2401 Mathematical Methods 3 Notes

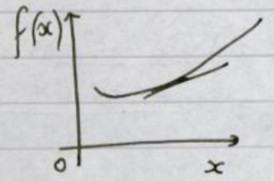
Based on the 2011 autumn lectures by Dr R I Bowles

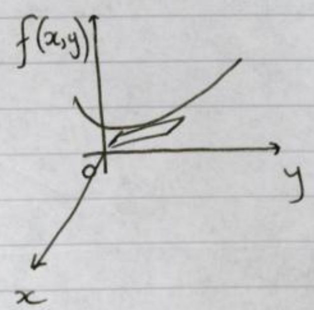
The Author has made every effort to copy down all the content on the board during lectures. The Author accepts no responsibility what so ever for mistakes on the notes nor changes to the syllabus for the current year. The Author highly recommends that reader attends all lectures, making their own notes and to use this document as a reference only

Methods

Differentiability $\approx$ locally linear

$$f(a+h) \approx f(a) + hf'(a)$$





We ~~seen~~ say $f'(a)$ exists if

$$f(a+h) = f(a) + hf'(a) + |h|\phi(h)$$

where $\phi(h) \to 0$ as $|h| \to 0$

in two dimensions $\mathbb{R}^2 \to \mathbb{R}$ the function $f(x,y)$ is differentiable at the point (a,b) if $\exists$ two numbers L_x, L_y such that $f(\cancel{x}\, a+h, b+k) = f(a,b) + L_x h + L_y k + \sqrt{h^2+k^2}\,\phi(h,k)$

where $\phi(h,k) \to 0$ as $h \to 0$ and $k \to 0$

We can write this differently, (more generally)

~~the say~~ $\cancel{f(\underline{x})}$

We say $f(\underline{x} + \underline{\delta x}) = f(\underline{x}) + \underline{L} \cdot \underline{\delta x} + |\underline{\delta x}|\,\phi(\underline{\delta x})$ and claim that f is differentiable at $\underline{x}$ if such $\underline{L}$ exists and $|\phi(\underline{\delta x})| \to 0$ as $|\underline{\delta x}| \to 0$

Consider now $\mathbb{R}^3 \to \mathbb{R}$, then it turns out $\underline{L} = \nabla f = \operatorname{grad} f$

since e.g. $\dfrac{df}{dx} = \lim_{h \to 0} \dfrac{f(x+h,y,z) - f(x,y,z)}{h}$

$$= \lim_{h \to 0} \left(\frac{L_x h + 0 + 0 + |h|\,\phi(h)}{h} \right) = L_x \quad \text{since } \phi(h) \to 0 \text{ as } h \to 0$$

So $\underline{L} = \begin{pmatrix} df/dx \\ df/dy \\ df/dz \end{pmatrix} = \underline{\nabla} f$

Consider now functions $\mathbb{R}^m \to \mathbb{R}^n$

we say f is differentiable at point $\underline{x}$ (vector in $\mathbb{R}^m$) if $\exists$ a $n \times m$ matrix $\underline{\underline{L}}$: $\underline{f}(\underline{x} + \underline{h}) = \underline{f}(\underline{x}) + \underline{\underline{L}}\,\underline{h} + |\underline{h}|\,\underline{\phi}(\underline{h})$

with $|\underline{\phi}(\underline{h})| \to 0$ as $|\underline{h}| \to 0$

Where $\underline{\underline{L}}$ is :

$$
\begin{array}{cc}
n \updownarrow & \begin{pmatrix} \delta f_1/\delta x_1 & \delta f_1/\delta x_2 & \cdots & \delta f_1/\delta x_m \\ \delta f_2/\delta x_1 & & & \\ \vdots & \cdots & \delta f_i/\delta x_j & \\ \delta f_n/\delta x_1 & & & \delta f_n/\delta x_m \end{pmatrix} \\
 & \xleftrightarrow[\quad m \quad]{}
\end{array}
$$

<u>This is the Jacobian</u>

$$L_{ij} = \frac{\delta f_i}{\delta x_i}$$

<u>The chain rule</u>

Consider functions formed by the composition of others.

In one dimension we might consider $F(t) = f(x(t))$.

We have $t \to x \to F$, $\mathbb{R} \to \mathbb{R} \to \mathbb{R}$.

Consider the change in F caused by a change in t.

$$F(t+\delta t) - F(t) = f(x(t+\delta t)) - f(x(t))$$

$$= f\left(x(t) + \delta t\, x'(t) + |\delta t|\,\phi(\delta t) + \ldots\right) - f(x(t))$$ as $x(t)$ is differentiable

$$= f(x(t)) + f'(x(t))\,\delta t\, x'(t) + \ldots - f(x(t))$$

$$= \delta t\, f'(x(t))\,x'(t)$$

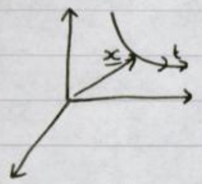
Compare this now with the statement that $F(t)$ is differentiable, in the form

$$F(t+\delta t) = F(t) + \delta t\, F'(t)$$

$$\therefore\ F'(t) = f'(x(t))\,x'(t)$$

New Office
Writer

$R \to \mathbb{R}^3 \to R$



$\underline{x}(t)$ $\quad f(\underline{x}(t))$
$\underline{r}(t)$

Consider $\quad F(t) = f(x(t), y(t), z(t))$

$$F(t+\delta t) - F(t) = f(x(t+\delta t), y(t+\delta t), z(t+\delta t)) - f(x(t), y(t), z(t))$$

$$= f(x(t) + \delta t\, x'(t) + \dots, y(t) + \delta t\, y'(t) + \dots, z(t) + \delta t\, z'(t) + \dots)$$

$$- f(x(t), y(t), z(t))$$

$f(\underline{x}+\underline{h}) \approx f(\underline{x})$

$$+ \quad \underline{\nabla} f \cdot \underline{h}$$

$$\approx f(x(t), y(t), z(t)) + \frac{df}{dx}\delta t\, x'(t) + \frac{df}{dy} dt\, y'(t) + \frac{\delta f}{\delta z}\delta t\, z'(t)$$

$$+ \dots - f(x(t), y(t), z(t))$$

So we identify $\quad F'(t) = \frac{\delta f}{dx} x'(t) + \frac{df}{dy} y'(t) + \frac{df}{dz} z'(t)$

$\therefore$ If we use a $\cdot$ for time derivatives and $\underline{r}(t)$

instead of $x(t)$,

$$\dot{F} = \dot{\underline{r}} \cdot \underline{\nabla} f \quad = \underbrace{\begin{pmatrix} \frac{df}{dx} & \frac{df}{dy} & \frac{df}{dz} \end{pmatrix}}_{\text{Jacobian for } f(\underline{x})} \left.\begin{pmatrix} \frac{dx}{dt} \\ \frac{dy}{dt} \\ \frac{dz}{dt} \end{pmatrix}\right\} \text{Jacobian for } \underline{x}(t)$$

$F(t) = f(\underline{x}(t))$

~~The chain rule~~

The Jacobian for f is obtained by matrix multiplication i.e. composition of the Jacobians for $\underline{x}(t)$ and $f(\underline{x})$.

The Jacobian of the composition is the composition of the Jacobians.

We can generalise this observation.

$$\mathbb{R}^l \longrightarrow \mathbb{R}^m \longrightarrow \mathbb{R}^n$$

If the two functions $\underline{x}(\underline{u})$ and $f(\underline{x})$ are from $\mathbb{R}^l \to \mathbb{R}^m$ and $\mathbb{R}^m \to \mathbb{R}^n$ respectively then the composition $\underline{F}(\underline{u}) = f(\underline{x}(\underline{u}))$ is from $\mathbb{R}^l \to \mathbb{R}^n$

$\mathbb{R}^l \to \mathbb{R}^m \to \mathbb{R}^n$. These mappings $\underline{x}, f, \underline{F}$ have Jacobians :

$$\underline{x} : \begin{pmatrix} \frac{dx_1}{du_1} & \frac{dx_1}{du_2} & \cdots & \frac{dx_1}{du_l} \\ \vdots & & & \\ \frac{dx_n}{du_1} & \cdots & & \frac{dx_n}{du_l} \end{pmatrix} = \underline{\underline{U}} \quad m \times l \text{ matrix}$$

$$f : \begin{pmatrix} \frac{df_1}{dx_1} & \cdots & \frac{df_1}{dx_m} \\ \vdots & & \\ \frac{df_n}{dx_1} & & \frac{df_n}{dx_m} \end{pmatrix} = \underline{\underline{T}} \quad \text{an } n \times m \text{ matrix}$$

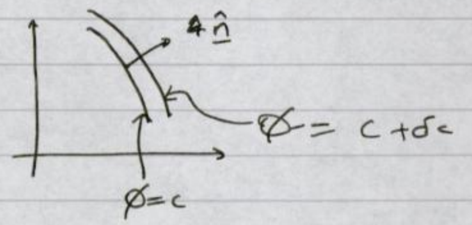
$$\underline{F} : \begin{pmatrix} \frac{dF_1}{du_1} & \cdots & \frac{dF_1}{du_l} \\ \vdots & & \vdots \\ \frac{dF_n}{du_1} & & \frac{dF_n}{du_2} \end{pmatrix} = \underline{\underline{S}} \quad \text{an } n \times l \text{ matrix}$$

Chain rule says $\underline{\underline{S}} = \underline{\underline{T}}\,\underline{\underline{U}}$

A geometric interpretation of the gradient ∇f

Consider $\phi(\underline{x}) = \phi(x,y,z)$ with x,y,z independent variables.

If x,y,z are chosen such that $\phi(x,y,z) = \text{constant} = c$ then this imposes a constraint on our choice of points (x,y,z) satisfying $\phi(x,y,z) = c$ lie on a surface, called a level surface of ϕ.



Consider a neighbouring surface given by $\phi = c + \delta c$.
Consider too a unit normal, $\hat{\underline{n}}$ to the surface. $\phi = c$

We ask how much ϕ changes if we move a distance δn in the direction of $\hat{\underline{n}}$. Call this change $\delta\phi$ and consider $\lim_{\delta n \to 0} : \frac{\delta\phi}{\delta n} = \frac{\partial\phi}{\partial n}$ i.e. the rate of change of ϕ measured in a direction normal to a level surface.

We define the gradient of the function ϕ to be the vector

$\underline{\nabla}\phi = \hat{\underline{n}} \frac{\partial\phi}{\partial n}$. We shall see $\underline{\nabla}\phi = \begin{pmatrix} \phi_x \\ \phi_y \\ \phi_z \end{pmatrix}$

$$\frac{d\phi}{ds} = \lim_{\delta s \to 0} \frac{\delta\phi}{\delta s} = \lim_{\delta s \to 0} \frac{\delta c}{\delta n / \cos\theta} = \cos\theta \lim_{\delta n \to 0} \frac{\delta\phi}{\delta n} = \cos\theta \frac{\partial\phi}{\partial n}$$

But $\cos\theta = \hat{\underline{s}} \cdot \hat{\underline{n}}$, so $\frac{\delta\phi}{ds} = \hat{\underline{s}} \cdot \hat{\underline{n}} \frac{d\phi}{dn} = \hat{\underline{s}} \cdot \underline{\nabla}\phi$

We call $\frac{d\phi}{ds}$ the directional derivative in the direction of $\hat{\underline{s}}$

and we see $\frac{d\phi}{ds} = \hat{\underline{s}} \cdot \underline{\nabla}\phi$

If we choose $\hat{\underline{s}} = i$, this becomes $\frac{d\phi}{dx} = \underline{i} \cdot \underline{\nabla}\phi$

= the first component of $\nabla\phi$

etc..

So we conclude that the 3 components of

$$\underline{\nabla}\phi = \begin{pmatrix} \phi_x \\ \phi_y \\ \phi_z \end{pmatrix}$$

Methods 3

$\frac{df}{ds} = \hat{\underline{s}} \cdot \underline{\nabla} f$ Defined at a point, but that point can be anything. So $\frac{df}{ds}$ is a function of position, assuming it exists. Consider $f(x,y) = (x+1)(y-1)$ and find the directional derivatives in the directions of $\underline{i}+\underline{j}$ and $\underline{i}-\underline{j}$. Unit vectors in these directions are: $\frac{1}{\sqrt{2}}(\underline{i}+\underline{j})$ and $\frac{1}{\sqrt{2}}(\underline{i}-\underline{j})$.

$$\underline{\nabla} f = \begin{pmatrix} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \end{pmatrix} = \begin{pmatrix} y-1 \\ x+1 \end{pmatrix}$$ and so if $\underline{s} = \underline{i} + \underline{j}$ we have:

$$\frac{df}{ds} = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix} \cdot \begin{pmatrix} y-1 \\ x+1 \end{pmatrix} = \frac{1}{\sqrt{2}}(x+y)$$

$s = \underline{i} - \underline{j}$ we have $$\frac{df}{ds} = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{pmatrix} \cdot \begin{pmatrix} y-1 \\ x+1 \end{pmatrix} = \frac{1}{\sqrt{2}}(y-x-2)$$

We can ask for the rate of change of these directional derivatives in different directions. Most useful will be

$$\frac{\partial}{\partial s}\left(\frac{\partial f}{\partial s}\right) = \hat{\underline{s}} \cdot \underline{\nabla}\left(\frac{\partial f}{\partial s}\right) = (\hat{\underline{s}} \cdot \underline{\nabla})(\hat{\underline{s}} \cdot \underline{\nabla} f) = (\hat{\underline{s}} \cdot \underline{\nabla})^2 f \text{ say}$$

For our example, the second derivatives are:

for $s = \underline{i} + \underline{j}$ $$\begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix} \cdot \begin{pmatrix} \frac{\partial}{\partial x}\left\{\frac{1}{\sqrt{2}}(x+y)\right\} \\ \frac{\partial}{\partial y}\left\{\frac{1}{\sqrt{2}}(x+y)\right\} \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix} \cdot \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix} = 1$$

$\underline{s} = \underline{i} - \underline{j}$ $$\begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix} \cdot \begin{pmatrix} \frac{\partial}{\partial x} \frac{1}{\sqrt{2}}(y-x-2) \\ \frac{\partial}{\partial y}(y-x-2) \end{pmatrix}$$

Need this for Taylor's theorem in several dimensions.

We know that $f: \mathbb{R} \to \mathbb{R}$ we have, under certain circumstances : ~~ff~~a

$$f(a+h) = f(a) + hf'(a) + \frac{1}{2}h^2 f''(a) + \dots = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)\,h^n}{n!}$$

with a radius of convergence.

What about $f(\underline{a}+\underline{h})$?

$= \underline{\nabla} f$ if just one number.

We know $f(\underline{a}+\underline{h}) = f(\underline{a}) + \underline{L} \cdot \underline{h} + \dots$

What next ?

To extend this and find the subsequent terms in the statement of Taylor's theorem for $f: \mathbb{R}^n \to \mathbb{R}$, we imagine fixing the direction of $\underline{h}$ (to be $\hat{\underline{s}}$ say) and the problem is then reduced to one in just one dimension, with variable $|\underline{h}|$, the distance travelled in the direction of $\underline{h}$ and we can use Taylors theorem for $f: \mathbb{R} \to \mathbb{R}$.

We see that the $f^n(a)$ need to be replaced by

$$\frac{d^n f}{ds^n} = (\hat{\underline{s}} \cdot \underline{\nabla})^n f$$

So we get $f(\underline{a}+\underline{h}) = f(\underline{a}) + |\underline{h}|(\hat{\underline{s}} \cdot \underline{\nabla}) f + \frac{1}{2}|\underline{h}|^2 (\hat{\underline{s}} \cdot \underline{\nabla})^2 f + \dots$

$$+ \frac{1}{n!}|\underline{h}|^n (\hat{\underline{s}} \cdot \underline{\nabla}^n) f + \dots$$

but $\underline{h} = |\underline{h}|\hat{\underline{s}}$

$\therefore$ it becomes $f(\underline{a}+\underline{h}) = f(\underline{a}) + (\underline{h} \cdot \underline{\nabla}) f + \frac{1}{2}(\underline{h} \cdot \underline{\nabla})^2 f + \dots + \frac{1}{n!}(\underline{h} \cdot \underline{\nabla})^n f$

$$\therefore f(\underline{a}+\underline{h}) = \sum_{n=0}^{\infty} \frac{(\underline{h} \cdot \underline{\nabla})^n}{n!} f$$

[Within some radius of convergence h]

not the modulus of h

If $f: \mathbb{R}^2 \to \mathbb{R}$ and $\underline{h} = \begin{pmatrix} h \\ k \end{pmatrix}$, then

$$\underline{h} \cdot \nabla f = \begin{pmatrix} h \\ k \end{pmatrix} \cdot \begin{pmatrix} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \end{pmatrix} = h\frac{\partial f}{\partial x} + k\frac{\partial f}{\partial y}$$

$$(\underline{h} \cdot \underline{\nabla})^2 f = \begin{pmatrix} h \\ k \end{pmatrix} \cdot \begin{pmatrix} \frac{\partial}{\partial x}\left(h\frac{\partial f}{\partial x} + k\frac{\partial f}{\partial y}\right) \\ \frac{\partial}{\partial y}\left(h\frac{\partial f}{\partial x} + k\frac{\partial f}{\partial y}\right) \end{pmatrix} = \begin{pmatrix} h \\ k \end{pmatrix} \cdot \begin{pmatrix} hf_{xx} + kf_{xy} \\ hf_{xy} + kf_{yy} \end{pmatrix}$$

$$= h^2 f_{xx} + 2hk f_{xy} + k^2 f_{yy}$$

Similarly,

$$(\underline{h} \cdot \underline{\nabla})^3 f = h^3 f_{xxx} + 3h^2 k f_{xxy} + 3hk^2 f_{xyy} + k^3 f_{yyy}.$$

If $f: \mathbb{R}^3 \to \mathbb{R}$, $h = \begin{pmatrix} h \\ k \\ l \end{pmatrix}$ then $\underline{h} \cdot \underline{\nabla} = h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y} + l\frac{\partial}{\partial z}$

$$\therefore (\underline{h} \cdot \underline{\nabla})^2 = h^2\frac{\partial^2}{\partial x^2} + k^2\frac{\partial^2}{\partial y^2} + l^2\frac{\partial^2}{\partial z^2} + 2hk\frac{\partial^2}{\partial x \partial y} + 2hl\frac{\partial^2}{\partial x \partial z} + 2kl\frac{\partial^2}{\partial y \partial z}$$

coefficients are found by considering $(a+b+c)^n$ etc.

Express $f(x,y) = x^2 y + 3y - 2$ in powers of $(x-1)$ and $(y+2)$. We will do this by finding Taylor series for $f(x,y)$ about the point $(1,-2)$

$$\left[\underline{a} = \begin{pmatrix} 1 \\ -2 \end{pmatrix} \text{ and } \underline{h} = \begin{pmatrix} h \\ k \end{pmatrix} = \begin{pmatrix} x-1 \\ y+2 \end{pmatrix} \therefore \begin{pmatrix} x \\ y \end{pmatrix} = \underline{a} + \underline{h}\right]$$

$$\frac{\partial f}{\partial x} = 2xy \quad \frac{\partial f}{\partial y} = x^2 + 3 \quad \frac{\partial^2 f}{\partial x^2} = 2y \quad \frac{\partial^2 f}{\partial y^2} = 0$$

$$\frac{\partial^2 f}{\partial x \partial y} = 2x \quad \frac{\partial^3 f}{\partial x^3} = 0 \quad \frac{\partial^3 f}{\partial x^2 \partial y} = 2 \quad \frac{\partial^3}{\partial x \partial y^2} = 0 \quad \frac{\partial^3 f}{\partial y^3} = 0.$$

And ~~a~~ higher derivatives are zero.

$$x^2y + 3y - 2 = f(x,y)$$

$$= f(1,-2) + \left(\overset{x-1}{h} \underset{-4}{\frac{\partial f}{\partial x}} + \overset{y+2}{k} \underset{4}{\frac{\partial f}{\partial y}} \right) \Bigg|_{(1,-2)}$$

$$+ \frac{1}{2}\left(h^2 f_{xx} + 2hk f_{xy} + k^2 . 0 \right) \Bigg|_{(1,-2)}$$

$$+ \frac{1}{6}\left(h^3 . 0 + 3h^2k \frac{\partial^3 f}{\partial x^2 \partial y} + 3hk^2 0 + k^3 . 0 \right) \Bigg|_{(1,-2)}$$

$$= -10 - 4(x-1) + 4(y+2) - 2(x-1)^2 + 2(x-1)(y+2)$$
$$+ (x-1)^2(y+2)$$

When you multiply all this out you get the original expression.

Examples:

(1) $s = \underline{i} + \underline{j}$

$$\hat{s} = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix} \cdot \underbrace{\begin{pmatrix} \frac{d}{dx}\left[\frac{1}{\sqrt{2}}(x+y)\right] \\ \frac{d}{dy}\left[\frac{1}{\sqrt{2}}(x+y)\right] \end{pmatrix}}_{\nabla f} = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix} \cdot \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix} = 1$$

(2) $s = \underline{i} - \underline{j}$

$$\begin{pmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{pmatrix} \cdot \begin{pmatrix} \frac{d}{dx}\left(\frac{1}{\sqrt{2}}(y-x-2)\right) \\ \frac{d}{dy}\left(\frac{1}{\sqrt{2}}(y-x-2)\right) \end{pmatrix} = -1$$

Taylor's Theorem

We know that for $f: \mathbb{R} \to \mathbb{R}$ we have, under certain conditions

$$f(a+h) = f(a) + hf'(a) + \frac{h^2}{2}f''(a) + \frac{h^3}{6}f'''(a) + \ldots = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!} h^n$$ with a radius of convergence.

$$f(\underline{a}+\underline{h}) = f(\underline{a}) + \underbrace{\overset{\text{function of } f}{\underline{L}}}_{\nabla f} \cdot \underline{h} + \ldots$$

To extend the expansion and find the subsequent terms in a statement of Taylor's Theorem of $f: \mathbb{R}^n \to \mathbb{R}$, we imagine f is in the direction of $\underline{h}$ (take $\hat{s}$ say) & the problem is then reduced to one in one dimension, with variable $|\underline{h}|$. The distance travelled in the direction of $\underline{h}$ & we can use Taylor's Theorem for $f: \mathbb{R} \to \mathbb{R}$

We see that the $f^n(a)$ need to be replaced by $\frac{d^n f}{ds^n} = (\hat{s} \cdot \nabla)^n f$ and the h needs to be replaced by $|\underline{h}|$

So we get: $f(\underline{a}+\underline{h}) = f(\underline{a}) + \overset{\text{length of } h}{|\underline{h}|} \overset{\text{direction of } h}{(\hat{s} \cdot \nabla)} f + \frac{1}{2}|\underline{h}|^2(\hat{s} \cdot \nabla)^2 f + \ldots + \frac{1}{n!}|\underline{h}|^n(\hat{s} \cdot \nabla)^n f + \ldots$

But $\underline{h} = |\underline{h}|\hat{s}$

$$\text{So, } f(\underline{a}+\underline{h}) = f(\underline{a}) + (\underline{h} \cdot \nabla)f + \frac{1}{2}(\underline{h} \cdot \nabla)^2 f + \ldots + \frac{1}{n!}(\underline{h} \cdot \nabla)^n f + \ldots = \sum_{n=0}^{\infty} \frac{(\underline{h} \cdot \nabla)f}{n!}$$

If $f: \mathbb{R}^2 \to \mathbb{R}$ $\quad \underline{h} = \begin{pmatrix} h \\ k \end{pmatrix}$

$$\underline{h} \cdot \nabla f = \begin{pmatrix} h \\ k \end{pmatrix} \cdot \begin{pmatrix} \frac{df}{dx} \\ \frac{df}{dy} \end{pmatrix} = h\frac{df}{dx} + k\frac{df}{dy}$$

$$(\underline{h} \cdot \nabla)^2 f = \begin{pmatrix} h \\ k \end{pmatrix} \cdot \begin{pmatrix} \frac{d}{dx}\left(h\frac{df}{dx} + k\frac{df}{dy}\right) \\ \frac{d}{dy}\left(h\frac{df}{dx} + k\frac{df}{dy}\right) \end{pmatrix} = \begin{pmatrix} h \\ k \end{pmatrix} \cdot \begin{pmatrix} hf_{xx} + kf_{xy} \\ hf_{xy} + kf_{yy} \end{pmatrix} = h^2 f_{xx} + 2hkf_{xy} + k^2 f_{yy}$$

$$(\underline{h} \cdot \nabla)^3 f = h^3 f_{xxx} + 3h^2 k f_{xxy} + 3hk^2 f_{xyy} + k^3 f_{yyy}$$

If $f:\mathbb{R}^3 \to \mathbb{R}$ $\quad h = \begin{pmatrix} h \\ k \\ l \end{pmatrix}$ then

$$\underline{h}.\underline{\nabla} = h\frac{d}{dx} + k\frac{d}{dy} + l\frac{d}{dz}$$

$$(\underline{h}.\underline{\nabla})^2 = h^2\frac{d^2}{dx^2} + k^2\frac{d^2}{dy^2} + l^2\frac{d^2}{dz^2} + 2hk\frac{d^2}{dxdy} + 2hl\frac{d^2}{dxdz} + 2kl\frac{d^2}{dydz}$$

(coefficients are found by considering $(a+b+c)^n$)

Examples:

Express $f(x,y) = x^2y + 3y - 2$ in powers of $(x-1)$ & $(y+2)$, we will do this by finding a Taylor Series for $f(x,y)$ about the point $(1,-2)$

$$\underline{a} = \begin{pmatrix} 1 \\ -2 \end{pmatrix}, \ \underline{h} = \begin{pmatrix} h \\ k \end{pmatrix} = \begin{pmatrix} x-1 \\ y+2 \end{pmatrix} \text{ so } \begin{pmatrix} x \\ y \end{pmatrix} = \underline{a} + \underline{h}$$

$$\frac{df}{dx} = 2xy \qquad \frac{df}{dy} = x^2+3 \qquad \frac{d^2f}{dx^2} = 2y \qquad \frac{d^2f}{dy^2} = 0 \qquad \frac{d^2f}{dxdy} = 2x$$

$$\frac{d^3f}{dx^3} = 0 \quad \frac{d^3f}{dx^2dy} = 2 \quad \frac{d^3f}{dxdy^2} = 0 \quad \frac{d^3f}{dy^3} = 0 \quad \text{Any higher orders are zero}$$

$$x^2y + 3y - 2 = f(x,y) = f(1,-2) + \left(\overset{x-1}{h}\underset{-4}{\frac{df}{dx}} + \overset{y+2}{k}\underset{4}{\frac{df}{dy}}\right)\Bigg|_{(1,-2)}$$

$$+\frac{1}{2}\left(h^2 f_{xx} + 2hk f_{xy} + k^2 0\right)\Big|_{(1,-2)}$$

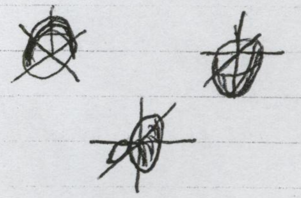
$$+\frac{1}{6}\left(h^3 0 + 3h^2k f_{xxy} + 3hk^2 . 0 + k^3 . 0\right)\Big|_{(1,-2)} + \overset{\text{higher orders}}{0}$$

$$= -10 - 4(x-1) + 4(y+2) - 2(x-1)^2 + 2(x-1)(y+2) + (x-1)^2(y+2)$$

<u>Extreme values & critical points of functions of several (mainly two) variables</u>

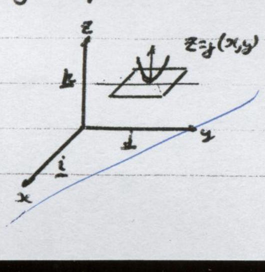
If $f:\mathbb{R} \to \mathbb{R}$

At critical pt. the tangent to the curve is horizontal (i.e. // to x-axis) & test by finding positions where $f'(x) = 0$



$$\left[\begin{array}{l} \text{Max pt. at } (x_0,y_0) \text{ if } f_{xx} < 0 \quad f_{xx}f_{yy} - f_{xy} > 0 \\ \text{Min pt. at } (x_0,y_0) \quad f_{xx} > 0 \quad f_{xx}f_{yy} - f_{xy} > 0 \\ \text{Saddle pt. at } (x_0,y_0) \text{ if } f_{xx}f_{xy} - f_{xy} < 0 \end{array}\right.$$

If we have a funct. ~~f(x)~~ $z = f(x,y)$ then a pt. where f has a local max/min/saddle pt. the tangent-plane to the surface $z = f(x,y)$ is // to the (x,y) plane, or has a normal // to $\underline{k}$.



