2401 Mathematical Methods 3
Based on the autumn 2011 lectures by
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5/10/11

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Wed 12:30 → 4
Fri 12:30 → 4
Tue 1:30 → 4

For help the UCL'S people.

Partial Differentiation

Simple case $\mathbb{R}^3 \to \mathbb{R}$

Let $f(x,y)$ be a function of two indepenant variables x, y. The two partial derivatives of f at a point (a,b) are the limits.

$$\frac{\partial f}{\partial x} = \lim_{h \to 0} \frac{f(a+h,b) - f(a,b)}{h}$$

$$\frac{\partial f}{\partial y} = \lim_{k \to 0} \frac{f(a,b+k) - f(a,b)}{k}.$$

if these limits exist.

$$\frac{\partial f}{\partial x}, \quad \left.\frac{\partial f}{\partial x}\right|_y, \quad f_x, \quad f_x(a,b).$$

If these dirivatives exist at every point in the regeon of $\mathbb{R}^2$ then we have a function derived from $f(x, y)$.

$$\frac{\partial f}{\partial x}(x, y) = \lim_{\delta x \to 0} \frac{f(x+\delta x, y) - f(x, y)}{\delta x}$$

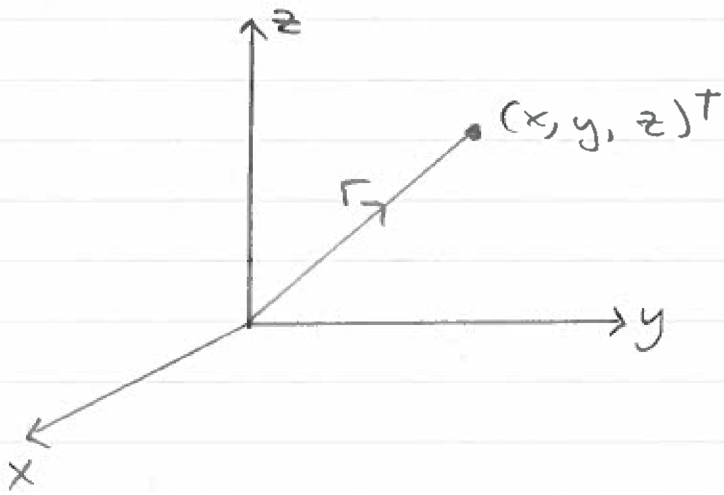
These function themselves may be differentiable and we can find higher derivatives.

$$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = f_{xy}.$$

$$\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = f_{xy}$$

$$f_{xy} = f_{yx}.$$

<u>Example:</u> $\mathbb{R}^3 \to \mathbb{R}$.



$$(x, y, z)^T = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

If $\underline{r} = (x, y, z)^T$ then $f(x, y, z) = f(\underline{r}$, is $f(\underline{r}) = |\underline{r}| = $ (distance of point (x, y, z) from O) $= \sqrt{x^2+y^2+z^2}$

$$\frac{\partial f}{\partial x} = \frac{x}{\sqrt{x^2+y^2+z^2}} = \frac{x}{|\underline{r}|}.$$

$$f_y = \frac{y}{|\underline{r}|}$$

$$f_z = \frac{z}{|\underline{r}|}$$

$$(\nabla f) = \begin{pmatrix} \partial f/\partial x \\ \partial f/\partial y \\ \partial f/\partial z \end{pmatrix} = \frac{\underline{r}}{|\underline{r}|}.$$

Differentiability of function of serval variables.

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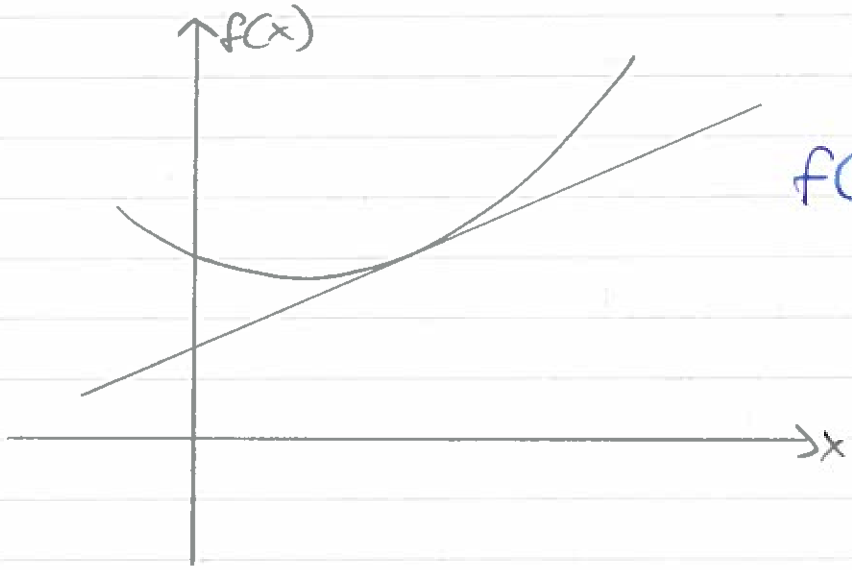
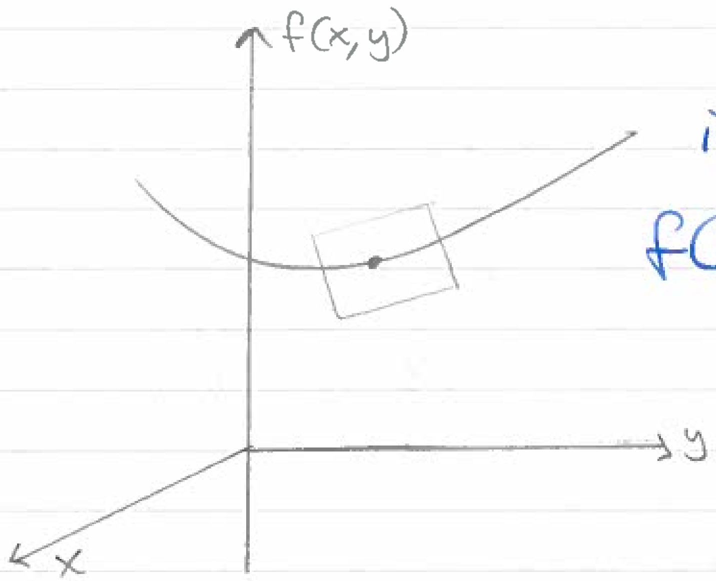
Differentiabity $\approx$ locally linear.



$f(a+h) \approx f(a) + hf'(a$



We say $f'(a)$ exist if:

$$f(a+h) = f(a) + hf'(a) + |h|\,\phi(h)$$

where $\phi(h) \to 0$ as $|h| \to 0$.

$$L = \begin{pmatrix} L_x \\ L_y \end{pmatrix}, \quad x = \begin{pmatrix} a \\ b \end{pmatrix}, \quad \delta x = \begin{pmatrix} h \\ k \end{pmatrix}$$

In two dimenseon $\mathbb{R}^2 \to \mathbb{R}$ the function $f(x,y)$ is differentiable at the point (a,b) if there exist two numbers L_x, L_y st $f(a+h, b+k) = f(a,b) + L_x h + L_y k + \sqrt{h^2+k^2}\,\phi(h,k)$ where $\phi(h,k) \to 0$

as $h, k \to 0$.

We can write this slightly differently and extend to $\mathbb{R}^n \to \mathbb{R}$ and say

$$f(x+\delta x) = f(\underline{x}) + \underline{L} \cdot \delta \underline{x} + |\delta \underline{x}| \phi(\delta \underline{x})$$

and claim that f is differentiable at $\underline{x}$ if such $\underline{L}$ exist and $|\phi(\delta \underline{x})| \to 0$ as $|\delta \underline{x}| \to \underline{0}$.

$$\underline{L} \cdot \delta \underline{x} = \begin{pmatrix} L_x & L_y \end{pmatrix} \begin{pmatrix} \delta x \\ \delta y \end{pmatrix}$$

Consider now, three dimension $\mathbb{R}^3 \to \mathbb{R}$ then it turns out that $\underline{L} = \underline{\nabla} f = \text{grad } f$ since; for example

$$\frac{\partial f}{\partial x} = \lim_{h \to 0} \left(\frac{f(x+h, y, z) - f(x, y, z)}{h} \right)$$

$$= \lim_{h \to 0} \left(\frac{L_x h + 0 + 0 + |h| \phi(h)}{h} \right)$$

f is differentiable

$$= L_x \quad \text{as } \phi(h) \to 0 \text{ as } h \to 0$$

So $$\underline{L} = \begin{pmatrix} \partial f/\partial x \\ \partial f/\partial y \\ \partial f/\partial z \end{pmatrix} = \nabla f.$$

Consider now function $\mathbb{R}^m \to \mathbb{R}^n$, we say $\underline{f}$ (vector in $\mathbb{R}^n$) is differentiable at point $\underline{x}$ (vector in $\mathbb{R}^n$) if there exist a $n \times m$ matrix $\underline{\underline{L}}$ st:

$$f(\underline{x}+\underline{h}) = f(\underline{x}) + \underline{\underline{L}}\,\underline{h} + |\underline{h}|\,\phi(\underline{h})$$

as $|\phi(\underline{h})| \to 0$ as $|\underline{h}| \to 0$.

Where $\underline{\underline{L}}$ is:

$$\begin{array}{cc}
\begin{pmatrix}
\partial f_1/\partial x_1 & \partial f_1/\partial x_2 & \cdots & \partial f_1/\partial x_m \\
\partial f_2/\partial x_1 & \ddots & & \vdots \\
\vdots & & \partial f_i/\partial x_j & \vdots \\
\partial f_n/\partial x_1 & & \cdots & \partial f_n/\partial x_m
\end{pmatrix} & \updownarrow n \\
\xleftrightarrow{\quad m \quad} &
\end{array}$$

$$L_{ij} = \frac{\partial f_i}{\partial x_j}$$

and is called the Jacobian matrix.

<u>The chain rule:</u>

Consider function formed by the composition of others. In one dimension we might consider $F(t) = f(x(t))$. We have $t \to x \to F$, $\mathbb{R} \to \mathbb{R} \to \mathbb{R}$.

Consider the change in F caused

by a change in t.

$$F(t+\delta t) - F(t) = f(x(t+\delta x)) - f(x(t))$$

$$= f(x(t)) + \underbrace{\delta t\, x'(t)} + \underbrace{|\delta t|\, \phi(\delta t)} - f(x(t))$$

+ ... the higer order terms that arent linear.

As $x(t)$ is differentiable.

$$= f(x(t)) + f'(x(t))\, \delta x\, x'(t)$$

$$+ \dots - f(x(t))$$

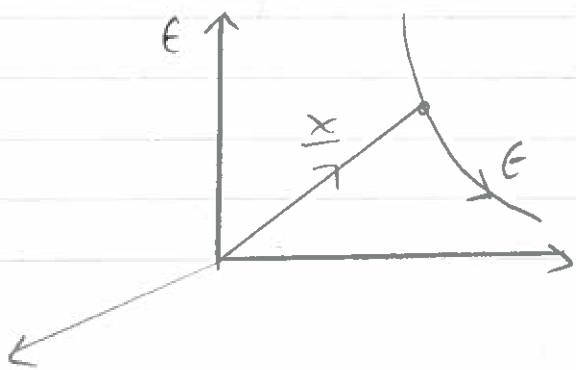
$$= \delta t (f'(x))\, x'(t) + \dots$$

Compare this now with the statement that $F(t)$ is differentible, in the form.

$$F(t+\delta t) = F(t) + \delta t\, F'(t) + \dots$$

and $F'(t) = f'(x(t))\, x'(t)$.

$$\mathbb{R} \to \mathbb{R}^3 \to \mathbb{R}.$$



$\underline{x}(t) = \underline{r}(t)$

$f(\underline{x}(t))$.

Consider $F(\epsilon) = f(x(\epsilon), y(\epsilon), z(\epsilon))$

$F(\epsilon+\delta t) - F(\epsilon) = f(x(\epsilon+\delta t), y(\epsilon+\delta t), z(\epsilon+\delta t)) - f(x(\epsilon), y(\epsilon), z(\epsilon))$.

$= f(x(\epsilon)+\delta\epsilon\, x'(\epsilon)+\dots, y(\epsilon)+\delta\epsilon\, y'(\epsilon)+\dots, z(\epsilon)+\delta\epsilon\, z'(\epsilon)) - f(x(\epsilon), y(\epsilon), z(\epsilon))$.

$$= f(x(\epsilon), y(\epsilon), z(\epsilon)) + \frac{\partial f}{\partial x}\delta\epsilon\, x'(\epsilon) + \frac{\partial f}{\partial y}\delta\epsilon\, y'(\epsilon) + \frac{\partial f}{\partial z}\delta\epsilon\, z'(\epsilon) + \dots - f(x(\epsilon), y(\epsilon), z(\epsilon))$$

Note: $f(x+h) \approx f(x) + \nabla f \cdot h$

So we identify:

$$F'(\epsilon) = \frac{\partial f}{\partial x}x'(\epsilon) + \frac{\partial f}{\partial y}y'(\epsilon) + \frac{\partial f}{\partial z}z'(\epsilon).$$

If we used a $\cdot$ for time derivatives and $\underline{r}'(\epsilon)$ instead of $x(\epsilon)$.

$$\dot{F} = \dot{\underline{r}} \cdot \nabla f.$$

Jacobian for $x(\epsilon)$ (3×1)

$$\dot{F} = \nabla f \cdot \dot{x} = \underbrace{\begin{pmatrix} \partial f/\partial x & \partial f/\partial y & \partial f/\partial z \end{pmatrix}}_{\text{Jacobian for } f(x)\ (1\times3)} \begin{pmatrix} \partial x/\partial t \\ \partial y/\partial t \\ \partial z/\partial t \end{pmatrix}$$

Jacobian for $F(\epsilon)$ a (1×1) matrix

$F(\epsilon) = f(\underline{x}(\epsilon))$

The Jacobian for F, the composition of $x(\epsilon)$ and $f(\underline{x})$ is obtained by matrix multiplication (i.e composition) of the Jacobian for $\underline{x}(\epsilon)$ and $f(\underline{x})$.

We can generalise this observation.

$$\mathbb{R}^L \longrightarrow \mathbb{R}^m \longrightarrow \mathbb{R}^n.$$

If the two function $\underline{x}(\underline{u})$ and $f(\underline{x})$ are from $\mathbb{R}^L \to \mathbb{R}^m$ and $\mathbb{R}^m \longrightarrow \mathbb{R}^n$ respectively then the composition $\underline{F}(\underline{u}) = \underline{f}(\underline{x}(\underline{u}))$ is from $\mathbb{R}^L \longrightarrow \mathbb{R}^n$. $\mathbb{R}^L \to \mathbb{R}^m \to \mathbb{R}^n$ These mappings $\underline{x}$, $\underline{f}$ and $\underline{F}$ have Jacobians.

$$\underline{x}: \begin{pmatrix} \partial x_1/\partial u_1 & \partial x_1/\partial u_2 & \cdots & \partial x_1/\partial u_L \\ \vdots & & & \vdots \\ \partial x_m/\partial u_1 & & & \partial x_m/\partial u_L \end{pmatrix}$$

$= \underline{\underline{u}}$ ($m \times L$ matrix).

$$f: \begin{pmatrix} \partial f_1/\partial x_1 & \cdots & \partial f_1/\partial x_m \\ \vdots & \ddots & \\ \partial f_n/\partial x_1 & \cdots & \partial f_n/\partial x_m \end{pmatrix} = \underline{\underline{T}} \text{ (a } n \times m \text{ matrix)}$$

$$F: \begin{pmatrix} \partial F_1/\partial u_1 & \cdots & \partial F_1/\partial u_L \\ \vdots & \ddots & \\ \partial F_n/\partial u_1 & & \partial F_n/\partial u_L \end{pmatrix} = \underline{\underline{S}} \text{ (a } n \times L \text{ matrix)}$$

The chain rule says:

$$\underline{\underline{S}}' = \underline{\underline{I}}\,\underline{\underline{u}}.$$

<u>A geometric interpretation of gradient ∇f.</u>

Consider $\phi(\underline{x}) = \phi(x, y, z)$ with x, y, z independent variables. If x, y, z are chosen so that $\phi(x, y, z) = \text{constant} = c$ then this impose a constraint on our choice and points (x, y, z) satisfying $\phi(x, y, z) = c$ lie on the surface, called a level surface, called a level surface of ϕ (3D + 1 constraint = 2D)

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$f(x,y,z) = x^2 + y^2 + z^2$, sphere centered at origin $\phi(x, y, z) = ax + by + cz = d$ is a plane with the normal (a,b,c) the distance $d/\sqrt{x^2+y^2+z^2}$

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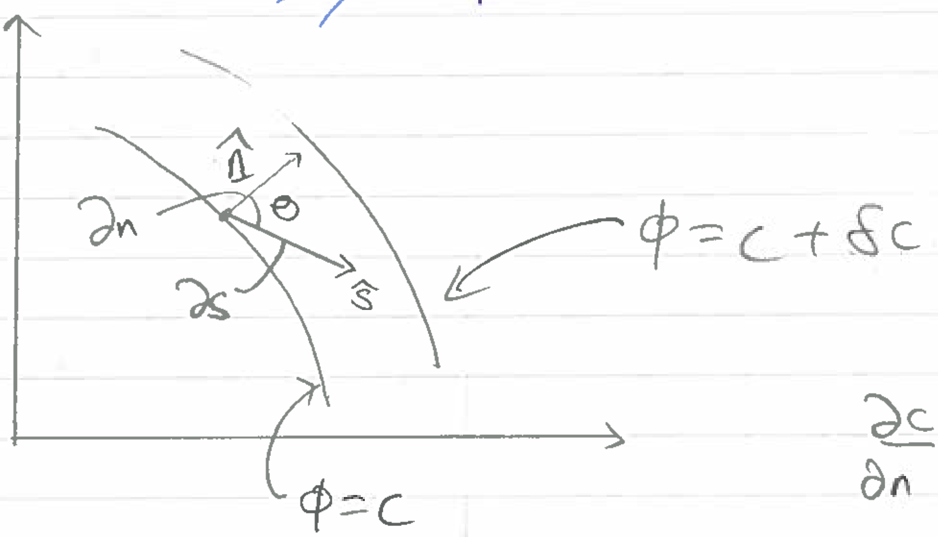

$$\frac{\partial c}{\partial n}$$

$$\delta s = \frac{\delta n}{\cos\theta}.$$

Consider a neighbouring surface given by $\phi = c + \delta c$. Consider too a unit normal $\hat{\underline{n}}$ to the surface $\phi = c$.

We can ask how much ϕ changes if we move a distance δn in the direction of $\hat{\underline{n}}$. Call this change $\delta\phi$ and consider $\lim_{\delta n \to 0} (\delta\phi/\delta n) = \partial\phi/\partial n$. i.e the rate of change of ϕ measured in a direction normal to a level surface.

We define the gradient of the function ϕ to be the vector:

$$\underline{\nabla}\phi = \hat{\underline{n}} \frac{\partial \phi}{\partial n}$$

We shall see $\underline{\nabla}\phi = \begin{pmatrix} \phi_x \\ \phi_y \\ \phi_z \end{pmatrix}$

Suppose I want the rate of change of ϕ in a different direction given by $\hat{\underline{s}}$, $\partial\phi/\partial s$ say

$$\frac{\partial \phi}{\partial s} = \lim_{\delta s \to 0} \frac{\delta \phi}{\delta s} = \lim_{\delta s \to 0} \frac{\delta c}{\delta n / \cos\theta}$$

$$= \cos\theta \lim_{\delta n \to 0} \frac{\delta \phi}{\delta n} = \cos\theta \frac{\partial \phi}{\partial n}$$

But $\cos\hat{\theta} = \hat{\underline{s}} \cdot \hat{\underline{n}}$ so $\partial\phi/\partial s = \hat{\underline{s}} \cdot \hat{\underline{n}}\, \partial\phi/\partial n$ $= \hat{s} \cdot \underline{\nabla} p$.

We call $\partial\phi/\partial s$ the directional derivative of $\hat{\underline{s}}$ and we see $\partial\phi/\partial s$ $= \hat{\underline{s}} \cdot \underline{\nabla}\phi$.

If we choose $\hat{\underline{s}} = \underline{i}$, this becomes $\partial\phi/\partial x$ $= \hat{\underline{c}} \cdot \underline{\nabla}\phi =$ the first component of $\underline{\nabla}\phi$

So we conclude:

$$\underline{\nabla}\phi = \begin{pmatrix} \phi_x \\ \phi_y \\ \phi_z \end{pmatrix}$$

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$$\frac{\partial f}{\partial s} = \hat{\underline{s}} \cdot \underline{\nabla} f.$$

Defined at a point, but that point can be anything. So $\partial f/\partial s$ is a function of position assuming its exist.

Consider $f(x,y)=(x+1)(y-1)$ and find the directional derivatives in the direction of $\underline{i}+\underline{j}$ and $\underline{i}-\underline{j}$.

Unit vector in these direction are

$$\frac{1}{\sqrt{2}}(\underline{i}+\underline{j}) \quad \text{and} \quad \frac{1}{\sqrt{2}}(\underline{i}-\underline{j}).$$

$$\underline{\nabla} f = \begin{pmatrix} \partial f/\partial x \\ \partial f/\partial y \end{pmatrix} = \begin{pmatrix} y-1 \\ x+1 \end{pmatrix}$$

and so if $s=\underline{i}+\underline{j}$ we have:

$$\frac{\partial f}{\partial s} = \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix} \cdot \begin{pmatrix} y-1 \\ x+1 \end{pmatrix} = \frac{1}{\sqrt{2}}(y-x-2)$$

We can also for the rate of change of these directional dirivaties in different direction. Most useful will be:

$$\frac{\partial}{\partial s}\left(\frac{\partial f}{\partial s}\right) = (\hat{\underline{s}} \cdot \nabla f)(\hat{\underline{s}} \cdot \nabla f)$$

$= (\hat{s} \cdot \nabla)^2 \cdot f$ say.

For our example the second derivates are for $s = \underline{i} + j$.

$$\begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix} \cdot \begin{pmatrix} \frac{\partial}{\partial x} \frac{1}{\sqrt{2}}(x+y) \\ \frac{\partial}{\partial y} \frac{1}{\sqrt{2}}(x+y) \end{pmatrix} = \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix} \cdot \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix}$$

$$= +1$$

for $\underline{s} = i - j$.

$$\begin{pmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{pmatrix} \cdot \begin{pmatrix} \partial/\partial x \left(1/\sqrt{2} (y - x - 2) \right) \\ \partial/\partial y \left(1/\sqrt{2} (y - x - 2) \right) \end{pmatrix} = -1$$

Taylor's theorem:

We know that $f : \mathbb{R} \to \mathbb{R}$ we have, under certain conditions

$$f(a+h) = f(a) + hf'(x) + \frac{1}{2}h^2 f''(a) + \frac{1}{6}h^3 f'''(a) + \dots$$

$$= \sum_{n=0}^{\infty} \frac{f^n(a) h^n}{n!}$$

within a radius of convergence.

$$f(\underline{a}+\underline{h}) = f(\underline{a}) + \underline{L} \cdot \underline{h} + \dots$$

∇f

To extend this and the subsequent terms in a statment of Taylor's theorem for $f: \mathbb{R}^n \to \mathbb{R}$, we imagine fixing the problem is then reduced to one dimension, with variable $|\underline{h}|$ the distance travelled in the direction of h and we can use Taylor's theorem for $f: \mathbb{R} \to \mathbb{R}$.

We see that the $f^n(a)$ need to replaced by:

$$\frac{\partial^n f}{\partial s^n} = (\hat{\underline{s}} \cdot \underline{\nabla})^n f.$$

and the h needs to be replaced by $|\underline{h}|$.

So we get:

$$f(\underline{a}+\underline{h}) = f(\underline{a}) + |\underline{h}|(\hat{\underline{s}} \cdot \underline{\nabla})f + \frac{1}{2}|\underline{h}|^2(\hat{\underline{s}} \cdot \nabla)^2 f + \dots$$

$$+ \frac{1}{n!}|\underline{h}|^n(\hat{\underline{s}} \cdot \underline{\nabla})^n f + \dots$$

But $\underline{h} = |\underline{h}|\hat{\underline{s}}$ and so

$$f(\underline{a}+\underline{h}) = f(\underline{a}) + (\underline{h} \cdot \underline{\nabla})f + \frac{1}{2}(\underline{h} \cdot \underline{\nabla})^2 f$$

$$+ \dots + \frac{1}{n!}(\underline{h} \cdot \underline{\nabla})^n f + \dots$$

$$= \sum_{n=0}^{\infty} \frac{(\underline{h} \cdot \underline{\nabla})^n f}{n!}$$

If $f:\mathbb{R}^2 \to \mathbb{R}$ and $\underline{h} = \binom{h}{k}$ then

$$\underline{h} \cdot \underline{\nabla} f = \begin{pmatrix} h \\ k \end{pmatrix} \cdot \begin{pmatrix} \partial f/\partial x \\ \partial f/\partial y \end{pmatrix} = h \frac{\partial f}{\partial x} + k \frac{\partial f}{\partial y}.$$

$$(\underline{h} \cdot \underline{\nabla})^2 f = \begin{pmatrix} h \\ k \end{pmatrix} \cdot \begin{pmatrix} \partial/\partial x \, (h \, \partial f/\partial x + k \, \partial f/\partial y) \\ \partial/\partial y \, (h \, \partial f/\partial x + k \, \partial f/\partial y) \end{pmatrix}$$

$$= \begin{pmatrix} h \\ k \end{pmatrix} \cdot \begin{pmatrix} h f_{xx} + k f_{xy} \\ h f_{xy} + k f_{yy} \end{pmatrix}$$

$$= h^2 f_{xx} + 2hk f_{xy} + k^2 f_{yy}.$$

$$(h \cdot \nabla)^3 f = h^3 f_{xxx} + 3h^2 k f_{xxy} + 3hk^2 f_{xyy} + k^3 f_{yyy}$$

If $f:\mathbb{R}^3 \to \mathbb{R}$, $\underline{h} = \begin{pmatrix} h \\ k \\ l \end{pmatrix}$ then:

$$h \cdot \nabla = h \frac{\partial}{\partial x} + k \frac{\partial}{\partial y} + l \frac{\partial}{\partial z}$$

$$(h \cdot \nabla)^2 = h^2 \frac{\partial^2}{\partial x^2} + k^2 \frac{\partial^2}{\partial y^2} + l^2 \frac{\partial^2}{\partial z^2} + 2hk \frac{\partial^2}{\partial x \partial y}$$

$$+ 2hl \frac{\partial^2}{\partial x \partial z} + 2kl \frac{\partial^2}{\partial y \partial z}$$

Coefficients are found by considering $(a+b+c)^n$.

Ex: Express $f(x,y) = x^3y + 3y - 2$ in powers of $(x-1)$ and $(y-1)$

We will do this by finding a Talyor series for $f(x,y)$ about the point $(1,-2)$.

$$\underline{a} = \begin{pmatrix} 1 \\ -2 \end{pmatrix}, \text{ and } \underline{h} = \begin{pmatrix} h \\ k \end{pmatrix} = \begin{pmatrix} x+1 \\ y+2 \end{pmatrix} \text{ so}$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \underline{a} + \underline{h}.$$

$$\frac{\partial f}{\partial x} = 2xy, \qquad \frac{\partial^3 f}{\partial x^3} = 0.$$

$$\frac{\partial f}{\partial y} = x^3 + 3, \qquad \frac{\partial^3 f}{\partial x^2 \partial y} = 2$$

$$\frac{\partial^2 f}{\partial x^2} = 2y, \qquad \frac{\partial^3 f}{\partial x \partial y^2} = 0.$$

$$\frac{\partial^2 f}{\partial y^2} = 0, \qquad \frac{\partial^3 f}{\partial y^3} = 0$$

$$\frac{\partial^2 f}{\partial x \partial y} = 2x.$$

and higer derivatives are zero:

$$x^3y + 3y - 2 = f(x,y)$$

$$\ldots = f(1,-2) + \left(\overset{x-1}{h} \underset{-4}{\frac{\partial f}{\partial x}} + \overset{y+2}{k} \underset{4}{\frac{\partial f}{\partial y}} \right)\Bigg|_{(1,-2)}$$

$$+ \frac{1}{2}\left(h^2 f_{xx} + 2hk f_{xy} + k^3 \cdot 0 \right)\Bigg|_{(1,-2)}$$

$$+ \frac{1}{6}\left(h^3 \cdot 0 + 3h^2 \frac{\partial f}{\partial x^2 \partial y} + 3hk^2 \cdot 0 + k^3 \cdot 0 \right)\Bigg|_{(1,-2)}$$

$$+ 0$$

$$= -10 - 4(x-1) + 4(y+2) - 2(x-1)(y+2)$$
$$+ (x-1)^2(y+2).$$

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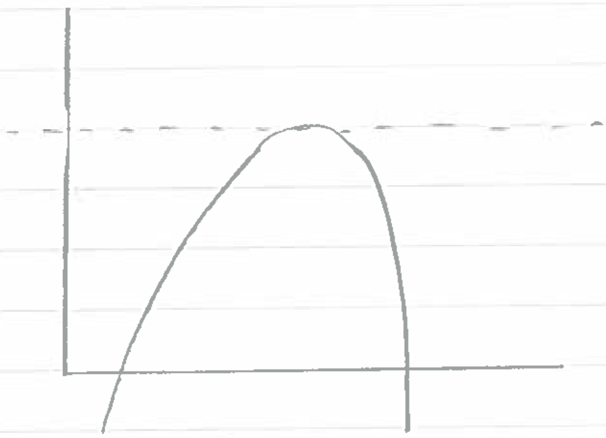
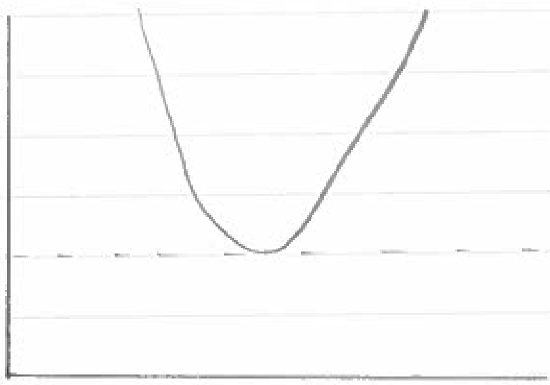
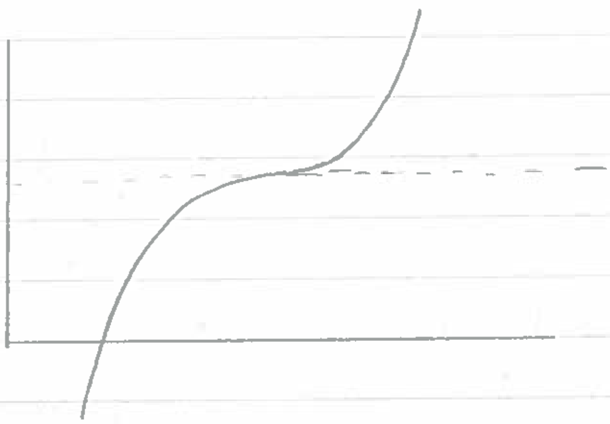
Extreme values and critical points of functions of several (mainly two) turning points.

