3109 Multivariable Analysis Notes

Based on the 2011 autumn lectures by Dr I Petridis

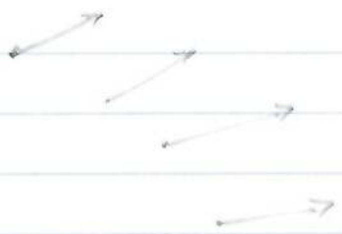
04/09/2011

Introduction

$F: \mathbb{R}^n \to \mathbb{R}^m$

if $m = 1$, F is called a **scalar field**
if $m > 1$, F is called a **vector field**

Example A fluid flow



force field

Differential forms: ω

$$\int_{\partial M} \omega = \int_M d\omega$$

→ essential in Stokes theorem for differential form

what is ω? what is $d\omega$? What is a manifold M?
what is ∂M

Motto: Differential forms are meant to be integrated

Newton $f'(x)$
Leibnitz $\frac{df}{dx} = f'(x)$ ← not a quotient

1899
Elie Cartan
Henri Poincaré

"Les méthodes nouvelles de la mécanique céleste"

$F : \mathbb{R} \to \mathbb{R}$ differentiable

$$\int_a^b F'(x)\, dx = F(b) - F(a)$$

1-dim $\mathbb{R}^1$

A diff. form looks like $g(x)\, dx$

$[a, b]$

$\int_a^b g(x)\, dx$ is a real number

2-dim $\mathbb{R}^2$

Let $\bar{F}$ be a constant vector field

work is $= \bar{F} \cdot \overline{AB}$

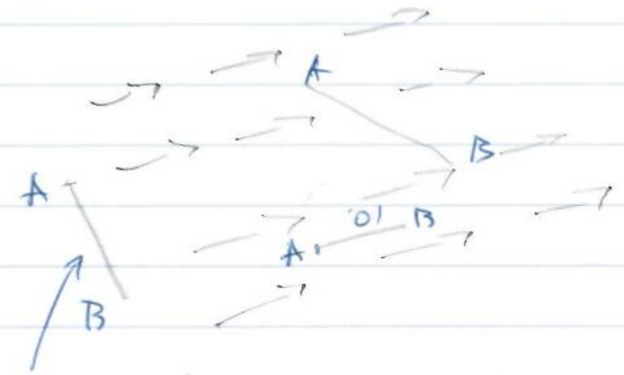
$\bar{F} = (a, b) = a\bar{i} + b\bar{j}$

$\overline{AB} = (x, y) = x\bar{i} + y\bar{j}$

$$\int_A^B \bar{F}\, d\bar{r} = \int_A^B a\,dx + b\,dy$$

displacement in x direction

Fluid flow



max flow $\vec{F} = v_1 \vec{i} + v_2 \vec{j}$

$\vec{AB} = x\vec{i} + y\vec{j}$

Area of rectangle

||

to calculate the flow you need

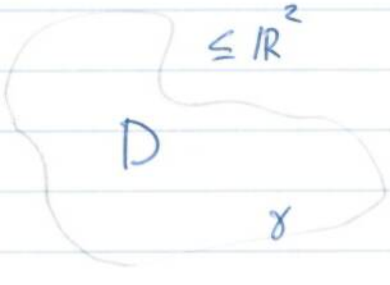
Area is the determinant

$$\begin{vmatrix} v_1 & v_2 \\ x & y \end{vmatrix} = v_1 y - v_2 x$$

$$\int_A^B -v_2\,dx + v_1\,dy$$

displacement in x dir.

green's thm (1828)



$$\int_\gamma \overbrace{f\,dx + g\,dy}^{w} = \iint_D \underbrace{\left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right) dx\,dy}_{dw}$$

M. Ostrogradsi 1831

In $\mathbb{R}^3$

$f(x,y,z)$ 0-form to be "integrated" summed up over 0-chains which is a collection of pts = points

$f(x,y,z)\,dx \wedge dy \wedge dz$ 3-form to be integrated over solids.

$\vec{F} = (f, g, h) \longrightarrow$ 2-form

$f\,dy \wedge dz + g\,dz \wedge dx + h\,dx \wedge dy$
to be integrated over a surface

$\downarrow$

1-form
$f\,dx + g\,dy + h\,dz$
be integrated over a curve

operators

$$\underset{\text{gradient}}{\nabla f} = \frac{\partial f}{\partial x}\bar{i} + \frac{\partial f}{\partial y}\bar{j} + \frac{\partial f}{\partial z}\bar{k}$$

$w = f$ 0-form

$$dw = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz$$

do not have machinery to explain!

$\vec{F} = (f, g, h)$

$$\text{curl } \vec{F} = \nabla \times \vec{F} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f & g & h \end{vmatrix} =$$

$$= \left(\frac{\partial h}{\partial y} - \frac{\partial g}{\partial z}\right)i + \left(-\frac{\partial h}{\partial x} + \frac{\partial f}{\partial z}\right)j + \frac{\partial g}{\partial x}$$

curl w similar to ω we will investigate later

$$\omega = f\,dx + g\,dy + h\,dz$$

$$d\omega = \left(\frac{\partial h}{\partial y} - \frac{\partial g}{\partial z}\right) dy \wedge dz + \left(-\frac{\partial h}{\partial x} + \frac{\partial f}{\partial z}\right) dz \wedge dx + \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right) dx \wedge dy$$

$$\bar{F} = (f, g, h)$$

divergente of ~~the~~ vector field

$$\nabla \cdot \bar{F} \quad \mathrm{div}(\bar{F}) = \frac{\partial f}{\partial x} + \frac{\partial g}{\partial y} + \frac{\partial h}{\partial z}$$

2-form

$$\omega = f\,dy \wedge dz + g\,dz \wedge dx + h\,dx \wedge dy$$

$$d\omega = \left(\frac{\partial f}{\partial x} + \frac{\partial g}{\partial y} + \frac{\partial h}{\partial z}\right) dx \wedge dy \wedge dz$$

Theorem in fluid $\to$

f potential $\Rightarrow$ ∇f conservative v.f. $\Rightarrow \mathrm{curl}(\nabla f) = 0$

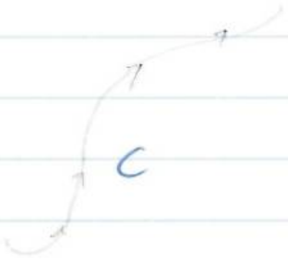
$\omega = f$ 0-form $\quad d\omega$ 1-form $\quad d(d\omega) = 0$ 2-form

$\left[\vec{F} \Rightarrow \mathrm{curl}(\vec{F}) \Rightarrow \mathrm{div}(\mathrm{curl}\,\vec{F}) = 0 \right]$

$\left[\omega \text{ 1-form} \Rightarrow \underset{\text{2-form}}{d\omega} \qquad d(d\omega) = 0 \right]$

understand in weeks

$F = (f, g, h)$

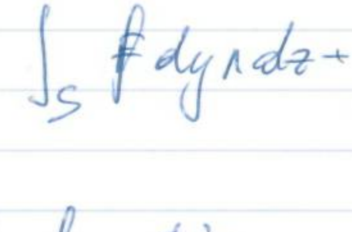


$$\int_C \vec{F} \cdot d\vec{r}$$

line integral

$$\int_C f\,dx + g\,dy + h\,dz$$

Surface integral

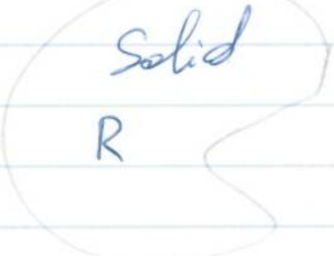


$$\int_S \vec{F} \cdot \vec{n}\, d\sigma$$

$$\int_S f\,dy\wedge dz + g\,dz\wedge dx + h\,dx\wedge dy$$

f function $\displaystyle\int_S \frac{\partial f}{\partial n}\, d\sigma = \int_S \frac{\partial f}{\partial x} dy\wedge dz + \frac{\partial f}{\partial y} dz\wedge dx + \frac{\partial f}{\partial z} dx\wedge dy$

Triple integral



$$\iiint_R f\,dV \qquad \int_R f\,dx\wedge dy\wedge dz$$

If $\cancel{f}$ $\vec{F}$ has a potential i.e. $\vec{F} = \nabla f$

$$\int_A^B \nabla f\, d\vec{r} = f(B) - f(A)$$

Work done by a conservative field

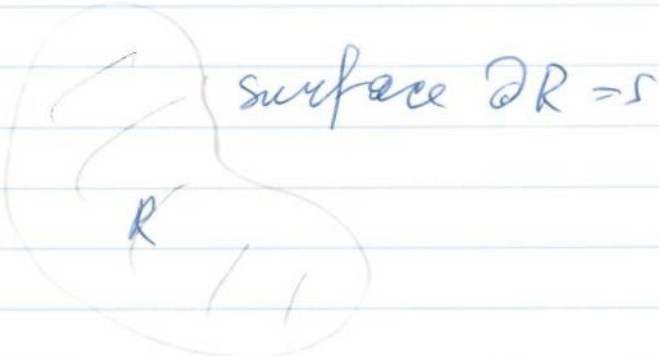
$$\int_C df = \int_{\partial C} f$$

Gauss Theorem (divergence Thm)

$$\int_R \nabla \cdot \vec{F}\, dV = \int_{\partial R} \bar{F} \cdot \bar{n}\, d\sigma$$

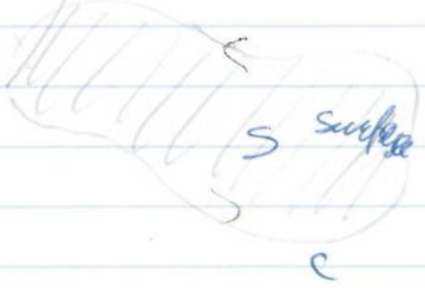
solid

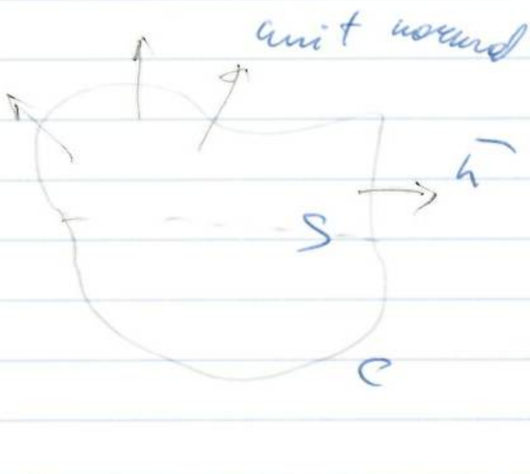
flux of $\bar{F}$ through the boundary of R



$$\int_R dw = \int_{\partial R} w$$

Classical Stokes Thm





$$\int_C \bar{F}\, d\bar{r} = \int_S \operatorname{curl} \bar{F} \times \vec{n}\, d\sigma$$

$$w = f\,dx + g\,dy + h\,dz$$

$$dw = \left(\frac{\partial h}{\partial y} - \frac{\partial g}{\partial x}\right) dy \wedge dz$$
$$+ \left(\frac{\partial f}{\partial z} - \frac{\partial h}{\partial x}\right) dz \wedge dx$$
$$+ \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right) dx \wedge dy$$

$$\int_C w = \int_C dw$$

Notation:

$$\mathbb{R}^n \ni x = (x^1, x^2, \dots, x^n)$$
$$x^i \in \mathbb{R}$$

$\mathbb{R}^n$ is a vector space

length - norm $|x| = \sqrt{x^{1^2} + x^{2^2} + \dots + x^{n^2}}$

if $x, y \in \mathbb{R}^n$ $\quad y = (y^1, y^2, \dots, y^n)$

$$x \cdot y = x^1 y^1 + x^2 y^2 + \dots + x^n y^n$$

dot product

Standard basis

$e_j = (0, 0, 0, \ldots, \underset{j-1}{0}, \underset{j}{1}, \underset{j+1}{0}, \ldots)$ $\quad j = 1, 2, 3 \ldots$

e_1 in $\mathbb{R}^2 = (1, 0)$
e_1 in $\mathbb{R}^3 = (1, 0, 0)$

Properties of norm:

$|x| \geq 0$

$|x| = 0$ iff $x = \vec{0}$

$|\lambda x| = |\lambda| \cdot |x| \quad x \in \mathbb{R}^n \quad \lambda \in \mathbb{R}$

Linear transformations

$T : \mathbb{R}^n \to \mathbb{R}^m$

(i) $T(x + y) = T(x) \overset{+ \text{ in } \mathbb{R}^m}{+} T(y)$ $\in \mathbb{R}^m$

add in $\mathbb{R}^n$ $\qquad$ $\mathbb{R}^m$

$\forall x, y \in \mathbb{R}^n$

$\forall \lambda \in \mathbb{R}$

(ii) $T(\lambda \cdot x) = \lambda \cdot T(x)$

Matrix representation of T w.r.t the standard bases of $\mathbb{R}^n$ and $\mathbb{R}^m$

$\mathbb{R}^m \ni T(e_j) = \sum_{i=1}^{m} a_{ji} e_i$

in $\mathbb{R}^n$ $\qquad$ in $\mathbb{R}^m$

$[T]_{\mathcal{E}}^{\mathcal{E}} = A = (a_{ij})_{\substack{i=1,\dots m \\ j=1,\dots n}}$ of size $m \times n$

$T : \mathbb{R}^n \to \mathbb{R}^m$ has matrix A $m \times n$

$T : \mathbb{R}^n \to \mathbb{R}^m$

$S : \mathbb{R}^n \to \mathbb{R}^m$

$[T + S] = [T] + [S]$

λ scalar

$[\lambda T] = \lambda \cdot [T]$

$U : \mathbb{R}^m \to \mathbb{R}^k$

$U \circ T : \mathbb{R}^n \to \mathbb{R}^k$

$\underset{k \times n}{[U \circ T]} = \underset{k \times m}{[U]}\underset{m \times n}{[T]}$

$T : \mathbb{R}^n \to \mathbb{R}^m$

$x \in \mathbb{R}^n \quad , \quad y = T(x) \in \mathbb{R}^m$

$x = (x^1, x^2, \dots, x^n)$

$y = (y^1, y^2, \dots, y^m)$

$$\underset{m \times n}{[T]} \underset{n \times 1}{\begin{pmatrix} x^1 \\ x^2 \\ \vdots \\ x^n \end{pmatrix}} = \underset{m \times 1}{\begin{pmatrix} y^1 \\ y^2 \\ \vdots \\ y^m \end{pmatrix}}$$

Def: A function from one set to another is injective (one-to-one) iff. $f(x) = f(y) \Rightarrow x = y$.
In other words, only one value of x gives any one value of y.

A function from one set to another is surjective (onto) iff for every y in the range set, $\exists\, x$ in the domain s.t. $f(x) = y$.
In other words, there are no "left over" members of the range set.

$f : V \to W$

domain = область определения

codomain (or range) = = область значений.

$T : V \to W$ is lin. map

$\dim(\operatorname{im} T) + \dim(\ker T) = \dim V$

$\operatorname{rank}(T) + \operatorname{null}(T) = \dim V$

Functions and Continuity

$\boxed{f: \mathbb{R}^n \to \mathbb{R}^m}$ **vector valued functions**

$f: A \to \mathbb{R}^m$ where $A \subseteq \mathbb{R}^n$ (nice open set)

f has components which are scalar fields

$f(x) = (f^1(x), f^2(x), f^3(x) \dots f^m(x))$, where $f^i: A \to R$
and these are scalar fields

$\Pi^i: \mathbb{R}^m \to \mathbb{R}$ check it is linear transformation

$\Pi^i(x^1, x^2, \dots, x^m) = x^i$

is a linear transf $i = 1, 2, \dots, m$

$$\boxed{\begin{tikzcd} \mathbb{R}^n \arrow[r, "f"] \arrow[dr, "f^i"'] & \mathbb{R}^m \arrow[d, "\Pi^i"] \\ & \mathbb{R} \end{tikzcd}}$$

$f: \mathbb{R}^n \to \mathbb{R}^m$

Def-n $\lim\limits_{\mathbb{R}^n \ni x \to a \in \mathbb{R}^n} f(x) = l \in \mathbb{R}^m$ means

norm in $\mathbb{R}^n$

$$\boxed{\forall \varepsilon > 0 \ \exists \delta > 0 \text{ s.t. } 0 < |x-a| < \delta \Rightarrow |f(x) - l| < \varepsilon}$$

norm in $\mathbb{R}^m$

Def-n: f is called **continuous at a** if $\boxed{\lim\limits_{x \to a} f(x) = f(a)} \iff \lim\limits_{h \to 0} f(a+h) = f(a)$

f is called **continuous on the set** A if it is continuous at a, for $\forall\, a \in A$

Combinational theorem

Assume, $\lim\limits_{x \to a} f(x) = b$

$\lim\limits_{x \to a} g(x) = c$

then, (i) $\lim\limits_{x \to a} \left(f(x) + g(x) \right) = b + c$

add in $\mathbb{R}^m$

Proof (i),(ii) (iv)

(ii) $\lim\limits_{x \to a} \left(\lambda \cdot f(x) \right) = \lambda \cdot b$

scalar mult. in $\mathbb{R}^m$, $\lambda \in \mathbb{R}$

(iii) $\lim\limits_{x \to a} \left(\underset{\mathbb{R}^m}{f(x)} \cdot \underset{\mathbb{R}^m}{g(x)} \right) = b \cdot c$

dot product in $\mathbb{R}^m$

(iv) $\lim\limits_{x \to a} |f(x)| = |b|$

norm in $\mathbb{R}^m$

Proof of (iii)

$$f(x) \cdot g(x) - b \cdot c = f(x) \cdot g(x) - b \cdot g(x) + b \cdot g(x) - b \cdot c =$$
$$= (f(x) - b) \cdot g(x) + b \cdot (g(x) - c) =$$

stick | |

$|f(x) \cdot g(x) - b \cdot c| = |(f(x) - b) \cdot g(x) + b \cdot (g(x) - c)| \leq$

triangle thm

modulus in $\mathbb{R}$ $\leq |(f(x) - b) g(x)| + |b \cdot (g(x) - c)| \leq$

Cauchy – Schwartz $|x \cdot y| \leq |x| \cdot |y|$

$|x^1 y^1 + x^2 y^2 + \dots + x^n y^n| \leq \sqrt{(x^1)^2 + \dots + (x^n)^2}\sqrt{(y^1)^2 + \dots (y^n)^2}$

$\leq \underbrace{|f(x) - b|}_{\to 0} |g(x)| + |b| \underbrace{|g(x) - c|}_{\to 0}$

Since $\lim_{x \to a} g(x) = c$

g is bounded in a neighborhood of a

i.e. $\exists M \geq 0 \ \exists \delta > 0$, $|g(x)| \leq M$ for $|x - a| < \delta$

Q.E.D.

(iv) $\lim_{x \to a} |f(x)| = |b|$

$||f(x)| - |b|| \leq |f(x) - b|$

triangle

06/10/11

Remark

1) $f : \mathbb{R}^n \to \mathbb{R}^n$ is continuous iff
$f^i : \mathbb{R}^n \to \mathbb{R}$ is continuous for $i = 1, 2, \dots, n$

2) Polynomial f-ns in n-variables
$f(x^1, \dots, x^n)$ are continuous

3) Rational f-ns $R(x) = \dfrac{P(x)}{Q(x)}$ are continuous,
where defined, i.e. $Q(x) \neq 0$
P, Q polynomials in n var

Ex. $\dfrac{(x^1)^2 + 5x^2}{(x^1)^2 - (x^2)^2}$

$Q(x) = (x^1)^2 - (x^2)^2 = 0$ hyperbola in (x^1, x^2) plane

Theorem Linear transformations are continuous.

Proof $T : \mathbb{R}^n \to \mathbb{R}^m$ let $a \in \mathbb{R}^n$

<u>To show:</u> $\lim_{h \to 0} T(a+h) = T(a)$ $\quad h = (h^1, \dots, h^n)$

$$|T(a+h) - T(a)| \overset{T}{\underset{\text{lin.}}{=}} |T(h)| = |T(h^1 e_1 + h^2 e_2 + \dots + h^n e_n)| =$$

$$\overset{T \text{ lin.}}{=} |(h^1 T(e_1) + h^2 T(e_2) + \dots + h^n T(e_n))| \overset{\Delta}{\leq}$$

$$\overset{\Delta}{\leq} |h^1|\,|T(e_1)| + |h^2|\,|T(e_2)| + \dots + |h^n|\,|T(e_n)| \leq$$

$$\leq |h|\,|T(e_1)| + |h|\,|T(e_2)| + \dots + |h|\,|T(e_n)| =$$

$$= (|T(e_1)| + |T(e_2)| + \dots + |T(e_n)|)\,|h|$$

$|T(a+h) - T(a)| \leq M|h|$

with $M = \sum_{i=1}^{n} |T(e_i)|$

Given $\varepsilon > 0$ choose $\delta = \frac{\varepsilon}{M}$

$|h| < \delta \Rightarrow |T(a+h) - T(a)| < \varepsilon$ $\square$

E-① $f(x,y) = \frac{x^2 - y^2}{x^2 + y^2}$, $(x,y) \neq (0,0)$

$\lim_{(x,y)\to(0,0)} f(x,y)$

the limit does not exist !!!

Assume $\lim_{(x,y)} f(x,y) = l$

$\forall \varepsilon > 0 \quad \exists \delta > 0 \quad$ s.t. $0 < |(x,y)| < \delta \Rightarrow |f(x,y) - l| < \varepsilon$

Plug $(x,0)$ into f $\quad f(x,0) = \frac{x^2 - 0^2}{x^2 + 0^2} = \frac{x^2}{x^2} = 1$

$(0,y)$ into f $\quad \cancel{f\frac{0^2-y^2}{0^2+y^2}}$

$f(0,y) = \frac{0^2 - y^2}{0^2 + y^2} = -\frac{y^2}{y^2} = -1$

(x,x) $\quad f(x,x) = \frac{x^2 - x^2}{x^2 + x^2} = 0$

If $|x| < \delta$ $\quad |(x,0)| < \delta \Rightarrow |f(x,0) - l| < \varepsilon$

$1 - l < \varepsilon$

If $|y| < \delta$, $\|(0,y)\| < \delta$

$|f(0,y) - l| < \varepsilon$

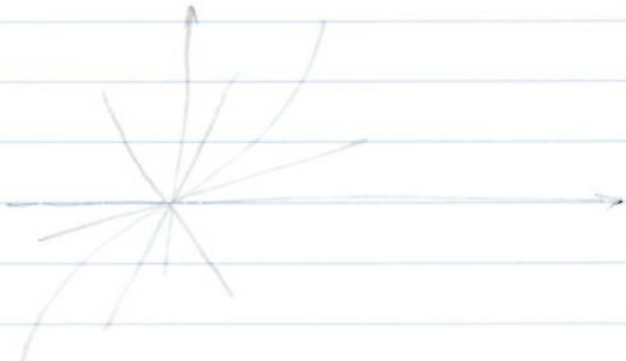
$|-1 - l| < \varepsilon$

$\varepsilon = \frac{1}{2}$ contradiction

$y = mx \qquad m \in \mathbb{R}$

$$f(x, mx) = \frac{x^2 - m^2x^2}{x^2 + m^2x^2} = \frac{1-m^2}{1+m^2}\,\frac{x^2}{x^2} = \frac{1-m^2}{1+m^2}$$

31/10 **Remark** checking along lines is not enough



$$\lim_{\substack{x \to 0 \\ x \neq 0}} \left(\lim_{y \to 0} f(x,y) \right) = \lim_{x \to 0} 1 = 1$$

fix x, look at $\lim_{y \to 0} f(x,y) = \lim_{y \to 0} \frac{x^2 - y^2}{x^2 + y^2} = \frac{x^2 - 0}{x^2 + 0} = 1$

$$\lim_{\substack{y \to 0 \\ y \neq 0}} \left(\lim_{x \to 0} f(x,y) \right) = \lim_{y \to 0} -1 = -1$$

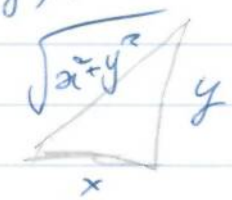
$$\lim_{x \to 0} \frac{x^2 - y^2}{x^2 + y^2} = -1$$

Ex(2) $f(x,y) = \begin{cases} \dfrac{xy}{\sqrt{x^2+y^2}}, & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$

show f is continuous at $(0,0)$

$|f(x,y)| < \varepsilon$ if $\|(x,y)\| < \delta$

$\left|\dfrac{xy}{\sqrt{x^2+y^2}}\right| < \varepsilon$



$\left|\dfrac{xy}{\sqrt{x^2+y^2}}\right| \le \dfrac{|x|\cdot|y|}{\sqrt{x^2+y^2}} \le \dfrac{\sqrt{x^2+y^2}\sqrt{x^2+y^2}}{\sqrt{x^2+y^2}} = \sqrt{x^2+y^2} =$

$= \|(x,y)\|$ given $\varepsilon > 0$ choose $\delta = \varepsilon$

If i had x^2y in numerator $|x^2y| \le \sqrt{x^2+y^2}^{\,2}\sqrt{x^2+y^2}$

If the total degree of each monomial in numerator is > than the total degree in denominator the limit should be ϕ.

Theorem

If f is con-us at a and g is con-us at $f(a)$

then $g \circ f$ is con-us at a.

Partial derivative

Let $f: \mathbb{R}^n \to \mathbb{R}$, $a \in \mathbb{R}^n$

Define: $$D_i f(a) = \lim_{h \to 0} \frac{f(a^1, a^2, \dots, a^{i-1}, a^i + h, a^{i+1}, \dots, a^n) - f(a)}{h}$$

If $\underset{(a,b)}{f}: \mathbb{R}^2 \to \mathbb{R}$

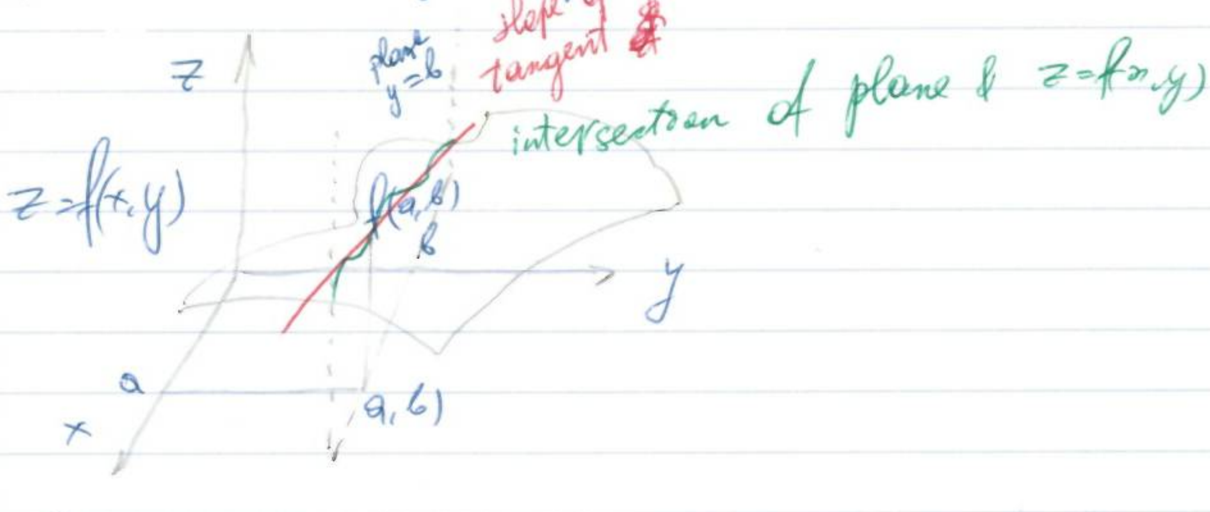
$$\frac{\partial f}{\partial x}(a,b) = D_1 f(a,b)$$

$$\frac{\partial f}{\partial y}(a,b) = D_2 f(a,b)$$

Ex. $f(x,y) = x^y$

$$\frac{\partial f}{\partial x} = y\, x^{y-1}, \quad \frac{\partial f}{\partial y} = x^y \cdot \ln(x)$$

Geometric meaning of partial derivative



plane $y = b$

Ex.① $f(x,y) = \begin{cases} \dfrac{x^2-y^2}{x^2+y^2} & (x,y) \neq (0,0) \\ 1 & (x,y) = (0,0) \end{cases}$

$$\frac{\partial f}{\partial x}(0,0) = \lim_{x\to 0} \frac{f(x,0)-f(0,0)}{x} = \lim_{x\to 0} \frac{\frac{x^2-0^2}{x^2+0^2}-1}{x} =$$

$$= \lim_{x\to 0} \frac{0}{x} = 0$$

$$\frac{\partial f}{\partial y}(0,0) = \lim_{y\to 0} \frac{f(0,y)-f(0,0)}{y} =$$

$$= \lim_{y\to 0} \frac{\frac{0^2-y^2}{0^2+y^2}-1}{y} = \lim_{y\to 0} \frac{-1-1}{y} = \lim_{y\to 0} -\frac{2}{y} = \pm\infty$$

Note: In 1-dim $f: \mathbb{R} \to \mathbb{R}$

$$f'(a) = \lim_{h\to 0} \frac{f(a+h)-f(a)}{h}$$

not allowed to div vectors in different dim.

Try it in higher dim $f: \mathbb{R}^n \to \mathbb{R}^m$

$$\cancel{\lim_{h\to 0} \frac{f(a+h)-f(a)}{h}}$$

$a \in \mathbb{R}^n$
$h \in \mathbb{R}^n$
$f(a) \in \mathbb{R}^m$
$f(a+h) \in \mathbb{R}^m$

$$= \lim_{h\to 0} \left[\frac{f(a+h)-f(a)}{h} - f'(a) \right] = \lim_{h\to 0} \left| \frac{f(a+h)-f(a)-h\cdot f'(a)}{h} \right| =$$

$$= \lim_{h\to 0} \frac{|f(a+h)-f(a)-hf'(a)|}{|h|}$$

11/10/11 Derivative, total Derivative.

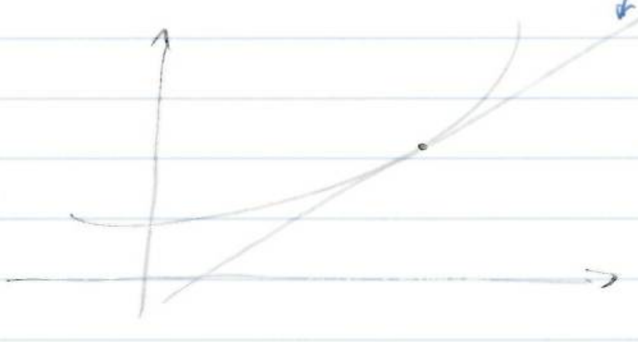
$f = \mathbb{R} \to \mathbb{R}$
$a \in \mathbb{R}$

$$\lim_{h \to 0} \frac{f(a+h) - f(a) - hf'(a)}{h} = 0$$

$f: \mathbb{R}^n - \mathbb{R}^m$

Tangent line at a
$y = f(a) + f'(a)(x-a)$

call $x - a = h$
$f(a) + h \cdot f'(a)$



Not a lin. trans
it is affine trans

Look at the map
$h \xrightarrow{\lambda} hf'(a) \quad h \in \mathbb{R}$
This is a lin. map
$h_1 + h_2 \to (h_1+h_2) f'(a) = h_1 f'(a) + h_2 f'(a)$
$\lambda(h_1+h_2) \quad = \lambda(h_1) + \lambda(h_2)$

$$\lim_{h \to 0} \frac{|f(a+h) - f(a) - \lambda(h)|}{|h|} = 0$$

Def $f : \mathbb{R}^n \to \mathbb{R}^m$ ~~fon~~

($f : A \to \mathbb{R}^m$, A open in $\mathbb{R}^m$)

is differentiable at a ($a \in A$)

if we can find a lin transformation

$\lambda : \mathbb{R}^n \to \mathbb{R}^m$

$$\lim_{h \to 0} \frac{|f(a+h) - f(a) - \lambda(h)|}{|h|} = 0$$

norm in $\mathbb{R}^n$

$|h| \leftarrow$ norm in $\mathbb{R}^n$

the lin transf. λ is called the (total) derivative of f at a and denoted

$Df(a)$ s.t. $Df(a)(h) = \lambda(h)$

Examples

! 1) $f : \mathbb{R}^n \to \mathbb{R}^m$ which is const $f(x) = k$

is differentiable at $a \in \mathbb{R}^m$ with

! $\boxed{Df(a) = 0}$ which is the 0 lin. trans

$0 : \mathbb{R}^n \to \mathbb{R}^m$

i.e. $0(h) = 0 \in \mathbb{R}^m$
$h \in \mathbb{R}^n$

$$\frac{|f(a+h) - f(a) - 0(h)|}{|h|} = \frac{|k - k - 0|}{|h|} = 0 \underset{|h| \to 0}{\to} 0$$

! 2) $f: \mathbb{R}^n \to \mathbb{R}^m$ is a lin. trans.
is differentiable at $a \in \mathbb{R}^n$

! $\boxed{Df(a) = f}$

$$f(a+h) - f(a) - Df(a)(h) = f(a+h) - f(a) - f(h) =$$

f lin.

$$= f(a+h-a-h) = f(0) = 0$$

3) $f: \mathbb{R} \to \mathbb{R}$ is a lin. transf.

$f: \mathbb{R} \to \mathbb{R}$

$$f(a) = ma$$
$$f'(a) = m$$

$Df(a)$ is a lin transf.
$\mathbb{R} \to \mathbb{R}$

$$Df(a) = f$$
$$Df(a)(h) = f(h) = mh$$

$$Df(a)(h) = m \cdot h$$

$$f'(a) = m = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h} =$$

$$= \frac{f(a+h) - f(a) - hf'(a)}{h} \to 0$$

<u>Theorem</u> if f is diff. at a then there exists a <u>unique</u> $\lambda: \mathbb{R}^n \to \mathbb{R}^m$ linear trans. s.t. $\lim_{h \to 0} \frac{|f(a+h) - f(a) - \lambda(h)|}{|h|} = 0$

Proof ① Suppose $\mu : \mathbb{R}^n \to \mathbb{R}^m$ is another l. trans

s.t. $0 = \lim_{h \to 0} \frac{|f(a+h) - f(a) - \mu(h)|}{|h|}$

② Deduce $\lambda = \mu$ i.e. $\forall h \in \mathbb{R}^n$ $\lambda(h) = \mu(h)$

$$\frac{|\lambda(h) - \mu(h)|}{|h|} = \frac{|\lambda(h) + f(a) - f(a+h) + f(a+h) - f(a) - \mu(h)|}{|h|} \leq$$

$$\leq \frac{|f(a+h) - f(a) - \lambda(h)|}{|h|} + \frac{|f(a+h) - f(a) - \mu(h)|}{|h|} \xrightarrow[h \to 0]{} 0$$

Conclusion $\boxed{\lim_{h \to 0} \frac{|\lambda(h) - \mu(h)|}{|h|} = 0}$ $*$

③ Let $h = 0$ $\lambda(h) = 0 = \mu(h)$ since λ, μ linear $0 \to 0$

④ Shrink!

Let $t \in \mathbb{R}$ $th \in \mathbb{R}^n$, $t \to 0 \in \mathbb{R}$

$th \to 0 \in \mathbb{R}^n$

Plug ~~t~~ th in $*$

$$0 = \lim_{t \to 0} \frac{|\lambda(th) - \mu(th)|}{|th|} = \lim \frac{|t\lambda(h) - t\mu(h)|}{|t|\,|h|} =$$

$$= \lim_{t \to 0} \frac{\cancel{|t|}\,|\lambda(h) - \mu(h)|}{\cancel{|t|}\,|h|} = \frac{|\lambda(h) - \mu(h)|}{|h|}$$ i.e. const.

