_{n=0}^{\infty} B_n \cos nx \, e^{(-n^2 t)}$$

$$\sum_{n=0}^{\infty} B_n \cos nx = \sin^2 x = \frac{1}{2} - \frac{1}{2}\cos 2x$$

$$\underbrace{B_0 + B_1 \cos x + B_2 \cos 2x + B_3 \cos 3x + \dots}$$

$$B_0 \to \tfrac{1}{2} \qquad B_2 \to -\tfrac{1}{2}$$

$$B_0 = \frac{1}{2} \;, \quad B_2 = -\frac{1}{2}$$

$B_m = 0$ otherwise.

So $\phi(x,t) = \sum_{n=0}^{\infty} B_n \cos(nx) e^{-n^2 t}$

$= B_0 + B_2 \cos 2x \, e^{-4t}$

$= \frac{1}{2} - \frac{1}{2}\cos 2x \, e^{-4t}$

Wave Equation

$$\frac{1}{c^2}\frac{\partial^2 \phi}{\partial t^2} = \frac{\partial^2 \phi}{\partial x^2}$$

$\phi(x,t) = X(x)T(t)$

$\frac{\partial^2 \phi}{\partial t^2} = X(x)\,T''(t)$

$\frac{\partial^2 \phi}{\partial x^2} = X''(x)\,T(t)$

$\Rightarrow \frac{1}{c^2} X(x) T''(t) = X''(x) T(t)$

So $\underbrace{\frac{1}{c^2}\frac{T''(t)}{T(t)}}_{\text{does not depend on } t} = \underbrace{\frac{X''(x)}{X(x)}}_{\text{does not depend on } x} = \text{constant} = -\lambda$

ODE: $\begin{cases} X''(x) + \lambda X(x) = 0 \\ T''(x) + \lambda c^2 T(t) = 0 \end{cases}$

Boundary conditions

Dirichlet BC: $\phi(0,t) = 0 \quad \phi(L,t) = 0$

Neumann BC: $\frac{\partial \phi}{\partial x}(0,t) = 0 \quad \frac{\partial \phi}{\partial x}(L,t) = 0$

$\Rightarrow X'(0) = 0, \quad X'(L) = 0$

$X(0)T(t) = 0 \rightarrow X(0) = 0$

$X(L)T(t) = 0 \rightarrow X(L) = 0$

MATH 2401

14-10-16

$$\begin{cases} X'' + \lambda X = 0 \\ X(0) = 0, \ X(L) = 0 \end{cases}$$

positive $\lambda = p^2$, $X(x) = A\cos(px) + B\sin(px)$ ①

$\lambda = 0$, $X(x) = Ax + B$ ②

$\lambda = -p^2$, $X(x) = A\cosh(px) + B\sinh(px)$ ③

② & ③ are trivial.

①: $X(0) = 0 \Rightarrow A = 0$

$X(L) = B\sin(pL) = 0$

so $pL = n\pi$, $n = 1, 2, 3, \ldots$

$\Rightarrow p = \frac{n\pi}{L}$

so $\lambda = \frac{n^2\pi^2}{L^2}$

So $X(x) = B\sin\left(\frac{n\pi x}{L}\right)$, $n = 1, 2, 3, \ldots$

$$T''(t) + \frac{m^2\pi^2}{L^2} c^2 T(t) = 0$$

$$T(t) = C\cos\left(\frac{n\pi c t}{L}\right) + D\sin\left(\frac{n\pi c t}{L}\right)$$

$$X(x) = \sin\left(\frac{n\pi x}{L}\right)$$

$$\phi(x,t) = \sum_{n=1}^{\infty} \left[C_n \cos\left(\frac{n\pi ct}{L}\right) + D_n \sin\left(\frac{n\pi ct}{L}\right)\right] \sin\left(\frac{n\pi x}{L}\right)$$

IC $\phi(x,0) = F(x)$, $\frac{\partial\phi}{\partial t}(x,0) = G(x)$

$$= \sum_{n=1}^{\infty} C_n \sin\left(\frac{n\pi x}{L}\right)$$

$$= \sum_{n=1}^{\infty} \left[C_n\left(-\frac{n\pi c}{L}\right)\sin\left(\frac{n\pi ct}{L}\right) + D_n\left(-\frac{n\pi c}{L}\right)\cos\left(\frac{n\pi ct}{L}\right)\right]\sin\left(\frac{n\pi x}{L}\right)$$

So from the second condition

$$\sum_{n=1}^{\infty} D_n\left(\frac{n\pi c}{L}\right)\sin\left(\frac{n\pi x}{L}\right) = G(x)$$

(Fourier)

$$\frac{1}{c^2}\frac{\partial^2\phi}{\partial t^2} = \frac{\partial^2\phi}{\partial x^2} \qquad 0 < x < \pi \qquad L = \pi$$

B.C. $\phi(0,t) = 0 \qquad \phi(\pi, t) = 0$

I.C. $\phi(x,0) = x(\pi - x) \qquad \frac{\partial\phi}{\partial t}(x,0) = 0$

$F(x) = x(\pi - x) \qquad G(x) = 0$

$\Rightarrow D_n = 0.$

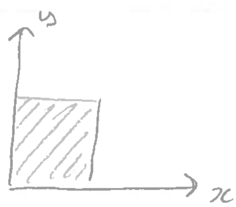
$$\sum_{n=1}^{\infty} C_n \sin\left(\frac{n\pi x}{L}\right) = x(\pi - x)$$

$$C_n = \frac{4}{n^3\pi}\left[(-1)^{n+1} + 1\right]$$

$$\phi(x,t) = \sum_{n=1}^{\infty} \frac{4}{n^3\pi}\left[(-1)^{n+1} + 1\right]\cos(nct)\sin(nx).$$

<u>Laplace equation</u>

$$\phi(x,y) = \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} = 0$$



$\phi(x,y) = X(x)\,Y(y)$

$\frac{\partial^2\phi}{\partial x^2} = X''(x)\,Y(y)$

$\frac{\partial^2\phi}{\partial y^2} = X(x)\,Y''(y)$

so $\phi(x,y) = X''(x)Y(y) + X(x)Y''(y) = 0$

MATH 2401

14-10-16

$$\underbrace{\frac{X''(x)}{X(x)}}_{\text{does not depend on } y} = - \underbrace{\frac{Y''(y)}{Y(y)}}_{\text{does not depend on } x} = \text{constant} = -\lambda$$

$$\begin{cases} X''(x) + \lambda X(x) = 0 \\ Y''(y) - \lambda Y(y) = 0 \end{cases}$$

$$X(x) = \begin{cases} A\cos(px) + B\sin(px), & \text{if } \lambda = p^2 \\ Ax + B, & \text{if } \lambda = 0 \\ A\cosh(px) + B\sinh(px), & \text{if } \lambda = -p^2 \end{cases}$$

$$Y(y) = \begin{cases} C\cosh(py) + D\sinh(py), & \text{if } \lambda = p^2 \\ Cy + D, & \text{if } \lambda = 0 \\ C\cos(py) + D\sin(py), & \text{if } \lambda = -p^2 \end{cases}$$

MATH 2401

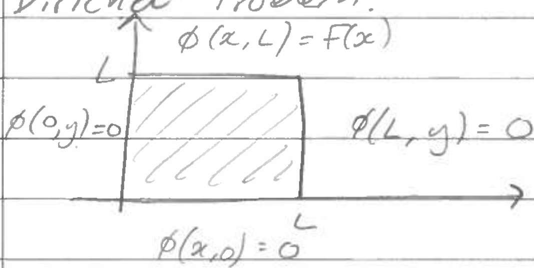
21-10-16

Laplace's equation

$\phi(x,y)$, $\dfrac{\partial^2 \phi}{\partial x^2} + \dfrac{\partial^2 \phi}{\partial y^2} = 0$

$\left[\phi(x,y,z) , \dfrac{\partial^2 \phi}{\partial x^2} + \dfrac{\partial^2 \phi}{\partial y^2} + \dfrac{\partial^2 \phi}{\partial z^2} = 0 \right]$

Dirichlet Problem.



$\phi(x,y) = X(x)\,Y(y)$

$\Rightarrow X''(x)\,Y(y) + X(x)\,Y''(y) = 0$

$\Rightarrow \dfrac{X''(x)}{X(x)} = -\dfrac{Y''(y)}{Y(y)} = \text{constant} = -\lambda$

So $\begin{cases} X''(x) + \lambda X(x) = 0 \\ Y''(y) - \lambda Y(y) = 0 \end{cases}$

So $X(x) = \begin{cases} A\cos(px) + B\sin(px) & , \lambda = p^2 \\ Ax + B & , \lambda = 0 \\ A\cosh(px) + B\sinh(px) & , \lambda = -p^2 \end{cases}$

$Y(y) = \begin{cases} C\cosh(py) + D\sinh(py) & , \lambda = p^2 \\ Cy + D & , \lambda = 0 \\ C\cos(py) + D\sin(py) & , \lambda = -p^2 \end{cases}$

Lemma 2.15

The only seperated solution satisfying

$\phi(x,0)=0 \rightarrow X(x)Y(0)=0 \rightarrow Y(0)=0$

$\phi(0,y)=0 \rightarrow X(0)Y(y)=0 \rightarrow X(0)=0$

$\phi(L,y)=0 \rightarrow X(L)Y(y)=0 \rightarrow X(L)=0$

is $D_n \sin\left(\frac{n\pi x}{L}\right)\sinh\left(\frac{n\pi y}{L}\right)$, $n=1,2,3,\ldots$

$$\begin{cases} X'' + \lambda X = 0 \\ X(0)=0, \ X(L)=0 \end{cases}$$

$\Longrightarrow X(x) = B\sin\left(\frac{n\pi x}{L}\right)$, $\lambda = \frac{n^2\pi^2}{L^2}$ (Lemma 2.5)

[$X(x) = A\cos(px) + B\sin(px)$ is non trivial.

$X(0)=0 \rightarrow A=0$

$X(L)=0 \rightarrow B\sin(pL)=0$

$\Rightarrow \sin(pL)=0 \Rightarrow pL = n\pi \Rightarrow p=\left(\frac{n\pi}{L}\right)$, $n=1,2,3,\ldots$]

$Y(y) = C\cosh\left(\frac{n\pi y}{L}\right) + D\sinh\left(\frac{n\pi y}{L}\right)$

$Y(0)=0 \rightarrow C=0$

So, $XY = D\sin\left(\frac{n\pi x}{L}\right)\sinh\left(\frac{n\pi y}{L}\right)$

$$\phi(x,y) = \sum_{n=1}^{\infty} D_n \sin\left(\frac{n\pi x}{L}\right)\sinh\left(\frac{n\pi y}{L}\right)$$

$\phi(x,L) = f(x)$

MATH 2401

21-10-16

$$\Rightarrow \sum_{n=1}^{\infty} D_n \sin\left(\frac{n\pi x}{L}\right) \sinh(n\pi) = f(x)$$

$$= \sum_{n=1}^{\infty} f_n \sin\left(\frac{n\pi x}{L}\right)$$

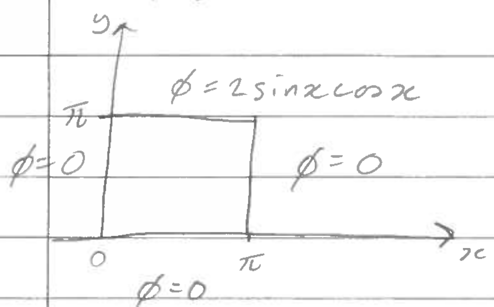
where $f_n = D_n \sinh(n\pi)$

So $D_n = \dfrac{f_n}{\sinh(n\pi)}$

$$\therefore \phi(x,y) = \sum_{n=1}^{\infty} \frac{f_n}{\sinh(n\pi)} \sin\left(\frac{n\pi x}{L}\right) \sinh\left(\frac{n\pi y}{L}\right)$$

Example

$f(x) = 2\sin x\cos x$, $L = \pi$



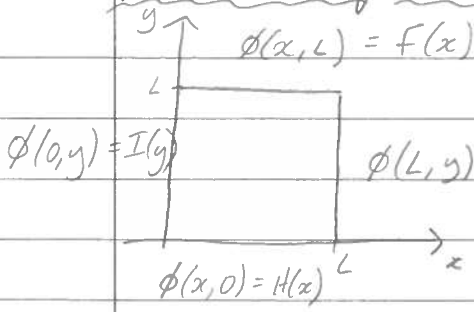
$$f(x) = \sum_{n=1}^{\infty} f_n \sin nx = \sin 2x$$

So $f(x) = f_1 \sin x + f_2 \sin 2x + \ldots = \sin 2x$

$\Rightarrow f_2 = 1$, $f_n = 0 \quad n \neq 2$

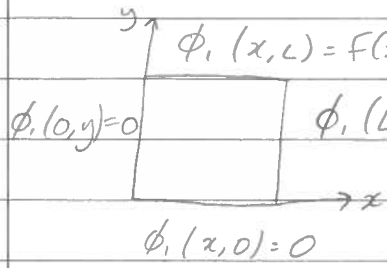
So $\phi(x,y) = \dfrac{1}{\sin 2\pi} \sin 2x \sinh 2y$

More complicated case

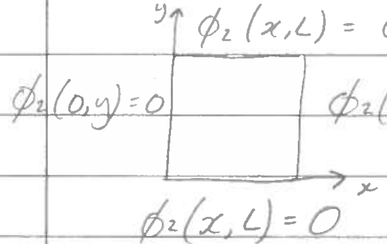


$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0$$

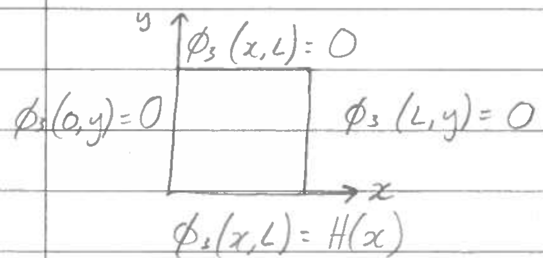
$$\phi(x,y) = \phi_1(x,y) + \phi_2(x,y) + \phi_3(x,y) + \phi_4(x,y)$$



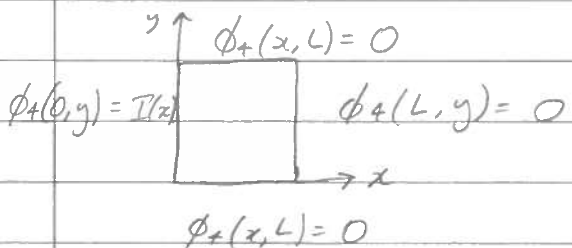
$$\frac{\partial^2 \phi_1}{\partial x^2} + \frac{\partial^2 \phi_1}{\partial y^2} = 0$$



$$\frac{\partial^2 \phi_2}{\partial x^2} + \frac{\partial^2 \phi_2}{\partial y^2} = 0$$



$$\frac{\partial^2 \phi_3}{\partial x^2} + \frac{\partial^2 \phi_3}{\partial y^2} = 0$$



$$\frac{\partial^2 \phi_4}{\partial x^2} + \frac{\partial^2 \phi_4}{\partial y^2} = 0$$

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = \frac{\partial^2}{\partial x^2}\left(\phi_1 + \phi_2 + \phi_3 + \phi_4\right) + \frac{\partial^2}{\partial y^2}\left(\phi_1 + \phi_2 + \phi_3 + \phi_4\right)$$

MATH 2401

21-10-16

So $$\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2}=\overbrace{\left(\frac{\partial^2\phi_1}{\partial x^2}+\frac{\partial^2\phi_1}{\partial y^2}\right)}^{0}+\overbrace{\left(\frac{\partial^2\phi_2}{\partial x^2}+\frac{\partial^2\phi_2}{\partial y^2}\right)}^{0}$$
$$+\underbrace{\left(\frac{\partial^2\phi_3}{\partial x^2}+\frac{\partial^2\phi_3}{\partial y^2}\right)}_{0}+\underbrace{\left(\frac{\partial^2\phi_4}{\partial x^2}+\frac{\partial^2\phi_4}{\partial y^2}\right)}_{0}=0$$

$$\phi(x,L)=\phi_1(x,L)+\phi_2(x,L)+\phi_3(x,L)+\phi_4(x,L)$$
$$=F(x)+0+0+0=F(x)\ \checkmark$$

⋮

etc.

