3103 Functional Analysis Notes

Based on the 2018 spring lectures by Prof A Sobolev

MATH 3103 Books: - Kolmogorov & Fomin
- Friedman

08-01-18 Functional Analysis Prof. Alex Sobolev 710
Office Hour Monday 1pm

§1 Chapter 1 - Introduction

Sets, functions

- Sets, operations $A \cup B$, $A \cap B$, $A \setminus B$, $A \Delta B = (A \setminus B) \cup (B \setminus A)$
- complement $CA = A^c = E \setminus A$, $A \subset E$
- Empty set $\emptyset$

- Functions $f : X \to Y$
- $\operatorname{Ran} f = f(X)$ = range = image
$= \{f(x), x \in X\}$
- Inverse image = pre-image $= f^{-1}(y) = \{x \in X : f(x) = y, y \in f(X)\}$
- Identity $Id_X : X \to X$, $Id_X x = x$

- Injection: f is injective if $f^{-1}(y)$ consists of one point only $\forall y \in f(X)$. (one-to-one)
- Surjection: f is surjective if $f(X) = Y$.
- Bijection: f is bijective if it is both injective and surjective.

Thus the equation $f(x) = y$ has a solution for all y Y (by surjectivity). Since f is injective, this solution is unique. Therefore we can define the inverse function $f^{-1} : Y \to X$ (bijection) s.t. $f^{-1}(y) = x$. In other words $(f^{-1} \circ f)(x) = x$ $\forall x \in X$, $(f \circ f^{-1})(y) = y$ $\forall y \in Y$.

Example

$f(x) = x^2$, $X = Y = \mathbb{R}$ not injective or surjective
$X = \mathbb{R}_+$, $Y = \mathbb{R}$ injective not surjective
$X = \mathbb{R}_+$, $Y = \mathbb{R}_+$ bijective
The inverse map: $f^{-1}(y) = \sqrt{y}$, $y \in \mathbb{R}_+$
$X = \mathbb{R}_-$, $Y = \mathbb{R}_+$ bijective, $f^{-1}(y) = -\sqrt{y}$, $y \in \mathbb{R}_+$

Def^n 1.1

X is said to be equivalent to Y if there is a bijection $f: X \to Y$.

Then we say that X and Y have the same cardinality, $|X| = |Y|$ "card"(X)

A set X ~~is~~ equivalent to $\mathbb{N}$ is said to be countable.

Notation $|\mathbb{N}| = \aleph_0$, $0 \notin \mathbb{N}$.

Observations:

1). Every infinite set has a countable subset.

2). Every subset of a countable set is either countable or finite.

Examples 1.2

1). If X is finite then, then $|X|$ = number of elements.

2). $|\mathbb{Z}| = |\mathbb{N}| = \aleph_0$

Bijection $g: \mathbb{Z} \to \mathbb{N}$, $g(m) = \begin{cases} 2m, & m > 0 \\ 1 - 2m, & m \leq 0 \end{cases}$

3). $|\mathbb{Q}| = |\mathbb{N}|$, if $x \in \mathbb{Q}$, then $x = \frac{p}{q}$, $p \in \mathbb{Z}$, $q \in \mathbb{N}$

Assume that $(p,q) = 1$ (coprime).

Let $a_{pq} = \frac{p}{q}$:

$$\begin{array}{ccccc} a_{11} \to & a_{12} & a_{13} \to & a_{14} & \ldots \\ a_{21} & a_{22} & a_{23} & a_{24} & \ldots \\ a_{31} & a_{32} & a_{33} & a_{34} & \ldots \\ \vdots & \vdots & \vdots & \vdots & \end{array}$$

labeling ("counting")

The same for negative rationals

4). Define: $|\mathbb{R}| = \aleph$ – cardinality of continuum.

Claim: $|(0,1)| = |\mathbb{R}|$.

Indeed define the bijection $h: \mathbb{R} \to (0,1)$ by

$h(x) = \frac{1}{\pi} \arctan x + \frac{1}{2}$

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Theorem 1.3
The set $\mathbb{R}$ is not countable.

Proof (by contradiction)
Assume $\mathbb{R}$ is countable.
Then $[0,1]$ is also countable.
Let $f : \mathbb{N} \to [0,1]$ be a bijection.
Thus all numbers $x \in [0,1]$ are contained in this list

$f(1) = a_1 = .a_{11}a_{12}a_{13}\ldots,$

$f(2) = a_2 = .a_{21}a_{22}a_{23}\ldots,$

$\vdots$

$f(n) = a_n = .a_{n1}a_{n2}a_{n3}\ldots;$ where $a_{mn} = 0, 1, \ldots, 9.$

Want to construct a number $b \in [0,1]$, which is not on the list.
"Cantor Diagonalisation Procedure"
Seek $b = .b_1b_2b_3\ldots$

If $a_{11} = 1$, set $b_1 = 2$, otherwise $b_1 = 1$,

If $a_{22} = 1$, set $b_2 = 2$, otherwise $b_2 = 1$,

$\vdots$

If $a_{nn} = 1$, set $b_n = 2$, otherwise $b_n = 1$

Therefore $b = .b_1b_2b_3\ldots$ is not on the list and hence $[0,1]$ is not countable as required. $\square$

[Need to be careful since $0.0\bar{9} = 0.1$]

Remark 1.4
Let X be the set of all sequences $a = \{a_1, a_2, \ldots, a_n, \ldots\}$, consisting of "0"s and "1"s only, e.g. $(0,1,1,1,0,1,0,1,\ldots)$.
This set is not countable.
(Proof by contradiction using the Cantor Diagonalisation Procedure).

How do we compare cardinalities?

Def^n 1.5

Let X, Y be sets. We say that $|X| \leq |Y|$ $(|Y| \geq |X|)$ if there is an injection $f: X \to Y$.

If $|X| \leq |Y|$ and there is no bijection between X and Y, then we say that $|X| < |Y|$.

From Theorem 1.3: $\aleph_0 < \aleph$

Theorem 1.6 (Cantor - Bernstein Thm)

If $|X| \leq |Y|$ and $|Y| \leq |X|$ then $|X| = |Y|$.

Example

Let $X = (0,1)$, $Y = [0,1]$.

Then $|X| = |Y| = \aleph$.

Proof

Already know $|X| = \aleph$.

Define $g: (0,1) \to [0,1]$, $g(x) = x$ injection, so $|X| \leq |Y|$.

$h: [0,1] \to \mathbb{R}$, $h(x) = x$ injection, so $|Y| \leq |\mathbb{R}| = \aleph$

Therefore $\aleph = |X| \leq |Y| \leq |\mathbb{R}| = \aleph$,

so $|Y| \geq \aleph$ and $|Y| \leq \aleph$, so by Thm 1.6 $|Y| = \aleph$ as claimed. $\square$

§2 Chapter 2 - Metric and normed spaces

Metric space (X, ρ) = a set X, and a metric $\rho : X \times X \to \mathbb{R}_+$, with the properties:

1). $\rho(x,y) = 0$ iff $x = y$ ← non-degeneracy

2). $\rho(x,z) \leq \rho(x,y) + \rho(y,z)$ ← triangle inequality

3). $\rho(x,y) = \rho(y,x)$ ← symmetry

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Examples

1). $X = [0,1]$, $\rho(x,y) = |x-y|$

2). Discrete space:

$$\rho(x,y) = \begin{cases} 0, & x = y, \\ 1, & x \neq y. \end{cases}$$

Normed spaces

Normed space = a linear space X over $\mathbb{R}$ or $\mathbb{C}$ and a function $\|\cdot\| : X \to \mathbb{R}_+$ called "norm" with the properties:

1). $\|x\| = 0$ iff $x = 0$

2). $\|\alpha x\| = |\alpha| \|x\|$, $\alpha \in \mathbb{R}$ or $\mathbb{C}$

3). $\|x+y\| \leq \|x\| + \|y\|$.

On X (normed space) we can define the metric $\rho(x,y) = \|x-y\|$.

Examples

1). $X = \mathbb{R}^n$, $n \geq 1$, $x = (x_1, x_2, \dots, x_n)$

The norm: $\|x\|_2 = \left[\sum_{k=1}^{n} |x_k|^2\right]^{1/2}$ ← Euclidean norm.

It is a norm since 1)., 2). are trivially satisfied, and $\|x+y\|_2 \leq \|x\|_2 + \|y\|_2$ $\forall x,y \in \mathbb{R}^n$.

It is derived from the Cauchy-Schwarz inequality: (look it up!) $\left|\sum_{j=1}^{n} x_j y_j\right| \leq \|x\|_2 \|y\|_2$

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Examples

1). $\mathbb{R}^n$, $x = (x_1, x_2, \dots, x_n)$

$$\|x\|_2 = \left[\sum_{k=1}^{n} |x_k|^2\right]^{1/2}$$

p-norms on $\mathbb{R}^n$. Let $p \in [1, \infty]$:

$$\|x\|_p = \left[\sum_{k=1}^{n} |x_k|^p\right]^{1/p}, \quad 1 \leq p < \infty$$

• non-degenerate ✓
• homogeneous ✓

$$\|x\|_\infty = \max_{1 \leq k \leq n} |x_k|$$

• non-degenerate ✓
• homogeneous ✓

Triangle inequality?

$p = 1$ or ∞ – easy cases

Conjugate parameters: $p \in [1, \infty]$, $q \in [1, \infty]$:

$$\frac{1}{p} + \frac{1}{q} = 1 \quad ; \left[\begin{array}{l} \text{e.g. } p = 1,\ q = \infty \\ \qquad p = 2,\ q = 2 \end{array}\right] \quad (p-1)(q-1) = 0, \quad 1 + \frac{p}{q} = p, \quad 1 + \frac{q}{p} = q$$

Lemma 2.1 (Young's inequality)

Let p, q be conjugate, and let $a, b \geq 0$.

Then $ab \leq \frac{a^p}{p} + \frac{b^q}{q}$

Proof

Define $g(t) = \frac{t^p}{p} + \frac{t^{-q}}{q}$, $t > 0$.

Then $g'(t) = t^{p-1} - t^{-q-1}$.

Thus $g'(t) < 0$, $t < 1$ & $g'(t) > 0$, $t > 1$.

Therefore $\min g(t) = g(1) = 1$ since p, q are conjugate,

so $1 \leq \frac{t^p}{p} + \frac{t^{-q}}{q}$, $\forall t > 0$.

Assuming $a > 0$, $b > 0$, take $t = a^{\frac{1}{q}} b^{-\frac{1}{p}}$:

$$1 \leq \frac{a^{\frac{p}{q}} b^{-1}}{p} + \frac{a^{-1} b^{\frac{q}{p}}}{q}$$

$$\Rightarrow ab \leq \frac{a^{p/q+1}}{p} + \frac{b^{q/p+1}}{q} = \frac{a^p}{p} + \frac{b^q}{q} \quad \square$$

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Let $x = (x_1, \dots, x_n)$, $y = (y_1, \dots, y_n)$.

Theorem 2.2 (Hölder's Estimate)
Let p, q be conjugate.
Then $\sum_{k=1}^{n} |x_k y_k| \leq \|x\|_p \|y\|_q$ (Cauchy-Schwarz if $p = q = 2$)

Proof
Assume $\|x\|_p = \|y\|_q = 1$. Assume also that $p > 1$ ($\Rightarrow q \neq \infty$)
The case $p = 1$, $q = \infty$ is straightforward:
$$\sum_{k=1}^{n} |x_k||y_k| \leq \max_k |y_k| \sum |x_k| = \|y\|_\infty \|x\|_1$$

So, $p > 1$: Estimate:
$$\sum_{k=1}^{n} |x_k||y_k| \leq \frac{\sum_k |x_k|^p}{p} + \frac{\sum_k |y_k|^q}{q} = \frac{1}{p} + \frac{1}{q} = 1$$
$$= \|x\|_p \|y\|_q$$

General case:
Define $\tilde{x} = \frac{x}{\lambda}$, $\lambda = \|x\|_p$, $\tilde{y} = \frac{y}{\mu}$, $\mu = \|y\|_q$.

So $\|\tilde{x}\|_p = \|\tilde{y}\|_q = 1$,
subsequently $\sum |\tilde{x}_k \tilde{y}_k| \leq \|\tilde{x}\|_p \|\tilde{y}\|_q = 1$
Substitute:
$$\sum \frac{|x_k y_k|}{\lambda \mu} \leq 1$$
$$\Rightarrow \sum |x_k y_k| \leq \lambda \mu = \|x\|_p \|y\|_q \text{ as claimed.}$$
$\square$

Theorem 2.3 (Minkowski's inequality)
Let $p \in [1, \infty]$. Then $\|x + y\|_p \leq \|x\|_p + \|y\|_p$.

Proof
Exercise.

$\therefore$ The p-norm is indeed a norm!

Examples cont.

2). Extend example 1 to infinite sequences,

i.e. $x = (x_1, x_2, \dots)$, where $x_k \in \mathbb{R}$ or $\mathbb{C}$

Define the set l_p, $p \in [1, \infty]$ as the set of all sequences x s.t. the series

$$\|x\|_p = \left[\sum_{k=1}^{\infty} |x_k|^p\right]^{1/p} \quad \text{converges.}$$

$p \geq 1$

[e.g. $x_k = \frac{1}{k}$ $k = 1, 2, \dots$, $x \in l_p \ \forall p > 1$, $x \notin l_1$]

$$\|x\|_\infty = \sup_k |x_k| < \infty.$$

Hölder's inequality, Minkowski inequality hold:

- If $x \in l_p$, $y \in l_q$, where p, q are conjugate, then the sequence $\{x_k y_k\}$ is in l_1 and
$$\sum_{k=1}^{\infty} |x_k y_k| \leq \|x\|_p \|y\|_q$$

- If $x, y \in l_p$, $p \geq 1$, then $x + y \in l_p$ and
$$\|x + y\|_p \leq \|x\|_p + \|y\|_p$$

This proves that l_p is a linear space, and $\|\cdot\|_p$ is a norm!

3). Let $c \subset l_\infty$ be the set of all convergent sequences

c is a subspace.

$c_0 \subset c \subset l_\infty$ is a subspace of sequences; converging to zero

e.g. $x = \{1, \frac{1}{2}, \frac{1}{3}, \dots\}$, $x_k = 1/k$, $k = 1, 2, \dots$

$c_{00} \subset c \subset l_\infty$ is a subspace of finite sequences,

i.e. $x \in c_{00}$ iff $x = (x_1, x_2, \dots, x_N, 0, 0, 0, \dots)$ with some $N = N(x)$.

Note: $c_{00} \subset l_p$, $\forall p \geq 1$.

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4). Continuous functions on $[a,b]$, $-\infty < a < b < \infty$.

Notation: $C[a,b] = (C[a,b], \|\cdot\|_{sup})$

Norm: $\|f\|_C = \sup_{x\in[a,b]} |f(x)| = \max_{x\in[a,b]} |f(x)|$ [cts. fns on $[a,b]$ attain their maximum value]

$C[a,b]$ is a normed space.

For $f \in C[a,b]$ define the norm

$$\|f\|_p = \left[\int_a^b |f(t)|^p \, dt\right]^{1/p}, \quad p \geq 1, \quad p \neq \infty.$$

non-degenerate & homogeneous

As for sequences, Hölder's inequality holds,

$$\int_a^b |f(t)g(t)|^p dt \leq \left(\int_a^b |f(t)|^p dt\right)^{1/p} \left(\int_a^b |g(t)|^p dt\right)^{1/p} \text{ where } \frac{1}{p} + \frac{1}{q} = 1.$$

Also, $\|f+g\|_p \leq \|f\|_p + \|g\|_p$, $p \geq 1$.

<u>Topology of metric spaces</u> (see handout)

$B(x_0, r)$ open ball

$B[x_0, r]$ closed ball

<u>Examples</u>

1). $A = (0,1) \subset \mathbb{R}$ A is open $A =$ its interior (A°)

Acc. points of $A = [0,1]$

$B = (0,1) \cup \{2\}$, $B^\circ = (0,1)$ (interior)

$\{2\}$ is an isolated point.

Acc. points of $B = [0,1]$ $\{2\}$ is not an accumulation point since no elements from the set approach it without hitting it.

$(\overline{B} =) [B] = [0,1] \cup \{2\}$ (closure)

2). $c_{00} \subset \ell_p$, $p \in [1, \infty)$

Claim: $[c_{00}] = \ell_p$

Proof

Need to prove $[c_{00}] \subset l_p$ and $l_p \subset [c_{00}]$

$[c_{00}] \subset l_p$ is trivial.

Let $x \in l_p$. Want to show that x is an accumilation point of c_{00}.

Precisely, want to show that $\forall \varepsilon > 0 \ \exists y \in c_{00}$, s.t. $\|x - y\|_p < \varepsilon$.

Let $x = (x_1, x_2, \dots)$.

Let N be the number s.t. $\left[\sum_{k=N+1}^{\infty} |x_k|^p\right]^{1/p} < \varepsilon$.

Such N exists, since $x \in l_p$.

Define $y = (x_1, x_2, \dots, x_N, 0, 0, \dots)$.

Then $\|x - y\|_p = \|(\underbrace{0, 0, \dots, 0}_{N}, x_{N+1}, x_{N+2}, \dots)\|_p$

$$= \left[\sum_{k=N+1}^{\infty} |x_k|^p\right]^{1/p} < \varepsilon \quad \text{as required.}$$

Thus x is an accumilation point of c_{00}. □

3). What if $p = \infty$?

If $c_{00} \subset l_\infty$, ie. $\|x\|_\infty = \sup_k |x_k|$,

then $[c_{00}] = c_0$.

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l_p, $p \in [1, \infty]$

$\|x\|_p = \left[\sum_{k=1} |x_k|^p\right]^{1/p}$, $1 \leq p < \infty$

$\|x\|_\infty = \sup_k |x_k|$

c space of all sequences which have a limit.

c_0 " " " " " tend to zero.

c_{00} " " " " " have finitely many non-zero terms.

$[c_{00}] = l_p$, $c_{00} \subset l_p$, $1 \leq p < \infty$.

$[c_{00}] = c_0$, $c_{00} \subset l_\infty$

Proof

NTP: $^{(i)}[c_{00}] \subset c_0$, $^{(ii)} c_0 \subset [c_{00}]$

(i) Let $x \in l_\infty$ be an accumilation point of c_{00}, i.e.

$\forall \varepsilon > 0 \ \exists y \in c_{00}$ s.t. $y \neq x$, $\|x-y\|_\infty < \varepsilon$.

Want: $\forall \varepsilon > 0 \ \exists N$ s.t. $\sup_{n > N} |x_n| < \varepsilon$, i.e. $x_n \to 0$ as $n \to \infty$.

Since $y = (y_1, y_2, \dots, y_N, 0, 0, \dots)$, we have

$\sup_{k \geq N+1} |x_k| = \sup_{k \geq N+1} |x_k - y_k| \leq \sup_{k \geq 1} |x_k - y_k| = \|x-y\|_\infty < \varepsilon$.

This means that $x_n \to 0$ as $n \to \infty$ as claimed.

(ii) Want: each $x \in c_0$ is an accumilation point of c_{00}. Fix an $\varepsilon > 0$. and find N: $\sup_{n \geq N+1} |x_n| < \varepsilon$.

Define $\tilde{x} = (x_1, x_2, \dots, x_N, 0, 0, \dots) \in c_{00}$ and therefore

$\|x - \tilde{x}\|_\infty = \sup_{k \geq 1} |x_k - \tilde{x}_k| = \sup_{k \geq N+1} |x_k| < \varepsilon$, so $x \in [c_{00}]$

as claimed.

$\square$

Dense sets, seperability

Def 2.4

Let (X, ρ) be a metric space.

(i) Let $A, B \subset X$. We say that A is dense in B if $B \subset [A]$

(ii) We say that A is nowhere dense, if for every open ball $B \subset X$ we have $B \not\subset [A]$, i.e. for every open ball B there is another open ball $B' \subset B$, we have $B' \cap A = \emptyset$

(iii) The space X is said to be seperable if it contains a countable dense subset,
i.e. $\exists$ countable set $A \subset X$ s.t. $[A] = X$.

Examples

1). $X = \mathbb{R}$, $\rho(x,y) = |x-y|$
Since $[\mathbb{Q}] = \mathbb{R}$, $\mathbb{R}$ is seperable

2). $\mathbb{R}^n$ is seperable $(n \geq 1)$ since $[\mathbb{Q}^n] = \mathbb{R}^n$

3). l_p, $1 \leq p < \infty$, is seperable.

