## 3103 Functional Analysis Notes

Based on the 2014 spring lectures by Prof A Sobolev

The Author(s) has made every effort to copy down all the content on the board during lectures. The Author(s) accepts no responsibility whatsoever for mistakes on the notes nor changes to the syllabus for the current year. The Author(s) highly recommends that the reader attends all lectures, making their own notes and to use this document as a reference only.

	MATH3103 - Functional Avalysis.	
		13 January 2014
	Rof A. Sobolen, from 7-10, office hour 9 ann Thursday	Roof Alexander Stor Gardon Sq (23) 107.
0	Outline of course: (1) set theory, (2) metric spaces - compactness, completeness, (2) nonmed spaces - Barach spaces.	
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	depler 1.	
	SET THEORY.	
	SOB: M, B, C, XEA if element x belongs to A, ACB if YXEA we have xEB, moreover we say A=B if ACB and BCA.	
	(Definition 1.1 (set operations) A B A B A	,
	let A, B be sets; then the intersections is A (B= 1x: xEA and xEB); the union is AUB= 1x: XEA or XEB); the difference in A (B= 1xEA)	۱: × & B',
	A $\Box B^{B}$ the symmetric difference is $A \Delta B = f A   B + U + B   A + and the complement of ACE is A^{C} = E   A E [ \Box B ]$	
	Union, indersection can be taken for any number of sets: Un Ad, D Bd.	
	e.g. let $A \subset \mathbb{R}$ be $A = \mathbb{E}_{0,1}$ , then $A^{c_{\mu}}(-\infty, \sigma) \cup \mathbb{I}_{1,\infty}$ .	
	$\frac{1}{1000000112}$	
	Proof-left do exenite.	
0		XO SY
	Functions: let X, Y be two set. let f: X -> Y be a function (a nopping). The function of is a rule which associates to each x & X a uniquely defined	(x <sub>1</sub> ,
	exement y=fu)eY.	W-J
	Equition 1.3 the image (or earliest of f: X-7 ) is the set f(X)= 4y E Y: 3x E X s.t. f(x) = 4y'. The image (or preimage) of the element y E Y is the set	f-1(y)= tx e X : f(x)=y's.
	$if y \notin f(V_{i}), f^{-1}(y) = \phi  (empty \ set).$	
	to find the A mapping is said to be an injection (ONE-to-one) if f(x1)=f(x2) ⇒ x1=x2. In other nords, if y ∈ f(x), then f-ty) consists of only one point.	
	f is said to be a surjection (surjective) if f(k)=7. f is a bijection if it is simultaneously an injection and surjection.	X D + DY
	to inverse function P-1: tox : topiced by 2 f-1(f(x))= x VxeX	F-1
	the inverse function f <sup>-1</sup> : 1 > x is defined by 1 f (f-1(y)) = y y y e y.	f-1
5 1 1	the inverse function $f^{-1}: Y \rightarrow \chi$ is defined by $1 f(f^{-1}(y)) = y  \forall y \in Y$ . e.g if $f(x) = \chi^2$ , $(O = \chi^2, R, Y = R, f$ is neither an injection or surjection. $(Z = R, Y = R_+ = [O_1 \circ O])$ is surjection, not an injection. (	f-1
	the inverse function $f^{-1}: Y \to \chi$ is defined by $\int f^{-1}(f(x)) = \chi  \forall \chi \in \chi$ . e.g If $f(\chi) = \chi^{2}$ , $O = \chi^{2} R$ , $f$ is neither an injection or surjection. $O = \chi^{2} R$ , $Y = R_{+} = [0, \infty)$ is surjection, not an injection. (	f-1 3 X=Y=R+ is a bijection, f
	the inverse function $f^{-1}: Y \to \chi$ is defined by $\int f^{-1}(f(x)) = \chi  \forall \chi \in \chi$ . e.g If $f(\chi) = \chi^2$ , $\odot  \chi = \mathbb{R}$ , $f$ is neither an injection or surjection. $\bigodot  \chi = \mathbb{R}$ , $Y = \mathbb{R}_+ = [0, \infty)$ is surjection, not an injection. ( e.g If $f(\chi) = \chi^2$ , $\odot  \chi = \mathbb{R}$ , $f$ is neither an injection or surjection. $\bigodot  \chi = \mathbb{R}_+ = [0, \infty)$ is surjection, not an injection. ( <b>Equator</b> 15 two sets $\chi$ and $\Upsilon$ are said to be equivalent if there exists a bijection $f: \chi \to \Upsilon$ . In this case, $\chi$ and $\Upsilon$ have the same candinality, which is der	f-1 3 X=Y=R+ is a bijection, f
	the inverse function $f^{-1}: Y \rightarrow \chi$ is defined by $\int f^{-1}(f(x)) = \chi  \forall \chi \in \chi$ e.g if $f(\chi) = \chi^{2}$ , $(O \times = \mathbb{R}, Y = \mathbb{R}, f)$ is norther an injection or surjection. $(O \times = \mathbb{R}, Y = \mathbb{R}, f)$ is norther an injection or surjection. $(O \times = \mathbb{R}, Y = \mathbb{R}, f)$ is norther an injection or surjection. $(O \times = \mathbb{R}, Y = \mathbb{R}, f)$ is norther an injection of surjection $(O \times = \mathbb{R}, Y = \mathbb{R}, f)$ is norther an injection of surjection $(O \times = \mathbb{R}, Y = \mathbb{R}, f)$ is norther an injection of surjection $(O \times = \mathbb{R}, Y = \mathbb{R}, f)$ is norther an injection of surjection $(O \times = \mathbb{R}, Y = \mathbb{R}, f)$ is surjective, not an injection. $(O \times = \mathbb{R}, Y = \mathbb{R}, f)$ is norther an injection of $(O \times = \mathbb{R}, Y = \mathbb{R}, f)$ is norther an injection of $(O \times = \mathbb{R}, Y = \mathbb{R}, f)$ is norther an injection. $(O \times = \mathbb{R}, Y = \mathbb{R}, f)$ is norther an injection of $(O \times = \mathbb{R}, Y = \mathbb{R}, f)$ is surjective, $(O \times = \mathbb{R}, Y = \mathbb{R}, f)$ is norther an injection of $(O \times = \mathbb{R}, Y = \mathbb{R}, f)$ is norther an injection of $(O \times = \mathbb{R}, Y = \mathbb{R}, f)$ is surjective, $(O \times = \mathbb{R}, Y = \mathbb{R}, f)$ is surjective, $(O \times = \mathbb{R}, Y = \mathbb{R}, f)$ is surjective, $(O \times = \mathbb{R}, Y = \mathbb{R}, f)$ is surjective, $(O \times = \mathbb{R}, Y = \mathbb{R}, f)$ in this case. $(O \times = \mathbb{R}, Y = \mathbb{R}, f)$ is surjective, $(O \times = \mathbb{R}, Y = \mathbb{R}, f)$ in this is and $(O \times = \mathbb{R}, Y = \mathbb{R}, f)$ in this is a field to be called the field of $(O \times = \mathbb{R}, f)$ in this case. $(O \times = \mathbb{R}, Y = \mathbb{R}, f)$ is a field to be called the field of $(O \times = \mathbb{R}, f)$ in this is a field to be called the field of $(O \times = \mathbb{R}, f)$ . In this case, $(O \times = \mathbb{R}, f)$ is a field to be called the field of $(O \times = \mathbb{R}, f)$ is a field to be called the field of $(O \times = \mathbb{R}, f)$ in this is a field to be called the field of $(O \times = \mathbb{R}, f)$ is a field to be called the field of f).	f-1 3 X=Y=R+ is a bijection, f
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	the inverse function $f^{-1}: Y \rightarrow \chi$ is defined by $\int f^{-1}(f(x)) = \chi  \forall \chi \in \chi$ e.g if $f(\chi) = \chi^{2}$ , $(O \times = \mathbb{R}, Y = \mathbb{R}, f)$ is norther an injection or surjection. $(O \times = \mathbb{R}, Y = \mathbb{R}, f)$ is norther an injection or surjection. $(O \times = \mathbb{R}, Y = \mathbb{R}, f)$ is norther an injection or surjection. $(O \times = \mathbb{R}, Y = \mathbb{R}, f)$ is norther an injection of surjection $(O \times = \mathbb{R}, Y = \mathbb{R}, f)$ is norther an injection of surjection $(O \times = \mathbb{R}, Y = \mathbb{R}, f)$ is norther an injection of surjection $(O \times = \mathbb{R}, Y = \mathbb{R}, f)$ is norther an injection of surjection $(O \times = \mathbb{R}, Y = \mathbb{R}, f)$ is surjective, not an injection. $(O \times = \mathbb{R}, Y = \mathbb{R}, f)$ is norther an injection of $(O \times = \mathbb{R}, Y = \mathbb{R}, f)$ is norther an injection of $(O \times = \mathbb{R}, Y = \mathbb{R}, f)$ is norther an injection. $(O \times = \mathbb{R}, Y = \mathbb{R}, f)$ is norther an injection of $(O \times = \mathbb{R}, Y = \mathbb{R}, f)$ is surjective, $(O \times = \mathbb{R}, Y = \mathbb{R}, f)$ is norther an injection of $(O \times = \mathbb{R}, Y = \mathbb{R}, f)$ is norther an injection of $(O \times = \mathbb{R}, Y = \mathbb{R}, f)$ is surjective, $(O \times = \mathbb{R}, Y = \mathbb{R}, f)$ is surjective, $(O \times = \mathbb{R}, Y = \mathbb{R}, f)$ is surjective, $(O \times = \mathbb{R}, Y = \mathbb{R}, f)$ is surjective, $(O \times = \mathbb{R}, Y = \mathbb{R}, f)$ in this case. $(O \times = \mathbb{R}, Y = \mathbb{R}, f)$ is surjective, $(O \times = \mathbb{R}, Y = \mathbb{R}, f)$ in this is and $(O \times = \mathbb{R}, Y = \mathbb{R}, f)$ in this is a field to be called the field of $(O \times = \mathbb{R}, f)$ in this case. $(O \times = \mathbb{R}, Y = \mathbb{R}, f)$ is a field to be called the field of $(O \times = \mathbb{R}, f)$ in this is a field to be called the field of $(O \times = \mathbb{R}, f)$ . In this case, $(O \times = \mathbb{R}, f)$ is a field to be called the field of $(O \times = \mathbb{R}, f)$ is a field to be called the field of $(O \times = \mathbb{R}, f)$ in this is a field to be called the field of $(O \times = \mathbb{R}, f)$ is a field to be called the field of f).	f-1 3 X=Y=R+ is a bijection, f
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	the inverse function $f^{-1}: Y \rightarrow \chi$ is defined by $\int f'(f(x)) = \chi  \forall \chi \in \chi$ e.g if $f(\chi) = \chi^2$ , $(\mathfrak{O} \times = \mathbb{R}, Y = \mathbb{R}, f$ is norther an injection or surjection. $(\mathfrak{D} \times = \mathbb{R}, Y = \mathbb{R}, f(\chi) = \chi^2, f(\chi) = \chi^2, f(\chi) = \chi^2, \chi = \mathfrak{R}, \chi = \mathfrak{R}, f(\chi) = \mathfrak{R}, \chi = \mathfrak{R}, f(\chi) = \chi^2, \chi = \mathfrak{R}, \chi = $	f=1 3 X=Y=R+ is a bijection, f noted by  X =  Y .
	the inverse function f <sup>-1</sup> : 1→x, is defined by f <sup>-1</sup> (f(x))= x Vx e X e.g if f(N=x <sup>2</sup> , O X=R, Y=R, f is neither an injection or surjection. @ X=R, Y=R <sub>+</sub> = [0,00] is surjection, not an injection. ( INITED TWO SETS X and Y are said to be equivalent if there exists a bijection f: X→Y. In this case, X and Y have the same candinality, which is do A set equiplent to IN is said to be causable, and its cardinality is expressed by the notation [INI] = N o. Choose on (i) every infinite set nos a countable subset, (ii) every subset of a countable set is either countable or finite. Example 1.5 (1) of X is finite, then IXI is the number of elements (by definition). (2) IZI = INNI. g(m) = 1-2m+1 m ≤ 0 is dearly a bijection from Z to RN	f-1 3. X=Y=R, is a bijection, f noted by  X  =  Y . q. First court numbers of 16. Universet
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	<ul> <li>the inverse function f<sup>-1</sup>: 1→x, is defined by ∫ f<sup>-1</sup>(f(x))= x ∀x ∈ X</li> <li>e.g if f(x)=x<sup>2</sup>, O X=R, Y=R, f is neither an injection or surjection. O X=R, Y=R, f is neither an injection or surjection. O X=R, Y=R, f is neither an injection or surjection. O X=R, Y=R = [0,00] is surjection, not an injection. (</li> <li>Induced 115 Two sets X and Y are said to be equivalent if there exists a bijection f: X→Y. In this case, X and Y have the same candinality, which is der</li> <li>A set equivalent to IN is said to be causable, and its cardinality is expressed by the notation INII= No.</li> <li>Chowstons - (i) every infinite set has a countable subset, (ii) every subset of a countable set is either countable or finite.</li> <li>Example 1.6 (1) of X is finite, then IXI is the number of elements (by definition).</li> <li>(a) IZI=INI. g(m)=1-2m+1 m ≤ 0 is dearly a bijection from Z to RN</li> <li>(a) Q = { 1/2, g ∈ INI. Write all rational numbers in the form f with coprime p.g. Define the height of f to be h=1plt.</li> </ul>	f-1 3. X=Y=R, is a bijection, f noted by  X  =  Y . q. First court numbers of 16 January 2011 frof Mecander 5080
	<ul> <li>the inverse function f<sup>-1</sup>: 1.→x is defined by 1 f(f(f(x))= x Y × eX</li> <li>e.g f(f(x)=x, O X=R, Y=R, f is nother an injection or surjection. D X=R, Y=R, = [0,0] is surjection, not an injection. (</li> <li>(1) (1) (1) (1) (1) (1) (1) (1) (1) (1)</li></ul>	f-1 3. X=Y=R, is a bijection, f noted by  X  =  Y . q. First court numbers of 16 January 2011 frof Mecander 5080
	<ul> <li>the inverse function f<sup>-1</sup>: 1→x is defined by ∫f<sup>-1</sup>(f(x))= x ∀x ∈ X</li> <li>e.g. = f<sup>-1</sup>(f(x)=x<sup>2</sup>, O) x=R, Y=R, f<sup>-1</sup>(x notifier an injection or surjection. (2) X=R, Y=R += [0, ∞) is surjection, not an injection. (1)</li> <li>(1)</li> <li>(2)</li> <li>(2)</li> <li>(3)</li> <li>(4)</li> <li>(4)</li> <li>(4)</li> <li>(5)</li> <li>(5)</li> <li>(6)</li> <li>(7)</li> <li>(7)</li> <li>(8)</li> <li>(9)</li> <li>(9)</li> <li>(1)</li> <li>(1)</li> <li>(1)</li> <li>(1)</li> <li>(2)</li> <li>(2)</li> <li>(3)</li> <li>(4)</li> <li>(4)</li> <li>(4)</li> <li>(4)</li> <li>(4)</li> <li>(4)</li> <li>(4)</li> <li>(5)</li> <li>(6)</li> <li>(7)</li> <li>(7)</li> <li>(7)</li> <li>(8)</li> <li>(8)</li> <li>(9)</li> <li>(9)</li> <li>(9)</li> <li>(1)</li> <li>(1)</li> <li>(1)</li> <li>(2)</li> <li>(3)</li> <li>(4)</li> <li>(4)</li> <li>(4)</li> <li>(5)</li> <li>(5)</li> <li>(6)</li> <li>(7)</li> <li< td=""><td>f-1 3. X=Y=R, is a bijection, f noted by  X  =  Y . 9. First count numbers of 16 bowing 2014 Rof Mexinder 5080 Natur 706.</td></li<></ul>	f-1 3. X=Y=R, is a bijection, f noted by  X  =  Y . 9. First count numbers of 16 bowing 2014 Rof Mexinder 5080 Natur 706.
	<ul> <li>the inverse function f<sup>-1</sup>: 1→x is defined by 1 f<sup>-1</sup>(f(x))=x ∀x ∈ X</li> <li>e.g # fth=x<sup>+</sup>, O x=R, Y=R, f is neither an injection or surjection. D x=R, Y=R, f=10,00) is surjection, not an injection. (</li> <li>(a) - # fth=x<sup>+</sup>, O x=R, Y=R, f is neither an injection or surjection. D x=R, Y=R, f=10,00) is surjection, not an injection. (</li> <li>(a) - # fth=x<sup>+</sup>, O x=R, Y=R, f is neither an injection or surjection. D x=R, Y=R, f=10,00) is surjection, not an injection. (</li> <li>(b) - # fth=x<sup>+</sup>, O x=R, Y=R, f is neither an injection or surjection. D x=R, Y=R, f=10,00) is surjection, not an injection. (</li> <li>(c) - # fth=x<sup>+</sup>, O x=R, Y=R, f is neither an injection or surjection or surjection. Note: the same cardinality, which is do a constant of the constant. (</li> <li>(c) - # fth=x<sup>+</sup>, I = N = xontinues of elements. (</li> <li>(c) - # fth=fth  = X = - # fth=x<sup>+</sup>. (</li> <li>(c) - # fth=x<sup>+</sup>, finite, then 1X1 is the number of elements. (</li> <li>(c) - # fth=x<sup>+</sup>. (</li></ul>	f <sup>-1</sup> 3 X=Y=R, is a bijection, f noted by  X  =  Y . 9. First count numbers of 16 bourseport frof Maxinder 5080 Mather 706. 0. anap, an = f(n)= 0.a
	<ul> <li>the interve function f<sup>-1</sup>: 1→x is defined by ∫ f<sup>-1</sup>(f(x))=x<sup>-</sup> ∀ x e X</li> <li>e.g if f(x-x<sup>+</sup>, O) x = R, Y = R, f is notifier an injection or surjection. (a) X=R, Y = R + = [0, x) is surjection, not an injection.</li> <li>(a) X=R, Y = R, f is notifier an injection or surjection. (b) X=R, Y = R + = [0, x) is surjection, not an injection.</li> <li>(b) X=R, Y=R, f is notifier a injection f(X-Y). In this case, X and Y have the same cardinality, when is do A cot equivalent to the is said to be cauceade, and it cardinality is expressed by the notation. INI = X o.</li> <li>(c) A cot equivalent to the is said to be cauceaded, and it cardinality is expressed by the notation. INI = X o.</li> <li>(c) A X is finite; then 1XI is the number of element toy adjustion.</li> <li>(c) A X is finite; then 1XI is the number of element toy adjustion.</li> <li>(c) A X is finite; then 1XI is the number of element toy adjustion.</li> <li>(c) A X is finite; then 1XI is the number of element toy adjustion.</li> <li>(c) A I = [N]. (c) (f X is the tot of X) is the antiple of it dearly a bijection from II to RI</li> <li>(c) A = (f 1, p ∈ II, g ∈ [N]. while all reduced toy adjustion.</li> <li>(c) A = (f 1, p ∈ II, g ∈ [N]. while all reduced toy notures and thus, 1 (R] = X o.</li> <li>(c) A = (f 1, P ∈ II, g ∈ [N]. while all reduced by notures and thus, 1 (R] = [N] = X o.</li> <li>(c) A = [R] × (continues. Bijection 1 g: R &gt; (c)(1), g(t) = f(x) + to a.</li> <li>(c) A = [R] × (continues. Bijection 1 g: R &gt; (c)(1), g(t) = f(x) + to a.</li> <li>(c) A = (f 1, C) = (f</li></ul>	P-1 3. X=Y=R, is a bijection, f noted by  X  =  Y . q. First course numbers of 16 boursey 2014 Frof Mexinder SOBO Mather Fob. 0. anap, an = f(n)= 0.a b if an=1, then let b
	<ul> <li>the interve function f<sup>-1</sup>: 1→x is defined by 1 f(f<sup>-1</sup>(g))=y. Yie?</li> <li>e.g f(f(0)=t, O) X=R, Y=R, f is notiver an injection or surjection. (2) X=R, Y=R += [0,00] is surjection, not an injection. (2)</li> <li>(a) X=R, Y=R, Y=R, f is notiver an injection or surjection. (2) X=R, Y=R += [0,00] is surjection, not an injection. (3)</li> <li>(a) X=R, Y=R, Y=R, Y=R, f is notiver an injection or surjection. (2) X=R, Y=R += [0,00] is surjection, not an injection. (4)</li> <li>(a) X=R, Y=R, Y=R, Y=R, Y=R, Y=R, f is notiver an injection or surjection. (3) X=R, Y=R += [0,00] is surjection, not an injection. (4)</li> <li>(a) X=R, Y=R, Y=R, Y=R, Y=R, Y=R, f is notiver an injection or surjection. (3) X=R, Y=R += [0,00] is surjection, not an injection. (4)</li> <li>(a) X=R, Y=R, Y=R, Y=R, f is notiver exists a bijection fixer fixe one, X and Y have the same cardinating, which is do A for equivalent to NA is said to be consisted a, and it cardinating is expressed by the notation. [NI]= No.</li> <li>(b) One (1) are one infinite at two a constable ender, (ii) or equipation form Z to RA</li> <li>(b) Q = (1/2), PEZ, gENT, whithe AI various numbers in the form f with coprime p.g. Define the beight of f to be h= [p]t.</li> <li>(a) Q = (1/2), PEZ, gENT, whithe AI various numbers in the form f with coprime p.g. Define the beight of f to be h= [p]t.</li> <li>(b) Q = (1/2), PEZ, gENT, whithe AI various numbers is a figure form the two of the local to f f to be h= [p]t.</li> <li>(b) Q = (1/2), PEZ, gENT, whithe AI various numbers is a subjection from Z to RA</li> <li>(b) Q = (1/2), PEZ, QENT, whithe AI various numbers is the form f with coprime p.g. Define the beight of f to be h= [p]t.</li> <li>(a) (0) (10) = [R] = X: continuon. Figure 1: generates by notures and thus, 1 Q[1] = [N][ = No.</li> <li>(b) (0) (1) = [R] = X: continuon. Figure 1: generates by notures and thus, 1 Q[1] = [N].</li> <li>(b) (0) (1) = [R] = X: continuon. Figure 1: generates a subjection from figure 1: generates a bijection from figure 1: generate</li></ul>	P-1 3. X=Y=R, is a bijection, f <sup>2-</sup> noted by  X  =  Y . 9. First courd numbers of 16 ionizy2011 Frof Mexander SOBOI Mather Fob. 0. anap, an = f(n)= 0.a. b if an=1, then let b lity    q.e.d.
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	<ul> <li>the interver function f<sup>-1</sup>: 1-3x is defined by ∫f<sup>+1</sup>(f(3))=3<sup>-</sup> Y × K<sup>2</sup>.</li> <li>e.g. = # f(0=x<sup>2</sup>, O) X= R, Y= R, f is neither an injection or surjection. (2) Xo R, Y= R + = [0, n) is surjection, not an injection. (3)</li> <li>(2) = # f(0=x<sup>2</sup>, O) X= R, Y= R, f is neither an injection or surjection. (2) Xo R, Y= R + = [0, n) is surjection, not an injection. (4)</li> <li>(4) The surjective of the solution is equivalent if dreve exists a bijection f: X=Y. In this case, X and Y have the solve cardinating, which is do A not equivalent to the is solve constable. For different of the solution to be equivalent to the constable, and it is constable, and it construction [1k] = N o.</li> <li>(4) Oreganize (1) every infinite set has a conversable solver, (i) or equivalent of a conversable solver on the solver conversable or finite.</li> <li>(5) Or f X is finite, then [X] is the number of element tog definition.</li> <li>(6) If Z[=[N], g(EN], white all rational numbers is infection from II to Ri</li> <li>(5) Or = f(1, p EI, g(EN], white all rational numbers is the form fight copy into p.g. Define the beight of fight to be the light of the to be the light (1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1</li></ul>	P-1 3. X=Y=R, is a bijection, f noted by  X  =  Y . q. First count numbers of 16 bonony 2011 fref Mexander SOBOI Mather 706. 0. anap, an = f(n)= 0.a b if an=1, then let b lity    q.e.d.
	<ul> <li>the interce function f<sup>-1</sup>: 1-3x is defined by ∫ f (f(f))=3 y yes.</li> <li>e.g. = if f(n = t, 0) x = R, f = R, f is notiver an injection or surjection. De YoR, y = R, = [0, no) is surjection, not an injection. (</li> <li>E.g. = if f(n = t, 0) x = R, f = R, f is notiver an injection or surjection. De YoR, y = R, = [0, no) is surjection, not an injection. (</li> <li>E.g. = if f(n = t, 0) x = R, y = R, f is notiver as injection or surjection. De YoR, y = R, = [0, no) is surjection, not an injection. (</li> <li>E.g. = if f(n = t, 0) x = R, y = R, f is notiver as injection or surjection. De YoR, y = R, = [0, no) is surjection, not an injection of the equivalent to Rive constanting, not an injection of the constantion of the equivalent to Rive to solve constanting. (ii) or ony subset of a constantiate or finite.</li> <li>Example to (n) f(X) is finite; then a constant of demonst (by definition).</li> <li>(a) R = f(n) = f(X) = f(X) = (X) = (X)</li></ul>	P-1 3. X=Y=R, is a bijection, f noted by  X  =  Y . q. First count numbers of 16 bonony 2011 fref Mexander SOBOI Mather 706. 0. anap, an = f(n)= 0.a b if an=1, then let b lity    q.e.d.
	<ul> <li></li></ul>	P-1 3. X=Y=R, is a bijection, f <sup>2-</sup> noted by  X  =  Y . 9. First courd numbers of 16 ionizy2011 Frof Mexander SOBOI Mather Fob. 0. anap, an = f(n)= 0.a. b if an=1, then let b lity    q.e.d.