The normal to a surface written as a level surface of $g(x,y,z) = z - f(x,y) = c$ is given by $\underline{\nabla} g = \begin{pmatrix} -\frac{df}{dx} \\ -\frac{df}{dy} \end{pmatrix}$ & so at a critical pt. $f_x = f_y = 0$.

A critical pt. (x_0, y_0) is s.t. $\frac{df}{dx}(x_0,y_0) = \frac{df}{dy}(x_0,y_0) = 0$

Using Taylor's Thm. about (x_0,y_0) & with $\binom{x_0}{y_0} = \underline{x}_0$

$$f(\underline{x}_0+\underline{h}) = f(\underline{x}_0) + \underline{h}\cdot\underbrace{\underline{\nabla} f\big|_{\underline{x}_0}}_{\left.\begin{pmatrix} df/dx \\ df/dy \end{pmatrix}\right|_{x_0} = 0} + \frac{1}{2}(\underline{h}\cdot\underline{\nabla}^2) f\big|_{\underline{x}_0} + \ldots$$

and so

$$f(\underline{x}_0+\underline{h}) - f(\underline{x}_0) = \frac{1}{2}\left(h^2\frac{d^2f}{dx^2} + 2hk\frac{d^2f}{dxdy} + k^2\frac{d^2f}{dy^2}\right)\Bigg|_{(x_0,y_0)}$$

If $f : \mathbb{R}^n \to \mathbb{R}$ then with $\underline{h} = (h_1, h_2, \ldots, h_n)^T$ this is

$$\underbrace{(h_1, h_2, \ldots, h_n)\underbrace{\begin{pmatrix} \frac{d^2f}{dx_1^2} & \frac{d^2f}{dx_1 dx_2} & \cdots & \frac{d^2f}{dx_1 dx_n} \\ \vdots & \ddots & & \vdots \\ \frac{d^2f}{dx_n dx_1} & \cdots & \cdots & \frac{d^2f}{dx_n^2} \end{pmatrix}}_{\text{hessian of } f}\begin{pmatrix} h_1 \\ h_2 \\ \vdots \\ h_n \end{pmatrix}}_{\text{A quadratic form}} = f(\underline{x}_0+\underline{h}) - f(\underline{x}_0)$$

Testing whether this quadratic form is always +ve/-ve or either depending on coefficients in $(h_1 \ldots h_n)$ reduces to seeing if the eigen values of the Hessian are all +ve/-ve/mixed in sign. Here though $(f : \mathbb{R}^2 \to \mathbb{R})$ we proceed by completing the sqr.

We will assume $f_{xx} \neq 0$. If $f_{xx} = 0$, then proceed using f_{yy} instead of f_{xx}. If $f_{xx} = f_{yy} = 0$ then it is clear we have a saddle pt. since the product hk can be made of either sign by choosing h, k appropriately.

$$f(\underline{x}_0+\underline{h}) - f(\underline{x}_0) = \frac{1}{2} f_{xx}\left[h^2 + 2hk\frac{f_{xy}}{f_{xx}} + k^2\frac{f_{yy}}{f_{xx}}\right]$$

$$= \frac{1}{2} f_{xx}\left[\underbrace{\left(h + k\frac{f_{xy}}{f_{xx}}\right)^2}_{\text{always +ve or 0}} + \underbrace{k^2\left(\frac{f_{yy}}{f_{xx}} - \frac{f_{xy}^2}{f_{xx}^2}\right)}_{\frac{k^2}{(f_{xx})^2}\underbrace{\left(f_{xx}f_{yy} - f_{xy}^2\right)}_{\text{may be +ve/-ve depending on the signs of } f_x}}\right]$$

$$f_{xx}f_{yy} - f_{xy}^2 = \Delta = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix} \text{ (determinant)}$$

$$= \frac{1}{2} f_{xx}\left[\left(h + k\frac{f_{xy}}{f_{xx}}\right)^2 + \frac{k^2}{(f_{xx})^2}\Delta\right]$$

If $\Delta > 0$ this has the same sign as f_{xx}. So if $\Delta > 0$, $f_{xx} > 0$ we have a minimum

$\Delta > 0$, $f_{xx} < 0$ we have a maximum

$\Delta < 0$, we have a saddle pt. bcs if we choose $k = 0$, the term is +ve, but if we choose $k = -h\frac{f_{xx}}{f_{xy}}$, then the term is < 0. (dependent upon $f_{xx} > 0$, $f_{xx} < 0$)

Example:

Find the critical pts. of $f(x,y) = \frac{1}{3}(x^3+y^3) - (x^2+y^2)$ & determine their nature.

a) To find critical pts. solve simultaneously

$$0 = \frac{df}{dx} = x^2 - 2x \quad \Rightarrow \quad x = 0, 2$$
$$0 = \frac{df}{dy} = y^2 - 2y \quad \Rightarrow \quad y = 0, 2$$

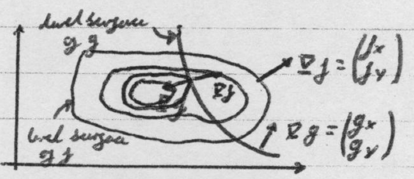
Critical pts. are $(0,0)$, $(2,0)$, $(0,2)$ and $(2,2)$

b) To determine their nature we need

$$f_{xx} = 2(x-1) \qquad f_{xy} = 0$$
$$f_{yy} = 2(y-1)$$
$$\Delta = (f_{xx}f_{yy} - f_{xy}^2) = 4(x-1)(y-1)$$

	$(0,0)$	$(2,0)$	$(0,2)$	$(2,2)$
$f_{xx} = 2(x-1)$	-2	2	-2	2
$f_{yy} = 2(y-1)$	-2	-2	2	2
$f_{xy} = 0$	0	0	0	0
$\Delta = 4(x-1)(y-1)$	4	-4	-4	4
	max $\Delta > 0$ $f_{xx} < 0$	saddle pts. $\Delta < 0$	saddle pts. $\Delta < 0$	min $\Delta > 0$ $f_{xx} > 0$

Constrained Optimisation



Consider level surfaces of $f(x,y)$ - i.e. lines in (x,y) plane along which f = const.

Consider too a line given by $g(x,y) = c$

If we ask what is the extreme value of $f(x,y)$ subject to the constraining $g(x,y) = c$. We see geometrically that this is achieved where a level surface of g, (given by $g(x,y) = c$) is tangential to a level surface of f.

If the normals to these curves are given by ∇f & ∇g then this occurs where $\nabla f / \nabla g$ i.e. $\nabla f = \lambda \nabla g$ (Lagrange multip.)

$$\nabla f = \lambda \nabla g \equiv \nabla(f - \lambda g) = \underline{0}$$
$$\equiv \nabla(f - \lambda(g - a)) = \underline{0} \quad a = \text{const.}$$

This condition tells us that only a level surface of f is tangential to a level surface of g. There are obviously many such pts., each given by a particular value of λ. To find the one we want we add in the constraint equation: $g(x,y) = c$

i.e. we need to solve the 3 equations

$\frac{d}{dx}(f-\lambda g)=0$ i.e. $\frac{df}{dx} - \lambda\frac{dg}{dx} = 0$

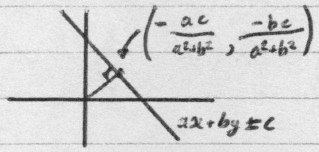
$\frac{d}{dy}(f-\lambda g)=0$ i.e. $\frac{df}{dy} - \lambda\frac{dg}{dy} = 0$

(If $h(x,y,\lambda) = f - \lambda(g-c)$ then these are $\frac{dh}{dx} = \frac{dh}{dy} = \frac{dh}{d\lambda} = 0$, $\underline{\nabla} h = 0$)

$g(x,y) = c$ to give values of x, y where f has a local max/min subject to the constr $g=c$.

Example:

Find the shortest distance of the line $ax+by+c=0$ to the origin



We will find the extreme value of $\sqrt{x^2+y^2}$. To make the algebra easy we will take $f(x,y) = x^2+y^2$

The constraint is $g(x,y) = ax+by+c=0$

Let $h(x,y,\lambda) = f - \lambda g = x^2+y^2 - \lambda(ax+by+c)$

Need to solve: $\frac{dh}{dx} = 2x - \lambda a = 0 \Rightarrow x = \lambda a/2$

$\frac{dh}{dy} = 2y - \lambda b = 0 \Rightarrow y = \frac{\lambda b}{2}$

add the constraint $ax+by=-c$ $\quad \frac{\lambda a^2}{2} + \frac{\lambda b^2}{2} = -c \Rightarrow \lambda = \frac{-2c}{a^2+b^2}$

So $x = \frac{-ac}{a^2+b^2}$ $\quad y = \frac{-bc}{a^2+b^2}$ & the distance can now be found as $\sqrt{x^2+y^2} = \frac{|c|}{\sqrt{a^2+b^2}}$

More Generally $f : \mathbb{R}^n \to \mathbb{R}$

If $f : \mathbb{R}^n \to \mathbb{R}$ is defined $f(x)$, $\underline{x} \in \mathbb{R}^n = (x_1, \dots, x_n)$ then we can have upto $n-1$ constraints.

Suppose we have m constraints $g_i(\underline{x}) = 0$, $1 \le i \le m$

We form the funct., the LAGRANGIAN:

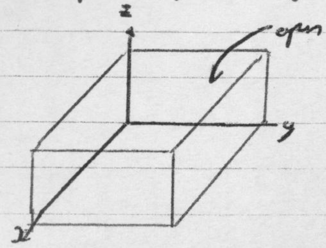
$$L(\underline{x}, \underset{\text{vector of lagrange multipliers } (\lambda_1,\lambda_2,\dots,\lambda_m)}{\underline{\lambda}}) = f(\underline{x}) - \sum_{i=1}^{m} \lambda_i g_i(\underline{x})$$

We then solve the n-equations: $\frac{dL}{dx_i} = 0$ ($\nabla_x L = 0$) $i=1,\dots,n$ together with the m constraints $g_i(\underline{x}) = 0$

or $\frac{dL}{d\lambda_i} = 0$ ($\nabla_\lambda L = 0$) $i=1,\dots,m$. $\quad \underline{\nabla} L = 0$

Example:

Construct an open-topped rectangular box of volume V, minimising its surface area.



Volume is $V = xyz$

Surface area is $S = 2xz + 2yz + xy$

min/max S subject to $V = xyz$

Minimize $S = 2xz + 2yz + xy$
subject to $xyz = V$ - [V is constant]

$$\therefore L(\cancel{x,y,z}\, x, y, z, \lambda) = 2xz + 2yz + xy - \lambda(xyz - V)$$

$$\frac{dL}{dx} = 2z + y - \lambda yz = 0, \quad \lambda = \frac{2z+y}{yz}$$

$$\frac{dL}{dy} = 2z + x - \lambda xz = 0 \quad \lambda = \frac{2z+x}{xz}$$

$$\frac{dL}{dz} = 2x + 2y - \lambda xy = 0 \quad \lambda = \frac{2x+2y}{xy}$$

$$\frac{dL}{d\lambda} = xyz - V = 0 \quad \therefore V = xyz.$$

$$\cancel{\frac{2z+y}{yz} = \frac{2z+x}{xz}}$$

$$\lambda xz = 2z + x$$

$$\therefore \lambda xz - 2z = x$$

$$\therefore z(\lambda x - 2) = x \quad \therefore z = \frac{x}{\lambda x - 2}$$

$$z = \frac{\frac{V}{yz}}{\lambda \cdot \frac{V}{yz} - 2} = \frac{V}{\lambda V - 2yz}$$

$\therefore$

Methods 3

Minimise Surface $S(x,y,z) = 2yz + 2xz + xy$

subject to Volume $V = xyz$ fixed i.e. $V(x,y,z) =$ constant $= V$

use Lagrange multipliers and consider

$$H(x,y,z) = S(x,y,z) - \lambda V(x,y,z)$$

$$= 2yz + 2xz + xy - \lambda xyz$$

we set $\frac{\partial H}{\partial x} = \frac{\partial H}{\partial y} = \frac{\partial H}{\partial z} = 0$

$2z + y - \lambda yz = 0$ (1)

$2z + x - \lambda xz = 0$ (2)

$2y + 2x - \lambda xy = 0$ (3)

Solve ①-③ together with the 4th equation $xyz = V$.

Consider (1)-(2) $\Rightarrow (y-x) - \lambda z(y-x) = 0$ $(y-x)(1-\lambda z) = 0$

$\therefore y - x = 0$ or $\lambda z = 1$.

in (1), $\lambda z = 0 \Rightarrow 2z + y - \cancel{\lambda z} y = 2z = 0 \Rightarrow z = 0$ $\lambda = [\infty]$

but $z = 0 \Rightarrow V = 0$ but we presume $V \neq 0$ so discount this solution.

other solution is $y = x$

then ③ $\Rightarrow$ $4x = \lambda x^2$ $\therefore x = 0$ (discount)

or $x = \frac{4}{\lambda}$ and $y = \frac{4}{\lambda}$

$y = x$ in ① gives $z = \frac{x}{\lambda x - 2} = \frac{4}{\lambda} \cdot \frac{1}{4-2} = \frac{2}{\lambda}$

λ is found from constraint $xyz = V$ i.e. $\frac{4}{\lambda} \cdot \frac{4}{\lambda} \cdot \frac{2}{\lambda} = \frac{1}{2}\left(\frac{4}{\lambda}\right)^3 = V$

$\therefore \frac{4}{\lambda} = \sqrt[3]{2V}$

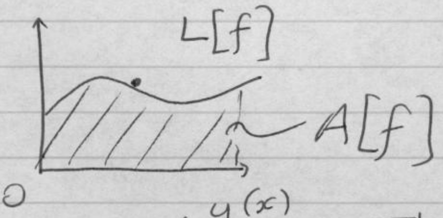
$x = y = \sqrt[3]{2V}$

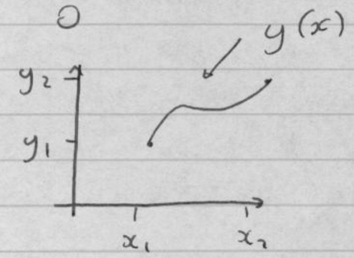
$z = \frac{1}{2}\sqrt{2V}$

New topic: Calculus of Variations

This is concerned with finding extreme values of functionals. Functionals are functions which map from a set of functions into the number δ: $\delta(f) = f(z)$

1) $A[f] = \int_0^1 f(x)\,dx$ 2) $L[f] = \int_0^1 \sqrt{1+(f')^2}\,dx$





The function y which makes these functionals take on extreme values is called the extremal and generates the extreme value.

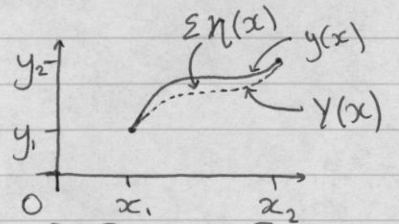
Generally the functionals we will consider are

$$I[y] = \int_{x_1}^{x_2} F(x,y,y')\,dx. \quad \therefore \text{ in } 1),\ F(x,y,y') = y$$

$$2)\ F(x,y,y') = \sqrt{1+(y')^2}$$

We wish to find an extremal curve satisfying boundary conditions, (*) $y(x_1) = y_1$ and $y(x_2) = y_2$

Assume that this extremal curve $y(x)$ exists.



& $\eta(x_1) = \eta(x_2) = 0$

$$Y(x) = y(x) + \varepsilon\eta(x)$$

$$I[Y] = I[y, \varepsilon, \eta] = \int_{x_1}^{x_2} F(x, y+\varepsilon\eta, y'+\varepsilon\eta')\,dx$$

If y is the extremal curve then, $\frac{dI}{d\varepsilon}[y,\varepsilon,\eta]_{\varepsilon=0} = 0$

$$\therefore 0 = \int_{x_1}^{x_2} \frac{d}{d\varepsilon} F(x, y+\varepsilon\eta, y'+\varepsilon\eta')\,dx \Big|_{\varepsilon=0}$$

(*) $\therefore\ 0 = \int_{x_1}^{x_2} \left(\frac{\partial F}{\partial y} \eta + \frac{\partial F}{\partial y'} \eta' \right) dx \Big|_{\varepsilon = 0}$

setting $\varepsilon = 0$.

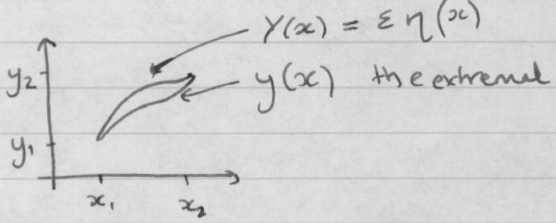
$$0 = \int_{x_1}^{x_2} \left(\frac{\partial F}{\partial y}(x, y, y')\, \eta + \frac{\partial F}{\partial y'}(x, y, y')\, \eta' \right) dx$$

This needs to be true independently of η //

Methods 3

$$I[y] = \int_{x_1}^{x_2} F(x, y, y')\,dx$$

$$g(x_1) = y_1 \quad , g(x_2) = y_2$$



$$\eta(x_1) = \eta(x_2) = 0$$

$y(x)$ is extremal $\Rightarrow \left.\dfrac{dI}{d\varepsilon}\right|_{\varepsilon=0} = 0 \Rightarrow \displaystyle\int_{x_1}^{x_2} \left(\frac{\partial F}{\partial y}\eta + \frac{\partial F}{\partial y'}\eta'\right)dx = 0$

$\forall \eta$

Use integration by parts on $\dfrac{\partial F}{\partial y'}\eta'$

$$\Rightarrow \int_{x_1}^{x_2} \frac{\partial F}{\partial y'}\eta'\,dx = \left[\eta\frac{\partial F}{\partial y'}\right]_{x_1}^{x_2} - \int_{x_1}^{x_2} \eta\frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right)dx$$

$$\therefore \int_{x_1}^{x_2} \left(\frac{\partial F}{\partial y}\eta + \frac{\partial F}{\partial y'}\eta'\right)dx = 0$$

$$\equiv \left[\eta\frac{\partial F}{\partial y'}\right]_{x_1}^{x_2} + \int_{x_1}^{x_2} \left(\frac{\partial F}{\partial y}\eta - \eta\frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right)\right)dx = 0$$

$= 0$ since $\eta(x_2) = \eta(x_1) = 0$

$$\Rightarrow \int_{x_1}^{x_2} \eta\left[\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right)\right]dx = 0$$

We conclude that for the extremal curve $y(x)$

$$\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = 0$$

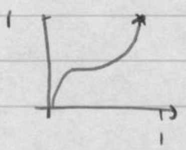
This is the Euler-Lagrange equation and is a second order differential equation for $y(x)$, the extremal curve, to be solved with the boundrey conditions $y(x_1) = y_1$, $y(x_2) = y_2$. Once the extremal is known, it can be substituted into $I[y]$ to find the extreme value.

In general the 2nd OD can't be solved analytically.

$\Rightarrow$ not many choices for exam questions.

e.g. Find the extremal curve for $\int_0^1 \underbrace{(y^2 - 2xy - y'^2)}_{F(x,y,y')} dx$:

$y(0) = 0 \quad , \quad y(1) = 1$



$$\frac{\partial F}{\partial y} = 2y - 2x$$

$$\frac{\partial F}{\partial y'} = -2y'$$

$\therefore$ Euler Lagrange Equation is:

$$(2y - 2x) - \frac{d}{dx}(-2y') = 0$$

Solve this

$\Longrightarrow y'' + y - x = 0 \qquad y(0) = 0 \;, \; y(1) = 1$

do this Solve this using 1401

$\Longrightarrow y = A\cos x + B\sin x + x$

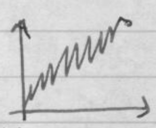
choose now A and B to satisfy boundrey conditions.

$\Longrightarrow A = B = 0. \quad \Longrightarrow y = x.$

The extreme value of the integral :

$$\int_0^1 x^2 - 2x^2 - 1 = -\int_0^1 x^2 + 1 = -\left[\frac{x^3}{3} + x\right]_0^1 = -\left(\frac{1}{3} + 1\right) = -\frac{4}{3}$$

We can argue that if we can make the value of $I[y]$ as large and negative as we like by choosing an appropriate $y(x)$, the $-\frac{4}{3}$ ~~gives~~ indicates a maximum value.

Choose a y which does this:  "$y = x + \sin nx$"

$y' = 1 + n\cos nx$

$(y')^2$ goes like $\approx n^2\cos^2 nx$ for large n.

So due to the $-y'^2$ term in the integrand, we can make the integral as large and as negative as we like by choosing a sufficiently oscillatory $y(x)$.

The shortest distance between two points.

In Euclidean space the length of a curve $y(x)$ is $\int_{x_1}^{x_2} \sqrt{1+y'^2}\,dx$. Here $F(x,y,y') = \sqrt{1+y'^2}$

$\therefore \cancel{\frac{\partial F}{\partial y} = 0}$.

$\left[\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right)=0\right]$ $\therefore$ the E,L equation gives

$$0 - \frac{d}{dx}\left[y'(1+y'^2)^{-\frac{1}{2}} \right] = 0$$

← don't need to diff then int.

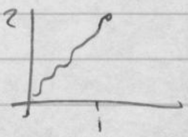
$\therefore$ we can deduce $\frac{y'}{\sqrt{1+y'^2}} = C$ [C is a constant]

and this is true only for $y' = \cancel{\text{constant}}\ m$ [m is a constant]

i.e. the extremal : $y = mx + c$ i.e. the shortest distance between two points is a straight line.

