3019 Multivariable Analysis Notes

Based on the 2017 autumn lectures by Dr E Zatorska

MATH 3109

02-10-17 Multivariable Analysis

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Office hour: Mondays 14:00 - 14:45

Lectures: Monday 11:00 - 13:00

Wednesday 9:00 - 11:00

Exceptions: No class Oct 4th.

October 11th, 18th, 25th 10-11 am only.

Homework: On moodle, Wednesday mornings

Due: Wednesday 12:00, in 3109 drop box.

Best 8 of 9 homeworks

90% exam, 10% coursework

Recommended texts: Michael Spivak: Calculus on Manifolds

(Walter Rudin: Principles of Mathematical Analisis (Ch 9-10)

Overview

1). Functions on Euclidean Space
- norm & inner product
- subsets of Euclidean space
- functions and continuity

2). Differentiation
- partial derivatives
- derivatives
- Inverse Function Theorem
- Implicit Function Theorem

3). Integration in higher dimensions
- measure zero and content zero
- integrable functions
- Fubini's Theorem
- change of variables.

4). Integration on chains
- algebraic preliminaries
- fields and forms
- geometric preliminaries
- fundamental theorem of calculus

5). Integration on Manifolds
- (sub) manifolds
- fields and forms on manifolds
- Stokes Theorem on manifolds
- the volume element
- Classical Stokes' Theorem

Differentiation

Recall that in 1D functions $f: (a,b) \to \mathbb{R}$,
f diff at $x_0 \in (a,b)$ if $\lim_{h \to 0} \left(\frac{f(x_0+h) - f(x_0)}{h} \right) := f'(x_0)$ exists.

$f: A \subset \mathbb{R}^n \to \mathbb{R}^m$ $\quad n,m > 1$ (prev. def. doesn't work here)
We can rewrite the 1D defn. into an equivalent one:
$f(x_0+h) = f(x_0) + f'(x_0) \cdot h + R(x_0, h)$ such that
$\lim_{|h| \to 0} \left| \frac{R(x_0, h)}{h} \right| = 0.$

In multi-d
$(a,b) \dashrightarrow \overset{\text{open}}{U} \subset \mathbb{R}^m$
$f: U \to \mathbb{R}^k$ is diff at $x_0 \in U$ if $\exists$ a linear map
$L_x : \mathbb{R}^m \to \mathbb{R}^k$ s.t. $\forall h \in \mathbb{R}^n$
$f(x+h) = f(x) + L_x(h) + R(x,h)$
s.t. $\lim_{|h| \to 0} \left| \frac{R(x,h)}{h} \right| = 0.$

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Integration in higher dimensions

Let $I^n \subset \mathbb{R}^n$ be the unit cube

and $f : I^n \mapsto \mathbb{R}$ be "integrable".

* Fubini's theorem states that $(x \in I^m)$

$$\int_{I^m} f(x)\,dx = \underbrace{\int_0^n \cdots \left(\int_0^n}_{n} f(x_n, \ldots x_1)\,dx_1 \right) \ldots\, dx_n$$

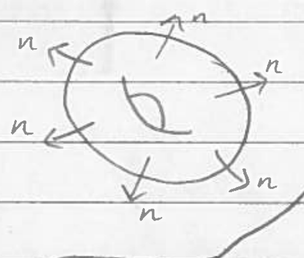
Integration on Manifolds

Theorem (Gauss / Divergence Thm)

Let $\Omega \subset \mathbb{R}^n$ open, bounded, with smooth boundary, and let $\vec{n}$ be the unit outer normal.

Let $X : \overline{\Omega} \to \mathbb{R}^n$ be a differentiable vector field. $\overline{X} = (X_1, \ldots, X_n)$

Then $$\int_\Omega \operatorname{div} X \, \underset{\substack{\uparrow \\ \text{volume element}}}{dV} = \int_{\partial\Omega} X \cdot \vec{n} \, \underset{\substack{\uparrow \\ \text{surface element}}}{d\sigma} \qquad \left[\operatorname{div} X = \sum_{i=1}^n \frac{\partial X_i}{\partial x_i}(x) \right]$$

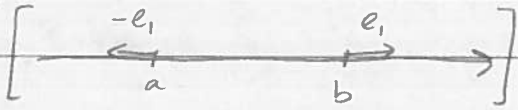


1D case:

$\overline{\Omega} \to [a,b]$ $\quad \vec{n} \dashrightarrow \begin{cases} -e_1 & \text{at } x = a \\ e_1 & \text{at } x = b \end{cases}$

$$X = f(x) e_1$$

$$\int_a^b f'(x)\,dx = \int_{(a,b)} \operatorname{div}(f e_1)\,dV = \int_\Omega \operatorname{div} X\,dV = \int_{\partial\Omega} X \cdot \vec{n}\,d\sigma$$



$$= \int_{\partial\Omega} f\vec{e}_1 \cdot (\pm \vec{e}_1)\,d\sigma$$

$$= f(a)(-1) + f(b)(1) = f(b) - f(a)$$

In general $M \subset \mathbb{R}^n$ is a "k-dimensional" "submanifold" "orientable" "compact" "with boundary" and ω is a "$(k-1)$-form" on M. Then the Stokes theorem becomes

$$\int_M d\omega = \int_{\partial M} \omega .$$ The straight forward computation in a complex language.

↑ last chapter

§1 Functions in Euclidean Space

$\mathbb{R}^n = \{(x^1, \ldots, x^n) \mid x^i \in \mathbb{R}, \; i = 1, \ldots, n\}$

n-tuples of real numbers.

Some properties

- $\mathbb{R}^n$ is a vector space
- the norm $|x| = \left(\sum_{i=1}^{n} (x^i)^2\right)^{1/2}$
- the inner product $\langle x, y\rangle = \sum_{i=1}^{n} x^i y^i$
- So, $\mathbb{R}^n$ is the inner-product vector space normed.
- $\mathbb{R}^n$ is also a metric space.

Notation

- The vectors (elements or points in $\mathbb{R}^n$) will be denoted by a single letter $x = (x^1, \ldots, x^n)$
- The vector $(0, \ldots, 0)$ will be denoted by 0
- The standard basis $e_1, \ldots, e_n$ with $e^i = (0, \ldots, 0, \underset{\substack{\uparrow \\ \text{ith place.}}}{1}, 0, \ldots, 0)$

Linear transformations

Let $T : \mathbb{R}^n \mapsto \mathbb{R}^m$ be linear.

(i.e. $T(x+y) = T(x) + T(y)$ and $T(\lambda x) = \lambda T(x)$.)

We assign to T a matrix $A = (a_{ij}) \in M_{m \times n}$

w.r.t. the standard basis.

$T(e_i) = \sum_{j=n}^{m} a_{ji} e_j$

Thus if $T(x) = y$, $y^j = \sum_{i=n}^{m} a_{ji} x^i$.

$$\begin{pmatrix} y^1 \\ \vdots \\ y^m \end{pmatrix} = \begin{pmatrix} a_{11} & \cdots & a_{1m} \\ \vdots & & \vdots \\ a_{m1} & \cdots & a_{mm} \end{pmatrix} \begin{pmatrix} x^1 \\ \vdots \\ x^n \end{pmatrix}$$

Let $S : \mathbb{R}^m \mapsto \mathbb{R}^p$ with matrix $B = (b_{rs}) \in M_{p \times m}$

$S \circ T : \mathbb{R}^n \mapsto \mathbb{R}^p$ (linear map)

with the matrix $B \cdot A$

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Subsets of Euclidean Space

Recall that the open ball centred at $x \in \mathbb{R}^n$ with radius $r > 0$ is $B_r(x) = \{y \in \mathbb{R}^n : |x - y| < r\}$

Definition

A set $\Omega \in \mathbb{R}^n$ is called open if $\forall x \in \Omega\ \exists r > 0$ s.t. $B_r(x) \subset \Omega$

Definition

A set $\Omega \in \mathbb{R}^n$ is called closed if $\Omega^c = \mathbb{R}^n \setminus \Omega$ is open (then $\Omega = \bar{\Omega}$)

Definition

A set $\Omega \in \mathbb{R}^n$ is called compact if it satisfies:
For every family of open sets in $\mathbb{R}^n$
$\{\Gamma_\alpha\}_{\alpha \in \Lambda}$ covering the set Ω ($\Omega \subset \bigcup_{\alpha \in \Lambda} \Gamma_\alpha$),

there exists a finite number $N \in \mathbb{N}$ and $\alpha_1, \dots, \alpha_N \in \Lambda$ s.t. $\Omega \subset \bigcup_{i=1}^{N} \Gamma_{\alpha_i}$

Theorem

The following statements are equivalent for $\Omega \subset \mathbb{R}^n$:
1). Ω is compact
2). Ω is closed and bounded
3). Ω is sequentially compact
[every sequence of $(x_i)_{i \in \mathbb{N}}$, $x_i \in \Omega$, $\forall i \in \mathbb{N}$, contains a compact subsequence].

Remark: The proof is from Analysis 4.

Functions and Continuity

Let $A \subset \mathbb{R}^n$ $f : A \mapsto \mathbb{R}^m$ is
- vector valued
- $m = 1 \Rightarrow$ scalar function
- $m = n$ vector field.

We can write $\forall x \in A$

$$f(x) = \left(f^1(x), \dots, f^m(x)\right)$$

with $f^i : A \mapsto \mathbb{R}$ $\quad i = 1, \dots, m$

i.e. $f^i(x) = (\pi^i \circ f)(x)$, where

$\pi^i : \mathbb{R}^m \to \mathbb{R}$ is a projection on the ith coordinate

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[dr, "f^i"'] & \mathbb{R}^m \arrow[d, "\pi^i"] \\
& \mathbb{R}
\end{tikzcd}$$

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Def

We will call graph(f), $f: A \subset \mathbb{R}^n \mapsto \mathbb{R}^m$

graph$(f) := \{(x, f(x)) : x \in A\} \subset \mathbb{R}^{n+m}$

Def

We say that $\lim_{x \to a} f(x) = b$ for some $a \in A$, $b \in \mathbb{R}^m$

if: $\forall \varepsilon \ \exists \delta$ s.t. $\forall x \in A$, $|x-a| < \delta \Rightarrow |f(x) - b| < \varepsilon$.

We say that f is continuous at $x = a$ if: $b = f(a)$

f continuous $\Rightarrow$ f cont. at all $a \in A$.

Theorem

Function $f: A \mapsto \mathbb{R}^m$ is continuous iff

$\forall U \in \mathbb{R}^m$ open, $\exists$ $V \in \mathbb{R}^n$ open s.t. $f^{-1}(U) = V \cap A$

Proof

See Analysis 4.

Theorem

If $A \subset \mathbb{R}^n$ is compact and f is continuous then $f(A)$ is compact.

Proof

Exercise.

Proposition

Let g, f be functions defined on $A \subset \mathbb{R}^n$, $g, f: A \mapsto \mathbb{R}^n$

$\lim_{x \to a} f(x) = b$, $\lim_{x \to a} g(x) = c$, then we have

i). $\lim_{x \to a} (f + g) = b + c$

ii). $\forall \lambda \in \mathbb{R}$, $\lim_{x \to a} (\lambda f(x)) = \lambda b$

iii). $\lim_{x \to a} \langle g(x), f(x) \rangle = b \cdot c$

iv). $\lim_{x \to a} |f(x)| = |b|$

Proof

i, ii as exercise.

iii). $<f(x), g(x)> = <f(x)-b, g(x)> + <b, g(x)>$

$= <f(x)-b, g(x)> + <b, g(x)-c> + <b, c>$

$|<f(x), g(x)> - b\cdot c| \leq |<f(x)-b, g(x)>| + |<b, g(x)-c>|$

$\leq |f(x)-b||g(x)| + |b||g(x)-c|$ by Cauchy Schwarz

$$\lim_{x\to a} |<f(x), g(x)> - bc| \leq \lim_{x\to a}\Big[\underset{\downarrow 0}{|f(x)-b|}\underset{\downarrow |c|}{|g(x)|} + \underset{\downarrow |b|}{|b|}\underset{\downarrow 0}{|g(x)-c|}\Big]$$

$= 0$

iv). $0 \leq \big||f(x)| - |b|\big| \leq |f(x)-b|$

$$\lim_{x\to a}\big||f(x)|-|b|\big| \leq \lim_{x\to a}|f(x)-b| = 0$$

Remarks

1). Linear transformations are continuous

2). A function f is continuous iff all f^i are continuous.

3). All polynomials of n variables are continuous functions

4). $p(x)/q(x)$ are continuous on the subset $\{x \in \mathbb{R}^n : q(x) \neq 0\}$

5). Be careful: $f(x,y) = \begin{cases} \dfrac{x^2 y}{x^4+y^2}, & (x,y)\neq(0,0) \\ 0 & \text{otherwise} \end{cases}$ has limit for (rx, ry), $r \to 0$

Theorem

Let $f: A \subset \mathbb{R}^n \to \mathbb{R}^m$ be continuous and $g: B \subset \mathbb{R}^m \mapsto \mathbb{R}^p$ be continuous, then $g \circ f : A \subset \mathbb{R}^n \mapsto \mathbb{R}^p$ is also continuous

Proof

Exercise.

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§2 Differentiation

Motivation: 1D case

$$\lim_{h \to 0} \frac{f(x_0+h) - f(x_0)}{h} := f'(x_0)$$

$$\begin{cases} f(x_0+h) = f(x_0) + f'(x_0)\cdot h + R(x_0, h) \\ \lim_{h\to 0} \frac{R(x_0,h)}{h} = 0 \end{cases}$$

Def

We say that $f: A \subset \mathbb{R}^n \mapsto \mathbb{R}^m$ is differentiable at $x_0 \in A$ if $\exists$ a linear mapping $L_{x_0} : \mathbb{R}^n \mapsto \mathbb{R}^m$ s.t.

$$* \begin{cases} f(x_0+h) = f(x_0) + L_{x_0}(h) + R(x_0,h) \\ \text{and } \lim_{h\to 0} \left| \frac{R(x_0,h)}{h} \right| = 0. \end{cases}$$

Note:

$$f(x) = \begin{cases} 0 & x \neq 1 \\ 1 & x = 1 \end{cases}$$

$$\forall x \in A, \underline{\underline{x \neq a}}, \quad \lim_{\substack{x\to 1 \\ x \neq 1}} f(x) = 0$$

Def

We call this linear transformation (L_{x_0}) a derivative of f at point x_0 and we denote $L_{x_0} = Df(x_0) = f'(x_0)$.

Theorem

If f is differentiable at $x_0 \in A$ then there exists a unique L_{x_0} satisfying $(*)$ (the definition).

Proof

$$\overset{L_{x_{0_1}}}{\downarrow} \qquad \overset{L_{x_{0_2}}}{\downarrow}$$

Assume we have two mappings L_1 and $L_2 : \mathbb{R}^n \mapsto \mathbb{R}^m$, $h \in \mathbb{R}^n$

$f(x_0 + h) = f(x_0) + L_1(h) + R_1(x_0, h)$

$f(x_0 + h) = f(x_0) + L_2(h) + R_2(x_0, h)$

$$|L_1(h) - L_2(h)| = |R_1(x_0, h) - R_2(x_0, h)|$$

$$\lim_{h \to 0} \frac{R_1(x_0, h)}{h} = 0 \quad \text{and} \quad \lim_{h \to 0} \frac{R_2(x_0, h)}{h} = 0$$

So $\displaystyle \lim_{h \to 0} \left| \frac{L_1(h) - L_2(h)}{h} \right| = \lim_{h \to 0} \frac{|R_1(x_0, h) - R_2(x_0, h)|}{|h|}$

$$\leq \lim_{h \to 0} \frac{|R_1(x_0, h)|}{|h|} + \lim_{h \to 0} \frac{|R_2(x_0, h)|}{|h|} = 0$$

$$\lim_{h \to 0} \frac{|L_1(h) - L_2(h)|}{|h|} = 0 \quad \forall h \in \mathbb{R}^n$$

Let $x \in \mathbb{R}^n$ (fixed), $x \neq 0$, $h = \lambda x$, $\lambda \in \mathbb{R}$,

$h \to 0$ consider $\lambda \to 0$

$$\lim_{\lambda x \to 0} \frac{|L_1(\lambda x) - L_2(\lambda x)|}{|\lambda x|} = \lim_{\lambda \to 0} \frac{|\lambda| |L_1(x) - L_2(x)|}{|\lambda| |x|}$$

$$= \lim_{\lambda \to 0} \frac{|L_1(x) - L_2(x)|}{|x|} = 0$$

$\Rightarrow L_1(x) = L_2(x) \quad \forall x \neq 0.$

$\square$

Example

$f(x, y) = \sin(x)$, $f : \mathbb{R}^2 \mapsto \mathbb{R}$

Claim is: $Df(x_0, y_0)(x, y) = \cos(x_0)(x)$.

$$0 \leq \lim_{(h,k) \to 0} \frac{|R((x_0, y_0), (h, k))|}{|(h, k)|} = \lim_{(h,k) \to 0} \frac{|f((x_0, y_0) + (h, k)) - f(x_0, y_0) - Df(x_0, y_0)(h, k)|}{|(h, k)|}$$

$$= \lim_{(h,k) \to 0} \frac{|\sin(x_0 + h) - \sin(x_0) - \cos(x_0) h|}{|(h, k)|}$$

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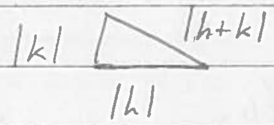

So $0 \leq \lim_{(h,k)\to 0} \dfrac{|R((x_0,y_0),(h,k))|}{\|(h,k)\|}$

$\leq \lim_{(h,k)\to 0} \dfrac{|\sin(x_0+h) - \sin(x_0) - \cos(x_0)\cdot h|}{|h|} = 0$

because $\sin(x)$ is a differentiable function of one variable.

So $f'(x_0, y_0) = (\cos(x_0), 0)$.

<u>Def</u>

The $(m \times n)$-matrix $Df(x_0) : \mathbb{R}^n \mapsto \mathbb{R}^m$ w.r.t. the standard basis of $\mathbb{R}^n$ and $\mathbb{R}^m$ is called the Jacobian matrix of f in the point x_0 and is denoted by $f'(x_0)$.

In our example, this matrix was equal to $(\cos(x_0), 0)$.

<u>Theorem</u>

Let $f : A \subset \mathbb{R}^n \longrightarrow \mathbb{R}^n$ be differentiable

$\Rightarrow f$ is continuous.

<u>Proof</u>

Exercise (hw 1, 26).

[1D $[f(g(x_0))]' = f'(g(x_0)) \cdot g'(x_0)$]

<u>Theorem 2.2</u> (Chain Rule)

Let $f : \mathbb{R}^n \mapsto \mathbb{R}^m$ be differentiable at $x_0 \in \mathbb{R}^n$, and $g : \mathbb{R}^m \mapsto \mathbb{R}^p$ be differentiable at $y_0 = f(x_0) \in \mathbb{R}^m$.

Then $g(f) = g \circ f$ is a differentiable function from $\mathbb{R}^n \mapsto \mathbb{R}^p$ at $x_0 \in \mathbb{R}^n$. Moreover

$D(g \circ f)(x_0) = Dg(f(x_0)) \circ Df(x_0)$

Proof

We know that f and g are differentiable.

$$f(x_0+h) = f(x_0) + Df(x_0)(h) + R_f(x_0,h)$$

$$g(y_0+k) = g(y_0) + Dg(y_0)(k) + R_g(y_0,k).$$

We know $\lim_{h\to 0} \frac{|R_f(x_0,h)|}{|h|} = 0$, $\lim_{k\to 0} \frac{|R_g(y_0,k)|}{|k|} = 0$

$$(g\circ f)(x_0+h) = g(f(x_0+h))$$

$$= g\Big(\underbrace{f(x_0)}_{y_0} + \underbrace{Df(x_0)(h) + R_f(x_0,h)}_{k}\Big)$$

$$= g(f(x_0)) + D_g(f(x_0))\big(Df(x_0)(h) + R_f(x_0,h)\big) + R_g(f(x_0),k)$$

$$= (g\circ f)(x_0) + D_g(f(x_0))\cdot Df(x_0)(h) + \underbrace{D_g(f(x_0))\cdot R_f(x_0,h) + R_g(f(x_0),k)}_{\tilde{R}(x_0,\tilde{h})}$$

we will show that $\lim_{\tilde{h}\to 0} \frac{|\tilde{R}(x_0,\tilde{h})|}{|\tilde{h}|} = 0.$

11-10-17 $|Df(x_0)(h)| \le M_1|h|$, $|Dg(y_0)(k)| \le M_2|k|$

- $|R_g(y_0)(k)| \le \varepsilon|k|$ for $|k| < \delta$
- $|R_f(x_0)(h)| \le \varepsilon|h|$ for $|h| < \delta$

$$\frac{|Dg(y_0)\,R_f(x_0,h)|}{|h|} \le \frac{M_2\,|R_f(x_0,h)|}{|h|} \xrightarrow{h\to 0} 0$$

$$\frac{|R_g(y_0, Df(x_0)(h) + R_f(x_0,h))|}{|h|} \overset{\text{if } |h|<\delta}{\le} \varepsilon\big(|Df(x_0)(h) + |R_f(x_0,h)|\big)/|h|$$

$$\le \varepsilon\big(M_1|h| + \varepsilon|h|\big)/|h| \xrightarrow{h\to 0} 0$$

So we proved that

$$\lim_{h\to 0} \frac{|(g\circ f)(x_0+h) - (g\circ f)(x_0) - (Dg(f(x_0))\circ Df(x_0))(h)|}{|h|} = 0$$

$$\Rightarrow D(g\circ f)(x_0) = Dg(f(x_0))\circ Df(x_0)$$

$\square$

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Theorem 2.3

1). If $f: \mathbb{R}^n \mapsto \mathbb{R}^m$ is a constant function then $Df(x_0) = 0$, $x_0 \in \mathbb{R}^n$

2). If $f: \mathbb{R}^n \mapsto \mathbb{R}^m$ is linear then $Df(x_0) = f$

3). If $s: \mathbb{R}^2 \mapsto \mathbb{R}$ is defined as $s(x,y) = x + y$, then $Ds(x_0, y_0) = s$

4). If $p: \mathbb{R}^2 \mapsto \mathbb{R}$ is defined as $p(x,y) = x \cdot y$, then

$Dp(x_0,y_0)(x,y) = y_0 x + x_0 y$. ← on problem sheet 1.

Proof

1). $f(x_0 + h) = f(x_0)$ because f is constant

$$\lim_{h \to 0} \frac{|f(x_0+h) - f(x_0) - 0|}{|h|} = \lim_{h\to 0} \frac{|0|}{|h|} = 0 \Rightarrow Df(x_0) = 0$$

2). f linear.

$$\lim_{h\to 0} \frac{|f(x_0+h) - f(x_0) - f(h)|}{|h|}$$

$$= \lim_{h\to 0} \frac{|f(x_0) + f(h) - f(x_0) - f(h)|}{|h|} = \lim_{h\to 0} \frac{|0|}{|h|} = 0$$

$\Rightarrow Df(x_0) = f$

3). follows from 2

4). exercise.

Theorem 2.4

Let $f: \mathbb{R}^n \mapsto \mathbb{R}^m$, then f is differentiable at $x_0 \in \mathbb{R}^n$ iff each f^i is differentiable at x_0 and

$Df(x_0) = (Df^1(x_0), \dots, Df^m(x_0))$.

Remark

$Df(x_0)$ is $n \times m$ matrix whose i-th row is $Df^i(x_0)$.

Proof

[$\Rightarrow$]

If f is differentiable then so is $\pi^i \circ f$.

