3103 Functional Analysis Notes

Based on the 2014 spring lectures by Prof A Sobolev

MATH3103 – Functional Analysis.

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Outline of course: ① set theory, ② metric spaces – compactness, completeness, ③ normed spaces – Banach spaces.

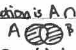
Chapter 1.
SET THEORY.

Sets: $A, B, C, \ldots$ $x \in A$ if element x belongs to A, $A \subset B$ if $\forall x \in A$ we have $x \in B$, moreover we say $A = B$ if $A \subset B$ and $B \subset A$.

Definition 1.1 (set operations)



Let A, B be sets, then the <u>intersection</u> is $A \cap B = \{x : x \in A \text{ and } x \in B\}$, the <u>union</u> is $A \cup B = \{x : x \in A \text{ or } x \in B\}$, the <u>difference</u> is $A \setminus B = \{x \in A : x \notin B\}$,



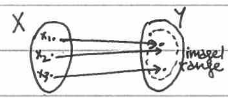
the <u>symmetric difference</u> is $A \Delta B = \{A \setminus B\} \cup \{B \setminus A\}$ and the <u>complement</u> of $A \subset E$ is $A^c = E \setminus A$

Union, intersection can be taken for any number of sets: $\bigcup_\alpha A_\alpha$, $\bigcap_\alpha B_\alpha$.

e.g. let $A \subset \mathbb{R}$ be $A = [0,1)$, then $A^c = (-\infty, 0) \cup [1, \infty)$.

Lemma 1.2 $(A \cup B) \cup C = A \cup (B \cup C)$, $(A \cap B) \cap C = A \cap (B \cap C)$. $(A \cup B) \cap C = (A \cap C) \cup (B \cap C)$, $(A \cap B) \cup C = (A \cup C) \cap (B \cup C)$.

Proof – left as exercise.

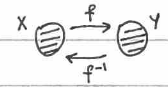


Functions: let X, Y be two sets. let $f : X \to Y$ be a function (a mapping). The function f is a rule which associates to each $x \in X$ a <u>uniquely defined</u> element $y = f(x) \in Y$.

Definition 1.3 The <u>image</u> (or <u>range</u>) of $f : X \to Y$ is the set $f(X) = \{y \in Y : \exists x \in X \text{ s.t. } f(x) = y\}$. The <u>inverse image</u> (or <u>preimage</u>) of the element $y \in Y$ is the set $f^{-1}(y) = \{x \in X : f(x) = y\}$.

If $y \notin f(X)$, $f^{-1}(y) = \emptyset$ (empty set).

Definition 1.4 A mapping is said to be an <u>injection</u> (one-to-one) if $f(x_1) = f(x_2) \Rightarrow x_1 = x_2$. In other words, if $y \in f(X)$, then $f^{-1}(y)$ consists of only one point.



f is said to be a <u>surjection</u> (surjective) if $f(X) = Y$. f is a <u>bijection</u> if it is simultaneously an injection and surjection.

the <u>inverse function</u> $f^{-1} : Y \to X$ is defined by $\begin{cases} f^{-1}(f(x)) = x & \forall x \in X \\ f(f^{-1}(y)) = y & \forall y \in Y. \end{cases}$

e.g. – if $f(x) = x^2$, ① $X = \mathbb{R}$, $Y = \mathbb{R}$, f is neither an injection or surjection. ② $X = \mathbb{R}$, $Y = \mathbb{R}_+ = [0, \infty)$ is surjection, not an injection. ③ $X = Y = \mathbb{R}_+$ is a bijection, $f^{-1}(y) = \sqrt{y}$.

Definition 1.5 Two sets X and Y are said to be <u>equivalent</u> if there exists a bijection $f : X \to Y$. In this case, X and Y have the same <u>cardinality</u>, which is denoted by $|X| = |Y|$.

A set equivalent to $\mathbb{N}$ is said to be <u>countable</u>, and its cardinality is expressed by the notation $|\mathbb{N}| = \aleph_0$.

Observations – (i) every infinite set has a countable subset, (ii) every subset of a countable set is either countable or finite.

Example 1.6 (1) If X is finite, then $|X|$ is the number of elements (by definition).

(2) $|\mathbb{Z}| = |\mathbb{N}|$. $g(m) = \begin{cases} 2m & m > 0 \\ -2m+1 & m \leq 0 \end{cases}$ is clearly a bijection from $\mathbb{Z}$ to $\mathbb{N}$

(3) $\mathbb{Q} = \{\frac{p}{q}, p \in \mathbb{Z}, q \in \mathbb{N}\}$. Write all rational numbers in the form $\frac{p}{q}$ with coprime p, q. Define the height of $\frac{p}{q}$ to be $h = |p| + q$. First count numbers of height 1, then 2, etc. This labels all rational numbers by naturals and thus, $|\mathbb{Q}| = |\mathbb{N}| = \aleph_0$.

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(4) $|(0,1)| = |\mathbb{R}| = \aleph$: <u>continuum</u>. Bijection: $g : \mathbb{R} \to (0,1)$, $g(t) = \frac{2}{\pi} \arctan t + \frac{1}{2}$.

Theorem 1.7 $\mathbb{R}$ is not countable. i.e. $\aleph \neq \aleph_0$

Proof – Suppose $\mathbb{R}$ is countable then so is $[0,1]$. Write each $a \in [0,1]$ as a decimal assuming that there is a bijection fraction. $f : \mathbb{N} \to [0,1]$: $a_1 = f(1) = 0.a_{11}a_{12}\cdots$, $a_n = f(n) = 0.a_{n1}a_{n2}\cdots$

Note that $1 = 0.999\ldots$ is also represented. Aim: to construct a real number $b \in [0,1]$ which is not listed: diagonalisation procedure. $b = 0.b_1b_2\cdots$ If $a_{11} = 1$, then let $b_1 = 2$. If $a_{11} \neq 1$, $b_1 = 1$. Likewise, in general, if $\begin{matrix} a_{nn} = 1, & b_n = 2 \\ a_{nn} \neq 1, & b_n = 1. \end{matrix}$ Thus $b \neq a_k$ for all $k = 1, 2, \ldots$. Thus f is not a bijection $\Rightarrow$ no countability // q.e.d.

Remark 1.8 let X be the set of all sequences of the type $a = (a_1, a_2, a_3, \ldots)$ s.t. $a_k = 0$ or $a_k = 1$ $\forall k = 1, 2, \ldots$, then X is not countable – prove it! (left as exercise)

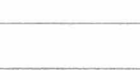
Definition 1.9 Let X, Y be sets. Assume that there is an injection $g : X \to Y$. Then we say that $|X| \leq |Y|$ (or $|Y| \geq |X|$).

If $|X| \leq |Y|$ and there is no bijection from X into Y, then $|X| < |Y|$ (or $|Y| > |X|$).

By Theorem 1.7, $\aleph_0 < \aleph$.

Theorem 1.10 (Cantor–Bernstein theorem).

If $|X| \leq |Y|$ and $|X| \geq |Y|$, then $|X| = |Y|$.



Example – Let $X = (0,1)$, $Y = [0,1]$. Show that $|X| = |Y| = \aleph$: observe that $X \subset Y$, and $Id : X \to Y$ is the identity map is an injection, so $|X| \leq |Y|$, $\aleph \leq |Y|$.

On the other hand, $Id: Y \to \mathbb{R}$ is an injection, so $|Y| \le |\mathbb{R}| = \aleph$, so $|Y| \le \aleph$. By Theorem 1.10, $|Y| = \aleph$.

Chapter 2.
METRIC SPACES.

Let X be a set. Then $X \times X$ is the product set of ordered pairs (x,y) where $x, y \in X$.

Definition 2.1 Let $\rho: X \times X \to \mathbb{R}$. The pair (X, ρ) is said to be a metric space if

[non-degeneracy] (1) $\rho(x,y) \ge 0$ and $\rho(x,y) = 0$ iff $x = y$, [symmetry] (2) $\rho(x,y) = \rho(y,x)$ [triangle inequality] (3) $\rho(x,z) \le \rho(x,y) + \rho(y,z)$ $\forall x,y,z \in X$.

In this case, ρ is called a metric. For any subset $M \subset X$, the pair (M, ρ) is called a metric subspace.

Observation – from (3), $\rho(x,z) - \rho(x,y) \le \rho(y,z)$. Moreover, $\rho(x,y) \le \rho(x,z) + \rho(z,y) \Rightarrow \rho(x,y) - \rho(x,z) \le \rho(y,z) \Rightarrow |\rho(x,y) - \rho(x,z)| \le \rho(y,z)$.

Therefore, $|\rho(x,y) - \rho(x_0,y_0)| \le g(x,x_0) + g(y,y_0)$.

Examples – ① Let X be a set. Define $\rho(x,y) = \begin{cases} 0 & x = y \\ 1 & x \ne y \end{cases}$. This is the space of isolated points.

② $X = \mathbb{R}$, $\rho(x,y) = |x-y|$ – All three properties of a metric are satisfied.

③ $X = \mathbb{R}^n = \underbrace{\mathbb{R} \times \dots \times \mathbb{R}}_{\text{n-fold products}}$ – set of all n-tuples of real numbers. If $x = (x_1, x_2, \dots, x_n)$, $\rho(x,y) = \sqrt{\sum_k |x_k - y_k|^2}$ is the Euclidean distance.

Lemma 2.2 Let $a_k, b_k \in \mathbb{R}$, $k = 1, \dots, n$. Then $\left(\sum_{k=1}^n a_k b_k\right)^2 = \sum_{k=1}^n a_k^2 \sum_{k=1}^n b_k^2 - \frac{1}{2}\sum_{k,j=1}^n (a_k b_j - a_j b_k)^2$. Such that $\left(\sum_{k=1}^n a_k b_k\right)^2 \le \sum_{k=1}^n a_k^2 \sum_{k=1}^n b_k^2$ (Cauchy-Schwartz inequality).

The Cauchy-Schwartz inequality implies triangle inequality for this metric.

④ Take $X = \mathbb{R}^n$, and define $\rho_p(x,y) = \left[\sum_k |x_k - y_k|^p\right]^{\frac{1}{p}}$, $p > 0$. Clearly, ρ_p is non-degenerate and symmetric. We however need to establish the triangle inequality.

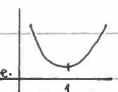
We will show that ρ_p defines a metric for all $p \in [1, \infty)$. Here, ρ_∞ is defined by $\rho_\infty(x,y) = \max_{1 \le k \le n} |x_k - y_k|$. If $p < 1$, this is not a metric space. We use notation $\mathbb{R}^n_p$.

We say that two numbers $p, q > 0$ are conjugate to each other (or form a conjugate pair) if $\frac{1}{p} + \frac{1}{q} = 1$. Note: this implies that $p, q \ge 1$. We also include $p = \infty$, $q = 1$ (or vice versa)

then $1 + \frac{p}{q} = p$, $1 + \frac{q}{p} = q$.

Lemma 2.3 (Young's inequality)

Let $a, b \ge 0$ and let p, q be a conjugate pair. Then $ab \le \frac{a^p}{p} + \frac{b^q}{q}$.



Proof – Define $h(t) = \frac{t^p}{p} + \frac{t^{-q}}{q}$, $t > 0$. By taking derivatives, $h(t)$ is decreasing on $(0,1]$, increasing on $[1, \infty)$. Therefore $h(t) \ge h(1) = 1$ since p, q conjugate.

Thus, $1 \le \frac{t^p}{p} + \frac{t^{-q}}{q}$ $\forall t > 0$. Take $t = a^{\frac{1}{q}} b^{-\frac{1}{p}}$, $a, b > 0$. Then $t > 0$, so $1 \le \frac{1}{p} a^{\frac{p}{q}} b^{-1} + \frac{1}{q} a^{-1} b^{\frac{q}{p}}$. Multiply by ab, $ab \le \frac{1}{p} a^{\frac{p}{q}+1} + \frac{1}{q} b^{\frac{q}{p}+1} = \frac{a^p}{p} + \frac{b^q}{q}$.

If $a = 0$ or $b = 0$, inequality is also trivially proved // q.e.d.

Theorem 2.4 (Hölder's inequality)

Let p, q be as before, and let $a = (a_1, a_2, \dots, a_n)$, $b = (b_1, \dots, b_n)$. Then $\sum_{k=1}^n |a_k b_k| \le \left(\sum_{k=1}^n |a_k|^p\right)^{\frac{1}{p}} \left(\sum_{k=1}^n |b_k|^q\right)^{\frac{1}{q}}$.

Proof – Assume each factor on right $\left(\sum_k^n |a_k|^p\right)^{\frac{1}{p}} = \left(\sum_k^n |b_k|^q\right)^{\frac{1}{q}} = 1$. Estimate LHS: $|a_k b_k| \le \frac{|a_k|^p}{p} + \frac{|b_k|^q}{q}$ by Lemma 2.3. Then summing over terms gives us

$\sum |a_k b_k| \le \frac{1}{p} \sum |a_k|^p + \frac{1}{q} \sum |b_k|^q = 1$. Then for general case, define new (scaled) vectors by $\tilde{a} = \lambda^{-1} a$, $\tilde{b} = \mu^{-1} b$ with $\lambda = \left(\sum_k |a_k|^p\right)^{\frac{1}{p}}$, $\mu = \left(\sum_k |b_k|^q\right)^{\frac{1}{q}}$. Then

clearly, $\left(\sum |\tilde{a}_k|^p\right)^{\frac{1}{p}} = \left(\sum |\tilde{b}_k|^q\right)^{\frac{1}{q}} = 1$ and inequality holds for these vectors. Therefore, $\sum |\tilde{a}_k \tilde{b}_k| \le 1 \Rightarrow \frac{1}{\lambda\mu} \sum_k |a_k b_k| \le 1 \Rightarrow \sum_k |a_k b_k| \le \lambda\mu$ // q.e.d.

Theorem 2.5 (Minkowski's inequality)

Let $p \in [1, \infty]$. Then $\left(\sum_{k=1}^n |a_k + b_k|^p\right)^{\frac{1}{p}} \le \left(\sum_{k=1}^n |a_k|^p\right)^{\frac{1}{p}} + \left(\sum_{k=1}^n |b_k|^p\right)^{\frac{1}{p}}$.

Proof – Estimate $\sum_k |a_k + b_k|^p \le \sum_k |a_k + b_k|^{p-1} |a_k| + \sum_k |a_k + b_k|^{p-1} |b_k|$. By Hölder's inequality, $\sum_k |a_k + b_k|^{p-1} |a_k| \le \left(\sum_k |a_k|^p\right)^{\frac{1}{p}} \left(\sum_k |a_k + b_k|^{(p-1)q}\right)^{\frac{1}{q}}$. Recall that $\frac{p-1}{p} = \frac{1}{q}$

so $(p-1)q = p$, then $\sum_k |a_k + b_k|^{p-1} |a_k| \le \left(\sum |a_k|^p\right)^{\frac{1}{p}} \left(\sum |a_k + b_k|^p\right)^{\frac{1}{q}}$. Also by Hölder's inequality, $\sum_k |a_k + b_k|^{p-1} |b_k| \le \left(\sum |b_k|^p\right)^{\frac{1}{p}} \left(\sum |a_k + b_k|^p\right)^{\frac{1}{q}}$

Thus, $\sum_k |a_k + b_k|^p \le \left(\sum_k |a_k + b_k|^p\right)^{\frac{1}{q}} \left[\left(\sum |a_k|^p\right)^{\frac{1}{p}} + \left(\sum |b_k|^p\right)^{\frac{1}{p}}\right]$. Divide by $\left(\sum_k |a_k + b_k|^p\right)^{\frac{1}{q}}$ to get solution // q.e.d.

⑤ Let $x = (x_1, x_2, \dots)$ be an infinite sequence of real (or complex) numbers. Let $\boxed{\ell_p,\ p \in [1, \infty] \text{ be the set of all sequences } x \text{ s.t. } \sum_{k=1}^\infty |x_k|^p < \infty}$.

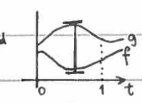
e.g. $x = \{x_k = \frac{1}{k}\}$. Then $x \in \ell_p$ with $p > 1$ but not with $p = 1$! Note – Hölder's, Minkowski's inequalities hold e.g. if $x \in \ell_p$, $y \in \ell_q$, $\frac{1}{p} + \frac{1}{q} = 1$, then Hölder's

inequality holds – $\sum_{k=1}^\infty |x_k y_k| \le \left(\sum_{k=1}^\infty |x_k|^p\right)^{\frac{1}{p}} \left(\sum_{k=1}^\infty |y_k|^q\right)^{\frac{1}{q}}$.

sup rather than max.

⑥ $C \subset \ell_\infty$ consists of all convergent sequences. $\rho_c(x,y) = \rho_\infty(x,y) = \sup_k |x_k - y_k|$. Then $C_0 \subset C$ is the subspace of sequences converging to 0.

since interval is compact, extrema are attained



⑦ (space of functions) Let $C[0,1]$ be the space of all continuous functions on the closed interval $[0,1]$ with the metric $\rho_C(f,g) = \max_{t \in [0,1]} |f(t) - g(t)|$

ℓ_p, $1 \le p \le \infty$. $C[0,1]$, $C[a,b]$.

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⑧ Take the set of all continuous functions on $[a,b]$ and define $\rho_p(f,g) = \left[\int_a^b |f(t) - g(t)|^p dt\right]^{\frac{1}{p}}$, $1 \le p \le \infty$. $\rho_\infty(f,g) = \sup_{t \in [a,b]} |f(t) - g(t)|$

Notation for this metric space: $C_p[a,b]$. Is this indeed a metric? Yes!. As for ℓ_p, we have $\left(\int_a^b |f+g|^p dt\right)^{\frac{1}{p}} \le \left(\int_a^b |f|^p dt\right)^{\frac{1}{p}} + \left(\int_a^b |g|^p dt\right)^{\frac{1}{p}}$ $\forall f, g \in C_p[a,b]$,

$\int_a^b |fg|\, dt \le \left(\int_a^b |f(t)|^p dt\right)^{\frac{1}{p}} \left(\int_a^b |g(t)|^q dt\right)^{\frac{1}{q}}$, $\frac{1}{p} + \frac{1}{q} = 1$.

Continuous functions on metric space: Let (X, ρ_X), (Y, ρ_Y) be two metric spaces.

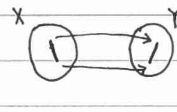
Definition 2.6 Let $f: X \to Y$ be a function. We say that f is continuous at $x_0 \in X$ if for any $\varepsilon > 0$, there is a $\delta > 0$ s.t. $\rho_Y(f(x), f(x_0)) < \varepsilon$ as soon as $\rho_X(x, x_0) < \delta$.

If f is continuous at all points $x \in X$, then we say that f is continuous on X.

e.g. – Look at $X \times X$ as a metric space with the metric $d((x,y),(w,z)) = \rho(x,w) + \rho(y,z)$ [check properties to establish that this is a metric].

Then the function $f: X \times X \to \mathbb{R}$ defined by $f(x,y) = \rho(x,y)$ is continuous. Indeed, recall that $|\rho(x,y) - \rho(x_0,y_0)| \leq \rho(x,x_0) + \rho(y,y_0)$.

