3113 Differential Geometry Notes

Based on the 2017 autumn lectures by Dr B Lambert

MATH3113

03-10-17 Differential Geometry

Dr Ben Lambert , Room 600 , b.lambert@ucl.ac.uk

Office hour: Mon 704 3-4pm

90% exam, 10% coursework

Hand in h/w on Tuesdays

Lecture notes available at the end of each chapter

Book: Do Carmo - Differential Geometry

Theory of Curves

Def (Parameterised Curve)

- A parameterised curve is a continuously differentiable mapping from an interval $I \subset \mathbb{R}$ to $\mathbb{R}^n$, i.e. $\gamma : I \longmapsto \mathbb{R}^n$.
- The trace of γ $tr(\gamma) := \gamma(I) \subset \mathbb{R}^n$
- γ is regular if $\gamma'(t) = \dfrac{d\gamma(t)}{dt} \neq 0 \quad \forall t \in I$
- The tangent vector at $t \in I$ of γ is $\gamma'(t)$

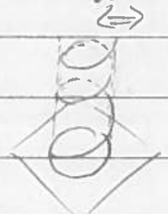
Examples

i). $\gamma(t) = (a\cos t, a\sin t, bt)$, $t \in \mathbb{R}$

tangent vector : $\gamma'(t) = (-a\sin t, a\cos t, b) \neq 0$

iff $a^2 + b^2 \neq 0$

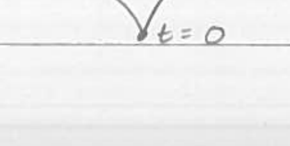
$\Leftrightarrow a \neq 0$ or $b \neq 0$



ii). $\gamma(t) = (t^3, t^2)$

not a regular curve

$\gamma'(t) = (3t^2, 2t) = 0$ if $t = 0$



Remarks:

- We could also define a C^k-curve, that is one that is k times continuously differentiable, $k \geq 1$.

Def (Homeo- and Diffeomorphisms)
Let $I, J \subset \mathbb{R}$ and suppose $\phi: I \mapsto J$ is bijective.
Then ϕ is called
i). a homeomorphism if both ϕ and ϕ^{-1} (inverse fn.) are continuous
ii). a diffeomorphism if both ϕ and ϕ^{-1} are continuously differentiable.

Remark: By the Inverse Function Theorem, we can see that a bijection $\phi: I \mapsto J$ is a diffeo (diffeomorphism) iff ϕ is continuously differentiable and $\phi'(t) \neq 0 \ \forall t \in I$

Def (Reparameterisation)
- Let $\gamma_1: I \mapsto \mathbb{R}^n$ and $\gamma_2: J \mapsto \mathbb{R}^n$ be parameterised curves. If there is a diffeomorphism $\phi: I \mapsto J$ satisfying $\gamma_2 = \gamma_1 \circ \phi$ then γ_2 is a reparameterisation of γ_1
- A curve in $\mathbb{R}^n$ is an equivalence class of parameterised curves, where two curves are equivalent $\Leftrightarrow$ one is a reparameterisation of the other.

Example
$\gamma(t) = (\cos t, \sin t, t)$
$\tilde{\gamma}(t) = (\cos 2t, \sin 2t, 2t)$
$\underbrace{\phi(t) = 2t}_{\text{diffeo}}$ then $\tilde{\gamma}(t) = \gamma(\phi(t))$

MATH 3113

03-10-17

Remarks:

- Regularity does not depend on the parameterisation:

  $\phi : I \mapsto J$ diffeo

  $\gamma_1 : I \mapsto \mathbb{R}^n$

  $\gamma_2 : J \mapsto \mathbb{R}^n$ a reparameterisation of γ.

  then $\gamma_1 = \gamma_2 \circ \phi$

  $$\frac{d}{dt}(\gamma_1) = \frac{d}{dt}(\gamma_2 \circ \phi) = \gamma_2'\big|_{\phi} \cdot \phi'(t)$$

  $$\Rightarrow \quad |\gamma_2'| \neq 0 \iff |\gamma_1'| \neq 0$$

Arc Length

Def

The arc length of a parameterised curve $\gamma : I \mapsto \mathbb{R}^n$ is $l(\gamma) := \int_I |\gamma'(t)|\,dt$.

If we have that $l\left(\gamma\big|_{[t_1, t_2]}\right) = \int_{[t_1, t_2]} |\gamma'(t)|\,dt = t_2 - t_1$

for all $t_2 > t_1$, $t_1, t_2 \in I$, then γ is parameterised by arc length.

Notation

Denote a curve γ parameterised by arc length by $\gamma(s)$, derivatives w.r.t. s will be denoted $\dot{\gamma}(s) = \frac{d}{ds}(\gamma(s))$.

Prop

i). Arc length does not depend on parameterisation.

ii). γ is parameterised by arc length $\iff |\gamma'(t)| = 1 \;\; \forall t \in I$

iii). Any regular curve can be parameterised by arc length.

Proof

i) Suppose $I = [a,b]$ and $\phi : I \mapsto J$ be some diffeomorphism s.t. $\phi(a) < \phi(b)$.

$\gamma_2 = \gamma_1 \circ \phi$, then

$$\int_a^b \gamma_1(\phi) \cdot \phi' dt$$

$$l(\gamma_2) = \int_a^b |\gamma_2'(t)| dt = \int_a^b |\gamma_1'|_{\phi(t)} \cdot \phi'(t)| dt$$

$$= \int_{\phi^{-1}(a)}^{\phi^{-1}(b)} \gamma_1'(r) dr = l(\gamma_1).$$

If $\phi(a) > \phi(b)$ then $\phi'(t)$ is negative then the change of sign is cancelled out by change of orientation.

ii). Suppose $|\gamma'(t)| = 1$ then

$$l(\gamma|_{[t_1, t_2]}) = \int_{t_1}^{t_2} |\gamma'(t)| dt = t_2 - t_1$$

Suppose $l(\gamma|_{[t_1, t]}) = t - t_1$

then $$\frac{d}{dt} l(\gamma|_{[t_1, t]}) = \frac{d}{dt}\int_{t_1}^{t} |\gamma'(r)| dr$$

$$= |\gamma'(t)| = 1$$

iii) $\gamma : I \mapsto \mathbb{R}^n$ is regular.

$\gamma'(t) \neq 0 \quad \forall t \in I = [a,b]$

$\psi(t) := l(\gamma|_{[a,t]})$, $\psi : I \mapsto [0, l(\gamma)]$

$\psi'(t) \neq 0$

$\Rightarrow \psi$ is a diffeomorphism.

$\bar{\gamma}(s) = \gamma \circ \psi^{-1}(s)$. Choose $t_1, t_2 \in I$, set $s_1 = \psi(t_1)$, $s_2 = \psi(t_2)$ assume for now that $s_1 < s_2$, so:

$$s_1 - s_2 = \psi(t_1) - \psi(t_2) = l(\gamma|_{[a, t_2]}) - l(\gamma|_{[a, t_1]})$$

$$= l(\gamma|_{[t_1, t_2]}) = l(\bar{\gamma}|_{[s_1, s_2]})$$

therefore $\bar{\gamma}$ is a reparameterisation of γ by arc length.

Case of $s_2 < s_1$ similar. $\square$

MATH3113

03-10-17

Examples

i) $\gamma(t) = (a\cos t, a\sin t, bt)$, $t \in \mathbb{R}$

tangent vector: $\gamma'(t) = (-a\sin t, a\cos t, b)$

$$\boxed{\bar{\gamma}(s) = \gamma \circ \psi^{-1}(s), \quad \psi(t) = \ell(\gamma|_{[a,t]})}$$

$$|\gamma'(t)|^2 = a^2(\sin^2 t + \cos^2 t) + b^2 = a^2 + b^2$$

$$\psi(t) = \int_0^t |\gamma'(t)|\, dt = \sqrt{a^2+b^2}\, t \quad \Rightarrow \psi^{-1}(s) = \frac{s}{\sqrt{a^2+b^2}}$$

$$\bar{\gamma}(s) = \gamma\left(\frac{s}{\sqrt{a^2+b^2}}\right)$$

$$|\bar{\gamma}'| = \frac{1}{\sqrt{a^2+b^2}}|\gamma'| = \frac{\sqrt{a^2+b^2}}{\sqrt{a^2+b^2}} = 1$$

ii). $\gamma(t) = (\cos^2 t, \sin^2 t, 1 + \sin^2 t)$ $[= (r, 1-r, 2-r)]$

$$\gamma'(t) = (-2\cos t \sin t, 2\sin t\cos t, 2\sin t \cos t)$$
$$= \sin 2t\,(-1, 1, 1)$$

$\Rightarrow |\gamma'(t)| = |\sin 2t|\sqrt{3}$

γ is only regular if $t \in \mathbb{R} / \frac{\pi}{2}\mathbb{Z}$. $t < \pi/2$

$$\psi(t) = \int_0^t \sqrt{3}\sin 2r\, dr = \frac{\sqrt{3}}{2}(\cos 2t - 1)$$

$$\psi^{-1}(s) = \tfrac{1}{2}\cos^{-1}\left(1 - \frac{2}{\sqrt{3}}s\right)$$

$[\sin\arccos(s) = \sqrt{1-s^2}]$

$$\bar{\gamma}(s) = \gamma(\psi^{-1}(s))$$

$$|\dot{\bar{\gamma}}| = |(\psi^{-1}(s))' \cdot \gamma'|_{\psi(s)}|$$

$$= \frac{\sqrt{3}\,|\sin(\arccos(1 - \frac{2}{\sqrt{3}}s))|}{\sqrt{3}\sqrt{1 - (1 - \frac{2}{\sqrt{3}}s)^2}} = 1$$

The Local theory of Curves

- Specialise: Regular curves in $\mathbb{R}^3$
- Everything is parameterised by arc length.
- At least twice differentiable.

