3113 Differential Geometry Notes

Based on the 2013 autumn lectures by Prof R Halburd

MATH3113 – Differential Geometry.

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Chapter 1
LOCAL THEORY OF CURVES.

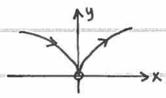
$\boxed{\text{Definition}}$ A (parameterised) differentiable curve is a differentiable map $\gamma : I \to \mathbb{R}^3$. The set $\gamma(I) \subset \mathbb{R}^3$ is called the trace of γ.

$\boxed{\text{Definition}}$ A differentiable curve γ is said to be regular if $\gamma'(t) \neq 0 \quad \forall t \in I$

Remark – Here, $\gamma'(t)$ is a tangent vector.

Examples –

- The helix $\gamma : \mathbb{R} \to \mathbb{R}^3$ given by $\gamma(t) = (a\cos t, a\sin t, bt)$ where $a, b \neq 0$ is a regular curve $\because \gamma'(t) = (-a\sin t, a\cos t, b) \neq 0$.
- $\gamma : (-1,1) \to \mathbb{R}^3$ given by $\gamma(t) = (t^3, t^2, 0)$ is not regular, since $\gamma'(t) = (3t^2, 2t, 0) = 0$ if $t = 0$.



For any curve $\gamma : I \to \mathbb{R}^3$ and any $t_0 \in I$, the arclength of γ from $\gamma(t_0)$ is $s = s(t) = \int_{t_0}^{t} |\gamma'(u)|\, du$.

Here, if $\gamma(t) = (x(t), y(t), z(t))$, then $|\gamma'| = \sqrt{\dot{x}^2 + \dot{y}^2 + \dot{z}^2}$.

Example – Let $\gamma(t) = (a\cos t, a\sin t, bt)$, then $s = \int^t |\gamma'(u)|\, du = \sqrt{a^2+b^2}\, t$ (plus constant of integration).

We can re-parameterise to get $\bar{\gamma}(s) = \gamma(t) = \left(a\cos\frac{s}{\sqrt{a^2+b^2}}, a\sin\frac{s}{\sqrt{a^2+b^2}}, \frac{bs}{\sqrt{a^2+b^2}}\right)$ in terms of arc length.

Frenet frame.

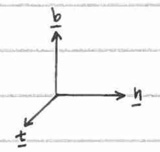
Consider $s = \int |\gamma'(t)|\, dt \Rightarrow s = \int |\gamma'(s)|\, ds \Rightarrow 1 = |\gamma'(s)|$. Then let the unit tangent vector be $\underline{t} = \gamma'(s)$ [notation – also denoted $\underline{e}_t$].

Then $\underline{t} \cdot \underline{t} = 1 \Rightarrow \underline{t} \cdot \underline{t}' = 0 \Rightarrow \underline{t}'$ is orthonormal to $\underline{t}$.

Let $k(s) = |\underline{t}'(s)|$ be the curvature. If $k(s) \neq 0$, we define the principal normal $\underline{n}(s) = \frac{\underline{t}'(s)}{k(s)}$.

With these two vectors, we can also define the unit binormal vector, $\underline{b} = \underline{t} \times \underline{n}$.

This gives us the basis $\{\underline{t}, \underline{n}, \underline{b}\}$ that defines a Frenet frame, which is a right-handed orthonormal frame.



Moreover, $\underline{b} = \underline{t} \times \underline{n} \Rightarrow \underline{b}' = \underline{t}' \times \underline{n} + \underline{t} \times \underline{n}' = k\underline{n} \times \underline{n} + \underline{t} \times \underline{n}' = \underline{t} \times \underline{n}'$. Dotting both sides with $\underline{t}$, we get $\underline{b}' \cdot \underline{t} = 0$

Also, $\underline{b}' \cdot \underline{b} = 0$, so $\underline{b}' = \tau\underline{n}$ for some scalar $\tau(s)$, which is called the torsion.

Then, $\underline{n} = \underline{b} \times \underline{t} \Rightarrow \underline{n}' = \underline{b}' \times \underline{t} + \underline{b} \times \underline{t}' = \tau\underline{n} \times \underline{t} + \underline{b} \times k\underline{n} = -k\underline{t} - \tau\underline{b}$

Together, these combine to give the Frenet-Serret formulas:

$$\boxed{\begin{aligned} \underline{t}' &= && k\underline{n} && \\ \underline{n}' &= -k\underline{t} && && -\tau\underline{b} \\ \underline{b}' &= && \tau\underline{n} && \end{aligned}} \quad \Leftrightarrow \quad \underline{F}(s) = \begin{pmatrix} -\underline{t}- \\ -\underline{n}- \\ -\underline{b}- \end{pmatrix} = \left(\underline{t} \,\vdots\, \underline{n} \,\vdots\, \underline{b}\right)^T, \text{ then if } A(s) = \begin{pmatrix} 0 & k(s) & 0 \\ -k(s) & 0 & -\tau(s) \\ 0 & \tau(s) & 0 \end{pmatrix}, \quad \underline{F}'(s) = A(s)\underline{F}(s).$$

$\boxed{\text{Ex}}$ Recall the earlier helix paramerised by $\gamma(s) = \left(a\cos\frac{s}{\sqrt{a^2+b^2}}, a\sin\frac{s}{\sqrt{a^2+b^2}}, \frac{bs}{\sqrt{a^2+b^2}}\right)$, with $a > 0, b \neq 0$.

Find $\underline{t}, \underline{n}, \underline{b}$ and the curvature and torsion of the system.

Soln. Unit tangent $\underline{t}$ is found by normalisation. $\underline{t}(s) = \gamma'(s) = \frac{1}{\sqrt{a^2+b^2}}\left(-a\sin\frac{s}{\sqrt{a^2+b^2}}, a\cos\frac{s}{\sqrt{a^2+b^2}}, b\right)$ //

$\underline{t}'(s) = -\frac{1}{a^2+b^2}\left(a\cos\frac{s}{\sqrt{a^2+b^2}}, a\sin\frac{s}{\sqrt{a^2+b^2}}, 0\right) = -\frac{a}{a^2+b^2}\left(\cos\frac{s}{\sqrt{a^2+b^2}}, \sin\frac{s}{\sqrt{a^2+b^2}}, 0\right)$. Thus, since $\underline{t}' = k\underline{n}$,

$k(s) = \frac{a}{a^2+b^2}$ // and $\underline{n} = -\left(\cos\frac{s}{\sqrt{a^2+b^2}}, \sin\frac{s}{\sqrt{a^2+b^2}}, 0\right)$ //

$\underline{b}(s) = \underline{t}(s) \times \underline{n}(s) = \frac{1}{\sqrt{a^2+b^2}}\left(b\sin\frac{s}{\sqrt{a^2+b^2}}, -b\cos\frac{s}{\sqrt{a^2+b^2}}, a\right)$ // then $\underline{b}'(s) = \frac{b}{a^2+b^2}\left(\cos\frac{s}{\sqrt{a^2+b^2}}, \sin\frac{s}{\sqrt{a^2+b^2}}, 0\right) = -\frac{b}{a^2+b^2}\underline{n} \Rightarrow \tau(s) = -\frac{b}{a^2+b^2}$ //

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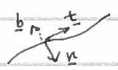
If $\gamma : I \to \mathbb{R}^3$ is a curve (regular curve, i.e. $\gamma'(t) = 0$), we have defined arclength $\frac{ds}{dt} = |\gamma'(t)|$. We have also defined the Frenet frame, which is a right-handed coordinate system with unit coordinate vectors $\underline{t}, \underline{n}, \underline{b}$.



$\boxed{\text{Theorem}}$ (Fundamental Theorem of Local Theory of Curves).

Given differential functions $k : I \to \mathbb{R}_{>0}$ and $\tau : I \to \mathbb{R}$, there exists a regular curve $\gamma : I \to \mathbb{R}^3$ s.t. $k(s)$ and $\tau(s)$ are the curvature and torsion respectively of γ as functions of arclength. Furthermore, γ is unique up to a rigid motion in $\mathbb{R}^3$. (i.e. $\underline{x} \mapsto \underset{\substack{\in SO(3) \\ \text{(rotation)}}}{\rho}\underline{x} + \underset{\substack{\text{constant vector} \\ \text{(translation)}}}{\underline{c}}$)

Proof – We start out with the Frenet equations to construct an orthonormal frame. $\underline{F}' = \underline{A}\,\underline{F}$, i.e. $\begin{array}{l} \underline{t}' = k\underline{n} \\ \underline{n}' = -k\underline{t} - \tau\underline{b} \\ \underline{b}' = \tau\underline{n} \end{array}$ — ①.

① has a unique solution with specified initial values. Let $(\underline{t}_0, \underline{n}_0, \underline{b}_0)$ be any three orthonormal vectors with $\underline{b}_0 = \underline{t}_0 \times \underline{n}_0$ (right-handed).

The initial value problem ① with $\underline{t}(s_0) = \underline{t}_0$, $\underline{n}(s_0) = \underline{n}_0$ and $\underline{b}(s_0) = \underline{b}_0$ for some $s_0 \in I$. has a unique solution $(\underline{t}(s), \underline{n}(s), \underline{b}(s))$.

This is from the theory of differential equations. We need to show that these solution vectors remain orthonormal:

Consider $M = \begin{pmatrix} \underline{t}\cdot\underline{t} & \underline{t}\cdot\underline{n} & \underline{t}\cdot\underline{b} \\ \underline{n}\cdot\underline{t} & \underline{n}\cdot\underline{n} & \underline{n}\cdot\underline{b} \\ \underline{b}\cdot\underline{t} & \underline{b}\cdot\underline{n} & \underline{b}\cdot\underline{b} \end{pmatrix}$ — ② - We must establish that $M = I_3$. Since $\underline{F} = \begin{pmatrix} -\underline{t}- \\ -\underline{n}- \\ -\underline{b}- \end{pmatrix}$, $\underline{F}^T = \begin{pmatrix} | & | & | \\ \underline{t} & \underline{n} & \underline{b} \\ | & | & | \end{pmatrix}$ $\Rightarrow M = \underline{F}\,\underline{F}^T$.

Then $M' = (FF^T)' = F'F^T + F(F^T)' \overset{①}{=} AFF^T + F(F')^T = AFF^T + F(AF)^T = AFF^T + FF^TA^T = AM + M\underbrace{A^T}_{\begin{pmatrix} 0 & k & 0 \\ -k & 0 & -\tau \\ 0 & \tau & 0 \end{pmatrix}^T} = AM - MA$

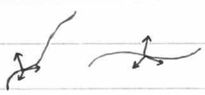
Hence, $M' = AM - MA$ — ③, which is a linear differential equation. Then $M(s_0) = I$ — ④ by definition.

Thus ③, ④ is a regular IVP. Note that I is a solution of ③, ④, so by uniqueness, $M(s) = I \Rightarrow$ system is orthonormal.

We are only left with proving right-handedness, i.e. NTP: $\det(\underline{F}) = +1$. Since $\underline{t}, \underline{n}, \underline{b}$ are orthonormal, $\det \underline{F} = \pm 1$.

Also, $\det(F(s_0)) = 1$. Since $\det(\underline{F})$ is continuous, then $\det \underline{F} = 1 \Rightarrow$ right-handed system ~~proved~~.

Now, $\gamma(s) = \int_{s_0}^{s} \underline{t}(\hat{s})\, d\hat{s}$, and Frenet equations are satisfied. This $\gamma(s)$ is a desired regular curve.// q.e.d.



For uniqueness, suppose we have two curves $\gamma, \tilde{\gamma} : I \to \mathbb{R}^3$ that satisfy the conditions of theorem.

Let $(\underline{t}, \underline{n}, \underline{b})$ and $(\tilde{\underline{t}}, \tilde{\underline{n}}, \tilde{\underline{b}})$ be Frenet frames. Choose $s_0 \in I$. $\exists$ rotation $\rho \in SO(3)$ s.t. $\tilde{\underline{t}}(s_0) = \rho \circ \underline{t}(s_0)$, $\tilde{\underline{n}}(s_0) = \rho \circ \underline{n}(s_0)$, $\tilde{\underline{b}}(s_0) = \rho \circ \underline{b}(s_0)$

Define $(\hat{\underline{t}}(s), \hat{\underline{n}}(s), \hat{\underline{b}}(s)) = (\rho^{-1} \circ \tilde{\underline{t}}(s), \rho^{-1} \circ \tilde{\underline{n}}(s), \rho^{-1} \circ \tilde{\underline{b}}(s))$. Note that $\hat{\underline{t}}(s_0) = \underline{t}(s_0)$, $\hat{\underline{n}}(s_0) = \underline{n}(s_0)$, $\hat{\underline{b}}(s_0) = \underline{b}(s_0)$.

Consider $\frac{d}{ds}\{\underbrace{|\underline{t}(s) - \hat{\underline{t}}(s)|^2 + |\underline{n}(s) - \hat{\underline{n}}(s)|^2 + |\underline{b}(s) - \hat{\underline{b}}(s)|^2}_{f(s)}\} = \frac{d}{ds}\{(\underline{t} - \hat{\underline{t}})\cdot(\underline{t} - \hat{\underline{t}}) + (\underline{n} - \hat{\underline{n}})\cdot(\underline{n} - \hat{\underline{n}}) + (\underline{b} - \hat{\underline{b}})\cdot(\underline{b} - \hat{\underline{b}})\}$

$= 2\{(\underline{t} - \hat{\underline{t}})\cdot(\underline{t}' - \hat{\underline{t}}') + (\underline{n} - \hat{\underline{n}})\cdot(\underline{n}' - \hat{\underline{n}}') + (\underline{b} - \hat{\underline{b}})\cdot(\underline{b}' - \hat{\underline{b}}')\} = 2\{(\underline{t} - \hat{\underline{t}})\cdot k(\underline{n} - \hat{\underline{n}}) + (\underline{n} - \hat{\underline{n}})\cdot\{-k(\underline{t} - \hat{\underline{t}}) - \tau(\underline{b} - \hat{\underline{b}})\} + (\underline{b} - \hat{\underline{b}})\cdot\tau(\underline{n} - \hat{\underline{n}})\} = 0$.

Then $f(s) = |\underline{t} - \hat{\underline{t}}|^2 + |\underline{n} - \hat{\underline{n}}|^2 + |\underline{s} - \hat{\underline{s}}|^2$ is ~~constant~~ constant. But $f(s_0) = 0 \Rightarrow f(s) \equiv 0$ everywhere $\Rightarrow \hat{\underline{t}} \equiv \underline{t}$, $\hat{\underline{b}} \equiv \underline{b}$, $\hat{\underline{n}} \equiv \underline{n}$.

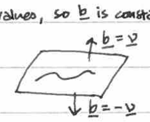
$\Rightarrow \tilde{\underline{t}}(s) = \rho \circ \hat{\underline{t}}(s) = \rho \circ \underline{t}(s)$, $\tilde{\underline{n}}(s) = \rho \circ \underline{n}(s)$, $\tilde{\underline{b}}(s) = \rho \circ \underline{b}(s)$. Then $\tilde{\gamma}(s) = \rho \circ \gamma(s) + \underline{c}$. [$\because$ $\tilde{\gamma}' = \tilde{\underline{t}}$, $\gamma'(s) = t(s)$] // q.e.d.

$\boxed{\text{Theorem}}$ The torsion of a regular curve vanishes if and only if the trace of the curve lies in a plane.

Proof — Suppose $\gamma(I)$ is contained in a plane. Then $\underline{t}$ and $\underline{n}$ are parallel to that plane, and $\underline{b}$ is a unit normal to the plane.

Hence, $\underline{b} = \underline{v}$ or $\underline{b} = -\underline{v}$ for some unit normal (constant) $\underline{v}$. However, $\underline{b}(s)$ is continuous but takes only discrete values, so $\underline{b}$ is constant.

Thus, $\underline{b}' = \underline{0} \Rightarrow \tau\underline{n} = 0 \Rightarrow \tau = 0$.// q.e.d.



Now suppose $\tau \equiv 0$. Using Frenet equation, $\underline{b}' = \tau\underline{n} \equiv \underline{0}$, then $\underline{b}(s) = \underline{b}_0$. Consider $\gamma(s)\cdot\underline{b}$.

$\frac{d}{ds}(\gamma(s)\cdot\underline{b}) = \frac{d}{ds}(\gamma(s)\cdot\underline{b}_0) = \gamma'(s)\cdot\underline{b}_0 = \underline{t}\cdot\underline{b}_0 = 0$ by orthogonality $\Rightarrow \gamma(s)\cdot\underline{b} = \text{const} \Rightarrow (x, y, z)\cdot\underline{b} = \text{const}$.

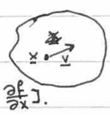
This is an equation for a plane $\Rightarrow$ trace of $\gamma(s)$ lies in plane perpendicular to $\underline{b}_0$.// q.e.d.

Chapter 2.
SURFACES.

Differentiable functions $f : \mathbb{R}^m \to \mathbb{R}^n$.