## $\text{On the other hand, Id}: \underline{Y} \rightarrow R \text{ is an injection, so } |Y| \leq |R| = X, \text{ so } |Y| \leq X. \text{ By Theorem 1.10, } |Y| = X.$

chapter2. METRIC SPACES.

14 VI a set the VVV a dependence of sheet start and the set of V
let Shes set. Then XXX is the product set of ordered psils (x, y) where x, y & Z.
Infuition 21 let p: X×X→ IR. The pair (X,p) is said to be a metric space if Inon-degeneracy I Esymmetry I [triangle_inequality]
(1) $p(x,y) \ge 0$ and $p(x,y) = 0$ iff $x \ge y$ , (2) $p(x,y) = p(y,x)$ (3) $p(x,z) \le p(x,y) + p(y,z)$ $\forall x,y,z \in \mathbb{Z}$ .
In this case, p is called a metric. For any subset MCX, the pir (M, p) is called a metric subspace.
dependion - from (3), $p(x,z) - p(x,y) \leq p(y,z)$ . Moreover, $p(x,y) \leq p(x,z) + f(z,y) \Rightarrow p(x,y) - p(x,z) \leq p(y,z) \Rightarrow  p(x,y) - p(x,z)  \leq p(y,z)$ .
therefore, $ p(x,y) - p(x_0,y_0)  \leq g(x_0, x_0) + g(y_0, y_0)$ .
Essemptes - Q let X be a set. Define $e(x_1y) = 11$ $x \neq y$ . This is the space of isolated points.
(2) X=R, p(x,y)=  x-y . All three properties of a metric are solitified.
$(T - P^h = R Y_{,} Y R_{,} = 1 $
$\frac{\left(2 - \frac{1}{k}\right)^{2}}{\left(2 - \frac{1}{k}\right)^{2}} = \frac{1}{k + 1} a_{k}^{2} a_{k}^$
the couchy-schnortz inequality implies triangle inequality for this metric.
Tote X=R, and define pp (x,y) = [ = [x y-y ], p>0. Clearly, fp is non-degenerate and symmetric. We however need to establish the triangle inequality.
WE NIL Show that fp defines a metric for all p E [1,00). Here, foo is defined by foo (X,y) = Max IXE-YKI. If p<1, this is not a metric space. We use notation IRp.
Nessy that two numbers p, 9,20 are conjugate to each other (or form a conjugate pair) if $\frac{1}{p} + \frac{1}{q} = 1$ . Note: this implies that p, 9, 21. We also include p=00, 9=1
then $1+\frac{p}{q}=p$ , $1+\frac{p}{p}=q$ .
$\frac{1}{120000123} (42000g) i (4200g) i (4200g)$
let a, b >0 sha let $p, q$ be a carrient poir. Then at $p = 1$ . $t^{p} + \frac{t^{2}}{t^{2}}$
$\frac{t^{p}}{t^{p}} + \frac{t^{-1}}{q} + \frac{t^{-2}}{q} + \frac$
if a=0 or b=0, inequality is also trivially proved f q.e.d.
Record 2.4 (Hölder's inequality)
Let p,q be so before, and let a=(a_1, a_2,, a_n), b=(b_1,, b_n), then $\sum_{k=1}^{n}  a_k b_k  \le (\sum_{k=1}^{n}  a_k ^p)^{\frac{1}{p}} (\sum_{k=1}^{n}  b_k ^q)^{\frac{1}{q}}$ .
Roof-Assume each factor on night ( $\sum_{k}^{p}  a_{k} ^{p})^{\frac{1}{p}} = (\sum_{k}^{p}  b_{k} ^{q})^{\frac{1}{2}} = 1$ . Estimate LHS= $ a_{k}b_{k}  \leq \frac{ a_{k} ^{r}}{p} + \frac{ b_{k} ^{r}}{q}$ by Lemma 2.3. Then summing over terms gives us
Z  akbk  = = Z  akl <sup>1</sup> + = Z  bkl <sup>2</sup> = 1. then for general cose, define new (scaled) vectors by $\tilde{a} = \lambda^{-1}a$ , $\tilde{b} = \mu^{-1}b$ with $\lambda = (\sum_{k}  a_{kl}^{0} ^{\frac{1}{2}}, \mu = (\sum_{k}  b_{kl} ^{0})^{\frac{1}{2}}$ . Then
ubordy, $(\Sigma  \tilde{a}_k ^p)^{\frac{1}{p}} = (\Sigma  \tilde{b}_k ^q)^{\frac{1}{p}} = 1$ and inequality holds for these vectors. Therefore, $\Sigma  \tilde{a}_k _{k} \leq 1 \Rightarrow \frac{1}{p} \sum_{k}  a_k b_k  \leq 1 \Rightarrow \sum_{k}  a_k b_k  \leq \lambda \mu$ .
$\frac{(1+ence)(2.5)}{(1+ence)(2.5)} (Mintervalis's inequality) = \left(\sum_{k=1}^{\infty}  a_k ^p\right)^{\frac{1}{p}} \leq \left(\sum_{k=1}^{\infty}  a_k ^p\right)^{\frac{1}{p}} + \left(\sum_{k=1}^{\infty}  b_k ^p\right)^{\frac{1}{p}}.$
$\begin{array}{c} \text{let } p \in \mathbb{H}_{1} \text{ od } \text{. then } \left( \sum_{k=1}^{k}  a_{k}  + b_{k}   \right) = \left( \sum_{k=1}^{k}  a_{k}  \right) + \left( \sum_{k=1}^{k}  b_{k}  \right) .$
$\frac{1}{k} = \frac{1}{k} + \frac{1}{k} = \frac{1}{k} + \frac{1}$
$\sum_{k=1}^{\infty} (p-1)q = p, \text{ then } \sum_{k=1}^{k}  a_{k}t b_{k} ^{p-1}  a_{k}  \leq (\sum  a_{k}t b_{k} ^{p})^{\frac{1}{p}} (\sum  a_{k}t b_{k} ^{p}) (\sum  a_{k}t $
Thus, I akt bk 1 < ( I akt bk 1) = [( I akt) + + ( I bk 1) + ( I bk 1) + ( I bk 1) + ]. Divide by ( I lak + bk 1) + to get solution 1 q.e.d.
() let x = (x1, x1,) be on inflicte sequence of red (or complex) numbers. Let lp, pe [1, or] be the set of all sequences x set. Exit  xx  < 00.
e.g. x= 1× K= t.y. Then x ∈ 2p with p>1 but not with p=1! Note- Hölder's, Minkowski's inequalities hold e.g. if x ∈ 2p, y ∈ 2q, t = =1, then Hölder's
inequality holds = $\sum_{k=1}^{\infty}  X_k U_k  \le (\sum_{k=1}^{\infty}  X_k ^p)^{\frac{1}{p}} (\sum_{k=1}^{\infty}  Y_k ^q)^{\frac{1}{2}}$ . sup rather than max.
O C C los consists of all convergent sequences. C(x,y) = po(x,y) = K [xx-yk]. Then Co C C is the subspace of sequences converging to 0.
since interval is compact,
(space of functions) let C[0,1] be the space of all continuous functions on the closed interval [0,1] with the metric fc (f,g)= te(0,1] [f(t)-g(t)]
$\ell_{p,1} \leq p \leq \omega$ . $C[c_0,1], C[a,b].$ Prof. Nersonal Lev
(2) Take the set of sil continuous functions on (a,b) and define {r(f,q) [ ∫a [f(t)-g(t)] (dt]], 1≤p≤oa. po(f(g) = telaps) [f(t)-g(t)] Hother Tob.
Notation for this metric space: Cp [a,b]. Is this indeed a metric? Yes! As for lp, we have $(\int_{a}^{b}  f+g ^{2} dt)^{\frac{1}{p}} \leq (\int_{a}^{b}  f ^{2} dt)^{\frac{1}{p}} + (\int_{a}^{b}  g ^{2} dt)^{\frac{1}{p}} \forall f, g \in Cp[a,b],$
$\int_{a}^{b} H_{gl} dt \leq \left(\int_{a}^{b}  f(t) ^{2} dt\right)^{\frac{1}{2}} \left(\int_{a}^{b}  g(t) ^{2} dt\right)^{\frac{1}{2}} = 1.$

s.t. fro (X,X) = sup fro (X,X) = sup from (X,X) =	1	
# f. is antihumen at a polit. X. I., elim est by due f. is antihume. It. J. e. f. in article. X. X. S. an antihum open site der der der K. (Ken). Just J. E. (Dill 2 (1992). Elime proteine in medile der der is is a method. J. anne f. (Y. X. F. S. defands by (Star) / Ken) is antihume. Inder der is in the article. Samme f. (Y. X. F. S. defands by (Star) / Ken) is antihume. Inder der is in the article. Samme f. (Y. X. F. S. defands by (Star) / Ken) is antihume. Inder der is in the article. Samme f. (Y. X. F. S. defands by (Star) / Ken) is a starter method. Samme f. (Y. X. F. S. defands by (Star) / Ken) is a starter method. Samme f. (Y. X. F. S. defands by (Star) / Ken) is a starter method. Samme f. (Y. Y. S. S. defands and Y. Ken) (Star) is a starter method. Samme f. (Y. Y. S. S. defands and the intervention. Samme f. (Y. S. S. defands and the intervention. Samme f. (Y. S. S. defands and the intervention. Samme f. (Y. S. S. defands and the intervention. Samme f. (Y. S. S. defands and the intervention. Samme f. (Y. S. S. defands and the intervention. Samme f. (Y. S. S. defands and the intervention. Samme f. (Y. S. S. defands and the intervention. Samme f. (Y. S. S. defands and the intervention. Samme f. (Y. S. S. defands and the intervention. Samme f. (Y. S. S. defands and the intervention. Samme f. (Y. S. S. defands and the intervention. Samme f. (Y. S. S. defands and the intervention. Samme f. (Y. S. S. defands and the intervention. Samme f. (Y. S. S. defands and the intervention. Samme f. (Y. S. S. defands and the intervention		Constitutions functions on metric space: Let (X, FW), (Y, Fy) be two metric spaces.
<pre>equds_z 1.2 %, or sumic pipe size de metris al (eqs), (eqs) = (f(s) 2 (</pre>		(opportund 2.6 let f: X->Y be a function. We say that f is continuous at XDE X if for any EZO, there is a S>O sit. By (f(X), f(Xa) < E do soon as Bx(X,Xa) <s.< td=""></s.<>
<ul> <li>Anne a forder, f. (14) 24. E. Appende Sa. (1409) * f. 100 % a modelance, a baked a read wat in (1400) * for (141) 24. (241) 24.</li> <li>BERRED &amp; F. (141) A. (2000) * f. (141) (140) (</li></ul>		If f is contributous at all points x e X, then we say that f is contributors on X.
<ul> <li>Information of first in a biggering of another and fir another and the first homeomorphism.</li> <li>E. J. S. Station of S. S.</li></ul>		e.g- book at X×X as a memic space with the metric d((x,y), (w,z)) = p(x,w) + p(y, ≥) [Check properties to establish that this is a metric].
$e_{1} = he \ [ antom le life 2 E(3) depud hy Mail \frac{1}{2} and he is in anomaly in higher, the influence intermediate.e_{1} = e_{2} = e_$	1	then the fluction $f: X \times X \rightarrow \mathbb{R}$ defined by $f(x,y) = p(x,y)$ is continuous. Indeed, recall that $ p(x,y) - p(x_0,y_0)  \le p(x,x_0) + p(y,y_0)$ .
Experience of an eff a hord is an increase of fir (b)		
$c_{1} - c_{2} + c_{2} + S_{2}^{-1}, Y \in \mathbb{L}$ selface see region of X = p_{2}^{-1} (S_{2}^{-1} (S_{2		
$ \left\{ \begin{array}{c} \left\{ \left\{ \left\{ p \in p \in \mathcal{A} \mid \{ p \in \mathcal{A} \mid p \in A$		
$\begin{aligned} \begin{array}{c} & & & & & & & & & & & & & & & & & & &$		
$ \begin{aligned} & \left  $		
$\begin{aligned} & \left  $		
A set MCX is toke who second of the isomethod by some $B(s_0, t)$ . (2) $e_1 \rightarrow B_{1-}^{-1}$ (1) $e_1 \rightarrow B_{1-}^{-1}$ (1) is a kine of radius T. Lie. B(CR) is a proved define $A_{1-}$ (1) $E_{1-}^{-1}$ (1) $E_{1-}^{-1}$ (1) $E_{1-}^{-1}$ $h_{1-}^{-1}$ (1) $E_{1-}^{-1}$ (1) is a kine of radius T. Lie. B(CR) is a proved define $A_{1-}$ (1) $E_{1-}^{-1}$		
Are then have been been been been been been been be		710 (14)
h. (LG11: B(11) is de cert of as forming ranging between 1 and 1. Definition is a second as forming the second of the second o		A set MC X is sold to be bounded if it is covering it some D(NoIN).
$ \begin{array}{c} \label{eq: constraints} \\ \begin{tabular}{lllllllllllllllllllllllllllllllllll$		
Monumentation prior if for a 520, the bill B(s, b) and bit a for the prior if for a billed prior for some for a communitation prior if for a 520, the bill B(s, b) and bit a for the for a billed prior for a bill bill bill bill bill bill bill bi		x is an
$ \frac{1}{10} + \frac{1}{10$		point.
$\begin{array}{c} (\operatorname{der} \operatorname{chol} \operatorname{chol} \operatorname{sens} \operatorname{print} \operatorname{sens} \operatorname{print} \operatorname{de} \operatorname{sens} \operatorname{sens}$		
$ \begin{aligned} & \left( \sum_{k=1}^{\infty} \sum_{j=1}^{k} \sum_{j=1}^{k} \sum_{k=1}^{k} \sum_{j=1}^{k} \sum_{j=1}^$		
tent B(x_0) is juiced an e-meightoutered of to. Mother: B(x_0, 2) = O(2(3). Heat B(x_0, 2) is juiced an e-meightoutered of to. Mother: B(x_0, 2) = juic B(2(3)). The dest to A C X. then Th I = 1 × C X : Abit (A, X) = 0 }. More dive (A, X) = juic B(2(3)). The dest B = 1 × E X : Abit (A, X) = 0 }. More dive (A, X) = juic B(2(3)). The dest B = 1 × E X : Abit (A, X) = 0 }. More dive (A, X) = juic B(2(3)). The dest B = 1 × E X : Abit (A, X) = 0 }. More dive (A, X) = juic B = 1 A = 0 }. Heat for any 500, there is a part get ST = (bx_0) < C . More dive (A, X) = 0 E CAI. Thus B = EAI = que dive Heat for any 500, there is a part get ST = (bx_0) < C . More dive (A, X) = 0 }. Heat for any 500, there is a part get ST = (bx_0) < C . More dive (A, X) = 0 }. Heat for any 500, there is a part get ST = (bx_0) < C . More dive (C = 0) }. Heat for any 500, there is a part get ST = (bx_0) < C . More dive (C = 0) }. Heat for any 500, there is a part get ST = (bx_0) < C . More dive (C = 0) }. Heat for any 500, there is a part get ST = for an extend all for the conduction prived (co, ise . Y=20). Side (C = 0) $F = 0$ for the text of all for a formation of contragenees. Let $\tilde{T} = [b_1(x_0), (x_0), (x_0), (x_0) = 0$ . Heat $T(x_0) = (b_1 - b_1) = (b_1) =$		
$ \begin{array}{c} \begin{array}{c} \begin{array}{c} \begin{array}{c} \begin{array}{c} \begin{array}{c} \begin{array}{c} \begin{array}{c}$		olio
$\begin{aligned} & \operatorname{Red}_{r} \operatorname{Rever}_{r} \mathbb{D} = \{x \in X : A \forall r (A, x) = 0^{h}. Suppose_{r} A A A C X \in A A A A A A A A$		
Hon for any \$20, there is a part yet \$1, by \$1, \$2, Again by Definition 210, \$2, \$2, \$2, \$3, \$5, \$2, \$2, \$3, \$2, \$2, \$2, \$2, \$2, \$2, \$2, \$2, \$2, \$2		
Pomork - die (A, x) = die (IA), N. Pomork - die (A, x) = die (IA), N. Pomork - die (A, x) = die (IA), N. Pomork - die (IA), N. Pomor		
$\begin{array}{c} \begin{array}{c} \begin{array}{c} \begin{array}{c} \begin{array}{c} \begin{array}{c} \begin{array}{c} \begin{array}{c} $		
$ \begin{aligned} & \qquad \qquad$		ob mile et.
Let $p \in [1, \infty)$ . When $[Cool] = 2p$ . Roof we want to show that onemy $x \in 2p$ is an accumulation point of coo, i.e. $\forall 2 \ge 0$ , $\exists \tilde{x} \in Coo set. [p(k, \tilde{x}] < \varepsilon.$ slinic $x \in (x_1, x_2,) \in Cp$ , there is a number $N_{\varepsilon}$ st. $\frac{1}{k \times N_{\varepsilon}} = \frac{1}{k \times N_{\varepsilon$		po Brof Alexander Solar EV.
since $X = (X_1, X_2,, ) \in \mathbb{C}_p$ , there is a number $N_2$ st. $K_{2}^{-}$ by definition of convergence. Let $\tilde{X} = (X_1, X_2,, X_N, Q, Q,, ) \in Coo. Let P_1(X_1, \tilde{X}_1) = \begin{bmatrix} \sum_{i=1}^{n}  X_N ^2 \end{bmatrix}^{\frac{1}{p}} < (e^{i})^{\frac{1}{p}} = e_{p_1}e_{i}d_{i}P_1(X_1, \tilde{X}_1) = \begin{bmatrix} \sum_{i=1}^{n}  X_N ^2 \end{bmatrix}^{\frac{1}{p}} < (e^{i})^{\frac{1}{p}} = e_{p_1}e_{i}d_{i}P_2(X_1, \tilde{X}_1) = \begin{bmatrix} \sum_{i=1}^{n}  X_N ^2 \end{bmatrix}^{\frac{1}{p}} < (e^{i})^{\frac{1}{p}} = e_{p_1}e_{i}d_{i}P_2(X_1, \tilde{X}_1) = \begin{bmatrix} \sum_{i=1}^{n}  X_N ^2 \end{bmatrix}^{\frac{1}{p}} < (e^{i})^{\frac{1}{p}} = e_{p_1}e_{i}d_{i}P_2(X_1, \tilde{X}_1) = \begin{bmatrix} \sum_{i=1}^{n}  X_N ^2 \end{bmatrix}^{\frac{1}{p}} < (e^{i})^{\frac{1}{p}} = e_{p_1}e_{i}d_{i}P_2(X_1, \tilde{X}_1) = \begin{bmatrix} \sum_{i=1}^{n}  X_N ^2 \end{bmatrix}^{\frac{1}{p}} < (e^{i})^{\frac{1}{p}} = e_{p_1}e_{i}d_{i}P_2(X_1, \tilde{X}_1) = \sum_{i=1}^{n}  X_N ^2 \end{bmatrix}^{\frac{1}{p}} < (e^{i})^{\frac{1}{p}} = e_{p_1}e_{i}d_{i}P_2(X_1, \tilde{X}_1) = \sum_{i=1}^{n}  X_N ^2 \end{bmatrix}^{\frac{1}{p}} < (e^{i})^{\frac{1}{p}} = e_{p_1}e_{i}d_{i}P_2(X_1, \tilde{X}_1) = \sum_{i=1}^{n}  X_N ^2 \end{bmatrix}^{\frac{1}{p}} < (e^{i})^{\frac{1}{p}} = e_{p_1}e_{i}d_{i}d_{i}d_{i}d_{i}d_{i}d_{i}d_{i}d$		
$\begin{aligned} \left\{ p(x_{1}, \tilde{x}) = \left[ \sum_{k=m}^{m}  k_{k} ^{p} \right]^{\frac{1}{p}} < (e^{k})^{\frac{1}{p}} = \varepsilon_{\beta,q,e,k} \\ & @ consider concernent to c$		
$\frac{\operatorname{Proof}-\operatorname{We}\operatorname{will}\operatorname{show}\operatorname{that}[\operatorname{Cool} \subseteq \operatorname{Co} : \operatorname{e.}\operatorname{suppingt} \operatorname{of}\operatorname{Coo}\operatorname{helong}\operatorname{to}\operatorname{Co}, \operatorname{ide}\operatorname{we}\operatorname{sh}\operatorname{secconsultation}\operatorname{paint}\operatorname{of}\operatorname{Coo}, \operatorname{i.e.}\operatorname{We}, \exists X = (X_1, X_2,, X_N, o, c)$ $s.t.$ $\int_{\operatorname{Proof}}\operatorname{We}(X, X) = \frac{\operatorname{Sup}}{\operatorname{W}}  X_K - \tilde{X}_K  \leq \varepsilon. \operatorname{Theorepres}, \operatorname{Kann+1}  X_K  = \frac{\operatorname{Sup}}{\operatorname{Kan+1}}  X_K - \tilde{X}_K  \leq \frac{\operatorname{Sup}}{\operatorname{K}}  X_K - \tilde{X}_K  = \frac{\operatorname{Pro}(X, X)}{\operatorname{K}} \cdot \varepsilon. \operatorname{Thuo}, X_K \to 0 \text{ do } K \to \infty, so [Cool}$ $\operatorname{We}\operatorname{just}\operatorname{nead}\operatorname{to}\operatorname{shout}\operatorname{that}  \operatorname{Co} \subset [\operatorname{Cool}]  i.e. \operatorname{energy}\operatorname{element} \operatorname{of}\operatorname{Co}\operatorname{is}\operatorname{son}\operatorname{sccconsultation}\operatorname{point}.$ $\operatorname{We}\operatorname{just}\operatorname{nead}\operatorname{to}\operatorname{shout}\operatorname{that} \operatorname{Co}_2[\operatorname{Cool}]  i.e. \operatorname{energy}\operatorname{element} \operatorname{of}\operatorname{co}\operatorname{is}\operatorname{son}\operatorname{sccconsultation}\operatorname{point}.$ $\operatorname{We}\operatorname{just}\operatorname{nead}\operatorname{to}\operatorname{shout}\operatorname{that} \operatorname{Co}_2[\operatorname{Cool}]  i.e. \operatorname{energy}\operatorname{element} \operatorname{of}\operatorname{co}\operatorname{is}\operatorname{son}\operatorname{sccconsultation}\operatorname{sccconsultation}\operatorname{point}.$ $\operatorname{We}\operatorname{just}\operatorname{scc}_{2}(\operatorname{shout}\operatorname{co}\operatorname{shout}$		sition $x = (x_1, x_2,) \in \mathbb{C}_p$ , there is a number $N_2$ st. $k = N+1$ $ X_K ^2 < \varepsilon'$ by definition of convergence. Let $\tilde{x} = (x_1, x_2,, x_N, 0, 0,) \in C_{00}$ be a flate: $\int p(x_1, \tilde{x}) = \left[\frac{\varepsilon}{k_{1}} + \frac{1}{k_{1}} + \frac{1}{k_{2}} + 1$
$\frac{s_{1}, \ldots}{p_{1}} = \frac{s_{1}p_{1}}{p_{2}} \left[ \frac{s_{1}p_{1}}{p_{2}} - \frac{s_{1}p_{1}}{p_{2}} \right] \left[ \frac{s_{1}p_{1}}{p_{2}} - \frac{s_{2}p_{1}}{p_{2}} \right] \left[ \frac{s_{1}p_{1}}{p_{2}} - \frac{s_{1}p_{1}}{p_{2}} \right] \left[ \frac{s_{1}p_{1}}{p_{2}} - \frac{s_{1}p_{2}}{p_{2}} \right] \left[ \frac{s_{1}p_{2}}{p_{2}} - \frac{s_{1}p_{2}}{p_{2}} \right] \left[$		@ consider coo as a subset of los= {x= (x1,x2,): sw 1xx < 00 }. (doin: [Coo]= Co (all sequences convergent to 0, i.e. xx -> 0 as k-> 00).
We just need to show that Co C [Coo] i.e. energy element of Co is an accumulation point. (a) If An CA2, then [An] C [A2]. (b) [[A]] = [A]. (c) If An CA2, then [An] C [A2]. (c) [[A]] = [A]. (c) If An CA2, then [An] C [A2]. (c) [[A]] = [A]. (c) If An CA2, then [An] C [A2]. (c) If An CA2, then [An] C [A2] = [An] C [A2]. (c) If An CA2, then [An] C [A2] = [An] C [A2] = (An) C [A2]	a a f	Proof- we will show that [Coo] ⊂ co i.e. all points of coo belong to Co. let x ∈ los be an accumulation point of coo, i.e. Ve, ∃X «(Xi, ix,, Xi), o, o, s.t. for (X, X) = sup for (X, X) = sup for (X, X) = sup for (X, X) = sup
Broof- (1) and (2) stretrively straightforward. (3): dist ([A], x) = dist (A, x). ∀x ∈ X. Then [[A]] = {x ∈ X : dist ([A], x) = 0} = {x ∈ X: dist (A, x) = 0} = [A]//q.e         (4) left to exercise.         Evention:       [A1∩A2] = [A1∩[A2]? No! Indeed, let A1 = (0,1), A2 = (1,2). Then A1∩A2 = φ. However: [A1]∩[A2] = [0,1]∩[1,2] = 1} ≠ [φ].		
(4) left & exense. (4) left & exense. Election: [A10A2] = [A10 [A2]? No! Indeed, let A1 = (0,1), A2=(1,2). Then A10A2 = \$\$\$\$. However [A1]0[A2] = [0,1]0[1,2] = 11} \$		Moment 212 (1) A C [A] (2) If A1 C A2, then [A1] C [A2]. (3) [[A]]=[A]. (4) [A1UA2]=[A1] U [A2].
(4) left to exercise. Election: [A10A2] = [A10 [A2]? No! Indeed, let A1 = (0,1), A2=(1,2). Then A10A2= \$\vec{p}\$. However, [A1]0[A2] = [0,1]0[1,2] = 11] \$\nothermathcal{p}\$ [\$\vec{p}\$].		(Thm 2-11) Broof - (1) and (2) are trivial/ straightforward. (3): dist ([A], x) = dist (A, x) $\forall x \in X$ . Then [[A]] = $f x \in X$ : dist ([A], x) = $of = f \times (A, x) = of = f \times $
		(4) left » exemise.
UNIT REPORTS WE HAVE IN CHAILING C CHAILING TO THE MORE MORE AND THE		