Assume (at first) that the corner values $\phi(0,0)$, $\phi(0,L)$, $\phi(L,0)$ and $\phi(L,L)$ are all zero.

$$\phi_1(x,y)=\sum_{n=1}^{\infty}F_n\sin\left(\frac{n\pi x}{L}\right)\frac{\sinh\left(\frac{n\pi y}{L}\right)}{\sinh(n\pi)}$$ ✓ (derived above)

$$\phi_2(x,y)=\sum_{n=1}^{\infty}G_n\sin\left(\frac{n\pi y}{L}\right)\frac{\sinh\left(\frac{n\pi x}{L}\right)}{\sinh(n\pi)}$$

$$\phi_3(x,y)=\sum_{n=1}^{\infty}H_n\sin\left(\frac{n\pi x}{L}\right)\frac{\sinh\left(\frac{n\pi(L-y)}{L}\right)}{\sinh(n\pi)}$$

$$\phi_4(x,y)=\sum_{n=1}^{\infty}I_n\sin\left(\frac{n\pi y}{L}\right)\frac{\sinh\left(\frac{n\pi(L-x)}{L}\right)}{\sinh(n\pi)}$$

$\underline{\phi_2}$

$\phi_2(x,L) = 0$, $\phi_2(0,y) = 0$, $\phi_2(x,0) = 0$

$\downarrow \quad \downarrow \quad \downarrow$

$X(x)Y(L) = 0 \quad X(0)Y(y) = 0 \quad X(x)Y(0) = 0$

$\downarrow \quad \downarrow \quad \downarrow$

$Y(L) = 0 \quad X(0) = 0 \quad Y(0) = 0$

$$\begin{cases} Y'' - \lambda Y = 0 \\ Y(0) = 0, \quad Y(L) = 0 \end{cases}$$

$Y(y) = C\cos(py) + D\sin(py)$ is non-trivial solution.
This gives $Y(y) = D\sin\left(\frac{n\pi y}{L}\right)$ $\left[p = \frac{n\pi}{L}, \quad \lambda = -\frac{n^2\pi^2}{L^2} < 0\right]$

$X(x) = A\cosh\left(\frac{n\pi x}{L}\right) + B\sinh\left(\frac{n\pi x}{L}\right)$

$X(0) = 0 \Rightarrow A = 0$

So $X(x)Y(y) = B_n \sinh\left(\frac{n\pi x}{L}\right)\sin\left(\frac{n\pi y}{L}\right)$

So $\phi_2(x,y) = \sum_{n=1}^{\infty} B_n \sinh\left(\frac{n\pi x}{L}\right)\sin\left(\frac{n\pi y}{L}\right)$

$\phi_2(L,y) = G(y)$

$\Rightarrow \sum_{n=1}^{\infty} B_n \sinh(n\pi)\sin\left(\frac{n\pi y}{L}\right) = G(y) = \sum_{n=1}^{\infty} G_n \sin\left(\frac{n\pi y}{L}\right)$

So $B_n = \dfrac{G_n}{\sinh n\pi}$

MATH
2401

21-10-16

$\underline{\phi_3}$

$\phi_3(x, L) = 0 \longrightarrow X(x)Y(L) = 0 \longrightarrow Y(L) = 0$

$\phi_3(0, y) = 0 \longrightarrow X(0)Y(y) = 0 \longrightarrow X(0) = 0$

$\phi_3(L, y) = 0 \rightarrow X(L)Y(y) = 0 \rightarrow X(L) = 0$

$$\begin{cases} X'' + \lambda X = 0 \\ X(0) = 0, \ X(L) = 0 \end{cases}$$

This gives $X(x) = A\sin\left(\frac{n\pi x}{L}\right)$ $\left[p = \frac{n\pi}{L}, \ \lambda = \frac{n^2\pi^2}{L^2} > 0\right]$

$Y(y) = C\cosh\left(\frac{n\pi}{L}y\right) + D\sinh\left(\frac{n\pi y}{L}\right)$

$Y(L) = C\cosh(n\pi) + D\sinh(n\pi) = 0$

$D = -\dfrac{C\cosh(n\pi)}{\sinh(n\pi)}$

$$Y(y) = C\cosh\left(\frac{n\pi y}{L}\right) - \frac{C\cosh(n\pi)}{\sinh(n\pi)}\sinh\left(\frac{n\pi y}{L}\right)$$

$$= \frac{C}{\sinh(n\pi)}\left[\sinh(n\pi)\cosh\left(\frac{n\pi y}{L}\right) - \cosh(n\pi)\sinh\left(\frac{n\pi y}{L}\right)\right]$$

$$= \frac{C}{\sinh(n\pi)}\left[\sinh\left(n\pi - \frac{n\pi y}{L}\right)\right]$$

$\left[\sinh(A - B) = \sinh A\cosh B - \sinh B\cosh A\right]$

So $\phi_3 = \displaystyle\sum_{n=1}^{\infty} C_n \sin\left(\frac{n\pi x}{L}\right) \frac{\sinh\left(n\pi - \frac{n\pi y}{L}\right)}{\sinh(n\pi)}$

$\phi_3(x,0) = H(x)$

So $\displaystyle\sum_{n=1}^{\infty} C_n \sin\left(\frac{n\pi x}{L}\right) = H(x) = \sum_{n=1}^{\infty} H_n \sin\left(\frac{n\pi x}{L}\right)$

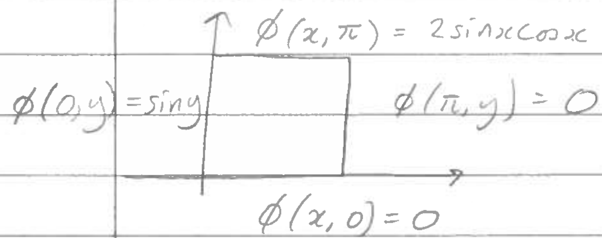
$\Rightarrow C_n = H_n.$

ϕ_4

Similar to ϕ_3 but with x and y interchanged.

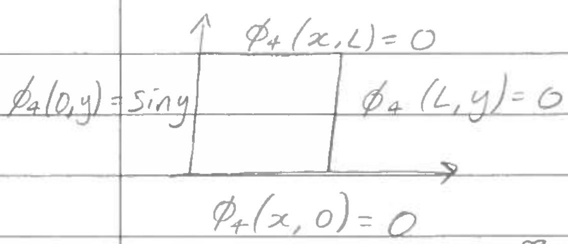
Example

$L = \pi$



Here $\phi = \phi_1 + \phi_4$

$\phi_1 = \sin(2x)\dfrac{\sinh(2y)}{\sinh(2\pi)}$ (done before)



$I(y) = \sin y = \displaystyle\sum_{n=1}^{\infty} I_n \sin ny$

$= I_1 \sin y + I_2 \sin 2y + \ldots$

$\Rightarrow I_1 = 1, \quad I_n = 0 \quad n \neq 1.$

MATH 2401

21-10-16

So $\phi_4 = \sin y \dfrac{\sinh(\pi - x)}{\sinh(\pi)}$

So $\phi = \phi_1 + \phi_4$

$$= \frac{\sin 2x \sinh 2y}{\sinh 2\pi} + \frac{\sin y \sinh(\pi - x)}{\sinh \pi}$$

Remark about the heat equation

$$\frac{\partial \phi}{\partial t} = \frac{\partial^2 \phi}{\partial x^2}$$

B.C. $\phi(0,t) = M, \quad \phi(L,t) = N$

I.C $\phi(x,0) = f(x)$

Case 1 $M = N = 0$

$$\phi(x,t) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{L}\right) e^{\frac{-n^2\pi^2}{L^2} t}$$

Case 2 $M \neq 0, N \neq 0$

$\phi_0(0,t) = M \quad \phi_0(L,t) = N, \quad \phi_0(x,t) = M + \dfrac{M-N}{L} x \quad ?$

Define $\theta(x,t) = \phi(x,t) - \phi_0(x,t)$

$$\frac{\partial \theta}{\partial t} = \frac{\partial^2 \theta}{\partial x^2}$$

$\theta(0,t) = 0, \quad \theta(L,t) = 0$

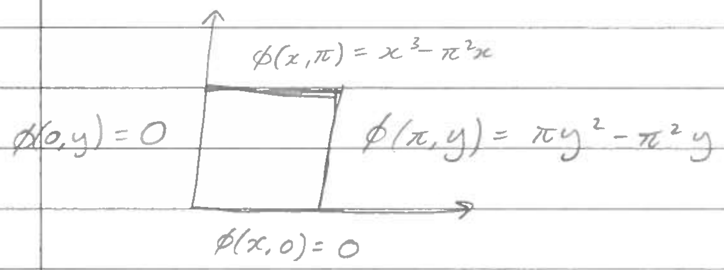
$\theta(x,0) = \phi(x,0) - \phi_0(x,0)$

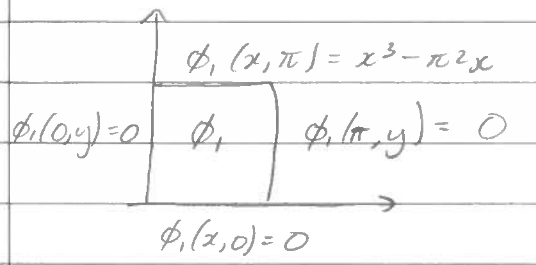
$$\theta(x,t) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{L}\right) e^{\frac{-n^2\pi^2}{L^2} t}$$

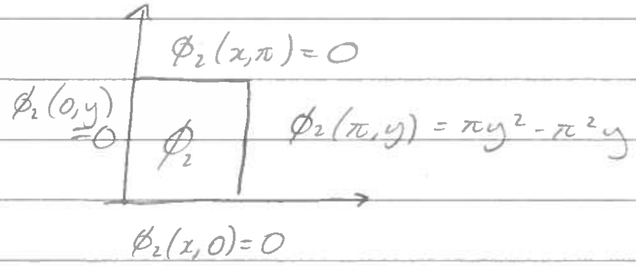
$$\theta(x,0) = \phi(x,0) - \phi_0(x,0) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{L}\right)$$

Example

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0 \quad , \quad L = \pi$$







$$\phi = \phi_1 + \phi_2$$

$$\phi_1 = \sum_{n=1}^{\infty} F_n \sin(nx) \cdot \frac{\sinh(ny)}{\sinh(n\pi)}$$

$$\phi_2 = \sum_{n=1}^{\infty} G_n \sin(ny) \frac{\sinh(nx)}{\sinh(n\pi)}$$

$$x^3 - \pi^2 x = \sum_{n=1}^{\infty} F_n \sin(nx) \rightarrow F_n = \frac{2}{\pi}\int_0^{\pi} (x^3 - \pi^2 x) \sin(nx)\,dx$$

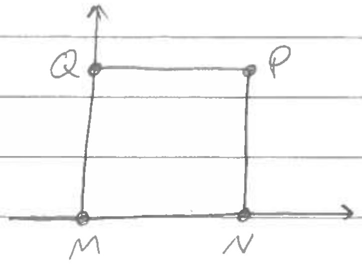
$$\pi y^2 - \pi^2 y = \sum_{n=1}^{\infty} G_n \sin(ny) \rightarrow \ldots$$

$$\phi(x,y) = \phi_1(x,y) + \phi_2(x,y)$$

MATH 2401

21-10-16

Case when the values at the corners are non zero



$\phi(0,0) = M \qquad \phi(0,L) = Q$

$\phi(L,0) = N \qquad \phi(L,L) = P$

$$\phi_0(x,y) = Axy + Bx + Cy + D$$

We want to find A, B, C and D such that

$\phi_0(0,0) = M \rightarrow D = M$

$\phi_0(0,L) = Q \rightarrow CL + D = Q$

$\phi_0(L,0) = N \rightarrow BL + D = N$

$\phi_0(L,L) = P \rightarrow AL^2 + BL + CL + D = P$

so $D = M$, $C = \dfrac{Q-M}{L}$, $B = \dfrac{N-M}{L}$,

$$A = \frac{P - BL - CL - D}{L^2}$$

Want to know if $\phi_0(x,y)$ satisfies the Laplace equation:

$\dfrac{\partial \phi_0}{\partial x} = Ay + B \qquad \dfrac{\partial \phi_0}{\partial y} = Ax + C$

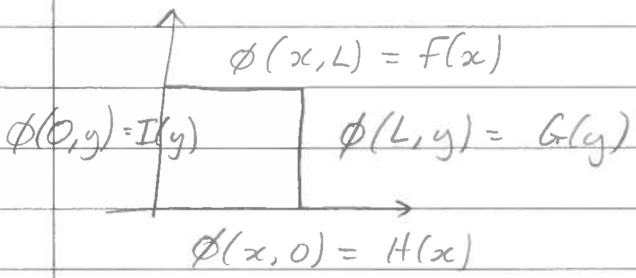
$\dfrac{\partial^2 \phi}{\partial x^2} = 0 \qquad \dfrac{\partial^2 \phi}{\partial y^2} = 0$

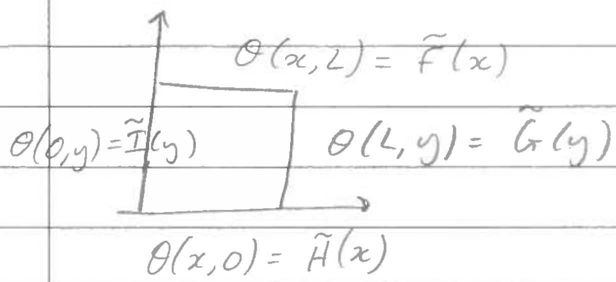
$$\Rightarrow \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0 \quad \checkmark$$

Define $\theta(x,y) = \phi(x,y) - \phi_0(x,y)$

$$\frac{\partial^2 \theta}{\partial x^2} + \frac{\partial^2 \theta}{\partial y^2} = 0$$

$\theta = 0$ at the corners.



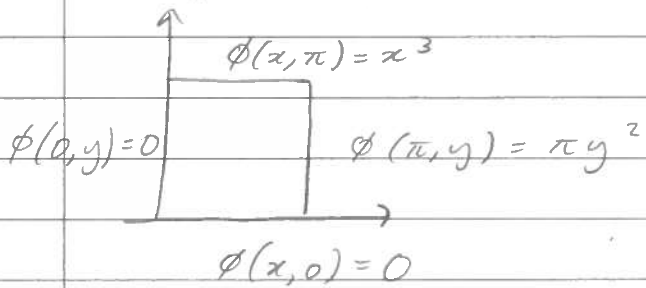


where

$$\tilde{F}(x) = F(x) - \phi_0(x, L)$$
$$\tilde{G}(y) = G(y) - \phi_0(L, y)$$
$$\tilde{H}(x) = H(x) - \phi_0(x, 0)$$
$$\tilde{I}(y) = I(y) - \phi_0(0, y)$$

Example

$L = \pi$



$\phi(\pi, \pi) = \pi^3$

$\phi(0, 0) = 0$

$\phi(0, \pi) = 0$

$\phi(\pi, 0) = 0$

MATH
2401

21-10-16

$$\phi_0(x,y) = Axy + \cancel{B}x + Cy + \cancel{D}$$

$\phi_0(\pi,\pi) = \pi^3 \rightarrow A\pi^2 = \pi^3 \rightarrow A = \pi$

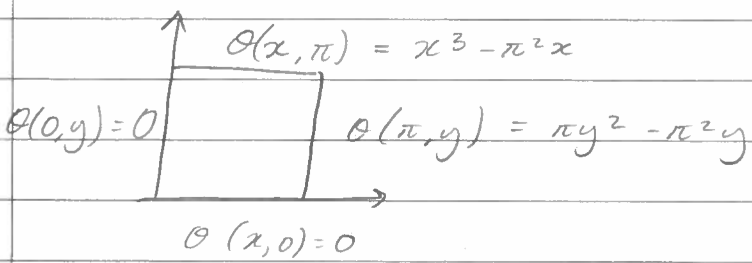
$\phi_0(0,0) = 0 \rightarrow D = 0$

$\phi_0(0,\pi) = 0 \rightarrow C\pi = 0 \rightarrow C = 0$

$\phi_0(\pi,0) = 0 \rightarrow B\pi = 0 \rightarrow B = 0$

$$\phi_0(x,y) = \pi xy$$

Define $\theta(x,y) = \phi(x,y) - \pi xy$



$\theta(0,y) = \phi(0,y) - 0 = 0$

$\theta(x,0) = \phi(x,0) - 0 = 0$

$\theta(\pi,y) = \phi(\pi,y) - \pi^2 y = \pi y^2 - \pi^2 y$

$\theta(x,\pi) = \phi(x,\pi) - \phi_0(x,\pi) = x^3 - \pi^2 x$

$\theta(x,y)$ is $\phi(x,y)$ in previous example

then.

$$\phi(x,y) = \theta(x,y) + \pi xy.$$

MATH 2401

28-10-16

Part II of Course – Calculus of variations

Recall

Function $f(x)$: "associate a number $f(x)$ to the number x."
Local maxima / minima at $f'(x) = 0 \Rightarrow x_0$
(critical points).

$f''(x_0) > 0 \Rightarrow$ minimum

$f''(x_0) < 0 \Rightarrow$ maximum

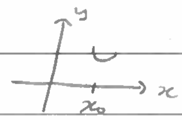
Taylor expansion:

$$f(x) = f(x_0) + \cancel{(x-x_0)f'(x_0)} + \frac{(x-x_0)^2}{2} f''(x_0) + \ldots$$

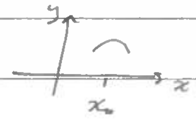
$[|x - x_0|$ small$]$

So $$f(x) - f(x_0) = \frac{(x-x_0)^2}{2} f(x_0)^2 + \ldots$$

$f''(x_0) > 0 \Rightarrow f(x) - f(x_0) > 0 \Rightarrow f(x) > f(x_0)$



$f''(x_0) < 0 \Rightarrow f(x) - f(x_0) < 0 \Rightarrow f(x) < f(x_0)$



$$A(y) = \int_a^b L(x, y(x), y'(x))\, dx$$

Example
$\boxed{L = x^2 y(x) + [y'(x)]^2}$

Functional: "associate a number $A(y)$ to the function $y(x)$"

Consider the space V consisting of functions $y:[a,b] \to \mathbb{R}$ satisfying the boundary conditions $y(a) = y_a$ and $y(b) = y_b$.

A function $L(p,q,r)$ of 3 variables is called a LAGRANGIAN.
It defines a functional $A : V \to \mathbb{R}$ by

$$A(y) = \int_a^b L(x, y(x), y'(x))\,dx$$

$$\begin{aligned} p &\longrightarrow x \\ q &\longrightarrow y(x) \\ r &\longrightarrow y'(x) \end{aligned}$$

Critical points of $A(y)$?

Let $y(x)$ be a critical point.
Consider $y(x) + \varepsilon t(x)$ where $t(a) = 0$ and $t(b) = 0$, and $|\varepsilon|$ is small.

$y(a) + \varepsilon t(a) = y_a$
$y(b) + \varepsilon t(b) = y_b$.
So $y(x) + \varepsilon t(x) \in V$.

$$T(\varepsilon) = \int_a^b L(x, y(x) + \varepsilon t(x), y'(x) + \varepsilon t'(x))\,dx$$

We want $T'(\varepsilon) = 0$ when $\varepsilon = 0$

$$\begin{aligned} T'(\varepsilon) &= \int_a^b \left[\frac{\partial L}{\partial y} t(x) + \frac{\partial L}{\partial y'} t'(x)\right] dx \qquad \text{(chain rule)} \\ &= \int_a^b \frac{\partial L}{\partial y} t(x)\,dx + \int_a^b \frac{\partial L}{\partial y'} t'(x)\,dx \\ &= \int_a^b \frac{\partial L}{\partial y} t(x)\,dx + \left[\frac{\partial L}{\partial y'} t(x)\right]_a^b - \int_a^b \frac{d}{dx}\left(\frac{\partial L}{\partial y'}\right) t(x)\,dx \\ &= \int_a^b \left(\frac{\partial L}{\partial y} - \frac{d}{dx}\frac{\partial L}{\partial y'}\right) t(x)\,dx = 0 \end{aligned}$$

MATH
2401

28-10-16

The last line $\Rightarrow \dfrac{\partial L}{\partial y} - \dfrac{d}{dx}\dfrac{\partial L}{\partial y'} = 0$.

$\leftrightarrow$ Euler-Lagrange equation.