$\boxed{\text{Definition}}$ Let U be an open subset of $\mathbb{R}^m$ and let $f : U \to \mathbb{R}$ be a real-valued function on U.

For any unit vector $\underline{v} \in \mathbb{R}^m$, the <u>directional derivative</u> of f at $\underline{x} \in U$ in the direction $\underline{v}$ is $\lim_{h \to 0} \frac{f(\underline{x} + h\underline{v}) - f(\underline{x})}{h}$.



If $\underline{v}$ is one of the coordinate vectors, then the directional derivative is called a <u>partial derivative</u>. [e.g. $\underline{v} = (1, 0, 0)$, d.d. is $\frac{\partial f}{\partial x}$].

e.g. —
The partial derivatives for $f(x, y) = \begin{cases} \frac{xy}{x^2 + y^2} & (x, y) \neq (0, 0) \\ 0 & (x, y) = (0, 0) \end{cases}$ exist $\forall (x, y) \in \mathbb{R}^2$. In particular, $f_x(0, 0) = 0$ and $f_y(0, 0) = 0$.

However, f is not continuous at $(0, 0)$: approach along line $\overset{(x \to 0)}{y = kx}$: for $x \neq 0$, $f(x, kx) = \frac{kx^2}{x^2 + k^2x^2} = \frac{k}{1 + k^2}$.

$\boxed{\text{Definition}}$ Let U be an open subset of $\mathbb{R}^m$ and let $f : U \to \mathbb{R}$ be a real-valued function on U.

We say that f is <u>(once) differentiable</u> at a point $\underline{a} = (a_1, \dots, a_m) \in U$ if $\exists\, b_1, \dots, b_m \in \mathbb{R}$ s.t. $\lim_{\underline{x} \to \underline{a}} \frac{f(\underline{x}) - f(\underline{a}) - \sum_{j=1}^{m} b_j (x_j - a_j)}{|\underline{x} - \underline{a}|} = 0$.

linearisation of taylor series expansion.

In fact, $b_j = \left.\frac{\partial f}{\partial x_j}\right|_{\underline{x} = \underline{a}}$.

$\boxed{\text{Theorem}}$ Suppose $U \overset{\text{open}}{\subset} \mathbb{R}^m$, and that $f : U \to \mathbb{R}^n$ and its first order partial derivatives are continuous throughout U.

Then f is once differentiable throughout U.

Proof — omitted, to be covered in other courses.

From this point onwards, we take "differentiable" to mean <u>infinitely differentiable</u> (i.e. C^∞).

Consider function $F: U \subset \mathbb{R}^m \to \mathbb{R}^n$. If all partial derivatives exist, we define the differential of F as follows: if $\underline{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_m \end{pmatrix} \in U$, $F(\underline{x}) = \begin{pmatrix} f_1(\underline{x}) \\ \vdots \\ f_n(\underline{x}) \end{pmatrix}$,

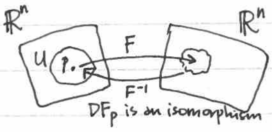
then $F(\underline{x} + \Delta \underline{x}) = F(\underline{x}) + \underbrace{(DF_{\underline{x}})}_{\text{differential of } F \text{ at } \underline{x}} \Delta x + R(\underline{x}, \Delta \underline{x})$ where $\Delta x \in \mathbb{R}^m$ and $\lim_{\Delta \underline{x} \to 0} \frac{R(x, \Delta x)}{|\Delta x|} = 0$ [i.e. $R(\underline{x}, \Delta \underline{x})$ goes to 0 faster than Δx].

Hence, the differential $DF_x : \mathbb{R}^m \to \mathbb{R}^n$ is a linear map. It can be represented by the matrix $\begin{pmatrix} \partial f_1/\partial x_1 & \cdots & \partial f_1/\partial x_m \\ \vdots & \ddots & \vdots \\ \partial f_n/\partial x_1 & \cdots & \partial f_n/\partial x_m \end{pmatrix}$.

This is called the Jacobian matrix.

In the special case where $m=n$, then the determinant of DF is denoted by $\frac{\partial(f_1, \ldots, f_n)}{\partial(x_1, \ldots, x_n)}$. This is called the Jacobian determinant.

$\boxed{\text{Theorem}}$ (Inverse Function Theorem)



Let $F: U \subset \mathbb{R}^n \to \mathbb{R}^n$ be a smooth map, and suppose that at $p \in U$, the differential DF_p is an isomorphism.

(i.e. the corresponding matrix has non-zero determinant). Then there is a neighbourhood V of p in U,

and a neighbourhood W of $F(p)$ in $\mathbb{R}^n$ s.t. the restriction of F to $V \to W$ has an inverse $F^{-1}: W \to V$.

Proof — Omitted, covered in other courses.

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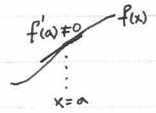
Consider the function $F: \mathbb{R}^n \to \mathbb{R}^n$, $\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \mapsto \begin{pmatrix} f_1(\underline{x}) \\ \vdots \\ f_n(\underline{x}) \end{pmatrix}$. Recall we defined Jacobian determinant $\frac{\partial(f_1, \ldots, f_n)}{\partial(x_1, \ldots, x_n)} = \det \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & & \vdots \\ \frac{\partial f_n}{\partial x_1} & \cdots & \frac{\partial f_n}{\partial x_n} \end{pmatrix}$.

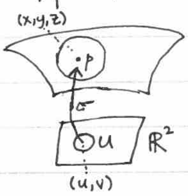
In 1 dimension, if $f'(a) \neq 0$, there exists an inverse function. Likewise, the Jacobian determinant plays a similar role.

This is outlined in the Inverse Function Theorem.



Regular surfaces

$\boxed{\text{Definition}}$ A non-empty set $\Sigma \subset \mathbb{R}^3$ is called a regular surface if for each $p \in \Sigma$, there is an open set $U \subset \mathbb{R}^2$ and an open neighbourhood V of p in $\mathbb{R}^3$, and an onto (surjective) map $\sigma: U \to V \cap \Sigma$ such that



(i) σ is a smooth function (C^∞). i.e. $\sigma(u,v) = (x(u,v), y(u,v), z(u,v)) \Leftrightarrow x, y, z$ are smooth functions.

(ii) σ is a homeomorphism ($\sigma^{-1}: V \cap \Sigma \to U$ exists and is continuous), and

(iii) The differential $D\sigma: \mathbb{R}^2 \to \mathbb{R}^3$ is one-to-one (injective). [Recall that $D\sigma = \begin{pmatrix} x_u & x_v \\ y_u & y_v \\ z_u & z_v \end{pmatrix}$.]

Remark — Condition (iii) can also be stated as $\sigma_u \times \sigma_v \neq 0$, or at least one of $\frac{\partial(x,y)}{\partial(u,v)}, \frac{\partial(y,z)}{\partial(u,v)}, \frac{\partial(z,x)}{\partial(u,v)}$ is non-zero.

Examples of regular surfaces —

- The paraboloid $z = x^2 + y^2$ is the image of $\sigma(u,v) = (u, v, u^2+v^2)$, $\sigma: \mathbb{R}^2 \to \mathbb{R}^3$ [Note: parametrisations are not unique!]

  Clearly (i) is true as coordinates are polynomials, so they are C^∞ functions. ✓ (ii) $\sigma^{-1}(x,y,z) = (x,y)$ exists and is smooth. ✓

  For (iii), $\frac{\partial(x,y)}{\partial(u,v)} = \frac{\partial(u,v)}{\partial(u,v)} = \det \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = 1 \neq 0$. ✓ Thus, this is a regular surface.

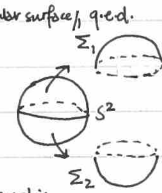
We can generalise this to a theorem:

$\boxed{\text{Theorem}}$ If $f: U \to \mathbb{R}$ is a smooth function on an open subset $U \subset \mathbb{R}^2$, then the graph of f [i.e. $\{(x,y,z): z = f(x,y), (x,y) \in U\}$] is a regular surface.

Proof — As above. (i) is true as f is smooth, it is locally invertible by $f^{-1}(x,y,z) = (x,y)$ and $\frac{\partial(x,y)}{\partial(u,v)} = 1$. All conditions are met $\Rightarrow$ regular surface. q.e.d.

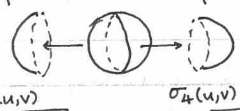
(Examples, cont'd)

- Sphere: $S^2 = \{(x,y,z): x^2+y^2+z^2 = 1\}$. We need to split the sphere to simplify topology. Cut at equator to produce two hemispheres.

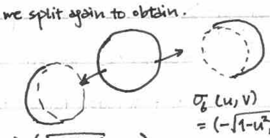


  Let $U = \{(x,y) \in \mathbb{R}^2 : x^2+y^2 < 1\}$. Then $\sigma_1(u,v) = (u, v, \sqrt{1-u^2-v^2})$, $\sigma_2(u,v) = (u, v, -\sqrt{1-u^2-v^2})$. However, this misses

  out the equator. We thus need to split it again, to take in all but 2 points, which we split again to obtain:

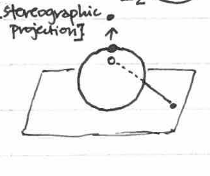


$\sigma_3(u,v) = (u, \sqrt{1-u^2-v^2}, v)$ $\quad$ $\sigma_4(u,v) = (u, -\sqrt{1-u^2-v^2}, v)$



$\sigma_5(u,v) = (\sqrt{1-u^2-v^2}, u, v)$ $\quad$ $\sigma_6(u,v) = (-\sqrt{1-u^2-v^2}, u, v)$.

[Stereographic projection]



At each point $p \in S^2$, the surface can be parametrised as a graph (smooth), so we see that: graph is smooth since $x^2+y^2 < 1$.

For σ_1, σ_2, $\frac{\partial(x,y)}{\partial(u,v)} \neq 0$. For σ_3, σ_4, $\frac{\partial(z,x)}{\partial(u,v)} \neq 0$ and for σ_5, σ_6, $\frac{\partial(y,z)}{\partial(u,v)} \neq 0$ $\Rightarrow$ regular surface.

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Recall the inverse function theorem — For $F: \underset{\mathbb{R}^n}{U} \to \mathbb{R}^m$, if $DF|_a \neq 0$, then locally around a, an inverse function $\mathbb{R}^m \to U$ exists.

$\boxed{\text{Theorem}}$ Let $\Sigma \subset \mathbb{R}^3$ be a regular surface. For each $p \in \Sigma$, $\exists$ a neighbourhood V of p in Σ s.t. V is the graph of a smooth function in one of the following forms:

$z = f(x,y)$, $y = f(z,x)$ or $x = f(y,z)$.

Proof — Let $\sigma: U \to \mathbb{R}^3$ be a parametrisation of Σ in a neighbourhood of p. Writing $\sigma(u,v) = (x(u,v), y(u,v), z(u,v))$, then WLOG, renaming axes if necessary,

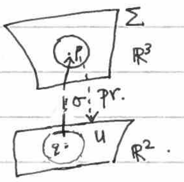
$\left.\frac{\partial(x,y)}{\partial(u,v)}\right|_q \neq 0$ where $q = \sigma^{-1}(p)$. Let $pr: \mathbb{R}^3 \to \mathbb{R}^2$ be the projection onto xy-plane: $pr(x,y,z) = (x,y)$.

Hence, $pr \circ \sigma: U \to \mathbb{R}^2$ and since $\left.\frac{\partial(x,y)}{\partial(u,v)}\right|_q \neq 0$, $pr\circ\sigma$ has a (local) differentiable inverse.

$(u,v) = (pr\circ\sigma)^{-1}(x,y) = (\tilde{u}(x,y), \tilde{v}(x,y))$. However, from our parametrisation,

$(u,v) \to \sigma(u,v) = (x(u,v), y(u,v), z(u,v))$ so $z = z(u,v) = z(\tilde{u}(x,y), \tilde{v}(x,y)) \Rightarrow z = f(x,y)$.

Thus, z is a function of x and y $\Rightarrow$ z is a graph.



$\boxed{\text{Ex}}$ Show that the cone $z = \sqrt{x^2+y^2}$, $(x,y) \in \mathbb{R}^2$ is not a regular surface.

Soln. If cone were a regular surface, it would be the graph of a regular function in the neighbourhood of any point, w.r.t. one of the coordinate axes. $z = f(x,y) = \sqrt{x^2+y^2}$, but this is not smooth at $(0,0)$.

For other options, $y = g(z,x)$ or $x = h(y,z)$, g and h are multi-valued and would not be functions. $\Rightarrow$ cone is not regular surface, q.e.d.

$\boxed{\text{Theorem}}$ Let $f: U \to \mathbb{R}$ be a smooth function on open set $U \subseteq \mathbb{R}^3$, and let $a \in f(U)$. If for all $p \in f^{-1}(a) = \{(x,y,z) \in U : f(x,y,z) = a\}$, if $f_x(p), f_y(p), f_z(p)$ are not all zero, then pre-image $f^{-1}(a)$ is a regular surface.

Example — Let $f: \mathbb{R}^3 \to \mathbb{R}$, $f(x,y,z) = x^2+y^2+z^2$. Sphere, $S^2 = f^{-1}(1) = \{(x,y,z): x^2+y^2+z^2 = 1\}$. Verify: $f_x = 2x$, $f_y = 2y$, $f_z = 2z$.

Now $f_x = f_y = f_z = 0 \Leftrightarrow (x,y,z) = (0,0,0) \notin S^2$. f is differentiable $\Rightarrow S^2$ is a regular surface.

Proof — WLOG, let $f_z(p) \neq 0$ for some $p \in U$, $p \neq 0$. Say $f_z(p) = 0$, and define $F: U \to \mathbb{R}^3$ by $F\begin{pmatrix}x\\y\\z\end{pmatrix} = \begin{pmatrix}x\\y\\f(x,y,z)\end{pmatrix}$

$DF = \det\begin{vmatrix}1&0&0\\0&1&0\\f_x&f_y&f_z\end{vmatrix}$ $\det(DF)|_p = f_z(p) = 0$. By the inverse function theorem, F^{-1} exists locally and is differentiable.

$\begin{pmatrix}x\\y\\z\end{pmatrix} = F^{-1}\begin{pmatrix}u\\v\\t\end{pmatrix}$ $\Rightarrow$ gives $x = u, y = v, z = g(u,v,t)$, where g is some smooth functions.

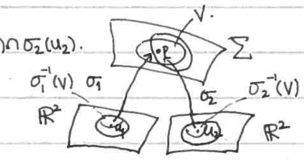
At $t = a$ i.e. $f(x,y,z) = a$, we have $z = g(x,y,a)$ i.e. a smooth graph $\Rightarrow$ regular surface.

$\boxed{\text{Theorem}}$ Let p be a point on a regular surface Σ and let $\sigma_1: U_1 \to \Sigma$ and $\sigma_2: U_2 \to \Sigma$. [i.e. 2 parametrisations s.t. $p \in V := \sigma_1(U_1) \cap \sigma_2(U_2)$].

Then the "change of coordinates" $f := \sigma_1^{-1} \circ \sigma_2 : \sigma_2^{-1}(V) \to \sigma_1^{-1}(V)$ is a diffeomorphism.

[i.e. differentiable function with a differentiable inverse].

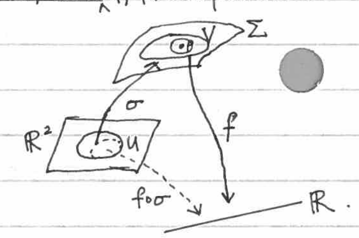
Proof — To be covered later.



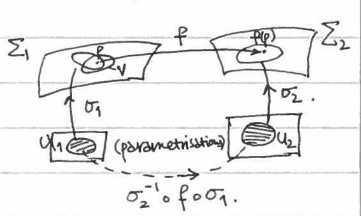
Functions on Surfaces.

$\boxed{\text{Definition}}$ Let $f: V \to \mathbb{R}$ be a function defined on an open subset V of a regular surface Σ. Then f is said to be ~~differentiable/smooth~~ at p if, for some parametrisation $\sigma: U \to \Sigma$ with $p \in \sigma(U) \subset V$, the composition $f \circ \sigma: U \to \mathbb{R}$ is differentiable at $\sigma^{-1}(p)$.

We say that f is differentiable if it is differentiable at all $p \in V$.



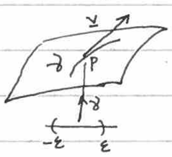
$\boxed{\text{Definition}}$ Let Σ_1, Σ_2 both be regular surfaces, and let V be a subset of Σ_1. A continuous map $f: V \to \Sigma_2$ is said to be differentiable at $p \in V$ if $\exists$ parametrisations $\sigma_1: U_1 \to \Sigma_1$ and $\sigma_2: U_2 \to \Sigma_2$ with $p \in \sigma_1(U_1)$ and $f(\sigma_1(U_1)) \subset \sigma_2(U_2)$ such that $\sigma_2^{-1} \circ f \circ \sigma_1: U_1 \to U_2$ is differentiable at $\sigma_1^{-1}(p)$.