Moreover $Df^i)(x_0) = D(\pi^i \circ f)(x_0) = D\pi^i(f(x_0)) \circ Df(x_0)$ (chain rule)

$= (\pi^i \circ Df(x_0))$

[$\Leftarrow$]

$$f^i(x_0+h) = f^i(x_0) + Df^i(x_0)(h) + R_i(x_0+h)$$

and $\dfrac{|R^i(x_0,h)|}{|h|} \xrightarrow{h\to 0} 0$

We define $L_{x_0} = (Df^1(x_0), \dots, Df^m(x_0))$

$$\lim_{h\to 0} \frac{|f(x_0+h) - f(x_0) - L_{x_0}(h)|}{|h|}$$

$$\leq \lim_{h\to 0} \sum_{i=1}^{n} \frac{|f^i(x_0+h) - f^i(x_0) - Df^i(x_0)(h)|}{|h|} = 0$$

$\square$

Corollary

Let $f,g : \mathbb{R}^n \mapsto \mathbb{R}$ be differentiable at $x_0 \in \mathbb{R}^n$, then

1). $D(f+g)(x_0) = Df(x_0) + Dg(x_0)$

2). $D(f \cdot g)(x_0) = g(x_0) Df(x_0) + f(x_0) Dg(x_0)$

3). $D(f/g)(x_0) = (g(x_0) Df(x_0) - f(x_0) Dg(x_0)) / (g(x_0))^2$, $g(x_0) \neq 0$.

Proof

1). $f+g = s \circ (f,g)$

$D(f+g)(x_0) = D(s \circ (f,g))(x_0) = Ds(f(x_0), g(x_0)) \circ (Df(x_0), Dg(x_0))$

$= s \circ (Df(x_0), Dg(x_0)) = Df(x_0) + Dg(x_0)$

2). exercise.

hint: use the definition of function p (from Thm 2.3 (4))

3). exercise.

hint: what is the derivative of $1/g$? $D(id)(x_0)$

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Def (Partial Derivatives)

Let $f: \mathbb{R}^n \mapsto \mathbb{R}$, $x_0 \in \mathbb{R}^n$, then we define

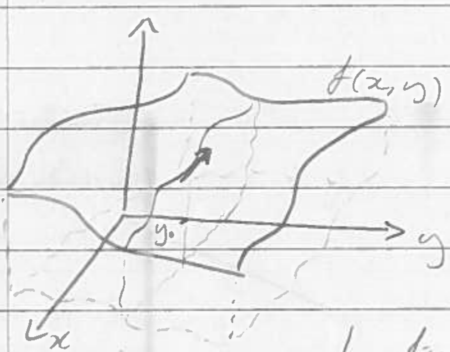
$$D_i f(x_0) = \lim_{h \to 0} \frac{f(x_0^1, \dots, x_0^{i-1}, x_0^i + h, x_0^{i+1}, \dots, x_0^n) - f(x_0^1, \dots, x_0^n)}{h}$$

If this limit exists it is called the i-th partial derivative of f.

Remark

$D_i f(x_0) = g'(x_0^i)$

$g(x) = f(x_0^1, \dots, x_0^{i-1}, x, x_0^{i+1}, \dots, x_0^n)$



| fix $y = y_0$

$D_i f(x_0)$ is slope of tangent line to graph(f) at $(x_0, f(x_0))$ cut by the plane $x^j = x_0^j \ \forall j \neq i$

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$f: \mathbb{R}^n \mapsto \mathbb{R}^m$

$f = (f^1, \dots, f^m)$, $f^i: \mathbb{R}^n \mapsto \mathbb{R}$

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Def (Partial Derivative) $f: \mathbb{R}^n \mapsto \mathbb{R}$

$$D_i f(x_0) := \lim_{h \to 0} \frac{f(x_0^1, \dots, x_0^i + h, \dots, x_0^n) - f(x_0^1, \dots, x_0^n)}{h}$$

$$g(x) = f(x_0^1, \dots, x_0^{i-1}, x, x_0^{i+1}, \dots, x_0^n)$$

$$g'(x_0^i) = D_i f(x_0).$$

Remark

$$D_i f(x_0) = \frac{\partial f}{\partial x_i}(x_0)$$

In $\mathbb{R}^3$ $\frac{\partial f}{\partial x}$, $\frac{\partial f}{\partial y}$, $\frac{\partial f}{\partial z}$

Examples

1). $f(x,y) = \sin(xy^2)$

$$D_1 f(x,y) = \frac{\partial f}{\partial x} = y^2 \cos(xy^2)$$

$$D_2 f(x,y) = \frac{\partial f}{\partial y} = 2xy \cos(xy^2)$$

2). $f(x,y) = x^y$ (PS 2 q5)

$$\frac{\partial f}{\partial x} = y x^{y-1} \qquad \frac{\partial f}{\partial y} = x^y \log x$$

3). $f(x,y) = \begin{cases} \frac{(x^2-y^2)^2}{(x^2+y^2)^2} & x,y \neq (0,0) \\ 1 & x,y = (0,0) \end{cases}$

$$D_1 f(0,0) = \lim_{h \to 0} \frac{f(h,0) - f(0,0)}{h} = \lim_{h \to 0} \left(\frac{h^4/h^4 - 1}{h} \right) = \lim_{h \to 0} \left(\frac{1-1}{h} \right) = 0$$

$$D_2 f(0,0) = \lim_{h \to 0} \dots = 0$$

$$f(t,0) = f(0,t) = 1 \quad \forall t \in \mathbb{R}$$

$$f(t,t) = 0 \quad \forall t \neq 0$$

Existence of partial derivatives $\not\Rightarrow$ differentiability or continuity.

Def

Let $f: \mathbb{R}^n \mapsto \mathbb{R}$. For $x \in \mathbb{R}^n$ then

$$\lim_{t \to 0} \frac{f(x_0 + tx) - f(x_0)}{t} := D_x f(x_0).$$

If this limit exists it is called the directional derivative of f at x_0 in direction x.

Remarks

a). Note that if we choose $x = e_i$ then

$$D_{e_i} f(x_0) = D_i f(x_0).$$

b). Note that for $\lambda \in \mathbb{R}$, $\lambda \neq 0$

$$D_{\lambda x} f(x_0) = \lim_{t \to 0} \frac{f(x_0 + tx) - f(x_0)}{t}$$

$$= \lambda \lim_{\lambda t \to 0} \frac{f(x_0 + t\lambda x) - f(x_0)}{t\lambda}$$

$$= \lambda D_x f(x_0).$$

c). Let f be a differentiable function at x_0 $f: \mathbb{R}^n \mapsto \mathbb{R}$.

Then $D_x f(x_0) = Df(x_0)(x)$

$g: \mathbb{R} \mapsto \mathbb{R}^n \quad t \mapsto x_0 + tx$

$$D_x f(x_0) = D(f \circ g)(0) = Df(g(0)) \cdot Dg(0)$$

$$= Df(x_0) \cdot x$$

d). $D_{x+y} f(x_0) = Df(x_0)(x+y) = D_x f(x_0) + D_y f(x_0)$

using (c).

Assume that for $f: \mathbb{R}^n \mapsto \mathbb{R}$, $D_i f(x)$ exists $\forall x \in \mathbb{R}^n$, $1 \leq i \leq n$.

This means that $D_i f(x): \mathbb{R}^n \mapsto \mathbb{R}$.

Def

We denote by $D_{i,j} f(x) = D_j(D_i f(x))$ the second order partial derivative of f at x.

$$\left[D_{i,j} f(x) = \frac{\partial^2}{\partial x^i \partial x^j} f(x) = \partial^2_{i,j} f(x). \right]$$

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Theorem 2.5

If $D_{i,j}f$ and $D_{j,i}f$ are continuous in an open set containing x_0, then $D_{i,j}f(x_0) = D_{j,i}f(x_0)$.

Proof

Later using Fubini's Theorem. $\square$

Theorem 2.6

Let $\Omega \subset \mathbb{R}^n$. If $f : \Omega \mapsto \mathbb{R}$ has a local minimum (maximum) at the interior point $x_0 \in \Omega$ and if $D_i f(x_0)$ exists then $D_i f(x_0) = 0$.

Proof

Exercise (PS2 q1b).

Theorem 2.7

If $f : \mathbb{R}^n \mapsto \mathbb{R}^m$ is differentiable at some point x_0, then $D_j f^i(x_0)$ exists for $1 \le i \le m$, $1 \le j \le n$ and $f'(x_0)$ is the $m \times n$ matrix $D_j f^i(x_0)$.

Proof

Assume that $m = 1$

$f : \mathbb{R}^n \mapsto \mathbb{R}$. Then $h : \mathbb{R} \mapsto \mathbb{R}^n$, $x \mapsto h(x) = (x_0^1, \dots, \underset{j\text{th place}}{x}, \dots, x_0^n)$

therefore $D_j f(x_0) = (f \circ h)'(x_0^j)$.

Using the chain rule: $(f \circ h)'(x_0^j) = f'(h(x_0^j)) \cdot h'(x_0^j) = f'(x_0) e_j$

$D_j f(x_0) = f'(x_0) e_j$.

The generalisation of this argument to $m > 1$ is a consequence of Thm 2.4 since this implies that each f^i $1 \le i \le m$ is differentiable at x_0 and the ith row of $f'(x_0)$ is equal to $(f^i)'(x_0)$.

$$f'(x_0) = \begin{pmatrix} \frac{\partial f^1}{\partial x_1}(x_0) & \cdots\cdots & \frac{\partial f^1}{\partial x_n}(x_0) \\ \vdots & & \vdots \\ \frac{\partial f^m}{\partial x_1}(x_0) & \cdots & \frac{\partial f^m}{\partial x_n}(x_0) \end{pmatrix}$$

$$= \left(D_j f^i(x_0) \right)_{i,j}.$$

$\square$

Corollary 2.8

Let $f : \mathbb{R}^n \mapsto \mathbb{R}^m$ be differentiable at $x_0 \in \mathbb{R}^n$.
Let $g : \mathbb{R}^m \mapsto \mathbb{R}^p$ and be differentiable at $f(x_0) \in \mathbb{R}^m$.

$$\left((g \circ f)'(x_0)\right)_{i,j} = \left(g'(f(x_0)) \circ f'(x_0)\right)_{i,j}$$

$$\left[(A \cdot B)_{i,j} = \sum_{l=1}^{m} A_{ij} B_{ij} \right]$$

$$\Rightarrow \left((g \circ f)'(x_0)\right)_{i,j} = \sum_{l=1}^{m} D_l g^i(f(x_0)) D_j f^l(x_0)$$

Theorem 2.9

Let $f : \mathbb{R}^n \mapsto \mathbb{R}^m$ and assume that all $D_j f^i$ exist in an open set containing x_0 and each of them is continuous at x_0. Then f is differentiable at x_0.

Proof

Again I reduce this problem to the case $m = 1$.
Now $f : \mathbb{R}^n \mapsto \mathbb{R}$.

$$\begin{aligned} f(x_0 + h) - f(x_0) &= f(x_0^1 + h^1, x_0^2 + h^2, \dots, x_0^n + h^n) - f(x_0^1, \dots, x_0^n). \\ &= f(x_0^1 + h^1, \dots, x_0^n + h^n) - f(x_0^1, x_0^2 + h^2, \dots, x_0^n + h^n) \\ &\quad + f(x_0^1, x_0^2 + h^2, \dots, x_0^n + h^n) - f(x_0^1, x_0^2, x_0^3 + h^3, \dots, x_0^n + h^n) \\ &\quad + \dots - f(x_0^1, x_0^2, \dots, x_0^{n-1}, x_0^n + h^n) - f(x_0^1, \dots, x_0^n). \end{aligned}$$

Recall that $D_1 f$ is a derivative of $g(x) = f(x, x_0^2 + h^2, \dots, x_0^n + h^n)$

$$D_1 f(x_0^1, x_0^2 + h^2, \dots, x_0^n + h^n) = g'(x_0^1)$$

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Mean value theorem

$\Rightarrow \exists\, \theta_1 \in (x_0^1, x_0^1+h^1)$ s.t.

$$f(x_0^1+h^1, x_0^2+h^2, \dots, x_0^n+h^n) - f(x_0^1, x_0^2+h^2, \dots, x_0^n+h^n)$$
$$= h^1 D_1 f(\theta_1, x_0^2+h^2, \dots, x_0^n+h^n)$$

Similarly we take any ith part of the sum above is equal to 0.

$$h^i D_i f(x_0^1, \dots, x_0^{i-1}, \theta_i, x_0^{i+1}+h^{i+1}, \dots, x_0^n+h^n)$$
$$:= h^i D_i f(c_i) \qquad c_i \in \mathbb{R}^n \quad \theta_i \in (x_0^i, x_0^i+h^i)$$

Note:
$c_i \to x_0$ for $|h| \to 0$.

Therefore we can write that

$$\frac{\left| f(x_0+h) - f(x_0) - \sum_{i=1}^{n} D_i f(x_0)\cdot h^i \right|}{|h|}$$

$$= \left| \sum_{i=1}^{m} D_i f(c_i) h^i - \sum_{i=1}^{n} D_i f(x_0) h^i \right| / |h|$$

$$\leq \frac{1}{|h|} \sum_{i=1}^{n} |D_i f(c_i) - D_i f(x_0)|\,|h^i|$$

$$\leq c \sum_{i=1}^{n} |D_i f(c_i) - D_i f(x_0)| \xrightarrow[|h|\to 0]{} 0$$

because $D_i f$ is continuous.

Conclusion is that $Df(x_0)(h) = \sum_{i=1}^{n} D_i f(x_0) h^i$.

$\square$

<u>Inverse Function Theorem</u>

<u>Motivation</u>: Let $f: \mathbb{R} \mapsto \mathbb{R}$ be continuously differentiable on $I \subset \mathbb{R}$, I open and $f'(x_0) \neq 0$ $x_0 \in I$.

- If $f'(x_0) > 0 \Rightarrow \exists$ an open subinterval $J \subset I$ and $x \in J$ s.t. $f'(x) > 0$
  $\Rightarrow \forall x \in J$ $f(x)$ is increasing
- If $f'(x_0) < 0$ .... $K \subset I$ s.t. $f(x)$ is decreasing on $x \in K$.

Then $W = f(J)$ s.t. it is possible to define the inverse $f^{-1}: W \mapsto J$.

$$(f^{-1})'(y) = \frac{1}{f'(f^{-1}(y))}$$

The observation needed for n-dimensional generalisation is that $l(h) = f'(x_0)h$ is invertible for $f'(x_0) \neq 0$.

Before formulating the Inverse Function Theorem, let us write:

<u>Lemma</u> 2.10

Let $B_r(x_0) \subset \mathbb{R}^n$ and $f : B_r(x_0) \mapsto \mathbb{R}^n$ be s.t. all $D_i f^j(x)$, $x \in B_r(x_0)$ exist and $|D_i f^j(x)| \leq M \ \forall x \in B_r(x_0)$, $i,j = 1, \dots, n$, then

$|f(x) - f(y)| \leq n^2 M |x-y| \ \forall x,y \in B_r(x_0)$.

<u>Proof</u>

Exercise.

Typical on exams

<u>Theorem</u> (Inverse Function theorem) 2.11

Assume that $f : \mathbb{R}^n \mapsto \mathbb{R}^n$ is continuously differentiable in some open set $\Omega \subset \mathbb{R}^n$ containing $x_0 \in \Omega$ and $\det(f'(x_0)) \neq 0$.

Then there is open set V s.t. $x_0 \in V \subset \Omega$ and $W \subset \mathbb{R}^n$ open s.t. $f(x_0) \in W$, where

$f : V \mapsto W$ has a continuous inverse

$f^{-1} : W \mapsto V$ which is differentiable $\forall y \in W$ and

$(f^{-1})'(y) = [f'(f^{-1}(y))]^{-1}$.

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Proof (Inverse Function Thm)

plan $\begin{cases} \text{Step 1: "} f \text{ is injective around } x_0 \text{"} \\ \text{Step 2: "} f \text{ is bijective around } x_0 \text{"} \\ \text{Step 3: "} f^{-1} \text{ is continuous, differentiable, and formula for } (f^{-1})' \text{"} \end{cases}$

Define $\lambda := Df(x_0) : \mathbb{R}^n \mapsto \mathbb{R}^n$, so λ^{-1} exists as $\det \lambda \neq 0$.

$$D(\lambda^{-1} \circ f)(x_0) = D(\lambda^{-1})(f(x_0)) \cdot Df(x_0)$$
$$= \lambda^{-1} \cdot Df(x_0) = Id$$

The inverse function theorem works for $f \iff$ it works for $\lambda^{-1} \circ f$.
Therefore, from now on I assume that $Df(x_0) = Id_{n \times n}$.

Step 1

$\exists\, \varepsilon > 0$ s.t. $\forall\, x \in B_\varepsilon(x_0) = U$

i). $f(x) \neq f(x_0)$

ii). $\det(Df(x)) \neq 0$

iii). $|D_j f^i(x) - D_j f^i(x_0)| \leq 1/2n^2 \quad \forall x, x_0 \in U$.

I introduce function $g(x) = f(x) - x$. $Dg(x_0) = 0$.
Apply Lemma 2.10 to function $g(x)$ and the property 3).
to show that $|g(x) - g(x_0)| \leq \frac{1}{2}|x - x_0|$

$\forall x_1, x_2 \in B_\varepsilon(x_0)$, $|f(x_1) - x_1 - (f(x_2) - x_2)| \leq \frac{1}{2}|x_1 - x_2|$

$$|x_1 - x_2| - |f(x_1) - f(x_2)| \overset{\Delta\text{ inequality}}{\leq} |f(x_1) - x_1 - f(x_2) + x_2| \leq \frac{1}{2}|x_1 - x_2|$$

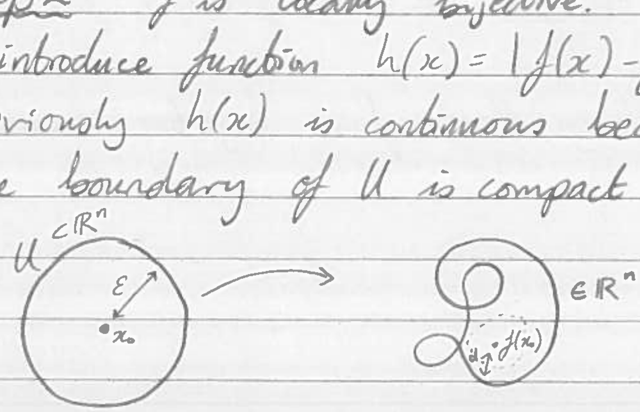
iv). $|x_1 - x_2| \leq 2|f(x_1) - f(x_2)| \quad \forall x_1, x_2 \in \overline{U}$.

Step 2 "f is locally bijective."

I introduce function $h(x) = |f(x) - f(x_0)|$

Obviously $h(x)$ is continuous because $f(x)$ is.

The boundary of U is compact, so $h(\partial U)$ is compact.



$\forall x \in \partial U, \ |f(x) - f(x_0)| \geq \varepsilon/2 = d$

$W = \{y : |y - f(x_0)| < d/2\} = B_{d/2}(f(x_0))$

v). $\forall y \in W \ \ |y - f(x_0)| < d/2 \leq |f(x) - y| \ \ \forall x \in \partial U$

<u>Claim</u>

$\forall y \in W \ \exists x \in U$ s.t. $f(x) = y$.

$g(x) = |f(x) - y|^2$ for some $y \in W$

$= \sum_{i=1}^{n} (f^i(x) - y^i)^2 \qquad g : \bar{U} \mapsto R$

g continuous on the compact set $\bar{U}$, so g attains its minimum on $\bar{U}$.

Let $x \in \partial U$, then

$g(x) > g(x_0) \Rightarrow x$ is not the point at which the minimum of $g(x)$ is attained.

$\exists x \in U$ s.t. $D_j g(x) = 0 \ \ \forall 1 \leq j \leq n$

$D_j g(x) = \sum_{i=1}^{n} 2(f^i(x) - y^i) \cdot D_j f^i(x) = 0$

$$= 2\left(Df^T(x)\right) \cdot \begin{pmatrix} f^1(x) - y^1 \\ \vdots \\ f^n(x) - y^n \end{pmatrix}$$

$\Rightarrow f^i(x) = y^i \ \ \forall i \quad \Rightarrow f(x) = y.^{x \in U}$ (this proves the claim.)

Moreover this is a <u>unique</u> x. This follows from iv).

So far we have shown that f is injective around x_0 and is bijective.

Define $V = U \cap f^{-1}(W)$

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$f: V \mapsto W$ and there exists $f^{-1}: W \mapsto V$

Step 3a "f^{-1} is continuous"
$\forall y \in W \; \exists! \, x \in V$ s.t. $f(x) = y$, s.t. $x = f^{-1}(y)$.
From property iv). $|f^{-1}(y_1) - f^{-1}(y_2)| \leq 2|y_1 - y_2|$
$\Rightarrow f^{-1}$ is Lipschitz continuous with the constant 2
$\Rightarrow f^{-1}$ is continuous

Step 3b "f^{-1} is differentiable"
Let us denote $\mu = Df(x) : \mathbb{R}^n \longmapsto \mathbb{R}^n$
Claim
f^{-1} is differentiable and that $D(f^{-1}) = \mu^{-1}$.

f is differentiable.
$$f(x_1) = f(x) + \mu(x_1 - x) + R(x, x_1 - x)$$
moreover $\displaystyle\lim_{x_1 \to x} \frac{|R(x, x_1 - x)|}{|x_1 - x|} = 0$

$$y_1 = y + \mu(f^{-1}(y_1) - f^{-1}(y)) + R(x, x_1 - x)$$

$$f^{-1}(y_1) = f^{-1}(y) + \mu^{-1}(y_1 - y) - \mu^{-1} R(x, x_1 - x)$$

WTS: for $y_1 \to y$, $\dfrac{|\mu^{-1} R(x, x_1 - x)|}{|y_1 - y|} \to 0$.

It is enough to prove that
$$\lim_{y_1 \to y} \frac{|R(x, x_1 - x)|}{|y_1 - y|} = 0.$$
$$(*) = \frac{|R(x, f^{-1}(y_1) - f^{-1}(y))|}{|y_1 - y|}, \quad x = f^{-1}(y)$$

vi). $|f^{-1}(y_1) - f^{-1}(y)| \leq 2|y_1 - y|$ (consequence of iv)).

$$\Rightarrow \frac{1}{|y_1 - y_2|} \leq \frac{2}{|f^{-1}(y_1) - f^{-1}(y)|}$$

$$(*) \leq c \frac{|R(x, f^{-1}(y_1) - f^{-1}(y))|}{|f^{-1}(y_1) - f^{-1}(y)|}$$

So because f^{-1} is continuous, if $y_1 \to y$ then $f^{-1}(y_1) \to f^{-1}(y)$.

so $$\lim_{f^{-1}(y_1) \to f^{-1}(y)} \frac{|R(x, f^{-1}(y_1) - f^{-1}(y))|}{|f^{-1}(y_1) - f^{-1}(y)|} = 0$$

$\square$

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Example

Consider the following function $f: \mathbb{R}^2 \to \mathbb{R}^2$, $f(x,y) = (xy, x^2+y^2)$

$z = f^1 = x \cdot y$, $w = f^2 = x^2+y^2$.

$$f'(x,y) = \begin{pmatrix} y & x \\ 2x & 2y \end{pmatrix}$$

$\det(f'(x,y)) = 2y^2 - 2x^2 \neq 0$ when $x \neq \pm y$.