Frenet - Serrin Apparatus

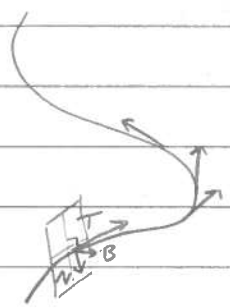
We know that $|\dot{\gamma}(s)| = 1$

Def

The unit tangent vector: $T(s) = \dot{\gamma}(s)$

Observe $0 = \frac{d}{ds} 1 = \frac{d}{ds} |T(s)|^2 = <\dot{T}, T> + <T, \dot{T}> = 2<\dot{T}, T>$

$\Rightarrow \dot{T}$ is perpendicular to T



Def (Frenet - Serrin Frame)

$\gamma : I \mapsto \mathbb{R}^3$ regular, parameterised by arc length.
Define the scalar function $\kappa : I \mapsto \mathbb{R}$,
$\kappa(s) := |\dot{T}|$ which is called the <u>curvature</u> of γ at s

If $\kappa(s) \neq 0$ then define $N(s) := \frac{1}{\kappa(s)} \dot{T}(s) = \frac{\dot{T}}{|\dot{T}|}$

(this is a unit vector on γ)
$N(s)$ is the normal vector to γ at s
$B(s) := T(s) \times N(s)$ (cross product) called the binormal of γ at s.

MATH3113

03-10-17

We have:

$\dot{T} = \kappa N$ and $B = T \times N$

so $\dot{B} = \dot{T} \times N + T \times \dot{N}$

$= \cancel{\kappa N \times N} + T \times \dot{N}$

$\Rightarrow \dot{B} = T \times \dot{N}$

Since $\dot{N}$ is a vector crossed with T, $\dot{B}$ is $\perp$ to T.

Differentiating $\frac{d}{ds}|\dot{B}(s)|^2$ (as with T), $\dot{B}$ is $\perp$ to B.

$\Rightarrow \dot{B} = \underset{\uparrow \text{ some real number}}{\tau(s)} N(s)$

<u>Def</u>

$\gamma : I \mapsto \mathbb{R}^3$ parameterised by arc length at points s.t. $\kappa(s) \neq 0$ then define $\tau(s)$ by $\dot{B} = \tau(s) N(s)$.

At points where $\kappa(s) = 0$ define $\tau(s) = 0$.

04-10-17

<u>Last time:</u>

- Parameterised curve
- homeo- and diffeomorphisms
- Reparameterisations
- Curves
- Parameterising by arc length
- Curves in $\mathbb{R}^3$

From here on $\gamma : I \mapsto \mathbb{R}^3$ regular, parameterised by arc length.

<u>Frenet-Serrin Frame</u>

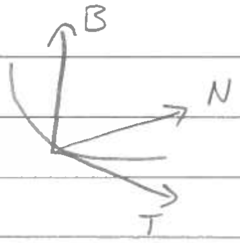
$T = \dot{\gamma}$

$\kappa = |\dot{T}|$ (curvature)

If $\kappa \neq 0$ $N = \dot{T}/|\dot{T}|$ (normal)

$B = T \times N$ (binormal)

T, N, B form a right-handed set.



$\dot{B} = \tau N$ (τ = torsion)

(only if $\varkappa \neq 0$, $\tau = 0$ otherwise).

$\dot{T} = \varkappa N$

$\dot{N} = ?$

$\left[\langle v, w \times z \rangle = \det \begin{pmatrix} | & | & | \\ w & z & v \\ | & | & | \end{pmatrix} \right]$

Observe $N = B \times T$

$\langle N, B \times T \rangle = \det \begin{pmatrix} | & | & | \\ B & T & N \\ | & | & | \end{pmatrix}$

$= (-1)^2 \det \begin{pmatrix} | & | & | \\ T & N & B \\ | & | & | \end{pmatrix} = 1 \cdot 1 = 1$

Def

An ordered basis $\{v_1, v_2, v_3\}$ is right handed $\Leftrightarrow \det \begin{pmatrix} | & | & | \\ v_1 & v_2 & v_3 \\ | & | & | \end{pmatrix} > 0$

$\dot{N} = \dot{B} \times T + B \times \dot{T}$

$= \tau N \times T + \varkappa B \times N$

$= -\tau T \times N - \varkappa N \times B$

$= -\tau B - \varkappa T$

So $\dot{N} = -\tau B - \varkappa T$

Frenet Trihedron

$\dot{B} = \tau N$, $\dot{T} = \varkappa N$, $\dot{N} = -\tau B - \varkappa T$

MATH 3113

04-10-17

We could write in Matrix form

$$F := \begin{pmatrix} -T- \\ -N- \\ -B- \end{pmatrix}$$

$$\dot{F}(s) = A(s)F(s) \quad , \quad A(s) := \begin{pmatrix} 0 & \varkappa & 0 \\ -\varkappa & 0 & -\tau \\ 0 & \tau & 0 \end{pmatrix}$$

$F \in SO(3)$

A is skew symmetric

Example

$$\gamma(s) = \left(a\cos\left(\frac{s}{\sqrt{a^2+b^2}}\right), a\sin\left(\frac{s}{\sqrt{a^2+b^2}}\right), \frac{b}{\sqrt{a^2+b^2}}s\right) \qquad a, b \geq 0$$

$$T(s) = \frac{1}{\sqrt{a^2+b^2}}\left(-a\sin\left(\frac{s}{\sqrt{a^2+b^2}}\right), a\cos\left(\frac{s}{\sqrt{a^2+b^2}}\right), \frac{b}{\sqrt{a^2+b^2}}\right)$$

$$\dot{T} = \frac{1}{a^2+b^2}\left(-a\cos\left(\frac{s}{\sqrt{a^2+b^2}}\right), -a\sin\left(\frac{s}{\sqrt{a^2+b^2}}\right), 0\right)$$

$$\varkappa = |\dot{T}| = \frac{a}{a^2+b^2} \qquad \text{if } a \neq 0$$

$$N = \frac{\dot{T}}{\varkappa} = \frac{1}{a}\left(-a\cos\left(\frac{s}{\sqrt{a^2+b^2}}\right), -a\sin\left(\frac{s}{\sqrt{a^2+b^2}}\right), 0\right)$$

$$= -\left(\cos\left(\frac{s}{\sqrt{a^2+b^2}}\right), \sin\left(\frac{s}{\sqrt{a^2+b^2}}\right), 0\right)$$

$$B = T \times N = \left(\frac{b}{\sqrt{a^2+b^2}}\sin\left(\frac{s}{\sqrt{a^2+b^2}}\right), \frac{-b}{\sqrt{a^2+b^2}}\cos\left(\frac{s}{\sqrt{a^2+b^2}}\right), \frac{a}{\sqrt{a^2+b^2}}\right)$$

$$\dot{B} = \frac{1}{a^2+b^2}\left(b\cos\left(\frac{s}{\sqrt{a^2+b^2}}\right), b\sin\left(\frac{s}{\sqrt{a^2+b^2}}\right), 0\right)$$

$$= \frac{-b}{a^2+b^2}N \quad \Rightarrow \quad \tau = \frac{-b}{a^2+b^2}$$

Proposition
Let $\gamma : I \mapsto \mathbb{R}^3$, regular, parameterised by arc length.
Then the torsion τ of γ vanishes identically
iff $tr(\gamma)$ is contained in a plane.

Proof
Suppose γ is contained in a plane P with unit normal ν. Then T and N are parallel to P.

$$<\gamma, \nu> = c \Rightarrow \underset{\Downarrow}{<\dot{\gamma}, \nu> = 0} \quad , \quad \underset{\parallel}{<\ddot{\gamma}, \nu> = 0}$$
$$<T, \nu> = 0 \qquad <\varkappa N, \nu> = 0$$

$\Rightarrow$ when it exists $B(s) = \pm \nu$
$\Rightarrow \dot{B}(s) = 0$ on any open interval
where B is defined. $\Rightarrow \tau = 0$.
Now suppose $\tau(s) \equiv 0$.
Then $\dot{B}(s) = \tau(s) N \equiv 0$ and $B(s) = B_0 \leftarrow$ constant!
Pick $s_0 \in I$ and consider $\frac{d}{ds}\left(<\gamma(s) - \gamma(s_0), B_0>\right) = <T(s), B_0> = 0$

$\Rightarrow <\gamma(s) - \gamma(s_0), B_0> = c$
sub in $s = s_0 \Rightarrow c = 0$
$<\gamma(s), B_0> = <\gamma(s_0), B_0>$
$\Rightarrow \gamma$ is contained in a plane
(normal to the plane is B_0, goes through $\gamma(s_0)$).
$\square$

A Euclidean rigid motion on $\mathbb{R}^n$ is a mapping
$T : \mathbb{R}^n \mapsto \mathbb{R}^n$ given by $T(x) = \rho x + v$
where v is some constant vector and $\rho \in SO(n)$

MATH3113

04-10-17

<u>Theorem</u> (The Fundamental theorem of Curves)
Let $I \subset \mathbb{R}$ be an interval and suppose there are continuous functions $\varkappa, \tau : I \mapsto \mathbb{R}$ s.t. $\varkappa(s) \neq 0 \ \forall s \in I$
Then

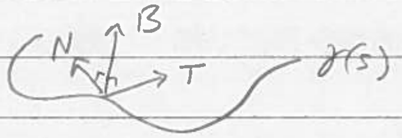
i). There is a regular curve parameterised by arc length s.t. the curvature & torsion are $\varkappa$ and τ respectively

ii). Given $s_0 \in I$, $\gamma_0, T_0, N_0 \in \mathbb{R}^3$ s.t. $T_0 \perp N_0$ then there is a unique curve $\gamma : I \mapsto \mathbb{R}^3$ parameterised by arc length, with curvature & torsion given by $\varkappa$ & τ respectively s.t. $\gamma(s_0) = \gamma_0$, $T(s_0) = T_0$, $N(s_0) = N_0$.