$\boxed{\text{Definition}}$ Let $\Sigma \subset \mathbb{R}^3$ be a regular surface. $\forall p \in \Sigma$, a vector $\underline{v} \in \mathbb{R}^3$ is called tangent to Σ and p if $\exists$ a curve $\gamma: (-\varepsilon, \varepsilon) \to \Sigma$ for some $\varepsilon > 0$ s.t. $\gamma(0) = p$, $\gamma'(0) = v$.

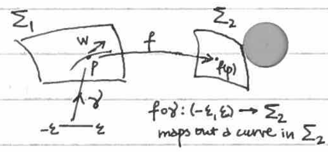
The set of all vectors tangent to Σ at p is called the tangent plane at p, which is denoted by $T_p\Sigma$.

Remark — A lot of differential geometry relates to the behaviour of the tangent plane.



$\boxed{\text{Definition}}$ Let $f: \Sigma_1 \to \Sigma_2$ be a differentiable function between regular surfaces Σ_1 and Σ_2. For any point $p \in \Sigma$ and vector $w \in T_p\Sigma$, let $\gamma: (-\varepsilon, \varepsilon) \to \Sigma$ be a curve s.t. $\gamma(0) = p$ and $\gamma'(0) = w$.

Then the map $(Df)_p: T_p\Sigma_1 \to T_{f(p)}\Sigma_2$ given by $(Df)_p\, w = (f\circ\gamma)'(0) = \left.\left(\frac{d}{dt}(f\circ\gamma)\right)\right|_{t=0}$ is called the differential of f at p.



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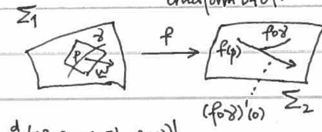
$w \in T_p\Sigma \Rightarrow \exists \gamma : (-\varepsilon,\varepsilon) \to \Sigma$ s.t. $\gamma(0)=p$, $\gamma'(0)=w$. So $(Df)_p : T_p\Sigma_1 \to T_{f(p)}\Sigma_2$, $(Df)_p(w) = (f\circ\gamma)'(0)$.



$\boxed{\text{Lemma}}$ The differential $(Df)_p$ defined above is independent of the choice of γ.

Proof – Choose $p \in \Sigma_1$, let $\sigma : U \to \Sigma_1$ be a parametrisation s.t. $p = \sigma(q)$ for some $q \in U$. We have $(Df)_p w = (f\circ\gamma)'(0) = \frac{d}{dt}\left((f\circ\gamma)\circ(\sigma^{-1}\circ\gamma(r))\right)\Big|_{r=0}$

$\Rightarrow (Df)_p(w) = (D(f\circ\sigma))_q\,(\sigma^{-1}\circ\gamma)'(0)$ — ⊛, which is the differential of a function from a subset of $\mathbb{R}^2$ to $\mathbb{R}^3$.

Now, $\sigma\circ(\sigma^{-1}\circ\gamma) = \gamma$ (rearranging to use chain rule) $\Rightarrow (D\sigma)_q \circ (\sigma^{-1}\circ\gamma)'(0) = \gamma'(0) = w$, which is an invertible mapping as it is a regular surface.

Thus from ⊛, $(Df)_p w = (D(f\circ\gamma))_q \circ (D\sigma)_q^{-1} w$, which is independent of σ, q.e.d.

We now return to the omitted proof from last lecture: to show that $f : \sigma_1^{-1}\circ\sigma_2$.

Proof – f is a homeomorphism (continuous with continuous inverse) because it is a composition of homeomorphisms. Let $\sigma_1(u,v) = (x(u,v), y(u,v), z(u,v))$, $(u,v) \in \sigma_1^{-1}(V)$.

WLOG, $\frac{\partial(x,y)}{\partial(u,v)} \neq 0$. Define $F : \sigma_1^{-1}(V) \times \mathbb{R} \to \mathbb{R}^3$. $F(u,v,t) = (x(u,v), y(u,v), z(u,v)+t)$. Then $DF\big|_q = \begin{pmatrix} \partial x/\partial u & \partial x/\partial v & 0 \\ \partial y/\partial u & \partial y/\partial v & 0 \\ \partial z/\partial u & \partial z/\partial v & 1 \end{pmatrix} \Rightarrow \det(DF) = \frac{\partial(x,y)}{\partial(u,v)} \neq 0$.

Apply the inverse function theorem, so F^{-1} exists (locally) and is differentiable. So $\sigma_2 : \sigma_2^{-1}(V) \to V$ and $F^{-1} : W \to \sigma_1^{-1}(W) \times \mathbb{R}$ where $W = F(U\times\mathbb{R})$.

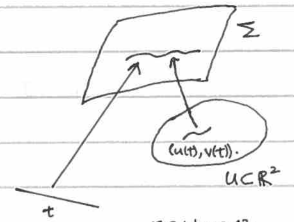
So f is the composition of these maps restricted to $t=0 \Rightarrow$ differentiable. So locally there is a differentiable inverse everywhere $\Rightarrow$ diffeomorphism, q.e.d.

$\boxed{\text{Theorem}}$ (Chain Rule) [Proof – in notes]

Let $f : \Sigma_1 \to \Sigma_2$ and $g : \Sigma_2 \to \Sigma_3$ be two differentiable maps where σ_1, σ_2 and σ_3 are regular surfaces in $\mathbb{R}^3$. For any $p \in \Sigma_1$, $(D(g\circ f))_p = (Dg)_{f(p)} \circ (Df)_p$.

Chapter 3

FIRST FUNDAMENTAL FORM.



Let $I_p(w) = \langle w,w\rangle = |w|^2$, $I_p : T_p\Sigma \to \mathbb{R}$, $\gamma(t) = \sigma(u(t), v(t))$. $w = \sigma'(0) = \sigma_u(u(0),v(0))\,\dot u(0) + \sigma_v(u(0),v(0))\,\dot v(0)$

$q = \sigma^{-1}(p)$. $(\sigma_u(q), \sigma_v(q))$ standard basis for $T_p\Sigma$

$\Rightarrow \langle w,w\rangle = \langle \sigma_u(q)\dot u + \sigma_v(q)\dot v,\ \sigma_u(q)\dot u + \sigma_v(q)\dot v\rangle$

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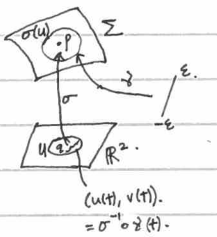
$\boxed{\text{Definition}}$ The <u>first fundamental form</u> (FFF) is the function $I_p : T_p\Sigma \to \mathbb{R}$ defined by $I_p(w) = \langle w,w\rangle = |w|^2$ for all $w \in T_p\Sigma$.



Let $p \in \Sigma$, $\sigma : U \to \Sigma$, $p \in \sigma(U)$. If $w \in T_p\Sigma$, $\exists \gamma : (-\varepsilon,\varepsilon) \to \Sigma$ s.t. $\gamma(0) = p$, $\gamma'(0) = w$.

Let $q = \sigma^{-1}(p)$ and $(u(t), v(t)) = \sigma^{-1}\circ\gamma(t)$. Then $\gamma(t) = \sigma(u(t), v(t))$. $\gamma'(t) = \sigma_u u' + \sigma_v v' \Rightarrow w = \gamma'(0) = \sigma_u(q)u'(0) + \sigma_v(q)v'(0)$.

$I_p(w) = \langle w,w\rangle = \langle \sigma_u u' + \sigma_v v',\ \sigma_u u' + \sigma_v v'\rangle = (u'(0))^2\langle\sigma_u(q),\sigma_u(q)\rangle + 2u'(0)v'(0)\langle\sigma_u(q),\sigma_v(q)\rangle + (v'(0))^2\langle\sigma_v(q),\sigma_v(q)\rangle$.

$= E(u'(0))^2 + 2F\,u'(0)v'(0) + G(v'(0))^2$ where $E = \langle\sigma_u,\sigma_u\rangle$, $F = \langle\sigma_u,\sigma_v\rangle$, $G = \langle\sigma_v,\sigma_v\rangle$ are called the <u>components</u> of FFF.

Often, this is written as $E\,du^2 + 2F\,du\,dv + G\,dv^2$, the metric.

$\boxed{\text{Ex}}$ Consider part of the unit sphere covered by the parametrisation $\sigma(\theta,\varphi) = (\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta)$. Find the FFF.

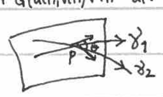
<u>soln.</u> $\sigma_\theta = (\cos\theta\cos\varphi, \cos\theta\sin\varphi, -\sin\theta)$, $\sigma_\varphi = (-\sin\theta\sin\varphi, \sin\theta\cos\varphi, 0)$. Then $E = \langle\sigma_\theta,\sigma_\theta\rangle = |\sigma_\theta|^2 = (\cos\theta)^2(\cos^2\varphi + \sin^2\varphi) + \sin^2\theta = 1$.

$F = \langle\sigma_\theta,\sigma_\varphi\rangle = 0$ and $G = \langle\sigma_\varphi,\sigma_\varphi\rangle = \sin^2\theta(\sin^2\varphi + \cos^2\varphi) = \sin^2\theta$. We could write this as $1\,d\theta^2 + \sin^2\theta\,d\varphi^2$.

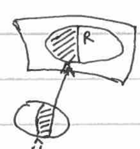
Any property or quantity that can be calculated from the FFF is called <u>intrinsic</u>.

Examples of intrinsic properties —

1. Lengths of curves: Let $\gamma(t) = \sigma(u(t),v(t)) = (x(t),y(t),z(t))$. Then $s = \int_{t_0}^{t_1}\sqrt{x'(t)^2 + y'(t)^2 + z'(t)^2}\,dt = \int_{t_0}^{t_1}\sqrt{E(u(t),v(t))\,u'(t)^2 + 2F(u(t),v(t))\,u'(t)v'(t) + G(u(t),v(t))\,v'(t)^2}\,dt$.



2. Angles between curves: Suppose we have two curves $\gamma_1 : (a_1,b_1) \to \Sigma$ and $\gamma_2 : (a_2,b_2) \to \Sigma$, and $\exists\, t_1 \in (a_1,b_1)$, $t_2 \in (a_2,b_2)$ s.t.
$\gamma_1(t_1) = \gamma_2(t_2) = p$ (i.e. intersection of curves at point p). Then, the angle between γ_1 and γ_2 at p is given by
$\theta = \cos^{-1}\left(\frac{\langle\gamma_1'(t_1),\gamma_2'(t_2)\rangle}{|\gamma_1'(t_1)||\gamma_2'(t_2)|}\right)$. Note that $\langle\gamma_1'(t_1),\gamma_2'(t_2)\rangle = Eu_1'(t_1)u_2'(t_2) + F\cdot\{u_1'(t_1)v_2'(t_2) + u_2'(t_2)v_1'(t_1)\} + G\,v_1'(t_1)v_2'(t_2)$.



3. Areas of regions: $A(R) = \iint_{\hat U}(\sigma_u\times\sigma_v)\,du\,dv$. To show that $\sigma_u\times\sigma_v$ follows from FFF, note that we have the identity:
$|\sigma_u\times\sigma_v|^2 + \langle\sigma_u,\sigma_v\rangle^2 = |\sigma_u|^2|\sigma_v|^2$ since $|\sigma_u|^2|\sigma_v|^2\sin^2\theta + |\sigma_u|^2|\sigma_v|^2\cos^2\theta = |\sigma_u|^2|\sigma_v|^2$. Then we obtain:
$|\sigma_u\times\sigma_v| = \sqrt{|\sigma_u|^2|\sigma_v|^2 - \langle\sigma_u,\sigma_v\rangle^2} = \sqrt{EG-F^2}$.

Ex The helicoid is the image of $\mathbb{R}^2$ under the mapping $\sigma(u,v) = (v\cos u, v\sin u, au)$ where a is a positive constant. Construct the FFF, and calculate its length and area of image.
(alternate method for FFF).

Soln. $(dx)^2 + (dy)^2 + (dz)^2$ where $x = v\cos u$, $y = v\sin u$, $z = au$. Then $(dx)^2 + (dy)^2 + (dz)^2 = (-v\sin u\, du + \cos u\, dv)^2 + (v\cos u\, du + \sin u\, dv)^2 + (a\, du)^2$

$= ((-v\sin u)^2 + (v\cos u)^2 + a^2)\, du^2 + 0\, du\, dv + (\cos^2 u + \sin^2 u)\, dv^2 = (v^2+a^2)\, du^2 + dv^2$. $[\Leftrightarrow E = v^2+a^2,\ F=0,\ G=1]$.

The image of the curve $\gamma(t) = (\cos t, \sin t, at)$, $0<t<2\pi$ lies on this helicoid. Use FFF to calculate its length —

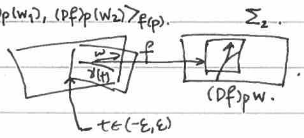
$s = \int_0^{2\pi} \sqrt{E\,u'(t)^2 + F\,u'(t)v'(t) + G\,v'(t)^2}\, dt$ where $u(t)=t$, $v(t)=1$ $= \int_0^{2\pi} \sqrt{(1+a^2) + 0 + 0} = 2\pi\sqrt{1+a^2}$.

Also, we can calculate area of image of region. $U = \{(u,v) : 0<u<2\pi,\ 0<v<1\}$. this is $A(U) = \iint_U \sqrt{EG-F^2}\, du\, dv = \int_0^{2\pi}\int_0^1 \sqrt{(v^2+a^2)\cdot 1 - 0}\, dv\, du$

$\Rightarrow A(U) = 2\pi \int_0^1 \sqrt{v^2+a^2}\, dv$.

Isometries.

Definition A diffeomorphism $f: \Sigma_1 \to \Sigma_2$ is called an isometry if $\forall p \in \Sigma_1$ and all $w_1, w_2 \in T_p\Sigma_1$, we have $\langle w_1, w_2\rangle_p = \langle (Df)_p(w_1), (Df)_p(w_2)\rangle_{f(p)}$.



The surfaces Σ_1 and Σ_2 are then said to be isometric.

Note — This is equivalent to $I_p(w) = I_{f(p)}((Df)_p w)$ $\forall w \in T_p\Sigma_p$.

Also, we observe that $\langle w_1+w_2, w_1+w_2\rangle$ can be expanded s.t. $2\langle w_1, w_2\rangle = \langle w_1+w_2, w_1+w_2\rangle - \langle w_1, w_1\rangle - \langle w_2, w_2\rangle$

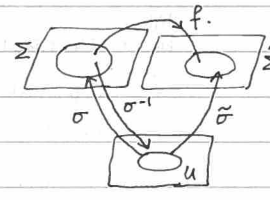
i.e. $2\langle w_1, w_2\rangle = I_p(w_1+w_2) - I_p(w_1) - I_p(w_2)$.

So if $I_p(w) = I_{f(p)}((Df)_p w)$ $\forall w \in T_p\Sigma_1$, then $\langle w_1, w_2\rangle_p = \langle (Df)_p w_1, (Df)_p w_2\rangle \Rightarrow$ isometries are diffeomorphisms that preserve FFF.

Definition A function $f: V \to \Sigma_2$ of a neighbourhood V of a point $p \in \Sigma_1$ is called a local isometry if $\exists$ a neighbourhood $\tilde{V}$ of $f(p)$ in Σ_2 s.t. $f: V \to \tilde{V}$ is an isometry.

If $\forall p \in \Sigma_1$, $\exists$ a local isometry to Σ_2, then Σ_1 is locally isometric to Σ_2.

If $f: \Sigma_1 \to \Sigma_2$ is a diffeomorphism and a local isometry $\forall p \in \Sigma$, then f is an isometry (globally).



Theorem Let $\sigma: U \to \Sigma$ and $\tilde{\sigma}: U \to \tilde{\Sigma}$ be parametrisations of the regular surfaces Σ and $\tilde{\Sigma}$ s.t. $E = \tilde{E}$, $F = \tilde{F}$, $G = \tilde{G}$.

Then the map $f := \tilde{\sigma} \circ \sigma^{-1} : \sigma(U) \to \tilde{\Sigma}$ is a local isometry.

Proof — Choose $p \in \sigma(U)$ and $w \in T_p\Sigma$. $\exists\ \gamma: (-\varepsilon, \varepsilon) \to \Sigma$ s.t. $\gamma(0) = p$, $\gamma'(0) = w$. We write $\gamma(t) = \sigma(u(t), v(t))$. $\Rightarrow$

$w = \gamma'(0) = \sigma_u(q)\, u'(0) + \sigma_v(q)\, v'(0)$ where $q = \sigma^{-1}(p) = (u(0), v(0))$. Therefore, $(Df)_p(w) = (f\circ\gamma)'(0) = \frac{d}{dt}(f\circ\sigma(u(t), v(t)))\big|_{t=0}$.

$(Df)_p(w) = \frac{d}{dt}(\tilde{\sigma}\circ\sigma^{-1}\circ\sigma(u(t), v(t)))\big|_{t=0} = \frac{d}{dt}(\tilde{\sigma}(u(t), v(t)))\big|_{t=0} = \tilde{\sigma}_u(q)\, u'(0) + \tilde{\sigma}_v(q)\, v'(0)$.