Want to find f^{-1} s.t. $(x,y) = f^{-1}(z,w)$

(so inverse exists when $x \neq \pm y$)

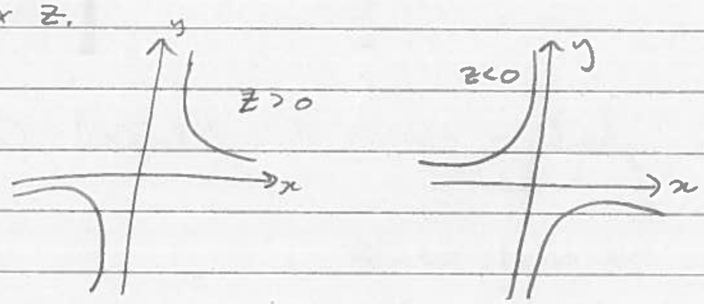
$y = \frac{z}{x}$, $w = x^2 + \left(\frac{z}{w}\right)^2$

$\Rightarrow x^4 + z^2 - wx^2 = 0$

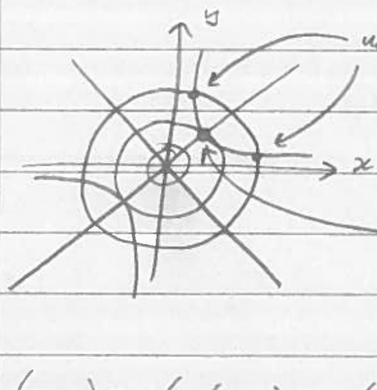
$\Rightarrow x = \pm\left(\frac{w \pm \sqrt{w^2-4z^2}}{2}\right)^{1/2}$

$\Rightarrow y = z \pm \left(\frac{w \pm \sqrt{w^2-4z^2}}{2}\right)^{-1/2}$ provided $w^2 - 4z^2 \geq 0$

Fix z.



Fix w



unique solution in a neighbourhood of each point.

not a unique solution in a neighbourhood of this point. $x \neq \pm y$ prevents this

$(x,y) = (x(z,w), y(z,w))$

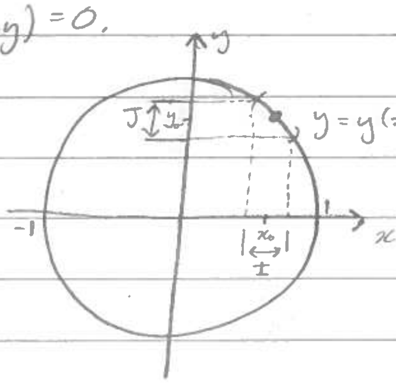
Computing $\begin{pmatrix} \frac{\partial x}{\partial z} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial z} & \frac{\partial y}{\partial w} \end{pmatrix}$ from the explicit formulas might be difficult.

$$(f^{-1})'(z,w) = (f'(x,y))^{-1}$$
$$= \begin{pmatrix} y & x \\ 2x & 2y \end{pmatrix}^{-1}$$
$$= \frac{1}{2(y^2-x^2)} \begin{pmatrix} 2y & -x \\ -2x & y \end{pmatrix}, \quad (x,y) = (x(z,w), y(z,w))$$

Motivation: Implicit Function Thm

$f: \mathbb{R}^2 \to \mathbb{R}$, $f(x,y) = x^2+y^2-1$

If $f(x_0,y_0) = 0$ (a constant but doesn't always have to be zero) and $x_0 \neq \pm 1$, there exists an open interval $x_0 \in I$ and $J \ni y_0$ s.t. $\forall\ x \in I$ $\exists!\, y \in J$ s.t. $(x,y) \in I \times J$ and $f(x,y) = 0$.



$y = y(x)$, $f(x, y(x)) = 0$

Near $x = \pm 1$, y is not unique.

We are looking for function $g(x) := y$ s.t. $f(x, g(x)) = 0$.

$x^2 + g(x)^2 = 1$

$\Rightarrow g(x)^2 = 1 - x^2 \quad \Rightarrow g(x) = \pm\sqrt{1-x^2}$

Let $x = \pm 1$. We know $f(x, g(x)) = 0$

$$\frac{d}{dx} f(x, g(x)) = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} \cdot \frac{\partial g(x)}{\partial x} = 0$$

$$\frac{\partial g(x)}{\partial x} = -\frac{\partial f}{\partial x} \Big/ \frac{\partial f}{\partial y} = -\frac{x}{y}$$

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General situation:

Take $f^i : \mathbb{R}^n \times \mathbb{R}^m \longrightarrow \mathbb{R}$ $\quad 1 \le i \le m$.

In the linear situation $L : \mathbb{R}^{n+m} \to \mathbb{R}^m$ instead of function f and I would write the condition $f = 0$ as $L = 0$,

$L = [S, T]^{\text{matrix}}$, $L(z) = S(x) + T(y)$.

If $T(y) + S(x) = 0$, then $y = T^{-1}(S(x))$

Most general situation:

Let us take $f : \underset{= \mathbb{R}^n \times \mathbb{R}^m}{\mathbb{R}^{n+m}} \longrightarrow \mathbb{R}^m$ $\quad z = (x, y)$, $x \in \mathbb{R}^n$, $y \in \mathbb{R}^m$

So each $f^i : \mathbb{R}^{n+m} \longrightarrow \mathbb{R}$

$f = 0$ means $f^i(x^1, \dots, x^n, y^1, \dots, y^m) = 0 \quad \forall\, 1 \le i \le m$.

Q: When can we find for each $(x^1, \dots, x^n)$ near $(x_0^1, \dots, x_0^n)$ a unique $(y^1, \dots, y^m)$ near $(y_0^1, \dots, y_0^m)$ s.t.

$f^i(x^1, \dots, x^n, y^1, \dots, y^m) = 0$?

Theorem Implicit Function Theorem.

Suppose $f : \mathbb{R}^n \times \mathbb{R}^m \longrightarrow \mathbb{R}^m$ is continuously differentiable in an open set containing (x_0, y_0) and $f(x_0, y_0) = 0$.

If $M = D_{n+j} f^i(x_0, y_0)$ $\quad i, j = 1, \dots, m$ has a full rank (i.e. $\det M \neq 0$), then there is an open set $B \subset \mathbb{R}^m$ containing y_0 and an open set $A \subset \mathbb{R}^n$ containing x_0 s.t.

$\forall x \in A \;\; \exists!\; g(x) \in B$ s.t. $f(x, g(x)) = 0$.

Moreover $g(x)$ is differentiable.

Proof

Let $F : \mathbb{R}^n \times \mathbb{R}^m \longrightarrow \mathbb{R}^n \times \mathbb{R}^m$ be given by $F(x, y) = (x, f(y))$.

The determinant $\det(F'(x_0, y_0)) \neq 0$ by assumption.

$\det(F'(x_0, y_0)) = \det M$.

By Inverse fn Thm (2.11) $\exists\, W \subset \mathbb{R}^n \times \mathbb{R}^m$ open containing $F(x_0, y_0) = (x_0, 0)$ and an open set $V \subset \mathbb{R}^n \times \mathbb{R}^m$ containing (x_0, y_0) which can be taken in the form $A \times B = V$ s.t. $F : A \times B \to W$ has a differentiable inverse $h : W \to A \times B$.

Clearly $h(x,y) = (x, k(x,y))$ and k is differentiable.
Recall π is defined $\pi(x,y) = y$
$\Rightarrow \pi \circ F = f$
Therefore $f(x, k(x,y)) = f \circ h(x,y) = (\pi \circ F) \circ (h(x,y))$
$= \pi \circ (F \circ h)(x,y) = y$

$\Rightarrow f(x, k(x,y)) = y$
Taking $y = 0$ we see $f(x, k(x,0)) = 0$
so, we take $g(x) = k(x,0)$. $\square$

Remark
The above theorem implies that $f(x, g(x)) = 0$.
Taking the (partial) derivative of this expression we get
$0 = D_j f^i(x, g(x)) + \sum_{l=1}^{m} D_{n+l} f^i(x, g(x)) \cdot D_j g^l(x), \quad i = 1, \dots, m, \ j = 1, \dots, n.$

Let $M = \left(D_{n+l} f^i(x, g(x))\right)_{i,l = 1, \dots, m}$

We know that ~~A~~ is invertible

$$M \cdot \begin{pmatrix} D_j g^1(x) \\ \vdots \\ D_j g^m(x) \end{pmatrix} = - \begin{pmatrix} D_j f^1(x, g(x)) \\ \vdots \\ D_j f^m(x, g(x)) \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} D_j g^1(x) \\ \vdots \\ D_j g^m(x) \end{pmatrix} = -M^{-1} \begin{pmatrix} D_j f^1(x, g(x)) \\ \vdots \\ D_j f^m(x, g(x)) \end{pmatrix}$$

Corollary 2.13
Let $f: \mathbb{R}^n \to \mathbb{R}^p$ be continuously differentiable in an open set containing x_0, and $p \leq n$.
If $f(x_0) = 0$ and the $p \times n$ matrix $D_j f^i(x_0)$ has rank p then there is an open set $A \subset \mathbb{R}^n$, $x_0 \in A$, and a differentiable function $h: A \to \mathbb{R}^n$ with differentiable inverse s.t.
$f \circ h(x^1, \dots, x^n) = (x^{n-p+1}, \dots, x^n)$

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Proof

We consider f as a function $f: \mathbb{R}^{n-p} \times \mathbb{R}^p \to \mathbb{R}^p$.

Assume that $p \times p$ matrix $M = (D_{n-p+j} f^i(x_0))_{ij=1,\dots,p}$ has a full rank $p \Rightarrow \det M \neq 0$.

Then we are precisely in the framework of the proof of the previous theorem, then exists h s.t.

$f \circ h(x^1, \dots, x^n) = (x^{n-p+1}, \dots, x^n)$.

In general case, $f'(x_0)$ has rank p, we find $j_1 < \dots < j_p$ s.t. $D_j f^i(x_0)$ $i = 1, \dots, p$, $j = j_1, \dots, j_p$ has a full rank.

If $g: \mathbb{R}^n \to \mathbb{R}^m$ permutes x^j so that $g(x^1, \dots, x^n) = (\dots, x^{j_1}, \dots, x^{j_p})$, then $f \circ g$ is as considered above, so $\exists$ k s.t.

$$(f \circ g) \circ k(x^1, \dots, x^n) = (x^{n-p+1}, \dots, x^n)$$
$$= f \circ (g \circ k)(x^1, \dots, x^n)$$

and we call $h = g \circ k$. $\square$

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§3

Def

i). Let $a, b \in \mathbb{R}^n$ s.t. $a^i < b^i$ $\forall$ $1 \le i \le n$.

We will call the set $\{x \in \mathbb{R}^n : a^i \le x^i \le b^i, 1 \le i \le n\} = R^{a,b}$ a rectangle.

ii). Recall that a partition of an interval $[c, d] \subset \mathbb{R}$ is a sequence $t_0, \dots, t_k$ s.t. $c = t_0 \le t_1 \le \dots \le t_k = d$ $\Rightarrow$ k subintervals.

A partition of a rectangle $[a^1, b^1] \times \dots \times [a^n, b^n]$ is a collection $P = (P_1, \dots, P_n)$ where each P_i is a partition of the edge $[a^i, b^i]$.

Suppose that P_i is a partition of $[a^i, b^i]$ into N_i intervals. Then P divides $R^{a,b}$ into $N = N_1 \cdot N_2 \cdot \dots \cdot N_n$ subrectangles.

iii). Let A be a rectangle, $A \subset \mathbb{R}^n$, $f: A \to \mathbb{R}$, bounded, let P denote

partition of A. Then for each subrectangle S of P
let $m_S = \inf\{f(x) : x \in S\}$

$M_S = \sup\{f(x) : x \in S\}$
and let $v(S)$ be the volume of $S = R^{p,q}$ ← $q^i \geq p^i$ by assumption
$v(S) = (q^1 - p^1)(q^2 - p^2)\cdots(q^n - p^n)$
The lower and upper sums of f w.r.t. P are
$$L(f, P) = \sum_{S \in P} m_S \cdot v(S)$$

$$U(f, P) = \sum_{S \in P} M_S \cdot v(S)$$

Clearly $L(f, P) \leq U(f, P)$.

<u>Lemma 3.1</u>
Suppose that the partition P' refines P.
(i.e. each rectangle of P' is contained in a rectangle of P)
Then $L(f, P) \leq L(f, P')$ and $U(f, P') \leq U(f, P)$.

<u>Proof</u>
Each subrectangle S of P is now divided into
$S_1, \dots, S_\alpha$ of P', so
$v(S) = v(S_1) + \dots + v(S_\alpha)$.
Now $m_S(f) \leq m_{S_i}(f)$
$$m_S(f)\, v(S) = m_S(f) \sum_{i=1}^{\alpha} v(S_i) \leq \sum_{i=1}^{\alpha} v(S_i) \cdot m_{S_i}(f)$$

So $L(f, P) = \sum_{S \in P} m_S(f)\, v(S) \leq \sum_{S \in P} \sum_{i=1}^{\alpha} m_{S_i}(f)\, v(S_i) = L(f, P')$

Other inequality similar. $\square$

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Remark
For any two partitions P and P' there exists another partition P'' that refines both of the partitions P and P'.

Corollary 3.2
If P and P' are any two partitions of a rectangle A, then $L(f, P') \leq U(f, P)$.

Proof
Let P'' be a refinement of P and P'.
$L(f, P') \leq L(f, P'') \leq U(f, P'') \leq U(f, P)$ $\square$

Def
$$\underline{\int_A} f = \sup_P L(f, P)$$
$$\overline{\int_A} f = \inf_P U(f, P)$$

f is called integrable if $\underline{\int_A} f = \overline{\int_A} f = \int_A f$

Theorem 3.3 (Riemann's Criterion)
Let A be a rectangle. A bounded function $f: A \to \mathbb{R}$ is integrable if $\forall \varepsilon > 0$ $\exists$ partition P of A s.t.
$U(f, P) - L(f, P) < \varepsilon$ $(*)$

Proof
If condition $(*)$ is satisfied then $\lim_{\varepsilon \to 0}$ gives
$\underline{\int_A} f = \overline{\int_A} f$ $\Rightarrow$ f is integrable.

Now suppose f is integrable.

Then for any ε there exists partitions P and P' s.t.

$$U(f, P) - L(f, P') < \varepsilon$$

If P'' refines both P and P' then

$$U(f, P'') - L(f, P'') \leq U(f, P) - L(f, P') < \varepsilon$$

which is condition $(*)$.

$\square$

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$m_S = \inf \{ f(x) \mid x \in S \}$

$M_S = \sup \{ f(x) \mid x \in S \}$

Lower and upper sums:

$$L(f, P) = \sum_{S \in P} m_S(f) \, v(S)$$

$$U(f, P) = \sum_{S \in P} M_S(f) \, v(S)$$

The function is integrable $\Leftrightarrow$

$$\underline{\int_A} f = \sup_P L(f, P) = \overline{\int_A} f = \inf_P U(f, P) = \int_A f$$

Examples

1). Let $f: A \to \mathbb{R}$, let $f(x) = c \quad \forall x \in A$

Then for every partition P and subrectangle S

$m_S(f) = M_S(f) = c$

$$\Rightarrow L(f, P) = U(f, P) = \sum_{S \in P} c \cdot v(S)$$

$$\Rightarrow \int_A f = \sum_{S \in P} c \cdot v(S) = c \cdot v(A)$$

2). The continuous function $f: A \to \mathbb{R}$ is always integrable.

(Exercise)

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3). Let $f : [0,1] \times [0,1] \to \mathbb{R}$ s.t.

$$f(x,y) = \begin{cases} 0 & x \in \mathbb{Q} \\ 1 & x \in \mathbb{R} \setminus \mathbb{Q} \end{cases}$$

Is this function integrable?

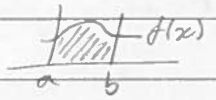
$U(f,P) = 1$, $L(f,P) = 0$ $\quad \forall P$

so the function f is not integrable.

Computing the integral.

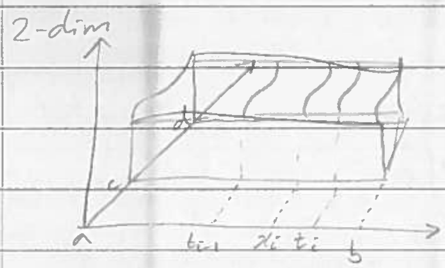
Motivation for Fubini's Theorem

1-dim



$\int_a^b f(x) =$ area under the graph.

2-dim



$f(x,y) : A \to \mathbb{R}$

$A = [a,b] \times [c,d]$

Let $t_0 < \dots < t_n$ be a partition of $[a,b]$

It divides $[a,b] \times [c,d]$ into n strips

$[t_{i-1}, t_i] \times [c,d]$. Let $g_x(y) = f(x,y)$, then the area under the graph of f above $\{x\} \times [c,d]$ is

$$\int_c^d g_x(y)\,dy = \int_c^d f(x,y)\,dy$$

So $\displaystyle\iint_{[a,b]\times[c,d]} f \approx \sum_{i=1}^{n} \int_{[t_{i-1},t_i]\times[c,d]} f \approx \sum_{i=1}^{n} (t_i - t_{i-1}) \int_c^d g_x(y)\,dy$

Sums of this type appear in the definition of integrals in 1-dim, so this means that we can split the 2-dim integral into a chain of 1-dim integrals

$$\int_{[a,b]\times[c,d]} f \approx \int_a^b \Big(\underbrace{\int_c^d f(x,y)\,dy}_{g_x(y)} \Big) dx$$

Problem

f might be integrable on $[a,b]\times[c,d]$ but not continuous at $\{x_0\}\times[c,d]$, then $h(x)=\int_c^d f(x,y)\,dy$ may not even be defined.

Theorem 3.4 (Fubini's Theorem)

Let $A\subset\mathbb{R}^n$, $B\subset\mathbb{R}^m$ be two closed rectangles; $f:A\times B\to\mathbb{R}$ be integrable.

Let $\forall x\in A$ $\quad g_x: B\to\mathbb{R}$, $g_x(y)=f(x,y)$.

Let $\mathcal{L}(x)=\underline{\int_B} g_x(y)\,dy=\underline{\int_B} f(x,y)\,dy$

$\mathcal{U}(x)=\overline{\int_B} g_x(y)\,dy=\overline{\int_B} f(x,y)\,dy$

then both $\mathcal{L}(x)$ and $\mathcal{U}(x)$ are integrable on A and

$$\int_{A\times B} f=\int_A \mathcal{L}=\int_A\left(\underline{\int_B} f(x,y)\,dy\right)dx$$

$$\int_{A\times B} f=\int_A \mathcal{U}=\int_A\left(\overline{\int_B} f(x,y)\,dy\right)dx.$$

Remarks

i) The integrals on the r.h.s. are called the iterrated integrals

ii). The statement of the theorem holds when x is interchanged with y.

Proof

Let P_A be a partition of A, P_B a partition of B

$\Rightarrow P=(P_A,P_B)$ is a partition of $A\times B$.

we denote the subrectangles S of P by $S_A\times S_B$

$\Rightarrow L(f,P)=\sum_{S\in P} m_S(f)v(S)=\sum_{S_A,S_B} m_{S_A\times S_B}(f)v(S_A\times S_B)$.

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$$= \sum_{S_A}\left(\sum_{S_B} m_{S_A\times S_B}(f)\,v(S_B)\right)v(S_A) \quad (*)$$

Clearly if $x \in S_A$ then $m_{S_A\times S_B}(f) \le m_{S_B}(g_x)$

$$\Rightarrow \sum_{S_B} m_{S_A\times S_B}(f)\,v(S_B) \le \sum_{S_B} m_{S_B}(g_x)\,v(S_B) \le \underline{\int_B} g_x = \mathcal{L}(x)$$

$$\overset{(*)}{\Rightarrow} \sum_{S_B} m_{S_A\times S_B}(f)\,v(S_B) \le m_{S_A}(\mathcal{L}(x))$$

take the inf on S_A

$$L(f,P) \le \sum_{S_A} m_{S_A}(\mathcal{L}(x))\,v(S_A) = L(\mathcal{L}, P_A)$$

I obtained that $L(f,P) \le L(\mathcal{L}, P_A)$

similarly I can prove $U(\mathcal{U}, P_A) \le U(f,P)$.

This implies that

$$L(f,P) \le L(\mathcal{L},P_A) \le U(\mathcal{L},P_A) \le U(\mathcal{U},P_A) \le U(f,P)$$

But f is an integrable function.

Therefore, the supremum of the l.h.s is equal to the infimum of th r.h.s. $\forall P$.

$$\sup_P\{L(f,P)\} = \inf_P\{U(f,P)\} = \int_{A\times B} f$$

Then $\sup_{P_A}\{L(\mathcal{L},P_A)\} = \inf_{P_A}\{U(\mathcal{L},P_A)\} = \int_{A\times B} f$ $\left[\Rightarrow \mathcal{L} \text{ is integrable and } \int_A \mathcal{L} = \int_{A\times B} f\right]$?

I will now do the same for $\mathcal{U}$.

$$L(f,P) \le L(\mathcal{L},P_A) \le L(\mathcal{U},P_A) \le U(\mathcal{U},P_A) \le U(f,P)$$

then $\sup_{P_A}\{L(\mathcal{U},P_A)\} = \inf_{P_A}\{U(\mathcal{U},P_A)\} = \int_{A\times B} f$

$\Rightarrow \mathcal{U}$ is integrable, $\int_A \mathcal{U} = \int_{A\times B} f$

$\square$

Remarks

1). If $\forall x \in [a,b]$, $g_x(y) = f(x,y)$ is integrable then $\mathcal{L}(x) = \mathcal{U}(x) = \int_B f(x,y)\,dy$ and

$$\int_{A\times B} f = \int_A \left(\int_B f(x,y)\,dy \right) dx.$$

This occurs when, for example, $f: A\times B \to \mathbb{R}$ continuous.

2). Often g_x is not integrable for finitely many $x \in A$. In this case we have

$$\mathcal{L}(x) = \underline{\int_B} g_x(y)\,dy = \int f(x,y)\,dy$$

for all but a finite number of points x. Since this does not matter in the integral we can still write exactly the same expression (*).

3). There might be some pathological examples why this doesn't always work

$f: [0,1]\times[0,1] \to \mathbb{R}$ st.

$$f(x,y) = \begin{cases} 1 & , x \in \mathbb{R}\setminus\mathbb{Q} \\ 1 & , x \in \mathbb{Q},\ y \in \mathbb{R}\setminus\mathbb{Q} \\ 1-1/q & , x = p/q,\ y \in \mathbb{Q} \end{cases}$$

hw: the function $\int_{[0,1]\times[0,1]} f(x,y) = 1$

and $\int_0^1 f(x,y) = 1$ if $x \in \mathbb{R}\setminus\mathbb{Q}$ but this integral does not exist if x is rational.

(in Spivak).

4). If $A = [a_1,b_1]\times\ldots\times[a_n,b_n]$ and $f: A\to\mathbb{R}$ is sufficiently nice then $\int_A f = \int_{a_n}^{b_n} \left(\ldots \left(\int_{a_1}^{b_1} f(x^1,\ldots,x^n)\,dx^1\right)\ldots\right) dx^n$.

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Measure 0 and content 0

Def

$A \subset \mathbb{R}^n$ has (n-dim) measure 0 if $\forall \varepsilon > 0$ $\exists$ a cover $\{U_1, U_2, \dots\}$ of A by closed intervals s.t.

$$\sum_{i=1}^{\infty} v(U_i) < \varepsilon.$$

Remark

1). Equivalently we can take open sets.

2). If A has measure 0 then $B \subset A$ has measure 0.

3). If A is countable $\Rightarrow$ A has a measure 0.

We can take U_i to be closed rectangles containing each point, $v(U_i) < \varepsilon/2^i$, then

$$\sum_i v(U_i) < \sum_i \varepsilon/2^i = \varepsilon$$

Theorem

If $A = A_1 \cup A_2 \cup \dots$ and each A_i has measure 0 then A has also measure 0.

Problem from PS3

Suppose $f(x,y,z) = 0$ for f differentiable function

Assume that each variable can be expressed as a differentiable function of the two other variables.

Assume $\frac{\partial f}{\partial w} \neq 0$ $w = x, y, z$

$$\left[\frac{d}{dx} f(x, y(x,z), z) = 0 \Rightarrow \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial x} = 0 \right]$$

(see Spivak p44?)