Definition 2.7 If $f: X \to Y$ is a bijection, f is continuous and f^{-1} is continuous, then we say that f is a homeomorphism.



e.g. – The function $h: \mathbb{R} \to (-1,1)$ defined by $h(x) = \frac{2}{\pi} \arctan x$ is a homeomorphism.

Definition 2.8 We say that $f: X \to Y$ is an isometry if $\rho_Y(f(x), f(y)) = \rho_X(x,y)$. If an isometry is a bijection, then it is called an isomorphism.

e.g. – Let $X = \mathbb{R}^2$, $Y = \mathbb{R}$. Define the metric on X by $\rho_X((x_1,x_2),(y_1,y_2)) = |x_1 - y_1|$, and the metric on Y by $\rho_Y(z,w) = |z-w|$. Let $g: X \to Y$ be $g((x_1,x_2)) = x_1$.

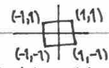
(projection of vector onto horizontal axis). This map is isometric.

Observations – Isometric $\Rightarrow$ continuous, isomorphic $\Rightarrow$ homeomorphic

Topology of Metric Spaces.

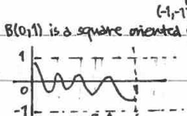
Definition 2.9 The set $B(x_0, r) = \{x \in X : \rho(x_0, x) < r\}$ is called the open ball centred at x_0 of radius $r > 0$. The set $B[x_0, r] = \{x \in X : \rho(x_0,x) \leq r\}$ is called the closed ball centred at x_0 of radius $r > 0$.

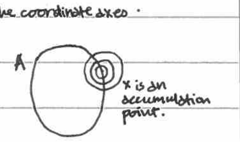
A set $M \subset X$ is said to be bounded if it is contained in some $B(x_0, r)$.



e.g. – In $\mathbb{R}^2_2$: $B(0,r)$ is a disc of radius r. [i.e. $B(0,1)$ is the round disc of radius 1]. In $\mathbb{R}^2_\infty$: $B(0,1)$ is a square oriented parallel to the coordinate axes.

In $C[0,1]$: $B(0,1)$ is the set of all functions ranging between -1 and 1.





Definition 2.10 We say that $x \in X$ is an accumulation point of the set $A \subset X$ if for any $\varepsilon > 0$, the ball $B(x, \varepsilon)$ contains infinitely many points of A.

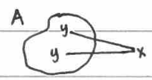
Alternatively, x is an accumulation point if for all $\varepsilon > 0$, the ball $B(x,\varepsilon)$ contains at least one point of A distinct from x.

The union of A and all its accumulation points is called the closure of A, denoted by $\bar{A} = [A]$. We say that $z \in A$ is an isolated point of A if there is a radius $r > 0$ s.t. $B(z,r) \cap A = \{z\}$.

Observation – Any point $x \in A$ is either an accumulation point or an isolated point of A.

Examples – ① $\overline{(0,1)} = [0,1]$ ② $B = [-1,1] \cup \{2\}$. Accumulation points: $[-1,1]$, isolated point: $\{2\}$, $[B] = [-1,1] \cup \{2\}$.

The set $B(x_0, \varepsilon)$ is also called an ε-neighbourhood of x_0. Notation: $B(x_0, \varepsilon) = O_\varepsilon(x_0)$.

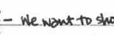


Theorem 2.11 Let $A \subset X$. Then $[A] = \{x \in X : \text{dist}(A,x) = 0\}$ where $\text{dist}(A,x) = \inf_{y \in A} \rho(x,y)$.

Proof – Denote $B = \{x \in X : \text{dist}(A,x) = 0\}$. By Definition 2.10, $[A] \subset B$. For the other inclusion, let $x \in B$. Then either $x \in A$ or $x \notin A$ and $\inf_{y \in A} \rho(x,y) = 0$. Suppose $x \notin A$,

then for any $\varepsilon > 0$, there is a point $y \in A$ s.t. $\rho(x,y) < \varepsilon$. Again by Definition 2.10, $x \in [A]$, so $B \subset [A]$. Thus $B = [A]$ // q.e.d.

Remark – $\text{dist}(A,x) = \text{dist}([A],x)$.

Examples – ①: $(0,1) \subset \mathbb{R}$. Closure of $(0,1)$ is $[0,1]$.  Then $\underset{\text{infimum}}{\text{dist}}((0,1), 0) = 0$; $\text{dist}([0,1], 0) = 0$. (0 is in the set.)

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② $[\mathbb{Q}] = \mathbb{R}$ ③ Let $c_{00} \subset \ell_p$ be the set of all finite sequences (recall ℓ_p is the set of all sequences s.t. $\sum_{k=1}^{\infty} |x_k|^p < \infty$).

Let $p \in [1, \infty)$. Claim: $[c_{00}] = \ell_p$. Proof – We want to show that every $x \in \ell_p$ is an accumulation point of c_{00}, i.e. $\forall \varepsilon > 0$, $\exists \tilde{x} \in c_{00}$ s.t. $\rho_p(x, \tilde{x}) < \varepsilon$.

Since $x = (x_1, x_2, \dots) \in \ell_p$, there is a number N_ε s.t. $\sum_{k=N+1}^{\infty} |x_k|^p < \varepsilon^p$ by definition of convergence. Let $\tilde{x} = (x_1, x_2, \dots, x_N, 0, 0, \dots) \in c_{00}$ be a finite sequence.

$\rho_p(x, \tilde{x}) = \left[\sum_{k=N+1}^{\infty} |x_k|^p\right]^{1/p} < (\varepsilon^p)^{1/p} = \varepsilon$ // q.e.d.

④ Consider c_{00} as a subset of $\ell_\infty = \{x = (x_1, x_2, \dots) : \sup_k |x_k| < \infty\}$. Claim: $[c_{00}] = c_0$ (all sequences convergent to 0, i.e. $x_k \to 0$ as $k \to \infty$).

Proof – We will show that $[c_{00}] \subset c_0$ i.e. all points of c_{00} belong to c_0. Let $x \in \ell_\infty$ be an accumulation point of c_{00}, i.e. $\forall \varepsilon$, $\exists \tilde{x} = (\tilde{x}_1, \tilde{x}_2, \dots, \tilde{x}_N, 0, 0, \dots) \in c_{00}$ s.t. $\rho_\infty(x, \tilde{x}) = \sup_k |x_k - \tilde{x}_k| < \varepsilon$. Therefore, $\sup_{k \geq N+1} |x_k| = \sup_{k \geq N+1} |x_k - \tilde{x}_k| \leq \sup_k |x_k - \tilde{x}_k| = \rho_\infty(x, \tilde{x}) < \varepsilon$. Thus, $x_k \to 0$ as $k \to \infty$, so $[c_{00}] \subset c_0$.

We just need to show that $c_0 \subset [c_{00}]$ i.e. every element of c_0 is an accumulation point.

Theorem 2.12 (1) $A \subset [A]$ (2) If $A_1 \subset A_2$, then $[A_1] \subset [A_2]$. (3) $[[A]] = [A]$. (4) $[A_1 \cup A_2] = [A_1] \cup [A_2]$.

Proof – (1) and (2) are trivial/straightforward. (3): $\text{dist}([A], x) = \text{dist}(A, x)$ $\forall x \in X$. Then $[[A]] \overset{(\text{Thm 2.11})}{=} \{x \in X : \text{dist}([A],x) = 0\} = \{x \in X : \text{dist}(A,x) = 0\} = [A]$ // q.e.d.

(4) left as exercise.

Question: $[A_1 \cap A_2] = [A_1] \cap [A_2]$? No! Indeed, let $A_1 = (0,1)$, $A_2 = (1,2)$. Then $A_1 \cap A_2 = \phi$. However $[A_1] \cap [A_2] = [0,1] \cap [1,2] = \{1\} \neq [\phi]$.

Only relation we have is $[A_1 \cap A_2] \subset [A_1] \cap [A_2]$ (can be proven).

Convergence

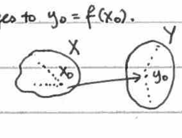
$\boxed{\text{Definition}}$ 2.13 Let x_k, $k=1,2,\dots$ be a sequence in a metric space (X,ρ). We say that x_k converges to $x\in X$ as $k\to\infty$ i.e. $x=\lim_{k\to\infty} x_k$, or $x_k\to x$ as $k\to\infty$, if $\rho(x_k,x)\to 0$ as $k\to\infty$. In other words, for any $\varepsilon>0$, there is a number $N=N_\varepsilon$ s.t. $\rho(x_k,x)<\varepsilon$ as soon as $k>N$.

$\boxed{\text{Theorem}}$ 2.14 The point $x\in X$ is an accumulation point of the set $A\subset X$ $\Leftrightarrow$ there is a sequence $x_k\in A$ s.t. $\lim_{k\to\infty} x_k = x$, and $x_k\neq x$.

($\Rightarrow$)
Proof - Suppose $x\in X$ is an accumulation point. Let $\varepsilon_n=\frac{1}{n}$ be the sequence of ε terms, $n=1,2,\dots$. By definition, for ε_1, $\exists$ element $x_1\in A$, $x_1\neq x$ and $\rho(x_1,x)<\varepsilon_1=1$. Repeat for ε_2 etc. For ε_n, $\exists$ a point $x_n\in A$ s.t. $x_n\neq x$ and $\rho(x_n,x)<\varepsilon_n=\frac{1}{n}$. Let $n\to\infty$, $\frac{1}{n}\to 0$. Thus we have constructed a sequence where we have $\rho(x_n,x)\to 0$ as $n\to\infty$; this proves claim.

($\Leftarrow$) Let x_k be a sequence s.t. $x_k\neq x$ and $x_k\to x$, $k\to\infty$. This means that $\forall\varepsilon>0$, $\exists N$ s.t. $\rho(x_k,x)<\varepsilon$, $\forall k>N$. In other words, $x_k\in O_\varepsilon(x)$ for all $k>N$. By definition of accumulation points, x is an accumulation point.

$\boxed{\text{Theorem}}$ 2.15 Let $f:X\to Y$ be a function. Then f is continuous at $x_0\in X$ iff for any sequence $x_k\in X$ convergent to x_0, the sequence $f(x_k)\in Y$ converges to $y_0=f(x_0)$.

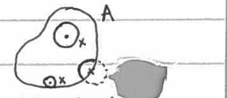


Remark - This can also be viewed as the sequential definition of continuity.

Proof - Simple, omitted.

Open and closed sets.

$\boxed{\text{Definition}}$ 2.16 We say that $x\in A$ is an interior point of A if there is an $\varepsilon>0$ s.t. $B(x,\varepsilon)\subset A$. The set of all interior points is denoted by $\mathring{A}$.



Remark - clearly, $\mathring{A}\subset A$.

$\boxed{\text{Definition}}$ 2.17 We say that A is open if $\mathring{A}=A$. A is closed if $A=[A]$.

Note - A is open iff $\mathrm{dist}(A^c,x)>0$ for all $x\in A$ [recall that $A^c=X\setminus A$].

$\boxed{\text{Theorem}}$ 2.18 The union of any number of open sets is open. The intersection of finite number of open sets is open. [Proof omitted].

e.g. - Let $A_k=(-\frac{1}{k},\frac{1}{k})$, $k=1,2,\dots$ Then $\bigcup_{k=1}^{\infty} A_k=(-1,1)$ open. $\bigcap_{k=1}^{\infty} A_k=\{0\}$ which is not open.

$\boxed{\text{Theorem}}$ 2.19 The set $A\subset X$ is open $\Leftrightarrow$ $A^c=X\setminus A$ is closed.

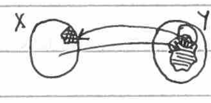
($\Rightarrow$)
Proof - Let A be open. NTP: $[A^c]=A^c$. i.e. $A^c=\{x:\mathrm{dist}(A^c,x)=0\}$. Suppose there is an element $x\notin A^c$, but $x\in[A^c]$ i.e. $\mathrm{dist}(A^c,x)=0$. Thus, $x\in A$, and $\mathrm{dist}(A^c,x)=0$. However, A is open, so $\mathrm{dist}(A^c,x)>0$ $\Rightarrow$ contradiction $\Rightarrow$ claim proven // q.e.d.

($\Leftarrow$)
Suppose $A^c=X\setminus A$ is closed, i.e. $A^c=[A^c]=\{x:\mathrm{dist}(A^c,x)=0\}$. Thus if $z\notin A^c$, then $\mathrm{dist}(A^c,z)>0$. $\Rightarrow$ $z\in A$ implies $\mathrm{dist}(A^c,z)>0$ $\Rightarrow$ A is open // q.e.d

$\boxed{\text{Proposition}}$ 2.20 Any open set A of the real line $\mathbb{R}$ is the union of (or finitely) countably many open intervals, i.e. $A=\bigcup_{k=1}^{\infty} A_k$. Where $A_k=(a_k,b_k)$ with $-\infty\le a_k<b_k\le\infty$.

Note - Sets $(-\infty,b)$, (a,∞) are considered open intervals.

Exercise - Show that open ball $B(x_0,r)$ is open, closed ball $B[x_0,r]$ is closed. $(r>0)$.



$\boxed{\text{Theorem}}$ 2.21 A function $f:X\to Y$ is continuous on X $\Leftrightarrow$ for any open set $A\subset Y$, the preimage $f^{-1}(A)$ is open. Proof - omitted.

Let $A\subset X$, then A is closed since $[A]=[[A]]$. We know that $A\subset[A]$. Let $B=[B]$ be a closed set s.t. $A\subset B$. What can we conclude about the relationship?

$\boxed{\text{Theorem}}$ 2.22 $[A]$ is the smallest closed set containing A, i.e. in above notation, $[A]\subset B$.

Example - Take $M=(0,1)\cap\mathbb{Q}$. Accumulation points are $[0,1]$, closure of M is $[0,1]$. Interior points of M are $\mathring{M}=\phi$, interior points of $[M]$ are $(0,1)$.

27 January 2014
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Maths 706.

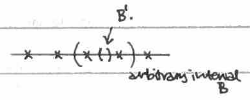
Dense sets, separability.

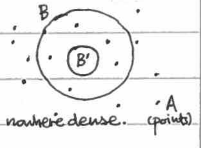
$\boxed{\text{Definition}}$ 2.23 Let $A,B\subset X$. We say that A is dense in B if $B\subset[A]$. The set A is everywhere dense (i.e. A is dense in X). if $X=[A]$. The set A is said to be nowhere dense if for any open ball $B\subset X$, $B\not\subset[A]$.

e.g. - The set $M=(0,1)\cap\mathbb{Q}$ is dense in $[0,1]$. Is M dense in $(0,\frac{1}{2})$? Yes, because $(0,\frac{1}{2})\subset[M]=[0,1]$. M is not dense in $(-\frac{1}{2},\frac{1}{2})$.

In other words, for any open ball B, there is another open ball $B'\subset B$ s.t. $A\cap B'=\phi$ (nowhere dense).

e.g. - For $\mathbb{R}=X$, the set $A=\mathbb{N}$ is nowhere dense. $[A]=\mathbb{N}$





The metric space X is called separable if it has a dense countable subset. (If A is everywhere dense and countable, then X is separable).

e.g. - ① $M=(0,1)\cap\mathbb{Q}$, $X=[0,1]$ with standard metric. We know that $[M]=X$, and M is countable as a subset of $\mathbb{Q}$. Therefore X is separable.

② $\mathbb{R}$ is separable, since $[\mathbb{Q}]=\mathbb{R}$. $\mathbb{R}^n$ is separable, since $[\mathbb{Q}^n]=\mathbb{R}^n$.

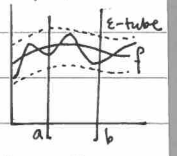
③ Consider ℓ_p, $p\in[1,\infty)$. ℓ_p is separable - Indeed we know that $[c_{00}]=\ell_p$. This means that $\forall x\in\ell_p$ and any $\varepsilon>0$, $\exists\tilde{x}\in c_{00}$ s.t. $\rho_p(x,\tilde{x})<\frac{\varepsilon}{2}$. Furthermore, there is a sequence $y\in c_{00}$ with rational components s.t. $\rho_p(\tilde{x},y)<\frac{\varepsilon}{2}$. By the triangle inequality, $\rho_p(x,y)\le\rho_p(x,\tilde{x})+\rho_p(\tilde{x},y)<\frac{\varepsilon}{2}+\frac{\varepsilon}{2}=\varepsilon$. $\Rightarrow$ accumulation point of sequences with rational components $\Rightarrow$ countable, so ℓ_p is separable.

④ ℓ_∞ is not separable. Indeed, let $M\subset\ell_\infty$ be the set which consists of all sequences of the form $a=(a_1,a_2,\dots)$ where $a_n=1$ or 0. Then M is not countable. In ℓ_∞, $\mathrm{dist}(x,y)=\sup_n|x_n-y_n|=1$ if x,y are distinct $\Rightarrow$ distinct elements are separated by a distance 1. Suppose that A is dense in M, then $\forall a\in M$, there is at least 1 element $y\in A$ s.t. $\rho_\infty(a,y)<1$. (neighbourhood of arbitrary size, we pick 1). Therefore, as there are uncountably many balls, A is uncountable.

103-04.

⑤ Let $X = C[a,b]$, with distance $\rho(f,g) = \max_{t\in[a,b]} |f(t)-g(t)|$. By Stone-Weierstrass theorem, for any $f \in X$ and $\varepsilon>0$, there exists a polynomial p s.t. $\rho(f,p) < \frac{\varepsilon}{2}$.

Moreover, there is a polynomial $\tilde{p}$ with rational coefficients s.t. $\rho(p,\tilde{p}) < \frac{\varepsilon}{2}$. By triangle inequality $\rho(f,\tilde{p}) \le \rho(f,p) + \rho(p,\tilde{p}) < \varepsilon$.



Consider $M = \mathbb{Q} \cap (0,1) \subset \mathbb{R}$. This is neither open nor closed. Is M dense in $\mathbb{R}$? $[M] = [0,1]$, $\mathbb{R} \not\subset [M]$ so it is not dense in $\mathbb{R}$.

This does not mean that M is nowhere dense: it is in fact dense in some intervals e.g. $(0,\frac{1}{2})$.

30 January 2014
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Complete Metric spaces & completions.

Compare $\mathbb{Q}$ and $\mathbb{R}$. We know $[\mathbb{Q}] = \mathbb{R}$. $\mathbb{Q}$ has "gaps", whereas we see $\mathbb{R}$ as the entire real line. Then compare $[0,1]$ and $(0,1)$. Take $x_n = \frac{1}{n}$, $n=1,2,\dots$. Then $x_n \to 0 \in [0,1]$ but $0 \notin (0,1)$.

$\boxed{\text{Definition}}$ 2.24 A sequence $x_j \in X$ is said to be Cauchy if $\forall \varepsilon>0$, $\exists N = N(\varepsilon)$ s.t. $\rho(x_n,x_m) < \varepsilon$ as soon as $m,n > N$.