	(Priergenez
	(Definition] 2.13.40t xk, k=1/2, be a sequence in a metric space (X, p). We say that Xk contracts to XE X do K > 00 i.e. X= K 300 Xk, or Xk > X do K > 00, if pUx, N 30
	as k→60. In other words, for any 200, there is a number N=NE st. p(xk, x) <e as="" k="" soon="">N.</e>
	Theorem 2.14 The point x & X is an accumulation point of the set ACX ( +> there is a sequence XKEA st K + an XK = X, and XK + X.
	(→) nonf-suppose X ∈ X is an accumulation point. Let En= to be the sequence of E terms, n=1,2, By definition, for E1, 3 element X1 ∈ A, X1 ≠ X and p(X1, X) < E1=
	Repet for E2 etc. For En, I a point Xn EA it. Xn # X and p(Xn, X) < En = 1. lor n > 00, n > 0. Thus we have ignormized a sequence where we have
	$(\Leftarrow)$ $p(x_{n}, x) \rightarrow 0  \text{do } n \rightarrow \infty; \text{ this proves claim. let } x_{k} \text{ be a sequence st. } x_{k} \mp x \text{ and } x_{k} \rightarrow x_{k} \text{ k} \rightarrow \infty. \text{ This means that } \forall \varepsilon > 0, \exists N \text{ st. } p(x_{k}, n) < \varepsilon, \forall k > N.$
	In other words, $\forall k \in O_{\mathcal{E}}(b)$ for all $k > N$ . By definition of securitybion point, $\chi$ is an accumulation point.
	town 12:15 let f: X > Y be & function. Then f is contributions at X0 EX iff for any sequence XK EX convergent to X0, the sequence f(x) EY converges to y0 = f(X0).
	Remark - This can also be viewed as the sequential definition of continuity.
	ProofSimple, onatted.
	Open and dord sets.
	1 Internation 1.216 Wessey that XEA is an indexinant point of A if there is an E>O st. B(X,E) C.A. The set of all interior points is denoted by A.
	Remark-desity, ÅcA.
States I al A	To the set that a is room if A = A. A is closed if A = [A].
	Note - A is open iff dist (A <sup>c</sup> , x)>0 for all xEA [recall that A <sup>c</sup> = X \A].
	thronow 2.18 The union of any number of open sols is open. The intersection of finite number of open sets is open. [Proof anisted].
	e.g Let $A_{k} = (-\frac{1}{k}, \frac{1}{k}), \ k = 1, 2, \ Then \ k = 1 \ A_{k} = (-1, 1) \ open \ \ k = 1 \ A_{k} = 10$ which is not open .
	theory 2:19 Theset A CX is open $\iff A^{C_{\pm}} X \setminus A$ is doed.
	(⇒) Troof - let A be open. NTP: [A <sup>C</sup> ] = A <sup>C</sup> . i.e. A <sup>C</sup> = { x: dist (A <sup>C</sup> , x) = 0 <sup>T</sup> . suppose there is an element × ¢ A <sup>C</sup> , but x ∈ [A <sup>C</sup> ] i.e. dist (A <sup>C</sup> , x) = 0. Thus, x ∈ A,
	and dist $(A_{i}^{c}, x) = 0$ . However, A is open, so dist $(A_{i}^{c}, x) > 0 \Rightarrow$ constradiction $\Rightarrow$ doin proven $j_{i}q$ -e.d.
	( =) Suppose A <sup>c</sup> = X\A is closed, i.e. A <sup>c</sup> = [A <sup>c</sup> ] = {x: dist (A <sup>c</sup> , x)=0}. Thus if z ∉ A <sup>c</sup> , then dist (A <sup>c</sup> , z) > 0. ⇒ z ∈ A implies dist (A <sup>c</sup> , z) > 0 ⇒ A is open 194
	( Parts Da
	Toportion 20 Any gen set A of the real line R is the which of courted by many open indewals, i.e. A = K=1 Ak. where AK=(RK, bk) with -os < aK < bK < 00.
	Note-sets (-00, b), (0, 00) are considered open intensis.
	Alchuse - show that open but blio, r) is open, aloned den blio, is bland. (170).
	Microsontizzy A function f: X→Y is continuous on X ⇔ for any open set ACY, the preimage f <sup>1</sup> (A) is open. Proof-outled.
	let ACX, then A is closed since EAJ= EEAJ]. We know that ACEAJ. Let B=EBJ be a closed set st. ACB. What can we conclude about the relationship?
- M.	theorem 2:22 CAJ is the smallest doxed set containing A, i.e. in above notation, CAJCB.
10 × 14	Bample - Take M=(0,1)∧Q. Accumulation paints are [0,1], closure of M is [0,1]. Interior paints of M are M= q, interior paints of [M] are (0,1). 27 January 2014 Arof Mexander 5080281 Martis 706.
	Dense sole, separability.
	(i.e. A is dense in X). Republid 2.23 Let A, B C X. We say that A is dense in B if BCIAJ. The set A is exerginhere dense, if X = IAJ. The set A is said to be nontrove dense if for any open wall BCX, B & IAJ.
	e:g The set M=(0,1) n Q is dense in E0,1]. 13 M dense in (0,1)? Yes, because (0,1) C [M]=[0,1]. Mis not dense in (-1,1).
And the second second	In other words, for any open hall B1 there is another open hall B'CB 5t. ANB'= \$ (wonthere dense). B.
	e.g For R=X, the set A=IN is nonhere dense. [A]=IN - + + + + + + + + + + + + + + + + + +
5	The metric spice. X is called separable of it has a dense constrable subset. (If his everywhere dense and countrable, then X is separable).
	e.g O M= (0, 170 D), X= [0, 1] with standard metric. We know that DU]=X, and M is countable as a subset of Q. Therefore X is separable.
	3 R is separable, since [O-]=R. R <sup>n</sup> is separable, since [Q <sup>n</sup> ]=R <sup>n</sup> .
	3 covider Lp., pE [1,00]. Lp is repeable - Indeed we know that [Coo]= Lp. This means that $\forall x \in Lp$ and any $\epsilon > 0$ , $\exists \hat{x} \in coo \ st$ . $fp(x, \hat{x}) < \frac{\epsilon}{a}$ .
	Turthermore, there is a sequence y∈ Coo with retrional components at. fp(x,y)< ±. By the triangle inequality, fp(x,y) ≤ fp(x,y) < ±+±=ε.
	⇒ secumulation point of sequences with rational components => countrable, so lp is separable.
	(a) how is not separable. Indeed, let M ⊂ how be the set which compists of all sequences of the form a = (a1, a2,) where an =1 or 0. Then M is not countrable.
	there is at least 1 element yEA st. foold, y) <1. (neighbourhood of arbitrary size, mepick 1). Therefore, as there are uncountably many balls, A is uncountable.

103-04-

	D Let X = C[a,b], with distance p(f,g) = max     [f(f)=g(t)]. By slone-weierotrans theorem, for any fex and \$>0, there exists a polynomial p.st. p(f, p) < <sup>6</sup> / <sub>2</sub> .
	Moreover, there is a polynomial with rational arefficience at $p(p; p') < \frac{\pi}{2}$ . By tridingle inequality $p(f; p) < p(p; p) < \epsilon$ .
	Consider M= Q n (0,1) C.R. This is neither open nor clock. K. M. denne in R? [M] = [0,1], R. & [M] so it is not dense in R.
	This does not meet that M is nowhere dense: it is in fact danse in some intends e.g. (9, ±). 30 somesny 2014
	Prof Accorder safalev North 706.
	complete Metric spaces & completions.
	comptre Q and R. We know [Q]=R. Q has "gaps", whereas we see R as the entire real line. Then comptre [0,1] and (0,1), Take $x_n = t_1$ , $n = 1, 2,, Then x_n \rightarrow 0 \in [0, 1] but 0 \notin (0, 1).$
	Betweed 224 A sequence X; E.X is said to be Cauchy if $\forall E>0$ , $\exists N=N(E)$ st. $P(Xn, Xm) as soon as m,n>N.$
	Remonta- there is no mention here of the concept of a "limit".
	theread 225 If X; j=1,2, is a convergent sequence, then it is eachy.
	hoof-leftss exercise.
	Note-ingeneral, the converse does not hold. For instance, Xn= In, n=1,2, is cauchy menot convergent in the space (0,1) as 73-limit is not constained.
	Definition 226 The metric space (X, P) is said to be complete f every Guidy sequence has a limit in X.
	Trimples -
	Ø X=(0,1) is not complete. X=[0,1] is complete.
	(2) CEO.13 is complete: recall that p(fg) = telo13 [fit)-g(t)]. If p(fn, fm) → 0 do n, m → 00, Afut is and to them fn has a limit do n→00, which is also a condition ⇒ CEO.13 complete.
	(a) Cp [-1,1], p & [1, b0] is incomplete. Hore, p & p & [1] dt ] P. Ne constructs specific Guides sequence and sim to show it is incomplete. Define for 1 thetest to the state of the thetest of the state o
	then $p_{l}(f_{n},f_{n}) = \left[\int_{-1}^{1}  f_{n}(t) ^{2} dt\right]^{\frac{1}{p}} = \left[\int_{-1/n}^{1/n}  f_{n}(t)-f_{n}(t) ^{2} dt\right]^{\frac{1}{p}} \leq 2\left[\int_{-1/n}^{1/n} dt\right]^{\frac{1}{p}} = 2\left(\frac{2}{n}\right)^{\frac{1}{p}} \rightarrow 0 \Rightarrow n \rightarrow \infty$ . Thus, $f_{n}(t) \leq 2 \operatorname{conchus} conchu$
	1-1, -1≤t≤0 14 fit)=11, o <t≤1 (heamybidestep="" (p[-1,1],="" 0="" 02.="" as="" complete.<="" continuous,="" cp[-1,1]="" dearly="" f)="" fife="" fix="" function),="" is="" not="" n→="" p(fn,="" so="" th="" then="" thus="" tomercer,="" →=""></t≤1>
	Note - trust by changing & metric, we have made the space incomplet.
	$ \begin{array}{c} \textcircled{P}_{p} \in [\mathcal{A}_{1} \text{ for }] \text{ is complete.}  f_{p}(x_{1}y) = \begin{bmatrix} \sum_{k=1}^{\infty}  x_{k}-y_{k} ^{p} \end{bmatrix}^{y_{p}}, \text{ where } X = (x_{1}, x_{2}, \ldots), y = [y_{1}, y_{2}, \ldots),  be s (such y sequence in $p$ (sequence of sequences) i.e. we have \\ & & & & \\ & & & \\ & & & & \\ & & & \\ & & & \\ & & & & \\ & & & & \\ & & & \\ & & & & \\ & & & & \\ & & & \\ & & & & \\ & &$
	$\frac{\mathcal{B}}{\mathcal{B}} = N(\varepsilon) \text{ st},  \frac{\mathcal{B}}{\mathcal{B}} = 1   x_{\mathcal{B}}^{(n)} - x_{\mathcal{B}}^{(n)}  ^{2} < \varepsilon^{2} \text{ for } \mu, m > \mathcal{N} - (\mathfrak{B}). \text{ this means that } x_{\mathcal{B}} \text{ is Gueday for each } \mathcal{K} \text{ (each term in sum } <\varepsilon) \Rightarrow x_{\mathcal{B}}^{(n)} \text{ bod a limit so } n \rightarrow \varepsilon_{n}, \text{ denoted } x_{\mathcal{B}}.$
	Then define $x = (x_1, x_2,)$ . No check that $x \in \mathbb{P}_p$ . Gooder @ for a finite sum of tome: $\sum_{k=1}^{M}  x_k^{(n)} - x_k^{(n)} ^p < \mathfrak{E}$ with artitizing MEIN, clearly. $-\mathfrak{B}$ . So now represent limit to Mass
	$\frac{\sum_{k=1}^{M}  \mathbf{h}_{k}^{(k)} - \mathbf{x}_{k} ^{2} < \frac{1}{\epsilon} \cdot \mathbf{h}_{k} \text{ Minbowski} (\text{Theorem 2.5)},  [\sum_{k=1}^{M}  \mathbf{h}_{k}^{(k)} ^{2}]^{\frac{1}{2}} + [\sum_{k=1}^{M}  \mathbf{h}_{k}^{(k)} - \mathbf{x}_{k} ^{2}]^{\frac{1}{2}} \leq [\sum_{k=1}^{M}  \mathbf{h}_{k}^{(k)} ^{2}]^{\frac{1}{2}} + \varepsilon \Rightarrow \sum_{k=1}^{M}  \mathbf{h}_{k}^{(k)} < \varepsilon \Rightarrow \mathbf{x} \in \mathbb{P}_{\epsilon},  \text{thus it follows from (3D) that}$
	p (x <sup>(n)</sup> , x) = [x <sup>(n)</sup> - x <sub>K</sub> ] <sup>p</sup> ] <sup>4</sup> < ε. Therefore, for any ε>0, we found N=N(ε)_s.t. p(x <sup>(n)</sup> , x) < ε. as soon, do n>N. so n>oo (x <sup>(n)</sup> ) = x in lp ⇒ lp is complete.
	Etwaceulz. 27. Let (X, p) be a complete metric space, and let ACX. Then the subspace (A, p) is complete <> A is closed (i.e. A=[A]).
	Noof-seft se exensise (just compère definitione).
	Remnal 2.28 let X be a complete methicspace. Let Kj= B[Xj, Yj] with Xj & X, Yj>0, j=1,2, suppose that Yj >0 as j >00 and that Kj is a nested sequence of closed balls (i.e. K1>K2>K1>).
2.3	Then the set K = AK; is not empty.
	Proof - Consider the sequence X1, j=1,2, This is a conclussequence. Indeed, Forany m 3 h, me have Xm E B[Xn, Yn] 5.T. p(Xn, Xm) < Th -> 0 30 n->00. As X is complete, 17hit has a Darit
	let x= him xn be this limit. Observe that xm ∈ Km ⊂ Kn Ymzn. Thus, more xm ∈ Kn do Kn is closed = x ∈ Kn ∀M = x ∈ of Kj do required / q.e.d.
	Remarks - The converse is truce if for any nested sequence of closed balls B[xj, y] as 1; +20, ; +20, , + the intersection is non-empty => X complete. (proof of this not assessed).
	Moreover from the estier proof, x is the unique Emilt for any such sequence, so j=1 kg has exactly only one element.
	Theorem 2.27 (Boire Cotepony Theorem).
	A somplete metric space connot be 2 countable union of nonhere dence cets.
	Broof - By contradiction. Suppose me con write X= 2 Ak with Ak nowhere dense \$ KEIN (since k runs over IN, this is a countable union). Then let Ko be a closed will of radius 1. since Ay
-	is nowhere dense, 3 & dosed boll K1 CK0 st. K1 A1= \$ WLOG, we can choose to assume that its radius is < 1. Since A2 is nowhere dense, 3 K2 CK1 st. K2 A2 = \$.
	Then WLOG assume radius of the is < 5. continuing this way, we construct a rested sequence of closed balls they, j=1,2, s.t. radius of the is bostion it and then the of the is the they be the is the they be the is the stand the they be the the they be the the the they be the they be the they be the the they be the the the the the the the the the th
	· in (K; n An)= \$ 4n. Therefore (in K;) n ( UAn) = \$. From Lemma 228, in K; is non-empty, so = x < in K; > x & UAn for restoment to hold. By definit
	X = U An, so X & X which is a constratiction as space is complete. Thus, the assumption is falle 1. q.e.d.
	Emology 2:30. A complete metric space without isolated points connetable.
1. J. H. K.	indeed, if X = 401 time, then time is nowhere dense. By Brine Category theorem, this is impossible.
	3103-0