Example.

~~Con~~ Consider $I[y] = \int_0^1 (y'-y)^2\,dx$ $y(0) = 0$ $y(1) = 2$



$\underbrace{(y'-y)^2}_{}$ $F(x,y,y') = (y'-y)^2$

E-L equation gives $-2(y'-y) - \frac{d}{dx}\left[\frac{\partial F}{\partial y'}\right] = 0$

$$= -2(y'-y) - \frac{d}{dx}2(y'-y) = 0$$

$$\Rightarrow y'' - y = 0 \quad \therefore y(x) = A\cosh x + B\sinh x$$

with B.C $\Rightarrow y = \frac{2\sinh(x)}{\sinh(1)}$ ← it is possible to prove that this extremal curve is a minimum.

$y = f$, where $f'' - f = 0$, $f(0) = 0$, $f(1) = 2$. gives a minimum value for the integral. This is by considering $I[f+g]$ where $g(0) = 0$, $g(1) = 0$ and showing $I[f+g] \geq I[f]$

$$I[f] = \int_0^1 (f'-f)^2 dx$$

$$I[f+g] = \int_0^1 (f'+g'-f-g)^2 dx$$

$$= \int_0^1 (f'-f+g'-g)^2 dx$$

$$I[f+g] = \int_0^1 \left[(f'-f)^2 + 2(f'-f)(g'-g) + (g'-g)^2\right] dx$$

$$= I[f] + \int_0^1 2(f'-f)(g'-g)\,dx + k \quad [k \geq 0]$$

Consider $2\int_0^1 \left((f'-f)g' - (f'-g)g\right) dx$ and integrate

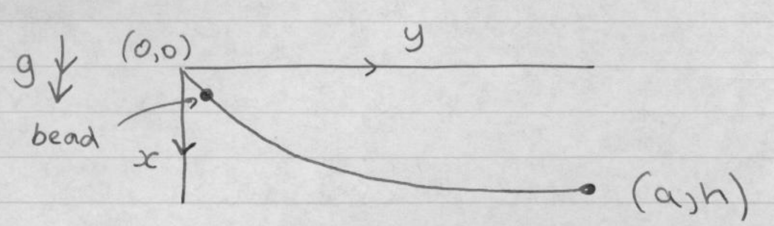
~~...~~ by parts on to give:

$g(0) = g(1) = 0 \rightarrow$ $\left[2(f'-f)g\right]_0^1 - 2\int_0^1 \left[(f''-f')g + (f'-f)g\right] dx$

$= 0 \quad + 0 \quad$ since $f''-f = 0$ [extremal curve]

$$\Rightarrow I[f+g] = I[f] + \int_0^1 (g'-g)^2 dx \geq I[f]$$

The Brachistrochrone Problem



What shape gives the shortest time from (0,0) to (a,h)

find $y(x)$: time taken for a bead to fall ~~...~~ a due to gravity from (0,0) to (a,h) on the wire $y(x)$ is a minimum.

must be a ~~maximum~~ minimum since we can choose the wire shape to give a time as large as we like.

$s = vt$

$$T = \int dt = \int \frac{ds}{v}.$$

KE. gained = PE lost gives

$$\frac{1}{2}mv^2 = mgx \Rightarrow v = \sqrt{2gx}$$

$$ds = \sqrt{1+y'^2}\,dx$$

$$\left[\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right)\right] = 0$$

All this $\Rightarrow$

$$T[y] = \frac{1}{\sqrt{2g}}\int_0^a \underbrace{\frac{\sqrt{1+y'^2}}{\sqrt{x}}}_{F(x,y,y')}\,dx$$

because of our choice of axis.

Here $\frac{\partial F}{\partial y} = 0$

$\therefore$ E.L eq $= \frac{d}{dx}\left[\frac{\partial F}{\partial y'}\right] = 0$

$$\Rightarrow \frac{\partial F}{\partial y'} = C$$

[C is a constant]

$$\frac{\partial F}{\partial y'} = \frac{1}{\sqrt{x}}\cdot\frac{y'}{\sqrt{1+y'^2}} = C$$

$$\Rightarrow y'^2 = C^2x(1+y'^2)$$

$$y'^2(1-c^2x) = c^2x$$

be careful as two options (+/-) for $\frac{dy}{dx}$.

$$\Rightarrow \left(\frac{dy}{dx}\right)^2 = \frac{c^2x}{1-c^2x}$$

In this case it won't matter but in some cases it does.

in this case $\Rightarrow \frac{dy}{dx} = c\sqrt{\frac{x}{1-c^2x}}$

[we expect y' to be ≥ 0]

$$\frac{dy}{dx} = c\sqrt{\frac{x}{1-c^2x}} = \sqrt{\frac{x}{\frac{1}{c^2}-x}} = \sqrt{\frac{x}{\alpha - x}} \quad \left[\frac{1}{c^2} = \alpha\right]$$

$$\Rightarrow \int dy = \int \sqrt{\frac{x}{\alpha - x}}\, dx \qquad \left[\alpha \text{ is a constant } = \frac{1}{c^2}\right]$$

use substitution
$x = \alpha \sin^2\theta$

$\therefore \frac{dx}{d\theta} = 2\alpha \sin\theta\cos\theta$

$$\therefore y + k = \int \sqrt{\frac{\alpha\sin^2\theta}{\alpha - \alpha\sin^2\theta}} \cdot 2\alpha\sin\theta\cos\theta \, d\theta$$

$$\Rightarrow y + k = \int \sqrt{\frac{\alpha\sin^2\theta}{\alpha\cos^2\theta}} \cdot 2\alpha\sin\theta\cos\theta\, d\theta$$

$$\Rightarrow y + k = \int \frac{\sin\theta}{\cos\theta} \cdot 2\alpha\sin\theta\cos\theta\, d\theta$$

$$\Rightarrow y + k = \int 2\alpha\sin^2\theta\, d\theta = \alpha\int(1-\cos 2\theta)\, d\theta$$

$$= \alpha\left[\theta - \tfrac{1}{2}\sin 2\theta\right] = \alpha\left[\theta - \sin\theta\cos\theta\right]$$

so $y + k = \alpha\left[\sin^{-1}\sqrt{\frac{x}{\alpha}} - \sqrt{\frac{x}{\alpha}}\sqrt{1-\frac{x}{\alpha}}\right]$

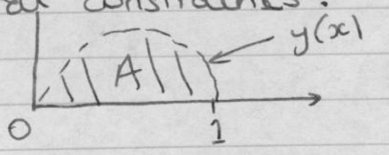
Need to choose constants : $y(0) = 0$, $y(a) = h$

$y(0) = 0 \Rightarrow k = 0$. $\therefore y(a) = h \Rightarrow \alpha$ can be found numerically.

Curve is the same shape as a cycloid.

<u>Next Topic</u> <u>Isoperimetric Problem</u> Calculus of variations with integral constraints.

e.g.



$L = \int_0^1 \sqrt{1+y'^2}\, dx = \text{fixed}$

We want to maximise $A[y] = \int_0^1 y\, dx$

Method: Use Lagrange multipliers. Form another functional,
$H[y,\lambda] = \int_0^1 \left(y - \lambda\sqrt{1+y'^2}\right) dx$. Then do E-L equation on integrand.

Gives λ hanging around in the solution.
Then find λ by imposing the constraint.

Methods 3

Special Forms of the Euler-Lagrange equation.

$$\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = 0 \quad , F = F(x, y, y')$$

i) no y' in F i.e. $\frac{\partial F}{\partial y'} = 0 \implies \frac{\partial F}{\partial y} = 0$

ii) No y in F i.e. $\frac{\partial F}{\partial y} = 0$

$$\therefore \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = 0$$

$$\implies \frac{\partial F}{\partial y'} = C$$ a first integral of EL equations.

iii) No x in F i.e. $\frac{\partial F}{\partial x} = 0$, then $F - y'\frac{\partial F}{\partial y'} = \text{constant}$

The first integral is called the Beltrami equation.

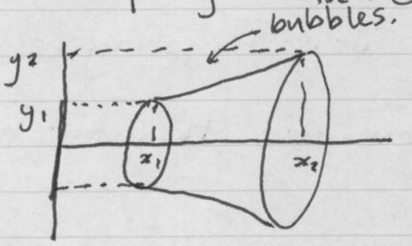
since

$$\frac{d}{dx}\left[F - y'\frac{\partial F}{\partial y'}\right] = \underbrace{\frac{\partial F}{\partial x}}_{=0} + \frac{\partial F}{\partial y}\underbrace{\frac{dy}{dx}}_{y'} + \frac{\partial F}{\partial y'}\underbrace{\frac{dy'}{dx}}_{y''} - \frac{d^2y}{dx^2}\frac{\partial F}{\partial y'} - y'\frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right)$$

$F(x,y,y')$ but in this case $F(y,y')$

$$= y'\left(\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right)\right) = 0$$

e.g.



Minimise the surface area produced by rotating the curve $y = y(x)$ about the x-axis.

not important for finding f

$$y(x_1) = y_1 \, , y(x_2) = y_2 \qquad A[y] = 2\pi\int_{x_1}^{x_2} y\sqrt{1+y'^2}\,dx$$

$F(x,y,y') = y\sqrt{1+y'^2}$. We see $\frac{\partial F}{\partial x} = 0$ so that we can go immediately to the first integral.

$$F - y'\frac{\partial F}{\partial y'} = C \quad \therefore y(1+y'^2)^{\frac{1}{2}} - \frac{y'y'y}{(1+y'^2)^{\frac{1}{2}}}$$

$$\implies \frac{1}{\sqrt{1+y'^2}}\left[y + yy'^2 - yy'^2\right] = C$$

$$\implies \frac{y}{\sqrt{1+y'^2}} = C$$

$y^2 = c^2(1+y'^2)$

$\Longrightarrow \frac{dy}{dx} = \pm \frac{1}{c}\sqrt{y^2-c^2}$

$\Longrightarrow \int \frac{dy}{\sqrt{y^2-c^2}} = \pm \int \frac{1}{c}\,dx \Longrightarrow \cosh^{-1}\left(\frac{y}{c}\right) = \pm\left(\frac{x}{c}+\frac{D}{c}\right)$

$\Longrightarrow \quad y = c\cosh\left(\pm\frac{(x+D)}{c}\right) = c\cosh\left(\frac{x+D}{c}\right)$

Found by $y(x_1)=y_1$, $y(x_2)=y_2$ done.

<u>Back to isoparametric problems</u>

Find extremal ~~exact~~ for the integral $\int_0^1 y'^2+2yy'$ $y(0)=y(-1)=0$

subject to constraint $\int_0^1 y\,dx = \frac{1}{6}$

Min/Max $\int F\,dx$ subject to $\int G\,dx$ = constant. Form $\int (F-\lambda G)\,dx$

Solve EL equations for the new functional. And apply constraint to find λ.

Consider $\int_0^1 (\underbrace{y'^2+2yy'-\lambda y}_{h(x,y,y',\lambda)})\,dx$ ⟵ use E-L equations on this

We see $\frac{dh}{dx}=0$ and it is tempting to use Beltrami equation. $\left(F-y'\frac{dH}{dy'}\right)$ = constant) However the algebra is very difficult.

Instead, use E-L equations. i.e. $\frac{dH}{dy} - \frac{d}{dx}\left(\frac{dH}{dy'}\right)=0$

$\Longrightarrow 2y-\lambda - \frac{d}{dx}\left[2y'+2y\right] = 0$

$\Longrightarrow \quad y'' = -\frac{\lambda}{2} \Longrightarrow y = -\frac{\lambda}{4}x^2+Ax+B$

Apply boundary conditions. $y(0)=0 \Rightarrow B=0$
$y(1)=0 \Rightarrow A=\frac{\lambda}{4}$

$\Longrightarrow y(x) = \frac{\lambda}{4}x(1-x)$ however the integral of this must be $\frac{1}{6}$

$\therefore$ this $\Rightarrow \lambda = 4$.

$\therefore y = \cancel{4}\, x(1-x)$

Methods 3

Don't always Jump to Beltrami's eq. when no xs in $F(x,y,y')$

Beltrami eq : $H - y'\frac{\partial H}{\partial y} = \text{const}$

$\therefore$ for $H = y'^2 + 2yy' - \lambda y$

$$\Rightarrow (y'^2 + 2yy' - \lambda y) - y'(2y' + 2y) = c$$

$$\Rightarrow y'^2 + \lambda y = c$$

$$\Rightarrow \frac{dy}{dx} = \pm\sqrt{c - \lambda y}$$

$$\therefore \int \frac{dy}{\sqrt{c-\lambda y}} = \pm\int dx$$

$$\Rightarrow -\frac{2}{\lambda}\sqrt{c - \lambda y} = \pm x + A$$

$$\Rightarrow \sqrt{c - \lambda y} = \pm\frac{\lambda}{2}(x + A)$$

Choose C ~~&~~ A : $y(0) = 0$, $y(1) = 0$ ← $\pm$ important

$$\Rightarrow \sqrt{c} = \pm\frac{\lambda A}{2} \quad , \quad \Rightarrow \sqrt{c} = \pm\frac{\lambda}{2}(1 + A)$$

What we need is $-A = (1 + A) \Rightarrow A = -\frac{1}{2}$

or $A^2 = (1 + A)^2$

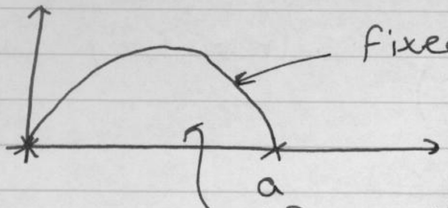
$\Rightarrow A^2 = 1 + 2A + A^2 \Rightarrow A = -\frac{1}{2}$ ✓

$$\therefore \quad c = \frac{\lambda^2 A^2}{4} = \frac{\lambda^2}{16}$$

$\therefore c - \lambda y = \frac{\lambda^2}{4}(x + A)^2 \Rightarrow y = \frac{\lambda}{4}x(1 - x)$ Same as before.

When using Beltrami N.B $\pm$ signs.

The Sheep Pen Problem



fixed Length $L = \int_0^a \sqrt{1+y'^2}\,dx$

$$A[y] = \int_0^a y\,dx$$

Maximise A subject to constraint of fixed L.

Form $\int_0^a \left(y - \lambda\sqrt{1+y'^2}\right)dx$. This time use Beltrami's

$$H - y'\frac{\partial H}{\partial y'} = C$$

$$\therefore \left(y - \lambda\sqrt{1+y'^2}\right) - y'\left(\frac{-\lambda y'}{\sqrt{1+y'^2}}\right) = C$$

$$\Longrightarrow \frac{-\lambda}{\sqrt{1+y'^2}}\left[1+y'^2-y'^2\right] = C-y$$

$$\Longrightarrow \frac{-\lambda}{\sqrt{1+y'^2}} = C-y \Longrightarrow \frac{-\lambda}{C-y} = \sqrt{1+y'^2}$$

$$\Longrightarrow \frac{dy}{dx} = \pm\sqrt{\frac{\lambda^2-(c-y)^2}{(c-y)^2}}$$

top is deriv. of bottom

$$\frac{dy}{dx} = \frac{\pm\sqrt{\lambda^2-(c-y)^2}}{(c-y)}$$

$$\Longrightarrow \int \frac{c-y}{\sqrt{\lambda^2-(c-y)^2}}\,dy = \int \pm\,dx$$

$$\Longrightarrow \lambda^2 = (x+A)^2 + (y-c)^2$$ B.C give A and C constraint gives λ.

$y(0)=0$, $y(a)=0$. $\Rightarrow \lambda^2 = A^2+C^2$

$y(a)=0 \Rightarrow \lambda^2 = (a+A)^2 + c^2$

$\Rightarrow a^2 + 2aA = 0$

$\therefore a(a+2A) \Rightarrow A = -\dfrac{a}{2}$

$C^2 = \lambda^2 - \dfrac{a^2}{4}$.

To find λ we use the constraint

$$L = \int_0^a \sqrt{1+y'^2}\,dx = \pm\int_0^a \frac{\lambda}{c-y}\,dx$$

$$= \int_0^a \frac{\lambda}{\sqrt{\lambda^2 + \left(x-\frac{a}{2}\right)^2}}\,dx \implies L = 2\lambda \sin^{-1}\left(\frac{a}{2\lambda}\right)$$

$$\implies \sin\left(\frac{L}{2\lambda}\right) = \frac{a}{2L} \Rightarrow \lambda \Rightarrow c$$

Partial Differential Equations

A partial different equation (PDE) is a relation between a function of several variables $u(x, y, \ldots)$ and its partial derivatives. $u_x, u_y \ldots$ $u_{xx}, u_{yy}, u_{xxx}, u_{xxy}$, etc. –

e.g. $u\frac{du}{dx} + xu = \frac{d^2u}{dy^2}$ for $u = u(x,y)$

second order PDE. (order of the highest derivative occuring)

If the differential equation can be written as $L[u] = f$ where f does not depend on u. and L is a linear operator
i.e. $L(\alpha u + \beta w] = \alpha L[u] + \beta L[w]$

e.g. $a(x,y)\frac{du}{dx} + b(x,y)\frac{du}{dy} = c(x,y)u + d(x,y)$

$L[u] = a\frac{du}{dx} + b\frac{du}{dy} - cu$, $f = d$. the the eq. is linear.

Quasi-Linear. → no product of highest order derivatives.

e.g. $(x+y)\frac{du}{dy} + y\frac{du}{dy} = 1 + xu$ linear

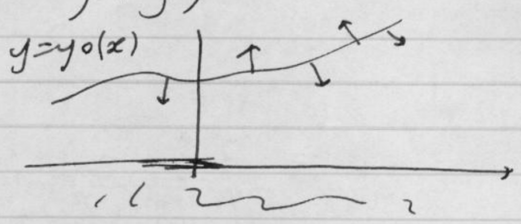
$u^2\frac{du}{dx} + u_y = 1$ is quasi linear.

but there are no product of the highest order derivative occuring. Such equations are called quasi-linear.

$u_{xx} + u_{yy} = 1$ is second order and linear

A simple equation is:

$$\frac{\partial u}{\partial x} = 0 \implies u = f(y) \qquad I$$



$$\left[\frac{dy}{dx} = 0 \implies y = \text{const.} \right]$$
in normal differential equations

We observe that the general solution of pdes contains arbitrary functions.

general solution of PDE contains arbitrary functions.

<u>A typical problem</u>:

Given a first order pde valid in a region D of the (x,y) plane. (e.g. $D = \mathbb{R}^2$), $x \geq 0$, $x^2+y^2 \leq a^2$.

and some knowledge of the solution $u(x,y)$ on a curve I in D. E.g. $y = y_0(x)$ or more generally, $g(x,y) = c$

D has $y \geq 0$

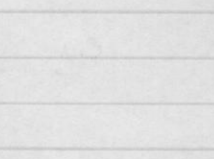
e.g. $u = u_0(x,y)$ on $y_0(x)$

Can we find $u(x,y)$ inside all of D?

If this can be done we will call the problem well-posed. If it cannot it will be ill-posed.
We will see examples that illustrate when a problem is ill-posed.

We often describe lines in the (x,y) plane in parametric form. E.g. the Line I can be described as
$x = x(s)$, $y = y(s)$, $u = u(s)$.

e.g. if we are told that $u = x^2$ on $y = 0$, this would be parameterised as $x = s$, $y = 0$ $u = s^2$



$I(s)$ in 3D is a line in the solution surface $u = u(x,y)$

<u>Example</u>: Solve $\frac{\partial u}{\partial x} = 0$ with I being the y axis and on this

$u = e^y$ i.e. $y = s$, $x = 0$, $y = e^s$

To find λ we use the constraint

$$L = \int_0^a \sqrt{1+y'^2}\, dx = \pm \int_0^a \frac{\lambda}{c-y}\, dx$$

$$= \int_0^a \frac{\lambda}{\sqrt{\lambda^2 + \left(x - \frac{a}{2}\right)^2}}\, dx \implies L = 2\lambda \sin^{-1}\left(\frac{a}{2\lambda}\right)$$

$$\implies \sin\left(\frac{L}{2\lambda}\right) = \frac{a}{2L} \implies \lambda \implies c$$

Partial Differential Equations

A partial different equation (PDE) is a relation between a function of several variables $u(x, y, \ldots)$ and its partial derivatives. $u_x, u_y \ldots$ $u_{xx}, u_{yy}, u_{xxx}, u_{xxy}$, etc. –

e.g. $$u \frac{du}{dx} + xu = \frac{d^2u}{dy^2} \quad \text{for } u = u(x,y)$$

second order PDE. (order of the highest derivative occuring)

If the differential equation can be written as $L[u] = f$ where f does not depend on u. and L is a linear operator

i.e. $L(\alpha u + \beta w] = \alpha L[u] + \beta L[w]$

e.g. $$a(x,y)\frac{du}{dx} + b(x,y)\frac{du}{dy} = c(x,y)u + d(x,y)$$

$$L[u] = a\frac{du}{dx} + b\frac{du}{dy} - cu, \quad f = d.$$ the eq. is linear.

Quasi-Linear. → no product of highest order derivatives.

e.g. $$(x+y)\frac{du}{dy} + y\frac{du}{dy} = 1 + xu \qquad \text{linear}$$

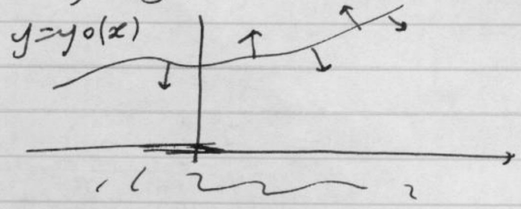
$$u^2\frac{du}{dx} + u_y = 1 \qquad \text{is quasi linear.}$$

but there are no product of the highest order derivative occuring. Such equations are called quasi-linear.

$u_{xx} + u_{yy} = 1$ is second order and linear

A simple equation is:

$$\frac{\partial u}{\partial x} = 0 \implies u = f(y)$$ I



$\left[\frac{dy}{dx} = 0 \implies y = \text{const.} \right.$ in normal differential equations $\left. \right]$

We observe that the general solution of pdes contains arbitrary functions.

general solution of PDE contains arbitrary functions.

<u>A typical problem</u>:

Given a first order pde valid in a region D of the (x,y) plane. (e.g. $D = \mathbb{R}^2$), $x \geq 0$, $x^2+y^2 \leq a^2$.

and some knowledge of the solution $u(x,y)$ on a curve I in D. E.g. $y = y_0(x)$ or more generally, $g(x,y) = c$

D has $y \geq 0$

e.g. $u = u_0(x,y)$ on $y_0(x)$

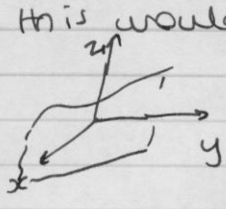
can we find $u(x,y)$ inside all of D?

If this can be done we will call the problem well-posed. If it cannot it will be ill-posed.

We will see examples that illustrate when a problem is ill-posed.

We often describe lines in the (x,y) plane in parametric form. E.g. the Line I can be described as $x = x(s)$, $y = y(s)$, $u = u(s)$.

e.g. if we are told that $u = x^2$ on $y = 0$, this would be parameterised as $x = s$, $y = 0$ $u = s^2$



I(s) in 3D is a line in the solution surface $u = u(x,y)$

<u>Example</u>: Solve $\frac{\partial u}{\partial x} = 0$ with I being the y axis and on this $u = e^y$ i.e. $y = s$, $x = 0$, $y = e^s$

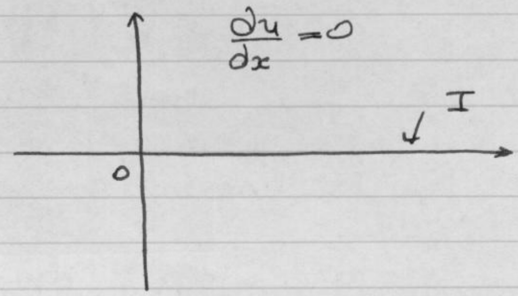
Integrating gives $u = f(y)$ but on I,

$u = e^s,\ y = s,\ x = 0$, so that substituting $e^s = f(s)$

i.e. $f(y) = e^y$ and our solution is $u(x,y) = e^y$

e.g. to Solve $\frac{du}{dx} = 0$ with I as the x-axis

e.g. $u = f(\overset{y}{\cancel{x}})$ on $y = 0$



ill-posed. $u = 1 + y$ is a solution.

but $u = 1 + g(y) - g(0)$ is still a solution.

$\therefore$ solution is not unique.