Notation:

$f(x(y,z), y, z) = 0$

$$0 = \frac{\partial f}{\partial y}(x(y,z), y, z) = \frac{\partial f(x,y,z)}{\partial x}\frac{\partial x(y,z)}{\partial y} + \frac{\partial f}{\partial y}(x,y,z)$$

f is a different function on the r.h.s. and on the l.h.s.

It was enough to assume that $\frac{\partial f}{\partial \omega} \neq 0$ for one of $\omega = x, y,$ or z.

This implies $\frac{\partial f}{\partial w} \neq 0 \ \forall w = x, y, z$.

$$\frac{\partial f}{\partial x} \cdot \frac{\partial x}{\partial y} = -\frac{\partial f}{\partial y}$$

$$\frac{\partial f}{\partial y} \cdot \frac{\partial y}{\partial z} = -\frac{\partial f}{\partial z}$$

$$\frac{\partial f}{\partial z} \cdot \frac{\partial z}{\partial x} = -\frac{\partial f}{\partial x}$$

$$\frac{\partial f}{\partial y} = 0 \Rightarrow \frac{\partial f}{\partial z} = 0 \Rightarrow \frac{\partial f}{\partial x} = 0$$

Theorem

If $A = A_1 \cup A_2 \cup \ldots$ and each A_i has measure 0 then A has also a measure 0.

Proof

Let $\varepsilon > 0$. Since A_i has measure 0, $\exists$ a cover of A_i $\{U_i^1, U_i^2, U_i^3, \ldots\}$ s.t.

$$\sum_{j=1}^{\infty} v(U_i^j) < \varepsilon/2^i.$$

Then the collection $\{U_i^j\}$ is a cover of A and

$$\sum_{i=1}^{\infty} \varepsilon/2^i = \varepsilon$$

$\square$

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Def
We say that $A \subset \mathbb{R}^n$ has (n-dim) content 0 if $\forall \varepsilon > 0$ $\exists$ a finite cover $\{U_1, \dots, U_n\}$ of A s.t. $\sum_{i=1}^{n} v(U_i) < \varepsilon$.

If A is of content zero then A is of measure zero.

Theorem
If A is compact and has measure zero then A has content zero.
[Proof as exercise]

Theorem 3.5
Let A be a closed rectangle and $f : A \to \mathbb{R}$, f bounded. Let $B = \{x : f \text{ is not continuous at } x\}$.
Then f is integrable $\Leftrightarrow$ B has measure 0.
[Proof: see Spivak 3.8]

Now let $C \subset \mathbb{R}^n$. We define the characteristic function χ_C of C as

$$\chi_C = \begin{cases} 0 & x \notin C \\ 1 & x \in C. \end{cases}$$

If $C \subset A$ where A is a closed rectangle in $\mathbb{R}^n$ and $f : A \to \mathbb{R}$ is bounded then we define

$$\int_C f = \int_A f \cdot \chi_C \quad \text{provided } f \cdot \chi_C \text{ is integrable.}$$

Theorem 3.6
The function $\chi_C : A \to \mathbb{R}$ is integrable $\Leftrightarrow$ ∂C has measure 0 (and hence content 0).
[Proof: see Spivak 3.9, follows from Thm 3.5].

Change of variables formula

Motivation: (1D) (substitution)

Assume $g : [a,b] \to \mathbb{R}$, and is continuously differentiable and $f : \mathbb{R} \to \mathbb{R}$ is continuous and 1-1.

Then $$\int_{g(a)}^{g(b)} f = \int_a^b (f \circ g) g'. \quad (*)$$

If g is 1-1 (inj.), then $g' < 0$ or $g' > 0$.

Then $(*)$ can be rewritten as

$$\int_{g((a,b))} f = \int_{(a,b)} f \circ g \, |g'|$$

Equivalent of this formula in 1-dim for multi-dim case is:

Theorem 3.7 (Change of variables formula).

Let $A \subset \mathbb{R}^n$ be open, $g : A \to \mathbb{R}^n$ be injective and continuously differentiable s.t. $\det g'(x) \neq 0 \ \forall x \in A$.

If $f : g(A) \to \mathbb{R}$ is integrable then

$$\int_{g(A)} f = \int_A (f \circ g) |\det g'|.$$

Proof (1-dim)

Let F be such that $F' = f$.

Then $(F \circ g)' = (f \circ g) g'$.

$$\int_{g(a)}^{g(b)} f = F(g(b)) - F(g(a))$$

$$\int_a^b (f \circ g) g' = (F \circ g)(b) - (F \circ g)(a)$$

$\square$

The proof in multi-dim case is much more complex.

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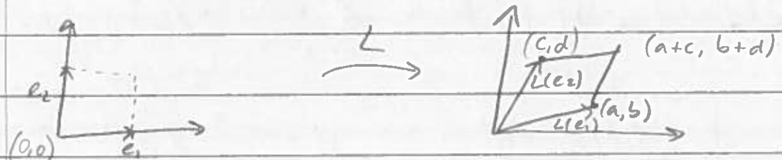
We will present only an idea of the proof.

Step 1:
By the definition of the integral we can approximate $\int_{g(A)} f$ by $\int_{g(A)} f_n$ where f_n is a sequence of functions that are constant on the family of rectangles.
This allows us to reduce the problem to the case $f = \text{const.}$
Step 2:
The theorem is true for g that is a linear function (see q5 PS5).
So to prove that $\int_{g(A)} f = \int_{g(A)} (f \circ g)|\det g'|$ we will consider $f = 1$ (from step 1) and we will approximate g by linear / affine functions.
We will also assume that $A = I^n = (0,1)^n$

$L : \mathbb{R}^2 \to \mathbb{R}^2 \quad M_L = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$



$v(L(I^2)) = \det M_L$

Let $C_x(\varepsilon)$ denote a cube in $\mathbb{R}^n$ of side length ε centred at x.
Let me introduce $\tilde{g}_{x_0}(h) = g(x_0) + Dg(x_0)(h) \leftarrow$ affine.
$\Rightarrow v(g_{x_0}(C_x(\varepsilon))) = v(Dg(x_0)(C_x(\varepsilon)))$
$= |\det g'(x_0)| \cdot \varepsilon^n$

Step 3 :

We show that

$$v(g(C_{x_0}(\varepsilon))) = v(\tilde{g}_{x_0}(C_{x_0}(\varepsilon))) + r(x_0, \varepsilon)$$

where $\lim\limits_{\varepsilon \to 0} \dfrac{r(x_0, \varepsilon)}{\varepsilon^n} \to 0$

it follows from the fact that $g(x) = \tilde{g}_{x_0}(x - x_0) + R(x, x - x_0)$.
Since g is continuously differentiable, this means that g' is uniformly continuous on I^n, so we find that $\lim\limits_{\varepsilon \to 0} \sup\limits_{x \in I^n} \dfrac{|r(x_0, \varepsilon)|}{\varepsilon^n} \to 0$.

Step 4:

We split the cube $I^n = (0,1)^n$ into k^n cubes $C_{x^i}(1/k)$, $1 \le i \le k^n$ (subcubes of I^n).

$$\int_{g(I^n)} 1 = \sum_{i=1}^{k^n} \int_{g(C_{x^i}(1/k))} 1 = \sum_{i=1}^{k^n} v(g(C_{x^i}(1/k)))$$

$$= \sum_{i=1}^{k^n} v(\tilde{g}_{x^i}(C_{x^i}(1/k))) + r(x^i, 1/k) \quad \text{from Step 3}$$

$$= \sum_{i=1}^{k^n} |\det(g'(x^i))| \, (1/k)^n + \sum_{i=1}^{k^n} r(x^i, 1/k)$$

$$\downarrow k\to\infty \qquad\qquad \downarrow k \to \infty$$

$$\int_{I^n} |\det(g'(x))| \qquad\qquad 0$$

$\square$

<u>Remark</u>

The proof of this theorem in Spivak includes step 1.

2). show that g can be w.l.o.g. reduced, $g'(x) = \mathrm{Id}$.

3). we then use induction w.r.t. dimension n and Fubini's thm.

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Example

Define $f : \{ r \mid r > 0 \} \times (0, 2\pi) \to \mathbb{R}^2$
by $f(r, \theta) = (r\cos\theta, r\sin\theta)$ (let $f^1 = r\cos\theta$, $f^2 = r\sin\theta$)

i). Show that f is 1-1.

ii) Compute $f'(r, \theta)$

iii). Compute $\det f'(r, \theta)$

i). Suppose $f(r, \theta) = f(s, \varphi)$

$\Rightarrow (r\cos\theta, r\sin\theta) = (s\cos\varphi, s\sin\varphi)$

$\Rightarrow (r\cos\theta - s\cos\varphi, r\sin\theta - s\sin\varphi) = 0$

$\Rightarrow r\cos\theta = s\cos\varphi$ and $r\sin\theta = s\sin\varphi$

$(f^1)^2 + (f^2)^2 = r^2 \Rightarrow r = \sqrt{(f^1)^2 + (f^2)^2}$ (*)

From definition we take $r_1 \neq r_2$, $\theta_1 \neq \theta_2$

$f(r_1, \theta_1) = f(r_2, \theta_2) \overset{?}{\Rightarrow} r_1 = r_2$, $\theta_1 = \theta_2$

We definitely know that $r_1 = r_2$ by (*).

Suppose $\theta_1 \neq \theta_2$.

$\cos\theta_1 = \cos\theta_2$ and $\sin\theta_1 = \sin\theta_2$

$\Rightarrow \theta_2 = 2\pi - \theta_1$ or $\theta_1 = 2\pi - \theta_2$

$\sin\theta_1 = \sin\theta_2 = \sin(2\pi - \theta_1) = -\sin\theta_1$

$\Rightarrow \theta_1 = \pi$

$\Rightarrow \theta_2 = \pi \Rightarrow \theta_1 = \theta_2$ $\therefore$ f is injective.

ii). $f'(r, \theta) = \begin{pmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{pmatrix}$

iii). $\det f'(r, \theta) = r\cos^2\theta + r\sin^2\theta = r \neq 0$

because in domain, $r > 0$.

iv). Show that $f(\{r \mid r>0\} \times (0,2\pi))$ is a set A,

$A = \{(x,y) \in \mathbb{R}^2 : x<0 \text{ or } (x \geq 0, y \neq 0)\}$.

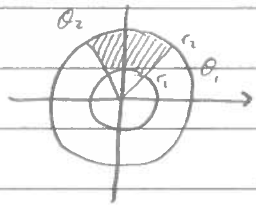
Exercise. (by contradiction).

v). Let $C \subset A$ be a region between the circles of radii r_1, r_2 and the half lines through O which make angles θ_1 and θ_2 with x-axis.

If $h : C \to \mathbb{R}$ is integrable and $h(x,y) = g(r(x,y), \theta(x,y))$ show that

$$\int_C h = \int_{r_1}^{r_2} \int_{\theta_1}^{\theta_2} r\, g(r,\theta)\, d\theta\, dr.$$

Let $\theta_1 \leq \theta_2$, $r_1 \leq r_2$



Define change of variables

$c : [r_1, r_2] \times [\theta_1, \theta_2] \longrightarrow C$

$c(r,\theta) = (\underbrace{r\cos\theta}_{x}, \underbrace{r\sin\theta}_{y})$

$h \circ c = g$

$\det c' = r$

$\therefore$ using the theorem we get

$$\int_C h = \int_{[r_1,r_2]\times[\theta_1,\theta_2]} (h \circ c)|\det c'| = \int_{r_1}^{r_2}\int_{\theta_1}^{\theta_2} g\, r\, d\theta\, dr.$$

$$\int_{B_r} e^{-(x^2+y^2)} = ?$$

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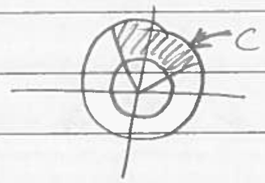
Recall: change of variables formula.

$f : \{r : r > 0\} \times (0, 2\pi) \to \mathbb{R}^2$

$f(r, \theta) = (r\cos\theta, r\sin\theta)$

$$\int_C h = \int_{r_1}^{r_2} \int_{\theta_1}^{\theta_2} r g(r,\theta)\, d\theta\, dr \quad , \; r_1 \leq r_2 \; , \; \theta_1 \leq \theta_2$$

$h : C \to \mathbb{R}$



$h(x,y) = g(r(x,y), \theta(x,y))$

Apply it to compute:

$$\int_{B_r} e^{-(x^2+y^2)} \qquad \text{(Exercise)}.$$

§4 Submanifolds of $\mathbb{R}^n$.

(Remark: Spivak first covers the integration on chains.)

Def

We will call a C^∞-function smooth. (Spivak calls this differentiable, we say differentiable $= C^1$).

Def

Let $U, V \subset \mathbb{R}^n$ be open, $h : U \to V$ smooth, st. $h^{-1} : V \to U$ is smooth.

h is called a diffeomorphism.

Def

A subset $M \subset \mathbb{R}^n$ is called a k-dim submanifold of $\mathbb{R}^n$ if $\forall x \in M$ the following condition holds:

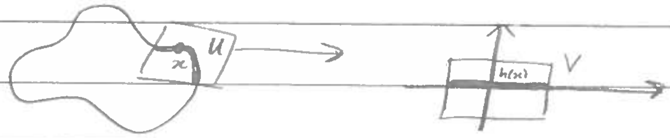
(M) $\exists\, U \subset \mathbb{R}^n$ open, $x \in U$, $V \subset \mathbb{R}^n$ open, and
$h: U \to V$ a diffeomorphism s.t.
$h(U \cap M) = V \cap \{\mathbb{R}^k \times \{0\}\} = \{y \in V : y^{k+1} = y^{k+2} = \dots = y^n = 0\}$.
(coordinates from $k+1$ to n equal 0)

We sometimes say that,
"up to diffeomorphism" $U \cap M$ is $\mathbb{R}^k \times \{0\}$.

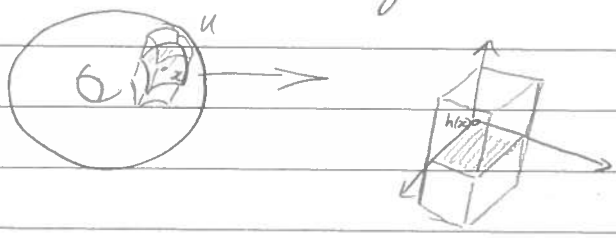
Examples
1). A point in $\mathbb{R}^n$ is a zero-dimensional submanifold.

2). Any open subset of $\mathbb{R}^n$ is an n-dimensional submanifold.

3). 1-dimensional submanifold in $\mathbb{R}^2$



4). 2-dimensional submanifold in $\mathbb{R}^3$



Theorem 4.1
Let $A \subset \mathbb{R}^n$ be open and let $g: A \to \mathbb{R}^p$, $p \leq n$,
be a smooth function s.t. $g'(x)$ has rank p whenever
$g(x) = 0$.
Then $g^{-1}(0)$ is an $(n-p)$-dimensional submanifold (in $\mathbb{R}^n$).

Proof
Follows from Corollary 2.13.
Let $f: \mathbb{R}^n \to \mathbb{R}^p$ be continuously differentiable in an open
set containing x_0, and $p \leq n$.
If $f(x_0) = 0$ and the $p \times n$ matrix $D_j f^i(x_0)$ has rank p,

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then there is an open set $A \subset \mathbb{R}^n$, $x_0 \in A$, and a differentiable function $h: A \to \mathbb{R}^n$ with differentiable inverse s.t. $(g \circ h)(x^1, \dots, x^n) = (x^{n-p+1}, \dots, x^n)$. $\square$

Examples

1). $S^{n-1} \subset \mathbb{R}^n$, $S^{n-1} = \{x \in \mathbb{R}^n : |x| = 1\}$

is an $(n-1)$-dimensional submanifold in $\mathbb{R}^n$.

Take $g(x) = |x|^2 - 1$, $g(x): \mathbb{R}^n \to \mathbb{R}$.

$$= \sum_{i=1}^{n} (x^i)^2 - 1$$

$g^{-1}(0)$ is our S^{n-1}.

Note $g'(x) = (2x^1, \dots, 2x^n)$ has rank 1 unless $x = 0$, this means that by Thm 4.1 $S^{n-1} = g^{-1}(0)$ is an $(n-1)$ dimensional submanifold in $\mathbb{R}^n$.

2). Hyperboloid

$H^n = \{x \in \mathbb{R}^{n+1}, x_1 > 0, x_1^2 - (x_2^2 + \dots + x_{n+1}^2) = 1\}$

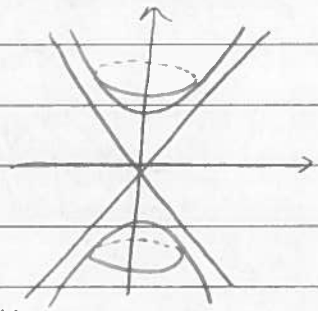
Again take

$g(x) = x_1^2 - (x_2^2 + \dots + x_{n+1}^2) - 1$

$g'(x) = (2x_1, -2x_2, \dots, -2x_{n+1})$

$g'(x)$ has rank $1 \Leftrightarrow x \neq 0$.

this means that H^n is an n-dimensional submanifold of $\mathbb{R}^{n+1}$.



3). Ellipsoid in $\mathbb{R}^3$ $\quad \dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} + \dfrac{z^2}{c^2} = 1$

with the points $(a,0,0)$, $(0,b,0)$, $(0,0,c)$ on the surface.

2-dim submanifold of $\mathbb{R}^3$ (Exercise).

Let $f: \mathbb{R}^n \to \mathbb{R}$ be differentiable, $M = \text{graph}(f) \subset \mathbb{R}^{n+1}$

$x' \in \mathbb{R}^n$, $x \in \mathbb{R}^{n+1}$, $x = (x', x^{n+1})$

$g: \mathbb{R}^{n+1} \to \mathbb{R}$, $g(x) = f(x') - x^{n+1}$

$g'(x) = (D_1 f, \dots, D_n f, -1)$

rank of $g'(x)$ is always 1

$\Rightarrow M = g^{-1}(0)$ is an n-dimensional submanifold in $\mathbb{R}^{n+1}$

$0 = f(x') - x^{n+1}$, $x^{n+1} = f(x')$

Thm 4.2

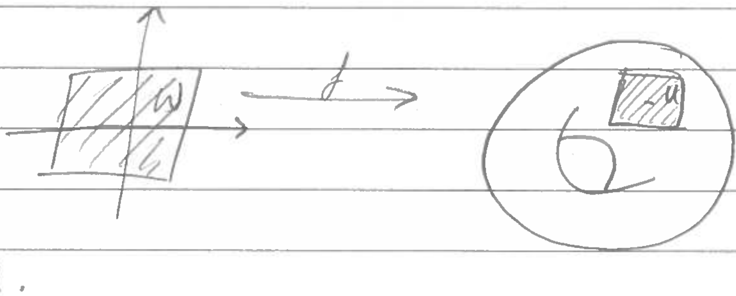
$M \subset \mathbb{R}^n$ is a k-dimensional submanifold in $\mathbb{R}^n$ iff $\forall x \in M$ the following "coordinate condition" is satisfied:

(C) There is $U \subset \mathbb{R}^n$ open, $x \in U$, $W \subset \mathbb{R}^k$ open, and $f: W \to \mathbb{R}^n$ smooth, 1-1, such that

(i) $f(W) = M \cap U$

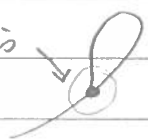
(ii) $f'(y)$ has rank k for each $y \in W$

(iii) $f^{-1}: f(W) \to W$ is continuous



Remark:

The condition (iii) is to avoid this



i.e. no intersections allowed.

Proof

[$\Rightarrow$] Let $x \in M$ k-dimensional submanifold in $\mathbb{R}^n$.

choose $h: U \to V$ s.t. condition (M) holds.

Take $W = \{ a \in \mathbb{R}^k \mid (a, 0) \in h(M) \}$ and define $f: W \to \mathbb{R}^n$ by $f = h^{-1}(a, 0)$.

Clearly $f(W) = U \cap M$ and f, f^{-1} are continuous.

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Let $\pi : \mathbb{R}^n \to \mathbb{R}^k$, $x = (x^1, \dots, x^n) \longmapsto (x^1, \dots, x^k)$

$\Rightarrow (\pi \circ h) \circ f(y) = y \quad \forall y \in W$, therefore

$(\pi \circ h)'(f(y)) \cdot f'(y) = I_k$

$\Rightarrow f'(y)$ has rank k.

Consider [$\Leftarrow$].

Assume $f : W \to \mathbb{R}^n$ satisfying condition (C).

By rearranging coordinates, we can assume that

$(D_j f^i(y))_{1 \le i, j \le k}$ has det $\neq 0$.

Define $g : W \times \mathbb{R}^{n-k} \to \mathbb{R}^n$ by $g(a,b) = f(a) + (\underbrace{0}_{k \text{ coords.}}, \underbrace{b}_{n-k \text{ coords.}})$

$$g'(y) = \begin{array}{c} k\{ \\ \\ n-k\{ \end{array} \begin{pmatrix} D_j f^i & 0 \\ & \\ D_j f^l & \begin{matrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{matrix} \end{pmatrix}$$
$\qquad\qquad\qquad\quad \underbrace{\qquad\quad}_{k} \underbrace{\qquad\quad}_{n-k}$

In particular I know that $g'(x) \neq 0$.

By the inverse function theorem

$\exists$ open set V_1' s.t. $(y, 0) \in V_1'$ and

V_2' (open) containing $g(y, 0) = x$ s.t.

$f(y) = x$ and $h : V_2' \to V_1'$.

<u>Claim</u> that $\{f(a) : (a, 0) \in V_1'\} = U \cap f(W)$

with U an open subset of $\mathbb{R}^n$.

This follows from the fact that $f^{-1} : f(W) \to \mathbb{R}^k$

is continuous.

Take $V_2 = V_2' \cap U$ and $V_1 = g^{-1}(V_2)$.

Then $V_2 \cap M$ is exactly $\{f(a) : (a, 0) \in V_1\} = \{g(a, 0) : (a, 0) \in V_1\}$.

[Denote $V_2 = U$, $V_1 = V$]

$$\begin{aligned} h(V_2 \cap M) &= g^{-1}(V_2 \cap M) \\ &= g^{-1}(\{g(a, 0) : (a, 0) \in V_1\}) \\ &= V_1 \cap \{\mathbb{R}^k \times \{0\}\}. \quad \square \end{aligned}$$

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Theorem 4.2

$M \subset \mathbb{R}^n$ is a k-dim. submanifold iff for every $x \in M$ the following "coordinate condition" is satisfied:

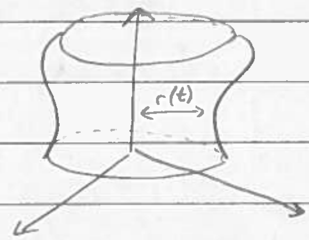
(C) - $\exists\ U$ open, $U \subset \mathbb{R}^n$, $x \in U$, $W \subset \mathbb{R}^k$ open and $f: W \to U$ smooth.

- f is injective (1-1)

(i) $f(W) = M \cap U$

(ii) $f'(y)$ is of full rank (k) $(y \in W)$

(iii) f^{-1} is continuous, $f^{-1}: f(W) \to W$.



Example Surface of revolution in $\mathbb{R}^3$.

Let $I \subset \mathbb{R}$, open interval.

Let $j: I \to \mathbb{R}^2$, $t \longmapsto (r(t), z(t))$

- injective $r(t) > 0$
- smooth
- has continuous inverse
- $(r'(t), z'(t)) \neq 0$. (no stopping point)

Define $f: I \times (-\pi, \pi) \longrightarrow \mathbb{R}^3$,

$(t, \theta) \mapsto (\underbrace{r(t)\sin\theta}_{x}, \underbrace{r(t)\cos\theta}_{y}, z(t))$

Claim:

$f(I \times (-\pi, \pi)) \subset \mathbb{R}^3$ is a 2-dim submanifold.