To check that it is a local isometry, we note that $I_{f(p)}((Df)_p w)$ $\cdots$ $= \langle \tilde{\sigma}_u u' + \tilde{\sigma}_v v', \tilde{\sigma}_u u' + \tilde{\sigma}_v v'\rangle = \langle \tilde{\sigma}_u, \tilde{\sigma}_u\rangle (u')^2 + 2\langle \tilde{\sigma}_u, \tilde{\sigma}_v\rangle (u'v') + \langle \tilde{\sigma}_v, \tilde{\sigma}_v\rangle (v')^2$.

Then, $I_{f(p)}((Df)_p w, (Df)_p w) = \tilde{E}(u')^2 + 2\tilde{F}u'v' + \tilde{G}(v')^2 = E(u')^2 + 2Fu'v' + G(v')^2 = \langle \sigma_u u' + \sigma_v v', \sigma_u u' + \sigma_v v'\rangle = \langle w, w\rangle = I_p(w)$.

$\Rightarrow f$ is a local isometry // q.e.d.

Ex Consider the cone (without its vertex) given by $z = a\rho$ in polar coordinates $(\rho > 0)$ where a is a constant. (Note: If $a=0$, this is a plane).

Using the parametrisation $\sigma(\rho, \theta) = (\rho\cos\theta, \rho\sin\theta, a\rho)$, find the FFF. Show that all cones are locally isometric to plane $\mathbb{R}^2$.

Soln. $\sigma_\rho = (\cos\theta, \sin\theta, a)$, $\sigma_\theta = (-\rho\sin\theta, \rho\cos\theta, 0)$. The FFF is $(a^2+1)\, d\rho^2 + \rho^2 d\theta^2$.

In terms of rescaled variables $\hat{\rho} = \sqrt{a^2+1}\,\rho$, $\hat{\theta} = \frac{\theta}{\sqrt{a^2+1}}$, we get FFF becoming $d\hat{\rho}^2 + \hat{\rho}^2 d\hat{\theta}^2$. i.e. $\hat{E} = 1$, $\hat{F} = 0$, $\hat{G} = \hat{\rho}^2$.

$\Rightarrow$ by previous theorem, all cones are locally isometric to a plane.

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Chapter 4
CURVATURE AND THE SECOND FUNDAMENTAL FORM.

Definition An orientation on a surface Σ is a continuous map $N: \Sigma \to \mathbb{R}^3$ s.t. $\forall p \in \Sigma$, $N(p)$ is a unit normal to $T_p\Sigma$.

Möbius strip —
non-orientable surface.

If a surface Σ admits an orientation, then it is called orientable.

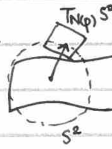
From here on, all surfaces that we consider are orientable.

Any coordinate neighbourhood $\sigma(U)$ is always orientable. [Recall — let $\gamma_1(t) = (u_0+t, v_0)$, $\gamma_2(t) = (u_0, v_0+t) \Rightarrow \gamma_1'(0) = \sigma_u(u_0, v_0)$, $\gamma_2'(0) = \sigma_v(u_0, v_0)$].

Define $N = \pm \frac{\sigma_u \times \sigma_v}{|\sigma_u \times \sigma_v|}$, where the +ve sign denotes the "standard" orientation.

Since we defined $N: \Sigma \to \mathbb{R}^3$, and N is the set of unit vectors, we can think of N as a map $N: \Sigma \to S^2$ (the 2-sphere), with points in S^2 identified with their position vectors.

Consider the differential $(DN)_p : T_p\Sigma \to T_{N(p)}S^2$. Note that $T_{N(p)}S^2 \cong T_p\Sigma$, as they are parallel planes, which are the same as vector spaces.



Considered as a map from Σ to S^2, N is called the <u>Gauss map</u>. The differential $(DN)_p$ is an endomorphism on $T_p\Sigma$, i.e. $(DN)_p : T_p\Sigma \to T_p\Sigma$.

<u>Self-adjoint maps.</u>

Let V be a real 2D vector space with an inner product $\langle , \rangle$ (e.g. $\mathbb{R}^2$).

Definition A linear map $A: V \to V$ is <u>self-adjoint</u> if $\langle Av, w\rangle = \langle v, Aw\rangle$ $\forall v, w \in V$. To each self-adjoint map $A: V \to V$, there is a symmetric bilinear map $B: V\times V \to \mathbb{R}$, defined by $B(v,w) = \langle Av, w\rangle$. If $\{e_1, e_2\}$ is an orthonormal basis for V, then the matrix $(b_{ij})_{2\times 2}$ given by $b_{ij} = \langle Ae_i, e_j\rangle$ is symmetric.

Furthermore, to each symmetric bilinear form B on V, there is a quadratic form $Q: V \to \mathbb{R}$ given by $Q(v) = B(v,v)$.

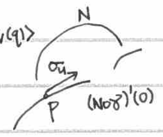
Q determines B uniquely by $B(u,v) = \frac{1}{2}(Q(u+v) - Q(u) - Q(v))$, so $\exists$ a 1-1 correspondence between symmetric bilinear maps and quadratic forms.

Theorem Let $A: V \to V$ be a self-adjoint linear map on V. Then the unit eigenvectors of A, e_1 and e_2, form an orthonormal basis for V.

The corresponding eigenvalues λ_1, λ_2 are real and are the maximum and minimum values of $Q(v) = \langle Av, v\rangle$ lie on the unit circle of V.

Theorem The differential $(DN)_p : T_p\Sigma \to T_p\Sigma$ of the Gauss map is self-adjoint. $\therefore$ NTP $\langle v, (DN)_p w\rangle = \langle (DN)_p v, w\rangle$.

Proof – Let $q = \sigma^{-1}(p) = (u_0, v_0)$. Since $\{\sigma_u(q), \sigma_v(q)\}$ is a basis for $T_p\Sigma$, it is sufficient to show WLOG that $\langle (DN)_p \sigma_u(q), \sigma_v(q)\rangle = \langle \sigma_u(q), (DN)_p \sigma_v(q)\rangle$



Let $\gamma(t) = \sigma(u_0+t, v_0)$, $\gamma(0) = p$. Then $\gamma'(0) = \sigma_u(u_0, v_0) = \sigma_u(t)$, $\therefore$ $(DN)_p \sigma_u(q) = (N\circ\gamma)'(0) = \frac{d}{dt}(N\circ\sigma(u_0+t, v_0))\big|_{t=0} = (N\circ\sigma)_u(q)$

$= \tilde{N}_u(q)$, where $\tilde{N} = N\circ\sigma$, $\tilde{N}: \underset{\mathbb{R}^2}{U} \to S^2$. Since $\tilde{N} \perp T_p\Sigma$, $\sigma_u \in T_p\Sigma$, then $\langle \tilde{N}, \sigma_u\rangle = 0$. Differentiating w.r.t. v,

$\langle \tilde{N}_v, \sigma_u\rangle + \langle \tilde{N}, \sigma_{uv}\rangle = 0$. Likewise, $\langle \tilde{N}, \sigma_v\rangle = 0 \Rightarrow \langle \tilde{N}_u, \sigma_v\rangle + \langle \tilde{N}, \sigma_{vu}\rangle = 0$. $\Rightarrow$ together, this gives that

$\langle \tilde{N}_u, \sigma_v\rangle = \langle \sigma_u, \tilde{N}_v\rangle \Leftrightarrow \langle (DN)_p \sigma_u, \sigma_v\rangle = \langle \sigma_u, (DN)_p \sigma_v\rangle$, q.e.d.

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Definition The quadratic form $\mathrm{II}_p : T_p\Sigma \to \mathbb{R}$ given by $\mathrm{II}_p(w) = -\langle (DN)_p w, w\rangle$ $\forall w \in T_p\Sigma$ is called the <u>2^{nd} fundamental form</u>.

The eigenvalues k_1, k_2 of $-(DN)_p$ are called the <u>principal curvatures</u> of Σ at p. Also, $K = k_1k_2 = \det((DN)_p)$ is called the <u>Gauss curvature</u>., and the quantity $H = \frac{1}{2}(k_1+k_2) = -\frac{1}{2}\operatorname{Tr}[(DN)_p]$ is the <u>mean curvature</u>.

For any $w \in T_p\Sigma$, $\gamma(0) = p$, $\gamma'(0) = w$, $\gamma(t) = \sigma(u(t), v(t))$, $w = \gamma'(0) = u'(0)\sigma_u(q) + v'(0)\sigma_v(q)$. Then $\mathrm{II}_p(w) = -\langle (DN)_p w, w\rangle$, and expanding it gives us

$\mathrm{II}_p(w) = -\langle u'(0)(DN)_p\sigma_u + v'(0)(DN)_p\sigma_v, u'(0)\sigma_u + v'(0)\sigma_v\rangle = -\langle u'\tilde{N}_u + v'\tilde{N}_v, u'\sigma_u + v'\sigma_v\rangle$ as $(DN)_p\sigma_u = \tilde{N}_u$; $= -(u')^2\langle \tilde{N}_u, \sigma_u\rangle - \underbrace{u'v'\langle \tilde{N}_u, \sigma_v\rangle - v'u'\langle \tilde{N}_v, \sigma_u\rangle}_{\text{same by self-adjointness}} - (v')^2\langle \tilde{N}_v, \sigma_v\rangle$

$= -[(u')^2\langle \tilde{N}_u, \sigma_u\rangle + 2u'v'\langle \tilde{N}_u, \sigma_v\rangle + (v')^2\langle \tilde{N}_v, \sigma_v\rangle] = e(u')^2 + 2f(u')(v') + g(v')^2$ where $e = -\underbrace{\langle (DN)_p\sigma_u, \sigma_u\rangle}_{=\langle \tilde{N}_u, \sigma_u\rangle}$, $f = -\langle (DN)_p\sigma_u, \sigma_v\rangle = -\langle \tilde{N}_u, \sigma_v\rangle = -\langle \tilde{N}_v, \sigma_u\rangle$.

Then e, f, g are called the components of the second fundamental form, which can also be expressed as $e\,du^2 + 2f\,du\,dv + g\,dv^2$.

Recall that $\langle \tilde{N}, \sigma_u\rangle = 0 \Rightarrow \langle \tilde{N}_u\sigma_u + \tilde{N}, \sigma_{uu}\rangle = 0 \Rightarrow e = -\langle \tilde{N}_u, \sigma_u\rangle = \langle \tilde{N}, \sigma_{uu}\rangle$. Likewise, $f = -\langle \tilde{N}_u, \sigma_v\rangle = \langle \tilde{N}, \sigma_{uv}\rangle$, $g = -\langle \tilde{N}_v, \sigma_v\rangle = \langle \tilde{N}, \sigma_{vv}\rangle$.

Also, remember that $\tilde{N} = \pm\frac{\sigma_u\times\sigma_v}{|\sigma_u\times\sigma_v|}$. $\tilde{N}$ is a unit vector $\Rightarrow \langle \tilde{N}, \tilde{N}\rangle = 1 \Rightarrow \langle \tilde{N}_u, \tilde{N}\rangle = \langle \tilde{N}_v, \tilde{N}\rangle = 0$, $\{\sigma_u, \sigma_v, \tilde{N}\}$ is a basis for $\mathbb{R}^3$. So,

$\exists$ functions $a_{ij}(u,v)$ s.t. $\tilde{N}_u = a_{11}\sigma_u + a_{21}\sigma_v$ –①, $\tilde{N}_v = a_{12}\sigma_u + a_{22}\sigma_v$ –②. Note for any $w = \alpha\sigma_u + \beta\sigma_v \in T_p\Sigma$, $(DN)_p w = \alpha(DN)_p\sigma_u + \beta(DN)_p\sigma_v$

$\Rightarrow (DN)_p w = \alpha\tilde{N}_u + \beta\tilde{N}_v = (\alpha a_{11} + \beta a_{12})\sigma_u + (\alpha a_{21} + \beta a_{22})\sigma_v \Rightarrow (DN)_p$ maps $(\alpha, \beta) \mapsto (\alpha a_{11} + \beta a_{12}, \alpha a_{21} + \beta a_{22})$, i.e. $\begin{pmatrix}\alpha\\ \beta\end{pmatrix} \mapsto \begin{pmatrix}a_{11} & a_{12}\\ a_{21} & a_{22}\end{pmatrix}\begin{pmatrix}\alpha\\ \beta\end{pmatrix}$.

The Gauss and mean curvatures are: $K = \det(a_{ij})$ and $H = -\frac{1}{2}(a_{11} + a_{22})$. Then $\langle ①, \sigma_u\rangle$: $\langle \tilde{N}_u, \sigma_u\rangle = a_{11}\langle \sigma_u, \sigma_u\rangle + a_{21}\langle \sigma_v, \sigma_u\rangle \Rightarrow -e = a_{11}E + a_{21}F$.

We end up with four equations: $\langle ①, \sigma_v\rangle$: $-f = a_{11}F + a_{21}G$; $\langle ②, \sigma_u\rangle$: $-f = a_{12}E + a_{22}F$ and $\langle ②, \sigma_v\rangle$: $-g = a_{12}F + a_{22}G$.

Four equations in four unknowns can be expressed in matrices: $\begin{pmatrix}E & F\\ F & G\end{pmatrix}\begin{pmatrix}a_{11} & a_{12}\\ a_{21} & a_{22}\end{pmatrix} = -\begin{pmatrix}e & f\\ f & g\end{pmatrix} \Rightarrow \begin{pmatrix}a_{11} & a_{12}\\ a_{21} & a_{22}\end{pmatrix} = \frac{-1}{EG-F^2}\begin{pmatrix}G & -F\\ -F & E\end{pmatrix}\begin{pmatrix}e & f\\ f & g\end{pmatrix}$

Then $K = \frac{eg-f^2}{EG-F^2}$, and $H = \frac{1}{2}\frac{eG - 2fF + gE}{EG-F^2}$. Also, $\tilde{N}_u\times\tilde{N}_v = (a_{11}\sigma_u + a_{21}\sigma_v)\times(a_{12}\sigma_u + a_{22}\sigma_v) = a_{11}a_{22}\sigma_u\times\sigma_v + a_{21}a_{12}\sigma_v\times\sigma_u = \det(A)\,\sigma_u\times\sigma_v$
$= K\,\sigma_u\times\sigma_v$.

$1 \leftrightarrow u$, $2 \leftrightarrow v$

$(\sigma_u)_v$, $(\sigma_v)_u$ are in $\mathbb{R}^3$, which is spanned by $\{\sigma_u, \sigma_v, \tilde{N}\}$. Thus, $\exists$ scalar functions of (u,v), Γ_{ij}^{k}, λ, μ, ν s.t.

- $\sigma_{uu} = \Gamma_{11}^{1}\sigma_u + \Gamma_{11}^{2}\sigma_v + \lambda\tilde{N}$ –③
- $\sigma_{vu} = \sigma_{uv} = \Gamma_{12}^{1}\sigma_u + \Gamma_{12}^{2}\sigma_v + \mu\tilde{N}$ – ④
- $\sigma_{vv} = \Gamma_{22}^{1}\sigma_u + \Gamma_{22}^{2}\sigma_v + \nu\tilde{N}$. –⑤.

Γ_{ij}^{k} are called <u>Christoffel symbols</u>. Then $\langle ③, \tilde{N}\rangle$: $\langle \sigma_{uu}, \tilde{N}\rangle = 0 + 0 + \lambda\langle \tilde{N}, \tilde{N}\rangle = \lambda \Rightarrow \lambda = e$ $\because$ $e = -\langle \tilde{N}_u, \sigma_u\rangle = \langle \tilde{N}, \sigma_{uu}\rangle$.

Similarly, we can get that $\mu = f$, $\nu = g$. For the Christoffel symbols, note for instance that $\langle ③, \sigma_u\rangle$: $\langle \sigma_{uu}, \sigma_u\rangle = \Gamma_{11}^{1}\langle \sigma_u, \sigma_u\rangle + \Gamma_{11}^{2}\langle \sigma_v, \sigma_u\rangle = \Gamma_{11}^{1}E + \Gamma_{11}^{2}F$.

$\Rightarrow \frac{1}{2}E_u = \Gamma_{11}^{1}E + \Gamma_{11}^{2}F$. $\langle ③, \sigma_v\rangle$: $\langle \sigma_{uu}, \sigma_v\rangle = \Gamma_{11}^{1}F + \Gamma_{11}^{2}G$. Note $F_u = \langle \sigma_u, \sigma_v\rangle_u = \langle \sigma_{uu}, \sigma_v\rangle + \langle \sigma_u, \sigma_{vu}\rangle = \langle \sigma_{uu}, \sigma_v\rangle + \frac{1}{2}E_v$.

