2401 Mathematical Methods 3 Notes

Based on the 2011 autumn lectures by Dr R I Bowles

MATH 2401 : Methods 3 — 5th october 2011

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PARTIAL DIFFERENTIATION (Revision and extension)

simple case $\mathbb{R}^2 \to \mathbb{R}$

Let $f(x,y)$ be a function of the two independent variables x, y. The two partial derivatives of f at a point (a,b) are the limits

$$\frac{\partial f}{\partial x} = \lim_{h \to 0} \frac{f(a+h,b) - f(a,b)}{h}$$

$$\frac{\partial f}{\partial y} = \lim_{k \to 0} \frac{f(a,b+k) - f(a,b)}{k}$$

if these limits exist.

notation $\frac{\partial f}{\partial x}$, $\left.\frac{\partial f}{\partial x}\right|_y$, f_x, $f_x(a,b)$

If these derivatives exist at every point in a region of $\mathbb{R}^2$ then we have a function derived from $f(x,y)$

$$\frac{\partial f}{\partial x}(x,y) = \lim_{\partial x \to 0} \frac{f(x+\partial x, y) - f(x,y)}{\partial x}$$

These functions themselves may be differentiable and we can find higher derivatives

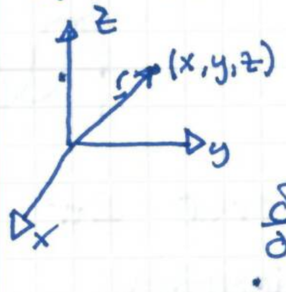
$$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = f_{xx}$$

$$f_{xy} = f_{yx}$$

$$\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = f_{xy}$$

example

$\mathbb{R}^3 \to \mathbb{R}$



If $\underline{r} = (x,y,z)^T = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ then $f(x,y,z) = f(\underline{r})$

is $f(\underline{r}) = |\underline{r}|$ = distance of point (x,y,z) from O

$$= \sqrt{x^2+y^2+z^2}$$

$$\frac{\partial f}{\partial x} = \frac{x}{\sqrt{x^2+y^2+z^2}} = \frac{x}{|\underline{r}|}$$

$$f_y = \frac{y}{|\underline{r}|}, \quad f_z = \frac{z}{|\underline{r}|}$$

$$\nabla f = \begin{pmatrix} \partial f/\partial x \\ \partial f/\partial y \\ \partial f/\partial z \end{pmatrix} = \frac{1}{|\underline{r}|}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \frac{r}{|\underline{r}|}$$

Differentiability of functions of several variables

Problem class

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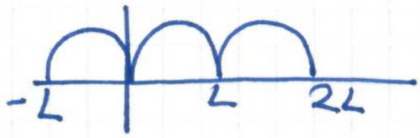
Problem sheet 1

FOURIER SERIES

$f(x) = f(x+L) \quad L \in \mathbb{R}, L \neq 0$



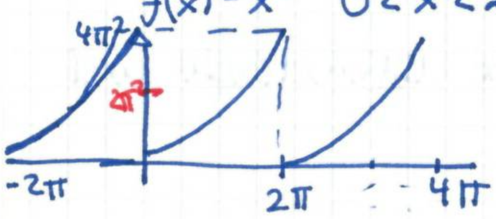
$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{2n\pi x}{L}\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{2n\pi x}{L}\right)$$

$$a_0 = \frac{1}{L}\int_0^L f(x)\,dx,$$

$$a_n = \frac{2}{L}\int_0^L f(x)\cos\left(\frac{2n\pi x}{L}\right)dx, \quad b_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{2n\pi x}{L}\right)dx$$

Qu 1

$f(x) = x^2 \quad 0 < x < 2\pi$



Periode is 2π

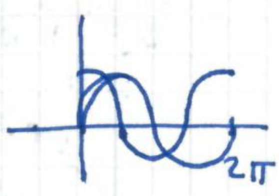
(i) $x^2 = a_0 + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$

integrating (i) over $[0, 2\pi]$ we get

$$\int_0^{2\pi} x^2\,dx = \int_0^{2\pi} a_0\,dx + \sum_{n=1}^{\infty} a_n \int_0^{2\pi}\cos nx\,dx + \sum_{n=1}^{\infty} b_n \int_0^{2\pi}\sin(nx)\,dx$$

$$\left.\frac{x^3}{3}\right|_0^{2\pi} = a_0 2\pi + 0 + 0 \Rightarrow$$

$$\frac{8\pi^3}{3} = a_0 2\pi \Rightarrow a_0 = \frac{4\pi^2}{3}$$



Multiplying (i) by $\cos mx$ and integrating over 2π we get

$$\int_0^{2\pi} x^2 \cos mx\,dx = a_0 \underbrace{\int_0^{2\pi}\cos mx\,dx}_{=0} + \sum_{n=1}^{\infty} a_n \int_0^{2\pi}\cos(nx)\cos(mx)\,dx + \sum_{n=1}^{\infty} b_n \int_0^{2\pi}\sin(nx)\cos(mx)\,dx$$

$$\int_0^{2\pi} x^2\cos(mx)\,dx = a_m \int_0^{2\pi}\cos^2 mx\,dx$$

$$\Rightarrow a_m = \frac{1}{\pi}\int_0^{2\pi} x^2\cos mx$$

similarly $b_m = \frac{1}{\pi}\int_0^{2\pi} x^2\sin mx\,dx$

let us evaluate a_m and b_m. Since $e^{imx} = \cos(mx)+i\sin(mx)$ we get

$$(a+ib) = \frac{1}{\pi}\int_0^{2\pi} x^2 e^{imx}\,dx \qquad u=x^2 \quad v'=e^{imx}$$

$$\pi(a_m+ib_m) = \int_0^{2\pi} x^2 e^{imx}\,dx = \int_0^{2\pi}\frac{x^2\,de^{imx}}{im}$$ ✓ integration by parts

$$= \frac{x^2}{im}e^{imx}\Big|_0^{2\pi} - \int_0^{2\pi}\frac{2x}{im}e^{imx}\,dx$$

$$= \frac{(2\pi)^2}{im} - \frac{2}{m}\underbrace{\int_0^{2\pi} x e^{imx}\,dx}_{=\int_0^{2\pi}\frac{x\,de^{imx}}{im}}$$

$$= \frac{4\pi^2}{im} + \frac{2}{m^2}\int_0^{2\pi} x\,de^{imx} = \frac{4\pi^2}{im} - \frac{2}{m^2}\underbrace{\int_0^{2\pi} e^{imx}\,dx}_{=0}$$

$$+ \frac{2}{m^2}xe^{imx}\Big|_0^{2m} = \frac{4\pi^2}{im} + \frac{2}{m^2}2\pi = \frac{4\pi}{m^2} - i\frac{4\pi^2}{m}$$

$$\Rightarrow a_m = \frac{4}{m^2}, \quad b_m = -\frac{4\pi}{m}$$

need to prove:

$$\frac{\pi^2}{6} = \sum_{k=1}^{\infty}\frac{1}{k^2} \qquad (3)$$

$$x^2 = \frac{4\pi^2}{3} + \sum_{n=1}^{\infty}\frac{4}{n^2}\cos(nx) - \frac{4\pi}{n}\sin(nx)$$

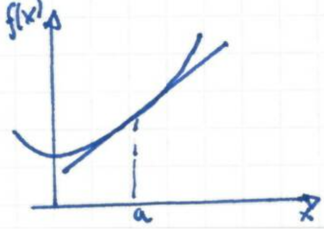
- only true where function is continuous, we can't use it

$$\overset{2\pi^2}{\cancel{0}} = \frac{4\pi^2}{3} + \sum_{n=1}^{\infty}\frac{4}{n^2} \qquad (2) \qquad (2)\Leftrightarrow(3)$$

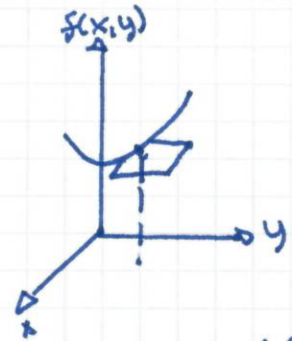
2. $f(x)=x \quad 0<x<2$

7th october 2011

Differentiability ≈ locally linear



$$f(a+h) \approx f(a) + hf'(a)$$



we say $f'(a)$ exists if

$$f(a+h) = f(a) + hf'(a) + |h|\varphi(h) \text{ where}$$
$$\varphi(h) \to 0 \text{ as } |h| \to 0$$

In two dimensions $\mathbb{R}^2 \to \mathbb{R}$ the function $f(x,y)$ is differentiable at the point (a,b) if there exist two numbers: L_x, L_y such that

tangent plane: $f(a+h, b+k) = f(a,b) + L_x h + L_y k + \sqrt{h^2+k^2}\,\varphi(h,k)$
wher $\varphi(h,k) \to 0$ as $h,k \to 0$.

we can write this slightly differently, and extend to $\mathbb{R}^n \to \mathbb{R}$ and say

$$f(\underline{x}+\underline{\delta x}) = f(\underline{x}) + \underline{L}\cdot\underline{\delta x} + |\underline{\delta x}|\,|\varphi(\underline{\delta x})|$$ and claim that f is differentiable at $\underline{x}$ if such $\underline{L}$ exists and $|\varphi(\underline{dx})| \to 0$ as $|\underline{\delta x}| \to 0$

$$\underline{L}\cdot\underline{\delta x} = (L_x \; L_y)\begin{pmatrix}\delta x\\ \delta y\end{pmatrix}$$

consider now three dimensions $\mathbb{R}^3 \to \mathbb{R}$, then it turns out that $\underline{L} = \underline{\nabla} f = \text{grad} f$ since, for example,

$$\frac{\partial f}{\partial x} = \lim_{h\to 0} \frac{f(x+h,y,z) - f(x,y,z)}{h} \overset{f \text{ is differentiable}}{=} \lim_{h\to 0} \left(\frac{L_x h + 0 + 0 + |h|\varphi(h)}{h}\right)$$

$$= L_x \text{ as } \varphi(h) \to 0 \text{ as } h \to 0$$

so $$\underline{L} = \begin{pmatrix}\delta f/\delta x\\ \partial f/\delta y\\ \delta f/\delta z\end{pmatrix} = \underline{\nabla} f$$

consider now functions $\mathbb{R}^m \to \mathbb{R}^n$, we say that $\underline{f}$ (vector in $\mathbb{R}^n$) is differentiable at the point $\underline{x}$ (vector in $\mathbb{R}^m$) if there exists a $n \times m$ matrix $\underline{\underline{L}}$ (matrix) such that

$$\underline{f}(\underline{x}+\underline{h}) = \underline{f}(\underline{x}) + \underline{\underline{L}}\,\underline{h} + |\underline{h}|\,\underline{\varphi}(\underline{h}) \text{ with } |\underline{\varphi}(\underline{h})| \to 0 \text{ as}$$
$|\underline{h}| \to 0$ where $\underline{\underline{L}}$ is

$$n \left\{ \begin{pmatrix} \delta f_1/\delta x_1 & \delta f_1/\delta x_2 & \dots & \delta f_1/\delta x_m \\ \partial f_2/\delta x_1 & & & \\ \vdots & & \delta f_i/\delta x_j & \\ \delta f_n/\delta x_1 & & & \delta f_n/\delta x_m \end{pmatrix} \right. \atop \underbrace{\hspace{6cm}}_{m}$$

and is called the Jacobian matrix.

$$L_{ij} \text{ is } = \frac{\delta f_i}{\delta x_j}$$

The chain rule

Consider functions formed by the composition of others. In one dimension we might consider $F(t) = f(x(t))$. We have $t \to x \to F$, $\mathbb{R} \to \mathbb{R} \to \mathbb{R}$

Consider the change in F caused by the change in t

$$F(t+\delta t) - F(t) = f(x(t+\delta t)) - f(x(t))$$

$$= f(x(t) + \delta t\, x'(t) + \underbrace{|\delta t|\,\varphi(\delta t)}_{+\dots \text{ the higher order terms that aren't linear}}) - f(x(t)),$$

as $x(t)$ is differentiable

$$= f(x(t)) + f'(x(t))\delta t\, x'(t) + \dots - f(x(t))$$

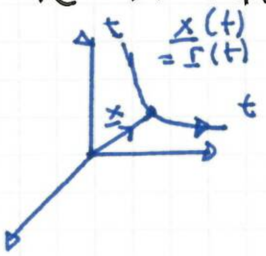
$$= \delta t\,\big(\underline{f'(x(t))\,x'(t)}\big) + \dots$$

Compare this now with the statement that $F(t)$ is differentiable, in the form

$$F(t+\delta t) = F(t) + \delta t\, \underline{F'(t)} + \dots$$

and $\boxed{F'(t) = f'(x(t))\,x'(t)}$

$\mathbb{R} \to \mathbb{R}^3 \to \mathbb{R}$



$$f(\underline{x}(t))$$

consider $F(t) = f(x(t), y(t), z(t))$

$$F(t+\delta t) - F(t) = f(x(t+\delta t), y(t+\delta t), z(t,\delta t)) - f(x(t), y(t), z(t))$$

$$= f(x(t) + \delta t\, x'(t) + \dots, y(t) + \delta t\, y'(t) + \dots, z(t) + \delta t\, z'(t) + \dots) - f(x(t), y(t), z(t))$$

$$f(\underline{x}+\underline{h}) \simeq f(\underline{x}) + \nabla f \cdot \underline{h}$$

$$= \cancel{f(x(t), y(t)}, z(t)) + \frac{\delta f}{\delta x}\delta t\, x'(t) + \frac{\delta f}{\delta y}\delta t\, y'(t) + \frac{\delta f}{\delta z}\delta t\, z'(t) + \dots - \cancel{f(x(t), y(t)}, z(t))$$

so we identify

$$F'(t) = \frac{\delta f}{\delta x}\cdot x'(t) + \frac{\delta f}{\delta y}\, y'(t) + \frac{\delta f}{\delta z}\cdot z'(t)$$

If we use a $\dot{}$ for time derivatives and $r(t)$ instead of $x(t)$

$$\dot{F} = \dot{\underline{r}} \cdot \nabla f$$

$$\dot{F} = \nabla f \cdot \dot{\underline{x}} = \underbrace{(\delta f/\delta x \quad \delta f/\delta y \quad \delta f/\delta z)}_{\text{Jacobian for } f(\underline{x})\ (1\times 3)} \overbrace{\begin{pmatrix} \delta x/\delta t \\ \delta y/\delta t \\ \delta z/\delta t \end{pmatrix}}^{\text{Jacobian for } \underline{x}(t)\ (3\times 1)}$$

Jacobian for $F(t)$, a 1×1 matrix

$$F(t) = f(\underline{x}(t))$$

The Jacobian for F, the composition of $\underline{x}(t)$ & $f(\underline{x})$, is obtained by matrix multiplication (i.e. composition) of the Jacobians for $\underline{x}(t)$ & $f(\underline{x})$

We can generalise this observation

$\mathbb{R}^l \to \mathbb{R}^m \to \mathbb{R}^n$

The two functions $\underline{x}(\underline{u})$ & $\underline{f}(\underline{x})$ are from $\mathbb{R}^l \to \mathbb{R}^m$ & $\mathbb{R}^m \to \mathbb{R}^n$ respectively, then the composition $\underline{F}(\underline{u}) = \underline{f}(\underline{x}(\underline{u}))$ is from $\mathbb{R}^l \to \mathbb{R}^n$. $\mathbb{R}^l \to \mathbb{R}^m \to \mathbb{R}^n$. These mappings $\underline{x}, \underline{f}$ & $\underline{F}$ have Jacobians

$$\underline{x}: \begin{pmatrix} \partial x_1/\delta u_1, \partial x_1/\delta u_2, \ldots, \partial x_1/\delta u_l \\ \vdots \\ \vdots \\ \partial x_m/\delta u_1 \quad \delta x_m/\delta u_l \end{pmatrix} = \underline{\underline{U}} \quad m\times l \text{ matrix}$$

$$\underline{f}: \begin{pmatrix} \delta f_1/\delta x_1 & \cdots & \delta f_1/\delta x_m \\ \vdots & & \vdots \\ \delta f_n/\delta x_1 & \cdots & \delta f_n/\delta x_m \end{pmatrix} = \underline{\underline{T}} \quad \text{a } n\times m \text{ matrix}$$

$$\underline{F}: \begin{pmatrix} \delta F_1/\delta u_1 & \cdots & \delta F_1/\delta u_l \\ \vdots & \ddots & \\ \delta F_n/\delta u_1 & \cdots & \delta F_n/\delta u_l \end{pmatrix} = \underline{\underline{S}} \quad \text{a } n\times l \text{ matrix}$$

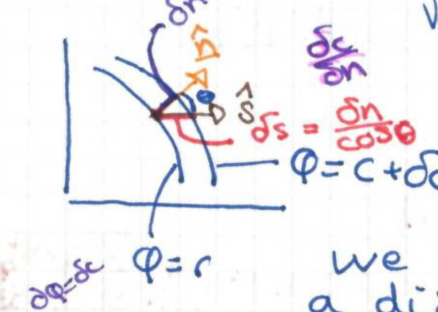
The Chain Rule says $\underline{\underline{S}} = \underline{\underline{T}}\,\underline{\underline{U}}$

A geometric interpretation of the gradient ∇f

Consider $\varphi(\underline{x}) = \varphi(x, y, z)$ with x, y, z independent variables. If x, y, z are chosen so that $\varphi(x,y,z) = \text{constant} = c$ then this imposes a constraint on our choice & points (x,y,z) satisfying $\varphi(x,y,z) = c$ lie on a surface, called a level surface of φ (3D + 1 constrant = 2D)

$f(x,y,z) = x^2+y^2+z^2$ sphere centered at origin

$\varphi(x,y,z) = ax+by+cz = d$ is a plane with the normal $\begin{pmatrix} a \\ b \\ c \end{pmatrix}$ and the distance $\frac{d}{\sqrt{a^2+b^2+c^2}}$



Consider a neighbouring surface given by $\varphi = c + \delta c$. Consider too a unit normal, $\hat{n}$, to the surface $\varphi = c$

We can ask how much φ changes if we move a distance δn in the direction of $\hat{n}$. Call this change $\delta\varphi$ & consider $\lim_{\delta n \to 0} \frac{\delta\varphi}{\delta n} = \frac{\partial\varphi}{\partial n}$ (if it exists)

i.e the rate of change of φ measured in a direction normal to a level surface.

we define the gradient of the function φ to be the vector

$$\underline{\nabla}\varphi = \hat{\underline{n}}\frac{\partial\varphi}{\partial n} \qquad \text{we shal se } \underline{\nabla}\varphi = \begin{pmatrix}\varphi_x\\ \varphi_y\\ \varphi_z\end{pmatrix}$$

suppose I want the rate of change of φ in a different direction given by $\hat{\underline{s}}$, df/ds say.

$$\frac{\partial\varphi}{\partial s} = \lim_{\delta s\to 0}\frac{\delta\varphi}{\delta s} = \lim_{\delta s\to 0}\frac{\delta c}{\delta n/\cos\theta} = \cos\theta \lim_{\delta n\to 0}\frac{\delta\varphi}{\delta n}$$

$$= \cos\theta\frac{\partial\varphi}{\partial n}$$

But $\cos\theta = \hat{\underline{s}}\cdot\hat{\underline{n}}$, so $\frac{\partial\varphi}{\partial s} = \hat{\underline{s}}\cdot\hat{\underline{n}}\frac{\partial\varphi}{\partial n} = \hat{\underline{s}}\cdot\underline{\nabla}\varphi$

we call $\frac{\partial\varphi}{\partial s}$ the directional derivative the directions of $\hat{\underline{s}}$ and we see that this directional derivative is obtained by

$$\boxed{\frac{\partial\varphi}{\partial s} = \hat{\underline{s}}\cdot\underline{\nabla}\varphi}$$

If we choose $\hat{\underline{s}} = \underline{i}$, this becomes $\frac{\partial\varphi}{\partial x} = \underline{i}\cdot\underline{\nabla}\varphi$ = the first component of $\underline{\nabla}\varphi$

so we conclude

$$\underline{\nabla}\varphi = \begin{pmatrix}\varphi_x\\ \varphi_y\\ \varphi_z\end{pmatrix}$$

12/10/2011

$$\frac{\partial f}{\partial s} = \hat{\underline{s}}\cdot\underline{\nabla}f$$

Defined at a point, but that point can be anything. So $\frac{\partial f}{\partial s}$ is a function of position, assuming it exists.

consider $f(x,y) = (x+1)(y-1)$ & find the directional derivatives in the directions of $\underline{i}+\underline{j}$ & $\underline{i}-\underline{j}$ unit vectors in these directions are

$$\frac{1}{\sqrt{2}}(\underline{i}+\underline{j}) \quad \& \quad \frac{1}{\sqrt{2}}(\underline{i}-\underline{j})$$

$$\underline{\nabla}f = \begin{pmatrix}\partial f/\partial x\\ \partial f/\partial y\end{pmatrix} = \begin{pmatrix}y-1\\ x+1\end{pmatrix}$$

& so if $\underline{s} = \underline{i}+\underline{j}$ we have $\frac{\partial f}{\partial s} = \begin{pmatrix}1/\sqrt{2}\\ 1/\sqrt{2}\end{pmatrix}\begin{pmatrix}y-1\\ x+1\end{pmatrix}$

$s = \underline{i}-\underline{j}$ $\quad \frac{\partial f}{\partial s} = \begin{pmatrix}1/\sqrt{2}\\ -1/\sqrt{2}\end{pmatrix}\begin{pmatrix}y-1\\ x+1\end{pmatrix} = \frac{1}{\sqrt{2}}(y-x-2)$

we can ask for the rate of change of these directional derivatives in different directions. Most useful will be

$$\frac{\partial}{\partial s}\left(\frac{\partial f}{\partial s}\right) = \hat{\underline{s}}\cdot\underline{\nabla}\left(\frac{\partial f}{\partial s}\right) = (\hat{\underline{s}}\cdot\underline{\nabla})(\hat{\underline{s}}\cdot\underline{\nabla}f) = (\hat{\underline{s}}\cdot\underline{\nabla})^2 f \text{ say}$$

For our example the second derivatives are, for $s = \underline{i}+\underline{j}$

$$\begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix} \circ \begin{pmatrix} \partial/\partial x \; (1/\sqrt{2}(x+y)) \\ \partial/\partial y \; (1/\sqrt{2}(x+y)) \end{pmatrix} = \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix} \circ \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix} = 1$$

for $s = \underline{i} - \underline{j}$

$$\begin{pmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{pmatrix} \cdot \begin{pmatrix} \partial/\delta x \; (1/\sqrt{2}(y-x-2)) \\ \partial/\partial y \; (1/\sqrt{2}(y-x-2)) \end{pmatrix} = -1$$

Taylor's theorem

We know that for $f: \mathbb{R} \to \mathbb{R}$ we have, under certain conditions

$$f(a+h) = f(a) + hf'(a) + \frac{1}{2}h^2 f''(a) + \frac{1}{6}h^3 f'''(a) + \dots = \sum_{n=0}^{\infty} \frac{f^n(a)}{n!} h^n$$

within a radius of convergence.

$$f(\underline{a}+\underline{h}) = f(\underline{a}) + \underset{\substack{\uparrow \\ \nabla f}}{\underline{L}} \cdot \underline{h} + \dots$$

To extend this & find the subsequent terms in a statement of Taylors Theorem for $f: \mathbb{R}^n \to \mathbb{R}$, we imagine fixing the direction of $\underline{h}$ (take $\hat{\underline{s}}$ say) & the problem is then reduced to one in one dimension, with variable $|\underline{h}|$ the distance travelled in the direction of $\underline{h}$ & we can use Taylors theorem for $f: \mathbb{R} \to \mathbb{R}$

We see that the $f^n(a)$ need to be replaced by $\frac{\partial^n f}{\partial s^n} = (\hat{\underline{s}} \circ \nabla)^n f$ and the h needs to be replaced by $|\underline{h}|$.