This is true for all $t(x)$ satisfying $t(a) = 0$ and $t(b) = 0$.

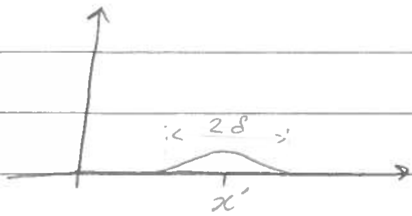
We show this by contradiction.
Assume that the Euler-Lagrange eqn is not true. Then $\exists$ at least one value $x = x'$ $(a < x' < b)$ for which

$$\frac{\partial L}{\partial y} - \frac{d}{dx}\left(\frac{\partial L}{\partial y'}\right) \neq 0.$$

W.l.o.g assume $\dfrac{\partial L}{\partial y} - \dfrac{d}{dx}\left(\dfrac{\partial L}{\partial y'}\right) > 0$

Then by continuity $\dfrac{\partial L}{\partial y} - \dfrac{d}{dx}\left(\dfrac{\partial L}{\partial y'}\right) > 0$ is in some neighborhood $[x' + \delta < x < x' + \delta]$ of x'.

Choose $t(x)$ such that $t(x) > 0$ for $x' - \delta < x' < x' + \delta$ and $t(x) = 0$ otherwise



Look at $\displaystyle\int_a^b \left[\frac{\partial L}{\partial y} - \frac{d}{dx}\frac{\partial L}{\partial y'}\right] t(x)\,dx$ for that $t(x)$.

$$\int_a^b \left[\frac{\partial L}{\partial y} - \frac{d}{dx}\frac{\partial L}{\partial y'}\right] t(x)\,dx = \int_{x'-\delta}^{x'+\delta} \underbrace{\left[\frac{\partial L}{\partial y} - \frac{d}{dx}\frac{\partial L}{\partial y'}\right]}_{>0} \underbrace{t(x)}_{>0}\,dx > 0$$

Contradiction. (as integral must $= 0$) $\square$

Summary

$$\int_a^b L(x, y(x), y'(x))\,dx$$

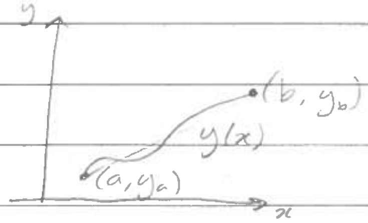
$$y(a) = y_a \quad , \quad y(b) = y_b$$

$$\frac{\partial L}{\partial y} - \frac{d}{dx}\frac{\partial L}{\partial y'} = 0$$

Example

$$\int_a^b \sqrt{1+y'^2}\,dx$$

$$L(x, y(x), y'(x)) = \sqrt{1+y'^2} \qquad \left[L(p,q,r) = \sqrt{1+r^2}\right]$$



$\int_a^b \sqrt{1+y'^2}\,dx$ is the length of the curve $y(x)$.

$$\frac{\partial L}{\partial y} = 0 \quad , \quad \frac{\partial L}{\partial y'} = \frac{y'}{\sqrt{1+y'^2}} \quad \Rightarrow \frac{d}{dx}\frac{\partial L}{\partial y'} = \frac{d}{dx}\left(\frac{y'}{\sqrt{1+y'^2}}\right) = 0$$

So we need to solve $\frac{d}{dx}\left(\frac{y'}{\sqrt{1+y'^2}}\right) = 0$, $y(a) = y_a$, $y(b) = y_b$

$$y' = c\sqrt{1+y'^2}$$
$$y'^2 = c^2(1+y'^2)$$
so $$y'^2(1-c^2) = c^2$$
$$y' = \frac{c}{\sqrt{1-c^2}} = \bar{C}$$

So $y = \bar{C}x + D$

MATH 2401

28-10-16

$$y(a) = y_a = \bar{C}a + D$$
$$y(b) = y_b = \bar{C}b + D$$

$$\bar{C}(a-b) = y_a - y_b$$
$$\Rightarrow \bar{C} = \frac{y_a - y_b}{a-b}$$
$$D = y_a - \left(\frac{y_a - y_b}{a-b}\right)a$$

So $y = \frac{y_a - y_b}{a-b} x + y_a - \left(\frac{y_a - y_b}{a-b}\right)a$

$$\Rightarrow y = \frac{y_a - y_b}{a-b}(x - a) + y_a$$

Example

$L(\underset{x}{p}, \underset{y}{q}, \underset{y'}{r}) = \frac{1}{2}\left[mr^2 - kq^2\right]$, m, k constants (+ve)

Let $\omega^2 = \frac{k}{m}$

$$A(y) = \frac{1}{2}\int_a^b \left(m(y')^2 - ky^2\right)dx$$

$$L = \frac{1}{2}(my'^2 - ky^2)$$

$$\frac{\partial L}{\partial y} = -ky, \quad \frac{\partial L}{\partial y'} = my'$$

So $-ky - \frac{d}{dx}(my') = 0$

$$-ky - my'' = 0$$
$$my'' + ky = 0$$
$$y'' + \omega^2 y = 0$$

So $y = A\sin\omega x + B\cos\omega x$

$y(a) = y_a = A\sin\omega a + B\cos\omega a$
$y(b) = y_b = A\sin\omega b + B\cos\omega b$

then solve for A & B.

Beltrami Identity

When L does not depend explicitly on x.

$$L - y'\frac{\partial L}{\partial y'} = c \quad \text{(constant)} \quad \text{1st order ODE}$$

We want to show that

$$\frac{d}{dx}\left[L - y'\frac{\partial L}{\partial y'}\right] = 0 \quad \text{note } L(y, y') \quad (\text{no } x!)$$

$$\frac{d}{dx}\left[L - y'\frac{\partial L}{\partial y'}\right] = \left[\frac{\partial L}{\partial y}y' + \cancel{\frac{\partial L}{\partial y'}y''}\right] - \left[\cancel{y''\frac{\partial L}{\partial y'}} + y'\frac{d}{dx}\left(\frac{\partial L}{\partial y'}\right)\right]$$

$$= \frac{\partial L}{\partial y}y' - y'\frac{d}{dx}\left(\frac{\partial L}{\partial y'}\right)$$

$$= y'\underbrace{\left(\frac{\partial L}{\partial y} - \frac{d}{dx}\left(\frac{\partial L}{\partial y'}\right)\right)}_{0 \text{ by Euler Lagrange eqn}}$$

$$= 0$$

Example

$L = \sqrt{1 + y'^2}$ from earlier

Using Beltrami's identity:

$$L - y'\frac{\partial L}{\partial y'} = c$$

we have $\sqrt{1+y'^2} - y'\left(\dfrac{y'}{\sqrt{1+y'^2}}\right) = C$

So $\dfrac{1 + \cancel{y'^2} - \cancel{y'^2}}{\sqrt{1+y'^2}} = c$

MATH 2401

28-10-16

So $c\sqrt{1+y'^2} = 1$

$c^2(1+y'^2) = 1$

So $y' = \sqrt{\frac{1}{c^2} - 1} = \bar{C}$

So $y = \bar{C}x + D.$

Example

$L = \frac{1}{2}[my'^2 - ky^2]$ from before

Using Beltrami's identity:

$$L - y'\frac{\partial L}{\partial y'} = C$$

So $\frac{1}{2}[my'^2 - ky^2] - y'(my') = C$

So $-\frac{1}{2}my'^2 - \frac{1}{2}ky^2 = C$

$$my'^2 + ky^2 = -2C$$

$$y'^2 + \frac{k}{m}y^2 = -\frac{2C}{m}$$

$$y'^2 = -\frac{k}{m}y^2 - \frac{2C}{m}$$

$$\frac{y'^2}{-\frac{k}{m}y^2 - \frac{2C}{m}} = 1$$

$$\frac{y'}{\sqrt{-\frac{k}{m}y^2 - \frac{2C}{m}}} = 1$$

$$\frac{y'}{\sqrt{-y^2 - \frac{2C}{k}}} = \sqrt{\frac{k}{m}}$$

So $\displaystyle\int \frac{dy}{\sqrt{\frac{-2C}{k} - y^2}} = \int \sqrt{\frac{k}{m}}\, dx$

let $y = \sqrt{\frac{-2C}{k}} \sin\theta$

So $\displaystyle\int \frac{\sqrt{\frac{-2C}{k}} \cos\theta\, d\theta}{\sqrt{\frac{-2C}{k} + \frac{2C}{k}\sin^2\theta}} = \int \sqrt{\frac{k}{m}}\, dx$

$\Rightarrow \displaystyle\int \frac{\cancel{\cos\theta}\, d\theta}{\cancel{\sqrt{1-\sin^2\theta}}} = \int \sqrt{\frac{k}{m}}\, dx$

$\Rightarrow \theta = \sqrt{\frac{k}{m}}\, x + D$

So $y = \sqrt{\frac{-2C}{k}} \sin\left[\sqrt{\frac{k}{m}}\, x + D\right]$

MATH 2401

28-10-16

Calculus of variations

$$\int_a^b L(x, y(x), y'(x))\, dx$$

$$y(a) = y_a \quad , \quad y(b) = y_b$$

Euler-Lagrange eqn: $\frac{\partial L}{\partial y} - \frac{d}{dx}\left(\frac{\partial L}{\partial y'}\right) = 0$

If L does not depend on x we can use Beltrami's identity: $L - y'\frac{\partial L}{\partial y'} = C$

Example 1

$$\int_a^b y\sqrt{1+(y')^2}\, dx \qquad y(a) = y_a \quad , \quad y(b) = y_b$$

$$L = y\sqrt{1+(y')^2} \qquad \frac{\partial L}{\partial y'} = \frac{yy'}{\sqrt{1+(y')^2}}$$

Beltrami's identity:

$$y\sqrt{1+(y')^2} - \frac{y(y')^2}{\sqrt{1+(y')^2}} = C$$

So $\frac{y}{\sqrt{1+(y')^2}} = C$

$$\frac{y^2}{1+(y')^2} = C^2$$

So $y' = \sqrt{\frac{y^2}{C^2} - 1}$

So $\int \frac{dy}{\sqrt{\frac{y^2}{C^2}-1}} = \int dx$

using $y = C\cosh\theta$ we get

$$\int \frac{C\sinh\theta\, d\theta}{\sqrt{\cosh^2\theta - 1}} = \int dx$$

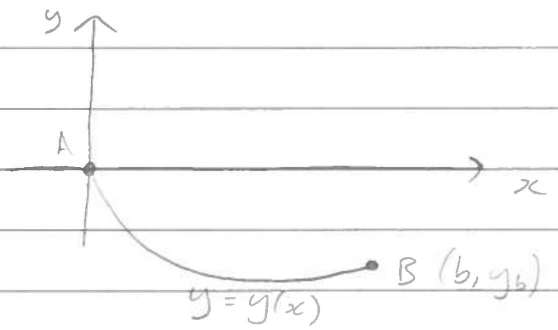
So $\theta = \dfrac{x+D}{C}$

So $y = C\cosh\left(\dfrac{x+D}{C}\right)$

Also $\begin{cases} y_a = C\cosh\left(\dfrac{a+D}{C}\right) \\ y_b = C\cosh\left(\dfrac{b+D}{C}\right). \end{cases}$

Brachistochrone problem

"shortest time".



$(a=0)$

$\left[ds = v\,dt \quad v = \dfrac{ds}{dt} \quad dt = \dfrac{ds}{v} \right]$

$$\int_a^b \frac{ds}{v(x)}$$

Conservation of energy

$\frac{1}{2}mv^2 + mgy = E = \text{const} = 0$

mass

$\Rightarrow v^2 = -2gy$

$\Rightarrow v = \sqrt{-2gy}$ note $y \le 0$

Also $ds = \sqrt{1+(y')^2}\,dx$

So $$\int_a^b \frac{ds}{v(x)} = \int_a^b \frac{\sqrt{1+(y')^2}}{\sqrt{-2gy}}\,dx$$

MATH 2401

28-10-16

$$L = \frac{\sqrt{1+(y')^2}}{\sqrt{-2gy}} \qquad \frac{\partial L}{\partial y'} = \frac{1}{\sqrt{-2gy}} \frac{1}{2} \frac{1}{\sqrt{1+(y')^2}} 2y'$$

Using Belbrani's inequality:

$$\frac{\sqrt{1+(y')^2}}{\sqrt{-2gy}} - \frac{(y')^2}{\sqrt{-2gy}\sqrt{1+(y')^2}} = C$$

So $$\frac{1+\cancel{(y')^2}-\cancel{(y')^2}}{\sqrt{-2gy}\sqrt{1+(y')^2}} = C$$

So $$\frac{1}{(-2gy)(1+(y')^2)} = C^2$$

So $$\frac{1}{-2gyC^2} - 1 = (y')^2$$

$$y' = \sqrt{\frac{1}{-2gyc^2} - 1}$$

So $$\int \frac{dy}{\sqrt{\frac{1}{-2gyc^2} - 1}} = \int dx$$

$$\int \frac{\sqrt{-y}\,dy}{\sqrt{\frac{1}{2gc^2} + y}} = \int dx$$

Let $y = -\frac{\sin^2\theta}{2gC^2}$ $\qquad \sqrt{\frac{1}{2gc^2} + y} = \sqrt{\frac{1}{2gc^2}(1-\sin^2\theta)}$

So $$\int \frac{\frac{\sin\theta}{\sqrt{2gc^2}}\left(-\frac{1}{2gc^2}\right) 2\sin\theta\cos\theta\, d\theta}{\sqrt{\frac{1}{2gc^2}}\cos\theta} = \int dx$$

So $\frac{-1}{gc^2}\int \sin^2\theta \, d\theta = \int dx$

$\Rightarrow \frac{-1}{2gc^2}\int (1-\cos 2\theta)\, d\theta = \int dx$

$-\frac{1}{2gc^2}\left(\theta - \frac{1}{2}\sin 2\theta\right) = x + D$

So $\begin{cases} x + D = \frac{-1}{2gc^2}\left(\theta - \frac{1}{2}\sin 2\theta\right) \\ y = \frac{-\sin^2\theta}{2gc^2} \end{cases}$

Example

$\int_a^b (1+x)(y')^2 \, dx \qquad y(a) = y_a \,, \; y(b) = y_b$

$L = (1+x)(y')^2 = (y')^2 + x(y')^2$

$\frac{\partial L}{\partial y} - \frac{d}{dx}\left(\frac{\partial L}{\partial y'}\right) = 0$

$\frac{\partial L}{\partial y} = 0 \,, \quad \frac{\partial L}{\partial y'} = 2y' + 2xy'$

So $-\frac{d}{dx}\left[2y' + 2xy'\right] = 0$

So $y' + xy' = C$

So $y'(1+x) = C$

$y' = \frac{C}{1+x}$

$y = C\log(1+x) + D$

MATH
2401

28-10-16

$$\begin{cases} y(a) = y_a = C\log(1+a) + D \\ y(b) = y_b = C\log(1+b) + D \end{cases}$$

$$y_a - y_b = C\log\left(\frac{1+a}{1+b}\right)$$

$$\Rightarrow C = \frac{y_a - y_b}{\log\left(\frac{1+a}{1+b}\right)}$$

$$D = y_a - C\log(1+a)$$

$$\Rightarrow D = y_a - \frac{(y_a - y_b)}{\log\left(\frac{1+a}{1+b}\right)}\log(1+a)$$

MATH 2401

04-11-16

$$A(y) = \int_a^b L(x, y, y')\,dx$$

is minimised / maximised by y satisfying

$$\frac{\partial L}{\partial y} = \frac{d}{dx}\left(\frac{\partial L}{\partial y'}\right)$$

If $\frac{\partial L}{\partial x} = 0$ then $L - y'\frac{\partial L}{\partial y'} = \text{const.}$

Constraints

We wish to find the extremal for the functional $A(y) = \int_a^b L(x, y, y')\,dx$ among the functions y that additionally satisfy a constraint of the form

$$G(y) = \int_a^b M(x, y, y')\,dx = 0.$$

Example

Find a curve of a given length that encloses a maximum area.

$$A(y) = \int_a^b y\,dx \quad \& \quad \int_a^b \sqrt{1+(y')^2}\,dx = L$$

$$\left\{ \int_a^b \sqrt{1+(y')^2} - \frac{L}{b-a}\,dx = 0 = G(y) \right\}$$

We use Lagrange multipliers.

method of Lagrange multipliers

Find the extreme values of $f(x,y)$ subject to the constraint $g(x,y) = C$.

Form the new function $h(x,y,\lambda) = f(x,y) - \underset{\text{Lagrange multiplier}}{\lambda}\, g(x,y)$ & then find the extreme values of $h(x,y,\lambda)$ as functions of x & y.

ie. solve $\frac{\partial h}{\partial x} = \frac{\partial h}{\partial y} = 0$ in conjunction with the third equation $g(x,y)=c$.

e.g. find the minimum value of x^2+y^2, subject to the constraint $x+y=1$.