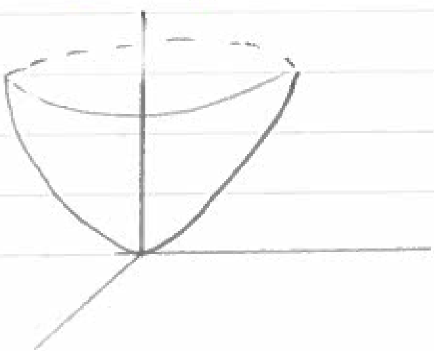
If f is from $\mathbb{R} \to \mathbb{R}$.



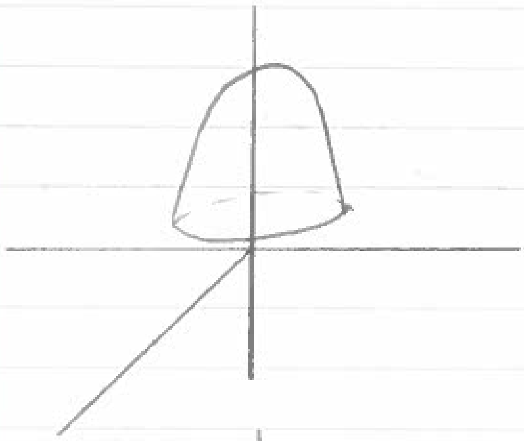




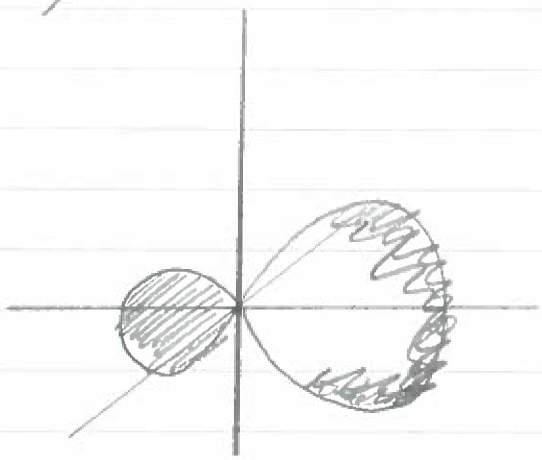
At critical point, the tangent to the curve is horizontal (i.e // to the x-axis) and we test for this by finding position where $f'(x) = 0$.



Minimum at (x_0, y_0)

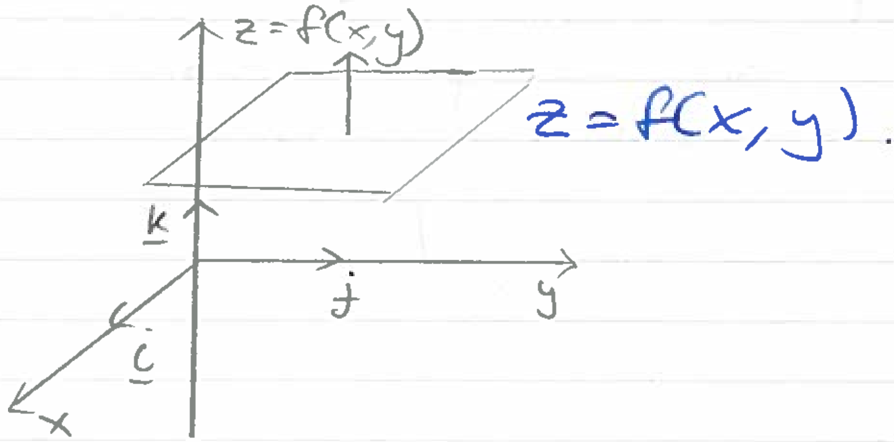


Maximum at (x_0, y_0)



Saddle point at (x_0, y_0).

If we have a function given by $z = f(x,y)$ then at a point where f has a local maximum/minimum/saddle point, the tangent-plane to the surface $z = f(x,y)$ is // to the (x,y) plane, or to the surface $z = f(x,y)$ is // to the $f(x,y)$ plane, or has a normal // $\underline{k}$.



The normal to a surface written as a level surface of $g(x,y) = z - f(x,y) = c$ is given by:

$$\underline{\nabla} g = \begin{pmatrix} -\partial f/\partial x \\ -\partial f/\partial y \\ +1 \end{pmatrix}$$

and so at a critial point (x_0, y_0) is such that:

$$\frac{\partial f}{\partial x}(x_0, y_0) = \frac{\partial f}{\partial y}(x_0, y_0) = 0.$$

Using Taylor's theorem about (x_0, y_0) and with $\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} = \underline{x}_0$.

$$f(\underline{x}_0 + \underline{h}) = f(\underline{x}_0) + \underline{h} \cdot \underline{\nabla} f \Big|_{x_0} + \frac{1}{2}(\underline{h} \cdot \nabla^2 f)\Big|_{x_0} + \dots$$

$$\begin{pmatrix} \partial f/\partial x \\ \partial f/\partial y \end{pmatrix}\Bigg|_{x_0} = 0$$

and so:

$$f(\underline{x}_0 + \underline{h}) = f(x_0) = \frac{1}{2}\left(h^2 \frac{\partial^2 f}{\partial x^2} + 2hk \frac{\partial^2 f}{\partial x \partial y} + k^2 \frac{\partial^2 f}{\partial y^2}\right)\Bigg|_{(x_0, y_0)}$$

If $f: \mathbb{R}^n \to \mathbb{R}$ then with $\underline{h} = (h_1, h_2, h_3, \ldots, h_n)^T$. this is:

$$\underbrace{(h_1, h_2, \ldots, h_n) \underbrace{\begin{pmatrix} \partial^2 f/\partial x_1^2 & \partial^2 f/\partial x_1 \partial x_2 & \ldots & \partial^2 f/\partial x_1 \partial x_n \\ \vdots & & \ddots & \\ \partial^2 f/\partial x_n \partial x_1 & \ldots & & \partial^2 f/\partial x_n^2 \end{pmatrix}}_{\text{The Hessian of } f} \begin{pmatrix} h_1 \\ h_2 \\ \vdots \\ h_n \end{pmatrix}}_{\text{A Quadratic form.}}$$

A 2 by 2 case: $(h\ k) \begin{pmatrix} f_{xx} & f_{xy} \\ f_{xy} & f_{yy} \end{pmatrix} \begin{pmatrix} h \\ k \end{pmatrix}$

Testing whether this quadratic form is always +ve/−ve/or either depending on coefficients in $(h_1, h_2, \ldots, h_m)$ reduces to seeing if the eigenvalues of the Hessian are all +ve/−ve/or mixed in sign. Here though we proceed by completing the square.

We will assume $f_{xx} \neq 0$. If $f_{xx} = 0$, then we proceed as below using $f_{xx} = f_{xy} = 0$ then it is clear we have a saddle point, since the product hk can be made of either sign by choosing h and k appropriatly.

$$f(\underline{x}_0 + \underline{h}) - f(x) = \frac{1}{2} f_{xx} \left[h^2 + 2hk \frac{f_{xy}}{f_{xx}} + k^2 \frac{f_{yy}}{f_{xx}} \right]$$

$$= \frac{1}{2} f_{xx}\left[\underbrace{\left(h + k\frac{f_{xy}}{f_{xx}}\right)^2}_{\text{always +ve or zero}} + k^2\left(\frac{f_{yy}}{f_{xx}} - \frac{f_{xy}^2}{f_{xx}^2}\right)\right]$$

$$\underbrace{\frac{k^2}{(f_{xx})^2}\left(f_{xx}f_{yy} - f_{xy}^2\right)}_{\text{May be either sign depending on sign of } f_{xx}}$$

$$\Delta = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix}$$ called the Discriminant

If $\Delta > 0$ then this sign has the sign as does f_{xx} so if

$\Delta > 0$ $f_{xx} > 0$ we have a minimum
$\Delta > 0$ $f_{xx} < 0$ we have a maximum
$\Delta < 0$ we have a saddle point.

because if we choose $k = 0$, the term is > 0 but if we choose $k = -h f_{xx}/f_{xy}$ then the term is < 0 ($f_{xx} > 0$)

Example: Find the critical point of

$$f(x,y) = \frac{1}{3}(x^3 + y^3) - (x^2 + y^2)$$

and determine their nature.

To find the critical points we slove simultanously

$$0 = \frac{\partial f}{\partial x} = x^2 + 2x \Rightarrow x = 0 \text{ or } x = 2.$$

$$0 = \frac{\partial f}{\partial y} = y^2 - 2y \Rightarrow y = 0 \text{ or } y = 2.$$

and the critical points are

$(0, 0)$ $\quad$ $(2, 0)$

$(0, 2)$ $\quad$ $(2, 2)$.

To determine their nature we need

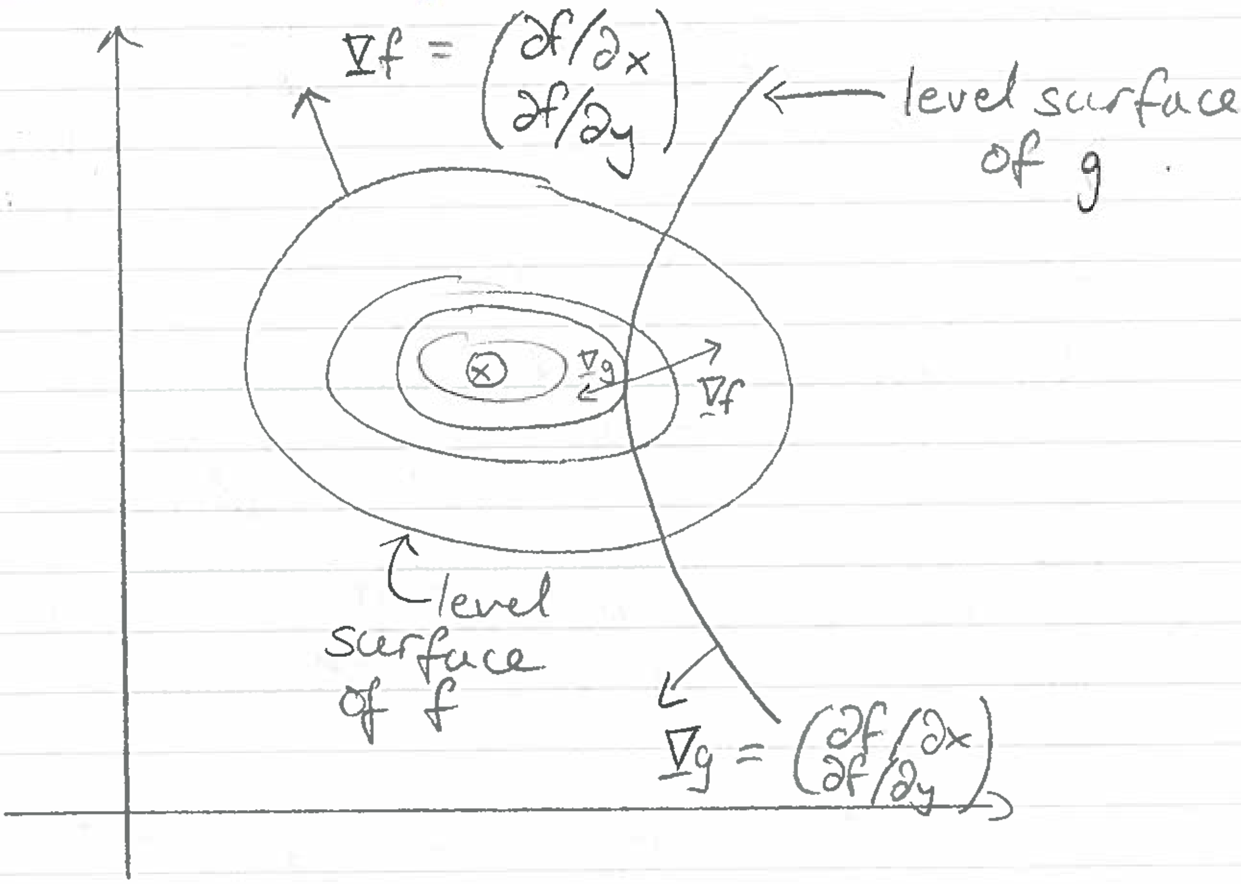
$f_{xx} = 2(x-1)$ and $f_{xy} = 0$

$f_{yy} = 2(y-1)$

$$\Delta = f_{xx} f_{xy} - f_{xy}^2 = 4(x-1)(y-1)$$

	$(0,0)$	$(2,0)$	$(0,2)$	$(2,2)$
$f_{xx} = (2)(x-1)$	-2	$+2$	-2	$+2$
$f_{yy} = 2(y-1)$	-2	-2	2	2
$f_{xy} = 0$	0	0	0	0
$\Delta = f_{xx} f_{xy} - f_{xy}^2$	4	-4	-4	4
	max	Saddle point	Saddle point	min

Constrained optimisation.



Consider level surface of $f(x,y)$ – i.e lines in (x,y) plane along which $f = \text{const}$.

Consider too a line given by $g(x,y) = c$.

If we ask what is the extreme value of $f(x,y)$ subject to the constraining $g(x,y) = c$ we see geometrically that this is achieved where a level surface of g (given by $g(x,y) = c$) is tangential to a level surface of f

If the normal to these curves are given by $\underline{\nabla} f$ and $\underline{\nabla} g$ then this occurs where $\underline{\nabla} f \,//\, \underline{\nabla} g$ i.e $\underline{\nabla} f = \lambda \underline{\nabla} g$

λ Lagrange Multiplier

$$\underline{\nabla}(f - \lambda g) = \underline{0}$$
$$\underline{\nabla}(f - \lambda(g - c)) = 0$$

This conditions tells as only that a level surface of f is tangental to a level surface of g. There are obviously many such points, each given by a particular value of λ. To find the one we want we add in the constraint equation.

$$\frac{\partial}{\partial x}(f - \lambda g) = 0 \quad \text{i.e} \quad \frac{\partial f}{\partial x} - \lambda \frac{\partial g}{\partial x} = 0.$$

$$\frac{\partial}{\partial y}(f - \lambda g) = 0 \quad \text{i.e} \quad \frac{\partial f}{\partial y} - \lambda \frac{\partial g}{\partial y} = 0.$$

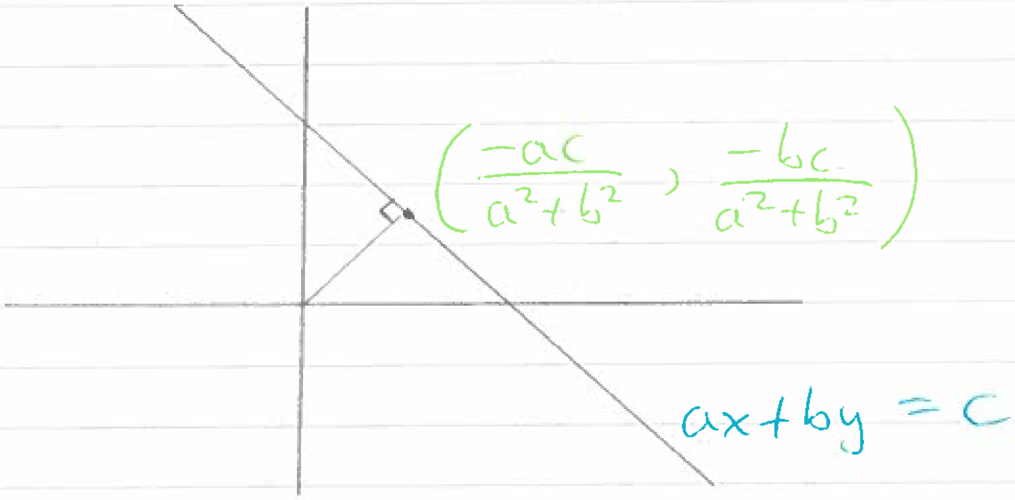
$$g(x,y) = c.$$

Aside: If $h(x,y) = f - \lambda(g - c)$ then these are:

$$\frac{\partial h}{\partial y} = \frac{\partial h}{\partial x} = \frac{\partial h}{\partial \lambda} = 0$$

$$\underline{\nabla} h = 0.$$

Example: Find the shortest distance of the line $ax + by + c = 0$ to the origin.



We will find the extreme value of $\sqrt{x^2+y^2}$. To make the algebra easier, we will take $f(x,y) = x^2 + y^2$ and the constraint is $g(x,y) = ax + by + c = 0$ and form:

$$h(x,y,\lambda) = f - \lambda g$$
$$= x^2 + y^2 - \lambda(ax + by + c)$$

We need to solve:

$$\frac{\partial h}{\partial x} = 2x - \lambda a = 0 \Rightarrow x = \frac{\lambda a}{2}$$

$$\frac{\partial h}{\partial y} = 2y - \lambda b = 0 \Rightarrow y = \frac{\lambda b}{2}.$$

and we add the constraint: $ax + by = -c$

$$\frac{\lambda a^2}{2} + \frac{\lambda b^2}{2} = -c.$$

$$\Rightarrow \lambda = \frac{-2c}{a^2+b^2}.$$

So $x = \dfrac{-ac}{a^2+b^2}$, $y = \dfrac{-bc}{a^2+b^2}$

and the distance can now be found as: $\sqrt{x^2+y^2} = \dfrac{|c|}{\sqrt{a^2+b^2}}$

<u>More generally</u>:

If $f: \mathbb{R}^n \to \mathbb{R}$ is written $f(\underline{x})$, $\underline{x} \in \mathbb{R}^n = (x_1, x_2, \ldots, x_n)^T$ then we can have up to $n-1$ constraints. Suppose we have m.

$$g_i(\underline{x}) = 0 \quad i \text{ runs from } 1 \text{ to } m.$$

We form the function, the <u>Lagrangian</u>

$$L(\underline{x}, \underline{\lambda}) = f(\underline{x}) - \sum_{i=1}^{m} \lambda_i g_i(\underline{x}).$$

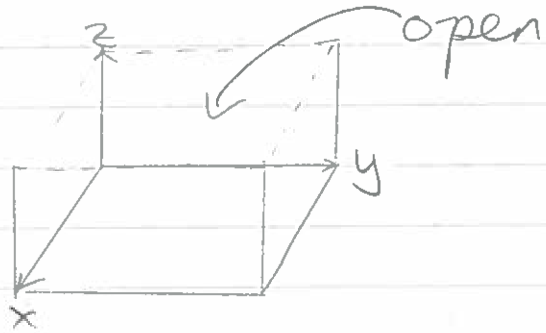
$(\lambda_1, \lambda_2, \ldots, \lambda_m)^T$

We then solve the n equations:

$$\frac{\partial L}{\partial x_i} = 0 \quad i = 1, \ldots, n, \quad \nabla_x \underline{L} = 0$$

together with the constraints $g_i(\underline{x}) = 0$
or $\partial L / \partial \lambda_i = 0 \quad i = 1, \ldots, m \quad \nabla_\lambda L = 0$
$\underline{\nabla} L = 0$.

<u>Example</u>: Construct an open-topped rectangluar box of volume V, minimising its surface aera.



Volume is $V = xyz$.

Surface aera is $S = 2xz + 2yz + xy$

Min/Max S subject to $V = xyz$.

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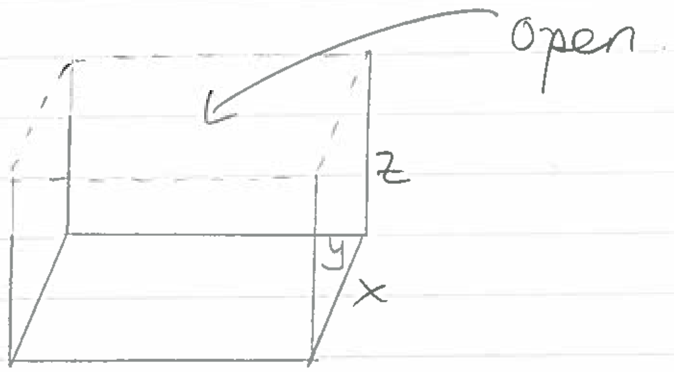


Minimise Surface $S(x,y,z) = 2yz + 2xz + xy$

Subject to volume $V = xyz$ fixed i.e. $V(x,y,z) = \text{Const} = V$.

$\lambda \neq 0$

We use Lagrange Multipliers and consider:

$$H(x,y,z) = S(x,y,z) - \lambda V(x,y,z)$$
$$= 2yx + 2xz + xy - \lambda xyz.$$

We set $\frac{\partial H}{\partial x} = \frac{\partial H}{\partial y} = \frac{\partial H}{\partial z} = 0$.

$$2z + y - \lambda yz = 0 \quad (1)$$
$$2z + x - \lambda xz = 0 \quad (2)$$
$$2y + 2x - \lambda xy = 0 \quad (3)$$

Solve (1) − (3) together with the fourth equation $xyz = V$ (4)

(1) − (2) $\Rightarrow (y-x) - \lambda z(y-x) = 0$

So $y - x = 0$ i.e $y = x$

<u>OR</u> $\lambda z = 1$.

[In ① $\lambda z = 1 \Rightarrow z = 0$, ($\lambda =$ "∞") $z = 0$
$\Rightarrow V = 0$ and we presume $V \neq 0$].

So discount $\lambda z = 1$ and follows $y = x$.

if $y = x$ then ③ gives $4x = \lambda x^2$

$\Rightarrow x = 0$ (discount as then $V = 0$).

or $x = \frac{4}{\lambda}$, $y = \frac{4}{\lambda}$

$y = x$ in ② gives:

$$z = \frac{x}{\lambda x - 2} = \frac{4}{\lambda}\left[\frac{1}{4-2}\right] = \frac{2}{\lambda}$$

λ is found from constraint

$$xyz = V \quad \text{i.e} \quad \frac{4}{\lambda} \cdot \frac{4}{\lambda} \cdot \frac{4}{\lambda} = \frac{1}{2}\left(\frac{4}{\lambda}\right)^3 = V.$$

$$\frac{4}{\lambda} = \sqrt[3]{2V}$$

$$x = y = \sqrt[3]{2V}, \quad z = \frac{1}{2}\sqrt[3]{2V}$$

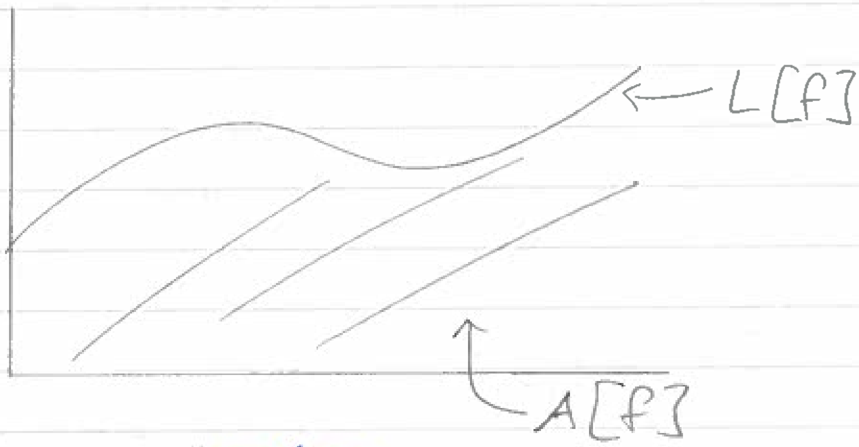
Calculus of Variations

This is concerned with finding extreme values of functionals. Functionals are functions which map from a set of functions onto the numbers.

$$\delta : \delta(f) = f(0)$$

$$A[f] = \int_0^1 f(x)\, dx.$$

$$L[f] = \int_0^1 \sqrt{1-(f')^2}\, dx.$$



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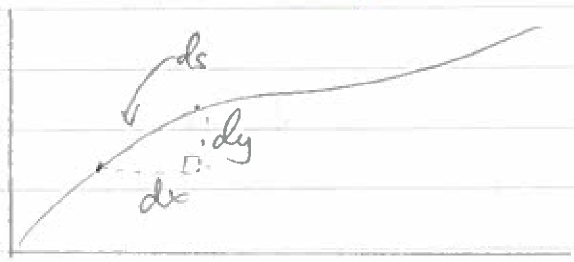

$$ds^2 = dx^2 + dy^2 = dx^2\left(1 + \left(\frac{dy}{dx}\right)^2\right)$$

$$\Rightarrow ds = \sqrt{1+y'^2}\, dx.$$

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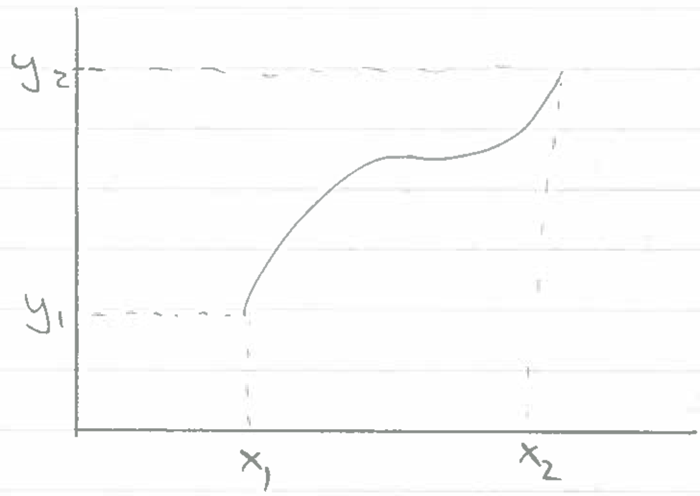


The functions of which makes these functionals on extreme values is called the <u>extremals</u>.

Generally the functionals we will consider are:

$$I[y] = \int_{x_1}^{x_2} F(x, y, y') \, dx.$$

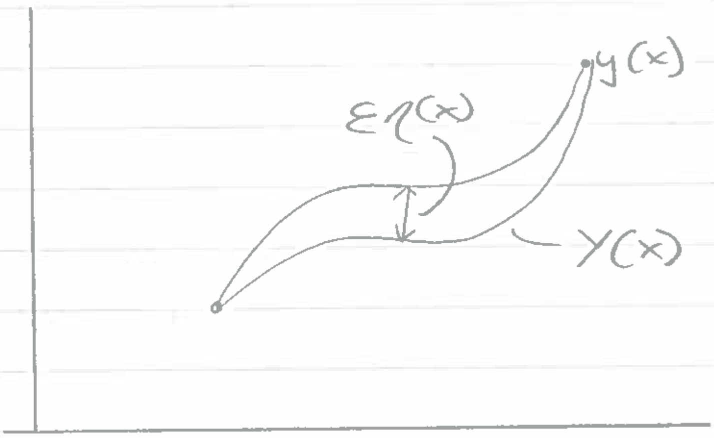
$$F(x, y, y') = y.$$

$$F(x, y, y') = \sqrt{1+(y')^2}$$

and we wish to find an external curve satisfying boundary conditions:

$$y(x_1) = y_1 \ , \ y(x_2) = y_2 .$$

We will assume that the extremal curve $g(x)$ exist.



$$Y(x) = y(x) + \varepsilon\eta(x)$$

$$\eta(x_1) = \eta(x_2) = 0.$$

$$I[y] = I[y, \varepsilon, \eta]$$
$$= \int_{x_1}^{x_2} F(x, y+\varepsilon\eta, y' + \varepsilon\eta')\, dx$$

If y is the extremal then

$$\frac{dI}{d\varepsilon}[y; \varepsilon, \eta]\Big|_{\varepsilon=0} = 0.$$

i.e.

$$0 = \int_{x_1}^{x_2} \frac{\partial}{\partial x} F(x, y+\varepsilon\eta, y'+\varepsilon\eta')\, dx \Big|_{\varepsilon=}$$

$$\Rightarrow \quad 0 = \int_{x_1}^{x_2} \frac{\partial F}{\partial y}\eta + \frac{\partial F}{\partial y'}\eta'\, dx \Big|_{\varepsilon=0}$$

$$0 = \int_{x_1}^{x_2} \frac{\partial F}{\partial y}(x,y,y')\eta + \frac{\partial F}{\partial y'}(x,y,y')\eta'\, d$$

independently of η.

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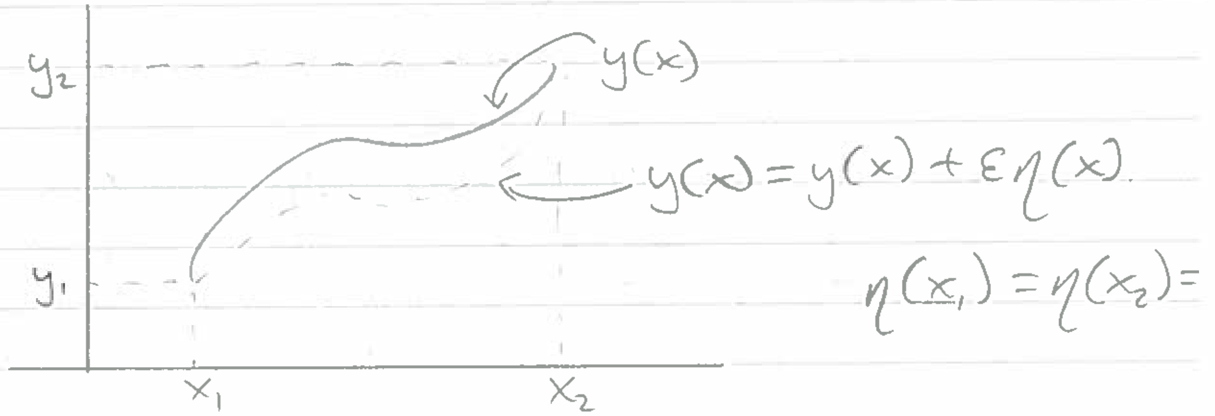
$$I[y] = \int_{x_1}^{x_2} F(x, y, y') \, dx$$

$$y(x_1) = y_2 \quad , \quad y(x_2) = y_2$$



$y(x)$ is external $\Rightarrow \left.\dfrac{dI}{d\varepsilon}\right|_{\varepsilon=0} = 0$

$$\Rightarrow \int_{x_1}^{x_2} \frac{\partial F}{\partial y}\eta + \frac{\partial F}{\partial y'}\eta' \, dx = 0 \quad \text{for all } \eta.$$

Use $\int$ by parts

$$\Rightarrow \left[\eta \frac{\partial F}{\partial y'}\right]_{x_1}^{x_2} + \int_{x_1}^{x_2} \frac{\partial F}{\partial y}\eta - \eta\frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) dx = 0$$

$= 0$ as $\eta(x_1) = \eta(x_2) = 0$.

$$\int_{x_1}^{x_2} \eta\left[\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right)\right] dx = 0.$$

for all η.

we conclude that for the external curve $y(x)$.