So we get $f(\underline{a}+\underline{h}) = f(\underline{a}) + |\underline{h}|(\hat{\underline{s}} \circ \nabla) f + \frac{1}{2}|\underline{h}|^2 (\hat{\underline{s}} \circ \nabla)^2 f + \dots$

$$+ \frac{1}{n!}|\underline{h}|^n (\hat{\underline{s}} \circ \nabla)^n f + \dots$$

but $\underline{h} = |\underline{h}|\hat{\underline{s}}$ and so $f(\underline{a}+\underline{h}) = f(\underline{a}) + (\underline{h} \circ \nabla) f + \frac{1}{2}(\underline{h} \cdot \nabla)^2 f$

$$+ \frac{1}{n!}(\underline{h} \circ \nabla)^n f + \dots = \sum_{n=0}^{\infty} \frac{(\underline{h} \circ \nabla)^n f}{n!}$$

If $f: \mathbb{R}^2 \to \mathbb{R}$ & $\underline{h} = \begin{pmatrix} h \\ k \end{pmatrix}$, then $\underline{h} \cdot \nabla f = \begin{pmatrix} h \\ k \end{pmatrix} \begin{pmatrix} \partial f/\partial x \\ \partial f/\partial y \end{pmatrix}$

$$= h\frac{\partial f}{\partial x} + k\frac{\partial f}{\partial y}$$

$$(\underline{h} \cdot \nabla)^2 f = \begin{pmatrix} h \\ k \end{pmatrix} \circ \begin{pmatrix} \partial/\partial x \; (h \, \partial f/\partial x + k \, \partial f/\partial y) \\ \partial/\partial y \; (h \, \partial f/\partial x + k \, \partial f/\partial y) \end{pmatrix} =$$

$$= \begin{pmatrix} h \\ k \end{pmatrix} \circ \begin{pmatrix} hf_{xx} + kf_{xy} \\ hf_{xy} + kf_{yy} \end{pmatrix}$$

$$= h^2 f_{xx} + 2hkf_{xy} + k^2 f_{yy}$$

$$(\underline{h} \cdot \nabla)^3 f = h^3 f_{xxx} + 3h^2 k f_{xxy} + 3hk^2 f_{xyy} + k^3 f_{yyy}$$

If $f: \mathbb{R}^3 \to \mathbb{R}$, $h = \begin{pmatrix} h \\ k \\ l \end{pmatrix}$ then $\underline{h} \cdot \underline{\nabla} = h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y} + l\frac{\partial}{\partial z}$

$$(\underline{h} \cdot \underline{\nabla})^2 = h^2\frac{\partial^2}{\partial x^2} + k^2\frac{\partial^2}{\partial y^2} + l^2\frac{\partial^2}{\partial z^2} + 2hk\frac{\partial^2}{\partial x \partial y} + 2hl\frac{\partial^2}{\partial x \partial z} + 2kl\frac{\partial^2}{\partial y \partial z}$$

coefficients are found by considering the expansion of $(a+b+c)^n$

Express

$f(x,y) = x^2y + 3y - 2$ in powers of $(x-1)$ & $(y+2)$

we will do this by finding a Taylor Series for $f(x,y)$ about the point $(1,-2)$

$\left[\underline{a} = \begin{pmatrix} 1 \\ -2 \end{pmatrix}\right.$ and $\underline{h} = \begin{pmatrix} h \\ k \end{pmatrix} = \begin{pmatrix} x-1 \\ y+2 \end{pmatrix}$ so $\begin{pmatrix} x \\ y \end{pmatrix} = \underline{a} + \underline{h}$

$\frac{\partial f}{\partial x} = 2xy$ $\quad \frac{\partial^2 f}{\partial x^2} = 2y$ $\quad \frac{\partial^2 f}{\partial x \partial y} = 2x$

$\frac{\partial f}{\partial y} = x^2 + 3$ $\quad \frac{\partial^2 f}{\partial y^2} = 0$ $\quad \frac{\partial^3 f}{\partial x^3} = 0$

$\frac{\partial^3 f}{\partial x^2 \partial y} = 2$ $\quad \frac{\partial^3 f}{\partial x \partial y^2} = 0$ $\quad \frac{\partial^3 f}{\partial y^3} = 0$

and higher derivatives are zero

$$x^2y + 3y - 2 = f(x,y) = f(1,-2) + \left(\overset{x-1}{h}\underbrace{\frac{\partial f}{\partial x}}_{-4} + \overset{y+2}{k}\underbrace{\frac{\partial f}{\partial y}}_{4}\right)\Bigg|_{(1,-2)}$$

$$+ \frac{1}{2}\left(h^2 f_{xx} + 2hk f_{xy} + k^2 0\right)\Big|_{(1,-2)}$$

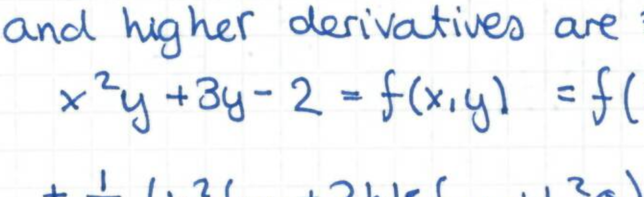
$$+ \frac{1}{6}\left(h^3 \cdot 0 + 3h^2k\frac{\partial^3 f}{\partial x^2 \partial y} + 3hk^2 \cdot 0 + k^3 \cdot 0\right)\Big|_{(1,-2)} + 0$$

$$= -10 - 4(x-1) + 4(y+2) - 2(x-1)^2 + 2(x-1)(y+2) + (x-1)^2(y+2)$$

14/10-2011

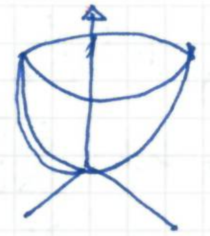
Extreme values & critical points of functions of several (mainly two variables

If f is from $\mathbb{R} \to \mathbb{R}$

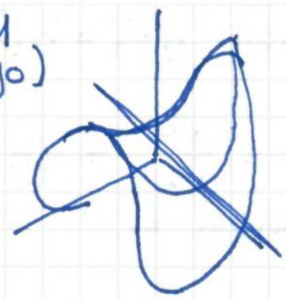


At critical point, the tangent to the curve is horizontal (parallell to x-axis) and we test for this by finding positions where $f'(x) = 0$

MAXIMUM at (x_0, y_0) if

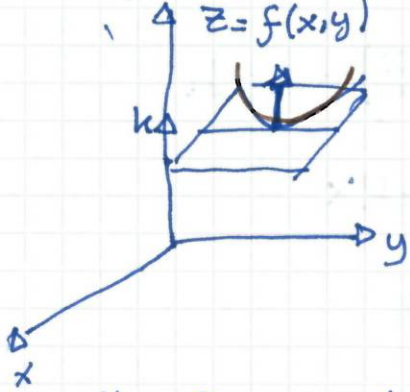


MINIMUM at (x_0, y_0)



SADDLE POINT at (x_0, y_0)

If we have a function given by $z = f(x,y)$ then at a point where f has a local maximum / minimum / saddle point, the tangent plane to the surface is parallel to the (x,y) plane, or has a normal parallel to $\underline{k}$



The normal to a surface written as a level surface of $g(x,y,z) = z - f(x,y) = 0$ is given by $\nabla g = \begin{pmatrix} -\partial f/\partial x \\ -\partial f/\partial y \\ 1 \end{pmatrix}$ and so at a critical point $f_x = f_y = 0$

A critical point (x_0, y_0) is such that $\frac{\partial f}{\partial x}(x_0, y_0) = \frac{\partial f}{\partial y}(x_0, y_0) = 0$.
Using Taylors theorem about (x_0, y_0) & with $\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} = \underline{x_0}$

$$f(\underline{x_0} + \underline{h}) = f(\underline{x_0}) + \underbrace{\underline{h} \cdot \nabla f \big|_{\underline{x_0}}}_{\begin{pmatrix} \partial f/\partial x \\ \partial f/\partial y \end{pmatrix} \big|_{\underline{x_0}} = 0} + \frac{1}{2} (\underline{h} \cdot \nabla)^2 f \big|_{\underline{x_0}} + \dots$$

and so $f(\underline{x_0} + \underline{h}) - f(\underline{x_0}) = \frac{1}{2} \left(h^2 \frac{\partial^2 f}{\partial x^2} + 2hk \frac{\partial^2 f}{\partial x \partial y} + k^2 \frac{\partial^2 f}{\partial y^2} \right) \Big|_{(x_0, y_0)}$

If $f : \mathbb{R}^n \to \mathbb{R}$ then with $\underline{h} = (h_1, h_2, h_3, \dots h_n)^T$ this is

$$\underbrace{(h_1, h_2 \dots h_n) \underbrace{\begin{pmatrix} \frac{\partial^2 f}{\partial x_1^2} & \frac{\partial^2 f}{\partial x_1 \partial x_2} & \dots & \frac{\partial^2 f}{\partial x_1 \partial x_n} \\ \vdots & & \ddots & \vdots \\ \frac{\partial^2 f}{\partial x_n \partial x_1} & \dots & \dots & \frac{\partial^2 f}{\partial x_n^2} \end{pmatrix}}_{\text{The Hessian of } f} \begin{pmatrix} h_1 \\ h_2 \\ \vdots \\ h_n \end{pmatrix}}_{\text{A quadratic form}} \qquad (h \ k) \begin{pmatrix} f_{xx} & f_{xy} \\ f_{xy} & f_{yy} \end{pmatrix} \begin{pmatrix} h \\ k \end{pmatrix}$$

Testing whether this quadratic form is always +ve / -ve / or either depending on coefficients in $(h_1, h_2 \dots h_n)$ reduces to seeing if the eigenvalues of the Hessian are all +ve / -ve / or mixed in sign. Here though we proceed by completing the square.

We will assume $f_{xx} \neq 0$. If $f_{xx} = 0$, then we proceed as below using f_{yy} instead of f_{xx}. If $f_{xx} = f_{yy} = 0$ then it is clear we have a saddle point since the product hk can be made of either sign by choosing h & k appropriately

$$f(\underline{x_0} + \underline{h}) - f(\underline{x_0}) = \frac{1}{2} f_{xx} \left[h^2 + 2hk \frac{f_{xy}}{f_{xx}} + k^2 \frac{f_{yy}}{f_{xx}} \right]$$

$$= \frac{1}{2} f_{xx} \left[\underbrace{\left(h + k \frac{f_{xy}}{f_{xx}} \right)^2}_{\text{always +ve or 0}} + \underbrace{k^2 \left(\frac{f_{yy}}{f_{xx}} - \frac{f_{xy}^2}{f_{xx}^2} \right)}_{\frac{k^2}{(f_{xx})^2} \cdot (f_{xx} f_{yy} - f_{xy}^2)} \right]$$

may be either sign depending on sign of $\Delta = | f_{xx}$

$$k^2\left(\frac{f_{yy}}{f_{xx}} - \frac{f_{xy}^2}{f_{xx}^2}\right)$$

$$\frac{k^2}{(f_{xx})^2}\left(\underbrace{f_{xx}f_{yy}-f_{xy}^2}\right)$$

may be either sign depending of sign of $\Delta = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{xy} & f_{yy} \end{vmatrix}$ called the Discriminant

$$= \frac{1}{2} f_{xx}\left[\left(h + k\frac{f_{xy}}{f_{xx}}\right)^2 + \frac{k^2}{(f_{xx})^2}\Delta\right]$$

If $\Delta > 0$ then this has the same sign as does f_{xx}. So if $\Delta > 0$ and $f_{xx} > 0$ we have a minimum

if $\Delta > 0$ and $f_{xx} < 0$ we have a maximum

if $\Delta < 0$ we have a saddle point because if we choose $k = 0$, the term is > 0, but if we choose $k = -h\frac{f_{xx}}{f_{xy}}$, then the term is < 0 $(f_{xx} > 0)$ $(f_{xx} > 0)$

example

Find the critical points of $f(x,y) = \frac{1}{3}(x^3+y^3) - (x^2+y^2)$ and determine their nature.

To find the critical points we solve simultaneously

$$0 = \frac{\partial f}{\partial x} = x^2 - 2x \quad \Rightarrow x = 0 \text{ or } x = 2$$

$$0 = \frac{\partial f}{\partial y} = y^2 - 2y \quad \Rightarrow y = 0 \text{ or } y = 2$$

and critical points are (0,0) (2,0) (0,2) (2,2)

To determine their nature we need:

$$f_{xx} = 2(x-1) \qquad \& \quad f_{xy} = 0$$
$$f_{yy} = 2(y-1)$$

$$\Delta = f_{xx} \cdot f_{yy} - f_{xy}^2 = 4(x-1)(y-1)$$

	(0,0)	(2,0)	(0,2)	(2,2)
$f_{xx}=2(x-1)$	-2	2	-2	2
$f_{yy}=2(y-1)$	-2	-2	2	2
$f_{xy}=0$	0	0	0	0
$\Delta=f_{xx}f_{yy}-f_{xy}^2$	4	-4	-4	4
	maximum	saddle points	saddle points	minimum

Constrained optimisation

consider level surface of $f(x,y)$ - i.e lines in (x,y) plane along which $f = \text{const}$.

Consider too a line given by $g(x,y) = c$
If we ask what is the extreme value of $f(x,y)$ subject to the constraint $g(x,y) = c$ we see geometrically that this is achieved where a level surface of g, (given by $g(x,y) = c$) is tangential to a level-surface of f.

If the normals to these curves are given by ∇f & ∇g then this occurs where $\nabla f \parallel \nabla g$ i.e $\nabla f = \lambda \nabla g$ ← Lagrange multiplier

$$\nabla(f - \lambda g) = 0$$
$$\nabla(f - \lambda(g - c)) = 0$$

This condition tells us only that a level surface of f is tangential to a level surface of g. There are obviously many such points, each given by a particular value of λ! To find the one we want we add in the constraint equation $g(x+y) = c$ i.e we need to solve the three equations

$$\frac{\partial}{\partial x}(f - \lambda g) = 0 \quad \text{i.e} \quad \frac{\partial f}{\partial x} - \lambda \frac{\partial g}{\partial x} = 0$$

$$\frac{\partial}{\partial y}(f - \lambda g) = 0 \quad \text{i.e} \quad \frac{\partial f}{\partial y} - \lambda \frac{\partial g}{\partial y} = 0$$

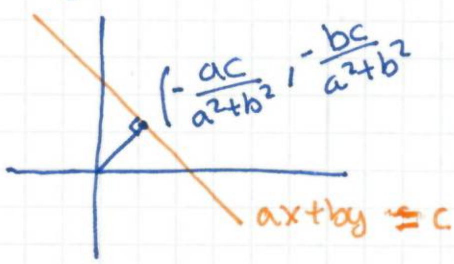
$$g(x,y) = c$$

to give values of x & y, where f has a local max/min subject to the constraint $g = c$

If $h(x,y,\lambda) = f - \lambda(g - c)$ then these equations are

$$\frac{\partial h}{\partial x} = \frac{\partial h}{\partial y} = \frac{\partial h}{\partial \lambda} = 0 \quad \nabla h = 0$$

example
Find the shortest distance of the line $ax + by + c = 0$ to the origin $\left(\frac{-c}{\sqrt{a^2+b^2}}\right)$



we will find the extreme value of $\sqrt{x^2+y^2}$. To make the algebra easier, we will take $f(x,y) = x^2 + y^2$ and the constraint is $g(x,y) = ax + by - c = 0$ and form $h(x,y,\lambda) = f - \lambda g = x^2 + y^2 - \lambda(ax + by + c)$

we need to solve

$$\frac{\partial h}{\partial x} = 2x - \lambda a = 0 \Rightarrow x = \frac{\lambda a}{2}$$

$$\frac{\partial h}{\partial y} = 2y - \lambda b = 0 \Rightarrow y = \lambda \frac{b}{2}$$

& we add the constraint $ax+by=-c$

$$\frac{\lambda a^2}{2} + \lambda \frac{b^2}{2} = -c \Rightarrow \lambda = \frac{-2c}{a^2+b^2}$$

so $x = -\frac{ac}{a^2+b^2}$ $\quad y = -\frac{bc}{a^2+b^2}$

and the distance can now be found as $\sqrt{x^2+y^2} = \frac{|c|}{\sqrt{a^2+b^2}}$

<u>More generally</u>

If $f: \mathbb{R}^n \to \mathbb{R}$ is written $f(x)$, $x \in \mathbb{R}^n = (x_1, x_2 \ldots, x_n)^T$ then we can have up to $n-1$ constraints. Suppose we have m. $g_i(\underline{x}) = 0$ $\quad i$ runs from 1 to m.

we form the function, the Lagrangian

$$L(\underline{x}, \underline{\lambda}) \quad \underline{\lambda} = (\lambda_1, \lambda_2, \ldots \lambda_n)^T$$
$$= f(\underline{x}) - \sum_{i=1}^{m} \lambda_i g_i(\underline{x})$$

We then solve the n equations

$$\frac{\partial L}{\partial x_i} = 0 \quad i = 1 \ldots n \quad \nabla_{\underline{x}} L = 0$$

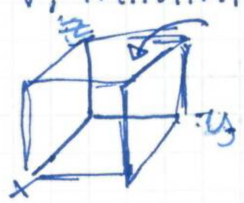
together with the m constraints $g_i(\underline{x}) = 0$ or

$$\frac{\partial L}{\partial \lambda_i} = 0 \quad i = 1 \ldots m \quad \nabla_{\underline{\lambda}} L = 0$$

$$\underline{\nabla} L = 0$$

example

Construct an open-topped rectangular box of volume V, minimising its surface area.

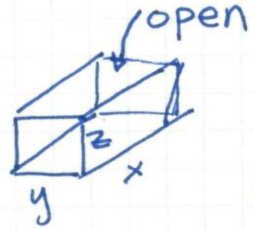


Volume is $V = xyz$

Surface area is $S = 2xz + 2yz + xy$

Max/Min S subject to $V = xyz$

19/10-2011



minimise surface $S(x,y,z) = 2yz + 2xz + xy$ subject to volume $V = xyz$ fixed, ie $V(x,y,z) = \text{const} = v \quad v \neq 0$

we use Lagrange multipliers & consider

$$H(x,y,z) = S(x,y,z) - \lambda V(x,y,z)$$
$$= 2yz + 2xz + xy - \lambda xyz$$

we set $\frac{\partial H}{\partial x} = \frac{\partial H}{\partial y} = \frac{\partial H}{\partial z} = 0$

$2z + y - \lambda yz = 0$ (1)
$2z + x - \lambda xz = 0$ (2)
$2y + 2x - \lambda xy = 0$ (3)

Solve (1) - (3) together with the fourth equation $xyz = V$ (4)

(1) - (2) $\Rightarrow (y-x) - \lambda z(y-x) = 0$
so $y - x = 0$ ie $y = x$
or $\lambda z = 1$ [In ① $\lambda z = 1 \Rightarrow z = 0$, $\lambda =$ "∞"
$z = 0 \Rightarrow V = 0$ & we presume $V \neq 0$]
so discount $\lambda z = 1$ & follow $y = x$

if $y = x$, then (3) gives $4x = \lambda x^2 \Rightarrow x = 0$ (discount as then $V = 0$)
or $x = \frac{4}{\lambda}$, $y = \frac{4}{\lambda}$

$y = x$ in (2) gives $z = \frac{x}{\lambda x - 2} = \frac{4}{\lambda} \frac{1}{4-2} = \frac{2}{\lambda}$

λ is found from the constraint $xyz = V$, ie

$\frac{4}{\lambda} \cdot \frac{4}{\lambda} \cdot \frac{2}{\lambda} = \frac{1}{2}\left(\frac{4}{\lambda}\right)^3 = V$, $\frac{4}{\lambda} = \sqrt[3]{2V}$

$x = y = \sqrt[3]{2V}$ $z = \frac{1}{2}\sqrt[3]{2V}$

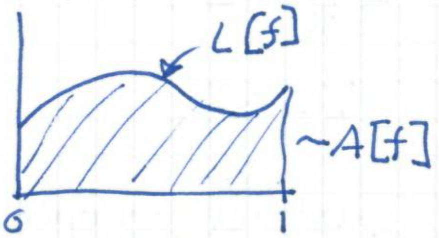
Calculus of Variation

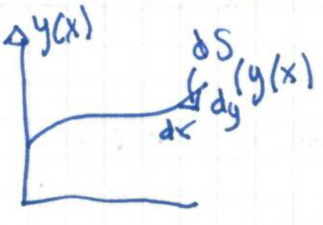
This is concerned with finding extreme values of <u>functionals</u>. Functionals are functions which map from a set of functions into the numbers.

δ: $\delta(f) = f(0)$

we might have $A[f] = \int_0^1 f(x)\,dx$

$A[f] = \int_0^1 \sqrt{1+f'^2}\,dx$

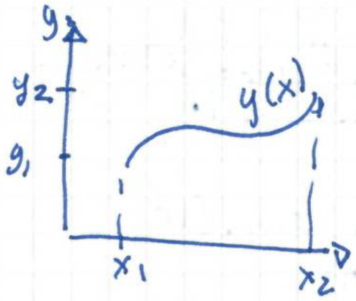




$\int ds$

$ds^2 = dx^2 + dy^2 = dx^2\left(1 + \left(\frac{dy}{dx}\right)^2\right)$

$ds = \sqrt{1+y'^2}\,dx$



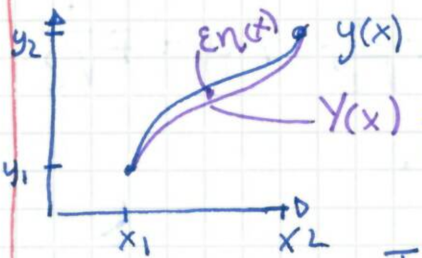
The function y which makes these functionals take on extreme values is called the extremal

Generally the functionals we will consider are

$$I[y] = \int_{x_1}^{x_2} F(x, y, y')\,dx \qquad F(x,y,y') = y$$
$$F(x,y,y') = \sqrt{1+y'^2}$$

and we wish to find an extremal curve satisfying boundary conditions $y(x_1) = y_1$, $y(x_2) = y_2$

We will assume that this extremal curve $y(x)$ exists



$$Y(x) = y(x) + \varepsilon\eta(x)$$

$$\eta(x_1) = \eta(x_2) = 0$$

$$I[Y] = I[y, \varepsilon, \eta] = \int_{x_1}^{x_2} F(x, y+\varepsilon\eta, y'+\varepsilon\eta')\,dx$$

If y is the extremal curve then $\frac{\partial I}{\partial \varepsilon}[y,\varepsilon,\eta]\Big|_{\varepsilon=0} = 0$

i.e $0 = \int_{x_1}^{x_2} \frac{\partial}{\partial \varepsilon} F(x, y+\varepsilon\eta, y'+\varepsilon\eta')\,dx \Big|_{\varepsilon=0}$

$$\Rightarrow \quad 0 = \int_{x_1}^{x_2} \frac{\partial F}{\partial y}\eta + \frac{\partial F}{\partial y'}\eta' \Big|_{\varepsilon=0}$$

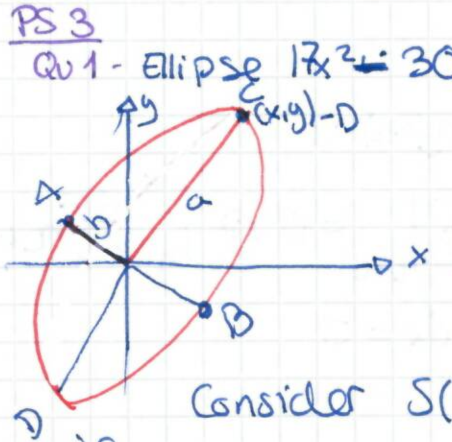
$$0 = \int_{x_1}^{x_2} \frac{\partial F}{\partial y}(x,y,y')\eta + \frac{\partial F}{\partial y'}(x,y,y')\eta'\,dx$$

independently of η

Problem class

PS 3

Qu 1 - Ellipse $17x^2 - 30xy + 17y^2 = 32$



find a, b & area

$$F(x,y) = x^2 + y^2$$

Find max (min) $F(x,y)$, subject to constraint $17x^2 - 30xy + 17y^2 = 32$

Consider $S(x,y,\lambda) = x^2 + y^2 + \lambda(17x^2 - 30xy + 17y^2 - 32)$

$$\frac{\partial S}{\partial x} = 2x + \lambda 34x - \lambda 30y = 0$$

$$\frac{\partial S}{\partial y} = 2y + \lambda(-30x + 34y) = 0$$

$$\Downarrow$$

$$A = \begin{pmatrix} 2+34\lambda & -30\lambda \\ -30\lambda & 2+34\lambda \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$(2+34\lambda)^2 = (30)^2\lambda^2 \iff 2+34\lambda = \pm 30\lambda$ det of $A = 0$