Remark – there is no mention here of the concept of a "limit".

$\boxed{\text{Theorem}}$ 2.25 If x_j, $j=1,2,\dots$ is a convergent sequence, then it is Cauchy.

Proof – left as exercise.

Note – in general, the converse does not hold. For instance, $x_n = \frac{1}{n}$, $n=1,2,\dots$ is Cauchy but not convergent in the space $(0,1)$ as its limit is not contained.

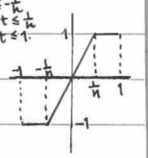
$\boxed{\text{Definition}}$ 2.26 The metric space (X,ρ) is said to be complete if every Cauchy sequence has a limit in X.

Examples –

① $X = (0,1)$ is not complete. $X = [0,1]$ is complete.

② $C[0,1]$ is complete: recall that $\rho(f,g) = \max_{t\in[0,1]} |f(t)-g(t)|$. If $\rho(f_n,f_m) \to 0$ as $n,m\to\infty$, $\{f_n\}$ is Cauchy. Then f_n has a limit as $n\to\infty$, which is also a continuous function $\Rightarrow$ $C[0,1]$ complete.

③ $C_p[-1,1]$, $p \in [1,\infty)$ is incomplete. Here, $\rho_p(f,g) = \left[\int_{-1}^{1} |f(t)-g(t)|^p\,dt\right]^{1/p}$. We construct a specific Cauchy sequence and aim to show it is incomplete. Define $f_n = \begin{cases} -1 & -1 \le t \le -\frac{1}{n} \\ nt & -\frac{1}{n} \le t \le \frac{1}{n} \\ 1 & \frac{1}{n} \le t \le 1 \end{cases}$



Then $\rho_p(f_n,f_m) = \left[\int_{-1}^{1} |f_n(t)-f_m(t)|^p\,dt\right]^{1/p} \overset{\substack{\text{WLOG}\\ n\ge m}}{=} \left[\int_{-1/n}^{1/n} |f_n(t)-f_m(t)|^p\,dt\right]^{1/p} \le 2\left[\int_{-1/n}^{1/n} dt\right]^{1/p} = 2\left(\frac{2}{n}\right)^{\frac{1}{p}} \to 0$ as $n\to\infty$. Thus, $\{f_n\}$ is a Cauchy sequence.

If $f(t) = \begin{cases} -1, & -1 \le t \le 0 \\ 1, & 0 < t \le 1 \end{cases}$ (Heaviside step function), then clearly $\rho_p(f_n,f) \to 0$ as $n\to\infty$. However, f is not continuous, so $f \notin C_p[-1,1]$, thus $C_p[-1,1]$ is not complete.

Note – Just by changing a metric, we have made the space incomplete.

④ ℓ_p, $p \in [1,\infty]$ is complete. $\rho_p(x,y) = \left[\sum_{k=1}^{\infty} |x_k - y_k|^p\right]^{1/p}$, where $x = (x_1, x_2, \dots)$, $y = (y_1, y_2, \dots)$. Let $x^{(n)} = (x_1^{(n)}, x_2^{(n)}, \dots)$ be a Cauchy sequence in ℓ_p (sequence of sequences) i.e. we have $\forall \varepsilon>0$, $\exists N = N(\varepsilon)$ s.t. $\sum_{k=1}^{\infty} |x_k^{(n)} - x_k^{(m)}|^p < \varepsilon^p$ for $n,m>N$ — ⊛. This means that $x_k^{(n)}$ is Cauchy for each k (each term in sum $<\varepsilon$) $\Rightarrow$ $x_k^{(n)}$ has a limit as $n\to\infty$, denoted x_k.

Then define $x = (x_1, x_2, \dots)$. We check that $x \in \ell_p$. Consider ⊛ for a finite sum of terms: $\sum_{k=1}^{M} |x_k^{(n)} - x_k^{(m)}|^p < \varepsilon^p$ with arbitrary $M \in \mathbb{N}$, clearly. — (⊛⊛). So now we pass to limit as $M\to\infty$.

$\sum_{k=1}^{M} |x_k^{(n)} - x_k|^p < \varepsilon^p$. By Minkowski (Theorem 2.5), $\left[\sum_{k=1}^{M} |x_k|^p\right]^{1/p} \le \underbrace{\left[\sum_{k=1}^{M} |x_k^{(n)}|^p\right]^{\frac{1}{p}}}_{\text{in } \ell_p} + \left[\sum_{k=1}^{M} |x_k^{(n)} - x_k|^p\right]^{\frac{1}{p}} \le \underbrace{\left[\sum_{k=0}^{\infty} |x_k^{(n)}|^p\right]^{\frac{1}{p}}}_{\text{finite}} + \varepsilon \Rightarrow \sum_{n=1}^{\infty} |x_k|^p < \infty \Rightarrow x \in \ell_p$. Thus it follows from (⊛⊛) that $\rho_p(x^{(n)},x) = \left[\sum_{k=1}^{\infty} |x_k^{(n)} - x_k|^p\right]^{1/p} < \varepsilon$. Therefore, for any $\varepsilon>0$, we found $N = N(\varepsilon)$ s.t. $\rho(x^{(n)},x) < \varepsilon$ as soon as $n>N$. So $\lim_{n\to\infty}(x^{(n)}) = x$ in ℓ_p $\Rightarrow$ ℓ_p is complete.

$\boxed{\text{Theorem}}$ 2.27 Let (X,ρ) be a complete metric space, and let $A \subset X$. Then the subspace (A,ρ) is complete $\Leftrightarrow$ A is closed (i.e. $A = [A]$).

Proof – left as exercise (just compare definitions).

$\boxed{\text{Lemma}}$ 2.28 Let X be a complete metric space. Let $K_j = B[x_j, r_j]$ with $x_j \in X$, $r_j > 0$, $j=1,2,\dots$ Suppose that $r_j \to 0$ as $j\to\infty$ and that K_j is a nested sequence of closed balls (i.e. $K_1 \supset K_2 \supset K_3 \supset \dots$).

Then the set $K = \bigcap_{j=1}^{\infty} K_j$ is not empty.

Proof – Consider the sequence x_j, $j=1,2,\dots$ This is a Cauchy sequence. Indeed, for any $m \ge n$, we have $x_m \in B[x_n, r_n]$ s.t. $\rho(x_n,x_m) < r_n \to 0$ as $n\to\infty$. As X is complete, $\{x_n\}$ has a limit.

Let $x = \lim_{n\to\infty} x_n$ be this limit. Observe that $x_m \in K_m \subset K_n$ $\forall m \ge n$. Thus, $\lim_{m\to\infty} x_m \in K_n$ as K_n is closed $\Rightarrow$ $x \in K_n$ $\forall n$ $\Rightarrow$ $x \in \bigcap_{j=1}^{\infty} K_j$ as required // q.e.d.

Remarks – The converse is true: If for any nested sequence of closed balls $B[x_j, r_j]$ as $r_j \to 0$, $j\to\infty$, the intersection is non-empty $\Rightarrow$ X complete. (important, but proof of this not assessed).

Moreover from the earlier proof, x is the unique limit for any such sequence, so $\bigcap_{j=1}^{\infty} K_j$ has exactly only one element.

$\boxed{\text{Theorem}}$ 2.29 (Baire Category Theorem).

A complete metric space cannot be a countable union of nowhere dense sets.

Proof – By contradiction. Suppose we can write $X = \bigcup_{k=1}^{\infty} A_k$ with A_k nowhere dense $\forall k \in \mathbb{N}$ (since k runs over $\mathbb{N}$, this is a countable union). Then let K_0 be a closed ball of radius 1. Since A_1 is nowhere dense, $\exists$ a closed ball $K_1 \subset K_0$ s.t. $K_1 \cap A_1 = \phi$. WLOG, we can choose to assume that its radius is $< \frac{1}{2}$. Since A_2 is nowhere dense, $\exists K_2 \subset K_1$ s.t. $K_2 \cap A_2 = \phi$.

Then WLOG assume radius of K_2 is $< \frac{1}{3}$. Continuing this way, we construct a nested sequence of closed balls K_j, $j=1,2,\dots$ s.t. radius of K_j is less than $\frac{1}{j+1}$ and $K_j \cap A_j = \phi$ $\forall j$.

$\therefore \bigcap_{j=1}^{\infty} (K_j \cap A_n) = \phi$ $\forall n$. Therefore $\left(\bigcap_{j=1}^{\infty} K_j\right) \cap \left(\bigcup_n A_n\right) = \phi$. From Lemma 2.28, $\bigcap_{j=1}^{\infty} K_j$ is non-empty, so $\exists x \in \bigcap_{j=1}^{\infty} K_j \Rightarrow x \notin \bigcup_n A_n$ for statement to hold. By definition $X = \bigcup_{n=1}^{\infty} A_n$, so $x \notin X$ which is a contradiction as space is complete. Thus, the assumption is false // q.e.d.

$\boxed{\text{Corollary}}$ 2.30 A complete metric space without isolated points cannot be countable.

Indeed, if $X = \bigcup_{n=1}^{\infty} \{x_n\}$, then $\{x_n\}$ is nowhere dense. By Baire Category theorem, this is impossible.

Completion of Metric Spaces.

$\boxed{\text{Definition}}$ 2.31 A complete metric space $\tilde{X}$ is said to be a completion of the metric space X if

(1) There is an isometry $\varphi: X \to \tilde{X}$ (i.e. distances are preserved) (2) $\varphi(X)$ is dense in $\tilde{X}$.

$\boxed{\text{Theorem}}$ 2.32 Any metric space has a completion. This completion is unique up to an isomorphism (i.e. if $\tilde{X}, \tilde{\tilde{X}}$ are completions, $\exists$ an isomorphism $\psi: \tilde{X} \to \tilde{\tilde{X}}$).

Proof (incomplete) – let X be a metric space. We seek to construct $\tilde{X}$.

- Let $\{x_n\}, \{y_n\}$ be Cauchy sequences in X. We call them equivalent if $\rho(x_n, y_n) \to 0$ as $n \to \infty$ (i.e. reflexive, symmetric, transitive :- follows from triangle inequality). Consider equivalence classes $\tilde{x}$ which consist of equivalent sequences (i.e. $\tilde{x} \sim x$). Define $\tilde{X}$ as the set of all equivalence classes. Metric: for $\tilde{x}, \tilde{y} \in X$ define $\tilde{\rho}(\tilde{x}, \tilde{y}) = \lim_{k\to\infty} \rho(x_k, y_k)$, where we have $\{x_k\} \in \tilde{x}$, $\{y_k\} \in \tilde{y}$. This limit exists; indeed $|\rho(x_k, y_k) - \rho(x_m, y_m)| \le \rho(x_k, x_m) + \rho(y_k, y_m)$ [from earlier remarks on Δ ineq] $\longrightarrow 0$ as $k, m \to \infty$ since $\{x_k\}, \{y_n\}$ are Cauchy.
- The definition of $\tilde{\rho}$ is independent of the choice of representatives $\{x_n\} \in \tilde{x}$, $\{y_n\} \in \tilde{y}$. Let $\{x_n'\} \in \tilde{x}$, $\{y_n'\} \in \tilde{y}$. Then $|\rho(x_n', y_n') - \rho(x_n, y_n)| \le \rho(x_n', x_n) + \rho(y_n', y_n) \to 0$ as $n \to \infty$, since $\{x_n\}, \{x_n'\} \in \tilde{x}$ and $\{y_n\}, \{y_n'\} \in \tilde{y}$ (i.e. same classes).

3 February 2014
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Example – Recall that $C_p[0,1]$, $1 \le p < \infty$ is incomplete, with the metric $\rho(f,g) = \left[\int_0^1 |f(t) - g(t)|^p \, dt\right]^{\frac{1}{p}}$; the completion of $C_p[0,1]$ is defined to be the space $L_p[0,1]$.

Contraction Mapping Theorem

$\boxed{\text{Definition}}$ 2.33 A function $A: X \to X$ is said to be a contraction mapping if there is a number $\alpha \in [0,1)$ s.t. for any $x, y \in X$ one has $\rho(A(x), A(y)) \le \alpha \rho(x,y)$.

The point $x \in X$ s.t. $A(x) = x$ is known as a fixed point.

(Banach Fixed Point Theorem)
$\boxed{\text{Theorem}}$ 2.34 If A is a contraction in a complete metric space X, then there exists a unique fixed point of A: i.e. $\exists$ an x_0 s.t. $A(x_0) = x_0$. [Remark – Here on, denote $Ax = A(x)$.]

Observation – $\rho(Ax_n, Ay) \to 0$ if $\rho(x_n, y) \to 0$ as $n \to \infty$. Moreover, $|\rho(Ax_n, z) - \rho(Ay, z)| \le \rho(Ax_n, Ay) \to 0$ as $n \to \infty$.

$\boxed{\text{Remark}}$ 2.35 – For any $x \in X$, we have $A^n x \to x_0$ as $n \to \infty$.

$\boxed{\text{Corollary}}$ 2.36 Suppose that for some t, the map $B = A^k$ is a contraction. Then x_0 is a fixed point of $A \iff x_0$ is a fixed point of B.

Proof – ($\Rightarrow$) Obviously $Bx_0 = A^k x_0 = A^{k-1}(Ax_0) \overset{\text{fixed pt of } A}{=} A^{k-1} x_0 = \dots = Ax_0 = x_0$, so x_0 is a fixed point of B.

($\Leftarrow$) Suppose $Bx_0 = x_0$, then $Ax_0 = AB^L x_0$ ($\forall L$) $= B^L(Ax_0)$ by commutativity (since $B = A^k$) $\to x_0$ by Remark 2.35. Hence x_0 is a fixed point of A // q.e.d.

Examples – (1) Picard's Theorem.

(2) Integral equations: Let $X = C[a,b]$. Let $K(x,y)$ be a continuous function on $[a,b] \times [a,b]$. Define $Kf(x) = \int_a^b K(x,y) + f(y) \, dy$ $\forall f \in X$. This is an integral operator with the kernel $K(x,y)$.

We want to solve $f = \varphi + \lambda Kf = Af$ where $\lambda \in \mathbb{C}$ is a number and $\varphi \in X$ is fixed [analogous to constant of integration]. Is this a contraction? $Af - Ag = \lambda K(f-g)$. We need $\rho(Af, Ag) \le \alpha \rho(f,g)$ for $\alpha \in [0,1)$. Thus, if $M = \max_{x,y} |K(x,y)|$, $\max_{t\in[a,b]} |\lambda K(f-g)(t)| = |\lambda| \max_{t \in [a,b]} \left| \int_a^b K(t,y) \, (f(y) - g(y)) \, dy \right| \le |\lambda| M \int_a^b |f(y) - g(y)| \, dy \le \overbrace{|\lambda| M (b-a)}^{\alpha} \rho(f,g)$. Thus, if $\alpha = |\lambda| M(b-a) < 1$, we have a unique soln by Fixed Point theorem.

(3) Define $T: C[a,b] \to C[a,b]$ by $Tf(x) = \int_a^x K(x,y) f(y) \, dy$, $x \in [a,b]$. This is the Volterra (integral) operator. We want to solve $f = \varphi + Tf$ for fixed points.

If k is sufficiently large, then T^k is a contraction (left as exercise). Use Corollary 2.36 to show that it has a unique fixed point: $\tilde{T}^2(f) = T(Tf) = \varphi + T\varphi + T^2\varphi$, $T^k(f) = \varphi + \dots + T^k\varphi$ (iterated equation).

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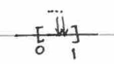
Compactness

$\boxed{\text{Definition}}$ 2.37 Let (X, ρ) be a metric space. Then a set $K \subset X$ is said to be relatively compact if any sequence in K contains a convergent subsequence.

If all possible limits of these subsequences belong to K, then we say that K is compact. A set that is both relatively compact and closed is compact.

Let K be infinite. It is relatively compact if any infinite subset of K has an accumulation point.



$\boxed{\text{Example}}$ 2.38 (Bolzano-Weierstraß theorem). Take closed interval $X = [0,1]$. Let $\{x_n\} \subset X$ be a sequence, then $\{x_n\}$ contains a convergent subsequence.

$\boxed{\text{Theorem}}$ 2.39 Let K_j, $j = 1, 2, \dots$ be a nested sequence of compact sets, i.e. $K_1 \supset K_2 \supset K_3 \supset \dots$ then the intersection $K = \bigcap_j K_j$ is non-empty.

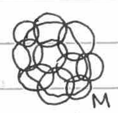
Proof – Let $x_1, x_2, \dots$ be a sequence s.t. $x_1 \in K_1, x_2 \in K_2, \dots, x_j \in K_j$. Thus $x_j \in K_1$ $\forall j = 1, 2, \dots \Rightarrow \exists$ a convergent subsequence x_{j_k}, $k = 1, 2, \dots \Rightarrow$ since K_1 is compact, $x_0 = \lim_{k\to\infty} x_{j_k} \in K_1$.

Since $x_{j_k} \in K_n$ $\forall n$ and sufficiently large k, we also have $x_0 \in K_n \Rightarrow x_0 \in \bigcap_j K_j$ as claimed // q.e.d.

Remark – Unlike closed ball theorem, in this case, we do not require that the space was complete.

Two Criteria for Compactness.

Consider $M \subset X$, $\varepsilon > 0$. We say that a set $N \subset X$ is an ε-net for M if for any $x \in M$, one can find an $x_\varepsilon \in N$ s.t. $x \in B(x_\varepsilon, \varepsilon)$.



set of centres of balls will be set N.

$\boxed{\text{Theorem}}$ 2.40 If $M \subset X$ is relatively compact, then for any $\varepsilon > 0$ it has a finite ε-net.

If the space X is complete, the converse also holds (i.e. $M \subset X$ has a finite ε-net $\forall \varepsilon > 0 \Rightarrow M$ is relatively compact).

Remark – If $\forall \varepsilon > 0$ there is a finite ε-net, then the set is called totally bounded.

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Proof – ($\Rightarrow$) Let M be relatively compact. Fix $\varepsilon>0$. Take $x_1 \in M$. Then either $M \subset B(x_1,\varepsilon)$, or there are points of M outside this ball. In the latter case, take $x_2 \notin B(x_1,\varepsilon)$ s.t. $x_2 \in M$.

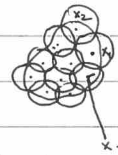
Then either $M \subset B(x_1,\varepsilon) \cup B(x_2,\varepsilon)$ or $M \not\subset B(x_1,\varepsilon)\cup B(x_2,\varepsilon)$. Continue the process: this will produce a collection of points $x_1, x_2, \dots, x_k$ s.t. $|x_j - x_k| \geq \varepsilon$.

Claim: This process is finite. Suppose otherwise, i.e. $x_1, x_2, \dots$ continues infinitely without terminating. Since $x_k \in M$ which is compact, $\{x_k\}$ must contain a convergent subsequence.

However, $|x_j - x_k| \geq \varepsilon \ \forall j,k$, so the subsequence cannot possibly be convergent $\Rightarrow$ contradiction. Thus, a finite ε-net is constructed. // q.e.d.