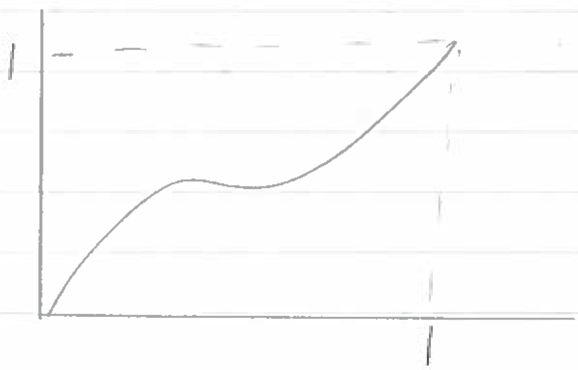
$$\boxed{\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = 0}.$$

This is the Euler – Lagrange equation and is a second order differential equation for $y(x)$, the extremal curve, to be solved with the boundary conditions $y(x_1) = y_1$, $y(x_2) = y_2$. Once the extremal is known it can be substituted into $I[y]$ to find the extreme value.

Example
Find the extremal curve for

$$I[y] = \int_0^1 \underbrace{y^2 - 2xy - y'^2}_{F(x,y,y')} \, dx$$

$y(0) = 0$, $y(1) = 1$.



$$\left.\begin{aligned}\frac{\partial F}{\partial y} &= 2y-2x \\ \frac{\partial F}{\partial y'} &= -2y'\end{aligned}\right\} \Rightarrow (2y-2x) - \frac{d}{dx}(-2y') = 0$$

$$\Rightarrow y'' + y = x \quad , \; y(0) = 0, \; y(1) = 1.$$

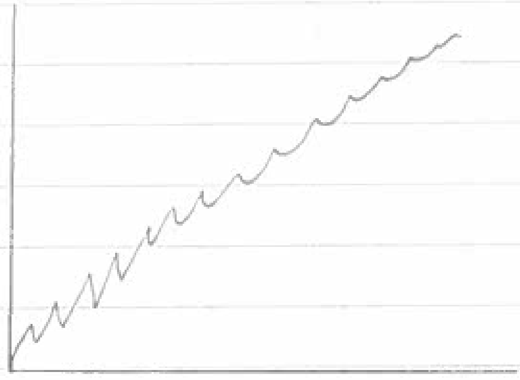
$$y = A\cos x + B\sin x + x.$$

and choosing A and B to satisfy BC $y(0)=0$, $y(1)=1$ gives $A=B=0$, $y=x$.

The extreme values of the integral:

$$\int_0^1 x^2 - 2x^2 - 1 \, dx = -\frac{4}{3}$$

We can argue that if we can make the value of $I(y)$ as large and $-$ve as we like by choosing an appropriate $y(x)$, then $-4/3$ indicates a maximum value for the integral.



$$y = x + \sin nx$$
$$y' = 1 + n\cos nx$$
$$y'' \approx n^2\cos^2 nx \quad \text{for large } n.$$

So due to the $-y'^2$ term in the integral as large and $-ve$ as we like by choosing a sufficiently oscillatory $y(x)$.

<u>The shortest distance between two points:</u>

In Euclidean space the length of a curve $y(x)$ is

$$\int_{x_1}^{x_2} \sqrt{1+(y')^2}\, dx$$

Here $F(x, y, y') = \sqrt{1+(y')^2}$ and $\frac{\partial F}{\partial y} = 0$.

$\left[\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = 0\right]$ and E-L equation is

$$0 - \frac{d}{dx}\left[\frac{y'}{\sqrt{1+(y')^2}}\right] = 0.$$

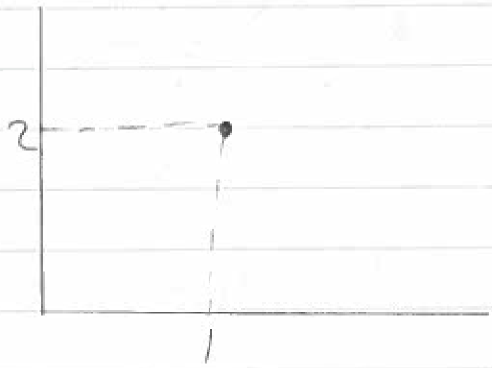
and we have

$$\frac{y'}{\sqrt{1+(y')^2}} = \text{constant}.$$

and thus is true only $y' = \text{const}$ i.e the extremal is a straight line.

<u>Example</u> : EXAM QUESTION

Consider $I[y] = \int_0^1 \underbrace{(y'-y)^2}_{F(x,y,y') = (y'-y)^2} dx \quad y(0) = 0,$ $y(1) = 2$



and EL gives

$$-2(y'-y) - \frac{d}{dx}\left(2(y'-y)\right) = 0.$$

$$\Rightarrow y'' - y = 0.$$

so $y(x) = A\cosh x + B\sinh x$

BC $\Rightarrow$ $y = \dfrac{2\sinh(x)}{\sinh(1)}$

It is possiable to prove, in this case, that this extremal curve $y = f$ where $f'' - f = 0$, $f(0) = 0$, $f(1) = 2$ gives a minimum value

for the integral. This is by considering $I[f+g]$ where $g(0)=0, g(1)=0$.
and $I[f+g] \geq I[f]$

$$I[f] = \int_0^1 (f' - f)^2 \, dx$$

$$I[f+g] = \int_0^1 \underbrace{(f'+g'-f-g')}_{((f'-f)+(g'-g))}{}^2 \, dx$$

$$I[f+g] = \int_0^1 (f'-f)^2 + 2(f'-f)(g'-g) + (g'-g)^2 \, dx$$

$$= I[f] + 2\int_0^1 (f'-f)(g'-g)\, dx$$

$$+ \text{ always } \geq 0.$$

Consider $2\int_0^1 (f'-f)g' - (f'-f)g \, dx$ and integrate by parts on the first integral to give

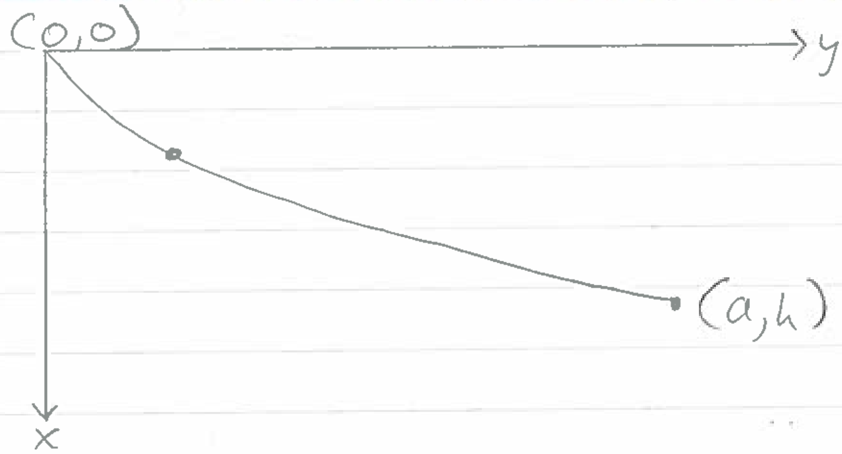
$$\left[2(f'-f)g\right]_0^1 - 2\int (f''-\cancel{f'})g + (\cancel{f'}-f)g \, dx$$

$g(0)=g(1)=0$ and $f'-f=0$ as f is extremal gives 0 for this integral and

$$I[f+g] = I[f] + \int_0^1 (g'-g)^2 \, dx \geq I[f]$$

so the extremal gives a minimum.

The Brachistrochrone Problem.



Find g(x) such that the time taken for a bead to fall due to gravity from (0, 0) to (a, h) on wire y(x) is a minimum.

$$T = \int dt = \int \frac{ds}{v}$$

KE gained = PE lost gives $\frac{1}{2} m v^2 = mgx$

so $v = \sqrt{2gx}$

$$ds = \sqrt{1 + y'^2}\, dx$$

$$T[y] = \frac{1}{\sqrt{2g}} \int \frac{\sqrt{1+(y')^2}}{\sqrt{x}}\, dx$$

$F(x, y, y')$

$$\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = 0.$$

Here $\dfrac{\partial F}{\partial y} = 0$ and $\dfrac{d}{dx}\left[\dfrac{\partial F}{\partial y'}\right] = 0$.

So $\dfrac{\partial F}{\partial y} = c$ (a constant)

$$\frac{\partial F}{\partial y'} = \frac{1}{\sqrt{x}}\left(\frac{y'}{\sqrt{1+(y')^2}}\right) = c.$$

$$\Rightarrow \quad y'^2 = c^2 x\left(1+(y')^2\right)$$

$$y'^2(1-c^2x) = c^2x.$$

$$\Rightarrow \quad \left(\frac{dy}{dx}\right)^2 = \frac{c^2x}{(1-c^2x)}$$

$$\Rightarrow \underbrace{\frac{dy}{dx} = c\sqrt{\frac{x}{1+c^2x}}}$$ (We expect $y' > 0$ have just the positive root)

$$\sqrt{\frac{c^2x}{1-c^2x}} = \sqrt{\frac{x}{\underbrace{(1/c^2)}_{\alpha} - x}}$$ where $\alpha = \dfrac{1}{c^2}$

So $$\int dy = \int \sqrt{\frac{x}{a-\underset{\uparrow\ \alpha\sin^2\theta}{x}}}\, dx$$

Put $x = \alpha \sin^2\theta$.

$$y + k = \int \sqrt{\frac{\alpha\sin^2\theta}{\cancel{\alpha - \alpha\sin^2\theta}}}\ 2\alpha\sin\theta\,\cancel{\cos\theta}\, d\theta$$

$$= \int 2\alpha \sin^2\theta \, d\theta$$

$$= \alpha \int 1 - \cos 2\theta \, d\theta$$

$$= \alpha\left[\theta - \tfrac{1}{2}\sin 2\theta\right]$$

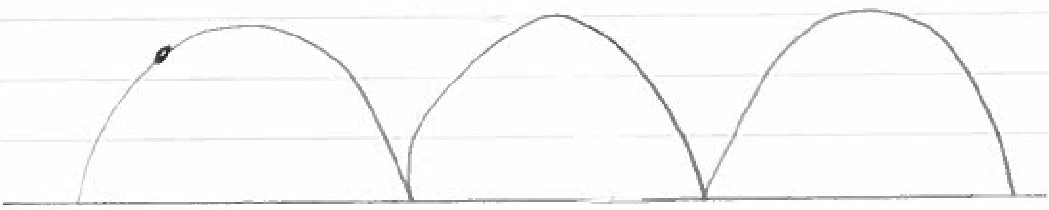
$$= \alpha\left[\theta - \sin\theta\cos\theta\right]$$

So $y + k = \alpha\left[\sin^{-1}\left(\sqrt{\frac{x}{\alpha}}\right) - \sqrt{\frac{x}{a}}\sqrt{1-\frac{x}{\alpha}}\right]$

$y(0) = 0 \Rightarrow k = 0$.

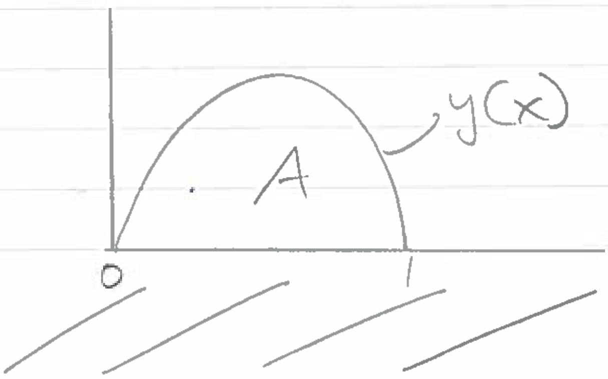
and α need to be such that

$y(\alpha) = h$.



Isoperimetric problems

[Calculus of variations with integral constants].



$$L = \int_0^1 \sqrt{1+(y')^2}\, dx \qquad \begin{array}{l} y(0) = 0 \\ y(1) = 0. \end{array}$$

$$A[y] = \int_0^1 y\, dx.$$

The method is to use Langrage Multipliers

Form $H[y, \lambda] = \int_0^1 y - \lambda \sqrt{1+(y')^2}\, dx$

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Special Forms of the E.L. equations

$$\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = 0 \quad , \quad F = F(x, y, y')$$

Case:

i) No y' in F i.e $\frac{\partial F}{\partial y'} = 0 \Rightarrow \frac{\partial F}{\partial y} = 0$

ii) No y in F i.e $\frac{\partial F}{\partial y} = 0 \Rightarrow \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = 0$

$$\Rightarrow \frac{\partial F}{\partial y'} = c$$

a first integral of E.L equation.

iii) No x in F i.e $\frac{\partial F}{\partial x} = 0$ then

$$F - y'\frac{\partial F}{\partial y'} = \text{constant}.$$

This first integral is called Beltrami Equation.

This is true as:

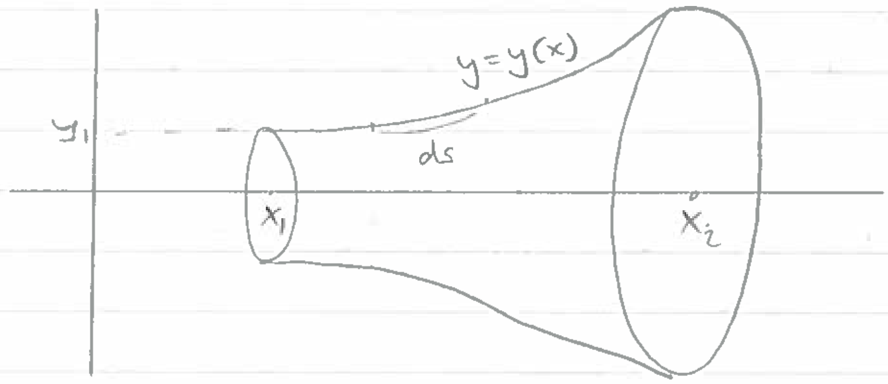
$$\frac{d}{dx}\Big[\underbrace{F}_{F(x,y,y')} - y'\frac{\partial F}{\partial y'}\Big] = \underbrace{\cancel{\frac{\partial F}{\partial x}}}_{0} + \frac{\partial F}{\partial y}\frac{dy}{dx} + \cancel{\frac{\partial F}{\partial y'}\frac{d}{dx}\left(\frac{dy}{dx}\right)}^{\,y''}$$

$$- \cancel{\frac{d^2y}{dx^2}\left(\frac{\partial F}{\partial y'}\right)} - \frac{dy}{dx}\frac{\partial}{\partial x}\left(\frac{\partial F}{\partial y'}\right) = 0$$

$$\Rightarrow \frac{dy}{dx}\left[\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right)\right] = \frac{dy}{dx}\cdot 0 = 0$$

$$\Rightarrow \frac{d}{dx}\left[F - y'\frac{\partial F}{\partial y'}\right] = 0 \Rightarrow F - y'\frac{\partial F}{\partial y'} = \text{const.}$$

—/—



$y(x_1) = y_1$
$y(x_2) = y_2$.

Minimise a surface area produced by rotating the curve $y = y(x)$ about the x-axis.

This is not important in finding y

$$A[y] = 2\pi \int_{x_2}^{x_1} y\sqrt{1+(y')^2}\,dx$$

$$ds^2 = dx^2 + dy^2$$

$$\frac{ds}{dx} = \sqrt{1+(y')^2}$$

$$A = \int \partial A = \int ds\, 2\pi y = \int 2\pi y \sqrt{1+(y')^2}\, dx$$

$$F(x, y, y') = y\sqrt{1+(y')^2}$$

We see $\partial f/\partial x = 0$ so that we can go immediately to the first integral.

$$F - y' \frac{\partial F}{\partial y'} = c$$

$$y\sqrt{1+(y')^2} - \frac{y' y y'}{\sqrt{1+(y')^2}} = c$$

$$\Rightarrow \frac{1}{\sqrt{1+(y')^2}}\left[y + y(y')^2 - y(y')^2\right] = c$$

$$\Rightarrow \frac{y}{\sqrt{1+(y')^2}} = c$$

$$\Rightarrow y^2 = c^2(1+(y')^2)$$

$$\Rightarrow \frac{y^2 - c^2}{c^2} = \left(\frac{dy}{dx}\right)^2$$

Note: Don't always use the Beltrami equation when you don't see a x

$$\frac{\partial F}{\partial y} - \frac{\partial}{\partial x}\left[\frac{\partial F}{\partial y'}\right] = 0$$

$$\sqrt{1+(y')^2} - \frac{d}{dx}\left[\frac{yy'}{\sqrt{1+(y')^2}}\right] = 0$$

$$\vdots$$

$$y'' - y = 0$$

$$\frac{dy}{dx} = \pm\sqrt{\frac{y^2-c^2}{c^2}} = \pm\frac{1}{c}\sqrt{y^2-c^2}.$$

$$\int \frac{dy}{\sqrt{y^2-c^2}} = \pm\int \frac{1}{c}\,dx.$$

$$\cosh^{-1}\left(\frac{y}{c}\right) = \pm\left(\frac{x}{c}+\frac{D}{c}\right)$$

$$y = C'\cosh\left[\pm\left(\frac{x+D}{c}\right)\right]$$

$$= C'\cosh\left[\frac{x+D}{c}\right]$$

Two constants of integration found so that $y(x_1)=y_1$, $y(x_2)=y_2$.

— / —

Back to isoperimetric problem.

Example: Find the extremal for the integral

$$\int_0^1 (y')^2 + 2yy'\,dx$$

$y(0)=y(1)=0$, subject to the constraint $\int_0^1 y\,dx = 1/6$.

Min/Max
$\int F\,dx$ subject to $\int G\,dx = \text{Const}$. Form
$\int F - \lambda G\,dx$ solve the EL for this new functional and apply constraint to find λ.

Consider $\underbrace{\int_0^1 (y')^2 + 2yy' - \lambda y\,dx}_{H(x,y,y',t)}$

We see $\partial H/\partial x = 0$ and it is tempting to use the Beltrami Equation:

$$H - y'\frac{\partial H}{\partial y'} = \text{const}.$$

However the algebra is ticky and instead we use the EL equations:

$$\frac{\partial H}{\partial y} - \frac{d}{dx}\left(\frac{\partial H}{\partial y'}\right) = 0$$

$$2y' - \lambda - \frac{d}{dx}\left[2y' + 2y\right] = 0$$

$$y'' = -\frac{\lambda}{2}$$

$$y' = -\frac{\lambda}{2}x - A$$

If use Beltrami Equation:

$$y'y'' = -\frac{\lambda}{2}y'$$

$$\frac{1}{2}(y')^2 = -\frac{\lambda}{2}y + C$$

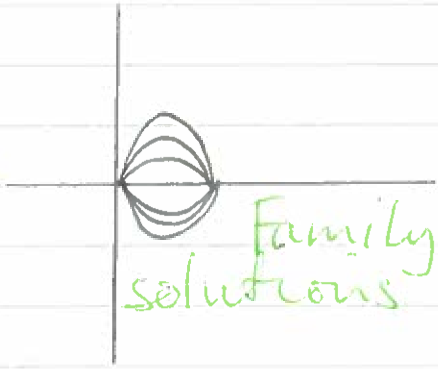
$$y = -\frac{\lambda}{4}x^2 + Ax + B$$

and since $y(0) = 0 \Rightarrow B = 0$

$y(1) = 0 \Rightarrow A = \frac{\lambda}{4}$

$$y(x) = \frac{\lambda}{4}x(1-x)$$

We use the constraint to give us a value of λ.



Family of solutions of λ

$$\int_0^1 \frac{\lambda}{4}(x - x^2)\,dx$$

$$= \int_0^1 y\,dx = \frac{1}{6}$$

i.e $\frac{1}{6} = \frac{\lambda}{4}\left[\frac{1}{2} - \frac{1}{3}\right] \Rightarrow \lambda = 4.$

$$y(x) = x(1-x).$$

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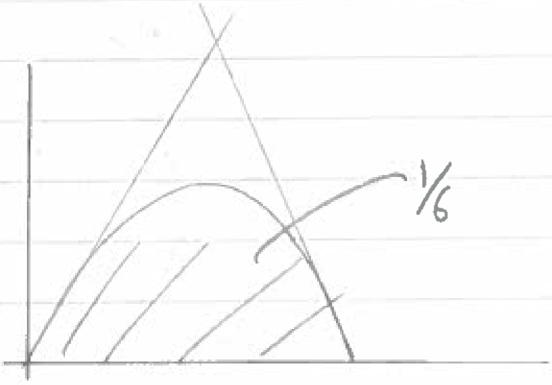
$\int_0^1 y'^2 + 2yy' \, dx \qquad y(0) = 0 = y(1).$

and $\int_0^1 y \, dx = \frac{1}{6}$ / $\underline{EL}$: $y'' = \frac{\lambda}{2}$, $y = x(1-x)$

$$\int_0^1 \underbrace{y'^2 + 2yy' - \lambda y}_{H} \, dx$$



Beltrami equation is $H - y' \frac{\partial H}{\partial y'} = \text{const}.$

$$(y'^2 + 2yy' - \lambda y) - y'(2y' + 2y) = c$$

$$-y'^2 - \lambda y = c$$

$$y'^2 + \lambda y = c$$

$$\left(\frac{dy}{dx}\right) = \pm\sqrt{c - \lambda y}$$

REMEMBER!

$$\int \frac{dy}{\sqrt{c - \lambda y}} = \pm \int dx$$

$$-\frac{2}{\lambda}\sqrt{c-\lambda y} = \pm(x+A)$$

$$\sqrt{c-\lambda y} = \pm\frac{\lambda}{2}(x+A).$$

Choose c and A so that $y(0)=0$, $y(1)=0$.

$$\sqrt{c} = \pm\lambda\frac{A}{2}.$$

$$\sqrt{c} = \pm\lambda[1+A]$$

If we had neglected the $\pm$ we might notice a contradicatcon here.

What we need is:

$$-A = (1+A) \Rightarrow A = -\frac{1}{2}.$$

$$A^2 = (1+A)^2 \Rightarrow A^2 = 1+2x+A^2, \; A = \frac{1}{2}$$

$$c = \frac{\lambda^2 A^2}{4} = \frac{1}{16}\lambda^2.$$

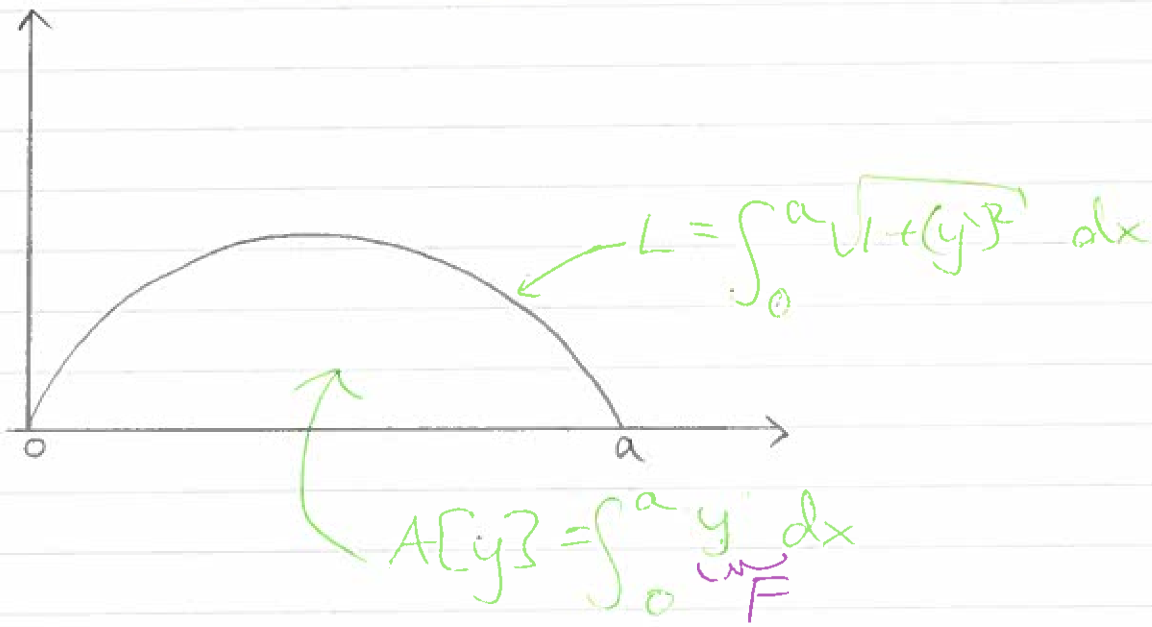
and finding y

$$c-\lambda y = \frac{\lambda^2}{4}(x+A)^2.$$

$$\frac{1}{16}\lambda^2 - \lambda y = \frac{\lambda^2}{4}\left(x - \frac{1}{2}\right)^2$$

$$\Rightarrow y = \frac{\lambda}{4}(x - x^2) \quad \text{as before.}$$

The Sheep pen problem.



Maximise A subject to constraint of fixed L.

$$F - \lambda G = H.$$

From:

$$\int_0^a \overbrace{y - \lambda\sqrt{1+(y')^2}}\, dx$$

$$\frac{\partial H}{\partial x} = 0 \quad \text{so we know:}$$

$$y - \lambda\sqrt{1+(y')^2} - y'\left(\frac{-\lambda y'}{\sqrt{1+(y')^2}}\right) = c$$

$$\frac{-\lambda}{\sqrt{1+(y')^2}}\left[1+y'^2-y'^2\right] = +c-y$$

$$\frac{\lambda^2}{1+y'^2} = (c-y)^2$$

$$1+y'^2 = \frac{\lambda^2}{(c-y)^2}.$$

$$y'^2 = \frac{\lambda^2-(c-y)^2}{(c-y)^2}$$

$$\frac{dy}{dx} = \pm\sqrt{\frac{\lambda^2-(c-y)^2}{(c-y)^2}} = \pm\sqrt{\frac{\lambda^2-(c-y)^2}{(c-y)^2}}$$

$$\int \frac{(c-y)}{\sqrt{\lambda^2-(c-y)^2}}\,dy = \int \pm\, dx$$

$$\sqrt{\lambda^2-(c-y)^2} = \pm(x+A)$$

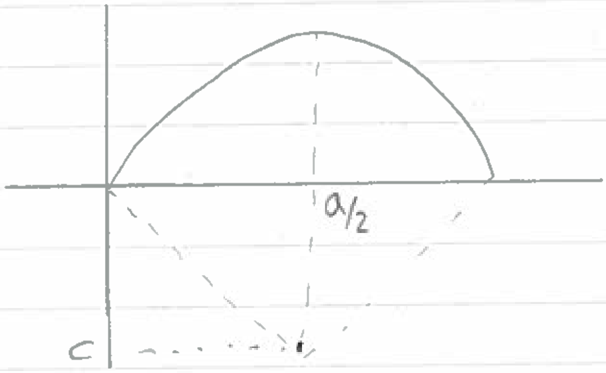
$$\Rightarrow \lambda^2 = (x+A)^2+(y-c)^2.$$

$\Rightarrow$ BC give A and c contraint gives λ

$y(0)=0, \quad \lambda^2 = A^2+C^2$

$y(a)=0, \quad \lambda^2 = (a+A^2)+c^2.$

$$\Rightarrow A^2 = (a+A)^2 \Rightarrow A = -\frac{a}{2}.$$



$$c^2 = \lambda^2 - \frac{a^2}{4}.$$

To find λ we use the constraint.

$$L = \int_0^a \sqrt{1+(y')^2}\, dx.$$

$$= \pm \int_0^a \frac{\lambda}{c-y}\, dx$$

$$= \pm \int_0^a \frac{\lambda}{\sqrt{\lambda^2 - \left(x - \frac{a}{2}\right)^2}}\, dx$$

$$\Rightarrow L = 2\lambda \sin^{-1}\left(\frac{a}{2\lambda}\right)$$

$$\Rightarrow \sin\left(\frac{L}{2\lambda}\right) = -\frac{a}{2\lambda}$$

$$\Rightarrow \lambda \quad \Rightarrow c.$$

Partial Differential Equation.

A partial differential equation (PDE) is a relation between a function of serveral variables $u(x, y, \ldots)$ and its partial derivatives, u_x, $u_y, \ldots, u_{xx}, u_{xy}, \ldots, u_{xxx}, u_{xxy}, \ldots$

$$u\frac{\partial u}{\partial x} + xu = \frac{\partial^2 u}{\partial y^2} \quad \text{order 2}$$

for $u = u(x,y)$.

The order of the PDE is the order of the highest derivative occuring.

If the differental equation can be written $L[u] = f$ where f does not depend on u and $L[u]$ is a linear operator.

$$L[\alpha u + \beta w] = \alpha L[u] + \beta L[w].$$

e.g.

$$a(x,y)\frac{\partial u}{\partial x} + b(x,y)\frac{\partial u}{\partial y} = c(x,y)u + d(x,y)$$

$$L[u] = a\frac{\partial u}{\partial x} + b\frac{\partial u}{\partial y} - cu$$

$f = d$. then the equation is linear

$(x+y)\frac{\partial u}{\partial y} + y\frac{\partial u}{\partial y} = 1 + xu$ is linear first order

$u^2\frac{\partial u}{\partial x} + uy = 1$ is not linear and first order but there are no products of the highest order derivative occuring. Such equations are called <u>quasi-linear</u>.

$u_x u_y = 1$ is not quasi linear

$u_{xx} + u_{yy} = 1$ is second order and linear.

A simple equation is $\partial u/\partial x = 0$ for $u(x,y)$ and has solutions $u = f(y)$ we observe that the general solution of PDE contains arbitrary functions.