Form $h(x,y,\lambda) = x^2+y^2 - \lambda(x+y-1)$

Solve
$$\begin{cases} \frac{\partial h}{\partial x} = 2x-\lambda = 0 \\ \frac{\partial h}{\partial y} = 2y-\lambda = 0 \\ g(x,y)=0 \Rightarrow \frac{\partial h}{\partial \lambda} = x+y-1=0 \end{cases}$$

$$\Rightarrow \begin{cases} x = \lambda/2 \\ y = \lambda/2 \end{cases}$$
$$\Rightarrow \lambda/2 + \lambda/2 - 1 = 0$$
$$\Rightarrow \lambda = 1$$

So $x^2+y^2 = \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$

So minimum distance $= \frac{1}{\sqrt{2}}$

We form the new functional
$$F(y,\lambda) = \int_a^b L - \lambda M \, dx$$

and then we solve $\frac{\partial(L-\lambda M)}{\partial y} = \frac{d}{dx}\left(\frac{\partial(L-\lambda M)}{\partial y'}\right)$

(a second order ode, with λ as a parameter) together with the constraint
$G(y)=0$ which fixes λ

MATH 2401

04-11-16

Example

Find the extreme value of

$\int_0^1 y'^2 + 2yy' \, dx$, $y(0) = y(1) = 0$

subject to the constraint $\int_0^1 y \, dx = \frac{1}{6}$

We consider the functional

$\int_0^1 (y'^2 + 2yy') - \lambda(y - \frac{1}{6}) \, dx$

and solve $\frac{\partial L}{\partial y} - \frac{d}{dx}\left(\frac{\partial L}{\partial y'}\right) = 0$

i.e. $(2y' - \lambda) = \frac{d}{dx}(2y' + 2y)$

So $\cancel{2y'} - \lambda = 2y'' + \cancel{2y'}$

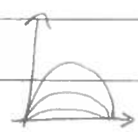
$\Rightarrow \quad y'' = -\lambda/2$

$y' = \frac{-\lambda}{2} x + A$

So $y = \frac{-\lambda}{4} x^2 + Ax + B$

$y(0) = y(1) = 0$ gives A & B

So $y = \frac{\lambda}{4}(x - x^2)$



Now use $\int_0^1 y \, dx = \frac{1}{6}$

which gives $\lambda = 4$

so $y = x - x^2$

OR using the Beltrami integral,

$$\left[\left(y'^2 + \cancel{2yy'}\right) - \lambda\left(y - \tfrac{1}{6}\right)\right] - y'\left[2y' + \cancel{2y}\right] = \text{Const.}$$

$$\Rightarrow \quad y'^2 + \lambda y = C$$

$$\frac{dy}{dx} = \pm\sqrt{C - \lambda y} \quad \Rightarrow \quad \int \frac{dy}{\sqrt{C-\lambda y}} = \int dx$$

can be brought back if necessary

$$-\frac{2}{\lambda}\sqrt{C - \lambda y} = x + A$$

Use $y(0) = y(1) = 0$ to find C & A

$$-\frac{2}{\lambda}\sqrt{C} = 0 + A$$

$$-\frac{2}{\lambda}\sqrt{C} = 1 + A$$

This can be fixed by introducing the two possible branches of $\sqrt{\ }$ or squaring.

$$\frac{4}{\lambda^2}(C - \lambda y) = (x + A)^2$$

giving
$$\begin{cases} \frac{4}{\lambda^2} C = A^2 & = A^2 \\ \frac{4}{\lambda^2} C = (1 - A)^2 & = 1 + 2A + A^2 \end{cases}$$

$$\Rightarrow \begin{cases} A = -1/2 \\ C = \lambda^2/16 \end{cases}$$

$$\frac{4}{\lambda^2}\left(\frac{\lambda^2}{16} - \lambda y\right) = \left(x - \tfrac{1}{2}\right)^2$$ and now completes as before.

Example

Minimise $\rho g \int_a^b y \sqrt{1+y'^2}\, dx$

subject to $\int_a^b \sqrt{1+y'^2}\, dx = L$

Form $\int_a^b \rho g y \sqrt{1+y'^2} - \lambda \left[\sqrt{1+y'^2} - \frac{L}{b-a} \right] dx$

Use the Beltrami identity: $L - y' \frac{\partial L}{\partial y'} = \text{constant}$.

giving $\left[\rho g y \sqrt{1+y'^2} - \lambda \left[\sqrt{1+y'^2} - \frac{L}{b-a} \right] \right]$

$$- y' \left[\frac{\rho g y y'}{\sqrt{1+y'^2}} - \frac{\lambda y'}{\sqrt{1+y'^2}} \right] = \text{const}$$

$$\frac{\rho g y}{\sqrt{1+y'^2}} \left[1 + \cancel{y'^2} - \cancel{y'^2} \right] - \frac{\lambda}{\sqrt{1+y'^2}} \left[1 + \cancel{y'^2} - \cancel{y'^2} \right] = C$$

$$\rho g y - \lambda = C \sqrt{1+y'^2} \quad (*)$$

(first order ODE for y with λ as a parameter)

$$\frac{dy}{dx} = \pm \sqrt{\left(\frac{\rho g y - \lambda}{C} \right)^2 - 1}$$

write $\frac{\rho g y - \lambda}{C} = \cosh v$

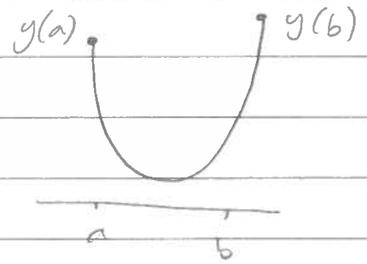
so that $\frac{\rho g}{C} \frac{dy}{dx} = \sinh v \frac{dv}{dx}$

& $\frac{C}{\rho g} \sinh v \frac{dv}{dx} = \cancel{\sqrt{\cosh^2 v - 1}}$

So $v = \frac{\rho g x}{C} + D$

$\cosh v = \cosh\left(\frac{\rho g x}{C} + D\right)$

$y = \frac{\lambda}{\rho g} + \frac{C}{\rho g}\cosh\left(\frac{\rho g}{C} + D\right)$



unknowns: C, D, λ
which can be found from boundary conditions on y & from the constraint

$\int_a^b \sqrt{1+y'^2}\,dx = L$

$= \int_a^b \frac{\rho g y - \lambda}{C}\,dx$ using $(*)$

$= \int_a^b \cosh\left(\frac{\rho g x}{C} + D\right)dx = L$

More Variables

So far we have
$A(y) = \int_a^b L(x, y, y')\,dx.$

Can we extend this to
$A(\underline{y}) = \int_a^b L(x, \underline{y}, \underline{y}')\,dx$, $\begin{cases} \underline{y} = (y_1(x), y_2(x), \dots, y_n(x)) \\ \underline{y}' = (y_1'(x), y_2'(x), \dots, y_n'(x)) \end{cases}$

$\Rightarrow A(y_1, y_2, \dots, y_n) = \int_a^b L(x, y_1, y_2, \dots, y_n, y_1', y_2', \dots, y_n')\,dx.$

We can treat this as $A(\underline{r}) = \int_{t_0}^{t_1} L(t, \underline{r}, \underline{r}')\,dt$

eg. $\int_{t_0}^{t_1} \frac{1}{2}m\underline{v}^2 - V(\underline{r})\,dt$

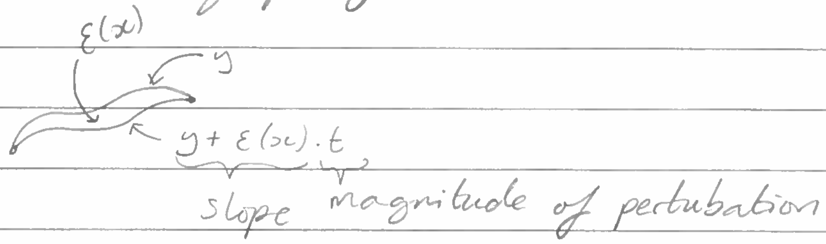
We have a system of n Euler Lagrange (EL) equations, essentially an EL equation for each variable.

$$\left.\begin{aligned} \frac{\partial L}{\partial y_1} &= \frac{d}{dx}\left(\frac{\partial L}{\partial y_1'}\right) \\ \frac{\partial L}{\partial y_2} &= \frac{d}{dx}\left(\frac{\partial L}{\partial y_2'}\right) \\ &\vdots \end{aligned}\right\} \quad \frac{\partial L}{\partial y_i} = \frac{d}{dx}\left(\frac{\partial L}{\partial y_i'}\right)$$

and if $\frac{\partial L}{\partial x} = 0$, we have the Beltrami identity: $L - y_1' \frac{\partial L}{\partial y_1'} - y_2' \frac{\partial L}{\partial y_2'} - \dots - y_n' \frac{\partial L}{\partial y_n'} = \text{const.}$

(we can write y rather than each y_i explicitly)

Motivation of proof



$$I = \int_a^b L(x, y, y')\,dx \qquad y \text{ is extremal}$$

$$y \to y + \varepsilon t$$
$$y' \to y' + \varepsilon' t$$

substitution, differentiation w.r.t. ___ which needs to be zero at $t = 0$

$$0 = \int_a^b \varepsilon \frac{\partial L}{\partial y} + \varepsilon' \frac{\partial L}{\partial y'}\,dx$$

$$0 = \int_a^b \varepsilon \frac{\partial L}{\partial y} - \varepsilon \frac{d}{dx}\left(\frac{\partial L}{\partial y'}\right) dx + 0 \qquad \text{as } \varepsilon = 0 \text{ at ends.}$$

MATH 2401

04-11-16

$$A(y) = \int_a^b L(t, y, y')\,dt$$

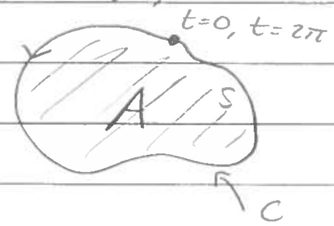
$$\frac{\partial L}{\partial y} = \frac{d}{dt}\left(\frac{\partial L}{\partial y'}\right)$$ ← Proved in exactly the same way as the single variable case.

Example

Consider a closed curve in two dimensions $(x(t), y(t))$, $t \in [0, 2\pi]$, enclosing an area A.
Find the maximum of A given that

$$\int_0^{2\pi} 2\pi(\dot{x}^2 + \dot{y}^2)\,dt = K \quad \text{(constraint)}$$

The area is $\iint_S dx\,dy$



slightly harder problem to do in own time

Obvious constraint

Maximise A given $2\pi\int_0^{2\pi} \sqrt{\dot{x}^2 + \dot{y}^2}\,dt = K$.

Maximising area enclosed in a curve of given length $\Rightarrow$ circle.

Green's Theorem

$$\oint_C (L\,dx + M\,dy) = \iint_S \left(\frac{\partial M}{\partial x} - \frac{\partial L}{\partial y}\right) dx\,dy$$

Pick $L = 0$, $M = x$

So $\frac{\partial M}{\partial x} - \frac{\partial L}{\partial y} = 1$

So $A = \iint_S dx\,dy = \oint_C (0\,dx + x\,dy) = \int_0^{2\pi} x\frac{dy}{dt}\,dt$

So we want to find the curve $(x(t), y(t))$ which maximises $\int_0^{2\pi} x \frac{dy}{dt}\, dt$ subject to $2\pi \int_0^{2\pi} (\dot{x}^2 + \dot{y}^2) dt = K$.

We use Lagrange multipliers and form the functional $\int_0^{2\pi} x \frac{dy}{dt} - \lambda(\dot{x}^2 + \dot{y}^2)\, dt$ (ignore constant K).

We expect two, inter-dependent, EL equations.

From x: $\frac{\partial L}{\partial x} = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) \Rightarrow \frac{dy}{dt} = \frac{d}{dt}(-2\lambda \dot{x})$

From y: $\frac{\partial L}{\partial y} = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{y}}\right) \Rightarrow 0 = \frac{d}{dt}(x - 2\lambda \dot{y})$

We have

$$\frac{dy}{dt} = -2\lambda \frac{d^2 x}{dt^2} \quad \text{and} \quad 0 = \frac{dx}{dt} - 2\lambda \frac{d^2 y}{dt^2}$$

take $\frac{d}{dt}$

take $\frac{d}{dt}$ and combine with other eqn.

with solutions $\begin{cases} y = a + r\cos(qt) \\ x = b + r\cos(qt) \end{cases}$ ← circles

Substitution gives

$$rq(-\sin(qt)) = -2\lambda(-q^2) r \sin qt$$

$$-q = 2\lambda q^2, \text{ i.e. } q = -\tfrac{1}{2}\lambda$$

and

$$0 = qr\cos(qt) - 2\lambda(-q^2) r\cos t \quad , \; q = -\tfrac{1}{2}\lambda$$

r can be found from the constraint

$$2\pi \int_0^{2\pi} \dot{x}^2 + \dot{y}^2 \, dt = K.$$

MATH 2401

04-11-16

Functions of several variables

$\varphi(\underline{x})$ in $\mathcal{U}$ $\leftarrow$ $\varphi = \varphi_0$, φ_0 known

We want to know which function $\varphi(\underline{x})$ makes the functional $A(\varphi) = \int_{\mathcal{U}} L(\underline{x}, \varphi, \underline{\nabla}\varphi) \underbrace{d\underline{x}\,dy}_{d\mathcal{U}}$ take an extreme value.

$d\mathcal{U} = dx_1 dx_2 \dots dx_n$ as $\underline{x} = (x_1, x_2, \dots, x_n)$

[$\mathcal{U}$, $T = T_0$ $\int(\varphi_x^{\,2} + \varphi_y^{\,2})\,dx\,dy$

gradients in $\mathcal{U}$]

The EL equation is $\dfrac{\partial L}{\partial \varphi} = \dfrac{\partial}{\partial x_1}\left(\dfrac{\partial L}{\partial \varphi_{x_1}}\right) + \dots + \dfrac{\partial}{\partial x_n}\left(\dfrac{\partial L}{\partial \varphi_{x_n}}\right)$

$\left(\varphi \to \varphi(x) + t\,\varepsilon(x), \quad \varepsilon = 0 \text{ on } \partial\mathcal{U}\right)$

This is sometimes written

$$\frac{\partial L}{\partial \varphi} = \underline{\nabla}\cdot\left(\frac{\partial L}{\partial \underline{\nabla}\varphi}\right)$$

$$\underline{\nabla} \to \begin{pmatrix} \partial/\partial x_1 \\ \partial/\partial x_2 \\ \vdots \\ \partial/\partial x_n \end{pmatrix} \qquad \frac{\partial L}{\partial \underline{\nabla}\varphi} \to \begin{pmatrix} \partial L/\partial \varphi_{x_1} \\ \partial L/\partial \varphi_{x_2} \\ \vdots \\ \partial L/\partial \varphi_{x_n} \end{pmatrix}$$

For our example, $\int_{\mathcal{U}} \underbrace{(\varphi_x^{\,2} + \varphi_y^{\,2})}_{L}\, d\mathcal{U}$

$$\frac{\partial L}{\partial \underline{\nabla}\varphi} = \begin{pmatrix} \partial L/\partial \varphi_x \\ \partial L/\partial \varphi_y \end{pmatrix} = \begin{pmatrix} 2\varphi_x \\ 2\varphi_y \end{pmatrix} = 2\underline{\nabla}\varphi$$

$\dfrac{\partial L}{\partial \varphi} = 0$ so the EL equation is $2\underline{\nabla}\cdot\underline{\nabla}\varphi = 0$

i.e. $\nabla^2\varphi = 0$.

MATH 2401

18-11-16

Part III
Method of characteristics for 1st order PDEs.

- Linear equations with constant coefficients

$$A\frac{\partial\phi}{\partial x} + B\frac{\partial\phi}{\partial y} + C(x,y) = 0 \quad , \quad \phi(x,y)$$

A and B are constant.

- Linear equations with variable coefficients

$$A(x,y)\frac{\partial\phi}{\partial x} + B(x,y)\frac{\partial\phi}{\partial y} + C(x,y) = 0$$

- Quasilinear equations

$$A(x,y,\phi)\frac{\partial\phi}{\partial x} + B(x,y,\phi)\frac{\partial\phi}{\partial y} + C(x,y,\phi) = 0$$

- Nonlinear equations

Example: $\left(\frac{\partial\phi}{\partial x}\right)^2 + \frac{\partial\phi}{\partial y} = \sin(x,y)$

Linear equations with constant coefficients

Example

$$\frac{\partial\phi}{\partial x} = 0 \quad \Rightarrow \quad \phi(x,y) = C(y)$$

Example

$$\frac{\partial\phi}{\partial x} - \frac{\partial\phi}{\partial y} = 0$$

Looks like chain rule: $\frac{\partial\phi}{\partial u} = \frac{\partial\phi}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial\phi}{\partial y}\frac{\partial y}{\partial u}$

with $\frac{\partial x}{\partial u} = 1$, $\frac{\partial y}{\partial u} = -1$

Let's change to new coordinates (u, v) satisfying $\frac{\partial x}{\partial u} = 1$, $\frac{\partial y}{\partial u} = -1$

This gives us many choices,

eg. $\begin{cases} x = u \\ y = -u + v \end{cases}$, $\begin{cases} x = u + 7v \\ y = -u \end{cases}$

$\Downarrow$ $\Downarrow$

$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix}$ $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 & 7 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix}$

note same first column

- 1st column contains the values of $\frac{\partial x}{\partial u}$ and $\frac{\partial y}{\partial u}$
- matrix non singular $\Leftrightarrow \det A \neq 0$

<u>Recall</u>

$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $A^{-1} = \frac{1}{ad - bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$

$\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$ $\begin{pmatrix} u \\ v \end{pmatrix} = \frac{1}{7}\begin{pmatrix} 0 & -7 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$

Using first choice:

$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix}$

$\phi(u, v)$, $\frac{\partial \phi}{\partial u} = \frac{\partial \phi}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial \phi}{\partial y}\frac{\partial y}{\partial u}$

$= \frac{\partial \phi}{\partial x} - \frac{\partial \phi}{\partial y} = 0$

$\frac{\partial \phi}{\partial u} = 0 \Rightarrow \phi(u, v) = C(v)$

$u = x$, $v = x + y$

So $\phi = C(x + y)$

MATH 2401

18-11-16

Using second choice:

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 & 7 \\ -1 & 0 \end{pmatrix}$$

$$\phi(u,v) = D(v)$$

$$u = -y \quad , \quad v = \frac{1}{7}(x+y)$$

So $\phi = D\left(\frac{1}{7}(x+y)\right)$

So different choices of u, v give the same answer.

Example

$$\frac{\partial \phi}{\partial x} - \frac{\partial \phi}{\partial y} = x$$

Using first choice: $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix}$, $\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$

from previous example (which is homogeneous case of this example).