Consider a curve in the x-y plane given parametrically by $x = x(t)$, $y = y(t)$

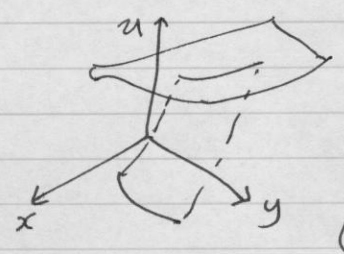
then $\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$, If we consider the ODE given by $\frac{dy}{dx} = \frac{b(x,y)}{a(x,y)} = F(x,y)$

then this equation has a solution given by the solution to $\frac{dy}{dt} = b(x,y)$, $\frac{dx}{dt} = a(x,y)$,

If $u = u(x,y)$ and $x = x(t)$, $y = y(t)$, then

$$\frac{du}{dt} = \frac{du}{dx}\frac{dx}{dt} + \frac{du}{dy}\frac{dy}{dt} = a(x,y)\frac{du}{dx} + b(x,y)\frac{du}{dy}$$

normal chain rule for a parameterised function



$$\frac{du}{dt} = a\frac{du}{dx} + b\frac{du}{dy}$$

Sometimes these equations may be written: $dx = a\,dt$ $dy = b\,dt$

or $\frac{dx}{a} = dt$ and $\frac{dy}{b} = dt$

parameterised

Characteristics Consider PDE given by:

$a(x,y)\frac{du}{dx} + b(x,y)\frac{du}{dy} = 0$ ← homogenous eq.

to be solved with a knowledge of u on a line I in x-y plane

Consider lines given by the solution to $\frac{dx}{dt} = a$, $\frac{dy}{dt} = b$, $\left[\frac{dx}{a} = \frac{dy}{b} = dt\right]$

then along this line

$$\frac{du}{dt} = \frac{du}{dx}\frac{dx}{dt} + \frac{du}{dy}\frac{dy}{dt} = a\frac{du}{dx} + b\frac{du}{dy} = 0$$

So that $u(x,y)$ is constant on ~~these~~ lines.

This lines are called characteristics or more accurately characteristic traces. And the equations

$\frac{dx}{a} = \frac{dy}{b} = \left(\frac{du}{0}\right) = dt$ are the characteristic equations

Partial Differential Equations

e.g. Solve $\frac{du}{dx} + x\frac{du}{dy} = 0$

$u = \sin y$ on $x = 0$

1) Parameterise $I(s)$ as $x = 0, y = s, u = \sin(s)$
solve ~~ea~~ charateristic eq. with initial conditions
$t = 0$, $y = s$, $u = \sin(s)$ at $t = 0$

can't integrate yet since x depends on t

these are $\frac{dx}{dt} = a = 1$ ① . $\frac{dy}{dt} = b = x$ ② $\left(\frac{du}{dt} = 0\right)$

① $\Rightarrow$ $x = t$ using $t = 0$ at $x = 0$

in ② $\frac{dy}{dt} = x = t$ $\therefore y = \frac{1}{2}t^2 + s$ using $y = s$ at $t = 0$.

$u = \sin(s)$

Now eliminate t and s in favour of x any y

$s = y - \frac{1}{2}t^2 = y - \frac{1}{2}x^2$ $\therefore u = \sin\left(y - \frac{1}{2}x^2\right)$

Also note that for any function $\mathbb{R} \to \mathbb{R} : f$ the function $u(x,y) = f\left(y - \frac{1}{2}x^2\right)$ satisfies the pde, but a choice of f is needed to satisfy the boundrey ~~e~~ conditions.

e.g. A quasi linear homogeneous.

$$u\frac{du}{dx} - \frac{du}{dy} = 0$$

$$u = x^2 \text{ on } y = 0$$

Solve the characteristic equations

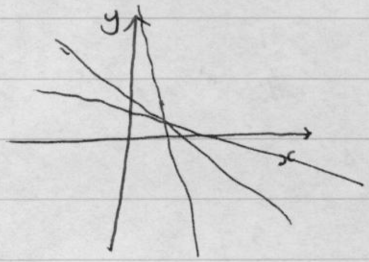
$$\frac{dx}{dt} = a = u \qquad \frac{dy}{dt} = b = -1 \qquad \frac{du}{dt} = 0$$

$t = 0,\ y = 0,\ x = s,\ u = s^2$

We can solve for y : $y = -t$ ($y = 0$ at $t = 0$)
We can't directly solve $\frac{dx}{dt} = u$ as we don't know what $u(t)$ is (yet). We do know however that $\frac{du}{dt} = 0$ on x(axis?) $\therefore$ u is constant. As $u = s^2$ at $t = 0$

$u = s^2$ so $\frac{dx}{dt} = u = s^2 \implies x = s^2 t + s$ as $x = s$ when $t = 0$

So we have a parametric solution, $x = s^2 t + s$
$y = -t$
$u = s^2$



So we have the parametric solution

x-traces are given by eliminating t,

$x = s - s^2 y$, $y = \frac{1}{s} - \frac{x}{s^2}$ The x-traces have an envelope $y = \frac{1}{4x}$. So we cannot find solutions for $u(x,y)$ in the region of $y > \frac{1}{4x}$ as no x which intersect $I(s)$ enters this region.

If we eliminate s and t, then:

(quadratic in u) $\quad x = u(-y) + \sqrt{u}$ assumes $s > 0$

$\implies \sqrt{u} = \frac{1 \pm \sqrt{4xy}}{2y}$ We need to decide whether we want $\pm$ $u = x^2$ i.e. $\sqrt{u} = x$ on $y = 0$

choose $-$. $\therefore \sqrt{u} = \frac{1 - \sqrt{4xy}}{2y}$ otherwise $y \to \infty$ as $x \to 0$

Methods

$f(x,y,s)=0 \qquad f = x+s^2y-s=0$

family of curves, parameters.

Eliminate s from $f=0$ and $\frac{\partial f}{\partial s}=0$
$\Rightarrow$ Envelope.

e.g. $f = x+s^2y-s=0$

$$\frac{\partial f}{\partial s} = 0 \Rightarrow 2sy-1=0 \Rightarrow s = \frac{1}{2y}$$

$$\therefore x + \frac{1}{4y^2}\cdot y - \frac{1}{2y} = 0$$

$$\Rightarrow x + \frac{1}{4y} - \frac{1}{2y} = 0 \Rightarrow x - \frac{1}{4y} = 0 \Rightarrow y = \frac{1}{4x}$$

The characteristic equations for ~~a(x,y,u)~~

$$a(x,y,u)\,u_x + b(x,y,u)\,u_y = 0$$

solutions of $\dfrac{dx}{a(x,y,u)} = \dfrac{dy}{b(x,y,u)}$ or $\dfrac{dy}{dx} = \dfrac{b(x,y,u)}{a(x,y,u)}$

If ~~PDE~~ PDE is linear, this becomes $\dfrac{dy}{dx} = \dfrac{b(x,y)}{a(x,y)}$

Hence for linear equations, if a and b are single valued $\frac{dy}{dx}$ is unique as a function of x and y and traces cannot cross.

This is not true for general quasilinear pde's

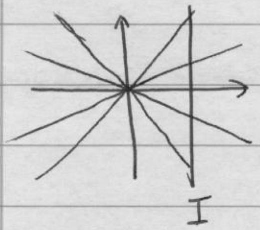
There are exceptions at points where both $a=0$, $b=0$ where $\frac{dy}{dx}$ is undetermined.

$$x\frac{du}{dx} + ~~~~ y\frac{du}{dy} = 0$$

$$\frac{dx}{x} = \frac{dy}{y} \implies \ln x = \ln y + \text{const.}$$

$$\therefore \; y = Cx \qquad [C \text{ is a constant}]$$

characteristics cross at origin when $a = b = 0$



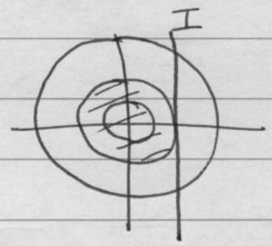
In this case we might expect a singularity in the solution to the pde at $x=0, y=0$ as the characteristics carrying contradictory information about the solution cross. This singularity could be avoided for particular I e.g. if $a = $ constant on I here.

characteristics: $\quad y\frac{du}{dx} - x\frac{du}{dy} = 0$

$$\frac{dx}{dy} = \frac{dy}{-x} \implies x\,dx + y\,dy = 0$$

$$\implies d\left(\tfrac{1}{2}x^2 + \tfrac{1}{2}y^2\right)$$

$\implies x^2 + y^2 = \text{const.}$



For I as shown we cannot find the solution inside the shaded region as no x which which cross I enter it.

Outside this region there are still problems with the characteristics if they cross ~~too~~ more than once. Unless the data on I is entirely consistent with the development of the solution ~~of~~ along the characteris the problem is ill-posed.

Can't use characteristic as data lines.
$\rightarrow$ have no information about the rest of the solution.

Def: Characteristic line: a line whereby ~~if the line is a data line~~ knowledge of the solution on which, tells us nothing about the solution elsewhere

Methods

$f(x,y,s)=0 \qquad f = x+s^2y-s=0$

family of curves, parameters.

Eliminate s from $f=0$ and $\frac{\partial f}{\partial s}=0$
$\Rightarrow$ Envelope.

e.g. $f = x+s^2y-s=0$

$$\frac{\partial f}{\partial s}=0 \Rightarrow 2sy-1=0 \Rightarrow s=\frac{1}{2y}$$

$$\therefore x+\frac{1}{4y^2}\cdot y-\frac{1}{2y}=0$$

$$\Rightarrow x+\frac{1}{4y}-\frac{1}{2y}=0 \Rightarrow x-\frac{1}{4y}=0 \Rightarrow y=\frac{1}{4x}$$

The characteristic equations for ~~$a(x,y,u)$~~

$$a(x,y,u)\,u_x + b(x,y,u)\,u_y = 0$$

solutions of $\dfrac{dx}{a(x,y,u)} = \dfrac{dy}{b(x,y,u)}$ or $\dfrac{dy}{dx} = \dfrac{b(x,y,u)}{a(x,y,u)}$

If PDE is linear, this becomes $\dfrac{dy}{dx} = \dfrac{b(x,y)}{a(x,y)}$

Hence for linear equations, if a and b are single valued $\frac{dy}{dx}$ is unique as a function of x and y and traces cannot cross.

This is not true for general quasilinear pde's

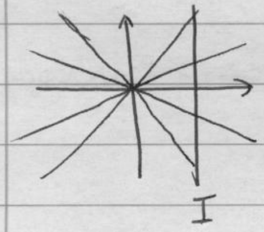
There are exceptions at points where both $a=0$, $b=0$ where $\frac{dy}{dx}$ is undetermined.

$$x\frac{du}{dx} + y\frac{du}{dy} = 0$$

$$\frac{dx}{x} = \frac{dy}{y} \implies \ln x = \ln y + \text{const.}$$

$$\therefore y = Cx \quad [C \text{ is a constant}]$$

characheristics cross at origin when $a = b = 0$



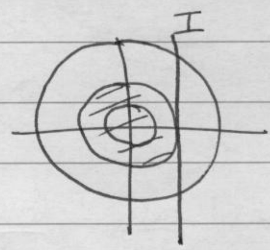
In this case we might expect a singularity in the solution to the pde at $x=0$, $y=0$ as the characteristics carrying contradictory information about the solution cross. This singularity could be avoided for particular I e.g. if a = constant on I here.

characteristics: $y\frac{du}{dx} - x\frac{du}{dy} = 0$

$$\frac{dx}{dy} = \frac{dy}{-x} \implies x\,dx + y\,dy = 0$$

$$\implies d\left(\tfrac{1}{2}x^2 + \tfrac{1}{2}y^2\right)$$

$\implies x^2 + y^2 = \text{const.}$



For I as shown we cannot find the solution inside the shaded region as no x which which cross I enter it.

Outside this region there are still problems with the characteristics if they cross more than once. Unless the data on I is entirely consistent with the development of the solution along the characteris the problem is ill-posed.

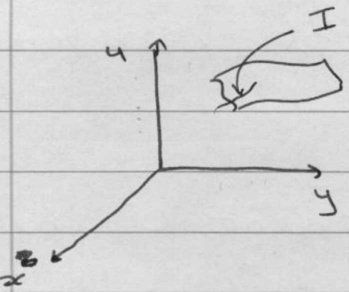
Can't use characteristic as data lines.

→ have no information about the rest of the solution.

Def: Characteristic line: a line whereby ~~if the line is a data line~~ knowledge of the solution on which, tells us nothing about the solution elsewhere

Characteristics for general inhomogeneous quasilinear equations :

$$a(x,y,u)\frac{du}{dx} + b(x,y,u)\frac{du}{dy} = c(x,y,u)$$



In solving this we are after a solution surface $u = u(x,y)$, containing I. The normal to this surface can be found by evaluating ∇g where $g(x,y,u) = u - u(x,y)$

$$\underline{n} = \begin{pmatrix} -u_x \\ -u_y \\ 1 \end{pmatrix}$$

Now consider $\begin{pmatrix} a \\ b \\ c \end{pmatrix}$ and consider $\begin{pmatrix} a \\ b \\ c \end{pmatrix} \cdot \begin{pmatrix} -u_x \\ -u_y \\ 1 \end{pmatrix}$

$$= -au_x - bu_y + c = 0$$

$$\Rightarrow \quad au_x + bu_y = c$$

Hence the vectors in the vector field $\begin{pmatrix} a \\ b \\ c \end{pmatrix}$ are normal to the solution surface and so are tangential to the solution surface.

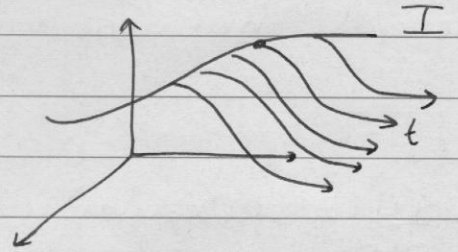
Now consider solutions to the equation

$$\frac{d\underline{r}}{dt} = \begin{pmatrix} a \\ b \\ c \end{pmatrix} \Rightarrow d\underline{r} = \begin{pmatrix} a \\ b \\ c \end{pmatrix} dt$$

Points in the direction of $\begin{pmatrix} a \\ b \\ c \end{pmatrix}$ so $d\underline{r}$ lies in the solution surface. But $d\underline{r} = \begin{pmatrix} dx \\ dy \\ du \end{pmatrix} = \begin{pmatrix} a\,dt \\ b\,dt \\ c\,dt \end{pmatrix}$

$\Rightarrow \dfrac{dx}{a} = \dfrac{dy}{b} = \dfrac{du}{c} \left(= dt\right)$ are the

charachsiss. (which lie in the solution surface and make up the solution surface.



$$\begin{pmatrix} a \\ b \\ c \end{pmatrix} \qquad \frac{dr}{dt} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$$

Solution surface is made up of characteristics coming from I.

Alternative Method The change of variables method. Consider $au_x + bu_y = c$

Consider linear equations, i.e. those of the form

$$a(x,y)\frac{\partial u}{\partial x} + b(x,y)\frac{\partial u}{\partial y} + c(x,y)u = d(x,y)$$

Consider characteristic traces, i.e. solutions to

$$\boxed{\frac{dx}{dt} = a, \; \frac{dy}{dt} = b \quad \text{or} \quad \frac{dy}{dx} = \frac{b}{a}}$$

Consider this as an ODE for $y(x)$. It has solutions given generally in the form $\phi(x,y) = \phi$ a constant

e.g. if $y = f(x) + \text{const.}$, $\phi(x,y) = y - f(x) = \phi$
i.e. the solution is a level curve of ϕ.

Use ϕ to identify particular characteristics and we need another variable to take you along the characteristic. say ξ. (Often choose $\xi = x$)
We make the change of variables ~~for~~ from x and y to ϕ and ξ.

We make the change of variables from x and y to ϕ and ξ.

E.g. Solve $x\frac{du}{dx} - 7y\frac{du}{dy} = x^2y$

solve for characteristic traces, i.e. solve $\frac{dy}{dx} = \frac{-7y}{x}$

$$\Rightarrow \int \frac{dy}{y} = -7\int \frac{dx}{x} \Rightarrow \ln y = -7\ln x + \text{const.}$$

i.e. $yx^7 = \phi$

Now make a change of varibles from from x and y to ϕ and ξ with $\phi(x,y) = yx^7$

using $x = \xi$ (doesn't have to be)

So $\frac{\partial u}{\partial x} = \frac{\partial u}{\partial \phi}\cdot\frac{\partial \phi}{\partial x} + \frac{\partial u}{\partial \xi}\cdot\frac{\partial \xi}{\partial x}$ now we consider $u = u(\phi, \xi)$

So $\frac{\partial u}{\partial x} = 7yx^6\frac{\partial u}{\partial \phi} + \frac{\partial u}{\partial \xi}$

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial \phi}\cdot\frac{\partial \phi}{\partial y} + \frac{\partial u}{\partial \xi}\cdot\frac{\partial \xi}{\partial y} = x^7\frac{\partial u}{\partial \phi}$$

So substituting,

$$x\left(7yx^6\frac{\partial u}{\partial \phi} + \frac{\partial u}{\partial \xi}\right) - 7yx^7\frac{\partial u}{\partial \phi} = x^2y$$

$$\Rightarrow \quad x\frac{\partial u}{\partial \xi} = x^2y \Rightarrow \frac{\partial u}{\partial \xi} = xy = x\cdot\frac{\phi}{x^7} = \frac{\phi}{\xi^6}$$

This equation tells you how u varies as you move along a characteristic i.e. for fixed ϕ.

arb function of ϕ

$$\Rightarrow \quad u = -\frac{1}{5}\phi\xi^{-5} + f(\phi)$$

$$\Rightarrow u = -\frac{1}{5}\frac{yx^7}{x^5} + f(yx^7)$$

$$\Rightarrow u = -\frac{1}{5}yx^2 + f(yx^7)$$

If the boundrey / initial conditions are, for example, $u=0$ on $y=x^2$, then we need

$$0 = -\frac{1}{5}x^2 \cdot x^2 + f(x^2 x^7)$$

$$f(x^9) = \frac{1}{5}x^4$$ $\Rightarrow$ $\therefore$ if we have $r = x^9$

$$f(r) = \frac{1}{5} r^{\frac{4}{9}}$$ and our solution is $u(x,y)$

$$= -\frac{1}{5}yx^2 + \frac{1}{5}(yx^2)^{\frac{4}{9}}$$

$\boxed{\text{CODE}}$ by seeking a parametric solution (s,t method)
by showing (some $\phi(x,y)$) is constant on characteristics, solve eq. (ϕ, ξ) method.

E.g.

$$x\frac{du}{dx} + (x^2+y^2)\frac{du}{dy} + \left(\frac{y}{x} - x\right)u = 1.$$

characteristic traces satisfy $\frac{dy}{dx} \quad \frac{x^2+y}{x} = \frac{y}{x} + x$

i.e. $\frac{dy}{dx} - \frac{y}{x} = x \qquad$ i.f $= e^{-\int \frac{1}{x}dx} = \frac{1}{x}$

$$\Rightarrow \frac{d}{dx}\left(\frac{y}{x}\right) = 1 \Rightarrow \frac{y}{x} = x + \text{const.}$$

So $\frac{y}{x} - x$ is constant on characteristics.

Make a change of variables from x and y to ϕ and ξ where $\phi(x,y) = \frac{y}{x} - x$ and $\xi(x,y) = x$

Lagrangés Method

Consider $a(x,y,u)\frac{\partial u}{\partial x} + b(x,y,u)\frac{\partial u}{\partial y} = c(x,y,u)$

Characteristic curves are tangent to the vectors $\begin{pmatrix} a \\ b \\ c \end{pmatrix}$ i.e. satisfy $\frac{dx}{dt} = a$, $\frac{dy}{dt} = b$, $\frac{du}{dt} = c$

$$\frac{dx}{a} = \frac{dy}{b} = \frac{du}{c}$$

Lagranges method asks you to find two constants of integration of these equations.

$$S_1(x,y,u) = C_1 \qquad S_2(x,y,u) = C_2$$

Then the general solution of the pde is

given by $C_1 = f(C_2)$ i.e.

$$S_1 = f(S_2)$$

Varying C_2 gives a family of surfaces S_2 given by $S_2(x,y,u) = C_2$. Varying C_1 gives a similar family of surfaces, $S_1(x,y,u) = C_1$. A surface S_1 intersects a surface S_2 is a line which is a characteristic line. If we relate C_1 to C_2 through $C_1 = f(C_2)$ + vary C_2, we get a one parameter set of lines of intersection, of surfaces S_1 + S_2

This one-parameter set defines another solution surface. We can choose f appropriately so that our boundrey conditions are satisfied.

Easy Ex. $\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} = 1$ $u = -x^2$ on $y = 0$

χ eq are $\frac{dx}{dt} = 1$, $\frac{dy}{dt} = 1$, $\frac{du}{dt} = 1$

$\frac{dy}{dx} = 1 \therefore y - x = C_1$ / $\frac{du}{dy} = 1 \Rightarrow u - y = C_2$ /

also $u - x = C_3$ /

The general solution is given by:

$$C_3 = f(C_2) \quad \text{(say)}$$

i.e. $u - x = f(u-y)$ ← general solution as can be seen

$\frac{\partial}{\partial x}$: $\frac{\partial u}{\partial x} - 1 = f'(u-y)\frac{\partial u}{\partial x}$

$\frac{\partial}{\partial y}$: $\frac{\partial u}{\partial y} = f'(u-y)\left(\frac{\partial u}{\partial y} - 1\right)$

eliminate $f'(u-y)$

$$\Rightarrow \frac{u_x - 1}{u_x} = \frac{u_y}{u_y - 1} \Rightarrow \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} = 1$$

Now choose so that $u = x^2$ on $y = 0$

$$u - x = f(u-y)$$
$$-x^2 - x = f(-x^2 - 0) \Rightarrow f(-x^2) = -x^2 - x$$
$$\Rightarrow f(r) = r \mp \sqrt{-r}$$

∴ solution is $(u-x) = (u-y) \pm \sqrt{y-u}$

$$\Rightarrow u = y - (x-y)^2$$

Solve $\quad x(y^2-u^2)\frac{\partial u}{\partial x} + y(u^2-x^2)\frac{\partial u}{\partial y} = u(x^2-y^2)$

Characteristic equations are:

$$\frac{dx}{dt} = x(y^2-u^2)$$

$$\frac{dy}{dt} = y(u^2-x^2)$$

$$\frac{du}{dt} = u(x^2-y^2)$$

Consider $\quad x\frac{dx}{dt} + y\frac{dy}{dt} + u\frac{du}{dt}$

$$= x^2(y^2-u^2) + y^2(u^2-x^2) + u^2(x^2-y^2)$$

$$= 0$$

$$\Rightarrow \frac{d}{dt}\left(\frac{x^2+y^2+u^2}{2}\right) = 0$$

$$\Rightarrow \frac{x^2+y^2+u^2}{2} = C_1$$

Methods

Eg. 1 $\dfrac{dx}{a} = \dfrac{dy}{b} = \dfrac{du}{c} = dt$

find constants $C_1 + C_2$ sol^n is $C_1 = f(C_2)$

$$\underbrace{x(y^2-u^2)}_{a}\frac{\partial u}{\partial x} + \underbrace{y(u^2-x^2)}_{b}\frac{\partial u}{\partial y} = \underbrace{u(x^2-y^2)}_{c}$$

$$\frac{\partial x}{\partial t} = x(y^2-u^2) \quad \frac{\partial y}{\partial t} = y(u^2-x^2) \quad \frac{\partial u}{\partial t} = u(x^2-y^2)$$

$$x\frac{\partial x}{\partial t} + y\frac{\partial y}{\partial t} + u\frac{\partial u}{\partial t} = 0 \Rightarrow \frac{1}{2}(x^2+y^2+u^2) = C$$

$$\therefore \; x^2+y^2+u^2 = C_1$$

$$yu\frac{\partial x}{\partial t} + xu\frac{\partial y}{\partial t} + xy\frac{\partial u}{\partial t} = xyu(y^2-u^2) + xyu(u^2-x^2)$$
$$+ xyu(x^2-y^2) = 0$$

$$\Rightarrow \frac{d}{dt}(xyu) = 0 \Rightarrow xyu = C_2$$

$$\therefore \; xyu = f(x^2+y^2+u^2)$$

then use boundrey conditions to find f.

E.g 2 $(y+u)\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} = x-y$

χ eqns are 1) $\dfrac{dx}{dt} = y+u$, 2) $\dfrac{dy}{dt} = y$, 3) $\dfrac{du}{dt} = x-y$

2) $\Rightarrow$ $y = Ae^t$

1) $\Rightarrow$ $\dfrac{d^2x}{dt^2} = \dfrac{dy}{dt} + \dfrac{du}{dt} = y + x - y = x \Rightarrow \dfrac{d^2x}{dt^2} = x.$

$\therefore$ $x = Ce^t + De^{-t}$

3) $\dfrac{d^2y}{dt^2} = \dfrac{dx}{dt} - \dfrac{dy}{dt} = y + u - y = u$

$\therefore$ $u = Ee^t + Fe^{-t}$. We have 5 constants of integration while we should only expect 3.