$\iff \lambda = -\frac{1}{32}$ or $\lambda = -\frac{1}{2}$

If $\lambda = -\frac{1}{32}$, then looking at first row

$\left(2-\frac{34}{32}\right)x + \frac{30}{32}y = 0 \iff +x+y=0$

$\iff -x = y$

Now from the constrain, we have

$17x^2 - 30x^2 + 17x^2 = 32$

$\iff 4x^2 = 32 \iff x^2 = \frac{1}{2} \iff x = \pm\frac{1}{\sqrt{2}}, \ y = \mp\frac{1}{\sqrt{2}}$

$A = \left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right), B = \left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) \iff b = 1.$

If $\lambda = -\frac{1}{2}$ we get

$C = (2\sqrt{2}, 2\sqrt{2})$ & $D = (-2\sqrt{2}, -2\sqrt{2})$

$\iff a = 4$

Area $= \pi ab = \underline{\underline{4\pi}}$

Q.3

Hint

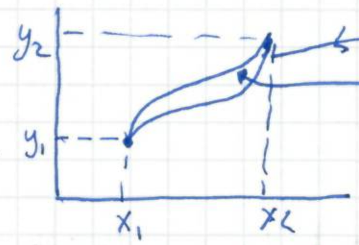
$\delta = [(x-x_0)^2 + (y-y_0)^2 + (z-z_0)^2] - \lambda(ax+by+cz)$

$\lambda = -2[d + ax_0 + by_0 + cz]/(a^2+b^2+c^2)$

20/10-2011

$I[y] = \int_{x_1}^{x_2} F(x,y,y')\,dx$

$y(x_1) = y_1, \ y(x_2) = y_2$



$y(x)$, the extremal

$y(x) = y(x) + \varepsilon\eta(x), \ \eta(x_1) = \eta(x_2) = 0$

use $\int$ by parts

$y(x)$ is extremal $\Rightarrow \left.\frac{\partial I}{\partial \varepsilon}\right|_{\varepsilon=0} = 0 \Rightarrow \int_{x_1}^{x_2} \frac{\partial F}{\partial y}\eta + \frac{\partial F}{\partial y'}\eta'\,dx = 0$

$\Rightarrow \left[\eta\frac{\partial F}{\partial y'}\right]_{x_1}^{x_2} + \int_{x_1}^{x_2} \frac{\partial F}{\partial y}\eta - \eta\frac{\partial}{\partial x}\left(\frac{\partial F}{\partial y'}\right)dx = 0$ for all

$F = \sqrt{1+y'^2}$

$= 0$ as $\eta(x_1) = \eta(x_2) = 0$

$\int_{x_1}^{x_2} \eta\left[\frac{\partial F}{\partial y} - \frac{\partial}{\partial x}\left(\frac{\partial F}{\partial y'}\right)\right]dx = 0$ for any η

we conclude that for the extremal curve $y(x)$

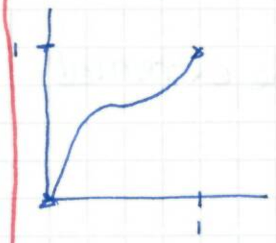
$$\boxed{\frac{\partial F}{\partial y} - \frac{\partial}{\partial x}\left(\frac{\partial F}{\partial y'}\right) = 0}$$

This is the Euler-Lagrange equation and is a second order differential equation for $y(x)$, the extremal curve, to be solved with the boundary conditions $y(x_1) = y_1$, $y(x_2) = y_2$. Once the extremal is known it can be substituted into $I[y]$ to find the extreme value.

example

Find the extremal curve for $I[y] = \int_0^1 \underbrace{y^2 - 2xy - y'^2}_{F(x,y,y')} dx$

$y(0) = 0$, $y(1) = 1$



$$\left.\begin{aligned}\frac{\partial F}{\partial y} &= 2y - 2x \\ \frac{\partial F}{\partial y'} &= -2y'\end{aligned}\right\} \Rightarrow (2y-2x) - \frac{d}{dx}(-2y') = 0$$

Fully differentiate $y = y(x)$

$$\Rightarrow y'' + y = x$$

$y(0) = 0$, $y(1) = 1$

$y = A\cos x + B\sin x + x$

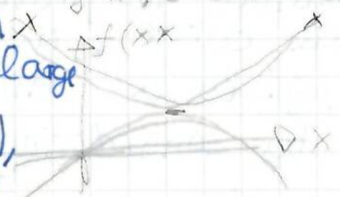
And choosing now A & B to satisfy boundary conditions $y(0) = 0$, $y(1) = 1$ gives $A = B = 0$ $\boxed{y = x}$

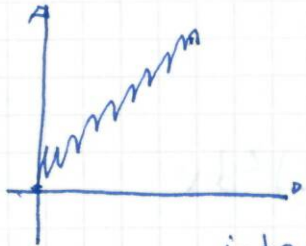
The extreme value of the integral

$$\int_0^1 x^2 - 2x^2 - 1 \, dx = -\frac{4}{3}$$

To find out if it is min/max you can look at large values of x, second derivative

We can argue that if we can make the value of $I[y]$ as large and ~~posi~~ negative as we like by choosing an appropriate $y(x)$, then $-4/3$ indicates a maximum value for the integral



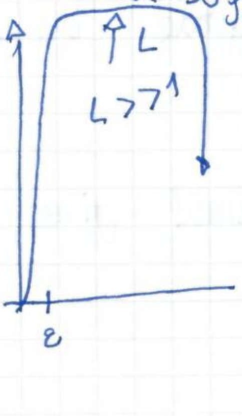


"$y = x + \sin nx$"

$y' = 1 + n\cos nx$

$y'^2 \approx n^2\cos^2 nx$ for large n

So due to the $-y'^2$ term in the integrand we can make the integral large and negative as we like by choosing a sufficiently oscilliatory $y(x)$.



$(L/\varepsilon)^2 \varepsilon$ - contribution for $-y'^2$

$L^2 \cdot 1$ - contribution for y^2

The shortest distance between two points

In Euclidian space the lenght of a curve $y(x)$ is $\int_{x_1}^{x_2} \sqrt{1+y'^2}\, dx$. Here $F(x,y,y') = \sqrt{1+y'^2}$ and $\frac{\partial F}{\partial y} = 0$

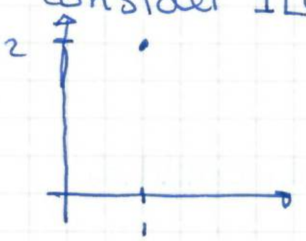
$\left[\frac{\partial F}{\partial y} - \frac{\partial}{\partial x}\left(\frac{\partial F}{\partial y'}\right) = 0\right]$ and the E-L equation is

$0 - \frac{\partial}{\partial x}\left[\frac{y'}{\sqrt{1+y'^2}}\right] = 0$ and we have $\frac{y'}{\sqrt{1+y'^2}} = \text{constant}$

and this is true only for $y' = \text{const}$ ie the extremal is a straight line.

example - $\boxed{\text{EXAM QUESTION}}$

Consider $I[y] = \int_0^1 \underbrace{(y'-y)^2}\, dx$ $\quad y(0) = 0$, $y(1) = 2$



$F(x,y,y') = (y'-y)^2$

and EL gives

$$-2(y'-y) - \frac{\partial}{\partial x}\left(2(y'-y)\right) = 0$$

$\Rightarrow y'' - y = 0$ so $y(x) = A\cosh(x) + B\sinh(x)$

Boundary conditions $\Rightarrow$ $y = \frac{2\sinh(x)}{\sinh(1)}$

It is possible to prove, in this case, that this extremal curve, $y = f$, where $f'' - f = 0$, $f(0) = 0$, $f(1) = 2$, gives a minimum value for the integral.

This is by considering $I[f+g]$

$$I[f] = \int_0^1 (f'-f)^2 dx$$

$$I[f+g] = \int_0^1 \underbrace{(f'+g'-f-g)^2}_{[(f'-f)+(g'-g)]^2} dx$$

$$I[f+g] = \int_0^1 (f'-f)^2 + 2(f'-f)(g'-g) + (g'-g)^2 dx$$

$$= I[f] + 2\int_0^1 (f'-f)(g'-g) + \underset{\geq 0}{\text{always}}$$

Consider $2\int (f'-f)g' - (f'-f)g\, dx$ and integrate by parts on first integral to give

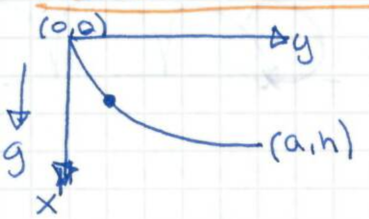
$$\left[2(f'-f)g\right]_0^1 - 2\int (f''-\cancel{f'})g + (\cancel{f'}-f)g\, dx$$

$g(0) = g(1) = 0$ and $f'' - f = 0$ as f is extremal gives 0 for this integral and

$$I[f+g] = I[f] + \int_0^1 (g'-g)^2 dx \geq I[f]$$

so the extremal gives a minimum.

The Brachistrochrone Problem



Find $y(x)$ such that the time taken for a bead to fall due to gravity from $(0,0)$ to (a,h) on the wire $y(x)$ is a minimom

$s = vt$

$$T = \int dt = \int \frac{ds}{v}$$

kE gained = PE lost gives $\frac{1}{2} m \dot{x}^2 = mgx$

kinetisk energy — potential energi

so $v = \sqrt{2gx}$

$ds = \sqrt{1 + y'^2}\, dx$ — $F(x, y, y')$

$$T[y] = \frac{1}{\sqrt{2g}} \int_0^a \underbrace{\frac{\sqrt{1+y'^2}}{\sqrt{x}}}_{\text{look at this part}} dx$$

forget about this

– Exam, just this part not the theory before

$$\frac{\partial F}{\partial y} - \frac{\partial}{\partial x}\left(\frac{\partial F}{\partial y}\right) = 0$$

Here $\frac{\partial F}{\partial y} = 0$ and $\frac{\partial}{\partial x}\left(\frac{\partial F}{\partial y}\right) = 0$

so $\frac{\partial F}{\partial y'} = c$ (a constant)

irst rder quation of y –

$$\frac{\partial F}{\partial y'} = \frac{1}{\sqrt{x}} \cdot \frac{y'}{\sqrt{1+y'^2}} = c$$

$\Rightarrow$ $y'^2 = c^2 x (1 + y'^2)$

$y'^2 (1 - c^2 x) = c^2 x$

$$\left(\frac{dy}{dx}\right)^2 = \frac{c^2 x}{1 - c^2 x}$$

3.
eed to consider ± hen taking the root

(we expect $y' \geq 0$ here) just the positive root

$$\frac{dy}{dx} = c\sqrt{\frac{x}{1-c^2 x}}$$

$$\sqrt{\frac{c^2 x}{1 - c^2 x}} = \sqrt{\frac{x}{\frac{1}{c^2} - x}}$$

$\frac{1}{c^2}$: α

nction

so $\int dy = \int \sqrt{\frac{x}{\underbrace{\alpha - x}_{\alpha \sin^2\theta}}}\, dx$

$\alpha = \frac{1}{c^2}$

th + and –
$(0) = 0$
$(1) = 0$

integration by substitution

Put $x = \alpha \sin^2\theta$

$$y + k = \int \sqrt{\frac{\alpha \sin^2\theta}{\alpha - \alpha\sin^2\theta}}\; 2\alpha \sin\theta\cos\theta\, d\theta$$

$$= \int 2\alpha \sin^2\theta\, d\theta = \alpha \int 1 - \cos 2\theta\, d\theta = \alpha\left[\theta - \tfrac{1}{2}\sin 2\theta\right]$$

$$= \alpha\left[\theta - \sin\theta\cos\theta\right]$$

so $y+k = a\left[\sin^{-1}\sqrt{\frac{x}{a}} - \sqrt{\frac{x}{a}}\sqrt{1-\frac{x}{a}}\right]$

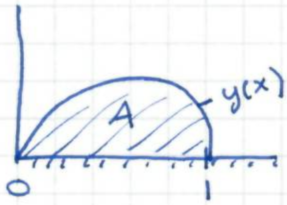
cycloid

$y(0)=0 \rightarrow k=0$

and a needs to be such that $y(a)=h$

ISOPERIMETRIC PROBLEMS

[calculus of variations with integral constraints]



$L = \int_0^1 \sqrt{1+y'^2}\,\partial x$

want to max.

$A[y] = \int_0^1 y\, dx$

$y(0)=0$
$y(1)=0$

The method is to use Lagrange multipliers.

form $H[y,\lambda] = \int_0^1 y - \lambda\sqrt{1+y'^2}\, dx$

26/10-2011

Special forms of the Euler-Lagrange equations

$$\frac{\partial F}{\partial y} - \frac{\partial}{\partial x}\left(\frac{\partial F}{\partial y'}\right) = 0 \quad , F = F(x,y,y')$$

case

(i) No y' in F, i.e. $\frac{\partial F}{\partial y'} = 0 \Rightarrow \frac{\partial F}{\partial y} = 0$

(ii) No y in F, i.e $\frac{\partial F}{\partial y} = 0 \Rightarrow \frac{\partial}{\partial x}\left(\frac{\partial F}{\partial y'}\right) = 0$

$\Rightarrow \frac{\partial F}{\partial y'} = c$ a first integral of the E-L equation.

(iii) No x in F, i.e $\frac{\partial F}{\partial x} = 0$, then $F - y'\frac{\partial F}{\partial y'} = \text{constant}$

This first integral is called the Beltrami equation.

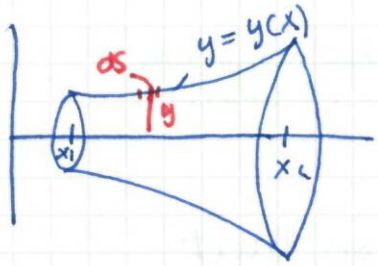
This is true as

$$\frac{\partial}{\partial x}\Big[\underset{F(x,y,y')}{F} - y'\frac{\partial F}{\partial y'}\Big] = \cancel{\frac{\partial F}{\partial x}}^{0} + \cancel{\frac{\partial F}{\partial y}\overbrace{\frac{\partial y}{\partial x}}^{y'}} + \cancel{\frac{\partial F}{\partial y'}\cdot\frac{\partial}{\partial x}\overbrace{\left(\frac{\partial y}{\partial x}\right)}^{y''}}$$

$$- \cancel{\frac{\partial^2 y}{\partial x^2}\left(\frac{\partial F}{\partial y'}\right)} - \cancel{\frac{\partial y}{\partial x}\frac{\partial}{\partial x}\left(\frac{\partial F}{\partial y'}\right)}$$

$$= y'\underbrace{\left(\frac{\partial F}{\partial y} - \frac{\partial}{\partial x}\left(\frac{\partial F}{\partial y'}\right)\right)}_{\text{E-L equation}} = y'\cdot 0 = 0$$

$\Rightarrow F - y'\frac{\partial F}{\partial y'} = \text{const}$



Minimize a surface are produced by rotating the curve $y=y(x)$ about the x-axis

$y(x_1) = y_1$
$y(x_2) = y_2$

$$A[y] = \underbrace{2\pi}_{\text{This is not important in finding } y} \int_{x_1}^{x_2} y\sqrt{1+y'^2}\, dx$$

$$\partial s^2 = \partial x^2 + \partial y^2 \quad A = \int \partial A = \int \partial s\, 2\pi y = \int 2\pi y \sqrt{1+y'^2}\, \partial x$$

$$\frac{\partial s}{\partial x} = \sqrt{1+y'^2}$$

$$F(x, y, y') = y\sqrt{1+y'^2}$$

we see $\frac{\partial F}{\partial x} = 0$ so that we <u>can</u> go immediately to the first integral

$$F - y'\frac{\partial F}{\partial y'} =$$

$$y\sqrt{1+y'^2} - y' \frac{y \cdot y'}{\sqrt{1+y'^2}} = c$$

$$\Rightarrow \frac{1}{\sqrt{1+y'^2}}\left[y + \cancel{yy'^2} - \cancel{yy'^2}\right] = c$$

$$\frac{y}{\sqrt{1+y'^2}} = c$$

Don't always use the Beltrami equation when you don't see an x.

? it is ard to solve, o back o E-L quation nd find he 2nd derivative

$$\frac{\partial F}{\partial y} - \frac{\partial}{\partial x}\left[\frac{\partial F}{\partial y'}\right] = 0$$

$$\sqrt{1+y'^2} - \frac{\partial}{\partial x}\left[\frac{yy'}{\sqrt{1+y'^2}}\right]$$

$$\vdots$$

$$y'' - y = 0 \quad ??$$

check

$$y^2 = c^2(1+y'^2)$$

$$\frac{y^2 - c^2}{c^2} = \left(\frac{\partial y}{\partial x}\right)^2$$

$$\frac{\partial y}{\partial x} = \pm\sqrt{\frac{y^2-c^2}{c^2}} = \pm\frac{1}{c}\sqrt{y^2-c^2}$$

$$\int \frac{\partial y}{\sqrt{y^2-c^2}} = \pm\int \frac{1}{c}\,\partial x$$

$$\cosh^{-1}\frac{y}{c} = \pm\left(\frac{x}{c} + \frac{D}{c}\right)$$

$$y = c \cdot \cosh\left[\pm\left(\frac{x+D}{c}\right)\right]$$ $\pm$ doesn't matter any more since cosh is even.

$$= c \cdot \cosh\left[\frac{(x+D)}{c}\right]$$

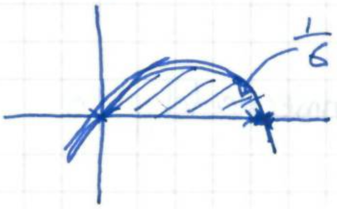
Two constants of integration found so that
$y(x_1) = y_1$ & $y(x_2) = y_2$

Back to isoperimetric problems

example Find the extremal for the integral

$\int_0^1 y'^2 + 2yy' \, dx$ $\quad y(0) = y(1) = 0$ subject to the constraint

$$\int_0^1 y \, dx = \frac{1}{6}$$



min/max

$\int F \, \partial x$ subject to $\int G \, \partial x = \text{constant}$.

Form $\int F - \lambda G \, \partial x$. Solve the EL for this new functional & apply constraint to find λ.

Consider

$$\int_0^1 \underbrace{y'^2 + 2yy' - \lambda y}_{H(x,y,y')} \, \partial x$$

We see $\frac{\partial H}{\partial x} = 0$ and it is tempting to use the Beltrami equatio

$H - y' \frac{\partial H}{\partial y'} = \text{constant}$.

However the algebra is tricky & instead we use the EL equations $\quad \frac{\partial H}{\partial y} - \frac{\partial}{\partial x}\left(\frac{\partial H}{\partial y'}\right) = 0$

$$\cancel{2y'} - \lambda - \frac{\partial}{\partial x}\left[2y' + \cancel{2y}\right] = 0$$

$$y'' = -\frac{\lambda}{2}$$

$$y' = -\frac{\lambda}{2} x + A$$

$y = -\frac{\lambda}{4} x^2 + Ax + B$ and since $y(0) = 0 \Rightarrow B = 0$

and $y(1) = 0 \Rightarrow A = \frac{\lambda}{4}$

$$y(x) = \frac{\lambda}{4} x(1-x)$$

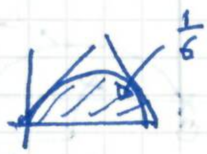
We use the constraint to give us the value of λ.

$$\int_0^1 \frac{\lambda}{4}(x - x^2) \, \partial x = \int_0^1 y \, dx = \frac{1}{6}$$

i.e $\frac{1}{6} = \frac{\lambda}{4}\left[\frac{1}{2} - \frac{1}{3}\right] \Rightarrow \lambda = 4$

$\underline{y(x) = x(1-x)}$

28/10-2011



$\int_0^1 y'^2 + 2yy' \, dx, \quad y(0) = y(1) = 0$

& $\int_0^1 y \, dx = 1/6 \;\Big|\; \overset{EL}{y''} = -\frac{\lambda}{2} \quad y = (1-x)x$

$\int_0^1 \underbrace{y'^2 + 2yy' - \lambda y}\, dx$

Beltrami equation is $H - y' \frac{\partial H}{\partial y} = \text{const}$

$(y'^2 + 2yy' - \lambda y) - y'(2y' + 2y) = c$

$-y'^2 - \lambda y = c$

$y'^2 + \lambda y = c^*$

$\left(\frac{dy}{dx}\right) = \pm\sqrt{c^* - \lambda y}$ NB: remember to keep $\pm$!!

$\int \frac{dy}{\sqrt{c^* - \lambda y}} = \pm \int dx$

$-\frac{2}{\lambda}\sqrt{c^* - \lambda y} = \pm x + A$

$\sqrt{c - \lambda y} = \pm \frac{\lambda}{2}(x + A)$

Choose c & A so that $y(0) = 0, \; y(1) = 0$

$\sqrt{c} = \pm \frac{\lambda A}{2}$

$\sqrt{c} = \pm \frac{\lambda}{2}(1 + A)$

If we had neglected the $\pm$ we might notice a contradiction here.

What we need is

$-A = 1 + A$

$\Rightarrow A = -1/2$

$A^2 = (1+A)^2 \Rightarrow \cancel{A^2} = 1 + 2A + \cancel{A^2}, \; A = -1/2$

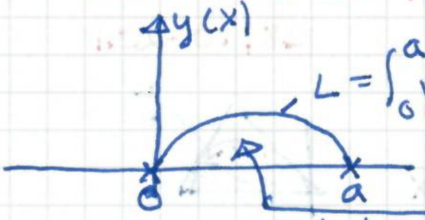
$c = \frac{\lambda^2 A^2}{4} = \frac{1}{16}\lambda^2$

& finding y

$c - \lambda y = \frac{\lambda^2}{4}(x + A)^2$

$$\frac{1}{16}\lambda^2 - \lambda y = \frac{\lambda^2}{4}\left(x-\frac{1}{2}\right)^2 \Rightarrow y = \frac{\lambda}{4}(x-x^2) \text{ as before}$$

The sheep pen problem



$$L = \int_0^a \underbrace{\sqrt{1+y'^2}}_{G}\,dx$$

~~want to maximize~~ $A[y] = \int_0^a \underbrace{y}_{F}\,dx$

Maximize A subject to constraint of fixed L.

Form $\int_0^a \underbrace{y - \lambda\sqrt{1+y'^2}}_{F-\lambda G = H}\,dx$

$\frac{\partial H}{\partial x} = 0$ so we know

$$H - y'\frac{\partial H}{\partial y'} = C$$

$$y - \lambda\sqrt{1+y'^2} - y'\left(\frac{-\lambda y'}{\sqrt{1+y'^2}}\right) = C$$

$$\frac{-\lambda}{\sqrt{1+y'^2}}\left[1+y'^2 - y'^2\right] = c - y$$

$$\frac{\lambda^2}{1+y'^2} = (c-y)^2$$

$$1+y'^2 = \frac{\lambda^2}{(c-y)^2} \quad (*)$$

$$y'^2 = \frac{\lambda^2 - (c-y)^2}{(c-y)^2}$$

$$\frac{dy}{dx} = \pm\sqrt{\frac{\lambda^2-(c-y)^2}{(c-y)^2}} = \frac{\pm\sqrt{\lambda^2-(c-y)^2}}{(c-y)^2}$$

$$\int \frac{(c-y)}{\sqrt{\lambda^2-(c-y)^2}}\,dy = \int \pm\,dx$$

$$\sqrt{\lambda^2-(c-y)^2} = \pm(x+A)$$

$$\Rightarrow \lambda^2 = (x+A)^2 + (y-c)^2$$

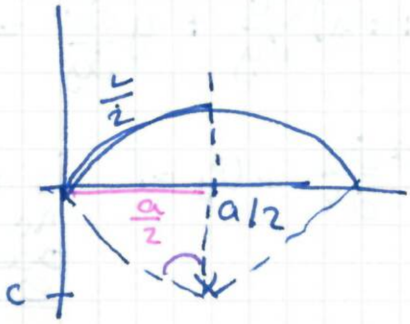
$\Rightarrow$ boundary conditions give A & C, constraint gives λ.

$y(0)=0$, $\lambda^2 = A^2 + C^2$

$y(a)=0$ $\lambda^2 = (a+A)^2 + C^2$

$\Rightarrow A^2 = (a+A)^2 \Rightarrow A = -\frac{a}{2}$

$C^2 = \lambda^2 - \frac{a^2}{4}$



To find λ we use the constraint

$$L = \int_0^a \sqrt{1+y'^2}\,dx = \pm\int_0^a \frac{\lambda}{C-y}\,dx = \pm\int_0^a \frac{\lambda}{\sqrt{\lambda^2 + (x-\frac{a}{2})^2}}\,dx$$

from differential equation earlier (*)

$\Rightarrow L = 2\lambda \sin^{-1}\left(\frac{a}{2\lambda}\right) \Rightarrow \sin\left(\frac{L}{2\lambda}\right) = \frac{a}{2\lambda}$

$\Rightarrow \lambda \Rightarrow C$

PARTIAL DIFFERENTIAL EQUATIONS

A partial differential equation (PDE) is a relation between a function of several variables $U(x,y,\ldots)$ & its partials derivatives, $U_x\ U_y \ldots$ $U_{xx}, U_{yy}, U_{xy}\ldots$ & $U_{xxx}, U_{yyy}, U_{xxy}\ldots$

$$U\frac{dU}{dx} + xy = \frac{\partial^2 U}{\partial y^2}$$

for $U = U(x,y)$

The order of the PDE is the order of the highest derivative occuring.