So far take I open.
Suppose not, $f : [a, b) \mapsto \mathbb{R}$
Then differentiable on this interval now means that $\exists \, \varepsilon > 0$ st. f may be extended to a differentiable function on $(a-\varepsilon, b)$.
I needs to be a connected interval with non-zero interior.

MATH 3113

10-10-17



$\dot{\gamma} = T, \quad \dot{T} = \varkappa N, \quad \dot{B} = \tau N$

Frenet - Serrin Trihedron

$\dot{T} = \varkappa N, \quad \dot{N} = -\varkappa T - \tau B, \quad \dot{B} = \tau N$

Alternatively $\dot{F}(s) = A(s) \circ F(s) \quad (*)$

$$F = \begin{pmatrix} T \\ N \\ B \end{pmatrix}, \quad A = \begin{pmatrix} 0 & \varkappa & 0 \\ -\varkappa & 0 & -\tau \\ 0 & \tau & 0 \end{pmatrix}$$

special orthogonal   anti symmetric.

[Existance of ODE's Picard-Lindlof $\Leftarrow$ revise]

Proof of Fundamental Thm of Curves

We have $(T_0, N_0, B_0)^T = F_0$ is a R-H orthonormal frame and $(*)$ is a linear system of 9 ODEs.

Picard-Lindlof Thm

$\Rightarrow$ Given initial data F_0 at s_0 then $\forall s \in I$ there exists a unique solution to $(*)$ s.t.

$T(s_0) = T_0, \quad N(s_0) = N_0, \quad B(s_0) = B_0$

Q: Is our solution a right handed orthonormal frame?

$\Leftrightarrow$ is $F(x) \in SO(3)$

$\Leftrightarrow M(s) = F(s) \circ F^T(s) \equiv I$

symmetric

$\dot{M}(s) = \dot{F} \circ F^T + F \circ \dot{F}^T$ anti symmetric

$= A \circ F \circ F^T + F \circ F^T \circ A^T$

$= A \circ M - M \circ A \quad (+)$

$M(s_0) = I$

However $M(s) = I$ is a solution to the above

$\Rightarrow$ by uniqueness of solution to $(+)$,

$M(s) = I$ is the only sol $\Rightarrow F(s) \circ F^T(s) \equiv I \ \forall s \in I$

Next: Is there a curve compatible with these frames?
A: Integrate
Define $\gamma : I \mapsto \mathbb{R}^3$ by $\gamma(s) = \int_{s_0}^{s} T(r)\, dr + \gamma_0$
(integral taken component by component).
- Fundamental Thm of calculus $\Rightarrow$ T is the tangent vector to γ.
- Since T is a solution to (*) then N is γ's unit normal and $\varkappa$ is its curvature
- Due to right-handedness we have that B is the binormal to γ.
- Due to (*) τ is the torsion of γ

Uniqueness:
Let $\bar{\gamma} : I \mapsto \mathbb{R}^3$ be a second curve with torsion and curvature given by τ and $\varkappa$ respectively. Write the Frenet-Serrin Frame w.r.t. $\bar{\gamma}$ by $\bar{F}$.
Since any two right-handed frames may be related by a rotation, we may write: there is a $\rho \in SO(3)$ s.t. $\bar{F}(s_0) = (\rho \circ F)(s_0)$
Define $\hat{F}(s) = (\rho^{-1} \circ \bar{F})(s)$
WTS: $\hat{F}(s) = F(s) \quad \forall s \in I$.

$Q(s) = |\hat{F}(s) - F(s)|^2 =: |\hat{T} - T|^2 + |\hat{N} - N|^2 + |\hat{B} - B|^2$

matrix norm. Euclidean norm on vectors

$$\frac{d}{ds} Q(s) = \frac{d}{ds}\left(|\hat{T} - T|^2 + |\hat{N} - N|^2 + |\hat{B} - B|^2\right)$$
$$= 2\left(\langle \hat{T} - T, \dot{\hat{T}} - \dot{T}\rangle + \langle \hat{N} - N, \dot{\hat{N}} - \dot{N}\rangle + \langle \hat{B} - B, \dot{\hat{B}} - \dot{B}\rangle\right)$$
$$= 2\big(\varkappa \langle \hat{T} - T, \hat{N} - N\rangle + \tau \langle \hat{B} - B, \hat{N} - N\rangle$$
$$+ \langle \hat{N} - N, -\varkappa(\hat{T} - T) - \tau(\hat{B} - B)\rangle\big)$$
$$= 0$$

$\Rightarrow Q(s) = c$ independent of s.
However $F(s_0) = \hat{F}(s_0) \Rightarrow c = 0$
$\Rightarrow F(s) = \hat{F}(s) \quad \forall s \in I$
Therefore $\bar{T}(s) = \rho T(s)$ and so $\dot{\bar{\gamma}}(s) = \rho \dot{\gamma}(s) \quad \forall s \in I$
Integrating w.r.t. s (as before)

MATH 3113

10-10-17

we see that $\tilde{\gamma}(s) = \rho\gamma(s) + v$ (for some constant vector $v \in \mathbb{R}^3$)

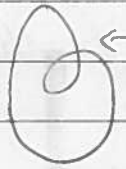
$\Rightarrow$ uniqueness up to Euclidean motion. $\square$

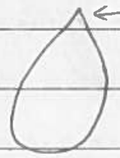
The Global Theory of Plane Curves

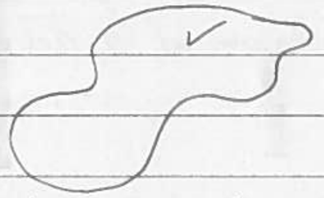
Def (simple closed curve)

Let $\gamma : [a,b] \mapsto \mathbb{R}^2$ be a regular parameterised curve (C^k-curve). We say that γ is simple if $\gamma|_{[a,b)}$ is injective and γ is closed if $\gamma(a) = \gamma(b)$.

It is a C^k-closed curve if additionally $\gamma'(a) = \gamma'(b)$ and $\gamma^{(l)}(a) = \gamma^{(l)}(b)$, $1 \le l \le k$.

 ← NOT simple

 ← NOT C^1



Three Classical Theorems

Theorem Jordan Curve Theorem (JCT)

Let γ be a simple C^1 closed plane curve.

Then the set $\mathbb{R}^2 \setminus tr(\gamma)$ is the disjoint union of two open path-connected sets, exactly one of which is bounded.

Def

The bounded set from the JCT will be called the interior of γ and will be denoted $int(\gamma)$

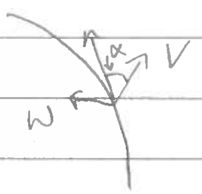
If γ is regular, closed, C^1, parameterized by arc length in such a way that $\exists$ a point $s_0 \in I$ s.t. $int(\gamma)$ lies on the side of $\{\gamma(s_0) + \lambda\gamma'(s_0) \mid \lambda \in \mathbb{R}\}$ that also contains the 90° rotation of γ; then the curve is said to be positively parameterized.

Q: what closed curve of a given length contains the maximum area?
A: a circle.

Theorem The Isoperimetric Inequality.
Let γ be a C^1, closed, simple, plane curve of length $l(\gamma)$. then $4\pi \, \text{Area}(\text{int}\,\gamma) \leq l(\gamma)^2$
and equality holds iff γ is a parameterized circle of radius $\frac{1}{2\pi} l$.

Def
An orthonormal frame on $\mathbb{R}^2$ is an ordered pair $\{V, W\}$ of vector valued continuously differentiable functions so that $\forall x \in \mathbb{R}^2$,
$\{V(x), W(x)\}$ is an oriented orthonormal frame.
(oriented $\Rightarrow \det\begin{pmatrix} -V- \\ -W- \end{pmatrix} > 0$. orthonormal $\Rightarrow \det\begin{pmatrix} -V- \\ -W- \end{pmatrix} = 1$).

Def
Let $\gamma : I \mapsto \mathbb{R}^2$ be a regular, C^1 parameterized curve, $\{V, W\}$ an orthonormal frame on $\mathbb{R}^2$.
A function $\alpha : I \mapsto \mathbb{R}$ is called an angular function of γ w.r.t. $\{V, W\}$ if it is continuously differentiable and $\dfrac{\gamma'(t)}{|\gamma'(t)|} = \cos(\alpha(t))V(\gamma(t)) + \sin(\alpha(t))W(\gamma(t))$.



Of course we have, $a, b \in I$,
$$\int_a^b \alpha'(t)\,dt = \alpha(b) - \alpha(a).$$

Theorem (Hopf's Umlaufsatz) (turning tangent)
Let $\gamma : [a,b] \mapsto \mathbb{R}^2$ be a regular, closed, C^2 plane curve and let α be an angular function of γ w.r.t. an orthonormal basis $\{V, W\}$ on $\mathbb{R}^2$.

MATH 3113

10-10-17

Then $\exists\, n_\gamma \in \mathbb{Z}$ so that
$\int_a^b \alpha'(t)\,dt = 2\pi n_\gamma$. If γ is simple then $n_\gamma = \pm 1$.