Using $\dfrac{dx}{dt} = y+u$ gives $Ce^t - De^{-t} = Ae^t + Ee^t + Fe^{-t}$
$\Rightarrow C = A+E$ and $F = -D$

$\frac{y}{A} = e^t, \quad \frac{A}{y} = e^{-t}$

$y = Ae^t, \quad x = Ce^t + De^{-t}, \quad u = (C-A)e^t - De^{-t}$

$x + u = (2C - A)e^t = (2C - A)\left(\frac{y}{A}\right)$

$\therefore \frac{x+u}{y} = \left(\frac{2C-A}{A}\right) = \text{constant.}$

$\therefore \frac{x+u}{y} = C_1$ /

Also $u + y = Ce^t - De^{-t}$

$\therefore x - (u+y) = 2De^{-t} = 2D\frac{A}{y}$

$\therefore y(x-u-y) = 2DA = \text{constant.}$

$y(x-u-y) = C_2$ /

$\therefore$ general solution can be written

$y(x-u-y) = f\left(\frac{x+u}{y}\right)$ [Sub back in and check]

$\boxed{\text{Slicker way}}$

① $dx = (y+u)dt$ ② $dy = y\,dt$ ③ $du = (x-y)\,dt$

① + ③ $dx + du = (x+u)dt$

$\therefore d(x+u) = (x+u)dt \Rightarrow$

$\frac{d}{dt}(x+u) = \frac{dx}{dt} + \frac{du}{dt}$

$\therefore \frac{d(x+u)}{x+u} = dt = \frac{dy}{y}$

$\therefore \ln(x+u) = \ln y + \text{constant}$

$\Rightarrow \frac{x+u}{y}$ is a constant /

① and ② give $dx = dy + u\,dt \quad \therefore d(x-y) = u\,dt = u \cdot \frac{du}{(x-y)}$

$\therefore (x-y)\,d(x-y) = u\,du$

$\therefore \frac{1}{2}(x-y)^2 = \frac{1}{2}u^2 + \text{Const}$

$\therefore (x-y)^2 - u^2 = \text{constant.}$

$\therefore \frac{x+u}{y} = f\left((x-y)^2 - u^2\right)$

$\boxed{\text{both acceptable answers}}$

Second order PDE's

The general second order quasilinear pde is

$$a(x,y,z,z_x,z_y) + b(x,y,z,z_x,z_y)\frac{\partial^2 z}{\partial x \partial y} + c(x,y,z,z_x,z_y)\frac{\partial^2 z}{\partial y^2}$$
$$= r(x,y,z,z_x,z_y)$$

The quantity $\Delta = b^2 - 4ac$ is the discriminant of the ~~differential~~ pde.

if $\Delta > 0$: equation is <u>hyperbolic</u> (waves)

if $\Delta < 0$: equation is <u>elliptic</u> (Steady temp. dist.)

if $\Delta = 0$: equation is <u>parabollic</u> (diffusion)

if $a, b, c = 0$ and $r = 0$

$$a z_{xx} + b z_{xy} + c z_{yy} = 0$$

Look for a solution of the form $z = f(y + mx)$

substitute to find.

$$\therefore a m^2 f'' + b m f'' + c f'' = 0$$

If $\Delta > 0$ (hyperbolic) we have two real roots for m.
If $\Delta > 0$ (elliptic) " two complex "
If $\Delta = 0$ (parabolic) " one real repeated root "

Lets call these m_1 and m_2

$\cancel{\frac{\partial z}{\partial x} =}$ introduce $s = y + m_1 x$, $t = y + m_2 x$ (canonical variables)

and make a change of variable from x and y to s and t

$z_x \rightarrow z_s$ and z_t etc.

$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial s}\cdot\frac{\partial s}{\partial x} + \frac{\partial z}{\partial t}\cdot\frac{dt}{dx} = m_1\frac{\partial z}{\partial s} + m_2\frac{\partial z}{\partial t}$$

$$\frac{\partial z}{\partial y} = \frac{\partial z}{\partial s}\cdot\frac{\partial s}{dy} + \frac{\partial z}{\partial t}\frac{dt}{dy} = \frac{\partial z}{\partial s} + \frac{\partial z}{\partial t}$$

$$\frac{\partial}{\partial x} = m_1\frac{\partial}{\partial s} + m_2\frac{\partial}{\partial t}$$

$$\frac{\partial}{\partial y} = \frac{\partial}{\partial s} + \frac{\partial}{\partial t}$$

$$\therefore z_{xx} = (m_1\partial_s + m_2\partial_t)(m_1 z_s + m_2 z_t)$$

$$= m_1^2 z_{ss} + 2m_1m_2 z_{st} + m_2^2 z_{tt} \quad /$$

$$z_{yy} = z_{ss} + 2z_{st} + z_{tt} \quad /$$

$$z_{xy} = (m_1\partial_s + m_2\partial_t)(z_2 + z_6)$$

$$= m_1 z_{ss} + (m_1+m_2) z_{st} + m_2 z_{tt}$$

Sub into $az_{xx} + bz_{xy} + cz_{yy} = 0$

to find:

$$z_{ss}(\underbrace{am_1^2 + bm_1 + c}_{=0}) + z_{st}(2am_1m_2 + b(m_1+m_2) + 2c)$$

[m_1, m_2 are roots of $ax^2+bx+c=0$]

aux. eq^n

$$+ z_{tt}(\underbrace{am_2^2 + bm_2 + c}_{=0}) = 0$$

also, we know $m_1 + m_2 = -\frac{b}{a}$, $m_1m_2 = \frac{c}{a}$

$$\therefore z_{st}\left(2c - \frac{b^2}{a} + 2c\right) = -\frac{\Delta}{a} z_{st} \qquad \Delta \neq 0 \text{ here}$$

This equation is now in <u>canonical</u> form.

$$\frac{\partial^2 z}{\partial s\partial t} = 0 \qquad f'(t) \qquad z(st) = f(t) + g(s)$$

$$= f(y + m_2x) + g(y + m_1x)$$

So, the general solution of a hyperbolic equation may be written in the form: $z = f(y+m_2x) + g(y+m_1x)$ with m_1, m_2 being the roots of $am^2 + bm + c$

The lines $y + m_2x = \text{cons.}$
$y + m_1x = \text{cons.}$ are called characteristics

Solutions of elliptic problems can also be written in this way, but the solutions are complex.

If $\Delta = 0$ and equation is parabolic we have one root $m = \frac{-b}{2a}$. Here we switch to variables $s = y + mx$ $t = x$ <u>Do this</u>

and find that $(\underbrace{2ma + b}_{=0}) z_{st} + z_{tt} = 0$

$\Rightarrow z_{tt} = 0.$

$z_t = g(s)$. $z(s,t) = t g(s) + f(s)$ and the general solution for parabolic equations is

$$z(x,y) = x\, g(y+mx) + f(y+mx)$$

<u>e.g.1</u> $z_{xx} - 3z_{xy} + 2z_{yy} = 0$

If $z = f(y+mx)$ then we need $m_1^2 - 3m_1 + 2 = 0$

[but if $z = f(x+my)$ then $2m_2^2 - 3m_2 + 1 = 0$, $m_1 = \frac{1}{m_2}$]

$(m-2)(m-1) = 0$

i.e. two distinct real values of m. $(1, 2)$.

and solution is $z = f(y+x) + g(y+2x)$

<u>e.g.2</u> $z_{xx} - 2z_{xy} + z_{yy} = 0$

$\therefore m^2 - 2m + 1 = 0.$ with $z = f(y+mx)$

$\therefore (m-1)^2 = 0.$

$\therefore$ equation is parabolic and the solution is

$$z = x f(y+x) + g(y+x)$$

<u>e.g.3</u> $z_{xx} - 3z_{xy} + 2z_{yy} = e^{x-y}$

As the equation is linear: solution has the form $y = C.F. + P.I.$ C.F is solution to LHS
P.I is anything that satisfies eq.

If $z = f(y+mx) \Rightarrow m^2 - 3m + 2 = 0$.

$\therefore m = 2, 1$

$\therefore z = f(y+x) + g(y+2x)$

the P.I. is the challenge.

try $z = Ae^{x-y}$

$z_x = Ae^{x-y} \quad \therefore z_{xx} = Ae^{x-y}$ /

$z_y = -Ae^{x-y} \quad z_{yy} = Ae^{x-y}$ /

$z_{xy} = -Ae^{x-y}$ /

sub in, $Ae^{x-y} - 3(-Ae^{x-y}) + 2Ae^{x-y} = e^{x-y}$

$\therefore e^{x-y}(6A) = e^{x-y} \Rightarrow A = \frac{1}{6}$ /

$\therefore z(x,y) = \underbrace{\frac{1}{6}e^{x-y}}_{\text{P.I.}} + \underbrace{f(y+x) + g(y+2x)}_{\text{C.F.}}$

<u>Alternatively</u> We can change varibles to the canonical variables. $s = x+y$. $t = x+2t$

LHS goes to $-\frac{\Delta}{a} z_{st}$ $\quad \Delta$ is disc. of $m^2 - 3m + 2$

$= 9 - 8 = 1$ / $\therefore \Delta > 0$.

$\Rightarrow -z_{st} = e^{x-y}$

$x = t - s$, $y = s - x = 2s - t$.

$\therefore -z_{st} = e^{(t-s) - (2s-t)} = e^{2t-3s}$

If P.I. is hard,
try to use canonical form.

$\therefore z = -\frac{1}{6} \cdot -\frac{1}{3} e^{2t-3s} + f'(t)$

$\therefore z = -\frac{1}{6} e^{2t-3s} + f(t) + g(s)$

$\frac{1}{6}$ comes out again

The Wave Equation

A simple physically relevant, second order hyperbolic equation, is the wave equation:

$$\frac{1}{c^2}\frac{\partial^2 z}{\partial t^2} = \frac{\partial^2 z}{\partial x^2}$$

for $z(x,t)$ for some constant c, known as the wave speed.

For the present we will consider $-\infty < x < \infty$, $t \geq 0$.

i.e. we look at initial value problems.

. We look for a solution of the form $z = f(x+mt)$.

Substitution gives $\frac{m^2}{c^2}f'' = f'' \Rightarrow m = \pm c$. ∴ equation is hyperbolic. (Two real roots for m) and so general solution.

∴ z is $z(x,t) = f(x-ct) + g(x+ct)$

This solution is made up of one wave travelling to the right without change of form. $[f(x-ct)]$ and one moving to the left $[f(x+ct)]$

To solve an initial value problem we need to find f and g such that $t=0$, $z(x,0) = F(x)$.

$\frac{\partial z}{dt}(x,0) = F(x)$

we know F and G. If $z(x,t) = f(x-ct) + g(x+ct)$

$z(x,0) = f(x) + g(x) = F(x)$. $z_t(x,t) = -cf'(x-ct) + cg'(x+ct)$

$\therefore z_t(x,0) = -cf'(x) + cg'(x) = G(x)$

$\therefore -f + g = \frac{1}{c}\int_\alpha^x G(\xi)\,d\xi$ for constant of integration α

$$f+g = F.$$

$$\therefore \quad g = \frac{1}{2}F + \frac{1}{2c}\int_{\alpha}^{x} G(\xi)\,d\xi, \quad f = \frac{1}{2}F - \frac{1}{2c}\int_{\alpha}^{x} G(\xi)\,d\xi$$

$$z(x,t) = f(x-ct) + g(x+ct)$$

$$= \frac{1}{2}\left\{F(x-ct) + F(x+ct)\right\} + \frac{1}{2c}\left\{\int_{x-ct}^{\alpha} G(\xi)\,d\xi + \int_{\alpha}^{x+ct} G(\xi)\,d\xi\right\}$$

$$= \frac{1}{2}\left\{F(x-ct) + F(x+ct)\right\} + \frac{1}{2c}\int_{x-ct}^{x+ct} G(\xi)\,d\xi.$$

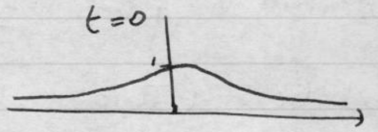
D'Alembert's solution to the wave equation.

$\boxed{\text{learn how to derive}}$

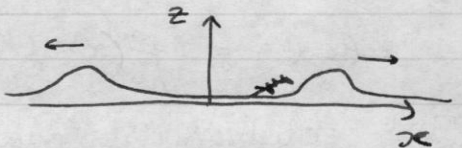
If $G=0$ and $F(x) = e^{-x^2}$

$$\therefore z(x,t) = \frac{1}{2}\left(e^{-(x-ct)^2} + e^{-(x+ct)^2}\right)$$

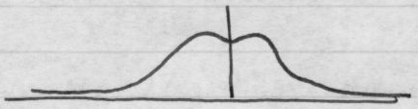
$t=0$



F



large t

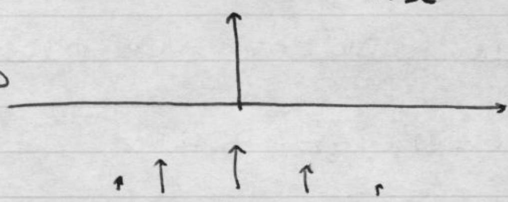


small t.

$F=0, \quad G(x) = \dfrac{1}{1+x^2}$

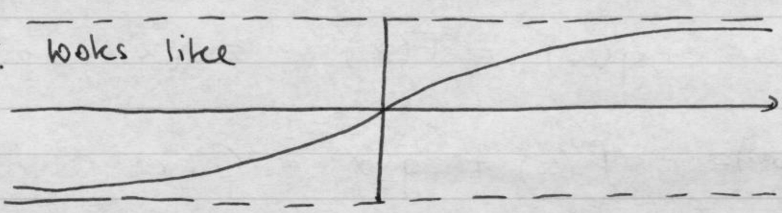
$t=0$

$z=0$



$$z(x,t) = \frac{1}{2c}\int_{x-ct}^{x+ct} \frac{1}{1+\xi^2}\,d\xi = \frac{1}{2c}\left[\tan^{-1}(x+ct) - \tan^{-1}(x-ct)\right]$$

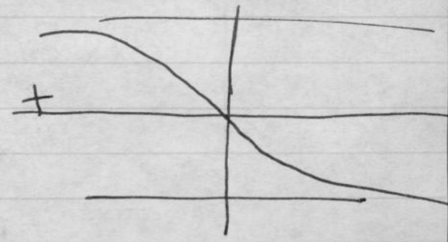
$\therefore$ looks like

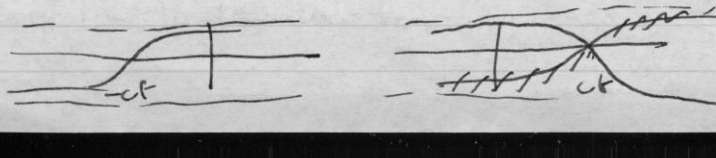


$\frac{\pi}{2}$

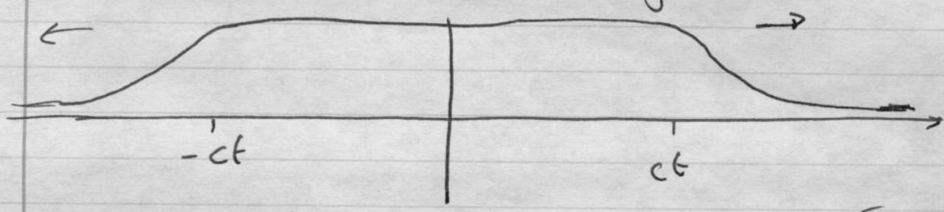
$-\frac{\pi}{2}$

$+$

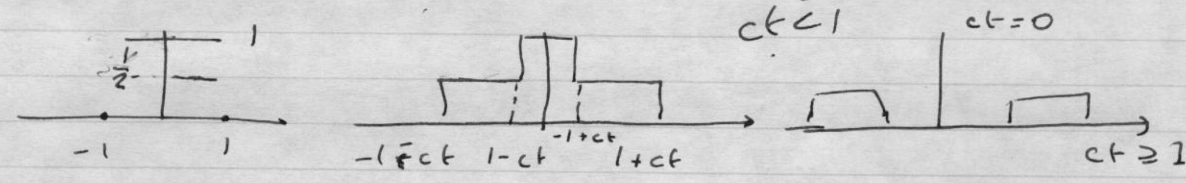


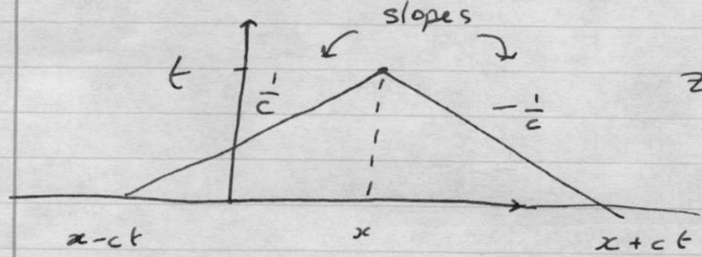


$\therefore$ the combination gives

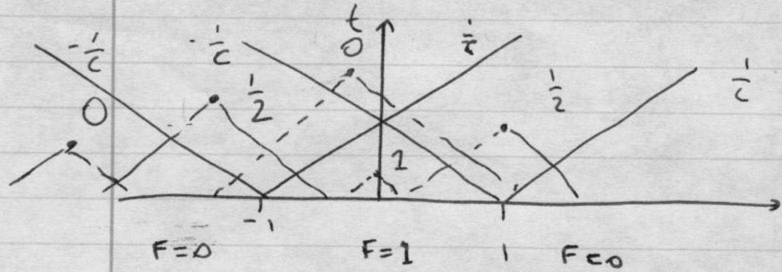


Example $\quad G = 0 \quad F(x) = \begin{cases} 1 & |x| \le 1 \\ 0 & |x| > 1 \end{cases}$





$$z = \frac{1}{2}\left(F(x-ct) + F(x+ct)\right)$$



Methods

$$\frac{1}{c^2}\frac{\partial^2 z}{\partial t^2} = \frac{\partial^2 z}{\partial x^2}$$

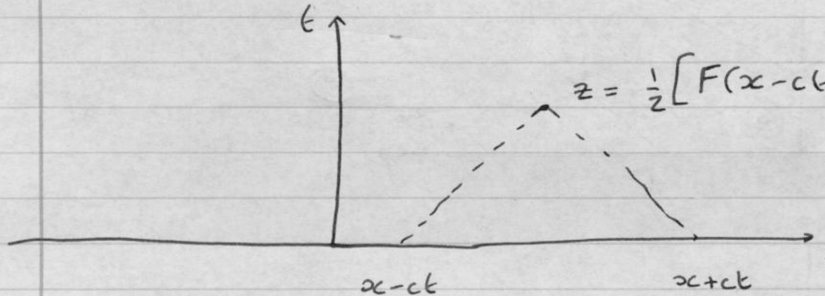
If $z = F(x)$ and $z_t = G(x)$ at $t=0$

$$z(x,t) = \frac{1}{2}\left(F(x+ct) + F(x-ct)\right) + \frac{1}{2c}\int_{x-ct}^{x+ct} G(\xi)\,d\xi$$ ← ~~A4~~ D'Alemberts solution.

$$= f(x-ct) + g(x+ct)$$

Information about solution travels at speed c.

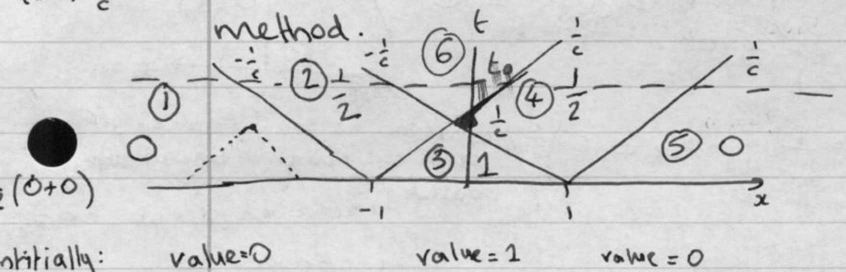
$$\boxed{\text{- two sets of characteristics} \quad x-ct = \text{const.} \quad x+ct = \text{const.}}$$

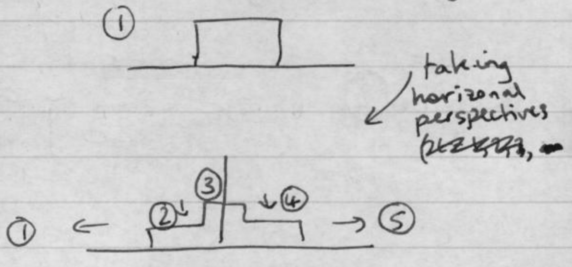


$z = \frac{1}{2}\left[F(x-ct) + F(x+ct)\right]$

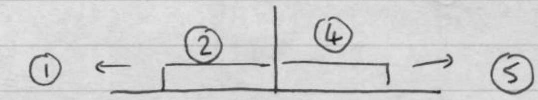
← any point z can be calculated by ξ creating ~~the~~ the characteristic lines of slope $\pm\frac{1}{c}$ and find the intersection with the x axis. (We know solution along this axis)

Can look at the blocks in a new way using this method.

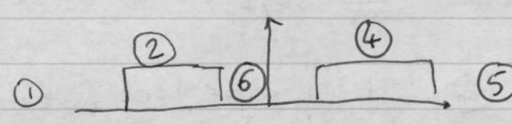


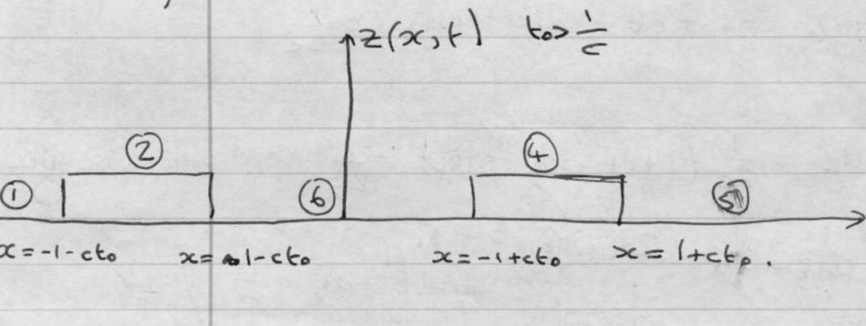


To find what the wave looks like for a given value of $t = t_0$, draw a horizontal line at t_0 and then read off regions



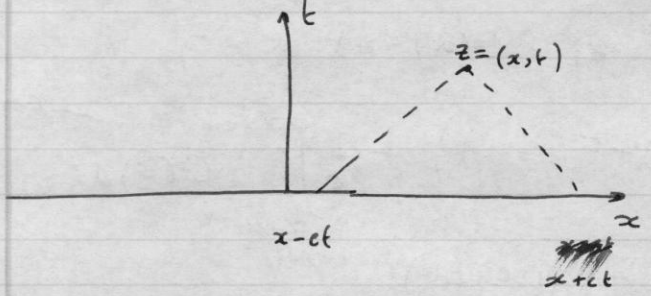
∴ for our t_0:





$F = 0, \quad G \neq 0.$

$$z(x,t) = \frac{1}{2c}\int_{x-ct}^{x+ct} G(\xi)\,d\xi$$

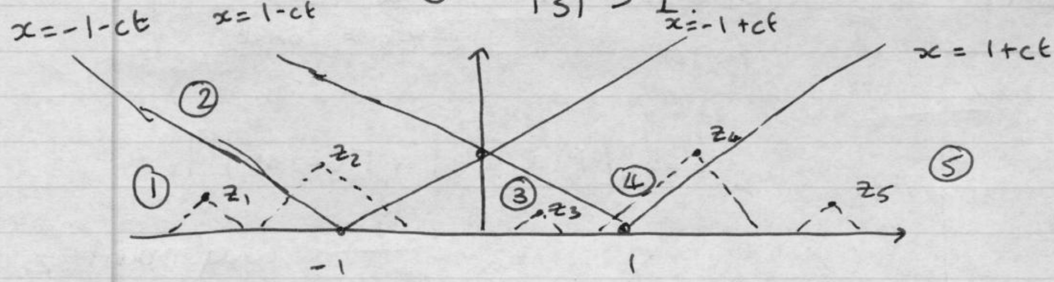


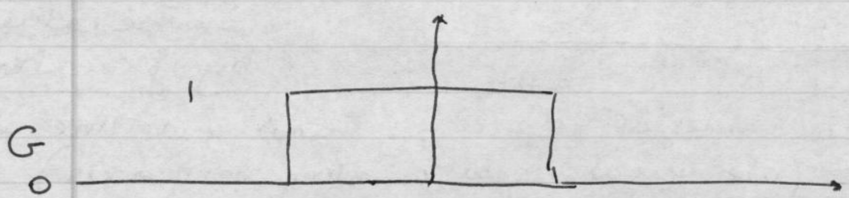
still same characteristics.

z is now <u>not</u> $\frac{1}{2}\left(F(x+ct)+F(x-ct)\right)$

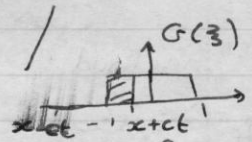
it is $\displaystyle\int_{x-ct}^{x+ct} g(\xi)\,d\xi.$

$$G(\xi) = \begin{cases} 1 & |\xi| < 1 \\ 0 & |\xi| > 1 \end{cases}$$





Region ① $\quad z_1 = \displaystyle\int_{x-ct}^{x+ct} G(\xi)\,d\xi = \int_{x-ct}^{x+ct} 0\,d\xi = 0$ /