Proof:

We will check the condition (C).

- Take $W = I \times (-\pi, \pi)$ and $U = \mathbb{R}^3$, $f: W \to U$, f smooth.

- WTS: f is injective.

Since $r = \sqrt{(x^1)^2 + (y^1)^2} = \sqrt{(x^2)^2 + (y^2)^2}$

so $r(t_1) = r(t_2)$, $z_1 = z_2$, $z(t_1) = z(t_2)$

so $r_1 = r_2$ from conditions on j.

It remains to show that $\theta_1 = \theta_2$.

We know that $\sin\theta_1 = \sin\theta_2$ & $\cos\theta_1 = \cos\theta_2$

$\Rightarrow \theta_1 = \theta_2$

(i) is clear due to the choice of $\mathcal{U}$ (since $M = f(W)$)

(ii) $f'(y)$ has rank k ? $k=2$ here.

$$f'(y) = \begin{pmatrix} r'(t)\sin\theta & r(t)\cos\theta \\ r'(t)\cos\theta & -r(t)\sin\theta \\ z'(t) & 0 \end{pmatrix}$$

Consider $\begin{pmatrix} r'(t)\sin\theta & r(t)\cos\theta \\ r'(t)\cos\theta & -r(t)\sin\theta \end{pmatrix} = A$

$\det A = 0 \iff r'r = 0$

$r(t) > 0$ so $r'(t) = 0$

$\Rightarrow z'(t) \neq 0$ (since $(r'(t), z'(t)) \neq 0$)

$\Rightarrow \operatorname{rank}(f'(y)) = 2$

(iii) $f^{-1} : f(W) \to W$ is continuous ?

[Remark:

$f(x)$ is continuous in x if for $x_i \to x$, $f(x_i) \to f(x)$.]

$(x_i, y_i, z_i) \to (x, y, z)$

i.e. $(x(t_i, \theta_i), y(t_i, \theta_i), z(t_i)) \to (x(\bar{t}, \bar{\theta}), y(\bar{t}, \bar{\theta}), z(\bar{t}))$

WTS: this $\Rightarrow t_i \to \bar{t}$ and $\theta_i \to \bar{\theta}$.

$r_i = \sqrt{(x_i)^2 + (y_i)^2} \to \sqrt{x^2+y^2}$

$\left.\begin{array}{l} r_i = r(t_i) \to r(\bar{t}) \\ z_i = z(t_i) \to z(\bar{t}) \end{array}\right\} \Rightarrow t_i \to \bar{t}$ (since γ has cont. inverse).

Checking that $\theta_i \to \bar{\theta}$ is just an observation that we can locally invert sin and cos.

$\therefore f(I \times (-\pi, \pi))$ is a 2-dim manifold.

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§5 Integration on chains.

Def

V - finite dimensional vector space on $\mathbb{R}$ $(\cong \mathbb{R}^k)$

$f: V \to \mathbb{R}$ linear map (called linear functional)

$V^* = \{f: V \to \mathbb{R}, \text{linear}\}$ is a dual space of V.

Prop

V^* is a vector space on $\mathbb{R}$

We will prove that if there is a scalar product defined on $V = \mathbb{R}^n$ then we can define a bijection between $\mathbb{R}^n \to (\mathbb{R}^n)^*$

Proof (of prop.)

$T \in V^*$, $T: V \to \mathbb{R}$ is linear. Let $u, v \in V$, $\alpha, \beta \in \mathbb{R}$.

Then $T(\alpha v + \beta u) = \alpha T(v) + \beta T(u)$.

Define for $S, T \in V^*$,

$(S+T)(v) = S(v) + T(v)$

This gives the additive group.

$a \in \mathbb{R}$

$(a \cdot S)(v) = a \cdot S(v)$.

We can also show that there exists $T: V \to V^*$ bijection if there is a scalar product on V.

$V \cong \mathbb{R}^k$, for some finite k.

Let $x \in \mathbb{R}^k$ and let us define $\varphi_x(y) = \langle x, y \rangle$.

$\varphi_x \in (\mathbb{R}^k)^*$.

φ_x is linear because $\langle \, , \rangle$ is bilinear. We will show $T: \mathbb{R}^k \to (\mathbb{R}^k)^*$ s.t. $T(x) = \varphi_x$ is a bijection.

Claim: T is injective.

Suppose $T(x) = T(x')$.

$\varphi_x(y) - \varphi_{x'}(y) = \langle x, y \rangle - \langle x', y \rangle = \langle x - x', y \rangle$

$\forall y \in \mathbb{R}^k$. Pick $y = x - x'$.

$\Leftrightarrow |x - x'|^2 = 0$

$\Rightarrow x - x' = 0 \Rightarrow x = x'.$

Claim: T is surjective.

Let $L \in (\mathbb{R}^k)^*$, $x = (x^1, \dots, x^k) \in \mathbb{R}^k$

$L(x) = \sum_{i=1}^{k} a^i x^i$, then take $a = (a^1, \dots, a^k)$

$L(x) = \langle a, x \rangle = \varphi_a.$

$\square$

Prop

$\dim_{\mathbb{R}} V^* = \dim_{\mathbb{R}} V$

Let $\{v_1, \dots, v_n\}$ be a basis of V, then we can define $\varphi_i \in V^*$ by $\varphi_i(v_j) = \delta_{ij}$. This is a basis of V^*. $\{\varphi_i\}_{i=1}^{n}$ is called a dual basis.

Proof

Let $x \in V \Rightarrow x = \sum_{i=1}^{n} x^i v_i$

$\Rightarrow \varphi_j(x) = \sum_{i=1}^{n} x^i \varphi_j(v_i) = x^j$ from the definition of φ.

We take $f \in V^*$. Let us denote $a^i = f(v_i)$.

Define $\phi = \sum_{i=1}^{n} a^i \varphi_i \in V^*$.

$$f(x) = f\left(\sum_{i=1}^{n} x^i v_i\right) = \sum_{i=1}^{n} x^i a^i$$

$$\phi(x) = \sum_{i=1}^{n} a^i \varphi_i(x) = \sum_{i=1}^{n} a^i \varphi_i\left(\sum_{j=1}^{n} x^j v_j\right)$$

$$= \sum_{i=1}^{n} a^i x^i = f(x)$$

So $\{\varphi_1, \dots, \varphi_n\}$ spans the vector space V^*

Claim

$\varphi_1, \dots, \varphi_n$ are linearly independent.

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Assume that $\sum_{i=1}^{n} a^i \varphi_i = 0$, then it has to hold for any $x \in \mathbb{R}^n$.

$0 = \sum_{i=1}^{n} a^i \varphi_i(x)$

We take $x = v_1, v_2, \dots, v_n$

Then this implies that $a_i = 0 \quad \forall i = 1, \dots, n$

$\Rightarrow \{\varphi_1, \dots, \varphi_n\}$ is a basis of V^*.

$\square$

Let us now consider $V^k = V \times \dots \times V$ k-fold product of V.

Def

$T: V^k \longrightarrow \mathbb{R}$ is called multilinear if $\forall i = 1, \dots, k$

$T(v_1, \dots, v_i + v_i', \dots, v_k) = T(v_1, \dots, v_i, \dots, v_k) + T(v_1, \dots, v_i', \dots, v_k)$

$a \in \mathbb{R}$, $T(v_1, \dots, a v_i, \dots, v_k) = a\, T(v_1, \dots, v_i, \dots, v_k)$.

Def

A multilinear function $T: V^k \to \mathbb{R}$ is called a k-tensor on V. The space of all k-tensors is denoted by $\mathcal{J}^k(V)$. It is a vector space over $\mathbb{R}$ with the following operations:

$(S+T)(v_1, \dots, v_k) = S(v_1, \dots, v_k) + T(v_1 + \dots + v_k)$.

$(a \cdot S)(v_1, \dots, v_k) = a \cdot S(v_1, \dots, v_k)$.

Def

We define the tensor product for $S \in \mathcal{J}^k(V)$, $T \in \mathcal{J}^l(V)$, $S \otimes T \in \mathcal{J}^{k+l}(V)$ by

$S \otimes T(v_1, \dots, v_k, v_{k+1}, \dots, v_{k+l}) = S(v_1, \dots, v_k)\, T(v_{k+1}, \dots, v_{k+l})$.

Remark

In general, $S \otimes T \neq T \otimes S$

We have the following properties:

(i) $S \otimes T \in \mathcal{J}^{k+l}$

(ii) $(S_1 + S_2) \otimes T = S_1 \otimes T + S_2 \otimes T$

(iii) $S \otimes (T_1 + T_2) = S \otimes T_1 + S \otimes T_2$

(iv) $(aS) \otimes T = a(S \otimes T) = S \otimes (aT)$

(v) $(S \otimes V) \otimes T = S \otimes (V \otimes T) = S \otimes V \otimes T$

Remark

$\mathcal{J}^1(V) = V^*$

20-11-17 Recall:

$S \in \mathcal{J}^k(V)$, $T \in \mathcal{J}^l(V)$ $\quad T \otimes S \in \mathcal{J}^{k+l}(V)$

$\mathcal{J}^1(V) = V^*$

V has a basis $v_1, \dots, v_k$, V^* has a dual basis $\varphi_1, \dots, \varphi_k$.

Theorem 5.1

Let $v_1, \dots, v_n$ be a basis of V and let $\varphi_1, \dots, \varphi_n$ be the dual basis $\varphi_i(v_j) = \delta_{ij}$. Then the set of all products $\varphi_{i_1} \otimes \cdots \otimes \varphi_{i_k}$ $\quad 1 \le i_1, \dots, i_k \le n$ is a basis for $\mathcal{J}^k(V)$ and thus has dimension n^k.

Proof

Note that $\varphi_{i_1} \otimes \dots \otimes \varphi_{i_k}(v_{j_1}, \dots, v_{j_k}) = \delta_{i_1 j_1} \delta_{i_2 j_2} \cdots \delta_{i_k j_k}$

$$= \begin{cases} 1 & \text{if } i_l = j_l, \ l = 1, \dots, k \\ 0 & \text{otherwise} \end{cases}$$

Take $T \in \mathcal{J}^k(V)$, let $w_1, \dots, w_k \in V$,

$$w_i = \sum_{j=1}^{n} a_{ij} v_j \qquad \text{then } \varphi_j(w_i) = a_{ij}$$

$$T(w_1, \dots, w_k) = \sum_{j_1, \dots, j_k = 1}^{n} a_{1,j_1} \cdots a_{k,j_k} T(v_{j_1}, \dots, v_{j_k})$$

$$= \sum_{j_1, \dots, j_k = 1}^{n} \underbrace{T(v_{j_1}, \dots, v_{j_k})}_{\text{coefficients}} \varphi_{j_1} \otimes \cdots \otimes \varphi_{j_k}(w_1, \dots, w_k)$$

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$\Rightarrow \varphi_{i_1} \otimes \cdots \otimes \varphi_{i_k}$ span $\mathcal{J}^k(V)$.

Now we need to show that they are linearly independent.

Assume that there are some numbers $a_{i_1, \dots, i_k}$ s.t.

$$\sum_{i_1, \dots, i_k = 1}^{n} a_{i_1, \dots, i_k} \varphi_{i_1} \otimes \dots \otimes \varphi_{i_k} = 0. \quad (*)$$

Apply both sides to $(v_{j_1}, \dots, v_{j_k})$

$\Rightarrow a_{j_1, \dots, j_k} = 0$ and we do the same for all coefficients in the sum $(*)$. $\square$

<u>Example</u>

$\dim V = 3$, $V^* = \mathcal{J}^1(V)$

if $\{v_1, v_2, v_3\}$ is a basis of V, we have $\{\varphi_1, \varphi_2, \varphi_3\}$ is a basis of V^*.

<u>$k=1$</u> every 1-tensor is equal to

$T = a_1 \varphi_1 + a_2 \varphi_2 + a_3 \varphi_3$ for some numbers a_1, a_2, a_3.

$\dim V^* = \dim \mathcal{J}^1(V) = 3$

<u>$k=2$</u> $T \in \mathcal{J}^2(V)$, the basis is

$\{\varphi_1 \otimes \varphi_1, \ \varphi_1 \otimes \varphi_2, \ \varphi_1 \otimes \varphi_3, \ \varphi_2 \otimes \varphi_1, \ \varphi_2 \otimes \varphi_2, \ \varphi_2 \otimes \varphi_3,$
$\varphi_3 \otimes \varphi_1, \ \varphi_3 \otimes \varphi_2, \ \varphi_3 \otimes \varphi_3\} \quad (\Delta)$

$\dim \mathcal{J}^2(V) = 9 = 3^2$

$T \in \mathcal{J}^2(V)$ is a linear combination of (Δ)

<u>$k=3$</u>

$\dim \mathcal{J}^3(V) = 27 = 3^3$

elements of basis have form $\varphi_i \otimes \varphi_j \otimes \varphi_k$, $i, j, k \in \{1, 2, 3\}$.

<u>Def</u>

Let $f: V \to W$ be a linear map, then a linear transformation $f^*: \mathcal{J}^k(W) \to \mathcal{J}^k(V)$ is defined via

$f^* T(v_1, \dots, v_k) = T(f(v_1), \dots, f(v_k))$

for $T \in \mathcal{J}^k(W)$ and $v_1, \dots, v_k \in V$.

One easily checks that $f^*(S \otimes T) = f^*(S) \otimes f^*(T)$.

Examples of tensors:

- $\langle \cdot , \cdot \rangle$ - scalar product $\mathcal{J}^2(V)$
- determinant of the set of vectors $h_1, \dots, h_k$

  $\det(h_1\, h_2 \dots h_k)$ is $\mathcal{J}^k(V)$

Def

(i) Symmetric 2-tensor T is such that $T(v_1, v_2) = T(v_2, v_1)$ $\forall v_1, v_2 \in V$

(ii) Inner product is a symmetric 2-tensor T s.t.

$T(v,v) \geq 0$ and $T(v,v) = 0$ iff $v = 0$.

(iii) k-tensor is called symmetric if $\forall i, j = 1, \dots, k$

$T(v_1, \dots, v_i, \dots, v_j, \dots, v_k) = T(v_1, \dots, v_{i-1}, v_j, v_{i+1}, \dots, v_{j-1}, v_i, v_{j+1}, \dots, v_k)$

(iv) k-tensor is called alternating if

$T(v_1, \dots, v_i, \dots, v_j, \dots, v_k) = -T(v_1, \dots, v_{i-1}, v_j, v_{i+1}, \dots, v_{j-1}, v_i, v_{j+1}, \dots, v_k)$

for any $i, j = 1, \dots, k$, $v_1, \dots, v_k \in V$

The set of all tensors satisfying definition (iv) is a subspace of $\mathcal{J}^k(V)$ and denoted by $\Lambda^k(V)$.

[Recall that S_k = set of all permutations of $1, 2, \dots, k$.

$$\operatorname{sign}(\sigma) = \begin{cases} +1 & \text{if } \sigma \text{ is even (even no. of transpositions)} \\ -1 & \text{if } \sigma \text{ is odd (odd no. of transpositions)} \end{cases}$$
]

Def

Let $T \in \mathcal{J}^k(V)$ we define $\operatorname{Alt}(T)$ by

$$\operatorname{Alt}(T)(v_1, \dots, v_k) = \frac{1}{k!} \sum_{\sigma \in S_k} \operatorname{sign}(\sigma) \cdot T(v_{\sigma(1)}, v_{\sigma(2)}, \dots, v_{\sigma(k)})$$

Theorem 5.3

1). If $T \in \mathcal{J}^k(V)$, then $\operatorname{Alt}(T) \in \Lambda^k(V)$

2). If $\omega \in \Lambda^k(V)$, then $\operatorname{Alt}(\omega) = \omega$

3). If $T \in \mathcal{J}^k(V)$, then $\operatorname{Alt}(\operatorname{Alt}(T)) = \operatorname{Alt}(T)$

Proof

$1 + 2 \Rightarrow 3.$

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1). $T \in \mathcal{J}^k(V)$.

Let (i,j) denote permutation $i \leftrightarrow j$

$\sigma \in S_k$, $\sigma' = \sigma \circ (i,j)$. $i<j$

$$Alt(T)(v_1, \dots, v_j, \dots, v_i, \dots, v_k) = \frac{1}{k!} \sum_{\sigma \in S_k} sign(\sigma) T(v_{\sigma(1)}, \dots, v_{\sigma(j)}, \dots, v_{\sigma(i)}, \dots, v_{\sigma(k)})$$

$$= \frac{1}{k!} \sum_{\sigma \in S_k} sign\, \sigma\, T(v_{\sigma'(1)}, \dots, v_{\sigma'(i)}, \dots, v_{\sigma'(j)}, \dots, v_{\sigma'(k)})$$

$$= \frac{1}{k!} \sum_{\sigma' \in S_k} -sign\, \sigma'\, T(v_{\sigma'(1)}, \dots, v_{\sigma'(k)})$$

$$= -Alt(T)(v_1, \dots, v_k)$$

2). $\omega \in \Lambda^k(V)$ and $\sigma = (i,j)$

then $\omega(v_{\sigma(1)}, \dots, v_{\sigma(k)}) = sign\, \sigma\, \omega(v_1, \dots, v_k)$ (*)

Since we can express any σ as a product of transpositions of the form (i,j) then (*) holds for $\sigma \in S_k$.

Therefore $Alt(\omega)(v_1, \dots, v_k) = \frac{1}{k!} \sum_{\sigma \in S_k} sgn(\sigma)\, \omega(v_{\sigma(1)}, \dots, v_{\sigma(k)})$

$$= \frac{1}{k!} \sum_{\sigma \in S_k} (sgn\, \sigma)^2 \omega(v_1, \dots, v_k) = \omega(v_1, \dots, v_k)$$

$\square$

sign = sgn

Note that if $\omega \in \Lambda^k(V)$ and $\eta \in \Lambda^l(V)$ then $\omega \otimes \eta \in \mathcal{J}^{k+l}(V)$ but usually not to $\Lambda^{k+l}(V)$

<u>Def</u>

The wedge product is defined as

$$\omega \wedge \eta = \frac{(k+l)!}{k!\,l!} Alt(\omega \otimes \eta)$$

for $\omega \in \Lambda^k(V)$, $\eta \in \Lambda^l(V)$.

<u>Properties</u>

(i) $(\omega_1 + \omega_2) \wedge \eta = \omega_1 \wedge \eta + \omega_2 \wedge \eta$

(ii) $\omega \wedge (\eta_1 + \eta_2) = \omega \wedge \eta_1 + \omega \wedge \eta_2$

(iii) $(a\omega) \wedge \eta = a \cdot (\omega \wedge \eta) = \omega \wedge (a\eta)$

(iii) $\omega \wedge \eta = (-1)^{kl} \eta \wedge \omega$

(iv) $f^*(\omega \wedge \eta) = f^*(\omega) \wedge f^*(\eta)$

Theorem 5.4

1). If $S \in \mathcal{J}^k(V)$ and $T \in \mathcal{J}^l(V)$ and $Alt(S) = 0$

$\Rightarrow Alt(S \otimes T) = Alt(T \otimes S) = 0$

2). $Alt(Alt(\omega \otimes \eta) \otimes \theta) = Alt(\omega \otimes \eta \otimes \theta) = Alt(\omega \otimes Alt(\eta \otimes \theta))$

3). If $\omega \in \Lambda^k(V)$, $\eta \in \Lambda^l(V)$, $\theta \in \Lambda^m(V)$, then

$$(\omega \wedge \eta) \wedge \theta = \omega \wedge (\eta \wedge \theta)$$
$$= \frac{(k+l+m)!}{k!\,l!\,m!} Alt(\omega \otimes \eta \otimes \theta)$$

Proof

1). $(k+l)!\, Alt(S \otimes T)(v_1, \dots, v_{k+l})$

$$= \sum_{\sigma \in S_{k+l}} sgn(\sigma) S(v_{\sigma(1)}, \dots, v_{\sigma(k)}) T(v_{\sigma(k+1)}, \dots, v_{\sigma(k+l)})$$

Therefore if $\sigma \in S_k$ then $\sum_{\sigma \in S_k} sgn(\sigma) S(v_{\sigma(1)}, \dots, v_{\sigma(k)}) = 0$

since $Alt(S) = 0$.

Define $G \subset S_{k+l}$ consists of all σ s.t. $\sigma(k+i) = k+i$, $i = 1, \dots, l$.

Then $\sum_{\sigma \in G} sgn(\sigma) S(v_{\sigma(1)}, \dots, v_{\sigma(k)}) T(v_{\sigma(k+1)}, \dots, v_{\sigma(k+l)}) = 0$

Assume that $\sigma_0 \notin G$ and consider

$G \cdot \sigma_0 = \{\sigma \circ \sigma_0 : \sigma \in G\}$ and denote

$v_{\sigma_0(1)}, \dots, v_{\sigma_0(k+l)} = w_1, \dots, w_{k+l}$

Then $\sum_{\sigma \in G \cdot \sigma_0} sgn(\sigma) S(v_{\sigma(1)}, \dots, v_{\sigma(k)}) T(v_{\sigma(k+1)}, \dots, v_{\sigma(k+l)})$

[Any σ is of the form $\sigma' \circ \sigma_0$ for some $\sigma' \in G$]

$$= sgn(\sigma_0) \sum_{\sigma' \in G} sgn(\sigma') S(w_{\sigma'(1)}, \dots, w_{\sigma'(k)}) T(w_{k+1}, \dots, w_{k+l})$$

$$= 0 \quad \text{(from previous step)}$$

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Note that $G \cap G \cdot \sigma_0 = \phi$ because if $\sigma \in G \cap G \cdot \sigma_0$, then $\sigma = \sigma' \cdot \sigma_0$ for some σ' in G

$\Rightarrow \sigma_0 = \sigma \cdot (\sigma')^{-1} \in G$ ※

We can thus "break" S_{k+l} into disjoint subsets of this form s.t. the sum over each subset is equal to 0, also the sum over S_{k+l} is 0.

$Alt(T \otimes S) = 0$ is proved similarly.

2). We now have

$$Alt(\underbrace{Alt(\eta \otimes \theta) - (\eta \otimes \theta)}_{S}) = Alt(\eta \otimes \theta) - Alt(\eta \otimes \theta) = 0$$

So by (1) we have

$$\begin{aligned} 0 &= Alt(\omega \otimes [Alt(\eta \otimes \theta) - \eta \otimes \theta]) \\ &= Alt(\omega \otimes Alt(\eta \otimes \theta)) - Alt(\omega \otimes \eta \otimes \theta) \end{aligned}$$

The other equality follows similarly.

22-11-17 3). $(\omega \wedge \eta) \wedge \theta = \dfrac{(k+l+m)!}{(k+l)!\, m!} Alt((\omega \wedge \eta) \otimes \theta)$

$$= \frac{(k+l+m)!}{\cancel{(k+l)!}\, m!} \frac{\cancel{(k+l)!}}{k!\, l!} Alt(\omega \otimes \eta \otimes \theta)$$

$$= \frac{(k+l+m)!}{k!\, l!\, m!} Alt(\omega \otimes \eta \otimes \theta)$$

other case similar. □

[In future, write $(\omega \wedge \eta) \wedge \theta = \omega \wedge \eta \wedge \theta = \omega \wedge (\eta \wedge \theta)$]

<u>Theorem 5.5</u>

The set of all $\varphi_{i_1} \wedge \ldots \wedge \varphi_{i_k}$, $1 \leq i_1 < i_2 < \ldots < i_k \leq n$, is a basis of $\Lambda^k(V)$, which therefore has a dimension $\binom{n}{k} = \frac{n!}{k!\,(n-k)!}$

Proof

For $\omega \in \Lambda^k(V) \subset \mathcal{J}^k(V)$, we can write

$$\omega = \sum_{i_1, i_2, \dots, i_k = 1}^{n} a_{i_1, i_2, \dots, i_k} \varphi_{i_1} \otimes \dots \otimes \varphi_{i_k}$$

$$\omega = Alt(\omega) = \sum_{i_1, i_2, \dots, i_k = 1}^{n} a_{i_1, i_2, \dots, i_k} \underbrace{Alt(\varphi_{i_1} \otimes \dots \otimes \varphi_{i_k})}_{const \cdot \varphi_{i_1} \wedge \dots \wedge \varphi_{i_k}} \qquad (*)$$

If there exist indices l and m s.t. $i_l = i_m$ then $\varphi_{i_1} \wedge \dots \wedge \varphi_{i_k} = 0$, so all elements on the r.h.s. of $(*)$ are of the form $C \cdot \varphi_{i_1} \wedge \dots \wedge \varphi_{i_k}$ with $i_1 < \dots < i_k$ after maybe a permutation.