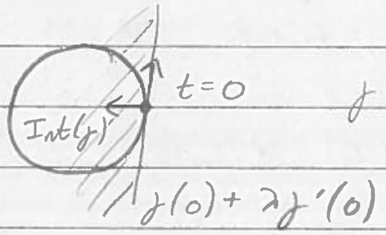
In fact: if $\{V, W\}$ are $\{e_1, e_2\}$ then $\dot{\alpha} = \mathcal{K}_\sigma = \pm \mathcal{K}$

$\left[\int_\gamma \mathcal{K}_\sigma = 2\pi n_\gamma\right]$

Remark:
The turning number of a Jordan curve is $+1$ iff it is positively parameterised.

Def
A Jordan curve is a simple closed curve.

Example
The circle: $\gamma(t) = (\cos t, \sin t)$, $t \in [-\pi, \pi]$
$int(\gamma) = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 < 1\}$
$\gamma'(t) = (-\sin t, \cos t) = (\cos(t + \frac{\pi}{2}), \sin(t + \frac{\pi}{2}))$
$R = 90°$ anticlockwise rotation
$R = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$



$\gamma' = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

so γ is positively parameterised.
α - angular function w.r.t. $\{e_1, e_2\}$
$\Rightarrow \gamma' = \cos(\alpha) e_1 + \sin(\alpha) e_2$
so $\alpha = t + \frac{\pi}{2}$
$\int_{-\pi}^{\pi} \alpha' = \int_{-\pi}^{\pi} 1 = 2\pi$ so the circle is a simple curve.

Chains and Turning Tangents

Def (Chains & polygons)

i). A piecewise C^k closed plane curve or a plane chain (of class C^k) is a continuous plane curve $\gamma : [a,b] \mapsto \mathbb{R}$ for which there are numbers
$a = t_1 < t_2 < \dots < t_n < t_{n+1} = b$
so that $\gamma|_{[t_i, t_{i+1}]}$ is a C^k-regular curve.

ii). The points $\gamma(t_i)$ are called vertices and the arcs $\gamma((t_i, t_{i+1}))$ are edges for $1 \leq i \leq n$.

iii). If γ is a simple plane chain we call $P_\gamma = int(\gamma)$ a generalised polygon.

Def Exterior Angles.

Let $\gamma : I \mapsto \mathbb{R}^2$ be a C^2 regular plane chain and $\{V, W\}$ an orthonormal frame on $\mathbb{R}^2$. An angular function w.r.t. $\{V, W\}$ is a function $\alpha : I \mapsto \mathbb{R}^2$

i). $\alpha|_{(t_i, t_{i+1})}$ is an angular function for $\gamma|_{(t_i, t_{i+1})}$ w.r.t. $\{V, W\}$ $\forall\ 1 \leq i \leq n$.

ii). $\theta_i := \lim\limits_{\varepsilon \to 0} (\alpha(t_i + \varepsilon) - \alpha(t_i - \varepsilon)) \in [-\pi, \pi]$ for $2 \leq i \leq n$

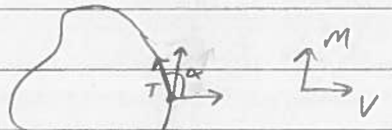
iii). If $|\theta_i| = \pi$ then θ_i is positive iff $\{\gamma'(t_i + \varepsilon), \gamma'(t_i - \varepsilon)\}$ is right handed $\forall\ 0 < \varepsilon < \varepsilon_0$ for some $\varepsilon_0 > 0$.

The angles θ_i are called exterior angles of the chain at $\gamma(t_i)$
the angle at $\gamma(a) = \gamma(b)$ is the (unique) representative $\theta_1 \in [-\pi, \pi]$ of $\lim\limits_{\varepsilon \to 0} (\alpha(a + \varepsilon) - \alpha(b - \varepsilon)) = \theta_1 \mod 2\pi$ with sign convention as above.

MATH 3113

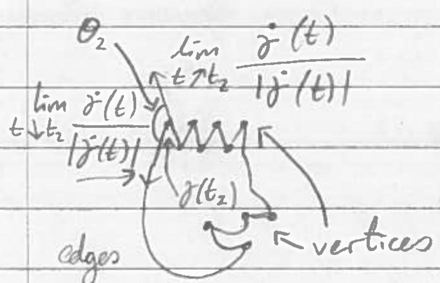
11-10-17

$$4\pi \, \text{Area}(\text{int}\,\gamma) \leq L(\gamma)^2$$



$$\int \alpha' \, dt = 2\pi n_\gamma$$

$n_\gamma = \pm 1$ if γ is simple & closed.



<u>Theorem</u> (Return of the Umlaufsatz)

Let $\gamma : [a,b] \mapsto \mathbb{R}^2$ be a closed plane chain of class C^2 with angular function α and exterior angles θ_i, $1 \leq i \leq n$, then there is an $n_\gamma \in \mathbb{Z}$ s.t.

$$\int_a^{b'} \alpha'(t)\,dt + \sum_{i=1}^{n} \theta_i = 2\pi n_\gamma .$$

<u>Proof</u> (+ve paramaterised simple curve case only) (sketch)

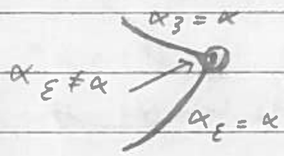
$\gamma : [a,b] \mapsto \mathbb{R}^2$ is the chain from the theorem.

We approximate γ by $\gamma_\varepsilon : [a,b] \mapsto \mathbb{R}^2$ (γ_ε is C^2, simple, closed) so that

1). $\gamma_\varepsilon \to \gamma$ uniformly on $[a,b]$ as $\varepsilon \to 0$

2). $\gamma_\varepsilon(t) = \gamma(t)$ if $|t - t_i| > \varepsilon$, $1 \leq i \leq n$

3). $\text{tr}(\gamma_\varepsilon) \subset \text{int}(\gamma)$



Let $U_i^\varepsilon = \{ t \in [a,b] \mid |t - t_i| < 2\varepsilon \}$

$V^\varepsilon = [a,b] \setminus \bigcup_{i=1}^{n+1} U_i$

Then $$2\pi \int_{[a,b]} \alpha'_3 = \sum_{i=1}^{n+1} \int_{U_i} \alpha'_\varepsilon + \int_{V^\varepsilon} \alpha'_\varepsilon \, dt$$

$$= \sum_{i=1}^{n+1} \left[\alpha_\varepsilon(t_i + 2\varepsilon) - \alpha_\varepsilon(t_i - 2\varepsilon) \right] + \int_{V^\varepsilon} \alpha_\varepsilon' \, dt$$

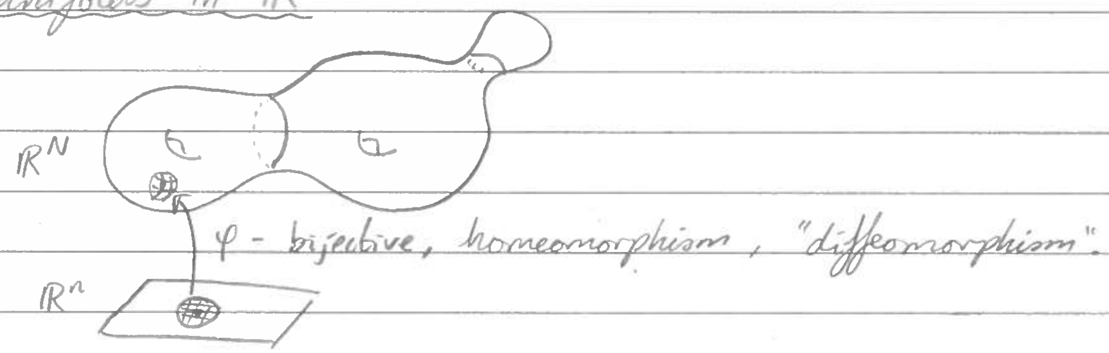
$$= \sum_{i=1}^{n+1} \left[\alpha(t_i + 2\varepsilon) - \alpha(t_i - 2\varepsilon) \right] + \int_{V^\varepsilon} \alpha' \, dt$$

$$\lim_{\varepsilon \to 0} \left[\alpha(t_i + 2\varepsilon) - \alpha(t_i - 2\varepsilon) \right] = \theta_i$$

$$\lim_{\varepsilon \to 0} \int_{V^\varepsilon} \alpha' \, dt = \int_V \alpha' \, dt \qquad \text{(from Measure Theory)}$$

So $2\pi = \sum_{i=1}^{n+1} \theta_i + \int_{[a,b]} \alpha' \, dt$ $\square$

(Sub) Manifolds in $\mathbb{R}^N$



Def (Submanifolds of $\mathbb{R}^N$)
A subset $M \subset \mathbb{R}^N$ is an n-dimensional submanifold of $\mathbb{R}^N$ if it is locally diffeomorphic to $\mathbb{R}^n$.
That is, $\forall p \in M$ $\exists$ an open neighbourhood $V \ni p$ and an open subset $U \subset \mathbb{R}^n$ and a map $\varphi : U \mapsto \mathbb{R}^N$ s.t.

i). $\varphi : U \mapsto \mathbb{R}^N$ is continuously differentiable

ii). $\varphi : U \mapsto V$ is a homeomorphism (continuous, bijective, continuous inverse)

iii). $D\varphi|_u : \mathbb{R}^n \mapsto \mathbb{R}^N$ is injective $\forall u \in U$.