	Completion of Metric Servers
	Infinitianal 231 A complete metric space X is said to, be a completion of the metric space X if
	(1) There is an isometry $P:X \to \tilde{X}$ (i.e. distances are preserved) (3) $P(X)$ is dense in $\tilde{X}$ .
	Throw 2.32 My metric space has a completion. This completion is unique up to an isomorphism line. if X, X are completions, I an isomorphism 4: X -> X).
- 100010-	(inapplete) froof-let X be a marie spice. We seak to construct X.
	· lot 1×n2, synt be savely sequences in X. We call them equivalent if flow, yw = 0 as n = 200 (i.e. referine, symmetric, transitive : follows from triangle inequality). consider equivale
	classes $\tilde{x}$ which consist of equivalent sequences (i.e. $\tilde{x} \sim x$ ). Refine $\tilde{\chi}$ as the set of all equivalence classes. Metric: for $\tilde{x}, \tilde{y} \in X$ define $\tilde{p}(\tilde{x}, \tilde{y}) = \frac{1}{K+2}$ on $p(x_{K+1}y_K)$ , where mean evolves remains on $\Delta$ ineq. There $\tilde{x}, 4y_K t \in \tilde{y}$ . This limit exists; indeed $[p(x_{K+y_K}) - p(x_{K+y_K})] \leq p(x_{K+2}y_K) + p(y_{K+y_K}) \longrightarrow 0$ as $k_1 m \rightarrow \infty$ since $1x_K t, 4y_K t$ are couchy.
	• The definition of $\tilde{\rho}$ is independent of the choice of representations that $\tilde{e}\tilde{x}$ , $1$ yet $\tilde{e}\tilde{y}$ . Let $1$ is $\tilde{x}$ , $1$ yet $\tilde{e}\tilde{y}$ . Then $ \rho(x'_n, y'_n) - \rho(x_n, y_n)  \leq \rho(x'_n, x_n) + \rho(y'_n, y_n) \rightarrow 0$
	$\partial n \rightarrow \omega_0$ , since $A \times n_1$ , $A \times n_1 \in \mathbb{X}$ and $A \in \mathbb{Y}_1$ , $A \in \mathbb{Y}_1$ i.e. some domes.
	3 February 2014. The Mexidue SoBolEV
	semple- Recall that CP[0,1], 1≤pcon is incomplete, with the metric p[f,g]= [[] [ft]-g[t]] [dt]]; the completion of cp[0,1] is defined to be the space Lp[0,1]. Matrix 0.
	_ Cartisection Mapping Theorem
	1 - 1 - 1 - 1 - 1 - 2 X is said to be a contraction suppling if shere is a number d∈ [0,1] s.t. for any x, y, ∈ X one has p(Ab), A(y)) ≤ d p(x,y).
	The point XeX st. A(s)=X is brown do a fixed point.
	(Banach Fixed Paint Theorem)
	Throad 234 if A is a contraction in a complete monic space X, then there exists a unique fixed point of A: i.e. 3 on to st. A(x)= Xo. [Pernox-stere on, denote Ax = A(x).].
	observation = p(AXn,Ay) → 0 if p(xn,y) → 0 as n→ as. Hencener, [p(AXn,2) = p(Ay,2)] ≤ p(AXn,Ay) → 0 as n→ 00.
	<b>Remain</b> 235 - For any $x \in X$ , we have $A^{i}x \rightarrow x_{0}$ as $n \rightarrow 0$ a.
1.1.2.1.2	toother 1236 suppose but for some t, the map B=16 is a contraction. Then to is a fixed point of A $\iff$ to is a fixed point of B.
	Roof-(7) Obviously BX0 = AK x0 = AK-1 (AXD) = AK-1 x0 = = AX0 = X0, 50 X0 is a fixed point of B.
1111	(4) suppose BX0=X0, then AX0= AB <sup>L</sup> X0 (VL) = B <sup>L</sup> (AX0) by commetativity (since B=A <sup>K</sup> ) -> X0 by Remark 2.35. Hence X0 is a fixed privator A/1, q.e.d.
5	Esymples - (1) Rissul's Theorem.
5 5 5 5	Esymples - (1) Rissul's Theorem.
2	Examples - (1) Ficeral's Theorem. (2) integral equations: Les X= C[a,b]. Let K(1x,y) be a continuous function on [a,b] × [a,b]. Tefine of fix) = [a, K(1x,y) + f(y) dy Yf (X. This is on integral operator with the termel K(1x,1y).
	Ersonples - (1) Prisonal's Theorem. (3) hatgoolephanisons: les X=C[a,b]. let K(Xny) be a continuous function on [a,b] X[a,b]. Tefine K(Fis) = Ja K(Xny) + fly) dy VfEX. This is an integral operator with the tennel K(Xny). (3) hatgood to constant of integration]. Ne next to solve f = 9+ NK f= Af where is 6 is a number and YEX is fixed. 13 this a constrabilities. Af-Ag = 1K(f-g). We need p(Af, Ag) < x p(Ag) for x(E(0,1). thus, if M= mig
	Examples - (1) Prisonals Theorem. (3) hatgoalephanisons: les X=C[a,b]. let K(X,y) be a continuous function on [a,b] X[a,b]. Tefine K(fis) = 1 a K(X,y) + f(y) dy Vf & X. this is an integral operator with the kennel K(X,y). (3) hatgoalephanisons: les X=C[a,b]. let K(X,y) be a continuous function on [a,b] X[a,b]. Tefine K(fis) = 1 a K(X,y) + f(y) dy Vf & X. this is an integral operator with the kennel K(X,y). (3) hatgoalephanisons: les X=C[a,b]. let K(X,y) be a continuous function on [a,b] X[a,b]. Tefine K(fis) = 1 a K(X,y) + f(y) dy Vf & X. this is an integral operator with the kennel K(X,y). (3) hatgoalephanisons: les X=C[a,b]. let K(X,y) be a contrained of integration]. Ne need to solve f= 9+ NK(f= Af nhore. Ne & is a number and Ye X is fixed. 13-this a contradiction]. Af-Ag = 1K(f-g). We need (Af, Ag) < < (P(A), for de[(0,1)]. thus, if M= may max. te(a,b]   1 k(lf-g)(t)  = 14 te(B,b)   1 a K(t;y) (f(y) - g(y)) dy   < (N M (a lf(y) - g(y)) dy < (N M (b-a) p(f,g). Thus, if d=1N M (b-a) <1, we have a unique sche hy. Fried Rind thear
	Examples - (1) Prisond's Theorem. (3) hotogradeprations: Les X=C[a,b]. Let K(1xy) be a continuous function on [a,b] X[a,b]. Teffice K(fis) = 1 a K(2xy) + f(y) dy ¥féX. This is an integral operator with the tennel K(1xy). (3) hotograde prations: Les X=C[a,b]. Let K(1xy) be a continuous function on [a,b] X[a,b]. Teffice K(fis) = 1 a K(2xy) + f(y) dy ¥féX. This is an integral operator with the tennel K(1xy). (3) hotograde praticus: Les X=C[a,b]. Let K(1xy) be a continuous function on [a,b] X[a,b]. Teffice K(fis) = 1 a K(1,0) dy ¥féX. This is an integral operator with the tennel K(1xy). (3) hotograde praticus: Les X=C[a,b]. Let K(1xy) be a contraduction on [a,b] X[a,b]. Teffice A find, A = A = A + A = A + A + A + A + A + A +
	Examples - (1) First & Theorem. (3) Integral persons: Les X= C[a,b]. Let \$(15,y) be a continuous function on [a,b] X[a,b]. Tefine \$(fs) = [b] (a) Integral persons: Les X= C[a,b]. Let \$(15,y) be a continuous function on [a,b] X[a,b]. Tefine \$(fs) = [b] (b) Integral persons: Les X= C[a,b]. Let \$(15,y) be a continuous function on [a,b] X[a,b]. Tefine \$(fs) = [b] (b) Integral persons: Les X= C[a,b]. Let \$(15,y) be a continuous function on [a,b] X[a,b]. Tefine \$(fs) = [b] (b) Integral persons: Les X= C[a,b]. Let \$(15,y) be a continuous function on [a,b] X[a,b]. Tefine \$(fs) = [b] (b) Integral persons: Les X= C[a,b]. Let \$(15,y) be a continuous function on [a,b] X[a,b]. Tefine \$(fs) = [b]. An end \$(15, 10, 10, 10, 10, 10, 10, 10, 10, 10, 10
	Examples - (1) Prisonal's Theorem. (2) Integrale positions: Les X= C[a,b]. Let K(15,y) be a continuous function on Eab]X[a,b]. Tefine K(fis) = $\int_{a}^{b} K(15,y) + f(y) dy$ $\forall f \in X$ . This is an integral operator with the kernel $K(X,y)$ . (3) Integrale positions: Les X= C[a,b]. Let K(15,y) be a continuous function on Eab]X[a,b]. Tefine K(fis) = $\int_{a}^{b} K(15,y) + f(y) dy$ $\forall f \in X$ . This is an integral operator with the kernel $K(X,y)$ . (3) Integrale positions: Les X= C[a,b]. Let K(15,y) be a continuous function on Eab]X[a,b]. Tefine K(fis) = $\int_{a}^{b} K(15,y) + f(y) dy$ . (3) Integral positions: Les X= C[a,b]. Let K(15,y) (f(y) - g(y)) dy] $\leq (X   M   \int_{a}^{b} (f(y) - g(y)) dy] \leq (X   M   \int_{a}^{b} (f(y) - g(y)) dy = (X   f(y) - g(y)  dy = (X   f(y) - g(y) ) dy] \leq (X   M   \int_{a}^{b} (f(y) - g(y)  dy = (X   f(y) - g(y) ) dy] \leq (X   M   \int_{a}^{b} (f(y) - g(y)  dy = (X   f(y) - g(y) ) dy] \leq (X   M   \int_{a}^{b} (f(y) - g(y)  dy = (X   f(y) - g(y) ) dy] \leq (X   M   \int_{a}^{b} (f(y) - g(y)  dy = (X   f(y) - g(y) ) dy] = (X   f(y) - g(y)  dy), x \in (a,b] - bic is the Voltoms (integral) operator. Ne want to solve f = (P + Tf - f(x) - g(x) $
	Examples - (1) Prismils Theorem. (3) integral persons: Les X= C[a,b]. Let \$(1x,y) be a continuous function on [a]b] X[a,b]. Teline \$(fis) = \int_{a}^{b} K(1x,y) + f(y) dy \$\forall f(X). This is an integral operator with the termed \$(1x,1y). (a) integral persons: Les X= C[a,b]. Let \$(1x,y) be a continuous function on [a]b] X[a,b]. Teline \$(fis) = \int_{a}^{b} K(1x,y) + f(y) dy \$\forall f(X). This is an integral operator with the termed \$(1x,1y). (a) integral equations: Les X= C[a,b]. Let \$(1x,y) be a continuous function on [a]b] X[a,b]. Teline \$(fis) = \int_{a}^{b} K(1x,y) + f(y) dy \$\forall f(X,1y). (b) integral equations: Les X= C[a,b]. Let \$(1x,y) be a contradiction on [a]b] X[a,b]. Teline \$(1x,1y) dy \$\forall f(X), \$(1x,1y). (b) integral to solve \$f = 9 + 10 K(f = Af none. A = 0.5 \$(1x a number and \$(2x is fixed. Us his a contradiction]. Af Ag = \$AK(f-g). We need \$(Af, Ag) \$\leftarrow \$(1x,1y). Thus, if M= \$(1x,1y) \$(f(y) - g(y)) dy \$\leftarrow \$(1x,1y) f(y) = g(y)] dy \$\leftarrow \$(1x,1y) f(y) dy \$\leftarrow \$(1x,1y) f(y) dy \$\leftarrow \$(1x,1y) f(y) dy \$\leftarrow \$(1x,1y) f(y) - g(y)] dy \$\leftarrow \$(1x,1y) f(y) g(y)] dy \$\leftarrow \$(1x,1y) f(y) g(y)] dy \$\leftarrow \$(1x,1y) f(y) dy \$\leftarrow \$(1x,1y) f(y) f(y) dy \$\leftarrow \$(1x,1y) f(y) f(y) dy \$\leftarrow \$(1x,1y) f(y) dy \$\leftarrow \$(1x,1y) f(y) dy \$\leftarrow \$(1x,1y) f(y) dy \$\leftarrow \$(1x,1y) f(y) f(y) dy \$\leftarrow \$(1x,1y) f(y) dy \$\leftarrow \$(1x,1y) f(y) dy \$\leftarrow \$(1x,1y) f(y) f(y) dy \$\leftarrow \$(1x,1y) f(y) dy \$\leftar
	Examples - (1) Firstelle Theorem. (3) Integrale participas: Les X= C[a,b]. Let K(X,y) be a continuous function on [a,b] X[a,b]. Tefine K(fis) = [b] (4) Integrale participas: Les X= C[a,b]. Let K(X,y) be a continuous function on [a,b] X[a,b]. Tefine K(fis) = [b] (5) Integrale participas: Les X= C[a,b]. Let K(X,y) be a continuous function on [a,b] X[a,b]. Tefine K(fis) = [b] (5) Integrale participas: Les X= C[a,b]. Let K(X,y) be a continuous function on [a,b] X[a,b]. Tefine K(fis) = [b] (5) Integrale participas: Les X= C[a,b]. Let K(X,y) be a continuous function on [a,b] X[a,b]. Tefine K(fis) = [b] (5) Integrale participas: Les X= C[a,b]. Let K(X,y) for x = [b]. Thus, if M= may terearbill   A K(f-g) (tr) = LiA treas   Ja] K(tr,y) (f(y)-g(y)) dy   ≤ [A] M [b] (F(y)-g(y)] dy < [A] M [b] (F(y)-g(y)] dy = [A] M(b-a) p(fig). Thus, if a = [A] M(b-a) < 1, we have a unique sch by Fried Rind theorem (5) Define T; C[a,b] → C[a,b] = [a] K(tr,y) f(y) dy, x = C[a,b]. This is the <u>Uttors</u> (integral) operator. Ne want to solve f= (P+TF for flue points. (5) Define T; C[a,b] → C[a,b] → [Tf(s) = [a] K(tr,y) f(y) dy, x = C[a,b]. This is the <u>Uttors</u> (integral) operator. Ne want to solve f= (P+TF for flue points. (6) Define T; C[a,b] → C[a,b] = [a] K(tr,y) f(y) dy, x = C[a,b]. This is the <u>Uttors</u> (integral) operator. Ne want to solve f= (P+TF for flue points. (6) Define T; C[a,b] → C[a,b] = [a] K(tr,y) f(y) dy, x = C[a,b]. This is the <u>Uttors</u> (integral) operator. Ne want to solve f= (P+TF for flue points. (6) Define T; C[a,b] → C[a,b] = [a] K(tr,y) f(y) dy, x = C[a,b]. This is the contradiction with the contradiction (eff to corrive). Use contradicts of the triangle field points: Tiff) = (P+TF+T <sup>2</sup> F), T*(F) = P+T·F Ref Meximular to contradiction to for the corrive.). Use contradicts of the triangle field points: Tiff) = (P+TF+T <sup>2</sup> F), T*(F) = P+T·F Ref Meximular to contradicts of the triangle point to form.
	Examples - (1) Fisind's Theorem. (3) Integral equations: Let X= C[a,b]. Let K(1+y) be a continuous function on Ea,b] XEa,b]. Refue K(fi) = 10 K(x) + f(y) dy YfEX. This is an integral operator with the kernel K(x,y). Endogous to contrad function on Ea,b] XEa,b]. Refue K(fi) = 10 K(f-q). We need P(AF, Aq) < < P(Aq) for de [0,1]. This, if M= max Ne neut to solve f = P+ NKF = AF, where he (C. is a number and YEX is fixed is this a contradictor). Af-Aq = NK(f-q). We need P(AF, Aq) < < P(Aq) for de [0,1]. This, if M= max refer, a) [A K(f-q)(t)] = [A] the B_{10}[] [a K(1;y) (f(y)-g(y)) dy] < (N M [a [P(Y)-g(y)] dy) < (N M [a [P(Y)-g(Y)]
	Examples - 11 Prisods Theorem. (3) hotgoslephsions: Les X=C[a,b]. Let K(1xy) be a continuous function on [a,b]X[a,b]. Table et f(b) = 1 (4) hotgoslephsions: Les X=C[a,b]. Let K(1xy) be a continuous function on [a,b]X[a,b]. Table et f(b) = 1 (5) hotgoslephsions: Les X=C[a,b]. Let K(1xy) be a continuous function on [a,b]X[a,b]. Table et f(b) = 1 (5) hotgoslephsions: Les X=C[a,b]. Let K(1xy) be a continuous function on [a,b]X[a,b]. Table et f(b) = 1 (5) hotgoslephsions: Les X=C[a,b]. Let K(1xy) be a continuous function on [a,b]X[a,b]. Table et al. (5) hotgoslephsions: Les X=C[a,b]. Let K(1xy) be a continuous function on [a,b]X[a,b]. Table et al. (6) hotgoslephsions: fee f= f+ there. X=C (1x). This is the fee f(1y) - g(1y) dy f(1y) - g(1y) dy f(1y) dy f(1y). g(1y) dy f(1y) - g(1y) dy f(1y) - g(1y) dy f(1y) dy f(1y). Thus, if d= [A] (A(1b)-a) <1, we have a unique side by Fried Rivel Theorem. (5) Tables - C(a,b] -> C(a,b] -> [a K(1yy) f(1y) dy, x=C (a,b]. Table is the boltom (integral) operator. Ne wave to solve f= f+ Tf for fleed point. (6) Tables - C(a,b] -> C(a,b] -> [a K(1yy) f(1y) dy, x=C (a,b]. Table is the boltom (integral) operator. Ne wave to solve f= f+ Tf for fleed point. (7) the contradiction deft to corrise). Use contradiction deft to corrise). Use contradiction the to solve the field from the field point field dequarter of the contradiction deft to corrise). Use contradiction to field to a unique field point field point f(f) = f(f) = T(f) = f(f) = T(f) = f(f) = f(f
	Examples - 19 Risseld Theorem. (3) hacquelequations: Les X=C[a,b]. Let f(12,y) be a continuous function on Lab JX[a,b]. Telue & f(b) = 1 <sup>b</sup> K(3,y) + f(y) dy + f(x). This is an integral operator with the kernel K(1,y). Durbanes to avoid of integration. Nervour to solve f = f+2Kf = Af none. No G is a number and 19 X is fired. Is this a contraction? Af Ag = 1K(f-g). No need (1/h, Ag) << q(1/g), for ele [0,1). This, if M= "Ly mages to solve f integration. Nervour to solve f = f+2Kf = Af none. No G is a number and 19 X is fired. Is this a contraction? Af Ag = 1K(f-g). No need (1/h, Ag) << q(1/g), for ele [0,1). This, if M= "Ly mages to solve f integration. (a) hole of (1/g) = (1/h the Brite] [1/a K(try) (f(y) - g(y)) dy] < (1/h M ) <sup>b</sup> (Hy)-g(y)) dy < (M/h (b-a) p(f,g). Thus, if d= 1/h M(b-a) <1, we have a unique solution for the definition (b) hole of T; C(a,b] -> (Ta,b) = 1 <sup>a</sup> K(by) f(y) dy. x < (a,b). This is the <u>bottom</u> (integral) operator. Ne want to solve f = (+Tf ff free float point. (f k is sufficiently larger, then Th is a contradiction seft as contraction. We construe to the solution field in the field point of the solut space. Then a set KCX is sold to be additively compart if any requese in K contract of a contractive d construct. (f all possible twice space. Then a set KCX is sold to be additively compart if any requese in K contract of a contractive d compart. (f all possible twice of these subsequences bedrag to K, then we say that K is secretar. A set that is both reditively compart and closed is compart. (at k, be infinite. It is reditively compart if any infinite solvest of K, hoo an accumulation print. (b) the infinite. It is reditively compart if any infinite solvest of K hoo an accumulation print. (c) 1
	Edensples - (1) Filewisk Theorem. 63 Hadgedepairones: Let Xe C [24, [1] . Let filewisk be a continuous function on [26, [2] XE [26]. Polye of for file [26]. His is an integral operator with the kernel K(x111). Enviropes to contest of integrations: We used to solve f= f+ XKf = Af where N & E is a number with fe X is first is their a contradiction. Af Ag = Ag = 1K(f-g). He need (phf, Ag) & x (E(D)). Thus, if M = #Wg (26) [1] K(f-g)(t)] = (1) the first [1] [1] K(trig) (fg) -g(y) ] dy (≤ 1X) M [1] [1] H(g) -g(y)] dy (≤ [X], M [2]) [1] H(g) = [X] (fg) = their of a unique soln by Fired Bind Item (3) Agles T: (C(a) (b) - b) [1] [2] K(trig) (fg) -g(y) ] dy (≤ 1X) M [2] (Fg) -g(y)] dy (≤ [X], M [2] (Fg) - thus, if d = [X] (H((b-a)) [1] fg) - thus d = [X] (H((b-a)) [1] fg) - thus d
	Everytes - 11. Fische Theorem. Everytes - 11. Fische Theorem.
	Estroples - 11. Field Theorem. (5) hoggelequations: Let XCC[a,b]. Let K(x,g) be a continuous function on [ab]X(a,b]. Phile K(fi) = 10 K(x,g) + fig) dy. Yf EX. This is an integral operate with the level K(x,g). (5) hoggelequations: Let XCC[a,b]. Let K(x,g) be a continuous function on [ab]X(a,b]. Phile K(fi) = 10 K(x,g) + fig) dy. Yf EX. This is an integral operate with the level K(x,g). (5) hoggelequations: Let XCC[a,b]. Let K(x,g) be a continuous function on [ab]X(a,b]. Phile K(fi) = 10 K(x,g) + fig) dy. Yf EX. This is an integral operate with the level K(x,g). (6) hoggelequations: Let XCC[a,b]. Let K(x,g) fig) dy. (2) [A] M [a [Ag]-g(g)] dy (2) [A] M [a [Ag]-g(g)] dy (2) [A] M [a [Ag]-g(g)] dy (2) [A] M [b - a) (fig) - thus, f a [A] M [b - a) (x, me here a unique sche by. Theod Band theorem (2) hogge T. (Cla,b]. Theorem (1) = 10 K (x,g) fig) dy. (3) (2 (A) M [a [Ag]-g(g)] dy (2 (A) M [b - a) (fig) - thus, f a [A] - Theorem (A
	Everytes - 11. Fische Theorem. Everytes - 11. Fische Theorem.
	Estroples - 11. Field Theorem. (5) hoggelequations: Let XCC[a,b]. Let K(xy) be a continuous function on Cab JXCa, b]. Polye K(fw) = 10 K(xy), = f(y) dy. Yf EX. This is an integral operate with the level K(xy). (5) hoggelequations: Let XCC[a,b]. Let K(xy) be a continuous function on Cab JXCa, b]. Polye K(fw) = 10 K(xy), = f(y) dy. Yf EX. This is an integral operate with the level K(xy). Notations to solve f = f + NK f = Af. hore. N = G: N a number of united function. (6) hoggelequations: Let XCC[a,b]. Let K(xy) f(y) dy. (2) (A M [a], F(y)-g(y)) dy (2) (A M [b], F(y)-g(y)) dy. (2) (A M [b], F(y)-g(y)) dy. (2) (A M [b], F(y)-g(y)) dy (2) (A M [b], F(y)-f(y)-f(y)-f(y)-f(y)-f(y)-f(y)-f(y)-f
	Bangles - 11 Risseld Theorem. (3) hogeslequerons: (ex Xe Claft). et Klarg) he a continuous function on Calif Starts. The fail of Klarg) field y VfeX. this is an integral opeober whethe bened Klarg). Darlingous to whethe function on Calif Starts. The fail of Klarg) - field y VfeX. this is an integral opeober whethe bened Klarg). Darlingous to whethe function of the control of the control of the fail of the fail of the fail of the fail of the control of the contr
	Biorgles - 111 Field Theorem. Is hadgelequations: Let XC (Eq.16). Let flag) be a continuous fluction on Edel IXEs 12. The KS 12 to Klags + flag) by Yf 6X. this is an integral greater vide the level Klassy. Independent on whith functions. Afrag - KC (Es.16). Let flags) be a continuous fluction on Edel IXEs 12. The Klassy + flag) by Yf 6X. this is an integral greater vide the level Klassy. Means to solve f. = F. + MKf = Afrage - XC (Es. 16 a mumber and 'PK's frad, 16 dis a contraction, 'Afrag = JK(f-g), the near flags, for set 0(1), thus, for 18 dis the flags is a flags of the flag
	Biorghan III Fileda Theorem. IS hadgeleguesions: Les Xe (Edit). Let Hisg) be a continuous function on Edit/SEG, 12. The King) - fig. by Yf (K twi is an integral product vide the bened King). Indexes to value f: - F: + Wh (- Af - Age - K - C & a number and YK & Field is this a contractions, - Af - Ag = Kinf-g, - the ends (MA, Ag) - for de Edit). Is use, filled for the field of the spectra
	Banquer 10 Frank Theorem. 60 hogsingurinous: 162 X : C(h, j). 162 K(sy) be a sometime function on Chill S (c, j). Take K (fil) = <sup>1</sup> / <sub>0</sub> M(sy) - f(s) dy. Yf (X. two an indexed queroper induce level (X, sy). The same a value of X = Y = XK = A = A = A = A = A = A = A = A = A =