From the change of variables, $x = u$

So $\frac{\partial \phi}{\partial u} = \frac{\partial \phi}{\partial x} - \frac{\partial \phi}{\partial y} = u$

$$\phi = \frac{u^2}{2} + C(v)$$

but $u = x$, $v = x + y$

So $\phi = \frac{x^2}{2} + C(x+y)$

$$\frac{\partial \phi}{\partial x} + 2\frac{\partial \phi}{\partial y} = \sin y$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix} \quad , \quad \frac{\partial x}{\partial u} = 1, \quad \frac{\partial y}{\partial u} = 2$$

$$\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} \qquad u = x, \; v = -2x + y$$

$$\frac{\partial \phi}{\partial u} = \frac{\partial \phi}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial \phi}{\partial y}\frac{\partial y}{\partial u} = \frac{\partial \phi}{\partial x} + 2\frac{\partial \phi}{\partial y} = \sin y$$

So $\dfrac{\partial \phi}{\partial u} = \sin(2u + v) \Rightarrow \phi = -\frac{1}{2}\cos(2u+v) + C(v)$

So $\phi = -\frac{1}{2}\cos(\cancel{2(x)} + (\cancel{-2x}+y)) + C(-2x+y)$

$\Rightarrow \phi = -\frac{1}{2}\cos(y) + C(-2x+y)$

<u>Example</u>

$$\frac{\partial \phi}{\partial x} + 2\frac{\partial \phi}{\partial y} = \sin y$$

<u>B.C.</u> $\phi(s, 0) = s^2 \Rightarrow x = s, \; y = 0, \; \phi = s^2$

$\phi = -\frac{1}{2}\cos y + \tilde{C}(y - 2x)$

$\phi(s, 0) = -\frac{1}{2} + \tilde{C}(-2s) = s^2$

$\Rightarrow \tilde{C}(-2s) = s^2 + \frac{1}{2}$

Let $w = -2s \Rightarrow \tilde{C}(w) = \dfrac{w^2}{4} + \dfrac{1}{2}$

Then $\tilde{C}(y - 2x) = \dfrac{(y-2x)^2}{4} + \dfrac{1}{2}$

So $\phi = -\frac{1}{2}\cos y + \dfrac{(y-2x)^2}{4} + \dfrac{1}{2}$

MATH 2401

18-11-16

General Case

$\phi(x_1, x_2, \ldots, x_n)$

New coordinates $u_1, u_2, \ldots, u_n$

$$\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} A_1 & \times & \times & \cdots & \times \\ A_2 & \times & \times & \cdots & \times \\ \vdots & & & & \vdots \\ A_n & \times & & \cdots & \times \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{pmatrix}$$

$$A_1 \frac{\partial \phi}{\partial x_1} + A_2 \frac{\partial \phi}{\partial x_2} + \ldots + A_n \frac{\partial \phi}{\partial x_n} = 0$$

$$\frac{\partial \phi}{\partial x_1} \underbrace{\frac{\partial x_1}{\partial u_1}}_{A_1} + \frac{\partial \phi}{\partial x_2} \underbrace{\frac{\partial x_2}{\partial u_1}}_{A_2} + \ldots + \frac{\partial \phi}{\partial x_n} \underbrace{\frac{\partial x_n}{\partial u_1}}_{A_n} = 0$$

$$\frac{\partial \phi}{\partial u_1} = 0 \quad , \quad \phi = C(u_2, u_3, \ldots, u_n)$$

Linear equations with variable coefficients

$$A_1(x_1, x_2, \ldots, x_n) \frac{\partial \phi}{\partial x_1} + A_2(x_1, x_2, \ldots, x_n) \frac{\partial \phi}{\partial x_2} + \ldots + A_n(x_1, x_2, \ldots, x_n) \frac{\partial \phi}{\partial x_n} = 0$$

Change of variables $u_1, u_2, u_3, \ldots, u_n$

such that $\dfrac{\partial x_i}{\partial u_i} = A_i(x_1, x_2, \ldots, x_n)$, $1 \le i \le n$.

$$\frac{\partial \phi}{\partial x_1} \frac{\partial x_1}{\partial u_1} + \frac{\partial \phi}{\partial x_2} \frac{\partial x_2}{\partial u_1} + \ldots + \frac{\partial \phi}{\partial x_n} \frac{\partial x_n}{\partial u_1} = \frac{\partial \phi}{\partial u_1}$$

Example

$$x\frac{\partial\phi}{\partial x} + y\frac{\partial\phi}{\partial y} = 0$$

$$\frac{\partial x}{\partial u} = x \qquad \frac{\partial y}{\partial u} = y$$

$$\downarrow \qquad\qquad \downarrow$$

$$\frac{\partial x}{\partial u} - x = 0 \qquad \frac{\partial y}{\partial u} - y = 0$$

$$\Rightarrow x = A(v)e^{u} \qquad \Rightarrow y = B(v)e^{u}$$

Choice: $A=1$, $B=v$

$$x = e^{u}, \quad y = ve^{u}$$

$$u = \ln x, \quad v = \frac{y}{x}$$

$$\frac{\partial\phi}{\partial u} = 0 \quad\Rightarrow\quad \phi = C(v) = C\left(\frac{y}{x}\right)$$

Back to previous example

$$\frac{\partial\phi}{\partial x} - \frac{\partial\phi}{\partial y} = x$$

$$\frac{\partial x}{\partial u} = 1, \qquad \frac{\partial y}{\partial u} = -1$$

$$\downarrow \qquad\qquad \downarrow$$

$$x = u + A(v) \qquad y = -u + B(v) \Rightarrow v = y + u = x + y$$

Choice: $A = 0$, $B = v$

So $x = u$, $y = -u + v$

$$\frac{\partial\phi}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial\phi}{\partial y}\frac{\partial y}{\partial u} = x$$

$$\Rightarrow \frac{\partial\phi}{\partial u} = x = u$$

$$\Rightarrow \phi = \frac{u^2}{2} + C(v) \longrightarrow \phi = \frac{x^2}{2} + C(x+y)$$

So this new method works in both cases.

MATH 2401

18-11-16

Example

$-y\frac{\partial \phi}{\partial x} + x\frac{\partial \phi}{\partial y} = 0 \Rightarrow \frac{\partial \phi}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial \phi}{\partial y}\frac{\partial y}{\partial u} = 0$

$\Rightarrow \frac{\partial \phi}{\partial u} = 0$

$\frac{\partial x}{\partial u} = -y \qquad \frac{\partial y}{\partial u} = x$

$\downarrow$

$\frac{\partial^2 x}{\partial u^2} = -\frac{\partial y}{\partial u} = -x$

$\Rightarrow \frac{\partial^2 x}{\partial u^2} + x = 0$

$\Rightarrow x = A(v)\cos u + B(v)\sin u$

$y = -\frac{\partial x}{\partial u} = A(v)\sin u - B(v)\cos u$

Choice : $B = 0$, $A = v$

So $x = v\cos u$, $y = v\sin u \Rightarrow x^2 + y^2 = v^2$

$\phi = C(v) = C(\sqrt{x^2+y^2})$

Example

$x\frac{\partial \phi}{\partial x} - \frac{\partial \phi}{\partial y} = 0 \Rightarrow \frac{\partial \phi}{\partial u} = 0 \Rightarrow \phi = C(v)$

$\frac{\partial x}{\partial u} = x$, $\frac{\partial y}{\partial u} = 1$

$\downarrow$

$\frac{\partial x}{\partial u} - x = 0$

$\downarrow$

$x = A(v)e^u \qquad y = -u + B(v)$

Choice: $B = 0$, $A = v$

$x = ve^u$, $y = -u$

$v = xe^{-u} = xe^y \Rightarrow \phi = C(xe^y)$

MATH 2201

18-11-16

Method of Characteristics (1st order PDE)

- linear
- quasilinear

Example

$\phi(x,y)$ $\quad \frac{1}{x}\frac{\partial\phi}{\partial x} - y\frac{\partial\phi}{\partial y} = 0$

$(x,y)\to(u,v)$: $\frac{\partial x}{\partial u} = \frac{1}{x} \quad \frac{\partial y}{\partial u} = -y$

$x\frac{\partial x}{\partial u} = 1$

$\frac{\partial\phi}{\partial u} = \frac{\partial x}{\partial u}\frac{\partial\phi}{\partial x} + \frac{\partial y}{\partial u}\frac{\partial\phi}{\partial y} = 0$

$\frac{\partial}{\partial u}\left(\frac{x^2}{2}\right) = 1$

$\frac{x^2}{2} = u + A(v)$

$\frac{\partial\phi}{\partial u} = 0 \longrightarrow \phi = C(v)$

$\frac{\partial y}{\partial u} + y = 0$, $y = B(v)e^{-u}$

Choice: $A=0$, $B=v$

$\frac{x^2}{2} = u$, $y = ve^{-u} \Rightarrow v = ye^{\frac{x^2}{2}}$

So $\phi = C\left(ye^{\frac{x^2}{2}}\right)$

Example

$\frac{\partial\phi}{\partial x} + 3y^{2/3}\frac{\partial\phi}{\partial y} = 2$, B.C. $\phi(x,1) = 1+x$

$\frac{\partial x}{\partial u} = 1 \rightarrow x = u + A(v)$

$\frac{\partial y}{\partial u} = 3y^{2/3} \quad y^{1/3} = u + B(v)$

Choice: $B=0$, $A=v$

$\begin{cases} x = u+v \\ y = u^3 \end{cases} \Rightarrow u = y^{1/3},\ v = x - y^{1/3}$

$\frac{\partial\phi}{\partial u} = 2$

$\phi = 2u + C(v)$

$= 2y^{1/3} + C(x - y^{1/3})$

B.C.

$$1 + x = 2 + C(x-1)$$

So $C(x-1) = x-1$

$w = x-1 \Rightarrow C(w) = w$

So $\phi = 2y^{1/3} + x - y^{1/3}$
$= x + y^{1/3}$

Quasilinear equations

$$A(x,y,\phi)\frac{\partial\phi}{\partial x} + B(x,y,\phi)\frac{\partial\phi}{\partial y} + C(x,y,\phi) = 0$$

$x(t)$ $\quad \dfrac{dx}{dt} = A(x,y,\phi)$

$y(t)$ $\quad \dfrac{dy}{dt} = B(x,y,\phi)$

$\phi(t)$ $\quad \dfrac{d\phi}{dt} = -C(x,y,\phi)$ $\qquad$ Why? $\underbrace{\frac{\partial\phi}{\partial x}\frac{dx}{dt} + \frac{\partial\phi}{\partial y}\frac{dy}{dt}}_{\frac{d\phi}{dt}} + C = 0$

Solve - find the constants of integration to satisfy the B.C. at $t=0$.

For previous example:

$$\frac{\partial\phi}{\partial x} + 3y^{2/3}\frac{\partial\phi}{\partial y} = 2 \qquad \text{B.C. } \phi(s,1) = 1+s$$

$\frac{dx}{dt} = 1 \longrightarrow x = t + D$

$\frac{dy}{dt} = 3y^{2/3} \longrightarrow y^{1/3} = t + E$

$\frac{d\phi}{dt} = 2 \longrightarrow \phi = 2t + F$

Using B.C. we get $D = s$, $E = 1$, $F = 1 + s$

MATH 220

18-11-16

So we have

$x = t + s$

$y^{1/3} = t + 1$

$\phi = 2t + 1 + s$

$$\phi = 2t + 1 + s = (t+s) + (t+1)$$
$$= x + y^{1/3}$$

Example

$$\frac{\partial \phi}{\partial x} + \phi \frac{\partial \phi}{\partial y} = 0$$

B.C. $\phi(0, s) = s^2$

$\frac{dx}{dt} = 1 \longrightarrow x = t + A$

$\frac{dy}{dt} = \phi \longrightarrow \frac{dy}{dt} = B \longrightarrow y = Bt + C$

$\frac{d\phi}{dt} = 0 \rightarrow \phi = B$

B.C. $\Rightarrow \quad A = 0, \quad C = s, \quad B = s^2$

So $\begin{cases} x = t \\ y = s^2 t + s \\ \phi = s^2 \end{cases}$

$y = s^2 x + s \longrightarrow xs^2 + s - y = 0$

$$s = \frac{-1 \pm \sqrt{1+4xy}}{2x}$$

$$\text{So } \phi = \left[\frac{-1 \pm \sqrt{1+4xy}}{2x}\right]^2$$

need to take +ve $\sqrt{\ }$ as $\sqrt{1+4xy} \sim 1 + 2xy + \ldots$ as $x \to 0$

So $\phi = \left[\frac{-1 + 1 + 2xy + \ldots}{2x}\right]^2 = y^2$

MATH 2401

25-11-16

First order PDE (method of characteristics)
- linear equations
- quasilinear equations
- fully nonlinear equations (chapter 9 - not examinable)

Chapter 10 - Linear second order hyperbolic equations with constant coefficients.

D'Alembat's method.
Wave equation

$$\frac{1}{C^2}\frac{\partial^2\phi}{\partial t^2} = \frac{\partial^2\phi}{\partial x^2}, \quad \phi(x,t)$$

Change of variables:

$$\begin{cases} x_+ = x + ct \\ x_- = x - ct \end{cases}$$

$$x = \tfrac{1}{2}(x_+ + x_-), \quad t = \frac{1}{2c}(x_+ - x_-)$$

$$\phi(x,t) = \phi\left[\tfrac{1}{2}(x_+ + x_-), \tfrac{1}{2c}(x_+ - x_-)\right] = \Phi(x_+, x_-)$$

$$\frac{\partial^2\Phi}{\partial x_-\partial x_+} = 0 \ ??$$

$$\frac{\partial\Phi}{\partial x_+} = \frac{\partial x}{\partial x_+}\frac{\partial\phi}{\partial x} + \frac{\partial t}{\partial x_+}\frac{\partial\phi}{\partial t} = \frac{1}{2}\frac{\partial\phi}{\partial x} + \frac{1}{2c}\frac{\partial\phi}{\partial t}$$

$$\frac{\partial^2\Phi}{\partial x_-\partial x_+} = \frac{\partial}{\partial x_-}\left[\frac{\partial\Phi}{\partial x_+}\right] = \frac{\partial}{\partial x_-}\left[\frac{1}{2}\frac{\partial\phi}{\partial x} + \frac{1}{2c}\frac{\partial\phi}{\partial t}\right]$$

$$= \frac{\partial x}{\partial x_-}\frac{\partial}{\partial x}[\ldots] + \frac{\partial t}{\partial x_-}\frac{\partial}{\partial t}[\ldots]$$

$$= \frac{1}{2}\frac{\partial}{\partial x}[\ldots] - \frac{1}{2c}\frac{\partial}{\partial t}[\ldots]$$

$$= \frac{1}{2}\left[\frac{1}{2}\frac{\partial^2\phi}{\partial x^2} + \cancel{\frac{1}{2c}\frac{\partial^2\phi}{\partial x\partial t}}\right] - \frac{1}{2c}\left[\cancel{\frac{1}{2}\frac{\partial^2\phi}{\partial t\partial x}} + \frac{1}{2c}\frac{\partial^2\phi}{\partial t^2}\right]$$

$$\frac{\partial^2 \Phi}{\partial x_- \partial x_+} = \frac{1}{4}\left[\frac{\partial^2 \phi}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2 \phi}{\partial t^2}\right] = 0$$

$$\frac{1}{c^2}\frac{\partial^2 \phi}{\partial t^2} = \frac{\partial^2 \phi}{\partial x^2}$$

$$\Rightarrow \frac{\partial^2 \Phi}{\partial x_- \partial x_+} = 0 \qquad \Phi(x_+, x_-)$$

$$\Rightarrow \frac{\partial}{\partial x_-}\left(\frac{\partial \Phi}{\partial x_+}\right) = 0$$

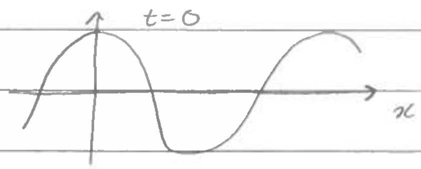
$$\frac{\partial \Phi}{\partial x_+} = A(x_+)$$

$$\Phi = \tilde{A}(x_+) + B(x_-)$$

$$\Rightarrow \phi = \underbrace{\tilde{A}(x+ct)}_{\text{'wave' moving to the left with velocity } c} + \underbrace{B(x-ct)}_{\text{'wave' moving to the right with velocity } c.}$$

Example

$\cos(x-ct)$



$\cos(x-ct_1)$: At $t=t_1$ we have the same picture but translated to the right by ct_1.

In general

$$\frac{\partial}{\partial x_+} = \frac{\partial x}{\partial x_+}\frac{\partial}{\partial x} + \frac{\partial t}{\partial x_+}\frac{\partial}{\partial t}$$

where $x_+ = x+ct$, $x_- = x-ct$

$\left(x = \frac{1}{2}(x_+ + x_-),\ t = \frac{1}{2c}(x_+ - x_-)\right)$

$$= \frac{1}{2}\frac{\partial}{\partial x} + \frac{1}{2c}\frac{\partial}{\partial t}$$

$$\frac{\partial}{\partial x_-} = \frac{\partial x}{\partial x_-}\frac{\partial}{\partial x} + \frac{\partial t}{\partial x_-}\frac{\partial}{\partial t}$$

$$= \frac{1}{2}\frac{\partial}{\partial x} - \frac{1}{2c}\frac{\partial}{\partial t}$$

MATH 2401

25-11-16

$$\frac{\partial^2}{\partial x_+ \partial x_-} = \left(\frac{\partial}{\partial x_+}\right)\left(\frac{\partial}{\partial x_-}\right)$$

$$= \left(\frac{1}{2}\frac{\partial}{\partial x} + \frac{1}{2c}\frac{\partial}{\partial t}\right)\left(\frac{1}{2}\frac{\partial}{\partial x} - \frac{1}{2c}\frac{\partial}{\partial t}\right)$$

$$= \frac{1}{4}\frac{\partial^2}{\partial x^2} - \cancel{\frac{1}{4c}\frac{\partial^2}{\partial x \partial t}} + \cancel{\frac{1}{4c}\frac{\partial^2}{\partial t \partial x}} - \frac{1}{4c^2}\frac{\partial^2}{\partial t^2}$$

$$= \frac{1}{4}\left[\frac{\partial^2}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right]$$

[same result as previous, but without defining ϕ.]