Proof

Fix an $\varepsilon > 0$. As $[c_{00}] = l_p$, for every $x \in l_p$ there is a sequence $\tilde{x} \in c_{00}$ s.t. $\|x - \tilde{x}\|_p < \varepsilon/2$.

Write $\tilde{x} = (\tilde{x}_1, \tilde{x}_2, \dots, \tilde{x}_N, 0, 0, \dots)$.

There is a sequence $y = (y_1, y_2, \dots, y_N, 0, 0, \dots)$ with rational elements s.t. $\|\tilde{x} - y\|_p < \varepsilon/2$.

Therefore, $\|x - y\|_p \leq \|x - \tilde{x}\|_p + \|\tilde{x} - y\|_p < \varepsilon/2 + \varepsilon/2 = \varepsilon$

The set of finite sequences with rational elements is countable, so l_p is seperable. $\square$

4). l_∞ is not seperable.

Proof

Assume l_∞ is seperable, i.e. let A be countable and let $[A] = l_\infty$.

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Let $M \subset l_\infty$ be the subset which consists of sequences containing only "1" and "0", e.g. $(1,0,0,1,1,1,0,\dots)$

Let $x, y \in M$, $x \neq y$, so $\|x-y\|_\infty \geq 1$, and hence $B(x, 1/2) \cap B(y, 1/2) = \phi$.

Since $[A] = l_\infty$, for each $x \in M$ there is an element $\tilde{x} \in A$ s.t. $\tilde{x} \in B(x, 1/2)$.

Since M is not countable, the set of all such elements $\tilde{x}$ is also uncountable. This gives a contradition, since A was assumed to be countable. $\square$

Q: Is c_0 seperable (as a subspace of l_∞)?

A: Yes, because $[c_{00}] = c_0$!

5). $X = C[a,b]$, $\|f\|_c = \max_{a \leq t \leq b} |f(t)|$.

X is seperable.

Proof

Fix an $\varepsilon > 0$. By the Weierstrass Thm, for every $f \in C[a,b]$ there is a polynomial g s.t. $\|f-g\|_c < \varepsilon/2$.

Write $g(t) = a_n t^n + a_{n-1} t^{n-1} + \dots + a_1 t + a_0$.

Thus there are $n+1$ rational coefficients $\tilde{a}_n, \dots, \tilde{a}_0$ st. the polynomial $\tilde{g}(t) = \tilde{a}_n t^n + \dots + a_0$ satisfies $\|g - \tilde{g}\|_c < \varepsilon/2$.

Therefore $\|f-\tilde{g}\|_c \leq \|f-g\| + \|g-\tilde{g}\|_c < \varepsilon/2 + \varepsilon/2 = \varepsilon$.

Thus X is seperable, as claimed. $\square$

Complete metric spaces

Reminder:

Let $\{x_j\} \subset X$.

- We say $x_j \to x$ if $\rho(x, x_j) \to 0$ as $j \to \infty$
- We say that the sequence $\{x_j\}$ is Cauchy if $\rho(x_n, x_m) \to 0$ as $n, m \to \infty$

If $\{x_j\}$ converges then it is Cauchy. The converse is not true in general.

If every Cauchy sequence converges in X then X is said to be complete.

Example

1). $X = (0,1)$, $\rho(x,y) = |x-y|$.

Let $x_j = \frac{1}{j}$, $j = 1, 2, \dots$

This sequence converges to 0, and hence, it does not have a limit in X (it is a Cauchy sequence in X however).

Thus X is not complete.

$[0,1]$ is complete $\left[\tilde{X} = [0,1]\right]$

A complete normed space is called a Banach space.

Look at the series $\sum_{k=1}^{\infty} x_k$.

Definition: $\sum_{k=1}^{\infty} x_k = \lim_{N \to \infty} \sum_{k=1}^{N} x_k$

Prop 2.5

Let X be a Banach space. Let $\{x_i\} \subset X$ be a sequence s.t. $\sum_{k=1}^{\infty} \|x_k\|$ converges.

Then the series $\sum_{k=1}^{\infty} x_k$ converges in X, ie. the limit $a = \lim_{N \to \infty} \sum_{k=1}^{N} x_k$ exists.

Proof

Define $a_m = \sum_{k=1}^{m} x_k$. Claim: the sequence $\{a_m\}$ is Cauchy.

Indeed, wlog take $m < n$, then

$$\|a_n - a_m\| = \left\| \sum_{k=m+1}^{n} x_k \right\| \underset{\Delta\text{-inequality}}{\leq} \sum_{k=m+1}^{n} \|x_k\| \longrightarrow 0 \text{ as } n, m \to \infty$$

since $\sum_{k=1}^{\infty} \|x_k\|$ converges.

By completeness of X, the sequence a_n has a limit as $n \to \infty$. $\square$

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Examples cont.

2). $C[a,b]$ is complete.

$\{f_n\}$ is Cauchy if $\max_t |f_n(t) - f_m(t)| \to 0$, $n,m \to 0$.

We know from Analysis 4 that such a sequence converges (uniformly) to a continuous function.

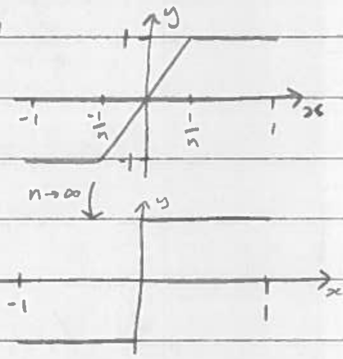
Therefore $C[a,b]$ is complete.

3). $X = C_p[a,b]$, $1 \le p < \infty$, is not complete.

$\|f\|_p = \left[\int_a^b |f(t)|^p dt\right]^{1/p}$.

Proof

Want to construct a sequence $f_n \in C_p[a,b]$ which is Cauchy, but doesn't have a limit.



Let $a=-1$, $b=1$.

Define $f_n(t) = \begin{cases} -1 & , -1 \le t \le -\frac{1}{n} \\ nt & , -\frac{1}{n} \le t \le \frac{1}{n} \\ 1 & , \frac{1}{n} \le t \le 1 \end{cases}$

Observe: $|f_m - f_n| \le 1$

$f_n(t) - f_m(t) = 0$ for $|t| \geq \frac{1}{m}$ $(n > m)$

Estimate:

$$\int_{-1}^{1} |f_n(t) - f_m(t)|^p dt = \int_{-\frac{1}{m}}^{\frac{1}{m}} |f_n(t) - f_m(t)|^p dt$$

$$\le \int_{-\frac{1}{m}}^{\frac{1}{m}} dt = \frac{2}{m} \to 0 \text{ as } n,m \to \infty$$

So f_n is Cauchy, but it converges pointwise to the step function $f(t) = \begin{cases} -1 & , -1 \le t < 0 \\ 1 & , 0 < t \le 1 \end{cases}$.

$\square$

4). l_p, $p \in [1, \infty]$, is complete.

Proof

Let $p \in [1, \infty)$

Let $\{x^{(n)}\}$ be a Cauchy sequence in l_p.

$\longrightarrow$

$x^{(n)} = (x_1^{(n)}, x_2^{(n)}, x_3^{(n)}, \dots)$,

ie. $\forall \varepsilon > 0 \ \exists N$ st. $\|x^{(n)} - x^{(m)}\|_p < \varepsilon$ if $n, m > N$.

ie. (*) $\sum_{k=1}^{\infty} |x^{(n)} - x^{(m)}|^p < \varepsilon^p$, if $n, m > N$.

This means that sequences $\{x_k^{(n)}\}$ are Cauchy for each $k = 1, 2, \dots$.

Thus they converge, ie.

$x_k = \lim_{n \to \infty} x_k^{(n)}$. I claim $x = (x_1, x_2, \dots) \in l_p$, and

$x^{(n)} \to x$, $n \to \infty$ in l_p.

It follows from (*):

$\sum_{k=1}^{M} |x_k^{(n)} - x_k^{(m)}|^p < \varepsilon^p$, $n, m > N$ for arbitrary finite M.

Take $m \to \infty$: $\sum_{k=1}^{M} |x_k^{(n)} - x_k|^p \leq \varepsilon^p$

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4). l_p, $1 \leq p < \infty$ is complete.

Proof

$x^{(n)} = (x_1^{(n)}, x_2^{(n)}, \dots)$ Cauchy sequence (of sequences).

ie. $\forall \varepsilon > 0 \ \exists N$ st. $\|x^{(n)} - x^{(m)}\|_p < \varepsilon$, $n, m > N$ $[N = N_\varepsilon]$

ie. (*) $\sum_{k=1}^{\infty} |x_k^{(n)} - x_k^{(m)}|^p < \varepsilon^p$, $n, m > N$

Thus $\{x_k^{(n)}\}$ is Cauchy for each k.

Denote $x_k = \lim_{n \to \infty} x_k^{(n)}$.

Let $x = (x_1, x_2, \dots)$.

Need to show that $x \in l_p$ and $x^{(n)} \to x$ as $n \to \infty$ in l_p.

It follows from (*):

$\sum_{k=1}^{M} |x_k^{(n)} - x_k^{(m)}|^p < \varepsilon^p$, $n, m > M$ for any finite M.

Pass to the limit as $m \to \infty$:

$\sum_{k=1}^{M} |x_k^{(n)} - x_k|^p \leq \varepsilon^p$, $n > N$. (**)

By Minkowski's inequality,

$$\left[\sum_{k=1}^{M} |x_k|^p\right]^{1/p} \leq \left[\sum_{k=1}^{M} |x_k^{(n)} - x_k|^p\right]^{1/p} + \left[\sum_{k=1}^{M} |x_k^{(n)}|^p\right]^{1/p} \leq \varepsilon + \|x^{(n)}\|_p$$

$\forall M$ and $n > N$.

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Therefore $x \in l_p$ as required.

Pass to the limit in $(**)$:

$$\left[\sum_{k=1}^{\infty} |x_k^{(n)} - x_k|^p \right]^{1/p} \le \varepsilon \quad , \quad n > N$$

$$\| x^{(n)} - x \|_p$$

Thus $x^{(n)} \to x$ in l_p, as claimed. $\square$

Def

Let $K_j = B[x_j, r_j]$ be a family of closed balls.
We say that $\{K_j\}$ is a nested sequence of closed balls if $K_1 \supset K_2 \supset K_3 \supset \dots$

Lemma 2·6

Let X be a metric space. Then the following two statements are equivalent:

(i) X is complete

(ii) For any nested sequence of closed balls $\{K_j\}$ s.t. $r_j \to 0$ as $j \to \infty$, the intersection $K = \bigcap_j K_j$ is not empty.

Proof

(i) $\Rightarrow$ (ii) [(ii) $\Rightarrow$ (i) omitted and not examinable]
will be in online notes

Take the sequence $\{x_j\}$. By definition

$\rho(x_j, x_m) \le r_j$ if $m \geq j$.

Since $r_j \to 0$ as $j \to \infty$, $\{x_j\}$ is Cauchy.

By completeness, $\{x_j\}$ has a limit $x = \lim_{j\to\infty} x_j$

For any n the limit x belongs to K_n, since K_n, being a closed set, contains all its accumulation points.

Thus $x \in \bigcap_n K_n = K$ as claimed.

$\square$

Theorem 2.7 (The Baire Category Theorem)
A complete metric space X cannot be represented as a finite or countable union of nowhere dense sets.

Reminder: We say that A is nowhere dense if for any closed ball K of positive radius there is another closed ball $K' \subset K$ of positive radius s.t. $K' \cap A = \emptyset$.

Proof (by contradiction).
Suppose that $X = \bigcup_k A_k$, where each A_k is nowhere dense.
Since A_1 is nowhere dense, there is a ball K_1 s.t. $A_1 \cap K_1 = \emptyset$. Assume $r_1 \leq 1$.
Since A_2 is nowhere dense, there is a ball $K_2 \subset K_1$ s.t. $A_2 \cap K_2 = \emptyset$. Assume $r_2 \leq 1/2$
⋮
Since A_n is nowhere dense, there is a ball $K_n \subset K_{n-1}$ s.t. $K_n \cap A_n = \emptyset$. Assume $r_n \leq 1/n$.
Thus $K_1 \supset K_2 \supset K_3 \supset \dots \supset K_n \supset \dots$, with $r_n \to 0$ as $n \to \infty$.
By Lemma 2.6 the set $K = \bigcap_{j=1}^{\infty} K_j$ is non-empty.
Let $x \in K$. By construction, $x \notin A_k$, $k = 1, 2, \dots$, and hence $x \notin \bigcup_{k=1}^{\infty} A_k = X$.
This contradiction proves the claim. $\square$

Remark
A complete metric space without isolated points cannot be countable.
Main point is that every one point set $\{x\}$, $x \in X$, is nowhere dense.

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Completion of a metric space

Recall:

Let (X, ρ) and (Y, μ) be two metric spaces.

Let $f: X \to Y$

- $\forall \varepsilon > 0 \ \exists \delta > 0$ s.t. $\mu(f(x), f(x_0)) < \varepsilon$ if $\rho(x, x_0) < \delta$, then f is continuous at $x_0 \in X$.
- f is continuous on X if f is continuous at every $x_0 \in X$.
- f is cont on X if for every open set $A \subset Y$, the set $f^{-1}(A)$ is open.
- If f is a bijection and f & f^{-1} are continuous then f is called a homeomorphism.

Example

$X = \mathbb{R}$, $Y = (-1, 1)$, $f(x) = \frac{2}{\pi} \arctan x$.

Definition 2.8

We say that $f: X \to Y$ is an isometry if

$\mu(f(x), f(y)) = \rho(x, y)$.

(Isometries are continuous and injective).

If f is a bijection and an isometry, then f is called an isomorphism. In this case, X and Y are isomorphic

Example

Let P_n be the space of all polynomials of degree n on the interval $[-1, 1]$:

$g(t) = a_n t^n + a_{n-1} t^{n-1} + \dots + a_0$.

metric : $\rho(g, \tilde{g}) = \left(\sum_{k=0}^{n} |a_k - \tilde{a}_k|^2 \right)^{1/2}$

This space is isomorphic to $\mathbb{R}^{n+1}$:

$W(g) = (a_0, a_1, \dots, a_n)$

Definition 2.9

A complete metric space $\tilde{X}$ is said to be the completion of the space X if there exists an isometry $\varphi : X \to \tilde{X}$ s.t. $[\varphi(X)] = \tilde{X}$.

Theorem 2.10

Any metric space has a completion.
All completions are isomorphic to each other.

Proof

Not examinable, will be in online notes (also see Friedman).

Definition

cont. fns with integral norm

The completion of $C_p[a,b]$ is called $L_p(a,b)$. $(1 \le p < \infty)$
(Continuous functions are dense in L_p).

Compactness

Let X be a metric space.

- A set $M \subset X$ is said to be relatively compact if every sequence in M contains a convergent subsequence.
- If all possible limits belong to M then M is called compact. (sequentially compact (Analysis 4)).
- Any finite set is compact.
- We will always assume that M is infinite.
- Equivalently M is relatively compact if every infinite subset of M has an accumulation point.
- If M is relatively compact, then $[M]$ is compact.

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- M is compact if every open cover of M contains a finite subcover.
- A compact set M is bounded and closed.
- If $X = \mathbb{R}^n$, then M compact $\Leftrightarrow$ M is closed and bounded.

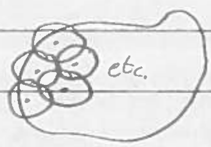
Criterion of compactness

Definition 2.11
Let $M \subset X$, and let $\varepsilon > 0$.
Then a set $N \subset X$ is called an ε-net for the set M if $M \subset \bigcup_{x \in N} B(x, \varepsilon)$

22-01-18 Recall: Compactness
- $M \subset X$ is relatively compact if every infinite sequence in M has a convergent subsequence.
- M is compact if every open cover of M admits a finite subcover.

Definition 2.11
Let $M \subset X$, and let $\varepsilon > 0$. Then a set $N \subset X$ is called an ε-net of M if $M \subset \bigcup_{x \in N} B(x, \varepsilon)$

(ε-net is the set of centres)



Theorem 2.12
If M is relatively compact, then for any $\varepsilon > 0$, M admits a finite ε-net.
If X is complete, then the converse is also true.
(i.e. X complete $\Rightarrow$ if $\forall \varepsilon > 0$ M admits a finite ε-net, then M is relatively compact).

Proof

Let M be relatively compact. Then $[M]$ is compact, and hence any open cover admits a finite subcover. The union $\bigcup_{x \in [M]} B(x, \varepsilon)$ is a cover for M.

A finite subcover provides a finite ε-net, as claimed.
Suppose now that X is complete and that for any $\varepsilon > 0$ $\exists$ a finite ε-net.
Let $\varepsilon_n > 0$, $\varepsilon_n \to 0$.
Let $N_n = \{x_1^{(n)}, x_2^{(n)}, \dots, x_{k_n}^{(n)}\}$ be an ε_n-net.
Let $T \in M$ be a sequence.
At least one of the balls $B(x_k^{(1)}, \varepsilon_1)$ contains infinitely many elements of T:
$T^{(1)} = T \cap B(x_j^{(1)}, \varepsilon_1)$
Do the same for $n=2$: $T^{(2)} = T^{(1)} \cap B(x_j^{(2)}, \varepsilon_2)$.
Repeat the construction for every n:
$T^{(n)} = T^{(n-1)} \cap B(x_j^{(n)}, \varepsilon_n)$.
Thus $T^{(1)} \supset T^{(2)} \supset T^{(3)} \supset \dots$
Take $\xi_n \in T^{(n)}$. Therefore $\rho(\xi_n, \xi_m) < 2\varepsilon_n$, if $m > n$.
Since $\varepsilon_n \to 0$, as $n \to \infty$ $\Rightarrow$ $\{\xi_n\}$ is Cauchy.
As X is complete, $\{\xi_n\}$ has a limit $\xi = \lim_{n\to\infty} \xi_n$
Thus M is relatively compact. $\square$

Corollary 2.13

If M is relatively compact, it is bounded.
(i.e. $\forall x_0 \in X$, $\exists r > 0$ s.t. $M \subset B(x_0, r)$).

Proof

Let $N_1 = \{x_1, x_2, \dots, x_n\}$ be a 1-net. Denote $d = \max_{1\le k\le n} \rho(x_0, x_k)$.
Then if $x \in B(x_k, 1)$, then $\rho(x_0, x) \le \rho(x_0, x_k) + \rho(x_k, x) \le d+1$
and hence $M \subset B(x_0, d+1)$.
$\square$

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Corollary 2.14 Let X be complete.
The set M is relatively compact iff M admits a relatively compact ε-net $\forall \varepsilon > 0$.

Corollary 2.15
If the metric space X is compact, then it is seperable i.e. it contains a dense countable subset.

Proof
Since X is compact, it admits a finite ε-net for each $\varepsilon > 0$. Let $\varepsilon_n > 0$, $\varepsilon_n \to 0$.
Let $N_n = \{x_1^{(n)}, x_2^{(n)}, \dots, x_k^{(n)}\}$ be an ε-net.
Let $A = \bigcup_{n=1}^{\infty} N_n$. A is countable (countable union of finite sets).
Want to prove: each $x \in X$ is either an element of A or an accumulation point of A.
By definition of N_n, for each $x \in X$ there is a $y_n \in N_n$ s.t. $\rho(x, y_n) < \varepsilon_n$.
Therefore the sequence $\{y_n\}$ converges to x as $n \to \infty$, as required. □

Corollary 2.16
Let $M \subset \mathbb{R}^n$. Then M is compact iff it is bounded and closed.

Proof
[⇒]
Compact ⇒ bounded by corollary 2.13
By compactness, $M = [M]$.
[⇐]
Suppose M is bounded and closed.
Let Q be a cube st. $M \subset Q$.
Q can be covered by finitely many cubes of size ε

for each $\varepsilon > 0$.
Thus by theorem 2.12, M is relatively compact.
Relatively compact + closed = compact, as claimed. $\square$

Examples
1). Let $X = l_p$, $p \in [1, \infty]$
The set $B[0,1]$ is not compact.
Indeed, let $x_1 = (1, 0, 0, \ldots)$
$x_2 = (0, 1, 0, \ldots)$
$\vdots$
nth place.
$x_n = (0, 0, \ldots, 1, 0, \ldots)$
Then $\|x_n - x_m\|_p = (1^p + 1^p)^{1/p} = 2^{1/p}$, $p < \infty$
$\|x_n - x_m\|_\infty = 1$ if $m \neq n$
This sequence cannot have a convergent subsequence.
Thus $B[0,1]$ is not compact.