$\Rightarrow F_u - \frac{1}{2}E_v = \Gamma_{11}^{1}F + \Gamma_{11}^{2}G$. The other four equations are eventually derived; together these give: $\begin{pmatrix}E & F\\ F & G\end{pmatrix}\begin{pmatrix}\Gamma_{11}^{1} & \Gamma_{12}^{1} & \Gamma_{22}^{1}\\ \Gamma_{11}^{2} & \Gamma_{12}^{2} & \Gamma_{22}^{2}\end{pmatrix} = \begin{pmatrix}\frac{1}{2}E_u & \frac{1}{2}E_v & F_v - \frac{1}{2}G_u\\ F_u - \frac{1}{2}E_u & \frac{1}{2}G_u & \frac{1}{2}G_v\end{pmatrix}$.

Thus we get $\begin{pmatrix}\Gamma_{11}^{1} & \Gamma_{12}^{1} & \Gamma_{22}^{1}\\ \Gamma_{11}^{2} & \Gamma_{12}^{2} & F_{22}^{2}\end{pmatrix} = \frac{1}{EG-F^2}\begin{pmatrix}G & -F\\ -F & E\end{pmatrix}\begin{pmatrix}\frac{1}{2}E_u & \frac{1}{2}E_v & F_v - \frac{1}{2}G_u\\ F_u - \frac{1}{2}E_u & \frac{1}{2}G_u & \frac{1}{2}G_v\end{pmatrix}$.

Remarks — $\cdot$ $EG - F^2 \neq 0$, so this is well-defined

- All Christoffel symbols depend solely on the 1^{st} fundamental form.
- We do not always need to calculate these for lower dimensions, but these are applicable in general.

We obtain three compatibility conditions on parameters: $(\sigma_{uu})_v = (\sigma_{uv})_u$, $(\sigma_{uv})_v, (\sigma_{vv})_u$, $(\tilde{N}_u)_v = (\tilde{N}_v)_u$. 31 October 2013 Prof Rod HALBURD. Gordon St (16-18)

Now, $(\sigma_{uu})_v = (\Gamma_{11}^1)_v \sigma_u + \Gamma_{11}^1(\Gamma_{21}^1 \sigma_u + \Gamma_{21}^2 \sigma_v + f\tilde{N}) + (\Gamma_{11}^2)_v \sigma_v + \Gamma_{11}^2(\Gamma_{22}^1 \sigma_u + \Gamma_{22}^2 \sigma_v + g\tilde{N}) + e_v \tilde{N} + e(a_{12}\sigma_u + a_{22}\sigma_v)$.

So, from $(\sigma_{uu})_v = (\sigma_{uv})_u$ condition, the coefficient of σ_v: $(\Gamma_{11}^2)_v - (\Gamma_{12}^2)_u + \Gamma_{11}^1\Gamma_{12}^2 - \Gamma_{12}^1\Gamma_{11}^2 + \Gamma_{11}^2\Gamma_{22}^2 - \Gamma_{12}^2\Gamma_{12}^2 = EK$ (the K = Gauss curvature).

A similar thing can be done to the other two compatibility conditions to create two other relationships between the 1st and 2nd Fundamental forms: $(e_v - f_u, f_v - g_u)$.

These relations are called the <u>Mainardi-Codazzi equations</u>.

$\boxed{\text{Ex}}$ Analyse the curve $z = \varphi(x,y)$, calculating its fundamental forms and Gauss curvature.

Soln. $\sigma(u,v) = (u, v, \varphi(u,v))$. $\sigma_u = (1, 0, \varphi_u)$, $\sigma_v = (0, 1, \varphi_v)$. $E = \langle \sigma_u, \sigma_u \rangle = 1 + \varphi_u^2$, $F = \langle \sigma_u, \sigma_v \rangle = \varphi_u \varphi_v$, $G = \langle \sigma_v, \sigma_v \rangle = 1 + \varphi_v^2$ thus, we obtain that the 1st Fundamental Form:

$FFF = (1+\varphi_u)^2 du^2 + 2\varphi_u\varphi_v\, du\, dv + (1+\varphi_v)^2 dv^2$ // $\sigma_u \times \sigma_v = (-\varphi_u, -\varphi_v, 1)$. $\tilde{N} = \frac{(-\varphi_u, -\varphi_v, 1)}{\sqrt{1+\varphi_u^2+\varphi_v^2}}$ $e = -\langle DN\sigma_u, \sigma_u \rangle = -\langle \tilde{N}_u, \sigma_u \rangle = \langle \tilde{N}, \sigma_{uu} \rangle$ $\left[\because \langle \tilde{N}, \sigma_u \rangle = 0 \atop \langle \tilde{N}_u, \sigma_u \rangle + \underbrace{\langle \tilde{N}, \sigma_{uu} \rangle}_{=0}\right]$

$\sigma_{uu} = (0, 0, \varphi_{uu})$, $\sigma_{vv} = (0, 0, \varphi_{vv})$, $\sigma_{uv} = (0, 0, \varphi_{uv})$. $e = \langle \tilde{N}, \sigma_{uu} \rangle = \frac{\varphi_{uu}}{\sqrt{1+\varphi_u^2+\varphi_v^2}}$, $f = \frac{\varphi_{uv}}{\sqrt{1+\varphi_u^2+\varphi_v^2}}$, $g = \frac{\varphi_{vv}}{\sqrt{1+\varphi_u^2+\varphi_v^2}}$

$SFF = \frac{\varphi_{uu}\, du^2 + 2\varphi_{uv}\, du\, dv + \varphi_{vv}\, dv^2}{\sqrt{1+\varphi_u^2+\varphi_v^2}}$ // and by Gauss, $K = \frac{eg - f^2}{EG - F^2} = \frac{\varphi_{uu}\varphi_{vv} - \varphi_{uv}^2}{(1+\varphi_u^2+\varphi_v^2)^2}$ //.

Chapter 5
GEODESICS.

<u>Covariant derivative</u>

$\boxed{\text{Definition}}$ Let V be an open set in a regular surface. A vector field on V is a smooth function $w: V \to \mathbb{R}^3$ s.t. $\forall p \in V$, $w(p) \in T_p\Sigma$. $w(t) = a(t)\sigma_u(u(t), v(t)) + b(t)\sigma_v(u(t), v(t))$

Then $\frac{dw}{dt} = \dot{a}\sigma_u + a(\sigma_{uu}\dot{u} + \sigma_{uv}\dot{v}) + \dot{b}\sigma_v + b(\sigma_{vu}\dot{u} + \sigma_{vv}\dot{v}) = \{\dot{a} + a\Gamma_{11}^1\dot{u} + a\Gamma_{12}^1\dot{v} + b\Gamma_{12}^1\dot{u} + b\Gamma_{22}^1\dot{v}\}\sigma_u + \{\dot{b} + \Gamma_{11}^2 a\dot{u} + \Gamma_{12}^2 a\dot{v} + \Gamma_{21}^2 b\dot{u} + \Gamma_{22}^2 b\dot{v}\}\sigma_v + \{ea\dot{u} + f(a\dot{v} + b\dot{u}) + gb\dot{v}\}\tilde{N}$

The projection of $\frac{dw}{dt}$ in the tangent plane is called the <u>covariant derivative</u> of w in the direction γ'. $\nabla_{\gamma'} w = (\dot{a} + \dots + \Gamma_{22}^1 b\dot{v})\sigma_u + (\dot{b} + \dots + \Gamma_{22}^2 b\dot{v})\sigma_v$. — ⊕

$\boxed{\text{Definition}}$ A smooth vector field is said to be <u>parallel along γ</u> if $\nabla_{\gamma'} w = 0$ $\forall t \in I$ $(\gamma: I \to \Sigma)$.

$\boxed{\text{Theorem}}$ Let w_1 and w_2 be parallel vector fields along $\gamma: I \to \Sigma$. Then $\langle w_1, w_2 \rangle$ is constant. In particular, $|w_1|$, $|w_2|$ and the angle between them is constant.

Proof – $w_1, w_2 \in T_p\Sigma$ but $w_1', w_2' \perp$ to $T_p\Sigma$. So $\langle w_1, w_2' \rangle = 0$. Now $\frac{d}{dt}\langle w_1, w_2 \rangle = \langle w_1', w_2 \rangle + \langle w_1, w_2' \rangle = 0$ // q.e.d.

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$\boxed{\text{Theorem}}$ Let $\gamma: I \to \Sigma$ be a parametrised curve, and choose $W_0 \in T_{\gamma(t_0)}\Sigma$ for some $t_0 \in I$. Then there is a unique parallel vector field $w(t)$ along $\gamma(t)$, with $w(t_0) = W_0$.

$\boxed{\text{Definition}}$ A non-constant parametrised curve $\gamma: I \to \Sigma$ is said to be geodesic if γ' is parallel along γ, i.e. $\nabla_{\gamma'}\gamma' = 0$.

Note – $\gamma' = \sigma_u u' + \sigma_v v'$ (i.e. $a = u'$, $b = v'$) from ⊕: $\ddot{u} + \Gamma_{11}^1\dot{u}^2 + 2\Gamma_{12}^1\dot{u}\dot{v} + \Gamma_{22}^1\dot{v}^2 = 0$, $\ddot{v} + \Gamma_{11}^2\dot{u}^2 + 2\Gamma_{12}^2\dot{u}\dot{v} + \Gamma_{22}\dot{v}^2 = 0$.

$\boxed{\text{Theorem}}$ Another form of the geodesic equation is $\frac{d}{dt}(E\dot{u} + F\dot{v}) = \frac{1}{2}(E_u\dot{u}^2 + 2F_u\dot{u}\dot{v} + G_u\dot{v}^2)$, $\frac{d}{dt}(F\dot{u} + G\dot{v}) = \frac{1}{2}(E_v\dot{u}^2 + 2F_v\dot{u}\dot{v} + G_v\dot{v}^2)$.

Note – $\nabla_{\gamma'}\gamma' = 0 \Leftrightarrow \gamma''$ has no component in tangent plane $\Leftrightarrow \gamma'' = (\dots)\tilde{N}$.

Proof – $0 = \ddot{\gamma}\cdot\sigma_u$ (no component of $\ddot{\gamma}$ in the σ_u direction) $= \{\frac{d}{dt}(\dot{u}\sigma_u + \dot{v}\sigma_v)\}\cdot\sigma_u = \frac{d}{dt}\{(\dot{u}\sigma_u + \dot{v}\sigma_v)\cdot\sigma_u\} - (\dot{u}\sigma_u + \dot{v}\sigma_v)\frac{d\sigma_u}{dt}$

$= \frac{d}{dt}\{\dot{u}E + \dot{v}F\} - (\dot{u}\sigma_u + \dot{v}\sigma_v)\cdot(\sigma_{uu}\dot{u} + \sigma_{uv}\dot{v}) = \frac{d}{dt}\{E\dot{u} + F\dot{v}\} - \{\overbrace{\sigma_u\cdot\sigma_{uu}}^{\frac{1}{2}E_u}\dot{u}^2 + \overbrace{(\sigma_u\cdot\sigma_{uv} + \sigma_v\cdot\sigma_{uu})}^{F_u}\dot{u}\dot{v} + \overbrace{\sigma_v\cdot\sigma_{uv}}^{\frac{1}{2}G_u}\dot{v}^2\} \Rightarrow \frac{d}{dt}\{E\dot{u} + F\dot{v}\} = \frac{1}{2}(E_u\dot{u}^2 + 2F_u\dot{u}\dot{v} + G_u\dot{v}^2)$ // q.e.d. //

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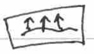
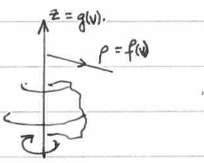
Recap: parallel $\Leftrightarrow \nabla_{\gamma'} w = 0$, geodesics are where $\nabla_{\gamma'}\gamma' = 0$. We know that $\nabla_\gamma w = 0 \Leftrightarrow |w|$ constant. $\left.\begin{matrix}\nabla_{\gamma'} w_1 = 0 \\ \nabla_{\gamma'} w_2 = 0\end{matrix}\right\}$ $w_1\cdot w_2 = 0$ $\because (w_1\cdot w_2)' = 0 + 0$. 

$\boxed{\text{Corollary}}$ Choose $w \in T_p\Sigma$. Then $\exists$ unique geodesic γ on Σ passing through p with tangent vector w.
$p \in \Sigma$,

<u>Geodesics on rotationally symmetric surfaces.</u>



Any surface that is rotationally symmetric about the z-axis has a parametrisation of the form $\sigma(u,v) = (f(v)\cos u, f(v)\sin u, g(v))$.

Curve in the ρ-z plane is unit speed by parametrisation, so $(f')^2 + (g')^2 = 1$. This simplifies our calculations.

$\sigma_u = (-f\sin u, f\cos u, 0)$, $\sigma_v = (f'\cos u, f'\sin u, g')$. $E = \langle \sigma_u, \sigma_u \rangle = f^2$, $F = \langle \sigma_u, \sigma_v \rangle = 0$, $G = \langle \sigma_v, \sigma_v \rangle = (f')^2 + (g')^2 = 1$. Then first fundamental form is $f^2 du^2 + dv^2$.

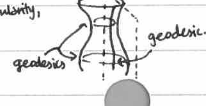
Apply "alternate" geodesic equations: $\frac{d}{dt}(E\dot{u} + F\dot{v}) = \frac{1}{2}(E_u\dot{u}^2 + 2F_u\dot{u}\dot{v} + G_u\dot{v}^2) \Leftrightarrow \frac{d}{dt}(f^2\dot{u}) = 0$ —① [as f is a function of v, no u term]. Also, our other equation is

$\frac{d}{dt}(F\dot{u} + G\dot{v}) = \frac{1}{2}(E_v\dot{u}^2 + 2F_v\dot{u}\dot{v} + G_v\dot{v}^2) \Leftrightarrow \frac{d}{dt}(\dot{v}) = \ddot{v} = f(v)f'(v)\dot{u}^2$ — ②. Start by considering geodesics of the form $u = u_0$ [i.e. a slice of the plane]. This satisfies ①.

② becomes $\ddot{v} = 0 \Rightarrow v = \alpha t + \beta$. The intersection of Σ with any plane containing the z-axis is the image of a geodesic.

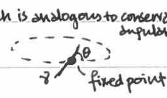
Next, we look for geodesics of form $v = v_0$. ① becomes $f(v_0)\ddot{u} = 0$, ②: $0 = f(v_0)f'(v_0)\dot{u}^2 \Rightarrow \dot{u} \neq 0$, else we have no curve, $f(v_0) \neq 0$ by non-singularity, $f'(v_0) = 0$

$\Rightarrow \ddot{u} = 0$ and $u = \alpha t + \beta$. $\Rightarrow$ hence geodesics occur where $f'(v_0) = 0 \Rightarrow v_0$ is a local extremum of f (max and min distances from z-axis).



In the general case, note $\langle \sigma_u, \dot{\gamma} \rangle = \langle \sigma_u, \dot{u}\sigma_u + \dot{v}\sigma_v \rangle = \dot{u}\langle \sigma_u, \sigma_u \rangle + \dot{v}\langle \sigma_u, \sigma_v \rangle = E\dot{u} + F\dot{v} = f^2(v)\dot{u}$ [Note: this is conserved by ①, which is analogous to conservation of angular momentum]

①: $\langle \sigma_u, \dot{\gamma} \rangle = \text{const} \Rightarrow |\sigma_u||\dot{\gamma}|\cos\theta = \text{const}$. (θ: angle between tangent vector to γ and the parallel (v = const) curve through the point.) $|\dot{\gamma}|$ is also constant (geodesic). Also, $|\sigma_u| = \sqrt{E} = f(v) =$ distance r from p to z-axis.



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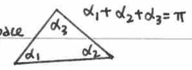
i.e. ① $\Leftrightarrow$ $\boxed{r\cos\theta = \text{const}}$. This is called Clairaut's relation. For the second relation, $\text{const} = |\dot{\gamma}|^2 = E\dot{u}^2 + 2F\dot{u}\dot{v} + G\dot{v}^2 = f^2\dot{u}^2 + \dot{v}^2$. $\dot{v}^2 = \text{const} - f^2\dot{u}^2$. Now $f^2\dot{u} = c$ const from ①.

$\Rightarrow \dot{v}^2 = \text{const} - \frac{c^2}{f^2(v)}$. Differentiating, $2\dot{v}\ddot{v} = 2\frac{c^2}{f^3(v)} f'(v)\dot{v} \Rightarrow \ddot{v} = \frac{c^2}{f^3(v)} f'(v) = f(v)f'(v)\dot{u}^2 \Leftrightarrow$ ②, so no further information/restriction is produced by ② to add to Clairaut's relation.

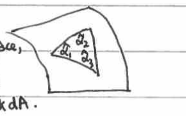
chapter 6

THE GAUSS-BONNET THEOREM



Consider a curve γ in Σ parametrised by arclength. A basis for $\mathbb{R}^3$ is $\{\dot{\gamma}, N, \dot{\gamma}\times N\}$. This is an orthonormal frame.

Now, $\ddot{\gamma} = k_n N + k_g(N\times\dot{\gamma})$. Note: γ is a geodesic $\Leftrightarrow k_g = 0$. k_g is called the geodesic curvature, k_n is called the normal curvature.