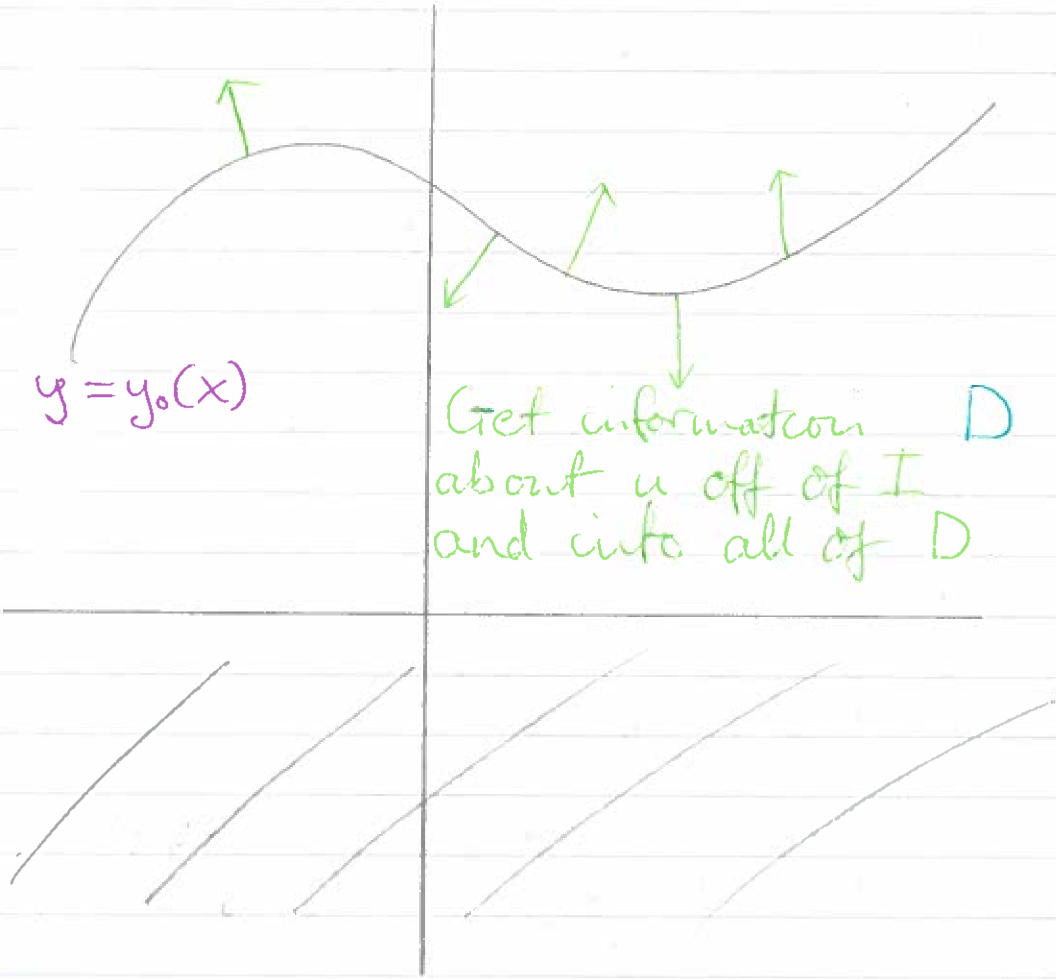
—/—

Take $\frac{\partial y}{\partial x} = 0$, $y =$ const.

—/—

<u>A typical problem.</u>

Given a first order pde valid in a region D of the (x,y) plane (e.g $D = \mathbb{R}^2$; $x \geq 0$ $x^2+y^2 \leq a$) and some knowledge of the solution $u(x,y)$ on a curve Γ in D (eg $y = y_0(x)$ or $g(x,y) = c$)



D has $y \geq 0$

eg. $u = u_0(x,y)$ on $y = y_0(x)$. Can we find $u(x,y)$ inside all of D?

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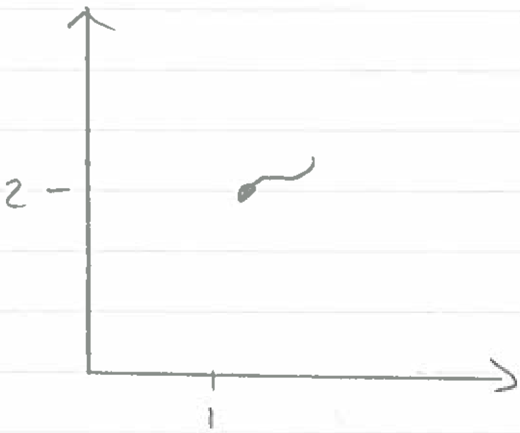


$\frac{dy}{dx} = f(x,y)$, $y(1) = 2$

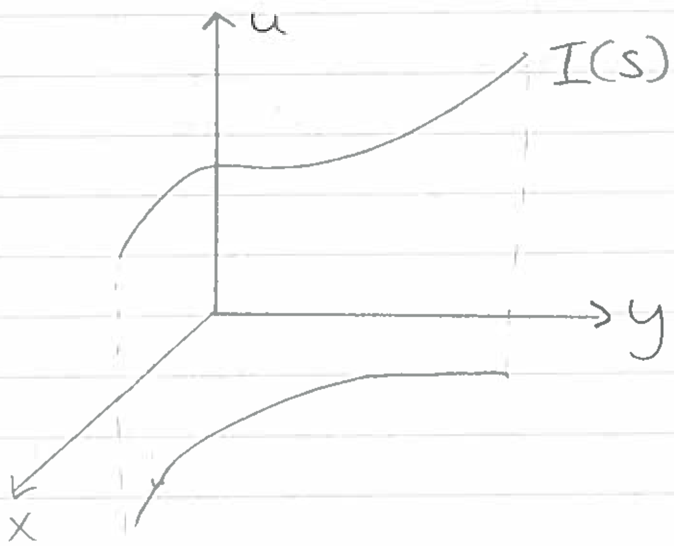
— / —

If this can be done obtaining a unique $u(x,y)$ we will call the problem well posed. If it cannot it will be ill-posed. We will see examples that illustrate when a problem is ill posed.

We often describe lines in the (x,y) plane in parametric form. Eg the line I can be described as

$$x = x(s)$$
$$y = y(s)$$
$$u = u(s).$$

So, for example if we are told that $u = x^2$ on $y = 0$ this would parameterised as: $x = s$, $y = 0$, $u = s^2$.



$I(s)$ in 3D is a line in the surface $u = u(x,y)$.

Example

Solve $\partial u/\partial x = 0$ with I being the y-axis on this $u = e^y$.

$y = s$
$x = 0$
$u = e^s$.

D

$\frac{\partial u}{\partial x} = 0$

u doesn't varies among the x-axis

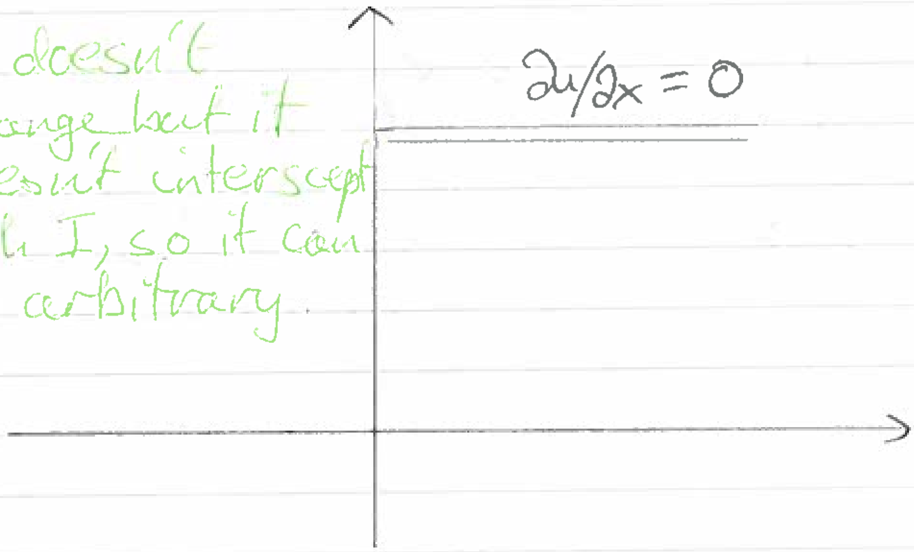
Intergating gives $u = f(y)$, where f is an arbitrary function but on I. $u = e^s$, $y = s$, $x = 0$ so that substituting $e^s = f(s)$ i.e. $f(y) = e^y$ and our solution is $u(x,y) = e^y$.

But to slove $\partial u/\partial x = 0$ with I with the x-axis e.g $u = 1$ on $y = 0$ is an ill-posed problem as

$u = 1 + y$ is a solution.
$u = 1 + g(y) - g(0)$ is a solution too

and the solution is not unique.

u doesn't change but it doesn't intercept with I, so it can be arbitrary.



Consider a curve in the x-y plane given parametrically $x = x(t)$, $y = y(t)$, then:

$$\frac{dy}{dt} = \frac{dy/dt}{dx/dt}$$

If we consider the ordinary differential equation.

$$\frac{dy}{dx} = \frac{b(x,y)}{a(x,y)} = F(x,y).$$

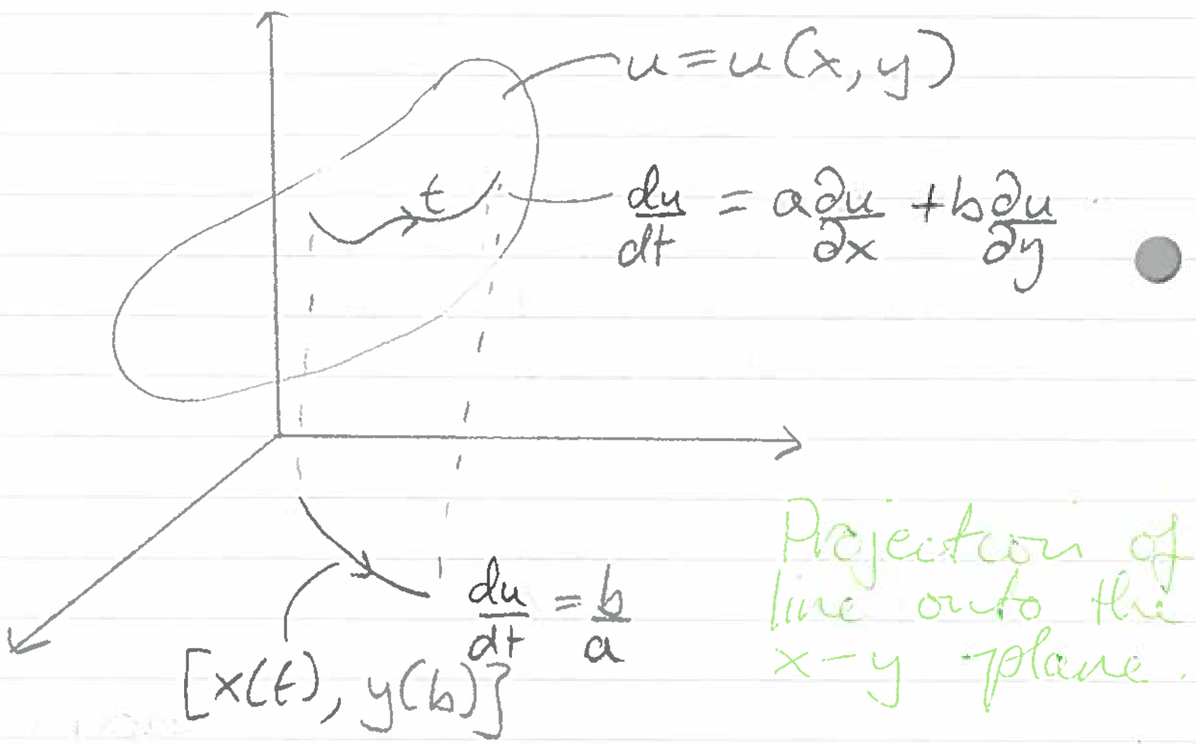
then this equation has a solution to

$$\frac{dy}{dt} = b(x,y) \quad , \quad \frac{dx}{dt} = a(x,y).$$

If $u = u(x,y)$ and $y = y(t)$, $x = x(t)$ then

$$\frac{du}{dx} = \frac{\partial u}{\partial x}\frac{dx}{dt} + \frac{\partial u}{\partial y}\frac{dy}{dt}.$$

$$= a(x,y)\frac{\partial u}{\partial x} + b(x,y)\frac{\partial u}{\partial y}.$$



Sometimes these equation may be written:

$$dx = a\,dt \quad , \quad dy = b\,dt$$

$$\frac{dx}{a} = dt \quad , \quad \frac{dy}{b} = dt$$

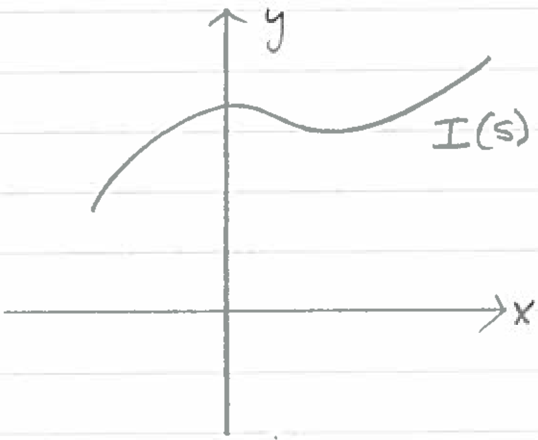
$$\frac{dx}{a} = \frac{dy}{b} = dt$$

Characteristics.

Consider the homogenous pde:

$$a(x,y)\frac{\partial u}{\partial x} + b(x,y)\frac{\partial u}{\partial y} = 0 .$$

to be solved with a knowledge of u on a line I in x-y plane.



Consider lines given by the solution

$$\frac{dx}{dt} = a, \quad \frac{dy}{dt} = b, \quad \frac{dx}{a} = \frac{dy}{b} \; (= dt)$$

then along this line:

$$\frac{du}{dt} = \frac{\partial u}{\partial x}\frac{dx}{dt} + \frac{\partial u}{\partial y} \cdot \frac{dy}{dt}$$

$$= a\frac{du}{dx} + b\frac{du}{dy}$$

So that $u(x,y)$ is constant on these

lines. The lines are called <u>characteristics</u>, more accurately <u>characteristic traces</u>

and the equations

$$\frac{dx}{a} = \frac{dy}{b} = \frac{du}{0} \quad (= dt)$$

are characteristics equations

A way of writing

$$\frac{du}{dt} = 0.$$

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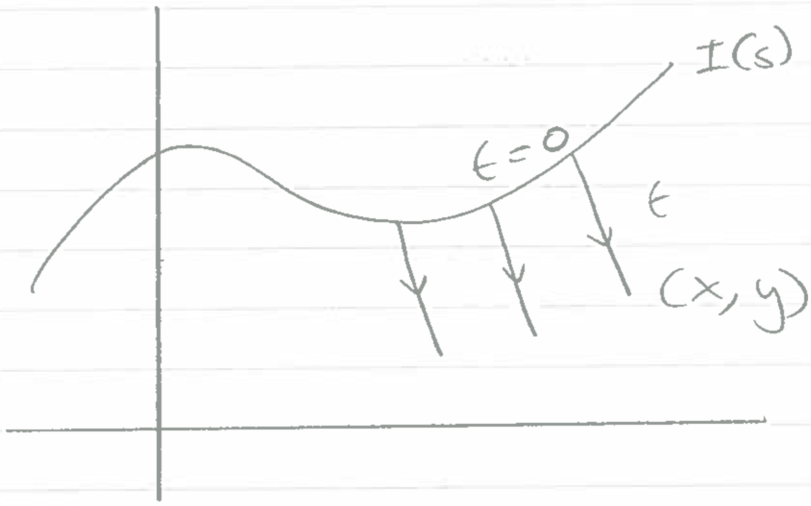
$au_x + bu_y = 0.$

on the lines given by

$$\frac{dx}{dt} = a \quad \text{and} \quad \frac{dy}{dt} = b.$$

We have seen

$$\frac{du}{dt} = 0 \quad , \frac{dx}{a} = \frac{dy}{b} = \frac{du}{0}.$$



We can use this as follows. If we know u on the curve I. $x = x(s)$, $y = y(s)$, $u = u(s)$ we can find the value of u at some point off of I by finding the characteristics trace passing through (x,y) – the solutions of $dx/dt = a$, $dy/dt = b$ and hopefully tracing it back to where it intersects I. Since u is constant on the ↓

($du/dt = 0$) we say $u(x,y)$ is equal to u at the point where ψ meets I. We slove the ψ equation $dx/dt = a$, $dy/dt = b$ subject to the initial conditions that $t=0$, $x = x(s)$, $y = y(s)$, $u = u(s)$ and the ψ intersects $I(s)$.

Example

Slove

$$\frac{\partial u}{\partial x} + x\frac{\partial u}{\partial y} = 0$$

$u = \sin y$ on $x = 0$.

1) Parameterise $I(s)$ as $x=0$, $y=s$, $u = \sin(s)$.

2) Solve the ψ equations with initial condition $x=0$ at $t=0$, $y=s$, $u = \sin s$.

These are: $\frac{dx}{dt} = a = 1$ ①

function of t!

$\frac{dy}{dt} = b = x$ ②

$\left(\frac{du}{dt} = 0\right)$

DON'T DO THIS!

$\therefore \frac{dy}{dt} = x \not\Rightarrow y = xt$ WRONG!

① $\Rightarrow x = t$ using $x = 0$ at $t = 0$.

$y = \frac{1}{2}x^2 + s$.

In ② : $\frac{dy}{dt} = x = t \Rightarrow y = \frac{1}{2}t^2 + s$.

using $y = s$ at $t = 0$.

$u = \sin(s)$ as $\frac{du}{dt} = 0$.

We now have a parametric form of our solution. The parameters are t and s:

$$x = t$$
$$y = \frac{1}{2}t^2 + s$$
$$u = \sin(s).$$

Now eliminate t and s in favour of x and y.

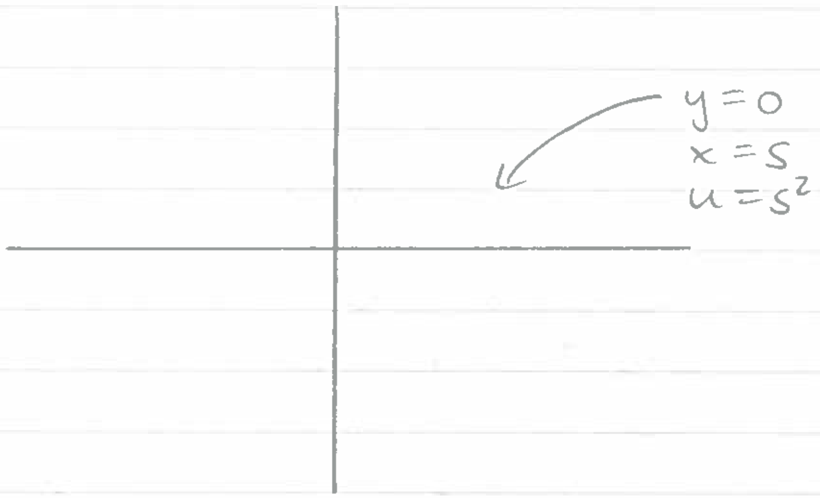
$$s = y - \frac{1}{2}t^2 = y - \frac{1}{2}x^2$$

and $u = \sin\left(y - \frac{1}{2}x^2\right)$

Also note that for any function $\mathbb{R} \to \mathbb{R}$ if the function $u(x,y) = f(y - \frac{1}{2}x^2)$ satisfies the pde but a choice of f is needed to satisfy the boundary conditions

Examples: A quasilinear homogeneous

$$u\frac{\partial u}{\partial x} - \frac{\partial u}{\partial y} = 0.$$



$u = x^2$ on $y = 0$ and solve the characteristic equations.

$$\frac{dx}{dt} = a = u$$

$$\frac{dy}{dt} = b = -1 \quad , \quad \frac{du}{dt} = 0.$$

with initial condition at

$$t=0, \quad y=0, \quad x=s, \quad u=s^2.$$

We can solve for y:

$$\underline{y=-t} \quad (y=0 \text{ at } t=0)$$

We can't solve $dx/dt=u$ as we don't know what $u(t)$ is (yet). We do know that as $du/dt=0$ on ψ then u is constant. As $u=s^2$ at $t=0$, $u=s^2$ so $dx/dt = u = s^2 \Rightarrow x = s^2t+s$ (as $x=s$ when $t=0$).

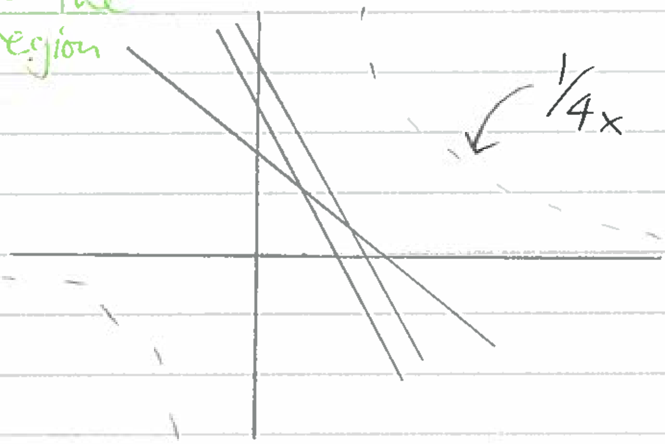
So we have the parametric solution

$$x = s^2t+s$$
$$y=-t$$
$$u=s^2.$$

The ψ-traces are given by eliminate t.

$$x = s - s^2 y$$
$$y = \frac{1}{s} - \frac{x}{s^2}$$

$x \Rightarrow$ characteristic will not enter the x-region



The ψ have an envelope $y = 1/4x$

So we cannot find solution for $u(x,y)$ in the region $y > 1/4x$ as no t which intersect $I(s)$ enter this region.

If we eliminate s and t then:

$$x = u(-y) + \sqrt{u} \quad (\text{assume } s > 0).$$

we have a quadratic for $\sqrt{u}$.

$$\sqrt{u} = \frac{1 \pm \sqrt{1 - 4xy}}{2y}.$$

We need to decide whether we want $+$ or $-$ we need $u = x^2$, i.e $\sqrt{u} = x$ on $y = 0$. This implies we need the $-$ sign so that as $y \to 0$ $\sqrt{u} \to x$ and not $\to \infty$.

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$$u\frac{\partial u}{\partial x} - \frac{\partial u}{\partial y} = 0, \quad u = x^2 \text{ on } y = 0$$

$$\frac{dx}{dt} = u, \quad \frac{dy}{dt} = -1, \quad \frac{dt}{dt} = 0.$$

$$\left.\begin{array}{l} x = s^2 t + s \\ y = -t \\ u = s^2 \end{array}\right\} \Rightarrow \sqrt{u} = \frac{1 - \sqrt{1 - 4xy}}{2y}$$

—/—

$f(x, y, s) = 0$, $f = x + s^2 y - s = 0$

Family of curves parameter s

Eliminate s for $f = 0$ and $\frac{\partial f}{\partial s} = 0$.

$\Rightarrow$ Envelope.

—/—.

The characteristic equations for

$$a(x, y, u)u_x + b(x, y, u)u_y = 0.$$

solutions of

$$\frac{dx}{a(x,y,z)} = \frac{dy}{b(x,y,z)}$$

or $$\frac{dy}{dx} = \frac{b(x,y,z)}{a(x,y,z)}.$$

If the pde is linear, this becomes

$$\frac{dy}{dx} = \frac{b(x,y)}{a(x,y)}$$

Hence for linear equations if a and b are single valued, dy/dx is unique as a function of x and y and Ψ traces cannot cross.

This is <u>not</u> true for general quasilinear pde's

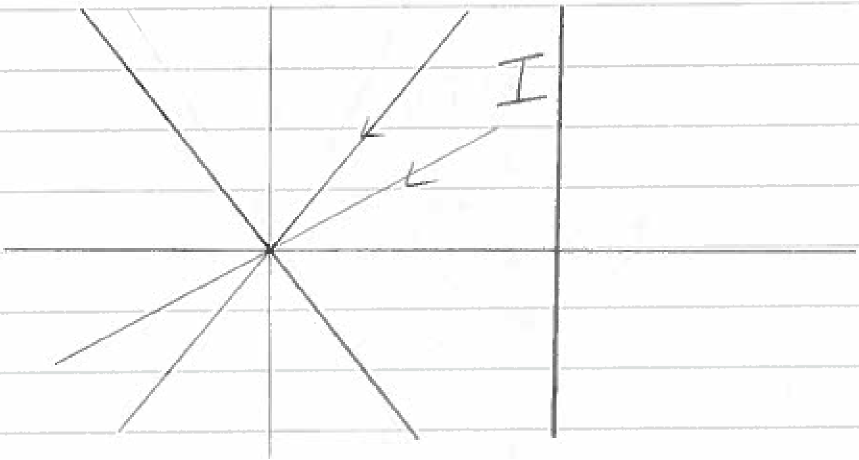
There are exceptions at points where both $a = 0$ and $b = 0$ where dy/dx is undetermined.

Look at $x\frac{du}{dx} + y\frac{du}{dy} = 0.$

$$\frac{dx}{x} = \frac{dy}{y} \Rightarrow \ln x + \ln y = \text{const}$$

$$\underline{y = Cx}$$

and x crosses at orgin when $a=b=0$



In this case, we might expect a singularity in the solution to the pde at $x=0, y=0$ as Ψ carrying contradictory information about the solution cross. This singularity could be avoided for particular I, e.g if $a = \text{const}$ on I here.

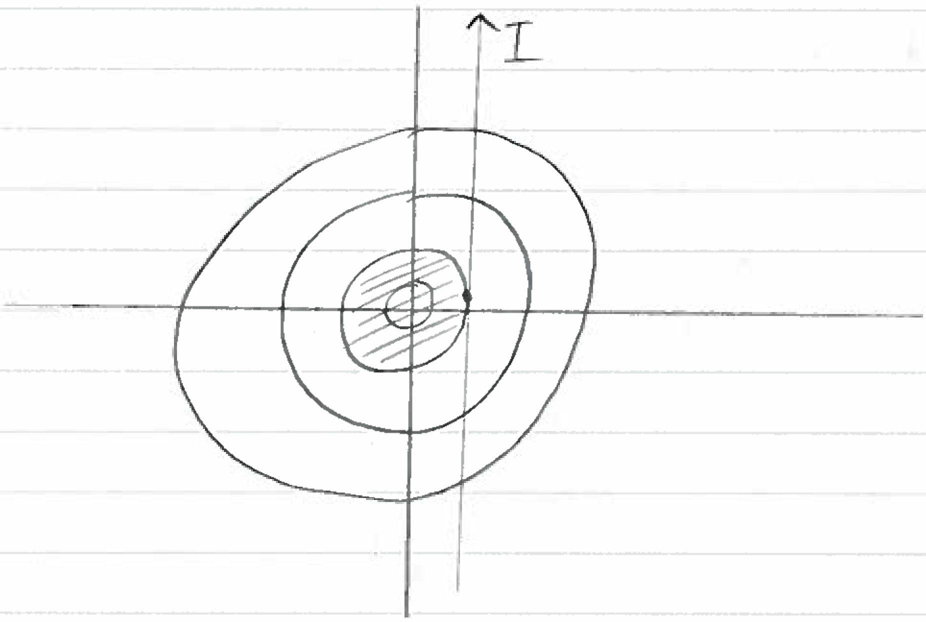
Consider : $$y\frac{\partial u}{\partial x} - x\frac{\partial u}{\partial y} = 0.$$

$$\Psi : \quad \frac{dx}{y} = \frac{dy}{-x} \qquad \left(\frac{dy}{dx} = -\frac{x}{y}\right)$$

$$\Rightarrow x\,dx + y\,dy = 0.$$

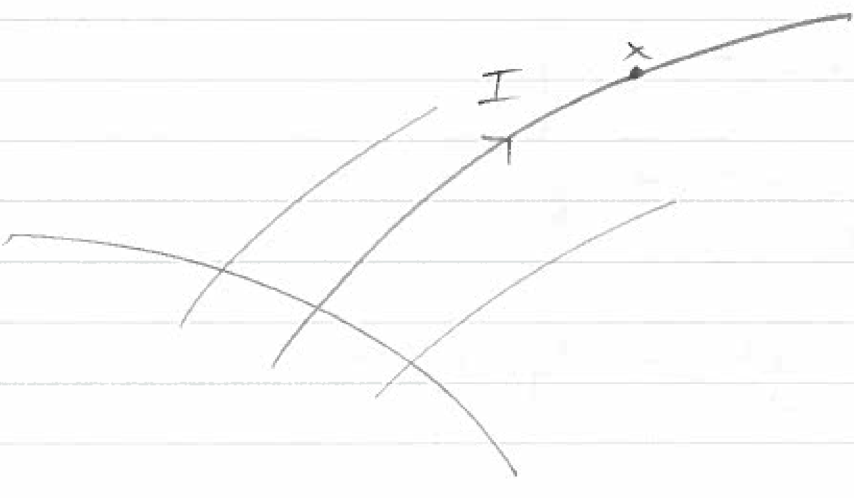
$$d\left(\frac{1}{2}x^2 + \frac{1}{2}y^2\right) = 0.$$

$$x^2 + y^2 = \text{const}.$$



For I as shown we cannot find the solution inside the shaded region as no characteristic which crosses I enters it.

Outside this region there are still problems if a Ψ crosses I more than once. Unless the data on I is entire consistent with the development of the solution along the Ψ, the problem is ill-posed.



An extreme case of Ψ crossing I is when I coincides with a Ψ. Then it is impossible to find information about the solution off of I. A good definition of a Ψ is a line, a knowlegde of the solution on which tell nothing about the solution elsewhere.

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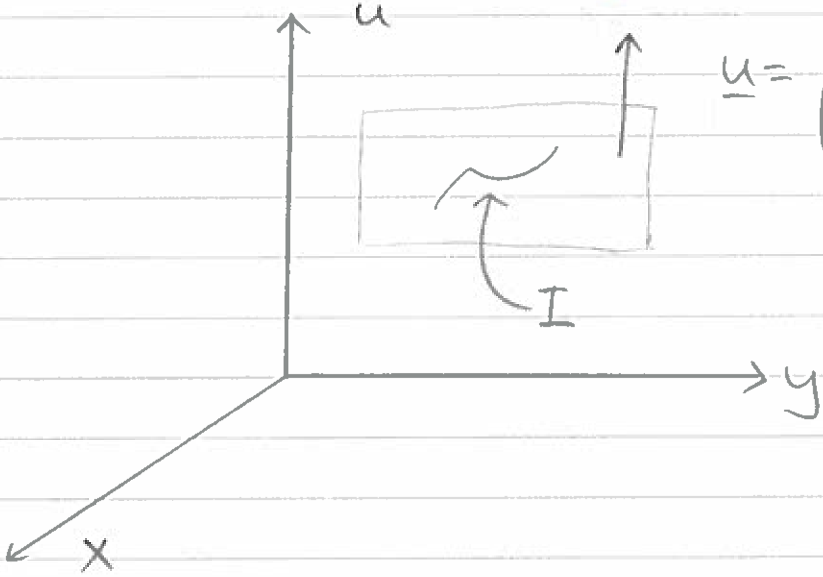
Characteristics for general inhomogeneous queslinear equation:

$$a(x,y,u)\frac{\partial u}{\partial x} + b(x,y,u)\frac{\partial u}{\partial y} = c(x,y,u).$$



$$\underline{u} = \begin{pmatrix} -u_x \\ u_y \\ 1 \end{pmatrix}$$

In solving this we are after a solution surface $u = u(x,y)$, containing I. The normal to thin surface can be found by evaluating $\underline{\nabla} g$ where $g(x,y,u) = u - u(x,y)$ i.e.