2). Let $M \subset l_2$ be the set $M = \{x \in l_2 : |x_k| \leq 1/k\}$
Assume that l_2 is the space of real sequences.
Claim: M is compact.
Proof
M is relatively compact.
We need to find a finite ε-net for every $\varepsilon > 0$.
Let n be a number s.t.
$\left(\sum_{k=n+1}^{\infty} |x_k|^2\right)^{1/2} < \varepsilon/2$ for all $x \in M$.

This is possible since $\sum_{k=n+1}^{\infty} |x_k|^2 \leq \sum_{k=n+1}^{\infty} \frac{1}{k^2} \leq \frac{1}{n}$.

In other words, if $\tilde{x} = (x_1, x_2, \ldots, x_n, 0, 0, \ldots)$
then $\|x - \tilde{x}\|_2 < \varepsilon/2$.
View $\tilde{x}$ as a vector in $\mathbb{R}^n$.
Let $S \subset \mathbb{R}^n$ be the set $\{z \in \mathbb{R}^n : \max_{1 \leq k \leq n} |z_k| \leq 1\}$

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The set S is bounded in $\mathbb{R}^n \Rightarrow S$ admits a finite $\varepsilon/2$-net: $N_{\varepsilon/2} = (z^{(1)}, z^{(2)}, \dots, z^{(m)})$, $m < \infty$,
i.e. there is a vector $z \in N_{\varepsilon/2}$ s.t. $\|\tilde{x} - z\|_2 < \varepsilon/2$.
Claim: $N_{\varepsilon/2}$ is an ε-net for M:

$$\|x - z\|_2 \le \|x - \tilde{x}\|_2 + \|\tilde{x} - z\|_2$$
$$< \varepsilon/2 + \varepsilon/2 = \varepsilon$$

Conclusion is that M is relatively compact.
To prove compactness, we need to show that $M = [M]$.
Let $x \in l_2$ be an accumulation point of M, i.e.
there is a sequence $x^{(n)} \in M$, $x^{(n)} \neq x$ s.t. $x^{(n)} \to x$, as $n \to \infty$.
This means that $\sum_{k=1}^{\infty} |x_k^{(n)} - x_k|^2 \to 0$ as $n \to \infty$.
Thus $x_k^{(n)} \to x_k$, as $n \to \infty$, for all k.
Since $|x_k^{(n)}| \le 1/k$, we also have $|x_k| \le 1/k$,
so $x \in M$.
$\square$

<u>Continuous functions on compact spaces.</u>

- Let $f: X \to Y$ be continuous. Then for any compact set $M \subset X$, the image $f(M)$ is also compact.
- Let X be compact, and let $f: X \to \mathbb{R}$ be continuous. Then f is bounded on X and it attains its maximum and minimum values.

<u>Proof</u>

Suppose that $|f|$ is not bounded on X,
i.e. for all $n \in \mathbb{N}$ there is a point $x_n \in X$ s.t.
$|f(x_n)| > n$. Let $\{x_{n_k}\}$ be a convergent subsequence of $\{x_n\}$, so $x_{n_k} \to x$ as $k \to \infty$.
By continuity of $|f|$, $|f(x_{n_k})| \to |f(x)|$ as $k \to \infty$.
The sequence $|f(x_{n_k})|$, $k = 1, 2, \dots$ is bounded,
but $|f(x_{n_k})| > n_k \to \infty$ as $k \to \infty$.
The contradiction $\Rightarrow |f|$ is bounded on X.
Max and min values: left as exercise.
$\square$

• Let $f: X \to Y$ be continuous. If X is compact, then f is uniformly continuous on X.

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$\forall \varepsilon > 0 \; \exists$ finite ε-net $\longrightarrow$ the set is rel. compact (if space is complete)

Continuous functions on compact spaces

Let X be compact space.
Let $f: X \to Y$ be continuous, then it is uniformly continuous.
$X = (X, \rho)$, $Y = (Y, \mu)$
Continuity:
f is continuous at $x \in X$ if $\forall \varepsilon > 0 \; \exists \delta > 0$ s.t.
$\mu(f(x), f(y)) < \varepsilon$ if $\rho(x, y) < \delta$. $\Big\}(*)$
δ depends on x, ε, f.

f is uniformly continuous on X if $(*)$ holds with a δ independent of x.

Let X be a compact space. Then the set of real-valued continuous functions on X with the norm $\|f\|_C = \max_{x \in X} |f(x)|$, is a banach space.

Proof
Let $\{f_n\}$ be Cauchy i.e. $\forall \varepsilon > 0 \; \exists N$ s.t. $\|f_n - f_m\| < \varepsilon/3$ if $n, m > N$.
i.e. $\max_{x \in X} |f_n(x) - f_m(x)| < \varepsilon/3$.
Thus for each $x \in X$ the sequence $f_n(x)$ converges to some number, denoted $f(x)$, so $\sup_{x \in X} |f_n(x) - f_m(x)| \leq \varepsilon/3$, $n > N$
Want: f is continuous on X.
Write: $|f(x) - f(y)| \leq |f(x) - f_n(x)| + |f_n(x) - f_n(y)| + |f_n(y) - f(y)|$.
Let $n > N$, so that terms 1 and 3 are bounded by $\varepsilon/3$.
Since f_n is continuous, there is a $\delta > 0$ s.t. $|f_n(x) - f_n(y)| < \varepsilon/3$ if $\rho(x, y) < \delta$.

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Therefore,

$|f(x) - f(y)| < \varepsilon/3 + \varepsilon/3 + \varepsilon/3 = \varepsilon$ if $\rho(x,y) < \delta$.

So f is continuous on $X \Rightarrow$ bounded and $\|f\|_C < \infty$

$\square$

Compactness in function spaces

Focus on $C(X, \mathbb{R})$

Definition 2.17

Let $A \subset X$. A family M of continuous functions on A is said to be equicontinuous if

$\forall \varepsilon > 0 \ \exists \delta$ s.t. $|f(x) - f(y)| < \varepsilon$ if $\rho(x,y) < \delta$,

$\forall x \in A$, $f \in M$. (δ is independent of $x \in A$ and of $f \in M$)

The family M is said to be pointwise bounded on A if the set $\{f(x), f \in M\}$ is bounded for each $x \in A$.

If M is an equicontinuous pointwise bounded family on a compact space X, then M is a bounded set, i.e. $\sup_{f \in M} \|f\| < \infty$

Theorem 2.18 (Arzela-Ascoli thm)

Let X be a compact space. Then a subset $M \subset C(X, \mathbb{R})$ is relatively compact in $C(X, \mathbb{R})$ iff M is equicontinuous and pointwise bounded.

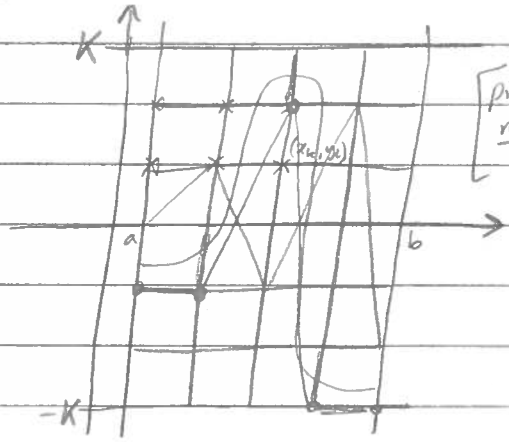
Proof [$\Leftarrow$ only.]

For simplicity assume $X = [a,b] \subset \mathbb{R}$. Let $M \subset C([a,b])$ be equicontinuous and pointwise bounded, so

$\sup_{f \in M} \|f\|_C \leq K < \infty$.

Want to prove that M is relatively compact.
Equicontinuity: $\forall \varepsilon > 0 \ \exists \delta > 0$ st. $|f(x) - f(y)| < \varepsilon/5$
if $|x-y| < \delta \quad \forall \ x,y \in [a,b], \ \forall f \in M$.

Partition the rectangle $[a,b] \times [-K, K]$
into equal smaller rectangles of
sizes $(< \delta) \times (< \varepsilon/5)$
Vertices (x_k, y_l) with
$a = x_0 < x_1 < \dots < x_n = b$
$-K = y_0 < y_1 < \dots < y_L = K$



We call a function $g \in C[a,b]$ piecewise linear if
it is linear on every interval $[x_k, x_{k+1}]$, and
$g(x_k) = y_p$ with some $p = 0, 1, \dots, L$.

There are finitely many such functions. Want to show that
they form an ε-net.
For each $f \in M$ we find a function g (piecewise linear):
Clearly, $f(x_k) \in [y_l, y_{l+1}]$ for some l.
Define g to satisfy $g(x_k) = y_l$
Thus $|f(x_k) - g(x_k)| < \varepsilon/5$ for all k.
Recall: $|f(x_k) - f(x_{k+1})| < \varepsilon/5$.
Therefore, $|g(x_k) - g(x_{k+1})| \le |g(x_k) - f(x_k)| + |f(x_k) - f(x_{k+1})| + |f(x_{k+1}) - g(x_{k+1})|$
$< \varepsilon/5 + \varepsilon/5 + \varepsilon/5 = \frac{3\varepsilon}{5}$.
Since g is linear on $[x_k, x_{k+1}]$, we also have
$|g(x_k) - g(x)| < 3\varepsilon/5, \quad x \in [x_k, x_{k+1}]$
Claim: $|f(x) - g(x)| < \varepsilon \ \forall \ x \in [a,b]$, i.e. $\|f - g\|_C < \varepsilon$.
Write: $|f(x) - g(x)| \le |f(x) - f(x_k)| + |f(x_k) - g(x_k)| + |g(x_k) - g(x)|$
(assuming $x \in [x_k, x_{k+1}]$) $< \varepsilon/5 + \varepsilon/5 + 3\varepsilon/5 = \varepsilon$
Conclusion: the set of piecewise linear functions forms a
finite ε-net for M.
Thus M is relatively compact, as required. $\square$

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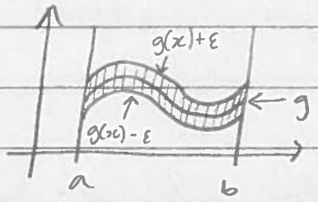
$g \in C[a,b]$, $B(g, \varepsilon)$

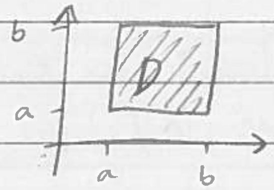


Illustration

Integral Operator.

Let $D = [a,b] \times [a,b]$ (compact)

Let $\mathcal{K} \in C(D, \mathbb{R})$.



Define the integral operator K:

$$(Kf)(x) = \int_a^b \mathcal{K}(x,y) f(y)\, dy$$

Theorem 2.19

Let $S \subset C[a,b]$ be such that $\|f\|_C \le R$, $f \in S$, i.e. $S \subset B[0,R]$ with some $R > 0$.

Then $K(S)$ is relatively compact in $C[a,b]$.

Proof

Want: $K(S)$ is equicontinuous and uniformly bounded.

Boundedness: $\forall f \in S$:

$$|(Kf)(x)| \le \int_a^b |\mathcal{K}(x,y)||f(y)|\, dy \le R_1 \int_a^b |f(y)|\, dy$$

$$\le R_1 R (b-a)$$

$\Rightarrow K(S)$ is bounded

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$\mathcal{K} \in C(D)$, $D = [a,b] \times [a,b]$

$$(Kf)(x) = \int_a^b \mathcal{K}(x,y) f(y)\, dy$$

$\mathcal{K}$ is called the integral kernel of the integral operator K.

Theorem 2.19

Let $S \subset C[a,b]$ be such that $\sup_{f \in S} \|f\|_C \le R$, with some $R > 0$.
Then the set $K(S)$ is relatively compact in $C[a,b]$.

Proof

Uniform boundedness of $K(S)$:

$$|(Kf)(x)| \le \int_a^b \underbrace{|\mathcal{K}(x,y)|}_{\le R_1} \underbrace{|f(y)|}_{\le R} dy$$

$$\le R_1 R \int_a^b dy = R_1 R (b-a) \quad \forall x \in [a,b], \ \forall f \in S$$

$\mathcal{K}$ is uniformly continuous on D:

$\forall \varepsilon > 0 \ \exists \delta$ s.t. $|\mathcal{K}(x_1,y) - \mathcal{K}(x_2,y)| < \frac{\varepsilon}{R(b-a)}$ if $|x_1 - x_2| < \delta$

Equicontinuity of $K(S)$:

$$|(Kf)(x_1) - (Kf)(x_2)| \le \int_a^b |\mathcal{K}(x_1,y) - \mathcal{K}(x_2,y)||f(y)|\, dy$$

$$< \frac{\varepsilon}{R(b-a)} \cdot R \int_a^b dy = \frac{\varepsilon}{(b-a)} (b-a) = \varepsilon$$

if $|x_1 - x_2| < \delta$

By Arzelà-Ascoli, $K(S)$ is relatively compact, as claimed.

$\square$

§3 **Linear Spaces**

- Linear Independence [$\hookrightarrow$ independent, $\not\hookrightarrow$ dependent]
- $\text{span}(v_1, v_2, \dots, v_n)$
- l_p is infinite dimensional

Indeed the vectors $e_1 = (1, 0, \dots)$, $e_2 = (0,1,0,\dots)$, ..., $e_n = (0, \dots, 0, 1, 0, \dots)$, ($n$th position)
are linearly independent

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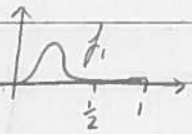
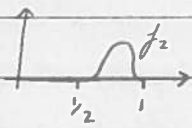
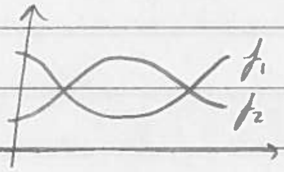
• $f_1, f_2 \in C[0,1]$





LI



LI (one is not a multiple of the other)

$x^2 + 2$ and x^3 are linearly independent.

Quotient space

Definition 3.1
Let $W \subset X$ be a subspace of the linear space X.
We say that $x \in X$ is equivalent to $y \in X$ if $x - y \in W$.
Let $\{x\}$ be the equivalence class of $x \in X$.
Then the set of all equivalence classes $\{x\}$ is called the quotient space X/W.
The addition and multiplication are defined as follows:
$\alpha\{x\} + \beta\{y\} = \{\alpha x + \beta y\}, \ \forall \alpha, \beta \in \mathbb{K}, \ \forall x, y \in X.$

Example
$X = \mathbb{R}^2, \ W = \{(\alpha, 0), \alpha \in \mathbb{R}\}$
Fix $x \in X$, $\{x\}$ = horizontal line
X/W = set of horizontal lines

Definition 3.2
Let $W \subset X$ be a subspace. Then the codimension of W is defined as the dimension of X/W.
$\operatorname{codim} W = \dim X/W$.

Example
If $\dim X = n$, $\dim W = k < n$, then $\operatorname{codim} W = n - k$.

• If $W \subset X$ (normed space) is a subspace, and W is closed, then call W a closed subspace of X.

• Let X be a normed space and $\|\cdot\|_1, \|\cdot\|_2$ are two norms on X. We say that these norms are equivalent if one can find two positive constants c_1, c_2 s.t.

$$c_1\|x\|_1 \leq \|x\|_2 \leq c_2\|x\|_1 , \text{ for all } x \in X.$$

Theorem 3.3

If $\dim X < \infty$, then all norms on X are equivalent to each other.

Proof (See Analysis 4).

Inner product spaces

Definition 3.4

Let X be a linear space. An inner product on X is a mapping $X \times X \longrightarrow \mathbb{K}$ with the following properties

(i) linearity: $\langle \alpha x + \beta y, z \rangle = \alpha \langle x, z \rangle + \beta \langle y, z \rangle \quad \forall \alpha, \beta \in \mathbb{K} \ \forall x, y, z \in X$

(ii) positive definite: $\langle x, x \rangle \geq 0$, $\langle x, x \rangle = 0 \Leftrightarrow x = 0$

(iii) symmetry: $\langle x, y \rangle = \overline{\langle y, x \rangle}$

Properties

$$\langle x, \alpha y + \beta z \rangle = \bar{\alpha} \langle x, y \rangle + \bar{\beta} \langle x, z \rangle$$

Thus the inner product is called a $1\frac{1}{2}$-linear form, or sesquilinear form

The functional $\|x\| = \sqrt{\langle x, x \rangle} \geq 0$ defines a norm on X. Indeed, $\|x\|$ is positive definite, $\|x\|$ is homogeneous. The Δ-inequality follows from Cauchy-Schwarz:

$|\langle x, y \rangle| \leq \|x\|\|y\| \ \forall x, y$, the equality holds iff x, y are linearly dependent.

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Δ-inequality: $\|x+y\| \leq \|x\| + \|y\|$.

- We say that x is orthogonal to y if $\langle x,y\rangle = 0$.
  Notation: $x \perp y$.
  If $x \perp y$, then $\|x+y\|^2 = \|x\|^2 + \|y\|^2$ (Pythagoras Thm).

Definition 3.5

An inner product space X is called a Hilbert Space if it is complete w.r.t. the norm induced by the inner product. If $\mathbb{K} = \mathbb{R}$, then X is a Euclidean Space.

Examples

1). Euclidean space $\mathbb{R}^n$: $\langle x,y\rangle = \sum_{k=1}^{n} x_k y_k$

2). Unitary space $\mathbb{C}^n$:

$$\langle x,y\rangle = \sum_{k=1}^{n} x_k \overline{y_k}$$

3). l_2 : $\langle x,y\rangle = \sum_{k=1}^{\infty} x_k \overline{y_k}$, $\forall x,y \in l_2$.

This series converges, since

$|\langle x,y\rangle| \leq (\sum |x_k|^2)^{1/2} (\sum |y_k|^2)^{1/2}$ by Hölder.

So l_2 is a Hilbert space.

4). $L_2(a,b)$: $\|f\|_2 = \left[\int_a^b |f(t)|^2 dt\right]^{1/2} < \infty$.

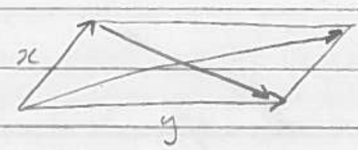
$\langle f,g\rangle = \int_a^b f(t)\overline{g(t)}\,dt$ $\leftarrow$ finite for all $f,g \in L_2(a,b)$.

Also a Hilbert Space.

Characteristic property of inner product spaces

Parallelogram equality:

(*): $\|x+y\|^2 + \|x-y\|^2 = 2(\|x\|^2 + \|y\|^2)$



Polarisation identity

(P1): $4\langle x,y\rangle = \|x+y\|^2 - \|x-y\|^2 + i\left(\|x+iy\|^2 - \|x-iy\|^2\right)$

(P2): If $\mathbb{K}=\mathbb{R}$, $4\langle x,y\rangle = \|x+y\|^2 - \|x-y\|^2$.

Theorem 3.6

Let X be a normed space then (P1) (or (P2)) defines an inner product on X iff the norm satisfies $(*)$.

Example

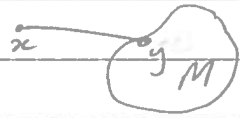
Let $X = l_p$, $1 \le p \le \infty$. $(*)$ holds $\forall\, x,y \in l_p$ iff $p=2$.
[hw 3 q 6]

Proof non-examinable, see online notes.

Projections

Distances to Sets

$\text{dist}(x, M) = \inf_{y\in M} \|x-y\|$



Want to find a minimising vector y s.t.
$\text{dist}(x,M) = \|x-y\|$.

Write $\delta(x,M) := \text{dist}(x,M)$.

Definition 3.7

We say that $M \subset X$ is a convex set if for any two points $x,y\in M$, the vector $tx+(1-t)y$ is also in M $\forall\, t\in[0,1]$.

Theorem 3.8

Let $M\subset H$ be a convex, closed subset of a Hilbert space H.
Then $\forall x\in H\ \exists!\ y\in M$ s.t. $\delta(x,M) = \|x-y\|$.

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Hilbert space, H

$\langle x, y \rangle$ inner product

$\|x\| = \sqrt{\langle x, x \rangle}$

$\|x+y\|^2 + \|x-y\|^2 = 2(\|x\|^2 + \|y\|^2)$ (parallelogram law)

$\delta(x, M) = dist(x, M) = \inf_{y \in M} \|x-y\|$

Minimising vector $y \in M : \|y-x\| = \delta(x, M)$

Theorem 3.8

Let $M \subset H$ be a closed convex set. Then for any $x \in H$ there exists a unique minimising vector $y \in M$, i.e. $\exists!\ y \in M$ s.t. $\|y-x\| = \delta(x, M)$.