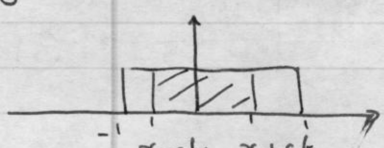
$x-ct < -1, \; -1 < x+ct < 1$

Region ② $\quad z_2 = \dfrac{1}{2c}\displaystyle\int_{-1}^{x+ct} 1\,d\xi = \frac{1}{2c}\left(1+x+ct\right)$ / which as a function of x is a straight line of slope $\frac{1}{2c}$.

valid $\quad -1-ct < x < -1+ct \quad t < \frac{1}{c}$

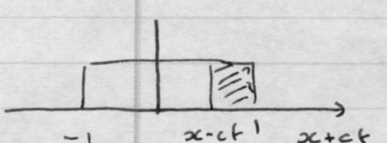
$-1-ct < x < 1-ct \quad t > \frac{1}{c}$

Region ③ $\quad -1+ct < 1-ct \qquad t < \frac{1}{c}$



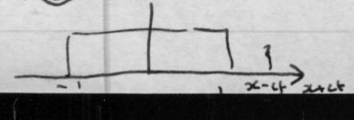
$\therefore\; z_3 = \dfrac{1}{2c}\displaystyle\int_{x-ct}^{x+ct} 1\,d\xi = t$ /

Region ④ $\quad 1-ct < x < 1+ct, \; t < \frac{1}{c}. \quad -1+ct < x < 1+ct \quad t > \frac{1}{c}$



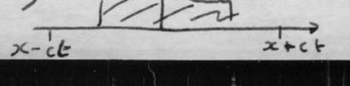
$z_4 = \dfrac{1}{2c}\displaystyle\int_{x-ct}^{1} 1\,d\xi = \frac{1}{2c}\left(1+ct-x\right)$ / slope $-\frac{1}{2c}$ but otherwise like ②

Region ⑤ $\quad 1 < x-ct < x+ct \quad \therefore\; z_5 = \dfrac{1}{2c}\displaystyle\int_{x-ct}^{x+ct} 0\,d\xi = 0$ /



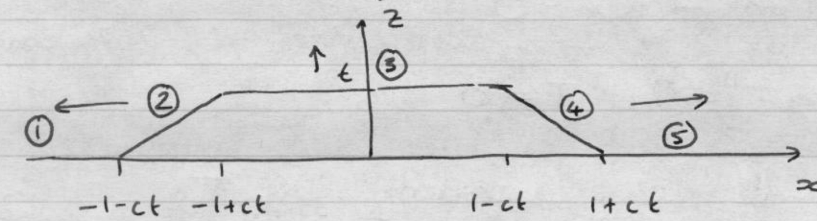
Region ⑥ $\quad t > \frac{1}{c} \quad x-ct < -1 < 1 < x+ct$

$\therefore\; z = \dfrac{1}{2c}\displaystyle\int_{-1}^{1} 1\,d\xi = \frac{1}{c}$ /

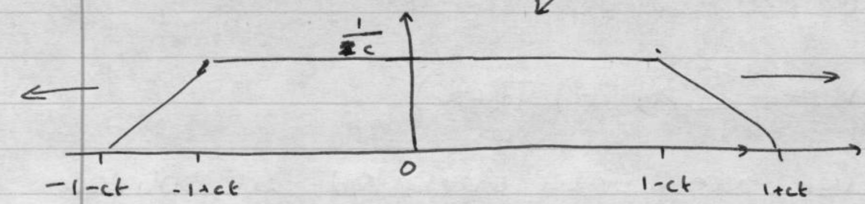


Drawing these for different values of $t = t_0$

$z(x,t) \qquad t_0 < \frac{1}{\cancel{2}c}$



$t_0 \geq \frac{1}{\cancel{2}c}$ top bit stops increasing.



Solution of Wave equation using the method of separation of variables

The wave equation is $\frac{1}{c^2}\frac{\partial^2 z}{\partial t^2} = \frac{\partial^2 z}{\partial x^2}$

Look for a solution $z = \underbrace{X(x)}_{\text{function of } x \text{ alone}}\underbrace{T(t)}_{\text{function of } t \text{ alone}}$

$$z_{tt} = X T''$$

$$z_{xx} = X'' T$$

sub in equation $\therefore \frac{XT''}{c^2} = X''T \implies \underbrace{\frac{T''}{c^2 T}}_{\text{function of } T} = \underbrace{\frac{X''}{X}}_{\text{function of } X}$

Imagine changing t but not x, LHS might change but the RHS must remain constant. We deduce that the LHS does not change and both $\frac{T''}{c^2T}$ and $\frac{X''}{X}$ are the same constant, independent of both x and t.

Let call this constant λ and refer to it as the separation constant.

Therefore $\frac{T''}{c^2T} = \frac{X''}{X} = \lambda$

$\therefore X'' - \lambda X = 0$ (λ could actually be complex)

$T'' - \lambda c^2 T = 0$

Any λ will do, generating X_λ, T_λ.

so $z_\lambda(x,t) = X_\lambda(x) T_\lambda(x)$ and as the wave equation is linear, any combination of these is also a solution.

$$\therefore z(x,t) = \sum_\lambda X_\lambda(t) T_\lambda(x)$$

However we are only interested in solutions satisfying particular initial conditions and, more relevant now, boundrey conditions. It is the boundrey conditions that restrict the values of λ.

string of finite length.

If we solve $z_{tt} = c^2 z_{xx}$ for $t \geq 0$, $0 \leq x \leq L$ and with boundrey conditions $z(0,t) = 0$ $z(L,t) = 0$.

$$z(x,t) = X(x)T(t)$$

then we need $X(0)T(t) = 0 \ \forall t \Rightarrow X(0) = 0$
and $X(L)T(t) = 0 \ \forall t \Rightarrow X(L) = 0$.

sub in wave equation, $XT'' = c^2 TX'' \Rightarrow \frac{T''}{c^2T} = \frac{X''}{X} = \lambda$

$$\therefore \boxed{X'' - \lambda X = 0 \qquad X(0) = X(L) = 0.}$$

It turns out that for particular values of λ, we get solutions to this other than the obvious $X = 0$.

3 Cases, $\lambda > 0$, $\lambda = 0$, $\lambda < 0$. $\lambda \in \mathbb{R}$

$\boxed{\text{Look at } \lambda \in \mathbb{C}?}$

$\underline{\lambda > 0}$, $\lambda = p^2$, $p \in \mathbb{R}$, $X'' - p^2 X = 0$

$\therefore$ $X'' - p^2 X = 0$ which has exponential solutions.

$X = Ae^{px} + Be^{-px}$ or $X = \tilde{A}\cosh px + \tilde{B}\sinh px$. ← use this form.

but $X(0) = 0$, $\Rightarrow$ $\tilde{A}.1 + \tilde{B}.0 = 0 \Rightarrow \tilde{A} = 0$ /

and $X(L) = 0$. $\Rightarrow$ $\tilde{B}\sinh pL = 0 \Rightarrow \tilde{B} = 0$ /

$\therefore$ <u>no none-zero solutions</u> $X = 0$

$\underline{\lambda = 0}$, $X'' = 0$, $X = Ax + B$ straight line

& This straight line must join $X(0) = X(L) = 0$

<u>Hence $X = 0$. no non-zero solutions</u>

$\lambda < 0$, $\lambda = -p^2$, $X'' + p^2 X = 0$

$\therefore X = A\cos px + B\sin px$

$X(0) = 0$, $A.1 + B.0 = 0 \Rightarrow A = 0$

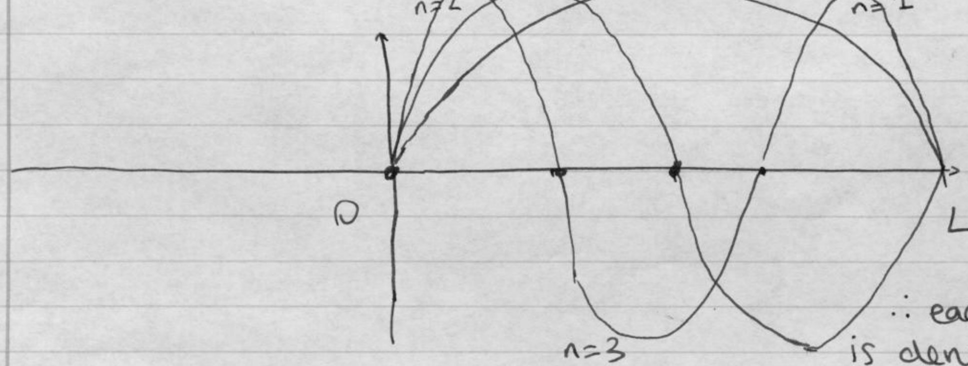
$X(L) = 0$, $\therefore B\sin(pL) = 0$ one way of doing this is having $B = 0$, $\Rightarrow X = 0$.

but if we look for none-zero solution, then $B \neq 0$.

$\therefore \sin(pL) = 0$ $\therefore$ $pL = n\pi$ $n \in \mathbb{N}$

$\Rightarrow$ $p = \frac{n\pi}{L}$ $n \in \mathbb{N}$. $\Rightarrow \lambda = -\frac{n^2\pi^2}{L^2}$

We have an infinite number of possible separation constants λ



etc.. countably infinite number of solutions therefore

$\therefore$ each different solution is denoted $\underline{\underline{X_n(x)}}$.

Now for T_n:

Recall $\dfrac{T_n''}{c T_n} = \lambda = -p^2 = -\left(\dfrac{n\pi}{2}\right)^2$

$$\therefore T_n'' + \frac{c^2 n^2 \pi^2}{L^2} T_n = 0$$

$$\therefore T_n = C\cos\left(\frac{cn\pi t}{L}\right) + D\cos\left(\frac{cn\pi t}{L}\right)$$

Fourier series

$$z(x,t) = \sum_n X_n(x)\, T_n(t) = \sum_{n=1}^{\infty} \sin\left(\frac{n\pi x}{L}\right)\left(C_n\cos\left(\frac{cn\pi t}{L}\right) + D_n\sin\left(\frac{cn\pi t}{L}\right)\right)$$

~~Fourier ser~~

2/12/11

Wave Equation.

$$\frac{1}{c^2}\frac{\partial^2 z}{\partial t^2} = \frac{\partial^2 z}{\partial x^2} \qquad \begin{aligned} z(0,t) &= 0 \\ z(l,t) &= 0 \end{aligned}$$

$z = X(x)\,T(t)$

$$\Rightarrow \frac{1}{c^2}\frac{T''}{T} = \frac{X''}{X} = \lambda = -p^2$$ so be satisfied.

$$X = A\sin px \quad , \quad p = \frac{n\pi}{L} \quad , \quad \begin{aligned} X(0) &= 0 \\ X(L) &= 0 \end{aligned}$$

$$T'' + c^2p^2T = 0.$$

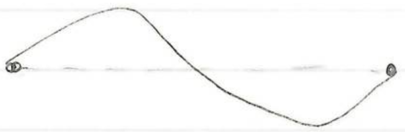
$$\Rightarrow T = C\cos\left(\frac{n\pi ct}{L}\right) + D\sin\left(\frac{n\pi ct}{L}\right).$$

$$X = \sum_1^\infty \sin\left(\frac{n\pi x}{L}\right)\left[C_n\cos\left(\frac{n\pi ct}{L}\right) + D_n\sin\left(\frac{n\pi ct}{L}\right)\right]$$

Each term in this sum is known as a normal mode oscillates with a normal frequency $\omega_n = n\left(\frac{\pi c}{L}\right)$. The mode $n=1$ is called the fundametal and the nodes $n = 2, 3, \ldots$ higher harmonics and the solution for $z(x,t)$ is a mixture sum of the fundermental and the harmonics.

$\sin\left(\frac{\pi x}{L}\right)$

fundamental



first harmonic

Second harmonic

The values of C_n and D_n, the amplitude of each mode is determine from initial condition. If $z(x,0) = F(x)$ and $z_t(x,0) = G(x)$ then putting $t = 0$ gives:

$$F(x) = \sum_{1}^{\infty} \sin\left(\frac{n\pi x}{L}\right) C_n .$$

$$\begin{pmatrix} \text{Sin} & \text{is} & \text{zero} & \text{at} & t = 0 \\ \text{Cos} & \text{is} & 1 & \text{at} & t = 0 \end{pmatrix}$$

Differentiating wrt t and putting $t = 0$ gives:

$$G(x) = \sum_{1}^{\infty} \sin\left(\frac{n\pi x}{L}\right) D_n \left(\frac{n\pi c}{L}\right).$$

(sin has non zero derivative at 0
cos has zero derivative at 0)

$\underline{r} = x\hat{\underline{i}} + y\hat{\underline{j}} + z\hat{\underline{k}}$

$x = \underline{r}\cdot\hat{\underline{i}}$.

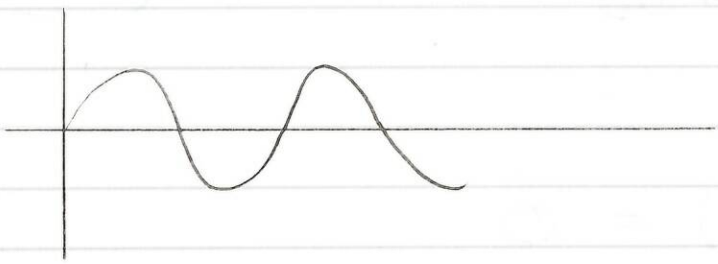
$\hat{\underline{i}}, \hat{\underline{j}}, \hat{\underline{k}}$ are othogonal ; $\hat{\underline{i}}\cdot\hat{\underline{j}} = 0$,
$\hat{\underline{i}}\cdot\hat{\underline{i}} = 0$

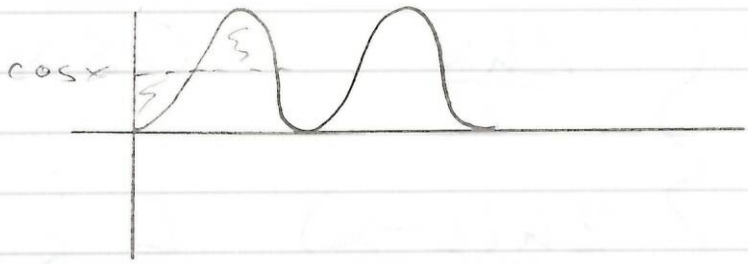
$$\int_0^L \sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{m\pi x}{L}\right) dx = 0.$$

if $m \neq n$.

when $m = n$.

$$\int_0^L \sin^2\left(\frac{n\pi x}{L}\right) dx = \frac{1}{2}L.$$





Take equation for $F(x)$, multiply by $\sin(m\pi x/L)$ and integrate in $[0, L]$.

$$\int_0^L \sin\left(\frac{m\pi x}{L}\right) F(x) \, dx.$$

$$= \sum_1^\infty C_n \int \sin\left(\frac{n\pi x}{L}\right) \sin\left(\frac{m\pi x}{L}\right) dx.$$

$$= C_m \frac{1}{2} L$$

$$C_n = \frac{2}{L} \int_0^L F(x) \sin\left(\frac{n\pi x}{L}\right) dx.$$

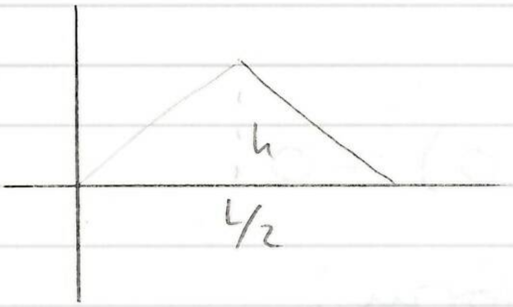
For $G(x)$ similarly

$$D_n = \frac{2}{n\pi c} \int_0^l G(x) \sin\left(\frac{n\pi x}{L}\right) dx.$$

<u>Example</u>:

$$G(x) = 0.$$

$$F(x) = \begin{cases} \dfrac{2hx}{L} & 0 \leq x \leq \dfrac{L}{2}. \\[2ex] \dfrac{2h(L-x)}{2} & \dfrac{L}{2} \leq x \leq L \end{cases}$$



i.e slove the wave equation

$$\frac{1}{c^2} z_{tt} = z_{xx}$$

on the interval $x \in [0, L]$, $z(0,t)=0$, $z(L,t) = 0$, $z(x,0) = F(x)$, $z_t(x,0) = G(x) = 0$.

$$z(x,t) = X(x)\,T(t).$$

$$\frac{T''}{c^2 T} = \frac{X''}{X} = -p^2.$$

so that we can satisfy $X(0) = X(L) = 0$.

$$X'' + p^2 X = 0.$$

$X = A \sin px$, so that $X(0) = 0$.

$$X(L) = 0 \Rightarrow p = \frac{n\pi}{L}.$$

$$T'' + p^2 c^2 T = 0.$$

$T(t) = \cos ptc$.

satisfying $T'(0) = 0$.

Solution has form:

$$z(x,t) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi x}{L}\right) \cos\left(\frac{nxct}{L}\right)$$

where A_n to be found so that $z(x,0) = F(x)$.

i.e $F(x) = \sum_{1}^{\infty} A_n \sin\left(\frac{n\pi x}{L}\right)$.

$$\Rightarrow A_n = \frac{2}{L}\int_0^L \sin\left(\frac{n\pi x}{L}\right) F(x)\, dx.$$

$$= \frac{2}{L}\left[\int_0^{L/2} \sin\left(\frac{n\pi x}{L}\right) \frac{2hx}{L}\, dx\right.$$

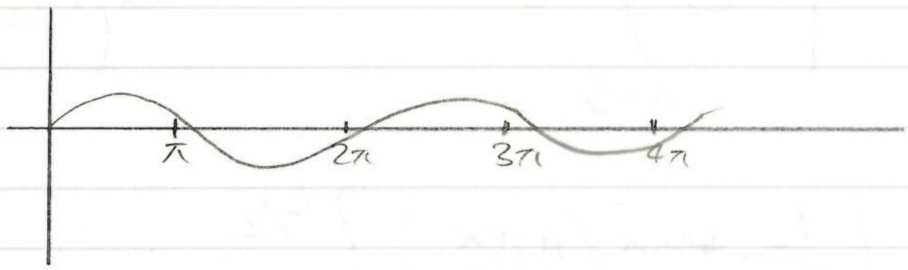
$$\left. + \int_{L/2}^{L} \sin\left(\frac{n\pi x}{L}\right) \frac{2h}{L}(L-x)\, dx\right].$$

In the second integral write $L - x = u$, $x = L - u$.

and it becomes:

$$\int_{L/2}^{0} \underbrace{\sin\left[\frac{n\pi}{L}(L-u)\right]}_{\sin(n\pi-\alpha)} \frac{2h}{L} u \ (-du).$$

$$\alpha = \frac{n\pi u}{L}.$$



$$\sin(n\pi - \alpha) = \sin\alpha \quad n = 1, 3, 5, \ldots$$

$$\sin(n\pi - \alpha) = -\sin\alpha \quad n = 2, 4, 6, \ldots$$

$$= (\pm)\int_0^{L/2} \sin\left(\frac{n\pi u}{L}\right)\left(\frac{2h}{L}\right) u \, du.$$

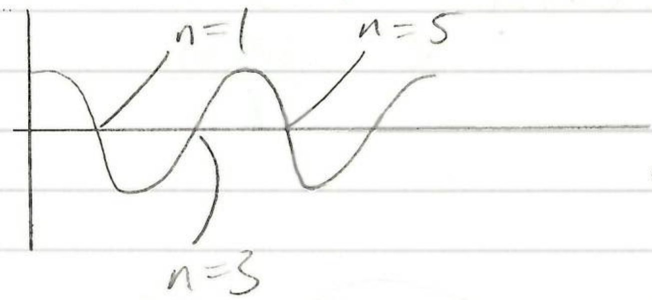
$+ \ n = 1, 3, 5$

$- \ n = 2, 4, 6.$

$A_n = 0, \quad n = 2, 4, 6, \ldots$

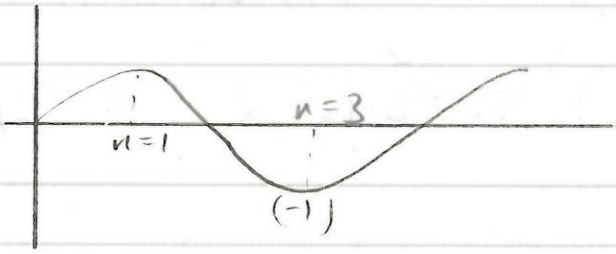
$$A_n = 2 \cdot \frac{2}{L} \cdot \frac{2h}{L} \int_0^{L/2} x \sin\left(\frac{n x \pi}{L}\right) dx \quad n = 1, 3, 5, \ldots$$

$$= \frac{8h}{L}\left\{\left[\frac{xL}{n\pi}(-1)\cos\left[\frac{n\pi x}{L}\right]\right]_0^{L/2} + \frac{L}{n\pi}\int_0^{L/2}\cos\left[\frac{n\pi x}{L}\right]dx\right\}.$$



$\cos\left(\frac{n\pi}{2}\right) = 0$ if n is odd

$$= \frac{8h}{m\pi L}\left[\frac{L}{n\pi}\sin\left(\frac{n\pi x}{L}\right)\right]_0^{L/2}.$$



$\sin\left(\frac{n\pi}{2}\right)$.

$$= \frac{8h}{n^2\pi^2} \begin{cases} 1 & \text{if } n = 1, 5, 9, 13, \dots \text{ i.e } j \text{ even} \\ (-1) & \text{if } n = 3, 7, 11, \dots \text{ i.e } j \text{ odd} \\ 0 & \text{if } n \text{ is even.} \end{cases}$$

where $n = 2j+1$, $j = 0, 1, 2, \dots$

$$z(x,t) = \frac{8h}{\pi^2}\sum_{j=0}^{\infty}\frac{(-1)^j}{(2j+1)^2}\left[\sin(2j+1)\frac{\pi x}{L}\right] \cdot \left[\cos\left[(2j+1)\frac{\pi c t}{L}\right]\right]$$

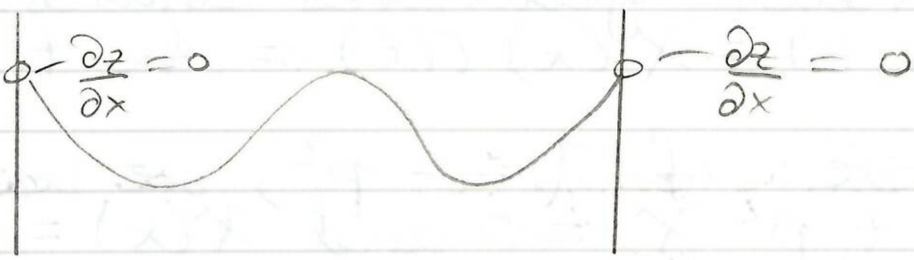
Different Boundary conditions and x-domains.

Solves:

$$\frac{z_{tt}}{c^2} = z_{xx}.$$

with $x \in [-L, L]$ and boundary conditions

$$\frac{\partial z}{\partial x} = 0, \quad \text{at } x = \pm L.$$



Look for a solution $z(x,t) = X(x)T(t)$ and we required

$$z_x(\pm L) = 0 = X'(\pm L)(T(t)).$$

$$\text{i.e. } X'(\pm L) = 0.$$

$$\frac{T''X}{c^2} = X''T$$

$$\frac{T''}{c^2 T} = \frac{X''}{X} = \lambda.$$

λ +ve, then we have exponential in X.

$$X'' - \lambda X = 0.$$

and the homogenous boundary conditions $X'(\pm L)$ cannot be satisfied.

$\lambda = 0$, $X'' = 0$ and $X = Ax + B$. and a solution with $X'(\pm L) = 0$ is just $x = \text{const}$. In this case $T'' = 0$. and $T = At + B$. So the zero separation constant generates solution $z(x,t) = X(x)T(t) = Ax + B$.