$\Rightarrow |\lambda(h) - \mu(h)| = 0 \to \lambda(h) = \mu(h)$ $\forall h \in \mathbb{R}^n$

$\therefore \lambda = \mu$. ∎

Def-n: $f\ \mathbb{R}^n \to \mathbb{R}^m$, $a \in \mathbb{R}^n$

f is diff at a

$Df(a) : \mathbb{R}^n \to \mathbb{R}^m$ is lin.
its matrix representation is denoted by

$$f'(a) \in M_{m \times n}$$

and is called the Jacobian of f at a
(Carl Gustave Jakob Jacob 1805 – 1851)

Ex. $f : \mathbb{R}^2 \to \mathbb{R}^2$ $\quad x, y \in \mathbb{R}$ $\quad f(x,y) = (x^2 y, x+5)$

Show that $Df(1,2)(h^1, h^2) = (4h^1 + h^2, h^1)$

HW Exercise Prove $Df(1,2)$ is linear

$$f((1,2) + (h^1, h^2)) - f(1,2) - Df(1,2)(h^1, h^2) =$$

$$= f((1+h^1), 2+h^2) - f(1,2) - (4h^1 + h^2, h^1) =$$

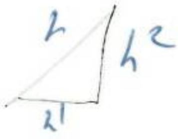
$$= ((1+h^1)^2 (2+h^2), 1+h^1+5) - (2,6) - (4h^1 + h^2, h^1)$$

$$= (2+h^2 + (h^1)^2 2 + (h^1)^2 h^2 + 2h^1 h^2 + 4h^1, 6+h^1) - (2,6) - (4h^1 + h^2, h^1)$$

$$= (\cancel{2} + \cancel{h^2} + 2(h^1)^2 + (h^1)^2 h^2 + 2h^1 h^2 + 4h^1 - 2 - 4h^1 - \cancel{h^2}, \cancel{6} + \cancel{h^1} - \cancel{6} - \cancel{h^1})$$

$$= (2(h^1)^2 + (h^1)^2 h^2 + 2h^1 h^2, 0)$$

Take length



$$| \quad | = |2(h^1)^2 + (h^1)^2 h^2 + 2hh^2| \le$$

$$\le 2\cdot|h|^2 + |h|^2|h| + 2|h||h| = 4|h|^2 + |h|^3$$

$$\frac{|f((1,2)+(h^1,h^2)) - f(1,2) - Df(1,2)(h^1,h^2)|}{|h|} \le \frac{4|h|^2+|h|^3}{|h|} =$$

$$= 4|h| + |h|^2 \xrightarrow[|h|\to 0]{} 0$$

<u>Find</u>: matrix reps of $Df(a)$

$f'(a)$ is matrix reps of $Df(a)$

$$\underset{\substack{m\times 1\\ \text{column vector}}}{Df(a)(h)} = \begin{pmatrix} y^1 \\ \vdots \\ y^m \end{pmatrix} = \underset{m\times n}{f'(a)} \begin{pmatrix} h^1 \\ h^2 \\ \vdots \\ h^n \end{pmatrix}$$

1402 $$f'(a) = \begin{pmatrix} D_1 f^1(a) & D_2 f^1(a) & \dots & D_n f^1(a) \\ D_1 f^2(a) & D_2 f^2(a) & \dots & D_n f^2(a) \\ \vdots & & & \\ D_1 f^m(a) & D_2 f^m(a) & \dots & D_n f^m(a) \end{pmatrix}$$

$f(x^2 y, x+5)$

$$\frac{\partial f^1}{\partial x} = 2xy \qquad \frac{\partial f^1}{\partial y} = x^2$$

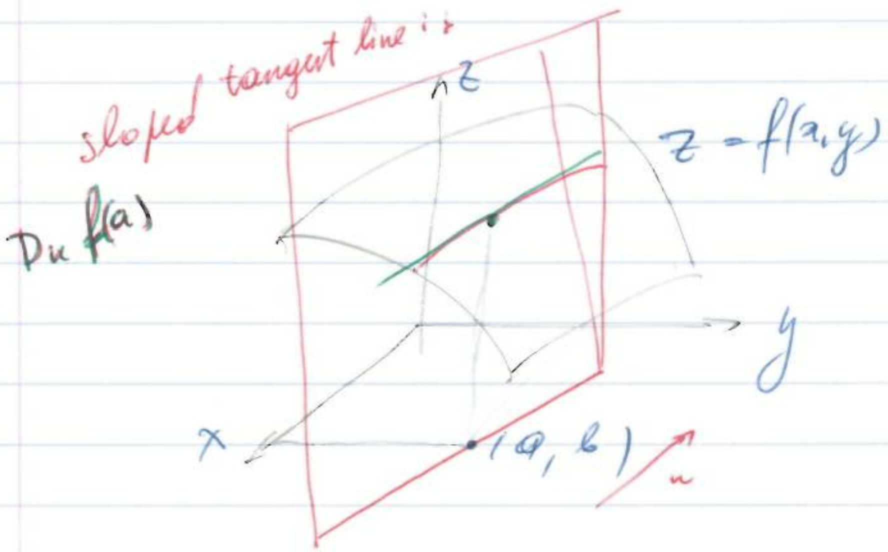
$$\frac{\partial f^2}{\partial x} = 1 \qquad \frac{\partial f^2}{\partial y} = 0$$

$$f'(1,2) = \begin{pmatrix} 4 & 1 \\ 1 & 0 \end{pmatrix}$$

$$f'(1,2)\begin{pmatrix} h^1 \\ h^2 \end{pmatrix} = \begin{pmatrix} 4 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} h^1 \\ h^2 \end{pmatrix} = \begin{pmatrix} 4h^1 + h^2 \\ h^1 \end{pmatrix}.$$

Def-n Let $u \neq 0$ $u \in \mathbb{R}^n$
~~f~~ $f : \mathbb{R}^n \to \mathbb{R}^n$
The directional derivative of f at a in the direction u is given by

$$D_u f(a) = \lim_{h \to 0} \frac{f(a+hu) - f(a)}{h} \quad h \in \mathbb{R}$$



$x = a$
$y = b$

Remark having dir. derivative in all dir-s $u \neq 0$ is not enough to guarantee $Df(a)$ exists

Theorem If f is diff. at a then f is continuous at a.

Proof: $\lim_{h \to 0} |f(a+h) - f(a)| = \lim_{h \to 0} |f(a+h) - f(a) - Df(a)h + Df(a)h|$

Trian.

$$\le \lim_{h \to 0} \left(\underbrace{\frac{|f(a+h) - f(a) - Df(a)(h)|}{|h|}}_{\to 0} \underbrace{|h|}_{\to 0} + |Df(a)(h)| \right)$$

$Df(a)h \to 0$

Since $Df(a)$ is lin tran $Df(a)$ is continuous

$$\lim_{h \to 0} |Df(a)(h)| = |Df(a)(0)| = |0| = 0$$

□

thm Chain Rule

$f : \mathbb{R}^n \to \mathbb{R}^m$ is diff. at a

$g : \mathbb{R}^m \to \mathbb{R}^k$ is diff. at $f(a)$

<u>then</u> $g \circ f : \mathbb{R}^n \to \mathbb{R}^k$ is diff. at a

$$\begin{tikzcd}
\mathbb{R}^n \arrow[r, "Df(a)", "\text{linear}"'] \arrow[dr, "D(g\circ f)(a)"'] & \mathbb{R}^m \arrow[d, "Dg(f(a))"] \\
& \mathbb{R}^k
\end{tikzcd}$$

$$D(g \circ f)(a) = Dg(f(a)) \circ Df(a)$$

composition — matrix multiplication

$$\underset{k \times n}{(g \circ f)'(a)} = \underset{k \times m}{g'(f(a))} \cdot \underset{m \times n}{f'(a)}$$ — jacobians

Remark $\lim_{h \to 0} \dfrac{|f(a+h) - f(a) - Df(a)(h)|}{|h|}$

Set $a + h = x$
$h = x - a$

set $\phi(x) = f(x) - f(a) - Df(a)(x - a)$

Then f is differentiable at a if we show

$$\lim_{x \to 0} \frac{|\phi(x)|}{|x - a|} = 0$$

Proof: ~~with this notation~~

call $Df(a) = \lambda$, $Dg(f(a)) = \mu$
$f(a) = b \in \mathbb{R}^m$

linear tr.

By the remark, f diff at a means

$f(x) - f(a) - \lambda(x - a) - \phi(x)$ with $\lim_{x \to a} \dfrac{|\phi(x)|}{x - a} = 0$

Similarly set $\psi(y) = g(y) - g(b) - \mu(y - b)$

s.t. g diff at b means $\lim_{y \to b} \dfrac{|\psi(y)|}{|y - b|} = 0$

We need to show that

$$\lim_{x \to a} \frac{|g(f(x)) - g(f(a)) - (\mu \circ \lambda)(x - a)|}{|x - a|} = 0$$

$$g(f(x)) - g(\overset{b}{\overset{\|}{f(a)}}) = g(f(a) + \lambda(x-a) + \phi(x)) - g(b)$$

$$= g(b + \lambda(x-a) + \phi(x)) - g(b)$$

$$= \mu(\lambda(x-a) + \phi(x)) + \psi(b + \lambda(x-a) + \phi(x))$$

μ lin

$$= \mu(\lambda(x-a)) + \mu(\phi(x)) + \psi(b + \lambda(x-a) + \phi(x))$$

$$g(f(x)) - g(b) - \mu(\lambda(x-a)) = \mu(\phi(x)) + \psi(b + \lambda(x-a) + \phi(x))$$

Therefore we need to prove

$$\lim_{x \to a} \frac{|\mu(\phi(x)) + \psi(b + \lambda(x-a) + \phi(x))|}{|x-a|} = 0$$

Suffices $\lim_{x \to a} \frac{|\mu(\phi(x))|}{|x-a|} = 0$

and $\lim_{x \to a} \frac{|\psi(b + \lambda(x-a) + \phi(x))|}{|x-a|} = 0$

~~But~~ (by triangl. ineq.)

proved on th lin. tr. is continuous

μ is lin. transf.

~~$\exists$~~ $\exists M \geq 0$ s.t. $|\mu(h)| < M|h|$

$\lim_{x \to a} \frac{|\phi(x)|}{|x-a|} = 0$

$$\frac{|\mu(\phi(x))|}{|x-a|} \leq \frac{M|\phi(x)|}{|x-a|} \xrightarrow[x \to a]{} 0$$

$\therefore \lim_{x \to a}$

enough to prove it is bounded

Set $y = b + \lambda(x-a) + \phi(x)$

$$\frac{|\psi(y)|}{|x-a|} = \frac{|\psi(y)|}{|y-b|}\frac{|y-b|}{|x-a|} =$$

$\to 0$ as $y \to b$

$$\frac{|y-b|}{|x-a|} = \frac{|\lambda(x-a) + \phi(x)|}{|x-a|} \leq \frac{|\lambda(x-a)| + |\phi(x)|}{|x-a|} =$$

$$= \frac{|\lambda(x-a)|}{|x-a|} + \underbrace{\frac{|\phi(x)|}{|x-a|}}_{\to 0} \leq$$

$$\leq M\frac{|x-a|}{|x-a|} + \dots$$

□

13/10/11 **Theorem**

(i) Define $s : \mathbb{R}^2 \to \mathbb{R}$
$s(x,y) = x+y$
then s is diff. and $Ds = s$

(ii) Define $p : \mathbb{R}^2 \to \mathbb{R}$
$p(x,y) = xy$
Then p is diff and $Dp(a,b) : \mathbb{R}^2 \to \mathbb{R}$ lin
$Dp(a,b)(h,k) = ak + bh$
$p'(a,b) = (b,a)$

Proof:

(i) s is lin

$s((x,y)+(x',y')) = s(x,y) + s(x',y')$ &
$s(\lambda(x,y)) = \lambda s(x,y)$

$S((x,y)+(x',y')) = S((x+x', y+y')) = x+x'+y+y' =$
$= S(x,y) + S(x',y')$

Hlw $S(\lambda(x,y)) = \lambda S(x,y)$

(ii') Use def of derivative

$p((a,b)+(h,k)) - p(a,b) - Dp(a,b)(h,k) =$

$= p(a+h, b+k) - p(a,b) - (ak + bh) =$
$= (a+h)(b+k) - a\cdot b - (ak + bh) =$
$= \cancel{ab} + \cancel{hb} + \cancel{ak} + hk - \cancel{a\cdot b} - \cancel{ak} - \cancel{bh} =$
$= hk$

$$\frac{|p((a,b)+(h,k)) - p(a,b) - Dp(a,b)(h,k)|}{|(h,k)|} = \frac{|hk|}{\sqrt{h^2+k^2}} \le$$

$$\le \frac{\sqrt{h^2+k^2}\,\sqrt{h^2+k^2}}{\sqrt{h^2+k^2}} = \sqrt{h^2+k^2} \to 0 \quad \text{as } (h,k) \to (0,0)$$

$Dp(a,b)(1,0) = b$
$Dp(a,b)(0,1) = a$
(b,a) Jacobian

Remark

(1) Let $T: \mathbb{R}^n \to \mathbb{R}^m$ to check it is lin we listed two properties

$T(x+y) = T(x) + T(y) \quad x, y \in \mathbb{R}^n$
$T(\lambda x) = \lambda T(x) \quad \lambda \in \mathbb{R}$

We can also check instead

$T(\lambda x + y) = \lambda T(x) + T(y)$

(2) Let $g^i : \mathbb{R}^n \to \mathbb{R}$ be a lin map.
Such a map is called a linear functional
The set of lin. functionals from $\mathbb{R}^n$ to $\mathbb{R}$
is called the dual space of $\mathbb{R}^n$, notation $(\mathbb{R}^n)^*$

Let now $g^1, g^2, \dots, g^m$ be lin f-als
$$g^i : \mathbb{R}^n \to \mathbb{R}$$

Then I can combine them to get a map
$$g : \mathbb{R}^n \to \mathbb{R}^m \quad \text{by}$$

$$g(a) = (g^1(a), g^2(a), \dots, g^m(a))$$

$g : \mathbb{R}^n \to \mathbb{R}^m$ is lin. $x, y \in \mathbb{R}^n$, $\lambda \in \mathbb{R}$
$$g(\lambda x + y) = \lambda g(x) + g(y)$$
$$g(\lambda x + y) \underset{def}{=} (g^1(\lambda x + y), g^2(\lambda x + y), \dots, g^m(\lambda x + y)) \overset{g^i \, lin.}{=}$$
$$= (\lambda g^1(x) + g^1(y), \lambda g^2(x) + g^2(y), \dots, \lambda g^m(x) + g^m(y)) =$$
$$= (\lambda g^1(x), \lambda g^2(x), \dots, \lambda g^m(x)) + (g^1(y), g^2(y), \dots, g^m(y))$$
$$= \lambda (g^1(x), g^2(x), \dots, g^m(x)) + (g^1(y), g^2(y), \dots, g^m(y)) =$$
$$\underset{def}{\overset{g}{=}} \lambda g(x) + g(y)$$
$\square$

Let $[g^i]$ be the matrix repr. of g^i $(1 \times n)$
$$[g^i] = (g^i_1, g^i_2, \dots, g^i_n)$$

11/10 $m \times n$
$$[g] = \begin{pmatrix} g^1_1, & g^1_2, & \dots, & g^1_n \\ g^2_1, & g^2_2, & \dots, & g^2_n \\ \vdots \\ g^m_1, & g^m_2, & \dots, & g^m_n \end{pmatrix}$$

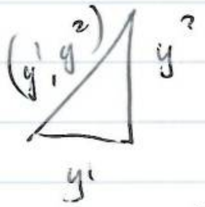
theorem $f: \mathbb{R}^n \to \mathbb{R}^m$ is diff at a iff.
f^i are diff at a, $i = 1, 2, \dots, m$ and
$Df(a) = (Df^1(a), Df^2(a), \dots, Df^m(a))$

Proof: $(\Rightarrow)$ Assume f is diff. at a
$f^i = \pi^i \circ f$

$\pi^i(x^1, \dots, x^n) = x^i$
π^i is lin.
$D\pi^i = \pi^i$
Chain Rule $\Rightarrow f^i$ is diff
$Df^i(a) = D\pi^i(f(a)) \circ Df(a)$
$Df^i(a) = \pi^i \circ Df(a)$ this is the eq.-n.
$Df(a) = (Df^1(a), \dots, Df^n(a))$

$(\Leftarrow)$ Assume f^i diff at a $i = 1, 2, \dots m$
$f(a+h) - f(a) - (Df^1(a)(h), Df^2(a)(h), \dots, Df^m(a)(h)) =$
$= (f^1(a+h), f^2(a+h), \dots, f^m(a+h)) - (f^1(a), \dots, f^m(a)) -$
$- (Df^1(a)(h), Df^2(a)(h), \dots, Df^m(a)(h)) =$
$= (f^1(a+h) - f^1(a) - Df^1(a)(h), f^2(a+h) - f^2(a) - Df^2(a)(h), \dots,$
$\dots, f^m(a+h) - f^m(a) - Df^m(a)(h))$

$$\frac{|f(a+h) - f(a) - Df(a)(h)|}{|h|} \leq$$



$$|(y^1, y^2)| \le |y^1| + |y^2|$$

$$|y| \le |y^1| + |y^2| + \dots + |y^m|$$

$$= \sqrt{(y^1)^2 + \dots + (y^m)^2}$$

$$\le \frac{|f^1(a+h) - f^1(a) - Df^1(a)(h)|}{|h|} + \dots + \frac{|f^m(a+h) - f^m(a) - Df^m(a)(h)|}{|h|}$$
$$\qquad \searrow 0 \qquad\qquad\qquad\qquad \searrow 0$$

Corollary $f, g : \mathbb{R}^n \to \mathbb{R}$ diff at a

(1) $D(f+g)(a) = Df(a) + Dg(a)$

Remark: if $T, S : \mathbb{R}^n \to \mathbb{R}^m$ are lin

$(T+S)(x) = T(x) + S(x)$ is lin

$T + S : \mathbb{R}^n \to \mathbb{R}^m$

if $\lambda \in \mathbb{R}$

$(\lambda T)(x) = \lambda T(x)$ $\quad$ $\lambda T : \mathbb{R}^n \to \mathbb{R}^m$ is also lin.

(2) Product rule

$$D(f \cdot g)(a) = \cancel{\overset{\text{lin map}}{Df(a)} \cdot \overset{\text{scalar}}{g(a)} + Dg(a) f(a)}$$

$$= g(a) \cdot Df(a) + f(a) \cdot Dg(a)$$

(3) Quotient Rule

if $g(a) \neq 0$ $\quad D\left(\frac{f}{g}\right)(a) = \frac{1}{g(a)^2}\left(g(a)\,Df(a) - f(a)\,Dg(a)\right)$

Proof

(1) $f, g : \mathbb{R}^n \to \mathbb{R}$

$$x \to (f(x), g(x)) \to f(x) + g(x)$$

with $(f(x), g(x)) \in \mathbb{R}^2$

$$\mathbb{R}^n \xrightarrow{(f,g)} \mathbb{R}^2 \xrightarrow{s} \mathbb{R}$$

$$f+g = s \circ (f,g)$$

$$D(f+g)(a) \underset{\text{Rule}}{\overset{\text{Chain}}{=}} \underbrace{Ds(f(a), g(a))}_{s \text{ is lin}} \circ \underbrace{D(f,g)(a)}_{\text{by Thm}} =$$

$$= s \circ (Df(a), Dg(a))$$

$$= Df(a) + Dg(a)$$

(2) $\mathbb{R}^n \xrightarrow{(f,g)} \mathbb{R}^2 \xrightarrow{p} \mathbb{R}$

$$f \cdot g = p \circ (f,g)$$

$$D(f\cdot g)(a) = \cancel{Dp \circ D(f,g)}\ Dp((f,g)(a)) \circ D(f,g)(a) =$$

$$= Dp(f(a), g(a)) \circ (Df(a), Dg(a)) =$$

$h^2 \in \mathbb{R}^2, \quad D(f\cdot g)(a) : \mathbb{R}^2 \to \mathbb{R}$

$$D(f\cdot g)(a)(h)$$

$$= Dp(f(a), g(a)) \circ (Df(a), Dg(a))(h) =$$

$\stackrel{\text{Th of}}{\underset{P}{=}}$ $\cancel{f(a)}$

$= Dp(f(a), g(a))(Df(a)(h), Dg(a)(h)) =$

$\stackrel{\text{Th of}}{\underset{P}{=}} f(a) \cdot Dg(a)(h) + g(a) Df(a)(h)$

18/10/11 $f: \mathbb{R}^n \to \mathbb{R}$, $a \in \mathbb{R}^n$

$$D_i f(a) = \lim_{h \to 0} \frac{f(a^1, a^2, \dots, a^{i-1}, a^i + h, a^{i+1}, \dots, a^n) - f(a)}{h}$$

If $D_i f(a)$ exists for all a in some open set U

$\Rightarrow$ we get a f-n $U \xrightarrow[D_i]{} \mathbb{R}$

$x \to D_i f(x)$

Then we can talk about the partial der. of $D_i f$

e.g. $D_j (D_i f)(x) = D_{i,j} f(x)$

If $D_j f(x)$ exists for $\forall x \in V$ this is a f-n of x and we can consider

$D_i (D_j f)(x) = D_{j,i} f(x)$ (in gen. $i \neq j$)

Example If $f(x,y) = x^3 y^5$

$D_1 f(x,y) = 3x^2 y^5$

$D_{1,1} f(x,y) = 6x y^5$

$\boxed{D_{2,1} f(x,y) = 15x^2 y^4}$

$D_2 f(x,y) = 5x^3 y^4$

$\boxed{D_2 (D_1 f) x = 15x^2 y^4}$

Thm If $D_{i,j} f$ & $D_{j,i} f$ are continuous on an open set containing a then

$$\boxed{D_{i,j} f(a) = D_{j,i} f(a)}$$

Proof In the ex of H/W 5

Thm If $A \subseteq \mathbb{R}^n$
If the max or min of $f : A \to \mathbb{R}$ occur at a point a in the ~~intersection~~ interior of A and $D_i f(a)$ exists
then $D_i f(a) = 0$

Proof:

one var. f-n

Consider $h(x) = f(a^1, a^2, \dots, a^{i-1}, x, a^{i+1}, \dots, a^n)$
x in an open interval around a^i

Since f has a max or min at a
h has a max or min at a^i

$$\frac{dh}{dx}(a^i) = D_i f(a)$$

By Analysis II, $\frac{dh}{dx}(a^i) = 0$

$$\Rightarrow D_i f(a) = 0 \qquad \square$$

Note Other way thm is not true
e.g. $y = x^3$ at $(0,0)$; $f(x,y) = x^2 - y^2$ at $(0,0)$

Recall: $f : \mathbb{R}^n \to \mathbb{R}^m \quad a \in \mathbb{R}^n$

$Df(a) : \mathbb{R}^n \to \mathbb{R}^m$ lin map, total der

Jacobian $f'(a) \in M_{m \times n}$ is the representation of $Df(a)$ in standard basis.

Thm If $f : \mathbb{R}^n \to \mathbb{R}^m$ is diff at a

then

1) $D_j f^i(a)$ exists for all $i = 1, \dots, m$

$j = 1, \dots, n$; ~~and~~

2) and the Jacobian matrix is

$$f'(a) = \left(D_j f^i(a)\right) \quad \begin{matrix} i = 1, \dots, m \\ j = 1, \dots, n \end{matrix}$$

e.g.

$$f'(a) = \begin{pmatrix} D_1 f^1(a) & D_2 f^1(a) & \dots & D_n f^1(a) \\ D_1 f^2(a) & D_2 f^2(a) & \dots & D_n f^2(a) \\ \vdots & & & \\ D_1 f^m(a) & D_2 f^m(a) & \dots & D_n f^m(a) \end{pmatrix} \in M_{m \times n}$$

Proof:

~~We can assume $m = 1$~~

case 1: $m = 1$

1) $h : \mathbb{R} \to \mathbb{R}^n \qquad h(t) = (a^1, \dots, a^{i-1}, t, a^{i+1}, \dots, a^n)$

$$\begin{tikzcd} \mathbb{R} \arrow[r, "h"] \arrow[dr] & \mathbb{R}^n \arrow[d, "f"] \\ & \mathbb{R} \end{tikzcd}$$

$$\frac{d}{dt}(f \circ h)\Big|_{t = a^i} = D_i f(a)$$

$$= \lim_{t \to a^i} \frac{(f \circ h)t - (f \circ h)(a^i)}{t - a^i} =$$

$$= \lim_{t \to a^i} \frac{f(a^1, a^2, \dots, a^{i-1}, t, a^{i+1}, \dots, a^n) - f(a^1, \dots, a^n)}{t - a^i}$$

2) h is differentiable
$h(t) = (a^1, a^2, \dots, a^{i-1}, t, a^{i+1}, \dots, a^n)$
because its component are diff.:

$$\left.\begin{array}{l} h^j(t) = a^j \text{ is const } \forall j \quad i \neq j \\ h^i(t) = t \text{ is lin. f-n} \Rightarrow \text{diff.} \end{array}\right\} \forall \text{ diff}$$

$$\therefore Dh(t) = (Dh^1(t), \dots, Dh^n(t)) =$$

$$= (0, 0, \dots, \underset{\uparrow}{0}, Id, \underset{\uparrow}{0}, \dots 0)$$
$\quad\quad i-1 \quad\quad i+1$

could be any gives same result

$$h'(a^i) = \begin{array}{c} \\ \\ \\ i-1 \rightarrow \\ i^{th} \rightarrow \\ i+1 \rightarrow \\ \\ \end{array} \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \end{pmatrix}$$

$h \to \frac{dg}{dt}(t_0) \cdot h$

Remark If $g : \mathbb{R} \to \mathbb{R}$ $\quad Dg(t_0) : \mathbb{R} \to \mathbb{R}$ lin map

$\frac{dg}{dt}(t_0)$ abuse of notation $\quad g'(t_0)$ Jacobian 1×1 matrix

$$g'(t_0) = \left(\frac{dg}{dt}(t_0)\right)$$

Since f, h are diff the chain rule implies

$$\underset{1\times 1}{(f\circ h)^{i\,\prime}(a^i)} = \underset{1\times n}{f'(h(a^i))} \cdot \underset{n\times 1}{h'(a^i)} = f'(a) \cdot \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} =$$

i^{th}

matrix multiplication

$$\| \qquad h(t) = (a^1, a^2, \dots, a^{i-1}, t, a^{i+1}, \dots, a^n)$$

$$\frac{d}{dt}(f\circ h)a^i \qquad h(a^i) = (a^1, a^2, \dots, a^{i-1}, a^i, a^{i+1}, \dots, a^n) = a$$

$$\|$$

$$D_i f(a)$$

$= (f'(a))^i$ i-entry of the Jacobian

$\therefore D_i f(a) = (f'(a))^i$

Conclusion: $f'(a) = (D_1 f(a), D_2 f(a), \dots, D_n f(a))$

Case $m > 1$

lin map. $\mathbb{R}^n \to \mathbb{R}$

$f : \mathbb{R}^n \to \mathbb{R}^m$

$f(x) = (f^1(x), \dots, f^m(x))$

In prev. lect.: $Df(a) = (Df^1(a), \dots, Df^m(a))$

$$f'(a) = \underbrace{\begin{pmatrix} (f^1)'(a) \\ (f^2)'(a) \\ \vdots \\ (f^m)'(a) \end{pmatrix}}_{m\times n \text{ matrix.}}$$

$(f^i)'(a)$ is a $1\times n$ matrix

By case $m = 1$

$$f'(a) = \begin{pmatrix} D_1 f^1(a) & D_2 f^1(a) & \dots & D_n f^1(a) \\ D_1 f^2(a) & D_2 f^2(a) & \dots & D_n f^2(a) \\ \vdots & & & \\ D_1 f^m(a) & D_2 f^m(a) & & D_n f^m(a) \end{pmatrix}$$

□ □

Example $g(x,y) = \begin{cases} \frac{x^2y}{x^4+y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$

Fix a vector $u \in \mathbb{R}^2$, $u(u^1,u^2) \neq (0,0)$, $\underline{u^2 \neq 0}$

directional $D_u\, g(0,0) = \lim_{\substack{h \to 0 \\ h \in \mathbb{R}}} \frac{g((0,0)+hu) - g(0,0)}{h} =$

$= \lim_{h\to 0} \frac{g(hu^1, hu^2) - 0}{h} = \lim_{h \to 0} \frac{(hu^1)^2(hu^2)}{(hu^1)^4+(hu^2)^2}\frac{1}{h} =$

$= \lim_{h \to 0} \frac{h^3(u^1)^2u^2}{h(h^4(u^1)^4+h^2(u^2)^2)} = \lim_{h\to 0} \frac{(u^1)^2u^2}{h^2(u^1)^4+(u^2)^2} =$

$= \frac{(u^1)^2u^2}{(u^2)^2} = \frac{(u^1)^2}{u^2}$

$\underline{u^2 = 0}$

$D_u\, g(0,0) = \lim_{h \to 0} \frac{g(hu^1, h\cdot 0)}{h} = \lim \left(\frac{\frac{(hu^1)^2 \cdot 0}{(hu^1)^4 + 0^2}}{h} \right) = 0$

$g(x,x^2) = \frac{x^2x^2}{x^4+x^4} = \frac{1}{2}$ $\quad g(x,0) = 0$ $\Rightarrow g$ is not diff.

Exercise $\boxed{\text{if } f \text{ is diff. at } a,\ f: \mathbb{R}^n \to \mathbb{R} \text{ then } D_u f(a) \text{ exists and } D_u f(a) = Df(a)\ (u)}$

NB! other way is not true

Thm ~~Df~~ Let $f: \mathbb{R}^n \to \mathbb{R}^m$

If ① $D_j f^i(x)$ exist $\quad \forall x \in U$, U open, $a \in U$ &

$j = 1, \dots, n$

$i = 1, \dots, m$ }

and ② continuous at a;

(i.e. $D_j f^i(x) \underset{x \to a}{\longrightarrow} D_j f^i(a)$)

then f is diff at a.

Proof:

I can assume $m = 1$ $\quad f: \mathbb{R}^n \to \mathbb{R}$

For simplicity, $n = 2$.

$$f(a^1 + h^1, a^2 + h^2) - f(a^1, a^2) - Df(a)(h^1, h^2)$$

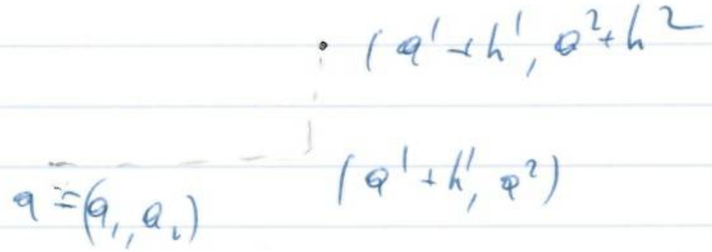
Candidate for $Df(a)$?

$$f'(a) = (D_1 f(a) \quad D_2 f(a))$$

$$Df(a)(h^1, h^2) = f'(a)\begin{pmatrix} h^1 \\ h^2 \end{pmatrix}$$

$$= D_1 f(a) h^1 + D_2 f(a) h^2$$

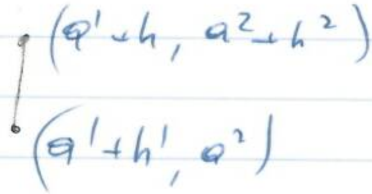
$$f(a^1 + h^1, a^2 + h^2) - f(a^1, a^2) - h^1 D_1 f(a) - h^2 D_2 f(a) =$$



$$= f(a^1 + h^1, a^2 + h^2) - f(a^1 + h^1, a^2) + \boxed{f(a^1 + h^1, a^2) - f(a^1, a^2)} -$$
$$- h^1 D_1 f(a) - h^2 D_2 f(a) = *$$

Since $D_2 f$ exists and is continuous on an open set around a.

$D_2 f$ —————— " —————— on segment



Apply MVT in second var.
There exist a ξ^2 between a^2 & a^2+h^2

$$f(a^1+h^1, a^2+h^2) - f(a^1+h^1, a^2) = D_2 f(a^1+h^1, \xi^2)\, h^2$$

MVT in first var.
there exists a ξ^1 between a^1 & a^1+h^1

$$f(a^1+h^1, a^2) - f(a^1, a^2) = D_1 f(\xi^1, a^2)\, h^1$$

$$* = D_2 f(a^1+h^1, \xi^2) h^2 + D_1 f(\xi^1, a^2) h^1 - D_1 f(a) h^1 - D_2 f(a) h^2$$

$$= h^2\left[D_2 f(a^1+h^1, \xi^2) - D_2 f(a)\right] + h^1\left[D_1 f(\xi^1, a^2) - D_1 f(a)\right]$$

Conclusion:

$$|f(a+h) - f(a) - Df(a)(h)| = |h^1[D_1 f(c_1) - D_1 f(a)] + h^2[D_2 f(c_2) - D_2 f(a)]|$$

with $c_1 = (\xi^1, a^2)$
$c_2 = (a^1+h^1, \xi^2)$

$$\frac{|f(a+h) - f(a) - Df(a)(h)|}{|h|} = \frac{|h^1[D_1 f(c_1) - D_1 f(a)] + h^2[D_2 f(c_2) - D_2 f(a)]|}{|h|}$$

triangle

$$\leq \frac{|h^1|\,|D_1 f(c_1) - D_1 f(a)| + |h^2|\,|D_2 f(c_2) - D_2 f(a)|}{|h|}$$

$$\leq \frac{|h| \, (\ldots)}{|h|} =$$

$$= |D_1 f(c_1) - D_1 f(a)| + |D_2 f(c_2) - D_2 f(a)| \to 0$$

Since ξ^2 between a^2 & $a^2 + h^2$
ξ^1 — " — a^1 & $a^1 + h^1$

as $(h^1, h^2) \to (0,0)$, $c_1 \to a$, $c_2 \to a$

We are given that $D_1 f$, $D_2 f$ are continuous at a

$$D_1 f(c_1) \to D_1 f(a), \quad D_2 f(c_2) \to D_2 f(a)$$

$\square$

Def if ① $f : \mathbb{R}^n \to \mathbb{R}^m$ has partial Derivatives $D_j f^i(x)$ $\forall x \in U$, U open, $a \in U$ and
② $D_j f^i(x)$ is cont. at a;
we say **f is continuously diff. at a**

Example: $f : \mathbb{R}^2 \to \mathbb{R}$ $\quad x : \mathbb{R} \to \mathbb{R}$
$\quad y : \mathbb{R} \to \mathbb{R}$

Define $g : \mathbb{R} \to \mathbb{R}$
$g(t) = f(x(t), y(t))$
$t \to x(t), y(t)$
$(x, y) \to f(x, y)$

$$\begin{tikzcd} & f \arrow[dl, dash] \arrow[dr, dash] & \\ x \arrow[dr] & & y \arrow[dl] \\ & t & \end{tikzcd}$$

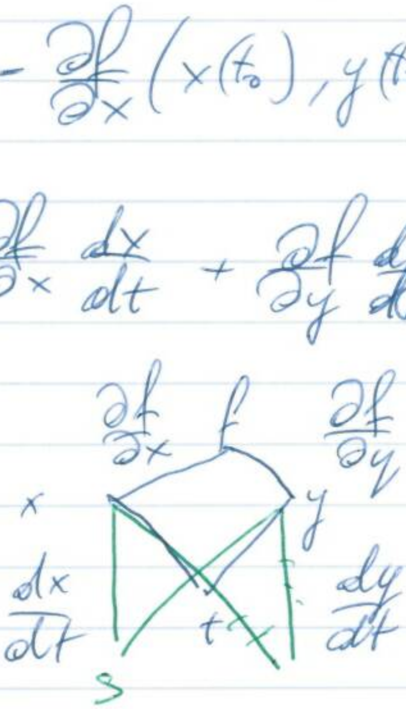
Jacobian

$$\frac{dg}{dt}(t_0) = (g'(t_0)) = f'(x(t_0), y(t_0)) \cdot \begin{pmatrix} x'(t_0) \\ y'(t_0) \end{pmatrix}$$

$$= \left(\frac{\partial}{\partial x} f(x(t_0), y(t_0)), \frac{\partial f}{\partial y}(x(t_0), y(t_0)) \right) \begin{pmatrix} x'(t_0) \\ y(t_0) \end{pmatrix}$$

$$\frac{dg}{dt}(t_0) = \frac{\partial f}{\partial x}(x(t_0), y(t_0)) \cdot \frac{dx}{dt}(t_0) + \frac{\partial f}{\partial y}(x(t_0), y(t_0)) \cdot \frac{dy}{dt} t_0$$

$$\frac{dg}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}$$



Ex. $f(x,y)$ $\quad x = 3s + 2t$ $\quad (s,t) \to (x,y) \to f(x,y)$

$\quad y = -s + 4t$ $\quad \mathbb{R}^2 \to \mathbb{R}^2 \to \mathbb{R}$

$$\frac{\partial f}{\partial s} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial s} = \frac{\partial f}{\partial x} 3 + \frac{\partial f}{\partial y}(-1)$$

$$\frac{\partial f}{\partial t} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial t} = \frac{\partial f}{\partial x} 2 + \frac{\partial f}{\partial y} 4$$

Show $14 \nabla f = (4 f_s + f_t)\vec{i} + (3 f_t - 2 f_s)\vec{j}$

19/10/11 **Inverse function Thm**

1 Dimension: Let $f:\mathbb{R}\to\mathbb{R}$ be diff. with continuous deriv f'

Assume $f'(a)\neq 0$

$\therefore$ By invariance principle $\exists$ interval $\mathcal{I}$ s.t. $a\in\mathcal{I}$ $\forall x\in\mathcal{I}$ $f'(x)\neq 0$

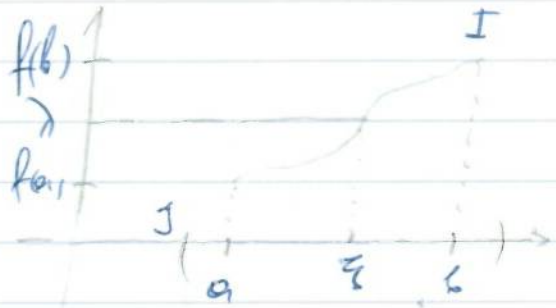
case 1 $f'(a)>0$; On $\mathcal{I}$ f is strictly increasing

$x,y\in\mathcal{I}$ $x>y \Rightarrow f(x)>f(y)$

MVT $\exists\ \xi\in(x,y)$:

$$\frac{f(x)-f(y)}{x-y}=f'(\xi)>0$$

$\mathcal{I}$ is an interval $\Rightarrow f(\mathcal{I})$ is an interval
$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\ \ \, \| $
$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\ \ \, J$



Intermediate value Thm

f is bijective from $\mathcal{I}$ to J
one-to-one & onto

$f^{-1}: J\to\mathcal{I}$

Concept. f^{-1} is diff and $(f^{-1})'(y)=\dfrac{1}{f'(f^{-1}(y))}$

Proof. $f^{-1}(y)=x$, $f^{-1}(y+h)=x+\delta$ for some δ

$y+h=f(x+\delta)$

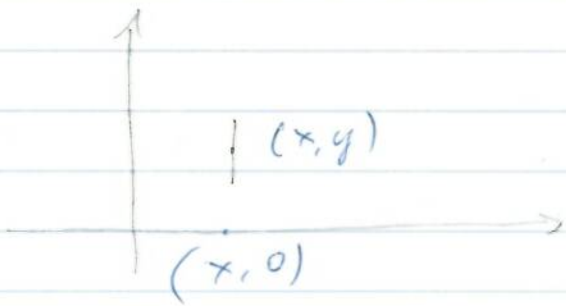
$$\lim_{h \to 0} \frac{f^{-1}(y+h) - f^{-1}(y)}{h} = \lim_{\delta \to 0} \frac{x+\delta-x}{f(x+\delta)-f(x)} =$$

$$= \lim_{\delta \to 0} \frac{\delta}{f(x+\delta)-f(x)} = \frac{1}{f'(x)}$$

{2 Dimension}

$f : \mathbb{R}^2 \to \mathbb{R}^2$ look at

$f(x, y) = (x, 0)$



$$f'(x,y) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \neq 0$$

$\det f'(a) \neq 0 \Rightarrow$ matrix is invertable
i.e. it is $Df(a)$ is invertable lin. map.