The curvature k of γ (as a curve in $\mathbb{R}^3$) is $k = |\dot{t}| = |\ddot{\gamma}|$. Then $k^2 = |\dot{t}|^2 = |\ddot{\gamma}|^2 = k_n^2 + k_g^2$

$\gamma'(s) \in T_{\gamma(s)}\Sigma$, $N\circ\gamma(s) \perp T_{\gamma(s)}\Sigma$. Thus, $\langle\gamma'(s), N\circ\gamma(s)\rangle = 0 \xRightarrow{\text{differentiate, set } s=0} \langle\gamma''(0), N(p)\rangle + \langle\gamma'(0), (N\circ\gamma)'(0)\rangle = 0 \Rightarrow \langle\gamma''(0), N(p)\rangle = -\langle(N\circ\gamma)'(0), w\rangle$

$\therefore \langle\gamma''(0), N(p)\rangle = -\langle(DN)_p w, w\rangle = \mathrm{II}_p(w) \Rightarrow k_n = \mathrm{II}_p(w)$. This gives us another interpretation of the second fundamental form.

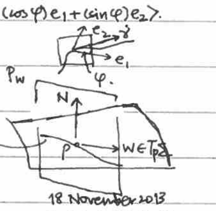
Recall that $-(DN)_p : T_p\Sigma \to T_p\Sigma$ is self-adjoint, i.e. it has real eigenvalues k_1, k_2 (principal curvatures) and eigenvectors e_1, e_2 which are orthonormal (principal directions).

Any unit vector $w \in T_p\Sigma$ ($|w|=1$) can be written as $w = e_1\cos\varphi + e_2\sin\varphi$, $k_n(p) = \mathrm{II}_p(w) = \langle -(DN)_p w, w\rangle = \langle(\cos\varphi)(-DN)e_1 + (\sin\varphi)(-DN)e_2, (\cos\varphi)e_1 + (\sin\varphi)e_2\rangle$. $(w = \gamma'(0))$

$= \langle k_1 e_1\cos\varphi + k_2 e_2\sin\varphi, e_1\cos\varphi + e_2\sin\varphi\rangle = k_1\cos^2\varphi + k_2\sin^2\varphi$. This gives us Euler's formula: $k_n = k_1\cos^2\varphi + k_2\sin^2\varphi$.

This enables us to calculate curvatures in components. In particular, $\varphi = 0 \Rightarrow k_n = k_1$, $\varphi = \frac{\pi}{2}$, $k_n = k_2$.

Note that we have thus far already defined several curvatures, which are all different. Do not mix them up!



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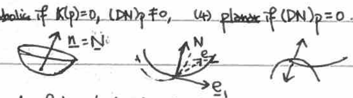
Definition For any $p \in \Sigma$, $w \in T_p\Sigma$, let P_w be the plane in $\mathbb{R}^3$ containing p, and parallel to N and w. The intersection $\Sigma \cap P_w$ is called the normal section of Σ at p in the direction w. We can parametrise this curve γ by arclength. Since γ lies in the plane, then so does its principal normal vector at p, $\underline{n}$; and $\underline{n} = \pm N$. Then $k_n = k\langle\underline{n}, N\rangle = \begin{cases} k & \text{if } \underline{n} = N \\ -k & \text{if } \underline{n} = -N. \end{cases}$

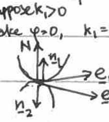
Recall that $k_n = k_1(\cos^2\varphi) + k_2(\sin^2\varphi)$, and Gauss curvature, $K = k_1 k_2$. If $K>0$, k_1, k_2 have same sign. In particular, k_n has same sign for any normal section.

Definition A point p on a regular surface Σ is called (1) elliptic if $K(p) > 0$, (2) hyperbolic if $K(p) < 0$, (3) parabolic if $K(p) = 0$, $(DN)_p \neq 0$, (4) planar if $(DN)_p = 0$.

Suppose we have case (1): elliptic at p. Then at p, $K>0 \Rightarrow$ $k_1, k_2 > 0$ or $k_1, k_2 < 0$. $\Rightarrow$ suppose $k_i > 0$ Take $\varphi = 0$, $k_1 = k_n = k\underbrace{\langle n, N\rangle}_{+1} \Rightarrow n = N$



Then for case (2), $K(p) < 0 \Rightarrow k_1, k_2$ have opposite signs. Take $k_1 > 0$, $k_2 < 0$.



Then $k_1 = k\langle n_1, N\rangle$, $\langle n_1, N\rangle = +1$. If $k_2 = k\langle n_2, N\rangle$, $\langle n_2, N\rangle = -1$.

Geodesic curvature

Until now, we have usually used the basis $\{\sigma_u, \sigma_v\}$ for $T_p\Sigma$. Now, we will instead use an orthonormal basis $\{e_1, e_2\}$ (not necessarily eigenvectors) e.g. $e_1 = \frac{\sigma_u}{\sqrt{E}}$, $e_2 = \frac{E\sigma_v - F\sigma_u}{\sqrt{E(EG-F^2)}}$ by orthogonal diagonalisation.

These are smooth, and we can verify that $|e_1| = |e_2| = 1$, $e_1 \cdot e_2 = 0$, $e_1 \times e_2 = \frac{\sigma_u \times \sigma_v}{|\sigma_u \times \sigma_v|}$.

Lemma Let Σ be an orientable regular surface with orientation N. Let e_1, e_2 be smooth functions s.t. at each $p \in \Sigma$, $\{e_1, e_2\}$ is an orthonormal basis of $T_p\Sigma$ and $N = e_1 \times e_2$.

Let θ be a smooth function s.t. $\dot{\gamma} = e_1\cos\theta + e_2\sin\theta$. Then $k_g = \dot{\theta} - e_1\cdot\dot{e}_2$ — ①.

Proof – $\ddot{\gamma} = \dot{e}_1\cos\theta + \dot{e}_2\sin\theta - e_1\sin\theta\,\dot{\theta} + e_2\cos\theta\,\dot{\theta}$ and $N\times\dot{\gamma} = -e_1\sin\theta + e_2\cos\theta$. Then we note that $e_1\cdot e_2 = 0$, $e_1\cdot e_1 = 1$ etc. Then $e_1\cdot\dot{e}_1 = 0$ from $e_1\cdot e_1 = 1$.

and $\dot{e}_1\cdot e_2 + e_1\cdot\dot{e}_2 = 0$ from $e_1\cdot e_2 = 0$, $\ddot{\gamma} = k_n N + k_g N\times\dot{\gamma} \Rightarrow k_g = \ddot{\gamma}\cdot(N\times\dot{\gamma}) = \dot{\theta} - e_1\cdot\dot{e}_2$, q.e.d.

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Lemma In the above notation, $(e_1)_u(e_2)_v - (e_1)_v(e_2)_u = \frac{eg-f^2}{\sqrt{EG-F^2}}$. — ②

Proof – $\{e_1, e_2, \tilde{N}\}$ is an orthonormal basis for $\mathbb{R}^3$. Then $e_1\cdot(e_1)_u = 0$ etc so $\exists$ scalars a, b, c, d s.t. $(e_1)_u = ae_2 + c\tilde{N}$, $(e_1) = be_2 + d\tilde{N}$. Similarly, noting that $e_1\cdot(e_2)_u = -(e_1)_u\cdot e_2$, $(e_2)_u = -a + \hat{c}\tilde{N}$ since $e_1\cdot(e_2)_u = -(e_1)_u\cdot e_2 = ae_2\cdot e_2$, $(e_2)_v = -be_1 + \hat{d}\tilde{N}$. Then $(e_1)_u\cdot(e_2)_v - (e_1)_v(e_2)_u = (ae_2 + c\tilde{N})\cdot(-be_1 + \hat{d}\tilde{N}) - (be_2 + d\tilde{N})\cdot(-ae_1 + \hat{c}\tilde{N})$

$= (c\hat{d} - d\hat{c}) = ((e_1)_u\cdot\tilde{N})((e_2)_v\cdot\tilde{N}) - ((e_1)_v\cdot\tilde{N})((e_2)_u\cdot\tilde{N}) = X$. Using $e_1\cdot\tilde{N} = 0$, $(e_1)_u\cdot\tilde{N} = -e_1\cdot\tilde{N}_u$ etc, $c\hat{d} - d\hat{c} = (e_1\cdot\tilde{N}_u)(e_2\cdot\tilde{N}_v) - (e_1\cdot\tilde{N}_v)(e_2\cdot\tilde{N}_u)$. Then, using the identity $(A\times B)\cdot(C\times D) = (A\cdot C)(B\cdot D) - (A\cdot D)(B\cdot C)$, $X = (\tilde{N}_u\times\tilde{N}_v)\cdot(e_1\times e_2) = (\tilde{N}_u\times\tilde{N}_v)\cdot\tilde{N} = \frac{eg-f^2}{\sqrt{EG-F^2}}$ from equation in Chapter 4.

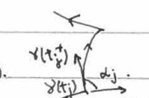
All these motivate our eventual statement of the Gauss-Bonnet theorem: $\sum_{j=0}^{n}\int_{s_j}^{s_{j+1}} k_g(s)\,ds + \iint_R K\,dA + \sum\alpha_j = 2\pi$.

Definition A map $\gamma : [0,1] \to \Sigma$ is a parametrised piecewise regular curve if γ is continuous and where $t_0, t_1, \dots, t_{n+1} \in [0,1]$ with $0 = t_0 < t_1 < \dots < t_{n+1} = \ell$ s.t. $j \in [0, 1, \dots, n]$.

The restriction of γ to $[t_j, t_{j-1}]$ is a regular curve (called a regular arc of γ). Furthermore, γ is called simple if $\gamma(a) \neq \gamma(b)$ $\forall$ distinct $a, b \in [0, \ell]$.

It is called closed if $\gamma(0) = \gamma(\ell)$. At each vertex $\gamma(t_j)$, the limits $\dot{\gamma}(t_j^+) = \lim_{t\to t_j^+}\dot{\gamma}(t)$ and $\dot{\gamma}(t_j^-)$ exist.

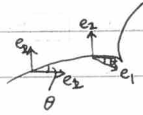
We define the exterior angle $\alpha_j \in [-\pi, \pi]$ at $\gamma(t_j)$ as follows. Let $|\alpha_j|$ be smallest determination of angle between $\dot{\gamma}(t_j^-)$ and $\dot{\gamma}(t_j^+)$.



If α_j is not 0 or π, then $w = \dot{\gamma}(t_j^-)\times\dot{\gamma}(t_j^+)$. If w points in the same direction as N, $\alpha_j > 0$. Otherwise, $\alpha_j < 0$. If $|\alpha_j| = \pi$, then α_j is chosen to be $\pi \Leftrightarrow$ the vector $\dot{\gamma}(t_j - \varepsilon)\times\dot{\gamma}(t_j + \varepsilon)$ points in the same direction as N for sufficiently small size.

Theorem (Turning tangents).

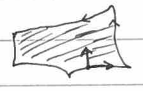
With above notation, $\sum_{j=0}^{n}[\theta(s_{j+1}^-)-\theta(s_{j+1}^+)]+\sum_{j=0}^{n}\alpha_i = \pm 2\pi$. — ③ (sign depends on orientation.)



where e_1, e_2 are any orthonormal frame in tangent plane.

Proof — Topology; omitted.

Definition A region R of an oriented surface Σ is said to be *simple* if it is homeomorphic to the unit circle, and its boundary ∂R is the trace of a simple closed piecewise regular curve $\gamma: I \to \Sigma$. At each point of ∂R, apart from the vertices, there is a unique unit vector (orthogonal to $\dot\gamma$) v s.t. the point $\gamma+\varepsilon v$ is in R for all sufficiently small $\varepsilon>0$. The curve γ is said to be *positively oriented* if $\dot\gamma \times v$ points in the same direction as the orientation N.



Theorem (Gauss-Bonnet Theorem — local version).

Let $U \subset \mathbb{R}^2$ be homeomorphic to an open disc and let Σ be a regular surface with an orientation compatible with a parametrisation $\sigma: U \to \Sigma$.

Let $R \subset \sigma(U)$ be a simple region of Σ and suppose that there is a closed simple piecewise regular curve $\gamma: I \to \Sigma$ parametrised by arclength, s.t. $\gamma(I)$ is the boundary ∂R of R. Let $\gamma(s_0), \dots, \gamma(s_n)$ and $\alpha_0, \dots, \alpha_n$ be the vertices and exterior angles of γ respectively. Then $\boxed{\sum_{j=0}\int_{s_j}^{s_{j+1}} k_g(s)\,ds + \iint_R K\,dA + \sum_{j=0}^{n}\alpha_j = 2\pi}$ — ④.

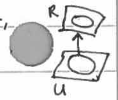
where k_g is the geodesic curvature of γ and K is the Gauss curvature.

Proof — From ①, $\sum_{j=0}^{n}\int_{s_j}^{s_{j+1}} k_g(s)\,ds = \sum_{j=0}^{n}\left(\int_{s_j}^{s_{j+1}}\dot\theta\,ds - \int_{s_j}^{s_{j+1}} e_1\cdot\dot e_2\,ds\right) = \sum_{j=0}^{n}[\theta(s_{j+1}^-)-\theta(s_j^+)] - \sum_{j=0}^{n}\int_{s_j}^{s_{j+1}} e_1\cdot\dot e_2\,ds$. Now use ③. Then ④ is equivalent to

$\sum_{j=0}^{n}\int_{s_j}^{s_{j+1}} e_1\cdot\dot e_2\,ds = \iint_R K\,dA$. Now $\sum_{j=0}^{n}\int_{s_j}^{s_{j+1}} e_1\cdot\dot e_2\,ds = \sum\int e_1\cdot[(e_2)_u\dot u + (e_2)_v\dot v]\,ds = \sum\int (e_1\cdot(e_2)_u)\dot u + (e_1\cdot(e_2)_v)\dot v\,ds$. Using Green's Theorem,

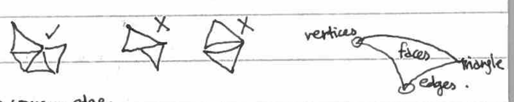
$\int P\,du + Q\,dv = \int P\frac{du}{ds}\,ds + Q\frac{dv}{ds}\,ds = \iint\left(\frac{\partial Q}{\partial u} - \frac{\partial P}{\partial v}\right)du\,dv$. Thus, ④ is $= \iint_{\sigma^{-1}(R)} (e_1\cdot(e_2)_v)_u - (e_1\cdot(e_2)_u)_v\,du\,dv$, and expanding terms out,

$= \iint_{\sigma^{-1}(R)} [(e_1)_u\cdot(e_2)_v - (e_1)_v\cdot(e_2)_u]\,du\,dv \overset{②}{=} \iint_{\sigma^{-1}(R)} \frac{eg-f^2}{\sqrt{EG-F^2}}\,du\,dv = \iint_{\sigma^{-1}(R)} K\cdot\underbrace{\sqrt{EG-F^2}\,du\,dv}_{dA = |\sigma_u\times\sigma_v|\,du\,dv} = \iint_R K\,dA$, q.e.d.



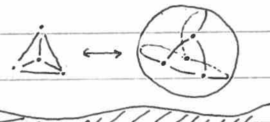
Definition A *triangulation* of a regular region $R \subset \Sigma$ is a finite family Υ of triangles $T_1, \dots, T_n$ s.t.

(1) $\bigcup T_i = R$ and (2) $\forall i \neq j$, $T_i \cap T_j$ is empty, or a single common vertex, or a single common edge.



Definition The *Euler characteristic* $\chi(R)$ is given by $\chi(R) = F - E + V$, where $F = \#$faces, $E = \#$edges, $V = \#$vertices.

Example — Consider a (topological) sphere, $S^2 \longleftrightarrow$ tetrahedron. Then by this choice of triangulation, $\chi(S^2) = 4-6+4 = 2$.



This allows us to generalise the Gauss-Bonnet theorem in more complicated topologies, then we have

$\sum\int_{C_i} k_g\,ds + \iint_R K\,dA + \sum\alpha_i = 2\pi\,\chi(R)$. This will be further elaborated upon.

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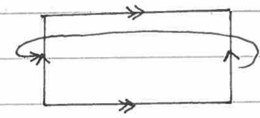
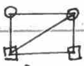
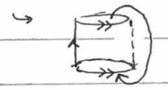
Consider the torus, T^2.  We attempt to find a triangulation; and first, a representation

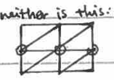


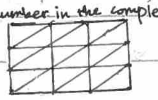
consider a rectangle. Then we identify edges (as denoted by arrows).

Then, we triangulate the rectangle:  this does not work, as more than two vertices are shared.



Likewise, neither is this: ...... However, for an odd number in the complexity, this works. Then $\chi(T^2) = F - E + V$, with $F = 18$, $E = 27$, $V = 9$.

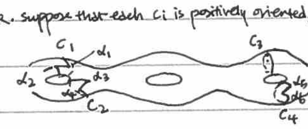




(note, one long horizontal & vertical edge of rectangle is repeated and should not be counted!).