	Proof-(=)) Let M be redshirely compace. Fix 670. The KIEM. Then either M C B(M, 6), or there are points of M subjude this built in the Bitter case, take	X2&B(X1,E)_5.T. X2€M·
	Than other MCF(x1,E) uB(x2,E) or M& B(x1,E) uB(x2,E). Continue the process this will produce a collection of possible x1, x2,, XIC S	t. h;-nk ≥e.
	Ubine this process is finite. Suppose otherwise, i.e. ×1, ×2, continues infinitely vithout terministing. Since Nr. 6M is compact, XXX	must contain a convergent subsequence.
	However, [xj-xk]≥ € ∀j.k, so the subsequence connectprosidy be convergent ⇒ contradiction. Thus, & finite sener is construed	
	(=) led X be complete, and let M have a finite E-net-for any E>0. Let En→0 be positive numbers and let Nn=1×1,, XEn t hear En-net	
	port of T. in one of B(XK, E1) containing infinitely many elements of T. Let IzeTo be a part of To in one of B(XK, E2) containing infinitely	
	T=T+=T_2==Tj==+tore, Tj CB(1/K, ej). choose a subsequence of T in the following user: \$1 < T1,, \$j & Tj. then p(\$10, \$n).	
	As En-20, the sequence 15; I is country. Since X is complete, 5; has a built = N is rebuirdy compactly que.d.	ул ти
		(N2)
	Eardbuy 2491 let the complete. Then a set MCX is relatively compact $\Leftrightarrow \forall \epsilon > 0$ , M has a relatively compact s-vet.	(C)
	Prof-reft so-exercise	(CSF
	I Combingle 72 If H is relatively compact, it is bounded.	×.
	Book-Let X1,, XK be 21-net of M. It is finite as Mis relatively compact. Rich any point XOEX and define d= 15jek p(XO)Xj). Then, V XEP	1, the manyle inequality
	yields that $p(x_0, x) \leq p(x_1, x) + p(x_1, x_0)$ for any $r = 1, \dots, k = dr 1 \Rightarrow M$ is boundary q.e.d. $x_1 = (1, 0, \dots)$	± 1
	Example - X=lp, 1=pcon. Define Minitus following way let M= 19k/k=1, be XK=10,	= 2, m =11. There is no convergent subsequence = not identively compact
	Emilon 2:43 of MCR", then M is completed in is bounded and dosed.	
	Broof-Leftos exercise.	
	Contemp 244 If A CM and M is compact, then A is compact <> A is closed	
an an an air air air an air	Construint 2.45 bot X be a complete complete space. Then X is separable.	
	Proof-let En=>0 bes positive sequence, and let Nn bes finite En-net for X. Then N= Nn is countable, and dense in X1. q.e.d.	
ta da		
	Befores 2th Acolection of open sets (Guir is called a concer for a set MCX if MC Q Gu.	-((00))
	Receltive traine-Bonel Lemma- Every set of corrors has a finite subscoror. We will show that this citerion is equivalent to compactness.	o 1
	Theren 247 Autorial set McX is complet (=) From any open conor M one can extract a finite subconor.	
	Boof-Omittal, noterominable.	<sup>™</sup> ),
1	throcal 24th let fix -> 1 be continuous. Then for any compart set ACX, the set f(A) is also compact.	
	Since A is compact, full is doved. (by continuity of f) Proof-let Abar bean open cover of f(A) it was, 1f <sup>-1</sup> (Gal)t is an open cover of A. Entrait a finite subcover as A is compact, so TTI,, That cover A and A	f(T1),, f(Tu)) is a finite cover of f(A). Also, this
	is a finite subcover of 1Get = flat is comparty g.c.d.	· · · · · · · · · · · · · · · · · · ·
	Bouldy 299 let X be a compact gale and let f: 1-2 18 be continuous. Then f is bounded and alloins its maximum and minimum values.	
		10 February 2014 Raf Mexander SOBOLEV Nathas 706 .
	We now pore one part of Theorem 2.47:	
	Boof-Assume that for any open cover-there is finite subcover. doin: ⇒ M is closed. Suppose M = [M], i.e. = xo. E [M] \M. Then p(x, xo)>0	CM
	Then $B(X_1, Y_1) \cap B(X_0, Y_1) = \beta$ . clearly, the balls $B(X_1, Y_1)$ , $X \in \mathbb{M}$ form an open cover of $M$ . Take a finite subcover : $B(X_1, Y_1), B(X_2, Y_2), \dots, B(X_N)$	
	Take r=15K5n K. Then (1) B(XK,YK), K=1,2,,n is a cover of M, (3) B(X0,17), B(XK,17K)=\$\$, Thue, dist(M, X0) ≥r>0. Since (N)	]= fy: dist (M, y)= 0] and NOE [M], we get
	s contradiction. Honce, M=[M]/, q.e.d.	
	there 250 set f: X > Y be continuous. Then for any compact set MCX, the function f is uniformly continuous on M. i.e. 7220 = 5>0 st. plfts).	fly))<& as soon as plx,y)<& tx,yEM.
	hoof-corollary of theorem 2.47 (exercise)	
	compartness in the space of continuous functions (Arzelà-Ascoli Theorem).	
	Defendent 251 Lot MC [[a,b]. Functions XEM and soid to be uniformly bounded if there is a constant C>O st. max [XII] < c for all XEM.	
	The functions XEM are said to be equication on of YE20, 3570 st. [X(ty)-X(ts)] <e [ty-tz]<8="" all="" and="" as="" for="" kem="" soon="" t<="" th="" ty,=""><th>e[a,b].</th></e>	e[a,b].
	theread 252 (Arzelà-Ascoli Theorem)	
	The set MC (Eq. 1] is relatively compare (=> M is uniformly baunded and equicantinuous.	- Alfi-
	(Portul, one wood) Roof - (=). Mis relatively compart. By conclising 242, Mis bounded. By Theonom 2,40, 1220, there is a finite = - net, i.e. there.	the -c
	x1,,Xn & Clarb] st. YXEM, there is a function Xx st. p(X/Xx)<5. Therefore, one can find 520 st. [X(ti)-X(ta)] < [X	ta) = × k(ta) + 1× k(ta) = × k(ta) + 1× k(ta) - ×(ta
	< =+ + /xk(ti)-xk(tz) + = < E as soon as 1ty-tz < Sk. if we take 8=15k cn Sk, then \$\$\$>0, 3 \$>0 st. 1ty-tz < S > 18(tz)-x	

	theorem 2.53 bet die C (taxb] × tab]) and let (kf) b) = (a d(xy) fy) dy, f e C (a,b]. then K: C (a,b] = C (a,b] is an integral operator. Let S=1 fe C (a,b]: (fi) < C1>0.
	Then K(S) is relatively complet
يت ال	Proof- (1) Uniform boundabless of K(S). Estimate: 1k f(x) = [a x/y 1k(x,y)] max 1f(x)   dz. = GS_1a dz = c1c2(b-a).
	(2) Equicontinuity. Neknow 4220 35 st. (K(14,9)-K(12,14))< & if 14-12/5. Therefore, ((Kf)(21)-(Kf)(22)) & [a (K(14,14)-K(12,14)] + (14,14) + (14,1
3	Thue, K(5) is equicontinuous. From (1) \$ (2), by Avida-Ascoli Theorem, K(S) is relatively compact of q.e.d."
	13 Telminy 2014 (Noplar 3 Prof. Mexicader SOBOLEV
2 4	NORMEDSPACES AND BANACH SPACES.
	linear restorspaces. X, K-field with the From C. we let elements of X be called vectors.
	Definition 231 Additions on X is defined as the operation t: X x X -> X with the properties (1) Xt Y= y+X, (2) (xt y) t z= xt (y+z) (3) = reador 0.5t. Xt 0=0t X=x, and
	(4) For each element x eX, 3 vector -x st. x+(-X)=0
	Befridoad 3.2 The propring ·: K X X -> X is called multiplication by a scaled if (1) of (BX) = (x B) X (2) 1.X = X (3) of (X+y) = of (x) + of (y) + of (x) + of (y) + of (x) + of (y) +
	$(4) (4+\beta)(x) = d(x) + \beta(x)$
	topinitian 33 A linear rector space X onor the field IK is a non-empty set X with operations of addition and multiplication by scalars defined on it. If IK = R, X is a real rector space (respectively C, complex).
	(1) R, R <sup>d</sup> , C <sup>d</sup> are v.s. Odisc of radius 1 is not a vis.
	(2) sequence spaces: Co, Coo are linear spaces. By Minkowski's inequality, two sequences in lp sum to another in lq, so lp, p∈[1, co] is a linear space. (3) c[a, b], Cp[a, b], Lp[a, b], 1≤p<00 c[a, b], Cp[a, b], cp[a, b], cp[a, b], cp[a, b], cp[a, b], the completion, is a linear space by Minkowski's inequality.
	Deputored 34 ∧ subspace of linear space X is a non-empty subset YeX which is closed under + and . (i.e. ∀X, y ∈ Y, d, β ∈ K, dX+ β y ∈ Y.).
	Termino 35 A lineor contribution of vectors X1, X2,, XN EX is the vector of the form $k=1, dK^{XK}$ with some $d_{1}, d_{2},, d_{N} \in K$ . The set of all lineor combinations of vectors $X \in M \subset X$ is called
	the space of M, denoted span M. This is a subspace.
	toginitions b A set of vectors X1, X2,, IN is said to be linearly independent if K= d K XK=0 implies d1=d2== dN=0. An influite set of rectors is linearly independent if each finite
	subset is linearly independent. if ID
	e.g consider X=CEO,1]. frinx, cosx t one LI, since d sinx + βcosx=0 VxeEO,1]. At x=0, β=0, d=0. Or, we consplit intovalino two pures Pi
	to construct the functions which are non-zero at different parts. Then they are LD.
	Infunto 327 A space X is said to be fuite diversioned if = de IL of . X contained live original pendent rectors, and any collection of dt 1. rectors is bready dependent. Notation d= dim X.
	If X is not finite dimensional, it is infinite dimensional . A basic of X is a set that is linearly independent and spans X.
	If dim X < 00, then any set of d= dim X LI rectors is a basis. claim: Every linear space has a basis (even infinite dimensional ones).
	Tora-finite dimensional space, x ∈ X ⇒ X = Z dxXx where 1×1,, xxt is a boxis. 1 dxt are uniquely defined.
-	Examples - 17) CTa/b] is infinite dimensional. Rite any fixed NGIN. then we split Ta/b] into N-equal intervals. Let Pi be a bump in it interval, O encymbere else. 1917 forms a LI set.
ан — — — — — — — — — — — — — — — — — — —	Since in was arbitrarily chosen, we can alway a construct N linearly independent functions. Lp (a, b) is infinite dimensional for the same reason.
	(2) $\beta_1$ pe [1, 12] is infinite dimensional, using $x_1 = (0, 0, \dots, 0, 1, 0, \dots)$ as a basis.
	TEREFUSERISS let WCX be a subspace. Then we say that X, y & X are equivalent if x-y & W. Then the space of all equivalence classes is called the question typice X/W.
	It becomes a lives space if one defines operations + and . s.t. (xx+ by) = alist + bly).
	Remore - It is easy to check that the requirements of Definitions 3.1, 3.2 are satisfied.
	Stephinion 329. Let WCX be a subspace. Then the codimension of W is defined to be dim X/W. Notation: sodim W = dim X/W.
	observation - If dim X=d, dim W= k <d, coolim="" then="" w="d-k&lt;/td"></d,>
	e.g Let X=R <sup>2</sup> , N=R
	[Recoil definition - Two linear spaces X, X, are isomorphic if 3 a bijection (9: K→X1 s.t. ((4×+βy) = d(1)) + β(1), ∀x, y ∈ X.
-	Nomed spices
	Described 210 A norm on rector space X is a real-valued function 11.11: X -> IR with the properties:
	(hon-hegativity) (non-degeneracy) (thiskyle indquality). (homogeneous function). (1)   x  ≥D (2)   x  =0 (→ x=0 (3)   x+y  ≤   x  +1 y   (4)   dx  =  d .1 x   ∀x, y.e. X, de K.
	If X has a norm defined on it, it is called a normed linear space.