Proof

By definition of $\delta(x, M)$, there is a sequence $y_n \in M$ s.t. $\delta_n = \|x-y_n\| \to \delta = \delta(x, M)$ as $n \to \infty$. Recall $\delta_n \geq \delta$.

WTS: y_n converges as $n \to \infty$.

Observe $\|(y_n + y_m)/2 - x\| \geq \delta$, by convexity.

By the parallelogram identity with $x - y_m$ and $x - y_n$:

$$\|2x - y_m - y_n\|^2 + \|y_n - y_m\|^2 = 2(\|x-y_m\|^2 + \|x-y_n\|^2) = 2(\delta_m^2 + \delta_n^2)$$

$$\|y_n - y_m\|^2 = -4\|x - (y_m+y_n)/2\|^2 + 2(\delta_m^2 + \delta_n^2)$$

$$\leq -4\delta^2 + 2(\delta_n^2 + \delta_m^2) \to 0 \text{ as } n, m \to \infty$$

Therefore $\{y_n\}$ is Cauchy.

As H is complete, $y_n \to y \in H$ as $n \to \infty$.

Since M is closed, $y \in M$. Claim: y is a minimising vector.

$$\delta \leq \|x-y\| \leq \underbrace{\|x-y_n\|}_{=\delta_n \to \delta} + \|y_n - y\| \to \delta \text{ as } n \to \infty$$

So $\delta \leq \|x-y\| \leq \delta \Rightarrow \|x-y\| = \delta$ as claimed.

Uniqueness: Suppose $\|x-y_1\| = \|x-y_2\| = \delta$, $y_1, y_2 \in M$

Then $\forall t \in [0, 1]$:

$$\delta \leq \|x - ty_1 - (1-t)y_2\| \leq t\|x - y_1\| + (1-t)\|x - y_2\|$$
$$= t\delta + (1-t)\delta = \delta$$

Therefore, by the parallelogram identity:

$$\|y_1 - y_2\|^2 = -4\|x - (y_1+y_2)/2\|^2 + 4\delta^2$$
$$= -4\delta^2 + 4\delta^2 = 0 \quad \Rightarrow y_1 = y_2$$

$\square$

For the set M take a closed subspace Y of H, obviously Y is convex.

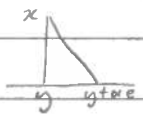
Theorem 3.9

Let Y be a closed subspace, let $x \in H$. Then $y \in Y$ is the minimising vector, i.e. $\|x-y\| = dist(x, Y)$ iff $x - y \perp Y$.

Proof

[$\Rightarrow$] Let y be the minimising vector. Assume that $x - y \not\perp Y$, i.e. $\exists$ a vector e st. $\langle x-y, e\rangle = \alpha \neq 0$.

Suppose wlog $\|e\| = 1$.



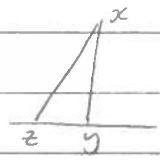
Calculate:

$$\|x - y - \alpha e\|^2 = \langle x-y-\alpha e, x-y-\alpha e\rangle$$
$$= \|x-y\|^2 - \alpha \underbrace{\langle e, x-y\rangle}_{=\bar{\alpha}} - \bar{\alpha}\overbrace{\langle x-y, e\rangle}^{\alpha} + |\alpha|^2\|e\|^2$$
$$= \|x-y\|^2 + |\alpha|^2 - 2|\alpha|^2 = \|x-y\|^2 - |\alpha|^2$$

Thus $x-y-\alpha e$ is shorter than $x-y$. $\#$

Therefore $\alpha = 0$, i.e. $x - y \perp Y$.

[$\Leftarrow$] Assume that $x - y \perp Y$.



Then for any vector $z \in Y$:

$$\|z - y\|^2 + \|x-y\|^2 = \|x - z\|^2$$
$$\Rightarrow \|x - z\| \geq \|x - y\|$$

So y is the minimising vector.

$\square$

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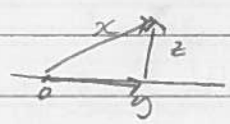
Definition 3.10
$\forall x \in H$, and any closed subspace Y, the uniquely defined minimising vector $y \in Y$ is called the orthogonal projection of x onto Y. Then the map $P: x \to y$ is called the orthogonal projection operator, so $Px = y$.

Rephrase:
$\forall x \in H$ and a closed subspace Y, there are uniquely defined vectors $y \in Y$ and $z \perp y$, s.t. $x = y + z$.



Alternatively $y = Px$, $z = (I-P)x$,
so $x = Px + (I-P)x$. By Pythagoras,
$\|Px\|^2 + \|(I-P)x\|^2 = \|x\|^2$.
Observe $\|Px\| \leq \|x\|$

Definition 3.11
A vector space X is said to be a direct sum of two subspaces X_1 and X_2 if every $x \in X$ is uniquely represented as $x = x_1 + x_2$ with $x_1 \in X_1$, $x_2 \in X_2$.
Notation: $X = X_1 \oplus X_2$

If $X_1 \oplus X_2 = X$, then $X_1 \cap X_2 = \{0\}$

If X is a Hilbert space, and $X_1 \perp X_2$, then the sum is said to be orthogonal

Definition 3.12
Let $Y \subset H$ be a closed subspace. Then the set $Y^\perp = \{z \in H : z \perp Y\}$ is called the orthogonal complement of Y.
Note: $Y^\perp$ is a closed subspace of H.

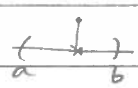
Indeed, let $z_n \to z$, with $z_n \in Y^\perp$, i.e.
$\langle z_n, y \rangle = 0 \quad \forall y \in Y$.
By Cauchy-Schwarz $|\langle z_n - z, y\rangle| \leq \|z_n - z\| \, \|y\| \to 0$ as $n \to \infty$
Therefore $\langle z, y\rangle - \langle z_n, y \rangle \to 0$ as $n \to \infty$,
so $\langle z, y \rangle = \lim\limits_{n\to\infty} \langle z_n, y \rangle = 0$, as claimed.

In other words, $Y^\perp$ is closed, by continuity of the inner product.
Observe $H^\perp = \{0\}$.

Theorem 3.13
For any closed subspace Y we have $H = Y \oplus Y^\perp$
(orthogonal sum)

Proof
Already know that for every vector $x \in H$ there is a uniquely defined pair $y \in Y$, $z \in Y^\perp$ s.t. $x = y + z$. $\square$

Note: $Y^{\perp\perp} = Y$.

Examples
1). $H = L_2(-1, 1)$
Let $Y = \{ f \in H : f(t) = 0, \ t \in (0,1) \}$
[Explanation:
If $\int_a^b |f(t)|^2 \, dt = 0$ thus $f = 0$ "almost everywhere" ]

Y is a subspace. It is closed:
Let $f_n \xrightarrow{\in Y} f$ as $n \to \infty$, i.e. $\int^1 |f_n(t) - f(t)|^2 \, dt \to 0$ as $n \to \infty$.
i.e. $\int_{-1}^{0} |f_n(t) - f(t)|^2 dt + \int_0^1 |f(t)|^2 dt \to 0$ as $n \to \infty$

$\Rightarrow \int_0^1 |f(t)|^2 dt = 0 \Rightarrow f(t) = 0$ almost everywhere $\forall t \in (0,1)$.

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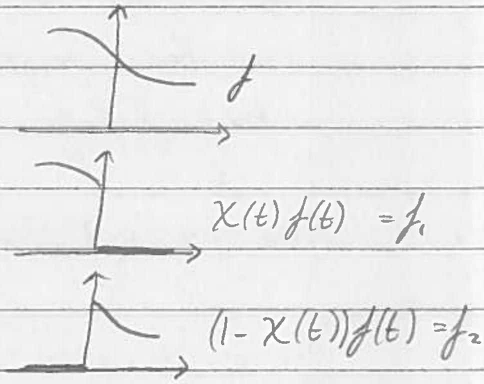
Define the indicator function:

$$\chi(t) = \begin{cases} 1, & -1 < t \leq 0 \\ 0, & 0 < t < 1 \end{cases}$$



Then the projection on Y:

$(Pf)(t) = \chi(t)f(t)$.

Indeed it is clear that

$$f(t) = \underbrace{\chi(t)f(t)}_{f_1} + \underbrace{(1-\chi(t))f(t)}_{f_2}$$

$f_1 \perp f_2$, indeed $\langle f_1, f_2 \rangle = \int_{-1}^{1} f_1(t)\overline{f_2(t)}\,dt$

$\Rightarrow \langle f_1, f_2 \rangle = \int_{-1}^{1} \chi(t)(1-\chi(t))|f(t)|^2\,dt = 0.$

2). Let $L \subset H$ be a subspace spanned by one vector, $e \in H$, $\|e\| = 1$.

Want to construct the projection P on L

Let $x \in H$. Then $Px = ce$ with some c.

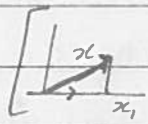
Find c:

$0 = \langle x - Px, e \rangle = \langle x - ce, e \rangle$

$= \langle x, e \rangle - c\|e\|^2 = \langle x, e \rangle - c$

Thus $c = \langle x, e \rangle$

$Px = \langle x, e \rangle e$ (1-dim projection)



$x_1 = \langle x, \binom{1}{0} \rangle$

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1). $e \in H$, $\|e\| = 1$, $L = \operatorname{span} e$

Projection on L: $Px = \langle x, e \rangle e$

2). Let $M = (e_1, e_2, \dots, e_n)$ be an orthonormal system,

i.e. $\|e_j\| = 1$, $\langle e_j, e_k \rangle = \delta_{jk}$.

$L = \operatorname{span} M$

M is LI: $\sum_{k=1}^{n} \alpha_k e_k = 0$

Multiply by e_j:

$$0 = \left\langle \sum_{k=1}^{n} \alpha_k e_k, e_j \right\rangle = \sum_{k=1}^{n} \alpha_k \langle e_k, e_j \rangle = \sum_{k=1}^{n} \alpha_k \delta_{kj} = \alpha_j, \quad 1 \leq j \leq n$$

Thus $\dim L = n$.

Find the projection operator on L:

$Px = \sum_{k=1}^{n} c_k e_k$. Want: $(x - Px) \perp L$,

i.e. $(x - Px) \perp e_j$, $1 \le j \le n$.

$$\langle x - Px, e_j \rangle = \langle x - \sum c_k e_k, e_j \rangle = \langle x, e_j \rangle - \sum c_k \langle e_k, e_j \rangle$$
$$= \langle x, e_j \rangle - \sum_{k=1}^{n} c_k \delta_{kj} = \langle x, e_j \rangle - c_j = 0$$

$\Rightarrow$ $c_j = \langle x, e_j \rangle$, $1 \le j \le n$.

Projection operator: $Px = \sum_{k=1}^{n} \langle x, e_k \rangle e_k$

3). Let $M = (v_1, v_2, \dots, v_n)$ be LI.
Let $L = \operatorname{span} M$, so $\dim L = n$.
How do we find the projection on L?
Use Gram-Schmidt to "orthonormalise" the system M.

4). Let $M = (e_1, e_2, \dots)$ be an infinite orthonormal sequence
i.e. $\langle e_k, e_j \rangle = \delta_{kj}$.
How do we find the projection on $L = [\operatorname{span} M]$?
Claim: the series $\sum_{k=1}^{\infty} \langle x, e_k \rangle e_k$ converges in H
and it defines the orthogonal projection on L.

Back to example 2:
$Px = \sum_{k=1}^{n} \langle x, e_k \rangle e_k$

Calculate: $\|Px\|^2 = \sum_{k,j=1}^{n} \langle x, e_k \rangle \overline{\langle x, e_j \rangle} \langle e_k, e_j \rangle$

$$= \sum_{k=1}^{n} |\langle x, e_k \rangle|^2 \qquad \forall x \in H$$

Recall $\|Px\|^2 + \|x - Px\|^2 = \|x\|^2$ so $\sum_{k=1}^{n} |\langle x, e_k \rangle|^2 \le \|x\|^2$
(Bessel inequality)

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$\langle x, e_k \rangle$ are called Fourier coefficients of x.

Back to example 4:

As in example 2 we have the Bessel inequality:
$$\sum_{k=1}^{\infty} |\langle x, e_k \rangle|^2 \le \|x\|^2, \quad \forall x \in H.$$

Proof of the claim:
Let $y_n = \sum_{k=1}^{n} \langle x, e_k \rangle e_k$
This sequence is Cauchy: Let $n > m$, then
$$\|y_n - y_m\|^2 = \sum_{k=m+1}^{n} |\langle x, e_k \rangle|^2 \to 0 \quad \text{as } n, m \to \infty$$
by Bessel inequality.
Denote $y = \lim y_n$. It remains to check that
$Px = \sum_{k=1}^{\infty} \langle x, e_k \rangle e_k \; (*)$ is the orthogonal projection on $L = [\text{span}\, M]$.
Want: $(x - Px) \perp L$, i.e. $(x - Px) \perp e_j$, $\forall j = 1, 2, \dots$.
Write:
$$\langle x - y_n, e_j \rangle = \langle x, e_j \rangle - \sum_{k=1}^{n} \langle x, e_k \rangle \langle e_k, e_j \rangle \quad (n > j)$$
$$= \langle x, e_j \rangle - \langle x, e_j \rangle = 0$$
By continuity of inner product,
$$\langle x - y, e_j \rangle = \lim_{n \to \infty} \langle x - y_n, e_j \rangle = 0$$

Thus $(x - y) \perp \text{span}\, M$ and $(x - y) \perp [\text{span}\, M]$ as required. $\square$

The infinite series $(*)$ is called the Fourier series of x w.r.t. the orthonormal sequence M.

5). Let $H = L_2(-\pi, \pi)$, and $e_k(x) = \frac{e^{ikx}}{\sqrt{2\pi}}$, $k \in \mathbb{Z}$, $x \in (-\pi, \pi)$
Then $\|e_k\|^2 = \int_{-\pi}^{\pi} |e_k(x)|^2 dx = 1$
and $(k \ne j)$
$$\langle e_k, e_j \rangle = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{ikx} e^{-ijx} dx = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{i(k-j)x} dx$$
$$= \frac{1}{2\pi} \left. \frac{e^{i(k-j)x}}{i(k-j)} \right|_{-\pi}^{\pi} = 0$$
(see Analysis 4).

Question: When do we have $x = Px$?

Complete (or total) orthonormal sequences

Definition 3.14
The orthonormal sequence $M = (e_1, e_2, \dots)$ is complete (or total) if span M is dense in H, i.e. $[\text{span}\, M] = H$.

Theorem 3.15
The orthonormal system is complete iff
$\sum_{k=1}^{\infty} |\langle x, e_k\rangle|^2 = \|x\|^2$ (*) (Parceval identity).
Moreover, if $x = \sum a_k e_k$, $y = \sum b_k e_k$, then
$\langle x, y\rangle = \sum_{k=1}^{\infty} a_k \overline{b_k}$

Proof
[⇒] If M is complete, then $x = Px$, and hence
$\|x\|^2 = \|Px\|^2 = \sum_{k=1}^{\infty} |\langle x, e_k\rangle|^2$ as claimed.
[⇐] Suppose that (*) holds, but assume that there is a vector $x \neq 0$, s.t. $x \perp \text{span}\, M$.
Therefore $\langle x, e_k\rangle = 0 \ \forall k$, but $\|x\| \neq 0$.
This contradicts (*) and hence proves the completeness.
$\langle x, y\rangle$ left as exercise.
□

In $H = L_2(-\pi, \pi)$, with $e_k(x) = \frac{1}{\sqrt{2\pi}} e^{ikx}$, $k \in \mathbb{Z}$.
The Fourier coefficients are $\quad [\overline{e^{i\alpha}} = e^{-i\alpha}]$
$\alpha_k = \langle f, e_k\rangle = \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} f(x) e^{-ikx}\, dx$

Then $\sum_{\infty}^{\infty} |\alpha_k|^2 = \|f\|^2$ (Analysis 4)

Therefore $\{e_k\}_{k \in \mathbb{Z}}$ is complete in $L_2(-\pi, \pi)$.

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A complete orthonormal sequence is called a basis of H.

Theorem 3.16
The Hilbert space H contains a complete orthonormal sequence iff it is seperable.

Theorem 3.17 (Riesz - Fisher Thm).
Let $\{c_n\}$ be a sequence from l_2.
Let $\{e_n\}$ be an orthonormal sequence in H.
Then there exists an element $x \in H$ s.t. $c_k = \langle x, e_k \rangle$, $k = 1, 2, \dots$

Proof
Let $x_n = \sum_{k=1}^{n} c_k e_k$.
This sequence is Cauchy: $(n > m)$
$$\|x_n - x_m\|^2 = \sum_{k=m+1}^{n} |c_k|^2 \to 0, \quad n, m \to \infty.$$
Thus $x_n \to x$, $n \to \infty$, and $x = \sum_{k=1}^{\infty} c_k e_k$,
so $c_k = \langle x, e_k \rangle$.
$\square$

Chapter 4 - Linear functions and Dual space

Let X be a normed space over $\mathbb{K}$.

Definition 4.1
We say that the mapping $f : X \to \mathbb{K}$ is a linear functional of X if $f(\alpha x + \beta y) = \alpha f(x) + \beta f(y)$ $\forall x, y \in X$, $\forall \alpha, \beta \in \mathbb{K}$.
A linear functional is said to be continuous if f is continuous.
" " " " " " " bounded if the image of the ball $B(0,1)$ is a bounded set.

Boundedness:

$f(B(0,1)) \subset B(0,R)$ for some $R>0$

By homogeneity, $f(B(0,t)) \subset B(0,tR)$, $\forall t \geq 0$.

This is equivalent to $|f(x)| < R\|x\|$, $\forall x \neq 0$.

Lemma 4.2

A linear functional is continuous iff it is continuous at $x = 0$.

Proof

[$\Leftarrow$] Suppose f is continuous at $x=0$: fix an $x \in X$.

Want: $\forall \varepsilon > 0$ $\exists \delta$ s.t. $f(y) \in B(f(x), \varepsilon)$ as soon as $y \in B(x, \delta)$

i.e. $|f(y) - f(x)| < \varepsilon$ if $\|x-y\| < \delta$.

In other words, $|f(y-x)| < \varepsilon$ if $\|x-y\| < \delta$

or $|f(z)| < \varepsilon$ if $\|z\| < \delta$, $z \in X$.

This is true since f is continuous at $x = 0$.

[$\Rightarrow$] Trivial.

$\square$

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Chapter 4 - Linear functionals

$f(\alpha x + \beta y) = \alpha f(x) + \beta f(y) \quad \forall \alpha, \beta \in \mathbb{K}, \ \forall x, y \in X$

$f : X \to \mathbb{K}$

f continuous

f bounded if $f(B(0,1)) \subset B(0,R)$, for some $R > 0$.

or $f(B(0,t)) \subset B(0,Rt)^{(*)} \ \forall t > 0$

Theorem 4.3

A linear functional f on X is continuous iff it is bounded.

Proof

[$\Rightarrow$] Suppose f is continuous, so f is continuous at $x = 0$,

i.e. $\forall \varepsilon > 0 \ \exists \delta$ st. $|f(x) - \underbrace{f(0)}_{=0}| < \varepsilon$ if $x \in B(0,\delta)$.

i.e. $f(x) \in B(0,\varepsilon)$ if $x \in B(0,\delta)$, $f(B(0,\delta)) \subset B(0,\varepsilon)$.

Taking $t = \delta$, $R = \varepsilon\delta^{-1}$ gives $(*)$. i.e. f is bounded.

[$\Leftarrow$] Assume $(*)$.

Let $Rt = \varepsilon$ and $t = \varepsilon/R$, so

$f(B(0, \varepsilon/R)) \subset B(0,\varepsilon) \ \forall \varepsilon > 0$.

This implies continuity. $\square$

Example

Let $X = \mathbb{R}$, $\mathbb{K} = \mathbb{R}$.

Linear functionals in X : $f(x) = \alpha x$.

Use this now:

$f(B[0,1]) \subset B[0,R]$, with some $R \geq 0$,

or $f(B[0,t]) \subset B[0,Rt]$, $\forall t \geq 0$.

$\to |f(x)| \leq R$, $\forall x \in B[0,1]$

Therefore $|f(x)| = \|x\| \left|f\left(\frac{x}{\|x\|}\right)\right| \leq R\|x\|$, $\forall x \in X$, $x \neq 0$

Definition 4.4
The norm of a bounded linear functional f on X is defined to be $\|f\| = \sup_{x : \|x\|=1} |f(x)|$.