{n Dim} Now : $f : \mathbb{R}^n \to \mathbb{R}^n$, $a \in \mathbb{R}^n$, & $\det f'(a) \neq 0$

$Df(a) : \mathbb{R}^n \to \mathbb{R}^n$ linear

Assume we have found f^{-1}
$f \circ f^{-1} = Id$

chain rule $f'(f^{-1}(b)) \cdot (f^{-1})'(b) = I$

$$(f^{-1})'(b) = [f'(f^{-1}(b))]^{-1}$$

$$Df(f^{-1}(b)) \circ (Df^{-1})(b) = Id$$

$(Df^{-1})(b)$ is the inverse linear maps to $Df(f^{-1}(b))$

$$b = f(a) \iff f^{-1}(b) = a$$

not a proof

Inverse f-n Thm

Thm Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be continuously diff. on open set containing a; and

Assume $\det f'(a) \neq 0$

Then (1) $\exists V$ open set $a \in V$

(2) $\exists W$ open set $f(a) \in W$ s.t.

- $f : V \to W$ is bijective (one-to-one & onto)
- $f^{-1} : W \to V$ is continuously differ.

and $(f^{-1})'(y) = [f'(f^{-1}(y))]^{-1}$ $\forall y \in W$

these are $n \times n$ matrix.

Example. $f : \mathbb{R}^2 \to \mathbb{R}^2$

$(z, w) = f(x, y) = (xy, x^2+y^2)$

$z = xy$

$w = x^2+y^2$

$$f'(x,y) = \begin{pmatrix} y & x \\ 2x & 2y \end{pmatrix}$$

$\det f'(x,y) = 2y^2 - 2x^2 = 2(y+x)(y-x)$

$\therefore \det f'(x,y) \neq 0$ iff. $y \neq \pm x$

Solve $\begin{cases} z = xy \\ w = x^2+y^2 \end{cases}$

$\therefore y = \frac{z}{x} \Rightarrow w = x^2 + \left(\frac{z}{x}\right)^2$

$$w = x^2 + \frac{z^2}{x^2}$$

$$x^2 w = x^4 + z^2$$

$x^4 - wx^2 + z^2 = 0$

$x^2 = t \quad \therefore t^2 - wt + z^2 = 0$

$$t_{1/2} = \frac{w \pm \sqrt{w^2 - 4z^2}}{2} =$$

$$\therefore \quad x = \pm\sqrt{\frac{w \pm \sqrt{w^2-4z^2}}{2}}$$

$$\therefore \quad y = \frac{\pm z}{\sqrt{\frac{w \pm \sqrt{w^2-4z^2}}{2}}}$$

Seen should be able diff. if

the same as
$\det f' \neq 0$

$w^2 - 4z^2 \neq 0$

$w^2 - 4z^2 = (x^2+y^2)^2 - 4x^2y^2 = (x^2-y^2)^2 = (x+y)^2(x-y)^2$

$(f^{-1})'(z,w) =$ Inverse function Thm $\begin{pmatrix} y & x \\ 2x & 2y \end{pmatrix}^{-1} =$ ...

$(z,w) = f(x,y)$

$$\begin{pmatrix} \frac{\partial x}{\partial z} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial z} & \frac{\partial y}{\partial w} \end{pmatrix} = \frac{1}{2(y^2-x^2)} \begin{pmatrix} 2y & -x \\ -2x & y \end{pmatrix}$$

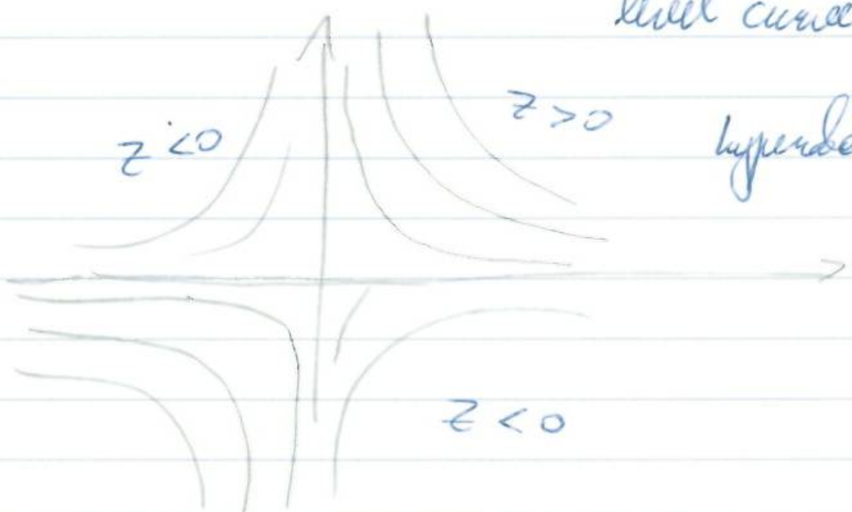
$$\left.\begin{aligned} \frac{\partial x}{\partial z} &= \frac{2y}{2(y^2-x^2)} \\ \frac{\partial x}{\partial w} &= \frac{-x}{2(y^2-x^2)} \end{aligned}\right\} \text{ etc.}$$

you still need to plug
$x = (z, w)$
$y = (z, w)$

Geometric Meaning

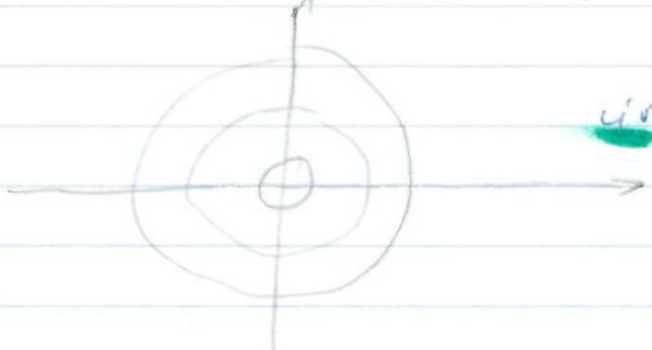
level curves of $z = x \cdot y$

hyperbola

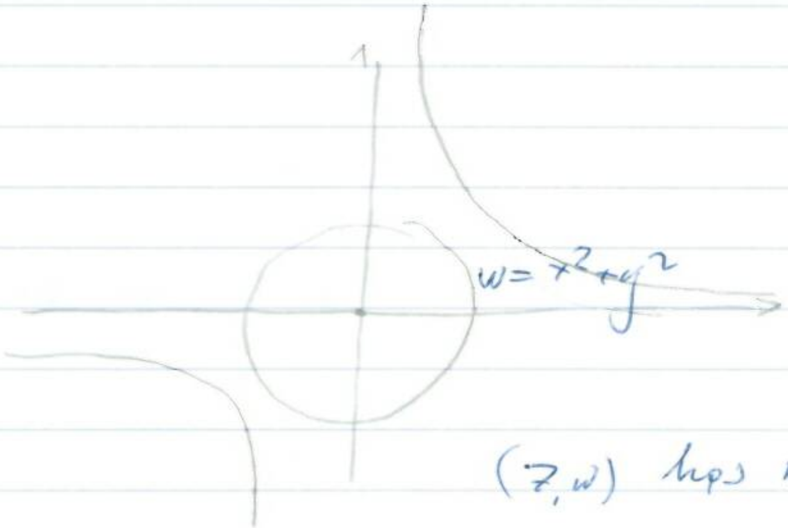


level curves of $w = x^2+y^2$

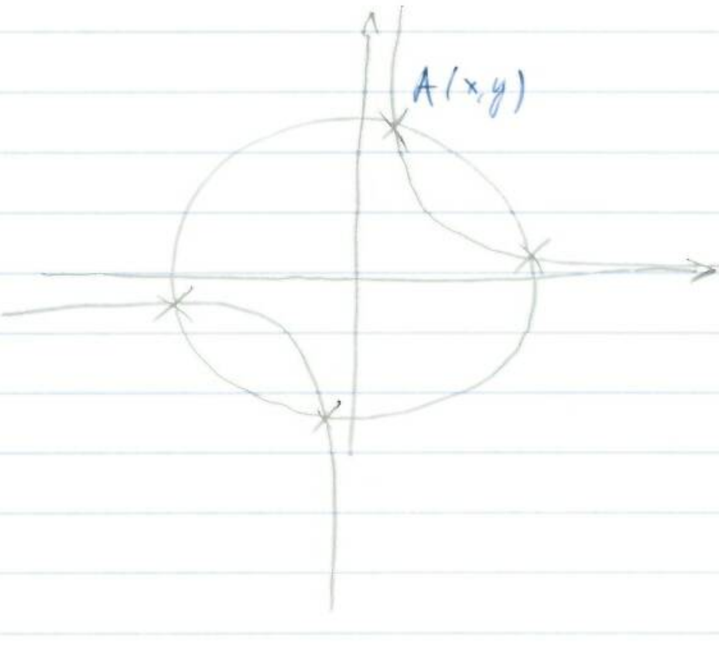
circles



$f(x,y) = (z,w)$
$f : \mathbb{R}^2 \Rightarrow \mathbb{R}^2$

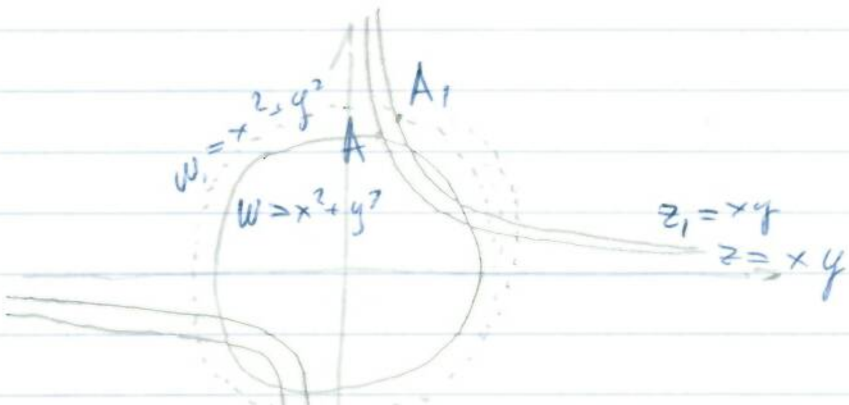


(z,w) has no preimage

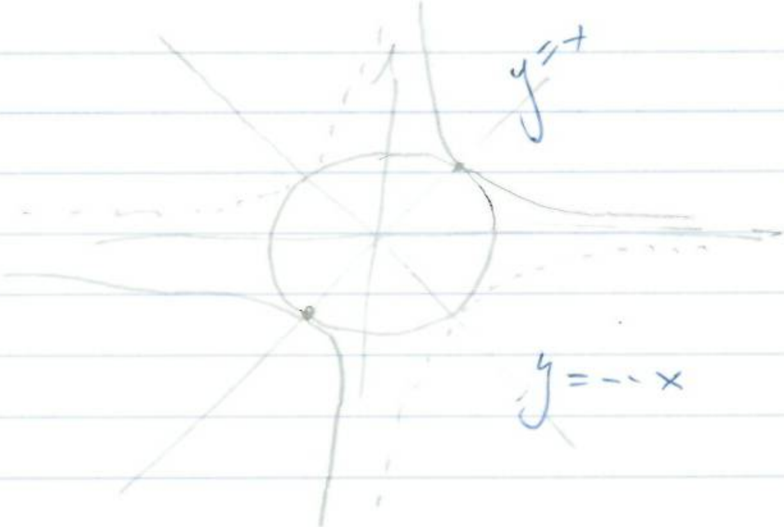


$x^2 + y^2 = w$
$z = xy$
$(z,w) = f(x,y)$
$(x,y) = f^{-1}(z,w)$

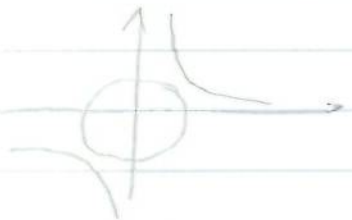
Take z_1, w_1 close to z, w

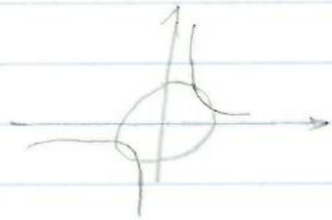


last case



$w = x^2 + y^2$
$z = x \cdot y$
if $x = y$
the circle & hyperbola
meet tangential.





24/10/11 Implicit function theorem

(1) $x^2 + y^2 = 1$

$y = g(x)$

$2x + 2y \frac{dy}{dx} = 0$

$\frac{dy}{dx} = \frac{dg}{dx} = -\frac{x}{y}, \quad y \neq 0$

(2) $y^2 + xz + z^2 - e^z - 4 = 0$

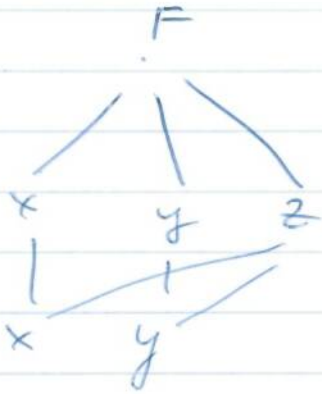
impossible to solve for z

$z = g(x, y)$

Set $F(x, y, z) = y^2 + xz + z^2 - e^z - 4$

$F(x, y, g(x, y)) = 0$

Gran' Chain rule



Diff F in x

$$\frac{\partial}{\partial x} F(x,y,g(x,y)) = 0$$

$$= \frac{\partial F}{\partial x}\overset{=1}{\frac{\partial x}{\partial x}} + \frac{\partial F}{\partial y}\overset{=0}{\frac{\partial y}{\partial x}} + \frac{\partial F}{\partial z}\frac{\partial z}{\partial x} = 0$$

$$= \frac{\partial F}{\partial x} + \frac{\partial F}{\partial z}\frac{\partial g}{\partial x} = 0$$

$$\frac{\partial g}{\partial x} = -\frac{\partial F/\partial x}{\partial F/\partial z} = -\frac{z}{x+2z-e^z}$$

$$\frac{\partial F}{\partial y} = 0 \overset{\text{chain}}{\underset{\text{rule}}{=}} \frac{\partial F}{\partial x}\frac{\partial x}{\partial y} + \frac{\partial F}{\partial y}\frac{\partial y}{\partial y} + \frac{\partial F}{\partial z}\frac{\partial z}{\partial y}$$

$$0 = \frac{\partial F}{\partial y} + \frac{\partial F}{\partial z}\frac{\partial g}{\partial y} \Rightarrow$$

$$\frac{\partial g}{\partial y} = -\frac{\partial F/\partial y}{\partial F/\partial z} = -\frac{2y}{x+2z-e^z}$$

e.g. $(0, e, 2)$ satisfies $F(x,y,z) = 0$

$$F(0,e,2) = e^2 + 0\cdot 2 + 2^2 - e^2 - 4 = 0$$

$$\left.\frac{\partial g}{\partial x}\right|_{(0,e)} = -\frac{z}{2z-e^z}$$

$$\left.\frac{\partial g}{\partial y}\right|_{(0,e)} = -\frac{2y}{x+2z-e^z} = -\frac{2\cdot e}{0+2\cdot 2-e^2}$$

$\boxed{\text{valid}}$: for $\frac{\partial F}{\partial z} \neq 0$

General situation:
m equations with m unknowns $y^1, y^2, \dots, y^m$
Depending on n parameters $x^1, x^2, \dots, x^n$.

i.e. $\begin{cases} f^1(x^1, x^2, \dots, x^n, y^1, y^2, \dots, y^m) = 0 \\ f^2(x^1, x^2, \dots, x^n, y^1, y^2, \dots, y^m) = 0 \\ \vdots \\ f^m(x^1, x^2, \dots, x^n, y^1, y^2, \dots, y^m) = 0 \end{cases}$

Try to solve for $y^1, y^2, \dots, y^m$.
$x = (x^1, \dots x^n)$, $y = (y^1, \dots, y^m)$

$f^1(x, y) = 0$
$f^2(x, y) = 0$
$\vdots$
$f^m(x, y) = 0$

$\underbrace{(0, 0, \dots, 0)}_{m \text{ zeros}}$

Define $f(x,y) = (f^1(x,y), f^2(x,y), \dots, f^m(x,y)) = 0$

Let $a \in \mathbb{R}^n$ $b \in \mathbb{R}^m$ s.t. $f(a, b) = 0$
When can we find for each $(x^1, \dots, x^n)$ near $a = (a^1, \dots, a^n)$
a <u>unique</u> $y = (y^1, \dots, y^m)$ near $(b^1, \dots, b^n) = b$ s.t.
$$f(x, y) = 0$$
$$f(x^1, \dots, x^n, y^1, \dots, y^m) = 0$$

<u>Theorem</u> Implicit f-n Theorem

<u>Let</u> $f: \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^m$ continuously diff on
an open set U containing (a, b), $a \in \mathbb{R}^n$, $b \in \mathbb{R}^m$

<u>Moreover</u> $f(a, b) = 0$

Consider ~~[illegible]~~ the matrix $\underset{m\times m}{M} = \left(D_{j+n} f^i(a,b) \right)_{j=1,\dots,m}^{i=1,\dots,m}$

Assume $\det M \neq 0$

Then there exists two open sets $A \subset \mathbb{R}^n$, $B \subset \mathbb{R}^m$, $a \in A$, $b \in B$ s.t.

$\forall x \in A$ $\exists$ unique $g(x) \in B$ s.t. $f(x, g(x)) = 0$

moreover $g : A \to B$ is diff.

proof. Increase the dim. of the target!

Define $F : \underset{\mathbb{R}^n \times \mathbb{R}^m}{U} \to \mathbb{R}^n \times \mathbb{R}^m$

$F(x^1, x^2, \dots, x^n, y^1, y^2, \dots, y^m) = (x^1, x^2, \dots, x^n, f^1(x,y), f^2(x,y), \dots, f^m(x,y))$

i.e. $F(x, y) = (x, f(x,y))$

I If F is diff? yes, because $x^1, \dots, x^n$ are diff. & $f^1(x,y), \dots, f^m(x,y)$ are dif. (as $f(x,y)$ is cont. diff.)

II $F(a,b) = (a, f(a,b)) = (a, 0)$

Jacobian

$$F'(a,b) = \begin{array}{c} \qquad\qquad\qquad\qquad \frac{\partial x^1}{\partial x^n} \; \frac{\partial x^1}{\partial y^1} \\ \begin{pmatrix} 1 & 0 & \dots & 0 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 & 0 & \dots & 0 \\ \vdots & \vdots & & \vdots & \vdots & & \vdots \\ 0 & 0 & \dots & 1 & 0 & \dots & 0 \\ \frac{\partial f^1}{\partial x^1} & \frac{\partial f^1}{\partial x^2} & \dots & \frac{\partial f^1}{\partial x^n} & \frac{\partial f^1}{\partial y^1} & \dots & \frac{\partial f^1}{\partial y^m} \\ \vdots & & & \vdots & \vdots & & \vdots \\ \frac{\partial f^m}{\partial x^1} & \dots & & \frac{\partial f^m}{\partial x^n} & \frac{\partial f^m}{\partial y^1} & \dots & \frac{\partial f^m}{\partial y^m} \end{pmatrix} \end{array}$$

i.e. $F'(a,b) = \left(\begin{array}{c|c} I_{n\times n} & 0_{n\times m} \\ \hline * & M_{m\times m} \\ \text{some matrix of size } m\times n & \end{array}\right)$

not zero? $\det F'(a,b) = \det M \neq 0$ so we can apply inverse f-n thm.

By the inv. f.n. thm $\exists$ open set, W, containing $F(a,b) = (a,0)$ and an open set V containing (a,b)

~~are~~ [which I can take to be rectangle $A\times B$, $a\in A$, $b\in B$ A open in $\mathbb{R}^n$ B open in $\mathbb{R}^m$]

$F: A\times B \to W$ is bijective

$\exists\ h = F^{-1}: W \to A\times B$ s.t. $F\circ h = Id$

h is continuously diff.

h must have the form:

$h(x,y) = (x, k(x,y))$ for some f-n $k(x,y)$

$k: W \to \mathbb{R}^m$

$W \to B$

k is cont. diff.

$F(h(x,y)) = (x,y)$

$(x, f(x, k(x,y))) = (x,y)$

$f(x, k(x,y)) = y$ set $y=0$

$f(x, k(x,0)) = 0$

the sol-n: $g(x) = k(x,0)$

01/10/11

$$f(x,y) = (xy,\ x^2+y^2) = (z, w)$$

$$f : \mathbb{R}^2 \to \mathbb{R}^2$$

$$\begin{pmatrix} \frac{\partial x}{\partial z} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial z} & \frac{\partial y}{\partial w} \end{pmatrix} = \begin{pmatrix} \frac{\partial z}{\partial x} & \frac{\partial z}{\partial y} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} \end{pmatrix}^{-1}$$

$$z = xy \Rightarrow y = \frac{z}{x}$$

$$w = x^2 + y^2 = x^2 + \frac{z^2}{x^2}$$

$$wx^2 = x^4 + z^2$$

$$x^4 - wx^2 + z^2 = 0 \qquad \Rightarrow \quad x = g(z,w)$$

? $\leftarrow 4x^3 \frac{\partial x}{\partial z} - w2x \frac{\partial x}{\partial z} + 2z = 0$ (implicit diff wrt z)

$$\frac{\partial x}{\partial z}\left(4x^3 - 2wx\right) = -2z$$

$$\frac{\partial x}{\partial z} = \frac{-\cancel{2}z}{\cancel{2}x(2x^2-w)} = *$$

valid for

$$\begin{cases} 2x^2 - w \neq 0 \\ 2x^2 - (x^2+y^2) = 0 \\ x^2 - y^2 \neq 0 \\ f(x,y) \neq 0 \end{cases}$$

$$* = \frac{-xy}{x(2x^2-w)} = \frac{-y}{2x^2-w}$$

$f(x,y) = 0 \qquad f(a,b) = 0$

$f(x, g(x)) = 0$ solving implicitly for y

$x \in \mathbb{R}^n,\ y \in \mathbb{R}^m \qquad g: \mathbb{R}^n \to \mathbb{R}^m$

} Set up ~~to~~ of Implicit function theorem

$f: \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^m$

$$i = 1, \dots, m \quad f^i(x^1, x^2, \dots, x^n, g^1(x^1, \dots, x^n), g^2(x^1, \dots, x^n), \dots, g^m(x^1, \dots, x^n)) = 0$$

How to compute $D_j g^i$?:

$D_j f^i(\dots) = 0$

$$D_1 f^i \cancel{\frac{\partial x^1}{\partial x^j}}^{\,0} + D_2 f^i \cancel{\frac{\partial x^2}{\partial x^j}}^{\,0} + \dots + D_j f^i \underbrace{\frac{\partial x^j}{\partial x^j}}_{1} + \dots + D_n f^i \cancel{\frac{\partial x^n}{\partial x^j}}^{\,0} +$$

$$+ D_{n+1} f^i \frac{\partial g^1}{\partial x^j} + D_{n+2} f^i \frac{\partial g^2}{\partial x^j} + \dots + D_{n+m} f^i \frac{\partial g^m}{\partial x^j} = 0$$

$$\therefore \quad D_{n+1} f^i \frac{\partial g^1}{\partial x^j} + D_{n+2} f^i \frac{\partial g^2}{\partial x^j} + \dots + D_{n+m} f^i \frac{\partial g^m}{\partial x^j} = -D_j f^i$$

$i = 1, \dots, m$

m unknowns $\qquad$ m eq.

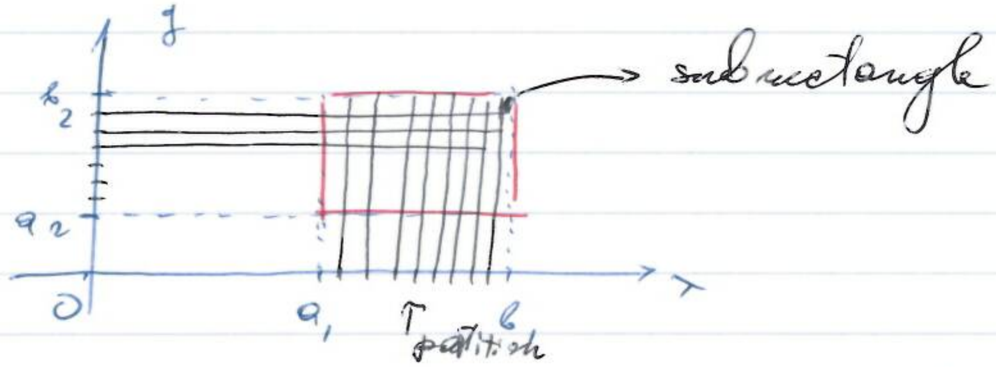
check det of coe-nts is $\neq 0$

$$\begin{vmatrix} D_{n+1} f^1 & D_{n+2} f^1 & \dots & D_{n+m} f^1 \\ D_{n+1} f^2 & D_{n+2} f^2 & \dots & D_{n+m} f^2 \\ \vdots & & & \\ D_{n+1} f^m & D_{n+2} f^m & \dots & D_{n+m} f^m \end{vmatrix} = M$$

Integration

$f: A \to \mathbb{R}$ $\quad$ A is rectangle in $\mathbb{R}^n$

$A = [a_1, b_1] \times [a_2, b_2] \times \ldots \times [a_n, b_n]$



Recall partition P of $[a, b]$ is a collection of points $t_0, t_1, t_2, \ldots, t_k$ with

$$a = t_0 < t_1 < t_2 < \ldots < t_k = b$$

A partition of the rectangle $[a_1, b_1] \times [a_2, b_2] \times \ldots \times [a_n, b_n]$ is a collection $P = (P_1, P_2, \ldots, P_n)$ with P_i a partition of $[a_i, b_i]$ $\quad i = 1, \ldots, n$.

Let f be bounded on the rectangle $[a_1, b_1] \times \ldots \times [a_n, b_n]$ / S be a subrectangle of the partition P.

Def-n. $m_S(f) = \inf\limits_{x \in S} f(x)$

$M_S(f) = \sup\limits_{x \in S} f(x)$

Lower Riemann sum: $L(f, P) = \sum\limits_S m_S(f) \cdot v(S)$

$v(S)$ is the volume of the rectangle

$S = [s_{i-1}, s_i] \times [t_{j-1}, t_j] \times \ldots \times [r_{k-1}, r_k]$

$v(S) = (s_i - s_{i-1})(t_j - t_{j-1}) \cdot \ldots (r_k - r_{k-1})$

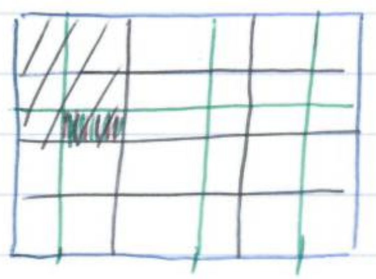
Upper Riemann Sum: $U(f,\mathcal{P}) = \sum_S M_S(f)\, v(S)$

NB! $\boxed{L(f,\mathcal{P}) \le U(f,\mathcal{P})}$

- Refinement

Def-n A refinement $\mathcal{P}'$ of the partition $\mathcal{P}$ is as follows
Given S a subrectangle of $\mathcal{P}'$
I can find a subrectangle T of $\mathcal{P}$ s.t.

$$S \subset T \quad \text{and}$$
$$T = \bigcup_{S \subset T} S \qquad S \text{ for } \mathcal{P}'$$



$\mathcal{P}$

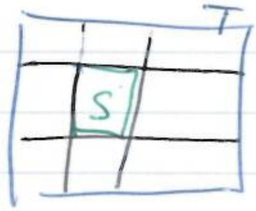
$\mathcal{P}'$

Lemma If $\mathcal{P}'$ is a refinement of $\mathcal{P}$

$$\boxed{\begin{aligned} L(f,\mathcal{P}) &\le L(f,\mathcal{P}') \\ U(f,\mathcal{P}) &\ge U(f,\mathcal{P}') \end{aligned}}$$

Proof. * Let S be subrectangle of $\mathcal{P}'$ and T be subrectangle of $\mathcal{P}$ s.t.

$$S \subset T$$



$$m_S(f) \geq m_T(f)$$

$$m_S(f)\, v(S) \geq m_T(f)\, v(S)$$

Sum up over all $S \subset T$ S for $\mathcal{P}'$

$$\sum_{S \subset T} m_S(f)\, v(S) \geq \sum_{S \subset T} m_T(f)\, v(S) = m_T(f)\, v(T)$$

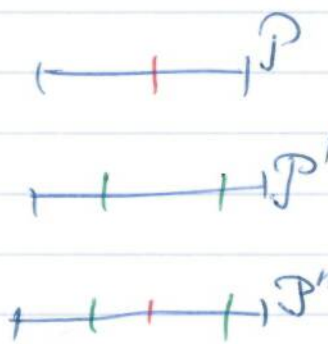
So

$$\underbrace{\sum_T \sum_{S \subset T} m_S(f)\, v(S)}_{= \sum_{S \text{ of } \mathcal{P}'} m_S(f) v(S) = L(f,\mathcal{P}')} \geq \sum_T m_T(f)\, v(T) = L(f, \mathcal{P})$$

$$\therefore \quad L(f, \mathcal{P}') \geq L(f, \mathcal{P})$$

Lemma For **any** two partitions $\mathcal{P}, \mathcal{P}^*$

$$\boxed{L(f,\mathcal{P}) \leq U(f, \mathcal{P}^*)}$$



Proof Take P'' a refinement of P and refinement of P'

$$\therefore L(f,P) \le L(f,P'') \le U(f,P'') \le U(f,P') \quad \blacksquare$$

Def-n The lower Riemann integral of f

$$\boxed{\underline{\int_A} f = \sup_P L(f,P)}$$

The upper Riemann integral of f

$$\boxed{\overline{\int_A} f = \inf_P U(f,P)}$$

where P partition of rectangle A

Def-n f is called integrable if

$$\boxed{\underline{\int_A} f = \overline{\int_A} f} \; ; \quad \text{and} \quad \boxed{\int_A f = \underline{\int_A} f = \overline{\int_A} f}$$

Theorem Riemann's integrability criterion
f is integrable over the rectangle A iff

$$\boxed{\begin{array}{c}\forall \varepsilon > 0 \ \exists \text{ partition } P \text{ of } A \text{ s.t.} \\ U(f,P) - L(f,P) < \varepsilon\end{array}}$$

Proof: $\Leftarrow$ $\underline{\int_A} f \overset{?}{=} \overline{\int_A} f$

$$\sup_{P} L(f,P) \overset{?}{=} \inf_{P} U(f,P)$$

from given: $\inf_{P} \left(U(f,P) - L(f,P) \right) = 0$

$$\inf_{P} U(f,P) - \sup_{P} L(f,P) = 0$$

$$\overline{\int_A} f - \underline{\int_A} f = 0 \implies f \text{ is Riemann integrable. } \blacksquare$$

$\Rightarrow$

Assume $\underline{\int_A} f = \overline{\int_A} f$

Fix $\varepsilon > 0$

Since $\underline{\int_A} f = \sup L(f,P) \implies \exists P$ s.t.

$$\underline{\int_A} f - \frac{\varepsilon}{2} < L(f,P)$$

Since $\overline{\int_A} f = \inf_{P} \{ U(f,P),$

some partition

$$\overline{\int_A} f + \frac{\varepsilon}{2} > U(f,P')$$

$$\therefore\ U(f, P'') - L(f, P'') < \left(\overline{\int_A} f + \frac{\varepsilon}{2}\right) - \left(\underline{\int_A} f - \frac{\varepsilon}{2}\right)$$

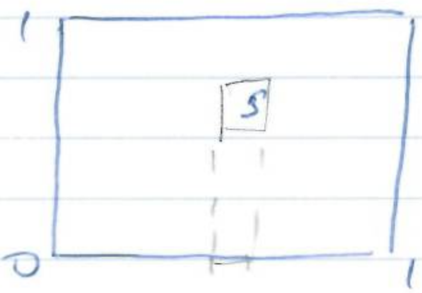
$$= \varepsilon \qquad \blacksquare$$

Example of non integrable f-ns

$f : \mathbb{R}^2 \to \mathbb{R}$

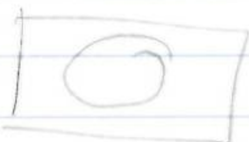
$$f(x,y) = \begin{cases} 1 & \text{if } x \in \mathbb{Q} \\ 0 & x \notin \mathbb{Q} \end{cases}$$

$A = [0,1] \times [0,1]$



$m_S(f) = 0 \quad M_S(f) = 1$

$L(f, P) = 0 \quad U(f, P) = 1$



Def-n if $C \subset \mathbb{R}^n$ define the characteristic f-n of C be

$$\chi_C(x) = \begin{cases} 1 & , x \in C \\ 0 & , x \in C \end{cases}$$

If f is bounded on $\overline{C}$ and C is contained in rectangle A we

define $\boxed{\int_C f = \int_A f \chi_C}$

Q How to compute integrals?
Use Fubini's thm.

$f : [a,b] \times [c,d] \to \mathbb{R}$

fix x, ~~a~~

consider $g_x : [c,d] \to \mathbb{R}$

$$g_x(y) = f(x,y)$$

$$I(x) = \int_c^d g_x \, dy = \int_c^d f(x,y)\, dy$$

$$\int_a^b I(x)\, dx = \int_a^b \left(\int_c^d f(x,y)\, dy \right) dx$$

fix y

define $h_y(x) = f(x,y)$

$$h_y : [a,b] \to \mathbb{R}$$

$$\int_a^b h_y(x)\, dx = \int_a^b f(x,y)\, dx = J(y)$$

$$\int_c^d J(y)\, dy = \int_c^d \left(\int_a^b f(x,y)\, dx \right) dy$$

but not always

$$= \int_{[a,b]\times[c,d]} f$$

Fubini's Thm.