	Examples - (1) Rd, then $  X   = \left[\sum_{k=1}^{d} \times k^{-1}\right]^{\frac{1}{2}}$ we can also consider different norms - p-room $  X  _p = \left[\sum_{k=1}^{d}   x_k  ^2\right]^{1/p}$
	By Minkowski's Inequality, both of these are norms.
	(2) Take \$p, 1 ≤ p ≤ 00. clearly, 11×11p = [ = [ + = 1×41P] <sup>yp</sup> is finite st. 11.11 is a norm (again by Mintonsti's).
	(3) C[a,b]. Hore IIf IIc = astsb lifter is a norm and Lp(a,b) with IIfIlip = [ [a ifth] b 1] is a normed space. (Note: [a ifth] b t is a Lesbeque integral.
	Arrows 1+11 we X be a normed linear space. Then the function g(x,y) = 1/X-y11 defines a metric on X.
/	Proof - Omitted, waightforward. Oneck. definitions.
2	Remote - We say that p is the metric induced by the norm.
X	Definited 3.12 X is complete if it is a complete metric space with the metric induced by the norm. If X is complete, it is called a Bonach space.
	Definition 3:13 let WCX be a subspace of a linear space. X. Then W is a subspace of the normed space X if W is closed, i.e. W=[W].
	where product cosces.
	Exploring 3:14 An inner product on the linear gave X is a mapping <>: X × X -> K with the following properties:
	(linearity) (homogenity) (1) <x+y,z>=<x,z>+<y,z> (2) &lt;0(x,y)= &lt;<x,y> (3) <x,y>= <y,x> and (4) <x,x>≥0 and <x,x>=0 ⇔ x=0.</x,x></x,x></y,x></x,y></x,y></y,z></x,z></x+y,z>
	The space X with an inner product is called a pre-Hilbert space. If IK=IR, it is also called <u>Euclidean</u> .
	2t Folonian 2019
	Ref Accuracy Republication of the second of
	Extendentists we say that x e X and y e X are anthreground to each other if (x, y)=0. Northigh: x Ly.
	permark- Recall the sythygoas's theorem if x 1 y, then $1 x+y ^2 =   x  ^2 +   y  ^2$
	We sho define a "norm" 11×11= J <x, x=""> Y X E.X. This is hanogeneous and non-degenerate.</x,>
	Terminal 7:16 (contry-schurove trequestity).
	VX14 E X, IXX1431 < 11X11 lly 11. Equility holds if and only if X14 are liveship dependent.
	Note- This implies the triangle inequality
	by submitted 377 If X is complete N.N.T. the norm induced by the inner product, it is colled a stillbed space. [NotHion: H].
	twortaintie popoly of inner podule spices.
r (	Ballelogion identity: 1 x+y 1 <sup>2</sup> + 1 x-y 1 <sup>2</sup> = 2.(1 x 1 <sup>2</sup> +1 y 1 <sup>2</sup> ). • identisation identity: 4(x,y) = 1 x+y 1 <sup>2</sup> - 1 x-y 1 <sup>2</sup> + i (1 x+iy 1 <sup>2</sup> - 1 x-iy 1 <sup>2</sup> ). If th=IR, 4(x,y) = 1 x+y 1 <sup>2</sup> - 1 x-y 1 <sup>2</sup> .
the second second	Theorem 3.18 let X be a normed space. The relation @ or @ defines an inner product @ parallelogram identity holds \$X,y & X.
	Approximation of the second space and let 11.112, 11.112 be two norms on X. We say that 11.112 are equivalent if there are two constants c2 3 c1, > 0.5.1. c1 11x112 & c2 11x12
1	Theorem 3.20 if dim X < 00, then all norme are equivalent to each other.
	Frankles-(1) Rd. this is finite-dimensional, so all norms are equivalent eg. IXIIp = [2] 1XKIP] . If p=2, this is a Hilbert space. If p=2, it cannot be itered as a tribert space as the
	proveled agreen identity does not hold for some X14 & TRd. For p=2, <x, *="" 47="K=1" <x,="" [for="" cd,="" k="1" k4k.="" k4k].<="" td="" ×=""></x,>
	(2) For Lp, 1 ≤ p ≤ 00, IIX IIp = [ $\sum_{k=1}^{\infty}$ 1×k  <sup>p</sup> ] <sup>P</sup> . Again, if p=2, this is a thibbert space, <x, y=""> = <math>\sum_{k=1}^{\infty}</math> × (IX. Here, since series is infinite, we do not talk about equivalent norms.</x,>
	(3) Lp(a,b), 1≤p<00. (closure of continuous functions). If p=2, (x,y)= Ja X(H y(t) dt. is on inner product.
	Let fe C(a,b] ⇒   f  c= te (a,b] +f(1). This is a Banach space, but does not satisfy the parallelegy an but. We require a different norm: C2 (a,b] is an inner product space, <x,y>= b x(+) y(+) d</x,y>
	By complexing it, we get hela. Which is a Hilbert space.
· · · · · · ·	By complexing it, we get 1212,16) which is s. Hilbert space.
· · · · · ·	By complexing it, we get 12(2,10) which is a Hilbert space.
	By complexing it, we get 12(2,10) which is s. Hilbert space.
	By complexing it, we get $L_2(a,b)$ which is a Hilbert space. By complexing it, we get $L_2(a,b)$ which is a Hilbert space. Betwee, convex sets and orthogonal complements. We MCH be a set, then we defined dist(M1X) = yief $  y-X   = S(x,M)$ . Is the infimum sensity stained, and if so, at how many points? Infinites 221 the set M is convex if $\forall x,y \in M$ , the linear combination $dx + (1-d)y$ is also in the set $\forall d \in (0,1)$ .
	By complexing it, we get $L_2(a,b)$ which is a Hilbert space. By complexing it, we get $L_2(a,b)$ which is a Hilbert space. By complexing it, we get $L_2(a,b)$ which is a Hilbert space. By complexing it, we get $L_2(a,b)$ which is a Hilbert space. By complexing it, we get $L_2(a,b)$ which is a Hilbert space. By complexing it, we get $L_2(a,b)$ which is a Hilbert space. We MCH be a set, then we defined dist $(M_1 \chi) = \frac{W^2}{Y_{ch}}$ . $\ U_1 - \chi\ _2 = S(\chi, M)$ . Is the infimum actually attained, and if so, at have many points? Definition 221. The set M is convex if $V_{X,Y} \in M$ , the linear combination $v(\chi + (1 - d)y)$ is also in the set $V d \in (0, 1)$ . Before 322. Let MCH be a convex dosed set. Then $V_X \in H$ , $\exists$ unique vector yet st. $S(\chi, M) = \ U_1 - \chi\ _2$ . Diamenulature - y is called a minimizing vector $J$ . $(\chi, \chi)$ .
	By complexing it, we get $L_2(a,b)$ which is a Hilbert space. By complexing it, we get $L_2(a,b)$ which is a Hilbert space. By complexing it, we get $L_2(a,b)$ which is a Hilbert space. By complexing it, we get $L_2(a,b)$ which is a Hilbert space. By complexing it, we get $L_2(a,b)$ which is a Hilbert space. By complexing it, we get $L_2(a,b)$ which is a Hilbert space. It was not if so, at how many points? It was not if so, at how many points? It is convex if $V_{X,Y} \in M$ , the linear combination $d_{X} + (1-d)_{Y}$ is also in the set $V d \in (O_1^{(1)})$ . It is convex if $V_{X,Y} \in M$ , the linear combination $d_{X} + (1-d)_{Y}$ is also in the set $V d \in (O_1^{(1)})$ . It is convex if $V_{X,Y} \in M$ , the linear combination $d_{X} + (1-d)_{Y}$ is also in the set $V d \in (O_1^{(1)})$ . It is convex if $V_{X,Y} \in M$ , the linear combination $d_{X} + (1-d)_{Y}$ is also in the set $V d \in (O_1^{(1)})$ . It is convex if $V_{X,Y} \in M$ , the linear combination $d_{X} + (1-d)_{Y}$ is also in the set $V d \in (O_1^{(1)})$ . It is convex if $V_{X,Y} \in M$ , the linear combination $d_{X} + (1-d)_{Y}$ is also in the set $V d \in (O_1^{(1)})$ . It is convex if $V_{X,Y} \in M$ , the linear combination $d_{X} + (1-d)_{Y}$ is also in the set $V d \in (O_1^{(1)})$ . It is convex if $V_{X,Y} \in M$ , the linear combination $d_{X} + (1-d)_{Y}$ is also in the set $V d \in (O_1^{(1)})$ . It is convex if $V_{X,Y} \in M$ , the linear combination $d_{X} + (1-d)_{Y}$ is also in the set $V d \in (O_1^{(1)})$ . From $C = S_{Y,Y} definition of inf, B is sequence V_{X} \in M is T_{Y,Y} = S(X,M) = \frac{1}{N + Y_N}. Denote S_N = 1 X-Y_N . We obtain that this sequence is cauchy is and then$
	By complexing it, we get 12(2,10) which is s. Hilbert opsice. By complexing it, we get 12(2,10) which is s. Hilbert opsice. Betone, convex sets and orthogonal complements. Wet MCH be a set, then we defined dist(M1X) = yieh IIy-XII = S(4,M). Is the infimum sensity stained, and if so, at how many points? InfiniteStat the set M is convex if $\forall x_i y \in M$ , the linear combination $dx + (1-d)y$ is also in the set $\forall d \in (o_1^n)$ . Benomed 322 Let MCH be a convex doed set. Then $\forall x \in H$ , $\exists$ unique vector yet st. $S(x_i,M) = IIy-XII$ . [Nomenclastice - y is called a minimising reaso.] Proof-By definition of inf, $\exists$ a sequence $y_u \in M$ st. $S(x_i,M) = \lim_{n \to \infty} IIx-y_n II$ . Needolimithat this sequence is cauchy, and then $Iy_n - U_jm(I] = -IIy_n + y_m - 2X II^2 + 2 (IIx-y_n III^2 + IIX-y_mII). We note that as M is convex, \frac{y_n + y_m}{2} \in M so II \frac{y_n + y_m}{2} - XII ≥ S(x_i, H) = S. Thus,$
	By completing it, we get $\lfloor_2 \lfloor a_1 \rfloor b \rfloor$ which is a Hilbert opace. Betwee, convex sets and orthogonial completenents. Bet MCH be a set, then we defined dist $(M_1X) = \frac{ih^2}{yeh}$ . Bet MCH be a set, then we defined dist $(M_1X) = \frac{ih^2}{yeh}$ . Bet MCH be a set, then we defined dist $(M_1X) = \frac{ih^2}{yeh}$ . Bet MCH be a set, then we defined dist $(M_1X) = \frac{ih^2}{yeh}$ . Bet MCH be a set, then we defined dist $(M_1X) = \frac{ih^2}{yeh}$ . Bet MCH be a set, then we defined dist $(M_1X) = \frac{ih^2}{yeh}$ . Bet MCH be a set, then we defined dist $(M_1X) = \frac{ih^2}{yeh}$ . Bet MCH be a set, then we defined dist $(M_1X) = \frac{ih^2}{yeh}$ . Bet MCH be a set, then we defined dist $(M_1X) = \frac{ih^2}{yeh}$ . Bet MCH be a set, then we defined dist $(M_1X) = \frac{ih^2}{yeh}$ . Bet MCH be a set, then we defined dist $(M_1X) = \frac{ih^2}{yeh}$ . Bet MCH be a set, then we defined dist $(M_1X) = \frac{ih^2}{yeh}$ . Bet MCH be a set, then we defined dist $(M_1X) = \frac{ih^2}{yeh}$ . Bet MCH be a set, then We defined in the set context doed set. then Yx eth, $\exists$ unique vector yeh str. $S(X,M) = \ Iy - X\ $ . Bet MCH be a set we defined dist $(M_1X) = \frac{ih^2}{yeh}$ . Bet MCH be a set $M = 1$ we defined this sequence is conclusion when $M = 1$ . Bet $M_1 = \frac{ih^2}{2}$ . Bet $M_2 = \frac{ih^2}{2}$ . Bet $M_2 = \frac{ih^2}{2}$ . Bet
	By complexing it, we get $L_2(a,b)$ which is a Hilbert space. By complexing it, we get $L_2(a,b)$ which is a Hilbert space. Betwee, convex sets and orthogonal complements. We MCH be a set, then we defined dist $(M_1X) = yich$ $  y-X   = S(4,M)$ . Is the infimum sensity stained, and if so, at how many points? InfiniteStat the set M is convex if $\forall x_i y \in M$ , the linear combination $dx + (1-d)y$ . Is also in the set $\forall d \in (o_1^{4})$ . InfiniteStat the set M is convex if $\forall x_i y \in M$ , the linear combination $dx + (1-d)y$ . Is also in the set $\forall d \in (o_1^{4})$ . InfiniteStat the set M is convex if $\forall x_i y \in M$ , the linear combination $dx + (1-d)y$ . Is also in the set $\forall d \in (o_1^{4})$ . InfiniteStat the set M is convex if $\forall x_i y \in M$ , the linear combination $dx + (1-d)y$ . Is also in the set $\forall d \in (o_1^{4})$ . InfiniteStat the set M is convex if $\forall x_i y \in M$ , the linear combination $dx + (1-d)y$ . Is also in the set $\forall d \in (o_1^{4})$ . InfiniteStat the set M is convex if $\forall x_i y \in M$ , the linear combination $dx + (1-d)y$ . Is also in the set $\forall d \in (o_1^{4})$ . InfiniteStat the set M is convex if $\forall x_i y \in M$ , the linear combination $dx + (1-d)y$ . Is also in the set $\forall d \in (o_1^{4})$ . InfiniteStat the set M is convex if $\forall x_i y \in M$ , the linear combination $dx + (1-d)y$ . Is also in the set $\forall d \in (o_1^{4})$ . InfiniteStat the set $M \in O_1^{4}$ is a sequence $\forall x \in M$ set. So $(x,M) =   y-x  $ . Denote $S_n =   x-y_n  $ . We obtain that this sequence is Couchy, and then $\sum_{\substack{x \in M_1 \\ y \in M_2^{4}}} \sum_{\substack{x \in M_1 \\ y \in M_2^{4}}} \sum_{x \in M_1 \\ y \in M_2^{4$
	By completing it, we get $\lfloor_2 \lfloor a_1 \rfloor b \rfloor$ which is a Hilbert opace. Bytome, convex sets and orthogonial completenents. Bytome, convex sets and orthogonial completenents. Bytome, convex sets and orthogonial completenents. Bytome, convex sets and dist $(M_1 \chi) = \int_{eff}^{104}   y - \chi   = S(\chi_1 M) \cdot b$ the infimum servally stained, and if so, at hop many points? Bytome vector yell is the set of the set on the set on the set of the infimum servally stained, and if so, at hop many points? Bytome vector yell set. S(\chi_1 M) - b the infimum servally stained, and if so, at hop many points? Bytome vector is a convex doed set. then $\forall x \in H$ , $\exists$ unique vector yell st. $S(\chi_1 M) =   y - \chi  $ . Fromenulsate – g is called a minimizing reason. From $\chi_1 = 0$ of the set of the sector is a convex doed set. then $\forall x \in H$ , $\exists$ unique vector yell st. $S(\chi_1 M) =   y - \chi  $ . Fromenulsate – g is called a minimizing reason. From $\chi_1 = 0$ of the set of the sector is a convex doed set. then $\forall x \in H$ , $\exists$ unique vector yell st. $S(\chi_1 M) =   y - \chi  $ . Note that this sequence is cauchy, and then $\int_{1}^{1} \int_{1}^{1} \int_{1}^$

and KEH	
Henren 273 let Meth be a closed subseptice. Then yem is the unique minimising vector $\Leftrightarrow$ X-y. L. M.	27. Hebrudy 2013. Auf Mexander So BOLEV
Boof- ut y be the minimising rector. support that there is a rector EEM st. sk-y, €2 =0. Assume   E =1. write   x-y-de  = 1x-y 1- d(x-y)e> -d(e)	ry Monwrob.
$  x-y  ^2 -  d ^2$ . There this vector is strictly less than $  x-y  ^2$ , which is a construction as y is induitivity $\Rightarrow \alpha = 0 \Rightarrow x-y \perp N$ .	Ale.
(4)	
suppose that yet is such that x-y $\perp$ M. For any $\neq$ , we have $  x-z  ^2 =   x-y  ^2 +   y-z  ^2 \ge   x-y  ^2$ . Hence, y is the minimizing rectarly q.	e.d. y Z.
Information to the provided subspace MCH and a rector XCH, the unique rector YCM St. X-y LM is called the addressing on the projection of X onto M. Ne write y=PX,	where this map P is called
	4
the adhogonal projection (operator) onto M.	
Properties of map P: (DP <sup>2</sup> =P (Idempotent operator) ③   PX   <sup>2</sup> +   X-PX   <sup>2</sup> =   X   <sup>2</sup> (Pythuggoras). thus,   PX   ≤   X  .	
Topono 2155 A linear space X is said to be a direct sum of two subspaces Y1Z if Grow XEX can be winned represented as X=y+Z, where yEY, ZEZ. If X	2 H and YLZ theory thricis
	I on i
colled on cathogonal surg. We write $X = Y \oplus Z$ .	
legenting 3.2. For a dosed subspace M, the set M <sup>L</sup> = 1×6H·××LM7 is colled the addressional complement of M.	
Note - Mt is a dozed subspace for general set M.	
Homewhild if Y is a dosed subspace, then $H = Y \oplus Y^{\perp}$ . Note $= Y^{\perp} = Y$ if Y is dosed.	
Annihilistas.	
Definition 28 let M be a non-empty set in H. then the annihilities of M is the set M <sup>1</sup> = (xEH: xLMY.	
the courses 3.29 The following properties hold for the sumiliestor.	
$  (3) M^{\perp} is a doxed subspace (3) M^{\perp} = [M^{\perp} (3) M^{\perp} = (span M)^{\perp} (3) M spans H \Leftrightarrow M^{\perp} = 10t. $	
orthornounal Sep.	
Individual 330k set MCH is called arthonormal if for any two vectors x, y ∈ M we have either (x, y>=0 (if x + y) or (x, y>=1 (if x=y).	
theoremissi IF H is separable, then every orthonormal set is countable.	
(Crou)	
$1 \times 1000$ . Proof - $1 \times 19 \times 19 \times 10^{-1} = 11 \times 10^{-1} \pm 11 \times 10^{-2} \Rightarrow 1000$ the balls $B(x, \frac{1}{2})$ and $B(y, \frac{1}{2})$ are disjoint. By separability, take an element of countable dance x	et por
each of these bollo	
orthonormal from now on, M= 191, 221T. ison requerer. For XEH, the numbers CK= (X1 ex) are called tornier coefficients with respect to M.	
theoremissic For any XEH, KET 1GK12 5 11×112 (Bessel's Inequality). In particular, 1CK2 El2. The series KET CKCK converges to a rector PX, where P is the	orthogonal projection and span MJ.
Nok-we require a Hilbert space so that it is complete, and thus infinite sums converge.	
Proof K=1CKCK is tested for convergence. Let Y n= K=1CKCK, then $\ Y_m = Y_n\ _{-\infty}^{2} = \ x_m - Y_n\ _{-\infty}^{2} = \ $	is Caller company than
	N
∃ lim yn= y∈H. MTP: x-y⊥ spanM; i.e. x-y⊥M. anear that x-y⊥ejfor ang j=12, Indeed, let n>j, then <x-yn,ej>= <x,e< td=""><td>j7-k=1 Ck (ek, ej2</td></x,e<></x-yn,ej>	j7-k=1 Ck (ek, ej2
= <x, ej="">- cj<ej, ej=""> = cj-cj=0. By continuity of inner product, https:// cj&gt;=0 = <x-4, ej=""> =&gt; x-4. + M/ ge.d.</x-4,></ej,></x,>	
Edulation and sequences	
intrational sequence is called total if [span M]=H. [i.e. M <sup>1</sup> = 507].	×
Microm 13.44 (Baserici's I dauging).	
M is total (=   x   <sup>2</sup> = = 1/chl <sup>2</sup> . In this case, for any x, y ∈ H, we have \$x, y) = x = char where ch, dk are forwire coefficients of x and y respect	tively.
(⇒) Proof-suppose Mistorial, so xn = K=1 CK CK converges to x do n→ 00 by Theorem 3.32. Therefore 1/x-Xnll2= 1/xll2- K=1 Chl2 → 0, Therefore 1/xll	00
Suppose @. Assume M is not total i.e. xLM and x=0. Therefore cx=0 but 11x11,=0 - contradiction 1, g.e.d. Secondpart laftas exerci	<u>че</u> ,
Alarman 3.75. A Hilbert-space contains a total orthonormal sequence 👄 it is separable.	
thraewist (Riesz-Fisher thoron)	
let take the les and let their be su orthonormal sequence in H. then there is an element XEH such that CK= (X1 CK), K=1,2,	
Most - Define Xn=K= ckek. Thus   Xm-Xn   = Zickl -> 0 as min-300 as 1ckt El2. Hene, 3 lim Xn=X. His straightforward to check that (X	$e_{k} = c_{k}$
Kalifi Kalifi	
Chapter 4-	
LINEAR-FUNCTIONALS MID DUAL SPACE	
functional	
Definition 4.1 Lot X be a normal space over the A mapping find the issuid to be a linear if flock + by) = a flock + b flog) VX, y E X, a BE IK	
Examples- (1) If X=1K=R, f(x)=x is a linear functional (2) g(x)=ax+b is a linear functional only if b=0.	
A linear functional is said to be continuous if f is continuous. A linear functional is said to be bounded if it maps a writhed B(0,1) into a bounded set.	