Lemma 4.5
Let f be a bounded linear functional. Then
$$\|f\| = \underbrace{\sup_{x \in B[0,1]} |f(x)|}_{= A_1} = \underbrace{\sup_{x \neq 0} \frac{|f(x)|}{\|x\|}}_{= A_2}.$$

Proof
Clear: $\|f\| \leq A_1$, $\|f\| \leq A_2$
Proof of $A_1 \leq \|f\|$:
Write for $x \in B[0,1]$, $x \neq 0$:
$$|f(x)| = \|x\| \left| f\left(\frac{x}{\|x\|}\right) \right| \leq \|x\| \|f\| \leq \|f\|$$
Thus $A_1 \leq \|f\|$.
Proof of $A_2 \leq \|f\|$:
Write for $x \neq 0$:
$$|f(x)| = \|x\| \left| f\left(\frac{x}{\|x\|}\right) \right| \leq \|x\| \|f\|, \quad \text{so} \quad \frac{|f(x)|}{\|x\|} \leq \|f\|$$
Thus $A_2 \leq \|f\|$. $\square$

It follows from Lemma 4.5:
$|f(x)| \leq \|f\| \|x\|$.
The constant $\|f\|$ is the best possible constant in the inequality $|f(x)| \leq R\|x\|$.
In other words, if we know that $|f(x)| \leq C\|x\|$ with some $C > 0$ then $\|f\| \leq C$.

Definition 4.6
The set $\{x : f(x) = 0\}$ is called the kernel of the functional f.
Notation: Ker f.

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<u>Theorem 4.7</u>

Let f be a bounded linear functional. Then the kernel of f is a closed linear subspace.
If $f \neq 0$ then $\operatorname{codim} \operatorname{Ker} f = 1$.

<u>Proof</u>

Hw 5.

<u>Example</u>

1). $X = H$. Let f be determined by $f(x) = \langle x, x_0 \rangle$ with some fixed $x_0 \in H$. Then f is a linear bounded functional and $\|f\| = \|x_0\|$.

<u>Proof</u>

f is linear since the inner product is linear.

$|f(x)| = |\langle x, x_0 \rangle| \leq \|x_0\| \|x\|$. Thus $\|f\| \leq \|x_0\|$.

Want to find a vector $z \in H$ st.

$|f(z)| = \|x_0\| \|z\|$. This would imply that $\|f\| = \|x_0\|$

Take $z = x_0$:

$$f(x_0) = \langle x_0, x_0 \rangle = \|x_0\|^2 = \|x_0\| \|x_0\|.$$

$\square$

2). Let $X = C[a,b]$. Define $l(u) = \int_a^b u(t)\,dt$, $u \in X$.

Then l is a bounded linear functional and $\|l\| = b-a$.

<u>Proof</u>

Estimate:

$$|l(u)| \leq \int_a^b |u(t)|dt \leq \|u\|_C \int_a^b dt = (b-a)\|u\|_C.$$

So $\|l\| \leq b-a$. Want to find $u_0 \in X$ st. $|l(u_0)| = (b-a)\|u_0\|_C$.

Take $u_0(t) = 1$, $t \in [a,b]$, so

$$l(u_0) = \int_a^b dt = b-a = (b-a)\|u_0\|_C.$$

Therefore $\|l\| = b-a$ as claimed.

$\square$

3). $X = C[-1,1]$. Let $l(u) = \int_0^1 u(t)\,dt - \int_{-1}^0 u(t)\,dt$

Clear that l is a linear functional.

Claim: l is bounded and $\|l\| = 2$.

Proof

Estimate:

$$|l(u)| \le \left|\int_0^1 u(t)\,dt\right| + \left|\int_{-1}^0 u(t)\,dt\right| \le \int_{-1}^1 |u(t)|\,dt \le 2\|u\|_C$$

so $\|l\| \le 2$.

Let $u_n(t) = \begin{cases} -1, & -1 \le t \le -1/n \\ nt, & -1/n \le t \le 1/n \\ 1, & 1/n < t \le 1 \end{cases}$ $\quad \|u_n\|_C = 1$

Then $l(u_n) = n\int_0^{1/n} t\,dt + \int_{1/n}^1 dt - \int_{-1}^{-1/n} (-1)\,dt - n\int_{-1/n}^0 t\,dt$

$$= \frac{n}{2}\frac{1}{n^2} + (1 - 1/n) + (1 - 1/n) + \frac{n}{2}\frac{1}{n^2} = 2 - 1/n$$

$\Rightarrow |l(u_n)| = (2 - \frac{1}{n})\|u_n\|_C$,

so $\frac{|l(u_n)|}{\|u_n\|} = 2 - 1/n$

Therefore $\sup_{u \ne 0} \frac{|l(u)|}{\|u\|} \ge 2 - 1/n \quad \forall n = 1, 2, \ldots$

Consequently $\|l\| \ge 2$. Together with $\|l\| \le 2$, this gives $\|l\| = 2$ as claimed. $\square$

4). Let $X = l_p$, $1 \le p < \infty$, fix a sequence $y \in l_q$, $\frac{1}{p} + \frac{1}{q} = 1$.

Define $f(x) = \sum_{k=1}^{\infty} x_k y_k$ for all $x = (x_1, x_2, \ldots) \in l_p$.

Claim: f is a bounded linear functional and $\|f\| = \|y\|_q$.

Proof

Estimate: $|f(x)| \le \sum_{k=1}^{\infty} |x_k||y_k| \le \|x\|_p \|y\|_q$ so $\|f\| \le \|y\|_q$.

[Note: f is linear since the sum is linear.]

Want to find a sequence $x \in l_p$ st.

$|f(x)| = \|x\|_p \|y\|_q$.

Pick $x_k = |y_k|^{q-1} \frac{\overline{y_k}}{|y_k|}$ if $y_k \ne 0$ and $x_k = 0$ if $y_k = 0$.

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since $\frac{1}{p}+\frac{1}{q}=1$

$$\left[\begin{array}{l}\frac{p}{q}+1=p \\ \Rightarrow p+q=pq \\ \Rightarrow qp-p=q\end{array}\right]$$

Then $|x_k| = |y_k|^{q-1}$ and $|x_k|^p = |y_k|^{(q-1)p} = |y_k|^q$

so $x \in l_p$ and $\|x\|_p^p = \|y\|_q^q$

Calculate: $f(x) = \sum_{y_k \neq 0} |y_k|^{q-1} \frac{\overline{y_k}}{|y_k|} y_k = \sum_{k=1}^{\infty} |y_k|^q = \|y\|_q^q$

$$= \|y\|_q \|y\|_q^{q-1} = \|y\|_q \|x\|_p$$

so $\|f\| = \|y\|_q$. $\square$

Theorem 4.8 (Riesz)

Let H be a Hilbert space. For any bounded linear functional on H there is a uniquely defined $x_0 \in H$ st. $f(x) = \langle x, x_0 \rangle$. Moreover $\|f\| = \|x_0\|$.

Proof

If $f \equiv 0$, then $x_0 = 0$, so $\|f\| = 0$.

Suppose $f \not\equiv 0$. Then $\mathrm{Ker} f$ is a closed subspace of codim $= 1$

Let $z \in (\mathrm{Ker} f)^\perp$, so $\alpha = f(z) \neq 0$.

The vector $x - \frac{f(x)}{\alpha} z$ belongs to $\mathrm{Ker} f$:

$f\left(x - \frac{f(x)}{\alpha} z\right) = f(x) - \frac{f(x)}{\alpha} f(z) = 0$. Thus $\left(x - \frac{f(x)}{\alpha} z\right) \perp z$:

i.e. $0 = \langle x - \frac{f(x)}{\alpha} z, z \rangle = \langle x, z \rangle - \frac{f(x)}{\alpha} \langle z, z \rangle$

$$= \langle x, z \rangle - \frac{f(x)}{\alpha} \|z\|^2$$

So $\frac{f(x)}{\alpha} \|z\|^2 = \langle x, z \rangle$

i.e. $f(x) = \frac{\alpha}{\|z\|^2} \langle x, z \rangle$

Take $x_0 = \frac{\overline{\alpha} z}{\|z\|^2}$, so that $f(x) = \langle x, x_0 \rangle$

x_0 is unique:

Suppose $f(x) = \langle x, x_1 \rangle = \langle x, x_2 \rangle$

$\Rightarrow \langle x, x_1 - x_2 \rangle = 0 \quad \forall x \in H \quad \Rightarrow \quad x_1 - x_2 = 0$

So x_0 is unique.

The identity $\|f\| = \|x_0\|$ was proved earlier. $\square$

19-02-18 $\|f\| = \sup_{\|x\|=1} |f(x)|$

If $|f(x)| \leq R\|x\| \quad \forall x \in X$, then $\|f\| \leq R$

Examples

- $X = l_p$, $1 < p < \infty$.
  Let $f(x) = \sum_{k=1}^{\infty} x_k y_k$ where $y \in l_q$, $q^{-1} + p^{-1} = 1$ (linear)
  Then f is bounded, and $\|f\| = \|y\|_q$.

- $X = H$, then any linear bounded functional is represented as $f(x) = \langle x, x_0 \rangle$ with some $x_0 \in H$.
  x_0 is uniquely defined and $\|f\| = \|x_0\|$.

Theorem 4.9

Any bounded linear functional on l_p $1 < p < \infty$, can be uniquely represented as $f(x) = \sum_{k=1}^{\infty} x_k y_k$ (*) with some $y \in l_q$, where $\frac{1}{p} + \frac{1}{q} = 1$. Moreover $\|f\| = \|y\|_q$

Proof

$\|f\| = \|y\|_q$ was proved earlier, assuming (*).

Suppose $x = (x_1, x_2, \ldots, x_n, 0, \ldots) \in l_p$.

Then $x = \sum_{k=1}^{n} x_k e^{(k)}$ where $e^{(k)} = (0, \ldots, 0, \underset{k\text{th place}}{1}, 0, \ldots)$.

Then $f(x) = \sum_{k=1}^{n} x_k f(e^{(k)}) = \sum_{k=1}^{n} x_k y_k$ by letting $y_k = f(e^{(k)})$.

Back to l_p. Let $x = (x_1, x_2, \ldots)$, and let $x^{(n)} = (x_1, x_2, \ldots, x_n, 0, 0, \ldots)$

Observe: $\|x - x^{(n)}\|_p^p = \sum_{k=n+1}^{\infty} |x_k|^p \to 0$ as $n \to \infty$

Then $x^{(n)} = \sum_{k=1}^{n} x_k e^{(k)}$, and hence $f(x^{(n)}) = \sum_{k=1}^{n} x_k f(e^{(k)}) = \sum_{k=1}^{n} x_k y_k$,

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$y_k = f(e^{(k)})$, $k = 1, 2, \dots$

Want $y = (y_1, y_2, \dots) \in l_q$.

Take $x_k = \begin{cases} |y_k|^{q-1} \overline{y_k} / |y_k| , & y_k \neq 0 \\ 0 , & y_k = 0 . \end{cases}$

Then $f(x^{(n)}) = \sum_{k=1}^{n} |y_k|^q = \|y^{(n)}\|_q^q$.

On the other hand, $|f(x^{(n)})| \leq \|f\| \, \|x^{(n)}\|_p$,

so $\underbrace{\|y^{(n)}\|_q^q}_{\|x^{(n)}\|_p \|y^{(n)}\|_q} \leq \|f\| \, \|x^{(n)}\|_p \quad$ if $x \neq 0$

Therefore $\|y^{(n)}\|_q \leq \|f\| \quad \forall n$

Therefore $y \in l_q$ (and $\|y\|_q < \|f\|$)

By continuity of the functional f:

$f(x) = \lim_{n \to \infty} f(x^{(n)}) = \lim_{n \to \infty} \sum_{k=1}^{n} x_k y_k = \sum_{k=1}^{\infty} x_k y_k$ as claimed.

Uniqueness left as exercise. $\square$

Extensions of bounded linear functionals

Definition 4.10

Let $L \subset X$ be a subspace and let f_0 be a bounded linear functional on L. Let f be a bounded linear functional on X. Then we say that f is an extension of f_0 to X if $f_0(x) = f(x)$ for $x \in L$.

Clearly $\|f_0\| \leq \|f\|$.

Lemma 4.11

Let $D \subset X$ be a subspace s.t. $[D] = X$.

Let f_0 be a bounded linear functional on D.

Then there exists a unique extension of f_0 to X and $\|f\| = \|f_0\|$.

Proof

For any $x \in X$ there is a sequence $x_n \in D$ s.t. $x_n \to x$ as $n \to \infty$.

Then the numerical sequence $f_0(x_n)$ is Cauchy.

$|f_0(x_n) - f_0(x_m)| \leq \|f_0\| \|x_n - x_m\| \to 0$ as $n, m \to \infty$.

Thus $f_0(x_n)$ converges.

$A = \lim_{n\to\infty} f_0(x_n)$.

The number A doesn't depend on the choice of the sequence $x_n \to x$ as $n \to \infty$. Indeed, if $\tilde{x}_n \to x$, $\tilde{x}_n \in D$, then $f_0(x_n) - f_0(\tilde{x}_n) \to 0$ as $n \to \infty$.

Denote: $A = f(x)$.

This map is linear: if $x_n \to x$, $y_n \to y$, $x_n, y_n \in D$, then $f(\alpha x + \beta y) = \lim_{n\to\infty} f_0(\alpha x_n + \beta y_n)$

$$= \lim_{n\to\infty} [\alpha f_0(x_n) + \beta f_0(y_n)] = \alpha f(x) + \beta f(y).$$

f is a bounded functional $\leftarrow$ WTS.

Indeed $\forall \varepsilon > 0 \ \exists N$ st. $|f(x) - f_0(x_n)| < \varepsilon$ if $n > N$.

Therefore, $|f(x)| < |f_0(x_n)| + \varepsilon \leq \|f_0\| \|x_n\| + \varepsilon$

$\longrightarrow \|f_0\| \|x\| + \varepsilon$ as $n \to \infty$.

$\Rightarrow |f(x)| \leq \|f_0\| \|x\| \quad \forall x \in X$.

Consequently, $\|f\| \leq \|f_0\|$.

On the other hand $\|f\| \geq \|f_0\| \Rightarrow \|f\| = \|f_0\|$.

Uniqueness left as exercise (follows from subspace being dense). $\square$

Theorem 4.12 (Hahn - Banach Thm)

Let $L \subset X$ be a subspace. For any bounded linear functional f_0 on L there is an extension f to X st. $\|f_0\| = \|f\|$.

Proof

Assume: X is real and seperable.

Take a vector $\xi \in X$, and define: $L_1 = \text{span}(L, \xi)$.

If $\xi \in L$, then $L_1 = L$, then we define $f_1(x) = f_0(x) \ \forall x \in L_1$.

Let $\xi \notin L$. Every $u \in L_1$ is represented (uniquely) as

$u = x + t\xi$, $x \in L$, $t \in \mathbb{R}$.

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Pick $x_1, x_2 \in L$, and write

$f_0(x_1) - f_0(x_2) = f_0(x_1 - x_2) \leq \|f_0\| (\|x_1 + \xi\| + \|x_2 + \xi\|)$.

We may assume $\|f_0\| = 1$, otherwise consider $f_0 / \|f_0\|$.

Therefore $f_0(x_1) - \|x_1 + \xi\| \leq f_0(x_2) + \|x_2 + \xi\| \quad \forall x_1, x_2 \in L$.

So $\sup_{x \in L} (f_0(x) - \|x_1 + \xi\|) \leq \inf_{x \in L} (f_0(x) + \|x + \xi\|)$

Let $a \in \mathbb{R}$ be a number between sup and inf:

$f_0(x) - \|x + \xi\| \leq a \leq f_0(x) + \|x + \xi\|, \ \forall x \in L$.

Hence $-\|x + \xi\| \leq f_0(x) - a \leq \|x + \xi\|, \ \forall x \in L$

$\Rightarrow |f_0(x) - a| \leq \|x + \xi\|$.

Define a functional f on L_1: $\quad (u = x + t\xi)$

$f(u) = f_0(x) - ta$. f is linear:

If $u_1 = x_1 + t_1 \xi$, $u_2 = x_2 + t_2 \xi$, then

$f(\alpha u_1 + \beta u_2) = f_0(\alpha x_1 + \beta x_2) - (\alpha t_1 + \beta t_2) a = \alpha f(u_1) + \beta f(u_2)$.

Boundedness:

$|f(u)| = |f_0(x) - at| = |t| \left| f_0\left(\frac{x}{t}\right) - a \right| \leq |t| \left\| \frac{x}{t} + \xi \right\| = \|x + t\xi\| = \|u\|$

So $\|f\| \leq 1$. Thus $\|f\| = 1$.

Since X is seperable, there is a countable dense set: $\xi_1, \xi_2, \xi_3, \ldots \in X$.

Let $L_1 = \text{span}(L, \xi_1)$, $L_2 = \text{span}(L_1, \xi_2), \ldots$

Using part 1 of the proof, we extend f_0 to all subspaces

L_n, $n = 1, 2, \ldots$ as functional f s.t. $\|f\| = \|f_0\|$.

Thus f is a bounded linear functional on $\bigcup L_n = D$.

This subspace is dense in X so by Lemma 4.11,

there is an extension to X and $\|f\| = \|f_0\|$. $\square$

Corollary 4.13

For any $x \in X$ there is a bounded (linear) functional f st.

$f(x) = \|x\|$ and $\|f\| = 1$.

Proof

If $x \neq 0$. Let $L = \text{span}(x)$.

Define $f_0(\alpha x) = \alpha \|x\|$, $\forall \alpha \in \mathbb{K}$.

If $|\alpha|\|x\| = 1$ then $|f_0(\alpha x)| = 1$, so $\|f_0\| = 1$.
By Hahn-Banach Thm there is an extension f st.
$\|f\| = \|f_0\| = 1$.
If $x = 0$, then $f(x) = 0 \ \forall f$. Take $\frac{f}{\|f\|}$.

$\square$

<u>Corollary 4.14</u>
Let $x_1, x_2 \in X$ be distinct vectors. Then there is a functional with $\|f\| = 1$ st. $f(x_1) \neq f(x_2)$.

<u>Proof</u>
Let $x = x_1 - x_2$. By corollary 4.13, there is a functional f, $\|f\| = 1$ st. $f(x) = \|x\|$, so $f(x_1) - f(x_2) = f(x) = \|x\| \neq 0$.

$\square$

<u>The dual space</u>
Introduce a linear structure on the set of bounded linear functionals.
$(f+g)(x) = f(x) + g(x) \quad \forall x \in X$, zero functional: $0(x) = 0 \ \forall x \in X$,
$(\alpha f)(x) = \alpha f(x) \quad \forall \alpha \in \mathbb{K}, \forall x \in X$.
Need to check that $\|f\|$ is a proper norm.
1). Non-degeneracy:
It is clear that if $f = 0$, then $\|f\| = 0$.
Suppose that $\|f\| = 0$, so $|f(x)| \leq \|f\|\|x\| = 0 \ \forall x \in X$
So $f(x) = 0 \ \forall x \in X$.

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1). $f_0 : D \quad [D] = X$

f_0 can be extended to X by continuity

2). Hahn-Banach Thm

$f_0 : L \to \mathbb{K} \Rightarrow$ extension $f : X \to \mathbb{K}$ st. $\|f_0\| = \|f\|$

Linear space of continuous linear functionals

$(f+g)(x) = f(x) + g(x)$

$(\alpha f)(x) = \alpha f(x)$

zero functional : $O(x) = 0$, $\forall x \in X$

$\|f\| = \sup\limits_{\|x\|=1} |f(x)|$

Norm properties

1). Non-degeneracy (done last time)

2). Homogeneity

$$\|\alpha f\| = \sup_{\|x\|=1} |\alpha f(x)| = \sup_{\|x\|=1} |\alpha||f(x)| = |\alpha| \sup_{\|x\|=1} |f(x)| = |\alpha|\|f\|$$

3). Δ-inequality:

$$|(f+g)(x)| = |f(x) + g(x)| \leq |f(x)| + |g(x)| \leq \|f\|\|x\| + \|g\|\|x\|$$
$$= (\|f\| + \|g\|)\|x\| \quad \forall x \in X$$

Therefore $\|f+g\| \leq \|f\| + \|g\|$.

Definition 4.15

The normed space of continuous linear functionals on X is called the dual space, X^*.

Theorem 4.16

The space X^* is complete.