The map φ^{-1} is called a chart and the neighbourhood V is

MATH 3113

11-10-17

called its <u>coordinate patch.</u>
The component functions $\varphi^{-1} = (u_1, \dots, u_n)$ are called local coordinates.
If $\{V_i\}$ are an open covering of M by coordinate patches & associated charts $\varphi_i : U_i \mapsto V_i$ then the collection of triples $\{(\varphi_i, U_i, V_i)\}$ is called an atlas.

17-10-17

<u>Def</u>
A subset $M \subset \mathbb{R}^N$ is an n-dim differential submanifold of $\mathbb{R}^N$ if it is "locally diffeomorphic to $\mathbb{R}^n$".
That is for $p \in M$ there is an open neighbourhood $V \subset M$ and an open set $U \subset \mathbb{R}^n$ and a map $\varphi : U \mapsto V$ s.t.

1). $\varphi : U \mapsto \mathbb{R}^N$ is continuously differentiable
2). $\varphi : U \mapsto V$ is homeomorphism
3). $D\varphi|_u : \mathbb{R}^n \mapsto \mathbb{R}^N$ is injective $\forall\, u \in U$.

Recall: For any set $M \subset \mathbb{R}^N$ a subset $V \subset M$ is open in M iff there is an open set $\tilde{V} \subset \mathbb{R}^N$ s.t. $\tilde{V} \cap M = V$.

<u>Notation</u>
$\varphi : U \mapsto V$ is a local parameterisation
$\varphi^{-1} : V \mapsto U$ is a chart
$\varphi^{-1} = (u_1, \dots, u_n)$ are called local coordinates
V - sometimes called a coordinate patch.
If $\{V_i\}$ is a covering of M by coordinate patches with local parameterisations $\varphi_i : U_i \mapsto V_i$ then the collection of triples $\{(\varphi_i, U_i, V_i)\}$ is called an atlas of M (atlas of parameterisation).

Examples

1). $n \leq N$, $\mathbb{R}^n$ can be considered as a submanifold of $\mathbb{R}^N$. Take $\iota : \mathbb{R}^n \mapsto \mathbb{R}^N$, $(x_1, \dots, x_n) \mapsto (x_1, \dots, x_n, 0, \dots, 0)$

2). (i) For I an open interval, $\gamma : I \mapsto \mathbb{R}^N$ is a regular simple curve then $\mathrm{tr}\,\gamma$ is a 1-dimensional sub manifold.

① ✓ ② ✓ ③ $\iff |\gamma'| \neq 0$ (regular) ✓

(ii) C^1 closed simple curves are also 1-dimensional submanifolds.

3). $\infty \subset \mathbb{R}^2$

Not a submanifold.

Need a mapping $\varphi : \ominus \longrightarrow \otimes$

$\left[\begin{array}{l} \ominus \text{ has 2 path connected pieces} \\ \otimes \;\; \text{"} \;\; 4 \;\; \text{"} \;\; \text{"} \;\; \text{"} \end{array} \right]$

4). The unit circle in $\mathbb{R}^2$

Local parameterisations $\varphi_i : (0, 2\pi) \mapsto \mathbb{R}^2$, $\varphi_1(t) = (\cos t, \sin t)$

$\varphi_2(t) = (-\cos t, -\sin t)$

Both are continuously differentiable.

$$D\varphi_1\Big|_t = \begin{pmatrix} -\sin t \\ \cos t \end{pmatrix} = -D\varphi_2\Big|_t \quad \text{are linear maps } \mathbb{R}^1 \mapsto \mathbb{R}^2$$

These are injective for $a \in \mathbb{R}$

$$\left[D\varphi_1\Big|_t (a) = a \cdot \begin{pmatrix} -\sin t \\ \cos t \end{pmatrix} = 0 \iff a = 0 \right]$$

$\varphi_1 : (0, 2\pi) \longmapsto V \overset{\text{open}}{=} S^1 \setminus \{(1,0)\}$

φ_1 maps open intervals to open intervals.

(calculation left as exercise).

$\iff \varphi_1$ is a homeomorphism

S_1 is clearly covered by $\varphi_1 \cup \varphi_2$.

MATH 3113

17-10-17

Remarks

1). Locally diffeomorphic to be properly defined later.

2). We may also define C^k manifolds (the above is C^1).
M^n is a C^k manifold if in addition to ①, ②, ③, φ needs to be k times continuously differentiable.

3). The following are equivalent:

(i) $D\varphi|_u$ is injective.

(ii) $J\varphi|_u$ has rank n $\quad J\varphi|_u = \left(\frac{\partial \varphi^I}{\partial u_i}\right)_{1 \le I \le N,\ 1 \le i \le n}$.

$$\begin{pmatrix} \frac{\partial \varphi^1}{\partial u_1} & \cdots & \frac{\partial \varphi^1}{\partial u_n} \\ \vdots & & \vdots \\ \frac{\partial \varphi^N}{\partial u_1} & \cdots & \frac{\partial \varphi^N}{\partial u_n} \end{pmatrix} = J\varphi|_u$$

(iii) $\frac{\partial \varphi}{\partial u_1}, \dots, \frac{\partial \varphi}{\partial u_n}$ are linearly independent.

Remark on homeomorphisms:

If we want to show φ is a homeomorphism, options include:

1). Showing φ is open, that is, a set O is open iff $\varphi(O) \subset V$ is open $(\varphi : U \mapsto V)$.

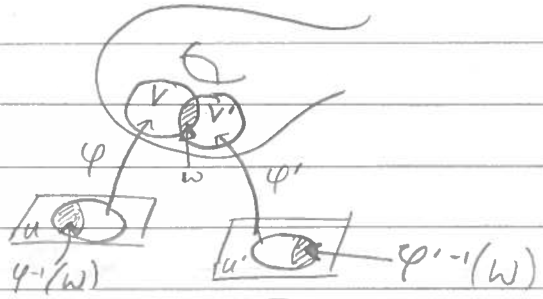
2). For any sequence $\{p_i\} \subset V$ converges in V, say to a point p, then the preimages $q_i = \varphi^{-1}(p_i)$ also converge to $q = \varphi^{-1}(p)$.
(prove bijective, continuous, then the above shows φ^{-1} is cont.).

3). Atlases are not unique
e.g. $\varphi : U \mapsto V$ and we have a diffeomorphism $\psi : U \mapsto U$ then $\varphi \circ \psi : U \mapsto V$ is another parameterisation.

Change of Coordinates

Suppose we have $\varphi : U \mapsto V$, $\varphi' : U' \mapsto V'$ local parameterisations.
Suppose $V' \cap V \neq \emptyset$. Let $W = V' \cap V$.
Since φ and φ' are homeomorphisms, define
$h = \varphi'^{-1} \circ \varphi : \varphi^{-1}(W) \mapsto \varphi'^{-1}(W)$ also a homeomorphism.
Write in local coordinates $u = (u_1, \dots, u_n)$ coordinates on
$\varphi^{-1}(W)$, $u' = (u_1', \dots, u_n')$ coords on $\varphi'^{-1}(W)$.

$$\begin{aligned} h \circ u(p) &= (\varphi'^{-1} \circ \varphi \circ \varphi^{-1})(p). \\ &= (\varphi'^{-1})(p) \\ &= u'(p) \end{aligned}$$



[coordinates : functions determining points]

Prop:
Changes of coordinates are diffeomorphisms.

Graphs:

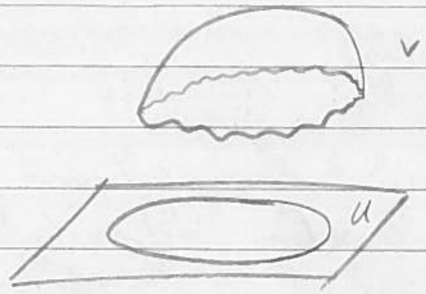
A graph of a function $g : \mathbb{R}^n \mapsto \mathbb{R}^m$ is the set
$\{(\underset{\in \mathbb{R}^n}{x}, \underset{\in \mathbb{R}^m}{y}) \in \mathbb{R}^{n+m} \mid g(x) = y\}$.

Prop (Submanifolds as graphs)
A set $M \subset \mathbb{R}^N$ is a n-dim submanifold of $\mathbb{R}^N$ iff
M is locally the graph of a continuously differentiable function, that is, iff: $\forall p \in M$ $\exists$ open neighbourhood
V of p in M and an open set $U \subset \mathbb{R}^n$ and a permutation
π of $\{1, \dots, N\}$ and a continuously differentiable function
$g : U \mapsto \mathbb{R}^{N-n}$ st. $V = \{(x_{\pi(1)}, \dots, x_{\pi(N)}) \mid (x_1, \dots, x_n) \in U, (x_{n+1}, \dots, x_N) = g(x_1, \dots, x_n)\}$

MATH 3113

17-10-17

i.e.



Prop (Submanifolds as level sets)
A set $M \subset \mathbb{R}^N$ is an n-dim submanifold of $\mathbb{R}^N$ iff M is locally given as the solution set of $N-n$ independent equations (at a regular point).
That is; $\forall p \in M$ there is an open set V of M and an open set W of $\mathbb{R}^N$ s.t. $V \subset W$ and a continuously differentiable function $f: W \mapsto \mathbb{R}^{N-n}$, $f = (f_1, \dots, f_{N-n})$ so that:
(i) $V = f^{-1}(\{0\}) = \{x \in W \mid f(x) = 0\}$
(ii) $Df|_p : \mathbb{R}^N \mapsto \mathbb{R}^{N-n}$ is surjective $\forall p \in V$.