The linear independence of tensors $\varphi_{i_1} \wedge \dots \wedge \varphi_{i_k}$ follows as in Thm 5.1.

□

Example

$\dim V = 3$

$k = 1$

$\dim \Lambda^1(V) = \dim V^* = 3$ $\quad \varphi_1, \varphi_2, \varphi_3$ is the dual basis to v_1, v_2, v_3 - the basis of V.

$k = 2$

$\dim \Lambda^2(V) = \binom{3}{2} = 3$

The basis is now $\varphi_1 \wedge \varphi_2, \ \varphi_2 \wedge \varphi_3, \ \varphi_1 \wedge \varphi_3$

$\varphi_1 \wedge \varphi_2 = (\varphi_1 \otimes \varphi_2 - \varphi_2 \otimes \varphi_1)$

$\varphi_1 \wedge \varphi_2 (v, w) = (\varphi_1(v) \cdot \varphi_2(w) - \varphi_2(v) \cdot \varphi_1(w))$, $\quad v, w \in V$

similarly $\varphi_2 \wedge \varphi_3 = (\varphi_2 \otimes \varphi_3 - \varphi_3 \otimes \varphi_2)$

$\varphi_1 \wedge \varphi_3 = (\varphi_1 \otimes \varphi_3 - \varphi_3 \otimes \varphi_1)$

$k = 3$

$\dim \Lambda^3(V) = \binom{3}{3} = 1$

The basis is $\varphi_1 \wedge \varphi_2 \wedge \varphi_3$

$$\begin{aligned} \varphi_1 \wedge \varphi_2 \wedge \varphi_3 &= 3! \ Alt(\varphi_1 \otimes \varphi_2 \otimes \varphi_3) \\ &= \varphi_1 \otimes \varphi_2 \otimes \varphi_3 - \varphi_1 \otimes \varphi_3 \otimes \varphi_2 - \varphi_2 \otimes \varphi_1 \otimes \varphi_3 - \varphi_3 \otimes \varphi_2 \otimes \varphi_1 \\ &\quad + \varphi_2 \otimes \varphi_3 \otimes \varphi_1 + \varphi_3 \otimes \varphi_1 \otimes \varphi_2 \end{aligned}$$

$\varphi_1 \wedge \varphi_2 \wedge \varphi_3 (v_1, v_2, v_3) = \varphi_1 \otimes \varphi_2 \otimes \varphi_3 (v_1, v_2, v_3) = \varphi_1(v_1) \cdot \varphi_2(v_2) \cdot \varphi_3(v_3) = 1$

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Note: $\Lambda^n(V)$, where $\dim V = n$, then
$\dim \Lambda^n(V) = \binom{n}{n} = 1$ and we know that

$\det([w_1, \dots, w_n]) \in \Lambda^n(V)$.

Theorem 5.6
Let $v_1, \dots, v_n$ be a basis of V and let $\omega \in \Lambda^n(V)$.
If $w_i = \sum_{j=1}^{n} a_{ij} v_j$ are n vectors in V, then

$$\omega(v_1, \dots, v_n) = \det(a_{ij})\, \omega(v_1, \dots, v_n)$$

Proof
Homework sheet 7. $\square$

Remarks
1) Let $\omega \in \Lambda^n(V)$ $(\dim V = n)$ $\omega \neq 0$.
Then this theorem splits the bases of V into two "groups": those with $\omega(v_1, \dots, v_n) > 0$ and $\omega(v_1, \dots, v_n) < 0$.
So, $v_1, \dots, v_n$ and $w_1, \dots, w_n$ are in the same "group" $\iff$ the determinant of a_{ij} is > 0 ($\det a_{ij} > 0$).
2) This criterion is independent of the choice of ω and can always be used to divide the bases of V in two disjoint groups.

Def
Either of these two groups is called an orientation of V.
One denotes the orientation of basis $v_1, \dots, v_n$ by $[v_1, \dots, v_n]$, and the other by $-[v_1, \dots, v_n]$.
In $\mathbb{R}^n$ the usual orientation is defined by $[e_1, \dots, e_n]$.

Def

We can define det. as the unique element $\omega \in \Lambda^n(\mathbb{R}^n)$ s.t. $\omega(e_1, \dots, e_n) = 1$.

Example

Let us consider $f : [0,1] \to (\mathbb{R}^n)^n$ and let f be continuous, moreover let $(f^1(t), \dots, f^n(t))$ be a basis of $\mathbb{R}^n$ for all $t \in [0,1]$.
We can show that

$$[f^1(0), \dots, f^n(0)] = [f^1(1), \dots, f^n(1)].$$

We will consider $\det \circ f$ which is a continuous function $[0,1] \to \mathbb{R}$, this is because $\det : (\mathbb{R}^n)^n \to \mathbb{R}$ is a polynomial w.r.t. coefficients of the vectors (entries of the matrix) and so is a continuous function. We also know that it does not take the value 0, because $\forall t \in [0,1]$ $(f^1(t), \dots, f^n(t))$ is a basis of $\mathbb{R}^n$ (thus system of linearly independent vectors). Applying the intermediate value theorem, the image of $\det \circ f$ contains numbers of the same sign.
So, all $f(t)$ have the same orientation because the division into two "groups" does not depend on the choice of $\omega \in \Lambda^n(\mathbb{R}^n)$ in particular $\omega = \det$.

So for $\mathbb{R}^n$ the definition of the determinant and the usual orientation is clear.
However for general V there is no such clear criterion.

Assume however that V has an inner product denoted by g. Take $v_1, \dots, v_n$ and $w_1, \dots, w_n$ two orthonormal bases w.r.t. g and $A = (a_{ij})$ is defined by $w_i = \sum_{j=1}^{n} a_{ij} v_j$ then $\delta_{ij} = g(w_i, w_j) = \sum_{k,l=1}^{n} a_{ik} a_{jl} g(v_k, v_l)$.

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$\Rightarrow \delta_{ij} = \sum_{k=1}^{n} a_{ik} a_{jk}$

$\Rightarrow A \cdot A^T = Id \Rightarrow \det A = \pm 1$.

So now assume $\omega(v_1, \dots, v_n) = \pm 1$, then $\omega(w_1, \dots, w_n) = \pm 1$.

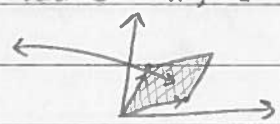
Assume that μ is an orientation for V, then there exists unique $\omega \in \Lambda^n(V)$ s.t. $\omega(v_1, \dots, v_n) = 1$ where $\underbrace{[v_1, \dots, v_n]}_{\text{the orthonormal basis of } V}$ is equal to μ.

Def

One calls ω the volume element of V determined by the inner product g and the orientation μ.

In particular determinant in $\mathbb{R}^n$ is the volume element of $\mathbb{R}^n$ defined by $\langle \cdot, \cdot \rangle$ and $[e_1, \dots, e_n]$.

If we now take $v_1, v_2 \in \mathbb{R}^2$, $|v_1| = |v_2| = 1$,

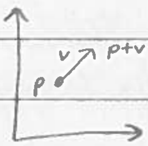
$|\det(v_1, v_2)| =$ 

§6 Fields and Forms

Def

Let $p \in \mathbb{R}^n$, define $\mathbb{R}^n_p = \{(p, v) : v \in \mathbb{R}^n\}$ to be the tangent space of $\mathbb{R}^n$ at p.

$\mathbb{R}^n_p$ is a vector space:



$(p, v) + (p, w) = (p, v+w)$

$\lambda(p, v) = (p, \lambda v)$

$p+v$ is the end point of (p, v).

We will denote $(p, v) = v_p$ (vector v at p).

The vector space $\mathbb{R}^n_p$ has similar structure to $\mathbb{R}^n$:
- the inner product $\langle v_p, w_p \rangle_p = \langle v, w \rangle$
- the usual orientation $\mathbb{R}^n_p$ is also induced from $\mathbb{R}^n$, it is $[(e_1)_p, \dots, (e_n)_p]$

Def
A vector field is a function $F : \mathbb{R}^n \to \bigcup_{p \in \mathbb{R}^n} \mathbb{R}^n_p$,
$p \mapsto (p, F(p))$
i.e. $F(p) = F^1(p)(e_1)_p + \dots + F^n(p)(e_n)_p$.
This gives n component functions $F^i : \mathbb{R}^n \to \mathbb{R}$.
$p \longmapsto (F^1(p), \dots, F^n(p))$

Remarks
1). The vector field F is called continuous / differentiable if all the component functions are continuous / differentiable.
2). All this terminology works if we replace $\mathbb{R}^n$ by an open subset of $\mathbb{R}^n$.

Def (of operations on vector fields)
Let F, G be vector fields on $\mathbb{R}^n$, $f : \mathbb{R}^n \to \mathbb{R}$.
- $(F + G)(p) = F(p) + G(p)$
- $\langle F, G \rangle(p) = \langle F(p), G(p) \rangle$
- $(f \cdot F)(p) = f(p) \cdot F(p)$

Examples
Grdient vector field, rotation and divergence.

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$\mathbb{R}^n_p$ ← tangent space

$[(e_1)_p, \dots, (e_n)_p]$ ← standard orientation

Def

A function $F : \mathbb{R}^n \longrightarrow \bigcup_{p \in \mathbb{R}^n} \mathbb{R}^n_p$

$$F(p) = F^1(p)(e_1) + \dots + F^n(p)(e_n)_p$$

The gradient vector field

Let's consider $f : \mathbb{R}^n \to \mathbb{R}$.

Def

A gradient vector field denoted by ∇f is a unique vector field whose scalar product with any unit vector v at each point of $\mathbb{R}^n$ is the directional derivative of f in the direction v.

$$\langle \nabla f(p), v_p \rangle_p = D_v f(p)$$

$$\nabla f(p) = \frac{\partial f(p)}{\partial x_1}(e_1)_p + \dots + \frac{\partial f(p)}{\partial x_n}(e_n)_p$$

Remark

$\nabla f(p)$ is a vector, it is not the same as the Jacobian matrix, $f'(p) = \left[\frac{\partial f}{\partial x_1}(p), \dots, \frac{\partial f(p)}{\partial x_n}\right]$

If we have a function $f : \mathbb{R}^n \longrightarrow \mathbb{R}^n$, then the Jacobian matrix is an $n \times n$ matrix,

$$f'(a) = \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \dots & \frac{\partial f_1}{\partial x_n} \\ \vdots & & \\ \frac{\partial f_n}{\partial x_1} & & \frac{\partial f_n}{\partial x_n} \end{pmatrix}$$

The Jacobian is the determinant of this matrix.

Def (Divergence)
div(F), where F is a vector field, is the divergence of F.
$div(F) = \sum_{i=1}^{n} D_i F^i$ "=" $\nabla \cdot F$ formally if we define

a formal symbol $\nabla = \sum_{i=1}^{n} D_i \cdot e_i$

Sometimes we denote $div(F) = \langle \nabla, F \rangle$

Def (Rotation / Curl in $\mathbb{R}^3$)

$$rot(F) = curl(F) = (\nabla \times F)(p) = \begin{vmatrix} (e_1)_p & (e_2)_p & (e_3)_p \\ D_1 & D_2 & D_3 \\ F^1 & F^2 & F^3 \end{vmatrix}$$

$$= (D_2 F^3 - D_3 F^2)(e_1)_p + (D_3 F^1 - D_1 F^3)(e_2)_p + (D_1 F^2 - D_2 F^1)(e_3)_p$$

The physical meaning of "divergence" and "rotation / curl" of a vector field will be given later after the Stokes Theorem.

Differential Forms

Def
The k-form (or differential form) is a function
$\omega : \mathbb{R}^n \longrightarrow \bigcup_{p \in \mathbb{R}^n} \Lambda^k(\mathbb{R}^n_p)$ s.t. $\omega(p) \in \Lambda^k(\mathbb{R}^n_p)$
i.e. $\omega(p) = \sum_{i_1 < \dots < i_k} \omega_{i_1 \dots i_k}(p) \left[\varphi_{i_1}(p) \wedge \dots \wedge \varphi_{i_k}(p) \right]$

where $\{\varphi_i(p)\}_{i=1}^{n}$ is the dual basis to $\{(e_i)_p\}_{i=1}^{n}$

and $\omega_{i_1 \dots i_k} : \mathbb{R}^n \to \mathbb{R}$

ω is continuous, differentiable, smooth
$\iff$ $\omega_{i_1 \dots i_k}$ are continuous, differentiable, smooth.

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Remarks

a). We define $\omega+\eta$, $f\cdot\eta$, $\omega\wedge\eta$ similarly as for the k-tensors on $\mathbb{R}^n$.

b). $f:\mathbb{R}^n\to\mathbb{R}$ then we say that f is a 0-form, $f\wedge\omega = f\cdot\omega$.

Def

Let $f:\mathbb{R}^n\to\mathbb{R}$ be differentiable, then $Df(p)\in\Lambda^1(\mathbb{R}^n)$; so we can define a 1-form df st.
$df(p)(v_p) = Df(p)(v)$.

Consider $\pi^i:\mathbb{R}^n\to\mathbb{R}$ st. $\pi^i(x^1,\dots,x^n)=x^i$.
We will use the notation $\pi^i := x^i$.

$$d\pi^i(p)(v_p) = Dx^i(p)(v_p)$$
$$= D\pi^i(p)(v) = v^i = \varphi_i(v)$$

So $\{dx^i(p)\}_{i=1}^n$ is the dual basis to $\{(e_i)_p\}_{i=1}^n$.

So we can write every k-form ω as

$$\omega = \sum_{i_1<\dots<i_k} \omega_{i_1\dots i_k}\, dx^{i_1}\wedge\dots\wedge dx^{i_k}.$$

This is called the canonical representation of the differential form.

Theorem 5.7

Let $f:\mathbb{R}^n\to\mathbb{R}$ smooth, then
$df = D_1 f\, dx^1 + \dots + D_n f\, dx^n$.

Proof

$$df(p)(v_p) = Df(p)(v) = \sum_{i=1}^n D_i f(p)\cdot v^i$$
where $v=(v^1,\dots,v^n)$)
$$\Rightarrow df(p)(v_p) = \sum_{i=1}^n dx^i(p)(v_p)\cdot D_i f(p). \quad \square$$

Example

$n = k+1$

k-forms can be represented as:

$$\omega = \sum_{i=1}^{k+1} \omega_i (-1)^{i+1} dx^1 \wedge \dots \wedge \widehat{dx^i} \wedge \dots \wedge dx^{k+1}$$

The notation $\widehat{dx^i}$ means that this element is missing.

Recall that for $f: V \to W$ linear we introduced

$f^*: \mathcal{J}^k(W) \to \mathcal{J}^k(V)$ by

$$f^* T(v_1, \dots, v_k) = T(f(v_1), \dots, f(v_k)), \quad v_i \in V, \ T \in \mathcal{J}^k(W).$$

Pull back of forms / Push forward of fields

Consider $f: \mathbb{R}^n \to \mathbb{R}^m$ smooth

$\Rightarrow Df(p): \mathbb{R}^n \to \mathbb{R}^m$ is linear.

Def

The push forward of field:

$f_*: \mathbb{R}^n_p \to \mathbb{R}^m_{f(p)}, \qquad f_*(v_p) = (Df(p)(v))_{f(p)}$

$(p, v_p) \longmapsto (f(p), Df(p)(v))$

Def

The pull back of form is:

$f^*: \Lambda^k(\mathbb{R}^m_{f(p)}) \to \Lambda^k(\mathbb{R}^n_p)$

$(f^*\omega)(p) = f^*(\omega(f(p)))$

$\forall \ v_1, \dots, v_k \in \mathbb{R}^n_p$

$(f^*\omega)(p)(v_1, \dots, v_k) = \omega(f(p))(f_*(v_1), \dots, f_*(v_k))$

ω is a k-form on $\mathbb{R}^m$

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Theorem 5.8

Let $f: \mathbb{R}^n \to \mathbb{R}^m$ be smooth, η an l-form and ω a k-form on $\mathbb{R}^m$, $g: \mathbb{R}^m \to \mathbb{R}^n$ smooth.

(i) $f^*(dx^i) = \sum_{j=1}^n D_j f^i dx^j = \sum_{j=1}^n \frac{\partial f^i}{\partial x^j} dx^j$

(ii) $f^*(\omega_1 + \omega_2) = f^*(\omega_1) + f^*(\omega_2)$

(iii) $f^*(g \cdot \omega) = (g \circ f) f^*(\omega)$

(iv) $f^*(\omega \wedge \eta) = f^*\omega \wedge f^*\eta$

Proof

(ii), (iii), (iv) as exercise

Hint for (iii): $f^*(g \cdot \omega) = f^*(g \wedge \omega) \overset{(iv)}{=} f^*(g) \wedge f^*(\omega)$.

(i) $f^*(dx^i)(p)(v_p) = dx^i(f(p))(f_* v_p)$

$= dx^i(f(p))\left(Df(p)(v)\right)_{f(p)}$

$= dx^i(f(p))\left(\sum_{j=1}^n v^j D_j f^1(p), \dots, \sum_{j=1}^n v^j D_j f^m(p)\right)_{f(p)}$

$= \sum_{j=1}^n v^j D_j f^i(p)$ by def of dx^i

$= \sum_{j=1}^n D_j f^i(p) dx^j(p)(v_p)$ by def of dx^i

$\square$

Examples

(a) $f^*(P dx^1 \wedge dx^2 + Q dx^2 \wedge dx^3)$

$= (P \circ f)[f^* dx^1 \wedge f^* dx^2] + (Q \circ f)[f^* dx^2 \wedge f^* dx^3]$

(b) $f: [0,1] \to \mathbb{R}^3$, $\omega = P dx^1 + Q dx^2 + R dx^3$

$f^*\omega$ is a 1-form on $(0,1)$

$f^*\omega(t)(v_t) = \omega(f(t))(f_*(v_t))$

$= \left[P(f(t)) dx^1_{(f(t))} + Q(f(t)) dx^2_{(f(t))} + R(f(t)) dx^3_{(f(t))}\right]$

$\cdot \left(Df^1(t)(v), Df^2(t)(v), Df^3(t)(v)\right)$

$$\Rightarrow f^*\omega(t)(v_t) = P(f(t))\frac{\partial f^1}{\partial t}\Big|_t \cdot v + Q(f(t))\frac{\partial f^2}{\partial t}\Big|_t \cdot v$$
$$+ R(f(t))\frac{\partial f^3}{\partial t}\Big|_t \cdot v$$

$v = dt(t)(v_t)$ where dt is from the dual basis to v

$$f^*\omega = \left[P\circ f(t)\frac{\partial f^1}{\partial t}(t) + Q\circ f(t)\frac{\partial f^2}{\partial t}(t) + R\circ f(t)\frac{\partial f^3}{\partial t}(t)\right]\cdot dt$$

29-11-17 - We introduced differential forms last time,

$\omega(p) \in \Lambda^k(\mathbb{R}^n_p)$

- push forward of fields $(f: \mathbb{R}^n \to \mathbb{R}^m)$

  $f_*: \mathbb{R}^n_p \to \mathbb{R}^m_{f(p)}$, $(p, v_p) \mapsto (f(p), (Df(p)v)_{f(p)})$
- pull back of differential forms

  $f^*: \Lambda^k(\mathbb{R}^m) \to \Lambda^k(\mathbb{R}^n)$, $f^*\omega(p)(v_1, \dots, v_k) = \omega(f(p))(f_*v_1, \dots, f_*v_k)$
- $df \Leftarrow$ 1-form coming from 0-form f.

Theorem 5.9

Let $f: \mathbb{R}^n \to \mathbb{R}^n$ be smooth, then

$f^*(h\,dx^1\wedge\dots\wedge dx^n) = (h\circ f)(\det f')\,dx^1\wedge\dots\wedge dx^n$

Proof

From Theorem 5.8 (3) we know that $f^*(g\cdot\omega) = (g\circ f)f^*\omega$,
therefore $f^*(h\,dx^1\wedge\dots\wedge dx^n) = (h\circ f)f^*(dx^1\wedge\dots\wedge dx^n)$.
So what we need to prove is that

$f^*(dx^1\wedge\dots\wedge dx^n) = \det f'\,dx^1\wedge\dots\wedge dx^n$

We denote $A = (a_{ij})$ the matrix $f'(p)$

Consider $f^*(dx^1\wedge\dots\wedge dx^n)(e_1, \dots, e_n) = dx^1\wedge\dots\wedge dx^n(f_*e_1, \dots, f_*e_n)$

$= dx^1\wedge\dots\wedge dx^n\left(\sum_{i=1}^n a_{i1}e_i, \dots, \sum_{i=1}^n a_{in}e_i\right)$

$= \det(a_{ij})\,dx^1\wedge\dots\wedge dx^n(e_1, \dots, e_n)$ (by Thm 5.6)

$\square$

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The operator "d" on k-forms

Recall:

For $f: \mathbb{R}^n \to \mathbb{R}$, $df = \sum_{j=1}^{n} D_j f \, dx^j$

So the symbol "d" changes 0-forms into 1-forms. For general k-form we can use the canonical form $\omega = \sum_{i_1 < \dots < i_k} \omega_{i_1 \dots i_k} dx^{i_1} \wedge \dots \wedge dx^{i_k}$.

Def

The k+1 form $d\omega$ is called a differential of ω and is given by

$$d\omega = \sum_{i_1 < \dots < i_k} d\omega_{i_1 \dots i_k} dx^{i_1} \wedge \dots \wedge dx^{i_k}$$

$$= \sum_{i_1 < \dots < i_k} \sum_{\alpha=1}^{n} d_\alpha \omega_{i_1 \dots i_k} dx^\alpha \wedge dx^{i_1} \wedge \dots \wedge dx^{i_k}$$

Examples

1). $d(P dx^1 + Q dx^2 + R dx^3)$

$$= \cancel{D_1 P dx^1 \wedge dx^1} + D_2 P dx^2 \wedge dx^1 + D_3 P dx^3 \wedge dx^1$$
$$+ D_1 Q dx^1 \wedge dx^2 + \cancel{D_2 Q dx^2 \wedge dx^2} + D_3 Q dx^3 \wedge dx^2$$
$$+ D_1 R dx^1 \wedge dx^3 + D_2 R dx^2 \wedge dx^3 + \cancel{D_3 R dx^3 \wedge dx^3}$$
$$= (D_1 Q - D_2 P) dx^1 \wedge dx^2 + (D_2 R - D_3 Q) dx^2 \wedge dx^3$$
$$+ (D_1 R - D_3 P) dx^1 \wedge dx^3$$

2). $k=1$, $n=1$, $f(t)$

$$d(f(t)dt) = \frac{\partial f(t)}{\partial t} dt \wedge dt = 0$$

3). for $k=n$

$$d(f(x) dx^1 \wedge \dots \wedge dx^n) = \sum_{\alpha=1}^{n} D_\alpha f \, dx^\alpha \wedge dx^1 \wedge \dots \wedge dx^\alpha \wedge \dots \wedge dx^n$$
$$= 0$$

4). $n=3,\ k=2,\quad F=(f_1, f_2, f_3)$ component functions of a vector field F.

$$d\big(f_1(x^1,x^2,x^3)\,dx^2\wedge dx^3 + f_2(x^1,x^2,x^3)\,dx^3\wedge dx^1 + f_3(x^1,x^2,x^3)\,dx^1\wedge dx^2\big)$$
$$= D_1 f_1(x^1,x^2,x^3)\,dx^1\wedge dx^2\wedge dx^3 + D_2 f_2(x^1,x^2,x^3)\,dx^2\wedge dx^3\wedge dx^1 + D_3 f_3(x^1,x^2,x^3)\,dx^3\wedge dx^1\wedge dx^2$$
$$= \big(D_1 f_1(x^1,x^2,x^3) + D_2 f_2(x^1,x^2,x^3) + D_3 f_3(x^1,x^2,x^3)\big)\,dx^1\wedge dx^2\wedge dx^3$$
$$= \operatorname{div}(F)\,dx^1\wedge dx^2\wedge dx^3$$

<u>Theorem 5.10</u> (Properties of the operator d) ω - k form, η - l form

(i) $d(\omega+\eta) = d\omega + d\eta$

(ii) $d(\omega\wedge\eta) = (d\omega)\wedge\eta + (-1)^k\,\omega\wedge d\eta$
for ω a k-form, η an l-form.