Proof

Let $f_n \in X^*$ be a cauchy sequence, ie. $\forall \varepsilon > 0 \; \exists N$ st. $\|f_n - f_m\| < \varepsilon$ if $n, m > N$.

In other words, $\forall x \in X$ we have:

$$|f_n(x) - f_m(x)| < \varepsilon \|x\| \quad , n, m > N. \quad (*)$$

Thus the sequence $f_n(x)$ is Cauchy, and hence it converges (it is a numerical Cauchy sequence):

$$f(x) = \lim_{n\to\infty} f_n(x) \, , \ \forall x \in X$$

f is linear:

$$f(\alpha x + \beta y) = \lim_{n\to\infty} f_n(\alpha x + \beta y)$$
$$= \lim_{n\to\infty} (\alpha f_n(x) + \beta f_n(y)) = \alpha \lim_{n\to\infty} f_n(x) + \beta \lim_{n\to\infty} f_n(y)$$
$$= \alpha f(x) + \beta f(y).$$

f is bounded:

Indeed, take $n \to \infty$ in $(*)$:

$$|f(x) - f_m(x)| \leq \varepsilon \|x\| \quad \forall x \in X \quad (**)$$

Therefore $|f(x)| \leq \varepsilon \|x\| + |f_m(x)|$
$$\leq (\varepsilon + \|f_m\|)\|x\| \quad \forall \ m > N$$

Now it follows from $(**)$ that

$$\|f - f_m\| \leq \varepsilon \text{ if } m > N.$$

Thus $f_m \to f$ as $m \to \infty$.

$\square$

Thus X^* is a Banach space.

Examples

1) $X = l_p$, $1 < p < \infty$.

We know that every $f \in X^*$ has the form $f(x) = \sum_{k=1}^{\infty} x_k y_k$ with some uniquely defined $y \in l_q$, $\frac{1}{p} + \frac{1}{q} = 1$.

Also $\|f\| = \|y\|_q$.

Therefore, the map $f \mapsto y$ is an isomorphism.

Consequently, l_p^* is isomorphic to l_q, $\frac{1}{p} + \frac{1}{q} = 1$.

2). $l_1^* = l_\infty$

3). c_0^* is isomorphic to l_1.

Want: (i) $\forall y \in l_1$ the sum $f(x) = \sum_{k=1}^{\infty} x_k y_k$ defines a bounded linear functional on c_0. Moreove $\|f\| = \|y\|_1$

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(ii) Every $f \in c_0^*$ can be represented as $f(x) = \sum_{k=1}^{\infty} x_k y_k$ with some uniquely defined $y \in l_1$.

<u>Proof</u>

(i) Estimate: $|f(x)| \leq \sum_{k=1}^{\infty} |x_k||y_k|$

$$\leq \sup_k |x_k| \sum_{k=1}^{\infty} |y_k| = \|x\|_\infty \|y\|_1$$

Thus $\|f\| \leq \|y\|_1$.

To prove that $\|f\| = \|y\|_1$, consider $x^{(n)} = (x_1, x_2, \ldots, x_n, 0, 0, \ldots)$.

Observe that $x^{(n)} \to x$ as $n \to \infty$.

Here we use the fact that $x_k \to 0$ as $k \to \infty$

$\|x^{(n)} - x\| = \sup_{k \geq n+1} |x_k| \to 0$ as $n \to \infty$.

Now, $x_k = \begin{cases} \overline{y_k}/|y_k| & , y_k \neq 0, \\ 0 & , y_k = 0 \end{cases}$

Assume $y \neq 0$. Then $f(x^{(n)}) = \sum_{k=1}^{n} x_k y_k = \sum_{k=1}^{n} |y_k|$

On the other hand,

$|f(x^{(n)})| \leq \|f\| \cdot 1$ (since $\|x_n\| = 1$ in l_∞)

and hence $\|y^{(n)}\|_1 \leq \|f\| \Rightarrow \|y\|_1 \leq \|f\|$

$\therefore \|f\| = \|y\|_1$.

(ii) Let $x^{(n)} = (x_1, x_2, \ldots, x_n, 0, \ldots)$. Then $x^{(n)} = \sum_{k=1}^{n} x_k e^{(k)}$.

Since $x \in c_0$, we have $x^{(n)} \to x$ as $n \to \infty$.

Then $f(x^{(n)}) = \sum_{k=1}^{n} x_k f(e^{(k)})$, write $y_k = f(e^{(k)})$.

Want: $y_k \in l_1$.

Take $x_k = \begin{cases} \overline{y_k}/|y_k| & \text{if } y_k \neq 0 \\ 0 & , y_k = 0 \end{cases}$

Assume $y \neq 0$.

Then $f(x^{(n)}) = \sum_{k=1}^{n} x_k y_k = \sum_{k=1}^{n} |y_k|$

On the other hand $|f(x^{(n)})| \leq \|f\| \cdot 1$

hence $\|y^{(n)}\|_1 \leq \|f\| \Rightarrow \|y\|_1 \leq \|f\| \Rightarrow y \in l_1$, as required.

Uniqueness left as exercise. $\square$

Conclusion: c_0^* is isomorphic to l_1.
(Write $c_0^* \sim l_1$ "$c_0^* = l_1$", similarly $l_p^* \sim l_q$, $1<p<\infty$).

The second dual space

Looking at X^* as a starting space, we can define the space of continuous linear functionals on X^*,
notation: X^{**}.

For every $x \in X$, define $F_x \in X^{**}$ by the formula
$F_x(f) = f(x)$, $\forall f \in X^*$ (f variable, x fixed).

F_x is a linear functional:
$F_x(\alpha f + \beta g) = (\alpha f + \beta g)(x) = \alpha f(x) + \beta g(x) = \alpha F_x(f) + \beta F_x(g)$.
F_x is bounded:
$|F_x(f)| = |f(x)| \leq \|f\| \|x\|$
Therefore $\|F_x\| \leq \|x\|$

Theorem 4.17
$\|F_x\| = \|x\|$

Proof
Need to find a functional $f \neq 0$ s.t. $|F_x(f)| = \|f\| \|x\|$.
By Corollary 4.13, $\forall x \in X$ $\exists$ a functional $f \in X^*$ s.t.
$\|f\| = 1$, $f(x) = \|x\|$.
Thus $F_x(f) = f(x) = \|x\| = \|x\| \|f\|$
$\Rightarrow \|F_x\| = \|x\|$. $\square$

Rephrase: Thm 4.17:
The map $F: X \to X^{**}$ defined by $F(x) = F_x$ is an isometry.

Definition 4.18
The map $F: X \to X^{**}$ is called the canonical map from X to X^{**}.

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If $\mathcal{F}$ is surjective, then we say that the space X is reflexive.

If X is reflexive, it is complete.

Examples

1). Assume $\dim X < \infty$.

X is linearly isomorphic to X^*,

X^* is linearly isomorphic to X^{**}.

So $\dim X^{**} = \dim X = n < \infty$.

Since $\dim X^{**} = \dim X < \infty$, every isometry is an isomorphism.

So X is reflexive.

2). Hilbert space H.

H is isomorphic to H^*, by Riesz Theorem.

Also H^* is isomorphic to H^{**}.

Thus H is isomorphic to H^{**}

Claim: H is reflexive.

3). l_p, $1 < p < \infty$, is reflexive.

$l_p^* \sim l_q$ and $l_q^* \sim l_p$

4). l_1 is not reflexive.

$l_1^* \sim l_\infty$, but $l_\infty^* \not\sim l_1$

5). c_0 is not reflexive, since $c_0^* \sim l_1$, $l_1^* \sim l_\infty$.

Theorem 4.19

The space X is reflexive iff X^* is reflexive.

Proof

Omitted, although in online notes.

Convergence in normed spaces

Definition 4.20
We say that x_n converges to x strongly, if $\|x_n - x\| \to 0$ as $n \to \infty$.
We say that x_n converges to x weakly, if $f(x_n) \to f(x)$ as $n \to \infty$, for all $f \in X^*$.

Notation:

$x_n \xrightarrow{s} x$ as $n \to \infty$, $x = \underset{n\to\infty}{s\text{-}\lim}\, x_n$ ← strong

$x_n \xrightarrow{w} x$ as $n \to \infty$, $x = \underset{n\to\infty}{w\text{-}\lim}\, x_n$ ← weak

If $x_n \xrightarrow[n\to\infty]{s} x$ then $x_n \xrightarrow[n\to\infty]{w} x$, by continuity of $x \in X^*$.

If $x_n \xrightarrow[n\to\infty]{w} x$ then one can't say anything about strong convergence.

Example
Let $X = l_p$, $1 < p < \infty$.
Define $x^{(n)} = (0, 0, \dots, 0, \overset{n^{th}\text{ place}}{1}, 0, \dots)$
Then $f(x) = \sum_{k=1}^{\infty} x_k y_k$, $y \in l_q$ for every $f \in l_p^*$

Then $f(x^{(n)}) = y_n \to 0$ as $n \to \infty$
Conclusion, $x^{(n)} \xrightarrow[n\to\infty]{w} 0$
On the other hand $\|x^n\|_p = 1 \neq 0$.

Lemma 4.21
If x_n converges weakly then the limit is unique.
If, in addition, x_n converges strongly, then these two limits coincide.

Proof
Suppose $x_n \xrightarrow{w} x$, $x_n \xrightarrow{w} \tilde{x}$, i.e. $f(x_n) \xrightarrow[n\to\infty]{} f(x)$ and $f(x_n) \xrightarrow[n\to\infty]{} f(\tilde{x})$

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$\forall f \in X^*$, so $f(x) = f(\tilde{x})$ $\forall f \in X^*$ (since they are numerical limits). By corollary 4.14, there is a $g \in X^*$ s.t. $g(x) \neq g(\tilde{x})$. ※
Thus $x = \tilde{x}$. □

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X, X^*, X^{**} dual spaces always complete

Convergence:

Strong: $x_n \xrightarrow[n\to\infty]{s} x \iff \|x_n - x\| \to 0$ as $n \to \infty$

weak: $x_n \xrightarrow[n\to\infty]{\omega} x \iff f(x_n) \to f(x)$ as $n \to \infty$ $\forall f \in X^*$

If $x_n \xrightarrow{s} x_0$, $x_n \xrightarrow{\omega} x_1$ then $x_0 = x_1$.

Examples

1). $X = H$. By Riesz Thm (Thm 4.8) $x_n \xrightarrow{\omega} x$ is equivalent to $\langle x_n, v\rangle \to \langle x, v\rangle$ as $n \to \infty$.
Let $\{e_k\}$ be an orthonormal sequence.
Then by Bessel inequality, $\sum_{k=1}^{\infty} |\langle x, e_k\rangle|^2 \le \|x\|^2$ $\forall x \in H$.
Since the series converges, $\langle x, e_k\rangle \to \langle x, 0\rangle$ as $k \to \infty$ $\forall x \in H$
Thus $\langle e_k, x\rangle \xrightarrow[k\to\infty]{} \langle 0, x\rangle \Rightarrow e_k \xrightarrow[k\to\infty]{\omega} 0$.

For instance, if $H = L_2(-\pi, \pi)$ and $e_k(x) = \frac{1}{\sqrt{2\pi}} e^{ikx}$, $k \in \mathbb{Z}$
then $\forall f \in L_2(-\pi,\pi)$: $\int_{-\pi}^{\pi} e^{-ikx} f(x)\, dx \to 0$ as $|k| \to \infty$.

2). Let $X = \mathbb{R}^m$. If $x^{(n)} \xrightarrow[n\to\infty]{\omega} x$, then $x^{(n)} \xrightarrow[n\to\infty]{s} x$.
WTS: $\|x^{(n)} - x\|^2 = \sum_{k=1}^{m} |x^{(n)}_k - x_k|^2 \xrightarrow[n\to\infty]{} 0$
i.e. $x_k^{(n)} \xrightarrow[n\to\infty]{} x_k$, $k = 1, 2, \dots, m$.

By example 1, $\langle x^n, v\rangle \xrightarrow[n\to\infty]{} \langle x, v\rangle$ $\forall v \in \mathbb{R}^m$.
Let $e^{(1)}, e^{(2)}, \dots, e^{(m)}$ be the canonical basis, so
$\langle x^{(n)}, e^{(k)}\rangle = x^{(n)}_k$, $k = 1, 2, \dots, m \Rightarrow x^{(n)}_k \xrightarrow[n\to\infty]{} x_k$. □

The uniform boundedness Thm

Theorem 4.22 (Banach - Steinhaus Thm)

Let X be a Banach space, and let $M \subset X^*$ be a set which is pointwise uniformly bounded, i.e.

$\forall x \in X$ $\exists$ a constant $c = c(x) > 0$, s.t. $|f(x)| \leq c$ $\forall f \in M$.

Then the set M is bounded in X^*, i.e. $\exists$ constant $c_1 > 0$ s.t. $\|f\| \leq c_1$ $\forall f \in M$.

Proof

Let $A_k = \{x \in X : |f(x)| \leq k, f \in M\}, k = 1, 2, \ldots$

Observe: $[A_k] = A_k$.

Indeed $|f|$ is continuous, so the preimage of a closed set is closed.

Also, $\forall x$ $\exists k$ s.t. $x \in A_k$. Indeed we know that $|f(x)| \leq c(x)$, $\forall f \in M$. Take $k \geq c(x)$, so $x \in A_k$.

Therefore $X = \bigcup_{k=1}^{\infty} A_k$

By Thm 2.7 (Baire Category Thm), at least one of the sets A_k is dense in some ball $B(x_0, \varepsilon)$, $x_0 \in X$, $\varepsilon > 0$.

This means that $B(x_0, \varepsilon) \subset [A_k] = A_k$, so

$|f(x)| \leq k$ $\forall x \in B(x_0, \varepsilon)$, $\forall f \in M$.

Rewrite: $x = x_0 + \varepsilon t$, $t \in B(0,1)$, so

$|f(x_0) + \varepsilon f(t)| \leq k$ $\forall t \in B(0,1)$, $\forall f \in M$.

Thus: $\varepsilon |f(t)| \leq k + |f(x_0)|$

$$|f(t)| \leq \frac{k + |f(x_0)|}{\varepsilon} \quad \forall t \in B(0,1), \forall f \in M$$

$$\leq \frac{k + c(x_0)}{\varepsilon} \Rightarrow \{\|f\|, f \in M\} \text{ is uniformly bounded as claimed.} \quad \square$$

Corollary 4.23

Assume that $x_n \xrightarrow[n \to \infty]{w} x$ in X.

Then $\|x_n\| \leq c$ with some $c > 0$

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Proof

$x_n \xrightarrow[n\to\infty]{\omega} x \iff f(x_n) \xrightarrow[n\to\infty]{} f(x) \quad \forall f \in X^*$

so $\sup_n |f(x_n)| \leq C = C(f) > 0$.

Rewrite: $\sup_n |F_{x_n}(f)| \leq c(f)$, $F_{x_n} \in X^{**}$

Since X^* is complete, by Thm 4.22,

there exists a constant c s.t. $\sup_n \| F_{x_n} \| \leq c$.

As $\|F_{x_n}\| = \|x_n\|$, we get the desired result. $\square$

Chapter 5 - linear operators

Definition 5.1

Let X, Y be linear spaces. Then a map $A: X \to Y$ is said to be a linear operator if

$A(\alpha x + \beta y) = \alpha Ax + \beta Ay \quad \forall \alpha, \beta \in \mathbb{K}, \ x, y \in X$.

Let X, Y be normed spaces. The linear operator A is continuous if the map A is continuous.

We say that A is bounded if A maps the unit ball $B_X(0,1)$ into a bounded set, i.e.

$A(B_X(0,1)) \subset B_Y(0,R)$ with some $R > 0$.

Theorem 5.2

A linear map A is continuous iff it is bounded.

(See Thm 4.3)

Definition 5.3

The norm of a bounded operator is defined as

$\|A\| = \sup_{\|x\|_X = 1} \|Ax\|_Y$.

Lemma 5.4

Let A be a bounded linear operator. Then

$\|A\| = \sup_{\|x\|_X \leq 1} \|Ax\|_Y = \sup_{x \neq 0} \frac{\|Ax\|_Y}{\|x\|_X}$. (See Lemma 4.5)

If $\|Ax\|_Y \leq R\|x\|_X$ $^{(*)}$ with some $R \geq 0$, then $\|A\| \leq R$. In other words, $\|A\|$ is the best constant in $(*)$.

Lemma 5.5

Let $A: X \to Y$, and let Y be Banach.
Let D be a subspace s.t. $[D] = X$.
Suppose that $A_0 : D \to Y$ is a bounded linear operator.
Then there exists a unique extension A to the entire space X, and $\|A_0\| = \|A\|$.
(See Lemma 4.11, although Y must be Banach in order to find extension).

Examples

1). The zero operator $O: X \to Y$ is defined as $Ox = 0_Y$ $\forall x \in X$. Clearly $\|O\| = 0$, and it is the only operator with norm 0.

2). Let $Y = X$. The identity operator $I: X \to X$ is defined as $Ix = x$ $\forall x \in X$. Clearly $\|I\| = 1$.

3). Let $X = H$, $Y = G$ with H, G Hilbert spaces.
The operator $U: H \to G$ is said to be an isometry if $\|Ux\|_G = \|x\|_H$ $\forall x \in H$.
If $R(U) = U(H) = G$, then U is said to be unitary.
If U is an isometry, then $\|U\| = 1$

$Im = R = $ range

The property $\|Ux\|_Y = \|x\|_H$ is equivalent to $\langle Ux, Uy \rangle_Y = \langle x, y \rangle_X$ $\forall x, y \in X$.
This follows from the polarisation identity.

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4). Let $X = H = L_2(0,1)$,
i.e. $\langle f, g \rangle = \int_0^1 f(t)\overline{g(t)}\,dt$

Let $m \in C[0,1]$ be a function.
Define A by $(Af)(t) = m(t)f(t)$ $\forall f \in L_2(0,1)$, $A: H \to H$
this is the multiplication operator.
A is linear, A is bounded: $\left[\|Ax\|_Y \leq R\|x\|_X \ \ \forall x \in X \right]$

$$\|Af\|^2 = \int_0^1 |m(t)|^2 |f(t)|^2\,dt \leq \max_{t\in[0,1]} |m(t)| \int_0^1 |f(t)|^2\,dt$$

$$= \|m\|_C^2 \|f\|^2$$

$\Rightarrow \|Af\| \leq \|m\|_C \|f\| \Rightarrow \|A\| \leq \|m\|_C$
In fact, $\|A\| = \|m\|_C$ (would need to find a sequence which approaches the desired value).

5). Let $X = C[a,b]$, and let $K: X \to X$ be the integral operator with the kernel
$\mathcal{K} \in C([a,b]\times[a,b])$:
$(Kf)(x) = \int_a^b \mathcal{K}(x,y)f(y)\,dy$. It is linear.
It is also bounded:

$$|(Kf)(x)| \leq \int_a^b |\mathcal{K}(x,y)||f(y)|\,dy$$

$$\leq \|\mathcal{K}\|_C \|f\|_C (b-a)$$

so $\|Kf\|_C \leq R\|f\|_C$, $R = \|\mathcal{K}\|_C(b-a)$
and hence $\|K\| \leq R$.

6). Let $X = C[a,b]$, $Y = L_2(a,b)$
Consider $K: C[a,b] \to L_2(a,b)$
K is linear, as before. Boundedness:

$$\|Kf\|_2^2 = \int_a^b \left| \int_a^b \mathcal{K}(x,y)f(y)\,dy \right|^2 dx \leq \int_a^b \left(\int_a^b |\mathcal{K}(x,y)||f(y)|\,dy \right)^2 dx$$

$$\underset{\text{Cauchy-Schwarz}}{\leq} \int_a^b \left(\int_a^b |\mathcal{K}(x,y)|^2 dy \int_a^b |f(y)|^2 dy \right) dx = \int_a^b\int_a^b |\mathcal{K}(x,y)|^2\,dx\,dy \int_a^b |f(y)|^2\,dy$$

$$\leq \|\mathcal{K}\|_c^2 (b-a)^2 \|f\|_c^2 (b-a) = \|\mathcal{K}\|_c^2 \|f\|_c^2 (b-a)^3$$

So $\|Kf\|_2 \leq R\|f\|_c$ with $R = \|\mathcal{K}\|_c (b-a)^{3/2}$

So $\|K\|_2 \leq \|\mathcal{K}\|_c (b-a)^{3/2}$

01-03-18 $A: X \to Y$, $\|A\| = \sup_{\|x\|_X = 1} \|Ax\|_Y$

7). Let $X = Y = H$. Let $w, z \in H$ and define

$Ax = \langle x, w \rangle z$, $R(A) = \text{span } z$

A is a "one-dimensional operator".