If the differential equation can be written $L[u] = f$ where f does not depend on U & $L[U]$ is a linear operator

$L[\alpha U + \beta W] = \alpha L[U] + \beta L[W]$

e.g $a(x,y)\frac{\partial y}{\partial x} + b(x,y)\frac{\partial u}{\partial y} = c(x,y)U + d(x,y)$

$L[U] = a\frac{\partial y}{\partial x} + b\frac{\partial u}{\partial y} - cU$

$f = d$

Then the equation is linear.

$(x+y)\dfrac{\partial u}{\partial y} + y\dfrac{\partial u}{\partial y} = 1 + xu$ is a linear first order equation

$u^2 \dfrac{\partial u}{\partial x} + u_y = 1$ is not linear & first order, but there are no products of the highest order derivative occuring. Such equations are called quasi linear

$u_x u_y = 1$ is not quasi linear

$u_{xx} + u_{yy} = 1$ is second order & linear

A simple equation is

$\dfrac{\partial u}{\partial x} = 0$ for $u(x,y)$ & has solution $u = f(y)$

similar $\left(\dfrac{dy}{dx} = 0\right)$ $y = \text{const}$

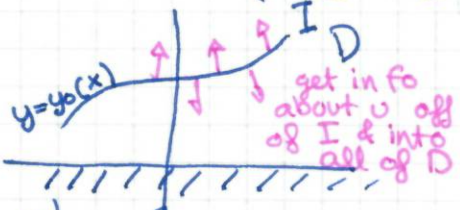
~~y=const~~

we observe that the general solution of PDEs contain arbitrary functions

A typical problem

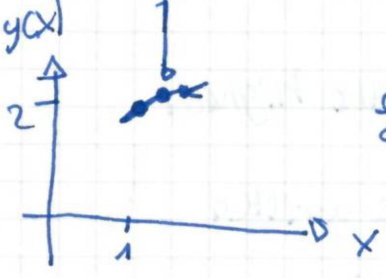
Given a first order PDE valid in a region D of the (x,y) plane (eg $D = \mathbb{R}^2$; $x \geq 0$; $x^2+y^2 \leq a^2$)

and some knowledge of the solution $u(x,y)$ on a curve Γ in D (e.g $y = y_0(x)$ or $g(x,y) = C$)



D has $y \geq 0$

e.g $u = u_0(x,y)$ on $y = y_0(x)$, can we find $u(x,y)$ inside all of D?

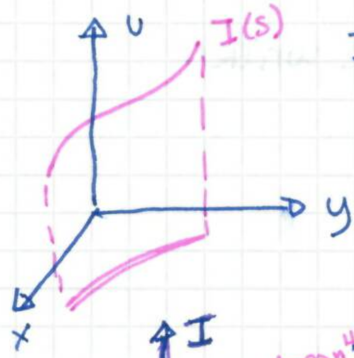


$\dfrac{dy}{dx} = f(x,y) \quad y(1) = 2$

If this can be done obtaining a unique $u(x,y)$ we will call the problem well-posed. If it cannot it will be ill-posed. We will see examples that illustrate when a problem is ill-posed.

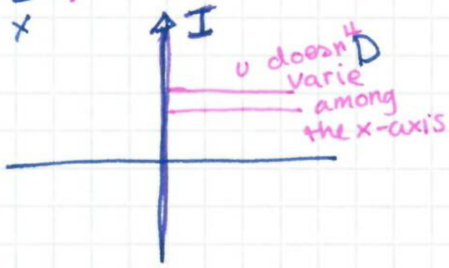
We often describe lines in the (x,y) plane in parametric form. Eg the line Γ can be described as $x = x(s)$
$y = y(s)$
$u = u(s)$

So, for example, if we are told that $u = x^2$ on $y = 0$, this would be parameterised as $x = s$, $y = 0$, $u = s^2$



I(s) in 3-D is a line in the solution surface $u = u(x,y)$

Solve $\frac{\partial u}{\partial x} = 0$ with I being the ~~y~~-axis & on this y-axis $u = e^y$.



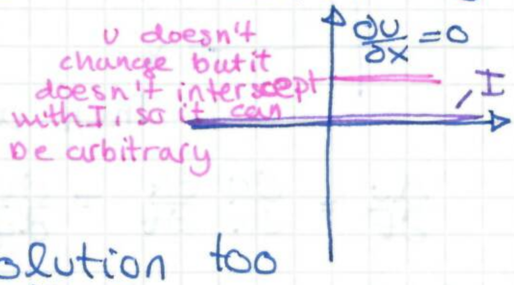
$y = s$
$x = 0$
$u = e^s$

Integrating gives $u = f(y)$ where f is an arbitrary function, but on I $u = e^s$, $y = s$, $x = 0$ so that substituting $e^s = f(s)$ i.e $f(y) = e^y$ & our solution is $u(x,y) = e^y$

<u>But</u> to solve $\frac{\partial u}{\partial x} = 0$ with I the x-axis, e.g $u = 1$ ~~u = f(x)~~ on $y = 0$ is an <u>ill-posed</u> problem as

$u = 1 + y$ is a solution

but $u = 1 + g(y) - g(0)$ is a solution too & the solution is not unique.



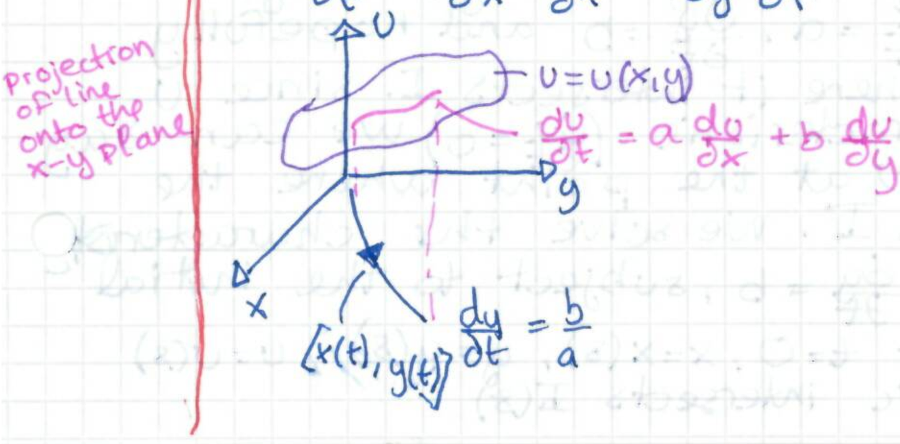
Consider a curve in the x-y plane given parametrically by $x = x(t)$, $y = y(t)$

then $\frac{dy}{dx} = \frac{\partial y/\partial t}{\partial x/\partial t}$

If we consider the ordinary differential equation $\frac{dy}{dt} = \frac{b(x,y)}{a(x,y)} = F(x,y)$ then this equation has a solution given by the solution to $\frac{\partial y}{\partial t} = b(x,y)$, $\frac{\partial x}{\partial t} = a(x,y)$

If $u = u(x,y)$ & $x = x(t)$, $y = y(t)$ then

$$\frac{\partial u}{\partial t} = \frac{\partial u}{\partial x} \cdot \frac{\partial x}{\partial t} + \frac{\partial u}{\partial y} \frac{\partial y}{\partial t} = a(x,y) \frac{\partial u}{\partial x} + b(x,y) \frac{\partial u}{\partial y}$$



Sometimes these equations may be written

$dx = a\,dt \quad , \quad dy = b\,dt$

$\frac{dx}{a} = dt \quad , \quad \frac{dy}{b} = dt$

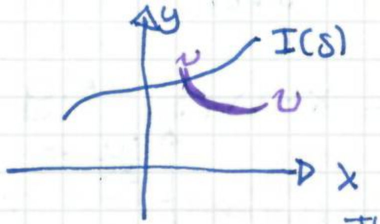
$$\frac{dx}{a} = \frac{dy}{b} \ (= dt)$$

Characteristics — homogeneous

Consider the PDE

$$a(x,y)\frac{\partial u}{\partial x} + b(x,y)\frac{\partial u}{\partial y} = 0$$

to be solved with a knowledge of u on a line I in x-y plan[e]



Consider lines given by the solution to

$$\frac{\partial x}{\partial t} = a \ , \ \frac{\partial y}{\partial t} = b \qquad \frac{\partial x}{a} = \frac{\partial y}{b} \ (= dt)$$

Then along this line

$$\frac{du}{\partial t} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial t} = a\frac{\partial u}{\partial x} + b\frac{\partial u}{\partial y} = 0$$

So that $u(x,y)$ is constant on these lines. These lines are called characteristics, more accurately characterist[ic] traces & the equations

$$\frac{dx}{a} = \frac{dy}{b} = \frac{du}{0}\overset{(du=0)}{} \ (= dt) \qquad \frac{du}{dt} = 0$$

are the characteristic equatio[ns]

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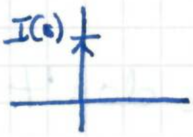
$$au_x + bu_y = 0$$

on lines given by $\frac{\partial x}{\partial t} = a$ and $\frac{dy}{\partial t} = b$ we have seen $\frac{du}{\partial t} = 0$.

We can use this as follows: If we know u on some curve I, $x = x(s)$, $y = y(s)$, $u = u(s)$. We can find the value of u at some point off of I, by finding the characteristic trace; passing through (x,y) - the solutions of $\frac{\partial x}{\partial t} = a$, $\frac{\partial y}{\partial t} = b$ and hopefully tracing it back to where it intersects I. Since u is constant on the characteristic $\left(\frac{du}{\partial t} = 0\right)$ we can say $u(x,y)$ is equal to u at the point where the characteristic meets I. We solve the characteristic equations $\frac{dx}{\partial t} = a$, $\frac{dy}{\partial t} = b$, subject to the initial conditions that at $t = 0$, $x = x(s)$, $y = y(s)$, $u = u(s)$ and the characteristic intersects $I(s)$

example

Solve $\frac{\partial u}{\partial x} + x\frac{\partial u}{\partial y} = 0$ $\quad (u = \sin(y)$ on $x=0)$



1) Parameterise $I(s)$ as $x=0$, $y=s$, $u=\sin(s)$
2) solve the characteristic equations with initial conditions $x=0$, $y=s$, $u=\sin(s)$ at $t=0$

These are $\overset{①}{\frac{\partial x}{\partial t} = a = 1}$, $\overset{②}{\frac{\partial y}{\partial t} = b = x}$ $\quad \left(\frac{\partial u}{\partial t} = 0\right)$

function of t!

DON'T DO THIS $\quad \therefore \frac{\partial y}{\partial t} = x \Rightarrow y = xt$ WRONG!

① $\Rightarrow x = t$ (using $x=0$ at $t=0$)

In ② $\Rightarrow \frac{\partial y}{\partial t} = x = t \Rightarrow y = \frac{1}{2}t^2 + s$ (using $y=s$ at $t=0$)

$u = \sin(s)$ as $\frac{\partial u}{\partial t} = 0$

we now have a parametric form of our solution. The parameters are t and s:

$$x = t$$
$$y = \frac{1}{2}t^2 + s \longrightarrow y = \frac{1}{2}x^2 + s$$
$$u = \sin(s)$$

Now eliminate t and s in favour of x and y

$$s = y - \frac{1}{2}t^2 = y - \frac{1}{2}x^2$$
$$\therefore \boxed{u = \sin\left(y - \left(\tfrac{1}{2}\right)x^2\right)}$$

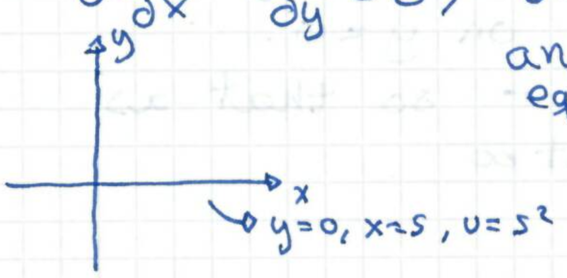
↳ solution of $\frac{\partial u}{\partial x} + x\frac{\partial u}{\partial y} = 0$

Also note that for any function $\mathbb{R} \to \mathbb{R}: f$ the function $u(x,y) = f(y - \frac{1}{2}x^2)$ satisfies the pde, but a choice of f is needed to satisfy the boundary conditions.

example

A quasilinear homogeneous

$$u\frac{\partial u}{\partial x} - \frac{\partial u}{\partial y} = 0; \quad u = x^2 \text{ on } y = 0$$



and solve the characteristic equations

$$\frac{\partial x}{\partial t} = a = u \qquad \frac{\partial u}{\partial t} = 0$$
$$\frac{\partial y}{\partial t} = b = -1$$

with initial conditions at $t=0$, $y=0$, $x=s$, $u=s^2$

• we can solve for y:

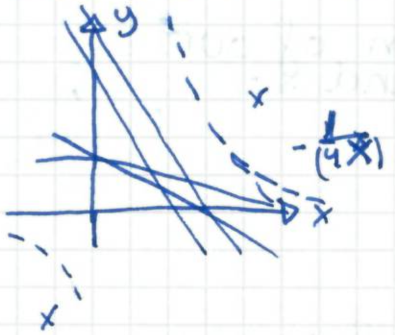
$\underline{y=-t}$: $y=0$ at $t=0$

we can't directly solve $\frac{\partial x}{\partial t}=u$ as we don't know what $u(t)$ is (yet).
We do know that as $\frac{du}{dt}=0$ on the characteristic, then u is constant. As $u=s^2$ at $t=0$, $u=s^2$
so $\frac{\partial x}{\partial t}=u=s^2 \Rightarrow x=s^2t+s$ (as $t=0$ at $x=s$)

So we have the parametric solution

$$x=s^2t+s$$
$$y=-t$$
$$u=s^2$$

The characteristic traces are given by eliminating t



$$x=s-s^2y$$
$$\Rightarrow y=\frac{1}{s}-\frac{x}{s^2}$$

x → characteristic will not enter the x region

The characteristics have an envelope $y=\frac{1}{4x}$ so we cannot find solutions for $u(x,y)$ in the region $y>\frac{1}{4x}$ as no characteristic which intersects $I(s)$ enters this region.

If we eliminate s and t, then

$$x=u(-y)+\sqrt{u} \quad (\text{assume } s>0)$$

we have a quadratic for $\sqrt{u}$:

$$\sqrt{u}=\frac{1\pm\sqrt{1-4xy}}{2y}$$

We need to decide whether we want + or −
We need $u=x^2$, i.e $\sqrt{u}=x$ on $y=0$.
This implies we need the − so that as $y\to0$, $\sqrt{u}\to x$ and not ∞

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$f(x,y,s)=0$, $f = x + s^2 y - s = 0$
Family of curves, parameter s.
Eliminate s from $f=0$ & $\frac{\partial f}{\partial s} = 0$

The characteristic equations for $a(x,y,u)u_x + b(x,y,u)u_y = 0$
Solutions of $\frac{\partial x}{a(x,y,u)} = \frac{\partial y}{b(x,y,u)}$ or $\frac{\partial y}{\partial x} = \frac{b(x,y,u)}{a(x,y,u)}$

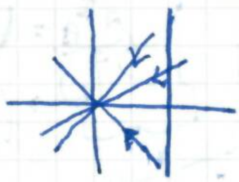
If the pde is linear, this becomes
$$\frac{dy}{\partial x} = \frac{b(x,y)}{a(x,y)}$$

Hence, for linear equations, if a and b are single valued, $\frac{\partial y}{\partial x}$ is unique as a function of x and y & the characteristic traces cannot cross

This is <u>not</u> true for general quasilinear pde's

There are exceptions at points where both $a=0$ and $b=0$, when $\frac{\partial x}{\partial y}$ is undetermined.

Look at $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 0$



$$\frac{dx}{x} = \frac{dy}{y} \Rightarrow \ln x = \ln y + \text{constant}$$

and the characteristics cross at origin when $a=b=0$

In this case, we might expect a singularity in the solution to the pde at $x=0$, $y=0$ as the characteristic carrying contradictory information about the solution cross. This singularity could be avoided for particular I, e.g if a=cost on I here.

<u>consider</u>:

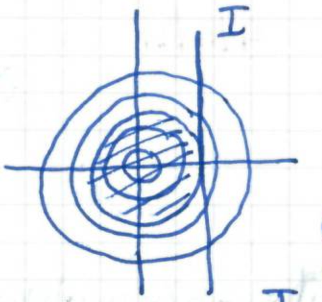
$$y\frac{\partial u}{\partial x} - x\frac{\partial u}{\partial y} = 0$$

The characteristic: $\frac{\partial x}{y} = -\frac{dy}{\partial x}$ $\left(\frac{\partial y}{\partial x} = -\frac{x}{y}\right)$

$\Rightarrow x\,dx + y\,dy = 0$

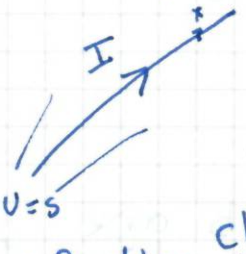
$$d\left(\tfrac{1}{2}x^2 + \tfrac{1}{2}y^2\right) = 0$$

$$x^2 + y^2 = 0$$



For I as shown, we cannot find the solution inside the shaded region as no characteristic which crosses I enters it.

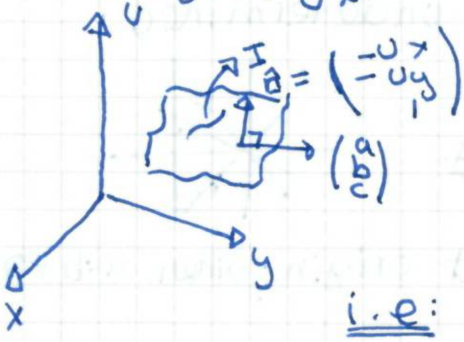
Outside this region, there are still problems if a characteristic crosses I more than once. Unless the data on I is entirely consistent with the development of the solution along the characteristic, the problem is ill-posed.



An extreme case of the characteristic crossing I is when I coincides with a characteristic. Then, it is impossible to find information about the solution off of I. A good definition of a characteristic is a line, a knowledge of the solution on which, tells us nothing about the solution elsewhere.

Characteristics for general inhomogeneous quasilinear equations:

$$a(x,y,u)\frac{\partial u}{\partial x} + b(x,y,u)\frac{\partial u}{\partial y} = c(x,y,u)$$



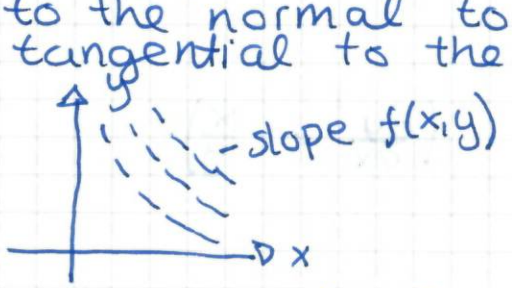
In solving this, we are after a solution surface $u = u(x,y)$ containing I. The normal to this surface can be found by evaluating ∇g where $g(x,y,u) = u - u(x,y)$,

i.e: $\underline{n} = \begin{pmatrix} -u_x \\ -u_y \\ 1 \end{pmatrix}$

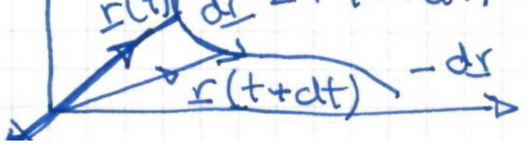
Now consider the vector $\begin{pmatrix} a \\ b \\ c \end{pmatrix}$ and consider $\begin{pmatrix} a \\ b \\ c \end{pmatrix} \cdot \begin{pmatrix} -u_x \\ -u_y \\ 1 \end{pmatrix} =$

$-au_x - bu_y + c = 0$ i.e: $au_x + bu_y = 0$

Hence, the vectors in the vector field $\begin{pmatrix} a \\ b \\ c \end{pmatrix}$ are normal to the normal to the solution surface, & so are tangential to the solution surface.



Now consider the solutions to the equation $\left(\frac{dr}{dt}\right)$



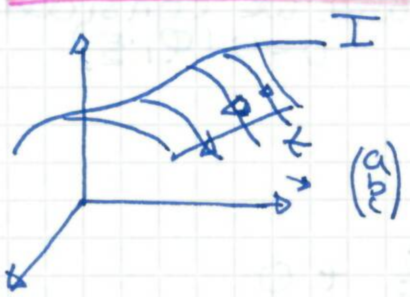
- dr is the tangential to this curve

$\frac{d\underline{r}}{dt} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$, then $d\underline{r} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$ at points in the direction of $\begin{pmatrix} a \\ b \\ c \end{pmatrix}$, so $d\underline{r}$ lies in the solution surface.

But $d\underline{r} = \begin{pmatrix} dx \\ dy \\ du \end{pmatrix} = \begin{pmatrix} a\,dt \\ b\,dt \\ c\,dt \end{pmatrix}$

and $\frac{dx}{a} = \frac{dy}{b} = \frac{du}{c}$ $(= dt)$ are the characteristics

$\Rightarrow$ The characteristics lie in the solution surface & make up the solution surface



$\frac{d\underline{r}}{dt} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$

The solution surface is made up of the characteristics coming from I

<u>An alternative methode</u>:

The change of variables method. Consider linear equations, <u>i.e</u>: those of the form

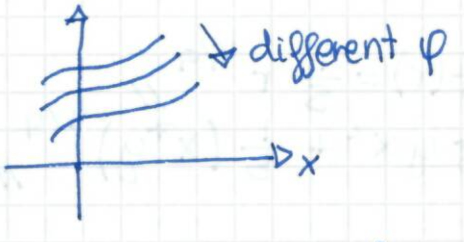
$$a(x,y)\frac{\partial u}{\partial x} + b(x,y)\frac{\partial u}{\partial y} + c(x,y)u = d(x,y)$$

& consider the characteristic traces, i.e. solutions to $\frac{dx}{dt} = a$, $\frac{dy}{dt} = b$ or $\frac{dy}{dx} = \frac{b}{a}$

Consider this as an ODE for $y(x)$. It has solutions given generally in the form $\varphi(x,y) = \varphi$ a constant.

• eg: If $y = f(x) + \underbrace{const}_{\varphi}$, $\varphi(x,y) = y - f(x) = \varphi$

<u>i.e</u>, The solution is a level curve for φ



we use φ instead of s to identify particular characteristics & we need another variable instead of t to take you along a characteristic, say ξ, often we would choose $\xi = x$.

We make the change of variables from x & y to φ & ξ

example

solve $x\frac{\partial u}{\partial x} - 7y\frac{\partial u}{\partial y} = x^2 y$

solve for the characteristic traces, i.e: solve $\frac{dy}{dx} = -\frac{7y}{x}$

$$\int \frac{dy}{y} = -7\int\frac{dx}{x} \Rightarrow \ln y = -7\ln x + c$$

$$yx^7 = \varphi$$

Now make a change of variables from x & y to φ & ξ with $\varphi(x,y) = yx^7$, $\xi(x,y) = x$

so $\frac{\partial u}{\partial x} = \frac{\partial u}{\partial \varphi}\cdot\frac{\partial \varphi}{\partial x} + \frac{\partial u}{\partial \xi}\cdot\frac{\partial \xi}{\partial x}$ and we consider $u = u(\varphi, \xi)$

$$= 7yx^6\frac{\partial u}{\partial \varphi} + \frac{\partial u}{\partial \xi}$$

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial \varphi}\cdot\frac{\partial \varphi}{\partial y} + \frac{\partial u}{\partial \xi}\cdot\frac{\partial \xi}{\partial y} = x^7\frac{\partial u}{\partial \varphi} + 0$$

so, substituting:

$$x\left(7yx^6\left(\frac{\partial u}{\partial \varphi}\right) + \frac{\partial u}{\partial \xi}\right) - 7yx^7\frac{\partial u}{\partial \varphi} = x^2 y$$

$\therefore\ x\frac{\partial u}{\partial \xi} = x^2 y$; an equation telling you how u varies as you make along a characteristic, i.e, for fixed φ.