Example

Solve $\frac{1}{c^2}\frac{\partial^2 \phi}{\partial t^2} = \frac{\partial^2 \phi}{\partial x^2}$ $\Rightarrow \phi = C_-(x-ct) + C_+(x+ct)$

$\phi(x,0) = e^{-x^2}$ $\quad \frac{\partial \phi}{\partial t}(x,0) = 0$

$C_-(x) + C_+(x) = e^{-x^2}$

$$\frac{\partial \phi}{\partial t} = -c\,C_-'(x-ct) + c\,C_+'(x+ct)$$

$$\frac{\partial \phi}{\partial t}(x,0) = -c\,C_-'(x) + c\,C_+'(x) = 0$$

Integrate:

$$-c\,C_-(x) + c\,C_+(x) = k \quad \text{(constant)}$$

$$C_+(x) = \frac{k}{c} + C_-(x)$$

$$C_-(x) + \frac{k}{c} + C_-(x) = e^{-x^2}$$

$$C_-(x) = \frac{1}{2}e^{-x^2} - \frac{k}{2c}$$

$$\Rightarrow C_+(x) = \frac{1}{2}e^{-x^2} + \frac{k}{2c}$$

$$\phi(x,t) = C_-(x-ct) + C_+(x+ct)$$

So $\phi(x,t) = \frac{1}{2}e^{-(x-ct)^2} - \cancel{\frac{k}{2c}} + \frac{1}{2}e^{-(x+ct)^2} + \cancel{\frac{k}{2c}}$

$$= \frac{1}{2}e^{-(x-ct)^2} + \frac{1}{2}e^{-(x+ct)^2}$$

$$\frac{1}{c^2}\frac{\partial^2\phi}{\partial t^2} = \frac{\partial^2\phi}{\partial x^2} \Rightarrow \phi(x,t) = C_-(x-ct) + C_+(x+ct)$$

Connection with separation of variables
(Page 26 of notes)

$$\phi(x,t) = \sum_{n=1}^{\infty}\left(C_n\cos\left(\frac{n\pi ct}{L}\right) + D_n\sin\left(\frac{n\pi ct}{L}\right)\right)\sin\left(\frac{n\pi x}{L}\right)$$

$L = \pi$

$$= \sum_{n=1}^{\infty} C_n\cos(nct)\sin nx + \sum_{n=1}^{\infty} D_n\sin(nct)\sin nx$$

$$\left[\begin{aligned} \cos a\sin b &= \tfrac{1}{2}\sin(a+b) - \tfrac{1}{2}\sin(a-b) \\ \sin a\sin b &= \tfrac{1}{2}\cos(a-b) - \tfrac{1}{2}\cos(a+b) \end{aligned}\right]$$

$$= \sum_{n=1}^{\infty}\frac{C_n}{2}\left[\sin n(x+ct) - \sin n(ct-x)\right] + \sum_{n=1}^{\infty}\frac{D_n}{2}\left[\cos n(ct-x) - \cos n(ct+x)\right]$$

$$= \overbrace{\sum_{n=1}^{\infty}\frac{C_n}{2}\sin n(x+ct) - \frac{D_n}{2}\cos n(x+ct)}^{C_+(x+ct)}$$

$$+ \underbrace{\sum_{n=1}^{\infty}\frac{C_n}{2}\sin n(x-ct) + \frac{D_n}{2}\cos n(x-ct)}_{C_-(x-ct)}$$

MATH 2401

25-11-16

Hyperbolic Equations

$$A\frac{\partial^2\phi}{\partial x^2} + B\frac{\partial^2\phi}{\partial x\,\partial y} + C\frac{\partial^2\phi}{\partial y^2} = D(x,y) \quad (*), \quad \phi(x,y)$$

A, B, C constants, D given.

Change of variables:

$$(x,y) \longrightarrow (s,t)$$

so that $(*)$ becomes

$$A\frac{\partial^2\phi}{\partial s\,\partial t} = D, \quad A \neq 0$$

$$\begin{cases} x = s + t \\ y = -\beta s - \alpha t \end{cases}$$

$$\frac{\partial}{\partial t} = \frac{\partial x}{\partial t}\frac{\partial}{\partial x} + \frac{\partial y}{\partial t}\frac{\partial}{\partial y} = \frac{\partial}{\partial x} - \alpha\frac{\partial}{\partial y}$$

$$\frac{\partial}{\partial s} = \frac{\partial x}{\partial s}\frac{\partial}{\partial x} + \frac{\partial y}{\partial s}\frac{\partial}{\partial y} = \frac{\partial}{\partial x} - \beta\frac{\partial}{\partial y}$$

$$\frac{\partial^2}{\partial s\,\partial t} = \left(\frac{\partial}{\partial s}\right)\left(\frac{\partial}{\partial t}\right) = \left(\frac{\partial}{\partial x} - \alpha\frac{\partial}{\partial y}\right)\left(\frac{\partial}{\partial x} - \beta\frac{\partial}{\partial y}\right)$$

$$= \frac{\partial^2}{\partial x^2} - \alpha\frac{\partial^2}{\partial x\,\partial y} - \beta\frac{\partial^2}{\partial y\,\partial x} + \alpha\beta\frac{\partial^2}{\partial y^2}$$

$$= \frac{\partial^2}{\partial x^2} - (\alpha+\beta)\frac{\partial^2}{\partial x\,\partial y} + \alpha\beta\frac{\partial^2}{\partial y^2}$$

We want $A\dfrac{\partial^2\phi}{\partial s\,\partial t} = A\dfrac{\partial^2\phi}{\partial x^2} + B\dfrac{\partial^2\phi}{\partial x\,\partial y} + C\dfrac{\partial^2\phi}{\partial y^2}$

$$\cancel{A\frac{\partial^2\phi}{\partial x^2}} - A(\alpha+\beta)\frac{\partial^2\phi}{\partial x\,\partial y} + A\alpha\beta\frac{\partial^2\phi}{\partial y^2} = \cancel{A\frac{\partial^2\phi}{\partial x^2}} + B\frac{\partial^2\phi}{\partial x\,\partial y} + C\frac{\partial^2\phi}{\partial y^2}$$

So $\left.\begin{aligned} -A(\alpha+\beta) &= B \\ A\alpha\beta &= C \end{aligned}\right\} \quad \begin{aligned} \alpha+\beta &= -B/A \\ \alpha\beta &= C/A \end{aligned}$

$\Rightarrow \alpha = \ldots, \ \beta = \ldots$

We see that α and β are the roots of the quadratic equation

$$AT^2 + BT + C = 0$$

$$\text{roots} = \frac{-B \pm \sqrt{B^2 - 4AC}}{2A}$$

$$\begin{cases} T_+ = \dfrac{-B + \sqrt{B^2 - 4AC}}{2A} \\ T_- = \dfrac{-B - \sqrt{B^2 - 4AC}}{2A} \end{cases}$$

$T_+ + T_- = -B/A$, $T_+ T_- = C/A$ $(B^2 - 4AC > 0)$

Example

$$\frac{\partial^2 \phi}{\partial x^2} + 5\frac{\partial^2 \phi}{\partial x \partial y} + 4\frac{\partial^2 \phi}{\partial y^2} = xy \qquad \begin{bmatrix} x = s + t \\ y = -\beta s - \alpha t \end{bmatrix}$$

$(A = 1, B = 5, C = 4)$ $\quad D(x, y) = xy$

$AT^2 + BT + C = 0$

$\Rightarrow T^2 + 5T + 4 = 0$

$$T = \frac{-5 \pm \sqrt{25 - 16}}{2} = \frac{-5 \pm 3}{2}$$

So $\alpha = -4$

$\beta = -1$. $\Rightarrow x = s + t$, $y = s + 4t$

So $A \dfrac{\partial^2 \phi}{\partial s \partial t} = D$

$$\Rightarrow \frac{\partial^2 \phi}{\partial s \partial t} = xy = (s+t)(s+4t) = s^2 + 5st + 4t^2$$

$$\frac{\partial \phi}{\partial t} = \frac{s^3}{3} + \frac{5}{2}s^2 t + 4t^2 s + C_1(t)$$

$$\Rightarrow \phi = \frac{s^3 t}{3} + \frac{5}{4} s^2 t^2 + \frac{4}{3} t^3 s + \tilde{C}_1(t) + C_2(s)$$

$$= \frac{1}{3}\left[\frac{1}{3}(4x - y)\right]^3 \left[\frac{1}{3}(y - x)\right] + \ldots$$

as $s = \frac{1}{3}(4x - y)$

$t = \frac{1}{3}(y - x)$

example boundary conditions:

$\phi(x, 0) = x$

$\frac{\partial \phi}{\partial y}(x, 0) = x^2$

MATH 2401

25-11-16

D'Alembert's method

Example

$$\frac{\partial^2\phi}{\partial x^2} + 5\frac{\partial^2\phi}{\partial x\partial y} + 4\frac{\partial^2\phi}{\partial y^2} = 0$$

B.C. $\phi(x,0) = x$, $\frac{\partial}{\partial y}\phi(x,0) = x^2$

$A=1$, $B=5$, $C=4$, $D=0$

$AT^2 + BT + C = 0$

$T^2 + 5T + 4 = 0$

$\Rightarrow T = -1, -4$

$\Rightarrow \beta = -1$, $\alpha = -4$

$$\begin{cases} x = s+t \\ y = -\beta s - \alpha t \end{cases} \Rightarrow \begin{cases} x = s+t \\ y = s+4t \end{cases} \longrightarrow \begin{cases} t = \frac{1}{3}(y-x) \\ s = \frac{1}{3}(4x-y) \end{cases}$$

$$\phi = C_1(s) + C_2(t) = C_1\left(\frac{4x-y}{3}\right) + C_2\left(\frac{y-x}{3}\right)$$

since $\frac{\partial^2\phi}{\partial s\partial t} = 0$

B.C.

$\phi(x,0) = x$, $C_1\left(\frac{4x}{3}\right) + C_2\left(\frac{-x}{3}\right) = x$ (i)

$$\frac{\partial\phi}{\partial y} = \frac{-1}{3}C_1'\left(\frac{4x-y}{3}\right) + \frac{1}{3}C_2'\left(\frac{y-x}{3}\right)$$

$$\frac{\partial\phi}{\partial y}(x,0) = -\frac{1}{3}C_1'\left(\frac{4x}{3}\right) + \frac{1}{3}C_2'\left(-\frac{x}{3}\right) = x^2$$

Integrate: $-\frac{1}{3}\frac{3}{4}C_1\left(\frac{4x}{3}\right) + \frac{1}{3}(-3)\,C_2\left(\frac{-x}{3}\right) = \frac{x^3}{3} + k$ (ii)

$$\Rightarrow -\frac{1}{4}C_1\left(\frac{4x}{3}\right) - C_2\left(\frac{-x}{3}\right) = \frac{x^3}{3} + k$$

Solving (i) and (ii) for C_1 and C_2 we get $\frac{3}{4}C_1\left(\frac{4x}{3}\right) = \frac{x^3}{3} + k + x$

$$\Rightarrow C_1\left(\frac{4x}{3}\right) = \frac{4}{9}\left(x^3 + 3x\right) + \frac{4}{3}k$$

$$C_2\left(\frac{-x}{3}\right) = x - C_1 = x - \frac{4}{9}(x^3 + 3x) - \frac{4}{3}k$$

$$= -\frac{4x^3 + 3x}{9} - \frac{4}{3}k$$

$$C_1\left(\frac{4x}{3}\right) = \frac{4}{9}(x^3 + 3x) + \frac{4}{3}k$$

Let $u = \frac{4x}{3} \rightarrow x = \frac{3u}{4}$

$$C_1(u) = \frac{4}{9}\left(\frac{27}{64}u^3 + \frac{9u}{4}\right) + \frac{4}{3}k$$

$$= \frac{3}{16}u^3 + u + \frac{4}{3}k$$

Let $w = -\frac{x}{3} \Rightarrow x = -3w$

$$C_2(w) = -3w - \frac{4}{9}\left[-27w^3 - 9w\right] - \frac{4}{3}k$$

$$= 12w^3 + w - \frac{4}{3}k$$

$$\phi = C_1\left(\frac{4x-y}{3}\right) + C_2\left(\frac{y-x}{3}\right)$$

So $\phi = \frac{3}{16}\left(\frac{4x-y}{3}\right)^3 + \left(\frac{4x-y}{3}\right) + \cancel{\frac{4}{3}k} + 12\left(\frac{y-x}{3}\right)^3 + \left(\frac{y-x}{3}\right) - \cancel{\frac{4}{3}k}$

Then simplify.

Note that interchanging β and α does <u>not</u> change the solution.

MATH 2401

25-11-16

Example

$$\frac{1}{c^2}\frac{\partial^2\phi}{\partial t^2} = \frac{\partial^2\phi}{\partial x^2}$$

I.C. $\phi(x,0) = f(x)$, $\frac{\partial\phi}{\partial t}(x,0) = g(x)$.

$$\phi(x,t) = C_+(x+ct) + C_-(x-ct)$$

I.C. $\Rightarrow$ $C_+(x) + C_-(x) = f(x)$

$$\frac{\partial\phi}{\partial t} = cC_+'(x+ct) - cC_-'(x-ct)$$

I.C. $cC_+'(x) - cC_-'(x) = g(x)$

Integrate $cC_+(x) - cC_-(x) = \int_{x_0}^{x} g(s)\,ds$

So $C_+(x) + C_-(x) = f(x)$

$C_+(x) - C_-(x) = \frac{1}{c}\int_{x_0}^{x} g(s)\,ds$

$$C_+(x) = \frac{f(x)}{2} + \frac{1}{2c}\int_{x_0}^{x} g(s)\,ds$$

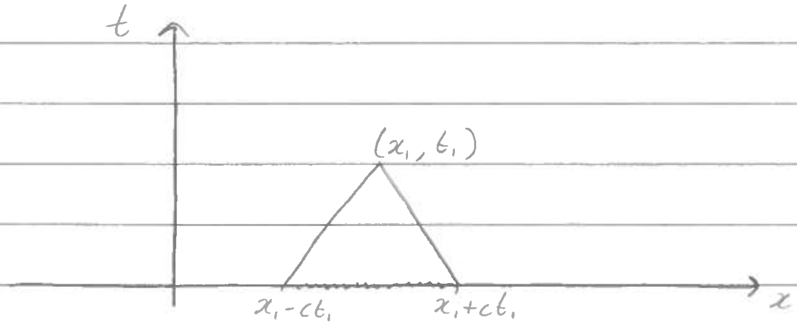
$$C_-(x) = f(x) - C_+(x) = \frac{f(x)}{2} - \frac{1}{2c}\int_{x_0}^{x} g(s)\,ds$$

$$\phi(x,t) = \underbrace{\frac{f(x+ct)}{2} + \frac{1}{2c}\int_{x_0}^{x+ct} g(s)\,ds}_{C_+(x+ct)} + \underbrace{\frac{f(x-ct)}{2} - \frac{1}{2c}\int_{x_0}^{x-ct} g(s)\,ds}_{C_-(x-ct)}$$

$$\Rightarrow \phi(x,t) = \frac{f(x+ct) + f(x-ct)}{2} + \frac{1}{2c}\int_{x-ct}^{x+ct} g(s)\,ds$$

$$\phi(x_1, t_1) = \frac{f(x_1 + ct_1) + f(x_1 - ct_1)}{2} + \frac{1}{2c}\int_{x_1 - ct_1}^{x_1 + ct_1} g(s)\,ds$$

(This is the value of the solution at a particular point).



lines $\begin{cases} x - ct = x_1 - ct_1 \\ x + ct = x_1 + ct_1 \end{cases}$

MATH 2401

02-12-16

Non-linear first order PDE

Not in Exam!

Weak nonlinear first order PDE

$$A(x,y,\phi)\frac{\partial \phi}{\partial x} + B(x,y,\phi)\frac{\partial \phi}{\partial y} + C(x,y,\phi) = 0$$

$$G\left(x, y, \phi, \frac{\partial \phi}{\partial x}(x,y), \frac{\partial \phi}{\partial y}(x,y)\right) = 0$$

Examples

$\phi_x^2 + \phi_y^2 = 1$

$\phi_x \phi_y - 1 = 0$

Notations

$u = \phi$, $p = \dfrac{\partial \phi}{\partial x}$, $q = \dfrac{\partial \phi}{\partial y}$

$G(x, y, u, p, q) = 0$

Examples become:

$p^2 + q^2 = 1$

$pq - 1 = 0$

Recall $\dfrac{\partial \phi}{\partial x} = \phi_x$, $\dfrac{\partial \phi}{\partial y} = \phi_y$

The general weak nonlinear first order PDE becomes $G(x,y,u,p,q) = A(x,y,u)p + B(x,y,u)q + C(x,y,u) = 0$

So $\dfrac{dx}{dt} = A(x,y,u) = \dfrac{\partial G}{\partial p}$

$\dfrac{dy}{dt} = B(x,y,u) = \dfrac{\partial G}{\partial q}$

$\dfrac{du}{dt} = -C(x,y,u)$

$$\boxed{\begin{aligned}
\frac{dx}{dt} &= \frac{\partial G}{\partial p} \\
\frac{dy}{dt} &= \frac{\partial G}{\partial q} \\
\frac{du}{dt} &= p\frac{\partial G}{\partial p} + q\frac{\partial G}{\partial q} \\
\frac{dp}{dt} &= -\frac{\partial G}{\partial x} - \frac{\partial G}{\partial u}p \\
\frac{dq}{dt} &= -\frac{\partial G}{\partial y} - \frac{\partial G}{\partial u}q
\end{aligned}}$$

in general depends on p, q and u

$$\frac{du}{dt} = \frac{\partial u}{\partial x}\frac{dx}{dt} + \frac{\partial u}{\partial y}\frac{dy}{dt}$$
$$= p\frac{dx}{dt} + q\frac{dy}{dt}$$
$$= p\frac{\partial G}{\partial p} + q\frac{\partial G}{\partial q}$$

Derivation of the last two equations above:

$$\frac{dp}{dt} = \frac{d}{dt}\left(\frac{\partial u}{\partial x}\right) = \frac{\partial^2 u}{\partial x^2}\frac{dx}{dt} + \frac{\partial^2 u}{\partial y\partial x}\frac{dy}{dt}$$
$$= p_x\frac{dx}{dt} + p_y\frac{dy}{dt}$$
$$= p_x\frac{\partial G}{\partial p} + p_y\frac{\partial G}{\partial q}$$

$$\frac{dq}{dt} = \frac{d}{dt}\left(\frac{\partial u}{\partial y}\right) = \frac{\partial^2 u}{\partial x\partial y}\frac{dx}{dt} + \frac{\partial^2 u}{\partial y^2}\frac{dy}{dt}$$
$$= q_x\frac{dx}{dt} + q_y\frac{dy}{dt}$$
$$= q_x\frac{\partial G}{\partial p} + q_y\frac{\partial G}{\partial q}$$

Eliminate p_x, p_y, q_x, q_y.