($\Leftarrow$) Let X be complete, and let M have a finite ε-net for any $\varepsilon>0$. Let $\varepsilon_n \to 0$ be positive numbers and let $N_n = \{x_1^{(n)}, \dots, x_{k_n}^{(n)}\}$ be an ε_n-net. Let $T \subset M$ be a sequence. Then let T_1 be the part of T in one of $B(x_k^{(1)}, \varepsilon_1)$ containing infinitely many elements of T. Let $T_2 \subset T_1$ be a part of T_1 in one of $B(x_k^{(2)}, \varepsilon_2)$ containing infinitely many elements. Continue the process to get $T \supset T_1 \supset T_2 \supset \dots \supset T_j \supset \dots$ Here, $T_j \subset B(x_k^{(j)}, \varepsilon_j)$. Choose a subsequence of T in the following way: $\xi_1 \in T_1, \dots, \xi_j \in T_j$. Then $\rho(\xi_m, \xi_n) < 2\varepsilon_n$ if $m \geq n$. This is true since $\xi_m \in T_n$.

As $\varepsilon_n \to 0$, the sequence $\{\xi_j\}$ is Cauchy. Since X is complete, ξ_j has a limit $\Rightarrow N$ is relatively compact. // q.e.d.



$\boxed{\text{Corollary}}$ 2.41 Let X be complete. Then a set $M \subset X$ is relatively compact $\Leftrightarrow \forall \varepsilon>0$, M has a relatively compact ε-net.

Proof – left as exercise.

$\boxed{\text{Corollary}}$ 2.42 If M is relatively compact, it is bounded.

Proof – Let $x_1, \dots, x_k$ be a 1-net of M. It is finite as M is relatively compact. Pick any point $x_0 \in X$ and define $d = \max_{1\leq j\leq k} \rho(x_0, x_j)$. Then, $\forall x \in M$, the triangle inequality yields that $\rho(x_0, x) \leq \rho(x_r, x) + \rho(x_r, x_0)$ for any $r = 1, \dots, k$ $= d+1 \Rightarrow M$ is bounded. // q.e.d.

Example – $X = \ell_p$, $1 \leq p < \infty$. Define M in the following way. Let $M = \{x_k\}_{k=1}^{\infty}$ be $x_1 = (1,0,\dots)$, $x_2 = (0,1,\dots)$, $x_k = (0,\dots,\underset{(k)}{1},\dots,0)$. Clearly, $x \in B[0,1]$. On the other hand, $\rho_p(x_m, x_n) = (1^p + 1^p)^{\frac{1}{p}} = 2^{\frac{1}{p}}$, $m \neq n$. There is no convergent subsequence $\Rightarrow$ not relatively compact.

$\boxed{\text{Corollary}}$ 2.43 If $M \subset \mathbb{R}^n$, then M is compact $\Leftrightarrow$ M is bounded and closed.

Proof – left as exercise.

$\boxed{\text{Corollary}}$ 2.44 If $A \subset M$ and M is compact, then A is compact $\Leftrightarrow$ A is closed.

$\boxed{\text{Corollary}}$ 2.45 Let X be a complete compact space. Then X is separable.

Proof – Let $\varepsilon_n \to 0$ be a positive sequence, and let N_n be a finite ε_n-net for X. Then $N = \bigcup_n N_n$ is countable, and dense in X. // q.e.d.

$\boxed{\text{Definition}}$ 2.46 A collection of open sets $\{G_\alpha\}$ is called a *cover* for a set $M \subset X$ if $M \subset \bigcup_\alpha G_\alpha$.

Recall the Heine-Borel Lemma – Every set of covers has a finite subcover. We will show that this criterion is equivalent to compactness.

$\boxed{\text{Theorem}}$ 2.47 A closed set $M \subset X$ is compact $\Leftrightarrow$ From any open cover M one can extract a finite subcover.

Proof – Omitted, not examinable.

$\boxed{\text{Theorem}}$ 2.48 Let $f: X \to Y$ be continuous. Then for any compact set $A \subset X$, the set $f(A)$ is also compact.

Proof – Since A is compact, $f(A)$ is closed. Let $\{G_\alpha\}$ be an open cover of $f(A)$. Thus, $\{f^{-1}(G_\alpha)\}$ is an open cover of A (by continuity of f). Extract a finite subcover as A is compact, so $\{T_1, \dots, T_n\}$ covers A and $\{f(T_1), \dots, f(T_n)\}$ is a finite cover of $f(A)$. Also, this is a finite subcover of $\{G_\alpha\} \Rightarrow f(A)$ is compact. // q.e.d.

$\boxed{\text{Corollary}}$ 2.49 Let X be a compact space and let $f: X \to \mathbb{R}$ be continuous. Then f is bounded and attains its maximum and minimum values.

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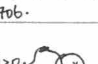
We now prove one part of Theorem 2.47:

Proof – Assume that for any open cover, there is a finite subcover. Claim: $\Rightarrow M$ is closed. Suppose $M \neq [M]$, i.e. $\exists x_0 \in [M] \setminus M$. Then $\rho(x, x_0) > 0 \ \forall x \in M$. Let $r_x = \frac{1}{2}\rho(x, x_0) > 0$.



Then $B(x, r_x) \cap B(x_0, r_x) = \phi$. Clearly, the balls $B(x, r_x)$, $x \in M$ form an open cover of M. Take a finite subcover: $B(x_1, r_1), B(x_2, r_2), \dots, B(x_n, r_n)$ for $r_k = r_{x_k}$, $k = 1, 2, \dots, n$.

Take $r = \min_{1\leq k\leq n} r_k$. Then (1) $B(x_k, r_k)$, $k = 1, 2, \dots, n$ is a cover of M, (2) $B(x_0, r) \cap B(x_k, r_k) = \phi$. Thus, $\text{dist}(M, x_0) \geq r > 0$. Since $[M] = \{y : \text{dist}(M, y) = 0\}$ and $x_0 \in [M]$, we get a contradiction. Hence, $M = [M]$ // q.e.d.

$\boxed{\text{Theorem}}$ 2.50 Let $f: X \to Y$ be continuous. Then for any compact set $M \subset X$, the function f is uniformly continuous on M. i.e. $\forall \varepsilon>0 \ \exists \delta>0$ s.t. $\rho(f(x), f(y)) < \varepsilon$ as soon as $\rho(x,y) < \delta \ \forall x, y \in M$.

Proof – Corollary of Theorem 2.47 (exercise)

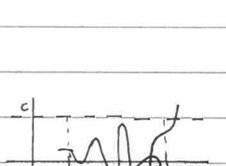
<u>Compactness in the space of continuous functions</u> (Arzelà–Ascoli Theorem).

$\boxed{\text{Definition}}$ 2.51 Let $M \subset C[a,b]$. Functions $x \in M$ are said to be <u>uniformly bounded</u> if there is a constant $c>0$ s.t. $\max |x(t)| \leq c$ for all $x \in M$.

The functions $x \in M$ are said to be <u>equicontinuous</u> if $\forall \varepsilon>0$, $\exists \delta>0$ s.t. $|x(t_1) - x(t_2)| < \varepsilon$ as soon as $|t_1 - t_2| < \delta$ for all $x \in M$ and $t_1, t_2 \in [a,b]$.

$\boxed{\text{Theorem}}$ 2.52 (Arzelà–Ascoli theorem)

The set $M \subset C[a,b]$ is relatively compact $\Leftrightarrow$ M is uniformly bounded and equicontinuous.



(Partial, one way)
Proof – ($\Rightarrow$). M is relatively compact. By Corollary 2.42, M is bounded. By Theorem 2.40, $\forall \varepsilon>0$, there is a finite $\frac{\varepsilon}{3}$-net, i.e. there are $x_1, \dots, x_n \in C[a,b]$ s.t. $\forall x \in M$, there is a function x_k s.t. $\rho(x, x_k) < \frac{\varepsilon}{3}$. Therefore, one can find $\delta_k > 0$ s.t. $|x(t_1) - x(t_2)| \leq |x(t_1) - x_k(t_1)| + |x_k(t_1) - x_k(t_2)| + |x_k(t_2) - x(t_2)|$
$< \frac{\varepsilon}{3} + \underbrace{|x_k(t_1) - x_k(t_2)|}_{\text{uniformly continuous}} + \frac{\varepsilon}{3} < \varepsilon$ as soon as $|t_1 - t_2| < \delta_k$. If we take $\delta = \min_{1\leq k\leq n} \delta_k$, then $\forall \varepsilon>0$, $\exists \delta>0$ s.t. $|t_1 - t_2| < \delta \Rightarrow |x(t_1) - x(t_2)| < \varepsilon \ \forall x \in M \Rightarrow$ equicontinuous // q.e.d.

$\boxed{\text{Theorem}}$ 2.53 Let $\mathcal{K} \in C([a,b] \times [a,b])$ and let $(Kf)(x) = \int_a^b \mathcal{K}(x,y) f(y)\,dy$, $f \in C[a,b]$. Then $K: C[a,b] \to C[a,b]$ is an integral operator. Let $S = \{f \in C[a,b] : |f(x)| \le C_1\}$, $C_1 > 0$.

Then $K(S)$ is relatively compact.

Proof – (1) Uniform boundedness of $K(S)$. Estimate: $|Kf(x)| \le \int_a^b \max_{x,y} |\mathcal{K}(x,y)| \max_t |f(t)|\,dz \le C_1 C_2 \int_a^b dz = C_1 C_2 (b-a)$.

(2) Equicontinuity. We know $\forall \varepsilon > 0\ \exists \delta$ s.t. $|\mathcal{K}(x_1,y) - \mathcal{K}(x_2,y)| < \varepsilon$ if $|x_1 - x_2| < \delta$. Therefore, $|(Kf)(x_1) - (Kf)(x_2)| \le \int_a^b |\mathcal{K}(x_1,y) - \mathcal{K}(x_2,y)| \max_t |f(t)|\,dy < \varepsilon\, C_1 (b-a)$ if $|x_1 - x_2| < \delta$.

Thus, $K(S)$ is equicontinuous. From (1) & (2), by Arzelà–Ascoli Theorem, $K(S)$ is relatively compact, q.e.d.

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Chapter 3

NORMED SPACES AND BANACH SPACES.

Linear vector spaces. X, $\mathbb{K}$ – field with $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. We let elements of X be called vectors.

$\boxed{\text{Definition}}$ 3.1 Addition on X is defined as the operation $+ : X \times X \to X$ with the properties (1) $x + y = y + x$, (2) $(x+y)+z = x+(y+z)$ (3) $\exists$ vector 0 s.t. $x + 0 = 0 + x = x$, and

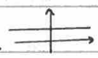
(4) For each element $x \in X$, $\exists$ vector $-x$ s.t. $x + (-x) = 0$

$\boxed{\text{Definition}}$ 3.2 The mapping $\cdot : \mathbb{K} \times X \to X$ is called multiplication by a scalar if (1) $\alpha(\beta x) = (\alpha\beta)x$ (2) $1 \cdot x = x$ (3) $\alpha(x+y) = \alpha(x) + \alpha(y)$ $\forall \alpha \in \mathbb{K}$, $x, y \in X$ and

(4) $(\alpha + \beta)(x) = \alpha(x) + \beta(x)$

$\boxed{\text{Definition}}$ 3.3 A linear vector space X over the field $\mathbb{K}$ is a non-empty set X with operations of addition and multiplication by scalars defined on it. If $\mathbb{K} = \mathbb{R}$, X is a real vector space (respectively $\mathbb{C}$, complex).

Examples –

(1) $\mathbb{R}, \mathbb{R}^d, \mathbb{C}^d$ are v.s.  disc of radius 1 is not a v.s.  In $\mathbb{R}^2$, not a vector space. Neither is 

(2) sequence spaces: c_0, c_{00} are linear spaces. By Minkowski's Inequality, two sequences in ℓ_p sum to another in ℓ_p, so ℓ_p, $p \in [1, \infty]$ is a linear space.

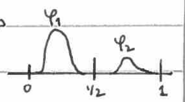
(3) $C[a,b]$, $C_p[a,b]$, $L_p(a,b)$, $1 \le p < \infty$ $C[a,b]$, $C_p[a,b]$ are linear spaces. $L_p[a,b]$, the completion, is a linear space by Minkowski's Inequality.

$\boxed{\text{Definition}}$ 3.4 A subspace of linear space X is a non-empty subset $Y \subset X$ which is closed under $+$ and $\cdot$. (i.e. $\forall x, y \in Y$, $\alpha, \beta \in \mathbb{K}$, $\alpha x + \beta y \in Y$.)

$\boxed{\text{Definition}}$ 3.5 A linear combination of vectors $x_1, x_2, \dots, x_N \in X$ is the vector of the form $\sum_{k=1}^N \alpha_k x_k$ with some $\alpha_1, \alpha_2, \dots, \alpha_N \in \mathbb{K}$. The set of all linear combinations of vectors $x \in M \subset X$ is called the span of M, denoted span M. This is a subspace.

$\boxed{\text{Definition}}$ 3.6 A set of vectors $x_1, x_2, \dots, x_N$ is said to be linearly independent if $\sum_{k=1}^N \alpha_k x_k = 0$ implies $\alpha_1 = \alpha_2 = \dots = \alpha_N = 0$. An infinite set of vectors is linearly independent if each finite subset is linearly independent.

e.g. – consider $X = C[0,1]$. $\{\sin x, \cos x\}$ are LI, since if LD $\alpha \sin x + \beta \cos x = 0$ $\forall x \in [0,1]$. At $x = 0$, $\beta = 0$, $\alpha = 0$. Or, we can split interval into two parts to construct two functions which are non-zero at different parts. Then they are LD.



$\boxed{\text{Definition}}$ 3.7 A space X is said to be finite dimensional if $\exists d \in \mathbb{Z}^+$ st. X contains d linearly independent vectors, and any collection of $d+1$ vectors is linearly dependent. Notation $d = \dim X$.

If X is not finite dimensional, it is infinite dimensional. A basis of X is a set that is linearly independent and spans X.

If $\dim X < \infty$, then any set of $d = \dim X$ LI vectors is a basis. Claim: Every linear space has a basis (even infinite dimensional ones).

For a finite dimensional space, $x \in X \Rightarrow x = \sum_{k=1}^d \alpha_k x_k$ where $\{x_1, \dots, x_k\}$ is a basis. $\{\alpha_k\}$ are uniquely defined.

Examples – (1) $C[a,b]$ is infinite dimensional. Pick any fixed $N \in \mathbb{N}$. Then we split $[a,b]$ into N equal intervals. Let φ_i be a bump in i^{th} interval, 0 everywhere else. $\{\varphi_i\}$ forms a LI set.

Since n was arbitrarily chosen, we can always construct N linearly independent functions. $L_p(a,b)$ is infinite dimensional for the same reason.

(2) ℓ_p, $p \in [1, \infty]$ is infinite dimensional, using $x_i = (0, 0, \dots, 0, \underset{i^{th}}{1}, 0, \dots)$ as a basis.

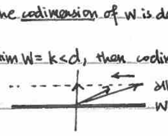
$\boxed{\text{Definition}}$ 3.8 Let $W \subset X$ be a subspace. Then we say that $x, y \in X$ are equivalent if $x - y \in W$. Then the space of all equivalence classes $\{x\}$ is called the quotient space X/W.

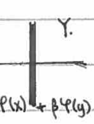
It becomes a linear space if one defines operations $+$ and $\cdot$ s.t. $\{\alpha x + \beta y\} = \alpha\{x\} + \beta\{y\}$.

Remark – It is easy to check that the requirements of Definitions 3.1, 3.2 are satisfied.

$\boxed{\text{Definition}}$ 3.9 Let $W \subset X$ be a subspace. Then the codimension of W is defined to be $\dim X/W$. Notation: $\operatorname{codim} W = \dim X/W$.

Observation – If $\dim X = d$, $\dim W = k < d$, then $\operatorname{codim} W = d - k$

e.g. – Let $X = \mathbb{R}^2$, $W = \mathbb{R}$. 

all vectors with endpoints along this line are equivalent. Let Y be as on right. 

Then X/W is isomorphic to Y.

[Recall definition – Two linear spaces X, X_1 are isomorphic if $\exists$ a bijection $\varphi : X \to X_1$ s.t. $\varphi(\alpha x + \beta y) = \alpha\varphi(x) + \beta\varphi(y)$ $\forall x, y \in X$.

Normed spaces.

$\boxed{\text{Definition}}$ 3.10 A norm on vector space X is a real-valued function $\|\cdot\| : X \to \mathbb{R}$ with the properties:

(1) $\|x\| \ge 0$ (non-negativity)
(2) $\|x\| = 0 \iff x = 0$ (non-degeneracy)
(3) $\|x + y\| \le \|x\| + \|y\|$ (triangle inequality).
(4) $\|\alpha x\| = |\alpha|\,\|x\|$ $\forall x, y \in X$, $\alpha \in \mathbb{K}$. (homogeneous function).

If X has a norm defined on it, it is called a normed linear space.

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Examples – (1) $\mathbb{R}^d$, then $\|x\| = \left[\sum_{k=1}^{d} x_k^2\right]^{\frac{1}{2}}$. We can also consider different norms – p-norm $\|x\|_p = \left[\sum_{k=1}^{d} |x_k|^p\right]^{1/p}$, $1 \le p < \infty$. infinity-norm $\|x\|_\infty = \max_{1 \le k \le d} |x_k|$.

By Minkowski's inequality, both of these are norms.

(2) Take ℓ_p, $1 \le p \le \infty$. Clearly, $\|x\|_p = \left[\sum_{k=1}^{\infty} |x_k|^p\right]^{1/p}$ is finite s.t. $\|\cdot\|$ is a norm (again by Minkowski's).

(3) $C[a,b]$. Here $\|f\|_c = \max_{a \le t \le b} |f(t)|$ is a norm and $L_p(a,b)$ with $\|f\|_{L_p} = \left[\int_a^b |f(t)|^p \, dt\right]^{1/p}$ is a normed space. (Note: $\int_a^b |f(t)|^p dt$ is a Lesbegue integral.

$\boxed{\text{Theorem}}$ 3.11 Let X be a normed linear space. Then the function $\rho(x,y) = \|x-y\|$ defines a metric on X.

Proof – Omitted, straightforward. Check definitions.

Remark – We say that ρ is the metric induced by the norm.

$\boxed{\text{Definition}}$ 3.12 X is <u>complete</u> if it is a complete metric space with the metric induced by the norm. If X is complete, it is called a <u>Banach space</u>.

$\boxed{\text{Definition}}$ 3.13 Let $W \subset X$ be a subspace of a linear space X. Then W is a <u>subspace of the normed space</u> X if W is closed, i.e. $W = [W]$.

<u>Inner product spaces.</u>

$\boxed{\text{Definition}}$ 3.14 An <u>inner product</u> on the linear space X is a mapping $\langle -,- \rangle : X \times X \to \mathbb{K}$ with the following properties:

(1) (linearity) $\langle x+y, z\rangle = \langle x,z\rangle + \langle y,z\rangle$ (2) (homogenity) $\langle \alpha x, y\rangle = \alpha\langle x,y\rangle$ (3) $\langle x,y\rangle = \overline{\langle y,x\rangle}$ and (4) $\langle x,x\rangle \ge 0$ and $\langle x,x\rangle = 0 \Leftrightarrow x = 0$.