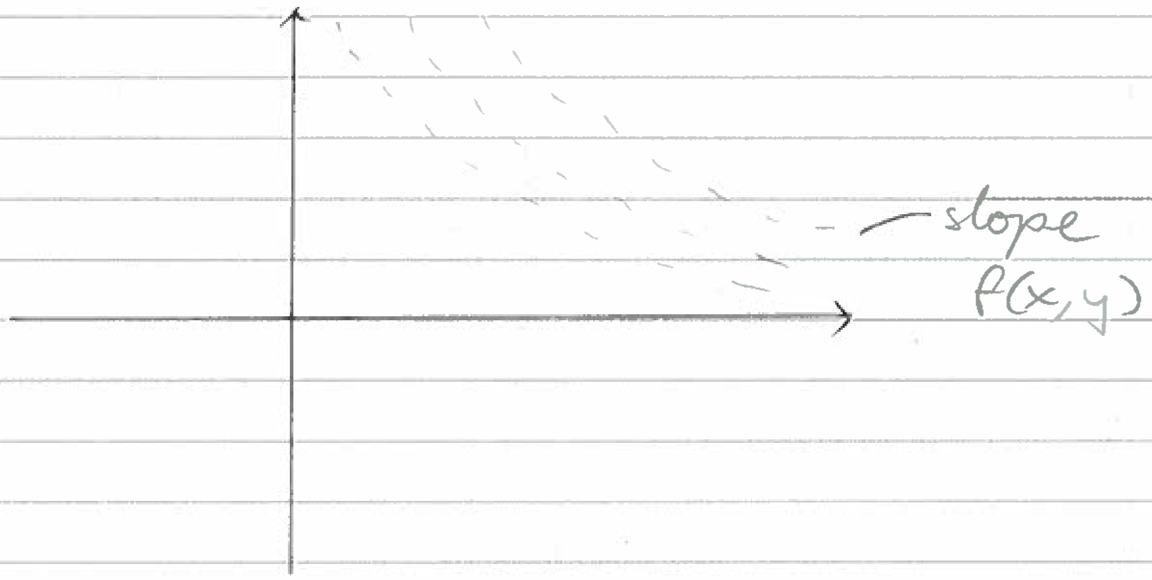
$$\underline{n} = \begin{pmatrix} -u_x \\ -u_y \\ 1 \end{pmatrix}$$

Now consider the vector $\left(\begin{smallmatrix} a \\ b \\ c \end{smallmatrix}\right)$ and consider

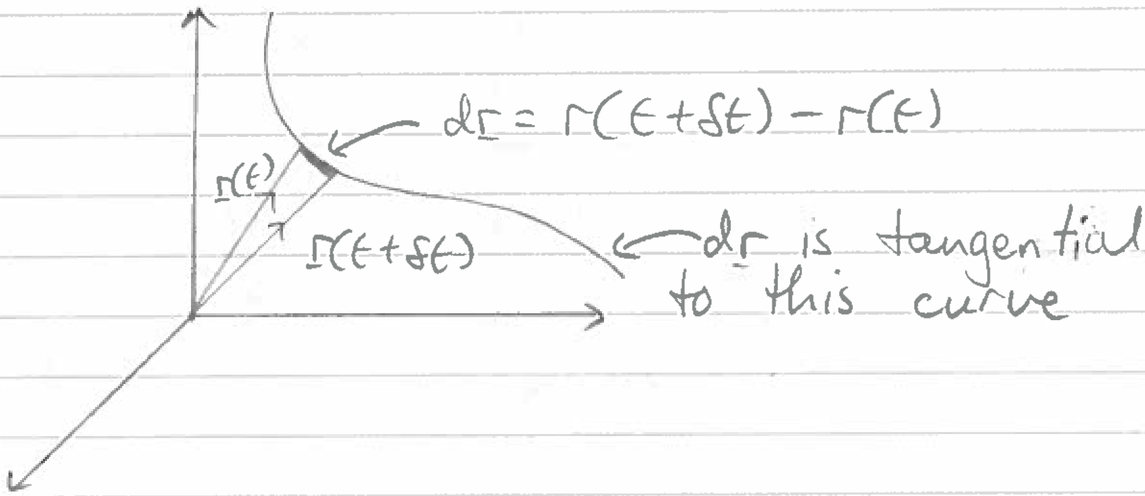
$$\begin{pmatrix} a \\ b \\ c \end{pmatrix} \cdot \begin{pmatrix} -u_x \\ -u_y \\ 1 \end{pmatrix} = -au_x - bu_y + c = 0$$

i.e. $au_x + bu_y = c$

Hence the vectors in the vector field $(a, b, c)^T$ are normal to the solution surface.



Now consider solution to the equation $\frac{d\underline{r}}{dt}$



$$\frac{d\underline{r}}{dt} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$$

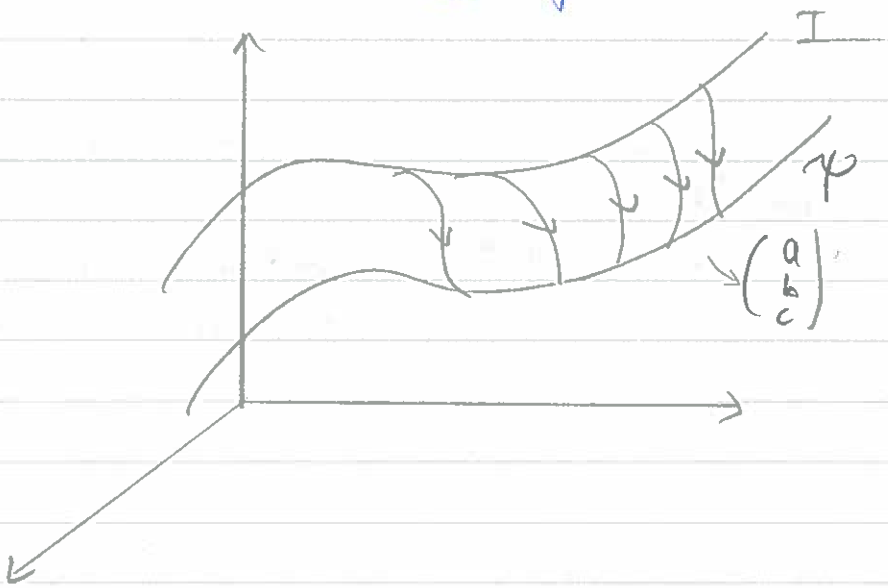
then $d\underline{r} = \begin{pmatrix} a \\ b \\ c \end{pmatrix} dt.$

points in the direction of $\begin{pmatrix} a \\ b \\ c \end{pmatrix}$ so $d\underline{r}$ lies in the solution surface

But $\frac{d\underline{r}}{dt} = \begin{pmatrix} dx \\ dy \\ du \end{pmatrix} = \begin{pmatrix} a\,dt \\ b\,dt \\ c\,dt \end{pmatrix}$

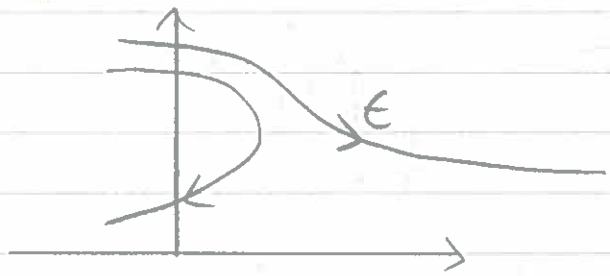
and $\frac{dx}{a} = \frac{dy}{b} = \frac{du}{c}$ $(= dt)$

are Ψ surfaces. So the Ψ lie in the solutions surface and make up the solution surface.



$$\frac{dr}{dt} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$$

The solution is made up of ψ coming from I.



An alternative method.

The change of variable method.

Consider linear equation i.e those of the form:

$$a(x,y)\frac{du}{dx} + b(x,y)\frac{du}{dy} + c(x,y)u$$
$$= d(x,y).$$

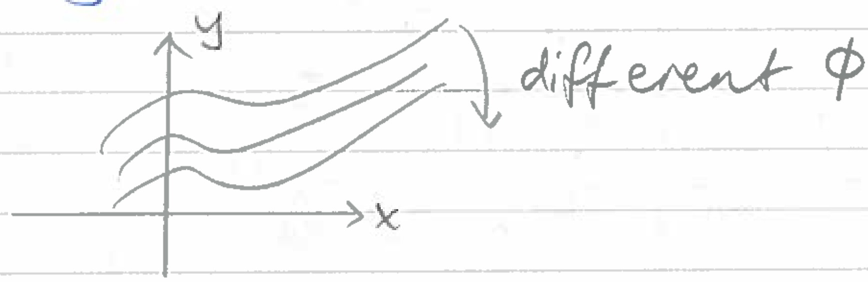
and consider the ψ - traces, i.e solution to

$$\frac{dx}{dt} = a \;, \quad \frac{dy}{dt} = b \quad \text{or} \quad \frac{dy}{dx} = \frac{b}{a} .$$

Consider this as an ode for $y(x)$. It has solution given generally in the form $\phi(x,y) = \phi$ a constant.

e.g if $y = f(x) + \underbrace{\text{const}}_{\phi}$.

$$\phi(x,y) = y - f(x) = \phi .$$



We use ϕ instead of s to identify particular ψ and we need another variable instead of t to take you along a ψ say ξ. Often we would choose $\xi = x$.

We make the change of variables from x and y to ϕ and ξ.

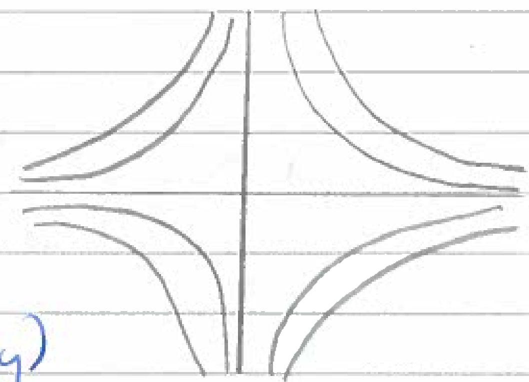
Example. Slove $x\frac{\partial u}{\partial x} - 7y\frac{\partial u}{\partial y} = x^2 y$.

Slove for the Ψ traces i.e slove

$$\frac{dy}{dx} = \frac{-7y}{x}$$

$$\int \frac{dy}{y} = -7\int \frac{dx}{x} \Rightarrow \ln y = -7\ln x + \text{const.}$$

$$yx^7 = \phi$$



Now make a change of variable from x and y to ϕ and ξ with $\phi(x,y) = yx^7$, $\xi(x,y) = x$.

So: $$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial \phi}\cdot\frac{\partial \phi}{\partial x} + \frac{\partial u}{\partial \xi}\cdot\frac{\partial \xi}{\partial x}$$

and we consider $u = u(\phi, \xi)$

$$\ldots = 7yx^6\frac{\partial u}{\partial \phi} + \frac{\partial u}{\partial \xi}$$

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial \phi}\cdot\frac{\partial \phi}{\partial y} + \frac{\partial u}{\partial \xi}\cdot\frac{\partial \xi}{\partial y}$$

$$= x^2\frac{\partial y}{\partial \phi} + 0.$$

So substituting

$$x\left(\cancelto{0}{7yx^6\frac{\partial u}{\partial \phi}} + \frac{\partial u}{\partial \xi}\right) - \cancelto{0}{7yx^7\frac{\partial u}{\partial \phi}} = x^2y.$$

$$\Rightarrow x\frac{\partial u}{\partial \xi} = x^2y;$$

an equation telling you how u varies as you make along ξ i.e fixed ϕ.

$$\frac{\partial u}{\partial \xi} = xy = x.\frac{\phi}{x^7} = \frac{\phi}{x^6}$$

$\Rightarrow$ integrating in ξ.

$$u = -\frac{1}{5}\frac{\phi}{\xi^2} + f(\phi)$$

$$u(x,y) = -\frac{1}{5}\frac{yx^7}{x^5} + f(yx^7)$$

$$= -\frac{1}{5}yx^2 + f(yx^7)$$

If the boundary/initial condition are for example, $u=0$ on $y=x^2$, then we need

$$0 = -\frac{1}{5}x^2x^2 + f(x^2x^7)$$

$$f(x^9) = \frac{1}{5}x^4.$$

and if we write $r = x^4$.

$$\Rightarrow f(r) = \frac{1}{5}r^{4/9}.$$

and our solution is:

$$u(x,y) = -\frac{1}{5}yx^2 + \frac{1}{5}(yx^7)^{4/9}$$

Example:

$$x\frac{\partial u}{\partial x} + (x^2+y)\frac{\partial u}{\partial y} + \left(\frac{y}{x} - x\right)u = 1$$

Ψ traces satisfy

$$\frac{dy}{dx} = \frac{x^2+y}{x} = \frac{y}{x} + x.$$

Integrating factor

i.e $\frac{dy}{dx} - \frac{y}{x} = x$. IF is $\exp\left(-\int \frac{1}{x}dx\right) = \frac{1}{x}$.

$$\frac{d}{dx}\left[\frac{y}{x}\right] = 1 \Rightarrow \frac{y}{x} = x + \text{const}.$$

So $\varphi = \frac{y}{x} - x$ is constant on Ψ.

Make a change of variable from x and y to ϕ and ξ where $\phi(x,y) = y/x - x$ and $\xi(x,y) = x$.

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$$x\frac{\partial u}{\partial x} + (x^2+y)\frac{\partial u}{\partial x} + \left(\frac{y}{x} - x\right)u = 1.$$

$$\frac{dx}{x} = \frac{dy}{x^2+y^2} \Rightarrow \frac{dy}{dx} = \frac{x^2+y}{x}$$

$$\frac{dy}{dx}\left(-\frac{y}{x}\right) = x$$

$$\phi = \frac{y}{x} - x, \quad \xi = x.$$

$$x\left(\frac{\partial u}{\partial \phi}\overbrace{\left(-\frac{y}{x^2}-1\right)}^{\partial\phi/\partial x} + \frac{\partial u}{\partial \phi}\cdot\overbrace{1}^{\partial\xi/\partial x}\right)$$

$$+(x^2+y)\left(\frac{\partial u}{\partial \phi}\underbrace{\frac{1}{x}}_{\frac{\partial\phi}{\partial y}} + \frac{\partial u}{\partial \xi}\cdot\underbrace{0}_{\frac{\partial\xi}{\partial y}}\right) + \left(\frac{y}{x} - x\right)u = 1$$

$$\Rightarrow x\frac{\partial u}{\partial \xi} + \left(\frac{y}{x} + x\right)u = 1$$

$$\xi\left(\frac{\partial u}{\partial \xi}\right) + \phi u = 1$$

We can solve this for $u(\xi)$ considering ϕ as a constant.

$$\frac{\partial u}{\partial \xi} + \frac{\phi}{\xi} . u = \frac{1}{\xi}$$

IF: is $e^{\int \frac{\phi}{\xi} d\xi} = \xi^{\phi}$

$$\frac{\partial}{\partial \xi}\left[u \xi^{\phi}\right] = \xi^{\phi - 1}$$

$$u \xi^{\phi} = \frac{\xi}{\phi} + f(\phi)$$

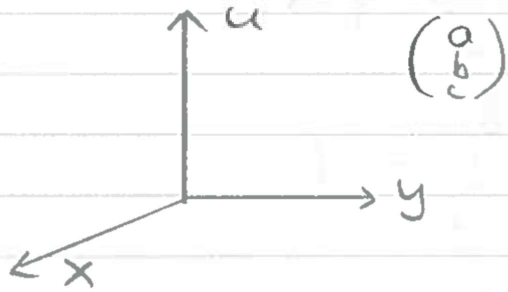
$$u = \frac{1}{\phi} + \xi^{-\phi} f(\phi)$$

$$u(x,y) = \frac{x}{y - x^2} + \left(\frac{1}{x}\right)^{\frac{y-x^2}{x}} f\left(\frac{y-x^2}{x}\right)$$

<u>Lagrange's method</u>:

Consider:

$$a(x,y,u)\frac{\partial u}{\partial x} + b(x,y,u)\frac{\partial u}{\partial y} = c(x,y,u)$$



The characteristic curves are tangent to the vectors $\begin{pmatrix} a \\ b \\ c \end{pmatrix}$ i.e satisfy:

$$\frac{dx}{dt} = a, \quad \frac{dy}{dt} = b, \quad \frac{du}{dt} = c.$$

$$\frac{dx}{a} = \frac{dy}{b} = \frac{du}{c} \, (= dt).$$

Lagrange's method asks you to find two constants of integration of these equations.

$$S_1(x,y,u) = C_1, \quad S_2(x,y,u) = C_2.$$

Then the general solution of the pde is given by:

$$C_1 = f(C_2)$$

$$S_1 = f(S_2)$$

$$[S_2 = f(S_1), \quad f(S_1, S_2) = 0]$$

Varying C_2 gives a family of surfaces S_2 given by $S_2(x,y,u) = C_2$. Varying C_1 gives a similar family of surface, $S_1(x,y,u) = C_1$. A surface S_1 intersects a surface S_2 in a line, which is a characteristic line. If we relate C_1 to C_2 through $C_1 = f(C_2)$ and vary C_2, we get a one parameter set of lines of intersection of intersection of surfaces S_1 and S_2.

This one-parameter set defines another solution surface. We can choose f appropriately so our boundary conditions are satisfied.

<u>Easy Example:</u>

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} = 1, \quad u = -x^2 \text{ on } y = 0.$$

Ψ eqn's are

$$\frac{dx}{dt} = 1$$

$$\frac{dy}{dt} = 1$$

$$\frac{du}{dt} = 1$$

$$\frac{dy}{dx} = 1 \Rightarrow y - x = C_1$$

$$\frac{du}{dy} = 1 \Rightarrow u - y = C_2$$

$$\frac{dx}{du} = 1 \Rightarrow u - x = C_3$$

The general solution is given by $C_1 = f(C_2)$ (say) i.e. $u - x = f(u-y)$

This is the general solution, as can be seen:

$$\frac{\partial}{\partial x}: \quad \frac{\partial u}{\partial x} - 1 = f'(u-y)\frac{\partial u}{\partial x}$$

$$\frac{\partial}{\partial y}: \quad \frac{\partial u}{\partial y} = f'(u-y)\left(\frac{\partial u}{\partial y} - 1\right)$$

Eliminate f':

$$\frac{u_x - 1}{u_x} = \frac{u_y}{u_y - 1}$$

$$\Rightarrow (u_x - 1)(u_y - 1) = u_y u_x$$

$$\Rightarrow \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} = 1$$

Now choose f so that $u = x^2$ on $y = 0$.

$$u - x = f(u-y)$$

$$-x^2 - x^2 = f(-x^2 - 0)$$

$$\text{i.e.} \quad f(-x^2) = -x^2 - x$$

$$r = -x^2, \qquad f(r) = r \pm \sqrt{-r}$$

i.e $u - x = (u - y) \pm \sqrt{y - u}$.

$\Rightarrow \quad u = y - (x - y)^2$.

—/—

Slove:

$$x(y^2-u^2)\frac{\partial u}{\partial x} + y(u^2-x^2)\frac{\partial u}{\partial y} = u(x^2-y^2)$$

The 4 equatcons are:

$$\frac{dx}{dt} = y(u^2-y^3)$$

$$\frac{dy}{dt} = y(u^2-x^2)$$

$$\frac{du}{dt} = u(x^2-y^2)$$

Consider:

$$x\frac{\partial x}{\partial t} + y\frac{\partial y}{\partial t} + u\frac{\partial u}{\partial t} = x^2(y^2-u^2)$$

$$+ y^2(u^2-x^2) + u^2(x^2-y^2)$$

$$= 0.$$

$$\frac{d}{dt}\left[\frac{x^2+y^2+u^2}{2}\right] = 0 \Rightarrow x^2+y^2+u^2 = C_1$$

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$$\frac{dx}{a} = \frac{dy}{b} = \frac{du}{c} \quad (= dt)$$

find constants C_1 and C_2 and soln is $C_1 = f(C_2)$.

$$\underbrace{x(y^2-u^2)}_{a}\frac{\partial u}{\partial x} + \underbrace{y(u^2-x^2)}_{b}\frac{\partial u}{\partial y} = \underbrace{u(x^2-y^2)}_{c}$$

$$\frac{dx}{dt} = x(y^2-u^2)\,dt,$$

$$\frac{dy}{dt} = y(u^2-x^2)\,dt,$$

$$\frac{du}{dt} = u(x^2-y^2)\,dt,$$

$$x\frac{dx}{dt} + y\frac{dy}{dt} + u\frac{du}{dt} = 0$$

$$\Rightarrow \frac{1}{2}(x^2+y^2+u^2) = C'.$$

$$C_1 = x^2+y^2+u^2.$$

$$yu\frac{dx}{dt} + xu\frac{dy}{dt} + xy\frac{du}{dt} = xyu(y^2-u^2) + xyu(u^2-x^2) + xyu(x^2-y^2) = 0$$

$$\Rightarrow \frac{d}{dt}(xyu) = 0 \Rightarrow xyu = C_2.$$

and the general solution is $C_2 = f(C_1)$

i.e. $xyu = f(x^2 + y^2 + u^2)$.

Example: $(y+u)\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = x - y.$

4 equations are:

$$\frac{dx}{dt} = y + u, \quad ①$$

$$\frac{dy}{dt} = y, \quad ②$$

$$\frac{du}{dt} = x - y, \quad ③$$

$$② \Rightarrow \quad y = Ae^t.$$

$$① \Rightarrow \frac{d^2x}{dt^2} = \frac{dy}{dt} + \frac{du}{dt} = y + x - y = x$$

$$x = Ce^t + De^{-t}$$

$$③ \Rightarrow \frac{d^2u}{dt^2} = \frac{dx}{dt} - \frac{dy}{dt} = y + u - y = u$$

$$\Rightarrow u = Ee^t + Fe^{-t}.$$

We have 5 constants, while we sould only expect 3. We have a system of 3 first order equations.

Using $\frac{dx}{dt} = y + u$ gives

$$Ce^t - De^{-t} = Ae^t + Ee^t + Fe^{-t}$$

$$C = A + E.$$

$$F = -D$$

$e^t = y/A$
$e^{-t} = A/y$

$$y = Ae^t \quad (4).$$

$$x = Ce^t + De^{-t} \quad (5)$$

$$u = (C - A)e^t = \left(\frac{2C - A}{A}\right) y$$

$$\frac{x+u}{y} = \frac{2C - A}{A} = \text{const}.$$

Also $u + y = Ce^t - De^{-t}$ (7)

Using (4) and (6) and substracting (5) and (7)

$$x - u - y = 2De^{-t} = \frac{2DA}{y}.$$

So $(x-u-y)y = \text{const}.$

and general soln can be written

$$(x-u-y)y = f\left(\frac{x+u}{y}\right)$$

Another Method.

① $dx = (y+u)\,dt$

② $dy = y\,dt$

③ $du = (x-y)\,dt$

① + ③ $dx + du = (x+u)\,dt.$

$$\Rightarrow \frac{d(x+u)}{x+u} = dt = \frac{dy}{y}$$

$$\Rightarrow \ln(x+u) = \ln y + \text{const}.$$

$$\Rightarrow \frac{x+u}{y} = \text{const}$$

① and ② $dx = dy + u\,dt$

$$\Rightarrow d(x-y) = u\,dt = \frac{u\,du}{(x-y)}$$

$$\Rightarrow (x-y)\,d(x-y) = u\,du.$$

$$\Rightarrow \frac{(x-y)^2}{2} = \frac{u^2}{2} + \text{const}.$$

So $(x-y)^2 - u^2 = \text{const}$

and $\frac{x+u}{y} = f((x-y)^2 - u)$.

Second order PDE.

The general second order quasilinear pde is:

$$a(x,y,z,z_x,z_y)\frac{\partial^2 z}{\partial x^2} + b(x,y,z,z_x,z_y)\frac{\partial^2 z}{\partial x \partial y}$$
$$+ c(x,y,z,z_x,z_y)\frac{\partial^2 z}{\partial y^2} = r(x,y,z,z_x,z_y).$$

The quantity $\Delta = b^2 - 4ac$ is the discriminant of the pde.

If $\Delta > 0$, the equation is said to be hyperbolic - waves

If $\Delta < 0$, the equation is said to be elliptic - steady temperature distribution

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If $\Delta = 0$, the equation is said to be parabolic - diffusion.

If a, b, c constant and $r = 0$.

$$a z_{xx} + b z_{xy} + c z_{yy} = 0.$$

Look for the solution of the form $z = f(y + mx)$ and substitute to find

$$a m^2 f'' + b m f'' + c f'' = 0$$

$$\Rightarrow a m^2 + b m + c = 0.$$

If $\Delta > 0$ (hyperbolic) we have two real roots for m.

If $\Delta < 0$ (elliptic) we have two complex roots for m.

If $\Delta = 0$ (parabolic) we have two real repeated roots for m

Lets call these m_1 and m_2

If $m_1 \neq m_2$, introduce

$$s = y + m_1 x$$
$$t = y + m_2 x$$

called canonical variables and make a change of variable from x and y to s and t.

$z_x \longrightarrow z_s$ and z_t etc.

$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial s} \cdot \frac{\partial s}{\partial x} + \frac{\partial z}{\partial t} \frac{\partial t}{\partial z}.$$

$$= m_1 \frac{\partial z}{\partial s} + m_2 \frac{\partial z}{\partial t}.$$

$$\frac{\partial z}{\partial y} = \frac{\partial z}{\partial s} \cdot \frac{\partial s}{\partial y} + \frac{\partial z}{\partial t} \cdot \frac{\partial t}{\partial y}$$

$$= \frac{\partial z}{\partial s} + \frac{\partial z}{\partial t}$$

$$\frac{\partial}{\partial y} = \frac{\partial}{\partial s} + \frac{\partial}{\partial t}$$

$\frac{\partial}{\partial s}$

$$z_{xx} = (m_1 \partial_s + m_2 \partial_t)(m_1 z_s + m_2 z_t)$$

$$= m_1^2 z_{ss} + 2 m_1 m_2 z_{st} + m_2^2 z_{tt}.$$

$$z_{yy} = z_{ss} + 2 z_{st} + z_{tt}.$$

$$z_{xy} = (m_1 \partial_s + m_2 \partial_t)(z_s + z_t)$$

$$= m_1 z_{ss} + (m_1 + m_2) z_{st} + m_2 z_{tt}.$$

Substitute into

$$a z_{xx} + b z_{xy} + c z_{yy} = 0.$$

to find

$$z_{ss}(\cancel{am_1^2 + bm_1 + c}) + z_{st}(a2m_1m_2 + b(m_1+m_2) + 2c) + z_{tt}(\cancel{am_2^2 + bm_2 + c}) = 0 \text{ or } f$$

m_1 and m_2 are roots of $am^2 + bm + c = 0$

$m_1 + m_2 = -b/a$

$m_1 m_2 = c/a$.

$(\Delta = b^2 - 4ac)$

So $z_{st}[2c - b^2/a + 2c]$

$$= \boxed{-\frac{\Delta}{a} z_{st} = 0 \text{ or } f} \quad \Delta \neq 0 \text{ here.}$$

This is in canonical form:

$$\frac{\partial^2 z}{\partial s \partial t} = 0 \Rightarrow \frac{\partial z}{\partial t} = f'(t).$$

$f'(t)$ is arbitrary

$$\Rightarrow z(s,t) = f(t) + g(s).$$
$$= f(y + m_2 x) + g(y + m_1 x)$$

So, the general solution of a hyperbolic equation may be written in the form $z = f(y + m_2 x) + g(y + mx)$ with m_1 and m_2 roots of $am^2 + bm + c$.

Auxiliary equation.

[The lines $y + m_2 x = \text{const}$, $y + m_1 x = \text{const}$ are called characteristics]

Solutions of elliptic problem can be written in this way, but m_1, m_2 are complex

If $\Delta = 0$ and the equation is parabolic we have one root $m = -b/2a$.

Here we switch to variables $s = y + mx$, $t = x$.

$\boxed{\text{YOU DO IT!}}$ and find:

$$\underbrace{(2ma+b)}_{0} z_{st} + z_{tt} = 0$$

So $z_{tt} = 0$.

$$\Rightarrow \quad z_t = g(s)$$

$$z(s,t) = g(s) + f(s).$$

and the general solution for parabolic equation is

$$z(x,y) = x\,g(y+mx) + f(y+mx)$$

Example:

a) $z_{xx} - 3z_{xy} + 2z_{yy} = 0.$

If $z = f(y+mx)$ then, we need:

$$m^2 - 3m + 2 = 0$$

$$z = g(x+my), \quad 1 - 3m + 2m^2 = 0$$
$$m = 1/m.$$

$$(m-2)(m-1) = 0.$$

i.e. two distinct real values of m: 1 and 2 and the solution is

$$z = f(y+x) + g(y+2x)$$

Example

b) $z_{xx} - 2z_{xy} + z_{yy} = 0.$

If $z = f(y+mx)$

$$m^2 - 2m + 1 = 0$$

$$(m-1)^2 = 0.$$

i.e the equation is parabolic and the solution is $z = xf(y+x) + g(y+x)$.

Example

c) $z_{xx} - 3z_{xy} + 2z_{yy} = e^{x-y}$.

As the equation is linear the solution has the form:

$$CF + PI.$$

A solution to $z_{xx} - 3z_{xy} + 2z_{yy} = 0$

Anything that satisfies the equation.