- $\frac{\partial u}{\partial \xi} = xy = x\frac{\varphi}{x^7} = \frac{\varphi}{x^6} = \frac{\varphi}{\xi^6} \Rightarrow$ Integrating in ξ;

i.e $u(x,y) = -\frac{1}{5}\frac{yx^7}{x^5} + f(yx^7) = -\frac{1}{5}yx^2 + f(yx^7)$

If the boundary/initial conditions are, for example $u = 0$ on $y = x^2$, then we need $0 = -\frac{1}{5}x^2 x^2 + f(x^2 x^7)$

$$\Rightarrow f(x^9) = \frac{1}{5}x^4 \ //$$

and if we write $r = x^9 \Rightarrow f(r) = \frac{1}{5}r^{4/9}$ //

and our solution is $u(x,y) = -\frac{1}{5}yx^2 + \frac{1}{5}(x^7 y)^{4/9}$ //

example

$$x\frac{\partial u}{\partial x} + (x^2+y)\frac{\partial u}{\partial y} + \left(\frac{y}{x} - x\right)u = 1$$

The characteristic traces satisfy $\frac{dy}{dx} = \frac{x^2+y}{x} = \frac{y}{x} + x$

i.e: $\frac{dy}{dx} - \frac{y}{x} = x$; I.f is $e^{-\int\frac{1}{x}dx} = \frac{1}{x}$ //

$$\frac{\partial}{\partial x}\left[\frac{y}{x}\right] = 1 \quad \Rightarrow \quad \frac{y}{x} = x + c$$

so $\varphi = \frac{y}{x} - x$ is constant on the characteristics

Make a change of variable from x & y to φ & ξ where $\varphi(x,y) = \frac{y}{x} - x$ & $\xi(x,y) = x$

example continued

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$$x\left(\cancel{\frac{\partial u}{\partial \varphi}\overbrace{\left(-\frac{y}{x^2} - 1\right)}^{\partial\varphi/\partial x}} + \frac{\partial u}{\partial \xi}\cdot\overbrace{1}^{\partial\xi/\partial x}\right) + (x^2+y)\left(\cancel{\frac{\partial u}{\partial \varphi}\underbrace{\frac{1}{x}}_{\partial\varphi/\partial y}} + \frac{\partial u}{\partial \xi}\cdot\underbrace{0}_{\partial\xi/\partial y}\right)$$

$$+ \left(\frac{y}{x} - x\right)u = 1$$

$$\Rightarrow x\frac{\partial u}{\partial \xi} + \left(\frac{y}{x} - x\right)u = 1$$

$$\xi\frac{\partial u}{\partial \xi} + \varphi u = 1$$

We can solve this for $u(\xi)$ considering φ as a constant

$$\frac{\partial u}{\partial \xi} + \frac{\varphi}{\xi}\cdot u = \frac{1}{\xi}$$

IF is $e^{\int \frac{\varphi}{\xi} d\xi} = \xi^{\varphi}$

$$\frac{\partial}{\partial \xi}\left[u\cdot\xi^{\varphi}\right] = \xi^{\varphi-1}$$

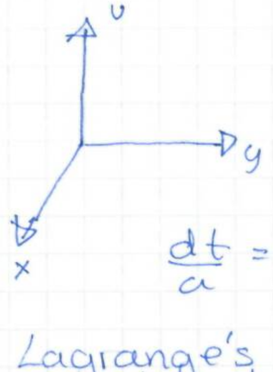
$$u\xi^{\varphi} = \frac{\xi^{\varphi}}{\varphi} + f(\varphi)$$

$$u = \frac{1}{\varphi} + \xi^{-\varphi}f(\varphi) \qquad \varphi = \frac{y}{x} - x = \frac{y-x^2}{x}$$

$$u(x,y) = \frac{x}{y-x^2} + \left(\frac{1}{x}\right)^{\frac{y-x^2}{x}} \cdot f\left(\frac{y-x^2}{x}\right)$$

LAGRANGE'S METHOD

Consider $a(x,y,u)\frac{\partial u}{\partial x} + b(x,y,u)\frac{\partial u}{\partial y} = c(x,y,u)$



$\begin{pmatrix} a \\ b \\ c \end{pmatrix}$ The characteristic curves are tangent to the vectors $\begin{pmatrix} a \\ b \\ c \end{pmatrix}$ ie satisfy $\frac{\partial x}{\partial t} = a$, $\frac{\partial y}{\partial t} = b$, $\frac{\partial u}{\partial t} = c$

$$\frac{dt}{a} = \frac{dy}{b} = \frac{du}{c} \; (= dt)$$

Lagrange's method asks you to find two constants of integration of these equations.

$$S_1(x,y,u) = C_1 \quad , \quad S_2(x,y,u) = C_2$$

Then the general solution of the pde is given by

$C_1 = f(C_2)$

$S_1 = f(S_2)$ $[S_2 = f(S_1)$, $f(S_1, S_2) = 0$

Varying C_2 gives a family of surfaces S_2 given by $S_2(x,y,u) = C_2$. Varying C_1 gives a similar family of surfaces, $S_1(x,y,u) = C_1$. A surface S_1 intersects a surface S_2 in a line, which is a characteristic line. If we relate C_1 to C_2 through $C_1 = f(C_2)$ & vary C_2, we get a one-parameter set of lines of intersection of surfaces S_1 & S_2.

This one-parameter set defines another solution surface. We can choose f appropriately so that our boundary conditions are satisfied.

easy example

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} = 1 \quad , u = -x^2 \text{ on } y = 0$$

χ eq^{ns} are $\frac{dx}{dt} = 1$, $\frac{dy}{dt} = 1$, $\frac{du}{dt} = 1$

$$\frac{dy}{dx} = 1 \Rightarrow \boxed{y - x = C_1}, \quad \frac{du}{dy} = 1 \Rightarrow \boxed{u - y = C_2}$$

$$\frac{dx}{du} = 1 \Rightarrow \boxed{u - x = C_3}$$

The general solution is given by $C_3 = f(C_2)$ (say)

i.e $\boxed{u - x = f(u - y)}$

This is the general solution, as can be seen

$$\frac{\partial}{\partial x} : \frac{\partial u}{\partial x} - 1 = f'(u-y)\frac{\partial u}{\partial x}$$

$$\frac{\partial}{\partial y} : \frac{\partial u}{\partial y} = f'(u-y)\left(\frac{\partial u}{\partial y} - 1\right)$$

Eliminate f' : $\quad \frac{u_x - 1}{u_x} = \frac{u_y}{u_y - 1}$

$$\Rightarrow (u_x - 1)(u_y - 1) = u_x u_y \Rightarrow \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} = 1$$

Now choose f so that $u = -x^2$ on $y = 0$

$u - x = f(u - y)$

$-x^2 - x = f(-x^2 - 0)$ i.e $f(-x^2) = -x^2 - x$

$r = -x^2$

$f(r) = r \pm \sqrt{-r}$

ie $\cancel{u-x} = (u-y) \pm \sqrt{y-u}$

$\Rightarrow \boxed{u = y-(x-y)^2}$

---

solve $x(y^2-u^2)\frac{\partial u}{\partial x} + y(u^2-x^2)\frac{\partial u}{\partial y} = u(x^2-y^2)$

inhomogeneous and quasi-linear so we have to use Lagrange

The ~~x~~ equations are

$\frac{\partial x}{\partial t} = x(y^2-u^2)$

$\frac{\partial y}{\partial t} = y(u^2-x^2)$

$\frac{\partial u}{\partial t} = u(x^2-y^2)$

Consider:

$x\frac{\partial x}{\partial t} + y\frac{\partial y}{\partial t} + u\frac{\partial u}{\partial t} = x^2(y^2-u^2)+y^2(u^2-x^2)+u^2(x^2-y^2)=0$

$\frac{\partial}{\partial t}\left[\frac{x^2+y^2+u^2}{2}\right] = 0 \quad \Rightarrow \boxed{x^2+y^2+u^2 = C_1}$

Problem class PS6

4 a) $z\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = z^2, \quad z=1, \quad \ln y = e^x$

$\frac{\partial x}{z} = \frac{\partial y}{y} = \frac{\partial z}{z^2}$

1) $\frac{\partial x}{z} = \frac{\partial z}{z^2} \Leftrightarrow \partial x = \frac{\partial z}{z}$

$\Rightarrow x + c = \ln z \Rightarrow z = Ce^x$

$ze^{-x} = \text{constant} = C_1$

2) $\frac{\partial y}{y} = \frac{\partial z}{z^2} \qquad \ln y = -z^{-1} + \text{Const}$

$y = e^{-\frac{1}{z}}C$, so $ye^{\frac{1}{z}} = C_2$, constant

The general solution $C_2 = f(C_1)$

$\Rightarrow ye^{\frac{1}{z}} = f(ze^{-x}) \qquad ey = f\left(\frac{1}{\ln y}\right) \quad \frac{1}{\ln y} = r$

when $z=1$, $\ln y = e^x$

$ye = f(e^{-x})$, so $f(r) = ee^{\frac{1}{r}}$ and $ye^{\frac{1}{z}} = ee^{\frac{1}{ze^{-x}}}$

$ye = f(e^{-x}), \; y = e^{e^x}$

$\Downarrow$

$ee^{e^x} = f(e^{-x})$

$\Leftrightarrow$

$ee^{\frac{1}{r}} = f(r)$

or $\ln y + \frac{1}{z} = 1 + \frac{1}{z e^{-x}} \Rightarrow$

$\frac{1}{z}\left(1 - \frac{1}{e^{-x}}\right) = 1 - \ln y \Rightarrow \boxed{z = \frac{e^x - 1}{\ln y - 1}}$ – True solution

<u>Check it</u>

$z = 1 \qquad 1 = \frac{e^x - 1}{\ln y - 1} = \frac{e^x - 1}{e^x - 1} = 1$ ✓

$$z\left(\frac{1}{\ln y - 1}\right) - \cancel{y}\left(\frac{e^x - 1}{(\ln y - 1)^2}\right)\frac{1}{\cancel{y}} = \frac{(e^x - 1)^2}{(\ln y - 1)^2}$$

$$\frac{(e^x - 1)}{\ln y}\left(\frac{e^x}{\ln y - 1}\right) - \left(\frac{e^x - 1}{(\ln y - 1)^2}\right) = \frac{(e^x - 1)^2}{(\ln y - 1)^2}$$ ✓

b) $z\frac{\partial z}{\partial x} + xy^2\frac{\partial z}{\partial y} = xz^3 \quad , x = 0 \quad y = 1$

$$\frac{\partial x}{z} = \frac{\partial y}{xy^2} = \frac{\partial z}{xz^3}$$

$$\frac{\partial y}{xy^2} = \frac{\partial z}{xz^3} \Rightarrow \frac{\partial y}{y^2} = \frac{\partial z}{z^3}$$

$$-\frac{1}{y} = -\frac{1}{2z^2} + C_1$$

$$C_{const} = \frac{1}{2z^2} - \frac{1}{y} \Rightarrow y - 2z^2 = C_1$$

$$\frac{\partial x}{z} = \frac{\partial z}{xz^3}$$

$$\int x\,\partial x = \int \frac{1}{z^2}\,\partial z$$

$$\frac{x^2}{2} = -\frac{1}{z} + \text{const}$$

$$C_2 = \frac{x^2}{2} + \frac{1}{z}$$

general solution:

$$\frac{1}{y} - \frac{1}{2z^2} = f\left(\frac{x^2}{2} + \frac{1}{z}\right) \qquad BC: x = 0 \quad y = 1$$

$$1 - \frac{1}{2z^2} = f\left(\frac{1}{z}\right) \qquad r = \frac{1}{z}$$

$$\Rightarrow f(r) = 1 - \frac{r^2}{2} \Rightarrow$$

$$\frac{1}{\frac{1}{y}-2z^2}$$

$$\frac{1}{\frac{1}{y}-2z^2}=1-\frac{1}{2}r^2=1-\frac{1}{2}\left(\frac{x^2}{2}+\frac{1}{z}\right)^2$$

$$\frac{1}{\frac{1}{y}-2z^2}=1-\frac{1}{2}\left(\frac{x^4}{4}+\frac{x^2}{z}+\frac{1}{z^2}\right)\cdot \cancel{z}$$

$$\frac{1}{y}-1+\frac{x^4}{8}=\cancel{\frac{1}{2z^2}}-\cancel{\frac{1}{2z^2}}-\frac{\cancel{2}x^2}{2z}$$

$$\frac{1}{y}-1+\frac{x^4}{8}=-\frac{x^2}{2z}\quad\Big|\cdot\left(\frac{2}{-x^2}\right)$$

$$-\frac{2}{x^2y}+\frac{2}{x^2}-\frac{x^2}{4}=\frac{1}{z}$$

1. a) $xz\frac{\partial z}{\partial x}-yz\frac{\partial z}{\partial y}=x^2-y^2$

$$\frac{dx}{xz}=\frac{dy}{-yz}=\frac{dz}{x^2-y^2}$$

$$\int\frac{dx}{x}=-\int\frac{dy}{y}$$

$$\ln x=-\ln y+\text{const}$$

$$\ln x+\ln y=C_1$$

$$\ln(xy)=C_1$$

$$\boxed{xy=C_1}$$

$$\frac{dx}{xz}=\frac{dz}{x^2-y^2}$$

$$\int\frac{(x^2-y^2)}{x}dx=\int z\,dz$$

$$\int x-\frac{y^2}{x}\,dx=\int z\,dz$$

$$\frac{x^2}{2}-y^2\ln x=\frac{z^2}{2}+\text{Const}$$

Take $x\frac{dx}{dt}+y\frac{dy}{dt}=x\cdot xz+y\cdot(-yz)=z(x^2-y^2)=z\frac{dz}{dt}$

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$$\frac{\partial}{\partial t}\left(\frac{x^2+y^2-z^2}{2}\right)=0$$

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$$\frac{d\cancel{x}}{a}=\frac{\partial y}{b}=\frac{\partial u}{c}(=dt)$$

find constants c_1 & c_2 & ~~soln~~ solution is $c_1=f(c_2)$

$$\underbrace{x(y^2-u^2)}_{a}\frac{\partial u}{\partial x}+\underbrace{y(u^2-x^2)}_{b}\frac{\partial u}{\partial y}=\underbrace{u(x^2-y^2)}_{c}$$

$$\frac{\partial x}{\partial t}=x(y^2-u^2)\ ,\ \frac{\partial y}{\partial t}=y(u^2-x^2)\ ,\ \frac{du}{\partial t}=u(x^2-y^2)$$

$$x\frac{\partial x}{\partial t}+y\frac{\partial y}{\partial t}+u\frac{\partial u}{\partial t}=0\ \Rightarrow\ \frac{1}{2}(x^2+y^2+u^2=c$$

$$\boxed{c_1=x^2+y^2+u^2}$$

$$yu\frac{\partial x}{\partial t}+xu\frac{\partial y}{\partial t}+xy\frac{\partial y}{\partial t}=xyu(y^2-u^2)+(xyu)(u^2-x^2)+xyu(x^2-y^2)$$

$$\Rightarrow\frac{\partial}{\partial t}(xyu)=0$$

$$\Rightarrow\boxed{xyu=c_2}$$

& the general solution is $c_2=f(c_1)$
i.e $xyu=f(x^2+y^2+u^2)$

example

$$(y+u)\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=x-y$$

χ eq^{ns} are $\frac{\partial t}{\partial t}=y+u$ ① , $\frac{dy}{dt}=y$ ② , $\frac{\partial u}{\partial t}=x-y$ ③

② $\Rightarrow y=Ae^t$

① $\Rightarrow\ \frac{\partial^2\cancel{x}}{\partial t^2}=\frac{\partial y}{\partial t}+\frac{\partial u}{\partial t}=y+x-y=x$

$$x=ce^t+De^{-t}$$

③ $\frac{\partial^2 u}{\partial t}=\frac{\partial x}{\partial t}-\frac{\partial y}{\partial t}=y+u-y=u$

$$\Rightarrow u=Ee^t+Fe^{-t}$$

We have 5 constants, while we should only expect 3 - we have a sistem of 3 first order equations.

using $\frac{\partial\cancel{x}}{\partial t}=y+u$ gives $ce^t-De^{-t}=Ae^t+Ee^t+Fe^{-t}$

$$c=A+E\ \ \&\ \ F=-D$$

$y = Ae^{t}$ (4) $\qquad e^{t} = \frac{y}{A} \qquad e^{-t} = \frac{A}{y}$

$x = Ce^{t} + De^{-t}$ (5)

$u = (C-A)e^{t} - De^{-t}$ (6)

$$x + u = (2C-A)e^{t} = \left(\frac{2C-A}{A}\right)y$$

$$\frac{x+u}{y} = \left(\frac{2C-A}{A}\right) = \text{constant}$$

Also $u + y = Ce^{t} - De^{-t}$ (7) using (4) in (6)

& subtracting (5) & (7) $x - u - y = 2De^{-t} = \frac{2DA}{y}$

so $(x-u-y)y = \text{constant}$

& general sol^n can be written

(*) $(x-u-y)y = f\left(\frac{x+u}{y}\right)$ —(substitute in and check answer)

Another method

(1) $dx = (y+u)\,dt$

(2) $dy = y\,dt$

(3) $du = (x-y)\,dt$

(1)+(3) $dx + du = (x+u)\,dt$

$$\Rightarrow \frac{d(x+u)}{(x+u)} = dt = \frac{dy}{y}$$

$$\Rightarrow \ln(x+u) = \ln y + \text{const}$$

$$\Rightarrow \frac{x+u}{y} = \text{const}$$

(1) & (2) give $dx = dy + u\,dt$

$$\Rightarrow d(x-y) = u\,dt = u\frac{du}{(x-y)}$$

$$\Rightarrow (x-y)\cdot d(x-y) = u\,du$$

$$\frac{(x-y)^2}{2} = \frac{u^2}{2} + c$$

also $C = (x-y)^2 - u^2$

& $\frac{x+u}{y} = f\left((x-y)^2 + u^2\right)$ (**)

(*) & (**) are not the same, but could be for different f's

Second order PDE'S

The general second order quasilinear pde is

$$a(x,y,z,z_x,z_y)\frac{\partial^2 z}{\partial x^2} + b(x,y,z,z_x,z_y)\frac{\partial^2 z}{\partial x \partial y} + c(x,y,z,z_x,z_y)\frac{\partial^2 z}{\partial y^2} = r(x,y,z,z_x,z_y)$$

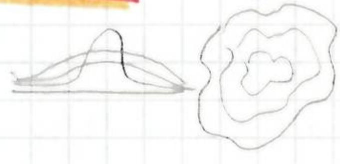
The quantity $\Delta = b^2 - 4ac$ is the discriminant of the pde.

If $\Delta > 0$, the equation is said to be hyperbolic – waves

If $\Delta < 0$, —— // —— elliptic – steady temperature distribution

If $\Delta = 0$ —— // —— parabolic – diffusion

If a, b, c are constant & $r = 0$



$$a z_{xx} + b z_{xy} + c z_{yy} = 0$$

Look for a solution of the form $z = f(y + mx)$ & substitute to find

$$am^2 f'' + bmf'' + cf'' = 0$$

$$\Rightarrow am^2 + bm + c = 0$$

If $\Delta > 0$ (hyperbolic) we have two real roots for m
$\Delta < 0$ (elliptic) — // — two complex — // —
$\Delta = 0$ (parabolic) — // — are real repeated root for m

Let's call these m_1 & m_2

If $m_1 \neq m_2$, introduce $s = y + m_1 x$, $t = y + m_2 x$, called canonical variables

& make a change of variable from x & y to s & t

$z_x \to z_s$ & z_t etc

$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial s}\cdot\frac{\partial s}{\partial x} + \frac{\partial z}{\partial t}\cdot\frac{\partial t}{\partial x} = m_1\frac{\partial z}{\partial s} + m_2\frac{\partial z}{\partial t}$$

$$\frac{\partial z}{\partial y} = \frac{\partial z}{\partial s}\cdot\frac{\partial s}{\partial y} + \frac{\partial z}{\partial t}\cdot\frac{\partial t}{\partial y} = \frac{\partial z}{\partial s} + \frac{\partial z}{\partial t}$$

$$\frac{\partial}{\partial x} = m_1\frac{\partial}{\partial s} + m\frac{\partial}{\partial t}, \quad \frac{\partial}{\partial y} = \frac{\partial}{\partial s} + \frac{\partial}{\partial t}$$

$z_{st} = z_{ts}$

$$z_{xx} = (m_1 \partial_s + m_2 \partial_t)(m_1 z_s + m_2 z_t) = m_1^2 z_{ss} + 2m_1 m_2 z_{st} + m_2^2 z_{tt}$$

∂_s = $\frac{\partial}{\partial s}$

$$z_{yy} = z_{ss} + 2z_{st} + z_{tt}$$

$$z_{xy} = (m_1 \partial_s + m_2 \partial_t)(z_s + z_t) = m_1 z_{ss} + (m_1 + m_2) z_{st} + m_2 z_{tt}$$

Substitute into $az_{xx} + bz_{xy} + cz_{yy} = 0$

to find

$$z_{ss}\left(am_1^2 + bm_1 + c\right) + z_{st}\left(a\,2m_1m_2 + b(m_1+m_2) + 2c\right) + z_{tt}\left(am_2^2 + bm_2 + c\right) = 0$$

m_1 & m_2 are roots of $am^2 + bm + c = 0$

$m_1 + m_2 = -\frac{b}{a}$ $\quad m_1m_2 = \frac{c}{a}$

so $z_{st}\left[2c - \frac{b^2}{a} + 2c\right] = -\frac{\Delta}{a} z_{st}$

$\Delta \neq 0$ here (we took $m_1 \neq m_2$, so we have two distinct roots)

This is in canonical form

(if $az_{xx} + bz_{xy} + cz_{yy} = f \Rightarrow -\frac{\Delta}{a} z_{st} = f$)

$\frac{\partial^2 z}{\partial s \partial t} = 0 \Rightarrow \frac{\partial z}{\partial t} = f'(t)$ $\quad f'(t)$ is arbitrary

$\Rightarrow z(st) = f(t) + g(s)$

$= f(y + m_2 x) + g(y + m_1 x)$

From Complex Analysis

$u + iv$

$\nabla^2 u = \nabla^2 v = 0$

$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$

so, the general solution of a hyperbolic equation may be written in the form

$z = f(y + m_2 x) + g(y + m_1 x)$ with m_1 & m_2 roots of $am^2 + bm + c$ ← auxiliary equation

[The lines $y + m_2 x = \text{const}$, $y + m_1 x = \text{const}$ are called characteristics]

Solutions of elliptic problems can also be written in this way, but m_1, m_2 are complex

If $\Delta = 0$ & the equation is parabolic, we have one root $m = -\frac{b}{2a}$

Here we switch to variables $s = y + mx$, $t = x$

NB you do it!