Thm Let A be a rectangle in $\mathbb{R}^n$
B — " — in $\mathbb{R}^m$

$f: A\times B \to \mathbb{R}$ be integrable over the rectangle $A\times B$

$\mathcal{L}(x) = \underline{\int}_B f(x,y)\,dy$
$U(x) = \overline{\int}_B f(x,y)\,dy$
} Defined for $\forall x \in A$ as $\underline{\int}$ & $\overline{\int}$ always exist

Then $\int_A \mathcal{L}(x) = \int_A U(x) = \int_{A\times B} f$ $\cancel{= \int_A \left(\overline{\int}_B f(x,y)\,d\right)}$

$$\int_A \left(\underline{\int}_B f(x,y)\,dy\right) dx = \int_A \overline{\int}_B f(x,y)\,dy\,dx$$

Remarks 1) if $\forall x\in A$ $\int_B f(x,y)\,dy$ exists i.e. $\mathcal{L}(x) = U(x)$
then Fubini reads as $\int_{A\times B} f = \int_A \left(\int_B f\right) =$
$= \int_A\left(\int_B f(x,y)\,dy\right)dx$

2) Similarly * define $\mathcal{L}(y) = \underline{\int}_A f(x,y)\,dx$, $\mathcal{U}(y) = \overline{\int}_A f(x,y)\,dx$

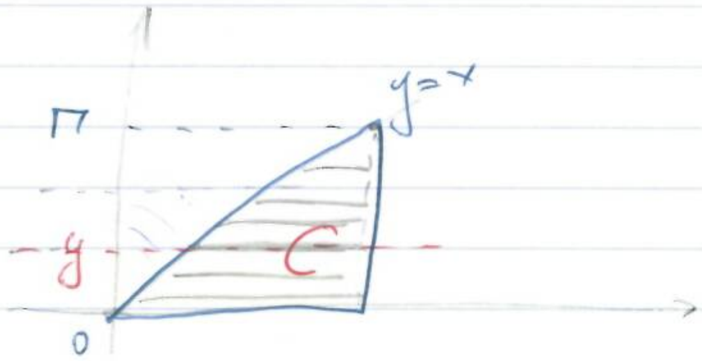
Fubini's: $\cancel{\int_{A\times B}}$ $\mathcal{L}(y)$, $\mathcal{U}(y)$ are integrable over B

and $\int_{A\times B} f = \int_B \mathcal{L}(y)\,dy = \int_B\left(\underline{\int}_A f(x,y)\,dx\right)dy =$
$= \int_B \mathcal{U}(y)\,dy = \int_B\left(\overline{\int}_A f(x,y)\,dx\right)dy$

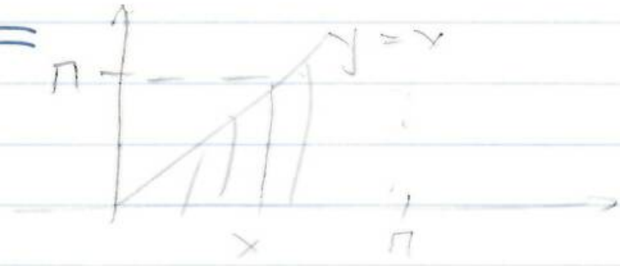
* If $\forall y \in B$ $f(x,y)$ is integrable i.e. $1(y)=1(x)$

then $\int_{A\times B} f = \int_B \left(\int_A f(x,y)\, dx \right) dy$

Example $\int_0^\pi \left(\int_{y=x}^{\pi=x} \frac{\sin x}{x}\, dx \right) dy \overset{\text{Fubini}}{=} \int_0^\pi \left(\int_0^\pi \frac{\sin x}{x} \chi_C(x,y)\, dx \right) dy =$



$\overset{\text{Fubini's}}{=} \int_{[0,\pi]\times[0,\pi]} \frac{\sin x}{x} \chi_C(x,y) \overset{\text{Fubini}}{=} \int_0^\pi \left(\int_0^x \frac{\sin x}{x}\, dy \right) dx =$

$=$ 

$= \int_0^\pi \frac{\sin x}{x} \cdot (x-0)\, dx = \int_0^\pi \sin x\, dx =$

$= -\cos x \Big|_0^\pi = -\cos \pi + \cos 0 = 1+1=2$

Recall Fubini's thm

Let A be a rectangle in $\mathbb{R}^n$
B — " — in $\mathbb{R}^m$

$f : A \times B \to \mathbb{R}$ be integrable

Define $g_x : B \to \mathbb{R}$ by $g_x(y) = f(x,y) \ \forall y \in B, \forall x \in A$

Let $\mathcal{L}(x) = \underline{\int}_B g_x = \underline{\int}_B f(x,y)\,dy$ } exist
$\mathcal{U}(x) = \overline{\int}_B g_x = \overline{\int}_B f(x,y)\,dy$ } $\forall x \in A$

then (1) $\mathcal{L}(x)$, $\mathcal{U}(x)$ are integrable over A and

(2) $\int_A \mathcal{L}(x)\,dx = \int_A \left(\underline{\int}_B f(x,y)\,dy \right) dx =$

$= \int_A \mathcal{U}(x)\,dx = \int_A \left(\overline{\int}_B f(x,y)\,dy \right) dx = \int_{A\times B} f$

Proof Let P_A be a partition of A,
P_B — " — of B,
S_A a subrectangle of P_A
S_B — " — P_B

then the rectangles $S_A \times S_B$ given a partition P of $A \times B$

we will prove:

$$(*) \quad L(f,P) \overset{(1)}{\leq} L(\mathcal{L},P_A) \overset{(2)}{\leq} U(\mathcal{L},P_A) \overset{(3)}{\leq} U(\mathcal{U},P_A) \leq U(f,P)$$

Since f is integrable over $A \times B$
given ~~B~~ $\varepsilon > 0$
Riemann's integrability criterion gives a partition P of $A \times B$ s.t.

$$U(f,P) - L(f,P) < \varepsilon.$$

then P defines P_A, P_B partitions of A & B respectively
By the inequality $*$:

$$U(\mathcal{L}, P_A) - L(\mathcal{L}, P_A) < \varepsilon$$

By Riemann's integrability criterion $\mathcal{L}$ is int-ble over A

Since $\sup L(f,P) = \inf_P U(f,P) = \int_{A \times B} f \Rightarrow$

$$\int_A \mathcal{L}(x)\,dx = \sup_{P_A} L(\mathcal{L}, P_A) = \inf_{P_A} U(\mathcal{L}, P_A) = \int_{A \times B} f$$

work similarly with

Proof (*):
(2) $L(\mathcal{L}, P_A) \leq U(\mathcal{L}, P_A)$ always true for a f-n $\mathcal{L}$, partition P_A that ~~the lower~~ the lower Riemann sum $\leq$ upper R. sum

(3) $$\left.\begin{aligned} \mathcal{L}(x) &= \underline{\int}_B f(x,y)\,dy \\ \mathcal{U}(x) &= \overline{\int}_B f(x,y)\,dy \end{aligned}\right\} \Rightarrow \mathcal{L}(x) \leq \mathcal{U}(x) \Rightarrow U(\mathcal{L}, P_A) \leq U(\mathcal{U}, P_A)$$

larger f-n has larger R sum

(4) similarly proceed as (1):

(1) Let $S_A \times S_B$ of P

$x \in S_A$

$$m_{(S_A \times S_B)} f \le m_{\{x\} \times S_B} f$$

! inf over a smaller set is larger

Multiply with $v(S_B)$ & sum over S_B's of P_B:

$$\sum_{S_B} m_{S_A \times S_B}(f)\, v(S_B) \le \sum_{S_B} m_{\{x\} \times S_B}(f)\, V(S_B) =$$

$$= L(g_x, P_B) \le \int_B g_x = \mathcal{L}(x)$$

Take inf over $x \in S_A$

$$\sum_{S_B} m_{S_A \times S_B}(f)\, v(S_B) \le \inf_{x \in S_A} \mathcal{L}(x) = m_{S_A}(\mathcal{L})$$

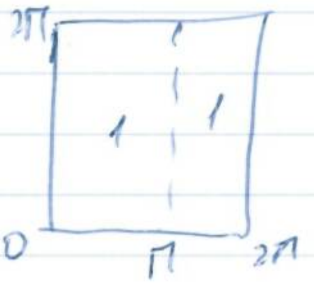
mult. by $v(S_A)$ sum over S_A

$$\Rightarrow \underbrace{\sum_{S_A} \sum_{S_B} m_{S_A \times S_B}(f)\, \underbrace{v(S_B)\, v(S_A)}_{v(S_A \times S_B)}}_{L(f, P)} \le \sum_{S_A} m_{S_A}(\mathcal{L})\, v(S_A) = L(\mathcal{L}, P_A)$$

Q.E.D. ∎

Warning ~~Ex~~ Let $f(x,y) = \begin{cases} 1 & \text{if } x \neq \pi \\ 0 & \text{if } x = \pi \ \& \ y \notin \mathbb{Q} \\ 1 & \text{if } x = \pi \ \& \ y \in \mathbb{Q} \end{cases}$

$A \times B = [0, 2\pi] \times [0, 2\pi]$



$$\int_{A\times B} f = \int_{A\times B} 1 = (2\pi)^2$$

<u>Notice</u> $g_\pi(y) = \begin{cases} 1 & y \in \mathbb{Q} \\ 0 & y \notin \mathbb{Q} \end{cases}$ is not integrability

thus $\mathcal{L}_M$, $\mathcal{U}_\pi$ $\to$ need to use them

<u>Thm.</u> <u>Let</u> 1) $A \subseteq \mathbb{R}^n$ be open
2) $g : A \to \mathbb{R}^n$ be ~~bijective~~ injective cont. diff. with $\det g'(x) \neq 0 \quad \forall x \in A$
3) $f : g(A) \to \mathbb{R}$ be integrable

<u>then</u> we have the change of vars form.

$$A \xrightarrow{g} g(A) \xrightarrow{f} \mathbb{R}$$

$$\boxed{\int_{g(A)} (f) = \int_A f \circ g \, |\det g'(x)| \, dx}$$

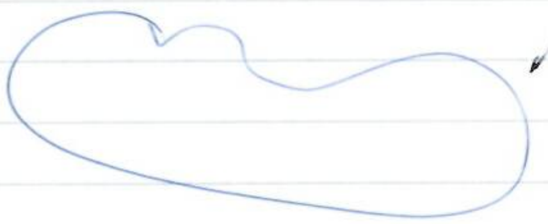
Manifolds

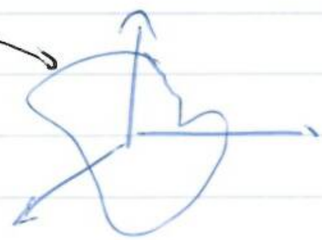
M k-dim manifold in $\mathbb{R}^n$

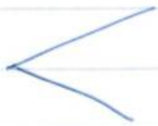
1-dim $\mathbb{R}^2$ 1-dim in $\mathbb{R}^3$

mf := manifold

curve



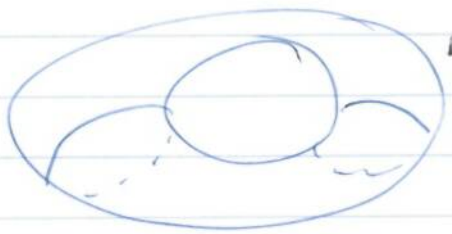




not 1-dim manifold in $\mathbb{R}^3$

2-dim surface in $\mathbb{R}^3$

surface



Torus

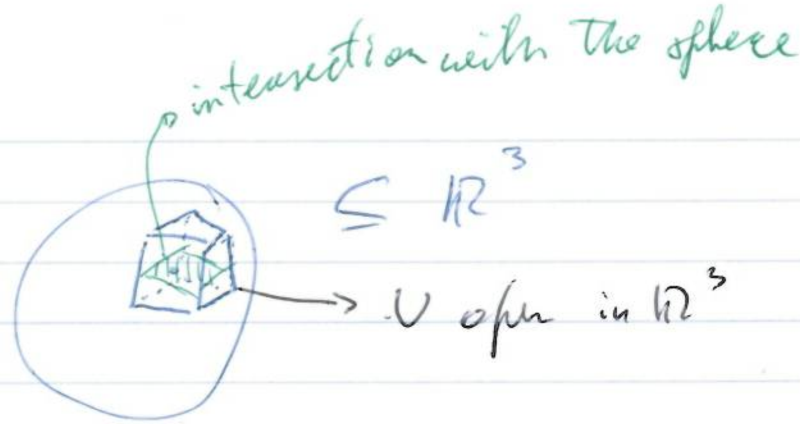
$S^2 \subseteq \mathbb{R}^3$

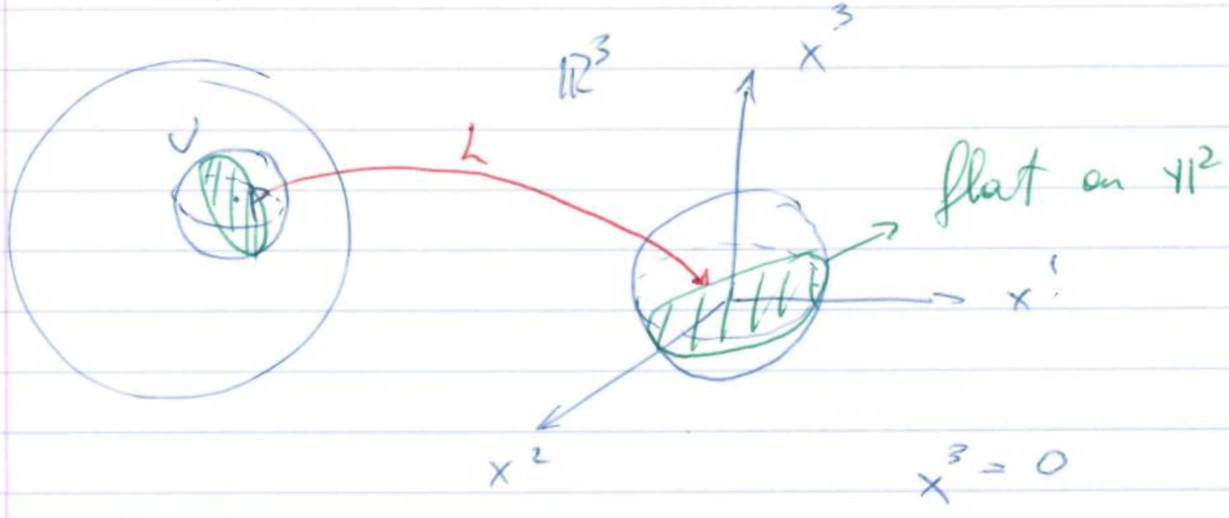
f-n.

hook line

ellipsoids

open





Def-n Let U, V be open sets in $\mathbb{R}^n$
$h : U \to V$ be a bijection which is ∞-differentiable
(all partials of all orders exist & cont.)
$h^{-1} : V \to U$ is also ∞-diff. (all part. ...)

then h is a diffeomorphism from U to V

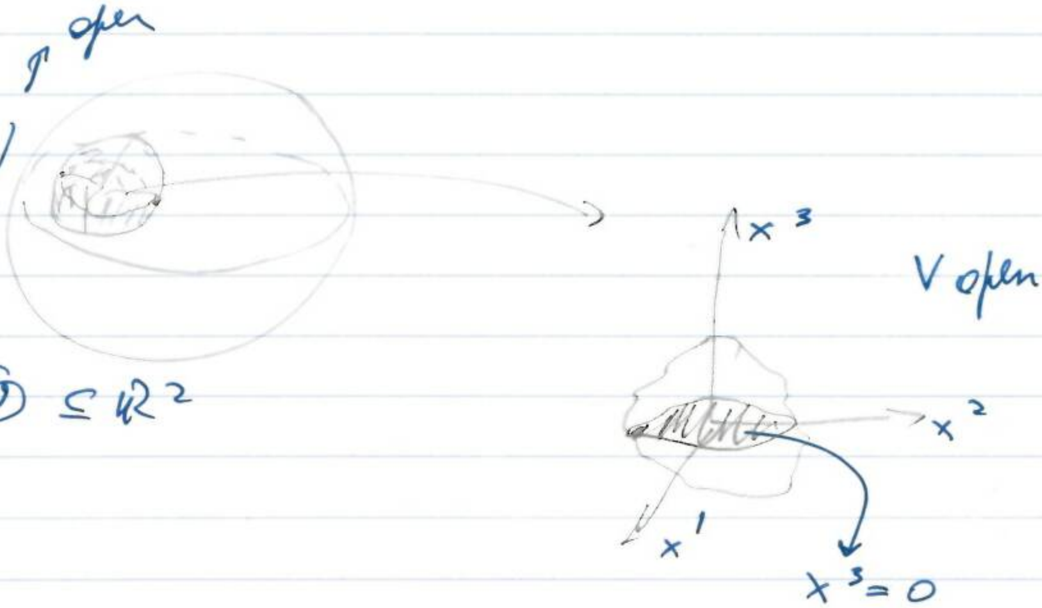
15/4/4
Recall A f-n $f : \mathbb{R}^n \to \mathbb{R}^m$ is called a C^∞ f-n
If all partial d-ves of all order of all components exist and continuous

$$\frac{\partial^k f^i}{\partial x_{i_1} \partial x_{i_2} \dots \partial x_{i_k}}$$

Def Let U, V be open sets in $\mathbb{R}^n$
$h : U \to V$ is C^∞ f-n, bijective, and
$h^{-1} : V \to U$ is also C^∞

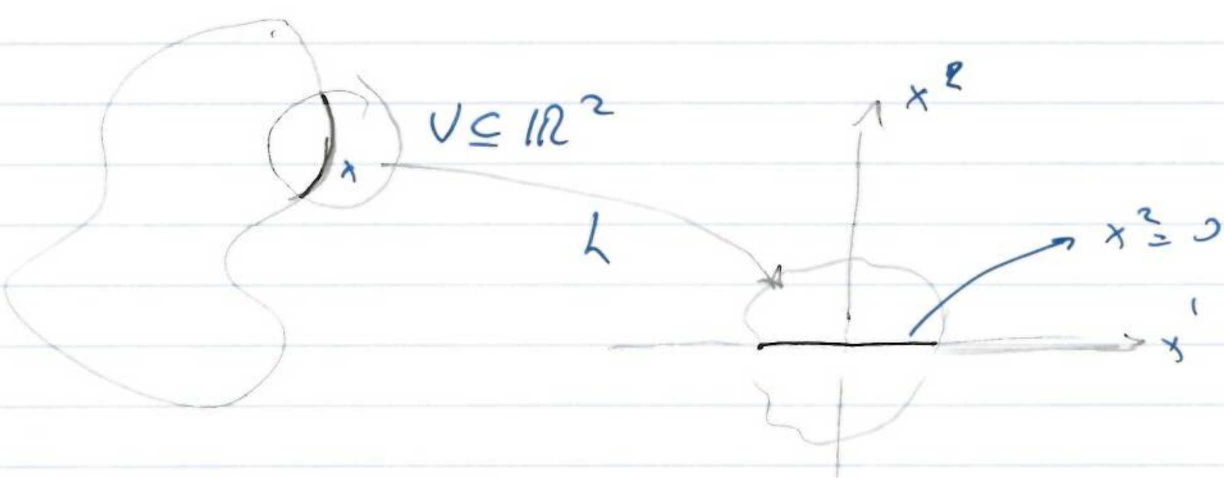
then ~~[illegible]~~ $h : U \to V$ is diffeomorphism from U to V

S^2 2 dim manifold in $\mathbb{R}^3$

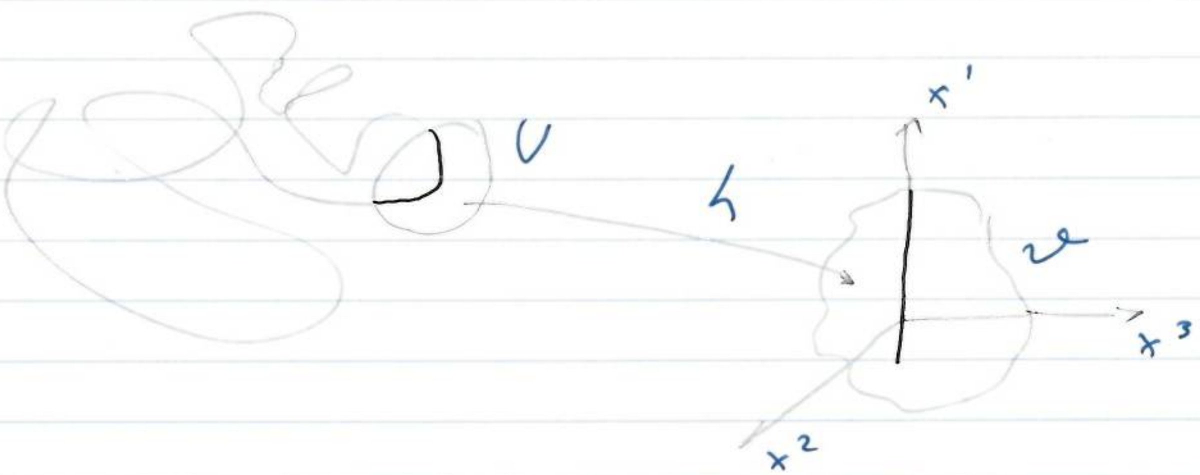


$\boxed{W}$ W $\subseteq \mathbb{R}^2$
W open

1 dim in $\mathbb{R}^2$



1-dim manifold in $\mathbb{R}^3$



$x^2 = 0$
$x^3 = 0$

Def A set M is a k-dim manifold in $\mathbb{R}^n$ if the following condition (M) for every $x \in M$

> (M) there exists 1) two open sets U, V of $\mathbb{R}^n$, $x \in U$,
> 2) a Diffeomorphism $h : U \to V$ s.t.
> $h(U \cap M) = \{ y \in V \text{ s.t. } y^{k+1} = y^{k+2} = \dots = y^n = 0 \}$

Thm 1

> Let $g : \mathbb{R}^n \to \mathbb{R}^p$, $n \geq p$ be C^∞ f-n
> Set $M = g^{-1}(0) = \{ x \in \mathbb{R}^n, \ g(x) = 0 \in \mathbb{R}^p \}$
>
> If $\forall x \in M$ the rank of $g'(x)$ is p
>
> then $M = g^{-1}(0)$ is an $(n-p)$-dim manifold in $\mathbb{R}^n$

Remainder from Alg. 1&2: $T : \mathbb{R}^n \to \mathbb{R}^p$ lin Transf.

I $\operatorname{rank}(T) = \dim T(\mathbb{R}^n) \leq p$

II $\operatorname{rank}[T]$ = max numb. of l. ind. of rows or columns

Recall: minor A_{ij} of an element a_{ij} of an n^{th}-order determinant is the determinant of order $(n-1)$ formed be deleting the i^{th} row and the j^{th} column of the original determinant

III $\text{rank}(T) \leq \min(n, p)$

$$[T] = \begin{pmatrix} a_{11} & \dots & a_{1n} \\ \vdots & & \\ a_{p1} & \dots & a_{pn} \end{pmatrix} \qquad [T] \in M_{p \times n}$$

IV Det of minors: if r is the max possible size of an $r \times r$ minor with non-zero determinant
Then $\text{rank}(T) = r$

Application 1) $S^2 = \{(x, y, z) \in \mathbb{R}^3 \; x^2 + y^2 + z^2 = 1\}$ 2-dim manifold

$$g : \mathbb{R}^3 \to \mathbb{R}$$

$$g(x, y, z) = x^2 + y^2 + z^2 - 1$$

$$S^2 = g^{-1}(0)$$

$$\underset{1 \times 3}{g^{*\prime}(x, y, z)} = \left(\frac{\partial g}{\partial x}, \frac{\partial g}{\partial y}, \frac{\partial g}{\partial z} \right) = (2x, 2y, 2z)$$

rank can be 0 or 1

$$\underset{n-p}{2} = \underset{n}{3} - \underset{p}{1}$$

Aim to show $\text{rank}\, g^* = 1$ on $M = g^{-1}(0)$

$\text{rank}\, g' = 0 \iff 2x = 2y = 2z = 0$
$\iff x = y = z = 0$

but $(0,0,0) \notin g^{-1}(0)$ because $g(0,0,0) = 0^2 + 0^2 + 0^2 = -1$

the sphere $S^n = \{(x^1, x^2, \dots, x^n, x^{n+1}) \; (x^1)^2 + (x^2)^2 + \dots + (x^{n+1})^2 = 1\}$
is a n-dim manifold in $\mathbb{R}^{n+1}$

$g : \mathbb{R}^{n+1} \to \mathbb{R} \qquad g(x^1, x^2, \dots, x^{n+1}) = (x^{n+1})^2 + (x^n)^2 + \dots + (x^1)^2 - 1$

$S^n = g^{-1}(0)$

$$g'(x^1, x^2, \dots, x^{n+1}) = \left(\frac{\partial g}{\partial x^1}, \frac{\partial g}{\partial x^2}, \dots, \frac{\partial g}{\partial x^{n+1}}\right) =$$
$$= (2x^1, 2x^2, \dots, 2x^{n+1}) =$$
$1 \times n+1$ matrix

rank $g' = 0$ iff $2x^1 = 2x^2 = \dots = 2x^{n+1} = 0$
$\Leftrightarrow x^1 = x^2 = \dots = x^{n+1} = 0$ but $(0, 0, \dots 0) \notin S^n$

Example Hyperbolic space $\mathbb{H}^n = \{(x^1, \dots, x^n, x^{n+1}) \in \mathbb{R}^{n+1},$
n-dim
$x^1 > 0$
$(x^1)^2 - [(x^2)^2 + (x^3)^2 + \dots + (x^{n+1})^2] = 1\}$

$$g(x^1, \dots, x^{n+1}) = (x^1)^2 - [(x^2)^2 + (x^3)^2 + \dots + (x^{n+1})^2] - 1$$

$\mathbb{H}^n = g^{-1}(0) \qquad g : A \to \mathbb{R}$
$A = \{x \in \mathbb{R}^{n+1} \; x^1 > 0\}$

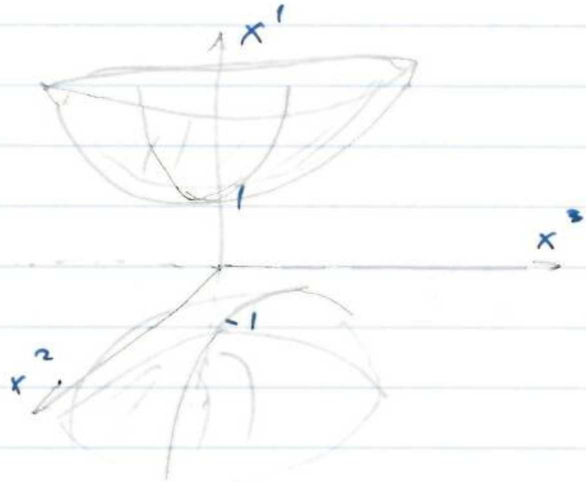
$$g'(x^1, \dots, x^{n+1}) = (2x^1, -2x^2, -2x^3, \dots, -2x^{n+1})$$

ranking $g' = 0$ iff $x^1 = x^2 = \dots = x^{n+1} = 0$

but $(0, 0, \dots, 0) \notin \mathbb{H}^n$

rank $g^{-1} = 1$ on $g^{-1}(0)$ so by thm
$g^{-1}(0)$ is an $(n+1)-1$ dim manifold in $\mathbb{R}^{n+1}$

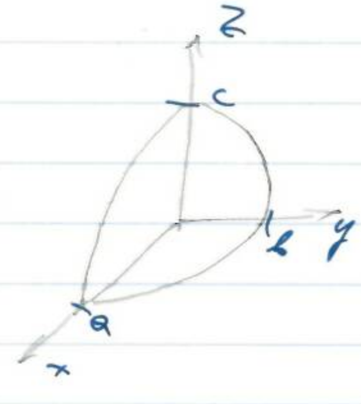
e.g. $\underline{n=2}$ $\mathbb{R}^3$ $(x^1)^2-(x^2)^2-(x^3)^2=1$ $x^1>0$



Fix $x^2=0$
$(x^1)^2-(x^3)^2=1$

Fix $x^3=0$
$(x^1)^2-(x^2)^2=1$

Ex. 3 Ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ $a,b,c>0$

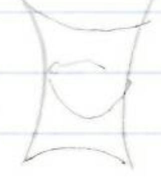


$g(x,y,z) = \frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}-1$

$g'(x,y,z) = \left(\frac{2x}{a^2}, \frac{2y}{b^2}, \frac{2z}{c^2}\right)$

$= 0$ iff $x=y=z=0$ but $(0,0,0)$ doesn't belong to the ellipsoid

Ex 4



$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1$

Ex 5 The graph of a diff-ble f-n $f: U \to \mathbb{R}$, $U \subseteq \mathbb{R}^2$

$M = \{(x,y,z) \in \mathbb{R}^3,\ z = f(x,y)\}$ (Monge patch)
is 2-dim manifold in $\mathbb{R}^3$

$$g(x,y,z) = f(x,y) - z$$

$$g'(x,y,z) = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, -1\right) \neq 0$$

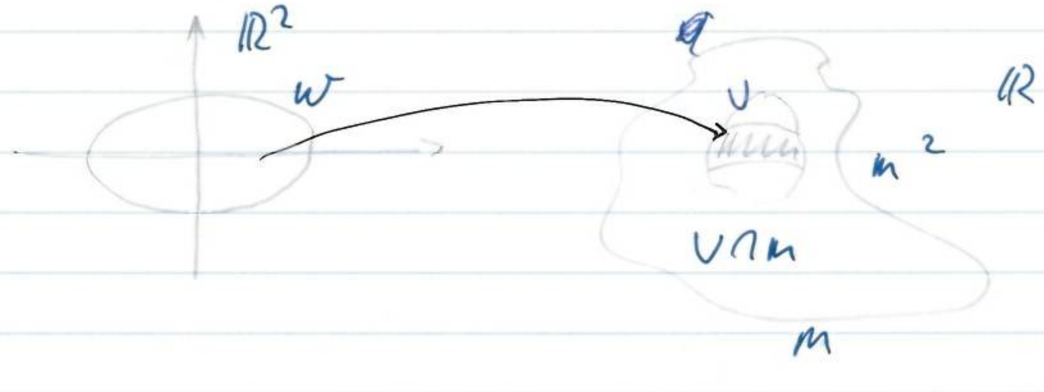
rank $g' = 1$

Thm 2 M is a k-dim manifold in $\mathbb{R}^n$ iff for every $x \in M$ the following conditions holds:

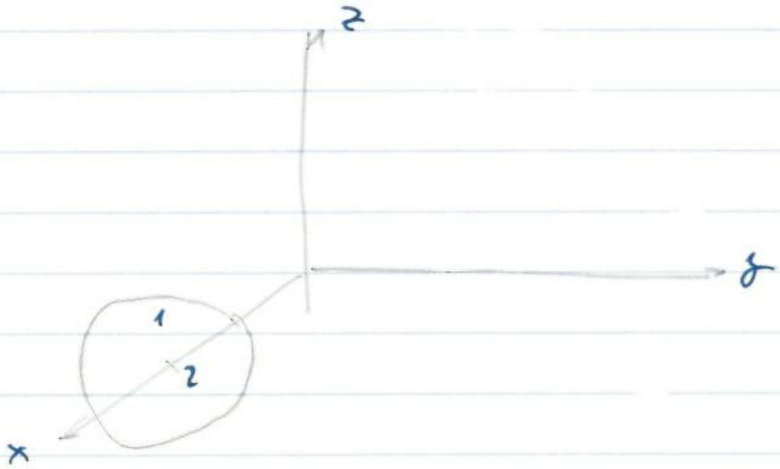
chart (C) there exists an open set $W \subseteq \mathbb{R}^k$, and open set $U \subseteq \mathbb{R}^n$,
$x \in U$
& $f: W \to \mathbb{R}^n$ is d-ble and injective s.t.

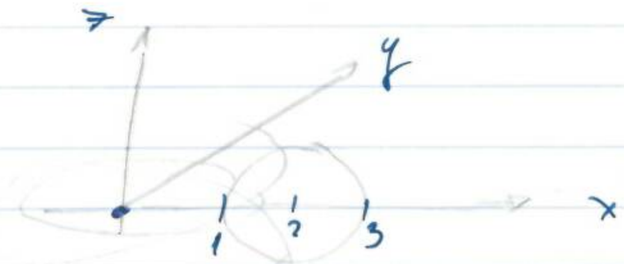
(i) $f(W) = U \cap M$

(ii) rank $f'(y) = k \quad \forall y \in W$



Ex 2 dim torus





$$(x-2)^2 + z^2 = 1$$

$$(r-2)^2 + z^2 = 1 \quad \text{cylindrical coord.}$$

$$z = \sin\phi$$

$$r - 2 = \cos\phi$$

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$$

$$\therefore \big((2+\cos\phi)\cos\theta,\ (2+\cos\phi)\sin\theta,\ \sin\phi\big) = f(\theta,\phi)$$

$$\left.\begin{array}{l} \theta \in (-\pi,\pi) \\ \phi \in (-\pi,\pi) \end{array}\right\} \rightarrow$$ not whole torus! exclude 2 circles to make W open => use these.

$W = (-\pi,\pi) \times (-\pi,\pi)$ open

1-1 for Thm 2

take $U = \mathbb{R}^3$

$f(W) = U \cap M$

$f(W) = M$

$f : W \to \mathbb{R}^3$

$\quad \cap \mathbb{R}^2$

$$f'(\theta, \phi) = \begin{pmatrix} (2+\cos\phi)(-\sin\theta) & -\sin\phi\cos\theta \\ (2+\cos\phi)(\cos\theta) & -\sin\phi\sin\theta \\ 0 & \cos\phi \end{pmatrix}$$

Does it have rank 2 on $W = (-\pi, \pi) \times (-\pi, \pi)$

2x2 minor $\begin{vmatrix} (2+\cos\phi)(-\sin\theta) & -\sin\phi\cos\theta \\ (2+\cos\phi)(\cos\theta) & -\sin\phi\sin\theta \end{vmatrix} =$

$= (2+\cos\phi)(\sin\phi) \begin{vmatrix} -\sin\theta & \cos\theta \\ \cos\theta & \sin\theta \end{vmatrix} = +(2+\cos\phi)\sin\phi$

$\|(2+\cos\phi) \neq 0\|$ $\qquad \neq 0$ iff $\sin\phi \neq 0$

$\iff \phi \neq 0$

rank $f' = 2$ whenever $\phi \neq 0$

when $\phi = 0$ $f'(\theta, \phi) = \begin{pmatrix} -3\sin\theta & 0 \\ 3\cos\theta & 0 \\ 0 & 1 \end{pmatrix}$

If $\theta = 0$ use $\begin{vmatrix} 3\cos\theta & 0 \\ 0 & 1 \end{vmatrix} = \begin{vmatrix} 3 & 0 \\ 0 & 1 \end{vmatrix} = 3$

if $\theta \neq 0$ use $\begin{vmatrix} -3\sin\theta & 0 \\ 0 & 1 \end{vmatrix} = -3\sin\theta \neq 0$

on $\theta \in (-\pi, \pi)$

$\theta \neq 0$

Any nice surface of revolution is a 2-dim m-d in $\mathbb{R}^3$

$$\gamma(t) = (r(t), z(t)), \quad t \in (a, b)$$

γ doesn't have self intersections $\quad r(t) > 0$

γ is differentiable

$$\gamma'(t) = (r'(t), z'(t)) \neq 0 \quad \forall t \in (a, b)$$

then when we rotate it around z-axis we get the surface

$$f(t, \theta) = (r(t)\cos\theta, r(t)\sin\theta, z(t)) \qquad t \in (a, b), \ \theta \in (-\pi, \pi)$$

$$f'(t, \theta) = \begin{bmatrix} r'\cos\theta & -r\sin\theta \\ r'\sin\theta & r\cos\theta \\ z' & 0 \end{bmatrix}$$

$$\begin{vmatrix} r'\cos\theta & -r\sin\theta \\ r'\sin\theta & r\cos\theta \end{vmatrix} = r \cdot r' \qquad \text{since } r > 0$$

$\cdot r' \neq 0 \rightarrow$ rank 2

If $r' = 0$, $z' \neq 0$

$\rightarrow$ not smooth at this point.