λ -ve. If $\lambda = -p^2$, we have $X'' + p^2 X = 0$ and $X(x) = A\cos px + B\sin px$ and we need to find p so that

$$X'(L) = 0.$$
$$X'(-L) = 0$$

and both A and B are zero.

$$X'(x) = pA\sin px + Bp\cos px.$$

and:

$$X'(L) = -Ap\sin pL + Bp\cos pL = 0.$$

$X'(-L) = -pA\sin(-pL) + Bp\cos(-pL) = 0$

$= pA\sin(pL) + Bp\cos pL = 0.$

Write this as:

$$\begin{pmatrix} -\sin pL & \cos pL \\ \sin pL & \cos pL \end{pmatrix} \begin{pmatrix} Ap \\ Bp \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Unless the determinant of this matrix is zero, then we have zero solutcoin $A = B = 0$

An alternative, non-zero solutcoin is possiable if the determiniant is zero

$-\sin(pL)\cos(pL) - \sin(pL)\cos(pL) = 0$

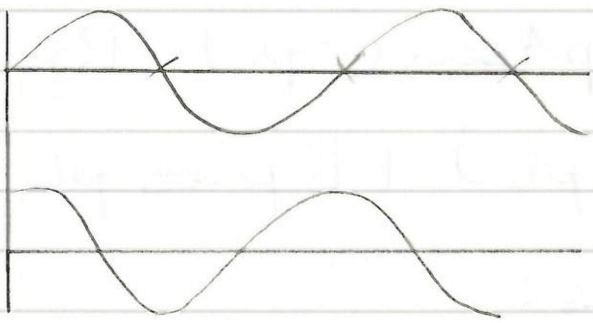
i.e $2\sin(pL)\cos(pL) = 0.$

$\sin(2pL) = 0.$

i.e. $2pL = n\pi$, $p = n\pi/2L$, $n = 1, 2, \ldots$

with $p = n\pi/2L$ we have

$$\begin{pmatrix} -\sin(\pi n/2) & \cos(\pi n/2) \\ \sin(\pi n/2) & \cos(n\pi/2) \end{pmatrix} \begin{pmatrix} Ap \\ Bp \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$



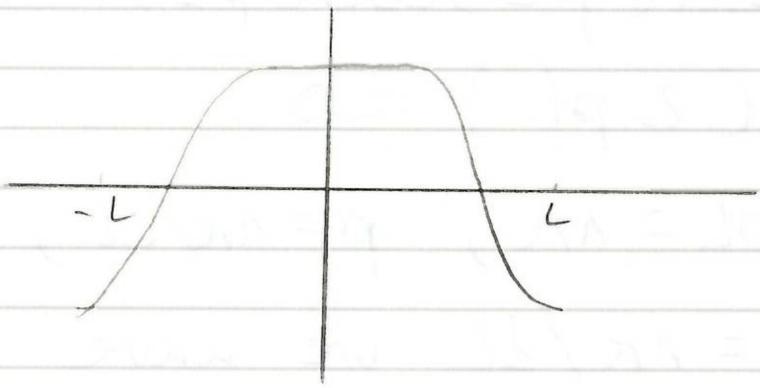
If n is even, $n = 2, 4, 6, \ldots$

$$\begin{pmatrix} 0 & \pm 1 \\ 0 & \pm 1 \end{pmatrix} \begin{pmatrix} A_p \\ B_p \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

i.e. A is anything, $B = 0$.

$$X(x) = A \cos\left(\frac{n\pi x}{L}\right) \quad n \text{ even}.$$

$$T(t) = C' \cos\left(\frac{n\pi c t}{2L}\right) + D \sin\left(\frac{n\pi c t}{2L}\right).$$



$n = 2$.

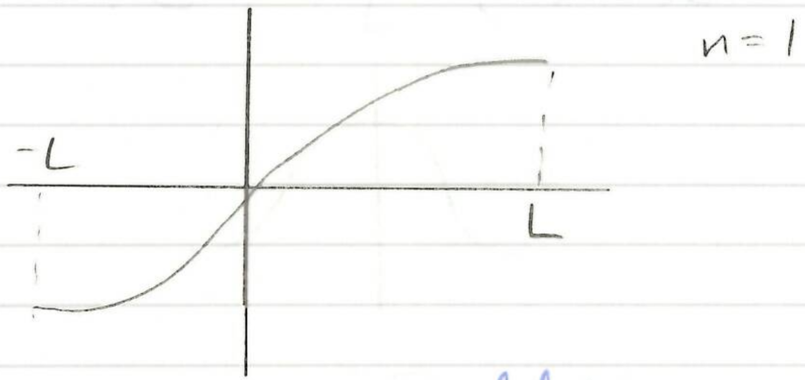
These are normal modes even in x.

If n is odd: $\begin{pmatrix} \mp 1 & 0 \\ \mp 1 & 0 \end{pmatrix} \begin{pmatrix} A_p \\ B_p \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$\Rightarrow$ B is anything, A is zero.

$$X(x) = B\sin\left(\frac{n\pi x}{2L}\right) \quad n \text{ odd.}$$

$$T(t) = C\cos\left(\frac{n\pi ct}{2L}\right) + D\sin\left(\frac{n\pi ct}{2L}\right).$$



odd normal modes.

So the general solution is.

$$z(x,t) = \underbrace{\overset{\text{even in } x}{\left(A_0 t + B_0\right)}}_{\text{zero sep const}}$$

odd in x

$$+\sum_{\substack{j=0 \\ n=2j+1}}^{\infty} \sin\left((2j+1)\frac{\pi x}{2L}\right)\left(C_j\cos\left(\frac{(2j+1)\pi ct}{2L}\right) + D_j\sin\left(\frac{(2j+1)\pi ct}{2L}\right)\right)$$

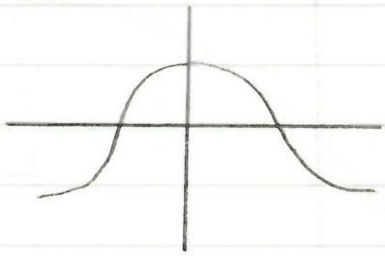
$$+\sum_{\substack{j=1 \\ n=2j}}^{\infty} \cos\left[\frac{j\pi x}{L}\right]\left(E_j\cos\left[\frac{j\pi ct}{L}\right] + F_j\sin\left[\frac{j\pi ct}{L}\right]\right).$$

even in x.

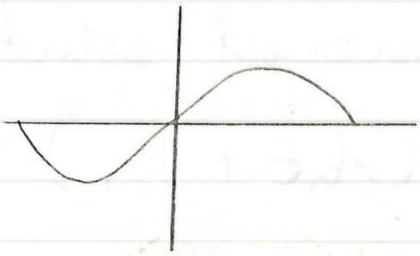
Initial Conditions $z_t = 0$ then we know D_j and $F_j = 0$, $A_0 = 0$.

If $z = 0$, C_j and $E_j = 0$, $B_0 = 0$.

If z = even, $C_j = 0$.



z = odd, $E_j = 0$



If z is even, $D_j = 0$
z is odd, $F_j = 0$, $A_0 = 0$.

7/12/11.

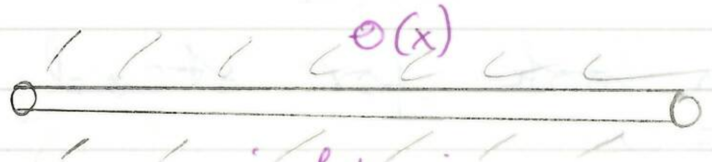
The heat / Diffusion equation:

$$\frac{\partial \theta}{\partial t} = k \frac{\partial^2 \theta}{\partial x^2} \quad \text{for } \theta(x, t).$$

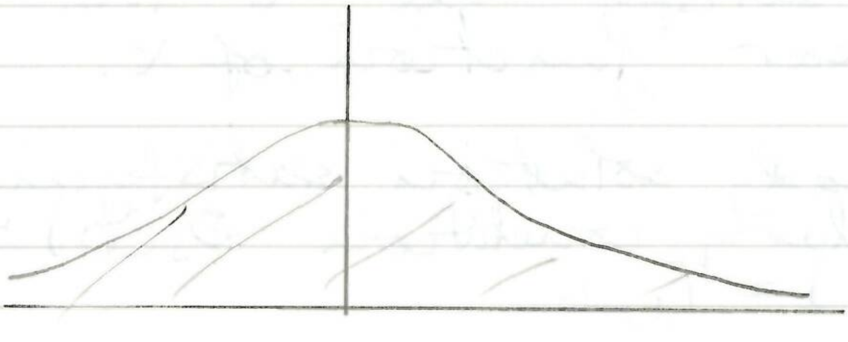
1 dim space — time

Constant, the thermal diffusion or diffusivity.

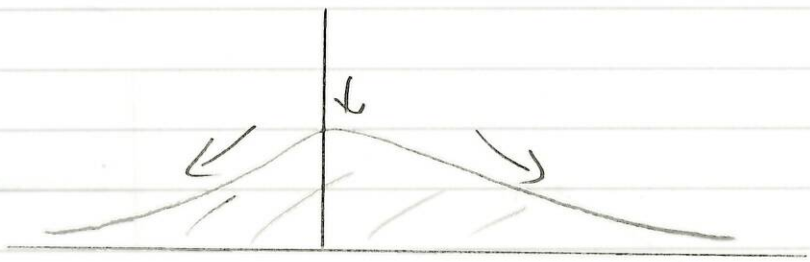
(In 3D: $\partial\theta/\partial t = k \nabla^2 \theta$).



Long / thin metal rod.



↓ t increases:



Typical boundary condition on a rod of finite length say $x \in [0, L]$
$\theta(0) = T_0$, $\theta(L) = T_1$.

Or we could improve insulating boundary condition $\partial\theta/\partial x = 0$ at $x = 0$ say.

— / —

Robin boundary conditions:

$$\frac{\partial \theta}{\partial x} = \alpha \theta$$

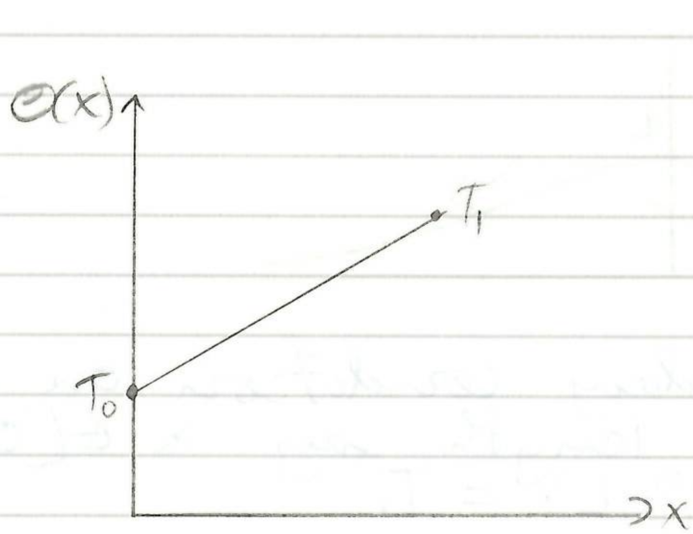
— / —

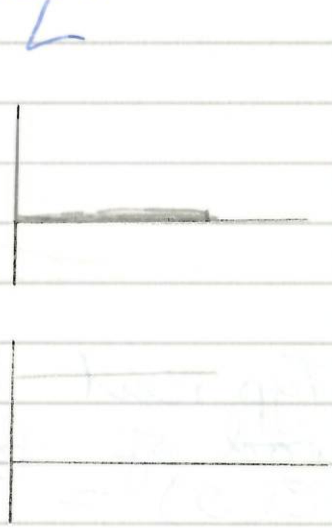
We can look for steady solution $\partial\theta/\partial t = 0$.

These satisfy $\theta_{xx} = 0$ i.e. $\theta_s = Ax + B$, a linear function of x.

A steady solution satisfying the boundary condition $\theta_s(0) = T_0$, $\theta_s(L) = T_1$, is

$$\theta_s = T_0 + (T_1 - T_0)\frac{x}{L}.$$





If one boundary conditcoin is insulating asks for $\partial\theta/\partial x = 0$, then the steady solutcoin requires $A = 0$ and $\theta_s = B$, with B obtained, maybe from the other boundary conditcoins.

We will look for time dependent solutcoins of the form $\theta(x,t) = X(x)T(t)$

$$XT' = kX''T$$

$$\Rightarrow \frac{T'}{kT} = \frac{X''}{X} = \text{const} = \lambda \quad \text{the separation constant}$$

$\lambda = 0$ gives $T' = 0$ i.e $T = \text{const}$
$X'' = 0$ $\quad X = Ax + B$

$\Rightarrow XT = Ax + B$; the steady solutcoin.

$\lambda > 0$

$T' = k\lambda T$

$T = Ae^{k\lambda t}$ and this grows in time $\lambda > 0$ which is unrealistic.

$X'' - \lambda X = 0$ which has expontial solutcoins and trigonometric solutcoin and we cannot satisfy homogenous b.c. if $\lambda > 0$.

$\lambda < 0$, $\lambda = -p^2$.

$X'' + p^2 X = 0 \Rightarrow X = A\sin px + B\cos px$ and find p from spatial boundary conditions.

$T' = -p^2 k T$, $T = Ae^{-p^2kt}$.

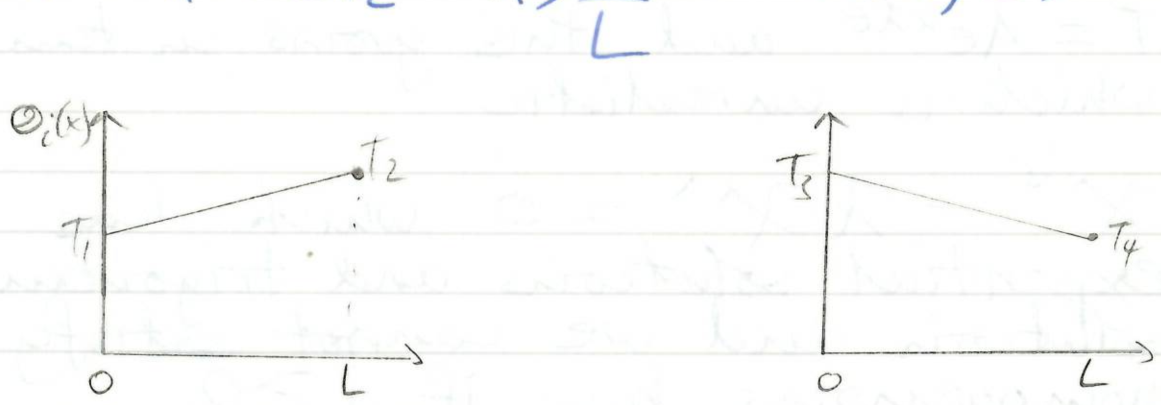
General Solution:

$$\Theta(x,t) = A_0 x + B_0$$

$$+ \sum_p (A_p \sin px + B_p \cos px)e^{-p^2kt}.$$

Example.
Solve the heat equation $\Theta_t = k\Theta_{xx}$ on the interval $x \in [0, L]$ with boundary conditions $\Theta(0) = T_3$, $\Theta(L) = T_4$ and intial conditions:

$$\Theta = T_1 + (T_2 - T_1)\frac{x}{L} = \Theta(x,0) = \Theta_i(x).$$



I "expect" that as $t \to \infty$ the solution to apporach the steady solution

$\theta = \theta_S = T_3 + (T_4 - T_3)x/L$. Note $\theta(0) = T_3$ and $\theta_3(L) = T_4$ i.e. θ satisfies over required boundary conditions

We write

unsteady

$$\theta(x,t) = \theta_S(x) + \theta_u(x,t) \quad ,$$

steady

$$\theta_{S_{xx}} = 0$$

and boundary condition on θ

if $\theta(0) = T_3$, $\theta_S(0) = T_3$

$\theta(L) = T_4$, $\theta_S(L) = T_4$.

So putting $x = 0$.

$$\begin{aligned} \theta(0,t) &= \theta_S(0) + \theta_u(0,t) \\ &= T_3 + \theta_u(0,t). \end{aligned}$$

and $\theta_u(0,t) = 0$.

Similarly $\theta_u(L,t) = 0$.

Since $\theta_u(L,t) = 0$.

Since $\theta_t = k\theta_{xx}$

$$\underbrace{\cancel{\theta_{S_t}}}_{=0} + \theta_{u_t} = k\Big[\underbrace{\cancel{\theta_{S_{xx}}}}_{=0} + \theta_{u_{xx}}\Big]$$

$$\theta_{u_t} = k\,\theta_{u_{xx}} \quad \text{and} \quad \begin{array}{l} \theta_u(0,t) = 0 \\ \theta_u(L,t) = 0. \end{array}$$

and we use the method of separation of variables to find $\theta_u(x,t)$

But what are the initial condition for θ_u?

$$\theta(x,0) = \theta_i(x) = \theta_s(x) + \theta_u(x,0)$$

So: $\theta_u(x,0) = \theta_i(x) - \theta_s(x).$

9/12/11

Heat equation

$$\theta_t = k\theta_{xx}$$

Solve for $x \in [0, L]$

Boundary Conditions:
$\theta(0) = T_3$
$\theta(L) = T_4$.

with intial conditions $\theta = T_1 + (T_2 - T_1)x/L$
$= \theta_i(x)$.

Write $\theta = \theta_s(x) + \theta_u(x,t)$

Satisfies $\theta_{s_{xx}} = 0$ and boundary conditions
$\Rightarrow \theta_s = T_3 + (T_4 - T_3)x/L$. Has nothing
to do with initial conditions

Then $\theta_{u_t} = k\theta_{u_{xx}}$, $\theta_u(0) = 0$, $\theta_u(L) = 0$.
and use sep variables $\theta_u(x,0) = \theta_i(x) - \theta_s(x)$

$$\theta_u(x,t) = X(x)T(t), \quad \frac{T''}{kT} = \frac{X''}{X} = \lambda$$

- Cannot have $\lambda > 0$ as this would
lead to exponential solution for X
and we required $\theta_u(0) = \theta_u(L) = 0$ i.e
$X(0) = X(L) = 0$ which exponential
cannot satisfy.

- Cannot have $\lambda = 0$ as this gives $X = Ax + B$ and we cannot satisfy $X(0) = X(L) = 0$ with a non-zero. solution for X.

- So for $\lambda < 0$, $\lambda = -p^2$ and $X(x) = A\cos px + B\sin px$, $T(t) = e^{-p^2kt}$.

Our boundary condition require:

$$X(0) = 0 \Rightarrow A = 0.$$
$$X(L) = 0 \Rightarrow \sin pL = 0 \Rightarrow p = \frac{n\pi}{L},\ n = 1,2,3,\ldots$$

$$\theta_u(x,t) = \sum_{n=1}^{\infty} A_n e^{-\frac{n^2\pi^2 kt}{L^2}} \sin\left(\frac{n\pi x}{L}\right)$$

Initially, i.e. putting $t = 0$, we need.

$$\theta_u(x,0) = \theta_i(x) - \theta_s(x).$$

i.e. $$\sum_1^{\infty} A_n \sin\left(\frac{n\pi x}{L}\right) = (T_1 - T_3) + \frac{x}{L}(T_2 - T_4 - T_1 + T_3)$$

$$= P + \frac{Qx}{L}.$$

Multiply by $\sin(m\pi x/L)$ and $\int_0^L$ and find:

$$A_m L \cdot \frac{1}{2} = \int_0^L \left(P + \frac{Qx}{L}\right) \sin\left(\frac{m\pi x}{L}\right) dx.$$

$$= \left\{ \left[\frac{-L}{\pi m} \cos\left(\frac{m\pi x}{L}\right)\left(P + \frac{Qx}{L}\right)\right]_0^L + \underbrace{\frac{L}{m\pi}\int_0^L \cos\left(\frac{m\pi x}{L}\right)\frac{Q\,dx}{L}}_{0} \right\}$$

$$= \frac{PL}{m\pi}\left(1 - (-1)^m\right) - \frac{L}{m\pi}\frac{QL}{L}(-1)^m.$$

$$= \frac{L}{m\pi}\left\{ P\left(\underbrace{1-(-1)^m}\right) - Q(-1)^m \right\}$$

2 if m is odd
0 if m is even.

So $\theta(x,t) = T_3 + (T_4 - T_3)x/L$

$$+ \sum_{n=1} \frac{2}{n\pi}\left\{ (T_1 - T_3)(1-(-1)^n) - (T_2 - T_4 + T_3 - T_1)(-1)^n \right\} \cdot$$

$$\cdot e^{\frac{-n^2\pi^2 kt}{L^2}} \sin\left(\frac{n\pi x}{L}\right)$$

—/—

Aside: $\int_0^L \sin\left(\frac{m\pi x}{L}\right)\sin\left(\frac{n\pi x}{L}\right) dx$ (only non zero when $n = m$)

$$\sum_1^\infty A_n \int_0^L \sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{m\pi x}{L}\right) dx$$

use double angle formula

$$= A_m \int_0^L \sin^2\left(\frac{\pi m x}{L}\right) dx = \frac{1}{2}L$$

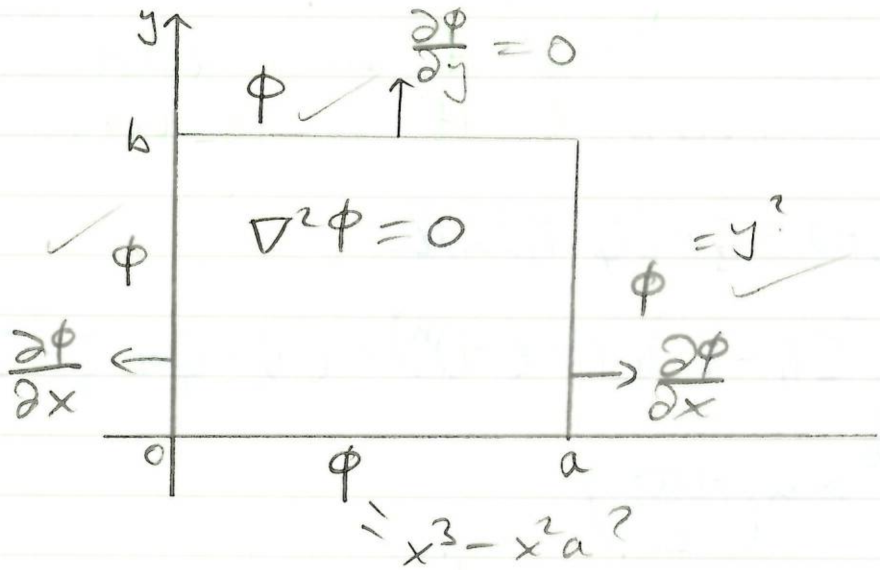
—/—

Laplace Equation for $\phi(x,y)$.

This is the elliptic equation:

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0.$$

i.e $[\underline{\nabla} \cdot \underline{\nabla}\phi = 0]$ or $\nabla^2 \phi = 0$.



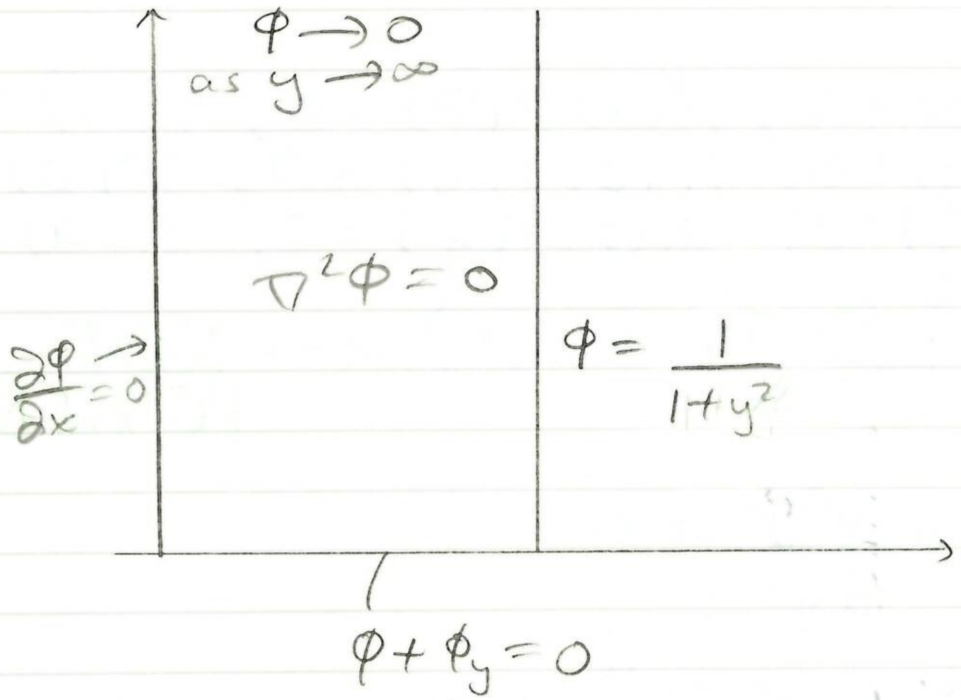
Boundary conditions for this problem are typically:

1) ϕ is specified on the boundary - Dirichlet boundary condition.
2) $\partial\phi/\partial n$ is specified
3) Robin condition:

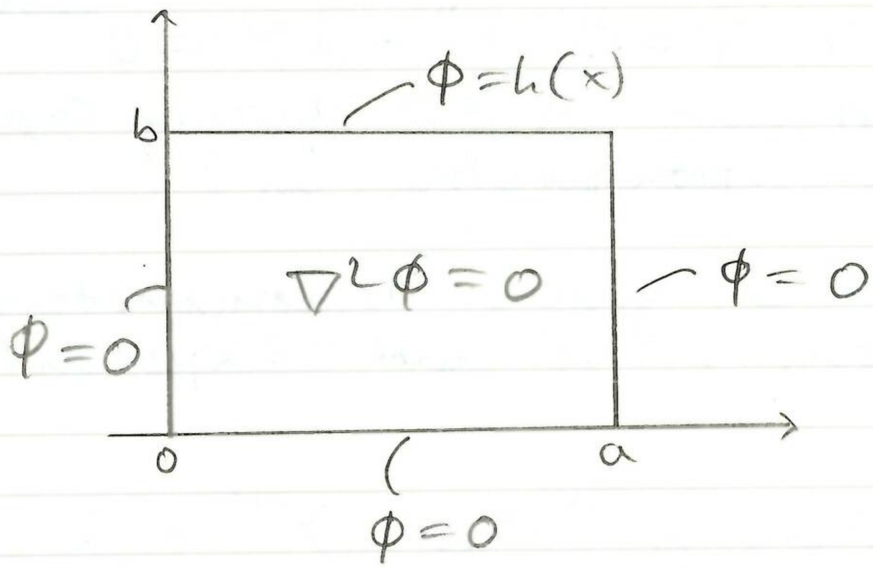
$$\phi + \beta \frac{\partial \phi}{\partial n} = 0.$$

These boundary conditions could be

different on different boundaires:



Laplaces equation in a rectangular domain



Slove $\nabla^2\phi = 0$ in the domain $0 \le x \le a$, $0 \le y \le b$.

with $\phi(x,0) = 0$, $\phi(0,y) = 0$, $\phi(x,b) = h(x)$, $\phi(a,y) = 0$

Look for a solution with $\phi(x,y) = X(x)Y(y)$

Since:

$\phi(x,0) = 0$, $X(x)\,Y(0) = 0$, $Y(0) = 0$.