Example S^2
The sphere is given by $\{(x,y,z) \mid x^2+y^2+z^2=1\}$
$f(x) = x^2+y^2+z^2-1$, $S^2 = f^{-1}(0)$
Here $W = \mathbb{R}^3 \setminus \{0\}$, $V = S^2$
f is continuously differentiable, $S^2 = f^{-1}(0)$

$$Df|_{(x,y,z)} = \begin{pmatrix} 2x \\ 2y \\ 2z \end{pmatrix}$$ (as a linear mapping $\mathbb{R}^3 \mapsto \mathbb{R}$).

$Df|_{(x,y,z)}$ is surjective iff $(x,y,z) \neq 0$, $0 \notin S^2$

$\Rightarrow S^2$ is a 2-dimensional manifold.

[Inverse fn Thm I.10
Implicit fn Thm I.12]

Proof (Graphs as submanifolds)

[$\Leftarrow$] Suppose $M \subset \mathbb{R}^N$ is given locally as a graph.

$\Leftrightarrow$ wlog $p \in M$, there is an open $V \subset M$, $p \in V$ and an open set $U \subset \mathbb{R}^n$ and a continuously differentiable mapping $g : U \mapsto \mathbb{R}^{N-n}$ so that for $x = (x', x'')$, $x' = (x_1, \dots, x_n)$, $x'' = (x_{n+1}, \dots, x_N)$

we have $V = \{x \in \mathbb{R}^N \mid x' \in U,\ x'' = g(x')\}$.

We define the map $\varphi : U \mapsto \mathbb{R}^N$, $\varphi(u) = (u, g(u))$. We now show φ is a local parameterisation.

Since g is continuously differentiable, so is φ.

$\varphi : U \mapsto V$ is a bijection. Moreover if $Q \subset U$ is open then $\varphi(Q) = (Q \times \mathbb{R}^{N-n}) \cap V$ is also open.

Also since φ is continuous, if W open in V then $\varphi^{-1}(W)$ is also open.

Finally, $D\varphi|_u = \begin{pmatrix} id \\ Dg|_u \end{pmatrix}$ $\forall\, u \in U$

which has rank n.

$\therefore \varphi$ is a local parameterisation.

[$\Rightarrow$] Suppose M is an n-dimensional submanifold of $\mathbb{R}^N$.

Pick $p \in M$, then there exists open V in M s.t. $p \in V$, and open $U \subset \mathbb{R}^n$ and a parameterisation $\varphi : U \mapsto V$.

Let $q \in U$ s.t. $\varphi(q)$ p.

By assumption (③ of def.)

$D\varphi|_q : \mathbb{R}^n \mapsto \mathbb{R}^N$ has rank n, and so w.l.o.g we may assume that the first n-rows are linearly independent.

[$D\varphi|_q = \begin{pmatrix} \boxed{} \\ \end{pmatrix}$ invertible.] ($\overset{n}{\rightarrow}$, $\downarrow N$)

Write $\varphi = (\varphi', \varphi'')$ where φ' is the first n components of φ and φ'' is the last $N-n$ components of φ.

$D\varphi'|_q : \mathbb{R}^n \mapsto \mathbb{R}^n$ is invertible, and so by inverse function theorem $\exists$ open sets $U', U'' \subset \mathbb{R}^n$ s.t. $\varphi' : U' \mapsto U''$ is a diffeomorphism.

MATH 3113

17-10-17

Let $g = \varphi'' \circ \varphi'^{-1} : U'' \mapsto \mathbb{R}^{N-n}$.
Then define $V' = \varphi(U')$.
So $V' = \varphi(U') = \left\{ \begin{pmatrix} \varphi'(u) \\ \varphi''(u) \end{pmatrix} \middle| \, u \in U' \right\}$

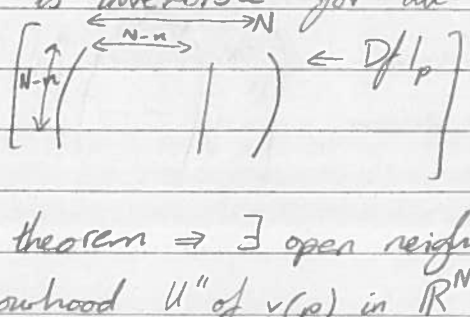
$$= \left\{ \begin{pmatrix} v \\ \varphi''(\varphi'^{-1}(v)) \end{pmatrix} \middle| \, v \in U'' \right\}$$

$\Leftrightarrow V'$ is locally a graph. $\square$

18-10-17

<u>Implicit Function Theorem</u>
Let $U \subset \mathbb{R}^N$ be open and $f : U \mapsto \mathbb{R}^n$ a continuously differentiable function. Suppose $x_0 \in \mathbb{R}^{N-n}$, $y_0 \in \mathbb{R}^n$ s.t. $(x_0, y_0) \in U$. Let $c = f(x_0, y_0)$.
If the matrix $\left(\frac{\partial f_i}{\partial y_j} \right)_{ij}$ is invertible then $\exists$ open set $x_0 \in U' \subset \mathbb{R}^{N-n}$, $y_0 \in U'' \subset \mathbb{R}^n$ and a continuously differentiable map $g : U' \mapsto U''$ s.t. $\forall\, x \in U'$, $y \in U''$,
$f(x,y) = c \Leftrightarrow y = g(x)$.

<u>Proof</u> (of Level set proposition).
[$\Leftarrow$] Suppose M is given locally as a level set in an open set $V \subset M$ as the level set of $f : W \mapsto \mathbb{R}^{N-n}$
$Df|_p : \mathbb{R}^N \mapsto \mathbb{R}^{N-n}$ is surjective.
So reordering coordinates and writing $p = (u, v)$, $u \in \mathbb{R}^n$, $v \in \mathbb{R}^{N-n}$, then $\left(\frac{\partial f_i}{\partial v_j} \right)_{ij}$ is invertible for all $p \in V$.



Implicit function theorem $\Rightarrow \exists$ open neighbourhood U' of $u(p)$ in $\mathbb{R}^n$ and open neighbourhood U'' of $v(p)$ in $\mathbb{R}^{N-n}$ and a continuously differentiable function $g : U' \mapsto U''$ s.t. $f(u,v) = 0 \Leftrightarrow v = g(u)$ $\forall u \in U'$ and $v \in V$.

[A neighbourhood U of a point p is a set such that $\exists$ an open set $V \subset U$ s.t. $p \in V \subset U$.]

Now M is locally written as a graph.

$\Rightarrow M$ may be locally written as a graph everywhere

$\Rightarrow M$ is a submanifold.

[$\Rightarrow$] Suppose M a submanifold, then write it locally as a graph.

$\exists V$ a neighbourhood of p in M, an open set $U \subset \mathbb{R}^n$, and a continuously differentiable $g: U \mapsto \mathbb{R}^{N-n}$ s.t.

$V = \{(u,v) \in \mathbb{R}^N \mid v = g(u)\}$.

Let $f(u,v) = v - g(u)$, $f: \underbrace{U \times \mathbb{R}^{N-n}}_{\subset \mathbb{R}^N \text{ open.}} \mapsto \mathbb{R}^{N-n}$.

Clearly $f^{-1}(0) = V$ and $Df|_p = \Big(\underbrace{-Dg|_u}_{N-n} \; \underbrace{id}_{n}\Big)$ which is surjective.

$\Rightarrow f$ is a suitable level set function. $\square$

<u>Proof</u> (Changes of coordinates are diffeomorphisms)

From proof of local graph proposition we know we can construct functions g & g' on suitably small neighbourhoods Z and Z' s.t. $\varphi(Z) = \{(x_1, \dots, x_N) \mid (x_1, \dots, x_n) \in Z, (x_{n+1}, \dots, x_N) = g(x_1, \dots, x_n)\}$

$\varphi(Z') = \{(x'_{\pi(1)}, \dots, x'_{\pi(N)}) \mid (x'_1, \dots, x'_n) \in Z', (x'_{n+1}, \dots, x'_N) = g(x'_1, \dots, x'_n)\}$

where π is a permutation.

w.l.o.g., $\varphi(Z) = \varphi(Z')$.

But now, change of coordinates is given by

$(x_1, \dots, x_n) \xmapsto{\varphi} (x_1, \dots, x_n, g(x_1, \dots, x_n)) = (y_1, \dots, y_N) \xmapsto{\varphi'^{-1}} (y_{\pi^{-1}(1)}, \dots, y_{\pi^{-1}(n)})$

φ is continuously differentiable, φ'^{-1} is a linear map

so h is continuously differentiable.

However, we could do the same with h^{-1}.

$\Rightarrow h^{-1}$ is also continuously differentiable.

$\therefore h$ is a diffeomorphism. $\square$

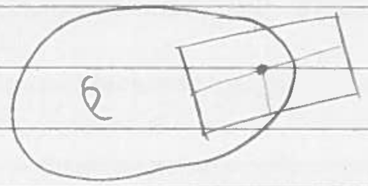
MATH 3113

18-10-17

Def

Let $M \subset \mathbb{R}^N$ be a submanifold, $p \in M$.

the set $T_pM := \{ v \in \mathbb{R}^N \mid \exists\, \varepsilon > 0,\ \gamma : (-\varepsilon, \varepsilon) \mapsto M,\ C^1,$ s.t. $\gamma(0) = p,\ \gamma'(0) = v \}$



In fact T_pM is a vector space spanned by $\left\{ \frac{\partial \varphi}{\partial u_1}\Big|_u, \dots, \frac{\partial \varphi}{\partial u_n}\Big|_u \right\}$ where $\varphi : U \mapsto M$ is a local parameterisation near p where $\varphi(u) = p$.