17/11/11

thm Let A be open in $\mathbb{R}^n$
$g : A \to \mathbb{R}^p$ be differentiable with
$g'(a)$ has rank p for $\forall x \in g^{-1}(0) = M$

then $g^{-1}(0)$ is an ~~n~~ $n-p$-dim manifold in $\mathbb{R}^n$

Proof Fix $x \in g^{-1}(0)$
According to previous thm (H/03)
$\exists$ open set V in $\mathbb{R}^n$ &
H a diffeomorphism $H : V \to U \ni x$ s.t.

$$\left[\underbrace{g(H(y^1, y^2, \dots, y^n)) = (y^{n-p+1}, y^{n-p+2}, \dots, y^n)}\right]$$

You need U & h diffeomorphism with
(*) $h(U \cap M) = \{y \in V : y^{n-p+1} = y^{n-p+2} = \dots = y^n = 0\}$
$h : U \to V$

$(f_1 \circ f_2)^{-1} = f_2^{-1} \circ f_1^{-1}$

Define $h = H^{-1} : U \to V$
Need to show (*)
Let $y \in U \cap M \Rightarrow y \in M$, $y \in g^{-1}(0)$, $g(y) = 0$
$h(g^{-1}(0)) = H^{-1}(g^{-1}(0)) = (H^{-1} \circ g^{-1})(0) =$
$h(y) \in \{y \in V, y^{n-p+1} = \dots = y^n = 0\} = (g \circ H)^{-1}(0) =$
$= \{y \in V, y^{n-p+1} = y^{n-p+2} = \dots = y^n = 0\}$

Let $y \in \{y \in V, y^{n-p+1} = \dots = y^n = 0\}$
Need to find $z \in U \cap M$ with $h(z) = y$
Define $z = H(y)$
Need to prove $z \in M$ i.e. $g(z) = 0$
$g(z) = g(H(y)) \underset{\text{by def}}{=} (y^{n-p+1}, y^{n-p+2}, \dots, y^n) = \overbrace{(0, 0, \dots, 0)}^{p \text{ zeroes}}$

$g(z) = 0$ Done.

11/03 Implicit f-n Thm

$f : \mathbb{R}^n \to \mathbb{R}^m \to \mathbb{R}^m$

$f(a, b) = 0 \qquad M = \left(D_j f^i(a, b)\right)_{\substack{j > n \\ 1 \le j \le m}}$

$F(x, y) = (x, f(x, y))$

h inverse $\quad F(h(x, y)) = (x, y) = F(x, k(x, y))$
$= (x, f(k(x, y)))$

$y = f(k(x, y))$

Thm Let $g : \mathbb{R}^n \to \mathbb{R}^p \qquad n \ge p$
cont-sly d-lle on an open set, containing a
$g(a) = 0 \qquad a \in \mathbb{R}^n$

If the rank $\left(D_j g^i(a)\right)_{p \times n}$ is exactly p at a

then there exists an open set $A \subset \mathbb{R}^n$ & open in $\mathbb{R}^n$
Diff-lle f-n $h : A \to V$ bijective
with diff inverse h^{-1}

(h is diffeomorphism $A \xrightarrow{h} V$) s.t.
$(g \circ h)(x^1, x^2, \ldots, x^n) = (x^{n-p+1}, \ldots, x^{n-2}, x^{n-1}, x^n)$

$$\begin{tikzcd} \mathbb{R}^n \arrow[r, "h"] \arrow[dr, "g \circ h"'] & \mathbb{R}^n \arrow[d, "g"] \\ & \mathbb{R}^p \end{tikzcd}$$

$g : \mathbb{R}^{n-p} \times \mathbb{R}^{p} \to \mathbb{R}^{p}$ Apply Implicit f-n thm & the remark:
$x^1, x^2, \dots x^{n-p}$ $x^{n-p+1} \dots x^n$

$f = g$ $\quad n \to p$

$k = n - p$

$x^{n-p+1} = y^1$

$x^{n-p+2} = y^2$

$\vdots$

$x^n = y^p$

gives h, k $\quad (f \circ h)(x, y) = y$

$$(g \circ h)(x^1, \dots, x^{n-p}, \dots, x^n) = (x^{n-p+1}, \dots, x^n)$$

$\operatorname{rank} \underset{p \times n}{g'(a)} = p$

Can find a minor $p \times p$ with $\det \neq 0$

$$\begin{pmatrix} \frac{\partial g^1}{\partial x^1} & \dots & \frac{\partial g^1}{\partial x^n} \\ \vdots & & \\ \frac{\partial g^p}{\partial x^1} & \dots & \frac{\partial g^p}{\partial x^n} \end{pmatrix}$$

if the last p colums are lin indep.
then the $\det \left(D_j \, g^i(a) \right)_{\substack{j > n-p \\ 1 \leq i \leq p}} \neq 0$

$\therefore$ you can apply Implicit f-n thm.

If this is not true, $j_1, \dots, j_p$ s.t. $\det D g^i_{j_k}(a) \neq 0$
$k = 1 \dots p$
$i = 1 \dots p$

then we relabel the variables to make
$x^{j_1} \dots x^{j_p}$ last

this is done as follows:
$$m(x^1, x^2, \dots, x^n) = (\dots, x^{j_1}, x^{j_2}, \dots, x^{j_p})$$

Consider $g \circ m$ and apply previous case
$\exists s : \mathbb{R}^n \to \mathbb{R}^n$ s.t. diffeo

$$g \circ m \circ s(x^1, \dots, x^n) = (x^{n-p+1}, x^{n-p+2}, \dots, x^n)$$

$$h = m \circ s \qquad x^{n-p+1} = y^1$$
$$\dots \quad x^n = y^n$$

Gives h, k
$$(f \circ h)(x, y) = y$$
$$(g \circ h)(x^1, \dots, x^{n-p}, \dots, x^n) = (x^{n-p+1}, \dots, x^n)$$

22/11/11

Recall thm c

$\forall x \in M$

(c) $\exists W$ open set in $\mathbb{R}^k$

$\exists f : W \to U$ injective

$\exists U$ open in $\mathbb{R}^n$

(i) $f(W) = U \cap M$

(ii) $f(y)$ has rank k $\forall y \in W$

(iii) $f^{-1} : U \cap M \to W$ is continuous

the updated version / other versions before are wrong (including 11/04)

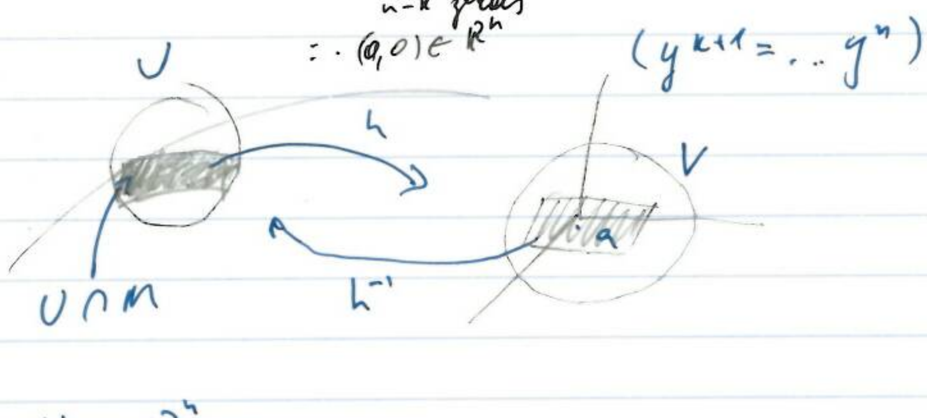
Thm $M \Rightarrow c$

$M \Rightarrow U$, V open both in $\mathbb{R}^n$, $x \in U$ $h : U \to V$ diffeom.

$h(U \cap M) = \{ y \in V,\ y^{k+1} = y^{k+2} = \dots = y^n = 0 \}$

$W = \{ a \in \mathbb{R}^k : (a, \underset{\substack{\uparrow \\ n-k \text{ zeroes}}}{0}) \in h(U \cap M) \}$

$\therefore (a, 0) \in \mathbb{R}^n$



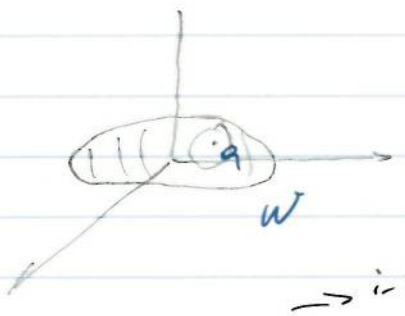
$f : W \to \mathbb{R}^n$

$f(a) = h^{-1}(a, 0)$

(iii) f^{-1} is continuous as it is "essentially" h which is continuous.

$f^{-1} : h | U \cap M$

(i) follow because h, h^{-1} are continuous, bijective



W open means point nearby
$a \in W$
are still in W

$\Rightarrow$ i.e. $|b-a|$ is sufficiently small

$\underbrace{\text{If } b \text{ is close to } a}$
then $(b,0)$ is close to $(a,0)$ &
h^{-1} is continuous ~~$h^{-1}(a,0)$~~
$h^{-1}(a,0) \in W$ because $a \in W$, $(a,0) \in V$, $h^{-1}: W \to U$

U is open so points nearby $h^{-1}(a,0)$ are in U
then points nearby $h^{-1}(a,0)$ and in M are in U
$h^{-1}(b,0)$ is close to $h^{-1}(a,0)$

$\Rightarrow b \in W$

(iii) $f'(y)$ has rank k for $y \in W$
define $H(\cancel{z^1, z^2, \dots, z^n}\ \tilde{y}) = (h^1(\tilde{y}), h^2(\tilde{y}), \dots, h^k(\tilde{y}))$
$H: \mathbb{R}^n \to \mathbb{R}^k$

k components

$(H \circ f)(z) = H(f(z)) = H(h^{-1}(z, 0)) = z$

$n-k$ comp.

$DH(f(z) \circ Df(z)) = Id_{\mathbb{R}^k}$

$\Rightarrow Df(z)$ is injective

Ker + Rank thm
$k = \dim \mathbb{R}^k = \dim \operatorname{Ker} Df(z) + \dim \operatorname{Im} Df(z)$
rank $Df(z)$
rank $f'(z)$

i.e. $k = \operatorname{rank} f'(z)$

Dual Spaces

V is n dim vector space

Def A **linear functional** f is a linear transf. $f: V \to \mathbb{R}$

// i.e. $f: V \to \mathbb{R}$ //

// $f(\lambda x + y) = \lambda f(x) + f(y)$, $\forall x, y \in V$ //

// $\forall \lambda \in \mathbb{R}$ //

Def The dual space

$$V^* = \{ f: V \to \mathbb{R},\ f \text{ linear functionals} \} \ni f, g$$

Prop: Let $f, g \in V^* \Rightarrow (f+g) \in V^*$ & $\lambda f \in V^*$ $\forall \lambda \in \mathbb{R}$:

Def $(f+g)(x) = f(x) + g(x)$ $\forall x \in V$, $(\lambda f)(x) = \lambda f(x)$

check Check $f+g \in V^*$ $\lambda \in \mathbb{R}$, $x, y \in V$

$$(f+g)(\lambda x + y) \overset{def}{=} f(\lambda x + y) + g(\lambda x + y)$$

$$\underset{\substack{f \in V^* \\ g \in V^*}}{=} \lambda f(x) + f(y) + \lambda g(x) + g(y)$$

$$= \lambda (f(x) + g(x)) + f(y) + g(y)$$

$$\overset{def}{=} \lambda (f+g)(x) + (f+g)(y) \quad \blacksquare$$

Example $V = P_{n-1}[x]$ fix $x_0 \in \mathbb{R}$

$f(p(x)) = p(x_0)$

show $f \in V^*$

Prop $\boxed{\dim V^* = \dim V}$

Proof We have $\{v_1, v_2, \dots, v_n\}$ basis of V

Defining ϕ_i | $\forall i = 1, \dots, n$ define $\phi_i : V \to \mathbb{R}$ as follows

given $x \in V$

$x \underset{=}{\text{uniquely}}\ x^1 v_1 + x^2 v_2 + \dots + x^n v_n \qquad x^i \in \mathbb{R}$

$\phi_i(x) = x^i$

φ_i is functional | ϕ_i is a lin functional

if $y = y^1 v_1 + y^2 v_2 + \dots + y^n v_n$

$\lambda \in \mathbb{R}$

$$\lambda x + y = (\lambda x^1 + y^1) v_1 + \dots v_i + \dots + (\lambda x^n + y^n) v_n$$

$$\phi_i(\lambda x + y) = \lambda x^i + y^i = \lambda \phi_i x + \phi_i y$$

Notice | $\boxed{\phi_i(v_j) = \delta_{ij} = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}}$

$\{\phi_1, \dots, \phi_n\}$ is a basis of V^*? | Now $\{\phi_1, \phi_2, \dots, \phi_n\}$ is a basis for V^*

1) Given $f \in V^*$ define $a_i \in \mathbb{R}$

$f(v_i) = a_i \in \mathbb{R}$

we will show $f = a^1 \phi_1 + a^2 \phi_2 + \dots + a^n \phi_n$

If f & $a^1 \phi_1 + \dots + a^n \phi_n$ agree on the basis $\{v_1, \dots, v_n\}$

then it is true

$k = 1, \dots, n$ | $f(v_k) = a_k \qquad (a^1 \phi_1 + \dots + a^n \phi_n)(v_k) = a^1 \underbrace{\phi_1(v_k)}_{0} + \dots + a^k \underbrace{\phi_k(v_k)}_{1} \dots a^n \underbrace{\phi_n(v_k)}_{0} = a^k$

2) $b^1 \phi_1 + b^2 \phi_2 + \dots + b^n \phi_n = 0 \overset{?}{\Rightarrow} b^k = 0 \ \forall k$

Apply to the basis vector v_k

$(b^1 \varphi_1 + \dots + b^n \varphi_n)(v_k) = b^1 \cdot 0 + b^2 \cdot 0 + \dots + b^k \cdot 1 + b^{k+1} \cdot 0 = \dots =$
$= b^k$

Multilinear Algebra

22/11/11 Recall

$f : V \to V$ lin map, lin functional

$V^* = \{ f : V \to \mathbb{R} \}$ lin. functions} dual space of V

If $\{v_1, v_2 \dots v_n\}$ is a basis of V,

then $\{\phi_1, \phi_2, \dots, \phi_n\}$ is a basis of V^*

$\phi_i(v_j) = \delta_{ij} = \begin{cases} 1 & i = j \\ 0 & i = j \end{cases}$

Exercise $V = P_n[x]$ with basis $\{1, x, \dots, x^n\}$

$f : V \to \mathbb{R}$ $\quad f(P) = P$

write f as a l combination of dual basis

Def: Let V be vector space over over $\mathbb{R}$

we define

$$V^k = \underbrace{V \times V \times V \times \dots \times V}_{k \text{ times}}$$

to be

$$V^k = \{(v_1, \dots, v_k),\ v_1, v_2, \dots, v_k \in V\}$$

this is a vector space with operations:

$$(v_1, v_2, \dots, v_k) + (w_1, w_2, \dots, w_k) = (v_1 + w_1, v_2 + w_2, \dots, v_k + w_k)$$

$$\lambda(v_1, v_2, \dots, v_k) = (\lambda v_1, \lambda v_2, \dots, \lambda v_k)$$

Check the 8 properties so that V^k is a vector space over $\mathbb{R}$

Def (1) $T : V^k \to \mathbb{R}$ is called multilinear if for all $i \in \{1, 2, \dots, k\}$:

* $T(v_1, v_2, \dots, v_{i-1}, v_i + v_i', v_{i+1}, \dots, v_k) =$
$= T(v_1, v_2, \dots, v_{i-1}, v_i, v_{i+1}, \dots, v_k) +$
$T(v_1, v_2, \dots, v_{i-1}, v_i', v_{i+1}, \dots, v_k)$

$\forall v_1, v_2, \dots, v_k, v_i' \in V$

* $T(v_1, v_2, \dots, v_{i-1}, \lambda v_i, v_{i+1}, \dots, v_k) =$
$= \lambda T(v_1, \dots, v_{i-1}, v_i, v_{i+1}, \dots, v_k)$

$\forall \lambda \in \mathbb{R}$

(2) A T like this is called a k-tensor on V

(3) Define $\mathcal{T}^k(V) = \{T : V^k \to \mathbb{R} \ k\text{-multilin}\}$

Example 1) $T(v_1 + v_2, w) = T(v_1, w) + T(v_1, w)$
$T(\lambda v, w) = \lambda T(v, w)$
$T(v, w_1 + w_2) = T(v, w_1) + T(v, w_2)$
$T(v, \lambda w) = \lambda T(v, w)$ bilinear form $k = 2$

2) T is symmetric if $T(v, w) = T(w, v)$
3) T is pos def $T(v, v) \geq 0$

Def T is a symmetric k-tensor $\forall v_1 \dots v_k \in V$
$$T(v_1, v_2, \dots, \underset{i^{th}}{v_i}, v_{i+1}, \dots, \underset{j^{th}\ slot}{v_j}, \dots, v_k) =$$
$$= T(v_1, v_2, \dots, \underset{i^{th}\ slot}{v_j}, \dots, \underset{j^{th}\ slot}{v_i}, \dots v_k)$$

Def T is an alternating k-tensor if
$$T(v_1, \dots, \underset{i}{v_i}, \dots, \underset{j}{v_j}, \dots, v_k) = -$$
$$- T(v_1, v_2, \dots, \underset{i}{v_j}, \dots, \underset{j}{v_i}, \dots, v_k)$$

Example $\mathbb{R}^2 = V$, $\mathbb{R}^2 \times \mathbb{R}^2 = V^2$
$$T(v_1, v_2) = v_1^1 v_2^2 - v_2^1 v_1^2$$
$$v_1 = (v_1^1, v_2^2)$$
$$v_2 = (v_2^1, v_2^2)$$

$$\begin{vmatrix} \lambda v_1 + v_1' \\ v_2 \end{vmatrix} = \lambda \begin{vmatrix} v_1 \\ v_2 \end{vmatrix} + \begin{vmatrix} v_1' \\ v_2 \end{vmatrix}$$

$$\begin{vmatrix} v_1 \\ \lambda v_2 + v_2' \end{vmatrix} = \lambda \begin{vmatrix} v_1 \\ v_2 \end{vmatrix} + \begin{vmatrix} v_1 \\ v_2' \end{vmatrix}$$

det on k matrices
(as function of k vectors in $\mathbb{R}^k$)
is an alternating k-tensor

Prop: $\boxed{\text{If } T, S \in \mathcal{J}^k(V) \quad (T+S) \in \mathcal{J}^k(V)}$

Def we define $(T+S)(v_1, v_2, \dots, v_k) =$
$$= T(v_1, v_2, \dots, v_k) + S(v_1, v_2, \dots, v_k)$$

~~Exerci~~ Similarly $\lambda \in \mathbb{R}$, $\lambda T \in \mathcal{T}^k(V)$

Def: $(\lambda T)(v_1, \dots, v_k) = \lambda T(v_1, v_2, \dots, v_k)$

$\forall v_1, v_2, \dots, v_k \in V$

Def $k, l \in \mathbb{N}$

Let $T \in \mathcal{T}^k(V)$ $\quad T : V^k \to \mathbb{R}$

$S \in \mathcal{T}^l(V)$ $\quad S : V^l \to \mathbb{R}$

Prop. Define $\boxed{T \otimes S \in \mathcal{T}^{k+l}(V)}$

real number multiplication

$$(T \otimes S)(v_1, v_2, \dots, v_k, \dots, v_{k+l}) = T(v_1, v_2, \dots, v_k) \cdot S(v_{k+1}, v_{k+2}, \dots, v_{k+l})$$

$\mathcal{T}^{l+k}(V) \ni S \otimes T \neq T \otimes S$

first plug l-vectors, remaining k

H/W

Properties

1. $T \otimes S \in \mathcal{T}^{k+l}(V)$
2. $(S_1 + S_2) \otimes T = S_1 \otimes T + S_2 \otimes T$
3. $S \otimes (T_1 + T_2) = S \otimes T_1 + S \otimes T_2$
4. $(\lambda S) \otimes T = \lambda (S \otimes T) = S \otimes (\lambda T)$
5. $(S \otimes T) \otimes U = S \otimes (T \otimes U)$
6. $\mathcal{T}^1(V) = V^*$

Thm Let $i_1, \dots, i_k \in \{1, 2, \dots, n\}$
V have basis $\{v_1, v_2, \dots, v_n\}$ $\dim V = n$

Let $\{\phi_1, \dots, \phi_n\}$ be the dual ~~space~~ basis of V^* $\phi_i(v_j) = \delta_{ij}$
not good representation

Consider $\phi_{i_1} \otimes \phi_{i_2} \otimes \dots \otimes \phi_{i_k}$ where $\{i_1, i_2, \dots, i_k\} \in \{1, 2, \dots, n\}$
form a basis of $\mathcal{J}^k(V)$

Therefore $\dim \mathcal{J}^k(V) = n^k$

Proof clearly $\phi_{i_1} \otimes \phi_{i_2} \otimes \dots \otimes \phi_{i_k} \in \mathcal{J}^k(V)$
since $\phi_{i_j} \in V^* = \mathcal{J}^1(V)$

The set spans $\mathcal{J}^k(V)$ and is linearly independent

Span: 1) Let $T \in \mathcal{J}^k(V)$.
Need to write
$$T = \sum_{\substack{i_1 = 1, \dots n \\ i_2 = 1, \dots n \\ i_3 = 1, \dots n \\ \vdots \\ i_k = 1 \dots n}} a^{i_1 i_2 \dots i_k} \phi_{i_1} \otimes \phi_{i_2} \otimes \dots \otimes \phi_{i_k}$$

Plug $(v_{j_1}, v_{j_2}, \dots, v_{j_k})$ into the suspected identity
$$T(v_{j_1}, \dots, v_{j_k}) = \sum_{i_1, \dots i_k} a^{i_1 \dots i_k} \phi_{i_1} \otimes \phi_{i_2} \otimes \dots \otimes \phi_{i_k}(v_{j_1}, v_{j_2}, \dots, v_{j_k})$$
$$= \sum_{i_1, \dots, i_k} a^{i_1 \dots i_k} \phi_{i_1}(v_{j_1}) \phi_{j_2}(v_{j_2}) \dots \phi_{i_k}(v_{j_k})$$
$$= \sum_{i_1, \dots, i_k} a^{i_1 \dots i_k} \delta_{i_1 j_1} \delta_{i_2 j_2} \dots \delta_{i_k j_k} =$$
$$= a^{j_1, j_2, \dots, j_k}$$

Define $a^{j_1 j_2 \dots j_k} = T(v_{j_1}, v_{j_2}, \dots, v_{j_k})$

Let $\underbrace{w_1, \dots, w_k}_{\text{arbitrary elements}} \in V$

$$w_1 = \sum_{j=1}^{n} a^{1j} v_j$$

$$w_2 = \sum_{j=1}^{n} a^{2j} v_j$$

$$\vdots$$

$$w_k = \sum_{j=1}^{n} a^{kj} v_j$$

$$T(w_1, w_2, \dots, w_k) = T\left(\sum_{j_1} a^{1 j_1} v_{j_1}, \sum_{j_2} a^{2 j_2} v_{j_2}, \dots, \sum_{j_k} a^{k j_k} v_{j_k}\right) =$$

$$\overset{T \text{ is multilin}}{=} \sum_{j_1, j_2, \dots, j_k = 1}^{n} a^{1j_1} a^{2j_2} \dots a^{kj_k} \cdot T(v_{j_1}, v_{j_2}, \dots, v_{j_k})$$

$$= \sum_{j_1, \dots, j_k} a^{1j_1} \dots a^{kj_k} \cdot a^{j_1 j_2 \dots j_k} \qquad (*)$$

$$\sum a^{i_1 \dots i_k} \phi_{i_1} \otimes \phi_{i_2} \otimes \dots \otimes \phi_{i_k}(w_1, w_2, \dots, w_k)$$

$$= \sum_{i_1 \dots i_k} a^{i_1 \dots i_k} \phi_{i_1}(w_1) \phi_{i_2}(w_2) \dots \phi_{i_k}(w_k)$$

$$= \sum_{i_1 \dots i_k} a^{i_1 \dots i_k} a^{1 i_1} a^{2 i_2} \dots a^{k i_k} \qquad (**)$$

Relabel $\begin{array}{l} i_1 \to j_1 \\ i_2 \to j_2 \\ \dots \end{array} \Rightarrow (*) = (**)$

$\square$

2) $\phi_{i_1} \otimes \dots \otimes \phi_{i_k}$ are l. independent

$$\sum_{i_1, i_2, \dots, i_k = 1}^{n} a^{i_1 i_2 \dots i_k} \phi_{i_1} \otimes \phi_{i_2} \otimes \dots \otimes \phi_{i_k} = 0$$

Plug $v_{j_1}, v_{j_2}, \dots, v_{j_k}$

$$\sum_{i_1, \dots i_k = 1} a^{i_1, \dots, i_k} \phi_{i_1} \otimes \phi_{i_2} \otimes \dots \otimes \phi_{i_k} \left(v_{j_1}, \dots, v_{j_k} \right) = 0$$

$$\sum_{i_1, \dots i_k = 1} a^{i_1, \dots, i_k} \delta_{i_1 j_1} \dots \delta_{i_k} \delta_{j_k} = a^{j_1, j_2 \dots, j_k} = 0$$

Repeat it for different ~~comb~~ of $j_1 j_2 \dots j_k$

$\Rightarrow a^{i_1 i_2, \dots i_k} = 0$

$\blacksquare$

Recall $f : \mathbb{R} \to \mathbb{R}$

f even $\quad f(-x) = f(x)$

f odd $\quad f(-x) = - f(x)$

Every $f : \mathbb{R} \to \mathbb{R}$ can be written as

$$f = \underset{\text{even}}{f_1} + \underset{\text{odd}}{f_2}$$

$$f_1(x) = \frac{f(x) + f(-x)}{2}$$

$$f_2(x) = \frac{f(x) - f(-x)}{2}$$

$g(x) = f(x) + f(-x)$

$g(-x) = f(-x) + f(x)$

even

$h(x) = f(x) - f - x$

odd

$g + h = 2 f(x)$

$x \longrightarrow -x$

σ is bijection on $\mathbb{R} \to \mathbb{R}$

$\sigma^2 = Id$

$\frac{f(x) + f(\sigma x)}{2}$, order of σ

Let S_k be symmetric group on k-letters

$S_k \longrightarrow \{\pm 1\}$

homomorphism — multiplicative gp

$$\sigma \longrightarrow \begin{cases} +1 & \text{if } \sigma \text{ even} \\ -1 & \text{if } \sigma \text{ odd} \end{cases}$$

$\sigma \longrightarrow sgn_k(\sigma)$

__Def__ If $T \in \mathcal{T}^k(V)$ $(w_1, w_2, \dots, w_k)$

we define $Alt(T) = \frac{1}{k!} \sum_{\sigma \in S_k} sgn_k(\sigma)\, T(w_{\sigma(1)}, w_{\sigma(2)}, \dots, w_{\sigma(k)})$

e.g. $k=2$ $Alt(T)(w_1, w_2) = \frac{1}{2!}\left(T(w_1, w_2) - T(w_2, w_1)\right)$

$$id \quad \begin{matrix} 1 & 2 \\ \downarrow & \downarrow \\ 1 & 2 \end{matrix} \qquad \sigma \quad \begin{matrix} 1 & 2 \\ \downarrow & \downarrow \\ 2 & 1 \end{matrix}$$

__Thm__ (a) If $T \in \mathcal{T}^k(V)$

$Alt(T) \in \mathcal{T}^k(V)$

$Alt\,T$ is alternating tensor

(b) If ω is alternating tensor, $Alt(\omega) = \omega$

(c) $Alt(Alt(T)) = Alt(T)$

Def The set of alternating k-tensors is denoted by $\Lambda^k(V)$

It is a subspace of $\mathcal{T}^k(V)$

Proof $\subset$ follows from b

use $w = Alt(T)$ which alternating by a

$Alt(T) = w = Alt(w) = Alt(Alt(T))$

(a) show $Alt(T) \in \mathcal{T}^k(V)$

I'll show it is alternating

$Alt(T)(w_1, \dots, w_i, \dots, w_j, \dots, w_k) =$

$\uparrow i^{th}$ $\uparrow j^{th}$

$= - Alt(T)(w_1, \dots, w_j, \dots, w_i, \dots, w_k)$

$\left.\begin{array}{l} i \to j \\ j \to i \end{array}\right\} \Rightarrow (ij)$ transposition

If $k \neq i, j$ $k \to k$

~~(i,j)~~

bijection

$S_k \to S_k$

$\sigma \to \sigma(ij) = \sigma'$

even $\to$ odd

odd $\to$ even

$\sigma_1 \to \sigma_1 (ij)$

$\sigma_2 \to \sigma_2 (ij)$

$\Rightarrow \sigma_1 = \sigma_2$

$$\mathrm{Alt}(T)(w_1, \dots, \overset{i^{th}}{w_j}, \dots, \overset{j^{th}}{w_i}, \dots, w_k) =$$

$$= \frac{1}{k!} \sum_{\sigma \in S_k} \mathrm{sgn}(\sigma)\, T(w_{\sigma(1)}, \dots, \overset{i^{th}}{w_{\sigma(j)}}, \dots, \overset{j^{th}}{w_{\sigma(i)}} \dots w_{\sigma(k)})$$

$$= \frac{1}{k!} \sum_{\sigma' \in S_k} -\mathrm{sgn}(\sigma')\, T(w_{\sigma'(1)}, \dots, w_{\sigma'(i)}, \dots, w_{\sigma'(j)} \dots w_{\sigma'(k)})$$

$$= -\frac{1}{k!} \sum_{\sigma \in S_k} \mathrm{sgn}(\sigma)\, T(w_{\sigma(1)}, \dots, w_{\sigma(i)}, \dots, w_{\sigma(j)}, \dots, w_{\sigma(k)}) =$$

$$= -\mathrm{Alt}(T)(w_1 \dots w_k)$$

(b) Let ω be alternating

$$\omega(w_1, \dots, \underset{i^{th}}{w_j}, \dots, \underset{j^{th}}{w_i}, \dots, w_k) = -\omega(w_1, \dots, \overset{i^{th}}{w_i}, \dots, \overset{j^{th}}{w_j}, \dots w_k)$$

$$\omega(w_{\sigma(1)}, w_{\sigma(2)}, \dots, w_{\sigma(k)}) = \mathrm{sgn}(\sigma)\, \omega(w_1, w_2, w_3, \dots, w_k)$$

$$\mathrm{Alt}(\omega)(w_1, \dots, w_k) = \frac{1}{k!} \sum_{\sigma \in S_k} \mathrm{sgn}(\sigma)\, \omega(w_{\sigma(1)}, \dots w_{\sigma(k)}) =$$

$$\underset{\text{alternating}}{\overset{\omega}{=}} \frac{1}{k!} \sum_{\sigma \in S_k} \mathrm{sgn}(\sigma)\,\mathrm{sgn}(\sigma)\, \omega(w_1, \dots, w_k) =$$

$$= \cancel{\frac{1}{k!}}\, |\cancel{S_k}|\, \omega(w_1, \dots, w_k)$$

So $\mathrm{Alt}(\omega) = \omega$ ∎

Remark: If $\omega \in \Lambda^k V$, $\eta \in \Lambda^l(V)$ then $\omega \otimes \eta \in \mathcal{T}^{k+l}(V)$,

~~We will prove that~~

Def Define $\boxed{\omega \wedge \eta = \frac{(k+l)!}{k!\,l!} \operatorname{Alt}(\omega \otimes \eta)}$ wedge product

$\omega \wedge \eta \in \Lambda^{k+l}(V)$

Do we need to have allways to use wedge prod. $\omega_1, \omega_2, \omega \in \Lambda^k(V)$ $\eta_1, \eta_2, \eta \in \Lambda^l$

Prop.
1. $(\omega_1 + \omega_2) \wedge \eta = \omega_1 \wedge \eta + \omega_2 \wedge \eta$
2. $\omega \wedge (\eta_1 + \eta_2) = \omega \wedge \eta_1 + \omega \wedge \eta_2$
3. $a\omega \wedge \eta = a(\omega \wedge \eta) = \omega \wedge (a\eta)$ $\quad a \in \mathbb{R}$

$\omega_1, \omega_2, \omega \in \Lambda^k(V) \quad \eta_1, \eta_2, \eta \in \Lambda^l(V)$

4. $\omega \wedge \eta = (-1)^{k \cdot l} \eta \wedge \omega$

13/11/11 **Prop:**

Let V, W be vector spaces
$f : V \to W$ linear transformation
If T is lin. transf. on W
$T : W \to \mathbb{R}$
then $T \circ f$ is a lin. functional on V

Def $f^*(T) = T \circ f$
$f^*(T)$ is called the pullback of T by f

$f^* : W^* \to V^*$
by $f^*(T) = T \circ f$

Pullback of tensors

If T is a k-tensor on W
i.e. $T \in \mathcal{J}^k(W)$

Prop. $\boxed{f^*(T) \in \mathcal{J}^k(V)}$

$v_i \in V$

Define $$f^*(T)(v_1, v_2, \dots, v_k) = T(f(v_1), f(v_2), \dots, f(v_k))$$

this is k-tensor on V

Proof <u>Need to show linearity in i-entry</u>

Let $v_i, v_i' \in V$, $\lambda \in \mathbb{R}$

$f^*(T)(v_1, v_2, \dots, \lambda v_i + v_i', v_{i+1}, \dots, v_k) \overset{\text{def}}{=}$
$= T(f(v_1), f(v_2), \dots f(\lambda v_i + v_i'), f(v_{i+1}), \dots, f(v_k)) \overset{f\ \text{lin}}{=}$
$= T(f(v_1), f(v_2), \dots, \lambda f(v_i) + f(v_i'), f(v_{i+1}), \dots, f(v_k)) \overset{T\ \text{lin}}{=}$
$= \lambda\, T(f(v_1), f(v_2), \dots, f(v_i), f(v_{i+1}), \dots, f(v_k)) +$
$+ T(f(v_1), f(v_2), \dots, f(v_i'), f(v_{i+1}), \dots, f(v_k)) =$
$= \lambda f^*(T)(v_1, v_2, \dots, v_k) + f^*(T)(v_1, v_2, \dots, v_i', \dots, v_k)$

<u>Properties</u>

a $f^*(T \otimes S) = f^*(T) \otimes f^*(S)$
$T \in \mathcal{J}^k(W)$, $S \in \mathcal{J}^l(W)$

b $f^*(\omega \wedge \eta) = f^*(\omega) \wedge f^*(\eta)$
for $\omega \in \Lambda^k(W)$, $\eta \in \Lambda^l(W)$

<u>Recall</u> If $T \in \mathcal{J}^k(V)$

$$\mathrm{Alt}(T)(w_1, \dots, w_k) = \frac{1}{k!} \sum_{\sigma \in S_k} \operatorname{sgn}(\sigma)\, T(w_{\sigma(1)}, w_{\sigma(2)}, \dots, w_{\sigma(k)})$$

seen to a basis of $\mathcal{J}^k(V)$

Consist of $\phi_{i_1} \otimes \phi_{i_2} \otimes \phi_{i_3} \otimes \dots \otimes \phi_{i_k}$

with $\{\phi_i\}$ dual basis of $\{v_i\}$
$i_1, i_2, \dots, i_k \in \{1, 2, \dots, n\}$
$n = \dim V$

$\Lambda^k(V)$ is a subspace of $\mathcal{J}^k(V)$
our difficulty
$(\omega \wedge \eta) \wedge \vartheta = \omega \wedge (\eta \wedge \vartheta)$

Thm (a) Let $S \in \mathcal{J}^k(V)$ $T \in \mathcal{J}^l(V)$
$\mathrm{Alt}(S) \stackrel{!}{=} 0$

then $\mathrm{Alt}(S \otimes T) = \mathrm{Alt}(T \otimes S) = 0$

(b) $\mathrm{Alt}(\mathrm{Alt}(\omega \otimes \eta) \otimes \vartheta) = \mathrm{Alt}(\omega \otimes \eta \otimes \vartheta) = \mathrm{Alt}(\omega \otimes \mathrm{Alt}(\eta \otimes \vartheta))$

(c) $\omega \in \Lambda^k(V)$,
$\eta \in \Lambda^l(V)$
$\vartheta \in \Lambda^m(V)$

(d) $\omega \wedge \eta \wedge \vartheta = \omega \wedge (\vartheta \wedge \eta) = \frac{(k+l+m)!}{k!\,l!\,m!} \mathrm{Alt}(\omega \otimes \eta \otimes \vartheta)$