$(\underbrace{2ma + b}_{=0}) z_{st} + z_{tt} = 0$

so $z_{tt} = 0$

$\Rightarrow z_t = g(s)$

$z(s,t) = t\,g(s) + f(s)$

& the general solution for parabolic equations is
$z(x,y) = x\, g(y+mx) + f(y+mx)$

Examples

a) $z_{xx} - 3z_{xy} + 2z_{yy} = 0$

if $z = f(y+mx)$ then, we need $m^2 - 3m + 2 = 0$ (1)

you can also take $z = g(x+my)$ $1 - 3m + 2m^2 = 0$ $m = \frac{1}{m}$

(1) $\Rightarrow (m-2)(m-1) = 0$

i.e two distinct real values of m: 1 & 2 & solution is
$z = f(y+x) + g(y+2x)$

b) $z_{xx} - 2z_{xy} + z_{yy} = 0$

If $z = f(y+mx)$
$m^2 - 2m + 1 = 0$
$(m-1)^2 = 0$

i.e the equation is parabolic & the solution is
$z = x\, f(y+x) + g(y+x)$

c) $z_{xx} - 3z_{xy} + 2z_{yy} = e^{x-y}$

As the equation is linear the solution has the form

$\underbrace{CF}_{\text{a solution to } z_{xx}-3z_{xy}+2z_{yy}=0} + PI \leftarrow$ anything that satisfies the equation

Complementary Function: If $z = f(y+mx)$
$m^2 - 3m + 2 = 0$
$(m-2)(m-1) = 0$

$z = f(y-x) + g(y+2x)$

Particular Integral: ODE – $y'' + y' + y = e^t$ PI: $y = \alpha e^t$

Try $z = Ae^{x-y}$

$z_x = Ae^{x-y}$ $z_y = -Ae^{x-y}$ $z_{xy} = -Ae^{x-y}$
$z_{xx} = Ae^{x-y}$ $z_{yy} = Ae^{x-y}$

Substituition gives
$Ae^{x-y} - 3(-Ae^{x-y}) + 2Ae^{x-y} = e^{x-y}$

$6A = 1$, $A = \frac{1}{6}$

so $z(x,y) = \underbrace{\frac{1}{6} e^{x-y}}_{PI} + \underbrace{f(y+x) + g(y+2x)}_{CF}$

Alternatively we can change variables to the canonical variables

~~s=x+y~~ $(m=1)$ $s = y + x$
~~t=x+2y~~ $(m=2)$ $t = y + 2x$

LHS goes to $-\frac{\Delta}{a} z_{st} = -z_{st} = e^{x-y} = e^{(t-s)-(2s-t)} = e^{2t-3s}$

$x = t - s$
$y = s - x = s - t + s = 2s - t$

$\Rightarrow -z_t = -\frac{1}{3} e^{2t-3s} + f'(t)$

$-z = -\frac{1}{6} e^{2t-3s} + f(t) + g(s)$

as before

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The wave equation

A simple physically relevant, second order hyperbolic equation, is the wave equation

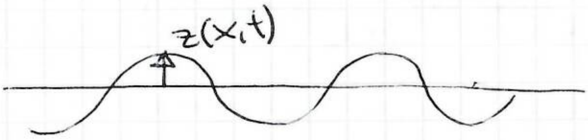
$c = \frac{l}{t}$

$$\frac{1}{c^2} \frac{\partial^2 z}{\partial t^2} = \frac{\partial^2 z}{\partial x^2}$$

-time -position

for $z(x,t)$ for some constant c, known as the wave speed.

For the present we will consider
$-\infty < x < \infty$, $t \geq 0$
i.e we look at initial value problems

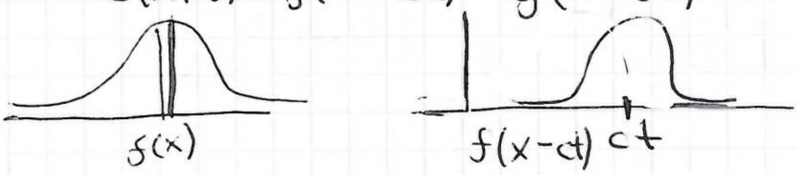


We look for a solution of the form $z = f(x+mt)$. Substitution gives

$$\frac{m^2}{c^2} f'' = f''$$

$m = \pm c$ & so the equation is hyperbolic & the general solution is

$$z(x,t) = f(x-ct) + g(x+ct)$$



This solution is made up of one wave travelling to the right without change of form $(f(x-ct))$ & one moving to the left $(f(x+ct))$.

To solve an initial value problem we need to find f & g so that at $t=0$, $z(x,0) = F(x)$

$\frac{\partial z}{\partial t}(x,0) = G(x)$ where we know F & G.

If $z(x,t) = f(x-ct) + g(x+ct)$

$z(x,0) = f(x) + g(x) = F(x)$ ←

$z_t(x,t) = -c f'(x-ct) + c g'(x+ct)$

$z_t(x,0) = -c f'(x) + c g'(x) = G(x)$ ←

or $-f + g = \frac{1}{c}\int_\alpha^x G(\xi)\,d\xi$ for constant of integration related to α

$f + g = F$

adding $g = \frac{1}{2}F + \frac{1}{2c}\int_\alpha^x G(\xi)\,d\xi$, $f = \frac{1}{2}F - \frac{1}{2c}\int_\alpha^x G(\xi)\,d\xi$

$$z(x,t) = f(x-ct) + g(x+ct)$$

$$= \frac{1}{2}\left[F(x-ct) + F(x+ct)\right] + \frac{1}{2c}\left[\int_{x-ct}^{\alpha} G(\xi)\,d\xi + \int_\alpha^{x+ct} G(\xi)\,d\xi\right]$$

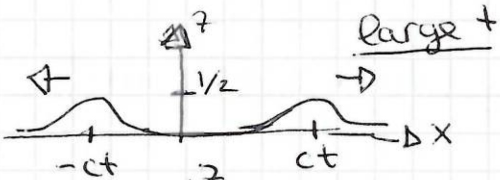
$$= \frac{1}{2}\left[F(x-ct) + F(x+ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} G(\xi)\,d\xi$$

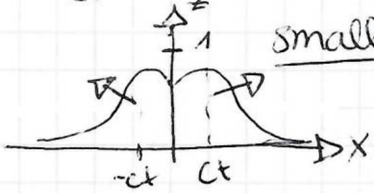
Exam

This is D'Alembert's solution to the wave equation.

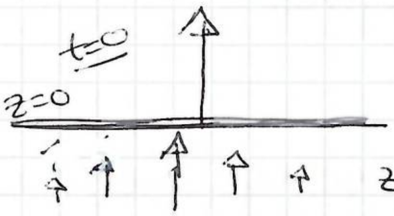
If $G = 0$ & $F(x) = e^{-x^2}$

$$z(x,t) = \frac{1}{2}\left(e^{-(x-ct)^2} + e^{-(x+ct)^2}\right)$$





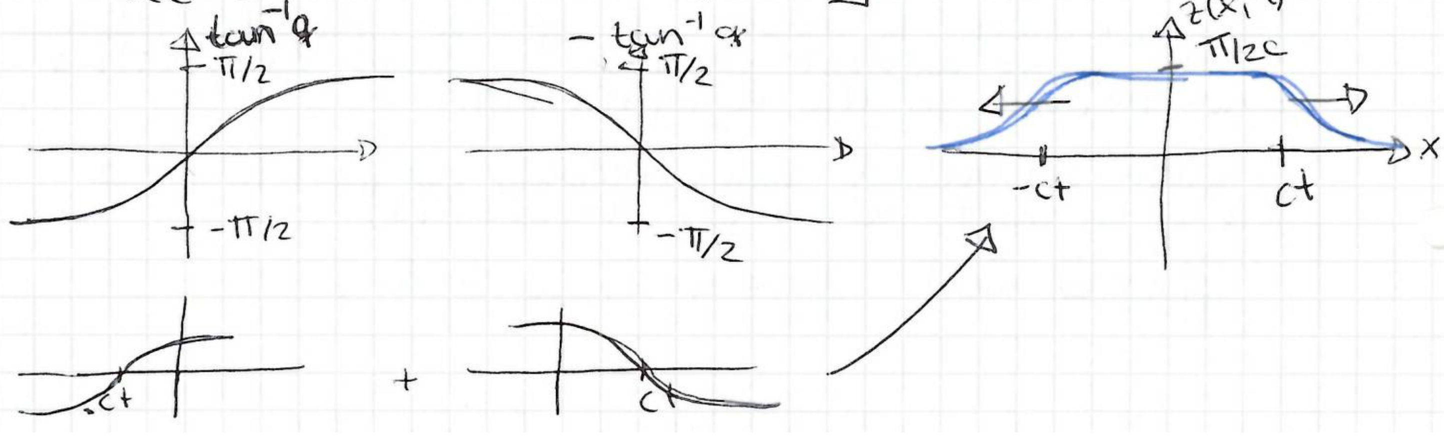
If $F = 0$, $G(x) = \frac{1}{1+x^2}$



$z_t = \frac{1}{1+x^2}$ (upwards velocity)

$$z(x,t) = \frac{1}{2c}\int_{x-ct}^{x+ct} \frac{1}{1+\xi^2}\,d\xi = \frac{1}{2c}\left[\tan^{-1}\xi\right]_{x-ct}^{x+ct}$$

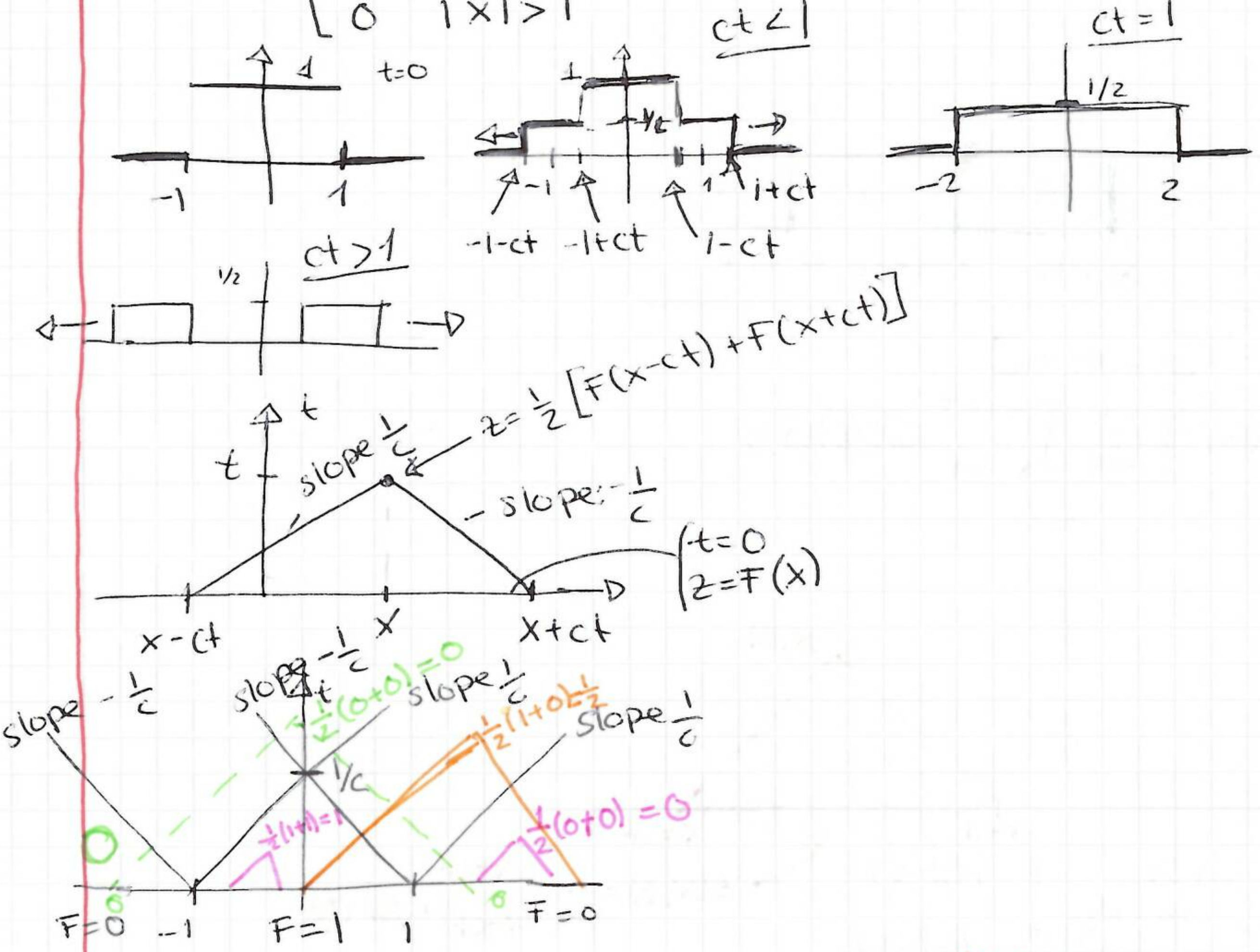
$$= \frac{1}{2c}\left[\tan^{-1}(x+ct) - \tan^{-1}(x-ct)\right]$$



example

$G = 0$

$$F(x) = \begin{cases} 1 & |x| \le 1 \\ 0 & |x| > 1 \end{cases}$$



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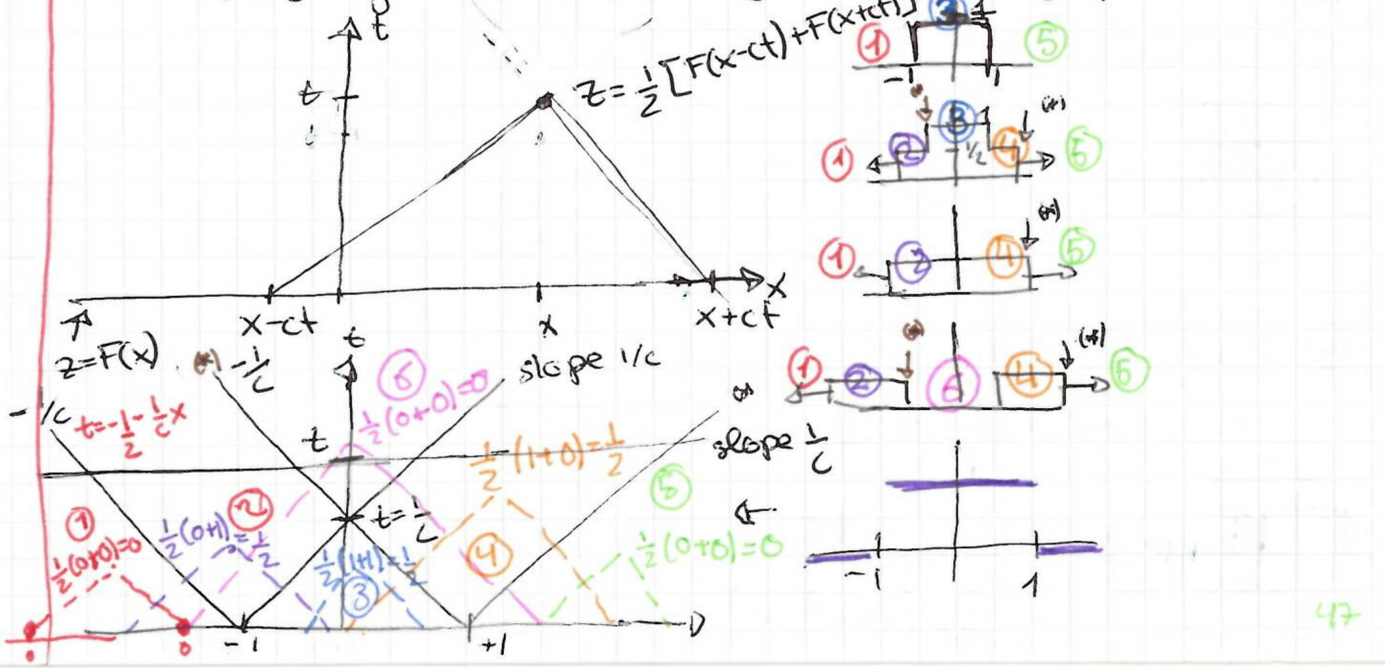
$$\frac{1}{c^2}\frac{\partial^2 z}{\partial t^2} = \frac{\partial^2 z}{\partial x^2} \quad \text{if } z = F(x) \ \& \ z_x = G(x) \text{ at } t=0$$

$$z(x,t) = \frac{1}{2}\left(F(x+ct) + F(x-ct)\right) + \frac{1}{2c}\int_{x-ct}^{x+ct} G(\xi)\,d\xi$$

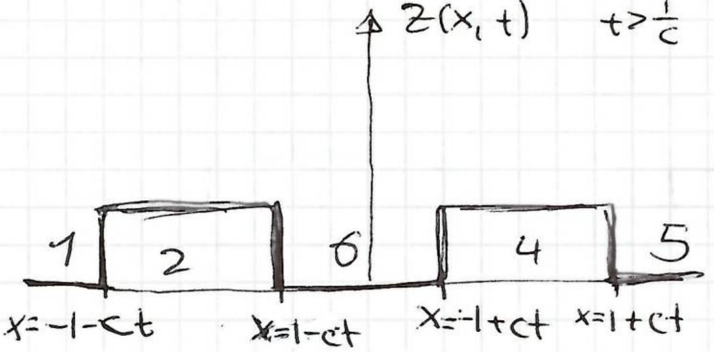
$$= f(x-ct) + g(x+ct)$$

Information about solution travels with speed c
two sets of characteristics $x-ct=\text{const}$, $x+ct=\text{const}$



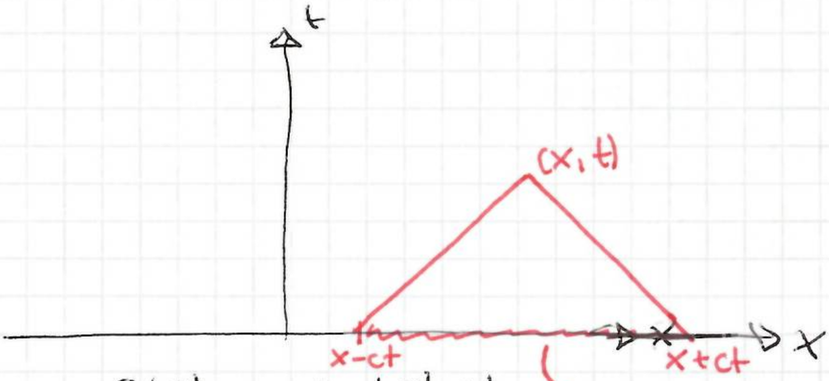
Find solutions of t & z

$t > \frac{1}{c}$



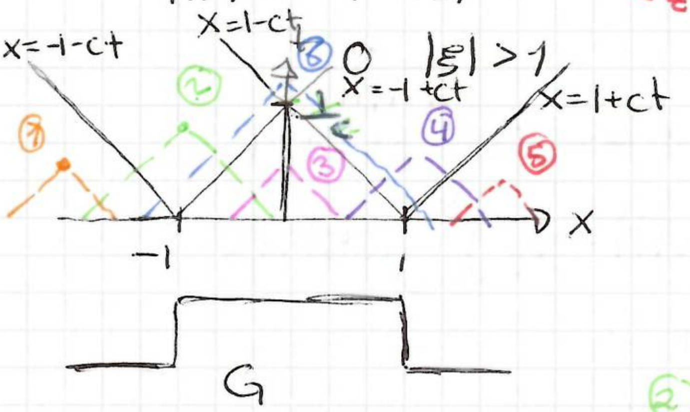
$F = 0, \quad G \neq 0$

$$z(x,t) = \frac{1}{2c}\int_{x-ct}^{x+ct} G(\xi)\,d\xi$$



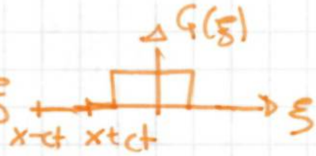
$$z(x,t) = \frac{1}{2c}\int_{x-ct}^{x+ct} G(\xi)\,d\xi$$

$G(\xi) = \begin{cases} 1 & |\xi| < 1 \\ 0 & |\xi| > 1 \end{cases}$



① $x < -1-ct$

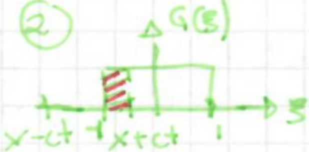
$$z = \frac{1}{2c}\int_{x-ct}^{x+ct} G(\xi)\,d\xi$$



$$= \frac{1}{2c}\int 0\,d\xi = 0$$

② $x - ct < -1$

$-1 < x + ct < 1$



$$z = \frac{1}{2c}\int_{-1}^{x+ct} 1\,d\xi = \frac{1}{2c}(1 + x + ct)$$

which, as a function of x is a straight line slope $\frac{1}{2c}$ valid between $t < \frac{1}{c}$

$x = -1-ct$ giving $z = 0$

$x = -1+ct$ giving $z = t$

& for $t > \frac{1}{c}$

$x = -1-ct$, gives $z = 0$

$x = 1-ct$, gives $z = \frac{1}{c}$

③ $-1+ct < 1-ct \quad t < \frac{1}{c}$

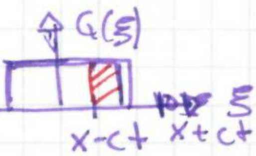
$-1 \leq x-ct \leq x+ct \leq 1$

$$z = \frac{1}{2c}\int_{x-ct}^{x+ct} 1\,d\xi$$

$$= t$$

④ $1-ct < x < 1+ct, \; t < \frac{1}{c}$

$-1+ct < x < 1+ct, \; t > \frac{1}{c}$

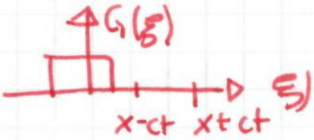


$$z = \frac{1}{2c}\int_{x-ct}^{1} 1\,d\xi$$

$$= \frac{1}{2c}(1 + ct - x)$$

slope $-\frac{1}{2c}$ but otherwise like ②.

⑤ $1 < x-ct < x+ct$



$$z = \frac{1}{2c}\int_{x-ct}^{x+ct} 0 \, d\xi = 0$$

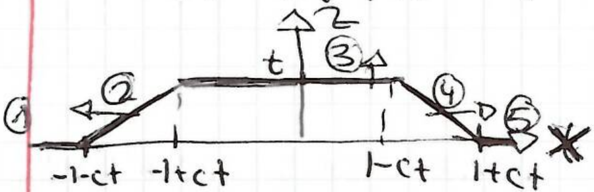
⑥ $t > \frac{1}{c}$

$$z = \frac{1}{2c}\int_{-1}^{1} 1 \, d\xi = \frac{1}{c}$$

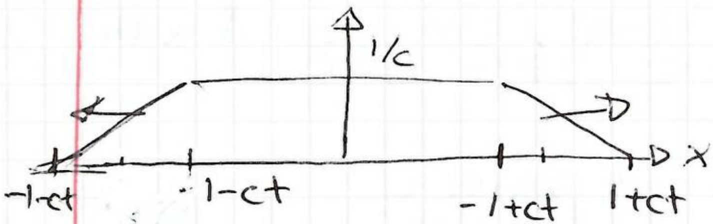
$x - ct < -1 < 1 < x+ct$

Choose a value of t: $t < \frac{1}{2c}$ and $t > \frac{1}{2c}$ we get different pictures

Draw $z(x,t)$ $t < \frac{1}{c}$



$t > \frac{1}{c}$



Solution of wave equation using the method of separation of variables

Fourier series

The wave equation is $\frac{1}{c^2}\frac{\partial^2 z}{\partial t^2} = \frac{\partial^2 z}{\partial x^2}$

We look for a solution $z = \underbrace{X(x)}_{\text{function of } x \text{ alone}}\underbrace{T(t)}_{\text{function of } t \text{ alone}}$

$z_{tt} = XT''$, $z_{xx} = X''T$

$$\frac{XT''}{c^2} = X''T \quad \Big| : XT$$

$$\underbrace{\frac{T''}{c^2 T}}_{\text{function of } t} = \underbrace{\frac{X''}{X}}_{\text{function of } x}$$

(x and t are enterly independent of each other)

(so we can't have a function of t equal a function of x, unless they are both constant)

Imagine changing t but not x. The LHS might change but the RHS must remain constant. We deduce LHS does not change & both $\frac{T''}{c^2T}$ & $\frac{X''}{X}$ are the same constant, independent of both x & t.

Let's call this constant λ, the separation constant.

$$\frac{T''}{c^2T} = \frac{X''}{X} = \lambda$$

$$X'' - \lambda X = 0$$

$$T'' - \lambda c^2 T = 0$$

Any λ will do generating X_λ, T_λ & so $z_\lambda(x,t) = X_\lambda(x)\,T_\lambda(x)$ & as the wave equation is linear any sum of these is also a solution.

$$z(x,t) = \sum_\lambda X_\lambda(x)\,T_\lambda(t)$$

However, we are only interested in solutions satisfying particular initial conditions &, more relevant now, boundary conditions. It is the boundary conditions that restrict values of λ.