$\frac{\partial \varphi}{\partial u_i}$ is contained in T_pM:

$$\gamma := \varphi(u_1, \dots, u_i + t, u_{i+1}, \dots, u_n)$$

$$\gamma'(0) = \frac{\partial \varphi}{\partial u_i}.$$

Conversely suppose γ is a curve s.t. $\gamma(0) = p$.

Then
$$\gamma'(0) = \frac{d}{dt}\Big|_{t=0} (\varphi \circ \varphi^{-1} \circ \gamma)(t)$$
$$= D\varphi|_u \frac{d}{dt}\Big|_{t=0} (\varphi^{-1} \circ \gamma)(t)$$
$$= \frac{\partial \varphi}{\partial u_j} \circ (u_j \circ \gamma)'(0)$$

$\Rightarrow \gamma'(0)$ is a linear combination of $\frac{\partial \varphi}{\partial u_i}$'s with coefficients $(u_j \circ \gamma')(0)$.

24-10-17

Tangent space is $T_p M = \{ v \in \mathbb{R}^N \mid \exists \varepsilon > 0,\ C^1$ function $\gamma : (-\varepsilon, \varepsilon) \to M : \gamma(0) = p,\ \gamma'(0) = v \}$

For γ s.t. $\operatorname{Im}\gamma \subset M$,

$$\gamma'(0) = \sum_{j=1}^{n} \frac{\partial \varphi}{\partial u_j}\Big|_u \circ (u_j \circ \gamma)'(0)$$

$\Rightarrow \dfrac{\partial \varphi}{\partial u_j}$ span $T_p M$

in fact they form a basis.

<u>Prop</u> (Bases of Tangent Spaces)
Let M be an n-dim manifold of $\mathbb{R}^N$. Then $T_p M$ is an n-dim linear subspace of $\mathbb{R}^N$.
Moreover:

i). If φ is a local chart of M near p satisfying $\varphi(u) = p$, then we have $T_p M = \text{range } D\varphi|_u$ and $\left\{ \dfrac{\partial \varphi}{\partial u_1}, \dots, \dfrac{\partial \varphi}{\partial u_n} \right\}$ is a basis of $T_p M$.

ii). If M is the graph of a function g near p, $p = (x, g(x))$ then a basis of $T_p M$ is given by $\left\{ \begin{pmatrix} e_1 \\ \frac{\partial g}{\partial x_1} \end{pmatrix}, \dots, \begin{pmatrix} e_n \\ \frac{\partial g}{\partial x_n} \end{pmatrix} \right\}$ where e_j is the jth standard coordinate in $\mathbb{R}^n$.

iii). If M is the level set of a function f near p, then we have $T_p M = \operatorname{Ker} Df|_p$.

$$\left[\begin{pmatrix} e_1 \\ \frac{\partial g}{\partial x_1} \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \\ \frac{\partial g_1}{\partial x_1} \\ \vdots \\ \frac{\partial g_{N-n}}{\partial x_1} \end{pmatrix} \right]$$

MATH 3113

24-10-17

Proof

i). $\left\{ \frac{\partial \varphi}{\partial u_1}, \dots, \frac{\partial \varphi}{\partial u_n} \right\}$ span T_pM

By ③ $D\varphi|_u$ is injective

$\iff$ $\frac{\partial \varphi}{\partial u_1}, \dots, \frac{\partial \varphi}{\partial u_n}$ are linearly independent

$\Rightarrow$ they are a basis

This $\Rightarrow \dim T_pM = n \quad \forall p \in M$

ii). $\exists$ local param. $U \subset \mathbb{R}^n$, $\varphi(x) = (x_1, \dots, x_n, g_1(x), \dots, g_{N-n}(x))^T$

Then $\frac{\partial \varphi}{\partial x_i} = \begin{pmatrix} 0 \\ \vdots \\ 1 \\ 0 \\ \vdots \\ 0 \\ \frac{\partial g_1}{\partial x_i} \\ \vdots \\ \frac{\partial g_{N-n}}{\partial x_i} \end{pmatrix}$ $\leftarrow$ i th component

iii). M is locally the level set of f, let γ be a continuously differentiable curve in M. $\gamma: (-\varepsilon, \varepsilon) \to M$, $\gamma(0) = p$.

$f(\gamma(t)) = 0$

$\Rightarrow Df|_p(\gamma'(0)) = 0.$

Since this holds for all γ we see that

$T_pM \subset \operatorname{Ker} Df|_p$

But $Df|_p : \mathbb{R}^N \to \mathbb{R}^{N-n}$ is surjective

$\Rightarrow \dim \operatorname{Ker} Df|_p = \dim \mathbb{R}^N - \dim \mathbb{R}^{N-n}$

$= N - (N-n) = n$

$\Rightarrow T_pM = \operatorname{Ker} Df|_p$ $\square$

Examples

i). $\gamma: I \to \mathbb{R}^N$ injective regular parameterised curve then γ is a parameterisation $\forall p \in \operatorname{tr} \gamma$. By Tangent Space Prop. (TSP) part i $T_p \operatorname{tr}(\gamma) = \operatorname{span}\left\{ \frac{\partial \gamma}{\partial t}\Big|_p \right\} = \{ \lambda \gamma'(t_0) \mid \lambda \in \mathbb{R}, \gamma(t_0) = p \}$

ii). $S^2 \cap \mathbb{R}^3_+$

$$\left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3 \;\middle|\; z > 0 \right\}$$

is a graph $g(x,y) = \sqrt{1 - x^2 - y^2}$
where $g : B_1(0) \to \mathbb{R}$.

$T_p S^2 \cap \mathbb{R}^3 = \text{span}\left\{ \begin{pmatrix} 1 \\ 0 \\ \frac{\partial g}{\partial x} \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ \frac{\partial g}{\partial y} \end{pmatrix} \right\}$ $\quad \left[\frac{\partial g}{\partial x} = \frac{-x}{\sqrt{1 - x^2 - y^2}} \right]$

g symmetric.

$T_{(0,0,1)} S^2 \cap \mathbb{R}^3_+ = \text{span}\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right\}$.

$\varphi(0,0) \cdot (x, y, g(x)) = \varphi(x,y)$

iii). Paraboloid $\quad z = x^2 + y^2$, $f : \mathbb{R}^3 \to \mathbb{R}$

$f(x, y, z) = x^2 + y^2 - z$

$$Df\Big|_{(x,y,z)}(v) = (2x, \; 2y, \; -1) \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}$$

Df is surjective ~~everywhere~~ on all $\mathbb{R}^3$.

Kernel:

$$T_{(x,y,z)} M = \text{Ker}\, Df\Big|_{(x,y,z)}$$

$$= \left\{ v \in \mathbb{R}^3 \;\middle|\; v \cdot \begin{pmatrix} 2x \\ 2y \\ -1 \end{pmatrix} = 0 \right\}$$

$$= \text{span}\left\{ \begin{pmatrix} 1 \\ 0 \\ 2x \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 2y \end{pmatrix} \right\}$$

Functions on Submanifolds

Def: Let M_j be n_j-dim submanifolds of $\mathbb{R}^{N_j}$ $j = 1, 2$.
Let $f : M_1 \to M_2$. We say f is continuously differentiable at $p \in M_1$ if for parameterisations φ_1 of M_1 near p and φ_2 of M_2 near $f(p)$, the composition $\varphi_2^{-1} \circ f \circ \varphi_1 = \tilde{f}$ is continuously differentiable at $\varphi^{-1}(p)$. f is continuously differentiable on all of M_1 if $\forall p \in M_1$ the above holds.

MATH 3113

24-10-17

More precisely: f is differentiable if
$\varphi_1 : U_1 \to V_1 \ni p$, $V_1 \subset M_1$,
$\varphi_2 : U_2 \to V_2 \ni f(p)$, $V_2 \subset M_2$,
then there exists open neighbourhoods $\tilde{U}_i \subset U_i$, $\tilde{V}_i \subset V_i$ s.t. $f(\tilde{V}_1) \subset \tilde{V}_2$ and $\tilde{f}$ is continuously differentiable where $\tilde{f}$ is given by

$$\begin{tikzcd} \mathbb{R}^{N_1} \supset \tilde{V}_1 \arrow[r, "f"] & \tilde{V}_2 \subset \mathbb{R}^{N_2} \\ \tilde{U}_1 \arrow[u, "\varphi_1"] \arrow[r, "\tilde{f}"] & \tilde{U}_2 \arrow[u, "\varphi_2"'] \end{tikzcd}$$

$\tilde{f} = \varphi_2^{-1} \circ f \circ \varphi_1 : \tilde{U}_1 \to \tilde{U}_2$.

Well-defined?
Need to show def is independent of choice of φ_is.
Suppose we have overlapping coordinates φ_i'.
Then we have coordinate changes
$h_1 = \varphi_1'^{-1} \circ \varphi_1$, $h_2 = \varphi_2'^{-1} \circ \varphi_2$.

$$\begin{aligned} \tilde{f}' &:= \varphi_2'^{-1} \circ f \circ \varphi_1' = \varphi_2'^{-1} \circ \varphi_2 \circ \varphi_2^{-1} \circ f \circ \varphi_1 \circ \varphi_1^{-1} \circ \varphi_1' \\ &= h_2 \circ \tilde{f} \circ h_1^{-1} \end{aligned}$$

But since h_1 and h_2 are diffeomorphisms then $\tilde{f}'$ continuously differentiable $\iff$ $\tilde{f}$ is.