CF: If $z = f(y+mx)$

$$m^2 - 3m + 2 = 0$$

$$(m-2)(m-1) = 0.$$

$$z = f(y+x) + g(y+2x).$$

PI:
$$z = Ae^{x-y}$$
$$z_x = Ae^{x-y}$$
$$z_{xx} = Ae^{x-y}$$
$$z_y = -Ae^{x-y}$$
$$z_{yy} = Ae^{x-y}$$
$$z_{xy} = -Ae^{x-y}$$

Substitution gives:

$$Ae^{x-y} - 3(-Ae^{x-y}) + 2(Ae^{x-y}) = e^{x-y}$$

$6A = 1 \Rightarrow A = 1/6.$

So

$$z(x,y) = \underbrace{\frac{1}{6}e^{x-y}}_{\text{PI}} + \underbrace{f(y+x) + g(y+2x)}_{\text{CF}}$$

Alternatively we can change variables to the canonical variables

$$s = y + x \quad (m = 1)$$
$$t = y + 2x \quad (m = 2)$$

$$m^2 - 3m + 2$$
$$\Delta = 9 - 8 = 1$$

This goes to $-\frac{\Delta}{a} z_{st} = -z_{st} = e^{x-y}$

$$x = t - s$$
$$y = s - x = s - t + s$$
$$= 2s - t$$

$$= e^{(t-s)-(2s-t)}$$
$$= e^{2t-3s}$$
$$\Rightarrow -z_{st} = \frac{1}{3}e^{2t-3s} + f'(t)$$
$$z = -\frac{1}{6}e^{2x-3s} + f(t) + g(s)$$

As before

25/11/11

$$\frac{1}{c^2}\frac{\partial^2 z}{\partial t^2} = \frac{\partial^2 z}{\partial x^2}$$

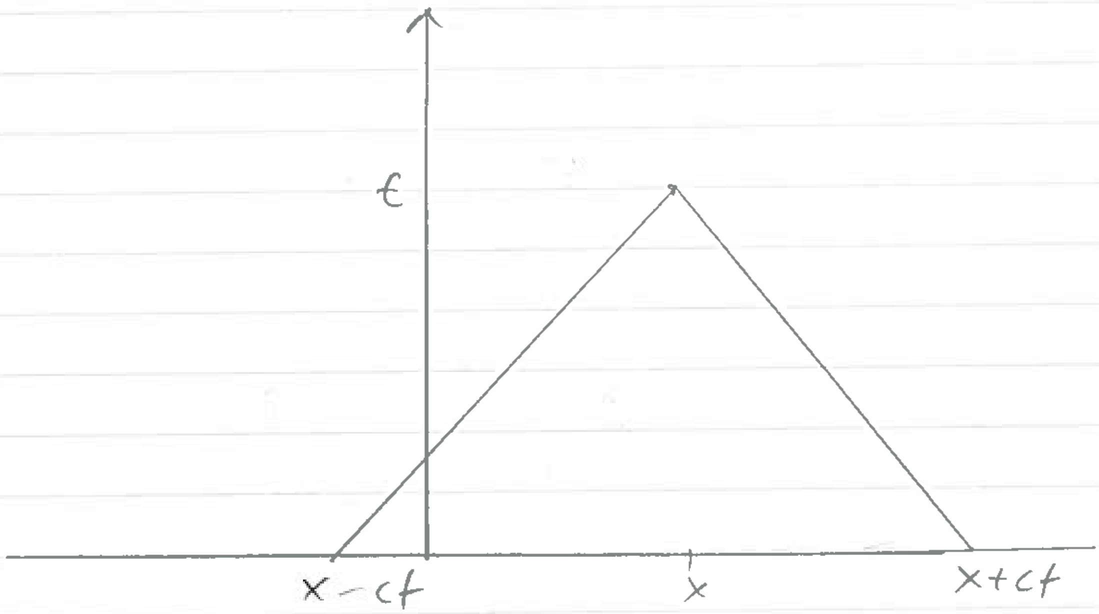
If $z = F(x)$ and $z_t = G(x)$ at $t=0$.

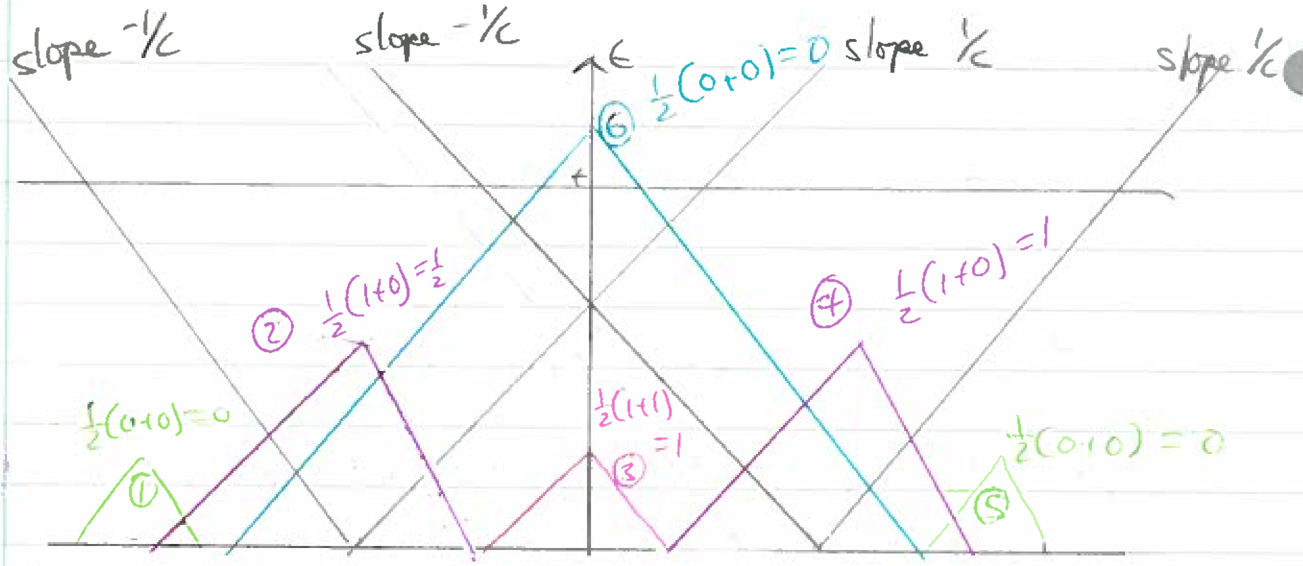
$$z(x,t) = \frac{1}{2}\Big(F(x+ct) + F(x-ct)\Big) + \frac{1}{2c}\int_{x-ct}^{x+ct} G(\xi)\,d\xi$$

$$= f(x-ct) + g(x+ct).$$

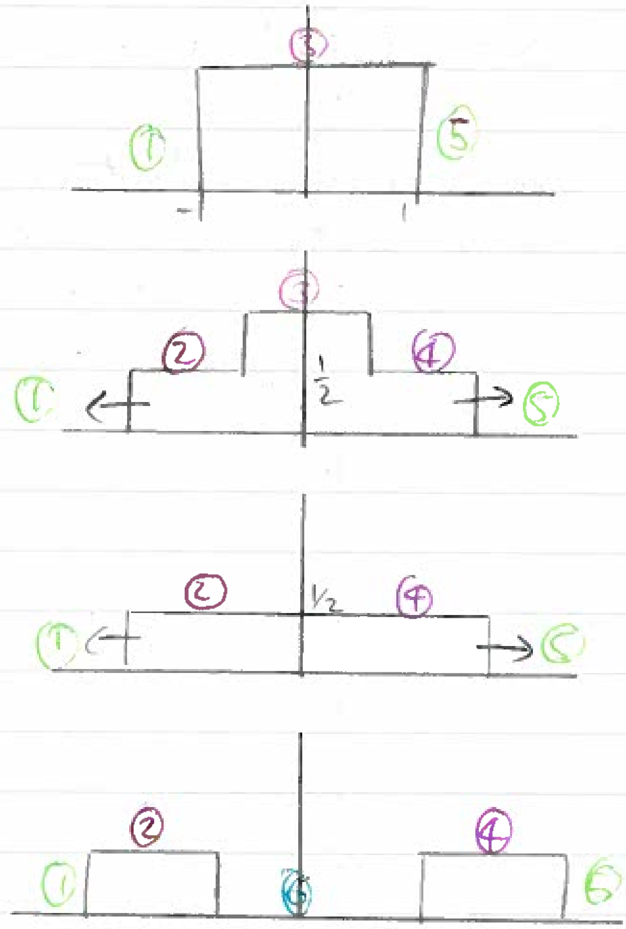
Information about solution travels with speed c – two sets of charateristics:

$$x - ct = \text{const}$$
$$x + ct = \text{const}$$





slope -1/c
slope -1/c
½(0+0)=0
slope 1/c
slope 1/c
⑥
②
½(1+0)=½
④
½(1+0)=1
½(0+0)=0
①
½(1+1)=1
③
½(0+0)=0
⑤



③
①
⑤
③
②
④
½
①
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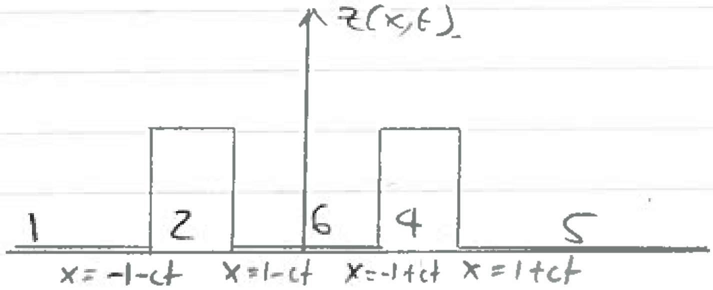
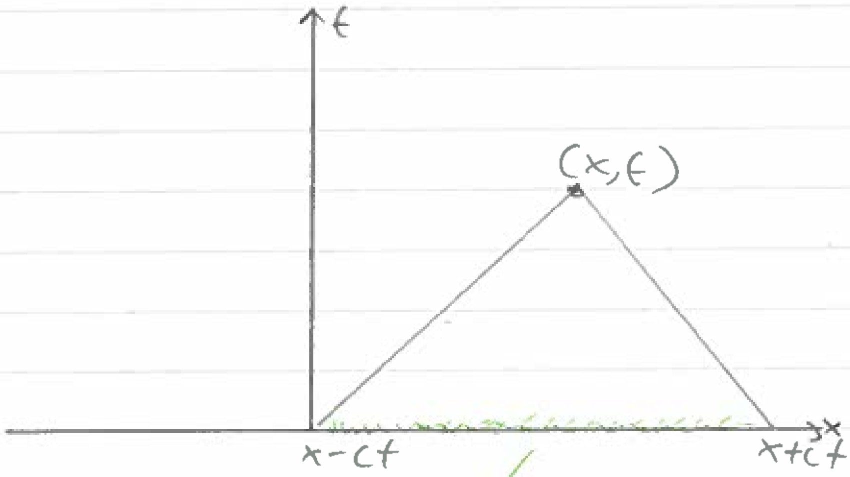
⑤

Find solutions of t and z.

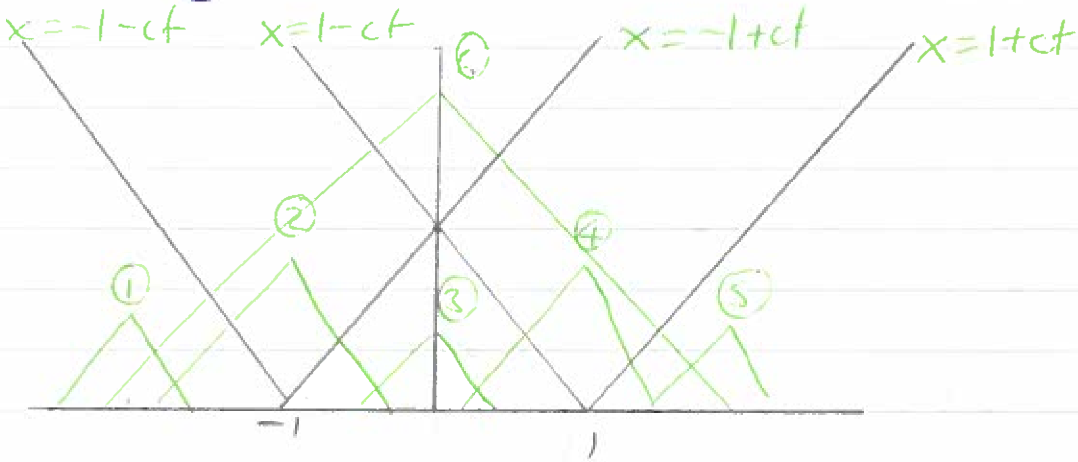


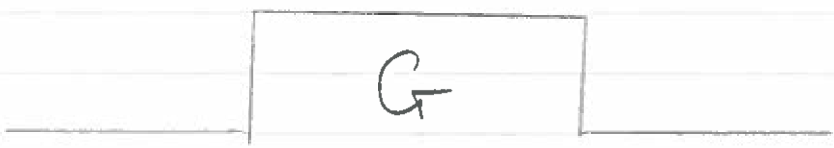
$$z(x,t) = \frac{1}{2c}\int_{x-ct}^{x+ct} G(\xi)\, d\xi.$$



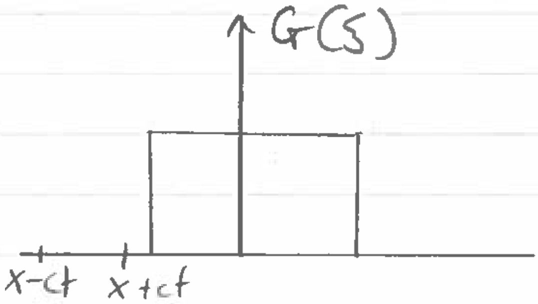
$$z(x,t) = \frac{1}{2c}\int_{x-ct}^{x+ct} G(\xi)\, d\xi$$

$$G(\xi) = \begin{cases} 1 & |\xi| < 1 \\ 0 & |\xi| > 0 \end{cases}$$





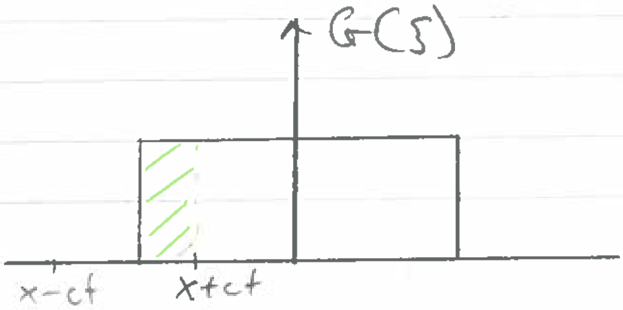
Region 1 $\quad x < -1 - ct$.



$$z = \frac{1}{2c} \int_{x-ct}^{x+ct} G(\xi)\, d\xi$$

$$= \frac{1}{2c} \int 0 \, d\xi = 0.$$

Region ② : $x - ct < 1$, $-1 < x + ct < 1$



$$z = \frac{1}{2c}\int_{-1}^{x+ct} 1 \, d\xi = \frac{1}{2c}(1 + x + ct)$$

which, as a function of x is a straight line slope $1/2c$ valid between $t < 1/c$.

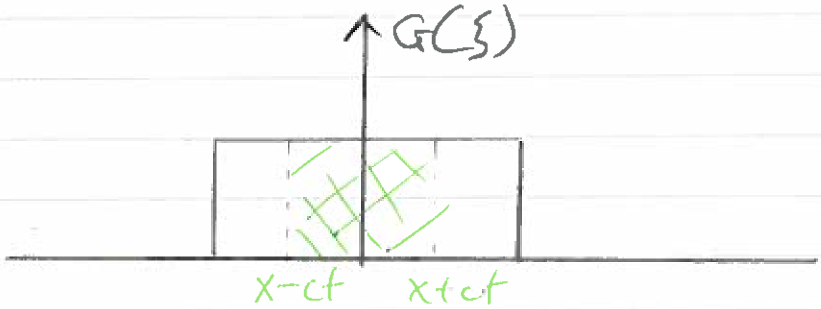
$1 = -x - ct$ giving $z = 0$
$x = -1 + ct$ giving $z = t$.

and for $t > 1/c$.

$x = -1 - ct$, gives $z = 0$
$x = 1 - ct$, gives $z = 1/c$.

Region ③ $-1 + ct < 1 + ct$, $t < 1/c$.

$$-1 \leq x - ct \leq x + ct \leq 1$$

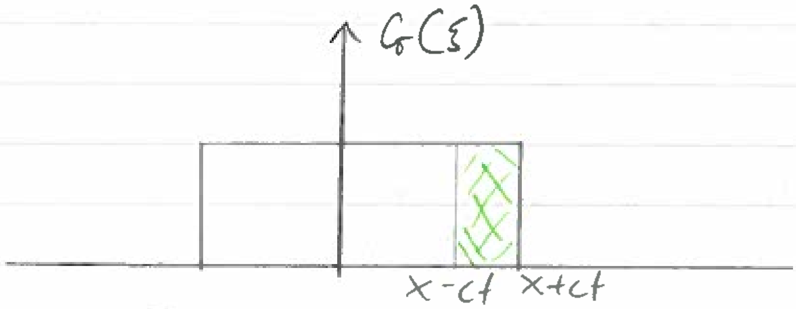


$$z = \frac{1}{2c} \int_{x-ct}^{x+ct} 1 \, d\xi = t.$$

Region ④:

$1 - ct < x < 1 + ct \quad , \quad t < 1/c$

$-1 + ct < x < 1 + ct \quad\quad t > 1/c$

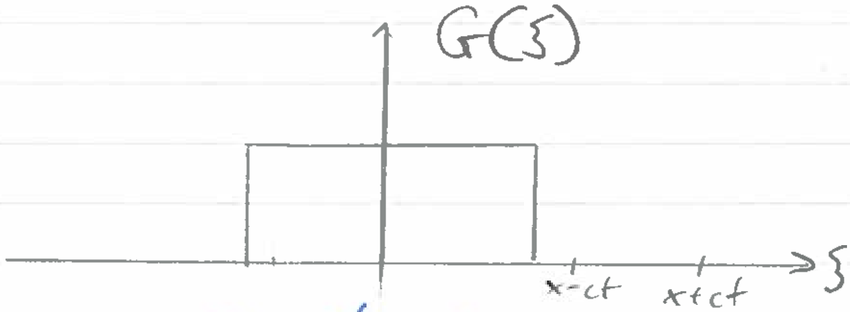


$$z = \frac{1}{2c} \int_{x-ct}^{1} 1 \quad d\xi$$

$$= \frac{1}{2c} \left[1 + ct - x \right]$$

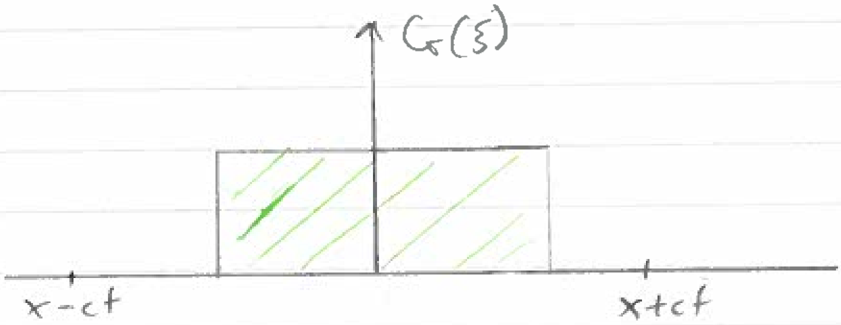
slope $-1/2c$ but otherwise like ②

Region ⑤ $\quad 1 < x - ct < x + ct$



$$z = \frac{1}{2c} \int_{x-ct}^{x+ct} 0 \, d\xi = 0.$$

Region ⑥ : $t > 1/c$, $x-ct < -1 < 1 < x+ct$

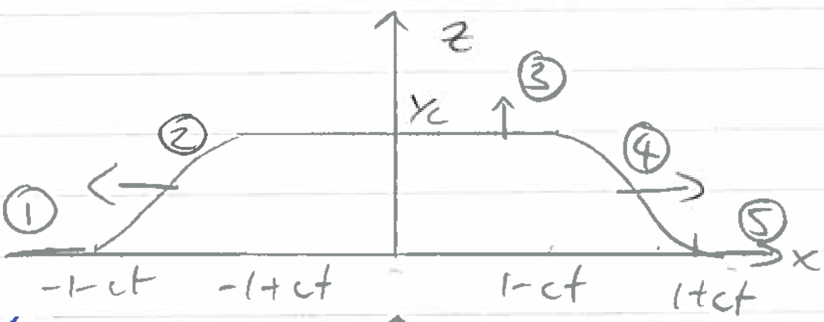


$$z = \frac{1}{2c}\int_{-1}^{1} 1 \, d\xi = \frac{1}{c}.$$

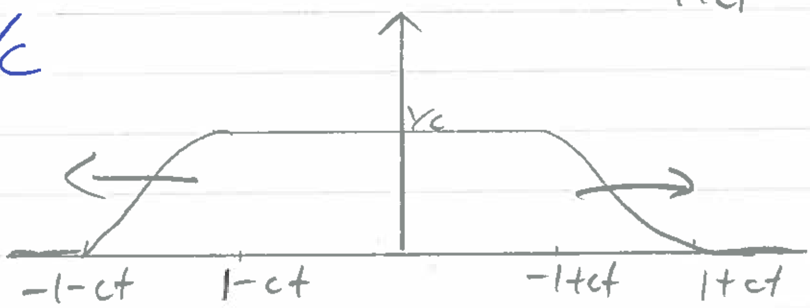
Choose a value of t; $t < 1/2c$ and $t > 1/2c$ we get different pictures

Draw $z(x,t)$ for

$t < 1/c$.



$t > 1/c$



Solution of wave equation using the method of separation of variables.

The wave equation:

$$\frac{1}{c^2}\frac{\partial^2 z}{\partial t^2} = \frac{\partial^2 z}{\partial x^2}$$

We look for a solution

$$z = X(x)\,T(t)$$

Function of x alone

Function of t alone

$$z_{tt} = XT'' \quad , \quad z_{xx} = X''T$$

$$\frac{XT''}{c^2} = X''T$$

$\div$ by XT

$$\underbrace{\frac{T''}{c^2T}}_{\text{Function of } t} = \underbrace{\frac{X''}{X}}_{\text{Function of } x}$$

Imagine changing t but not x. The l.h.s might change but the r.h.s must remain constant. We deduce l.h.s does not change and both T''/cT and X''/X are the same constant independent of both x and t.

Let's call constant λ, the separation constant:

$$\frac{T''}{c^2 T} = \frac{X''}{X} = \lambda$$

$$X'' - \lambda X = 0.$$

$$T'' - \lambda c^2 T = 0.$$

Any λ will do generating X_λ, T_λ and so $z_\lambda(x,t) = X_\lambda(x) T_\lambda(t)$ and as the wave equation is linear any sum of these is also a solution.

$$z(x,t) = \sum_\lambda X_\lambda(x) T_\lambda(t).$$

However we are interested in solution satisfying particular initial conditions that restrict values of λ.

If we slove $z_{tt} = c^2 z_{xx}$ for $t \geq 0$, $0 \leq x \leq L$ and with the boundary $z(0,t) = 0$, $z(L,t) = 0$.

If $z(x,t) = X(x)T(t)$ then we need

$$X(0)T(t) = 0 \quad \forall t \Rightarrow X(0) = 0$$
$$X(L)T(t) = 0 \quad \forall t \Rightarrow X(L) = 0.$$

$$XT'' = cTX''$$

$$\frac{T''}{c^2T} = \frac{X''}{X} = \lambda.$$

$$X'' - \lambda X = 0$$
$$X(0) = X(L) = 0.$$

It turns out that for a particular values of λ we get solutions to this other than the obvious $X = 0$.

$$\lambda > 0, \quad \lambda = 0, \quad \lambda < 0.$$

For $\underline{\lambda > 0}$, $\lambda = p^2$, p real,

$$X'' - p^2 X = 0$$

which has exponential solutions

$$X = Ae^{px} + Be^{-px}$$

or $X = \tilde{A}\cosh px + \tilde{B}\operatorname{senh} px$

But :

$X(0) = 0, \Rightarrow \tilde{A}\cdot 1 + \tilde{B}\cdot 0 = 0 \Rightarrow \tilde{A} = 0$
$X(L) = 0, \Rightarrow \tilde{B}\sinh pL \Rightarrow \tilde{B} = 0$ $\Big\}\, X = 0$

$\therefore$ No non-zero solution $X = 0$.

For $\lambda = 0$, $X'' = 0$, $X = Ax + B$
straight line.

But $X(0) = 0$, $X(L) = 0 \Rightarrow A = B = 0$
$\Rightarrow X = 0$.

This straight line must join $X(0) =$
$X(L) = 0$.

Hence $X = 0$. No non-zero solution

For $\boxed{\lambda < 0, \lambda = -p^2}$, $\boxed{X'' + p^2 X = 0}$

$X = A\cos(px) + \boxed{B\sin(px)}$

$X(0) = 0$, $A\cdot 1 + B\cdot 0 = 0$
$\Rightarrow A = 0$

We don't want cos since $X(0) = 0$ and $\cos 0 =$

$X(L) = 0$, $B\sin(pL) = 0$

One way of doing this is having $B = 0$
$\Rightarrow X = 0$.

so if $B \neq 0$ we conclude

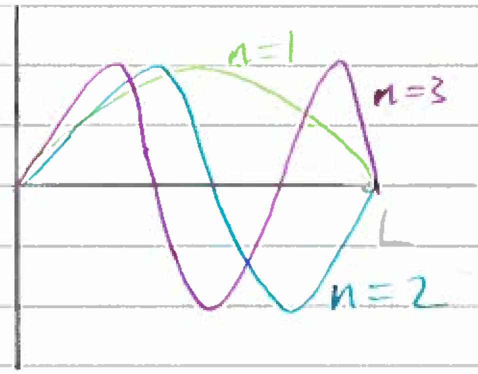
$\sin(pL) = 0$.

$\therefore \; pL = n\pi \quad , \quad n = 1, 2, 3, \ldots$

$$\boxed{p = \frac{n\pi}{L}} \qquad \lambda = \frac{n^2\pi^2}{L^2}$$

shortcut

We have an infinite number of possible seperation constants:



$X_n(x)$

Recall $T''/c^2T = \lambda = -p^2 = -\left(\frac{n\pi}{L}\right)^2$

$$T_n'' + \frac{c^2 n^2 \pi^2 T_n}{L^2} = 0.$$

$$T_n = C\cos\left[\frac{cn\pi t}{L}\right] + D\sin\left[\frac{cn\pi t}{L}\right]$$

$$z(x,t) = \sum_n X_n(x) T_n(t)$$

$$= \sum_{n=0}^{\infty} \sin\left(\frac{n\pi x}{L}\right)\left[C_n\cos\left[\frac{cn\pi t}{L}\right] + D_n\sin\left[\frac{cn\pi t}{L}\right]\right]$$

This is the Fourier Series.

2/12/11

Wave Equation.

$$\frac{1}{c^2}\frac{\partial^2 z}{\partial t^2} = \frac{\partial^2 z}{\partial x^2} \qquad \begin{aligned} z(0,t) &= 0 \\ z(l,t) &= 0 \end{aligned}$$

$$z = X(x)\,T(t)$$

$$\Rightarrow \frac{1}{c^2}\frac{T''}{T} = \frac{X''}{X} = \lambda = -p^2$$ so be satisfied.

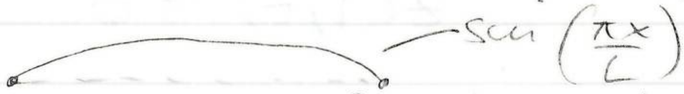
$$X = A\sin px \quad , \quad p = \frac{nx}{L} \quad , \quad \begin{aligned} X(0) &= 0 \\ X(L) &= 0 \end{aligned}$$

$$T'' + c^2 p^2 T = 0.$$

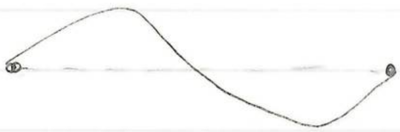
$$\Rightarrow T = C\cos\left(\frac{n\pi ct}{L}\right) + D\sin\left(\frac{n\pi ct}{L}\right).$$

$$X = \sum_1^\infty \sin\left(\frac{n\pi x}{L}\right)\left[C_n\cos\left(\frac{n\pi ct}{L}\right) + D_n\sin\left(\frac{n\pi ct}{L}\right)\right]$$

Each term in this sum is known as a normal mode oscillates with a normal frequency $\omega_n = n\left(\frac{\pi c}{L}\right)$. The mode $n=1$ is called the fendametal and the nodes $n = 2, 3, \ldots$ higher harmonics and the solution for $z(x, t)$ is a mixture sum of the fundermental and the harmonics.



fundamental



first harmonic

Second harmonic

The values of C_n and D_n, the amplitude of each mode is determine from initial condition. If $z(x, 0) = F(x)$ and $z_t(x, 0) = G(x)$ then putting $t = 0$ gives:

$$F(x) = \sum_{1}^{\infty} \sin\left(\frac{n\pi x}{L}\right) C_n .$$

$$\left(\begin{array}{llll} \text{Sin} & \text{is} & \text{zero} & \text{at } t = 0 \\ \text{Cos} & \text{is} & 1 & \text{at } t = 0. \end{array}\right)$$

Differentiating wrt t and putting $t = 0$ gives:

$$G(x) = \sum_{1}^{\infty} \sin\left(\frac{n\pi x}{L}\right) D_n \left(\frac{n\pi c}{L}\right).$$

(sin has non zero derivative at 0
cos has zero derivative at 0)

$\underline{r} = x\hat{\underline{i}} + y\hat{\underline{j}} + z\hat{\underline{k}}$

$x = \underline{r}\cdot\hat{\underline{i}}$.