$\phi(0,y) = 0$, $X(0)\,Y(y) = 0$, $X(0) = 0$.

$\phi(a,y) = 0$, $X(a)\,Y(y) = 0$, $X(a) = 0$.

$X(x)\cancel{Y}(b) = h(x)$ ⟸ NOT POSSIBLE

$$\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} = 0.$$

$$X''Y + XY'' = 0.$$

$$\frac{X''}{X} = -\frac{Y''}{Y} = \lambda$$, a separation constant

So $X'' + \lambda X = 0$, $Y'' - \lambda Y = 0$.

We can have $\lambda > 0$ giving exponentials X and trigonometric in Y.

or $\lambda < 0$, giving trigonometeric functions in X and exponentials in Y.

or $\lambda = 0$, $X'' = 0$, $Y'' = 0$ i.e linear functions in X and Y.

and ignoring the influence of boundary conditions the general solution is a combination of all these possibilites.

Considering these and especially the fact that $X = 0$ at both $x = 0$ and $x = a$ implies we must restrict ourselves to $\lambda < 0$.

Write $\lambda = p^2$.

$$X'' + p^2 X = 0.$$
$$Y'' - p^2 Y = 0.$$

$$X_p(x) = A \sin px + B \cos px$$
$$Y_p(y) = C \sinh py + D \cosh px.$$

$\frac{1}{2} \left. \begin{matrix} e^{py} \\ e^{-py} \end{matrix} \right\} = 1$ at $y=0$

Applying $X(0) = 0 \Rightarrow B = 0$.

$X(a) = 0 \Rightarrow \sin pa = 0$

$pa = \pi n \quad , n = 1, 2, 3, \ldots$

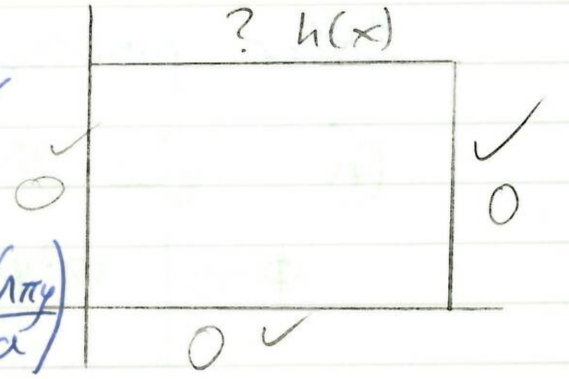
$Y(0) = 0 \Rightarrow D = 0$.

$$X_n(x) = A_n \sin\left(\frac{n\pi x}{a}\right),$$

$$Y_n(y) = C_n \sinh\left(\frac{n\pi y}{a}\right)$$ same p

and generally $\phi = \sum_n X Y$

$$\phi(x, y) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi x}{a}\right) \sinh\left(\frac{n\pi y}{a}\right)$$



A_n's are found so that $\phi(x, b) = h(x)$.

i.e.

$$\phi(x,b) = h(x) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi x}{a}\right) \sinh\left(\frac{n\pi b}{a}\right)$$

Multiply by $\sin\left(\frac{m\pi x}{a}\right)$ and $\int_0^a$.

$$\int_0^a \sin\left(\frac{m\pi x}{a}\right) h(x)\, dx = A_m \sinh\left(\frac{m\pi b}{a}\right) \frac{1}{2} a.$$

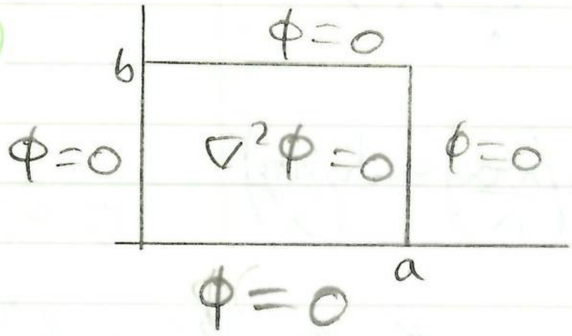
$$A_m = \frac{2\int_0^a \sin\left(\frac{m\pi x}{L}\right) h(x)\, dx}{a \sinh\left(\frac{m\pi b}{a}\right)}.$$

and

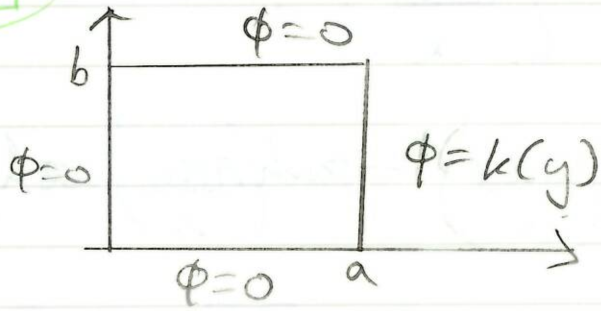
$$\phi(x,y) = \frac{2}{a} \sum_{n=1}^{\infty} \sin\left(\frac{n\pi x}{a}\right) \frac{\mathrm{sh}(n\pi y/a)}{\mathrm{sh}(n\pi b/a)} \int_0^a \sin\left(\frac{n\pi x}{a}\right) h(x)\, dx.$$

Other problems might be:

II

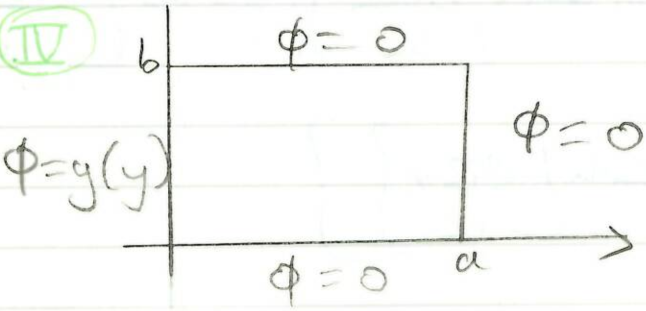


III

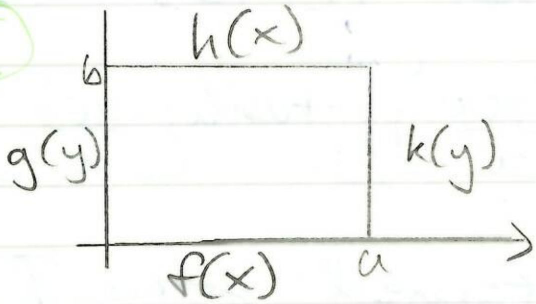


$$\sum_{n=1}^{\infty} A_n \sinh\left(\frac{n\pi x}{b}\right) \sin\left(\frac{n\pi y}{b}\right)$$

IV



V



II;
$$X(x) = \sin\left(\frac{n\pi x}{a}\right)$$

$$Y(y) = C\sinh\left(\frac{n\pi y}{a}\right) + D\cosh\left(\frac{n\pi y}{a}\right)$$

and $\phi(x, b) = 0$.

$$\Rightarrow \quad Y(b) = 0,$$

$$C\sinh\left[\frac{n\pi b}{a}\right] + D\cosh\left[\frac{n\pi b}{a}\right] = 0.$$

$$D = -C' \tanh\left[\frac{n\pi b}{a}\right].$$

$$Y(y) = C'\left(\sinh\left(\frac{n\pi y}{a}\right) - \tanh\left(\frac{n\pi b}{a}\right)\cosh\left(\frac{n\pi y}{a}\right)\right)$$

$$\phi(x,y) = \sum_{n=1}^{\infty} C_n \sin\left[\frac{n\pi x}{a}\right]\left[\sinh\left[\frac{n\pi y}{a}\right] - \tanh\left[\frac{n\pi b}{a}\right]\cosh\left[\frac{n\pi y}{a}\right]\right]$$

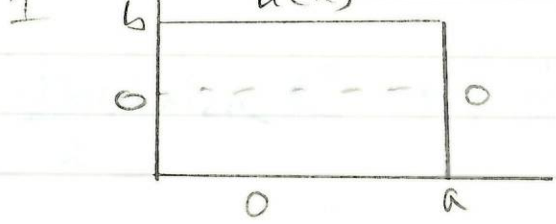
C_n's found so that $\phi(x,0) = f(x)$.

$$f(x) = \sum_{n=1}^{\infty} C_n \sin\left(\frac{n\pi x}{a}\right)\left[-\tanh\left[\frac{n\pi b}{a}\right]\right].$$

Problem II is obtained from I by reflecting in the line $y = b/2$ i.e $y \to b - y$ and

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0.$$

is unaltered under this transformation as it has a second order derivative, only in y.



So the solution for II is found from I be replacing y by $b-y$.

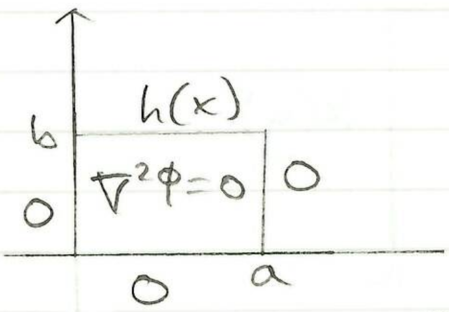
$$\phi(x,y) = \frac{2}{a}\sum_{n=1}^{\infty} \frac{\sin\left(\frac{n\pi x}{a}\right)\sinh\left(\frac{n\pi(b-y)}{a}\right)}{\sinh\left(\frac{n\pi b}{a}\right)} .$$

$$\cdot \int_0^a \sin\left(\frac{n\pi x}{a}\right) f(x)\, dx .$$

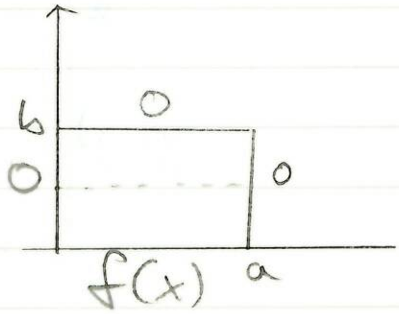
III can be mapped to I by reflecting in the diagonal of the rectangle, effected by $y \to x$, $x \to y$, $a \to b$, $b \to a$.

$$\phi(x,y) = \frac{2}{b}\sum_{n=1}^{\infty} \sin\left[\frac{n\pi y}{b}\right] \frac{\operatorname{sh}\left(\frac{n\pi x}{b}\right)}{\operatorname{sh}\left(\frac{n\pi a}{b}\right)} \int_0^b \sin\left(\frac{n\pi y}{b}\right) k(y)\, dy$$

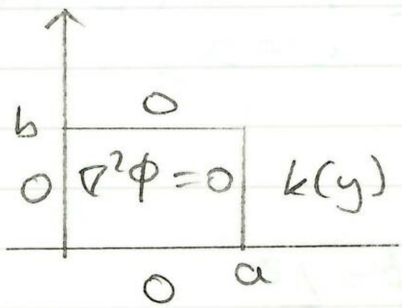
14/12/11.



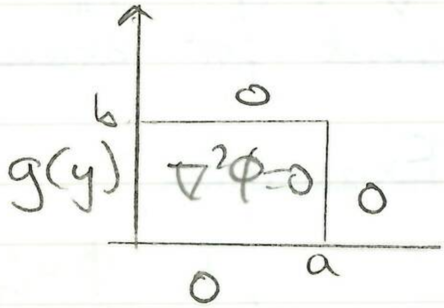
I



II



III



IV

I: $\phi_I = \sum_{n=1}^{\infty} \sin\left[\frac{n\pi x}{a}\right] sh\left[\frac{n\pi y}{a}\right] \frac{2}{a} \frac{1}{sh[n\pi b/a]} \int_0^a \sin\left[\frac{n\pi x}{a}\right] h(x)\,dx$

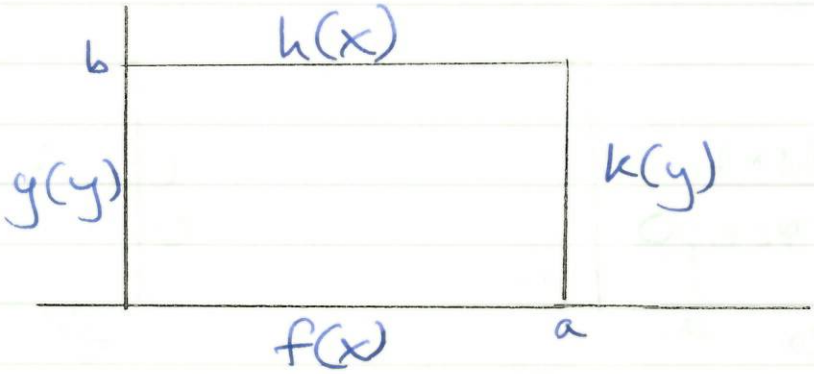
II: $\phi_{II} = \sum_{n=1}^{\infty}$ —//— $\sin\left[\frac{n\pi(y-b)}{a}\right]$ ——//—— $f(x)\,dx$

III $\phi_{III} = \sum_{n=1}^{\infty} \sin\left[\frac{n\pi y}{b}\right] sh\left[\frac{n\pi x}{b}\right] \frac{2}{b} \left(\frac{1}{sh[n\pi a/b]}\right) \int_0^b \sin\left[\frac{n\pi y}{a}\right] k(y)\,dy$

$a \to b$
$x \to y$

IV from II
$x \to a-x$

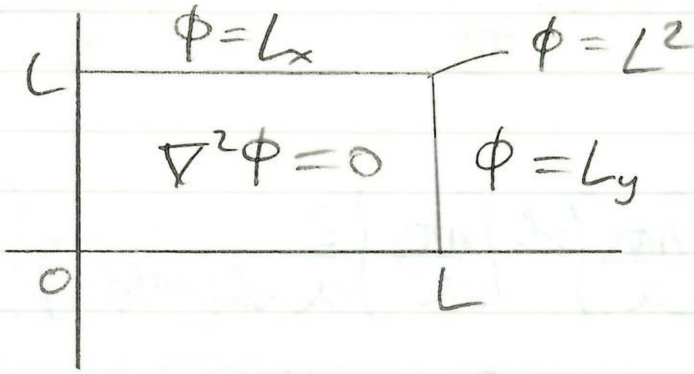
$\phi_{IV} = \sum_{1}^{\infty} \sin\left[\frac{n\pi y}{b}\right] sh\left[\frac{n\pi(a-x)}{b}\right] \left(\frac{2}{b}\right) \frac{1}{sh\left[\frac{n\pi a}{b}\right]} \int_0^b \sin\left[\frac{n\pi y}{b}\right] g(y)\,dy$



$$\nabla^2(\phi_{I} + \phi_{II} + \phi_{III} + \phi_{IV}) = 0 + 0 + 0 + 0 = 0$$

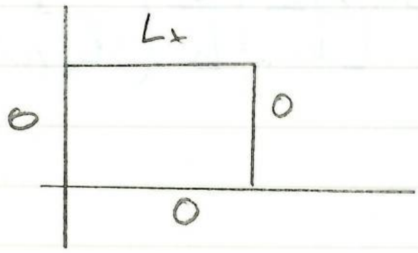
and for example on $x = a$.

<u>Example</u>:



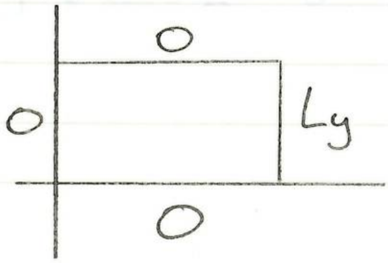
First slove:

①



Then slove:

②



$\nabla^2 \phi = 0, \quad \phi = X(x)\, Y(y).$

$$\frac{X''}{X} = -\frac{Y''}{Y} = \text{const} = -p^2.$$

as we want tri in x.

$$X'' + p^2 X = 0$$

$\Rightarrow X(x) = \sin px$ with $pL = n\pi$.
0 at $x=0$, 0 at $x = L$.

$$X(x) = \sin\left(\frac{n\pi x}{L}\right)$$

$$Y'' - p^2 Y = 0 \Rightarrow Y(y) = \sinh(py)$$

$= \text{sh}(py) = \text{sh}(n\pi y/L)$
(0 at $y=0$)

$$\phi(x,y) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi x}{L}\right) \text{sh}\left[\frac{n\pi y}{L}\right]$$

and $\phi(x, L) = L_x = \sum_{n=1}^{\infty} A_n \sin\left[\frac{n\pi x}{L}\right] \text{sh}\left[n\pi\right]$

requiring

$$\int_0^L L_x \sin\left[\frac{m\pi x}{L}\right] dx = A_m\, \text{sh}\left[m\pi\right] \int_0^L \sin\left[\frac{m\pi x}{L}\right] dx.$$

$$A_n = \frac{2}{L} \frac{1}{sh[n\pi]} \int_0^L x \sin\left[\frac{n\pi x}{L}\right] dx .$$

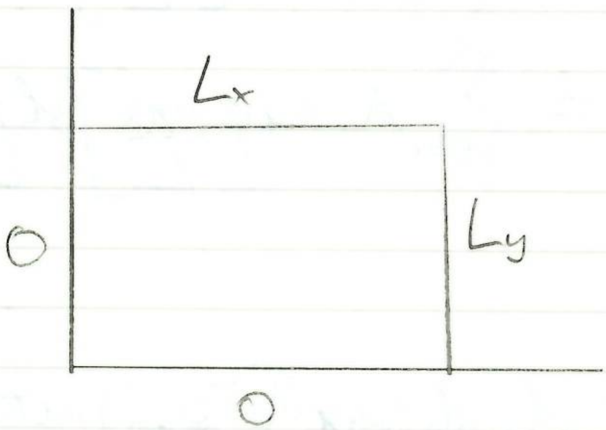
$$= \frac{2}{sh[n\pi]} \left\{ \left[x \frac{L}{n\pi} (-1) \cos\left[\frac{n\pi x}{L}\right] \right]_0^L \right.$$

$$\left. + \underbrace{\int_0^L \frac{L}{n\pi} \cos\left[\frac{n\pi x}{L}\right] dx}_{0.} \right\}$$

$$= \frac{-2}{sh[n\pi]} \frac{L^2}{n\pi} \cos\left[n\pi\right] .$$

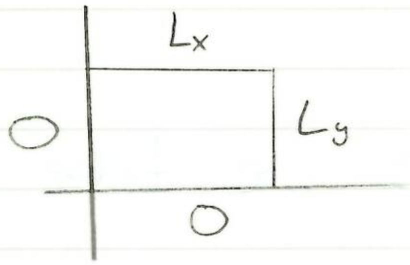
$$A_n = \frac{2(-1)^{n+1}}{n\pi sh[n\pi]} L^2 .$$

$$\phi(x,y) = \sum_{n=1}^{\infty} \frac{2L^2}{n\pi sh[n\pi]} (-1)^{n+1} \sin\left[\frac{n\pi x}{L}\right] sh\left[\frac{n\pi y}{L}\right]$$



The solution for this second problem is obtained by putting:

$$x \longrightarrow y$$
$$y \longrightarrow x .$$



has solution:

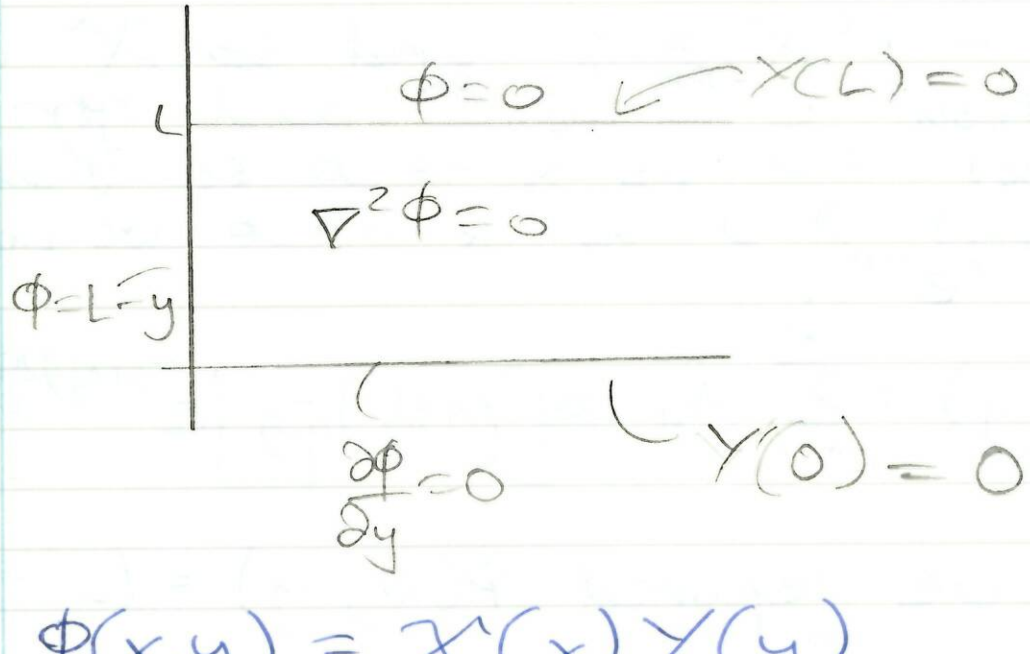
$$\phi(x,y) = \sum_{n=1}^{\infty} \frac{2L^2(-1)^{n+1}}{\sinh[n\pi]} \left\{ \sin\left[\frac{n\pi x}{L}\right] \sinh\left[\frac{n\pi y}{L}\right] + \sin\left[\frac{n\pi y}{L}\right] \sinh\left[\frac{n\pi x}{L}\right] \right\}$$

$$= xy.$$

The solution to this problem is in fact:

$\phi = xy$ $\quad$ [satisfy bc's and $\nabla^2\phi = 0$].

Slove:



$$\phi(x,y) = X(x)\,Y(y)$$

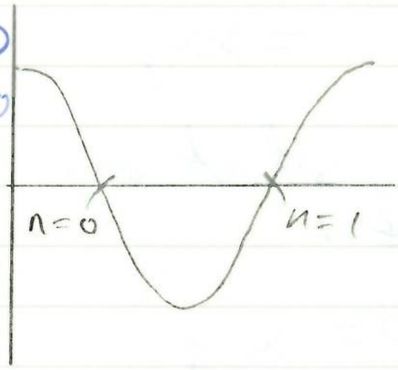
$$\frac{X''}{X} = -\frac{Y''}{Y} = \text{const} = -p^2.$$

trigonometric in y.

To satisfy $Y'(0) = 0$ we take the solution

$$Y(y) = \cos(py).$$

and satisfy $Y(L)=0$ we require $\cos(pL)=0$ requiring



$$pL = \left(n + \frac{1}{2}\right)\pi$$

$$p = \left(n + \frac{1}{2}\right)\frac{\pi}{L}.$$

$X'' - p^2 X = 0$ and so X has solution e^{-px}, e^{+px} and for $X(x) \to 0$ as $x \to \infty$ so that $\phi(x,y) \to 0$ as $x \to \infty$ we must take e^{-px}.

$$\phi(x,y) = \sum_{n=0}^{\infty} A_n \cos\left[\left(n+\frac{1}{2}\right)\frac{\pi y}{L}\right] e^{-(n+\frac{1}{2})\frac{\pi x}{L}}$$

and we required $\phi(0, y) = L - y$.

$$L - y = \sum_{n=0}^{\infty} A_n \cos\left[\left(n + \frac{1}{2}\right)\frac{\pi y}{L}\right]$$

Using the fact that $\cos[(n+1/2)\pi y/L]$ and $\cos[(m+1/2)\pi y/L]$ are orthogonal we find:

$$\int_0^L (L-y)\cos\left[\left(m+\frac{1}{2}\right)\frac{\pi y}{L}\right] dy$$

$$= A_m L \frac{L}{2}.$$

— / —

[illegible].

xxx.