If we solve $z_{tt} = c^2 z_{xx}$ for $t \geq 0$, $0 \leq x \leq L$ & with boundary conditions $z(0,t) = 0$, $z(L,t) = 0$

If $z(x,t) = X(x)\,T(t)$

then we need $X(0)\,T(t) = 0 \quad \forall t \Rightarrow X(0) = 0$

$X(L)\,T(t) = 0 \quad \forall t \Rightarrow X(L) = 0$

substitution

$$XT'' = c^2 T X''$$

$$\frac{T''}{c^2T} = \frac{X''}{X} = \lambda \qquad \boxed{\begin{array}{l} X'' - \lambda X = 0 \\ X(0) = X(L) = 0 \end{array}}$$

It turns out that for particular values of λ we get solutions to this other than the obvious $X = 0$

$\lambda > 0$, $\lambda = 0$, $\lambda < 0$ (real λ)

$\underline{\lambda > 0}$, $\lambda = p^2$, p real, $X'' - p^2 X = 0$

which has solutions $\cancel{X} = Ae^{px} + Be^{-px}$

or $\underline{X = \tilde{A}\cosh px + \tilde{B}\sinh px}$

but $X(0) = 0 \Rightarrow \tilde{A}\,1 + \tilde{B} \cdot 0 = 0 \Rightarrow \tilde{A} = 0$

$X(L) = 0 \Rightarrow \tilde{B}\sinh pL = 0 \Rightarrow \tilde{B} = 0$ $\Big\}$ $\underline{X = 0}$

don't want this

Look at $\underline{\lambda = 0}$

$X'' = 0 \quad , \quad X = Ax + B$

but $X(0) = 0 \; , \; X(L) = 0 \Rightarrow A = B = 0$ ← don't want this

$X = 0$

we can jump here if you explain

$\lambda < 0 \; , \; \lambda = -p^2 \; , \; \boxed{X'' + p^2 X = 0}$

& $\quad X = A\cos(px) + \boxed{B\sin(px)}$ ← we don't want cos since $X(0) = 0$ and $\cos 0 = 1$

$X(0) = 0 \quad , \quad A \cdot 1 + B \cdot 0 = 0 \Rightarrow A = 0$

$X(L) = 0 \; , \quad B\sin(pL) = 0$

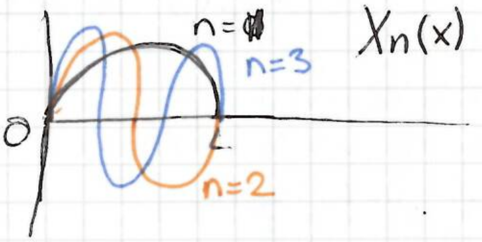
& if $B \neq 0$ we conclude

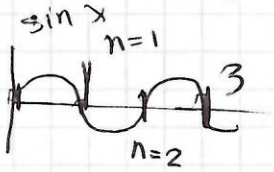
$\sin(pL) = 0$

$pL = n\pi \; , \qquad n = 1, 2, 3$

$p = \frac{n\pi}{L} \; , \; \boxed{\lambda = -\frac{n^2\pi^2}{L^2}}$

We have an infinite number of possible separation constants





Recall $\quad \frac{T_n''}{c^2 T_n} = \lambda = -p^2 = -\left(\frac{n\pi}{L}\right)^2$

$$T_n'' + \frac{c^2 n^2 \pi^2}{L^2} T_n = 0$$

$$T_n = C\cos\left(\frac{c^2 n^2 \pi^2}{L^2}\right) + D\sin\left(\frac{c^2 n^2 \pi^2}{L^2}\right)$$

$$z(x,t) = \sum_n X_n(x) T_n(t) = \sum_{n=1}^{\infty} \sin\left(\frac{n\pi x}{L}\right)\left(C_n \cos\left(\frac{c^2 n^2 \pi^2}{L^2}\right) + D_n \sin\left(\frac{c^2 n^2 \pi^2}{L^2}\right)\right)$$

02/11/12

Wave equation

$\frac{1}{c^2}\frac{\partial^2 z}{\partial t^2} = \frac{\partial^2 z}{\partial x^2} \quad , \; z(0,t) = 0, \; z(L,t) = 0$

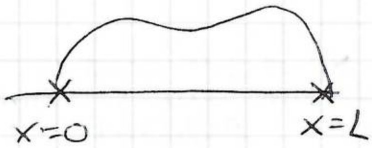
$z = X(x)T(t) \Rightarrow$

$\frac{1}{c^2}\frac{T''}{T} = \frac{X''}{X} = \lambda = -p^2 \quad$ so ~~bc~~ satisfied $\quad X(0) = 0, \; X(L) = 0$

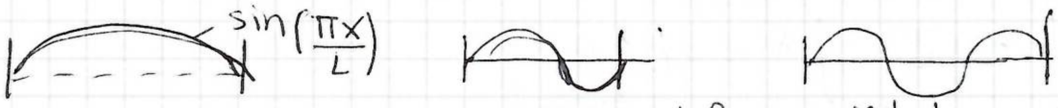
$X = A\sin px\,,\quad p = \frac{n\pi}{L}$ if you explain it, you can jump here (see last lecture)

$T'' + c^2p^2T = 0 \Rightarrow T = C\cos\left(\frac{n\pi ct}{2}\right) + D\sin\left(\frac{n\pi ct}{2}\right)$

$$X = \sum_{1}^{\infty} \sin\left(\frac{n\pi x}{L}\right)\left[C_n\cos\left(\frac{n\pi ct}{L}\right) + D_n\sin\left(\frac{n\pi ct}{L}\right)\right]$$



Each term in this sum is known as a normal mode. Each normal mode oscillates with a normal frequency $\omega_n = n\left(\frac{\pi c}{L}\right)$. The mode $n=1$ is called the fundamental & the modes $n=2,3,$ higher harmonics & the solution for $z(x,t)$ is a mixture sum of the fundamental and the harmonics



The values of C_n & D_n, the amplitude of each normal mode is determined from initial conditions. If $z(x,0) = F(x)$ & $z_t(x,0) = G(x)$, then putting $t=0$ gives

$F(x) = \sum_{1}^{\infty} \sin\left(\frac{n\pi x}{L}\right)\cdot C_n$ (sin is zero at $t=0$, cos is 1 at $t=0$)

Differentiating wrt t & putting $t=0$ gives

$G(x) = \sum_{1}^{\infty} \sin\left(\frac{n\pi x}{L}\right) D_n\left(\frac{n\pi c}{L}\right)$ (sin has non-zero derivative at 0, cos has zero derivative at 0)

if we know $F(x)$, we need to find C_n. Need to find the Fourier series of $F(x)$

vectors, $\underline{r} = x\underline{i} + y\underline{j} + z\underline{k}$ $\underline{i},\underline{j},\underline{k}$ are orthogonal

$x = \underline{r}\cdot\underline{i}$ $\underline{i}\cdot\underline{j} = 0 \quad \underline{i}\cdot\underline{i} = 1$

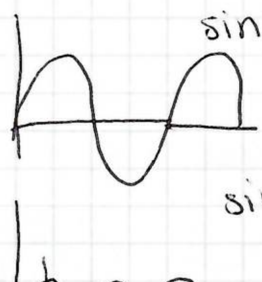
$\int_0^L \sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{m\pi x}{L}\right)dx = 0$ if $n \neq m$ - from Fourier series

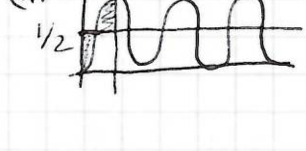
$\int_0^L \sin^2\left(\frac{n\pi x}{L}\right)dx \qquad n=m$

$= \frac{1}{2}L$ (*)   - same for $\int\cos(\;)\cos(\;)$.

- $\int\cos(\;)\sin(\;) = 0$ always

we are trying to write $F(x)$ in terms of a basis





The equation for $F(x)$, multiply by $\sin\left(\frac{m\pi x}{L}\right)$ & integrate in $[0,L]$

$$\int_0^L \sin\left(\frac{m\pi x}{L}\right)F(x)\,dx = \sum_1^\infty C_n \int_0^L \sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{m\pi x}{L}\right)dx$$

$$= C_m \tfrac{1}{2}L$$ (only non-zero element is when $m=n$)

$$C_n = \frac{2}{L}\int_0^L F(x)\sin\left(\frac{n\pi x}{L}\right)dx$$

For $G(x)$ similarly

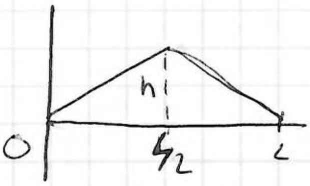
$$D_n = \frac{2}{n\pi C}\int_0^L G(x)\sin\left(\frac{n\pi x}{L}\right)dx$$

easy exam question

example

$G(x)=0$, $F(x) = \frac{2hx}{L}$, $0 \le x \le \frac{L}{2}$

$\frac{2h(L-x)}{L}$ $\quad \frac{L}{2} \le x \le L$



ie solve the wave equation

$\frac{1}{c^2} z_{tt} = z_{xx}$ on the integral

$x \in [0,L]$, $z(0,t)=0$, $z(L,t)=0$,

$z(x,0)=F(x)$, $z_t(x,0)=G(x)=0$

Look for $z(x,t)=X(x)T(t)$

$\Rightarrow \frac{T''}{c^2T} = \frac{X''}{X} = -p^2$ so that we can satisfy $X(0)=X(L)=0$

$X''+p^2X=0$

$X = A\sin px$ so that $x(0)=0 \Rightarrow p=\frac{n\pi}{L}$

$T''+p^2c^2T=0$

$T(t)=\cos pct$ satisfying $T'(0)=0$

Solution has the form $z(x,t) = \sum_{n=1}^\infty A_n \sin\left(\frac{n\pi x}{L}\right)\cos\left(\frac{n\pi ct}{L}\right)$

where A_n to be found so that $z(x,0)=F(x)$

i.e $F(x) = \sum_{n=1}^\infty A_n \sin\left(\frac{n\pi x}{L}\right)$

$$\Rightarrow A_n = \frac{2}{L}\int_0^L \sin\left(\frac{n\pi x}{L}\right)F(x)\,dx = \frac{2}{L}\left[\int_0^{L/2}\sin\left(\frac{n\pi x}{L}\right)\frac{2hx}{L}dx\right.$$

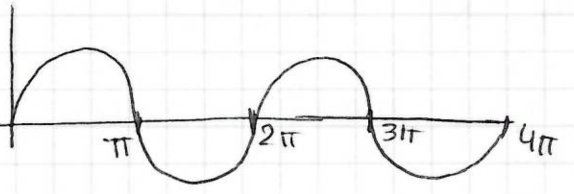
$$\left. + \int_{L/2}^L \sin\left(\frac{n\pi x}{L}\right)\frac{2h}{L}(L-x)\,dx\right]$$

~~Thus~~

rewrite to get one integral

In the second integral, write $L-x=u$

& it becomes $\int_{L/2}^0 \underbrace{\sin\left(\frac{n\pi}{L}(L-u)\right)}_{\sin(n\pi-\alpha)}\frac{2h}{L}u\,(-du)$ $\quad \alpha=\frac{n\pi}{L}$



$\sin(n\pi - \alpha) = \sin\alpha$, $n = 1,3,5,\ldots$

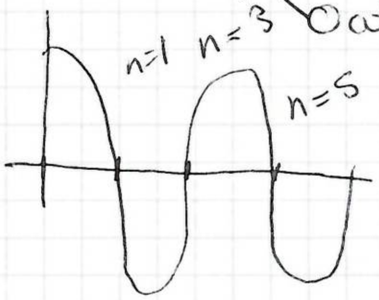
$\sin(n\pi - \alpha) = -\sin(\alpha)$, $n = 2,4,6\ldots$

$$= (\pm)\int_0^{L/2} \sin\left(\frac{n\pi u}{L}\right)\frac{2h}{L} u \, du \quad \begin{array}{l} + \; n=1,3,5 \\ - \; n=2,4,6 \end{array}$$

$A_n = 0$ if $n = 2,4,6\ldots$

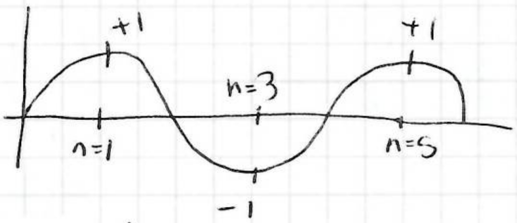
$$A_n = 2\cdot\frac{2}{L}\cdot\frac{2h}{L}\int_0^{L/2} x\cdot\sin\left(\frac{n\pi x}{L}\right)dx \;,\; n = 1,3,5,\ldots$$

$$= \frac{8h}{L^2}\left\{\left[\underbrace{x\frac{L}{n\pi}(-1)\cos\left(\frac{n\pi x}{L}\right)}_{0 \text{ at } x=0}\right]_0^{L/2} + \frac{L}{n\pi}\int_0^{L/2}\cos\left(\frac{n\pi x}{L}\right)dx\right\}$$



$\cos\left(\frac{n\pi}{2}\right) = 0$ for n=odd

$$= \frac{8h}{n\pi L}\left[\frac{L}{n\pi}\sin\left(\frac{n\pi x}{L}\right)\right]_0^{L/2}$$



$\sin\left(\frac{n\pi}{2}\right)$

$= \frac{8h}{n^2\pi^2}\cdot 1$ if $n = 1,5,9,13,\ldots$ ie j is even

$\cdot(-1)$ if $n = 3,7,11\ldots$ ie j is odd

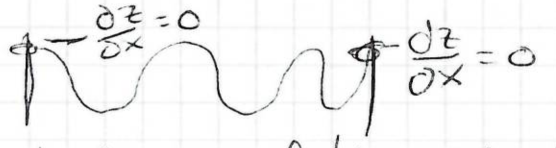
0 if n is even

$n = 2j+1$, $j = 0,1,2,\ldots$

$$z(x,t) = \frac{8h}{\pi^2}\sum_{j=0}^{\infty}\frac{(-1)^j}{(2j+1)^2}\sin\left[(2j+1)\frac{\pi x}{L}\right]\cos\left[(2j+1)\frac{\pi c t}{L}\right]$$

Different Boundary conditions & x-domains

Solve $\frac{z_{tt}}{c^2} = z_{xx}$ with $x \in [-L, L]$ and bc $\frac{\partial z}{\partial x} = 0$ at $x = \pm L$



Look for a solution $z(x,t) = X(x)T(t)$ & we require $z_x(\pm L, t) = 0 = X'(\pm L)T(t)$ i.e $X'(\pm L) = 0$

$$\frac{T''X}{c^2} = X''T \quad , \quad \frac{T''}{c^2 T} = \frac{X''}{X} = \lambda$$

λ +ive then we have exponential solutions in X $X'' - \lambda X = 0$ & the homogenous boundary conditions $X'(\pm L) = 0$ cannot be satisfied if $\lambda=0$, $\lambda''=0$ & $X=Ax+B$ and a solution with $X'(\pm L)=0$ is just $X=\text{const}$

In this case $T''=0$ & $T=At+B$. So the zero separation constant generates solution
$z(x,t)=X(x)T(t)=At+B$

λ -ve if $\lambda=-p^2$ we have $X''+p^2X=0$
& $X(x)=A\cos px+B\sin px$

& we need to find p so that $X'(L)=0$
$X'(-L)=0$
and not both of A & B are zero.

$X'(x)=-pA\sin px+Bp\cos px$

& $X'(L)=-pA\sin pL+Bp\cos pL=0$

$X'(-L)=-pA\sin(-pL)+Bp\cos(-pL)=0$

$=pA\sin pL+Bp\cos pL=0$

Write this as $\begin{pmatrix}-\sin pL & \cos pL\\ \sin pL & \cos pL\end{pmatrix}\begin{pmatrix}Ap\\ Bp\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix}$

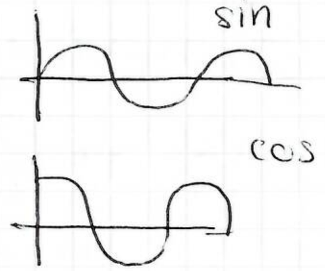
Unless the determinant of this matrix is zero, then we have the zero solution $A=B=0$. An alternative, non-zero solution is possible if the determinant is zero

$-\sin(pL)\cos(pL)-\sin(pL)\cos(pL)=0$
i.e. $2\sin(pL)\cos(pL)=0$

$\sin(2pL)=0$ i.e. $2pL=n\pi$ $\quad p=\frac{n\pi}{2L}$ $n=1,2,\ldots$

With $p=\frac{n\pi}{2L}$ we have

$$\begin{pmatrix}-\sin(\frac{n\pi}{2}) & \cos(\frac{n\pi}{2})\\ \sin(\frac{n\pi}{2}) & \cos(\frac{n\pi}{2})\end{pmatrix}\begin{pmatrix}Ap\\ Bp\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix}$$

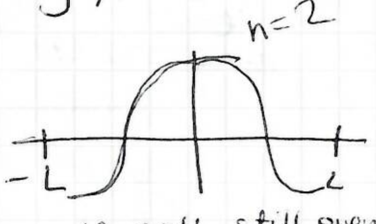


If n is even, $n=2,4,6$

$\begin{pmatrix}0 & \pm1\\ 0 & \pm1\end{pmatrix}\begin{pmatrix}Ap\\ Bp\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix}$ i.e. A is anything, $B=0$

$X(x)=A\cos\left(\frac{n\pi x}{2L}\right)$ n even

$T(t)=C\cos\left(\frac{n\pi ct}{2L}\right)+D\sin\left(\frac{n\pi ct}{2L}\right)$

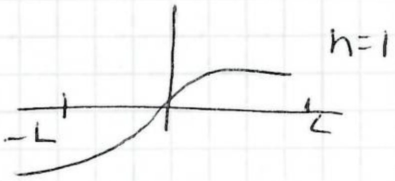


if $n=4$, still even but with extra wobbles

These are normal modes even in X

if n is odd

$$\begin{pmatrix} \mp 1 & 0 \\ \pm 1 & 0 \end{pmatrix}\begin{pmatrix} A_p \\ B_p \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \begin{matrix} B \text{ is anything,} \\ A \text{ is zero} \end{matrix}$$



$$X(x) = B\sin\left(\frac{n\pi x}{2L}\right) \quad n \text{ odd}$$

$$T(t) = C\cos\left(\frac{n\pi c t}{2L}\right) + D\sin\left(\frac{n\pi c t}{2L}\right)$$

odd normal modes

So the general solution is (without initial conditions, but with ...

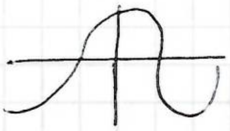
$$z(x,t) = \underbrace{(A_0 t + B_0)}_{\text{zero separation constant}} + \sum_{\substack{j=0 \\ n=2j+1}}^{\infty} \underbrace{\sin\left((2j+1)\frac{\pi x}{2L}\right)}_{\text{odd in } x}\left[C_j\cos\left((2j+1)\frac{\pi c t}{2L}\right) + D_j\sin\left((2j+1)\frac{\pi c t}{2L}\right)\right]$$

$$+ \sum_{\substack{j=1 \\ n=2j}}^{\infty} \underbrace{\cos\left(\frac{j\pi x}{L}\right)}_{\text{even in } x}\left(E_j\cos\left(\frac{j\pi c t}{L}\right) + F_j\sin\left(\frac{j\pi c t}{L}\right)\right)$$

Initial conditions ① $\dot{z} = 0$, then we know D_j & $F_j = 0$, $A_0 = 0$

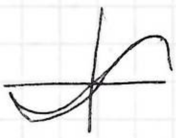
② if $z = 0$, then C_j & $E_j = 0$, $B_0 = 0$

① if z = even



, $C_j = 0$

if z = odd



$E_j = 0$, $B_0 = 0$

② if z is even, $D_j = 0$

z is odd $F_j = 0$ $A_0 = 0$

07/12/2011

The heat/diffusion equation

$$\frac{\partial\theta}{\partial t} = k\frac{\partial^2\theta}{\partial x^2} \quad \text{for } \theta(x,t)$$

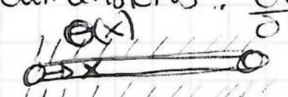
θ - temperature (heat) - density (diffusion)

$t \to$ time

$x \to$ 1 dimensional space

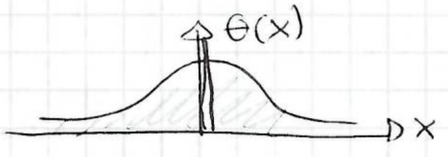
k - constant, the thermal diffusion or diffusivity

(3 dimensions: $\frac{\partial\theta}{\partial t} = k\nabla^2\theta$)

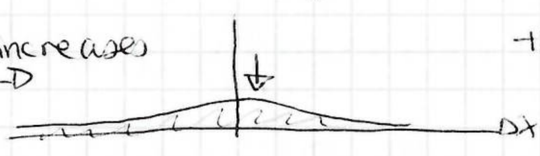


long/thin metal rod

insulation - so the heat will move along the rod

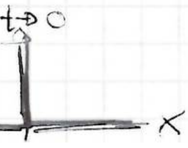


t increases →



the area remains constant, just flatter and more spread out

(heat term)



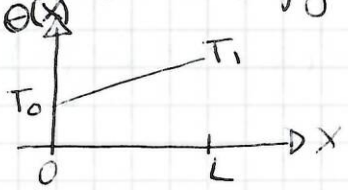
Typical boundary conditions on a rod of finite lenght say $x \in [0, L]$
are. $\Theta(0) = T_0$, $\Theta(L) = T_1$

or we could impose insulating boundary conditions
$\frac{\partial \Theta}{\partial x} = 0$ at $x = 0$ say

[Robin boundary condition
$\frac{\partial \Theta}{\partial x} = \alpha \Theta$]

we can look for steady solutions, $\frac{\partial \Theta}{\partial t} = 0$

These satisfy $\Theta_{xx} = 0$ ie $\Theta = Ax + B$, a linear function of x



A steady solution satisfying the boundary condition
$\Theta_s(0) = T_0$, $\Theta_s(L) = T_1$
is $\Theta_s = T_0 + (T_1 - T_0)\frac{x}{L}$

If one boundary condition is insulating, ie asks for $\frac{\partial \Theta}{\partial x} = 0$, then the steady solution requires

$A = 0$ & $\Theta_s = B$, with B obtained, maybe, from the other boundary condition.

We will look for time dependent solutions of the form
$\Theta(x,t) = X(x)\,T(t)$

$$XT' = k X'' T$$

$$\Rightarrow \frac{T'}{kT} = \frac{X''}{X} = \text{const} = \lambda$$, the separation constant

$\underline{\lambda = 0}$ gives $T' = 0$ $X'' = 0$ ie $T = \text{const}$ $X = Ax + B$

$XT = Ax + B$, the steady solution

$\underline{\lambda > 0}$
$T' = k\lambda T$, $T = Ae^{k\lambda t}$
and this grows in time if λ is positive, which is unrealistic

$X'' - \lambda X = 0$ which has exponential solutions, and not trigonometric & we cannot satisfy homogeneous bc if $\lambda > 0$

$\underline{\lambda < 0}$ $\lambda = -p^2$ $X'' + p^2 X = 0$ $\Rightarrow X = A\sin px + B\cos px$
$T' = -p^2 k T$

& find P from spatial boundary conditions (if these are homogeneous)

$T = Ae^{-P^2 kt}$

General solution:

$$\Theta(x,t) = A_0 x + B_0 + \sum_P (A_P \sin px + B_P \cos px) e^{-P^2 kt}$$

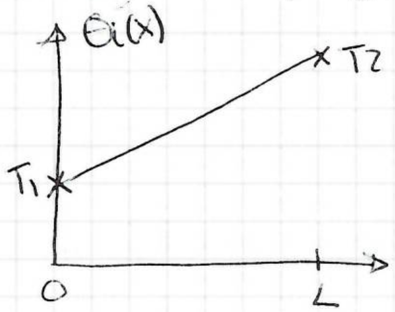
$(\lambda = 0)$

example

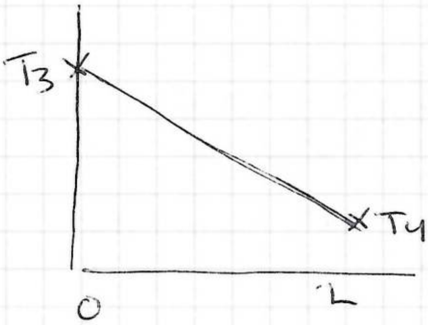
Solve the heat equation $\Theta_t = k\Theta_{xx}$ on the interval $x \in [0,L]$ with boundary conditions $\Theta(0) = T_3$, $\Theta(L) = T_4$ & initial conditions $\Theta = T_1 + (T_2 - T_1)\frac{x}{L}$ initial distribution of temp

$= \Theta(x,0) = \Theta_i(x)$

Θ initial of x



one end of rod in the fridge and one end in the oven, for long enough to get a steady solution



change the ends around

I "expect" that as $t \to \infty$ the solution to approach the steady solution

$$\Theta = \Theta_S = T_3 + (T_4 - T_3)\frac{x}{L}$$

(has nothing to do with initial conditions)

Note $\Theta_S(0) = T_3$ & $\Theta_S(L) = T_4$ i.e Θ_S satisfies our required boundary conditions.