The space X with an inner product is called a <u>pre-Hilbert space</u>. If $\mathbb{K} = \mathbb{R}$, it is also called <u>Euclidean</u>.

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$\forall x,y \in X$, $\alpha, \beta \in \mathbb{K}$, then $\langle \alpha x + \beta y, z\rangle = \alpha\langle x,z\rangle + \beta\langle y,z\rangle$. $\langle z, \alpha x + \beta y\rangle = \bar{\alpha}\langle z,x\rangle + \bar{\beta}\langle z,y\rangle$. The inner product is called $1\frac{1}{2}$-linear, or <u>sesquilinear</u>.

$\boxed{\text{Definition}}$ 3.15 We say that $x \in X$ and $y \in X$ are <u>orthogonal</u> to each other if $\langle x,y\rangle = 0$. Notation: $x \perp y$.

Remark – Recall the Pythygoras's Theorem: if $x \perp y$, then $\|x+y\|^2 = \|x\|^2 + \|y\|^2$.

We also define a "norm" $\|x\| = \sqrt{\langle x,x\rangle}$ $\forall x \in X$. This is homogeneous and non-degenerate.

$\boxed{\text{Lemma}}$ 3.16 (Cauchy-Schwarz Inequality).

$\forall x,y \in X$, $|\langle x,y\rangle| \le \|x\|\|y\|$. Equality holds if and only if x,y are linearly dependent.

Note – This implies the triangle inequality

$\boxed{\text{Definition}}$ 3.17 If X is complete w.r.t. the norm induced by the inner product, it is called a <u>Hilbert space</u>. [Notation: H].

<u>Characteristic property of inner product spaces.</u>

• Parallelogram identity: $\|x+y\|^2 + \|x-y\|^2 = 2(\|x\|^2 + \|y\|^2)$. • Polarisation identity: $\overset{(*)}{4\langle x,y\rangle = \|x+y\|^2 - \|x-y\|^2 + i(\|x+iy\|^2 - \|x-iy\|^2)}$. If $\mathbb{K} = \mathbb{R}$, $\overset{(**)}{4\langle x,y\rangle = \|x+y\|^2 - \|x-y\|^2}$.

$\boxed{\text{Theorem}}$ 3.18 Let X be a normed space. The relation $(*)$ or $(**)$ defines an inner product $\Leftrightarrow$ parallelogram identity holds $\forall x,y \in X$.

$\boxed{\text{Definition}}$ 3.19 Let X be a normed space and let $\|\cdot\|_1$, $\|\cdot\|_2$ be two norms on X. We say that $\|\cdot\|_1$ and $\|\cdot\|_2$ are equivalent if there are two constants $C_2 \ge C_1 > 0$ s.t. $C_1\|x\|_1 \le \|x\|_2 \le C_2\|x\|_1$ $\forall x \in X$.

$\boxed{\text{Theorem}}$ 3.20 If $\dim X < \infty$, then all norms are equivalent to each other.

Examples – (1) $\mathbb{R}^d$. This is finite-dimensional, so all norms are equivalent e.g. $\|x\|_p = \left[\sum_{k=1}^{d} |x_k|^p\right]^{\frac{1}{p}}$. If $p=2$, this is a Hilbert space. If $p \ne 2$, it cannot be viewed as a Hilbert space as the parallelogram identity does not hold for some $x,y \in \mathbb{R}^d$. For $p=2$, $\langle x,y\rangle = \sum_{k=1}^{d} x_k y_k$. [For $\mathbb{C}^d$, $\langle x,y\rangle = \sum_{k=1}^{d} x_k \overline{y_k}$].

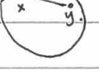
(2) For ℓ_p, $1 \le p \le \infty$, $\|x\|_p = \left[\sum_{k=1}^{\infty} |x_k|^p\right]^{\frac{1}{p}}$. Again, if $p=2$, this is a Hilbert space, $\langle x,y\rangle = \sum_{k=1}^{\infty} x_k \overline{y_k}$. Here, since series is infinite, we do not talk about equivalent norms.

(3) $L_p(a,b)$, $1 \le p < \infty$. (Closure of continuous functions). If $p=2$, $\langle x,y\rangle = \int_a^b x(t)\overline{y(t)}\,dt$. is an inner product.

Let $f \in C[a,b] \Rightarrow \|f\|_c = \sup_{t \in [a,b]} |f(t)|$. This is a Banach space, but does not satisfy the parallelogram law. We require a different norm: $C_2[a,b]$ is an inner product space, $\langle x,y\rangle = \int_a^b x(t)\overline{y(t)}\,dt$.

By completing it, we get $L_2(a,b)$ which is a Hilbert space.

<u>Distance, convex sets and orthogonal complements.</u>



Let $M \subset H$ be a set, then we defined $\text{dist}(M,x) = \inf_{y \in M} \|y-x\| = \delta(x,M)$. Is the infimum actually attained, and if so, at how many points?

$\boxed{\text{Definition}}$ 3.21 The set M is convex if $\forall x,y \in M$, the linear combination $\alpha x + (1-\alpha)y$ is also in the set $\forall \alpha \in (0,1)$.

$\boxed{\text{Theorem}}$ 3.22 Let $M \subset H$ be a convex closed set. Then $\forall x \in H$, $\exists$ unique vector $y \in M$ s.t. $\delta(x,M) = \|y-x\|$. [Nomenclature – y is called a <u>minimising vector</u>.]

Proof – By definition of inf, $\exists$ a sequence $y_n \in M$ s.t. $\delta(x,M) = \lim_{n\to\infty} \|x-y_n\|$. Denote $\delta_n = \|x-y_n\|$. We claim that this sequence is Cauchy, and then $\|y_n - y_m\|^2 \overset{\text{Paral.}}{=} -\|y_n + y_m - 2x\|^2 + 2(\underbrace{\|x-y_n\|^2}_{\delta_n} + \underbrace{\|x-y_m\|^2}_{\delta_m})$. We note that as M is convex, $\frac{y_n+y_m}{2} \in M$ so $\left\|\frac{y_n+y_m}{2} - x\right\| \ge \delta(x,M) = \delta$. Thus,

$\|y_n - y_m\|^2 \le -4\delta^2 + 2\delta_n^2 + 2\delta_m^2 \to 0$ as $n,m \to \infty$. $\Rightarrow$ y_n has a limit, $y = \lim_{n\to\infty} y_n$. As M is closed, $y \in M \Rightarrow$ Indeed, $\|x-y\| \le \|x-y_n\| + \|y_n - y\| \to \delta$ as $n \to \infty$.

Thus, RHS $\le \delta$, $\delta \le$ LHS, so $\delta(x,M) = \|x-y\|$ and y is a minimising vector. Suppose $\|x-y_1\| = \|x-y_2\| = \delta$. Then the same holds for $\alpha y_1 + (1-\alpha)y_2$, $\forall \alpha \in (0,1)$.

Hence, $\delta \le \|\alpha y_1 + (1-\alpha)y_2 - x\| \le \|\alpha y_1 - \alpha x\| + \|(1-\alpha)y_2 - (1-\alpha)x\| = \alpha\|y_1 - x\| + (1-\alpha)\|y_2 - x\| = \delta$. Therefore by parallelogram identity, we have that

$\|y_1 - y_2\|^2 = -\|y_1 + y_2 - 2x\|^2 + 2\|y_1 - x\|^2 + 2\|y_2 - x\|^2 = -4\delta^2 + 2\delta^2 + 2\delta^2 = 0$ $\Rightarrow$ $y_1 = y_2$, minimising vector is unique, q.e.d.

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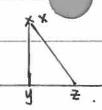
Theorem 3.23 Let $M \subset H$ be a closed subspace and $x \in H$. Then $y \in M$ is the unique minimising vector $\Leftrightarrow$ $x-y \perp M$.

($\Rightarrow$)
Proof – Let y be the minimising vector. Suppose that there is a vector $e \in M$ s.t. $\underbrace{\langle x-y, e\rangle}_{\alpha} \neq 0$. Assume $\|e\|=1$. Write $\|x-y-\alpha e\|^2 = \|x-y\|^2 - \bar{\alpha}\langle x-y, e\rangle - \alpha\langle e, x-y\rangle + |\alpha|^2$.
$\|x-y\|^2 - |\alpha|^2$. Then this vector is strictly less than $\|x-y\|^2$, which is a contradiction as y is minimising $\Rightarrow$ $\alpha = 0$ $\Rightarrow$ $x-y \perp M$.
($\Leftarrow$)
Suppose that $y \in M$ is such that $x-y \perp M$. For any z, we have $\|x-z\|^2 = \|x-y\|^2 + \|y-z\|^2 \geq \|x-y\|^2$. Hence, y is the minimising vector // q.e.d.



Definition 3.24 For a closed subspace $M \subset H$ and a vector $x \in H$, the unique vector $y \in M$ s.t. $x-y \perp M$ is called the <u>orthogonal projection</u> of x onto M. We write $y = Px$, where this map P is called the <u>orthogonal projection</u> (operator) onto M.

Properties of map P: ① $P^2 = P$ (idempotent operator) ② $\|Px\|^2 + \|x-Px\|^2 = \|x\|^2$ (Pythagoras). Thus, $\|Px\| \leq \|x\|$.

Definition 3.25 A linear space X is said to be a <u>direct sum</u> of two subspaces Y, Z if every $x \in X$ can be uniquely represented as $x = y+z$, where $y \in Y$, $z \in Z$. If $X = H$ and $Y \perp Z$, then this is called an <u>orthogonal sum</u>. We write $X = Y \oplus Z$.

Definition 3.26 For a closed subspace M, the set $M^{\perp} = \{x \in H : x \perp M\}$ is called the <u>orthogonal complement</u> of M.

Note – $M^{\perp}$ is a closed subspace for general set M.

Theorem 3.27 If Y is a closed subspace, then $H = Y \oplus Y^{\perp}$. Note – $Y^{\perp\perp} = Y$ if Y is closed.

<u>Annihilators.</u>

Definition 3.28 Let M be a non-empty set in H. Then the <u>annihilator</u> of M is the set $M^{\perp} = \{x \in H : x \perp M\}$.

Theorem 3.29 The following properties hold for the annihilator:

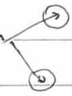
① $M^{\perp}$ is a closed subspace ② $M^{\perp} = [M]^{\perp}$ ③ $M^{\perp} = (\operatorname{span} M)^{\perp}$ ④ M spans H $\Leftrightarrow$ $M^{\perp} = \{0\}$.

<u>Orthonormal sets.</u>

Definition 3.30 A set $M \subset H$ is called <u>orthonormal</u> if for any two vectors $x, y \in M$ we have either $\langle x,y\rangle = 0$ (if $x \neq y$) or $\langle x,y\rangle = 1$ (if $x = y$).

Theorem 3.31 If H is separable, then every orthonormal set is countable.

(Sketch)
Proof – $\forall x, y \in M$, $x \neq y$: $\|x-y\|^2 = \|x\|^2 + \|y\|^2 = 2$. Thus the balls $B(x, \frac{1}{2})$ and $B(y, \frac{1}{2})$ are disjoint. By separability, take an element of countable dense set from each of these balls...



From now on, $M = \{e_1, e_2, \dots\}$ is an orthonormal sequence. For $x \in H$, the numbers $c_k = \langle x, e_k\rangle$ are called <u>Fourier coefficients</u> with respect to M.

Theorem 3.32 ① For any $x \in H$, $\sum_{k=1}^{\infty} |c_k|^2 \leq \|x\|^2$ (Bessel's inequality). In particular, $\{c_k\} \in \ell_2$. ② The series $\sum_{k=1}^{\infty} c_k e_k$ converges to a vector Px, where P is the orthogonal projection onto $[\operatorname{span} M]$.

Note – We require a Hilbert space so that it is complete, and thus infinite sums converge.

Proof – ② $\sum_{k=1}^{\infty} c_k e_k$ is tested for convergence. Let $y_n = \sum_{k=1}^{n} c_k e_k$, then (assume $m>n$) $\|y_m - y_n\|^2 = \|\sum_{k=n+1}^{m} c_k e_k\|^2 = \sum_{k=n+1}^{m} |c_k|^2 \to 0$ as $m,n \to \infty$ $\Rightarrow$ sequence is Cauchy, convergent. Then $\exists \lim_{n\to\infty} y_n = y \in H$. NTP: $x-y \perp \operatorname{span} M$; i.e. $x-y \perp M$. Check that $x-y \perp e_j$ for any $j = 1, 2, \dots$. Indeed, let $n > j$, then $\langle x-y_n, e_j\rangle = \langle x, e_j\rangle - \sum_{k=1}^{n} c_k \langle e_k, e_j\rangle = \langle x, e_j\rangle - c_j\langle e_j, e_j\rangle = c_j - c_j = 0$. By continuity of inner product, $\lim_{n\to\infty} \langle x-y_n, e_j\rangle = 0 = \langle x-y, e_j\rangle \Rightarrow x-y \perp M$ // q.e.d.

<u>Total orthonormal sequences</u>

Definition 3.33 An orthonormal sequence is called total if $[\operatorname{span} M] = H$. [i.e. $M^{\perp} = \{0\}$].

Theorem 3.34 (Parseval's Identity).

M is total $\Leftrightarrow$ $\|x\|^2 = \sum_{k=1}^{\infty} |c_k|^2$. — ⊛ In this case, for any $x, y \in H$, we have $\langle x, y\rangle = \sum_{k=1}^{\infty} c_k \bar{d}_k$ where c_k, d_k are Fourier coefficients of x and y respectively.

($\Rightarrow$)
Proof – Suppose M is total, so $x_n = \sum_{k=1}^{n} c_k e_k$ converges to x as $n \to \infty$ by Theorem 3.32. Therefore $\|x - x_n\|^2 = \|x\|^2 - \sum_{k=1}^{n} |c_k|^2 \to 0$ as $n \to \infty$, therefore $\|x\|^2 = \sum_{k=1}^{\infty} |c_k|^2$ as claimed.
($\Leftarrow$)
Suppose ⊛. Assume M is not total i.e. $x \perp M$ and $x \neq 0$. Therefore $c_k = 0$ but $\|x\| \neq 0$ – contradiction // q.e.d. Second part left as exercise.

Theorem 3.35 A Hilbert space contains a total orthonormal sequence $\Leftrightarrow$ it is separable.

Theorem 3.36 (Riesz–Fisher theorem)

Let $\{c_k\} \in \ell_2$ and let $\{e_k\}$ be an orthonormal sequence in H. Then there is an element $x \in H$ such that $c_k = \langle x, e_k\rangle$, $k = 1, 2, \dots$.

Proof – Define $x_n = \sum_{k=1}^{n} c_k e_k$. Thus $\|x_m - x_n\| = \sum_{k=n+1}^{m} |c_k|^2 \to 0$ (m>n) as $m,n \to \infty$ as $\{c_k\} \in \ell_2$. Hence, $\exists \lim_{n\to\infty} x_n = x$. It is straightforward to check that $\langle x, e_k\rangle = c_k$.

Chapter 4.
LINEAR FUNCTIONALS AND DUAL SPACE

Definition 4.1 Let X be a normed space over $\mathbb{K}$. A mapping $f : X \to \mathbb{K}$ is said to be a <u>linear functional</u> if $f(\alpha x + \beta y) = \alpha f(x) + \beta f(y)$ $\forall x, y \in X$, $\alpha, \beta \in \mathbb{K}$.

Examples – (1) If $X = \mathbb{K} = \mathbb{R}$, $f(x) = x$ is a linear functional (2) $g(x) = ax + b$ is a linear functional only if $b = 0$.

A linear functional is said to be <u>continuous</u> if f is continuous. A linear functional is said to be <u>bounded</u> if it maps a unit ball $B(0,1)$ into a bounded set.

Remark – An alternative formulation for boundedness: $f(B(0,1)) \subset B(0,R)$ for some $R>0$. Equivalently, $f(B(0,t)) \subset B(0,Rt)$ for any $t>0$, by linearity.

Equivalently, $|f(x)| \leq R\|x\|$ $\forall x \in X$.

$\boxed{\text{Lemma}}$ 4.2 A linear functional is continuous $\Leftrightarrow$ it is continuous at $x=0$.

Proof – $(\Rightarrow)$ is perfectly obvious. For $(\Leftarrow)$, suppose functional f is continuous at 0. NTP: $\forall \varepsilon>0$, $\exists \delta>0$ s.t. $f(y) \in B(f(x),\varepsilon)$ if $y \in B(x,\delta)$. Equivalently, $f(y-x) \in B(0,\varepsilon)$ if $y-x \in B(0,\delta)$

by linearity of function. This is satisfied by continuity of f at $z=0$ // q.e.d.

$\boxed{\text{Theorem}}$ 4.3 A linear functional is continuous $\Leftrightarrow$ it is bounded.

Proof – $(\Rightarrow)$ Let f be continuous (i.e. at 0). Then $\forall \varepsilon>0$, $\exists \delta>0$ so $x \in B(0,\delta) \Rightarrow f(x) \in B(0,\varepsilon)$. Then $f(B(0,\delta)) \subset B(0,\varepsilon) \Rightarrow f$ is bounded. $(\Leftarrow)$ Then assuming f is bounded, $f(B(0,t)) \subset B(0,Rt)$

for $t>0$, fixed $R>0$. In particular, this holds by defining $Rt=\varepsilon$ and setting $\delta = t = \frac{\varepsilon}{R}$. Then f is continuous at $z=0$ (and hence everywhere) //

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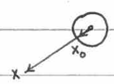
$\boxed{\text{Definition}}$ 4.4 The norm of f is defined to be $\|f\| = \sup_{x:\|x\|=1} |f(x)|$.

$\boxed{\text{Lemma}}$ 4.5 $\|f\| = \sup_{x:x\neq 0} \frac{|f(x)|}{\|x\|} = \sup_{x:\|x\|\leq 1} |f(x)|$

Proof – Let $A = \sup_{x:x\neq 0} \frac{|f(x)|}{\|x\|}$. Clearly, $\|f\| \leq A$. Prove that $A \leq \|f\|$. Let $t = \|x\|$, $x_0 = \frac{x}{t}$, so $\|x_0\|=1$. Then $\frac{|f(x)|}{\|x\|} = \frac{|f(tx_0)|}{t} = |f(x_0)| \leq \|f\|$. (Since $\|x_0\|=1$.)



Hence $A = \|f\|$. Same procedure for second equality // q.e.d.

Thus, $\frac{|f(x)|}{\|x\|} \leq \|f\| \Rightarrow |f(x)| \leq \|f\|\|x\|$. Then $\|f\|$ is the best (sharpest) constant we can use for R in the inequality $|f(x)| \leq R\|x\|$ $\forall x \in X$. i.e. if $|f(x)| \leq c\|x\|$, then $\|f\| \leq c$.