A is linear, since the inner product is linear.

Estimate: $\|Ax\| = |\langle x, w \rangle| \|z\| \leq \|x\| \|w\| \|z\|$

$\Rightarrow \|A\| \leq \|w\| \|z\|$

Take $x = w$, so $\|Aw\| = \|w\|^2 \|z\| = \|w\| \|z\| \|x\|$

Thus $\|A\| = \|w\| \|z\|$

8). Let $X = L_2(-\pi, \pi)$. Define $(Tu)(t) = -iu'(t)$ assuming that u is differentiable. $\left[\|T\| = \sup_{\|x\|=1} \|Tx\| \right]$

T is unbounded. Indeed, let

$e_n(t) = \frac{1}{\sqrt{2\pi}} e^{int}$, $n \in \mathbb{Z}$, so

$\|e_n\| = 1$, $Te_n = n e_n$

$\Rightarrow \|Te_n\| = |n| \to \infty$ as $|n| \to \infty$.

Thus T is unbounded, as claimed.

Need to identify a subspace where T makes sense.

Let $C'[-\pi, \pi] = \{ f \in C[-\pi, \pi] : f' \in C[-\pi, \pi] \}$

Define $D(T)$ (domain of definition, or simply, domain of T) to be $C_2'[-\pi, \pi]$ (C' with L_2 norm)

$D(T) = C_2'[-\pi, \pi] \to$ dense in $L_2(-\pi, \pi)$

Another choice of domain: $\{ f \in C_2'[-\pi, \pi] : f(-\pi) = f(\pi) \}$

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Algebra of bounded operators

Linear structure on the set of linear operators.

Let A, B be bounded. Then

$(A+B)x = Ax + Bx, \quad \forall x \in X.$

$(\alpha B)x = \alpha(Bx), \quad \forall \alpha \in \mathbb{K}$

Let's check that $\|A\|$ is a proper norm:

(i) non-degeneracy:

$\|A\| \geq 0$ always

If $\|A\| = 0$, then $\|Ax\| = \|A\|\|x\| = 0, \quad \forall x \in X.$

so $\|Ax\| = 0 \Rightarrow Ax = 0$, so $A = 0$.

(ii) homogeneity:

$$\|\alpha A\| = \sup_{\|x\|=1} \|\alpha Ax\| = |\alpha| \sup_{\|x\|=1} \|Ax\| = |\alpha|\|A\|.$$

(iii) Δ-inequality:

$\|(A+B)x\| \leq \|Ax\| + \|Bx\| \leq (\|A\| + \|B\|)\|x\|$

$\Rightarrow \|A+B\| \leq \|A\| + \|B\|.$

So we have a normed space of bounded linear operators from X to Y. Notation: $B(X,Y)$.

We also write $B(X) = B(X,X)$.

Theorem 5.6

If Y is Banach, then $B(X,Y)$ is complete

(See Thm 4.16)

Definition 5.7 (product of linear operators)

Let $A \in B(X,Y)$, $B \in B(Y,Z)$. Then the product BA is defined as $(BA)x = B(Ax), \quad \forall x \in X$

BA is linear and bounded

$\|(BA)x\| = \|B(Ax)\| \leq \|B\|\|Ax\| \leq \|B\|\|A\|\|x\|$

$\Rightarrow \|BA\| \leq \|A\|\|B\|.$

Consequently $BA \in B(X, Z)$.
Assume that $X = Y$, and $A, B, C \in B(X)$
By def. 5.7,

$$A(B+C) = AB + AC$$
$$(B+C)A = BA + CA$$
$$B(\alpha A) = \alpha BA$$
$$(AB)C = A(BC)$$

Thus $B(X)$ is an algebra.

Convergence of operators

Definition 5.8

Let $A_n, A \in B(X, Y)$.
We say that A_n converges to A in norm (or uniformly) if $\|A_n - A\| \to 0$ as $n \to \infty$. Notation: $A_n \xrightarrow{n} A$, $n \to \infty$
We say that A_n converges to A strongly, if $A_n x \to Ax$ as $n \to \infty$ for all $x \in X$. Notation: $A_n \xrightarrow{s} A$, $n \to \infty$.
We say that A_n converges to A weakly, if $f(A_n x) \to f(Ax)$ as $n \to \infty$ $\forall x \in X$, $\forall f \in Y^*$. Notation: $A_n \underset{n\to\infty}{\xrightarrow{\omega}} A$

$$A_n \xrightarrow{n} A \Rightarrow A_n \xrightarrow{s} A \Rightarrow A_n \xrightarrow{\omega} A$$

Examples

$H = l_2$

1). $x = (x_1, x_2, \dots)$

Define A_n : $A_n x = (\overbrace{0, \dots, 0}^{n}, x_{n+1}, x_{n+2}, \dots)$
It is clear that $\|A_n\| = 1$ (left as exercise)
On the other hand,

$$\|A_n x\|^2 = \sum_{k=n+1}^{\infty} |x_k|^2 \to 0 \text{ as } n \to \infty \text{ since } x \in l_2$$

Thus $A_n \underset{n\to\infty}{\xrightarrow{s}} 0$

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2). Let $S_n x = (\overbrace{0, \dots, 0}^{n}, x_1, x_2, \dots)$

Then $\|S_n x\| = \|x\| \quad \forall n, \ \forall x \in l_2$

On the other hand, $\forall \xi \in l_2$ we have

$$\langle S_n x, \xi \rangle = \sum_{k=n+1}^{\infty} x_{k-n} \overline{\xi_k} \overset{k-n=l}{=} \sum_{l=1}^{\infty} x_l \overline{\xi_{l+n}}$$

So $$|\langle S_n x, \xi \rangle| \leq \left| \sum_{l=1}^{\infty} x_l \overline{\xi_{l+n}} \right| \leq \|x\| \underbrace{\left(\sum_{l=1}^{\infty} |\xi_{l+n}|^2 \right)^{1/2}}_{\to 0 \text{ as } n \to \infty}$$

Thus $S_n \xrightarrow[n\to\infty]{w} 0$, but $S_n \overset{s}{\underset{n\to\infty}{\not\to}} 0$

<u>Theorem 5.9</u> (Banach - Steinhaus Thm)

Let X, Y be Banach and assume that $M \subset B(X,Y)$ is such that $\{Ax : A \in M\}$ is bounded,

i.e. $\exists C = C(x)$ s.t. $\sup_{A \in M} \|Ax\| \leq C(x)$ for all $x \in X$.

Then M is uniformly bounded, i.e. $\exists C_1 > 0$ s.t. $\sup_{A \in M} \|A\| \leq C_1$

(i.e. $\sup_{A \in M} \sup_{\|x\|=1} \|Ax\|$)

(See Thm 4.22)

<u>Corollary 5.10</u>

Let X, Y be Banach. Consider $A_n \in B(X,Y)$ s.t. $A_n x \xrightarrow[n\to\infty]{s} Ax$ for all $x \in X$ with some linear operator A.

Then the norms $\|A_n\|$ are uniformly bounded and A is bounded.

<u>Proof</u>

Since $A_n x \xrightarrow[n\to\infty]{} Ax$, we have the bound $\|A_n x\| \leq C(x)$ with some $C(x) > 0$.

Thus, by thm 5.9 $\|A_n\| \leq C_1$ for all n with some constant $C_1 > 0$.

As $\underbrace{\|A_n x\|}_{\leq \|A_n\|\|x\| \leq C_1 \|x\|} \xrightarrow[n\to\infty]{} \|Ax\|$, we conclude that $\|Ax\| \leq C_1 \|x\|$

so $\|A\| \leq C_1$ as claimed.

□

Adjoint Operator

Let $X = H$

Definition 5.11

A mapping $\phi : H \times H \to \mathbb{K}$ is said to be a sesquilinear functional if $\phi(\alpha x + \beta y, z) = \alpha \phi(x, z) + \beta \phi(y, z)$, and $\phi(z, \alpha x + \beta y) = \bar{\alpha} \phi(z, x) + \bar{\beta} \phi(z, y)$.

ϕ is said to be continuous (or bounded) if

$$\|\phi\| = \sup_{\|x\| = \|y\| = 1} |\phi(x, y)| < \infty$$

$\|\phi\|$ is called the norm of ϕ.

From the definition : $|\phi(x,y)| \leq \|\phi\| \|x\| \|y\|$.

? > or ≥ Also if $|\phi(x,y)| \leq C \|x\| \|y\|$ with some $C > 0$ then $\|\phi\| \leq C$.

Observe that $\psi(x,y) = \overline{\phi(y,x)}$ is also sesquilinear, and $\|\psi\| = \|\phi\|$

Theorem 5.12

Let ϕ be a bounded sesquilinear functional. Then there exist two uniquely defined operators, $S, T \in B(H)$ s.t. $\phi(x,y) = \langle Tx, y \rangle = \langle x, Sy \rangle$, $\forall x, y \in H$.

Moreover $\|\phi\| = \|S\| = \|T\|$.

Proof

For each fixed y define the linear functional $f_y(x) = \phi(x,y)$.

Observe : f_y is bounded:

$|f_y(x)| = |\phi(x,y)| \leq \|\phi\| \|x\| \|y\|$, so $\|f_y\| \leq \|\phi\| \|y\|$.

Therefore, there exists a uniquely defined $h = h_y$ s.t.

$f_y(x) = \langle x, h \rangle$ and $\|h\| = \|f_y\|$

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Denote $h = Sy$.

Let's prove that S is linear, i.e.

$S(\alpha_1 y_1 + \alpha_2 y_2) = \alpha_1 S y_1 + \alpha_2 S y_2$ (*), $\forall y_1, y_2 \in H$, $\alpha_1, \alpha_2 \in \mathbb{K}$.

By definition of h:

$$\begin{aligned}
\langle x, S(\alpha_1 y_1 + \alpha_2 y_2)\rangle &= \phi(x, \alpha_1 y_1 + \alpha_2 y_2) \\
&= \overline{\alpha_1}\,\phi(x, y_1) + \overline{\alpha_2}\,\phi(x, y_2) \\
&= \overline{\alpha_1}\langle x, Sy_1\rangle + \overline{\alpha_2}\langle x, Sy_2\rangle \\
&= \langle x, \alpha_1 Sy_1\rangle + \langle x, \alpha_2 Sy_2\rangle \\
&= \langle x, \alpha_1 Sy_1 + \alpha_2 Sy_2\rangle \qquad \forall x, y_1, y_2
\end{aligned}$$

As x is arbitrary this leads to (*)

12-03-18 Theorem 5.12 $\left[\|\phi\| = \sup_{\|x\|=\|y\|=1} |\phi(x,y)| < \infty\right]$

Let ϕ be a bounded sesquilinear functional.

Then there are two uniquely defined operators $T, S \in B(H)$ s.t. $\phi(x,y) = \langle Tx, y\rangle = \langle x, Sy\rangle$, $\forall x, y \in H$.

Moreover $\|\phi\| = \|T\| = \|S\|$.

Proof

Let $f_y(x) = \phi(x,y)$ for each fixed $y \in H$.

f is linear and bounded:

$|f_y(x)| \leq \|\phi\|\|y\|\|x\|$

so $\|f_y\| \leq \|\phi\|\|y\|$

By Thm 4.8 $\exists! h \in H$ s.t. $f_y(x) = \langle x, h\rangle$, $\|h\| = \|f_y\|$.

Define $h = Sy$, so $\phi(x,y) = \langle x, Sy\rangle$

S is linear (see previous lecture) and S is bounded:

$\|Sy\| = \|h\| = \|f_y\| \leq \|\phi\|\|y\|$. So $\|S\| \leq \|\phi\|$

On the other hand, $|\phi(x,y)| \leq \|x\|\|Sy\| \leq \|S\|\|x\|\|y\|$

Thus $\|\phi\| \leq \|S\|$. Therefore $\|\phi\| = \|S\|$.

Finally S is uniquely defined:

Indeed if $\langle x, S_1 y\rangle = \langle x, S_2 y\rangle \quad \forall x, y \in H$

$\Rightarrow \langle x, (S_1 - S_2)y\rangle = 0 \quad \forall x, y \in H$

Take $x = (S_1 - S_2)y \Rightarrow \|(S_1 - S_2)y\| = 0 \quad \forall y \in H$

and hence $S_1 - S_2 = 0 \Rightarrow S_1 = S_2$.
Considering the functional $\Psi(x,y) = \overline{\phi(y,x)}$
one proves the existence of T with the required properties. $\square$

Each of the three objects ϕ, T, S defines the other two uniquely.

Definition 5.13
Let $T \in B(H)$.
Then the form $\langle Tx, y \rangle$ is called the sesquilinear form of the operator T.
The operator S s.t. $\langle Tx, y \rangle = \langle x, Sy \rangle \ \forall x, y \in H$
is called the adjoint operator of T.
Notation: $S = T^*$.

If $T = T^*$, it is said to be self-adjoint, or symmetric: $\langle Tx, y \rangle = \langle x, Ty \rangle$.

Theorem 5.14
Let $T, T_1, T_2 \in B(H)$, then
(i) $\|T\| = \|T^*\|$
(ii) $(\alpha T_1 + \beta T_2)^* = \bar{\alpha} T_1^* + \bar{\beta} T_2^*$
(iii) $(T^*)^* = T^{**} = T$
(iv) $(T_1 T_2)^* = T_2^* T_1^*$

Proof: exercise & homework

Examples
1). Let $U \in B(H)$ be an isometry,
i.e. $\|Ux\| = \|x\|$, $\forall x \in H$.
from polarisation identity
Know: $\|U\| = 1$, $\langle Ux, Uy \rangle = \langle x, y \rangle^{(*)} \ \forall x, y \in H$
It follows from $(*)$: $\langle Ux, Uy \rangle = \langle U^*Ux, y \rangle = \langle x, y \rangle, \forall x, y \in H$
and hence $U^*U = I$.

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2). Let P be an orthogonal projection

This means that $R(P) = Y$ is a closed subspace,

and $(x - Px) \perp Y \quad \forall x \in H$.

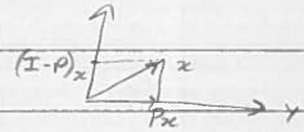
<u>Claim</u>: P is bounded and $P = P^*$.

Bounded: by Pythagoras:

$$\|Px\|^2 + \|(I-P)x\|^2 = \|x\|^2, \quad \forall x \in H$$

so $\|Px\| \leq \|x\|$, i.e. $\|P\| \leq 1$

$P = P^*$: want: $\langle Px, y \rangle = \langle x, Py \rangle, \quad \forall x, y \in H$.



Write:
$$\begin{aligned}
\langle Px, y \rangle &= \langle Px, Py + (I-P)y \rangle \\
&= \langle Px, Py \rangle + \underbrace{\langle Px, (I-P)y \rangle}_{= 0} \\
&= \langle Px, Py \rangle + \underbrace{\langle (I-P)x, Py \rangle}_{= 0} \\
&= \langle x, Py \rangle, \quad \forall x, y \in H
\end{aligned}$$

<u>Exercise</u>

A bounded operator P is an orthogonal projection iff $P^2 = P$ and $P = P^*$

3). Integral operator on $L_2(a,b)$.

Let $(Ku)(x) = \int_a^b \mathcal{K}(x,y) u(y)\, dy, \quad u \in L_2(a,b)$.

Assume that $\|\mathcal{K}\|_{HS} = \left[\int_a^b \int_a^b |\mathcal{K}(x,y)|^2 \,dx\,dy\right]^{1/2} < \infty$

Then K is bounded:

$$\|Ku\|^2 = \int_a^b \left| \int_a^b \mathcal{K}(x,y) u(y)\, dy \right|^2 dx$$

$$\overset{C-S}{\leq} \int_a^b \int_a^b |\mathcal{K}(x,y)|^2 \,dx\,dy \int_a^b |u(y)|^2 \,dy = \|\mathcal{K}\|_{HS}^2 \|u\|^2$$

So $\|K\| \leq \|\mathcal{K}\|_{HS}$

Operators with $\|\mathcal{K}\|_{HS} < \infty$ are called Hilbert-Schmidt operators.

Find K^*: Want: $\langle Ku, v \rangle = \langle u, K^* v \rangle \quad \forall u, v \in L_2$

Write:

$$\langle Ku, v\rangle = \int_a^b\int_a^b \mathcal{K}(x,y)\,u(y)\,\overline{v(x)}\,dy\,dx$$

$$= \int_a^b u(y)\left(\int_a^b \mathcal{K}(x,y)\,\overline{v(x)}\,dx\right)dy$$

$$= \int_a^b u(y)\left(\overline{\int_a^b \overline{\mathcal{K}(x,y)}\,v(x)\,dx}\right)dy$$

$$= \langle u, K^*v\rangle \quad \text{where } (K^*v)(y) = \int_a^b \overline{\mathcal{K}(x,y)}\,v(x)\,dx$$

or, which is the same as $(K^*v)(x) = \int_a^b \overline{\mathcal{K}(y,x)}\,v(y)\,dy$

Thus the kernel of the adjoint is $\overline{\mathcal{K}(y,x)}$.

Examples: $\sin(x-y) \longrightarrow \sin(y-x) = -\sin(x-y)$

$i\sin(x-y) \longrightarrow i\sin(x-y)$ self-adjoint.

$e^{x^2+y^2} \longrightarrow e^{x^2+y^2}$ self-adjoint.

$e^{i(x-y)} \longrightarrow e^{i(x-y)}$ self-adjoint.

4). $H = L_2(0,1)$

Let $(Tu)(x) = \int_0^x u(y)\,dy$, this is called a Voltera operator

Rewrite T using the function $\theta(s) = \begin{cases} 1 & \text{if } s>0 \\ 0 & \text{if } s\le 0\end{cases}$

Then $(Tu)(x) = \int_0^1 \underbrace{\theta(x-y)}_{t(x,y)}u(y)\,dy$,

so $\|t\|_{HS} = \frac{1}{\sqrt{2}}$, so $\|T\| \le \frac{1}{\sqrt{2}}$

The adjoint: $(T^*u)(x) = \int_0^1 \theta(y-x)\,u(y)\,dy = \int_x^1 u(y)\,dy$.

5). Let $H = L_2(a,b)$ and $(Tu)(x) = -iu'(x)$

on $D(T) = C_2^1[a,b]$

Want to find "the adjoint",

i.e. the operator T^* s.t. $\langle Tu, v\rangle = \langle u, T^*v\rangle$.

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Let $u, v \in D(T)$, and write

$$\langle Tu, v\rangle = \int_a^b -iu'(x)\overline{v(x)}\,dx$$

integrate by parts

$$= -iu(x)\overline{v(x)}\Big|_a^b + i\int_a^b u(x)\overline{v'(x)}\,dx$$

$$= -i\left(u(b)\overline{v(b)} - u(a)\overline{v(a)}\right) + \int_a^b u(x)\overline{(-iv'(x))}\,dx$$

$$= -i\left(u(b)\overline{v(b)} - u(a)\overline{v(a)}\right) + \langle u, Tv\rangle$$

Define a new domain $D'(T) = \{u \in C_2^1[a,b] : u(a) = u(b) = 0\}$
Then $\langle Tu, v\rangle = \langle u, Tv\rangle$ for all $u, v \in D'(T)$
Thus T is symmetric on $D'(T)$.

Theorem 5.15

(i) $A \in B(H)$ is weakly convergent.
i.e. $x_n \xrightarrow[n\to\infty]{\omega} x$, then $Ax_n \xrightarrow[n\to\infty]{\omega} Ax$

(ii) If $A_n \xrightarrow[n\to\infty]{\omega} A$, then $A_n{}^* \xrightarrow[n\to\infty]{\omega} A^*$

Proof

(i) Write for arbitrary $y \in H$:
$$\langle Ax_n, y\rangle = \langle x_n, A^*y\rangle \xrightarrow[n\to\infty]{} \langle x, A^*y\rangle = \langle Ax, y\rangle$$
so $Ax_n \xrightarrow{\omega} Ax$ as $n\to\infty$.

(ii) Have: $\langle A_n x, y\rangle \xrightarrow[n\to\infty]{} \langle Ax, y\rangle$, $\forall x, y \in H$

Write: $\langle A_n^* x, y\rangle = \langle x, A_n y\rangle \to \langle x, Ay\rangle = \langle A^*x, y\rangle$
so $A_n^* \xrightarrow{\omega} A^*$ as $n\to\infty$.