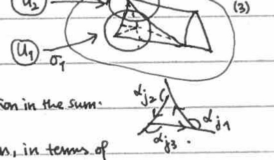
Theorem (Gauss-Bonnet Theorem — Global version).

Let $R \subset \Sigma$ be a regular region of an oriented surface, and let $C_1, \dots, C_p$ be closed regular curves which form the boundaries of R. Suppose that each C_i is positively oriented, and let $\alpha_1, \dots, \alpha_n$ be the set of external angles of the curves $C_1, \dots, C_p$. Then $\sum_{i=1}^{p}\int_{C_i} k_g(s) + \iint_R K\,dA + \sum_{l=1}^{n}\alpha_l = 2\pi\,\chi(R)$.

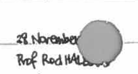


Remark — n, p have no correlation! Depends on smoothness of C_i

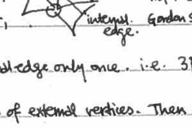
Proof — Consider a triangulation. We assume the following facts of the Euler characteristic: (1) every regular region admits a triangulation. (2) χ is independent of choice of triangulation. (3) Let Σ be an oriented surface, and $\{\sigma_\alpha\}$ be a parametrisation compatible with its orientation. Then $\exists$ a triangulation Υ of R s.t. for each triangle $T \in \Upsilon$, $T \in \sigma_\alpha(U_\alpha)$ for some α. Furthermore, if the boundary of every triangle is positively oriented, then adjacent triangles determine opposite directions on the common edge. Assume that R has a triangulation Υ as described in (1), (2), (3). Then let $\{\alpha_{j1}, \alpha_{j2}, \alpha_{j3}\}$ be the exterior angles of triangle T_j. Now apply local Gauss-Bonnet Theorem to each triangle,



and add up the results. The integrals of k_g along any common edge cancel because of the opposite directions of integration in the sum.



$\Rightarrow \underbrace{\sum_{i=1}^{p}\int_{C_i} k_g\,ds}_{\text{non-common edges}} + \iint_{\sum T_j = R} K\,dA + \sum_{j}^{F}\sum_{k=1}^{3}\alpha_{jk} = 2\pi\cdot F$ — ①. Thus, we need to understand the term $\sum_j\sum_{k=1}^{3}\alpha_{jK}$. Then, in terms of the usual internal angles $\varphi_{jk} = \pi - \alpha_{jk}$, we have $\sum_{j=1}^{F}\sum_{k=1}^{3}\alpha_{jk} = 3\pi F - \sum_{j=1}^{F}\sum_{k=1}^{3}\varphi_{jk}$. — ② Let $E_e = \#$external edges of Υ, $E_i = \#$internal edges of Υ, $V_e = \#$external vertices of Υ, $V_i = \#$internal vertices of Υ. Clearly, $E_e = V_e$ since C_j are closed curves. Moreover,



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suppose that we count number of edges (3) for each triangle, giving $3F$. Each internal edge has been counted twice, external edge only once. i.e. $3F = 2E_i + E_e$ — ③

②: $\sum_{j=1}^{F}\sum_{k=1}^{3}\alpha_{jk} = 2\pi E_i + \pi E_e - \sum_{j=1}^{F}\sum_{k=1}^{3}\varphi_{jk}$ by ③. We then analyse vertices into three types — internal, and two types of external vertices. Then we let

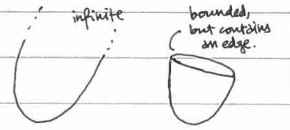
V_{et} = # $\overset{\text{external}}{\text{vertices}}$ from triangulation that are not vertices of C_js, and also V_{ec} = # external vertices that are vertices of $c_j = n$ = # vertices of C_js.

The sum of internal angles at each interior vertex is 2π, sum of internal angles at each external vertex of T (that is not a vertex of C) is π, sum at vertices of C_js is interior angle of vertex.

so $\sum_{j=1}^{F}\sum_{k=1}^{3}\varphi_{jk} = 2\pi V_i + \pi V_{et} + \sum_{l=1}^{n}(\pi - \alpha_l) = 2\pi V_i + \pi V_{et} + \pi V_{ec} - \sum_{l=1}^{n}\alpha_l = 2\pi V_i + \pi V_e - \sum_{l=1}^{n}\alpha_l$. then since $\sum_{j=1}^{F}\sum_{k=1}^{3}\alpha_{jk} = 2\pi E_i + \pi E_e - \sum_{j=1}^{F}\sum_{k=1}^{3}\varphi_{jk}$,

$\sum_{j=1}^{F}\sum_{k=1}^{3}\alpha_{jk} = 2\pi E_i + \pi E_e - 2\pi V_i - \pi V_e + \sum_{l=1}^{n}\alpha_l \overset{\text{add } 0=\pi(E_e-V_e)}{=} 2\pi E_i + 2\pi E_e - 2\pi V_i - 2\pi V_e + \sum_{l=1}^{n}\alpha_l = 2\pi(E-V) + \sum\alpha_l$. Substitute this into ①, then we get:

$\sum_{i=1}^{p}\int_{C_i} k_g\,ds + \iint_R K\,dA + \sum_{l=1}^{n}\alpha_l = 2\pi\chi(R) = 2(F-E+V)$ // q.e.d.



A compact surface is a surface that is bounded in $\mathbb{R}^3$ and has no edges. e.g. a sphere, torus but not a paraboloid.

Theorem For any compact surface, $2\pi\chi(\Sigma) = \iint_\Sigma K\,dA$.

Proof – From Gauss-Bonnet Theorem.

Applications: For any compact connected (i.e. one piece) surface Σ, the quantity $g = \frac{2-\chi(\Sigma)}{2}$ is called the genus. Roughly speaking, this is the number of holes.

Topologically, we have the following:

Theorem Let $\Sigma \subset \mathbb{R}^3$ be a compact connected surface, then $\chi(\Sigma)$ takes on one of the values $2, 0, -2, \ldots$, $(g = 0, 1, 2, \ldots \because g = \frac{2-\chi(\Sigma)}{2})$.

Furthermore, if $\tilde{\Sigma} \subset \mathbb{R}^3$ is a second compact connected surface s.t. $\chi(\tilde{\Sigma}) = \chi(\Sigma)$, then Σ is homeomorphic to $\tilde{\Sigma}$.

Proof – Omitted. (This theorem concerns deformation of structures).

Lemma (Jordan curve lemma)

Any simple closed curve in $\mathbb{R}^2$ is the boundary of two disjoint regions, one bounded (interior) and one unbounded (exterior).

Proof – Also assumed, proof omitted.

Corollary (of Local GBT) The local Gauss-Bonnet theorem holds even when $R \not\subset \sigma(U)$.

Proof – omitted.

Corollary Any compact connected surface with positive Gauss curvature is homeomorphic to the sphere.

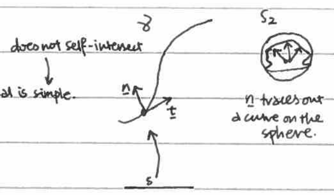
Proof – GB Theorem $\Rightarrow 2\pi\chi(\Sigma) = \iint_\Sigma K\,dA > 0$ since $K>0$. $2\pi\chi(\Sigma)>0 \Rightarrow$ since $\chi(\Sigma)$ can only ~~take~~ discrete values, only positive one is $2 \Rightarrow \chi(\Sigma)=2$, homeomorphic to S^2 // q.e.d.

Corollary Let Σ be an orientable surface with $K \le 0$. Two geodesics cannot meet twice in such a way that they form the boundaries of a simple region.

Proof – By contradiction. Trace the geodesics with a positive orientation. $k_g \equiv 0$, and $\chi(R) = 1 \Rightarrow \iint K\,dA + \alpha_1 + \alpha_2 = 2\pi$. Since $K \le 0$, $\iint K\,dA \le 0 \Rightarrow \alpha_1 + \alpha_2 \ge 2\pi$.

Since each exterior angle is between 0 and π by definition, $\alpha_1 = \alpha_2 = \pi \Rightarrow$ since geodesics are unique in direction at each point, the fact that they share a tangent vector (up to sign) at the vertices $\Rightarrow$ second geodesic is same as first, traced in opposite direction. Then $R = \phi$.

Corollary (Jacobi's Theorem)



Let $\gamma: I \to \mathbb{R}^3$ be a closed regular curve with non-zero curvature. Assume that the curve $n: I \to S^2$ traced by the principal normal is simple.

Then $n(I)$ divides S^2 into two regions of equal area.

Proof – Since this is the unit sphere S^2, $K \equiv 1$. Gauss-Bonnet theorem is $\int k_g\,d\hat{s} + \overbrace{\iint_R K\,dA}^{K=1,\ \text{area}} + \sum_l \alpha_l = 2\pi\chi(R) = \int k_g\,d\hat{s} + A + 0$.

Let $\hat{s}$ be the arclength of the curve $\gamma(\hat{s}) = \underline{n}$ (sphere curve). We apply Frenet formulae: $\underline{t}_s = k\underline{n}$, $\underline{n}_s = -k\underline{t} - \tau\underline{b}$, $\underline{b}_s = \tau\underline{n}$

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The geodesic curvature of $\underline{n} = \hat{\gamma}$ is given by $k_g = (N \times \dot{\hat{\gamma}}) \cdot (\ddot{\hat{\gamma}}) = (\underline{n} \times \frac{d\underline{n}}{d\hat{s}}) \cdot \frac{d^2\underline{n}}{d\hat{s}^2}$ since $\underline{N} = \underline{n}$, the position vector, dot wrt $\hat{s}$.

Now $\frac{d\underline{n}}{d\hat{s}} = \frac{ds}{d\hat{s}} \cdot \frac{d\underline{n}}{ds} = -(k\underline{t} + \tau\underline{b})\frac{ds}{d\hat{s}}$. Since $|\underline{\hat{n}}| = 1$, $\left|\frac{ds}{d\hat{s}}\right| = \frac{1}{\sqrt{k^2+\tau^2}}$ since $\hat{s}$ is arclength of $\underline{n}$. Then $\frac{d^2\underline{n}}{d\hat{s}^2} = -(k\underline{t} + \tau\underline{b})\frac{d^2s}{d\hat{s}^2} - (k_s\underline{t} + \tau_s\underline{b} + k\underline{t}_s + \tau\underline{b}_s)\left(\frac{ds}{d\hat{s}}\right)^2$

$= -(k\underline{t} + \tau\underline{b})\frac{d^2s}{d\hat{s}^2} - (k_s\underline{t} + \tau_s\underline{b} + (k^2+\tau^2)\underline{n})\left(\frac{ds}{d\hat{s}}\right)^2$. Then $k_g = (\underline{n} \times \frac{d\underline{n}}{d\hat{s}}) \cdot \frac{d^2\underline{n}}{d\hat{s}^2} = (k\underline{b} - \tau\underline{t})\frac{ds}{d\hat{s}} \cdot \left[-(k\underline{t} + \tau\underline{b})\frac{d^2s}{d\hat{s}^2} - (k_s\underline{t} + \tau_s\underline{b} + (k^2+\tau^2)\underline{n})\left(\frac{ds}{d\hat{s}}\right)^2\right] = -\left(\frac{ds}{d\hat{s}}\right)^3(k\tau_s - \tau k_s)$

$\Rightarrow k_g = \frac{\tau k_s - k\tau_s}{k^2+\tau^2}\frac{ds}{d\hat{s}}$. Then $\int k_g\,d\hat{s} + A(R) = 2\pi \Rightarrow \int k_g\,d\hat{s} = \int \frac{\tau k_s - k\tau_s}{k^2+\tau^2}\,ds = \int \frac{d}{ds}\left(\arctan\left(\frac{\tau}{k}\right)\right)ds = 0$ ($\because$ integral of an exact derivative around closed curve)

$\Rightarrow$ area of $R = \iint_R dA = 2\pi - \int k_g\,d\hat{s} = 2\pi = \frac{1}{2}(4\pi)$ // q.e.d.

Chapter 7.
GENERAL TOPOLOGY

Definition A topology on a set X is a collection Υ of subsets of X satisfying the following conditions:

1. $\phi \in \Upsilon$, $X \in \Upsilon$ 2. Υ is closed under arbitrary unions.   3. Υ is closed under finite intersections

$[\bigcap_{j=1}^{n} U_j$ is open, $\bigcup_{j \in \Lambda} U_j$ is open if each U_j is open$]$.

The subsets of X which lie in Υ are called the open sets of X.

Example – Two topologies on any open set X are (1) trivial topology $\Upsilon = \{\phi, X\}$ and (2) discrete topology $\Upsilon = 2^X$ = all subsets of X.

Let $X = \{1,2,3\}$, then $\Upsilon_1 = \{\phi, \{1,2,3\}\}$ is trivial topology, $\Upsilon_2 = \{\phi, \{1\}, \{1,2,3\}\}$ is also a topology; as is $\Upsilon_3 = \{\phi, \{1\}, \{1,2\}, \{1,2,3\}\}$.

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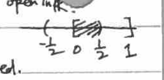
Note – A collection of subsets consisting of $\{\phi, \{1\}, \{2\}, \{1,2,3\}\}$ is not a topology as $\{1\} \cup \{2\} \notin \Upsilon$.

(X, Υ) is called a topological space.

Definition Let (X,T) be a topological space and let $Y\subset X$ be any subset (not necessarily open). Then $T_Y=\{Y\cap U : U\in T\}$ is a topology on Y called the subspace topology.

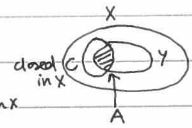
Example – Consider $X=[0,1]\subseteq\mathbb{R}$ with the subspace topology from $\mathbb{R}$ (with standard topology). Then $[0,\frac{1}{2})$ is open in X but not $\mathbb{R}$, since $[0,\frac{1}{2})=X\cap\underbrace{(-\frac{1}{2},\frac{1}{2})}_{\text{open in }\mathbb{R}}$.



Definition A subset $A\subset X$ is closed if $X\setminus A$ is open.

Lemma Let X be a topological space. Then (1) ϕ, X are closed, (2) arbitrary intersections of closed sets are closed, (3) finite unions of closed subsets are closed.

Lemma Let Y be a subset of the topological space X. Then $A\subset Y$ is closed in Y with respect to subspace topology $\Leftrightarrow$ $A=Y\cap C$, where C is closed in X.



Proof – Assume $A=Y\cap C$, C closed. $X\setminus C$ is open $\Rightarrow$ $Y\setminus A=(X\setminus C)\cap Y=(\text{open set in }X)\cap Y$ $\Rightarrow$ this is open in subspace topology.

Conversely, assume that A is closed in Y. So $Y\setminus A$ is open in Y. Then $Y\setminus A=Y\cap U$ where U is some open set in X. $\Rightarrow$ $X\setminus U$ is closed in X

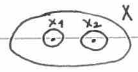
$\Rightarrow A=Y\cap(X\setminus U)=Y\cap C$ for closed C, q.e.d.

Definition Let Y be a subset of a topological space X. The closure of Y is the set $cl(Y)=\bar{Y}=\bigcap\{\text{all closed subsets containing }Y\}$. The interior of Y is $Int(Y)=\mathring{Y}=\bigcup\{\text{all open subsets of }Y\}$.

$x\in\bar{Y}\Leftrightarrow$ every open set U containing x intersects Y.

Definition For any x in a topological space X, a neighbourhood of x is any open set containing x.

Definition A topological space X is called Hausdorff (or a Hausdorff space) if for each pair of distinct points $x_1,x_2\in X$, $\exists$ neighbourhoods U_1,U_2 of x_1,x_2 respectively s.t. $U_1\cap U_2=\phi$.



Theorem Every finite point set in a Hausdorff space X is closed.

Proof – We only need to show that $\{x_0\}$ is closed for each $x_0\in X$. Take $x\neq x_0$. $\exists$ neighbourhood of x not containing x_0 $\therefore$ $X\setminus\{x_0\}$ = union of all such sets, which is open $\Rightarrow$ $\{x_0\}$ is closed.

Definition A sequence $x_1,x_2,\dots\in X$ is said to converge to $x\in X$ if given any neighbourhood U of x, $\exists N$ s.t. $x_n\in U$ $\forall n>N$.

Theorem If X is a Hausdorff space, then the sequence $\{x_n\}\subset X$ converges to at most one point of X.

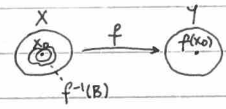
Proof – Suppose that there are two limit points $x,y\in X$. X is Hausdorff, so $\exists$ open sets U_1,U_2 s.t. $x\in U_1$, $y\in U_2$, $U_1\cap U_2=\phi$. $\exists N$ s.t. $\forall n>N$, $x_n\in U_1$ $\forall n>N$

$\Rightarrow x_n\notin U_2$ $\forall n>N\Rightarrow$ contradiction, q.e.d.

Theorem If X,Y are metric spaces, $f:X\to Y$ is continuous $\Leftrightarrow$ $f^{-1}(U)$ is open in X for all open subsets U of Y.