03-10

remont-an atomotive formulation for boundedness: f(B(0,1)) C. B(9, R) for some R>0. Equivalently, f(B(0,1)) C B(0, R) for any t>0, by lines	wity
Equinatentia, Ificil ≤ P. UKI X×EX.	1
Removal 4.2 A livest functional is constructed at <= 0.	
Avorf - (>) is perfectly obvious. For (<=), suppose-functional f is continuous at D. NTP: Y\$>0, 3 \$>0 st. f(y) € B(f(x), E) IF y € B(x, 8). Equivale	unduflu-x) & B(0,E) if u-xeB(0,S)
by linduity of function. This is satisfied by continuity of fat 2=0/1 qe.d.	
Buccound 43. A lower functional is constructed ⇒ it is bounded. (⇒) Proof - let f be continuous (1-e.st.0). Then 4670, 3570 so XEB(0,5)⇒firkEB(0,5). Then f(B(0,5))C B(0,6) ⇒ f is bounded. Then assuming f	
	1
for t>0, fixed R>0. In particular, this holds by refining $Rt=\epsilon$ and setting $S=t=\frac{\epsilon}{R}$ . Then f is continuous at $z=0$ (and hence exclymbare)	
	3 March 2013. Rof Alexander SOBULEV
Representation the manage of fix defined to be II fill = sup 1 fixed.	Marlis 706
$\frac{\sup_{x \in [1, \infty]} \frac{ f(x) }{ x   = x \cdot  x  } = \frac{ f(x) }{ x   = x \cdot  x  } $ $\frac{\sup_{x \in [1, \infty]} \frac{ f(x) }{ x   = x \cdot  x  } = \frac{ f(x) }{ x   = x \cdot  x  } $ $\frac{\sup_{x \in [1, \infty]} \frac{ f(x) }{ x   = x \cdot  x  } = \frac{ f(x) }{ x   = x \cdot  x  } $	
$\frac{ f(x) }{ x  } = \frac{ f(x) }{ x  } = \frac{ f(x) }{ x  } = \frac{ f(x_0) }{ x  } =  f(x_0) $	1fil. x #
Hence A = 11 fill. some procedure for second equality 1 g.e.d.	
Thus, It is if it is if it is if it is it fills in the left is the best (happent) constant we can use for R in the improving If (x) & R Wall \$\frac{1}{x \in X}\$. i.e. if (x) is the best (happent) constant we can use for R in the improvement of the improvement	f Ifixils clixil, then Infills c.
(CST) Examples-(1) Take X=H. Let Xo=H and define fix= \$x,xa>. It is linear. Then If xi1= ( <x,xa>1 ≤ 11×11.11×011, so it is bounded with 11fi1 ≤ 11×011. We prove that 11f</x,xa>	
To this end, it suffices to find a rector x sit. f(x)=   xo  . Toke x = 11xo  , then f(x,xo)= < 11xo  , xo> = 11xo   < xo, xo> = 11xo  . Hence, 11 fil	
(2). Take X=C[a,b], with   f  c= teta,b] f(t)]. let l(p)= la f(t) dt. This is linear. Is it bounded?  l(p)  < la l(f(t)) dt < 1 f(lc la dt =	Ilfllc (b-a). so it is bounded,
and   l   = b-a. Ne want in fact:   l  =b-a for some function f e X st.   f  c=1 and l(f)=b-a. If f(t)=1 Yte [a,b], the	. l(f)=b-a as required.
[variant]. The x= CE-TI, TT] with $l(p) = \int_{-TT} sint f(t) dt$ . Again this is linear. For boundedness, $ l(p)  \leq  lf   \leq \int_{-TT}  sint  dt$ . there, $ l(t)  \leq \int_{-TT}  sint  dt$ .	
then elf)=0! instead we could use f(t) = sgn(t), but it is not continuous. Instead, we need to approximate the step function by continuous or The state of This is lived.	<b>Λ</b>
(3). Take $X = l_{p_3} = \frac{1}{ X  _p} = \frac{1}{ X } \frac{1}{ X } = \frac{1}{$	≤ Z   Yullur
<   x  p.   y  q. Then   f   <   y  q. To show that   f   =   y  q, we need to find a sequence xelp s.t. f(x) =   y  q.  x  p. Take	
and we define $\frac{y_{k}}{ y_{k} } = 0$ for $y_{k} = 0$ . check that $x \in \mathbb{P}_{P}$ : $ x_{k} ^{P} =  y_{k} ^{q}$ so $  x_{1}  ^{p} =   y_{1}  ^{q} < \infty$ and $x \in \mathbb{P}_{P}$ . Then $f(x) = k$ .	
=   y  q   y  q =   y  q   x  p. Then the norm is indeed stained.	
Debutical 4.6 lef f be a bounded linear functional. Then the kernal of f is the set Ker f = 1x EX : 1(x)=05.	
non-trivial	
Theorem 4.7 For any bounded lines functional f, the hernel kerf is a closed subspace, and codim kerf=1.	
Ancorral 4.8 (Riesz's Theorem)	n Pri de se
let H bes Hilbert space, and let f be bounded linear functional on H. Then there exists a uniquely defined rector XOEH st. f(X)= (X) XOX Moreoror	
Roof-14 f=0, then f(x) is always f(x)= <x, x0=""> If f≠0, then ∃ &amp; rector = st. f(2)≠0, and = e(ker f), so = +0 ⇒ f(2) ≠0. 4×eH, x- For uniqueness, suppose = two rectors f(x)=<x, xi="" xi)="&lt;x,"> +2, xi-x2, f(x) d. ⇒ \$2(2;2) = <x, td="" z),<=""><td><math display="block">\frac{1}{\alpha} = \frac{\alpha}{1 z  ^{2}} \left\{ x_{1} \neq y_{2} \neq \frac{\alpha}{2} \neq \frac{1}{2} \neq \frac{1}{2} = 0 \\ f(x) = \frac{\alpha}{1 z  ^{2}} \left\{ x_{1} \neq y_{2} \neq \frac{\alpha}{2} + \frac{\alpha}{2} \neq \frac{\alpha}{2} \right\}</math></td></x,></x,></x,>	$\frac{1}{\alpha} = \frac{\alpha}{1 z  ^{2}} \left\{ x_{1} \neq y_{2} \neq \frac{\alpha}{2} \neq \frac{1}{2} \neq \frac{1}{2} = 0 \\ f(x) = \frac{\alpha}{1 z  ^{2}} \left\{ x_{1} \neq y_{2} \neq \frac{\alpha}{2} + \frac{\alpha}{2} \neq \frac{\alpha}{2} \right\}$
Theorem 19 let X= lp, 1 <p<00. <math="" as="" bounded="" for="" functional="" is="" linear="" represented="" say="" then="" x="">f(X) = \sum_{k=1}^{\infty} X_k Y_k with some uniquely defined <math>y = (y_1, y_2,) \in I</math></p<00.>	lg 6 Menh2014- Rof Alexander SoBolEV
where p <sup>-1</sup> +q <sup>-1</sup> =1. Horeover,   f  =1 y  q.	Hot Mexander Sobolev Hatho 706.
The Holm-Bonsch Theorems.	
let X be a normal space. We say that LCX is a linear set if L=spanL. (dees not have to be dosed, while subspace)	
tophilo let L be a livear set, and fo: L→K, f: X→K be the bounded livear functionals. We say that f is an extension of fo if fo(x)=f(x) ∀x e L. bunded livear fo onD Itemmol 4.11 let DEX be a livear set sit. [D]=X. Then any functional can be uniquely extended to a bounded livear functional fon X. Moreorer, 11 fo11=11 f 11.	
not - let trate be a convergent sequence with X = him on then the sequence fitm is couchy, and hence fo trai has a limit. Define f(X) = him for	
since for= lim form, V=>0 3N ST. How-form) < # 1 N. Thus [f(x)] = form) + for-form) ≤ [form) + [for-form] < [form) + t € []form) + for - form) < (form) + t € []form) + for - form) < (form) + t € []form) + for - form) + t € []form) + t € []form) + for - form) + t € []form) + t € []	
⇒ 19(3) < 1918/11+2 4270. Therefore 19(3) < 11for 11x11 4x € X ⇒ 11for 11 ≤ 11 for 11. On the other hands. 11 for 11 ≥ 11 for 1. Hence, 11 for 11= 11 for 1. 4. 9. e.d.	·
Remark- We say that fo is extended from D to X by constitutity. (Hahm-Banach theorem)	
Hereard the let LCX be a closed subspace and let fo be a bounded linear functional on L. Then there exists a bounded linear functional for X st. fis an extension	of fo, and (2) 11f11=11foll.
Proof - Mourne X is reportede, K=R. Assume Ilfoll=1 wLDG. Let $\xi \in X$ . We extend for to $L_1 = \text{span}(L,\xi)$ . If $\xi \in L_1$ then $L_1 = L$ and we define the extend	tion by f1=fo, suppose \$ \$ L.
Then any rector U.E.L.1, U=X+t&1, with XEL, tER. Take ¥1, X2EL. Thus, folx+36)=fo(x1)-fo(x2) ≤ 1(x1-X2)1=11(x1+&)-(x2+&)11 ≤ 11(x1+)	\$ 11 + 11×2+ \$11. Then remite
fo(x1)-   x1+ 5  ≤ fo(x2)+   x2+5   ∀x1, x2 ∈ L. Therefore, 549 (fo(x)-   x+5  ) ≤ a ≤ inf (fo(x)+   x+5  ) - @. for some a ∈ R. Then ∀x∈	$X_1$ for $-1 x+\xi_1  \leq \alpha \leq for +1 x+\xi_1 $

Then -IIXt \$11.5 for a <iixt \$11=""> Ifor y-al &lt; IIXt \$11. VKEL. For u=x+t\$ EL1, define the extension fito L1: for u= for u= the easy to deal that fit is</iixt>
linesr on L1. For the norm, Ifilia)= Ifolia-tal= It foli= It1 [foli= -al = It1·11 = II xt t [1= II xt t [1= II ull.] Ifall=1. since faison extension
Ilfall≥ Ilfall=1 => Ilfall=1= Ilfall=1= Ilfall. let \$1,52, be & countable dense set in X. Then we extend for from L first to L1= span (L, \$1) then to b2=span (L1, \$2),
continuing this process, we extend this to $L_n = span (L_{n-1}, \xi_n)$ , we end up with a functional $F$ defined on $\bigcup_{n=1}^{m} L_n = D$ , st. $  F   = 1$ . since $DI = X$ , we can extend
Fto X by continuity of e.e.d. (for separable spaces).
suppose X1,X2 E X are st. X1 = X2. Then there exists a bounded linear functional st. f(X1) = f(X2).

Roof-let 1= spsn(x1-x2). then define fo(h(x1-x2))= h ||x1-x2||. Vh ∈ K. II foll=1. let f be an extension to X, Il fil=1. then f(x1-x2)= fo(x1-x2)= ||x1-x2||. ⇒ f(x1)-f(x2)=||x1-x2||. ⇒ f(x1)=f(x2)+||x1-x2|| > f(x2)|| q.e.d. [this is true for R. for C, just note that f(x1)-f(x2)=0].

## Pustspace

Cop 14:13

Introduce these structure on the bounded linear functionals: (for t f2)(N=for t xex. (d f)(X)= d (fix) XxeX. Does the norm 11 fil satisfy the required properties?

Li) Non-degeneracy, 1|f1 ≥ 0. 1|f1|=0 ⇔ f=0. True as |f(x)| ≤ 1|f1||x|| (ii) Homogenerity. 1|x f1|=1d1 ||f1| is true by definition

(1))Thisogle inequality: [(for f2)(1)] = 1f1(1) + f2(1)) ≤ (1f1(1) + 1/f2(1)) = (1|f1|+1)f2(1).11×11, 50 1)f1+f2||≤|1|f1||+1)f2||.

topinition + 14 the worked linear space of bounded linear functionals is called the dual space, denoted N.

Theorem 4.15 X tis complete.

X is complete. Wear functional f, s.t. most-let fr. EXX, n=1,21... bes couchy sequence. i.e. YE>0, 3N st. If non-fmW/<EBX/ if m,n>N XXEX. Define fix. Then taking n-son gives

- HW-fmWl≤z11x11 Vm>N, Yxex. Thus HW1=1fmW+fW)-fm(x)1≤ 1fmW1+1f(x)-fmWl & 11fm111|x11+ €11x11=(11fm11+z)11x11 for some fixed m ⇒

fis bounded, UfII ≤ IIfmUtE. Ym>N ⇒ X\* is completely ged.

Examples-(1) X=H. then  $\forall f \in X^*$ ,  $f(x) = \langle x, x_0 \rangle$ ,  $\stackrel{?}{F} \leftrightarrow \stackrel{?}{x_0}$  is a 1-to-1 correspondence, then  $X^*$  is is concorplice to H. we write this as  $X^* = H$ . (2) If  $X = L_p$ ,  $1 , then <math>X^* = L_q$ , where  $p^{-1} + q^{-1} = 1$ . (3)  $l_1^* = L_{\omega_1}$ ,  $l_{\omega_2}^* = L_1$ . Instead,  $c_0^* = L_1$ .

(5) let d=dim X < 00. Then X\*=X.

second dual space.

let X<sup>\*\*</sup> be the space of all linear bounded functionals on X<sup>\*</sup>. Any rector XEX defines a functional on X<sup>\*</sup>: FX (f)=f(No VfeX<sup>\*</sup>. Fis a linear functional ea fis). It is bounded: 1FX(f)1=1f(N)1=11×11×11×11+ => 11FX11×11+ < 11×11×11+ =>

HILCONEMIALD 11FX 11 XXXX = 11X11X YXEX.

Prof-Need to finds functional f ∈ X\* s.t. Fx(f)= II.flx\*||X||X lot L= span(X) ⊆ X. Then define fo: follX) = X ||X||, X ∈ K. so that II foll=1. Let f be an extension to X s.t. II.fll=1 by Hahn-Banach. Thus, Fx(f)= f(X)= fo(X)=1|X|| ⇒ II.Fx||X\*\*= ||X||X.f. - g.e.d.

remark- The map FX: X > X\*\* is an isometry as it preserves norms (but not an isomorphism - may not be suggestive).

Deputer 1:17 The map Fx: X -> X \*\* is called a cononical map of X into X \*\*. If Fx is sujective, then X is said to be reflexione.

Examples-(1) d= dim X < 00. He know: dim X\* = dim X\* = d. since Fx is an isometry, it is an isomorphism. Indeed X is reflexive.

(2) X=His reflexive. (3) lp, 1<p<00. is reflexive, since lp=lq, lq=lp. (4) ly is not reflexive (5) co is not reflexive as co \$ los = co\*.

10 Month 2014 Rof Nexander SOBOLEV.

Theorem 9:18	A space y	is refle	ive iff	Xis	reflexi

Proof - (>) DIY, (=) omitted.

convergence in normed spaces

lefuine the sequence in converges should to x as n→00 if lixn-XII→0 as n→00. Xn converges to x in the norm it converges weakly to x if f(xn)→ f(x), n→00 for all f eX\*. Notation: w-lim xn=x, xn ~> x, n→00.

 $\frac{1}{2} \sum_{k=1}^{\infty} \frac{1}{2} \sum_{k=1}^{\infty} \frac{1}$ 

Remostly so if in converges restly, then the limit is unique. Insublition, if in converges strongly, then the strong limit coincides with the west one.

Broof-suppose that Xn→X, Xn→X, n→00. By cordiary 4.13, there is functional f ∈ X st. ft + f(x). Therefore f(xn) → f(x), f(x). since numerical convergence

sequences have uniquelimit, this is a constradiction = x=x. suppose that xn = z, xn = x. We know that xn = z. since the west limit is unique, z=x. / q.e.d.

rement= Suppose that X=H. Then Xn converges strongly to X <> Xn w> X and || Xn || -> || X||.

	suppose that $x_n \xrightarrow{\sim} x$ , $n \to \infty$ . Is $x_n$ bounded?
	Uniform boundedness the enter .
	Threadon 1+24 (Banach - Steinhaus Threaman)-
	Suppose X is a Banach space, and let M C X*. Assume that the set ffW, FEMY is bounded for each X EX i.e. I constand C= C(X) st. If (X) (S C(X) Y FEH. then M is uniformly
	bounded in the norm i.e. 3470 s.t. Ilfil 5 4 4f EM.
	Proof- Define AK= 1× EX: If (x) 1 ≤ K Vf ∈ M, K=1,2, J. cloim: AK is closed. Indeed, bt ×1->×, Xe ∈ AK. Ne what to show x ∈ AK. Thus If (Ke) 1 ≤ K V f ∈ M. By continuity of f,
	Ifall≤K ⇒ XEAK. Therefore AK=DAK]. Chim: X= $\bigvee_{k=1}^{m}$ Ak. Let XE X. We know that Ifall≤C(X) Vf EM. Take K= (X) so that Ifall≤K ⇒ XEAK.
	⇒ X C WAK. By the Baire (stegony theorem, at least one AK is dense in some ball B(x0, E), X0 EX, E>0. Let K0 be a number st. [AK0] > B(x0, E) i.e.
	If us 1 < Ko, YX & B(xo, E) uniformly in FEM, i.e. If (Xot EX) 1 < Ko YX & B(0,1) uniformly in FEM. Thus, & If us) = [f(xo) + f(xo) + f(xo)] (X.)
	<  f(xo)+  f(xotex)  < c(xo)+ Ko 4xeB(q1) 4fem. Therefore,  f(x)  < Ko + c(xo) × EB(0,1), 4fem. By Lemma 4.5, If    < Ko + c(xo) × fem. pre.d.
	Contray 1422 let ×n →x, n→∞. Then there is a constant C>0 st.   ×n   ≤ C Vn.
	Roof-By definition, f(xn) -> f(x) + f e X + Hence, I f(xn) < C - Yn mith some C=Cp. Therefore, IF xn (f) < Cp. Yn The space X is complete, so by the Banach-steinhans
	Theorem, II Fxn II < C1 minforming bounded with some C1 >0. Recollither II Fxn II = 11×n11; and hence 11×n11 < C1 ×n11 < C1 ×n11, q.e.d.
	13 March 2014
	Note: consider (2(0,1), where 50 lut at < 00. This is not the Riemann integral. It is the approximation of u by continuous functions (completion of Colo,1)). More solution of Colo,1).
	Worker 5 LINEAR OPERATORS -
	A war best out
	generalisation of theor functional. Interview of theory functiona
	∃R>0 st. Let X,Y be normed space. A linear operator A is said to be <u>continuous</u> if A is continuous on X. A is said to be <u>kounded</u> : fite image of the unit ball BX (0,1) is a bounded set i.e. A (Bx(0,1)) CB+(0,R) [i.e. 1 AX ly ≤ R   X  X]. [hereal] 52. A linear operator is continuous \$\$ it is bounded. (see theorem 4:3).
	Believed 53. The name of a bounded linear operator A is IIA II = x: IIX   x   V.
	$\frac{1}{100000154} = \frac{1}{10} \frac$
	Termine 55 Ter DCX be blivest set st. [D]=X. Let $A_0: D \rightarrow Y$ be a bounded livest operator. If Y is complete, then $A_0$ uniquely extends to X as a bounded livest operator A. Moreoror,   A  =  A_0  .
	Exemples-(1) The zero operator: Ax=0 4xeX, Notation:0, 1101=0. (2) Identity operator: I:X→X, Ix=x 4xeX. 1121=1.
	(3) Let X=12(0,1). Let me clo,1] and define (Au)(t)= m(t) u(t), u \in X is a multiplication operator. A: X -> X and is bounded. Io (m(t) u(t)) <sup>2</sup> at = [o(m(t)) <sup>2</sup> u(t)) <sup>2</sup> at = c <sup>2</sup> (w(t))
	$\Rightarrow   A_{11}  ^2 \leq c  u_1  ^2$ or $  A_{11}   \leq c  u_1  $ where $c = max = m(t)  t   \Rightarrow A$ is terms and $d$ .
	(4) $\chi = L_2(0/10)$ let $K = \frac{1}{2} \left[ \int_0^1 \frac{1}{2} \int_0^1 x_1(0,1) x_1(0$
	Then   Aull= Jo   Jo Klyy uy dy   dx. Then ince This kinn un = = The L(Z  kmn]? [ [un]") ] = The  kmn] = Z  un]?, we can proform a civiliar estimate for   Aull"
_	$\frac{(SI)}{(1 - 1)^{2}} = \int_{0}^{1} \int_{0}^{1} \mathcal{K}(x,y) u(y) dy \Big ^{2} dx. \text{ Then ince } \overline{\mathbb{R}} \Big  \frac{\Sigma}{2}   k_{nn} u_{n}  ^{2} = \overline{\mathbb{R}} \left[ \left( \frac{1}{2}   k_{nn}  ^{2} \right)^{2} \Big ^{2} = \frac{\Sigma}{nn}   k_{nn}   k_{nn} $

(5) Tilleoution operator. Let C<sup>1</sup>(Ia, b) = ffe C(Ia, b): f'e C(Ia, b): Let X = C's (Ia, b) be the space of C-functions with the L2-norm; [141] = [So 14(t)] dt]<sup>1/2</sup>, us C. the completion is L2(Ia, b). Let X= C2 [-T1,T], Y= L2[-T1,T]. Define (Tw)(t) = -iu(t), u ∈ X. claim: This quarter is unbounded! indeed let en (t)= 1/2T1, e int int e ther hand, Ten=nen.

⇒ ||Ten || = In | → as as n → so, so ||x||=1 ||Tx||= as. Tools on 12(-17,17) but it is defined on the smaller set CT-17,17].

Agebris of haunded livest operators.	
WAXIT be normed spaces. Grear structure: If A, Bare linear, bounded (dA+QB) x = dAx + QBx Vx EX, Vd, Q = 1K. Deco [IA11 hore all required properties]	
① Non-degeneracy: 11A11≥0 is clear. Then if 11A11=0, A=0 ⇒ so non-degeneracy holds. ✓	
@ Homogeneity: IldA II = sup IldAxII = Isup Lel IIAXII = 1 d ILXII=1 Lel IIAXII = 1 d ILAII. /	
@ Triangle inequality: 11(h+B)×11 = 11 A×+B×11 ≤ 11 A×11 + 11 BX11 ≤ 11 A1111×11 + 11 B111×11 = (11 A11+11B11)11×11. ⇒ 11 A+B11 ≤ 11 A11+11B11) √.	
The normal space of linear bounded operators A:X-Y is denoted by B(X;Y). Moe as a point on notation, me write B(X;X) = B(X).	
Strong 156 If Y is a Banach space, then B(X, Y) is a Banach space do well. (see Theorem 4:15).	_
1960 57 let A & B(X, Y), B & B(Y, Z), then the product of operators BA is defined by (BA) X = B. (AX) $\forall X \in X$ .	
Remork - BA & bounded.   (BA)x11 = 11.B(AX)11 ≤ 11.B11.   AX111 ≤ 11.B11.   A11.   x11. ∀x ∈ X. ⇒ 11.BA 11 ≤ 11.B11.   A11. ⇒ BA ∈ B(X,Z).	