Proof a. $\mathrm{Alt}(S \otimes T) \stackrel{?}{=} 0$
$(w_1, w_2, \dots, w_{k+l})$
$= \frac{1}{(k+l)!} \sum_{\sigma \in S_{k+l}} \mathrm{sgn}(\sigma) (S \otimes T)(w_{\sigma(1)}, w_{\sigma(2)}, \dots, w_{\sigma(k)}, w_{\sigma(k+1)}, \dots, w_{\sigma(k+l)})$

Let G be the subgroup of S_{k+l}
$G = \{\sigma \in S_{k+l},\ \sigma(k+1) = k+1$
$\sigma(k+2) = k+2, \dots, \sigma(k+l) = k+l\}$

The contribution of these to the sum is

$$\frac{1}{(k+l)!}\left[\sum_{\sigma\in G} sgn(\sigma)\, S(w_{\sigma(1)},\dots,w_{\sigma(k)})\right]\cdot T(w_{k+1}, w_{k+2},\dots w_{k+l})=$$

$$=\frac{1}{(k+l)!}\, k!\ Alt(S)(w_1,\dots,w_k)\cdot T(w_{k+1},\dots,w_{k+l})=0$$

Let $G\sigma_0$ be a coset of G in S_{k+l}

$\sigma_0 \neq id$

$G\sigma_0 = \{\sigma'\cdot\sigma_0,\ \sigma'\in G\}$

$\sigma = \sigma'\sigma_0$

The contribution of these elements is

$$\frac{1}{(k+l)!}\sum_{\sigma'\in S} sgn(\sigma'\cdot\sigma_0)\cdot S(z_{\sigma'(1)}, z_{\sigma'(2)},\dots, z_{\sigma'(k)})\cdot T(z_{\sigma'(k+1)},\dots, z_{\sigma'(k+l)})$$

homomorphism Define $(z_1,\dots,z_{k+l}) = (w_{\sigma_0(1)}, w_{\sigma_0(2)},\dots, w_{\sigma_0(k+l)})$

~~The contribution of these elements~~

but $\sigma'\in G$ $\quad \sigma'(k+1)=k+1$

$\sigma'(k+2)=k+2$ etc

$$\frac{1}{(k+l)!}\sum_{\sigma'\in G} sgn(\sigma')\, sgn(\sigma_0)\ S(z_{\sigma'(1)},\dots,z_{\sigma'(k)})\, T(z_{k+1},\dots z_{k+l})=$$

$$=\frac{1}{(k+l)!}\, sgn(\sigma_0)\, T(z_{k+1},\dots,z_{k+l})\ k!\, Alt\, S(z_1,\dots,z_k)=0$$

(b) $Alt(\omega\otimes\eta)-\omega\otimes\eta = S$

$Alt(S) = Alt(Alt(\omega\otimes\eta)-\omega\otimes\eta) =$

$= Alt(Alt(\omega\otimes\eta)) - Alt(\omega\otimes\eta) =$

$= Alt(\omega\otimes\eta) - Alt(\omega\otimes\eta) = 0$

Apply a with this s

$\mathrm{Alt}(S \otimes \vartheta) = 0$

$\mathrm{Alt}([\mathrm{Alt}(\omega \otimes \eta) - \omega \otimes \eta] \otimes \vartheta) = 0$

$\mathrm{Alt}(\mathrm{Alt}(\omega \otimes \eta) \otimes \vartheta) - \mathrm{Alt}(\omega \otimes \eta \otimes \vartheta) = 0$

(c) $\underbrace{(\omega \wedge \eta)}_{k+l} \wedge \underbrace{\vartheta}_{m} = \frac{(k+l+m)!}{(k+l)!\, m!} \mathrm{Alt}((\omega \wedge \eta) \otimes \vartheta) =$

alt. tensors

$= \frac{(k+l+m)!}{(k+l)!\, m!} \mathrm{Alt}\left(\frac{(k+l)!}{k!\, l!} \mathrm{Alt}(\omega \otimes \eta) \otimes \vartheta\right) =$

$= \frac{(k+l+m)!\,(k+l)!}{(k+l)!\, m!\, k!\, l!} \mathrm{Alt}(\mathrm{Alt}(\omega \otimes \eta) \otimes \vartheta) \overset{(b)}{=}$

$= \mathrm{Alt}(\omega \otimes \eta \otimes \vartheta) \cdot \frac{(k+l+m)!}{k!\, l!\, m!}$ ∎

thm Let $\dim V = n$

then the following is a basis of $\Lambda^k(V)$

$\phi_{i_1} \wedge \phi_{i_2} \wedge \dots \wedge \phi_{i_k}, \quad 1 \le i_1 < i_2 < \dots < i_k \le n$

{and here $\{\phi_i\}$ is dual basis of $\{v_i\}$ of V}

Therefore $\dim \Lambda^k(V) = \binom{n}{k}$

// take subset of size k from $\{1, \dots, n\}$ and order them incr.-ly

~~then~~

Corollary $\underline{k > n}$ $\quad \Lambda^k(V) = \{0\}$

$\underline{k = 1}$ $\quad \Lambda^1(V) = \binom{n}{1} = n$

Since Λ^1 has tensor with 1 slot

$\Lambda^1(V) = \mathcal{T}^1(V) = V^*$

$\underline{k=n}$ $\dim \Lambda^n(V) = \binom{n}{n} = 1$

$\det(v_1, \ldots, v_n) \in \Lambda^n(V)$ and
since $\det(I) = 1$
every n-alternating tensor is a multiply
of $\det(v_1, \ldots, v_n)$

Proof of thm $T \in \Lambda^k(V)$
then $Alt(T) = T$
Since $\phi_{i_1} \otimes \phi_{i_2} \otimes \ldots \otimes \phi_{i_k}$ is basis of $\mathcal{T}^k(V)$

$$T = \sum_{i_1, i_2, \ldots, i_k = 1, \ldots, n} a^{i_1 i_2 \ldots i_k} \phi_{i_1} \otimes \phi_{i_2} \otimes \ldots \otimes \phi_{i_k}$$

Apply Alt on both sides

$$T = \sum_{i_1, \ldots, i_k} a^{i_1 i_2 \ldots i_k} Alt(\phi_{i_1} \otimes \phi_{i_2} \otimes \ldots \otimes \phi_{i_k})$$

$Alt(\phi_{i_1} \otimes \phi_{i_2} \otimes \ldots \otimes \phi_{i_k})$ is a multiple of
$\phi_{i_1} \wedge \phi_{i_2} \wedge \ldots \wedge \phi_{i_k}$

Since $\phi_{i_j} \wedge \phi_{i_s} = -\phi_{i_s} \wedge \phi_{i_j}$ (H/W)

you can reorder to $\sum_{i_1 < i_2 < \ldots < i_k}$

So $\phi_{i_1} \wedge \phi_{i_2} \wedge \ldots \wedge \phi_{i_k}$ with $i_1 < i_2 < \ldots < i_k$
span generate $\Lambda^k(V)$

It is easy to see that they are l. independent

29/11/11

Example $\dim V = 3$

$k=1 \quad \dim \Lambda^1(V) = \binom{3}{1} = 3$

$\Lambda^1(V) = \mathcal{J}(V) = V^*$

$k=2 \quad \dim \Lambda^2(V) = \binom{3}{2} = 3$ basis is $\phi_1 \wedge \phi_2,\ \phi_1 \wedge \phi_3$ and $\phi_2 \wedge \phi_3$

// If $\{v_i\}_{i=1,2,3}$ is basis of V

// then the dual basis ϕ_1, ϕ_2, ϕ_3 is basis of $\Lambda^1(V)$

$$(\phi_1 \wedge \phi_2)(w_1, w_2) = \frac{(1+1)!}{1!\,1!} \mathrm{Alt}(\phi_1 \otimes \phi_2)(w_1, w_2) =$$

$$= \cancel{2!}\frac{1}{\cancel{2!}}\left(\phi_1 \otimes \phi_2(w_1, w_2) - \phi_1 \otimes \phi_2(w_2, w_1)\right) =$$

$$= \phi_1(w_1)\phi_2(w_2) - \phi_1(w_2)\phi_2(w_1) = \phi_1(w_1)\phi_2(w_2) - \phi_2(w_1)\phi_1(w_2) =$$

$$= (\phi_1 \otimes \phi_2)(w_1, w_2) - (\phi_2 \otimes \phi_1)(w_1, w_2) \quad *$$

i.e. $\phi_1 \wedge \phi_2 = \phi_1 \otimes \phi_2 - \phi_2 \otimes \phi_1$

$\phi_1 \wedge \phi_3 = \phi_1 \otimes \phi_3 - \phi_3 \otimes \phi_1$

$\phi_2 \wedge \phi_3 = \phi_2 \otimes \phi_3 - \phi_3 \otimes \phi_2$

$\phi_2 \wedge \phi_1 = \phi_2 \otimes \phi_1 - \phi_1 \otimes \phi_2 = -\phi_1 \wedge \phi_2$

i.e. $\boxed{\phi_2 \wedge \phi_1 = -\phi_1 \wedge \phi_2}$

$(\phi_1 \wedge \phi_1)(w_1, w_2) = \phi_1(w_1)\phi_1(w_2) - \phi_1(w_2)\phi_1(w_1) = 0$

i.e. $\boxed{\begin{aligned} \phi_1 \wedge \phi_1 &= 0 \\ \phi_2 \wedge \phi_2 &= 0 \\ \phi_3 \wedge \phi_3 &= 0 \end{aligned}}$

$k=3 \quad \dim \Lambda^3(V^3) = \binom{3}{3} = 1$ basis $\phi_1 \wedge \phi_2 \wedge \phi_3$

$$(\phi_1 \wedge \phi_2 \wedge \phi_3)(w_1, w_2, w_3) = 3!\ \mathrm{Alt}(\phi_1 \otimes \phi_2 \otimes \phi_3)(w_1, w_2, w_3) =$$

$$= \sum_{\sigma \in S_3} \operatorname{sgn}(\sigma)\left(\phi_1 \otimes \phi_2 \otimes \phi_3\right)\left(w_{\sigma(1)}, w_{\sigma(2)}, w_{\sigma(3)}\right) =$$

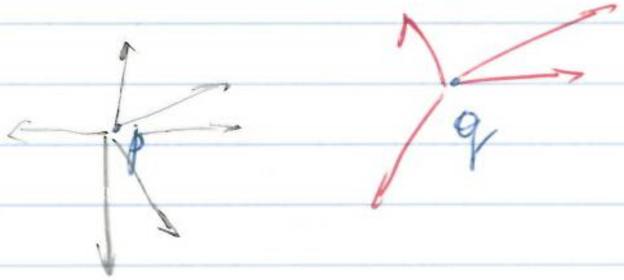
$$= \phi_1(w_1)\phi_2(w_2)\phi_3(w_3) \overset{(12)}{-} \phi_1(w_2)\phi_2(w_1)\phi_3(w_3) \overset{(13)}{-} \phi_1(w_3)\phi_2(w_2)\phi_3(w_1) -$$

$$\overset{(23)}{-} \phi_1(w_1)\phi_2(w_3)\phi_3(w_2) \overset{(123)}{+} \phi_1(w_2)\phi_2(w_3)\phi_3(w_1) \overset{(132)}{+} \phi_1(w_3)\phi_2(w_1)\phi_3(w_2) =$$

$$\overset{\text{rearrangement}}{=} \quad =$$

$$\therefore \quad \phi_1 \wedge \phi_2 \wedge \phi_3 = \phi_1 \otimes \phi_2 \otimes \phi_3 - \phi_2 \otimes \phi_1 \otimes \phi_3 - \phi_3 \otimes \phi_2 \otimes \phi_1$$

$$- \phi_1 \otimes \phi_3 \otimes \phi_2 + \phi_3 \otimes \phi_1 \otimes \phi_2 + \phi_2 \otimes \phi_3 \otimes \phi_1$$



Def $\mathbb{R}^n_p = \{(p, v),\ v \in \mathbb{R}^n\}$ this is the tangent space at p

Prop $\boxed{\begin{array}{l}(p, v) + (p, w) = (p, v + w) \\ \lambda(p, v) = (p, \lambda v)\end{array}}$ $\Bigg\}$ with this operations $\mathbb{R}^n_p$ is a vector space

NB! If $p \neq q$ it makes no sense $(p, v) \cancel{+} (q, w)$

Notation $v_p = (p, v)$

On $\mathbb{R}^n_p$ $\langle (p, v), (p, w) \rangle = \langle v, w \rangle$

Def A vector field in $\mathbb{R}^n$ is a f-n

$$p \xrightarrow{F} F(p) \in \mathbb{R}^n_p$$

$F(p) = (p, v)$

$\therefore v = (F^1(p), F^2(p), \dots, F^n(p))$

$p \to F^i(p)$

If the components F^i, $i \in \{1, \dots, n\}$ are continuous, the vector field is continuous

If the components are diff'ble the vector f. is differentiable

Prop: If F, g are vector fields in $\mathbb{R}^n$
$F+g$ is also a vector field in $\mathbb{R}^n$
$(F+g)(p) = F(p) + g(p)$

$\lambda \cdot F$ is a v.f. in $\mathbb{R}^n$, $\lambda \in \mathbb{R}$
$(\lambda F)(p) = \lambda \cdot F(p)$

Prop: If $f : \mathbb{R}^n \to \mathbb{R}$ is a function
then $f \cdot F$ is a new vector field on $\mathbb{R}^n$
$(f \cdot F)(p) = f(p) \cdot F(p)$

Def: If F is a vect. f.
then its divergence is

$$\boxed{\operatorname{div} F(p) = \sum_{i=1}^{n} D_i F^i(p)} \in \mathbb{R}$$

So $\operatorname{div} F : \mathbb{R}^n \to \mathbb{R}$

Notation $\operatorname{div} F = \nabla \cdot F$

Also $F : \mathbb{R}^3 \to \mathbb{R}^3$ you have seen a rotation or curling of the vect. f. defined by

$$\nabla \times F = \operatorname{curl}(F) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ F^1 & F^2 & F^3 \end{vmatrix} = \cdot$$
$$= \operatorname{rot}(F)$$

$$= \left(\frac{\partial F^3}{\partial y} - \frac{\partial F^2}{\partial z}\right)\vec{i} - \left(\frac{\partial F^3}{\partial x} - \frac{\partial F^1}{\partial z}\right)\vec{j} + \left(\frac{\partial F^2}{\partial x} - \frac{\partial F^1}{\partial y}\right)\vec{k}$$

Def Given $p \in \mathbb{R}^n$ let $\omega(p) \in \Lambda^k(\mathbb{R}^n_p)$

$$\boxed{\omega(p) = \sum_{i_1 < i_2 < \ldots < i_k} \omega_{i_1 i_2 \ldots i_k}(p)\, dx_{i_1}(p) \wedge dx_{i_2}(p) \wedge \ldots \wedge dx_{i_k}(p)}$$

this is a k-form on $\mathbb{R}^n$

It is determined by $\binom{n}{k}$ f-ns $p \to \omega_{i_1 i_2 \ldots i_k}(p)$, $i_1 < i_2 < \ldots < i_k$

- If these f-ns are continuous, then the k-form is continuous
- If these f-ns are d-ble, then ω is a differential k-form

- If ω, η are diff k-forms on $\mathbb{R}^n$ then $\omega + \eta$ is a diff k-form on $\mathbb{R}^n$

$$(\omega + \eta)(p) = \underbrace{\omega(p)}_{\Lambda^k(\mathbb{R}^n_p)} + \eta(p) \in \Lambda^k(\mathbb{R}^n_p)$$

- If $f : \mathbb{R}^n \to \mathbb{R}$ is a function differentiable then $f \cdot \omega$ is a diff k-form

$$(f \cdot \omega)(p) = f(p) \cdot \omega(p) \in \Lambda^k(\mathbb{R}^n_p)$$

- If ω is a diff. k-form & η is a diff l-form then $\omega \wedge \eta$ is a diff $(k+l)$ form

$$(\omega \wedge \eta)(p) = \omega(p) \wedge \eta(p)$$
$$\in \Lambda^k(\mathbb{R}^n_p) \quad \in \Lambda^l(\mathbb{R}^n_p)$$

Let $f: \mathbb{R}^n \to \mathbb{R}$ be diff-ble

then $Df(p): \mathbb{R}^n \to \mathbb{R}$ lin map

$$Df(p) \in (\mathbb{R}^n_p)^* = \mathcal{J}^1(\mathbb{R}^n_p) = \Lambda^1(\mathbb{R}^n_p)$$

Def We define df to be the following 1-form

$$df(p) \in \Lambda^1(\mathbb{R}^n_p)$$

where $\boxed{df(p)(v_p) = Df(p)(v)}$

$$v_p = (p, v)$$

Let $f = \pi^i$ the projection into i-component

$\pi^i(x^1, x^2, \dots, x^n) = x^i$ lin. map.

Sometimes it is denoted $x^i(x) = x^i$

$$d\pi^i(p)(v_p) = D\pi^i(p)(v) = \pi^i p)(v) = \pi^i(v) = v^i = \phi_i(v)$$

Def: $\therefore dx^i = d\pi^i = \phi_i$

A diff k-form on $\mathbb{R}^n$ we'll look like

$$\omega(p) = \sum_{i_1 < i_2 < \dots i_k} \omega_{i_1 i_2 \dots i_k}(p)\, dx^{i_1}(p) \wedge dx^{i_2}(p) \dots \wedge dx^{i_k}(p)$$

$$\omega = \sum_{i_1 < i_2 < \dots < i_k} \omega_{i_1 i_2 \dots i_k}\, dx^{i_1} \wedge dx^{i_2} \wedge \dots \wedge dx^{i_k}$$

Example 1) ~~[illegible]~~ $= \mathbb{R}^3 : (x^1, x^2, x^3) = (x, y, z)$

<u>$k=1$</u> $\omega = f(x,y,z)\,dx + g(x,y,z)\,dy + h(x,y,z)\,dz$

<u>$k=2$</u> $\omega = f(x,y,z)\,dx \wedge dy + g(x,y,z)\,dx \wedge dz +$
$+ h(x,y,z)\,dy \wedge dz$

<u>$k=3$</u> $\omega = f(x,y,z)\,dx \wedge dy \wedge dz$

<u>$k=0$</u> $\omega = f(x,y,z)$

$dx \wedge dx = 0$ $\quad$ $dx \wedge dy = -dy \wedge dx$
$dy \wedge dy = 0$ $\quad$ $dx \wedge dz = -dz \wedge dx$
$dz \wedge dz = 0$ $\quad$ $dy \wedge dz = -dz \wedge dy$

2) $n = 2$

<u>$k=1$</u> $\omega = f(x,y)\,dx + g(x,y)\,dy$

<u>$k=2$</u> $\omega = f(x,y)\,dx \wedge dy$

<u>$k=0$</u> $\omega = f(x,y)$

Thm

<u>Let</u>. $f : \mathbb{R}^n \to \mathbb{R}$ be diff-ble
<u>then</u> one form df is
$$df = D_1 f\,dx^1 + D_2 f\,dx^2 + \dots D_n f\,dx^n$$

(e.g $\mathbb{R}^3$ $df = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy + \frac{\partial f}{\partial z} dz$)

Proof $df(p) \in \Lambda^1(\mathbb{R}^n_p)$

$$df(p)(v_p) \overset{\text{def}}{=} Df(p)(v)$$
$$= (D_1 f(p), \dots, D_n f(p)) \begin{pmatrix} v^1 \\ v^2 \\ \vdots \\ v^n \end{pmatrix} =$$

$$= \sum_{i=1}^{n} D_i f(p)\, v^i =$$

$$\cancel{RHS} = (D_1 f\, dx^1 + \dots + D_n f\, dx^n)(p)(v_p) =$$
$$= [D_1 f(p)\, dx^1(p) + \dots + D_n f(p) \cdot dx^n(p)](v_p) =$$
$$= D_1 f(p)\, dx^1(p)(v_p) + \dots + D_n f(p)\, dx^n(p)(v_p)$$
$$= D_1 f(p) \cdot v^1 + \dots + D_n f(p) \cdot v^n$$

The operator d on k-forms

$k = 0 \quad \omega = f \quad \underset{1\text{-form}}{df} = \sum_i D_i f\, dx^i$

In general $\underset{k\text{ form}}{\omega} = \sum_{i_1 < i_2 < \dots < i_k} \omega_{i_1 i_2 \dots i_k} dx^{i_1} \wedge dx^{i_2} \wedge \dots \wedge dx^{i_k}$

Def. Define $\boxed{d\omega = \underset{k+1\text{ form}}{\sum_{i_1 < i_2 < \dots < i_k}} \sum_{i=1}^{k} D_i \omega_{i_1 i_2 \dots i_k} dx^i \wedge dx^{i_1} \wedge \dots \wedge dx^{i_k}}$

f-n $\quad x, y, z$

Ex (1) (2d) $n = 3 \quad k = 1 \quad \omega = f\, dx + g\, dy + h\, dz$

$$d\omega = \frac{\partial f}{\partial x} \cancel{dx \wedge dx} + \frac{\partial f}{\partial y} dy \wedge dx + \frac{\partial f}{\partial z} dz \wedge dx +$$
$$+ \frac{\partial g}{\partial x} dx \wedge dy + \frac{\partial g}{\partial z} \cancel{dy \wedge dy} + \frac{\partial g}{\partial z} dz \wedge dy +$$
$$+ \frac{\partial h}{\partial x} dx \wedge dz + \frac{\partial h}{\partial x} dy \wedge dz + \frac{\partial h}{\partial z} \cancel{dz \wedge dz} =$$

$$= -\frac{\partial f}{\partial y} dx \wedge dy - \frac{\partial f}{\partial z} dx \wedge dz + \frac{\partial g}{\partial x} dx \wedge dy - \cancel{\frac{\partial g}{}}$$

$$-\frac{\partial g}{\partial z} dy \wedge dz + \frac{\partial h}{\partial x} dx \wedge dz + \frac{\partial h}{\partial y} dy \wedge dz =$$

$$= \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right) dx \wedge dy + \left(\frac{\partial h}{\partial y} - \frac{\partial g}{\partial z}\right) dy \wedge dz + \left(\frac{\partial h}{\partial x} - \frac{\partial f}{\partial z}\right) dx \wedge dz$$

$$(*) \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f & g & h \end{vmatrix} = \left(\frac{\partial h}{\partial y} - \frac{\partial g}{\partial z}\right)\vec{i} - \left(\frac{\partial h}{\partial x} - \frac{\partial f}{\partial z}\right)\vec{j} + \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right)k$$

from (x) (*x) we can conclude:

~~ta~~ $\vec{i} \leftrightarrow dy \wedge dz$

$\vec{j} \leftrightarrow dz \wedge dx$

$\vec{k} \leftrightarrow dx \wedge dy$

(2) $n = 3 \quad k = 2$

$$\omega = f_1\, dy \wedge dz + f_2\, dz \wedge dx + f_3\, dx \wedge dy$$

$$0 = -\frac{\partial f_1}{\partial z}\, dy \wedge dz \wedge dz$$

$$d\omega = \frac{\partial f_1}{\partial x} dx \wedge dy \wedge dz + \frac{\partial f_1}{\partial y} \cancel{dy \wedge dy \wedge dz} + \frac{\partial f_1}{\partial z} dz \wedge dy \wedge dz$$

$$+ \frac{\partial f_2}{\partial y} dy \wedge dz \wedge dx + 0 + 0$$

$$+ \frac{\partial f_3}{\partial z} dz \wedge dx \wedge dy + 0 + 0 =$$

$$= \frac{\partial f_1}{\partial x} dx \wedge dy \wedge dz + \frac{\partial f_2}{\partial y} dx \wedge dy \wedge dz + \frac{\partial f_3}{\partial z} dx \wedge dy \wedge dz =$$

$$= \left(\frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z}\right) dx \wedge dy \wedge dz$$

$\longleftrightarrow F = (f_1, f_2, f_3)$

$$\operatorname{div} F = \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z} \longleftrightarrow d\omega$$

(3) $n = 3,\ k = 3$ $\quad \omega = f(x,y,z)\, dx \wedge dy \wedge dz$

$d\omega = 0$

4-form on $\mathbb{R}^3$ $\quad \binom{3}{4} = 0$

(4) $n = 2 \quad k = 1$ $\quad \omega = f(x,y)\, dx + g(x,y)\, dy$

$$d\omega = \cancel{\frac{\partial f}{\partial x} dx \wedge dx} + \frac{\partial f}{\partial y} dy \wedge dx + \frac{\partial g}{\partial x} dx \wedge dy + \cancel{\frac{\partial g}{\partial y} dy \wedge dy} = \left(-\frac{\partial f}{\partial y} + \frac{\partial g}{\partial x}\right) dx \wedge dy$$

Green's thm $\quad \int_\gamma f\, dx + g\, dy = \int \left(-\frac{\partial f}{\partial y} + \frac{\partial g}{\partial x}\right) dx\, dy$

$k = 2$ $\quad \omega = f(x,y)\, dx \wedge dy$

$d\omega = 0$

(5) $n = 1 \quad k = 1$ $\quad \omega = f(x)\, dx$

$d\omega = 0$

Thm a $\boxed{d(\omega+\eta) = d\omega + d\eta \quad \omega, \eta\ \kappa\text{-form}}$

b ω is κ-form η is l form
$\omega \wedge \eta$ is $\kappa+l$ form
$d(\omega\wedge\eta) = d\omega \wedge \eta + (-1)^{\kappa}\omega \wedge d\eta$

$\kappa+l+1$ form

$\kappa+l$

$\kappa+l$ form

$l+1$ form

c $\boxed{d(d\omega) = 0}$

Proof (c) $d(d\omega) =$

$\omega = \sum_{i_1 < \dots < i_\kappa} \omega_{i_1 \dots i_\kappa}\, dx^{i_1}\wedge \dots \wedge dx^{i_\kappa}$

$= d\left(\sum_{i_1<\dots<i_\kappa} \sum_{i=1}^{n} D_i \omega_{i_1\dots i_\kappa}\, dx^i \wedge dx^{i_1}\wedge\dots\wedge dx^{i_\kappa}\right) =$

$= \sum_{i_1<\dots<i_\kappa} \sum_{i=1}^{n} \sum_{j=1}^{n} D_j\left(D_i \omega_{i_1\dots i_\kappa}\right) dx^j \wedge dx^i \wedge dx^{i_1} \dots \wedge dx^{i_\kappa})$

if $i = j$ $\quad dx^i \wedge dx^j = 0$

$i \neq j$ $\quad (i,j)\ (j,i)$:

$D_j D_i \omega_{i_1\dots i_\kappa}\, dx^j dx^i + D_i D_j \omega_{i_1 \dots i_\kappa}\, dx^i \wedge dx^j$

$- \quad " \quad - D_i D_j \omega_{i_1\dots i_\kappa}\, dx^j \wedge dx^i$

$\cancel{D_j D} \quad " \quad - D_j D_i \omega_{i_1 \dots i_\kappa}\, dx^j \wedge dx^i =$

0

For f-ns like $\omega_{i_1\dots i_\kappa}$ which have continuous mixed partial derivatives we have proved

$D_i D_j \omega_{i_1\dots i_\kappa} = D_j D_i \omega_{i_1\dots i_\kappa}$

01/12/11

ω k - form

$p \in \mathbb{R}^n$ $\omega(p) \in \Lambda^k(\mathbb{R}^n_p)$

$$\omega = \sum_{i_1 < i_2 < \ldots < i_k} \omega_{i_1 i_2 \ldots i_k} dx^{i_1} \wedge dx^{i_2} \wedge \ldots \wedge dx^{i_k}$$

$$\underset{k+1 \text{ form}}{d\omega} = \sum_{i_1 < i_2 < \ldots < i_k} \sum_{\alpha=1}^{n} D_\alpha(\omega_{i_1 \ldots i_k}) dx^\alpha \wedge dx^{i_1} \wedge \ldots \wedge dx^{i_k}$$

$d(\omega + \eta) = d\omega + d\eta$

$d(d\omega) = 0$

$dx^i \wedge dx^j = -dx^j \wedge dx^i$

$dx^i \wedge dx^i = 0$

Product Rule ω k-form, η an l-form

$$d(\omega \wedge \eta) = d\omega \wedge \eta + (-1)^k \omega \wedge d\eta$$

Proof (b) Monomial term $\eta = \sum_{j_1 < j_2 < \ldots < j_l} \eta_{j_1 \ldots j_l} dx^{j_1} \wedge \ldots \wedge dx^{j_l}$

$$\omega \wedge \eta = \sum_{i_1 < i_2 \ldots < i_k} \sum_{j_1 < j_2 \ldots < j_l} \omega_{i_1 \ldots i_k} \eta_{j_1 j_2 \ldots j_l} dx^{i_1} \wedge dx^{i_2} \ldots \wedge dx^{i_k} \wedge dx^{j_1} \wedge \ldots \wedge dx^{j_l}$$

$$d(\omega \wedge \eta) = \sum_{i_1 < \ldots < i_k} \sum_{j_1 < \ldots < j_l} \sum_{\alpha=1}^{n} D_\alpha(\omega_{i_1 \ldots i_k} \eta_{j_1 \ldots j_l}) dx^\alpha \wedge dx^{i_1} \ldots \wedge dx^{i_k} \wedge dx^{j_1} \ldots \wedge dx^{j_l} =$$

$$= \sum_{i_1 < \ldots < i_k} \sum_{j_1 < \ldots < j_l}^{n} \sum_{\alpha}^{n} (D_\alpha \omega_{i_1 \ldots i_k} \eta_{j_1 \ldots j_l} + \omega_{i_1 \ldots i_k} D_\alpha \eta_{j_1 \ldots j_l}) dx^\alpha \wedge dx^{i_1} \ldots \wedge dx^{j_l}$$

$$= \sum_{i_1<\dots<i_k} \sum_{j_1<\dots<j_l} \sum_{\alpha=1}^{n} D_\alpha w_{i_1\dots i_k} \eta_{j_1\dots j_l} \, dx^\alpha \wedge dx^{i_1} \dots \wedge dx^{j_l} +$$

$$+ \sum_{i_1<\dots<i_k} \sum_{j_1<\dots<j_l} \sum_{\alpha=1}^{n} w_{i_1\dots i_k} D_\alpha \eta_{j_1\dots j_l} \, dx^\alpha \wedge dx^{i_1} \dots \wedge dx^{j_l} =$$

$$= \left(\sum_{i_1<\dots<i_k} \sum_{\alpha=1}^{n} D_\alpha w_{i_1\dots i_k} dx^\alpha \wedge dx^{i_1} \wedge \dots \wedge dx^{i_k} \right) \wedge$$

$$\wedge \left(\sum_{j_1<\dots<j_l} \eta_{j_1\dots j_l} \, dx^{j_1} \wedge \dots \wedge dx^{j_l} \right) +$$

$$+ \sum_{i_1<\dots<i_k} w_{i_1\dots i_k} \, dx^{i_1} \wedge \dots \wedge dx^{i_k} \wedge (-1)^k \sum_{j_1<\dots<j_l} \sum_{\alpha=1}^{n} D_\alpha \eta_{j_1\dots j_l} \, dx^\alpha \wedge dx^{j_1} \dots \wedge dx^{j_l}$$

$$= d\omega \wedge \eta + \omega \wedge (-1)^k d\eta$$

closed and exact forms

Let ω be a k-form

Def ω is called **closed** if $d\omega = 0$

ω is called **exact** if $\exists$ a $(k-1)$ form η s.t. $d\eta = \omega$

Prop $\boxed{\text{If } \omega \text{ is exact then it is closed}}$

Proof exact $\Rightarrow$ $\omega = d\eta$ then $d\omega = d(d\eta) = 0$

Ex. $n = 2$ $k = 1$ $\quad \omega = P(x,y)\,dx + Q(x,y)\,dy$

$$d\omega = \left(-\frac{\partial P}{\partial y} + \frac{\partial Q}{\partial x} \right) dx \wedge dy$$

ω is closed if $\dfrac{\partial Q}{\partial x} = \dfrac{\partial P}{\partial y}$

Recall: vector field $F = P\vec{i} + Q\vec{j}$

If $\dfrac{\partial P}{\partial y} = \dfrac{\partial Q}{\partial x}$ we call it conservative vector field

F is conservative if F has a potential function

$F = \text{grad}(f)$

when ω is exact?

0-form $\eta = f$

$\omega = d\eta \Rightarrow \omega = df = \dfrac{\partial f}{\partial x}dx + \dfrac{\partial f}{\partial y}dy$

i.e. $\omega = \dfrac{\partial f}{\partial x}dx + \dfrac{\partial f}{\partial y}dy$

$\text{grad}(f) = \dfrac{\partial f}{\partial x}\vec{i} + \dfrac{\partial f}{\partial y}\vec{j}$

Ex. (a) $\omega = xy^2\,dx + y\,dy$

$d\omega = \dfrac{\partial(xy^2)}{\partial y}dy \wedge dx + \dfrac{\partial y}{\partial x}dx \wedge dy$

$= 2xy\,dy \wedge dx \neq 0$ not closed

Prop $\Rightarrow$ not exact

(b) $\omega = xy^2\,dx + x^2y\,dy$

$$d\omega = 2xy\, dy \wedge dx + 2xy\, dx \wedge dy =$$
$$= -2xy\, dx \wedge dy + 2xy\, dx \wedge dy = 0$$

So it is closed

Is it exact?