We write $\Theta(x,t) = \underbrace{\Theta_S(x)}_{\text{steady}} + \underbrace{\Theta_U(x,t)}_{\text{unsteady, time dependant part}}$ - has nothing to do with initial conditions)

Θ_S satisfies $\Theta_{S_{xx}} = 0$ & the boundary conditions on Θ. if $\Theta(0) = T_3$, $\Theta_S(0) = T_3$

$\Theta(L) = T_4$, $\Theta_S(L) = T_4$

So putting $x = 0$

$$\Theta(0,t) = T_3 = \Theta_S(0) + \Theta_n(0,t) = T_3 + \Theta_U(0,t)$$

& $\Theta_U(0,t) = 0$ similarly $\Theta_U(L,t) = 0$

Since $\Theta_t = k\Theta_{xx}$

$$\underbrace{\cancel{\Theta_{St}}}_{=0} + \Theta_{Ut} = k(\underbrace{\cancel{\Theta_{Sxx}}}_{=0} + \Theta_{Uxx})$$

$\Theta_{Ut} = k\Theta_{Uxx}$ and $\Theta_U(0,t) = 0$ $\Theta_U(x,t) = 0$

and we use the method of separation of variables to find $\Theta_U(x,t)$

But what are the initial conditions for Θ_U?

$$\Theta(x,0) = \Theta_i(x) = \Theta_S(x) + \Theta_U(x,0)$$

so $\Theta_U(x,0) = \Theta_i(x) - \Theta_S(x)$

$\Theta_U(x,t) = X(x)T(t)$ 09/12/2011

$\frac{T'}{kT} = \frac{X''}{X} = \lambda$ - cannot have $\lambda > 0$ as this would lead to exponential solutions for X & we require $\Theta_U(0) = \Theta_U(L) = 0$ ie $X(0) = X(L) = 0$ which exponentials cannot satisfy

- cannot have $\lambda = 0$ as this gives $X = Ax + B$ & we cannot satisfy $X(0) = X(L) = 0$ with a non zero solution for X

- so $\lambda < 0$, $\lambda = -p^2$

& $X(x) = A\cos px + B\sin px$
$T(t) = e^{-p^2kt}$

our boundary conditions require
$X(0) = 0 \Rightarrow A = 0$
$X(L) = 0 \Rightarrow \sin pL = 0 \Rightarrow p = \frac{n\pi}{L}$, $n = 1, 2, 3 \ldots$

$$\Theta_U(x,t) = \sum_{n=1}^{\infty} A_n e^{-p^2kt} \sin pL$$

$$= \sum_{n=1}^{\infty} A_n e^{\frac{-n^2\pi^2kt}{L^2}} \sin\left(\frac{n\pi x}{L}\right)$$

Initially, ~~we~~ ie putting $t = 0$, we need
$\Theta_U(x,0) = \Theta_i(x) - \Theta_S(x)$

ie $\sum_1^{\infty} A_n \sin\left(\frac{n\pi x}{L}\right) = (T_1 - T_3) + \frac{x}{L}(T_2 - T_4 - T_1 + T_3)$

$= P + \frac{Qx}{L}$

multiply by $\sin\left(\frac{m\pi x}{L}\right)$ & $\int_0^L$ & find

$$A_m L \cdot \frac{1}{2} = \int_0^L \left(P + \frac{Qx}{L}\right) \sin\left(\frac{mpx}{L}\right) dx$$

$$= \left\{ \left[-\frac{L}{m\pi} \cos\left(\frac{m\pi x}{L}\right)\left(P + \frac{Qx}{L}\right)\right]_0^L + \underbrace{\frac{L}{m\pi}\int_0^L \cos\left(\frac{m\pi x}{L}\right)\frac{Q}{L}dx}_{=0 \text{ since } \sin(\ ) \text{ satisfies initial conditions}} \right\}$$

$$= P\frac{L}{m\pi}(1 - (-1)^m) - \frac{L}{m\pi}\frac{QL}{L}(-1)^m$$

$$= \frac{L}{m\pi}\left\{\underbrace{P(1-(-1)^m)}_{2 \text{ if } m \text{ odd}, \ 0 \text{ if } m \text{ even}} - Q(-1)^m\right\}$$

So $\Theta(x,t) = T_3 + (T_4 - T_3)\frac{x}{L} + \sum_{n=1}^{\infty} \frac{2}{n\pi}\left\{(T_1 - T_3)(1-(-1)^n) - (T_2 - T_4 + T_3 - T_1 \ldots \right.$ $(-1)^n$

$$\cdot e^{-\frac{n^2\pi^2 k t}{L^2}} \sin\left(\frac{n\pi x}{L}\right)$$

$\int_0^L \sin\left(\frac{m\pi x}{L}\right)\sin\left(\frac{n\pi x}{L}\right) dx$ only non-zero when $n=m$

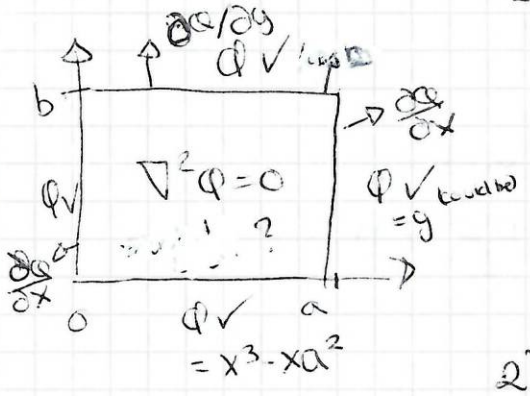
$\sum_1^{\infty} A_n \int_0^L \sin\frac{n\pi x}{L} \sin\frac{m\pi x}{L}\, dx = A_m \int_0^L \sin^2\frac{\pi m x}{L}\, dx$ - use double angle formula $= \frac{1}{2}L$

Laplace equation for $\varphi(x,y)$

This is the elliptic equation $\frac{\partial^2 \varphi}{\partial x^2} + \frac{\partial^2 \varphi}{\partial y^2} = 0$ in 2 dimensi…

(ie in several dimensions $[\underline{\nabla} \cdot \underline{\nabla}\varphi = 0]$) $\qquad y'' = 0$

$\nabla^2 \varphi = 0$



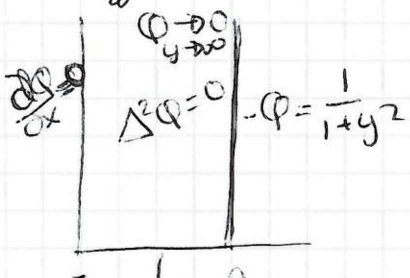
Boundary conditions for this problem are typically

1) φ is specified on the boundary
 - Dirichlet boundary condition

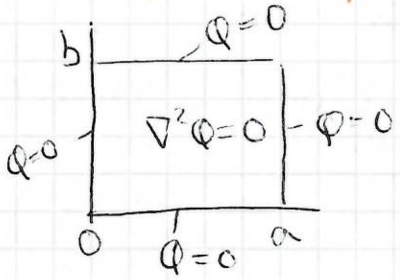
2) $\frac{\partial\varphi}{\partial n}$ is specified eg $\frac{\partial\varphi}{\partial n} = 0$

3) Robin condition $\varphi + \beta\frac{\partial\varphi}{\partial n} = 0$

These boundary conditions could be different on different boundaries



Laplace's equation in a rectangular domain



Solve $\nabla^2\varphi = 0$ in the domain $0 \le x \le a$, $0 \le y \le b$

with $\varphi(x,0) = 0$, $\varphi(0,y) = 0$, $\varphi(x,b) = h(x)$, $\varphi(a,y) = 0$

Look for a solution with $\varphi(x,y) = X(x)Y(y)$

Methods 09/12/2011

Since $\varphi(x,0)=0$, $X(x)Y(0)=0$, $Y(0)=0$

$\varphi(0,y)=0$, $X(0)Y(y)=0$, $X(0)=0$

$\varphi(a,y)=0$ $X(a)Y(y)=0$ $X(a)=0$

$\cancel{\varphi(x,b)=h(x)\quad X(x)Y(b)=h(x)}$ Look at later

$$\frac{\partial^2\varphi}{\partial x^2}+\frac{\partial^2\varphi}{\partial y^2}=0$$

$$X''Y+XY''=0$$

$$\frac{X''}{X}=-\frac{Y''}{Y}=\lambda,\text{ a separation constant}$$

$$X''-\lambda X=0$$

$$Y''+\lambda Y=0$$

We can have $\lambda>0$ giving exponentials in x and trigonometric functions in y, or $\lambda<0$ giving trigonometric functions in x and exponentials in y, or $\lambda=0$, $X''=0$, $Y''=0$ ie linear functions in X & Y

& ignoring the influence of boundary conditions the general solution is a combination of all these possibilities

Considering these & especially the fact that $X=0$ at both $x=0$ & $x=a$ implies we must restrict ourselves to $\lambda<0$.

with $\lambda=-p^2$ $\quad X''+p^2X=0$

$$Y''-p^2Y=0$$

$$X_p(x)=A\sin px+B\cos px$$

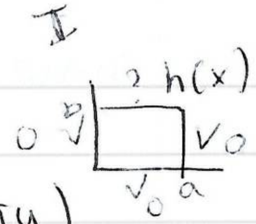
choose cosh and sinh when you have a finite range

$$Y_p(y)=C\sinh py+D\cosh py$$

Applying $X(0)=0$

same p

$$X(x)=A_n\sin\left(\frac{n\pi x}{a}\right),\quad Y(y)=C_n\sinh\left(\frac{n\pi y}{a}\right)$$



and generally $\varphi = \sum_n XY$

$$\varphi(x,y) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi x}{a}\right) \sinh\left(\frac{n\pi y}{a}\right)$$

A_n's are found so that $\varphi(x,b) = h(x)$

i.e $\varphi(x,b) = h(x) = \sum_{n=1}^{\infty} A_n \left(\sin\left(\frac{n\pi x}{a}\right) \sinh\left(\frac{n\pi b}{a}\right)\right)$

multiply by $\sin\left(\frac{m\pi x}{a}\right)$ & $\int_0^a$

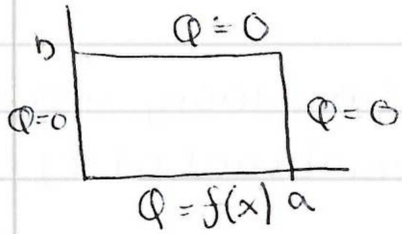
$$\int_0^a \sin\left(\frac{m\pi x}{a}\right) h(x)\, dx = A_m \sinh\left(\frac{m\pi b}{a}\right) \frac{1}{2} \cdot a$$

$$A_m = \frac{2 \int_0^a \sin\left(\frac{m\pi x}{a}\right) h(x)\, dx}{a \sinh\left(\frac{m\pi b}{a}\right)}$$

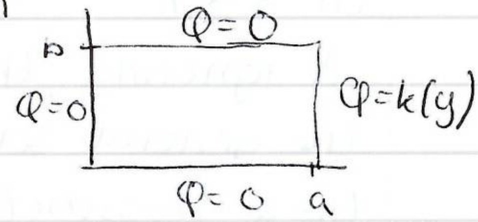
$$\& \quad \varphi(x,y) = \frac{2}{a}\sum_{n=1}^{\infty} \sin\left(\frac{n\pi x}{a}\right) \frac{\sinh\left(\frac{n\pi y}{a}\right)}{\sinh\left(\frac{n\pi b}{a}\right)} \int_0^a \sin\left(\frac{n\pi x}{a}\right) h(x)\, dx$$

Problem II

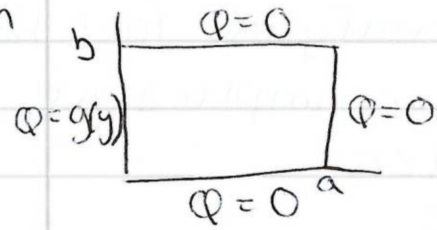
Other problems might be



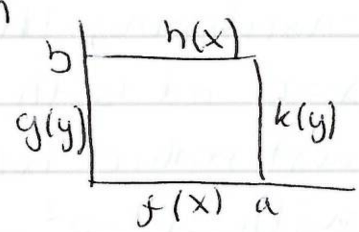
Problem III



Problem IV



Problem V



III: trig in y
sin at x = 0

$$\sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi y}{b}\right) \cdot \sinh\left(\frac{n\pi x}{b}\right)$$

IV : $X(x) = \sin\left(\frac{n\pi x}{a}\right)$

$Y(y) = C\sinh\left(\frac{n\pi y}{a}\right) + D\cosh\left(\frac{n\pi y}{a}\right)$

& $\varphi(x,b) = 0 \Rightarrow Y(b) = 0$

$C\sinh\left(\frac{n\pi b}{a}\right) + D\cosh\left(\frac{n\pi b}{a}\right) =$

[Trig in x
Exp in y

II cont: $D = -C\tanh\left(\frac{n\pi b}{a}\right)$

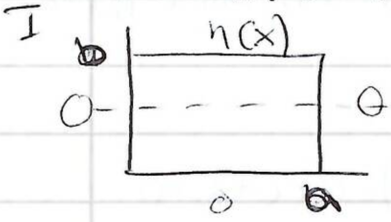
$$Y(y) = C\left(\sinh\left(\frac{n\pi y}{a}\right) - \tanh\left(\frac{n\pi b}{a}\right)\cosh\left(\frac{n\pi y}{a}\right)\right)$$

$$\varphi(x,y) = \sum_{n=1}^{\infty} C_n \sin\left(\frac{n\pi x}{a}\right)\left(\sinh\left(\frac{n\pi y}{a}\right) - \tanh\left(\frac{n\pi b}{a}\right)\cdot\cosh\left(\frac{n\pi y}{a}\right)\right)$$

C_n's found so that $\varphi(x,0) = f(x)$

$$f(x) = \sum_{n=1}^{\infty} C_n \sin\left(\frac{n\pi x}{a}\right)\left(-\tanh\left(\frac{n\pi b}{a}\right)\right)$$

~~we can turn I into II be reflecting I~~



Problem II is obtained from I by reflecting in the line $y = \frac{b}{2}$ ie $y \to b-y$ and $\frac{\partial^2\varphi}{\partial y^2} + \frac{\partial^2\varphi}{\partial x^2} = 0$

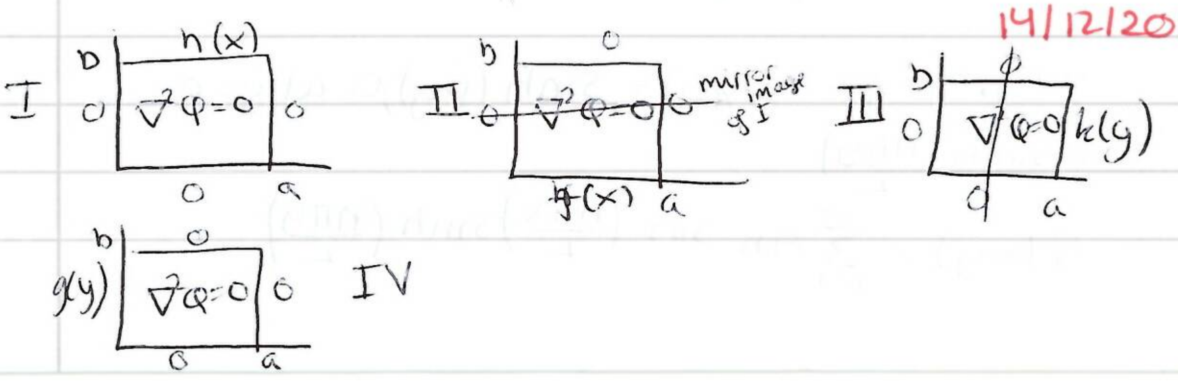
is unaltered under this transformation, at it has a second order derivative, only, in y
So the solution for II is found from I by replacing y by b−y

$$\varphi(x,y) = \frac{2}{a}\sum_{n=1}^{\infty} \frac{\sin\left(\frac{n\pi x}{a}\right)\sinh\left[\frac{n\pi(b-y)}{a}\right]}{\sinh\left(\frac{n\pi b}{a}\right)} \int_0^a \sin\left(\frac{n\pi x}{a}\right) dx$$

III can be mapped to I by reflecting in the diagonals of the rectangle, effected by $y \to x$, $x \to y$, $a \to b$, $b \to a$

$$\varphi(x,y) = \frac{2}{b}\sum_{n=1}^{\infty} \sin\left(\frac{n\pi y}{b}\right)\frac{\sinh\left(\frac{n\pi x}{b}\right)}{\sinh\left(\frac{n\pi a}{b}\right)} \int_0^b \sin\left(\frac{n\pi y}{b}\right) k(y)\, dy$$

14/12/2011



$$\text{I}: \varphi_{\text{I}} = \sum_{n=1}^{\infty} \sin\left(\frac{n\pi x}{a}\right) \sinh\left(\frac{n\pi y}{a}\right) \frac{2}{a} \frac{1}{\sinh\left(\frac{n\pi b}{a}\right)} \int_0^a \sin\left(\frac{n\pi x}{a}\right) h(x)\,dx$$

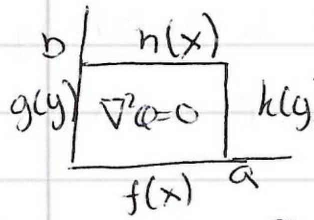
$$\text{II} \quad \varphi_{\text{II}} = \sum_{1}^{\infty} \quad - \, '' \, - \quad \sin\left(\frac{n\pi(y-a)}{a}\right) \quad - \, '' \quad \text{———} \quad f(x)\,dx$$

III, $a \to b$, $x \to y$

$$\varphi_{\text{III}} = \sum_{1}^{\infty} \sin\left(\frac{n\pi y}{b}\right) \sinh\left(\frac{n\pi x}{b}\right) \frac{2}{b} \frac{1}{\sinh\left(\frac{n\pi a}{b}\right)} \int_0^b \sin\left(\frac{n\pi y}{b}\right) k(y)\,dy$$

IV from III, $x \to a-x$

$$\varphi_{\text{IV}} = \sum_{1}^{\infty} \sin\left(\frac{n\pi y}{b}\right) \sinh\left(\frac{n\pi (a-x)}{b}\right) \frac{2}{b} \frac{1}{\sinh\left(\frac{n\pi a}{b}\right)} \int_0^b \sin\left(\frac{n\pi y}{b}\right) \cdot g(y)\,dy$$



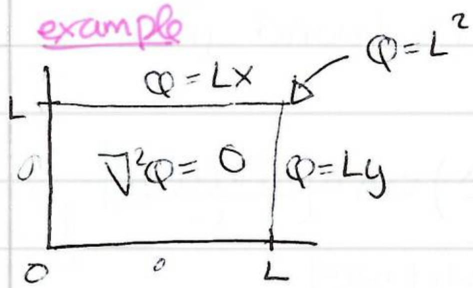
$$\nabla^2(\varphi_{\text{I}} + \varphi_{\text{II}} + \varphi_{\text{III}} + \varphi_{\text{IV}}) = 0 + 0 + 0 + 0$$

& for example on $x = a$

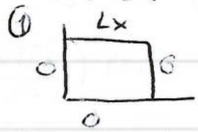
$$\varphi_1(a,y) + \varphi_{\text{II}}(a,y) + \varphi_{\text{III}}(a,y) + \varphi_{\text{IV}}(a,y) = 0 + 0 + k(y) - 0$$

$= k(y)$ as required

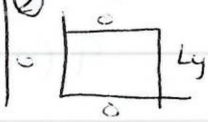
example



First solve:



Then solve:



① $\nabla^2 \varphi = 0$, $\varphi = X(x)Y(y)$

$$\frac{X''}{X} = -\frac{Y''}{Y} = \text{const} = -p^2 \quad \text{as we want trig in } x$$

(don't want cos since 0 at $x=0$)

$X'' + p^2 X = 0 \Rightarrow X(x) = \sin px$ 0 at $x = 0$

with $pL = n\pi$ 0 at $x = L$

$$X(x) = \sin\left(\frac{n\pi x}{L}\right)$$

$Y'' - p^2 Y = 0 \Rightarrow Y(y) = \sinh(py)$, 0 at $y = 0$

$$= \sinh\left(\frac{n\pi y}{L}\right)$$

$$\varphi(x,y) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi x}{L}\right) \sinh\left(\frac{n\pi y}{L}\right)$$

$$\& \quad \varphi(x,L) = Lx = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi x}{L}\right)\sinh(n\pi)$$

requiring

$$\int_0^L Lx \sin\left(\frac{m\pi x}{L}\right)dx = A_m \sinh(m\pi)\underbrace{\int_0^L \sin^2\left(\frac{m\pi x}{L}\right)dx}_{=L/2}$$

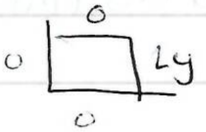
$$A_n = \frac{2}{L}\cdot\frac{1}{\sinh(n\pi)}\cdot L\int_0^L x\sin\left(\frac{n\pi x}{L}\right)dx$$

$$= \frac{2}{\sinh(n\pi)}\left\{\left[x\frac{L}{n\pi}(-1)\cos\left(\frac{n\pi x}{L}\right)\right]_0^L + \cancel{\int_0^L \frac{L}{n\pi}\cos\left(\frac{n\pi x}{L}\right)dx}_{=0}\right.$$

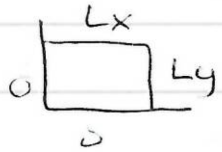
$$= \frac{-2}{\sinh(n\pi)}\frac{L^2}{n\pi}\cos(n\pi).$$

$$A_n = \frac{2\cdot(-1)^{n+1}}{n\pi\sinh(n\pi)}L^2$$

$$\varphi(x,y) = \sum_{n=1}^{\infty}\frac{2L^2}{n\pi\cdot\sinh(n\pi)}(-1)^{n+1}\cdot\sin\left(\frac{n\pi x}{L}\right)\sinh\left(\frac{n\pi y}{L}\right)$$



The solution for this second problem is obtained by putting $x \to y$, $y \to x$



has solution $\varphi(x,y) = \sum_{n=1}^{\infty}\frac{2L^2(-1)^{n+1}}{n\pi\sinh(n\pi)}$

$$\left[\sin\left(\frac{n\pi x}{L}\right)\sinh\left(\frac{n\pi y}{L}\right) + \sin\left(\frac{n\pi y}{L}\right)\sinh\left(\frac{n\pi x}{L}\right)\right]$$

*

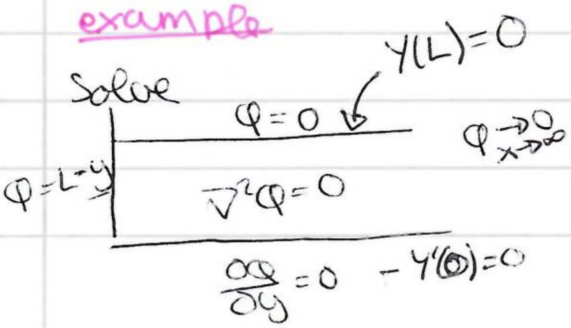
//

The solution to this problem is in fact $\varphi = xy$
[satisfies bc's & $\nabla^2\varphi = 0$]

$* = xy$

example

Solve

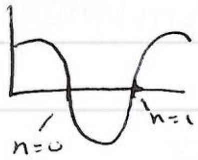


$$\varphi(x,y) = X(x)Y(y)$$

$$\frac{X''}{X} = \frac{-Y''}{Y} = \text{const} = p^2$$

trigonometric in y

To satisfy $Y'(0)=0$ we take the solution $Y(y)=\cos py$
and to satisfy $Y(L)=0$ we require $\cos pL = 0$,



requiring $pL=(n+\frac{1}{2})\pi$

$$p=(n+\tfrac{1}{2})\frac{\pi}{L}$$

$X''-p^2X=0$ & so X has solutions e^{-px}, e^{px} $(X=Ae^{-px}+Be^{px})$
& for $X(x)\to 0$ as $x\to\infty$ so that
$\varphi(x,y)\to 0$ as $x\to\infty$, we must take only e^{-px}

$$\varphi(x,y)=\sum_{n=0}^{\infty} A_n \cos\left((n+\tfrac{1}{2})\frac{\pi y}{L}\right)e^{-(n+\frac{1}{2})\frac{\pi x}{L}}$$

& we require $\varphi(0,y)=L-y$

$$L-y=\sum_{n=0}^{\infty} A_n \cos\left[(n+\tfrac{1}{2})\frac{\pi y}{L}\right]$$

$\therefore$ ($X''-p^2X=0$ homogeneous bc)

Using the fact that $\cos\left((n+\frac{1}{2})\frac{\pi y}{L}\right)$ & $\cos\left((m+\frac{1}{2})\frac{\pi y}{L}\right)$
are orthogonal, we find

$$\int_0^L (L-y)\cos\left((m+\tfrac{1}{2})\frac{\pi y}{L}\right)dy = A_m \frac{1}{2}L$$