	In B(X), we have (AB) C = A(BC). Cossociativity], (A+B) C = AC+ BC, A(B+C) = AB+AC [distribuctivity]. Thus B(X) is an algebra. If X is complete, B(X) is a Remach algebra.
	(1) (1) (1) (1) (1) (1) (1) (1) (1) (1) (1)
	(2) (3) Ne say that An converges to A strangly if 11 Anx-Ax11→0, n→∞ for all x ∈ X. We say that An converges to A strandy if f(Ax), n→∞ ∀x ∈ X, f ∈ Y*.
	These three forms of convorgence are commonly used, and are altranged in decreasing order of extrength. (1) ⇒ (2) ⇒ (3). [For (1) ⇒ (3), see (IAn X-AX II =    (An -A) X II =
	The revence implicitions do not hold in general; however if dim X<00, dim Y<00, then (1) (=> (2) (=> (3). Otherwise, consider the following counterexample:
	[ x  _= ( = ( +  x )^{1/2} < co] n times Net X= l Let A: X > X be the operator Ax = An (x1, X2,) = (0,0,, 0, Xn+1), Xn+2,). Then   An X  _2 = K=nH   X  _2 -> 0, do n > on. Hence, An convergeo to 0 strongly.
	On the other hand,   AnU=1 -> An does not converge to 0 withoundy.
	If (1) has a limit, it is unique. If (2) has a limit, it is unique: II AX-BXII = II AnX-AXII+ II AnX-BXII. And literize if (3) has a limit, it is unique.
	Theorem 159 (Brinsch-Heinhaus Theorem-Br. operations).
	let XI be Bondch spaces and let MCB(X). Suppose that for every XEX there is a constant C= C(X)>0 st. the 11AX 11 5 C(X) for all AEH. Then there is a constant c1>0 st.
	IIAII & CI For all AEM.
	Roof-see theorem 4.21.
1	17 Nowh soit
1	Ref. Necessian
+	Let X be & reflexive optice. Show that it is weathy complete ice. if f(M) is concluy for suy fex, then 3 X EX ST. f(XN) -> f(X) -> f(X
+	Example - Let H=l2, SNX = Sn(X1, X2,)= (0,,0, X1, X2,). Claim Sn - 0, n-200. discovation: IIS x112= (1×11)= (1\times11)= (1\times11)= (1\times11)= (1\times11)= (1\times11)= (1\times11)= (1\times11)= (1\times11)= (
-	$K_{S_{N}X}, \xi_{2} = \left  \sum_{k=int}^{\infty} x_{k-n} \overline{\xi}_{k} \right  \stackrel{\leq}{\leq} \left( \sum_{k=int}^{\infty}  x_{k-n} ^{2} \right)^{\frac{1}{2}} \left( \sum_{k=int}^{\infty}  \xi_{k} ^{2} \right)^{\frac{1}{2}}, \text{ As } n \rightarrow \infty, \left( \sum_{k=int}^{\infty}  \xi_{k} ^{2} \right)^{\frac{1}{2}} \rightarrow 0, \text{ so }  \langle S_{N}X, \xi_{2} \rangle  \rightarrow 0 \Rightarrow \text{ Sn } \xrightarrow{M \rightarrow 0} \text{ so downed}_{1} \text{ q.e.d.}$
-	toodby 1510 let X, Y be Barach spaces. let An & B(X, Y) - suppose that for some mapping A: X -> Y we have An x > A X os n - so - for every X & X. Then the normal I And are uniformly bounded,
-	and AEB(X, M.
-	Proof-wite Alex+ By) = him Anlex+ By) = him (xAnx+ BAny) = a him Anx + B him Any = aAx + pAy 4x, gex, Va, BEK. Thus A is lined. Observe also that
_	11 Anx 11 is bounded mighting in a. Thus by Bonoch-steinhous Theorom (5.9), 3 conserved c. st. 11 An11 5 C. Vn. Therefore, 11 Anx 11 5 C/1x/1 4xEX. Consequendly, see that
	1 AxII = Norma UAnxII ≤ CIIXII. Hence, A ∈ B(X,Y), -q.e.d.
	Adjoint operators
	in white the a difference of the adverter of the control of
1	ret X=H be a Hibert space (throughout this section). (linear in fat variable) (antilinear in second variable).
1	αφισουξη Α function φ: HXH→ K is could a scaquilinear functional form if φ(x1x1 + α2X2,y)= α1 φ(x1,y) + x2 φ(x2,y); φ(x1, d1y1 + α2y2) = a1 φ(x1,y1) + α2 φ(x1,y2)
1	\$ is said to be constinuous if its poron, II \$ II = 11×11= 11y1=1 (\$ (x,y)], is finite.
1	Note-If $\phi$ is continuous, $ \phi(x_1y)  \leq   \phi     X     y    \forall x, y \in H. Also, the functional given by \Psi(x,y) = \phi(y, X) is also seequitive at$
	Example - Let $\phi_1(x_1y) = (Tx_1y), T \in B(H)$ . $\phi_2(x_1y) = \langle x_1, Sy \rangle$ , $S \in B(H)$ . These we both sesquilinear. Are those any others? No! These examples exhaust all possibilities:
+	theored 5/2 let & be a sergification continuous form. Then there are two uniquely defined operators T, S = B(H) st. \$\frac{1}{2} \text{X}_1 y = \text{X}_1 y > \text{X}_1 y = \text{X}_1 y
-	most-let fy (x) = \$(xy). Then fy is linear, fy is bounded by continuity - Ifx(x) [= ]\$(xy) [ <   \$     x     y  , so   fy   <   \$
1	st f(x) = <x, (left="" 1="5y." 115y11="11h11=11" 11≤11¢111y11.="" 5="" as="" b).="" by="" check="" define="" easy="" exercise).<br="" fy="" is="" it="" linear="" mapping="" that="" the="" then="" thus,="" to="" very="">≤ 15111x111y11,</x,>
	dearly, $  S   \leq   q  $ . To prove that $  q  \leq   S  $ , write $\phi(x,y) = \langle x, Sy \rangle$ . So $  q  x_1y_2  _{A}$ thus $  q   \leq   S   \Rightarrow   S   =   q  $ . Hence, if we have
	<*1.54y>= <*, 52y> **, yeth, then sy=52y => 51=52, so sis unique. To check that \$(x,y)= <tx,y> with some T &amp; B(H), consider the functional</tx,y>
	Y(x,y)= \$(y,x), and use the first part of this proof to angue exactly the same usy 1/9-e.d. Alternations of the provident of the same of the proof to angul the same of the operator s defined by \$(x,y)= (x, Sy) is called the adjoint of T, denoted S=T*.
1	If $T = T^*$ , then T is called <u>self-signing</u> . [Remark: $\langle T_{X,Y} \rangle = \langle x_1 T^*_{Y} \rangle$ ]
-	$\frac{B_{10000055}tt}{B_{10000055}tt} = a_1 T_1^{*} + a_2 T_2^{*},  (4) (T_1 T_2)^{*} = T_2^{*} T_1^{*} + a_2 T_2^{*},  (4) (T_1 T_2)^{*} = T_2^{*} T_1^{*} + a_2 T_2^{*},  (4) (T_1 T_2)^{*} = T_2^{*} T_1^{*} + a_2 T_2^{*},  (4) (T_1 T_2)^{*} = T_2^{*} T_1^{*} + a_2 T_2^{*},  (4) (T_1 T_2)^{*} = T_2^{*} T_1^{*} + a_2 T_2^{*},  (4) (T_1 T_2)^{*} = T_2^{*} T_1^{*} + a_2 T_2^{*},  (4) (T_1 T_2)^{*} = T_2^{*} T_1^{*} + a_2 T_2^{*},  (4) (T_1 T_2)^{*} = T_2^{*} T_1^{*} + a_2 T_2^{*},  (4) (T_1 T_2)^{*} = T_2^{*} T_1^{*} + a_2 T_2^{*},  (4) (T_1 T_2)^{*} = T_2^{*} T_1^{*} + a_2 T_2^{*},  (4) (T_1 T_2)^{*} = T_2^{*} T_1^{*} + a_2 T_2^{*},  (4) (T_1 T_2)^{*} = T_2^{*} T_1^{*} + a_2 T_2^{*},  (4) (T_1 T_2)^{*} = T_2^{*} T_1^{*} + a_2 T_2^{*},  (4) (T_1 T_2)^{*} = T_2^{*} T_1^{*} + a_2 T_2^{*},  (4) (T_1 T_2)^{*} = T_2^{*} T_1^{*} + a_2 T_2^{*},  (4) (T_1 T_2)^{*} = T_2^{*} T_1^{*} + a_2 T_2^{*},  (4) (T_1 T_2)^{*} = T_2^{*} T_1^{*} + a_2 T_2^{*},  (4) (T_1 T_2)^{*} = T_2^{*} T_1^{*} + a_2 T_2^{*},  (4) (T_1 T_2)^{*} = T_2^{*} T_1^{*} + a_2 T_2^{*},  (5) (T_1 T_2)^{*} + T_2^{*} + a_2 T_2^{*},  (5) (T_1 T_2)^{*} + a_2 T_2^{*},  (5) (T_1 T_2)^{*} + T_2^{*} + a_2 T_2^{*},  (5) (T_1 T_2)^{*} + T_2^{*},  (5) (T_1 T_2)^{*} + a_2 T_2^{*},  (5) (T_1 T_2)^{*} + T_2^{*},  ($
-	Roof-(1),(2) left 30 exercises. (3): (left + d_2T2)x, y) = d1 (T1, X, y) + d2 (T2X, y) = d1 (X, T1, y) + d2 (X, T2, y) = (X, d1, y) + (X, d2, T2, y) 20 Morth 2014 Roof Alexander soBoiEV
	= <x, (4):="" (d,="" +="" 706.<="" <="" adjainty="" d,="" holds="" j2="" obtain="" q.e.d.="" so="" t,="" t1="" t2="" t2)y),="" th="" the="" we="" x,="" y)="&lt;x," y),=""></x,>
_	$Tor matrices, if A \rightarrow a_{jk}, A^* \rightarrow \overline{a_{kj}}, \text{ Hence it sub by } (AX)_j = k^{\frac{2}{2}} a_{jk} x_{K},  (A^* x)_j = k^{\frac{2}{2}} \overline{a_{kj}} x_{K}.$
	Examples=() H=ls, $  x  _2 = \left[\sum_{k=1}^{\infty}  x_k ^2\right]^{\frac{1}{2}}$ , $(Ax)_j = \sum_{k=1}^{\infty} a_{jk} x_k$ , $(A^*x)_j = \sum_{k=1}^{\infty} a_{kj} x_k$ . We do not know if operator is bounded, but if $\sum_{j=1}^{\infty}  k_{jk} ^2 < \omega \Rightarrow A$ is bounded.
	(2) H= L2(a,b). consider integral operator (Tw) (x) = la K(x,y) u(y) dy, u e.H. Assume K E C([a,b]x[a,b]). Then T is bounded, we seek to find T*
	(Tu, V) = [ b b K(x, y) u(y) V(x) dy dx = [ b b u(y) K(x,y) V(x) dy dx. Then T* has the kerned K (x,y) = K(y,x). To sheek for self-signituress, need K(x,y) = K(y,x).

	Hence e <sup>x2+y2</sup> is self-solipint, is in (x-y) is celf-solipint, e <sup>ixy</sup> is not self-solipint.	
	(3) $H = L_2(a,b)$ , $m \in C[a,b]$ . $(Au)(x) = m(x)u(x)$ , $u \in H$ , $(A^*u)(x) = \overline{m(x)}u(x)$ , $u \in H$ .	
	But lotally The Find and To by the first of the second states the	we attempt to
	$\frac{(b)}{(b)} H = L(u)(b),  (u = -u),  (u \in C_2, U, (b) = (e),  (e)  (f)  (f) $	›
	Hence, $\langle Tu, v \rangle = -i[u(b)v(b) - u(a)v(a)] + \langle u, Tv \rangle$ . Restrict the set of functions on which T is defined. $P(T) = \{u \in \mathcal{C} \mid a, b\} : u(a) = u(b) = 0\}$ .	
	Then on P(T), (TU, V) = <u, &="" [more="" a="" also="" but="" can="" d(t).="" different="" generally,="" gives="" just="" operator].<="" set="" td="" this="" tv)="" u(a)="u(b)," v="" we="" yu,=""><td></td></u,>	
	<b>Because</b> 5.75 (1) Every $A \in B(H)$ is weakly continuous i.e. if $x_n \xrightarrow{W} x_{ab} \to - \infty$ , then $A \times n \xrightarrow{W} A \times ab \to -\infty$ .	
	12) let An man, then An man, then An man i.e. <an x,="" y=""> &gt; <ax, y=""> + x, y <h, <an="" then="" x,="" y=""> -&gt; <ax, y=""> +x, y <h.< td=""><td></td></h.<></ax,></h,></ax,></an>	
	Broof - 11) write <axm, y=""> = <xm, a*y=""> → <x, a*y=""> = <ax, y=""> as n&gt;00. Yy EHA q.e.d. Downite <an*x, y=""> = <x, amy=""> → <x, ay=""> = <a*x, y=""> A q.e.d</a*x,></x,></x,></an*x,></ax,></x,></xm,></axm,>	۱
	Renote by R(A) the range of A: then,	
	Themews 516 let A & B(H), then H=[P(A)] @ ker A <sup>*</sup> .	
	Proof-we already know that Ker A* is a closed subspace, as A EB(H). Then we need to show that RIA) is a lineariset.	
	Dembed 5.17 RIAL is a linear set for any $A \in B(X,Y)_{*}$ .	
i est	Roof-ler y1, y2 E RIA). Work: diy1 + d2 y2 E RIA) & d1, 62 E K. Let Ax1= 41, Ax2= y2. Then d, y1+ d2 y2= d1 Ax1+ d2 Ax2 = Ald(x1+ d2 X2) E RIA),	g.e.d.
	(Theorem 5.14 undd) Proof-let Ax=y. Then YZEH, we have <4,37= <x, a*z="">. yER(A). What is RLA)? Look for those ZEH for which &lt;4,37=0 YyER(A), since &lt;4, A</x,>	*z>=0,
	A*==0 so z = Ker (A*), conversely, if z = Ker A*, then <x, a*="">=0 ∀x =H, so <ax, z="">=0 and hence z = R(A). Thus, R(A) = Ker A*.</ax,></x,>	
	Recall that $R(A)^{\perp} = [R(A)]^{\perp}$ , so $H = [R(A)] \oplus \ker A^{\star}_{\parallel}$ q.e.d.	
	jurene openstors.	
	Let X, Y be normed spaces. Suppose A & B(X,Y). We study equation Ax= Y. If there is a unique solution, then X= A <sup>-1</sup> y.	
	Induition 5.18 Let A & B(X,Y) be injestive. Then the inverse operator A <sup>-1</sup> is defined as an operator mapping each ye RIA) into the rector xe X uniquely defined by the relation AX=4	
	ATAX=X VXEX and AATy= y VyER(A).	g-in divernerous,
	Theorem 19. If A texistry, it is a linear operator.	
	$\frac{1}{1000} = \frac{1}{1000} = 1$	7
	1100= 111- A (a)(1 a 2)2)-21 A (y11 a 2A (y2 - Vy1)y2 = 1.01) Valiase 1. Let A 4- 91, A2- 92, 20 A- A 91, A2- A 92. 20 (3) A (a) A1 A LHS = A <sup>1</sup> (A(a)X1t d2X2)) = d1X1t d2X2 = RHS 20 required 9. e.d.	24127 - 2121 - 2222
	Atesen1520 let X be a Banach space, let A & B(X) be st. IIAII < 1. Then the inverse (I-A) <sup>-1</sup> exists, and it is given by the uniformly convergent series (I-A) <sup>-1</sup> = $\sum_{k=0}^{\infty} A^k$ , is	where A°=I
	Note-Inhener, R(I-A)=X. (entire your)	
	Proof- Let $S = \sum_{k=0}^{\infty} A^k$ . consider partial sums $Sn = \sum_{k=0}^{n} A^k$ . This is a cauchy sequence - assume $m > n_1$ then $  Sm = Sn   =   \sum_{k=0}^{n} A^k   \le \sum_{k=0}^{\infty}   A^k   \le \sum_{k=0}^$	k 111 1
	→ 0 do n → 00. Since X is Bonoch, B(X) is Bonoch do well = the series converges uniformly, so SE B(X). Not-finally need to show that S=(I-A) <sup>-1</sup> .	
	$(I-A)S_{N} = \sum_{k=0}^{N} A^{k} = I - A^{N+1}  (clescoping series). As n-2 m, LHS gives (I-A)S by uniform continuity, while RHS gives I-O as   A^{n+1}  $	
	$\frac{1}{100} \frac{1}{100} \frac{1}{100} \frac{1}{100} \frac{1}{100} \frac{1}{100} \frac{1}{100} \frac{1}{1000} \frac{1}{1000} \frac{1}{1000} \frac{1}{1000} \frac{1}{1000} \frac{1}{10000} \frac{1}{100000} \frac{1}{10000000000000000000000000000000000$	5 IIAII - 70.
		ler soboltar.
	Repaired at CReativent Identity". Matter and a la la a transfer at a transfer at at a transfer at transfer at a tr	
	Let $A_1 B \in \mathcal{B}(X,Y)$ , $A^{-1} \in \mathcal{B}(Y,X)$ , where $X_1 Y$ are normed spaces. Denote $V = B - A$ . Then $A^{-1} - B^{-1} = \overline{A}^{-1} V \overline{B}^{-1} = \overline{B}^{-1} V \overline{A}^{-1}$ .	
	Roof- left is exercise. Exercise - let A, B, A <sup>-1</sup> , B <sup>-1</sup> = B(H), H beings Hilbert space. Then $(A^{-1})^{\times} = (A^{\times})^{-1}$ , $(AB)^{-1} = B^{-1}A^{-1}$ .	
	b the invesse defined on the evolve space? Is the invesse bounded?	
	te open mapping theorem.	
	Togethistical 5-22 let X, X be metric spaces. Then the function f:X-> Y is said to be an even mapping if f maps open sets X into open sets in Y.	
	Huenzen 523 Open Happing Theorem	- <b>-</b>
	Let X,Y be Bonoch spoces. Suppose that A & B(X,1) is a surjection. Then A is an open mapping.	AT Y A(M)
	moof-Dwilled, non-examinate.	A S
	Cooling 5.24 let X, I be Bandh quees. Suppose that AE B(X, Y) to a bijection. Then A <sup>-1</sup> is bounded.	
	Proof- Yopenset MCX, the image A(M) CY is also open. (see theorem 5.23), then A <sup>1</sup> exists. Moreover, we deeve the set A(M) is the preimage of Mundar mapping A <sup>-1</sup> continuous => A <sup>-1</sup> is bounded as claimed/ g.e.d.	The map A <sup>-1</sup> is
	minimumo = 11 " unneuro os usinuali qe.d.	24-2-4

	Contrast State of the State of
	The operators are not assumed to be bounded. Let D(A) be the domain of the operator A, i.e. D(A) C X is a set of vectors where h matrices sense .
	An→x, xn ∈D(A) and x=y. (A)=u(b)=0 (A)=u(b). (A)=u(b)=0 (A)=u(b). (A)=u(b)=0 (A)=u(b). (A)=u(b)=0 (A)=u(b). (A)=(b)=0 (
	Hussen 52b bren AE B(X,T) is dosed.
	Roof-let ×n→×, ×n & D(A) = X. A×n→Y. Since A is continuous, so A×n→ A×. since the limit is unique, y=hx. Hence A is closed. Example-let ×=cton1. (Aplu) = f(x), fep(A) = c <sup>1</sup> ton1. (Asim: A is closed. Indeed. As the that fn→ fin ×, and that fn→ g ∈ ×., fn ∈ D(A). We want fecton13 and f'=g.
	write: fn(x) = fn(0) + Jo fn(t) dt (Fundamental Theorem of coludius). As no seq. fn(x) -> f(x), tx and fn(t) -> g(t) uniforming in t < [0,1]. Therefore, f(x) = f(0) + Jo g(t) dt,
	$\infty f \in C^{1}(0, 1]$ and $f(x_{2}, y_{3}) = (x_{1} + x_{2}, y_{3} + y_{2})$
	Ret X, Y be normed spokes. Define the direct sum X @ Y as follows: X @ Y is the set of ordered poins (X, y), x∈ X, y∈Y with the breat structure, i.e. To((X, y)=(dX, dy).
	Thus, XOY is a linear set. Define also the norm 11(X, y) 11= 1/X 1/X + 11/y 1/Y. If X, Y are Banach, then XOY is Banach as well.
	Behinool527 let A: P(A) → Y be a linear operator. Then the set G <sub>A</sub> = 1(x, Ax), x ∈ D(A) } C X ⊕ Y is called the graphs of the operator A.
	Permark - If A=Y=IR and Ax=ax, a GIR fixed then GA= {1x,ax}, x GR = Ax- GA C X @Y.
	$G_{A} is_{\partial} is_{\partial} is_{\partial} except, i.e. (x_{1}, Ax_{1}), (x_{2}, Ax_{2}) \in G_{A} \Rightarrow (x_{1}+x_{2}, A x_{1}+x_{2}) \in G_{A}  (ax_{1}, a Ax_{1}) \in G_{A}  27 \text{ Norm} 2019  27 \text{ Norm} $
1	Business of the generator A is closed as GA is closed.
	$\begin{array}{c} (\Rightarrow) \\ \chi_n \rightarrow \chi, \ \chi_n \in D(A) \\ \text{froof-ssame A is closed, i.e.}  (A_{X,n} \rightarrow y) \\ \Rightarrow [AX=y, I]^{2} (x_{n}, A_{X,n}] \rightarrow (x, y) \\ \Rightarrow [AX=y, I]^{2} (x_{n}, A_{X,n}] \rightarrow (x, y) \\ \text{there } x_{n} \in D(A), \ \text{then } (x, y) \in G_{A,y-i,e}, \ \chi \in p(A) \\ \text{out } y = A \\ \end{array}$
	However, we dready have obtained this exact result, so we are done of q.e.d. [Reverse done by revening steps, in some manner, to rephrase eprivalent deps].
	(⇐) Assume that GA is closed, i.e. if (xn, Axn) → (x, y) & n→00, Xn ∈ D(A), then (x, y) ∈ GA is x ∈ D(A), y=Ax. → A is closed y q.e.d.
	Recall that $A \in B(X, X) \Rightarrow A$ is closed. We now also seek to prove the converse.
	Theorem 5:23- (Coved Graph Theorem)
	Let X, Y be Banach. Let A: X-> Y be a linear operator, with D(A) = X. If A is closed, then A is bounded.
	Troof - A is dosed => GR is closed (Theorem 5.22). GR is Banach opsice. Define p: GR => X. by P(X, AX) = X [vetains only first component]. P is bounded: as me have
	$\ P( x Ax)\  = \ b_{x}\ _{X} \leq \ (x,Ax)\  = \ x\ _{X} + \ Ax\ _{Y} + \ ene_{x}\  \ P\  \leq 1$
	(divide by   x  ) . $(divide by   x  ) .$
	Here is an important corollary: let X be a Barrach space, and let X1Z be closed subspaces of X s.t. X=YOZ. For any XEX there is a uniquely defined pair yeY, ZEZ s.t. X=Y+Z. Define an
	operator $\Pi: X \to Y$ , $\Pi_X = y = projection operator.$
- 1	Condent 530 (Non-orthogonal projection).
in a side of the state	The operator This bounded.
	Proof- show that ∏ is doved, i.e. Gy is doved. Suppose that (Xr, ∏Xn) → (X;y) in X⊕Y. We wave to show (X;y) ∈ Gy, i.e. X ∈ X, y= [X. Indeed, the sequence Zn=Xn-∏Xn
	converges to Z = X-y e Z, so X=y+Z with ye 1, zeZ. This is a unique representation. By definition of [], []X=y > Gn is closed. By theorem 5.29, [] is bounded/gen
	Remark -    []    51 is not true in general.
4	of course though UTUN≥1.
	ENDOF SYLLABUS.
	END OF COURSE.
103-16	(c) A set of the se