$\exists f$ s.t. $df = \omega$

$$\frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy = \omega = xy^2\, dx + x^2 y\, dy$$

$$\frac{\partial f}{\partial x} = xy^2 \qquad \frac{\partial f}{\partial y} = x^2 y$$

$$f(x,y) = \int xy^2\, dx \qquad \frac{dc}{dy} = 0$$
$$= \frac{x^2y^2}{2} + c(y) \qquad c = k$$

$$\therefore \frac{\partial f}{\partial y} = x^2 y + \frac{dc}{dy} = x^2 y$$

$$\therefore f = \frac{x^2y^2}{2} + k \quad \Rightarrow \omega \text{ is exact.}$$

$n > 2 \qquad k = 1$

$\omega = \omega_1 dx^1 + \dots + \omega_n dx^n$

Is it exact $\omega = df = D_1 f dx^1 + \dots + D_n f dx^n$

$\omega_i = D_i f$

~~So~~ I can assume $f(0) = 0$

You can recover f by an integration in 1 var. t

$$f(x) - f(0) = \int_0^1 \frac{d}{dt}\left[f(tx)\right] dt \qquad x \in \mathbb{R}^n$$

$$\overset{\text{F.T. of calculus}}{=} f(tx)\Big|_{t=0}^{t=1}$$

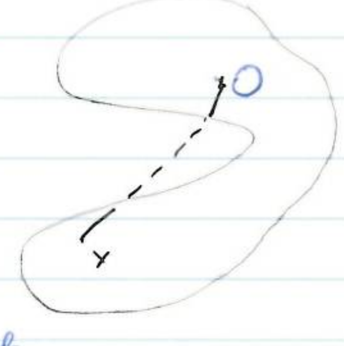
$$= f(x) - f(0)$$

$$f(x) = \int_0^1 \sum_{\alpha=1}^{n} D_\alpha f(tx) \frac{d}{dt}(t x^\alpha)\, dt$$

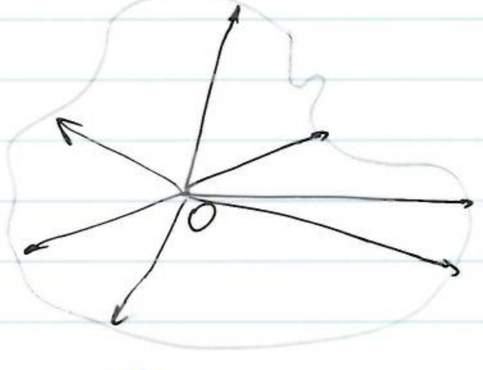
$$= \int_0^1 \sum_{\alpha=1}^{n} D_\alpha f(tx) \cdot x^\alpha\, dt$$

$$f(x) = \int_0^1 \sum_{\alpha=1}^{n} \omega_\alpha(tx)\, x^\alpha\, dt$$

Def A is a star-shaped region with respect to 0 (or p) if $\forall t \in [0,1]$, $\forall x \in A$ we have
$$t \cdot x \in A \qquad (p + t(x-p) \in A)$$



not



yes

Lemma Poincaré lemma

If A is star-shaped w.r.t. 0 and ω is a closed form on A

then ω is an exact form on A

Proof For any l-form I'll define $l-1$ form
$$I(\omega) \quad \text{s.t.} \quad I(\lambda\omega_1 + \omega_2) = \lambda I(\omega_1) + I(\omega_2)$$
$$I(0) = 0$$
$$\& \quad \underbrace{d\,\underbrace{I(\omega)}_{l-1}}_{l\text{ form}} + \underbrace{I(\underbrace{d\omega}_{l+1\text{ form}})}_{l\text{ form}} = \omega$$

$d(I(\omega)) + I(d\omega) = \omega$ (*) unproved yet

Then if ω is closed $d\omega = 0$ so $I(d\omega) = 0$

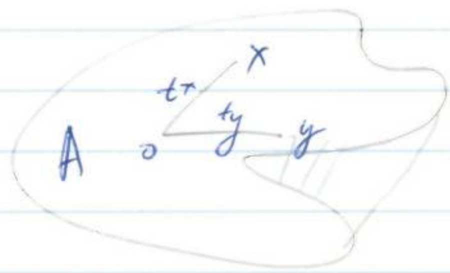
So we get $d(I(\omega)) = \omega \Rightarrow \omega$ is exact

$$\omega = \sum_{i_1 < \dots < i_l} \omega_{i_1 \dots i_l}\, dx^{i_1} \wedge \dots \wedge dx^{i_l}$$

$$I(\omega) = \sum_{i_1 < \dots < i_l} \sum_{\alpha=1}^{l} (-1)^{\alpha-1} \int_0^1 t^{l-1} \omega_{i_1 \dots i_l}(tx)\, \cancel{x^{i_\alpha}}\, dt\; x^{i_\alpha}\, dx^{i_1} \wedge dx^{i_2} \dots \wedge \widehat{dx^{i_\alpha}} \wedge \dots dx^{i_l}$$

this is removed

$$I(0) = 0$$



Because I is a lin ~~comb~~ operator it suffices to prove it for

$$\omega = f(x^1,\dots,x^n)\,dx^{i_1}\wedge\dots\wedge dx^{i_l}$$

$$d\omega = \sum_{\beta=1}^{n} D_\beta f(x^1,\dots,x^n)\,dx^\beta\wedge dx^{i_1}\wedge\dots\wedge dx^{i_l}$$

$$dI(\omega) = \sum_{\beta=1}^{n}\sum_{\alpha=1}^{l}(-1)^{\alpha-1}\int_0^1 t^{l-1} D_\beta\big(f(tx)\,x^{i_\alpha}\big)\,dt\; dx^\beta\wedge dx^{i_1}\dots\wedge\widehat{dx^{i_\alpha}}\dots =$$

$$= \sum_{\beta=1}^{n}\sum_{\alpha=1}^{l}(-1)^{\alpha-1}\int_0^1 t^{l-1}\big(\delta_{\beta,i_\alpha} f(tx) + x^{i_\alpha}(D_\beta f)(tx)\cdot t\big)dt\; dx^\beta\wedge dx^{i_1}\dots\wedge\widehat{dx^{i_\alpha}}\dots$$

$$= \sum_{\alpha=1}^{l}(-1)^{\alpha-1}\int_0^1 t^{l-1} f(tx)\,dt\; dx^{i_\alpha}\wedge dx^{i_1}\dots\wedge\widehat{dx^{i_\alpha}}\wedge\dots dx^{i_l} +$$

$$+ \sum_{\beta=1}^{n}\sum_{\alpha=1}^{l}(-1)^{\alpha-1}\int_0^1 t^{l+1} x^{i_\alpha}(D_\beta f)(tx)\,dt\; dx^\beta\wedge dx^{i_1}\dots\wedge\widehat{dx^{i_\alpha}}\dots$$

not depend on α

$$= \cancel{\sum_{\alpha=1}^{l}}\,\cancel{(-1)^{\alpha-1}}\,\cancel{(-1)^{\alpha-1}}\; l\int_0^1 t^{l-1} f(tx)\,dt\; dx^{i_1}\wedge\dots\wedge dx^{i_\alpha}\wedge\dots dx^{i_l} +$$

$$+ \sum_{\beta=1}^{n}\sum_{\alpha=1}^{l}(-1)^{\alpha-1}\int_0^1 t^{l}(D_\beta f)(tx)\,dt\; dx^{i_\beta}\wedge\dots dx^{i_\alpha}\dots dx^{i_l}$$

~~or~~ same as above

$$I(d\omega) = \sum_{J\in\{\beta,i_1,\dots,i_l\}}\sum_{\beta=1}^{n}(-1)^{j-1}\int_0^1 t^l D_\beta f(tx)\,dt\; x^j\,(dx^\beta\wedge dx^{i_1}\dots\wedge\widehat{dx^j}\dots\wedge dx^{i_l})$$

$$= \sum_{\beta=1}^{n}\cancel{(-1)^{\beta-1}}\int_0^1 t^l D_\beta f(tx)\,dt\; x^\beta\,dx^{i_1}\wedge\dots\wedge dx^{i_l} +$$

$=$

mistake

$$+\sum_{\alpha=1}^{\ell}\sum_{\beta=1}^{n}(-1)^{\alpha}\int_0^1 t^\ell (D_\beta f)(tx)\,dt\; x^{i_\alpha} dx^\beta \wedge \ldots \wedge \widehat{dx^{i_\alpha}} \ldots$$

$$dI(\omega)+I(d\omega)=\int_0^1 \ell t^{\ell-1} f(tx)\,dt\; dx^{i_1}\wedge\ldots\wedge dx^{i_\ell}+$$

$$+\sum_{\beta=1}^{n} x^\beta \int_0^1 t^\ell (D_\beta f)(tx)\,dt\; dx^{i_1}\wedge\ldots\wedge dx^{i_\ell}$$

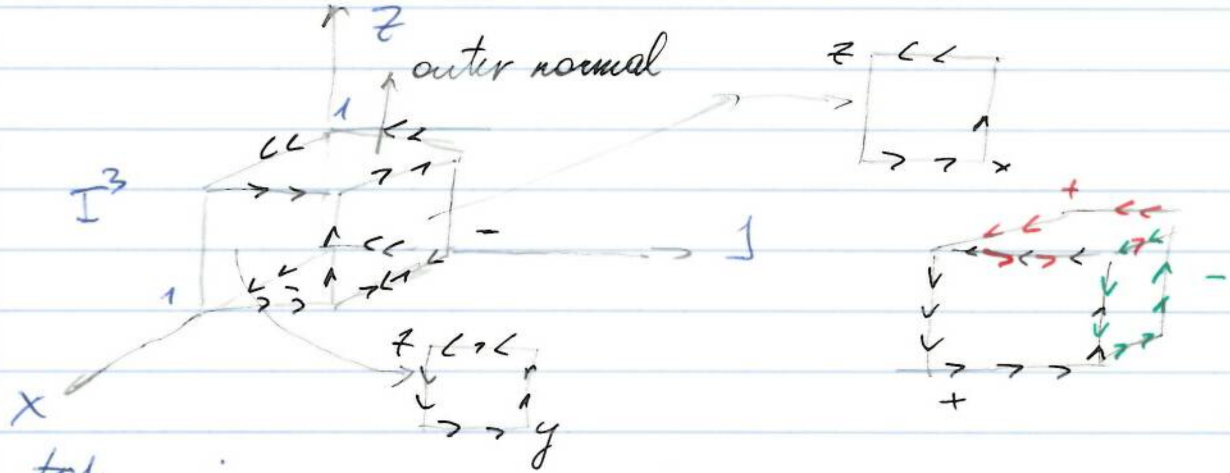
$$=\left(\int_0^1\left[\ell t^{\ell-1}f(tx)+\sum_{\beta=1}^{n}x^\beta t^\ell (D_\beta f)(tx)\right]dt\right)dx^{i_1}\wedge\ldots\wedge dx^{i_\ell}$$

$$=\int_0^1 \frac{d}{dt}\left(t^\ell f(tx)\right)dt\; dx^{i_1}\wedge\ldots\wedge dx^{i_\ell}$$

$$= t^\ell f(tx)\Big|_{t=0}^{t=1} dx^{i_1}\wedge\ldots dx^{i_\ell}$$

$$=\overset{t=1}{f(1\cdot x)}\,dx^{i_1}\wedge\ldots\wedge dx^{i_\ell}-\overset{t=0}{0}$$

$$=\omega$$



top

$$I^3_{(3,1)} = \{(x,y,1),\ 0 \le x \le 1,\ 0 \le y \le 1\}$$

base

$$I^3_{(3,0)} = \{(x,y,0),\ 0 \le x \le 1,\ 0 \le y \le 1\}$$

$$\partial I^3 = I^3_{(3,1)} - I^3_{(3,0)} + I^3_{(1,1)} + I^3_{(1,0)} + I^3_{(2,0)} - I^3_{(2,1)}$$

front face

$$I^3_{(1,1)} = \{(1,y,z),\ 0 \le y, z \le 1\}$$

back

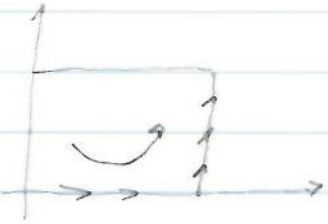
$$I^3_{(1,0)} = \{(0,y,z),\ 0 \le y, z \le 1\}$$

left

$$I^3(2,0) = \{(x,0,z),\ 0 \le x, z \le 1\}$$

right

$$I^3_{(2,1)} = \{(x,1,z),\ 0 \le x, z \le 1\}$$



Def. given an n-cube $I^n = [0,1]^n$
we define the various **faces** to be

$$I^n_{(i,0)} = \{(x^1, x^2, \dots, x^{i-1}, 0, x^{i+1}, \dots, x^n),\ 0 \le x^j \le 1\}$$

$$I^n_{(i,1)} = \{(x^1, x^2, \dots, x^{i-1}, 1, x^{i+1}, \dots, x^n),\ 0 \le x^j \le 1\}$$

$$j = 1, 2, \dots, i-1, i+1, \dots, n$$

and we define the **boundary** of I^n to be

$$\partial I^n = \sum_{i=1}^{n} \sum_{\alpha=0}^{1} (-1)^{i+\alpha} I^n_{(i,\alpha)}$$

We form formal sums of singular n-cubes with integer coefficients
(this is the contruction of a cetrain abelian gp or $\mathbb{Z}$-module)

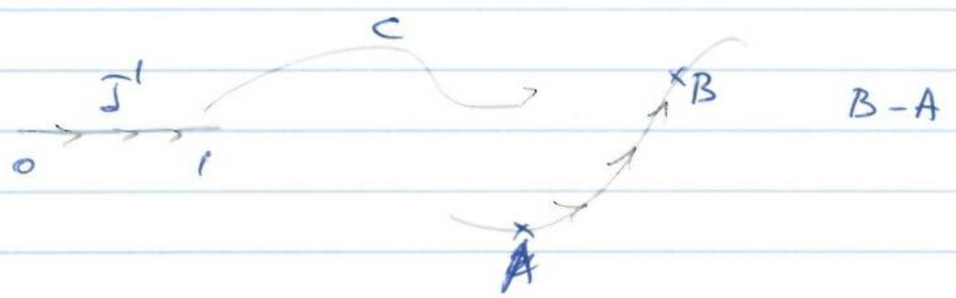
$c_1 : I^1 \to A$
$c_2 : I \to A$

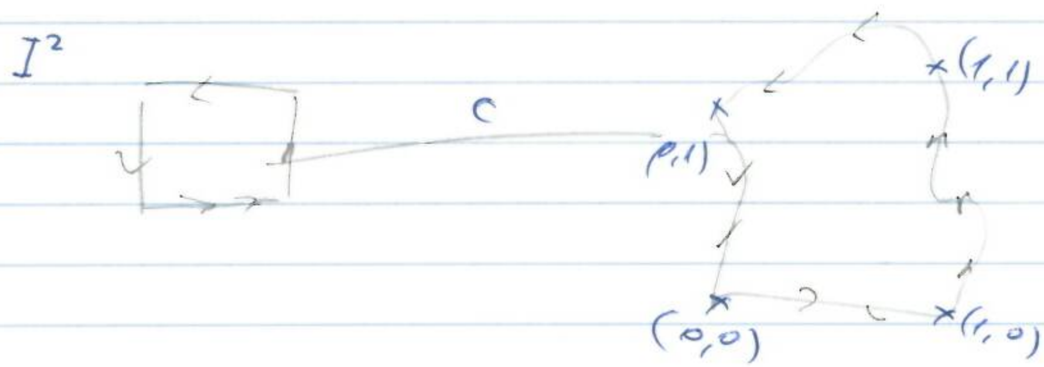
e.g. $c\,c_1 + 1 - 5\,c_2$

these are singular n-chains

Def A singular n-chain c in A is a (finite) lin. combination with integer coefficients of singular n-cubes

$$c = \sum_{j=1}^{m} m_j c_j \qquad m_j = \mathbb{Z}$$
$$c_j : I^n \to A$$





Def If c is a singular n-cube ($c: I^n \to A$)

then $\partial c = \sum_{i=1}^{n} \sum_{\alpha=0}^{1} (-1)^{i+\alpha} c(I^n_{(i,\alpha)})$

For a singular n-chain $c = \sum_{j=1}^{m} m_j c_j$, where c_j are singular n-cubes
$$\partial c = \sum_{j=1}^{n} m_j \partial(c_j)$$

~~Thm~~ $\cancel{\partial(\partial c) = 0}$ (see Algebraic topology)

$$\int_{\boxed{\partial c}} \boxed{\omega} = \int_{\boxed{c}} \boxed{d\omega}$$

if ω is $k-1$ form
$d\omega$ is a k form
c will be a singular k-chain
∂c is a singular $(k-1)$-chain

Today 1) on $\mathbb{R}^n$ we will define integration of a k form on a k-cube

2) $(k-1)$ form on a $(k-1)$-cube face

Let ω be a k form on I^k
$\forall p \in I^k$, $\omega(p) \in \Lambda^k(\mathbb{R}^k_p)$

$$\omega = f(x^1, \dots x^k) \; dx^1 \wedge dx^2 \wedge \dots \wedge dx^k$$

We define $\int_{I^k} \omega = \int_{I^k} f(x^1, \dots, x^k) \, dx^1 \wedge \dots \wedge dx^k$

$$\overset{\text{def}}{=} \int_{[0,1]^k} f(x^1, x^2, \dots, x^k) \, dx^1 \, dx^2 \dots dx^k$$

Riemann integral

can be evaluated by Fubini's Thm $= \underbrace{\int_0^1\left(\int_0^1\left(\int_0^1 \dots \int_0^1\right.\right.}_{k \text{ integrations}} f(x^1, x^2, \dots, x^k) \left.\left. dx^1\right) dx^2\right) \dots dx^k$

On $\mathbb{R}^k$

η be a $k-1$ form on $I^k_{(i,\alpha)}$

basis of $k-1$ forms in $\mathbb{R}^k$ is $dx^1 \wedge dx^2 \wedge \ldots \wedge \widehat{dx^j} \ldots \wedge dx^k$

$j = 1, \ldots, k$

Assume η is given by

$$\eta = g(x^1, x^2, \ldots, x^k)\, dx^1 \wedge dx^2 \wedge \ldots \wedge dx^{j-1} \wedge \widehat{dx^j} \wedge dx^{j+1} \wedge \ldots \wedge dx^k$$

$$\int_{I^k_{(i,\alpha)}} \eta = \begin{cases} \int_{[0,1]^{k-1}} g(x^1, x^2, \ldots, x^{j-1}, \alpha, x^{j+1}, \ldots, x^k)\, dx^1 dx^2 \ldots \widehat{dx^j} \ldots dx^k & i = j \\ 0 & i \neq j \end{cases}$$

1) $\underline{i = j}$

$x^j = x^i = \alpha$

$I^k_{(i,\alpha)}$

2) $\underline{i \neq j}$

$\int_{I^2_{(2,0)}} dy = 0$

$I^2_{(1,0)} \int dx = 0$

$x = 0$

$\int_{I^2_{(2,0)}} dx^2$ $i = 2$

missing dx^1 $1 = j$

$\int_{I^2_{(1,0)}} dx^1$ missing dx^2

$j = 2$

$i = 1$

If $\eta = \sum_{j=1}^{n} g_j(x^1,\dots,x^k)\, dx^1 \wedge \dots \wedge \widehat{dx^j} \wedge \dots \wedge dx^k$

then $\int_{I^k_{(i,\alpha)}} \eta = \sum_{j=1}^{n} \int_{I^k_{(i,\alpha)}} g_j(x^1,\dots,x^k)\, dx^1 \wedge \dots \wedge \widehat{dx^j} \wedge \dots \wedge dx^k$

If ω is a 0-form then ω is a f-n $f(x^1,\dots,x^k)$
0-cube is the point $\{0\}$

$$\int_{I^0} \omega = f(0,0,\dots,0)$$

If $c = \sum_{j=1}^{m} m_j c_j$ where c_j are all standard k-cubes

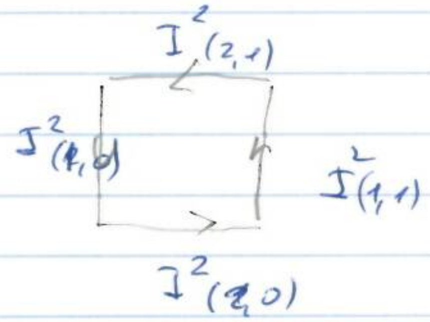
then $\int_c \omega = \sum_{i=1}^{n} m_j \int_{c_j} \omega$

if $c = \sum_{j=1}^{m} c_j$ where c_j are k-1 cubes

then $\int_c \eta = \sum_{j=1}^{n} m_j \int_{c_j} \eta$

$\int_{I^1} \omega$

$$\int_{5I^1} \omega = 5 \int_{I^1} \omega$$



1-form

$$\int_{\partial I^2} \eta = +\int_{I^2_{(2,0)}} \eta + \int_{I^2_{(1,1)}} \eta - \int_{I^2_{(2,1)}} \eta - \int_{I^2_{(1,0)}} \eta$$

8/12/14 **Stokes Theorem**

$$\int_{\partial c} \omega = \int_c d\omega$$

ω $k-1$ form
$d\omega$ k form
c k - singular chain
∂c $(k-1)$ sing. chain

We defined:

$\int_c \omega$ if ω is k form in $\mathbb{R}^k$
c k - cube (standard):
$c = I^k = [0,1]^k$

$\int_{I^k_{(i,\alpha)}} \eta$ $\quad \eta$ $k-1$ form

Proof on $\mathbb{R}^k$ for ω $k-1$ form
$c = I^k$ standard k - cube

$$\int_{\partial I^k} \omega = \int_{I^k} d\omega$$

We know $\int_c \eta$ is linear in η

i.e. $\int_c \lambda \eta_1 + \eta_2 = \lambda \int_c \eta_1 + \int_c \eta_2$

Therefore it suffices to prove it for

$$\omega = f(x^1, x^2, \dots, x^k)\, dx^1 \wedge dx^2 \wedge \dots \wedge \widehat{dx^j} \wedge \dots dx^k$$

$$d\omega = \sum_{\beta=1}^{k} D_\beta f\, dx^\beta \wedge dx^1 \wedge dx^2 \wedge \dots \wedge \widehat{dx^j} \wedge \dots \wedge dx^k$$

$$= D_j f\, dx^j \wedge dx^1 \wedge dx^2 \wedge \dots \wedge \widehat{dx^j} \wedge \dots \wedge dx^k =$$

$$(-1)^{j-1} D_j f\, dx^1 \wedge dx^2 \wedge \dots \wedge dx^k$$

$$\int_{I^k} d\omega = (-1)^{j-1} \int_{I^k} D_j f\, dx^1 \wedge dx^2 \wedge \dots \wedge dx^k =$$

$$= \int_{[0,1]^k} D_j f\, dx^1 dx^2 \dots dx^k (-1)^{j-1}$$

$$= (-1)^{j-1} \int_0^1 \int_0^1 \dots \left(\int_0^1 D_j f\, dx^j \right) dx^1 dx^2 \dots \widehat{dx^j} \dots dx^k$$

F.T.C. $$= (-1)^{j-1} \int_0^1 \int_0^1 \dots \left(f(x^1, x^2, \dots, x^k) \Big|_{x^j=0}^{x^j=1} \right) dx^1 dx^2 \dots \widehat{dx^j} \dots dx^k$$

$$= (-1)^{j-1} \int_0^1 \int_0^1 \dots f(x^1, x^2, \dots, 1, x^{j+1}, \dots, x^k)\, dx^1 \dots \widehat{dx^j} \dots dx^k -$$

$$- (-1)^{j-1} \int_0^1 \int_0^1 \dots f(x^1, \dots, 0, x^{j+1}, \dots x^k)\, dx^1 \dots \widehat{dx^j} \dots dx^k$$

$k-1$ form

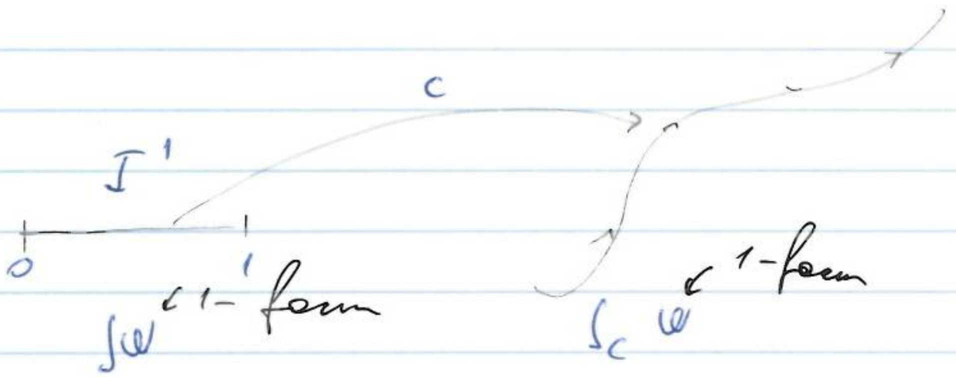
$$\int_{\partial I^k} \omega \overset{\text{def}}{=} \sum_{i=1}^{k} \sum_{\alpha=0,1} (-1)^{i+\alpha} \int_{I^k_{(i,\alpha)}} \omega$$

j slot

only $i=j$ remains $$\overset{\text{def}}{=} \sum_{\alpha=0,1} (-1)^{j+\alpha} \int_{[0,1]^{k-1}} f(x^1, x^2, \dots, \alpha, x^{j+1}, \dots x^k)\, dx^1 dx^2 \dots \widehat{dx^j} \dots dx^k$$

$$= \int_{[0,1]^{k-1}} f(x^1, x^2, \dots, 1, x^{j+1}, \dots x^k)\, dx^1 \dots \widehat{dx^j} \dots dx^k \, (-1)^{j+1} +$$

$$+ (-1)^j \int_{[0,1]^{k-1}} f(x^1, \dots, 0, x^{j+1}, \dots x^k)\, dx^1 \dots \widehat{dx^j} \dots dx^k$$



We will define c^* pull-back

Def If ω is a k-form on A containing sing. k-cube c $\quad c: I^k \to A$
then $\int_c \omega = \int_{I^k} c^*(\omega)$

How to define the pullback of a k form c?

Recall if $V \xrightarrow{S} W \downarrow T$ to $\mathbb{R}$, $\quad V, W$ vector spaces
S lin. $\quad T$ lin functions

$S^*(T) = T \circ S$ linear

Pullback of the tensors

$T \in \mathcal{J}^k(W)$ then $S^*(T) \in \mathcal{J}^q(V)$

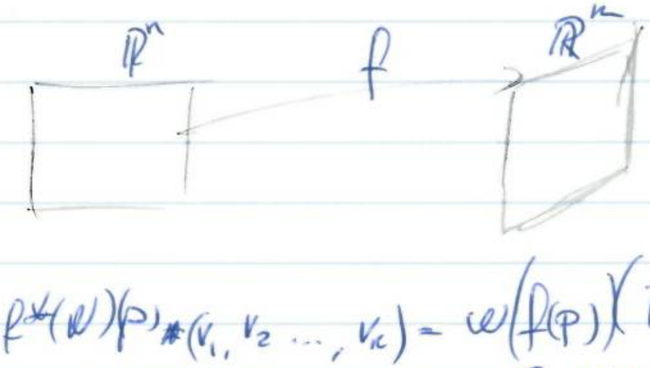
def.

$S^*(T)(v_1, \dots, v_k) = T(S(v_1), \dots, S(v_k))$

v_i's $\in V$

Let ω be a k-form on $\mathbb{R}^m$ and
$f : \mathbb{R}^n \to \mathbb{R}^m$

then we want to define:



$f^*(\omega)(p) \in \Lambda^k(\mathbb{R}^n_p)$
$\forall p \in \mathbb{R}^n$

$f^*(\omega)(p)_*(v_1, v_2, \dots, v_k) = \omega(f(p))(Df(v_1), Df(v_2), \dots, Df(v_k))$

$v_i \in \mathbb{R}^n_p$ $\quad \in \Lambda^k(\mathbb{R}^n_{f(p)})$

Let $f : \mathbb{R}^n \to \mathbb{R}^m$ be diff-ble
$p \in \mathbb{R}^n$
$Df(p) : \mathbb{R}^n \to \mathbb{R}^m$ lin. map

Define the push-forward of $\mathbb{R}^n_p$ to $\mathbb{R}^m_{f(p)}$

If $v_p \in \mathbb{R}^n_p$ $\quad v_p = (p, v) \quad v \in \mathbb{R}^n$

then $f_*(v_p) \in \mathbb{R}^m_{f(p)}$

$$f_*(v_p) = (f(p), Df(p)(v))$$

$f_* : \mathbb{R}^n_p \to \mathbb{R}^m_{f(p)}$ is linear

if $v_p, w_p \in \mathbb{R}^n_p$ $\quad \lambda \in \mathbb{R}$

$$f_*(\lambda v_p + w_p) = f_*(\lambda(p,v) + (p,w)) = f_*((p, \lambda v + w)) \stackrel{def}{=}$$

$$= (f(p), Df(p)(\lambda v + w))$$

$$\stackrel{Df(p) \text{ is linear}}{=} (f(p), \lambda Df(p)(v) + Df(p)(w))$$

$$\stackrel{def.}{=} \lambda(f(p), Df(p)(v)) + (f(p), Df(p)(w))$$

$$\stackrel{def.}{=} \lambda f_*(v_p) + f_*(w_p)$$

If $T \in \mathcal{J}^k(\mathbb{R}^m_{f(p)})$

then $f^*(T)$ will be $\mathcal{J}^k(\mathbb{R}^n_p)$ defined by

$$f^*(T)(v_1, v_2, \dots, v_k) = T(f_*(v_1), f_*(v_2), \dots, f_*(v_k))$$

$v_i \in \mathbb{R}^n_p$

If ω is a k-form on $\mathbb{R}^n$ then $f^*(\omega)$ is a k-form on $\mathbb{R}^n$ defined by.. $\forall p \in \mathbb{R}^n$

$$f^*(\omega)(p)(v_1, v_2, \dots, v_k) = \omega(f(p))(f_*(v_1), f_*(v_2), \dots, f_*(v_k))$$

$v_i \in \mathbb{R}^n_p$

Proposition $f: \mathbb{R}^n \to \mathbb{R}^m$ diff, $\quad g: \mathbb{R}^m \to \mathbb{R}$

(i) $f^*(dx^i) = \sum_{j=1}^{n} D_j f^i \, dx^j$

(ii) $f^*(\lambda \omega_1 + \omega_2) = \lambda f^*(\omega_1) + f^*(\omega_2)$

(iii) $f^*(g \cdot \omega) = g \circ f \; f^*(\omega)$

(iv) $f^*(\omega \wedge \eta) = f^*(\omega) \wedge f^* \eta$

Example ω 1-form in $\mathbb{R}^3$

$\omega = P(x,y,z)\,dx + Q(x,y,z)\,dy + R(x,y,z)\,dz$

$f: [0,1] \to \mathbb{R}^3$ parametrises a curve in $\mathbb{R}^3$ ($t \in$)

$f^*(\omega)$ 1-form on $[0,1]$

$f^*(\omega)$ has to be ?? dt

Let v_t be a tangent vector on $\mathbb{R}^1_t$ $\quad v_t = (t, v)$

$\underbrace{f^*(\omega)(t)}_{\Lambda^1(\mathbb{R}^1_t)}$ no space here $v_t = \omega(f(t))(f_*(v_t)) =$

$= (P\,dx + Q\,dy + R\,dz)(f(t))(f_*(v_t))$

$\overset{iii}{=} P(f(t))\, dx(f(t))(f_*(v_t)) + Q(f(t))\,dy(f(t))(f_*(v_t))$

$+ R(f(t))\, dz(f(t))(f_*(v_t)) =$

$= P(f(t))\; Df^1(t)(v) + Q(f(t)) Df^2(t)(v) + R(f(t))\, Df^3(t)(v)$

// $f_*(v_t) = (f(t), Df(t)(v)) = (f(t), Df^1(t)(v), Df^2(t)(v), Df^3(t)(v))$

// $f = (f^1, f^2, f^3)$

$\Rightarrow f^*(w) = (P\circ f)\frac{df^1}{dt}dt + (Q\circ f)\frac{df^2}{dt}dt + (R\circ f)\frac{df^3}{dt}dt$

$f^*(w) = f^*(P\,dx + Q\,dy + R\,dz) \overset{iii}{=}$

$\overset{iii}{=} (P\circ f)\, f^*dx + (Q\circ f)\, f^*(dy) + (R\circ f)\, f^*(dz)$

$\overset{(i)}{=} \boxed{(P\circ f)\, Df^1 dt + (Q\circ f)\, Df^2 dt + (R\circ f)\, Df^3 dt}$

Proof (i) $f^*(dx^i) = \sum_{j=1}^{n} D_j f^i dx^j$ ← 1 forms on $\mathbb{R}^n$

dx^i 1-form on $\mathbb{R}^m$

Take $p \in \mathbb{R}^n$

$f^*(dx^i)(p) \in \Lambda^1(\mathbb{R}^n_p)$ $\qquad v_p = (p, v) \in \mathbb{R}^n_p$

$f^*(dx^i)(p)(v_p) \overset{def}{=} dx^i(f(p))\,(f_*(v_p))$

$= dx^i(f(p))\,(f(p), Df(p)(v))$

dx^i pick up i-component of the vector

$= (f(p), Df(p)(v))^i$

$f : \mathbb{R}^n \to \mathbb{R}^m$

$$= \left[f'(p)\cdot v\right]^i = \left[\begin{array}{l} \\ i^{th} \text{ row} \rightarrow \end{array} \begin{pmatrix} D_1 f^1 & D_2 f^1 & \dots & D_n f^1 \\ \vdots & & & \\ D_1 f^m & D_2 f^m & \dots & D_n f^m \end{pmatrix} \begin{pmatrix} v^1 \\ v^2 \\ \vdots \\ v^n \end{pmatrix} \right]^i =$$

$$= \sum_{j=1}^{n} D_j f^i v^j$$

Compare

$$\left(\sum_{j=1}^{n} D_j f^i dx^j\right)(p)(v_p) = \sum_{j=1}^{n} D_j f^i(p)\, dx^j(p)(v_p)$$

picks up j^{th} comp.

$$= \sum_{j=1}^{n} D_j f^i(p)\, v^j$$

(iii) $f^*(g\cdot w) = g\circ f\; f^*(w)$

$f : \mathbb{R}^n \to \mathbb{R}^m$
$g : \mathbb{R}^m \to \mathbb{R}$
$p \in \mathbb{R}^n$
$v_1 \dots v_k \in \Lambda^m_p$

$$f^*(gw)(p)(v_1, \dots, v_k) = (gw)(f(p))(f_*(v_1), \dots f_*(v_k))$$
$$= g(f(p))\cdot w(f(p))(f_*(v_1), \dots, f_*(v_k))$$

Compute $(g\circ f)\cdot f^*(w)(p)(v_1, \dots, v_k) = (g\circ f)(p)\; w(f(p))(f_*(v_1), \dots f_*(v_k))$

Recall if $c : I^k \to A$ is a sing. k-cube in A
w is a k-form on A

$$\int_c w \overset{\text{def}}{=} \int_{I^k} c^*(w)$$

Ex. ω 1-form on $\mathbb{R}^2$

$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

$\omega = x\,dy$

, ellipse

$c : [0,1] \to \mathbb{R}^2$

$c(t) = \left(a\cos(2\pi t),\ b\sin(2\pi t)\right)$

$$\int_c x\,dy \overset{\text{def}}{=} \int_{[0,1]} c^*(x\,dy) = \int_0^1 (x\circ c(t)) \frac{dc^2}{dt}\,dt =$$

$$= \int_0^1 a\cos(2\pi t)\, b\, 2\pi \cos 2\pi t\,dt$$

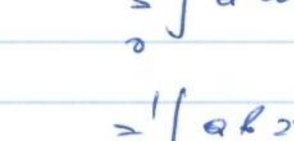
$$= \int_0^1 ab\,2\pi\, \frac{1+\cos 4\pi t}{2}\,dt = \pi\, a\cdot b.$$

Stokes' Theorem

$$\int_c \omega = \int_{\tilde{c}} d\omega = \int_{\tilde{c}} d(x\,dy) = \int_{\tilde{c}} dx\wedge dy$$

area of the region parametrized by $\tilde{c}$

Call $\tilde{c}$ the inside of the ellipse 2-cube



$\tilde{c}(u,t) = \left(a u\cos(2\pi t),\ b u\sin(2\pi t)\right)$

$t \in [0,1]$

$u \in [0,1]$

c

$\partial\tilde{c} = c$

If c is a singular k-chain

i.e. $c = \sum_{j=1}^{m} m_j c_j$ $\quad m_j \in \mathbb{Z}$, c_j sing. k cubes

$$\int_c \omega \overset{\text{def}}{=} \sum_{j=1}^{m} m_j \int_{c_j} \omega = \sum_{j=1}^{m} m_j \int_{I^k} c_j^*(\omega)$$

Thm Stokes' thm for singular k-chains

ω $k-1$ form on $\mathbb{R}^k$

$d\omega$ k form on $\mathbb{R}^k$

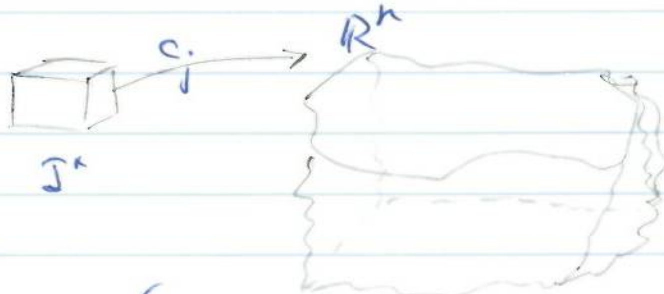
c k-singular chain

∂c $k-1$ sing. chain

then $\int_{\partial c} \omega = \int_c d\omega$

Proof // $c = \sum_{j=1}^{m} m_j c_j$ $\quad m_j \in \mathbb{Z}$ $\quad c_j$ sing. k-cube

// ∂c_j $\quad c_j : I^k \to \mathbb{R}^k$



$$\partial c_j = c_j(\partial I^k)$$

$$= \sum_{i=1}^{k} \sum_{\alpha=0,1} (-1)^{i+\alpha} c_j\left(I^k_{(i,\alpha)}\right)$$

$$\int_{\partial c} \omega = \sum_{j=1}^{n} m_j \int_{\partial c_j} \omega \overset{def}{=} \sum^{n}$$

$$= \sum_{j=1}^{m} \sum_{i=1}^{n} \sum_{\alpha=0,1} m_j \int_{c_j(I^k(i,\alpha))} \omega \, (-1)^{i+\alpha} =$$

$$= \sum_{j=1}^{m} \sum_{i=1}^{k} \sum_{\alpha=0,1} m_j (-1)^{\alpha+i} \int_{c_j(I^k_{(i,\alpha)})} \omega$$

$$\overset{def}{=} \sum_{j=1}^{m} \sum_{i=1}^{k} \sum_{\alpha=0,1} \left(m_j (-1)^{i+\alpha} \int_{I^k_{(i,\alpha)}} c_j^*(\omega) \right)$$

Now compute

$$\int_c d\omega = \sum_{j=1}^{m} m_j \int_{c_j} d\omega =$$

$c_j \leftarrow$ sing. k-cube

$$\overset{def}{=} \sum_{j=1}^{m} m_j \int_{I^k} c_j^* d(\omega) = \sum m_j \int_{I^k} d(c_j^*(\omega)) =$$

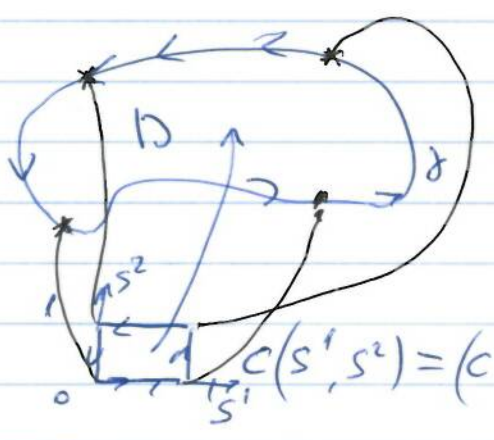
since $d(c_j^* \omega) = c_j^*(d\omega)$

Apply Stokes for stand. k-cube

$$= \sum_{j=1}^{m} m_j \int_{\partial I^k} c_j^*(\omega) = \sum_{j=1}^{m} m_j \sum_{i=1}^{k} \sum_{\alpha=0,1} (-1)^{i+\alpha} \int_{I^k(i,\alpha)} c_j^*(\omega)$$

$\boxtimes$

13/12/11 Classical Stokes theorem in $\mathbb{R}^2$



$$\int_\gamma P(x,y)\,dx + Q(x,y)\,dy = \iint_D \left(-\frac{\partial P}{\partial y} + \frac{\partial Q}{\partial x} \right) dx\,dy$$

$$c(s^1,s^2) = (c^1(s^1,s^2), c^2(s^1,s^2))$$

$$\gamma : [0,1] \to \mathbb{R}^2 \qquad \gamma(t) = (\gamma^1(t), \gamma^2(t))$$

$$\partial c \overset{def}{=} c(\partial I^2) = \gamma$$

$$\int_\gamma P\,dx + Q\,dy = \int_{c(\partial I^2)} P\,dx + Q\,dy \overset{def}{=} \int_{\partial I^2} c^*(P\,dx + Q\,dy)$$

$$= \int_{\partial I^2} P\left(c^1(s^1, s^2), c^2(s^1, s^2)\right) \underset{\gamma^*(dx)}{c^* dx} + Q\left(c^1(s^1, s^2), c^2(s^1, s^2)\right) \underset{\gamma^*(dy)}{c^*(dy)}$$

$$= \int_{\partial I^2} P \frac{d\gamma^1}{dt} dt + Q \frac{d\gamma^2}{dt} dt =$$

$$= \int_{\partial I^2} \left[P(\gamma^1(t), \gamma^2(t)) \frac{d\gamma^1}{dt} + Q(\gamma^1(t), \gamma^2(t)) \frac{d\gamma^2}{dt} \right] dt$$

$$\int_{\gamma} P dx + Q dy \overset{\text{stokes's Tm}}{\underset{\text{in } \mathbb{R}^2}{=}} \int_c \underbrace{d(P dx + Q dy)}_{d\omega}$$

γ singular 1-cube

$= \partial c$ which is the bndry of $c(I^2)$

$$\int_{\partial c} \omega = \int_c d\omega$$

$$\int d(P dx + Q dy) = \int_c P_x dx \wedge dx + P_y dy \wedge dx + Q_x dx \wedge dy + Q_y dy \wedge dx =$$

$$= \int_c - \frac{\partial P}{\partial y} dx \wedge dy + \frac{\partial Q}{\partial x} dx \wedge dy =$$

$$= \int_c \left(- \frac{\partial P}{\partial y} + \frac{\partial Q}{\partial x} \right) dx \wedge dy$$

$$\overset{\text{def}}{=} \int_{I^2} c^* \left(- \frac{\partial P}{\partial y} + \frac{\partial Q}{\partial x} \right) dx \wedge dy =$$

$$= \int_{I^2} \left(- \frac{\partial P}{\partial y}(c^1, c^2) + \frac{\partial Q}{\partial x}(c^1, c^2) \right) c^*(dx \wedge dy)$$

what is $c^*(dx \wedge dy)$?