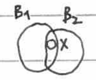
Proof – Let U be a neighbourhood of $f(x_0)$. $B=B_\varepsilon(f(x_0))\subset U$, $f^{-1}(B)$ is open and $x_0\in f^{-1}(B)$ $\therefore\exists\delta$ s.t. $B_\delta(x_0)\subset f^{-1}(B)$.

So x s.t. $|x-x_0|<\delta\Leftrightarrow B_\delta(x_0)$ $\therefore$ $f(x)\in B$ since $B_\delta(x_0)\subset f^{-1}(B)$ $\Leftrightarrow$ $|f(x)-f(x_0)|<\varepsilon$.

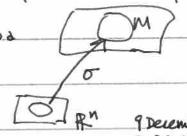


Definition Let X,Y be topological spaces. Then $f:X\to Y$ is said to be continuous if $f^{-1}(U)$ is open in X whenever U is open in Y.

Definition Let (X,T) be a topological space and let $\mathcal{B}$ be a collection of open subsets of X s.t. (1) $\forall x\in X$, $\exists B\in\mathcal{B}$ with $x\in B$, (2) if $x\in B_1\cap B_2$, $B_1,B_2\in\mathcal{B}$, then $\exists B_3\in\mathcal{B}$ s.t. $x\in B_3\subset B_1\cap B_2$. Then T is the topology generated by $\mathcal{B}$, and $\mathcal{B}$ is called a basis for T.



Definition A topological manifold of dimension n (n-manifold) is a topological space M s.t. (i) M is Hausdorff, (ii) M is locally Euclidean of dimension n (i.e. for $x\in M$, $\exists$ neighbourhood U of x s.t. $\exists$ a continuous function from U to an open set in $\mathbb{R}^n$ with continuous inverse) and (iii) M has a countable basis of open sets.



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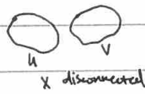
Connectedness and Compactness.

Definition Let X be a topological space. If $X=U\cup V$ where U,V are open, disjoint ($U\cap V=\phi$) and non-empty ($U\neq\phi$, $V\neq\phi$), then X is disconnected.

Otherwise, X is connected.



Theorem X is connected $\Leftrightarrow$ ϕ, X are the only sets that are both open and closed.

Theorem Suppose $X=U\cup V$ where U,V are open and disjoint. Then U,V are also closed.

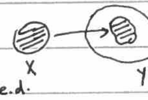
Proof – $V=X\setminus U$ is closed, $U=X\setminus V$ is closed, q.e.d.

Proof – ($\Leftarrow$) Suppose X were not connected, then we can find U,V as non-empty open and closed sets $\Rightarrow$ contradiction.

($\Rightarrow$) Let U be open and closed, $U\neq\phi$ and $U\neq X$. Then $X=U\cup(X\setminus U)$, but $X\setminus U$ is open since U is closed $\Rightarrow$ X is disconnected $\Rightarrow$ contradiction, q.e.d.

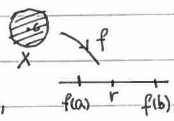
Theorem Let X be a connected topological space, and $f:X\to Y$ be continuous. Then $f(X)$ is connected.

Proof – Suppose otherwise. Then $\exists$ nonempty disjoint open sets U,V s.t. $f(X)=U\cup V$. Then $X=(f^{-1}(U))\cup(f^{-1}(V))$. Since f is continuous, $f^{-1}(U)$, $f^{-1}(V)$ are continuous in X and disjoint since U and V are disjoint. So X is not connected since $f^{-1}(U)$, $f^{-1}(V)$ are non-empty, q.e.d.



Theorem (Intermediate Value Theorem)

Let $f:X\to\mathbb{R}$ be a continuous function, where X is a connected topological space, and let $a,b\in X$. Then for all r s.t. $f(a)<r<f(b)$, $\exists c\in X$ s.t. $f(c)=r$.



Proof – Suppose otherwise, then $f(x)\neq r$ $\forall x\in X$. Then $f(X)=A\cup B$. $A=f(X)\cap(-\infty,r)$ open, $B=f(X)\cap(r,\infty)$ open. Clearly $A\cap B=\phi$. Moreover,

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$f(a) \in A$, $f(b) \in B \Rightarrow A, B \neq \phi$. $\Rightarrow$ By definition, $f(X)$ is not connected. However by previous theorem, the image of connected set X is connected $\Rightarrow$ contradiction // q.e.d.

Definition A collection of sets $\{A_\alpha\}_{\alpha \in \Lambda}$ is said to be a cover of a set A if $A \subset \bigcup_{\alpha \in \Lambda} A_\alpha$. If the A_α are all open, then $\{A_\alpha\}$ is called an open cover.

Definition A subset A of a topological space X is said to be compact if every open cover contains a finite subcover (i.e. if we have an open cover $\bigcup_{\alpha \in I} U_\alpha$, then we only need finitely many U_α).

Example – $\mathbb{R}$ (with the usual topology) is not compact. Cover $\mathbb{R}$ with $A_n = (n, n+2)$ $n \in \mathbb{Z}$. Then $\mathbb{R} = \bigcup A_n$ is an open cover. Then $n+1 \in A_n$ but $n+1 \notin A_j$ for $j \neq n$. So if we remove any A_n from the cover, we no longer have a cover $\Rightarrow$ not compact.

Theorem The image of a compact set under a continuous function is compact.

Proof – Let $f: X \to Y$ be continuous, X compact, and suppose that $f(X) \subset \bigcup_{\alpha \in \Lambda} U_\alpha$ is an open cover. Then $X = \bigcup_{\alpha \in \Lambda} f^{-1}(U_\alpha)$ is an open cover. Take a finite subcover, $f^{-1}(U_{\alpha_1}), \ldots, f^{-1}(U_{\alpha_n})$. So $U_{\alpha_1}, U_{\alpha_2}, \ldots, U_{\alpha_n}$ is a finite subcover for $f(X) \Rightarrow f(X)$ compact // q.e.d.

END OF SYLLABUS.

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Review of 2012-2013 Exam Paper

Q1(a) Given $\gamma(t)$, find $\underline{t}, \underline{n}, \underline{k}$. (A) Need to find $\frac{d}{ds}$, $\frac{d}{dt} = \dot{s}\frac{d}{ds}$ //

(b) For what values of constant c is $\Sigma = \{(x,y,z) : x^2+y^2-z^2 = c\}$ a regular surface? (A) Let $f(x,y,z) = x^2+y^2-z^2$, $\Sigma = f^{-1}(c)$. $\nabla f = (2x, 2y, -2z) = 0 \Leftrightarrow (x,y,z) = (0,0,0) \notin \Sigma$. If $c \neq 0$, $(0,0,0) \notin \Sigma$. So Σ is the regular preimage of a smooth function $\Rightarrow$ regular surface. If $c = 0$, $z = \pm\sqrt{x^2+y^2}$, which is a cone 

, not regular. If it were, then it could be written as $z = g(x,y)$, $y = g(x,z)$ or $x = g(y,z)$ for some g (smooth) regular locally. $z = \pm\sqrt{x^2-y^2}$ is not differentiable (and 2 valued). $x = \cdots$ $y = \cdots$ not functions //

(c) Show that if the 2nd FF of a regular surface Σ vanishes identically, then Σ is part of a plane. (A) $e = -\langle \tilde{N}_u, \sigma_u \rangle$, $f = -\langle \tilde{N}_v, \sigma_u \rangle = -\langle \tilde{N}_u, \sigma_v \rangle$ $g = -\langle \tilde{N}_v, \sigma_v \rangle$. $e = f = g = 0$. Then $\tilde{N}_u, \tilde{N}_v \in T_p\Sigma$ since $\tilde{N} \perp T_p\Sigma$. Then $\tilde{N}_u = \alpha\sigma_u + \beta\sigma_v$, but $\langle \tilde{N}_u, \sigma_u \rangle = \langle \tilde{N}_u, \sigma_v \rangle = 0$, $\tilde{N}_u = 0$, $\tilde{N}_v = 0 \Rightarrow \tilde{N} = \text{const} = N_0$. Take arbitrary point on surface, σ, then $\frac{\partial}{\partial u}\langle \sigma, \tilde{N} \rangle = \langle \overset{T_p\Sigma}{\sigma_u}, \tilde{N} \rangle + \langle \sigma, \overset{0}{\tilde{N}_u} \rangle = 0$. Likewise $\frac{\partial}{\partial v}\langle \sigma, \tilde{N} \rangle = 0 \Rightarrow \langle \sigma, \tilde{N} \rangle = \text{const} \Rightarrow$ equation of plane // [recall $\sigma(u,v) = (x(u,v), y(u,v), z(u,v))$].

Q2(a) For each $t \in I \subset \mathbb{R}$, let $\{\underline{e}_1(t), \underline{e}_2(t), \underline{e}_3(t)\}$ be a right-handed system of three orthonormal vectors in $\mathbb{R}^3$ (i.e. $\underline{e}_i \cdot \underline{e}_j = \delta_{ij}$, $\underline{e}_3 = \underline{e}_1 \times \underline{e}_2$). If each $e_j(t)$ is a smooth function of t, show that $\exists$ smooth functions $a(t), b(t), c(t)$ s.t. $\frac{d\underline{w}}{dt} = A(t)\underline{w}$ —① where $\underline{w}(t) = \begin{pmatrix} \underline{e}_1(t) \\ \underline{e}_2(t) \\ \underline{e}_3(t) \end{pmatrix}$ and $A(t) = \begin{pmatrix} 0 & c & b \\ -c & 0 & a \\ -b & -a & 0 \end{pmatrix}$ —②.

(A) Since $\dot{\underline{e}}_1, \dot{\underline{e}}_2, \dot{\underline{e}}_3 \in \mathbb{R}^3$, they can be expanded in basis $\underline{e}_1, \underline{e}_2, \underline{e}_3$. Then $\underline{w}$ satisfies an equation of form (1) for some $A(t)$. $\begin{pmatrix} \dot{e}_1 \\ \dot{e}_2 \\ \dot{e}_3 \end{pmatrix} = \begin{pmatrix} A_{11} & A_{12} & \cdots \\ \vdots & \ddots & \end{pmatrix}\begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix}$. $A_{11}e_1 + A_{12}e_2 + A_{13}e_3$. $A_{ij} = \dot{e}_i \cdot e_j$. Since $e_i \cdot e_j = \delta_{ij}$, take time derivative $\Rightarrow \dot{e}_i \cdot e_j + e_i \cdot \dot{e}_j = 0 \Rightarrow \dot{e}_i \cdot e_j = -e_i \cdot \dot{e}_j \Rightarrow A_{ij} + A_{ji} = 0 \Rightarrow A$ skew-symmetric //

(b) Let $M(t)$ be the 3x3 matrix with components $M_{ij} = \langle e_i(t), e_j(t) \rangle$. Show that $M'(t) = A(t)M(t) - M(t)A(t)$. (A) $M = \underline{w}\underline{w}^t = \begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix}(e_1\ e_2\ e_3)$ where products are taken as dot products. [or think of $\underline{w}$ as a 3x3 matrix, i^{th} row of which are components of $\underline{e}_i$. Then $\dot{\underline{M}} = \dot{w}w^t + w\dot{w}^t = (Aw)w^t + w(Aw)^t = Aww^t + ww^tA^t = AM + MA^T = AM - MA$ //.

(c) Suppose now that $\{e_1, e_2, e_3\}$ is any solution of the ODEs ①, ② $\forall t \in I$ s.t. $\{\underline{e}_1(t_0), \underline{e}_2(t_0), \underline{e}_3(t_0)\}$ is a right-handed system of three orthonormal vectors for some $t_0 \in I$. Show that $\{\underline{e}_1(t), \underline{e}_2(t), \underline{e}_3(t)\}$ is a right-handed system of 3 orthonormal vectors. (A) $M(t_0) = I_3$ —③ since $\{e_1, e_2, e_3\}$ is an orthonormal frame at $t = t_0$. Also, $\frac{d}{dt}M(t) = A(t)M(t) - M(t)A(t)$ —④. $\therefore$ There is a unique solutions of IVP ④ with ③. By uniqueness, $M \equiv I_3$ is the only solution, then $\underline{e}_i \cdot \underline{e}_j = \delta_{ij}$ so $\{\underline{e}_1, \underline{e}_2, \underline{e}_3\}$ orthonormal. For right-handedness, NTP: $\det w = +1$. $\det w(t_0) = 1$, $\det w(t) = \pm 1$ ($t \neq t_0$) (orthonormal). But det is continuous, so $\det w(t) = 1$.

Q3(a) Find the 1st and 2nd FFs of the surface parametrised by $\sigma(u,v) = ((a+b\cos u)\cos v, (a+b\cos u)\sin v, b\sin u)$ where $a > b > 0$ using standard orientation. Hence calculate mean curvature, and show that Gauss curvature is $K = \frac{\cos u}{b(a+b\cos u)}$. (A) $\sigma_u = (-b\sin u\cos v, -b\sin u\sin v, b\cos u)$. $\sigma_v = (-(a+b\cos u)\sin v, (a+b\cos u)\cos v, 0) = (a+b\cos u)(-\sin v, \cos v, 0)$. $E = \langle \sigma_u, \sigma_u \rangle = b^2$, $F = \langle \sigma_u, \sigma_v \rangle = 0$, $G = \langle \sigma_v, \sigma_v \rangle = (a+b\cos u)^2$ // [we can leave answer like this]. (standard orientation) $\sigma_u \times \sigma_v = (a+b\cos u)(-b\cos u\cos v, -b\cos u\sin v, -b\sin u) \Rightarrow \sigma_u \times \sigma_v = -b(a+b\cos u)(\cos u\cos v, \cos u\sin v, \sin u)$. So unit normal is just $\underline{N} = -(\cos u\cos v, \cos u\sin v, \sin u)$ [$\because \tilde{N} = \frac{\sigma_u \times \sigma_v}{|\sigma_u \times \sigma_v|}$]. Then $e = -\langle \tilde{N}_u, \sigma_u \rangle$ [$= \langle \tilde{N}, \sigma_{uu} \rangle$ to avoid differentiating sqrts normally]. $\tilde{N}_u = -(-\sin u\cos v, -\sin u\sin v, \cos u)$, $\tilde{N}_v = -(-\cos u\sin v, \cos u\cos v, 0)$. Then $e = -\langle \tilde{N}_u, \sigma_u \rangle = b(\sin^2 u\cos^2 v + \sin^2 u\sin^2 v + \cos^2 u) = b$. $f = -\langle \tilde{N}_u, \sigma_v \rangle = 0$. $g = -\langle \tilde{N}_v, \sigma_v \rangle = (a+b\cos u)\cos u$ //. $H = \frac{1}{2}\frac{eG - 2fF + gE}{EG - F^2}$, $K = \frac{eg - f^2}{EG - F^2}$. [formulae will be provided.]. $H = \frac{a + 2b\cos u}{2b(a+b\cos u)}$.

(b) Is this surface isometric to a sphere? (A) No. For a sphere of radius r, $K = \frac{1}{r^2}$ which is a positive constant $\neq K$ for above surface. Hence by Gauss's Theorema Egregium, they cannot be isometric – copy the theorem //.

Q4(a) Recall for a regular surface, $\sigma_{uu} = \Gamma_{11}^1\sigma_u + \Gamma_{11}^2\sigma_v + e\tilde{N}$ … etc for $\sigma_{uv}, \sigma_{vu}, \tilde{N}_u, \tilde{N}_v$. Derive the equation $e_v - f_u = e\Gamma_{12}^1 + f(\Gamma_{12}^2 - \Gamma_{11}^1) - g\Gamma_{11}^2$. (A) Use the fact that equations exhibit compatibility: $\underline{(\sigma_{uu})_v = (\sigma_{uv})_u}$, $(\sigma_{uv})_v = (\sigma_{vv})_u$, $(\tilde{N}_u)_v = (\tilde{N}_v)_u$. We can just use first underlined equation, and pay attention to $\tilde{N}$ term's coefficients only.

$(\sigma_{uu})_v = (\Gamma_{11}^1)_v\sigma_u + \Gamma_{11}^1\sigma_{uv} + (\Gamma_{11}^2)_v\sigma_v + \Gamma_{11}^2\sigma_{vv} + e_v\tilde{N} + e\tilde{N}_v = (\ldots)\sigma_u + (\ldots)\sigma_v + (\Gamma_{11}^1 f + \Gamma_{11}^2 g + e_v)\tilde{N}$. $(\sigma_{uv})_u = (\ldots)\sigma_u + (\ldots)\sigma_v + (\Gamma_{12}^1 e + \Gamma_{12}^2 f + f_u)\tilde{N}$. Equating coefficients, we get $e_v - f_u = \Gamma_{12}^1 e + \Gamma_{12}^2 f - \Gamma_{11}^1 f - \Gamma_{11}^2 g$ // q.e.d.

(b) Given some $E=\dots$, $F=\dots$, $G=\dots$, calculate the Gauss curvature etc.

END OF COURSE