Examples – (1) Take $X=H$. Let $x_0 \in H$ and define $f(x) = \langle x, x_0\rangle$. It is linear. Then $|f(x)| = |\langle x,x_0\rangle| \overset{(CSI)}{\leq} \|x\|\|x_0\|$, so it is bounded with $\|f\| \leq \|x_0\|$. We prove that $\|f\| = \|x_0\|$. We know $\|f\| = \sup_{\|x\|=1} |f(x)|$.

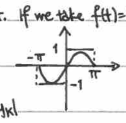
To this end, it suffices to find a vector x s.t. $f(x) = \|x_0\|$. Take $x = \frac{x_0}{\|x_0\|}$, then $f(x,x_0) = \langle \frac{x_0}{\|x_0\|}, x_0\rangle = \frac{1}{\|x_0\|}\langle x_0,x_0\rangle = \|x_0\|$. Hence, $\|f\| = \|x_0\|$.

(2). Take $X = C[a,b]$, with $\|f\|_c = \max_{t\in[a,b]} |f(t)|$. Let $\ell(f) = \int_a^b f(t)\,dt$. This is linear. Is it bounded? $|\ell(f)| \leq \int_a^b |f(t)|\,dt \leq \|f\|_c \int_a^b dt = \|f\|_c (b-a)$. So it is bounded,

and $\|\ell\| \leq b-a$. We want in fact: $\|\ell\| = b-a$ for some function $f \in X$ s.t. $\|f\|_c = 1$ and $\ell(f) = b-a$. If $f(t)=1$ $\forall t \in [a,b]$, then $\ell(f) = b-a$ as required.

[Variant].
Take $X = C[-\pi,\pi]$ with $\ell(f) = \int_{-\pi}^{\pi} \sin t\, f(t)\,dt$. Again this is linear. For boundedness, $|\ell(f)| \leq \|f\|_c \int_{-\pi}^{\pi} |\sin t|\,dt$. Hence, $\|\ell\| \leq \int_{-\pi}^{\pi} |\sin t|\,dt$. If we take $f(t)=1$,

then $\ell(f)=0$! Instead we could use $f(t) = \operatorname{sgn}(t)$, but it is not continuous. Instead, we need to approximate the step function by continuous ones.



(3). Take $X = \ell_p$, $\|x\|_p = \left[\sum_{k=1}^{\infty} |x_k|^p\right]^{1/p}$ $1<p<\infty$. Fix $y \in \ell_q$ s.t. $\frac{1}{p}+\frac{1}{q}=1$. Then define $f(x) = \sum_{k=1}^{\infty} x_k y_k$. This is linear. By Hölder's inequality, $|f(x)| \leq \sum_{k=1}^{\infty} |x_k||y_k|$

$\leq \|x\|_p \|y\|_q$. Then $\|f\| \leq \|y\|_q$. To show that $\|f\| = \|y\|_q$, we need to find a sequence $x \in \ell_p$ s.t. $f(x) = \|y\|_q \|x\|_p$. Take $x_k = |y_k|^{q-1} \frac{\overline{y_k}}{|y_k|}$ where $y_k \neq 0$,

and we define $\frac{\overline{y_k}}{|y_k|} \equiv 0$ for $y_k = 0$. Check that $x \in \ell_p$: $|x_k|^p = |y_k|^{(q-1)p} = |y_k|^q$ so $\|x\|_p^p = \|y\|_q^q < \infty$ and $x \in \ell_p$. Then $f(x) = \sum_{k=1}^{\infty} |y_k|^{q-1} \frac{\overline{y_k}}{|y_k|} y_k = \sum_{k=1}^{\infty} |y_k|^q$

$= \|y\|_q^q = \|y\|_q \|y\|_q^{q-1} = \|y\|_q \|x\|_p$. Then the norm is indeed attained.

$\boxed{\text{Definition}}$ 4.6 Let f be a bounded linear functional. Then the <u>kernel</u> of f is the set $\operatorname{Ker} f = \{x \in X : \ell(x) = 0\}$.

$\boxed{\text{Theorem}}$ 4.7 For any non-trivial bounded linear functional f, the kernel $\operatorname{Ker} f$ is a closed subspace, and $\operatorname{codim} \operatorname{Ker} f = 1$.

$\boxed{\text{Theorem}}$ 4.8 (Riesz's theorem)

Let H be a Hilbert space, and let f be a bounded linear functional on H. Then there exists a uniquely-defined vector $x_0 \in H$ s.t. $f(x) = \langle x, x_0\rangle$. Moreover, $\|f\| = \|x_0\|$.

Proof – If $f=0$, then $f(x)$ is always $f(x) = \langle x, x_0\rangle$. If $f \neq 0$, then $\exists$ a vector z s.t. $f(z) \neq 0$, and $z \in (\operatorname{Ker} f)^{\perp}$, so $z \neq 0 \Rightarrow \overset{\alpha =}{f(z)} \neq 0$. $\forall x \in H$, $x - \frac{f(x)}{\alpha} z \in \operatorname{Ker}(f)$, so $\langle x - \frac{f(x)}{\alpha} z, z\rangle = 0$

$\Rightarrow \frac{f(x)}{\alpha}\langle z,z\rangle = \langle x,z\rangle$, $f(x) = \frac{\alpha}{\|z\|^2}\langle x,z\rangle$. Take $x_0 = \frac{\bar{\alpha}}{\|z\|^2} z$.

For uniqueness, suppose $\exists$ two vectors $f(x) = \langle x,x_1\rangle = \langle x,x_2\rangle$ $\forall x \Rightarrow \langle x, x_1-x_2\rangle = 0 \Rightarrow x_1 = x_2$ // q.e.d.

$\boxed{\text{Theorem}}$ 4.9 Let $X = \ell_p$, $1<p<\infty$. Then any linear bounded functional f on X is represented as $f(x) = \sum_{k=1}^{\infty} x_k y_k$ with some uniquely defined $y = (y_1, y_2, \ldots) \in \ell_q$

where $p^{-1} + q^{-1} = 1$. Moreover, $\|f\| = \|y\|_q$.

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<u>The Hahn-Banach Theorems.</u>

Let X be a normed space. We say that $L \subset X$ is a linear set if $L = \operatorname{span} L$. (does not have to be closed, unlike subspace)

$\boxed{\text{Definition}}$ 4.10 Let L be a linear set, and $f_0 : L \to \mathbb{K}$, $f : X \to \mathbb{K}$ be two bounded linear functionals. We say that f is an extension of f_0 if $f_0(x) = f(x)$ $\forall x \in L$.

$\boxed{\text{Lemma}}$ 4.11 Let $D \subset X$ be a linear set s.t. $[D] = X$. Then any bounded linear functional f_0 on D can be uniquely extended to a bounded linear functional f on X. Moreover, $\|f_0\| = \|f\|$.

Proof – Let $\{x_n\} \subset D$ be a convergent sequence with $x = \lim_{n\to\infty} x_n$. Then the sequence $f_0(x_n)$ is Cauchy, and hence $f_0(x_n)$ has a limit. Define $f(x) = \lim_{n\to\infty} f_0(x_n)$. Clearly, f is a linear functional.

Since $f(x) = \lim_{n\to\infty} f_0(x_n)$, $\forall \varepsilon>0$ $\exists N$ s.t. $|f(x) - f_0(x_n)| < \varepsilon$ if $n>N$. Thus $|f(x)| = |f_0(x_n) + f(x) - f_0(x_n)| \leq |f_0(x_n)| + |f(x) - f_0(x_n)| < |f_0(x_n)| + \varepsilon \leq \|f_0\|\|x_n\| + \varepsilon \to \|f_0\|\|x\| + \varepsilon$ as $n\to\infty$.

$\Rightarrow |f(x)| \leq \|f_0\|\|x\| + \varepsilon$ $\forall \varepsilon>0$. Therefore $|f(x)| \leq \|f_0\|\|x\|$ $\forall x \in X \Rightarrow \|f\| \leq \|f_0\|$. On the other hand, $\|f\| \geq \|f_0\|$. Hence, $\|f\| = \|f_0\|$ // q.e.d.

Remark – We say that f_0 is extended from D to X by continuity.

$\boxed{\text{Theorem}}$ 4.12 (Hahn-Banach theorem) Let $L \subset X$ be a closed subspace and let f_0 be a bounded linear functional on L. Then there exists a bounded linear functional f on X s.t. (1) f is an extension of f_0, and (2) $\|f\| = \|f_0\|$.

Proof – Assume X is separable, $\mathbb{K} = \mathbb{R}$. Assume $\|f_0\|=1$ WLOG. Let $\xi \in X$. We extend f_0 to $L_1 = \operatorname{span}(L,\xi)$. If $\xi \in L$, then $L_1 = L$ and we define the extension by $f_1 = f_0$. Suppose $\xi \notin L$.

Then any vector $u \in L_1$, $u = x + t\xi$, with $x \in L$, $t \in \mathbb{R}$. Take $x_1, x_2 \in L$. Thus, $f_0(x_1 - x_2) = f_0(x_1) - f_0(x_2) \leq \|x_1 - x_2\| = \|(x_1+\xi) - (x_2+\xi)\| \leq \|x_1+\xi\| + \|x_2+\xi\|$. Then rewrite

$f_0(x_1) - \|x_1+\xi\| \leq f_0(x_2) + \|x_2+\xi\|$ $\forall x_1, x_2 \in L$. Therefore, $\sup_x (f_0(x) - \|x+\xi\|) \leq a \leq \inf_x (f_0(x) + \|x+\xi\|)$ – ⊛. for some $a \in \mathbb{R}$. Then $\forall x \in X$, $f_0(x) - \|x+\xi\| \leq a \leq f_0(x) + \|x+\xi\|$

then $-\|x+\xi\| \le f_0(x) - a \le \|x+\xi\| \Rightarrow |f_0(x)-a| \le \|x+\xi\|.\ \forall x \in L$. For $u = x + t\xi \in L_1$, define the extension f_1 to L_1: $f_1(u) = f_0(x) - ta$. It is easy to check that f_1 is linear on L_1. For the norm, $|f_1(u)| = |f_0(x) - ta| = |t\, f_0(\frac{x}{t}) - ta| = |t|\,|f_0(\frac{x}{t}) - a| \le |t| \cdot \|\frac{x}{t} + \xi\| = \|x + t\xi\| = \|u\|. \Rightarrow \|f_1\| \le 1$. Since f_1 is an extension $\|f_1\| \ge \|f_0\| = 1 \Rightarrow \|f_1\| = 1 = \|f_0\|$. Let $\xi_1, \xi_2, \dots$ be a countable dense set in X. Then we extend f_0 from L first to $L_1 = \mathrm{span}(L, \xi_1)$ then to $L_2 = \mathrm{span}(L_1, \xi_2), \dots$. Continuing this process, we extend this to $L_n = \mathrm{span}(L_{n-1}, \xi_n)$. We end up with a functional F defined on $\bigcup_{n=1}^{\infty} L_n = D$, s.t. $\|F\| = 1$. Since $[D] = X$, we can extend F to X by continuity // q.e.d. (for separable spaces).

Corollary 4.13 Suppose $x_1, x_2 \in X$ are s.t. $x_1 \neq x_2$. Then there exists a bounded linear functional s.t. $f(x_1) \neq f(x_2)$.

Proof - Let $L = \mathrm{span}(x_1 - x_2)$. Then define $f_0(\lambda(x_1 - x_2)) = \lambda \|x_1 - x_2\|.\ \forall \lambda \in \mathbb{K}$. $\|f_0\| = 1$. Let f be an extension to X, $\|f\| = 1$. Then $f(x_1 - x_2) = f_0(x_1 - x_2) = \|x_1 - x_2\|. \Rightarrow f(x_1) - f(x_2) = \|x_1 - x_2\|$ $\Rightarrow f(x_1) = f(x_2) + \|x_1 - x_2\| > f(x_2)$ // q.e.d. [This is true for $\mathbb{R}$. For $\mathbb{C}$, just note that $f(x_1) - f(x_2) \neq 0$].

Dual space

Introduce linear structure on the bounded linear functionals: $(f_1 + f_2)(x) = f_1(x) + f_2(x)\ \forall x \in X$. $(\alpha f)(x) = \alpha(f(x))\ \forall x \in X$. Does the norm $\|f\|$ satisfy the required properties?

(i) Non-degeneracy, $\|f\| \ge 0$. $\|f\| = 0 \Leftrightarrow f = 0$. True as $|f(x)| \le \|f\|\|x\|$ (ii) Homogeneity. $\|\alpha f\| = |\alpha|\,\|f\|$ is true by definition.

(iii) Triangle inequality: $|(f_1 + f_2)(x)| = |f_1(x) + f_2(x)| \le |f_1(x)| + |f_2(x)| \le (\|f_1\| + \|f_2\|)\|x\|$, so $\|f_1 + f_2\| \le \|f_1\| + \|f_2\|$.

Definition 4.14 The normed linear space of bounded linear functionals is called the dual space, denoted X^*.

Theorem 4.15 X^* is complete.

Proof - Let $f_n \in X^*$, $n = 1, 2, \dots$ be a Cauchy sequence. i.e. $\forall \varepsilon > 0$, $\exists N$ s.t. $|f_n(x) - f_m(x)| < \varepsilon \|x\|$ if $m, n > N$ $\forall x \in X$. Define linear functional f, s.t. $f(x) = \lim_{n\to\infty} f_n(x)$. Then taking $n \to \infty$ gives us $|f(x) - f_m(x)| \le \varepsilon\|x\|$ $\forall m > N$, $\forall x \in X$. Thus $|f(x)| = |f_m(x) + f(x) - f_m(x)| \le |f_m(x)| + |f(x) - f_m(x)| \le \|f_m\|\|x\| + \varepsilon\|x\| = (\|f_m\| + \varepsilon)\|x\|$ for some fixed $m \Rightarrow$ f is bounded, $\|f\| \le \|f_m\| + \varepsilon$. $\forall m > N \Rightarrow X^*$ is complete // qed.

Examples - (1) $X = H$. Then $\forall f \in X^*$, $f(x) = \langle x, x_0 \rangle$, $\underset{\in X^*}{f} \leftrightarrow \underset{\in H}{x_0}$ is a 1-to-1 correspondence, then X^* is isomorphic to H. We write this as $X^* \cong H$.

(2) If $X = \ell_p$, $1 < p < \infty$, then $X^* = \ell_q$, where $p^{-1} + q^{-1} = 1$.   (3) $\ell_1^* = \ell_\infty$, (4) $\ell_\infty^* \neq \ell_1$. Instead, $c_0^* = \ell_1$.

(5) Let $d = \dim X < \infty$. Then $X^* = X$.

Second dual space.

Let X^{**} be the space of all linear bounded functionals on X^*. Any vector $x \in X$ defines a functional on X^*: $F_x(f) = f(x)$, $\forall f \in X^*$. F is a linear functional (as f is). It is bounded: $|F_x(f)| = |f(x)| \le \|x\|_X \|f\|_{X^*}. \Rightarrow \|F_x\|_{X^{**}} \le \|x\|_X$.

Theorem 4.16 $\|F_x\|_{X^{**}} = \|x\|_X$ $\forall x \in X$.

Proof - Need to find a functional $f \in X^*$ s.t. $F_x(f) = \|f\|_{X^*}\|x\|_X$. Let $L = \mathrm{span}(x) \subset X$. Then define f_0: $f_0(\lambda x) = \lambda\|x\|$, $\lambda \in \mathbb{K}$, so that $\|f_0\| = 1$. Let f be an extension to X s.t. $\|f\| = 1$ by Hahn-Banach. Thus, $F_x(f) = f(x) = f_0(x) = \|x\| \Rightarrow \|F_x\|_{X^{**}} = \|x\|_X$. // q.e.d.

Remark - The map $F_x : X \to X^{**}$ is an isometry as it preserves norms (but not an isomorphism — may not be surjective).

Definition 4.17 The map $F_x : X \to X^{**}$ is called a <u>canonical map</u> of X into X^{**}. If F_x is surjective, then X is said to be <u>reflexive</u>.

Examples - (1) $d = \dim X < \infty$. We know: $\dim X^* = \dim X^{**} = d$. Since F_x is an isometry, it is an isomorphism. Indeed X is reflexive.

(2) $X = H$ is reflexive.   (3) ℓ_p, $1 < p < \infty$. is reflexive, since $\ell_p^* = \ell_q$, $\ell_q^* = \ell_p$.   (4) ℓ_1 is not reflexive   (5) c_0 is not reflexive as $c_0 \neq \ell_\infty = c_0^{**}$.

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Theorem 4.18 A space X is reflexive iff X^* is reflexive.

Proof - $(\Rightarrow)$ DIY, $(\Leftarrow)$ omitted.

Convergence in normed spaces.

Definition 4.19 A sequence x_n <u>converges strongly</u> to x as $n \to \infty$ if $\|x_n - x\| \to 0$ as $n \to \infty$. x_n converges to x in the norm. It <u>converges weakly</u> to x if $f(x_n) \to f(x)$, $n \to \infty$ for all $f \in X^*$.

Notation: $w\text{-}\lim_{n\to\infty} x_n = x$, $x_n \xrightarrow{w} x$, $n \to \infty$.

Example - Let $X = c_0$, $x^{(1)} = (1, 0, 0, \dots)$, $x^{(2)} = (0, 1, 0, \dots)$ etc. For any $f \in X^* = \ell_1$, $f(x^{(n)}) = \sum_{k=1}^{\infty} x_k^{(n)} y_k = y_n \to 0$ ($y \in \ell_1$). $\Rightarrow x^{(n)} \xrightarrow{w} 0$. However, $x^{(n)} \not\to 0$. as $n \to \infty$.

If $x_n \xrightarrow{s} x$, then $x_n \xrightarrow{w} x$.

Remark 4.20 If x_n converges weakly, then the limit is unique. In addition, if x_n converges strongly, then the strong limit coincides with the weak one.

Proof - Suppose that $x_n \xrightarrow{w} x$, $x_n \xrightarrow{w} \tilde{x}$, $n \to \infty$. By Corollary 4.13, there is a functional $f \in X^*$ s.t. $f(x) \neq f(\tilde{x})$. Therefore $f(x_n) \to f(x), f(\tilde{x})$. Since numerical convergent sequences have unique limit, this is a contradiction $\Rightarrow x = \tilde{x}$. Suppose that $x_n \xrightarrow{s} z$, $x_n \xrightarrow{w} x$. We know that $x_n \xrightarrow{w} z$. Since the weak limit is unique, $z = x$. // q.e.d.

Remark - Suppose that $X = H$. Then x_n converges strongly to $x \Leftrightarrow x_n \xrightarrow{w} x$ and $\|x_n\| \to \|x\|$.

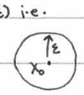
Suppose that $x_n \xrightarrow{w} x$, $n \to \infty$. Is x_n bounded?

<u>Uniform boundedness theorem.</u>

$\boxed{\text{Theorem}}$ 4.21 (Banach-Steinhaus Theorem).