Prop $c^*(dx) \wedge (dy)$

$$\overset{?}{=} \left(\frac{\partial c^1}{\partial s^1} ds^1 + \frac{\partial c^1}{\partial s^2} ds^2 \right) \wedge \left(\frac{\partial c^2}{\partial s^1} ds^1 + \frac{\partial c^2}{\partial s^2} ds^2 \right)$$

$$= \frac{\partial c^1}{\partial s^1}\frac{\partial c^2}{\partial s^2} ds^1 \wedge ds^2 + \frac{\partial c^1}{\partial s^2}\frac{\partial c^2}{\partial s^1} ds^2 \wedge ds^1 =$$

$$= \left(\frac{\partial c^1}{\partial s^1}\frac{\partial c^2}{\partial s^2} - \frac{\partial c^1}{\partial s^2}\frac{\partial c^2}{\partial s^1}\right) ds^1 \wedge ds^2$$

$$= \det c'(s^1, s^2)\, ds^1 \wedge ds^2 .$$

now it is ordinary double integral

can drop by def.

$$= \int_{I^2} \left(-\frac{\partial P}{\partial y}(c^1, c^2) + \frac{\partial Q}{\partial x}(c^1, c^2)\right) \cdot \det c'(s^1, s^2)\, ds^1 \cancel{\wedge} ds^2 =$$

1) Recall change of variables formula for n-dim integrals

$A \subseteq \mathbb{R}^n \quad g : A \to \mathbb{R}^n$ injective diff.

$\det g'(x) \qquad \forall x \in A$.

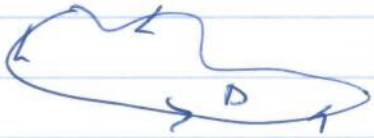
If $f : g(A) \to \mathbb{R}$ is integrable

$$\int_{g(A)} f = \int_A (f \circ g) |\det g'|$$

multiple integrals

$$A \xrightarrow{g} g(A) \subseteq \mathbb{R}^2 \xrightarrow{f} \mathbb{R}$$

2) Assuming $\int$ means going around



It can be shown that $\det c'(s^1, s^2) > 0$

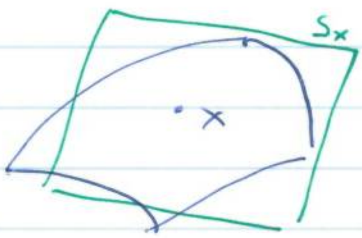
$$= \iint_D -\frac{\partial P}{\partial y}(x, y) + \frac{\partial Q}{\partial x}(x, y)\, dx\, dy$$

Gauss or divergence theorem

solid T in $\mathbb{R}^3$
with boundary surface S
vector field $\vec{F} = (F^1, F^2, F^3)$

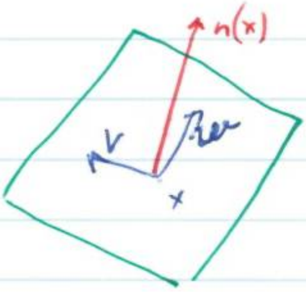
S_x = tangent plane to the solid ~~p~~ at point $x \in S$
$n(x)$ = is outward unit normal vector

$$\boxed{\int_S \underbrace{\langle \vec{F}, \vec{n} \rangle}_{\cancel{\text{scalar}}\ \text{dot product}} dA = \iiint_T \left(\operatorname{div} \vec{F}\right) dx\,dy\,dz}$$



S_x = $\underset{\text{}}{\underline{\text{also}}}$ is a vector space (has dim 2)

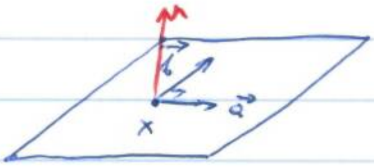
$\dim \Lambda^2(S_x) = \binom{2}{2} = 1$



$v, w \in S_x$

Def $\omega(v, w) = \langle \underset{\text{cross product}}{v \times w}, n \rangle = (v \times w) \underset{\text{dot product}}{\cdot} n$

$$= \begin{vmatrix} v^1 & v^2 & v^3 \\ w^1 & w^2 & w^3 \\ n^1 & n^2 & n^3 \end{vmatrix} = \begin{vmatrix} n^1 & n^2 & n^3 \\ v^1 & v^2 & v^3 \\ w^1 & w^2 & w^3 \end{vmatrix}$$



choose $\bar{a}, \bar{b} \in S_x$ s.t.
$\bar{a}, \bar{b}, \vec{n}$ are orthonormal system
right handed

Notation call $\omega(v, w) = dA(v, w)$

$v, w \in S_x$

Thm
$$\boxed{\begin{aligned} dA = {} & n^1\, dy \wedge dz \\ & + n^2\, dz \wedge dx \\ & n^3\, dx \wedge dy \end{aligned}}$$

$$dA(v, w) = \begin{vmatrix} n^1 & n^2 & n^3 \\ v^1 & v^2 & v^3 \\ w^1 & w^2 & w^3 \end{vmatrix} = n^1(v^2 w^3 - w^2 v^3) + n^2(-v^1 w^3 + v^3 w^1) + n^3(v^1 w^2 - w^1 v^2)$$

$$\begin{aligned} (dy \wedge dz)(v, w) &= (dy \otimes dz - dz \otimes dy)(v, w) = \\ &= dy(v)\, dz(w) - dz(v)\, dy(w) = \\ &= v^2 w^3 - v^3 w^2 \end{aligned}$$

$$(dz \wedge dx)(v, w) = v^3 w^1 - v^1 w^3$$
$$(dx \wedge dy)(v, w) = v^1 w^2 - v^2 w^1$$

thus $\omega(v, w) = dA(v, w)$

Thm
$$n^1 dA = dy \wedge dz \qquad (1)$$
$$n^2 dA = dz \wedge dx \qquad (2)$$
$$n^3 dA = dx \wedge dy \qquad (3)$$

Pf: (1) We know that
$$(dy \wedge dz)(v, w) = v^2 w^3 - v^3 w^2$$
where $v, w \in S_n$

$$dA(v,w) = \langle v \times w, n \rangle$$
since v and w are perp. to $\vec{n}$
i.e. $v \times w = \lambda n$, $\lambda \in \mathbb{R}$

LHS $n^1 dA(v,w) = n^1 \langle \lambda n, n \rangle = n^1 \cdot \lambda \qquad [1]$

RHS $\langle v \times w, i \rangle = \langle \lambda n, i \rangle$
$$= \lambda n^1$$
$$\| $$
$$\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ v^1 & v^2 & v^3 \\ w^1 & w^2 & w^3 \end{vmatrix} \cdot \vec{i} = v^2 w^3 - v^3 w^2 = \cancel{dA(v,w)} (dy \wedge dz)(v,w) \; [2]$$

LHS = RHS $\Leftrightarrow$ (1) is true

Proof Gauss theorem
Given $\vec{F} = (F^1, F^2, F^3) = F^1 \vec{i} + F^2 \vec{j} + F^3 \vec{k}$
$$div(\vec{F}) = \frac{\partial F^1}{\partial x} + \frac{\partial F^2}{\partial y} + \frac{\partial F^3}{\partial z}$$

To $\vec{F}$ we assign the 2-form η

$$\overset{\text{2-form}}{\eta} = F^1\, dy \wedge dz + F^2\, dz \wedge dx + F^3\, dx \wedge dy$$

Calculate $d\eta$ ← 3-form

$$d\eta = \frac{\partial F^1}{\partial x}\, dx \wedge dy \wedge dz + \frac{\partial F^2}{\partial y}\, dy \wedge dz \wedge dx + \frac{\partial F^3}{\partial z}\, dz \wedge dx \wedge dy =$$

$$= \left(\frac{\partial F^1}{\partial x} + \frac{\partial F^2}{\partial y} + \frac{\partial F^3}{\partial z}\right) dx \wedge dy \wedge dz$$

$$\int_T d\eta = \int_T (\operatorname{div} F)\, dx \wedge dy \wedge dz = \iiint \operatorname{div} F\, dx\, dy\, dz$$

T singular 3 cube

change of var. for I^3 to T

// Stokes'

$$\int_{\partial T} \eta = \int_S F^1\, dy \wedge dz + F^2\, dz \wedge dx + F^3\, dx \wedge dy =$$

$$= \int_S F^1 n^1\, dA + F^2 n^2\, dA + F^3 n^3\, dA =$$

$$= \int_S (F^1 n^1 + F^2 n^2 + F^3 n^3)\, dA =$$

$$= \int_S \vec{F} \cdot \vec{n}\, dA$$

□

Recall M is a k-dim manifold in $\mathbb{R}^n$ if for all $x \in M$

(M) $\exists$ U open set in $\mathbb{R}^n$, W open set in $\mathbb{R}^n$, $x \in U$

$\exists$ diffeomorphism $h: U \to \cancel{W}\, V$

s.t. $h(U \cap M) = V \cap \{y \in \mathbb{R}^n,\ y^{k+1} = y^{k+2} = \dots = y^n = 0\}$

Thm [R] M is a k-dim manifold in $\mathbb{R}^n$ iff.
$\forall x \in M$ condition c holds

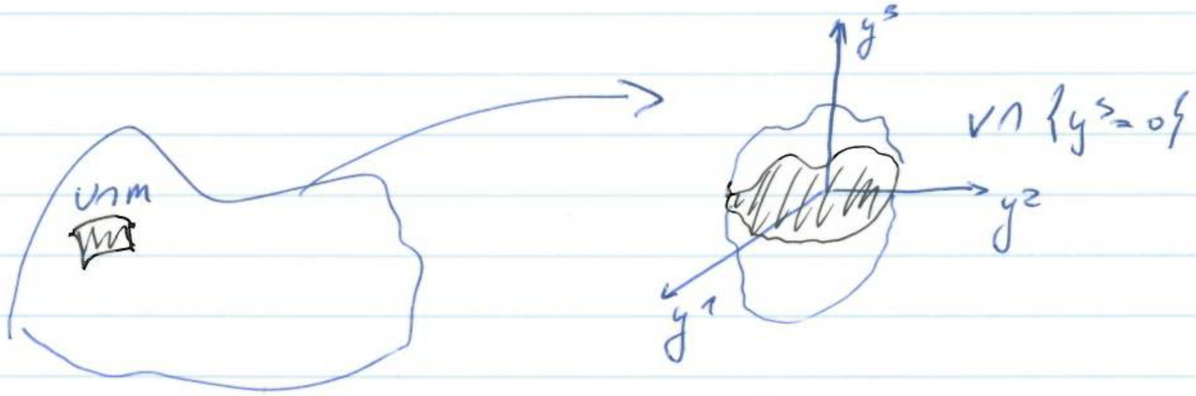
(c) $\exists W$ open in $\mathbb{R}^k$, $\exists U$ open in $\mathbb{R}^n$ $x \in U$

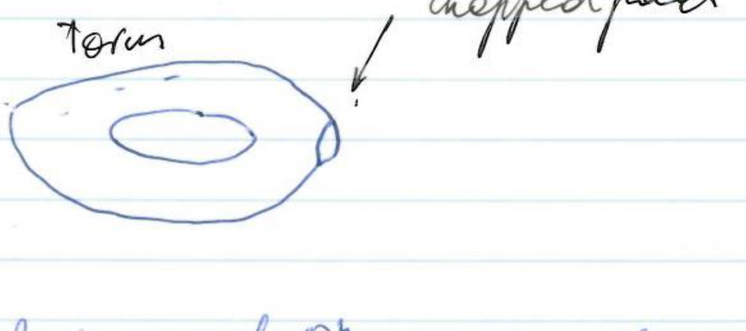
$\exists f: W \to U$ s.t. f is injective

$\operatorname{rank} f'(y) = k \quad \forall y \in W$

$f(W) = U \cap M$

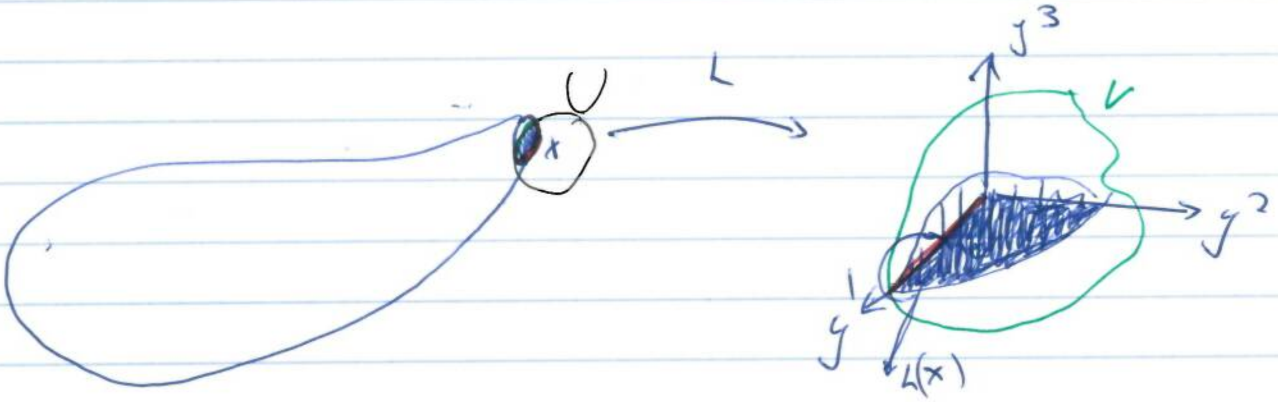
$f^{-1}: U \cap M \to W$ is continuous





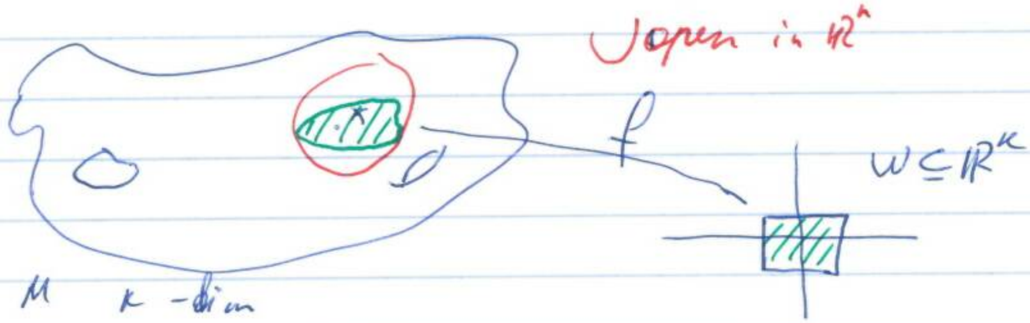
Def A subset M of $\mathbb{R}^n$ is a k-dim manifold with boundary if $\forall x \in M$ either
(M) holds or exclusive
(M') holds $\exists$ open set U of $\mathbb{R}^n$, $x \in U$

$\exists$ V open set in $\mathbb{R}^n$, h diffeomorphism
$h : U \to V$;
$h(U \cap M) = V \cap \{y,\ y^k \geq 0,\ y^{k+1} = y^{k+2} = \dots = y^n = 0\}$
$h^k(x) = 0$



The boundary ∂M of M is defined to be the set of points x where condition M' holds

15/12/11



$f(y)$ has rank k for all $y \in W$

nully + rank thm

dim $\mathbb{R}^k$ = dim ker $(Df(y))$ + rank $(Df(y))$

$Df(y) : \mathbb{R}^k \to \mathbb{R}^n$
since $f : \mathbb{R}^k \to \mathbb{R}^n$

$k = \text{nullity} + k$

$\text{nullity} = 0$

$\text{Ker}\,(Df(y)) = \{\vec{0}\}$

$Df(y)$ is injective.

$f : W \to U \cap M$

Let $a \in W$ s.t. $f(a) = x$

$\mathbb{R}^k_a = \{(a,v),\ v \in \mathbb{R}^k\}$

vector space

$\forall\, v \in \mathbb{R}^k$

$(a,v) \to (f(a),\ Df(a)(v)) \in \mathbb{R}^n_{f(a)} = \mathbb{R}^n_x$

(a,v) is pushed forward to give a vector $\mathbb{R}^n_x$

$v_a = (a,v)$

$f_*(v_a) = (x,\ Df(a)(v)) \in \mathbb{R}^n_x$

Def The tangent space of M at x is defined to be

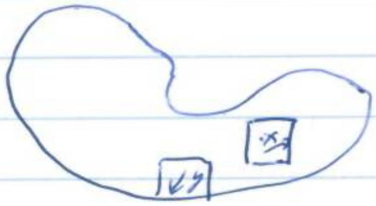
$$M_x = f_*(\mathbb{R}^k_a)$$

(given $x = f(a)$, f chart)

($\dim M_x = k$)

Def A vector field on M is a function F on M s.t.

$\forall x \in M, \ F(x) \in M_x$



Let $x = f(a)$ $\quad f: W \to U$

$f(W) = U \cap M$

Let $G(a) \in \mathbb{R}^k_a$ s.t. $f_*(G(a)) = F(f(a)) = F(x)$

Such $G(a)$ is unique, since $f_* : \mathbb{R}^k_a \to M_x$ is bijective

Def F a vector field on M is called continuous (or differentiable) if $\forall x \in M$ the vector field G on W is continuous (or diff-ble)

Def ω is a ~~f~~ (differential) P-form on M if $\forall x \in M \Rightarrow \omega(x) \in \Lambda^P(M_x)$

Note $f^*(\omega)$ is (differential) P-form on W

Prop. If $f^*(\omega)$ is differential then ω is differential on $W \subseteq \mathbb{R}^n$

If ω is a p-form on M, which k-dim in $\mathbb{R}^n$

$x \in M \quad \omega(x) = \sum_{i_1 < i_2 < \dots < i_p} \omega_{i_1 i_2 \dots i_p}(x)\, dx^{i_1} \wedge dx^{i_2} \dots \wedge dx^{i_p}$

ω is continuous if $f^*(\omega)$ is continuous on W
ω is differential if $f^*(\omega)$ is differentiable on W

We have difficulty with $\{D_j(\omega_{i_1 \dots i_p}(x))\}$ since $\omega_{i_1 \dots i_p}(x)$ is not defined on open set $V \ni x$

Thm Given a differential p-form on M which is a k-dim in $\mathbb{R}^n$

there exits a unique differential $(p+1)$-form $d\omega$ on M s.t. $\forall x \in M$ and $f: W \to V \cap M$ chart

$$d(f^*(\omega)) = f^*(d\omega)$$

$d\omega(x) \in \Lambda^{p+1}(M_x)$

$d\omega(x)(v_1, v_2, \dots, v_{p+1}) =$

Since f_x is bijection
$f_x : \mathbb{R}^k_a \xrightarrow{\sim} M_x$

$\exists$ unique vectors $w_1, w_2, \dots, w_{p+1} \in \mathbb{R}^k_a$ s.t. $f_*(w_i) = v_i$

$$d\omega(x)(v_1, \dots, v_{p+1}) = d\underbrace{f^*(\omega)(a)}_{\in \Lambda^{p+1}(\mathbb{R}^k_a)}(w_1, w_2, \dots, w_{p+1})$$

<u>Aim</u> To understand Stoke's theorem for k-dim manifold in $\mathbb{R}^n$ with boundary ∂M

$$\int_{\partial M} \omega = \int_M d\omega \qquad d\omega \text{ is a k form on } M$$

where ω is a $k-1$ differential form on M

Orientation on vector spaces
bases $\mathcal{F} = \langle v_1, v_2, \dots, v_n \rangle$ ordered
$\mathcal{B} = \langle w_1, w_2, \dots, w_n \rangle$

We say $\mathcal{F}$ & $\mathcal{B}$ define the same orientation if
if $\det [id]^{\mathcal{B}}_{\mathcal{F}} > 0$

opposite orientation

if $\det [id]^{\mathcal{B}}_{\mathcal{F}} < 0$

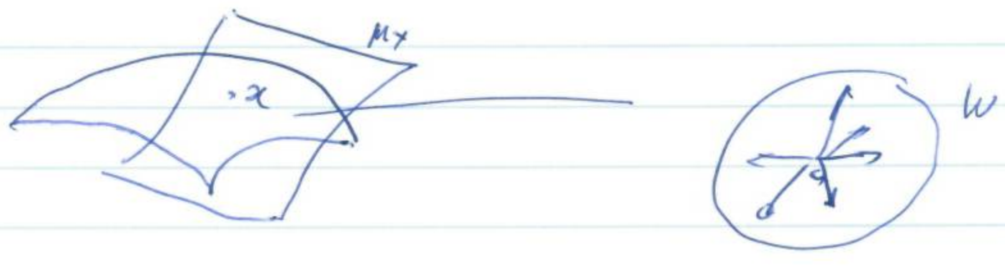
$$[Id]^{\mathcal{B}}_{\mathcal{F}} = \left([Id]^{\mathcal{F}}_{\mathcal{B}}\right)^{-1} \Rightarrow \det[Id]^{\mathcal{B}}_{\mathcal{F}} > 0 \Leftrightarrow \det[Id]^{\mathcal{F}}_{\mathcal{B}} > 0$$

$\mathcal{F} \sim \mathcal{B}$ iff they define the same orientation

This is an equivalence relation

Standard orientation $\mathbb{R}^n$ $\mathfrak{J} = \langle e_1, e_2, \dots, e_n \rangle$

it is denoted $\mu = [e_1, e_2, \dots, e_n]$

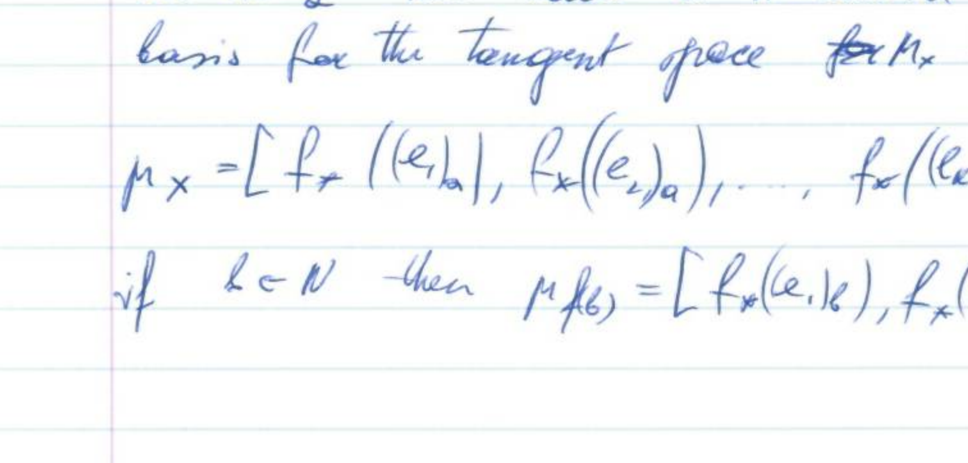


$f(a) = x$

On $\mathbb{R}^k_a$ we have the standard basis $\{(e_1)_a, (e_2)_a, \dots (e_k)_a\}$
basis for the tangent space ~~for~~ M_x : $\{f_*(e_1)_a, f_*(e_2)_a, \dots, f_*(e_k)_a\}$

$\mu_x = [f_*((e_1)_a), f_*((e_2)_a), \dots, f_*((e_k)_a)]$

if $b \in W$ then $\mu_{f(b)} = [f_*((e_1)_b), f_*((e_2)_b), \dots, f_*((e_k)_b)]$



$z = f(c)$
$z = g(d)$

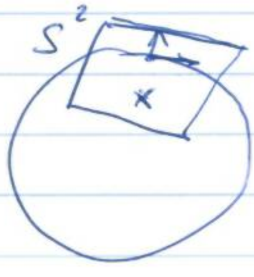
We assign two orientations at z

M_x $[f_*((e_1)_c), f_*((e_2)_c), \dots, f_*((e_k)_c)] =$

M_y $[g_*((e_1)_d), g_*((e_2)_d), \dots, g_*((e_k)_d)]$

If the two orientations are equal
i.e. $\det[Id] > 0$ on these two bases

then we say f and g define consistent orientation at point x



Hopefully this is true on $f(w) \cap g(w')$
then we call the two orientations consistent

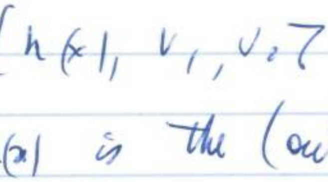
If there exists consistent orientation on all of M
we say M is orientable & the manifold is oriented once we fix orientation.

If S is a surface in $\mathbb{R}^3$ which is orientable let
$M_x \doteq [v_1, v_2]$ $\quad x \in S$ [2-manifold]

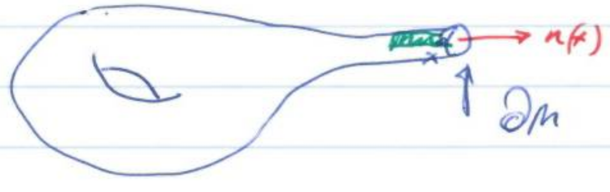
Draw the line perpendicular to S_x at x

Pick a unit vector $n(x)$ s.t.
$[n(x), v_1, v_2]$ = the standard orientation in $\mathbb{R}^3$

then $n(x)$ is the (outer) unit normal



M is n-dim manifold with boundary in $\mathbb{R}^n$



~~Case M' $\exists$ U & V open sets~~

$f(a) = x$

$y^k \geq 0$, $y^{k+1}_x = y^{k+2} = \dots 0$
$a^k = 0$

$(\partial M)_x$ has a basis $\mathcal{B} = \{f_*((e_1)_a), f_*((e_2)_a), \dots, f_*((e_{k-1})_a)\}$

then let $V_0 \in \mathbb{R}^k_a$ s.t.
$f_*(V_0)$ $(\perp)$ at p,

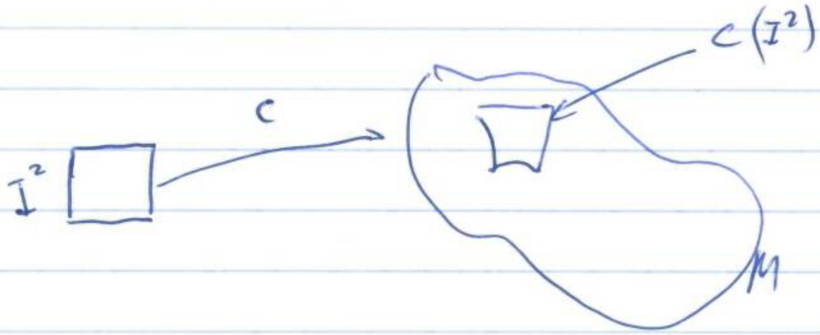
then $|f_*(V_0)| = 1$

then $n(x) = f_*(V_0)$

$\int_M \omega \qquad \int_{\partial M} \omega$

Integrals
Let c be a singular p-cube on M n-dim
$c: I^k \to M$

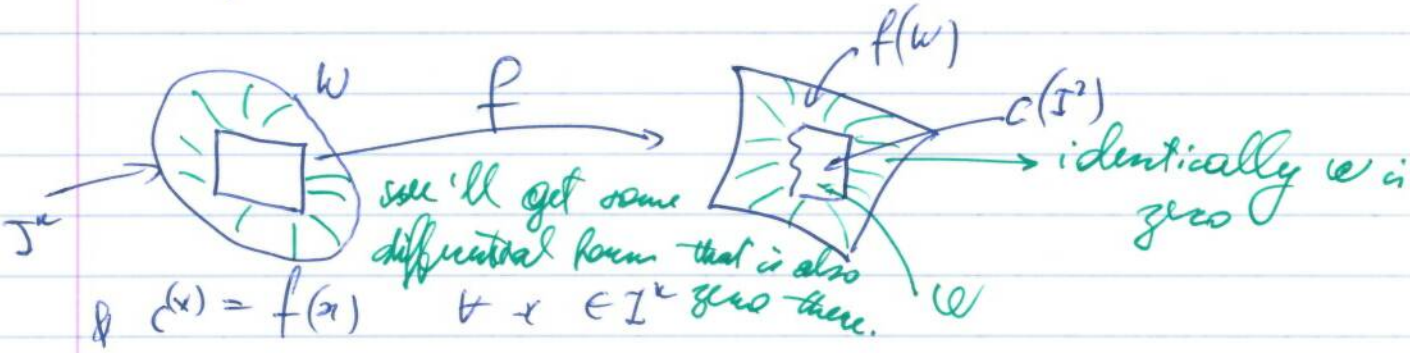
e.g.



Let ω be a p form
We define

$$\int_c \omega \underset{\text{by pullback}}{=} \int_{I^k} c^*(\omega)$$

If c is a k-cube in M k-manifold and $I^k \subseteq W$,
$f : W \to U \cap M$ is the chart

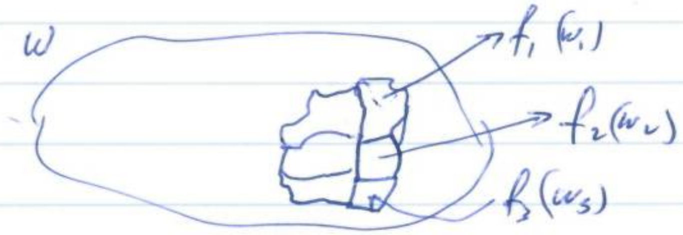


& $c(x) = f(x)$ $\forall x \in I^k$

f is preserving orientation
then we say c is orientation preserving singular k-cube on M

If ω is k-form on M with $\omega(y) = 0$ $\forall y \notin c(I^2)$
then we define

$$\int_M \omega = \int_c \omega$$



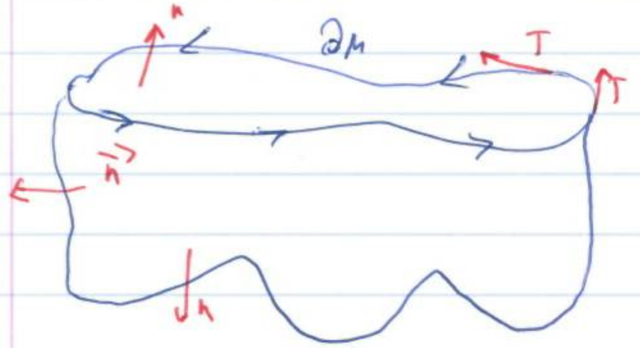
$$\int_M w = \int_{f_1 w_1} w + \int_{f_2 w_2} w + \dots$$

Use partitions of unity to define
$\int_M w^{k\text{-form}}$, $\int_M \eta^{k-1}$ form

Thm Let M be compact oriented k-manifold with boundary ∂M & w be a differential $k-1$ form on M

then $\int_{\partial M} w = \int_M dw$

Classical Stoke's theorem



$\mathbb{R}^3$ oriented
M is a 2 dim manifold with boundary
F diff. vector field on M

$$\int_{\partial M} \vec{F} \cdot \vec{T}\, ds = \int_M \operatorname{curl} \vec{F} \cdot \vec{n}\, dA$$

Let M be a compact oriented 2-dim manifold with boundary ∂M in $\mathbb{R}^3$

T be a vector field on ∂M s.t. $ds(T) = 1$ where ds is the length element of ∂M

$\vec{F}$ be a diff. vector field on an open set ~~containing~~ containing M

$\vec{n}$ be the outer unit normal on M

Then $$\int_{\partial M} \vec{F} \cdot \vec{T}\, ds = \int_M \operatorname{curl} \vec{F} \cdot \vec{n}\, dA$$

If $F = (F^1, F^2, F^3) = F^1 \vec{i} + F^2 \vec{j} + F^3 \vec{k}$
we define 1-form $\omega = F^1 dx + F^2 dy + F^3 dz$

then we calculate $$d\omega = \frac{\partial F^1}{\partial y} dy \wedge dx + \frac{\partial F^1}{\partial z} dz \wedge dx + \frac{\partial F^2}{\partial x} dx \wedge dy + \frac{\partial F^2}{\partial z} dz \wedge dy + \frac{\partial F^3}{\partial x} dx \wedge dz + \frac{\partial F^3}{\partial y} dy \wedge dz =$$

$$= G^1 dy \wedge dz + G^2 dz \wedge dx + G^3 dx \wedge dy$$

then $G^1 \vec{i} + G^2 \vec{j} + G^3 \vec{k} = \operatorname{curl}(\vec{F})$

last lecture $$dy \wedge dz = n^1 dA$$
$$dz \wedge dx = n^2 dA$$
$$dx \wedge dy = n^3 dA$$

$$\int_M g^1 dy \wedge dz + g^1 dz \wedge dx + g^3 dx \wedge dy =$$

$$= \int_M (g^1 n^1 + g^2 n^2 + g^3 n^3)\, dA =$$

$$= \int_M \vec{g} \cdot \vec{n}\, dA \underset{\text{of } g}{\overset{\text{def}}{=}} \int_M \text{curl } \vec{F} \cdot \vec{n}\, dA$$

According to general Stoke's Theorem

$$\int_{\partial M} \omega = \int_M d\omega = \int_M \text{curl } \vec{F} \cdot \vec{n}\, dA$$

Since $ds(T) = 1$

we can prove as in prev. lecture

$$dx = T^1 ds$$
$$dy = T^2 ds$$
$$dz = T^3 ds$$

$$\int_{\partial M} \omega = \int F^1 dx + \int F^2 dy + F^3 dz = \int_{\partial M} F^1 T^1 ds + F^2 T^2 ds + F^3 T^3 ds$$

$$= \int_{\partial M} \vec{F} \cdot \vec{T}\, ds$$