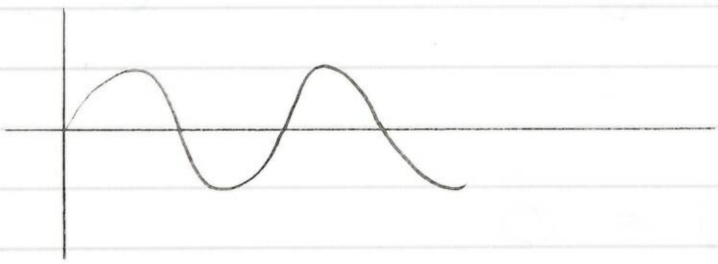
$\hat{\underline{i}}, \hat{\underline{j}}, \hat{\underline{k}}$ are othogonal ; $\hat{\underline{i}}\cdot\hat{\underline{j}} = 0$,
$\hat{\underline{i}}\cdot\hat{\underline{i}} = 0$

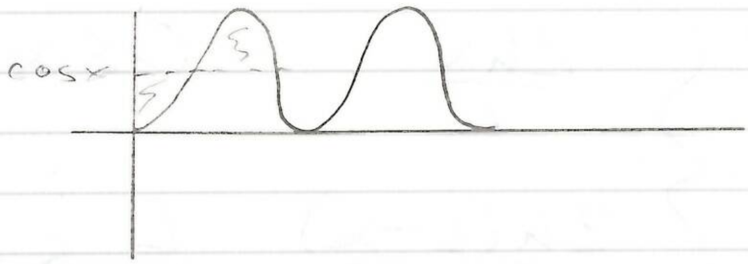
$$\int_0^L \sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{m\pi x}{L}\right) dx = 0 .$$

if $m \neq n$.

when $m = n$.

$$\int_0^L \sin^2\left(\frac{n\pi x}{L}\right) dx = \frac{1}{2}L .$$





Take equation for $F(x)$, multiply by $\sin(m\pi x/L)$ and integrate in $[0, L]$.

$$\int_0^L \sin\left(\frac{m\pi x}{L}\right) F(x) \quad dx.$$

$$= \sum_1^\infty C_n \int \sin\left(\frac{n\pi x}{L}\right) \sin\left(\frac{m\pi x}{L}\right) dx.$$

$$= C_m \quad \frac{1}{2} L$$

$$C_n = \frac{2}{L} \int_0^L F(x) \sin\left(\frac{n\pi x}{L}\right) dx.$$

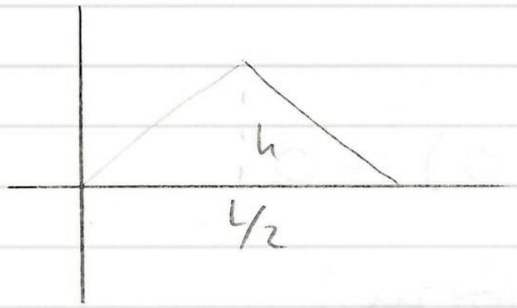
For $G(x)$ similarly

$$D_n = \frac{2}{n\pi c} \int_0^1 G(x) \sin\left(\frac{n\pi x}{L}\right) dx.$$

<u>Example</u>:

$$G(x) = 0.$$

$$F(x) = \begin{cases} \dfrac{2hx}{L} & 0 \leq x \leq \dfrac{L}{2}. \\[2ex] \dfrac{2h(L-x)}{2} & \dfrac{L}{2} \leq x \leq L \end{cases}$$



i.e slove the wave equation

$$\frac{1}{c^2} z_{tt} = z_{xx}$$

on the interval $x \in [0, L]$, $z(0,t)=0$, $z(L,t) = 0$, $z(x,0) = F(x)$, $z_t(x,0) = G(x) = 0$.

$$z(x,t) = X(x)\,T(t).$$

$$\frac{T''}{c^2 T} = \frac{X''}{X} = -p^2.$$

so that we can satisfy $X(0) = X(L) = 0$.

$$X'' + p^2 X = 0.$$

$X = A \sin px$, so that $X(0) = 0$.

$$X(L) = 0 \Rightarrow p = \frac{n\pi}{L}.$$

$$T'' + p^2 c^2 T = 0.$$

$T(t) = \cos ptc$.

satisfying $T'(0) = 0$.

Solution has form:

$$z(x,t) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi x}{L}\right) \cos\left(\frac{nxct}{L}\right)$$

where A_n to be found so that $z(x,0) = F(x)$.

i.e $F(x) = \sum_1^{\infty} A_n \sin\left(\frac{n\pi x}{L}\right)$.

$$\Rightarrow A_n = \frac{2}{L}\int_0^L \sin\left(\frac{n\pi x}{L}\right) F(x)\, dx.$$

$$= \frac{2}{L}\left[\int_0^{L/2} \sin\left(\frac{n\pi x}{L}\right) \frac{2hx}{L}\, dx\right.$$

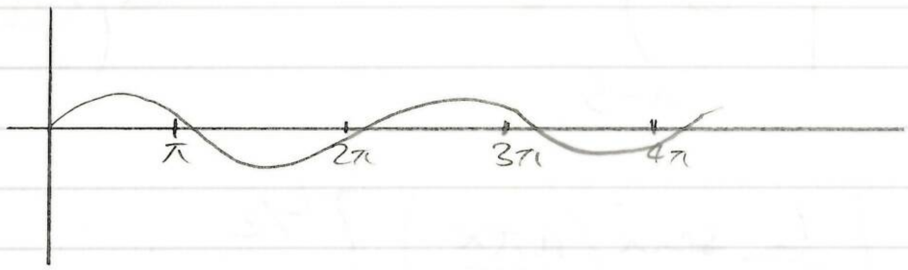
$$\left. + \int_{L/2}^{L} \sin\left(\frac{n\pi x}{L}\right) \frac{2h}{L}(L-x)\, dx\right].$$

In the second integral write $L - x = u$, $x = L - u$.

and it becomes:

$$\int_{L/2}^{0} \underbrace{\sin\left[\frac{n\pi}{L}(L-u)\right]}_{\sin(n\pi - \alpha)} \frac{2h}{L}u \; (-du).$$

$$\alpha = \frac{n\pi u}{L}.$$



$$\sin(n\pi - \alpha) = \sin\alpha \quad n = 1, 3, 5, \ldots$$

$$\sin(n\pi - \alpha) = -\sin\alpha \quad n = 2, 4, 6, \ldots$$

$$= (\pm)\int_{0}^{L/2} \sin\left(\frac{n\pi u}{L}\right)\left(\frac{2h}{L}\right) u \, du.$$

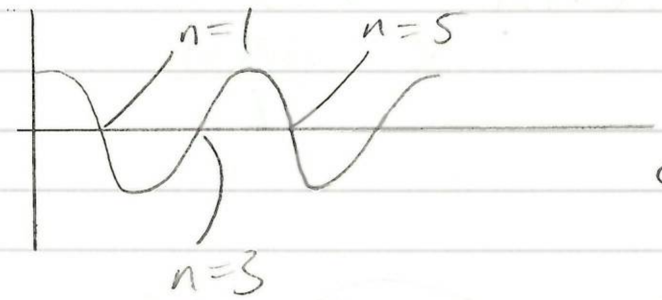
$$+ \; n = 1, 3, 5$$

$$- \; n = 2, 4, 6.$$

$$A_n = 0, \quad n = 2, 4, 6, \ldots$$

$$A_n = 2 \cdot \frac{2}{L} \cdot \frac{2h}{L} \int_{0}^{L/2} x \sin\left(\frac{n x \pi}{L}\right) dx \quad n = 1, 3, 5, \ldots$$

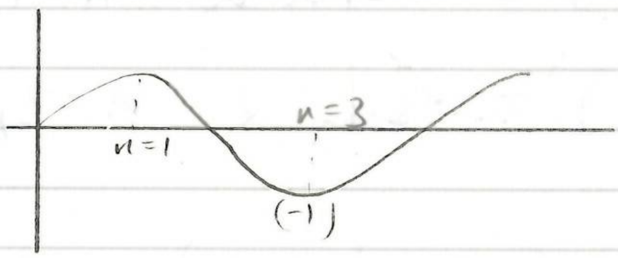
$$= \frac{8h}{L}\left\{\left[\frac{xL}{n\pi}(-1)\cos\left[\frac{n\pi x}{L}\right]\right]_0^{L/2} + \frac{L}{n\pi}\int_0^{L/2}\cos\left[\frac{n\pi x}{L}\right]dx\right\}.$$



$\cos\left(\frac{n\pi}{2}\right) = 0$

if n is odd

$$= \frac{8h}{m\pi L}\left[\frac{L}{n\pi}\sin\left(\frac{n\pi x}{L}\right)\right]_0^{L/2}.$$



$\sin\left(\frac{n\pi}{2}\right)$.

$$= \frac{8h}{n^2\pi^2}\begin{cases} 1 & \text{if } n = 1, 5, 9, 13, \ldots \text{ i.e } j \text{ even} \\ (-1) & \text{if } n = 3, 7, 11, \ldots \text{ i.e } j \text{ odd} \\ 0 & \text{if } n \text{ is even.} \end{cases}$$

where $n = 2j+1$, $j = 0, 1, 2, \ldots$

$$z(x,t) = \frac{8h}{\pi^2}\sum_{j=0}^{\infty}\frac{(-1)^j}{(2j+1)^2}\left[\sin(2j+1)\frac{\pi x}{L}\right] \cdot \left[\cos\left[(2j+1)\frac{\pi c t}{L}\right]\right]$$

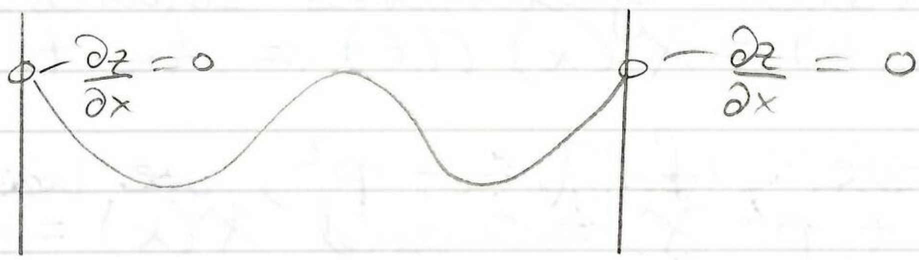
Different Boundary conditions and x-domains.

Solves:

$$\frac{z_{tt}}{c^2} = z_{xx}.$$

with $x \in [-L, L]$ and boundary conditions

$$\frac{\partial z}{\partial x} = 0, \quad \text{at } x = \pm L.$$



Look for a solution $z(x,t) = X(x)T(t)$ and we required

$$z_x(\pm L) = 0 = X'(\pm L)(T(t)).$$

$$\text{i.e. } X'(\pm L) = 0.$$

$$\frac{T''X}{c^2} = X''T$$

$$\frac{T''}{c^2 T} = \frac{X''}{X} = \lambda.$$

λ +ve, then we have exponential in X.

$$X'' - \lambda X = 0.$$

and the homogenous boundary conditions $X'(\pm L)$ cannot be satisfied.

$\lambda = 0$, $X'' = 0$ and $X = Ax + B$. and a solution with $X'(\pm L) = 0$ is just $x = \text{const}$. In this case $T'' = 0$. and $T = At + B$. So the zero separation constant generates solution $z(x,t) = X(x)T(t) = Ax + B$.

λ -ve. If $\lambda = -p^2$, we have $X'' + p^2 X = 0$ and $X(x) = A\cos px + B\sin px$ and we need to find p so that

$$X'(L) = 0.$$
$$X'(-L) = 0$$

and both A and B are zero.

$$X'(x) = pA\sin px + Bp\cos px.$$

and:

$$X'(L) = -Ap\sin pL + Bp\cos pL = 0.$$

$$X'(-L) = -pA\sin(-pL) + Bp\cos(-pL) = 0$$

$$= pA\sin(pL) + Bp\cos pL = 0.$$

Write this as:

$$\begin{pmatrix} -\sin pL & \cos pL \\ \sin pL & \cos pL \end{pmatrix} \begin{pmatrix} Ap \\ Bp \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Unless the determinant of this matrix is zero, then we have zero solution $A = B = 0$

An alternative, non-zero solution is possiable if the determinant is zero

$$-\sin(pL)\cos(pL) - \sin(pL)\cos(pL) = 0$$

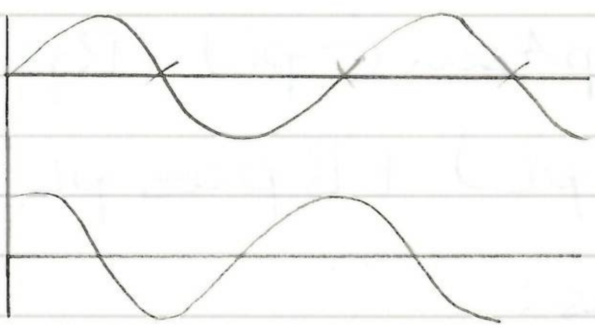
i.e $2\sin(pL)\cos(pL) = 0.$

$$\sin(2pL) = 0.$$

i.e. $2pL = n\pi$, $p = n\pi/2L$, $n = 1, 2, \ldots$

with $p = n\pi/2L$ we have

$$\begin{pmatrix} -\sin(\pi n/2) & \cos(\pi n/2) \\ \sin(\pi n/2) & \cos(n\pi/2) \end{pmatrix} \begin{pmatrix} Ap \\ Bp \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$



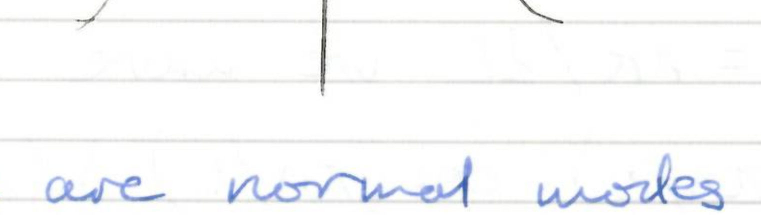
If n is even, $n = 2, 4, 6, \ldots$

$$\begin{pmatrix} 0 & \pm 1 \\ 0 & \pm 1 \end{pmatrix} \begin{pmatrix} A_p \\ B_p \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

i.e. A is anything, $B = 0$.

$$X(x) = A\cos\left(\frac{n\pi x}{L}\right) \quad n \text{ even}.$$

$$T(t) = C'\cos\left(\frac{n\pi ct}{2L}\right) + D\sin\left(\frac{n\pi ct}{2L}\right).$$



$n = 2$.

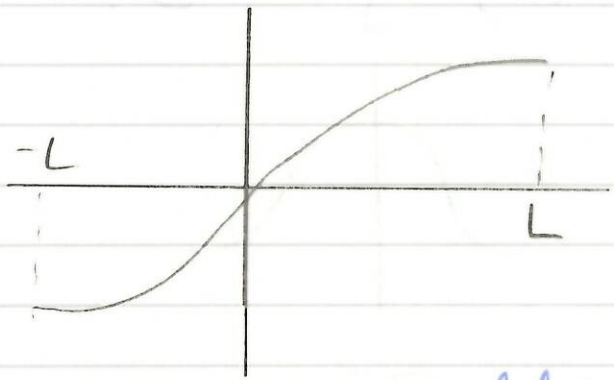
These are normal modes even in x.

If n is odd: $\begin{pmatrix} \mp 1 & 0 \\ \mp 1 & 0 \end{pmatrix} \begin{pmatrix} A_p \\ B_p \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$\Rightarrow$ B is anything, A is zero.

$$X(x) = B\sin\left(\frac{n\pi x}{2L}\right) \quad n \text{ odd.}$$

$$T(t) = C\cos\left(\frac{n\pi ct}{2L}\right) + D\sin\left(\frac{n\pi ct}{2L}\right).$$



$n=1$

odd normal modes.

So the general solution is.

even in x

$$z(x,t) = \underbrace{\left(A_0 t + B_0\right)}_{\text{zero sep const}}$$

odd in x

$$+\sum_{\substack{j=0\\ n=2j+1}}^{\infty} \sin\left((2j+1)\frac{\pi x}{2L}\right)\left(C_j\cos\left(\frac{(2j+1)\pi ct}{2L}\right) + D_j\sin\left(\frac{(2j+1)\pi ct}{2L}\right)\right)$$

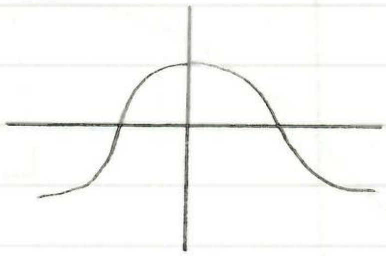
$$+\sum_{\substack{j=1\\ n=2j}}^{\infty} \cos\left[\frac{j\pi x}{L}\right]\left(E_j\cos\left[\frac{j\pi ct}{L}\right] + F_j\sin\left[\frac{j\pi ct}{L}\right]\right).$$

even in x.

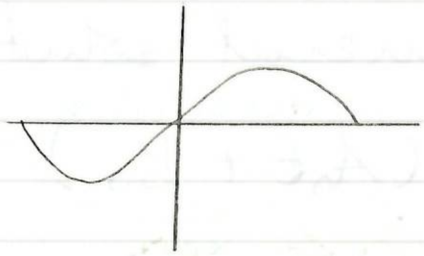
Initial Conditions $z_t = 0$ then
we know D_j and $F_j = 0$, $A_0 = 0$.

If $z = 0$, C_j and $E_j = 0$, $B_0 = 0$.

If $z =$ even, $C_j = 0$.



$z =$ odd, $E_j = 0$



If z is even, $D_j = 0$
z is odd, $F_j = 0$, $A_0 = 0$.

7/12/11.

The heat / Diffusion equation:

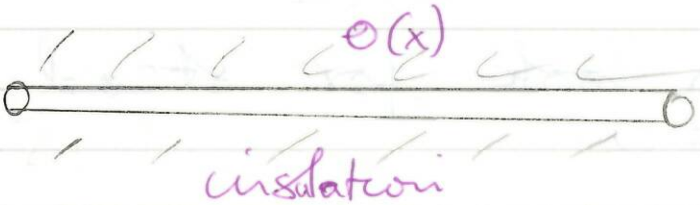
$$\frac{\partial \theta}{\partial t} = k \frac{\partial^2 \theta}{\partial x^2} \quad \text{for } \theta(x, t).$$

Constant, the thermal diffusion or diffusivity.

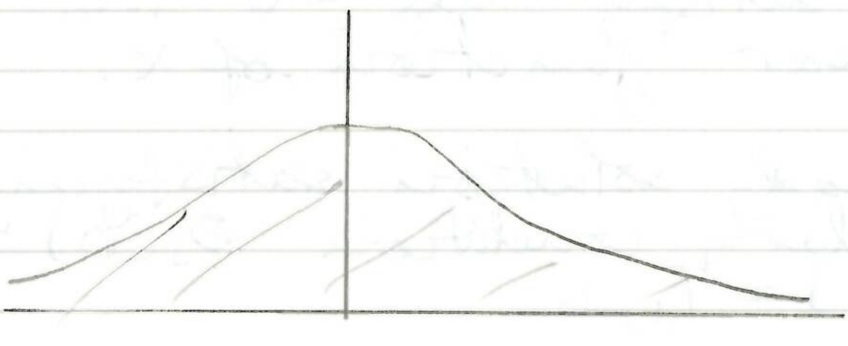
1 dim space

time

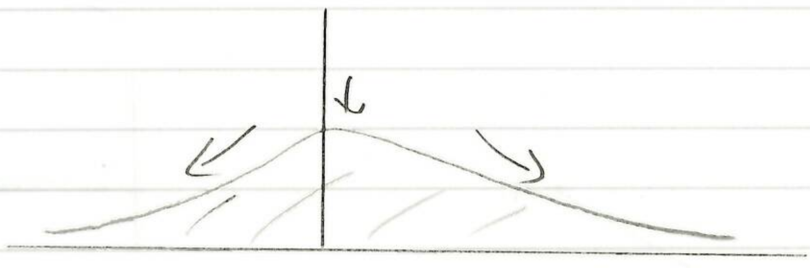
(In 3D: $\partial\theta/\partial t = k \nabla^2 \theta$).



Long/thin metal rod.



↓ t increases:



Typical boundary condition on a rod of finite length say $x \in [0, L]$
$\theta(0) = T_0$, $\theta(L) = T_1$.

Or we could improve insulating boundary condition $\partial\theta/\partial x = 0$ at $x = 0$ say.

— / —

Robin boundary conditions:

$$\frac{\partial\theta}{\partial x} = \alpha\theta$$

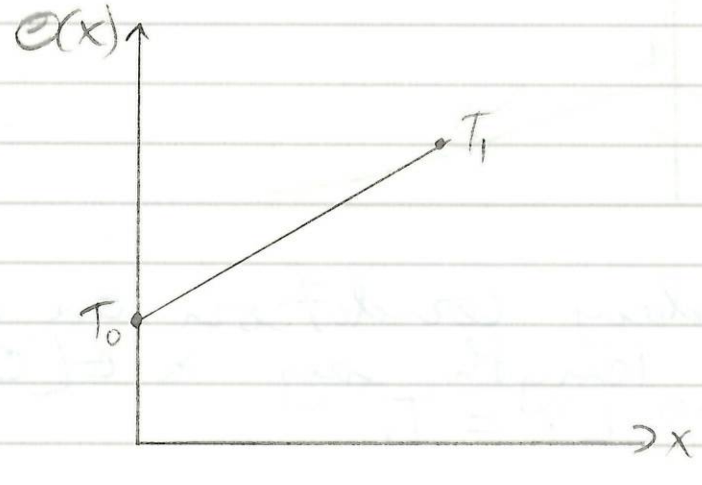
— / —

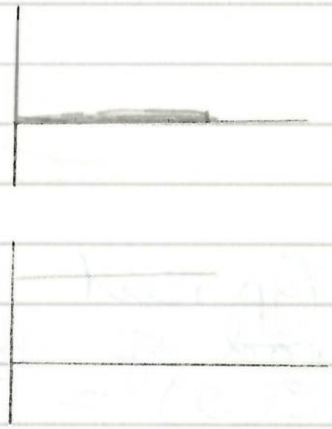
We can look for steady solution $\partial\theta/\partial t = 0$.

These satisfy $\theta_{xx} = 0$ i.e. $\theta_s = Ax + B$, a linear function of x.

A steady solution satisfying the boundary condition $\theta_s(0) = T_0$, $\theta_s(L) = T_1$, is

$$\theta_s = T_0 + (T_1 - T_0)\frac{x}{L}.$$





If one boundary conditcoin is insulating asks for $\partial\theta/\partial x = 0$, then the steady solutcoin requires $A = 0$ and $\theta_s = B$, with B obtained, maybe from the other boundary conditcoins.

We will look for time dependent solutcoins of the form $\theta(x,t) = X(x)T(t)$

$$X T' = kX'' T$$

$$\Rightarrow \frac{T'}{kT} = \frac{X''}{X} = \text{const} = \lambda \quad \text{the separation constant}$$

$\lambda = 0$ gives $T' = 0$ i.e $T = \text{const}$
$X'' = 0$ $\quad X = Ax + B$

$$\Rightarrow XT = Ax + B \text{ ; the steady solutcoin.}$$

$\lambda > 0$

$T' = k\lambda T$

$T = Ae^{k\lambda t}$ and this grows in time $\lambda > 0$ which is unrealistic.

$X'' - \lambda X = 0$ which has expontial solutcoins and trigonometric solutcoin and we cannot satisfy homogenous b.c if $\lambda > 0$.

$\lambda < 0$, $\lambda = -p^2$.

$X'' + p^2 X = 0 \Rightarrow X = A \sin px + B \cos px$ and find p from spatial boundary conditions.

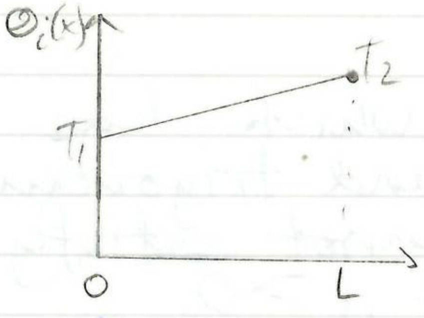
$T' = -p^2 k T$, $T = Ae^{-p^2 kt}$.

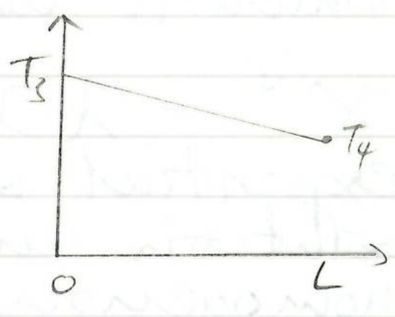
General Solution:

$$\theta(x,t) = A_0 x + B_0$$
$$+ \sum_p (A_p \sin px + B_p \cos px) e^{-p^2 kt}.$$

Example.
Solve the heat equation $\theta_t = k\theta_{xx}$ on the interval $x \in [0, L]$ with boundary conditions $\theta(0) = T_3$, $\theta(L) = T_4$ and intial conditions:

$$\theta = T_1 + (T_2 - T_1)\frac{x}{L} = \theta(x, 0) = \theta_i(x).$$





I "expect" that as $t \to \infty$ the solution to apporach the steady solution

$\theta = \theta_s = T_3 + (T_4 - T_3)x/L$ Note $\theta(0) = T_3$ and $\theta_3(L) = T_4$ i.e. θ satisfies over required boundary conditions

We write

$$\theta(x,t) = \theta_s(x) + \theta_u(x,t) \quad ,$$

unsteady

steady

$$\theta_{s_{xx}} = 0$$

and boundary condition on θ

if $\quad \theta(0) = T_3 \quad , \quad \theta_s(0) = T_3$

$\quad \theta(L) = T_4 \quad , \quad \theta_s(L) = T_4 .$

So putting $x = 0$.

$$\theta(0,t) = \theta_s(0) + \theta_u(0,t)$$
$$= T_3 + \theta_u(0,t).$$

and $\theta_u(0,t) = 0$.

Similarly $\quad \theta_u(L,t) = 0$.

Since $\quad \theta_u(L,t) = 0$.

Since $\quad \theta_t = k\theta_{xx}$

$$\underbrace{\cancel{\theta_{s_t}}}_{=0} + \theta_{u_t} = k\left[\underbrace{\cancel{\theta_{s_{xx}}}}_{=0} + \theta_{u_{xx}}\right]$$

$$\theta_{u_t} = k\,\theta_{u_{xx}} \quad \text{and} \quad \begin{array}{l} \theta_u(0,t) = 0 \\ \theta_u(L,t) = 0. \end{array}$$

and we use the method of separation of variables to find $\theta_u(x,t)$

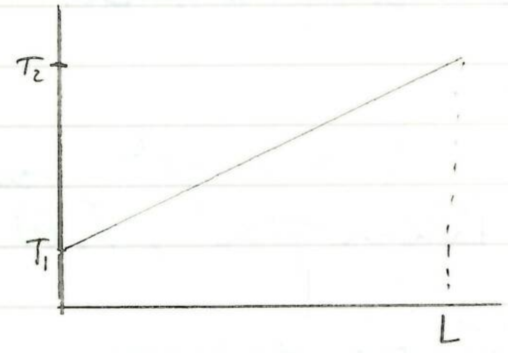
But what are the initial conditions for θ_u?

$$\theta(x,0) = \theta_i(x) = \theta_s(x) + \theta_u(x,0)$$

So: $\theta_u(x,0) = \theta_i(x) - \theta_s(x)$.

9/12/11

Heat equation



$$\theta_t = k\theta_{xx}$$

Solve for $x \in [0, L]$

Boundary Conditions:
$\theta(0) = T_3$
$\theta(L) = T_4$.

with intial conditions $\theta = T_1 + (T_2 - T_1)x/L$
$= \theta_i(x)$.

Write $\theta = \theta_s(x) + \theta_u(x,t)$

Satisfies $\theta_{s_{xx}} = 0$ and boundary conditions
$\Rightarrow \theta_s = T_3 + (T_4 - T_3)x/L$. Has nothing
to do with initial conditions

Then $\theta_{u_t} = k\theta_{u_{xx}}$, $\theta_u(0) = 0$, $\theta_u(L) = 0$.
and use sep variables $\theta_u(x, 0) = \theta_i(x) - \theta_s(x)$

$$\theta_u(x,t) = X(x)T(t), \quad \frac{T''}{kT} = \frac{X''}{X} = \lambda$$

- Cannot have $\lambda > 0$ as this would
lead to exponential solution for X
and we required $\theta_u(0) = \theta_u(L) = 0$ i.e
$X(0) = X(L) = 0$ which exponential
cannot satisfy.

- Cannot have $\lambda = 0$ as this gives $X = Ax + B$ and we cannot satisfy $X(0) = X(L) = 0$ with a non-zero solution for X.

- So for $\lambda < 0$, $\lambda = -p^2$ and $X(x) = A\cos px + B\sin px$, $T(t) = e^{-p^2kt}$.

Our boundary condition require:

$$X(0) = 0 \Rightarrow A = 0.$$
$$X(L) = 0 \Rightarrow \sin pL = 0 \Rightarrow p = \frac{n\pi}{L}, n = 1,2,3,\ldots$$

$$\theta_u(x,t) = \sum_{n=1}^{\infty} A_n e^{-\frac{n^2\pi^2kt}{L^2}} \sin\left(\frac{n\pi x}{L}\right)$$

Initially, i.e. putting $t = 0$, we need.

$$\theta_u(x,0) = \theta_i(x) - \theta_s(x).$$

i.e. $\sum_1^{\infty} A_n \sin\left(\frac{n\pi x}{L}\right) = (T_1 - T_3) + \frac{x}{L}(T_2 - T_4 - T_1 + T_3)$

$$= P + \frac{Qx}{L}.$$

Multiply by $\sin(m\pi x/L)$ and $\int_0^L$ and find:

$$A_m L . \frac{1}{2} = \int_0^L \left(P + \frac{Qx}{L}\right) \sin\left(\frac{m\pi x}{L}\right) dx.$$

$$= \left\{ \left[\frac{-L}{\pi m} \cos\left(\frac{m\pi x}{L}\right)\left(P + \frac{Qx}{L}\right)\right]_0^L + \underbrace{\frac{L}{m\pi}\int_0^L \cos\left(\frac{m\pi x}{L}\right)\frac{Q\,dx}{L}}_{0} \right\}$$

$$= \frac{PL}{m\pi}\left(1 - (-1)^m\right) - \frac{L}{m\pi}\frac{QL}{L}(-1)^m.$$

$$= \frac{L}{m\pi}\left\{ P\left(\underbrace{1-(-1)^m}_{\substack{2 \text{ if } m \text{ is odd} \\ 0 \text{ if } m \text{ is even.}}}\right) - Q(-1)^m \right\}$$

So $\theta(x,t) = T_3 + (T_4 - T_3)x/L$

$$+\sum_{n=1} \frac{2}{n\pi}\left\{ (T_1 - T_3)(1-(-1)^n) - (T_2 - T_4 + T_3 - T_1)(-1)^n \right\} \cdot e^{\frac{-n^2\pi^2 kt}{L^2}} \sin\left(\frac{n\pi x}{L}\right)$$

—/—

Aside: $\int_0^L \sin\left(\frac{m\pi x}{L}\right)\sin\left(\frac{n\pi x}{L}\right) dx$ (only non zero when $n = m$)

$$\sum_1^\infty A_n \int_0^L \sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{m\pi x}{L}\right) dx$$

Use double angle formula

$$= A_m \int_0^L \sin^2\left(\frac{\pi m x}{L}\right) dx = \frac{1}{2}L$$

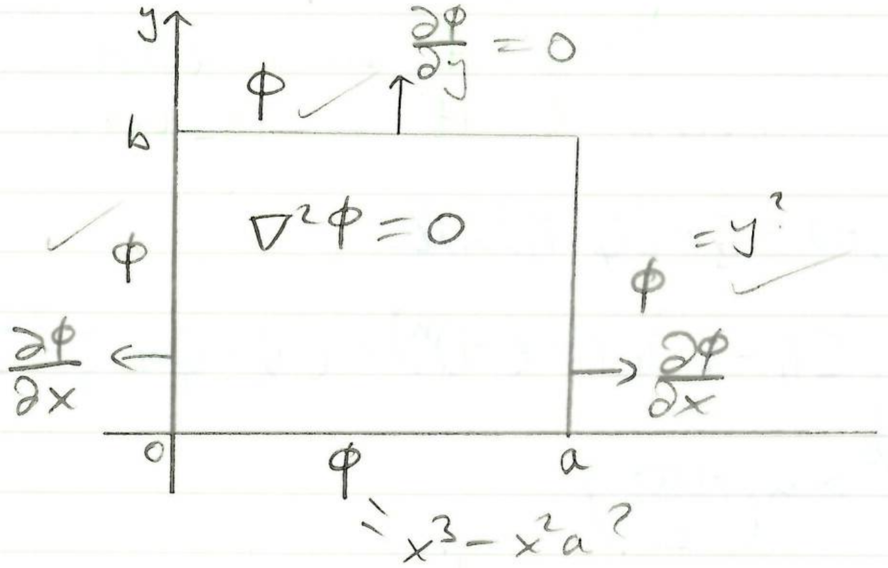
—/—

Laplace Equation for $\phi(x,y)$.