Suppose X is a Banach space, and let $M \subset X^*$. Assume that the set $\{f(x), f \in M\}$ is bounded for each $x \in X$ i.e. $\exists$ constant $C = C(x)$ s.t. $|f(x)| \leq C(x)$ $\forall f \in M$. Then M is uniformly bounded in the norm i.e. $\exists C_1 > 0$ s.t. $\|f\| \leq C_1$ $\forall f \in M$.

Proof – Define $A_k = \{x \in X : |f(x)| \leq k \ \forall f \in M, k = 1, 2, \dots\}$. Claim: A_k is closed. Indeed, let $x_\ell \to x$, $x_\ell \in A_k$. We want to show $x \in A_k$. Thus $|f(x_\ell)| \leq k$ $\forall f \in M$. By continuity of f, $|f(x)| \leq k \Rightarrow x \in A_k$. Therefore $A_k = [A_k]$. Claim: $X = \bigcup_{k=1}^{\infty} A_k$. Let $x \in X$. We know that $|f(x)| \leq C(x)$ $\forall f \in M$. Take $k \geq C(x)$ so that $|f(x)| \leq k \Rightarrow x \in A_k$. $\Rightarrow X \subset \bigcup_k A_k$. By the Baire Category theorem, at least one A_k is dense in some ball $B(x_0, \varepsilon)$, $x_0 \in X$, $\varepsilon > 0$. Let k_0 be a number s.t. $[A_{k_0}] \supset B(x_0, \varepsilon)$ i.e. $|f(x)| \leq k_0$, $\forall x \in B(x_0, \varepsilon)$ uniformly in $f \in M$, i.e. $|f(x_0 + \varepsilon x)| \leq k_0$ $\forall x \in B(0,1)$ uniformly in $f \in M$. Thus, $\varepsilon |f(x)| = |f(\varepsilon x) - f(x_0) + f(x_0)|$



$\leq |f(x_0)| + |f(x_0 + \varepsilon x)| \leq C(x_0) + k_0$ $\forall x \in B(0,1)$ $\forall f \in M$. Therefore, $|f(x)| \leq \frac{k_0 + C(x_0)}{\varepsilon}$ $\forall x \in B(0,1)$, $\forall f \in M$. By Lemma 4.5, $\|f\| \leq \frac{k_0 + C(x_0)}{\varepsilon}$, $\forall f \in M$ // q.e.d.

$\boxed{\text{Corollary}}$ 4.22 Let $x_n \xrightarrow{w} x$, $n \to \infty$. Then there is a constant $C > 0$ s.t. $\|x_n\| \leq C$ $\forall n$.

Proof – By definition, $f(x_n) \to f(x)$ $\forall f \in X^*$. Hence, $|f(x_n)| \leq C$ $\forall n$ with some $C = C_f$. Therefore, $|F_{x_n}(f)| \leq C_f$ $\forall n$. The space X^* is complete, so by the Banach-Steinhaus theorem, $\|F_{x_n}\| \leq C_1$ uniformly bounded with some $C_1 > 0$. Recall that $\|F_{x_n}\| = \|x_n\|$; and hence $\|x_n\| \leq C_1$ $\forall n$ // q.e.d.

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Note: consider $L_2(0,1)$, where $\int_0^1 |u|^2 dt < \infty$. This is not the Riemann integral! It is the approximation of u by continuous functions (completion of $C_2(0,1)$).

Chapter 5
LINEAR OPERATORS.

$\boxed{\text{Definition}}$ 5.1 A mapping $A : X \to Y$ between two linear spaces X, Y is said to be a <u>linear operator</u> if $A(\alpha x_1 + \beta x_2) = \alpha A x_1 + \beta A x_2$ for all $x_1, x_2 \in X$, $\alpha, \beta \in \mathbb{K}$. ← generalisation of linear functional.

Let X, Y be normed spaces. A linear operator A is said to be <u>continuous</u> if A is continuous on X. A is said to be <u>bounded</u> if the image of the unit ball $B_X(0,1)$ is a bounded set i.e. $\exists R > 0$ s.t. $A(B_X(0,1)) \subset B_Y(0,R)$ [i.e. $\|Ax\|_Y \leq R\|x\|_X$].

$\boxed{\text{Theorem}}$ 5.2 A linear operator is continuous $\Leftrightarrow$ it is bounded. (See theorem 4.3).

$\boxed{\text{Definition}}$ 5.3 The <u>norm</u> of a bounded linear operator A is $\|A\| = \sup_{x : \|x\|_X = 1} \|Ax\|_Y$.

$\boxed{\text{Lemma}}$ 5.4 $\|A\| = \sup_{\|x\|_X \leq 1} \|Ax\|_Y = \sup_{x \neq 0} \frac{\|Ax\|_Y}{\|x\|_X}$ (See Lemma 4.5).

$\boxed{\text{Lemma}}$ 5.5 Let $D \subset X$ be a linear set s.t. $[D] = X$. Let $A_0 : D \to Y$ be a bounded linear operator. If Y is complete, then A_0 uniquely extends to X as a bounded linear operator A. Moreover, $\|A\| = \|A_0\|$.

Examples – (1) The zero operator: $Ax = 0$ $\forall x \in X$. Notation: 0, $\|0\| = 0$. (2) Identity operator: $I : X \to X$, $Ix = x$ $\forall x \in X$. $\|I\| = 1$.

(3) Let $X = L_2(0,1)$. Let $m \in C[0,1]$ and define $(Au)(t) = m(t) u(t)$, $u \in X$ is a multiplication operator. $A : X \to X$ and is bounded. $\int_0^1 |m(t) u(t)|^2 dt = \int_0^1 |m(t)|^2 |u(t)|^2 dt \leq c^2 \int_0^1 u(t)^2 dt = c^2 \|u\|^2$ $\Rightarrow \|Au\|^2 \leq c^2 \|u\|^2$, so $\|Au\| \leq c\|u\|$ where $c = \max_{t \in [0,1]} |m(t)|$ $\Rightarrow A$ is bounded.

(4) $X = L_2(0,1)$. Let $\mathcal{K} = \mathcal{K}(x,y)$ be a function defined on $(0,1) \times (0,1)$ s.t. $\left[\int_0^1 \int_0^1 |\mathcal{K}(x,y)|^2 dx\, dy\right]^{\frac{1}{2}} = K < \infty$. Let $(Au)(x) = \int_0^1 \mathcal{K}(x,y) u(y)\, dy$. Claim: Au is bounded on X. Then $\|Au\|^2 = \int_0^1 \left| \int_0^1 \mathcal{K}(x,y) u(y)\, dy \right|^2 dx$. Then since $\sum_m \left| \sum_n k_{mn} u_n \right|^2 \overset{CSI}{=} \sum_m \left[\left(\sum_n |k_{mn}|^2 \right)^{\frac{1}{2}} \left(\sum_n |u_n|^2 \right)^{\frac{1}{2}} \right]^2 = \sum_{mn} |k_{mn}|^2 = \sum_n |u_n|^2$; we can perform a similar estimate for $\|Au\|^2$: $\|Au\|^2 = \int_0^1 \left| \int_0^1 \mathcal{K}(x,y) u(y)\, dy \right|^2 dx \overset{CSI}{\leq} \int_0^1 \left[\left(\int |\mathcal{K}(x,y)|^2 dy \right)^{\frac{1}{2}} \left(\int |u(y)|^2 dy \right)^{\frac{1}{2}} \right]^2 dx = \int_0^1 \int_0^1 |\mathcal{K}(x,y)|^2 dx\, dy \cdot \|u\|^2 = K^2 \|u\|^2 \Rightarrow \|A\| \leq K$.

Remark – Kernel $\mathcal{K}$ is called a Hilbert-Schmidt kernel.

(5) Differentiation operator. Let $C^1[a,b] = \{f \in C[a,b] : f' \in C[a,b]\}$. Let $X = C_2^1[a,b]$ be the space of C^1-functions with the L_2-norm: $\|u\| = \left[\int_a^b |u(t)|^2 dt\right]^{\frac{1}{2}}$, $u \in C^1$. The completion is $L_2(a,b)$. Let $X = C_2^1[-\pi,\pi]$, $Y = L_2[-\pi,\pi]$. Define $(Tu)(t) = -i u'(t)$, $u \in X$. Claim: This operator is unbounded! Indeed let $e_n(t) = \frac{1}{\sqrt{2\pi}} e^{int}$, $n \in \mathbb{Z}$, so $\|e_n\| = 1$. On the other hand, $Te_n = n e_n$. $\Rightarrow \|Te_n\| = |n| \to \infty$ as $n \to \infty$, so $\sup_{\|x\|=1} \|Tx\| = \infty$. T acts on $L_2(-\pi,\pi)$ but it is defined on the smaller set $C^1[-\pi,\pi]$.

<u>Algebra of bounded linear operators.</u>

Let X, Y be normed spaces. Linear structure: if A, B are linear, bounded $(\alpha A + \beta B)x = \alpha Ax + \beta Bx$ $\forall x \in X$, $\forall \alpha, \beta \in \mathbb{K}$. Does $\|A\|$ have all required properties?

① Non-degeneracy: $\|A\| \geq 0$ is clear. Then if $\|A\| = 0$, $A = 0 \Rightarrow$ so non-degeneracy holds. ✓

② Homogeneity: $\|\alpha A\| = \sup_{\|x\|=1} \|\alpha Ax\| = \sup_{\|x\|=1} |\alpha| \|Ax\| = |\alpha| \sup_{\|x\|=1} \|Ax\| = |\alpha| \|A\|$. ✓

③ Triangle inequality: $\|(A+B)x\| = \|Ax + Bx\| \leq \|Ax\| + \|Bx\| \leq \|A\|\|x\| + \|B\|\|x\| = (\|A\| + \|B\|)\|x\|$. $\Rightarrow \|A+B\| \leq \|A\| + \|B\|$ ✓.

The normed space of linear bounded operators $A : X \to Y$ is denoted by $\mathcal{B}(X,Y)$. Also as a point on notation, we write $\mathcal{B}(X,X) = \mathcal{B}(X)$.

$\boxed{\text{Theorem}}$ 5.6 If Y is a Banach space, then $\mathcal{B}(X,Y)$ is a Banach space as well. (See Theorem 4.15).

$\boxed{\text{Definition}}$ 5.7 Let $A \in \mathcal{B}(X,Y)$, $B \in \mathcal{B}(Y,Z)$, then the <u>product of operators</u> BA is defined by $(BA)x = B(Ax)$ $\forall x \in X$.

Remark – BA is bounded. $\|(BA)x\| = \|B(Ax)\| \leq \|B\|\|Ax\| \leq \|B\|\|A\|\|x\|$ $\forall x \in X$. $\Rightarrow \|BA\| \leq \|B\|\|A\| \Rightarrow BA \in \mathcal{B}(X,Z)$.

In $B(X)$, we have $(AB)C = A(BC)$ [associativity], $(A+B)C = AC + BC$, $A(B+C) = AB + AC$ [distributivity]. Thus $B(X)$ is an algebra. If X is complete, $B(X)$ is a Banach algebra.

Convergence

Definition 5.8 Let $A, A_n \in B(X,Y)$, $n = 1, 2, \dots$. (1) We say that A_n converges uniformly (or converges in the norm) to A if $\|A - A_n\| \to 0$ as $n \to \infty$.

(2) We say that A_n converges to A strongly if $\|A_n x - Ax\| \to 0$, $n \to \infty$ for all $x \in X$. (3) We say that A_n converges to A weakly if $f(A_n x) \to f(Ax)$, $n \to \infty$ $\forall x \in X$, $f \in Y^*$.

These three forms of convergence are commonly used, and are arranged in decreasing order of strength. (1) $\Rightarrow$ (2) $\Rightarrow$ (3). [For (1) $\Rightarrow$ (2), see $\|A_n x - Ax\| = \|(A_n - A)x\| \leq \|A_n - A\| \|x\|$).

The reverse implications do not hold in general; however if $\dim X < \infty$, $\dim Y < \infty$, then (1) $\Leftrightarrow$ (2) $\Leftrightarrow$ (3). Otherwise, consider the following counterexample:

$[\|x\|_2 = (\sum_{k=1}^{\infty} |x_k|^2)^{1/2} < \infty]$

Let $X = \ell_2$. Let $A: X \to X$ be the operator $Ax = A_n(x_1, x_2, \dots) = (\underbrace{0, 0, \dots, 0}_{n \text{ times}}, x_{n+1}, x_{n+2}, \dots)$. Then $\|A_n x\|_2^2 = \sum_{k=n+1}^{\infty} |x_k|^2 \to 0$, as $n \to \infty$. Hence, A_n converges to 0 strongly.

On the other hand, $\|A_n\| = 1 \Rightarrow A_n$ does not converge to 0 uniformly.

If (1) has a limit, it is unique. If (2) has a limit, it is unique: $\|Ax - Bx\| \leq \underbrace{\|A_n x - Ax\|}_{\to 0} + \underbrace{\|A_n x - Bx\|}_{\to 0}$. And likewise if (3) has a limit, it is unique.

Theorem 5.9 (Banach-Steinhaus theorem for operators).

Let X, Y be Banach spaces and let $M \subset B(X,Y)$. Suppose that for every $x \in X$ there is a constant $C = C(x) > 0$ s.t. the $\|Ax\| \leq C(x)$ for all $A \in M$. Then there is a constant $C_1 > 0$ s.t.

$\|A\| \leq C_1$ for all $A \in M$.

Proof – see theorem 4.21.

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Let X be a reflexive space. Show that it is weakly complete i.e. if $f(x_n)$ is Cauchy for any $f \in X^*$, then $\exists$ $x \in X$ s.t. $f(x_n) \to f(x)$ as $n \to \infty$.

Example – Let $H = \ell_2$, $S_n x = S_n(x_1, x_2, \dots) = (\overbrace{0, \dots, 0}^{n}, x_1, x_2, \dots)$. Claim $S_n \xrightarrow{w} 0$, $n \to \infty$. Observation: $\|S_n x\|_2 = \|x\|_2 = (\sum_{k=1}^{\infty} |x_k|^2)^{\frac{1}{2}}$. On the other hand, let $\xi \in \ell_2$ be an arbitrary vector:

$|\langle S_n x, \xi \rangle| = |\sum_{k=n+1}^{\infty} x_{k-n} \bar{\xi}_k| \overset{CSI}{\leq} (\sum_{k=n+1}^{\infty} |x_{k-n}|^2)^{\frac{1}{2}} (\sum_{k=n+1}^{\infty} |\xi_k|^2)^{\frac{1}{2}}$. As $n \to \infty$, $(\sum_{k=n+1}^{\infty} |\xi_k|^2)^{\frac{1}{2}} \to 0$, so $|\langle S_n x, \xi \rangle| \to 0 \Rightarrow S_n \xrightarrow{w} 0$ as claimed, q.e.d.

Corollary 5.10 Let X, Y be Banach spaces. Let $A_n \in B(X,Y)$. Suppose that for some mapping $A: X \to Y$ we have $A_n x \to Ax$ as $n \to \infty$ for every $x \in X$. Then the norms $\|A_n\|$ are uniformly bounded, and $A \in B(X,Y)$.

Proof – Write $A(\alpha x + \beta y) = \lim_{n \to \infty} A_n(\alpha x + \beta y) = \lim_{n \to \infty} (\alpha A_n x + \beta A_n y) = \alpha \lim_{n \to \infty} A_n x + \beta \lim_{n \to \infty} A_n y = \alpha Ax + \beta Ay$ $\forall x, y \in X$, $\forall \alpha, \beta \in \mathbb{K}$. Thus A is linear. Observe also that

$\|A_n x\|$ is bounded uniformly in n. Thus by Banach-Steinhaus theorem (5.9), $\exists$ constant C s.t. $\|A_n\| \leq C$ $\forall n$. Therefore, $\|A_n x\| \leq C\|x\|$ $\forall x \in X$. Consequently, see that

$\|Ax\| = \lim_{n \to \infty} \|A_n x\| \leq C\|x\|$. Hence, $A \in B(X,Y)$, q.e.d.

Adjoint operators

Let $X = H$ be a Hilbert space (throughout this section).

Definition 5.11 A function $\phi: H \times H \to \mathbb{K}$ is called a sesquilinear functional/form if $\phi(\alpha_1 x_1 + \alpha_2 x_2, y) = \alpha_1 \phi(x_1, y) + \alpha_2 \phi(x_2, y)$ (linear in first variable); $\phi(x, \alpha_1 y_1 + \alpha_2 y_2) = \bar{\alpha}_1 \phi(x, y_1) + \bar{\alpha}_2 \phi(x, y_2)$ (antilinear in second variable).

ϕ is said to be continuous if its norm, $\|\phi\| = \sup_{\|x\| = \|y\| = 1} |\phi(x,y)|$, is finite.

Note – If ϕ is continuous, $|\phi(x,y)| \leq \|\phi\| \|x\| \|y\|$ $\forall x, y \in H$. Also, the functional given by $\Psi(x,y) = \overline{\phi(y,x)}$ is also sesquilinear.

Example – Let $\phi_1(x,y) = \langle Tx, y \rangle$, $T \in B(H)$. $\phi_2(x,y) = \langle x, Sy \rangle$, $S \in B(H)$. These are both sesquilinear. Are there any others? No! These examples exhaust all possibilities:

Theorem 5.12 Let ϕ be a sesquilinear continuous form. Then there are exactly two uniquely defined operators $T, S \in B(H)$ s.t. $\phi(x,y) = \langle Tx, y \rangle = \langle x, Sy \rangle$ $\forall x, y \in H$. Furthermore, $\|S\| = \|T\| = \|\phi\|$.

Proof – Let $f_y(x) = \phi(x,y)$. Then f_y is linear, f_y is bounded by continuity – $|f_y(x)| = |\phi(x,y)| \leq \|\phi\| \|x\| \|y\|$, so $\|f_y\| \leq \|\phi\| \|y\|$. By the Riesz theorem, $\exists$ vector $h \in H$

s.t. $f_y(x) = \langle x, h \rangle$. Then define the mapping S by $h = Sy$. Thus, $\|Sy\| = \|h\| = \|f_y\| \leq \|\phi\| \|y\|$. It is very easy to check that S is linear (left as exercise).

Clearly, $\|S\| \leq \|\phi\|$. To prove that $\|\phi\| \leq \|S\|$, write $\phi(x,y) = \langle x, Sy \rangle$. So $|\phi(x,y)| \leq \|x\| \|Sy\|$ $\leq \|S\| \|x\| \|y\|$, thus $\|\phi\| \leq \|S\| \Rightarrow \|S\| = \|\phi\|$. Hence, if we have

$\langle x, S_1 y \rangle = \langle x, S_2 y \rangle$ $\forall x, y \in H$, then $S_1 y = S_2 y \Rightarrow S_1 = S_2$, so S is unique. To check that $\phi(x,y) = \langle Tx, y \rangle$ with some $T \in B(H)$, consider the functional

$\Psi(x,y) = \overline{\phi(y,x)}$, and use the first part of this proof to argue exactly the same way, q.e.d.

Definition 5.13 Let $T \in B(H)$. The functional $\phi(x,y) = \langle Tx, y \rangle$ is called the sesquilinear form of the operator T. The operator S defined by $\phi(x,y) = \langle x, Sy \rangle$ is called the adjoint of T, denoted $S = T^*$.