Differentiating G w.r.t. x and y

$$\frac{dG}{dx} = \frac{\partial G}{\partial x} + \frac{\partial G}{\partial u}u_x + \frac{\partial G}{\partial p}p_x + \frac{\partial G}{\partial q}q_x = 0$$

So $\displaystyle \frac{\partial G}{\partial x} + \frac{\partial G}{\partial u}p = -\left(\frac{\partial G}{\partial p}p_x + \frac{\partial G}{\partial q}q_x\right)$ but $q_x = p_y$

since $\displaystyle \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right) = \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}\right)$

MATH 2401

02-12-16

So $\frac{\partial G}{\partial p} p_x + \frac{\partial G}{\partial q} p_y = -\frac{\partial G}{\partial x} - \frac{\partial G}{\partial u} p$

$$\Rightarrow \frac{dp}{dt} = -\frac{\partial G}{\partial x} - \frac{\partial G}{\partial u} p$$

$$\frac{dG}{dy} = \frac{\partial G}{\partial y} + \frac{\partial G}{\partial u} u_y + \frac{\partial G}{\partial p} p_y + \frac{\partial G}{\partial q} q_y = 0$$

$$\Rightarrow \frac{\partial G}{\partial y} + \frac{\partial G}{\partial u} q = -\left(\frac{\partial G}{\partial p} p_y + \frac{\partial G}{\partial q} q_y\right)$$

$$\Rightarrow \frac{\partial G}{\partial p} q_x + \frac{\partial G}{\partial q} q_y = \frac{\partial G}{\partial y} + \frac{\partial G}{\partial u} q$$

So $\frac{dq}{dt} = -\frac{\partial G}{\partial y} - \frac{\partial G}{\partial u} q$

Initial conditions

$$\begin{cases} x = \gamma_1(s) \\ y = \gamma_2(s) \\ u = \tilde{\phi}(s) \\ t = 0 \end{cases}$$

Also $p = \psi_1(s)$, $q = \psi_2(s)$ which we can specify in terms of given conditions.

$G(x, y, u, p, q) = 0$

$G(\gamma_1(s), \gamma_2(s), \tilde{\phi}(s), \psi_1(s), \psi_2(s)) = 0$

$\frac{du}{ds} = \frac{\partial u}{\partial x}\frac{dx}{ds} + \frac{\partial u}{\partial y}\frac{dy}{ds}$ by chain rule $(u(x,y))$

$= p\frac{dx}{ds} + q\frac{dy}{ds}$

$\tilde{\phi}'(s) = \psi_1(s)\gamma_1'(s) + \psi_2(s)\gamma_2'(s)$

Example

$$\frac{\partial \phi}{\partial x} + a\frac{\partial \phi}{\partial y} = 0$$

$$\phi(0,y) = \sin y$$

$\frac{dx}{dt} = 1 \longrightarrow x = t + A \Rightarrow A = 0$

$\frac{dy}{dt} = a \longrightarrow y = at + B \Rightarrow B = s$

$$\frac{d\phi}{dt} = \frac{\partial \phi}{\partial x}\frac{dx}{dt} + \frac{\partial \phi}{\partial y}\frac{dy}{dt}$$

$$= \frac{\partial \phi}{\partial x} + a\frac{\partial \phi}{\partial y} = 0 \longrightarrow \phi = C \Rightarrow C = \sin s$$

So $x = t$

$y = at + s$

$\phi = \sin s$

$\Rightarrow \phi = \sin(y - ax)$

$\left[p = \frac{\partial \phi}{\partial x}, \ q = \frac{\partial \phi}{\partial y}\right]$

Using above method:

$G = p + aq = 0$

$\frac{dx}{dt} = 1$

$\frac{dy}{dt} = a$

$\frac{du}{dt} = p + aq$

$\frac{dp}{dt} = 0$

$\frac{dq}{dt} = 0$

$$\begin{cases} x = 0 \\ y = s \\ u = \sin s \\ p = \psi_1(s) \\ q = \psi_2(s) \end{cases}$$

we have $\psi_1(s) + a\psi_2(s) = 0$

$\cos s = \psi_2(s)$

MATH 2401

02-12-16

$$x = t + A \rightarrow A = 0$$
$$y = at + B \rightarrow B = s$$
$$p = C = -a\cos s$$
$$q = D = \cos s$$
$$u = \underbrace{(C + Da)}_{0} t + E$$

$$\Rightarrow \begin{cases} x = t \\ y = at + s \\ p = -a\cos s \\ q = \cos s \\ u = \sin s \end{cases}$$

$\left.\begin{matrix} x = t \\ y = at + s \end{matrix}\right\} s = y - at = y - ax$

$p = -a\cos s \rightarrow \begin{cases} p = -a\cos(y - ax) \\ q = \cos(y - ax) \end{cases}$

$u = \sin s \longrightarrow \phi = \sin(y - ax)$

Example

$\phi_x \phi_y = 1 \quad , \quad \phi(x, 0) = x$

$\Rightarrow pq = 1$

$G = pq - 1 = 0$

$$\begin{cases} x = s \\ y = 0 \\ \phi = s \\ p = \psi_1(s) \\ q = \psi_2(s) \end{cases}$$

$$\begin{cases} \dfrac{dx}{dt} = \dfrac{\partial G}{\partial p} = q \\[2ex] \dfrac{dy}{dt} = \dfrac{\partial G}{\partial q} = p \\[2ex] \dfrac{du}{dt} = p\dfrac{\partial G}{\partial p} + q\dfrac{\partial G}{\partial q} = pq + qp = 2pq \\[2ex] \dfrac{dp}{dt} = -\dfrac{\partial G}{\partial x} - \dfrac{\partial G}{\partial u} p = 0 \rightarrow p = A = 1 \\[2ex] \dfrac{dq}{dt} = -\dfrac{\partial G}{\partial y} - \dfrac{\partial G}{\partial u} q = 0 \rightarrow q = B = 1 \end{cases}$$

So $\frac{dx}{dt} = 1 \to x = t + C \to C = s \Rightarrow x = t + s$

$\frac{dy}{dt} = 1 \to y = t + D \to D = 0 \Rightarrow y = t$

$\Rightarrow s = x - y$

$\frac{du}{dt} = 2 \to u = 2t + E \to E = s \Rightarrow u = 2t + s$

$\Rightarrow u = 2y + x - y = x + y$

Example

$\phi_y + \phi_x^2 = 0 \quad , \quad \phi(x, 0) = ax$

$G = q + p^2 \quad , \quad \psi_2 + \psi_1^2 = 0 \, , \quad a = \psi_1$

$$t = 0 \begin{cases} x = s \\ y = 0 \\ \phi = as \\ p = \psi_1(s) = a \\ q = \psi_2(s) = -a^2 \end{cases}$$

$\frac{dx}{dt} = \frac{\partial G}{\partial p} = 2p = 2a \to x = 2at + A \Rightarrow A = s$

$\frac{dy}{dt} = \frac{\partial G}{\partial q} = 1 \to y = t + B \Rightarrow B = 0$

$\frac{du}{dt} = p\frac{\partial G}{\partial p} + q\frac{\partial G}{\partial q} = 2p^2 + q = 2a^2 - a^2 \to u = a^2 t + C \Rightarrow C = as$

$\frac{dp}{dt} = -\frac{\partial G}{\partial x} - \frac{\partial G}{\partial u} p = 0 \to p = A \to p = a$

$\frac{dq}{dt} = -\frac{\partial G}{\partial y} - \frac{\partial G}{\partial u} q = 0 \to q = B \to q = -a^2$

$$\begin{cases} x = 2at + s \\ y = t \\ u = a^2 t + as \end{cases} \to \phi = a^2 y + a(x - 2y) = ax - a^2 y$$

$\Rightarrow \phi = a(x - ay)$

MATH 2401

02-12-16

$$G\left(x, y, \phi(x,y), \frac{\partial \phi}{\partial x}(x,y), \frac{\partial \phi}{\partial y}(x,y)\right) = 0$$

$$u = \phi \quad , p = \frac{\partial \phi}{\partial x} \quad , \quad q = \frac{\partial \phi}{\partial y}$$

$$G(x, y, u, p, q) = 0$$

$$\begin{cases} \dfrac{dx}{dt} = \dfrac{\partial G}{\partial p} \\ \dfrac{dy}{dt} = \dfrac{\partial G}{\partial q} \\ \dfrac{du}{dt} = \dfrac{\partial G}{\partial p} p + \dfrac{\partial G}{\partial q} q \\ \dfrac{dp}{dt} = -\dfrac{\partial G}{\partial x} - p\dfrac{\partial G}{\partial u} \\ \dfrac{dq}{dt} = -\dfrac{\partial G}{\partial y} - q\dfrac{\partial G}{\partial u} \end{cases} \qquad \begin{cases} t = 0 \\ x = \gamma_1(s) \\ y = \gamma_2(s) \\ u = \tilde{\phi}(s) \\ p = \psi_1(s) \\ q = \psi(s) \end{cases}$$

$$G(\gamma_1(s), \gamma_2(s), \tilde{\phi}(s), \psi_1(s), \psi_2(s)) = 0$$

$$\tilde{\phi}'(s) = \psi_1(s)\gamma_1'(s) + \psi_2(s)\gamma_2'(s)$$

Example

$$\phi_y + \phi_x^2 = y \quad , \quad \phi(x,0) = 0$$

$$G = q + p^2 - y = 0$$

$$\begin{cases} \dfrac{dx}{dt} = 2p \\ \dfrac{dy}{dt} = 1 \\ \dfrac{du}{dt} = 2p^2 + q \\ \dfrac{dp}{dt} = 0 \rightarrow p = 0 \\ \dfrac{dq}{dt} = 1 \longrightarrow q = t \end{cases} \qquad \begin{cases} t = 0 \\ x = s \\ y = 0 \\ \tilde{\phi} = 0 \\ p = \psi_1(s) \\ q = \psi_2(s) \end{cases}$$

$$\psi_1^2 + \psi_2 = 0 \; , \; 0 = \psi_1 \Rightarrow \psi_2 = 0$$

$$\begin{cases} \dfrac{dx}{dt} = 0 \rightarrow x = s \\ \dfrac{dy}{dt} = 1 \rightarrow y = t \\ \dfrac{du}{dt} = t \rightarrow u = t^2/2 \end{cases}$$

So $\phi = \dfrac{y^2}{2}$

Example

$\phi_x^2 + \phi_y^2 = 1$

$\phi = 0$ on a circle of radius 1

$G = p^2 + q^2 - 1 = 0$

$$\begin{cases} \dfrac{dx}{dt} = 2p \\ \dfrac{dy}{dt} = 2q \\ \dfrac{du}{dt} = 2p^2 + 2q^2 \\ \dfrac{dp}{dt} = 0 \\ \dfrac{dq}{dt} = 0 \end{cases}$$

$$\begin{cases} x = \cos s \\ y = \sin s \\ \tilde{\phi} = 0 \\ p = \psi_1 \\ q = \psi_2 \end{cases}$$

$$\begin{cases} \psi_1^2 + \psi_2^2 = 1 \\ 0 = -\psi_1 \sin s + \psi_2 \cos s \end{cases}$$

$\psi_1 = \psi_2 \dfrac{\cos s}{\sin s}$

$\Rightarrow \psi_2^2 \dfrac{\cos^2 s}{\sin^2 s} + \psi_2^2 = 1$

$\Rightarrow \psi_2^2 = \sin^2 s$

$\Rightarrow \psi_2 = \pm \sin s, \ \psi_1 = \pm \cos s$

[+ve = 1st choice

-ve = 2nd choice]

1st choice

$\dfrac{dp}{dt} = 0 \rightarrow p = \psi_1(s) = \cos s$

$\dfrac{dq}{dt} = 0 \rightarrow q = \psi_2(s) = \sin s$

MATH 2401

02-12-16

So $\begin{cases} \dfrac{dx}{dt} = 2\cos s \rightarrow x = (2\cos s)t + \cos s = \cos s(2t+1) \\ \dfrac{dy}{dt} = 2\sin s \longrightarrow y = (2\sin s)t + \sin s = \sin s(2t+1) \\ \dfrac{du}{dt} = 2\cos^2 s + 2\sin^2 s = 2 \rightarrow u = 2t \end{cases}$

$$x^2 + y^2 = (2t+1)^2$$

So $\phi = -1 + \sqrt{x^2+y^2}$

<u>2nd choice</u>

$\phi = 1 - \sqrt{x^2+y^2}$

<u>Example</u>

$\phi_x \phi_y - u = 0, \quad \phi(0,y) = y^2$

$G = pq - u = 0$

$\begin{cases} \dfrac{dx}{dt} = q \\ \dfrac{dy}{dt} = p \\ \dfrac{du}{dt} = pq + pq = 2pq \\ \dfrac{dp}{dt} = p \rightarrow p = Ae^t \rightarrow p = \dfrac{s}{2}e^t \\ \dfrac{dq}{dt} = q \rightarrow q = Be^t \rightarrow q = 2se^t \end{cases}$

$\begin{cases} x = 0 \\ y = s \\ \phi = s^2 \\ p = \psi_1(s) = \dfrac{s}{2} \\ q = \psi_2(s) = 2s \end{cases}$

$\psi_1\psi_2 - s^2 = 0$

$2s = \psi_1(0) + \psi_2$

$\psi_2 = 2s, \quad \psi_1 = \dfrac{s}{2}$

So $\begin{cases} \dfrac{dx}{dt} = q = 2se^t \longrightarrow x = 2se^t - 2s \\ \dfrac{dy}{dt} = p = \dfrac{s}{2}e^t \rightarrow y = \dfrac{se^t}{2} + \dfrac{s}{2} \\ \dfrac{du}{dt} = 2pq = 2s^2e^{2t} \rightarrow u = s^2e^{2t} \end{cases}$

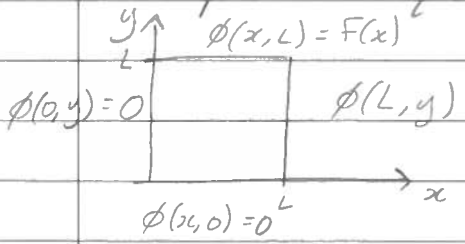
So $\phi = \left(\dfrac{x+4y}{4}\right)^2$

MATH 2401

09-12-16

Separation of variables

Laplace's equation



Example of problem

$$\phi_{xx} + \phi_{yy} = 0$$

$$\phi(x,y) = X(x)Y(y)$$

So $X''(x)Y(y) + X(x)Y''(y) = 0$

$$\frac{X''(x)}{X(x)} = -\frac{Y''(y)}{Y(y)} = -\lambda$$

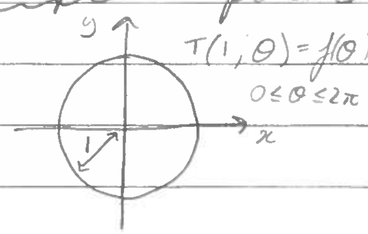
$$(\text{ODE}) \Rightarrow \begin{cases} X''(x) + \lambda X(x) = 0 \\ Y''(y) - \lambda Y(y) = 0 \end{cases}$$

$$\text{with} \begin{cases} \phi(0,y) = 0 \Rightarrow X(0)Y(y) = 0 \Rightarrow X(0) = 0 \\ \phi(x,0) = 0 \Rightarrow X(x)Y(0) = 0 \Rightarrow Y(0) = 0 \\ \phi(L,y) = 0 \Rightarrow X(L)Y(y) = 0 \Rightarrow X(L) = 0 \\ \phi(x,L) = F(x) \end{cases}$$

Solving Laplace's eqn with different coordinates

- Cartesian - use a square
- polars - use a circle
- spherical coords - use a sphere
- cylindrical coords - use a cylinder

Example in polars



$$\left[\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases} \quad \begin{cases} r = \sqrt{x^2+y^2} \\ \theta = \tan^{-1}\left(\frac{y}{x}\right) \end{cases} \right]$$

$$T(x,y): \quad \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 0$$

$$\tilde{T}(r,\theta) = T(r\cos\theta, r\sin\theta)$$

$$\frac{\partial T}{\partial x} = \frac{\partial \tilde{T}}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial \tilde{T}}{\partial \theta}\frac{\partial \theta}{\partial x}$$

$$\frac{\partial r}{\partial x} = \frac{x}{\sqrt{x^2+y^2}} = \frac{r\cos\theta}{r} = \cos\theta$$

$$\frac{\partial \theta}{\partial x} = \frac{1}{1+\frac{y^2}{x^2}}\left(-y/x^2\right) = \frac{-y}{x^2+y^2} = \frac{-r\sin\theta}{r^2} = -\frac{1}{r}\sin\theta$$

So $\frac{\partial T}{\partial x} = \frac{\partial \tilde{T}}{\partial r}\cos\theta - \frac{\partial \tilde{T}}{\partial \theta}\frac{\sin\theta}{r}$

$$\frac{\partial^2 T}{\partial x^2} = \frac{\partial}{\partial r}\left(\frac{\partial \tilde{T}}{\partial r}\cos\theta - \frac{\partial \tilde{T}}{\partial \theta}\frac{\sin\theta}{r}\right)\frac{\partial r}{\partial x} \leftarrow \cos\theta$$

$$+ \frac{\partial}{\partial \theta}\left(\frac{\partial \tilde{T}}{\partial r}\cos\theta - \frac{\partial \tilde{T}}{\partial \theta}\frac{\sin\theta}{r}\right)\frac{\partial \theta}{\partial x} \leftarrow -\frac{\sin\theta}{r}$$

$$= \frac{\partial^2 \tilde{T}}{\partial r^2}\cos^2\theta - \frac{\partial^2 \tilde{T}}{\partial r \partial \theta}\frac{\sin\theta\cos\theta}{r} + \frac{\partial \tilde{T}}{\partial \theta}\frac{\sin\theta\cos\theta}{r^2} - \frac{\partial^2 \tilde{T}}{\partial \theta \partial r}\frac{\sin\theta\cos\theta}{r}$$

$$+ \frac{\partial \tilde{T}}{\partial r}\frac{\sin^2\theta}{r} + \frac{\partial^2 \tilde{T}}{\partial \theta^2}\frac{\sin^2\theta}{r^2} + \frac{\partial \tilde{T}}{\partial \theta}\frac{\sin\theta\cos\theta}{r^2}$$

$$\frac{\partial T}{\partial y} = \frac{\partial \tilde{T}}{\partial r}\frac{\partial r}{\partial y} + \frac{\partial \tilde{T}}{\partial \theta}\frac{\partial \theta}{\partial y}$$

$$\frac{\partial r}{\partial y} = \frac{y}{\sqrt{x^2+y^2}} = \frac{r\sin\theta}{r} = \sin\theta$$

$$\frac{\partial \theta}{\partial y} = \frac{1}{1+\frac{y^2}{x^2}}\left(\frac{1}{x}\right) = \frac{x}{x^2+y^2} = \frac{r\cos\theta}{r^2} = \frac{\cos\theta}{r}$$

So $\frac{\partial T}{\partial y} = \frac{\partial \tilde{T}}{\partial r}\sin\theta + \frac{\partial \tilde{T}}{\partial \theta}\frac{\cos\theta}{r}$

$$\frac{\partial^2 T}{\partial y^2} = \frac{\partial}{\partial r}\left(\frac{\partial \tilde{T}}{\partial r}\sin\theta + \frac{\partial \tilde{T}}{\partial \theta}\frac{\cos\theta}{r}\right)\frac{\partial r}{\partial y} \leftrightarrow \sin\theta$$

$$+ \frac{\partial}{\partial \theta}\left(\frac{\partial \tilde{T}}{\partial r}\sin\theta + \frac{\partial \tilde{T}}{\partial \theta}\frac{\cos\theta}{r}\right)\frac{\partial \theta}{y} \leftrightarrow \frac{1}{r}\cos\theta$$

MATH
2401

09-12-16

$$\Rightarrow \frac{\partial^2 T}{\partial y^2} = \frac{\partial^2 \tilde{T}}{\partial r^2}\sin^2\theta + \frac{\partial^2 \tilde{T}}{\partial r\partial\theta}\frac{\sin\theta\cos\theta}{r} - \frac{\partial\tilde{T}}{\partial\theta}\frac{\sin\theta\cos\theta}{r^2} + \frac{\partial^2\tilde{T}}{\partial\theta\partial r}\frac{\sin\theta\cos\theta}{r}$$
$$+ \frac{\partial\tilde{T}}{\partial r}\frac{\cos^2\theta}{r} + \frac{\partial^2\tilde{T}}{\partial\theta^2}\frac{\cos^2\theta}{r^2} - \frac{\partial\tilde{T}}{\partial\theta}\frac{\sin\theta\cos\theta}{r^2}$$

So $\dfrac{\partial^2 T}{\partial x^2} + \dfrac{\partial^2 T}{\partial y^2} = \dfrac{\partial^2\tilde{T}}{\partial r^2} + \dfrac{1}{r}\dfrac{\partial\tilde{T}}{\partial r} + \dfrac{1}{r^2}\dfrac{\partial^2\tilde{T}}{\partial\theta^2}$

as most terms cancel.