(iii) $d(d\omega) = 0$ (sometimes write $d^2\omega = 0$)

(iv) let ω be a k-form on $\mathbb{R}^m$ and $f:\mathbb{R}^n\to\mathbb{R}^m$ smooth, then $f^*(d\omega) = d(f^*\omega)$.

<u>Proof</u>

Everything is smooth.

(i) follows from definition.

(ii) $$d(\omega\wedge\eta) = d\Big(\Big(\sum_{i_1<\dots<i_k}\omega_{i_1\dots i_k}\,dx^{i_1}\wedge\dots\wedge dx^{i_k}\Big)\wedge\Big(\sum_{j_1<\dots<j_l}\eta_{j_1\dots j_l}\,dx^{j_1}\wedge\dots\wedge dx^{j_l}\Big)\Big)$$

$$= d\Big(\sum_i\sum_j \omega_{i_1\dots i_k}\eta_{j_1\dots j_l}\,dx^{i_1}\wedge\dots\wedge dx^{i_k}\wedge dx^{j_1}\wedge\dots\wedge dx^{j_l}\Big)$$

$$\overset{\text{def of "d"}}{=} \sum_i\sum_j\sum_{\alpha=1}^{n}\big(D_\alpha(\omega_{i_1\dots i_k})\eta_{j_1\dots j_l} + \omega_{i_1\dots i_k}D_\alpha(\eta_{j_1\dots j_l})\big)\cdot dx^\alpha\wedge dx^{i_1}\wedge\dots\wedge dx^{i_k}\wedge dx^{j_1}\wedge\dots\wedge dx^{j_l}$$

$$= (d\omega)\wedge\eta + (-1)^k\omega\wedge d\eta$$

(iii) $$\omega = \sum_i \omega_{i_1\dots i_k}\,dx^1\wedge\dots\wedge dx^k$$

$$d\omega = \sum_i\sum_{\alpha=1}^{n} D_\alpha\,\omega_{i_1\dots i_k}\,dx^1\wedge\dots\wedge dx^k$$

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$$d^2\omega = \sum_i \sum_{\alpha=1}^{n} \sum_{\beta=1}^{n} D_\beta(D_\alpha \omega_{i_1 \dots i_k}) dx^\beta \wedge dx^\alpha \wedge dx^{i_1} \wedge \dots \wedge dx^{i_n}$$

- $D_{\alpha,\beta}\, \omega_{i_1 \dots i_k}\, dx^\beta \wedge dx^\alpha \wedge dx^{i_1} \wedge \dots \wedge dx^{i_k}$
- $D_{\beta,\alpha}\, \omega_{i_1 \dots i_k}\, dx^\alpha \wedge dx^\beta \wedge dx^{i_1} \wedge \dots \wedge dx^{i_k}$

$D_{\alpha,\beta} f = D_{\beta,\alpha} f$

$\Rightarrow d^2\omega = 0.$

(see problem sheet 5, q 2)

(iv) If ω is a 0-form then we know that $(\omega = g)$

$$f^*(dg)(v_p) = dg(f_*(v_p))$$
$$= Dg(f(p))(Df(p)v)$$
$$= D(g \circ f)(v)$$
$$= d(g \circ f)(p)(v_p)$$

I can now use Thm 5.8 (3) from which we get that this is equal to $d(f^*g)(v_p)$

We proceed by induction.

Assume the theorem holds for any k-form, ω.

WTS: it holds for any $k+1$-form.

We consider $k+1$ form $\omega \wedge dx^i$.

$$\underbrace{f^*(d\omega) = d(f^*\omega)}_{\text{for } k\text{-form } \omega} \overset{?}{\Rightarrow} f^*(d(\omega \wedge dx^i)) = d(f^*(\omega \wedge dx^i))$$

$$f^*(d(\omega \wedge dx^i)) = f^*(d\omega \wedge dx^i + (-1)^k \omega \wedge \underbrace{d^2x^i}_{=0})$$

by previous properties

$$= f^*(d\omega \wedge dx^i) = f^*(d\omega) \wedge f^*(dx^i)$$
$$= f^*(d\omega) \wedge f^*(dx^i) + \underbrace{(-1)^k f^*(\omega)\, d f^*(dx^i)}_{=0}$$
$$= d(f^*\omega) \wedge f^*(dx^i) + (-1)^k f^*(\omega)\, d f^*(dx^i)$$
$$\overset{2}{=} d((f^*\omega) \wedge (f^*(dx^i)))$$
$$= d(f^*(\omega \wedge dx^i))$$

$\square$

$$\left[\begin{array}{l} (-1)^k f^*(\omega)\, d f^*(dx^i) \\ f^*(ddx^i) = d(f^* dx^i) \quad \text{by I.H.} \\ \overset{\shortparallel}{0} = f^*(d^2 x^i) \end{array}\right]$$

Def

We call a k-form ω:

(i) closed $\iff d\omega = 0$

(ii) exact $\iff \exists\ (k-1)$-form η s.t. $\omega = d\eta$

Remark

If the form is exact ($\omega = d\eta$), then by the previous theorem (part 3) we know that ω is closed ($d\omega = d^2\eta = 0$).

In general, this might not be true for the opposite direction.

Examples

1). $n=2$, $k=1$

$\omega = P\,dx^1 + Q\,dx^2$

$d\omega = (D_1 Q - D_2 P)\,dx^1 \wedge dx^2$

ω is closed $\iff D_1 Q = D_2 P \quad (*)$

Recall problem sheet 5 q. 4.

$$f(x^1, x^2) = \int_0^{x^1} P(t,0)\,dt + \int_0^{x^2} Q(x^1, s)\,ds$$

We showed that $\begin{cases} D_2 f = Q(x^1, x^2) \\ D_1 f = P(x^1, x^2) \end{cases}$

only if $(*)$ holds.

$\Rightarrow \omega = D_1 f\,dx^1 + D_2 f\,dx^2 = df$

2). (warning!)

Consider on $\mathbb{R}^2 \setminus \{0\}$ the form

$$\omega = \frac{-y}{x^2+y^2}\,dx + \frac{x}{x^2+y^2}\,dy.$$

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Consider $\theta : \mathbb{R}^2 \setminus \{(x,0) \mid x<0\} \to (-\pi, \pi)$
the "angle function".
We can show that $\omega = d\theta$ wherever θ is defined,
but θ cannot be defined continuously on the
whole set $\mathbb{R}^2 \setminus \{0\}$.
Assume that $\omega = df$ on $\mathbb{R}^2 \setminus \{0\}$, then
$D_1\theta = D_1 f$ and $D_2\theta = D_2 f$ on $\mathbb{R}^2 \setminus \{(x,0) \mid x<0\}$
$\Rightarrow \theta = f + \text{const.}$ and therefore f would not be
continuous.

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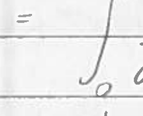
(3) $k=1$ in $\mathbb{R}^n$, $\omega = \sum_{i=1}^{n} \omega_i dx^i = df$

$\Rightarrow \omega = \sum_{i=1}^{n} D_i f \, dx^i$. Assume $f(0)=0$

$$f(x) = \int_0^1 \frac{d}{dt}(f(tx))dt$$
$$= \int_0^1 \sum_{i=1}^{n} D_i f(tx) x^i \, dt \qquad \text{(by chain rule)}$$
$$= \int_0^1 \sum_{i=1}^{n} \omega_i(tx) x^i dt$$

To find f in terms of ω we need to look at
$I\omega(x) = \int_0^1 \sum_{i=1}^{n} \omega_i(tx) x^i dt$ which is defined if every

point belonging to the domain the whole ray
connecting this point and 0 belongs to the domain.

 "star shaped domain"

Theorem 5.11 (Poincaré Lemma)
Let $A \subset \mathbb{R}^n$ be open and star-shaped w.r.t. 0, then every closed form on A is exact.

Proof
Define a function $I : \{l\text{-forms}\} \to \{(l-1)\text{-forms}\}$ s.t. $I(0) = 0$ and $\omega = I(d\omega) + dI(\omega)$ for any form ω.
Note that for $d\omega = 0$, $\omega = d(I\omega)$.
Let us consider the l-form ω

$$\omega = \sum_{i_1 < \dots < i_l} \omega_{i_1 \dots i_l} dx^{i_1} \wedge \dots \wedge dx^{i_l}.$$

We introduce the $I\omega$

$$I\omega(x) = \sum_{i_1 < \dots < i_l} \sum_{\alpha=1}^{l} (-1)^{\alpha-1} \left(\int_0^1 t^{l-1} \omega_{i_1 \dots i_l}(tx) dt \right) x^{i_\alpha} dx^{i_1} \wedge \dots \wedge \widehat{dx^{i_\alpha}} \wedge \dots \wedge dx^{i_l}$$

$$d(I\omega) = \sum_{i_1 < \dots < i_l} \sum_{\alpha=1}^{l} (-1)^{\alpha-1} \left(\int_0^1 t^{l-1} \omega_{i_1 \dots i_l}(tx) dt \right) dx^{i_\alpha} \wedge dx^{i_1} \wedge \dots \wedge \widehat{dx^{i_\alpha}} \wedge \dots \wedge dx^{i_l}$$

$$+ \sum_{i_1 < \dots < i_l} \sum_{\alpha=1}^{l} \sum_{j=1}^{n} (-1)^{\alpha-1} \left(\int_0^1 t^l D_j(\omega_i)(tx) dt \right) x^{i_\alpha} dx^j \wedge dx^{i_1} \wedge \dots \wedge \widehat{dx^{i_\alpha}} \wedge \dots \wedge dx^{i_l}$$

$$= l \sum_i \left(\int_0^1 t^{l-1} \omega_i(tx) dt \right) dx^{i_1} \wedge \dots \wedge dx^{i_l}$$

$$+ \sum_i \sum_\alpha \sum_j (-1)^{\alpha-1} \left(\int_0^1 t^l D_j(\omega_i)(tx) dt \right) x^{i_\alpha} dx^j \wedge dx^{i_1} \wedge \dots \wedge dx^{i_\alpha} \wedge \dots \wedge dx^{i_l}$$

$$d\omega = \sum_{i_1 < \dots < i_l} \sum_{j=1}^{n} D_j(\omega_i) dx^j \wedge dx^{i_1} \wedge \dots \wedge dx^{i_l}$$

Using the definition of I:

$$I(d\omega) = \sum_{i_1 < \dots < i_l} \sum_{j=1}^{n} \left(\int_0^1 t^l D_j(\omega_{i_1 \dots i_l})(tx) dt \right) x^j dx^{i_1} \wedge \dots \wedge dx^{i_l}$$

$$- \sum_{i_1 < \dots < i_l} \sum_{j=1}^{n} \sum_{\alpha=1}^{l} (-1)^{\alpha-1} \left(\int_0^1 t^l D_j \omega_i(tx) dt \right) x^{i_\alpha} dx^j \wedge dx^{i_1} \wedge \dots \wedge \widehat{dx^{i_\alpha}} \wedge \dots \wedge dx^{i_l}$$

$$d(I\omega) + I(d\omega) = \sum_{i_1 < \dots < i_l} \left(\int_0^1 l t^{l-1} \omega_{i_1 \dots i_l}(tx) dt \right) dx^{i_1} \wedge \dots \wedge dx^{i_l}$$

$$+ \sum_{i_1 < \dots < i_l} \sum_{j=1}^{n} \left(\int_0^1 t^l x^j D_j(\omega_{i_1 \dots i_l})(tx) dt \right) dx^{i_1} \dots dx^{i_l}$$

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$$\Rightarrow d(I\omega) + I(d\omega) = \sum_{i_1 < \dots < i_l} \left(\int_0^1 \frac{d}{dt}\left(t^l \omega_{i_1 \dots i_l}(tx) \right) dt \right) dx^{i_1} \wedge \dots \wedge dx^{i_l}$$

$$= \sum_{i_1 < \dots < i_l} \omega_{i_1 \dots i_l}\, dx^{i_1} \wedge \dots \wedge dx^{i_l} = \omega$$

$\square$

Geometric Preliminaries for Stoke's Theorem

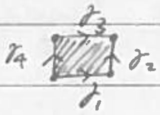
Def

- A standard n-cube is a function $I^n : [0,1]^n \longrightarrow \mathbb{R}^n$ s.t. $I^n(x) = x$.
- A singular n-cube in $A \subset \mathbb{R}^n$ is $c : [0,1]^n \longrightarrow A$ continuous

$$\begin{cases} \text{1-cube in } \mathbb{R}: \quad c : [0,1] \to \mathbb{R} \text{ (is a curve)} \\ \text{2-cube is a surface} \\ \text{0-cube in } \mathbb{R}^n: \quad c : [0,1]^0 = \{0\} \to A \text{ (is a point)} \end{cases}$$

Def

We call the formal sum of singular n-cubes in $A \subset \mathbb{R}^n$ an n-chain, i.e. $2c_1 + 3c_2 - 4c_3$ with coefficients in $\mathbb{Z}$.

$\partial I^2 = \gamma_1 + \gamma_2 - \gamma_3 - \gamma_4$

$\partial I^1 = 1 - 0$

Def

Let $1 \le i \le n$. Define $(n-1)$-cubes $I^n_{(i,0)}$ and $I^n_{(i,1)}$ as follows.

Let $x \in [0,1]^{n-1}$, $I^n_{(i,0)}(x) = I^n(x^1, \dots, x^{i-1}, 0, x^i, \dots, x^{n-1})$
$= (x^1, \dots, x^{i-1}, 0, x^i, \dots, x^{n-1})$

$I^n_{(i,1)}(x) = I^n(x^1, \dots, x^{i-1}, 1, x^i, \dots, x^{n-1})$
$= (x^1, \dots, x^{i-1}, 1, x^i, \dots, x^{n-1})$.

We call $I^n_{(i,0)}$ the $(i,0)$-face of I^n and $I^n_{(i,1)}$ the $(i,1)$-face of I^n.

The boundary of I^n is equal to

$$\partial I^n = \sum_{i=1}^{n} \sum_{\alpha=0,1} (-1)^{i+\alpha} I^n_{(i,\alpha)}.$$

Def

For a general singular n-cube $c: [0,1]^n \to A$ we define the (i,α)-face

$$c_{(i,\alpha)} = c \circ (I^n_{(i,\alpha)})$$

$$\partial c = \sum_{i=1}^{n} \sum_{\alpha=0,1} (-1)^{i+\alpha} c_{(i,\alpha)}$$

The boundary of the chain $\sum a_i c_i$ is defined as:

$$\partial(\sum a_i c_i) = \sum a_i \partial_i c_i.$$

Theorem 5.12

Let c be an n-chain in A, then $\partial(\partial c) = \partial^2 c = 0$

Proof as exercise.

Integration on Chains & the Fundamental Thm of Calculus

From now on:

n-chains are smooth

ω is a k-form on $[0,1]^k$, $\omega = f dx^1 \wedge \dots \wedge dx^k$

Def

(i) $$\int_{[0,1]^k} \omega = \int_{[0,1]^k} f(x^1, \dots, x^k)\, dx^1 dx^2 \dots dx^k$$

f is smooth

- Riemann integral
- Fubini's theorem can be applied

(ii) For ω a k-form in A and c singular k-cube in A

$$\int_c \omega = \int_{[0,1]^k} c^* \omega$$

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(iii) For $k=0$, a 0-form ω is a function
$c: \{0\} \to A$ is a singular 0-cube in A
$$\int_c \omega = \omega(c(0))$$

(iv) For a k-chain $c = \sum a_i c_i$ define
$$\int_c \omega = \sum a_i \int_{c_i} \omega.$$

Example
Consider ω a $(k-1)$-form on $[0,1]^k$
s.t. $\omega = f(x^1, \dots, x^k) dx^1 \wedge \dots \wedge \widehat{dx^i} \wedge \dots \wedge dx^k$
omitted

(k-1)-form

$$\int_{I^k_{(j,\alpha)}} \omega = \int_{[0,1]^{k-1}} \left(I^k_{(j,\alpha)}\right)^* \left(f(x^1, \dots, x^k) dx^1 \wedge \dots \wedge \widehat{dx^i} \wedge \dots \wedge dx^k\right)$$

(k-1)-dimensional

Recall: • $f^*(\eta \wedge \omega) = f^*(\eta) \wedge f^*(\omega)$
• $f^*(g\, dx^1 \wedge \dots \wedge dx^k) = (g \circ f)(f^*(dx^1 \wedge \dots \wedge dx^k))$

concerned only with jth place, rename other elements

$$\Rightarrow \int_{I^k_{(j,\alpha)}} \omega = \begin{cases} \displaystyle\int_{[0,1]^{k-1}} \overbrace{f(x^1, \dots, x^{j-1}, \alpha, x^{j+1}, \dots, x^k)}^{f(I^k_{(j,\alpha)})} & \text{for } i = j \\ 0 & \text{for } i \neq j \end{cases}$$

$$f^*(dx^i) = \sum_j D_j f^i dx^i, \quad \left(I^k_{(j,\alpha)}\right)^*(dx^i) = \begin{cases} dx^i & i \neq j \\ 0 & i = j \end{cases}$$

(very important Thm)
Theorem 5.13 (Stoke's Thm)
Let ω be a $(k-1)$-form on $A \subset \mathbb{R}^n$ open and c be a k-chain in A, then
$$\int_c d\omega = \int_{\partial c} \omega.$$

Proof

Assume first that $c = I^k$ and ω is a $(k-1)$-form on $[0,1]^k$. Then ω is a sum of $(k-1)$-forms of the type $f\,dx^1 \wedge \dots \wedge \widehat{dx^i} \wedge \dots \wedge dx^k$.

$$\int_{\partial I^k} f\,dx^1 \wedge \dots \wedge \widehat{dx^i} \wedge \dots \wedge dx^k \qquad \left[\int f\,dx^1 \wedge \dots \wedge dx^k = \int f\,dx^1 \dots dx^k\right]$$

$$= \sum_{j=1}^{k} \sum_{\alpha=0,1} (-1)^{j+\alpha} \int_{I^k_{(j,\alpha)}} f\,dx^1 \wedge \dots \wedge \widehat{dx^i} \wedge \dots \wedge dx^k$$

$$= \sum_{\alpha=0,1} (-1)^{i+\alpha} \int_{[0,1]^{k-1}} f(x^1, \dots, \alpha, \dots, x^k)\,dx^1 \wedge \dots \wedge \widehat{dx^i} \wedge \dots \wedge dx^k$$

$\int_0^1 \cdot\, dx^i = 1$?

$$= \sum_{\alpha=0,1} (-1)^{i+\alpha} \int_{[0,1]^{k-1}} f(x^1, \dots, \alpha, \dots, x^k)\,dx^1 \dots \widehat{dx^i} \dots dx^k$$

$$= \sum_{\alpha=0,1} (-1)^{i+\alpha} \int_{[0,1]^k} f(x^1, \dots, \alpha, \dots, x^k)\,dx^1 \dots dx^k$$

$$= (-1)^{i+1} \int_{[0,1]^k} f(x^1, \dots, 1, \dots, x^k)\,dx^1 \dots dx^k$$

$$+ (-1)^{i} \int_{[0,1]^k} f(x^1, \dots, 0, \dots, x^k)\,dx^1 \dots dx^k$$

$$d(f\,dx^1 \wedge \dots \wedge \widehat{dx^i} \wedge \dots \wedge dx^k)$$

$$= D_i f\,dx^i \wedge dx^1 \wedge \dots \wedge \widehat{dx^i} \wedge \dots \wedge dx^k$$

$$= (-1)^{i-1} D_i f\,dx^1 \wedge \dots \wedge dx^i \wedge \dots \wedge dx^k$$

$$\Rightarrow \int_{I^k} d\omega = (-1)^{i-1} \int_{I^k} D_i f\,dx^1 \wedge \dots \wedge dx^k$$

$$= (-1)^{i-1} \int_0^1 \dots \left(\int_0^1 D_i f\,dx^i\right) dx^1 \dots \widehat{dx^i} \dots dx^k$$

by FTC

$$= (-1)^{i-1} \int_0^1 \dots \left(f(x^1, \dots, 1, \dots, x^k) - f(x^1, \dots, 0, \dots, x^k)\right) dx^1 \dots \widehat{dx^i} \dots dx^k$$

$$= (-1)^{i-1} \int_{[0,1]^k} f(x^1, \dots, 1, \dots, x^k)\,dx^1 \dots dx^k + (-1)^i \int_{[0,1]^k} f(x^1, \dots, 0, \dots, x^k)\,dx^1 \dots dx^k$$

$\Rightarrow$ LHS = RHS for $c = I^k$.

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$c: [0,1]^k \to A$ $\qquad\qquad$ $\partial^2 c = 0$ (Thm 5.12)

$\omega = f\, dx^1 \wedge \dots \wedge dx^n$

$\int \omega := \int f\, dx^1 \dots dx^n$

Theorem (Stoke's)

Let ω be a $(k-1)$-form on $A \subset \mathbb{R}^n$ open, and let c be a k-chain in A, then

$$\int_c d\omega = \int_{\partial c} \omega$$

Proof

$\omega = f\, dx^1 \wedge \dots \wedge \widehat{dx^i} \wedge \dots \wedge dx^k$

We proved so far that the statement of the theorem for such an ω and c being a standard cube:

$$\int_{I^k} d\omega = \int_{\partial I^k} \omega$$

- Fubini Thm
- Fundamental Thm of Calculus
- $\omega = f\, dx^1 \wedge \dots \wedge \widehat{dx^i} \wedge \dots \wedge dx^k$

$$\int_{I^k_{(j,\alpha)}} \omega = \begin{cases} \int_{[0,1]^k} f(x_1, \dots, \underset{i\text{th place}}{\alpha}, \dots, x_k)\, dx^1 \dots dx^k & i = j \\ 0 & i \neq j \end{cases}$$

∂c, integral of the form, pullback, $\int_c \omega = \int_{[0,1]^k} c^* \omega$

$$\Rightarrow \int_{\partial c} \omega = \int_{\partial I^k} c^* \omega$$

$c: [0,1]^k \to A$, $c^*\omega \in \Lambda^k([0,1]^k)$

$I^k: [0,1]^k \to A$, $I^k(x) = x$

(exercise to show this).

c - k-cube

$$\int_c d\omega = \int_{I^k} c^*(d\omega) = \int_{I^k} d(c^*\omega) = \int_{\partial I^k} c^*\omega = \int_{\partial c} \omega$$

c k-chain : $c = \sum a_i c_i$

$$\int_c d\omega = \sum a_i \int_{c_i} d\omega = \sum a_i \int_{\partial c_i} \omega = \int_c \omega \qquad \square$$

<u>Remark</u>

Let $c : I^n \longrightarrow A \subset \mathbb{R}^n$ open, ω n-form on A. $\omega = f dx^1 \wedge \ldots \wedge dx^n$

c smooth and injective, $\det(c'(x)) > 0 \ \forall x \in [0,1]^n$.