If $T = T^*$, then T is called self-adjoint.

[Remark: $\langle Tx, y \rangle = \langle x, T^* y \rangle$]

Theorem 5.14 Let $T \in B(H)$. Then (1) $\|T^*\| = \|T\|$, (2) $T^{**} = (T^*)^* = T$, (3) $(\alpha_1 T_1 + \alpha_2 T_2)^* = \bar{\alpha}_1 T_1^* + \bar{\alpha}_2 T_2^*$, (4) $(T_1 T_2)^* = T_2^* T_1^*$

Proof – (1), (2) left as exercises. (3): $\langle (\alpha_1 T_1 + \alpha_2 T_2)x, y \rangle = \alpha_1 \langle T_1 x, y \rangle + \alpha_2 \langle T_2 x, y \rangle = \alpha_1 \langle x, T_1^* y \rangle + \alpha_2 \langle x, T_2^* y \rangle = \langle x, \bar{\alpha}_1 T_1^* y \rangle + \langle x, \bar{\alpha}_2 T_2^* y \rangle$

$= \langle x, (\bar{\alpha}_1 T_1^* + \bar{\alpha}_2 T_2^*) y \rangle$, so we obtain the adjoint, q.e.d. (4): $\langle T_1 T_2 x, y \rangle = \langle T_2 x, T_1^* y \rangle = \langle x, T_2^* T_1^* y \rangle$, q.e.d.

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For matrices, if $A \to a_{jk}$, $A^* \to \overline{a_{kj}}$. Hence it acts by $(Ax)_j = \sum_{k=1}^{d} a_{jk} x_k$, $(A^* x)_j = \sum_{k=1}^{d} \overline{a_{kj}} x_k$.

Examples – ① $H = \ell_2$, $\|x\|_2 = [\sum_{k=1}^{\infty} |x_k|^2]^{\frac{1}{2}}$, $(Ax)_j = \sum_{k=1}^{\infty} a_{jk} x_k$, $(A^* x)_j = \sum_{k=1}^{\infty} \overline{a_{kj}} x_k$. We do not know if operator is bounded, but if $\sum_{j,k=1}^{\infty} |a_{jk}|^2 < \infty \Rightarrow A$ is bounded.

② $H = L_2(a,b)$. Consider integral operator $(Tu)(x) = \int_a^b K(x,y) u(y)\, dy$, $u \in H$. Assume $K \in C([a,b] \times [a,b])$. Then T is bounded. We seek to find T^*.

$\langle Tu, v \rangle = \int_a^b \int_a^b K(x,y) u(y) \overline{v(x)}\, dy\, dx = \int_a^b \int_a^b u(y) \overline{\overline{K(x,y)} v(x)}\, dy\, dx$. Then T^* has the kernel $\tilde{K}(x,y) = \overline{K(y,x)}$. To check for self-adjointness, need $K(x,y) = \overline{K(y,x)}$.

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Hence $e^{x^2+y^2}$ is self-adjoint, $i\sin(x-y)$ is self-adjoint, e^{ixy} is not self-adjoint.

③ $H = L_2(a,b)$, $m \in C[a,b]$. $(Au)(x) = m(x)u(x)$, $u \in H$, $(A^*u)(x) = \overline{m(x)}\,u(x)$, $u \in H$.

④ $H = L_2(a,b)$, $Tu = -iu'$, $u \in C^1_2[a,b]$ i.e. C^1 functions with L_2-norm. Technically, we cannot define an adjoint based on what we know as T is unbounded. Nevertheless, we attempt to construct one. $\langle Tu, v\rangle = \int_a^b (-iu'(x))\overline{v(x)}\,dx \overset{IBP}{=} \underbrace{-iu(x)\overline{v(x)}\Big|_a^b}_{\text{does not have form of scalar product.}} - (-i)\int_a^b u(x)\overline{v'(x)}\,dx = -iu(x)\overline{v(x)}\Big|_a^b + \int_a^b u(x)\overline{(-iv'(x))}\,dx = -iu(x)\overline{v(x)}\Big|_a^b + \langle u, Tv\rangle$.

Hence, $\langle Tu, v\rangle = -i[u(b)\overline{v(b)} - u(a)\overline{v(a)}] + \langle u, Tv\rangle$. Restrict the set of functions on which T is defined. $D(T) = \{u \in C^1[a,b] : u(a) = u(b) = 0\}$.

Then on $D(T)$, $\langle Tu, v\rangle = \langle u, Tv\rangle$ $\forall u, v \in D(T)$. [More generally, we can also just set $u(a) = u(b)$, but this gives a different operator].

$\boxed{\text{Theorem}}$ 5.15 (1) Every $A \in B(H)$ is weakly continuous i.e. if $x_n \xrightarrow{w} x$ as $n \to \infty$, then $Ax_n \xrightarrow{w} Ax$ as $n \to \infty$.

(2) Let $A_n \xrightarrow{w} A$ as $n \to \infty$, then $A_n^* \xrightarrow{w} A^*$, i.e. $\langle A_n x, y\rangle \to \langle Ax, y\rangle$ $\forall x, y \in H$, then $\langle A_n^* x, y\rangle \to \langle A^* x, y\rangle$ $\forall x, y \in H$.

Proof – (1) Write $\langle Ax_n, y\rangle = \langle x_n, A^*y\rangle \xrightarrow{w} \langle x, A^*y\rangle = \langle Ax, y\rangle$ as $n \to \infty$. $\forall y \in H$ // q.e.d. (2) Write $\langle A_n^* x, y\rangle = \langle x, A_n y\rangle \xrightarrow[n\to\infty]{w} \langle x, Ay\rangle = \langle A^* x, y\rangle$ // q.e.d.

Denote by $R(A)$ the range of A: then,

$\boxed{\text{Theorem}}$ 5.16 Let $A \in B(H)$. Then $H = [R(A)] \oplus \operatorname{Ker} A^*$.

Proof – We already know that $\operatorname{Ker} A^*$ is a closed subspace, as $A \in B(H)$. Then we need to show that $R(A)$ is a linear set.

$\boxed{\text{Lemma}}$ 5.17 $R(A)$ is a linear set for any $A \in B(X, Y)$.

Proof – Let $y_1, y_2 \in R(A)$. Want: $\alpha_1 y_1 + \alpha_2 y_2 \in R(A)$ $\forall \alpha_1, \alpha_2 \in \mathbb{K}$. Let $Ax_1 = y_1$, $Ax_2 = y_2$. Then $\alpha_1 y_1 + \alpha_2 y_2 = \alpha_1 Ax_1 + \alpha_2 Ax_2 = A(\alpha_1 x_1 + \alpha_2 x_2) \in R(A)$ // q.e.d.

(Theorem 5.16, cont'd)

Proof – Let $Ax = y$. Then $\forall z \in H$, we have $\langle y, z\rangle = \langle x, A^*z\rangle$. $y \in R(A)$. What is $R(A)^\perp$? Look for those $z \in H$ for which $\langle y, z\rangle = 0$ $\forall y \in R(A)$, since $\langle \underset{\in H}{x}, A^*z\rangle = 0$, $A^*z = 0$ so $z \in \operatorname{Ker}(A^*)$. Conversely, if $z \in \operatorname{Ker} A^*$, then $\langle x, A^*z\rangle = 0$ $\forall x \in H$. So $\langle Ax, z\rangle = 0$ and hence $z \in R(A)^\perp$. Thus, $R(A)^\perp = \operatorname{Ker} A^*$.

Recall that $R(A)^\perp = [R(A)]^\perp$, so $H = [R(A)] \oplus \operatorname{Ker} A^*$ // q.e.d.

Inverse operators.

Let X, Y be normed spaces. Suppose $A \in B(X, Y)$. We study equation $Ax = y$. If there is a unique solution, then $x = A^{-1}y$.

$\boxed{\text{Definition}}$ 5.18 Let $A \in B(X, Y)$ be injective. Then the inverse operator A^{-1} is defined as an operator mapping each $y \in R(A)$ into the vector $x \in X$ uniquely defined by the relation $Ax = y$. In other words,

$A^{-1}Ax = x$ $\forall x \in X$ and $AA^{-1}y = y$ $\forall y \in R(A)$.

$\boxed{\text{Theorem}}$ 5.19 If A^{-1} exists, it is a linear operator.

Proof – NTP: $A^{-1}(\alpha_1 y_1 + \alpha_2 y_2) = \alpha_1 A^{-1}(y_1) + \alpha_2 A^{-1}(y_2)$ – ⊛ $\forall y_1, y_2 \in R(A)$ $\forall \alpha_1, \alpha_2 \in \mathbb{K}$. Let $Ax_1 = y_1$, $Ax_2 = y_2$, so $x_1 = A^{-1}y_1$, $x_2 = A^{-2}y_2$. So ⊛: $A^{-1}(\alpha_1 Ax_1 + \alpha_2 Ax_2) \overset{?}{=} \alpha_1 x_1 + \alpha_2 x_2$

$LHS = A^{-1}(A(\alpha_1 x_1 + \alpha_2 x_2)) = \alpha_1 x_1 + \alpha_2 x_2 = RHS$ as required // q.e.d.

$\boxed{\text{Theorem}}$ 5.20 Let X be a Banach space. Let $A \in B(X)$ be s.t. $\|A\| < 1$. Then the inverse $(I-A)^{-1}$ exists, and it is given by the uniformly convergent series $(I-A)^{-1} = \sum_{k=0}^{\infty} A^k$, where $A^0 = I$

Note – In here, $R(I-A) = X$. (entire space)

Proof – Let $S = \sum_{k=0}^{\infty} A^k$. Consider partial sums $S_n = \sum_{k=0}^{n} A^k$. This is a Cauchy sequence – assume $m > n$, then $\|S_m - S_n\| = \|\sum_{k=n+1}^{n} A^k\| \overset{(\Delta)}{\le} \sum_{k=n+1}^{n} \|A^k\| \le \sum_{k=1}^{n} \|A\|^k \le \|A\|^{n+1} \cdot \frac{1}{1-\|A\|}$ $\to 0$ as $n \to \infty$. Since X is Banach, $B(X)$ is Banach as well $\Rightarrow$ the series converges uniformly, so $S \in B(X)$. We finally need to show that $S = (I-A)^{-1}$. Write:

$(I-A)S_n = \sum_{k=0}^{n} A^k - \sum_{k=1}^{n+1} A^k = I - A^{n+1}$ (telescoping series). As $n \to \infty$, LHS gives $(I-A)S$ by uniform continuity, while RHS gives $I - 0$ as $\|A^{n+1}\| \le \|A\|^{n+1} \to 0$.

Likewise, $S_n(I-A) \to S(I-A) \to I$; so $S = (I-A)^{-1}$

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$\boxed{\text{Proposition}}$ 5.21 ("Resolvent Identity").

Let $A, B \in B(X, Y)$, $A^{-1}, B^{-1} \in B(Y, X)$, where X, Y are normed spaces. Denote $V = B - A$. Then $A^{-1} - B^{-1} = A^{-1}VB^{-1} = B^{-1}VA^{-1}$.

Proof – Left as exercise.

Exercise – Let $A, B, A^{-1}, B^{-1} \in B(H)$, H being a Hilbert space. Then $(A^{-1})^* = (A^*)^{-1}$, $(AB)^{-1} = B^{-1}A^{-1}$.

Is the inverse defined on the entire space? Is the inverse bounded?

The open mapping theorem.

$\boxed{\text{Definition}}$ 5.22 Let X, Y be metric spaces. Then the function $f: X \to Y$ is said to be an open mapping if f maps open sets in X into open sets in Y.

$\boxed{\text{Theorem}}$ 5.23 (Open Mapping Theorem)

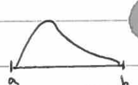
Let X, Y be Banach spaces. Suppose that $A \in B(X, Y)$ is a surjection. Then A is an open mapping.

Proof – Omitted, non-examinable.

$\boxed{\text{Corollary}}$ 5.24 Let X, Y be Banach spaces. Suppose that $A \in B(X, Y)$ is a bijection. Then A^{-1} is bounded.

Proof – $\forall$ open set $M \subset X$, the image $A(M) \subset Y$ is also open. (See theorem 5.23). Then A^{-1} exists. Moreover, we observe that the set $A(M)$ is the preimage of M under mapping A^{-1}. $\therefore$ The map A^{-1} is continuous $\Rightarrow$ A^{-1} is bounded as claimed // q.e.d.

Closed operators



$u(a)=u(b)=0$ or $u(a)=u(b)$.

The operators are not assumed to be bounded. Let $D(A)$ be the domain of the operator A, i.e. $D(A) \subset X$ is a set of vectors where A makes sense.

$\boxed{\text{Definition}}$ 5.25 A linear operator $A: D(A) \to Y$ is said to be closed if the assumptions $\begin{cases} x_n \to x, \ x_n \in D(A) \\ Ax_n \to y \end{cases}$ imply that $x \in D(A)$ and $Ax = y$. [Note – $A \in B(X,Y) \Rightarrow A$ is closed].

$\boxed{\text{Theorem}}$ 5.26 Every $A \in B(X,Y)$ is closed.

Proof – Let $x_n \to x$, $x_n \in D(A) = X$. $Ax_n \to y$. Since A is continuous, so $Ax_n \to Ax$. Since the limit is unique, $y = Ax$. Hence A is closed.

Example – Let $X = C[0,1]$, $(Af)(x) = f'(x)$, $f \in D(A) = C^1[0,1]$. Claim: A is closed. Indeed, assume that $f_n \to f$ in X, and that $f_n' \to g \in X$, $f_n \in D(A)$. We want $f \in C^1[0,1]$ and $f' = g$.

Write: $f_n(x) = f_n(0) + \int_0^x f_n'(t)\,dt$ (Fundamental theorem of calculus). As $n \to \infty$, $f_n(x) \to f(x)$, $\forall x$ and $f_n'(t) \to g(t)$ uniformly in $t \in [0,1]$. Therefore, $f(x) = f(0) + \int_0^x g(t)\,dt$,

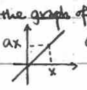
so $f \in C^1[0,1]$ and $f'(x) = g(x)$ $\forall x \in [0,1]$.

Let X, Y be normed spaces. Define the direct sum $X \oplus Y$ as follows: $X \oplus Y$ is the set of ordered pairs (x,y), $x \in X$, $y \in Y$ with the linear structure, i.e. $\begin{cases} (x_1,y_1) + (x_2,y_2) = (x_1+x_2, y_1+y_2) \\ \alpha(x,y) = (\alpha x, \alpha y). \end{cases}$

Thus, $X \oplus Y$ is a linear set. Define also the norm $\|(x,y)\| = \|x\|_X + \|y\|_Y$. If X, Y are Banach, then $X \oplus Y$ is Banach as well.

$\boxed{\text{Definition}}$ 5.27 Let $A: D(A) \to Y$ be a linear operator. Then the set $G_A = \{(x, Ax), x \in D(A)\} \subset X \oplus Y$ is called the graph of the operator A.

Remark – If $A = Y = \mathbb{R}$ and $Ax = ax$, $a \in \mathbb{R}$ fixed. Then $G_A = \{(x, ax), x \in \mathbb{R}\} \Rightarrow$



$G_A \subset X \oplus Y$.

G_A is a linear set, i.e. $(x_1, Ax_1), (x_2, Ax_2) \in G_A \Rightarrow (x_1+x_2, A(x_1+x_2)) \in G_A$ and $(\alpha x_1, \alpha A x_1) \in G_A$.

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$\boxed{\text{Theorem}}$ 5.28 The operator A is closed $\Leftrightarrow$ G_A is closed.

($\Rightarrow$)
Proof – Assume A is closed, i.e. $\begin{cases} x_n \to x, \ x_n \in D(A) \\ Ax_n \to y \end{cases} \Rightarrow \begin{cases} x \in D(A) \\ Ax = y \end{cases}$. If $(x_n, Ax_n) \to (x,y)$ where $x_n \in D(A)$, then $(x,y) \in G_A$, i.e. $x \in D(A)$ and $y = Ax$.

However, we already have obtained this exact result, so we are done, q.e.d. [Reverse done by reversing steps, in same manner, to rephrase equivalent defs].

($\Leftarrow$)
Assume that G_A is closed, i.e. if $(x_n, Ax_n) \to (x,y)$ as $n \to \infty$, $x_n \in D(A)$, then $(x,y) \in G_A$ i.e. $x \in D(A)$, $y = Ax$. $\Rightarrow$ A is closed, q.e.d.

Recall that $A \in B(X,Y) \Rightarrow A$ is closed. We now also seek to prove the converse.

$\boxed{\text{Theorem}}$ 5.29 (Closed Graph Theorem)

Let X, Y be Banach. Let $A: X \to Y$ be a linear operator, with $D(A) = X$. If A is closed, then A is bounded.

Proof – A is closed $\Rightarrow$ G_A is closed (Theorem 5.28). G_A is a Banach space. Define $P: G_A \to X$ by $P(x, Ax) = x$ [retains only first component]. P is bounded: as we have

$\|P(x,Ax)\| = \|x\|_X \leq \|(x,Ax)\| = \|x\|_X + \|Ax\|_Y$. Hence, $\|P\| \leq 1$. P is a bijection, by corollary 5.24, P^{-1} is bounded. Therefore $\|P^{-1}x\| = \|(x,Ax)\| \leq \|P^{-1}\|\,\|x\|$.

(divide by $\|x\|$).

i.e. $\|x\| + \|Ax\| \leq \|P^{-1}\|\,\|x\| \Rightarrow$ LHS $\geq \|Ax\|$, so $\|Ax\| \leq \|P^{-1}\|\,\|x\|$ and $\|A\| \leq \|P^{-1}\|$, q.e.d.

Here is an important corollary: Let X be a Banach space, and let Y, Z be closed subspaces of X s.t. $X = Y \oplus Z$. For any $x \in X$ there is a uniquely defined pair $y \in Y$, $z \in Z$ s.t. $x = y + z$. Define an operator $\Pi: X \to Y$, $\Pi x = y$. — projection operator.

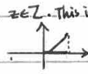
$\boxed{\text{Corollary}}$ 5.30 (Non-orthogonal projection).

The operator Π is bounded.

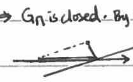
Proof – Show that Π is closed, i.e. G_Π is closed. Suppose that $(x_n, \Pi x_n) \to (x,y)$ in $X \oplus Y$. We want to show $(x,y) \in G_\Pi$, i.e. $x \in X$, $y = \Pi x$. Indeed, the sequence $z_n = x_n - \Pi x_n$

converges to $z = x - y \in Z$, so $x = y + z$ with $y \in Y$, $z \in Z$. This is a unique representation. By definition of Π, $\Pi x = y$ $\Rightarrow$ G_Π is closed. By Theorem 5.29, Π is bounded, q.e.d.

Remark – $\|\Pi\| \leq 1$ is not true in general.



orthogonal (true). But not true otherwise in general



, here the projection is longer!

Of course though $\|\Pi\| \geq 1$.

END OF SYLLABUS.

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