So $\dfrac{\partial^2 T}{\partial x^2} + \dfrac{\partial^2 T}{\partial y^2} = \dfrac{\partial^2\tilde{T}}{\partial r^2} + \dfrac{1}{r^2}\dfrac{\partial^2\tilde{T}}{\partial\theta^2} + \dfrac{1}{r}\dfrac{\partial\tilde{T}}{\partial r}$

[drop ~]

Separation of variables

$T(r,\theta) = R(r)G(\theta)$

we get $R''(r)G(\theta) + \frac{1}{r^2}R(r)G''(\theta) + \frac{1}{r}R'(r)G(\theta) = 0$

So $\left(r^2R''(r) + rR'(r)\right)G(\theta) = -R(r)G''(\theta)$

So $\dfrac{r^2R''(r) + rR'(r)}{R(r)} = -\dfrac{G''(\theta)}{G(\theta)} = k$

$$\begin{cases} G''(\theta) + kG(\theta) = 0 \\ r^2R''(r) + rR'(r) - kR(r) = 0 \end{cases}$$

Recall B.C. $T(1,\theta) = f(\theta)$, $0 \le \theta \le 2\pi$

we need $T(r,0) = T(r,2\pi)$, $0 \le r \le 1$

$G(\theta) = A\cos\sqrt{k}\,\theta + B\sin\sqrt{k}\,\theta$, $k \ge 0$

$G(0) = G(2\pi)$, so $\sqrt{k} = n$, $n = 0, 1, 2, \ldots$

So $G(\theta) = A\cos n\theta + B\sin n\theta$

$r^2R''(r) + rR'(r) + n^2R(r) = 0$ Euler

Change of variable $r = e^t \longrightarrow t = \log r$

$$\frac{d}{dr} = \frac{dt}{dr}\frac{d}{dt} = \frac{1}{r}\frac{d}{dt}$$

$$\frac{d^2}{dr^2} = -\frac{1}{r^2}\frac{d}{dt} + \frac{1}{r}\frac{d}{dr}\frac{d}{dt} = -\frac{1}{r^2}\frac{d}{dt} + \frac{1}{r}\overbrace{\frac{dt}{dr}}^{\frac{1}{r}}\frac{d^2}{dt^2}$$

$$= -\frac{1}{r^2}\frac{d}{dt} + \frac{1}{r^2}\frac{d^2}{dt^2} \qquad \tilde{R}(t) = R(e^t)$$

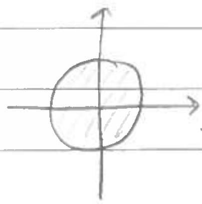
So $$-\frac{d}{dt}\tilde{R} + \frac{d^2}{dt^2}\tilde{R} + \frac{d\tilde{R}}{dt} - n^2\tilde{R} = 0$$

$$\Rightarrow \frac{d^2\tilde{R}}{dt^2} - n^2\tilde{R} = 0$$

$\boxed{n \neq 0}$ $\Rightarrow \tilde{R} = Ce^{nt} + De^{-nt}$ (recall $r = e^t$)

$\Rightarrow R(r) = Cr^n + Dr^{-n}$

$\boxed{n = 0}$ $$\frac{\partial^2\tilde{R}}{\partial t^2} = 0$$



$T(\pi, \theta)$ bounded inside the unit circle

$\Rightarrow \tilde{R} = Ft + H$

$\Rightarrow R(r) = F\log r + H$

$\Rightarrow D = 0,\ F = 0$

So $R(r) = Cr^n$, $n = 0, 1, 2, \ldots$

So $$T = \sum_{n=0}^{\infty} r^n\left[A_n\cos n\theta + B_n\sin n\theta\right]$$

$T(1, \theta) = f(\theta)$

$$\Rightarrow \sum_{n=0}^{\infty} r^n\left[A_n\cos n\theta + B_n\sin n\theta\right] = f(\theta)$$

use Fourier Series.

MATH 2401

09-12-16

Hw question 3. (Hyperbolic equations)

$$F_{xx} + 2F_{xy} + F_{yy} = 0$$
$$(AF_{xx} + BF_{xy} + CF_{yy} = 0)$$
$$\Rightarrow A=1, \quad B=2, \quad C=1$$
$$B^2 - 4AC = 4-4 = 0 \Rightarrow \text{Parabolic.}$$

$$u(x,y) = Px + Qy$$
$$v(x,y) = Rx + Sy \qquad , \; RS - QR \neq 0$$

$$F_{uu} = 0$$
$$\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} P & Q \\ R & S \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$
$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} P & Q \\ R & S \end{pmatrix}^{-1}\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix} \quad \Rightarrow \begin{cases} x = au + bv \\ y = cu + dv \end{cases}$$

$$\frac{\partial}{\partial u} = \frac{\partial x}{\partial u}\frac{\partial}{\partial x} + \frac{\partial y}{\partial u}\frac{\partial}{\partial y} = a\frac{\partial}{\partial x} + c\frac{\partial}{\partial y}$$
$$\frac{\partial}{\partial v} = \frac{\partial x}{\partial v}\frac{\partial}{\partial x} + \frac{\partial y}{\partial v}\frac{\partial}{\partial y} = b\frac{\partial}{\partial x} + d\frac{\partial}{\partial y}$$

$$\frac{\partial^2}{\partial u^2} = a^2\frac{\partial^2}{\partial x^2} + 2ac\frac{\partial^2}{\partial x \partial y} + c^2\frac{\partial^2}{\partial y^2}$$

So $\dfrac{\partial^2 F}{\partial u^2} = a^2\dfrac{\partial^2 F}{\partial x^2} + 2ac\dfrac{\partial^2 F}{\partial x \partial y} + c^2\dfrac{\partial^2 F}{\partial y^2}$

But $0 = \dfrac{\partial^2 F}{\partial x^2} + 2\dfrac{\partial^2 F}{\partial x \partial y} + \dfrac{\partial^2 F}{\partial y^2}$

$\Rightarrow a = 1, \quad c = 1$

choose $b = 0, d = 1$ (so matrix non singular)

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$$

$$\begin{pmatrix} P & Q \\ R & S \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}$$

$$\Rightarrow \begin{cases} u = x \\ v = y - x \end{cases}$$

$$\frac{\partial^2 F}{\partial u^2} = 0$$

$$\Rightarrow \frac{\partial F}{\partial u} = A(v)$$

So $F = A(v)u + B(v)$

$$F = A(y-x)x + B(y-x)$$

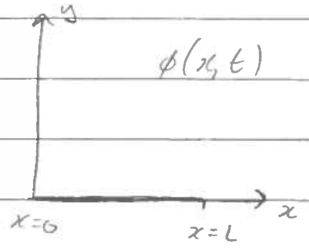
MATH 2401

16-12-16

PDE, More variables

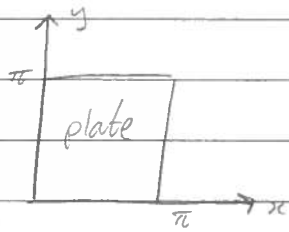
Heat equation

$$\frac{\partial \phi}{\partial t} = K \frac{\partial^2 \phi}{\partial x^2}$$

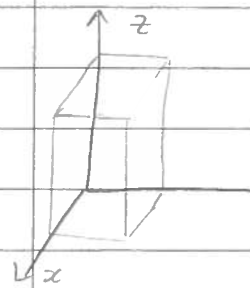


$\phi(x,t)$ B.C. $\begin{cases} \phi(0,t) = M \\ \phi(L,t) = N \end{cases}$

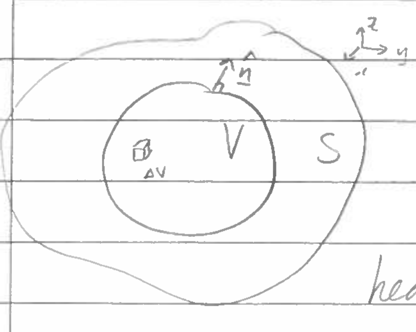
I.C. $\{ \phi(x,0) = f(x)$



$\phi(x,y,t)$



$\phi(x,y,z,t)$



V: arbitrary volume (bounded by S)

$\hat{n}$: outward unit normal

heat enenergy in ΔV: $c \rho \phi \Delta V$ (temperature; specific heat; density)

heat energy inside V : $\int_V c \rho \phi \, dV$

rate of change of this heat energy

$$\frac{d}{dt} \int_V c \rho \phi \, dV = \int_V c \rho \frac{d\phi}{dt} \, dV$$ = amount of heat entering V (through the surface S) per unit time.

Fourier Law

Flux of heat $\vec{q}$ is given by $\vec{q} = -\underset{\text{thermal conductivity}}{k}\, \text{grad}\,\phi$

amount of heat crossing ΔS per unit time is $\vec{q}\cdot\vec{m}\,\Delta S$

$$\int_V c\rho \frac{\partial \phi}{\partial t}\, dV = -\int_S \vec{q}\cdot\vec{m}\, dS = -\int_V \text{div}\,\vec{q}\, dV \quad \text{(divergence thm)}$$

$$\int_V \left(c\rho \frac{\partial \phi}{\partial t} + \text{div}\,\vec{q} \right) dV = 0 \qquad \forall V$$

$$\Rightarrow c\rho \frac{\partial \phi}{\partial t} + \text{div}\,\vec{q} = 0$$

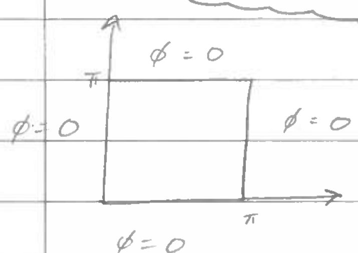
$$\begin{aligned} \text{div}\,\vec{q} &= -\text{div}(k\,\text{grad}\,\phi) \\ &= -k\,\text{div}(\text{grad}\,\phi) \\ &= -k\left[\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2} \right] \end{aligned}$$

$$\Rightarrow c\rho \frac{\partial \phi}{\partial t} = k\left[\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2} \right]$$

So $$\frac{\partial \phi}{\partial t} = K\left[\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2} \right], \quad K = \frac{k}{\rho c}$$

If $\phi(x,t)$, $\dfrac{\partial \phi}{\partial t} = K \dfrac{\partial^2 \phi}{\partial x^2}$

If $\phi(x,y,t)$, $\dfrac{\partial \phi}{\partial t} = K\left[\dfrac{\partial^2 \phi}{\partial x^2} + \dfrac{\partial^2 \phi}{\partial y^2} \right]$



$$\text{B.C.} \begin{cases} \phi(0,y,t) = 0 \\ \phi(\pi,y,t) = 0 \\ \phi(x,0,t) = 0 \\ \phi(x,\pi,t) = 0 \end{cases}$$

$$\text{I.C.} \; \{ \; \phi(x,y,0) = f(x,y)$$

MATH 2401

16-12-16

$\phi(x,y,t) = X(x)Y(y)T(t)$ assume $K=1$.

$X(x)Y(y)T'(t) = X''(x)Y(y)T(t) + X(x)Y''(y)T(t)$

So $\dfrac{T'(t)}{T(t)} = \dfrac{X''(x)}{X(x)} + \dfrac{Y''(y)}{Y(y)}$ by dividing by $X(x)Y(y)T(t)$

Let $\dfrac{X''(x)}{X(x)} = \dfrac{T'(t)}{T(t)} - \dfrac{Y''(y)}{Y(y)} = \alpha$

so $\dfrac{Y''(y)}{Y(y)} = \dfrac{T'(t)}{T(t)} - \alpha = \beta$

$$\text{ODE} = \begin{cases} X''(x) - \alpha X(x) = 0 & X(0)=0,\ X(\pi)=0 \\ Y''(y) - \beta Y(y) = 0 & Y(0)=0,\ Y(\pi)=0 \\ T'(t) - (\alpha+\beta)T(t) = 0 \end{cases}$$

$$\frac{\partial\phi}{\partial t} = K\left[\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2}\right]$$

$X''(x) - \alpha X(x) = 0$, $X(0) = X(\pi) = 0$

$\alpha > 0$ and $\alpha = 0$ trivial.

$\alpha < 0$, $\alpha = -p^2$

so $X''(x) + p^2 X(x) = 0$

$X(x) = A\cos px + B\sin px$

$X(0) = 0 \Rightarrow A = 0$

$X(\pi) = 0 \Rightarrow B\sin p\pi = 0 \rightarrow B = 0$ (trivial)

$\searrow \sin p\pi = 0$

so $p = n$, $n = 1, 2, 3, \dots$

So $X(x) = B\sin nx$, $n = 1, 2, 3, \dots$

Similarly $Y(y) = D\sin my$, $m = 1, 2, 3, \dots$

$$T'(t) - (\alpha+\beta)T(t) = 0$$
$$T'(t) + (n^2+m^2)T(t) = 0$$

$$T(t) = Fe^{-(n^2+m^2)t}$$

So $$\phi(x,y,t) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} C_{n,m} \sin nx \sin my \, e^{-(n^2+m^2)t}$$

$$\left[\begin{array}{l} \displaystyle\int_{-\pi}^{\pi} \sin nx \sin lx \, dx = \pi \delta_{m,l} \\ \displaystyle\int_{0}^{\pi} \sin nx \sin lx \, dx = \frac{\pi}{2}\delta_{m,l} \end{array}\right]$$

$t=0$ and integrating:

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} C_{n,m} \underbrace{\int_0^{\pi} \sin nx \sin lx \, dx}_{\frac{\pi}{2}\delta_{n,l}} \underbrace{\int_0^{\pi} \sin my \sin jy \, dy}_{\frac{\pi}{2}\delta_{m,j}} = \int_0^{\pi}\int_0^{\pi} f(x,y) \sin lx \sin jy \, dx\, dy$$

So $$C_{l,j} = \frac{4}{\pi^2}\int_0^{\pi}\int_0^{\pi} f(x,y) \sin lx \sin jy \, dx\, dy$$

Further generalisations

$\phi(x,y,z,t)$

$$\frac{\partial \phi}{\partial t} = \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2}$$

B.C. : $\phi = 0$ on the six sides of the cube

I.C. : $\phi(x,y,z,0) = f(x,y,z)$

Let $\phi(x,y,z,t) = X(x)Y(y)Z(z)T(t)$

$$\text{ODE} = \begin{cases} X''(x) - \alpha X(x) = 0 \\ Y''(y) - \beta Y(y) = 0 \\ Z''(z) - \gamma Z(z) = 0 \\ T'(t) - (\alpha+\beta+\gamma)T(t) = 0 \end{cases}$$

MATH 2401

16-12-16

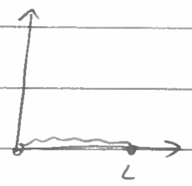
B.C. $\begin{cases} X(0) = X(\pi) = 0 \\ Y(0) = Y(\pi) = 0 \\ Z(0) = Z(\pi) = 0 \end{cases}$

So $\begin{cases} X(x) = A\sin nx \\ Y(y) = B\sin my \\ Z(z) = C\sin lz \\ T(t) = De^{-(n^2+m^2+l^2)t} \end{cases}$

So $\phi(x,y,z,t) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\sum_{l=1}^{\infty} C_{n,m,l} \sin nx \sin my \sin ly \, e^{-(n^2+m^2+l^2)t}$

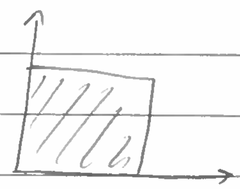
Wave Equation

$\phi(x,t)$ $\quad \frac{\partial^2 \phi}{\partial t^2} = c^2 \frac{\partial^2 \phi}{\partial x^2}$



B.C. $\begin{cases} \phi(0,t) = 0 \\ \phi(L,t) = 0 \end{cases}$

I.C. $\begin{cases} \phi(x,0) = f(x) \\ \frac{\partial \phi}{\partial t}(x,0) = g(x) \end{cases}$



$\phi(x,y,t)$

$\frac{\partial^2 \phi}{\partial t^2} = c^2 \left[\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} \right]$

B.C. $\phi = 0$ on sides

I.C. $\phi(x,y,0) = f(x,y)$

$\frac{\partial \phi}{\partial t}(x,y,0) = g(x,y)$