This is the elliptic equation:

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0.$$

i.e $[\underline{\nabla} \cdot \underline{\nabla} \phi = 0]$ or $\nabla^2 \phi = 0$.



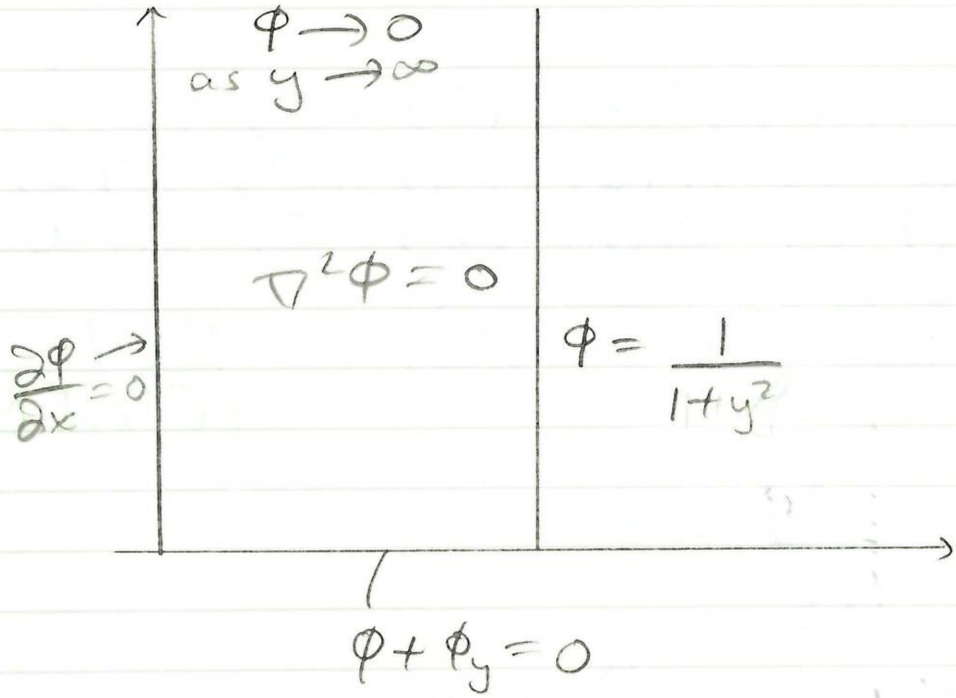
Boundary conditions for this problem are typically:

1) ϕ is specified on the boundary - Dirichlet boundary condition.
2) $\partial \phi / \partial n$ is specified
3) Robin condition:

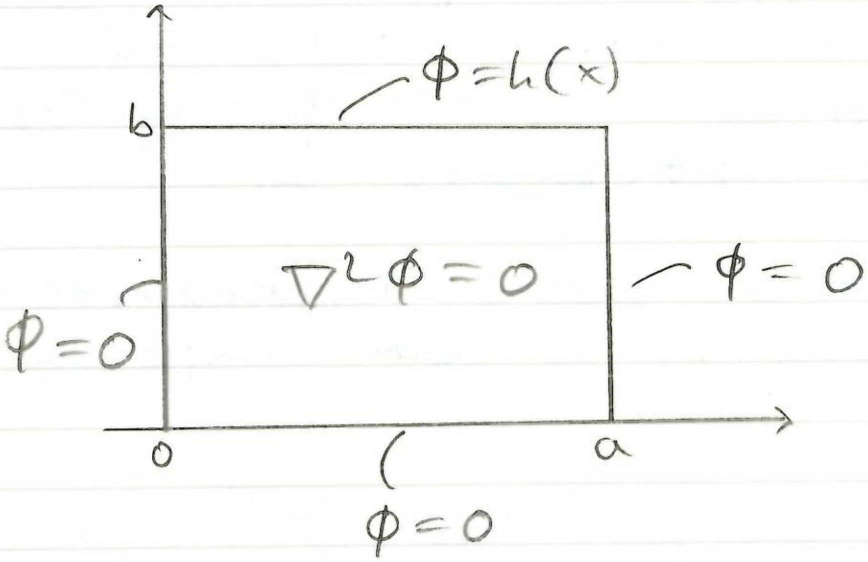
$$\phi + \beta \frac{\partial \phi}{\partial n} = 0.$$

These boundary conditions could be

different on different boundaires:



<u>Laplaces equation in a rectangular domain</u>



Slove $\nabla^2 \phi = 0$ in the domain
$0 \leq x \leq a$, $0 \leq y \leq b$.

with $\phi(x,0) = 0$, $\phi(0,y) = 0$, $\phi(x,b) = h(x)$, $\phi(a,y) = 0$

Look for a solution with $\phi(x,y) = X(x)Y(y)$

Since:

$\phi(x,0) = 0, \quad X(x)Y(0) = 0, \quad Y(0) = 0.$

$\phi(0,y) = 0, \quad X(0)Y(y) = 0, \quad X(0) = 0.$

$\phi(a,y) = 0, \quad X(a)Y(y) = 0, \quad X(a) = 0.$

$X(x)\cancel{Y}(b) = h(x) \leftarrow$ NOT POSSIBLE

$$\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} = 0.$$

$$X''Y + XY'' = 0.$$

$$\frac{X''}{X} = -\frac{Y''}{Y} = \lambda, \text{ a separation constant}$$

So $X'' + \lambda X = 0$, $Y'' - \lambda Y = 0$.

We can have $\lambda > 0$ giving exponentials X and trigonometric in Y.

or $\lambda < 0$, giving trigonometeric functions in X and exponentials in Y.

or $\lambda = 0$, $X'' = 0$, $Y'' = 0$ i.e linear functions in X and Y.

and ignoring the influence of boundary conditions the general solution is a combination of all these possibilites.

Considering these and especially the fact that $X = 0$ at both $x = 0$ and $x = a$ implies we must restrict ourselves to $\lambda < 0$.

Write $\lambda = p^2$.

$$X'' + p^2 X = 0.$$
$$Y'' - p^2 Y = 0.$$

$$X_p(x) = A \sin px + B \cos px$$
$$Y_p(y) = C \sinh py + D \cosh px.$$

$\frac{1}{2}\left.\begin{matrix} e^{py} \\ e^{-py} \end{matrix}\right\} = 1$ at $y=0$

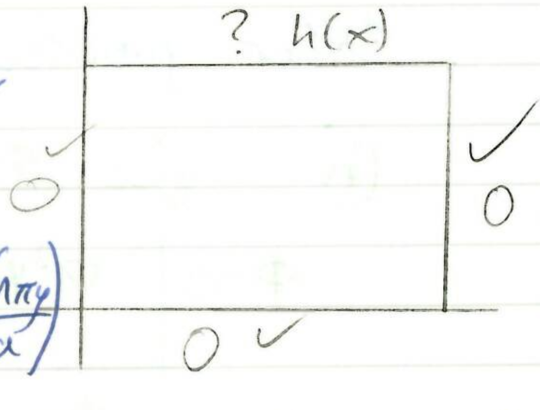
Applying $X(0) = 0 \Rightarrow B = 0.$
$X(a) = 0 \Rightarrow \sin pa = 0$
$pa = \pi n \quad , n = 1, 2, 3, \ldots$
$Y(0) = 0 \Rightarrow D = 0.$

$$X_n(x) = A_n \sin\left(\frac{n\pi x}{a}\right),$$

$$Y_n(y) = C_n \sinh\left(\frac{n\pi y}{a}\right)$$ ← same p

and generally $\phi = \sum_n X Y$

$$\phi(x, y) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi x}{a}\right) \sinh\left(\frac{n\pi y}{a}\right)$$



A_n's are found so that $\phi(x, b) = h(x)$.

i.e.

$$\phi(x,b) = h(x) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi x}{a}\right) \sinh\left(\frac{n\pi b}{a}\right)$$

Multiply by $\sin\left(\frac{m\pi x}{a}\right)$ and $\int_0^a$.

$$\int_0^a \sin\left(\frac{m\pi x}{a}\right) h(x)\, dx = A_m \sinh\left(\frac{m\pi b}{a}\right)\frac{1}{2}a.$$

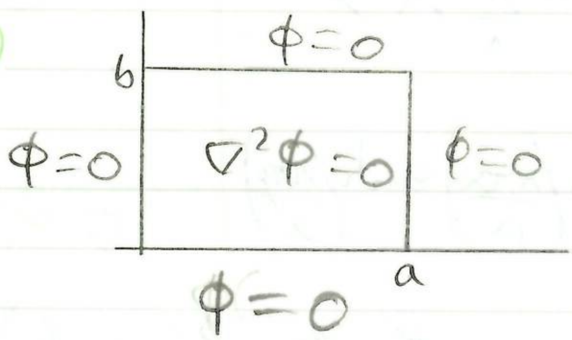
$$A_m = \frac{2\int_0^a \sin\left(\frac{m\pi x}{L}\right) h(x)\, dx}{a\sinh\left(\frac{m\pi b}{a}\right)}.$$

and

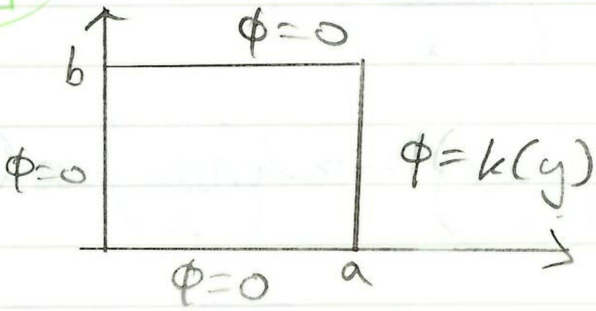
$$\phi(x,y) = \frac{2}{a}\sum_{n=1}^{\infty} \sin\left(\frac{n\pi x}{a}\right)\frac{\mathrm{sh}(n\pi y/a)}{\mathrm{sh}(n\pi b/a)}\int_0^a \sin\left(\frac{n\pi x}{a}\right) h(x)\, dx.$$

Other problems might be:

II

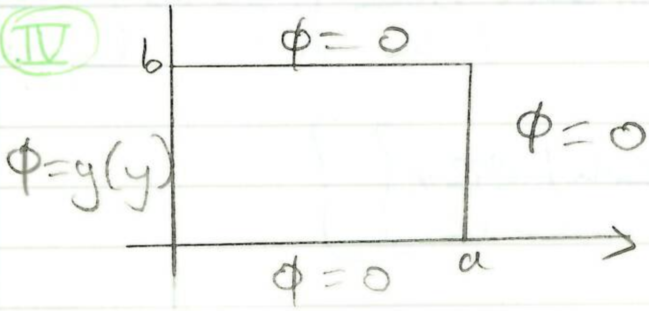


III

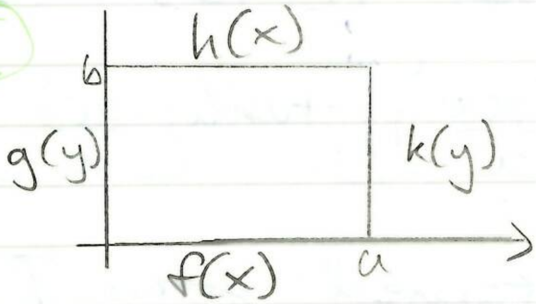


$$\sum_{n=1}^{\infty} A_n \sinh\left(\frac{n\pi x}{b}\right)\sin\left(\frac{n\pi y}{b}\right)$$

IV



V



II;
$$X(x) = \sin\left(\frac{n\pi x}{a}\right)$$

$$Y(y) = C\sinh\left(\frac{n\pi y}{a}\right) + D\cosh\left(\frac{n\pi y}{a}\right)$$

and $\phi(x,b) = 0$.

$$\Rightarrow \quad Y(b) = 0,$$

$$C\sinh\left[\frac{n\pi b}{a}\right] + D\cosh\left[\frac{n\pi b}{a}\right] = 0.$$

$$D = -C' \tanh\left[\frac{n\pi b}{a}\right].$$

$$Y(y) = C'\left(\sinh\left(\frac{n\pi y}{a}\right) - \tanh\left(\frac{n\pi b}{a}\right)\cosh\left(\frac{n\pi y}{a}\right)\right)$$

$$\phi(x,y) = \sum_{n=1}^{\infty} C_n \sin\left[\frac{n\pi x}{a}\right]\left[\sinh\left[\frac{n\pi y}{a}\right] - \tanh\left[\frac{n\pi b}{a}\right]\cosh\left[\frac{n\pi y}{a}\right]\right]$$

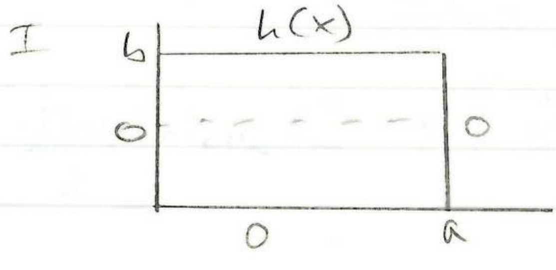
C_n's found so that $\phi(x,0) = f(x)$.

$$f(x) = \sum_{n=1}^{\infty} C_n \sin\left(\frac{n\pi x}{a}\right)\left[-\tanh\left[\frac{n\pi b}{a}\right]\right].$$

Problem II is obtained from I by reflecting in the line $y = b/2$ i.e $y \to b - y$ and

$$\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} = 0.$$

is unaltered under this transformation as it has a second order derivative, only in y.



So the solution for II is found from I be replacing y by $b-y$.

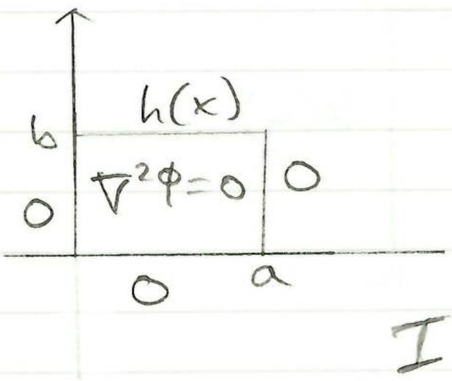
$$\phi(x,y) = \frac{2}{a}\sum_{n=1}^{\infty} \frac{\sin\left(\frac{n\pi x}{a}\right)\sinh\left(\frac{n\pi(b-y)}{a}\right)}{\sinh\left(\frac{n\pi b}{a}\right)} .$$

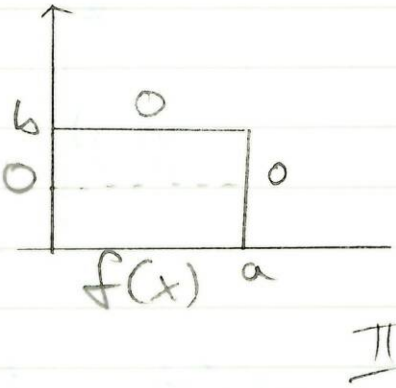
$$\cdot \int_0^a \sin\left(\frac{n\pi x}{a}\right) f(x)\, dx .$$

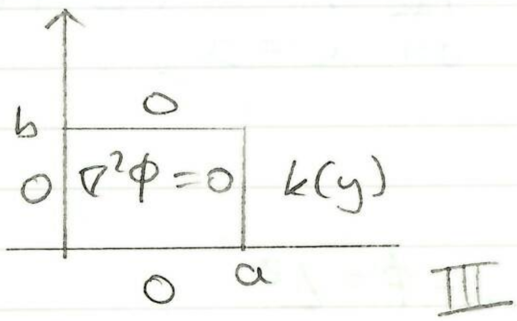
III can be mapped to I by reflecting in the diagonal of the rectangle, effected by $y \to x$, $x \to y$, $a \to b$, $b \to a$.

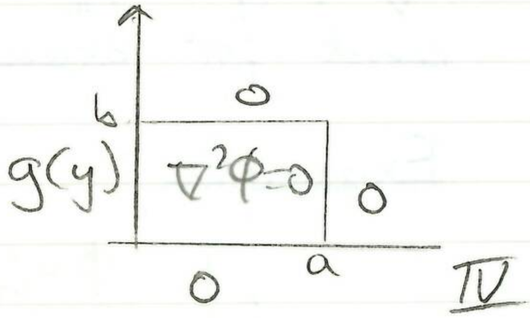
$$\phi(x,y) = \frac{2}{b}\sum_{n=1}^{\infty} \sin\left[\frac{n\pi y}{b}\right] \frac{\sh\left(\frac{n\pi x}{b}\right)}{\sh\left(\frac{n\pi a}{b}\right)} \int_0^b \sin\left(\frac{n\pi y}{b}\right) k(y)\, dy$$

14/12/11.









$$\mathrm{I}: \phi_{\mathrm{I}} = \sum_{n=1}^{\infty} \sin\left[\frac{n\pi x}{a}\right] sh\left[\frac{n\pi y}{a}\right] \frac{2}{a} \frac{1}{sh[n\pi b/a]} \int_0^a \sin\left[\frac{n\pi x}{a}\right] h(x)\,dx$$

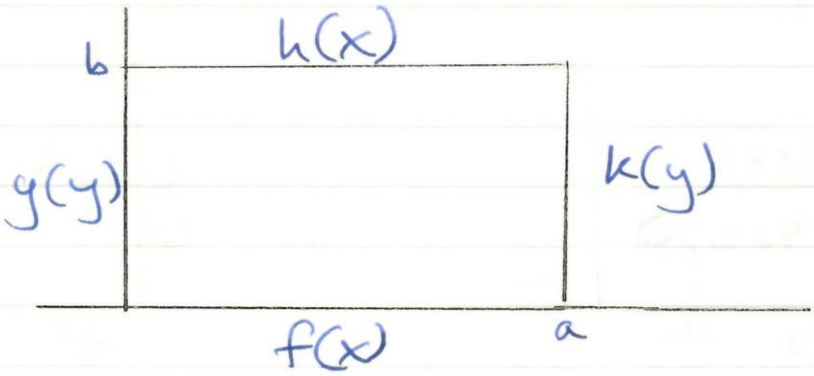
$$\mathrm{II}: \phi_{\mathrm{II}} = \sum_{n=1}^{\infty} \text{—//—} \sin\left[\frac{n\pi(y-b)}{a}\right] \text{———//———} f(x)\,dx$$

$$\mathrm{III} \quad \phi_{\mathrm{III}} = \sum_{n=1}^{\infty} \sin\left[\frac{n\pi y}{b}\right] sh\left[\frac{n\pi x}{b}\right] \frac{2}{b} \left(\frac{1}{sh[n\pi a/b]}\right) \int_0^b \sin\left[\frac{n\pi y}{a}\right] k(y)\,dy$$

$a \to b$
$x \to y$

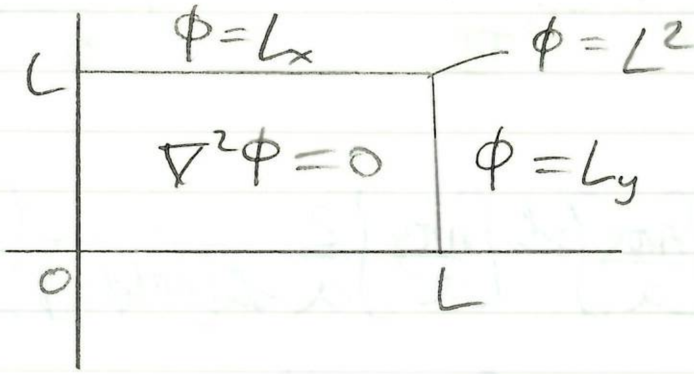
IV from II
$x \to a - x$

$$\phi_{\mathrm{IV}} = \sum_{1}^{\infty} \sin\left[\frac{n\pi y}{b}\right] sh\left[\frac{n\pi(a-x)}{b}\right] \left(\frac{2}{b}\right) \frac{1}{sh\left[\frac{n\pi a}{b}\right]} \int_0^b \sin\left[\frac{n\pi y}{b}\right] g(y)\,dy$$



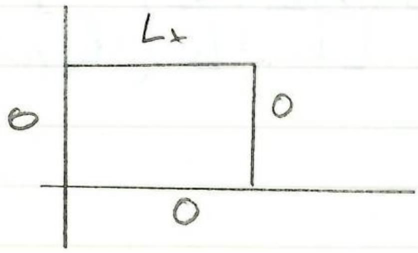
$\nabla^2(\phi_I + \phi_{II} + \phi_{III} + \phi_{IV}) = 0 + 0 + 0 + 0 = 0$
and for example on $x = a$.

Example:



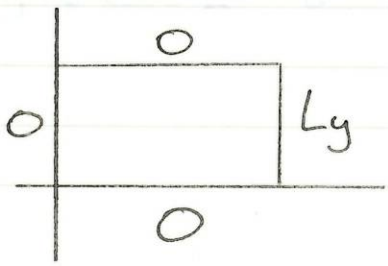
First slove:

①



Then slove:

②



$\nabla^2 \phi = 0, \quad \phi = X(x)\, Y(y).$

$$\frac{X''}{X} = -\frac{Y''}{Y} = \text{const} = -p^2.$$

as we want tri in x.

$X'' + p^2 X = 0$

$\Rightarrow X(x) = \sin px$ with $pL = n\pi$.

0 at $x = 0$, 0 at $x = L$.

$$X(x) = \sin\left(\frac{n\pi x}{L}\right)$$

$Y'' - p^2 Y = 0 \Rightarrow Y(y) = \sinh(py)$

$= \mathrm{sh}(py) = \mathrm{sh}(n\pi y/L)$

(0 at $y=0$)

$$\phi(x,y) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi x}{L}\right) \mathrm{sh}\left[\frac{n\pi y}{L}\right]$$

and $\phi(x, L) = L_x = \sum_{n=1}^{\infty} A_n \sin\left[\frac{n\pi x}{L}\right] \mathrm{sh}\left[n\pi\right]$

requiring

$$\int_0^L L_x \sin\left[\frac{m\pi x}{L}\right] dx = A_m \mathrm{sh}\left[m\pi\right] \int_0^L \sin\left[\frac{m\pi x}{L}\right] dx.$$

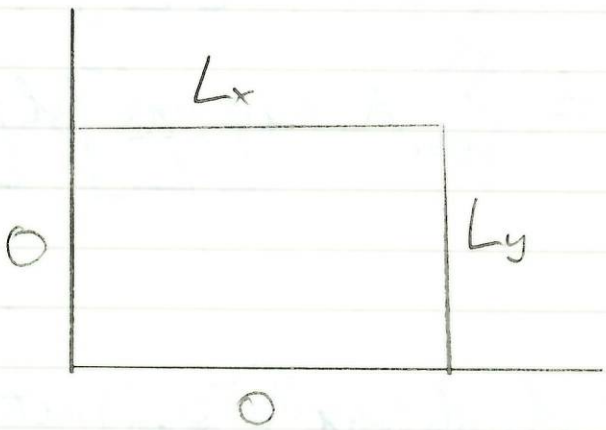
$$A_n = \frac{2}{L}\,\frac{1}{sh[n\pi]}\int_0^L x\sin\left[\frac{n\pi x}{L}\right] dx .$$

$$= \frac{2}{sh[n\pi]}\left\{\left[x\frac{L}{n\pi}(-1)\cos\left[\frac{n\pi x}{L}\right]\right]_0^L + \underbrace{\cancel{\int_0^L \frac{L}{n\pi}\cos\left[\frac{n\pi x}{L}\right] dx}}_{0.}\right\}$$

$$= \frac{-2}{sh[n\pi]}\,\frac{L^2}{n\pi}\cos\left[n\pi\right].$$

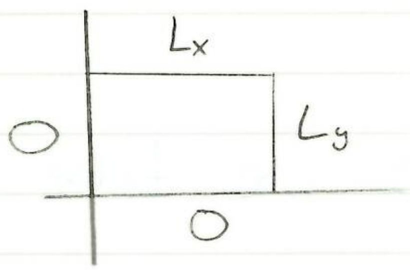
$$A_n = \frac{2(-1)^{n+1}}{n\pi sh[n\pi]}\,L^2 .$$

$$\phi(x,y) = \sum_{n=1}^{\infty} \frac{2L^2}{n\pi sh[n\pi]}(-1)^{n+1}\sin\left[\frac{n\pi x}{L}\right] sh\left[\frac{n\pi y}{L}\right]$$



The solution for this second problem is obtained by putting:

$$x \longrightarrow y$$
$$y \longrightarrow x .$$



has solutcon:

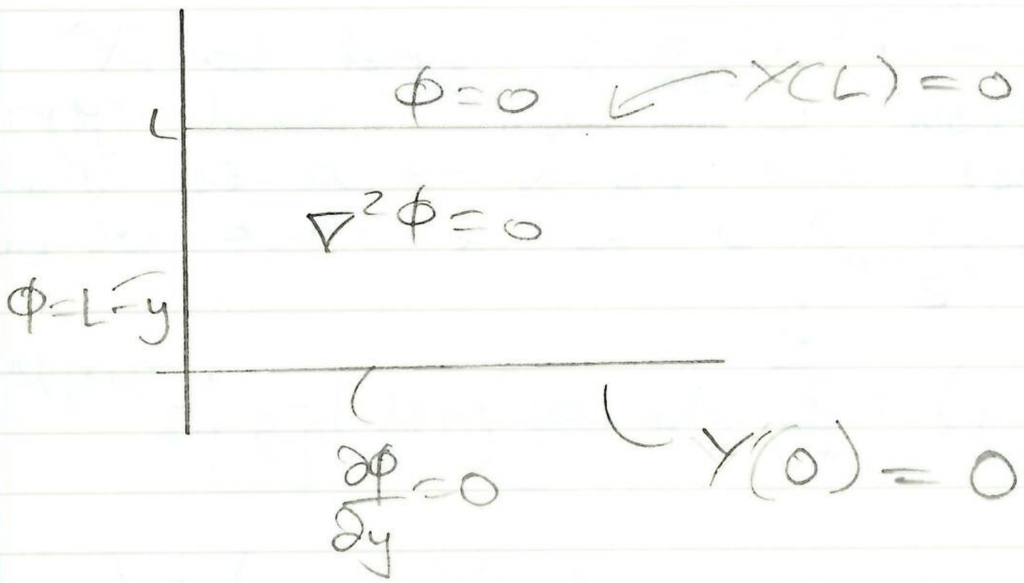
$$\phi(x,y) = \sum_{n=1}^{\infty} \frac{2L^2(-1)^{n+1}}{sh[n\pi]} \left\{ \sin\left[\frac{n\pi x}{L}\right] sh\left[\frac{nxy}{L}\right] + \sin\left[\frac{n\pi y}{L}\right] sh\left[\frac{n\pi x}{L}\right] \right\}$$

$$= xy.$$

The solutcon to this problem is in fact:

$\phi = xy$ $\left[\text{satisfy bc's and } \nabla^2\phi = 0\right]$.

<u>Slove</u>:



$$\phi(x,y) = X(x)\,Y(y)$$

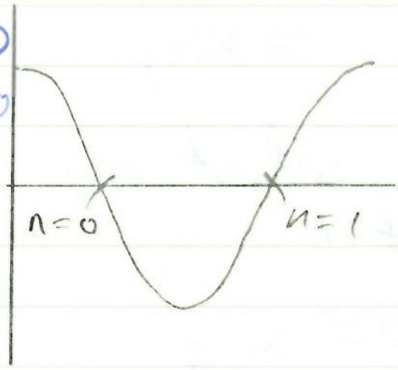
$$\frac{X''}{X} = -\frac{Y''}{Y} = \text{const} = -p^2.$$

trigonmetric in y.

To satisfy $Y'(0) = 0$ we take the solution

$$Y(y) = \cos(py) .$$

and satisfy $Y(L)=0$
we require $\cos(pL)=0$
requiring



$$pL = \left(n + \frac{1}{2}\right)\pi$$

$$p = \left(n + \frac{1}{2}\right)\frac{\pi}{L} .$$

$X'' - p^2 X = 0$ and so X has solution e^{-px}, e^{+px} and for $X(x) \to 0$ as $x \to \infty$ so that $\phi(x,y) \to 0$ as $x \to \infty$ we must take e^{-px}.

$$\phi(x,y) = \sum_{n=0}^{\infty} A_n \cos\left[\left(n+\frac{1}{2}\right)\frac{\pi y}{L}\right] e^{-\left(n+\frac{1}{2}\right)\frac{\pi x}{L}}$$

and we required $\phi(0, y) = L - y$.

$$L - y = \sum_{n=0}^{\infty} A_n \cos\left[\left(n+\frac{1}{2}\right)\frac{\pi y}{L}\right]$$

Using the fact that $\cos[(n+1/2)\pi y/L]$ and $\cos[(m+1/2)\pi y/L]$ are orthogonal we find:

$$\int_0^L (L-y)\cos\left[\left(m+\frac{1}{2}\right)\frac{\pi y}{L}\right] dy$$

$$= A_m L \frac{L}{2}.$$

— / —

[illegible].

xxx.