$\square$

Another example:
If $A_n \to A$ uniformly, then $A_n{}^* \to A^*$ uniformly.
Reason: $\|A_n - A\| = \|A_n{}^* - A\|$
(Note strong does not imply strong!)

Definition 5.16
Let $A \in \mathcal{B}(X, Y)$.
The set $\{x \in X : Ax = 0\}$ is called the kernel of A.
Notation: $\operatorname{Ker}(A)$.

Lemma 5.17
For any $A \in \mathcal{B}(X, Y)$, the kernel $\operatorname{Ker} A$ is a closed subspace, $R(A)$ is a subspace.
A is injective iff $\operatorname{Ker} A = \{0\}$.

Proof
Injectivity left as exercise.
$R(A)$ is a subspace:
Let $y_1, y_2 \in R(A)$. WTS: $\alpha_1 y_1 + \alpha_2 y_2 \in R(A)$, $\forall \alpha_1, \alpha_2 \in \mathbb{K}$.
Let $y_1 = Ax_1$, $y_2 = Ax_2$ with some $x_1, x_2 \in X$.
Therefore
$$\alpha_1 y_1 + \alpha_2 y_2 = \alpha_1 A x_1 + \alpha_2 A x_2 = A(\alpha_1 x_1 + \alpha_2 x_2) \in R(A).$$

15-03-18 X, Y, H
Take H:
$A \in \mathcal{B}(H)$, $\langle Ax, y \rangle = \langle x, A^* y \rangle$, $\forall x, y \in H$.

Lemma 5.17
Let $A \in \mathcal{B}(X, Y)$ Then $R(A)$ is a subspace,
$\operatorname{Ker} A$ is a closed subspace and A is injective $\Leftrightarrow$ $\operatorname{Ker} A = \{0\}$.

Proof
Let $x_n \xrightarrow[n \to \infty]{} x$, $x_n \in \operatorname{Ker} A$. Then
$0 = Ax_n \xrightarrow[n \to \infty]{} Ax$, so $Ax = 0$ i.e. $x \in \operatorname{Ker} A$.
(see above for rest of proof) $\square$

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Theorem 5.18
Assume $A \subset \mathcal{B}(H)$. Then
$[R(A)] \oplus \operatorname{Ker} A^* = H$

Proof
Let $x \in H$. To find $R(A)^\perp$ (the annihilator)
write $\langle Ax, z \rangle = 0 \quad \forall x \in H$.
$0 = \langle Ax, z \rangle = \langle x, A^* z \rangle \quad \forall x \in H$
This means that $z \in R(A)^\perp$ iff $z \in \operatorname{Ker} A^*$.
Since $R(A)^\perp = [R(A)]^\perp$, we have two closed subspaces, s.t. each of them is the orthogonal complement of the other, so $[R(A)] \oplus \operatorname{Ker} A^* = H$, as required. $\square$

The inverse operator
Want to solve the equation $Ax = y$ where
$A \in \mathcal{B}(X, Y)$, and $x \in X$ is the unknown, $y \in Y$ is given.
If $y \in R(A)$ then there is a solution

Definition 5.19
Let $A \in \mathcal{B}(X, Y)$ be injective.
Then the inverse operator A^{-1} is defined as the map that associates to every $y \in R(A)$ the vector $x \in X$ uniquely defined by the formula $Ax = y$.
i.e. $x = A^{-1} y$.
We say that A is invertible on $R(A)$.

Observe: $A^{-1} A x = x, \ \forall x \in X \quad , \ A A^{-1} y = y, \ \forall y \in R(A)$
$[A^{-1} A = I_X]$

Theorem 5.20
If A^{-1} exists, it is a linear operator on $R(A)$.

Proof
Want: $A^{-1}(\alpha_1 y_1 + \alpha_2 y_2) = \alpha_1 A^{-1} y_1 + \alpha_2 A^{-1} y_2 \quad \forall \alpha_1, \alpha_2 \in \mathbb{K}, \forall y_1, y_2 \in R(A)$

Let $x_1 = A^{-1} y_1$, $x_2 = A^{-1} y_2$, then
the r.h.s. $= \alpha_1 x_1 + \alpha_2 x_2$
the l.h.s. $= A^{-1}(\alpha_1 A x_1 + \alpha_2 A x_2)$
$= A^{-1} A (\alpha_1 x_1 + \alpha_2 x_2) = \alpha_1 x_1 + \alpha_2 x_2$
as claimed. □

Note: If B is a map from $R(A)$ to X s.t. $BA = Id_X$
then A is invertible and $B = A^{-1}$.

Indeed $BAx = x$, $x \in X$ means that $Ker(A) = \{0\}$
Thus A is injective.
Also, let $Ax = y$, so $By = x$ and hence $B = A^{-1}$.

Example
Let U be an isometry on H, so $U^* U = I$.
Thus $U^{-1} = U^*$.

Theorem 5.21
Let $A \in B(X, Y)$. Then A has an inverse
$A^{-1} \in B(Y, X)$ iff there is an operator $B \in B(Y, X)$
s.t. $BA = Id_X$ and $AB = Id_Y$.
Moreover $B = A^{-1}$.

Proof
$BA = I_X$ implies that A is injective, so $B = A^{-1}$.
$AB = I_Y$ implies that A is surjective
Indeed, for every $y \in Y$ we have $y = A(By) \in R(A)$.
□

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Lemma 5.22

Let $A, A^{-1} \in B(H)$. Then A^* is invertible and $(A^*)^{-1} = (A^{-1})^*$.

Proof

Let $B = (A^{-1})^*$, so

$(A^{-1})^* A^* = (AA^{-1})^* = I^* = I$

$A^*(A^{-1})^* = (A^{-1}A)^* = I^* = I$

By Thm 5.21 $(A^*)^{-1} = (A^{-1})^*$ □

Theorem 5.23

Let $A, B, A^{-1}, B^{-1} \in B(X)$.

Then (i) AB is invertible and $(AB)^{-1} = B^{-1}A^{-1}$

(ii) The resolvant identity holds:

if $V = B - A$, then $A^{-1} - B^{-1} = A^{-1}VB^{-1} = B^{-1}VA^{-1}$

Proof

(i) Use Thm 5.21: Let $T = B^{-1}A^{-1}$

$TAB = B^{-1}A^{-1}AB = B^{-1}B = I$

$ABT = ABB^{-1}A^{-1} = AA^{-1} = I$

(ii) Write $A^{-1} - B^{-1} = A^{-1}(I - AB^{-1}) = A^{-1}(B-A)B^{-1} = A^{-1}VB^{-1}$

$A^{-1} - B^{-1} = B^{-1}(BA^{-1} - I) = B^{-1}(B-A)A^{-1} = B^{-1}VA^{-1}$.

□

see 19.3.18 for correct statement

Theorem 5.24 (X is Banach).

Let $A \in B(X,Y)$, and assume that $\|Ax\| \geq c\|x\|$, $\forall x \in X$, with some $c > 0$. Then A is invertible, A^{-1} is bounded on $R(A)$ and $R(A)$ is closed.

Proof

It is clear that $\operatorname{Ker} A = \{0\}$, so A^{-1} exists.

Write for every $y \in R(A)$: $x = A^{-1}y$,

so $\|y\| \geq c\|A^{-1}y\|$, $\forall y \in R(A)$

So $\|A^{-1}y\| \le c^{-1}\|y\|$ so $\|A^{-1}\| \le c^{-1}$.

$R(A)$ closed?

Let $y_n \in R(A)$, $y_n \xrightarrow[n\to\infty]{} y$. Denote $x_n = A^{-1}y_n$.

Since y_n is Cauchy, x_n is also Cauchy, by continuity of A^{-1}.

Since X is Banach (i.e. complete), x_n has a limit, $x = \lim_{n\to\infty} x_n$. By continuity of A, $Ax_n \xrightarrow[n\to\infty]{} Ax$.

At the same time, $Ax_n = y_n \xrightarrow[n\to\infty]{} y$, so $y = Ax \in R(A)$

□

Theorem 5.25 (Lax-Milgram Thm)

Let H be a real Hilbert space.

Let $\phi(x,y)$ be a bounded bilinear form.

i.e. $|\phi(x,y)| \le c\|x\|\|y\|$, $c>0$, $\forall x,y \in H$.

Assume that for some positive $\beta > 0$ we have

$\phi(x,x) \ge \beta\|x\|^2$, $\forall x \in H$, i.e. ϕ is coercive.

Then for any $v \in H$, $\exists$ a unique vector $u \in H$ s.t. $\phi(u,y) = \langle v, y\rangle$ $(*)$, $\forall y \in H$.

Proof

Let $A \in B(H)$ be the uniquely defined operator s.t. $\phi(x,y) = \langle Ax, y\rangle$ $\forall x,y \in H$.

This operator is invertible. Indeed,

$\beta\|x\|^2 \le \langle Ax, x\rangle \le \|Ax\|\|x\|$ so $\|Ax\| \ge \beta\|x\|$ $\forall x \in H$

By Thm 5.24, A^{-1} is bounded on $R(A)$, and $R(A)$ is closed.

Claim: $R(A) = H$.

Indeed, let $w \in R(A)^\perp$. Then

$\beta\|w\|^2 \le \langle Aw, w\rangle = 0$

$\Rightarrow w = 0$ as claimed.

Rewrite $(*)$ $\langle Au, y\rangle = \langle v, y\rangle$, $\forall y \in H$

i.e. $Au = v$, so $u = A^{-1}v$ is uniquely defined.

□

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$\beta \|x\|^2 \leq \phi(x,x)$

If H is complex, then in order to have the coexivity we need to assume that the sesquilinear form $\phi(x,y)$ is symmetric, i.e. $\phi(x,y) = \overline{\phi(y,x)}$

In the real case no symmetry is required.

Let T be the operator $Tu = -u''$, for $u \in C_2^2[0,1]$, $u(0) = u(1) = 0$. Thus

$$\langle Tu, u \rangle = \int_0^1 -u'' u \, dx = \int_0^1 (u')^2 \, dx \qquad \text{(integration by parts)}$$

Then the coexivity holds:

$$\int_0^1 (u')^2 \, dx \geq \beta \int_0^1 u^2 \, dx \quad , \quad \forall u \in C_2^2[0,1] \, , \; u(0) = u(1) = 0$$

with some positive β.

19-03-18 Theorem 5.24 (correct statement!)

Let $A \in B(X,Y)$. If $\|Ax\| \geq c\|x\|$ $\forall x \in X$ with some $c > 0$, then A is invertible on $R(A)$ and $\|A^{-1}\| \leq c^{-1}$.

Moreover, if X is complete, then $R(A)$ is closed.

Theorem 5.26

Let X be Banach. Let $A \in B(X)$ be such that $\|A\| < 1$. Then the inverse $(I-A)^{-1}$ exists, it is bounded, and it is given by the formula

$$(I-A)^{-1} = \sum_{k=0}^{\infty} A^k, \text{ where the series converges uniformly.}$$

Here $A^0 = I$.

Proof

Let $S_n = \sum_{k=0}^{n} A^k$. WTP: S_n converges uniformly as $n \to \infty$.

Estimate: $\|S_n - S_m\| = \left\| \sum_{k=m+1}^{n} A^k \right\|$ assuming $m < n$

$$\leq \sum_{k=m+1}^{n} \|A\|^k \leq \|A\|^{m+1} \sum_{k=0}^{\infty} \|A\|^k = \|A\|^{m+1} \frac{1}{1 - \|A\|}$$

$\to 0$ as $n \to \infty$ since $\|A\| < 1$.

Since X is complete, by Thm 5.6 $B(X)$ is complete.
Thus $S_n \xrightarrow[n\to\infty]{} S \in B(X)$.

Write $S = \sum_{k=0}^{\infty} A^k$.

Now we need to check that $(I-A)S = I$ and $S(I-A) = I$.
By Thm 5.21 this would imply that $S = (I-A)^{-1}$
Write $S(I-A) = u\text{-}\lim_{n\to\infty} S_n(I-A)$.

Calculate:

$$S_n(I-A) = \sum_{k=0}^{n} A^k(I-A) = \sum_{k=0}^{n}(A^k - A^{k+1})$$

$$= \sum_{k=0}^{n} A^k - \sum_{k=1}^{n+1} A^k = A^0 - A^{n+1}$$

$$= I - A^{n+1} \xrightarrow{n\to\infty} I \quad \text{since } \|A\| < 1$$

So $S(I-A) = I$
$(I-A)S = I$ is proved similarly. □

The formula $(I-A)^{-1} = \sum_{k=0}^{\infty} A^k$ is called the von Neumann series.

The Open Mapping Theorem and its consequences

Definition 5.27
Let X, Y be metric spaces. Then the mapping $f: X \to Y$ is said to be open if it maps open sets into open sets.

Theorem 5.28 (The Open Mapping Theorem)
Let X, Y be Banach spaces. Suppose that $A \in B(X,Y)$ is a surjection. Then A is an open mapping.

Proof omitted.

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Corollary 5.29

Let X, Y be Banach spaces. Suppose that $A \in B(X,Y)$ is a bijection. Then A^{-1} is bounded.

Proof

By Thm 5.28, for each open $M \subset X$ the set $A(M)$ is also open: $M \xrightarrow{A} A(M) = D$.

For the operator A^{-1}: D is the pre-image of M.

The pre-image of every open M is open, and hence A^{-1} is continuous, i.e. it is bounded. $\square$

Closed operators, graphs

We do not assume that A is bounded.

Let $D(A) \subset X$ be its domain, i.e. $D(A)$ is a linear subspace, and $\|Ax\|_Y < \infty \quad \forall x \in D(A)$.

For bounded A: if $x_n \xrightarrow[n\to\infty]{} x$, then $Ax_n \xrightarrow[n\to\infty]{} Ax$.

For unbounded A. $x_n \xrightarrow[n\to\infty]{} x$, $x_n \in D(A)$, $Ax_n \to y$?

Definition 5.30

Let A be a linear operator with the domain $D(A)$.

Then the operator A is said to be closed if the convergences $x_n \to x$, $Ax_n \to y$, $x_n \in D(A)$ imply $x \in D(A)$ and $y = Ax$.

Bounded operators are closed:

For $A \in B(X,Y)$, if $x_n \to x$, then $Ax_n \to Ax$, so A is closed.

For arbitrary linear spaces X, Y, define the direct sum as the set of pairs (x,y) with $x \in X$, $y \in Y$ such that $(x_1 + x_2, y_1 + y_2) = (x_1, y_1) + (x_2, y_2)$ and $\alpha(x,y) = (\alpha x, \alpha y) \ \forall \alpha \in \mathbb{K}$.

Notation $X \oplus Y$.

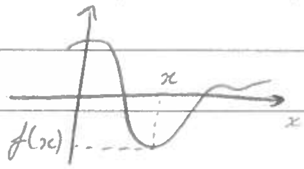
If X, Y are normed spaces, then $X \oplus Y$ is also a normed space with the norm $\|(x,y)\| = \|x\|_X + \|y\|_Y$

If X, Y are Banach, then $X \oplus Y$ is also Banach.

Definition 5.31

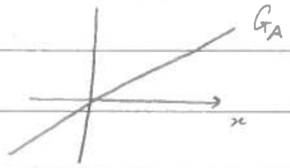
Let A be a linear operator with the domain $D(A)$. Then the graph of A is defined as the set
$G_A = \{(x, Ax) : x \in D(A)\} \subset X \oplus Y$.

Recall: if $f: \mathbb{R} \to \mathbb{R}$, then $\{(x, f(x)), x \in \mathbb{R}\}$ is the graph of f



Example

Let $A: \mathbb{R} \to \mathbb{R}$ be linear. Then $Ax = ax$ with some constant $a \in \mathbb{R}$.



Theorem 5.32

The graph G_A is a linear subspace.
The graph is closed iff A is closed.

[A is closed if $\left.\begin{matrix} x_n \to x, \ x_n \in D(A) \\ Ax_n \to y \end{matrix}\right\} \Rightarrow x \in D(A), \ Ax = y$]

Proof

Let $x_1, x_2 \in D(A)$, so $(x_1, Ax_1) + (x_2, Ax_2) = (x_1 + x_2, A(x_1 + x_2)) \in G_A$

Also $\forall \alpha \in \mathbb{K}$: $\alpha(x, Ax) = (\alpha x, \alpha Ax) = (\alpha x, A(\alpha x)) \in G_A$

Thus G_A is a subspace.

Recall: $\|(x, Ax)\| = \|x\|_X + \|Ax\|_Y$

Suppose that A is closed WTS: G_A is closed.

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i.e. if $(x_n, Ax_n) \xrightarrow[n\to\infty]{} (x,y)$, then $(x,y) \in G_A$, i.e. $\underset{\wedge}{}^{x\in D(A)}$ $y = Ax$.

If $x_n \xrightarrow[n\to\infty]{} x$, $x_n \in D(A)$, $Ax_n \xrightarrow[n\to\infty]{} y$ then by Def^n 5.30
$x \in D(A)$ and $y = Ax$ as required.
Suppose G_A is closed, i.e. if $(x_n, Ax_n) \xrightarrow[n\to\infty]{} (x,y)$, $x_n \in D(A)$,
then $x \in D(A)$ and $y = Ax$. $\left[\|(x,y)\| = \|x\|_X + \|y\|_Y\right]$
This is the def^n of a closed operator.
So A is closed. $\square$

<u>Theorem 5.33</u> (The Closed Graph Theorem)
<u>Let X, Y be Banach, and let A be an operator</u>
<u>with $D(A) = X$. Then if A is closed, then A is bounded.</u>

<u>Proof</u>
A is closed $\Leftrightarrow$ G_A is closed.
Thus G_A can be viewed as a Banach space.
Define the operator $P(x, Ax) = x \quad \forall x \in X$.
P is a bijection.
P is bounded:
$\|P(x, Ax)\| = \|x\|_X \leq \|x\|_X + \|Ax\|_Y = \|(x, Ax)\|$
So $\|P\| \leq 1$.
By Corollary 5.29, P^{-1} is bounded: $\|P^{-1}\| = K < \infty$
Thus $(x, Ax) = P^{-1}x$ and $\|(x, Ax)\| = \|x\|_X + \|Ax\|_Y \leq \|P^{-1}\|\|x\|_X = K\|x\|_X$
so that $\|Ax\|_Y \leq K\|x\|_X$
Hence A is bounded. $\square$

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<u>Corollary 5.29</u>
Let X, Y be Banach.
Suppose that $A \in B(X,Y)$ is bijective.
Then A^{-1} is bounded.

A bounded $\Rightarrow$ $x_n \xrightarrow[n\to\infty]{} x \Rightarrow Ax_n \xrightarrow[n\to\infty]{} Ax$

A general operator:
Domain $D(A) \subset X$. A is closed if:
$\underset{D(A)}{\overset{\cap}{x_n}} \xrightarrow[n\to\infty]{} x$, $Ax_n \xrightarrow[n\to\infty]{} y \Rightarrow x \in D(A)$ and $y = Ax$.

Graph: $G_A = \{(x, Ax), x \in D(A)\} \subset X \oplus Y$
$\|(x,y)\| = \|x\|_X + \|y\|_Y$

<u>Theorem 5.32</u>
G_A is a linear subspace of $X \oplus Y$.
G_A is closed iff A is closed.

<u>Theorem 5.33</u> (Closed Graph Thm)
Let X, Y be Banach. Suppose that $D(A) = X$, and A is closed. Then A is bounded.

<u>Non-orthogonal projection</u>
Let X be a Banach space, and let $Y, Z \subset X$ be closed subspaces s.t. $X = Y \oplus Z$, i.e. each x is uniquely represented as $x = y + z$ where $y \in Y$, $z \in Z$.
Then the mapping $\pi : x \mapsto y$ is called the (non-orthogonal) projection of x onto Y.

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Corollary 5.34
The projection π is a bounded operator.

Proof
Want to show that G_π is closed.
Write: $G_\pi = \{(x,y), x \in X\}$
Assume that $(x_n, y_n) \to (x,y)$ as $n \to \infty$.
Then $z_n = x_n - y_n \in Z$ also converges:
$\lim_{n\to\infty} z_n = z = x - y \in Z$
i.e. $x = \underset{\in Y}{y} + \underset{\in Z}{z}$

Thus by the definition of π, we have $y = \pi x$
so $(x,y) \in G_\pi \Rightarrow G_\pi$ is closed.
Since $D(\pi) = X$, G_π is closed, by the Closed Graph Thm (5.33), π is bounded, as claimed. $\square$

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