Then $$\int_c \omega = \int_{I^n} c^* \omega = \int_{I^n} (f \circ c) c^*(dx^1 \wedge \ldots \wedge dx^n)$$

$$= \int_{I^n} (f \circ c) \det(c') dx^1 \wedge \ldots \wedge dx^n$$

$$= \int_{I^n} (f \circ c) |\det(c')| dx^1 \ldots dx^n$$

change of variables formula

$$= \int_{c(I^n)} f \, dx^1 \ldots dx^n$$

<u>Integration on manifolds</u>

<u>Recall</u>

diffeomorphism: h is a diffeo $\Leftrightarrow$ $h : U \to V$, $U, V \subset \mathbb{R}^n$ open, h is smooth, 1-1, h^{-1} exists and is smooth.

<u>Definition of manifold:</u>

M is a k-dim submanifold in $\mathbb{R}^n$ if

(M) $\exists U \subset \mathbb{R}^n$ open, $V \subset \mathbb{R}^n$ open and diffeo $h : U \to V$

st. $h(U \cap M) = V \cap (\mathbb{R}^k \times \{0\}) = \{y \in V : y^{k+1} = \ldots = y^n = 0\}$

Thm 4.2 :

(c) $\exists U \subset \mathbb{R}^n$ open, $x \in M$, $W \subset \mathbb{R}^k$ open,

$f : W \to \mathbb{R}^n$, injective, smooth, such that

(i) $f(W) = M \cap U$

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(ii) $f'(y)$ has full rank (k) $\forall y \in W$

(iii) $f^{-1} : f(W) \to W$ is continuous

Def

We define the half space $H^k \subset \mathbb{R}^k$ by $\{x \in \mathbb{R}^k \mid x^k \geq 0\}$

Def

$M \subset \mathbb{R}^n$ is a k-dim submanifold-with-boundary if $\forall x \in M$ either (M) is satisfied or

(M') There are open sets U, $x \in U$, $V \subset \mathbb{R}^n$ and a diffeomorphism h s.t. $h(M \cap U) = V \cap (H^k \times \{0\})$

i.e. $h(M \cap U) = \{y \in V \mid y^k \geq 0,\ y^{k+1} = \dots = y^n = 0\}$

and $h(x)$ has kth component equal to zero.

Remark

For any point $x \in M$, x cannot satisfy both conditions (M) & (M') at the same time.

Def

We call the set of all points $x \in M$ for which (M') is satisfied, the boundary of M.

We denote it by ∂M.

The tangent space on a submanifold

$M \subset \mathbb{R}^n$ a k-dim submanifold, $\exists f : W \to \mathbb{R}^n$, W open in $\mathbb{R}^k$ around $x \in M$, $f(a) = x$ $a \in W$.

Note that $f'(a)$ has rank k

$\Rightarrow$ linear transformation $f_* : \mathbb{R}^k_a \to \mathbb{R}^n_x$,

$(a, v) \mapsto (x, Df(a)(v))$.

so f is injective $\Rightarrow f_*(\mathbb{R}^k_a)$ is a k-dim subspace of $\mathbb{R}^n_x$.

Def
We call $f_*(\mathbb{R}^k_a)$ the tangent space of M at $x = f(a)$.
We denote this space by M_x or $T_x M$.

Remark
a). This definition does not depend on the system of coordinates.

$g : V \to \mathbb{R}^n$, V is open in $\mathbb{R}^k$

$g(b) = f(a) = x$

$$g = f \circ f^{-1} \circ g$$

$$g_* = f_* (f^{-1} \circ g)_*$$

$$g_*(\mathbb{R}^k_b) = f_*(f^{-1} \circ g)_*(\mathbb{R}^k_b)$$
$$= f_*(\mathbb{R}^k_a)$$

$$f'(a) = \big(D_1 f(a), \ \dots, \ D_k f(a) \big)$$

$f_* \leftarrow$ multiplication by $Df(a)$.

b). $f_*(\mathbb{R}_a{}^k) = \{ (x, \sum_{i=1}^{k} \alpha_i D_i f(a)) \}$

$\Rightarrow \{ D_1 f, \dots, D_k f \}$ is the basis of $T_x M$.

Vector fields and forms on M
Assume that $M \subset A \subset \mathbb{R}^n$, A open.
Assume that $\exists$ a smooth vector field on A s.t.
$F(x) \in M_x (= T_x M)$ $\forall x \in M$.
Let $f : W \to \mathbb{R}^n$ be a coordinate system on n
$\Rightarrow \exists$ a unique vector field G on W s.t. for $a \in W$
$f_*(G(a)) = F(f(a))$

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Def

Let F be a function that assigns a vector $F(x) \in M_x$ for $x \in M$

i.e. $F: M \longrightarrow \bigcup_x M_x \quad F(x) \in M_x$.

F is called a vector field on M.

Def

Let $f: W \to \mathbb{R}^n$ be a coordinate system, then $\exists!\, G$, vector field on W, s.t.

$f_*(G(a)) = F(f(a)) \; \forall a \in W$.

$G(a) = (f^{-1})_* (F(f(a)))$.

We say that F is a smooth vector field on M iff G is a smooth vector field on W.

Def

A function $\omega: M \longrightarrow \bigcup_{x \in M} \Lambda^p(M_x)$ s.t. $\omega(x) \in \Lambda^p(M_x)$ is called a p-form on M.

Again if $\overset{W \to \mathbb{R}^n}{f}$ is a system of coordinates then $f^* \omega$ is a p-form on W.

pull back.

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$f: W \to M \quad W \subset \mathbb{R}^k \quad M \subset \mathbb{R}^n$

$\mathbb{R}^k_a \xrightarrow{f_*} M_x$ ← tangent space $\Lambda(\mathbb{R}^k_a) \xleftarrow{f^*} \Lambda^p(M_x)$

$v \in M_x$, $w \in \mathbb{R}_a^k$, $a \in W$, $f(a) = x$

$f_*(w) = v$

$$\int_c d\omega = \int_{\partial c} \omega \quad \leftarrow \text{Stoke's Thm}$$

Differential form on manifold is a function
$\omega : M \to \bigcup_x \Lambda^p(M_x)$ s.t. $\omega(x) \in \Lambda^p(M_x)$
Again if $f: W \to \mathbb{R}^n$, $f^*\omega$ is a p-form on W.
We express the forms on M as

$$\omega = \sum_{i_1 < \dots < i_p} \omega_i \, dx^{i_1} \wedge \dots \wedge dx^{i_p}$$

with the functions $\omega_{i_1 \dots i_p}$ defined on M only.

Note: the definition of $d\omega$ does not make sense since $D_j(\omega_{i_1 \dots i_p})$ does not make sense on M.

Theorem 4.3
There is a unique $(p+1)$-form $d\omega$ on M, such that for every coordinate system $f: W \to \mathbb{R}^n$ we have
$f^*(d\omega) = d(f^*\omega)$ $(*)$

Proof
Let $f: W \to \mathbb{R}^n$ with $x = f(a)$, $a \in W$ and let
$v_1, \dots, v_{p+1} \in M_x$
$\Rightarrow \exists!\, w_1, \dots, w_{p+1} \in \mathbb{R}^k_a$ s.t. $f_*(w_i) = v_i$
$d\omega(x)(v_1, \dots, v_{p+1}) := d(f^*\omega)(a)(w_1, \dots, w_{p+1})$
$\square$

Exercises
1). Check that this definition of $d\omega(x)$ does not depend on the choice of coordinate system ("f").
2). This form has to be defined like this so that $(*)$ holds!
1 $\Rightarrow$ well defined   2 $\Rightarrow$ unique.

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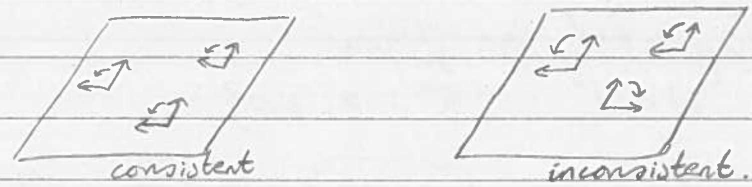
Orientation of the submanifold

Def

Let μ_x be a choice of orientation of M_x $\forall x \in M$. We say that this choice is consistent if for every coordinate system $f: W \to \mathbb{R}^n$, $\forall a, b \in W$,

$[f_*((e_1)_a), \ldots, f_*((e_k)_a)] = \mu_{f(a)}$ iff

$[f_*((e_1)_b), \ldots, f_*((e_k)_b)] = \mu_{f(b)}$.



Def

Assume that μ_x is chosen consistently.

If $f: W \to \mathbb{R}^n$ is s.t. $[f_*((e_1)_a), \ldots, f_*((e_k)_a)] = \mu_{f(a)}$ $\forall a \in W$ then f is called orientation-preserving.

Remarks

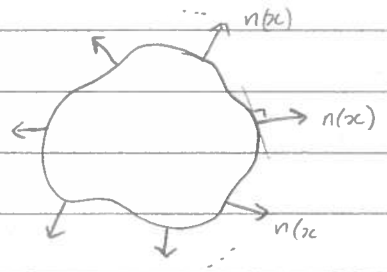
1). If f is not orientation-preserving and $T: \mathbb{R}^k \to \mathbb{R}^k$ linear transformation s.t. $\det T = -1$
$\Rightarrow f \circ T$ is orientation-preserving.

2). If we take f, g orientation-preserving, $f(a) = g(b) = x$, then
$[f_*(e_1)_a, \ldots, f_*(e_k)_a] = \mu_x = [g_*(e_1)_b, \ldots, g_*(e_k)_b]$
$[(g^{-1} \circ f)_*(e_1)_a, \ldots, (g^{-1} \circ f)_*(e_k)_a] = [(e_1)_b, \ldots, (e_k)_b]$
$\Rightarrow \det (g^{-1} \circ f)' > 0$!!!

Def

A submanifold for which orientations μ_x can be chosen consistently is called orientable. A choice of μ_x is called an orientation of M, and is denoted by μ. The pair M, μ is called an oriented submanifold.

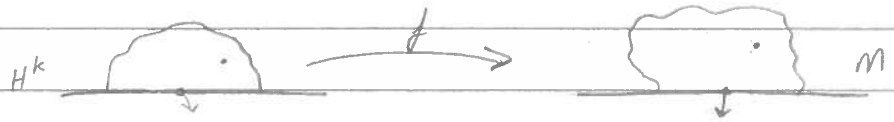
Recall that if M is a k-dim submanifold-with-boundary, $x \in \partial M \Rightarrow (\partial M)_x$ is a $(k-1)$-dim subspace of M_x.



$n(x) \in M_x$, $n(x) \notin (\partial M)_x$

$|n(x)| = 1$

Since $(\partial M)_x$ is a $(k-1)$-dim subspace of M_x
there exist exactly two unit vectors perpendicular to $(\partial M)_x$.
They can be distinguished as follows:
$f: W \to \mathbb{R}^n$ s.t. $W \subset H^k = \{\mathbb{R}^k : x_k \geq 0\}$



$f(0) = x$, then only one of these unit vectors is
$f_*(v_0)$, $v_0 \in \mathbb{R}^k$ s.t. $v^k < 0$

Let $n(x) \in M_x$ be the outward unit normal:

Def
If μ is the orientation of M, then to get orientation of ∂M we take $(v_1, \dots, v_{k-1}) \in (\partial M)_x$ so that
$[n(x), v_1, \dots, v_{k-1}] = \mu_x$.
We call $\partial\mu$ the induced orientation

Example
$M = H^k \subset \mathbb{R}^k$
Let us take $\mathbb{R}^k$ with the standard orientation $[e_1, \dots, e_k]$, then the induced orientation is $[n, e_1, \dots, e_{k-1}] = [-e_k, e_1, \dots, e_{k-1}]$
$= (-1)^k [e_1, \dots, e_k] \Rightarrow \partial\mu = (-1)^k [\text{standard orientation of } \mathbb{R}^k]$.

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Definition

Let M be $(n-1)$-dim submanifold in $\mathbb{R}^n$.

Let M be oriented.

We call $\nu(x) \in \mathbb{R}^n_x$, $x \in M$ s.t. $|\nu(x)| = 1$

the unit normal if $\forall x \in M$, $\nu(x) \perp M_x$

The outward unit normal is defined so that if

$[v_1, \dots, v_{n-1}] = \mu_x$ orientation of M_x, then

$[\nu(x), v_1, \dots, v_{n-1}]$ is the usual orientation $\mathbb{R}^n_x$.

Exercise

The choice of the outward unit normal in the continuous way determines (consistent) orientation of M.

Stokes Theorem on Submanifolds

Let $M \subset \mathbb{R}^n$ be k-dimensional submanifold-with-boundary, ω is a p-form on M, $c: [0,1]^p \to M \subset \mathbb{R}^n$ a singular p-cube in M.

Recall: $\int_c \omega = \int_{[0,1]^p} c^* \omega$.

Assumption: If $k = p$, $c: [0,1]^k \to M$ is a k-cube, we will assume that there is a coordinate system $f: W \to \mathbb{R}^n$ s.t. $[0,1]^k \subset W$ s.t. $f(x) = c(x)$ $\forall x \in [0,1]^k$

M is oriented $\Rightarrow$ c is called orientation-preserving if f is orientation preserving.

Theorem 4.4

If $c_1, c_2: [0,1]^k \to M$ are two orientation-preserving singular k-cubes in the oriented k-dim submanifold M of $\mathbb{R}^n$ and ω is a k-form on M s.t. $\omega = 0$ outside $c_1([0,1]^k) \cap c_2([0,1]^k)$, then $\int_{c_1} \omega = \int_{c_2} \omega$.

Proof

$$\int_{c_1} \omega = \int_{[0,1]^k} c_1^*(\omega) = \int_{[0,1]^k} (c_2^{-1} \circ c_1)^* c_2^*(\omega)$$

only defined on a subset of $[0,1]^k$

Claim: $\displaystyle\int_{[0,1]^k} (c_2^{-1} \circ c_1)^* c_2^*(\omega) = \int_{[0,1]^k} c_2^*(\omega) = \int_{c_2} \omega$.

Denote $c_2^*(\omega) = f\, dx^1 \wedge \dots \wedge dx^k$ and $g = c_2^{-1} \circ c_1 : [0,1]^k \to [0,1]^k$

$$\int_{[0,1]^k} (c_2^{-1} \circ c_1)^* c_2^*(\omega) = \int_{[0,1]^k} g^*(f\, dx^1 \wedge \dots \wedge dx^k)$$

Thm 5.9

$$= \int_{[0,1]^k} (f \circ g) \det g' \, dx^1 \wedge \dots \wedge dx^k$$

$$= \int_{[0,1]^k} (f \circ g) |\det g'| \, dx^1 \dots dx^k$$

$\det g' > 0$ since c_i orientation-preserving

Then we use change of variables to conclude.

□

Def

Let ω be a k-form on oriented k-dim submanifold M, c orientation-preserving singular k-cube on M, s.t. $\omega = 0$ outside $c([0,1]^k)$. The integral of the form ω over M is

$$\int_M \omega = \int_c \omega \quad \left(= \int_{[0,1]^k} c^* \omega \right).$$

Remark

The definition of integral of any k-form requires partition of unity.

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Handout

$c : [0,1]^k \to \mathbb{R}^n$

$c : [0,1]^k \to M^k \subset \mathbb{R}^n$ k-dim submanifold in $\mathbb{R}^n$

$f : W \subset \mathbb{R}^k \to \mathbb{R}^n$

$[0,1]^k \subset W$

$f(x) = c(x) \quad \forall x \in [0,1]^k$

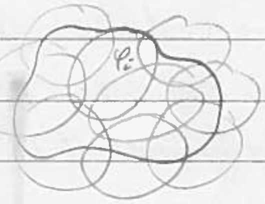
ω k-form on M $\quad (*)$

$\omega = 0$ outside $c[0,1]^k$

$$\int_M \omega = \int_c \omega$$

To define the integral of the general form on M

M - open, bounded subset of $\mathbb{R}^n$



$$\sum_i \varphi_i(x) = 1$$
$$0 \le \varphi_i(x) \le 1$$

ω general k-form $\quad \varphi_i \omega$ would be a form of type $(*)$

<u>Def</u> (3 on handout)

$$\int_M \omega = \sum_{\varphi \in \Phi} \int_M \varphi \cdot \omega$$

where the sum of integrals on the r.h.s. is finite.

<u>Theorem</u> (Stoke's Thm on submanifolds) (2 on handout)

If M is a compact oriented k-dim submanifold-with-boundary and ω is a $(k-1)$-form on M, then

$$\int_M d\omega = \int_{\partial M} \omega$$ where ∂M is taken with the induced orientation.

<u>Case 1</u>

There exists orientation-preserving singular k-cube c in $M \setminus \partial M$ st. $\omega = 0$ outside $c([0,1]^k)$

By Stoke's Thm on chains,

$$\int_c d\omega = \int_{[0,1]^k} c^*(d\omega) = \int_{[0,1]^k} d(c^*\omega) = \int_{\partial I^k} c^*\omega = \int_{\partial c} \omega$$

We know that

$$\int_M d\omega = \int_c d\omega = \int_{\partial c} \omega = 0 = \int_{\partial M} \omega$$

Case 2

$\exists$ orientation-preserving singular k-cube c in M s.t. $c_{(k,0)}$ is the only face in ∂M and $\omega = 0$ outside $c([0,1]^k)$

$$\int_{c_{(k,0)}} \omega = (-1)^k \int_{\partial M} \omega$$ follows from the example about induced orientation of ∂H^k.

On the other hand note that in def. ∂c, every face (k,α) was taken with coefficient $(-1)^{k+\alpha}$

Therefore

$$\int_{\partial c} \omega = (-1)^k \int_{c_{(k,0)}} \omega = (-1)^k(-1)^k \int_{\partial M} \omega = \int_{\partial M} \omega$$

(This is the point where orientations of ∂c and induced orientation of ∂M come together).

$$\Rightarrow \int_M d\omega = \int_c d\omega = \int_{\partial c} \omega = \int_{\partial M} \omega$$

Case 3

ω is now a general form.

$\mathcal{O}$ is the finite cover of M with some open sets.

ϕ – partition of unity for M w.r.t. $\mathcal{O}$

$\forall \varphi \in \phi$ $\varphi \cdot \omega$ is one of the forms analysed in case1 or case 2.

$$0 = d(1) = d\Big(\sum_{\varphi \in \phi} \varphi(x)\Big) = \sum_{\varphi \in \phi} d\varphi(x)$$

then we can write that

$$\sum_{\varphi \in \phi} (d\varphi(x) \wedge \omega) = 0$$

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$$\Rightarrow \sum_{\varphi \in \phi} \int_M (d\varphi(x) \wedge \omega) = 0$$

ω is $(k-1)$-form $\Rightarrow d\omega$ is k-form

$$\int_M d\omega = \sum_{\varphi\in\phi} \int_M \varphi \cdot d\omega = \sum_{\varphi\in\phi} \int_M (d\varphi \wedge \omega + \underbrace{\varphi \cdot d\omega}_{= \varphi \wedge d\omega})$$

since φ 0-form

$$= \sum_{\varphi\in\phi} \int_M d(\varphi \cdot \omega)$$

$$= \sum_{\varphi\in\phi} \int_{\partial M} \varphi \cdot \omega \overset{\text{def.}}{=} \int_{\partial M} \omega \quad \square$$

The volume element of submanifold

Recall that for V with an inner product g and orientation μ we called the unique form $\omega \in \Lambda^n(V)$ a volume element if

$\omega(v_1, \dots, v_n) = 1$ $v_1, \dots, v_n$ are orthonormal basis of V

$[v_1, \dots, v_n] = \mu$.

For k-dim submanifolds in $\mathbb{R}^n$ there is natural inner product T_x on M_x induced by that on $\mathbb{R}^n_x$

$T_x(v, w) = \langle v_x, w_x \rangle = \langle v, w \rangle$, $v, w \in M_x$

Def

Let M be a k-dim submanifold with orientation μ.

The volume element on M is the unique $\omega \in \Lambda^k(M_x)$ s.t. $\omega(v_1, \dots, v_k) = 1$ if $v_1, \dots, v_k$ is an orthonormal basis of M_x s.t. $[v_1, \dots, v_k] = \mu_x$.

It is denoted dV or $dVol$.

On a 2-dim submanifold, vol. element is a surface area and we denote it dA.

On a 1-dim submanifold, vol. element is the length and we denote it dS.

Theorem 4.7 (Gauss / Divergence Theorem)
Let $M \subset \mathbb{R}^3$ a compact 3-dim oriented manifold with boundary, and n is the unit outward normal to ∂M. Let F be a smooth vector field on M, then $\int_M \operatorname{div} F \, dV = \int_{\partial M} \langle F, n \rangle \, dA$.

Theorem 4.8 (Classical Stokes's Thm)
Let $M \subset \mathbb{R}^3$ be a compact oriented 2-dim manifold with boundary and n be unit outward normal on M.
Let ∂M have the induced orientation.
Let T be the vector field on ∂M with $dS(T) = 1$, and let F be a smooth vector field in an open set containing M.

$$\int_M \langle \nabla \times F, n \rangle \, dA = \int_{\partial M} \langle F, T \rangle \, dS$$

Before proof of 4.7.
2-dim surfaces in $\mathbb{R}^3$.
Let M be a 2-dim oriented submanifold in $\mathbb{R}^3$, and $n(x)$ be the unit outward normal to M at $x \in M$.
Define

$$\omega \in \Lambda^2(M_x)$$

$$\omega(v,w) = \det \begin{pmatrix} v \\ w \\ n(x) \end{pmatrix}$$

Note that $\omega(v,w) = 1$ if $v, w \in M_x$ and if $[v,w] = \mu_x$ on basis of M_x.
This means that $dA(v,w) = \omega(v,w)$

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$dA = \omega = \langle v\times w, n\rangle$ (from def of cross product)

but $v\times w \parallel n$ $\quad (v \perp n,\ w \perp n)$

$dA(v,w) = |v\times w|$

<u>Theorem</u>

Let M be as above. Then

(i) $dA = n^1 dy\wedge dz + n^2 dz\wedge dx + n^3 dx\wedge dy$

(ii) $n^1 dA = dy\wedge dz$

(iii) $n^2 dA = dz\wedge dx$

(iv) $n^3 dA = dx\wedge dy$

<u>Proof</u>

(i) follows from the fact that

$dA(v,w) = \det\begin{pmatrix} v \\ w \\ n\end{pmatrix}$

$dx\wedge dy = (dx\otimes dy - dy\otimes dx)$

To show (ii) to (iv) let us take $z \in \mathbb{R}_x^3$

$v\times w = \alpha n \quad , \alpha \in \mathbb{R}$

$\langle z, n\rangle \cdot \langle v\times w, n\rangle = \langle z, n\rangle \alpha$

$= \langle z, \alpha n\rangle = \langle z, v\times w\rangle$

taking $z = e_1, e_2, e_3$ gives the result. $\square$

<u>Proof of 4.7</u>

$F = (F^1, F^2, F^3)$

consider 2-form $\omega = F^1 dy\wedge dz + F^2 dz\wedge dx + F^3 dx\wedge dy$

$d\omega = \operatorname{div}(F)\, dx\wedge dy\wedge dz$

$= \operatorname{div}(F)\, dV$

$\langle F, n\rangle dA = F^1 n^1 dA + F^2 n^2 dA + F^3 n^3 dA$

$= F^1 dy\wedge dz + F^2 dz\wedge dx + F^3 dx\wedge dy = \omega$

$\Rightarrow \int_M d\omega = \int_{\partial M} \omega \quad \square$