Def (Differential)
Let $f : M_1 \to M_2$ be continuously differentiable where M_i is n_i-dimensional and is a submanifold of $\mathbb{R}^{N_i}$. For any $p \in M_1$ and $v \in T_pM_1$, let γ be a curve s.t. $\gamma(0) = p$, $\gamma'(0) = v$. ($\gamma : (-\varepsilon, \varepsilon) \to M_1$)
The map $Df|_p : T_pM_1 \to T_{f(p)}M_2$, $v \mapsto Df_p(v) := \left.\dfrac{d}{dt}\right|_{t=0} f(\gamma(t))$
is called the differential of f.

Well-defined?

If γ represents $v \in T_p M_1$, i.e. $\left.\frac{d}{dt}\right|_{t=0} \gamma = v$, $\gamma(0)=p$,

then $f(\gamma(t))$ is a curve in M_2 and so $\left.\frac{d}{dt}\right|_{t=0} f(\gamma(t))$ is a vector in $T_{f(p)} M_2$.

Also: need to check Df is independent of choice of γ.

$\varphi : U \to V$ is a local parameterisation near p

$u(p) = \varphi^{-1}(p)$.

$D\varphi|_u : \mathbb{R}^n \to T_p M$ is a linear isomorphism.

Differentiate $\gamma = \varphi \circ \varphi^{-1} \circ \gamma$ at $t=0$.

$v = \gamma'(0) = D\varphi|_u \left.\frac{d}{dt}\right|_{t=0} (\varphi^{-1} \circ \gamma)$

Then $Df|_p (v) = \left.\frac{d}{dt}\right|_{t=0} (f \circ \varphi \circ \varphi^{-1} \circ \gamma)$

$$= D(f \circ \varphi)|_u \circ (\varphi^{-1} \circ \gamma)'(0)$$

$$= \left(D(f \circ \varphi)|_u \circ D\varphi|_p^{-1}\right)(v)$$

$\Rightarrow$ well-defined.

Prop

Let $f : M_1 \to M_2$ be a continuously differentiable mapping and $p \in M_1$. $Df|_p$ is a linear map and if φ_1 and φ_2 are local parameterisations for M_1 near p and for M_2 near $f(p)$, then

$$Df|_p\left(\frac{\partial \varphi_1}{\partial u_j}\right) = \sum_{i=1}^{n_2} \frac{\partial \tilde{f}_i}{\partial u_j} \frac{\partial \varphi_2}{\partial v_i}$$

for all $1 \leq j \leq \dim M_1$ where $p = \varphi_1(u)$, $n_2 = \dim M_2$ and $\tilde{f} = \varphi_2^{-1} \circ f \circ \varphi_1$

MATH 3113

24-10-17

Proof

$\varphi^{-1} \circ \varphi = id$

Differentiating and using that $D\varphi(e_j) = \frac{\partial \varphi}{\partial u_j}$

$$D\varphi^{-1}\left(\frac{\partial \varphi}{\partial u_j}\right) = D(\varphi^{-1} \circ \varphi)(e_j) = e_j$$

$$\begin{aligned} Df|_p\left(\frac{\partial \varphi_1}{\partial u_j}\right) &= D\left(\varphi_2 \circ \tilde{f} \circ \varphi_1^{-1}\right)\Big|_p\left(\frac{\partial \varphi_1}{\partial u_j}\right) \\ &= \left(D\varphi_2\big|_{\tilde{f}(u)} \circ D\tilde{f}\big|_u \circ D\varphi_1^{-1}\big|_p\right)\left(\frac{\partial \varphi_1}{\partial u_j}\right) \\ &= \left(D\varphi_2\big|_{\tilde{f}(u)} \circ D\tilde{f}\big|_u\right)(e_j) \\ &= D\varphi_2\big|_{\tilde{f}(u)}\left(\frac{\partial \tilde{f}}{\partial u_j}\right) \\ &= \sum_{i=1}^{n_2} \frac{\partial \tilde{f}_i}{\partial u_j}\frac{\partial \varphi_2}{\partial v_i} \end{aligned}$$

Since $Df|_p$ may be written as matrices in this form it is indeed a linear mapping. $\square$

Prop (Chain rule):
Let M_j, $j=1,2,3$, be submanifolds of $\mathbb{R}^{N_j}$, and $f: M_1 \to M_2$, $g: M_2 \to M_3$ differentiable, then $g \circ f$ is differentiable at $p \in M_1$ and its derivative is given by

$$D(g \circ f)|_p = Dg|_{f(p)} \circ Df|_p$$

$$: T_pM_1 \longrightarrow T_{g(f(p))}M_3.$$

Proof

Using local parameterisations we see that $\widetilde{g \circ f} = \tilde{g} \circ \tilde{f}$

$\Rightarrow g \circ f$ is continuously differentiable.

$Df|_p : T_pM_1 \longrightarrow T_{f(p)}M_2$

$Dg|_{f(p)} : T_{f(p)}M_2 \longrightarrow T_{g(f(p))}M_3$

$\Rightarrow$ mappings are well-defined.

Let $\gamma : (-\varepsilon, \varepsilon) \longrightarrow M_1$ be a curve with $\gamma(0) = p$, $\gamma'(0) = v$.
$f \circ \gamma : (-\varepsilon, \varepsilon) \rightarrow M_2$ is a curve in M_2 s.t.
$f \circ \gamma(0) = f(p)$, $(f \circ \gamma)' = Df|_p$

$$D(g \circ f)|_p (v) = \frac{d}{dt}(g \circ f \circ \gamma)(0)$$
$$= Dg|_{f(p)} (f \circ \gamma)'(0)$$
$$= Dg|_{f(p)} \circ Df|_p (v). \quad \square$$

Examples

i). Let $\gamma : I \rightarrow \mathbb{R}^3$ be a regular C^1 curve, injective. Parameterised by arc length
Then $T, N, B : tr\,\gamma \longrightarrow \mathbb{R}^3$.
Parameterisation: $\varphi_1 = \gamma$

$$\left.\frac{\partial \varphi_1}{\partial s}\right|_p = \dot{\gamma}(s_0) \qquad p = \gamma(s_0)$$

Parameterisation for $\mathbb{R}^3$: $\varphi_2(x, y\, z) = x\,T(s_0) + y\,N(s_0) + z\,B(s_0)$

$$T_p\mathbb{R}^3 = \text{span}\,\{T(s_0), N(s_0), B(s_0)\}$$

$\tilde{T} : I \rightarrow \mathbb{R}^3$, $\tilde{T}(s) = (\langle T(s), T(s_0)\rangle, \langle T(s), N(s_0)\rangle, \langle T(s), B(s_0)\rangle)$
$\tilde{T} = \varphi_2^{-1} \circ T \circ \varphi_1$

$$D\tilde{T}|_s = \frac{\partial \tilde{T}}{\partial s}(s_0) = (\langle \dot{T}(s_0), T(s_0)\rangle, \langle \dot{T}(s_0), N(s_0)\rangle, \langle \dot{T}(s_0), B(s_0)\rangle)$$
$$= (0, \kappa(s_0), 0)$$

MATH3113

25-10-17

Example

Take S^2 with atlas of 6 graphs as in hw 3.

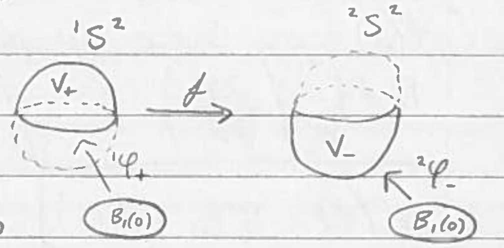
$f: S^2 \to S^2$, $x \mapsto -x$

(well-defined: $|x| = 1 \iff |-x| = 1$)

Continuously differentiable?

Pick $p = (x, y, z) \in S^2$.

w.l.o.g. assume $z > 0$.



We have local parameterisations

$\varphi_\pm : B_1(0) \to V_\pm$ $\qquad {}^i\varphi_\pm(u,v) = (u, v, \pm\sqrt{1-u^2-v^2})$

$V_+ = S^2 \cap \{(x,y,z) \in \mathbb{R}^3 \mid z > 0\}$

$V_- = S^2 \cap \{(x,y,z) \in \mathbb{R}^3 \mid z < 0\}$

$V_\pm$ is open.

Observe $p \in V_+$ and $f(p) = (-x, -y, -z) \in V_-$

In this case, $\tilde{f} : B_1(0) \to B_1(0)$

$$\begin{aligned}\tilde{f}(u,v) &= ({}^2\varphi_-^{-1} \circ f \circ {}^1\varphi_+)(u,v) \\ &= ({}^2\varphi_-^{-1} \circ f)(u, v, \sqrt{1-u^2-v^2}) \\ &= {}^2\varphi_-^{-1}(-u, -v, -\sqrt{1-u^2-v^2}) \\ &= (-u, -v)\end{aligned}$$

Clearly this is continuously differentiable.

$J\tilde{f} = -I$

$$Df|_p\left(\frac{\partial \varphi_+}{\partial u_j}\right) = \sum_{i=1}^{2} \frac{\partial \tilde{f}_i}{\partial u_j}\frac{\partial\, {}^2\varphi_-}{\partial u_i} = -\frac{\partial\, {}^2\varphi_-}{\partial u_j}$$

e.g. $p = (0,0,1) \Rightarrow f(p) = (0,0,-1)$

$$T_pS^2 = \text{span}\left\{\begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\end{pmatrix}\right\} = \text{span}\left\{\frac{\partial \varphi_+}{\partial u}, \frac{\partial \varphi_-}{\partial v}\right\}$$

$$T_{f(p)}S^2 = \text{span}\left\{\begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\end{pmatrix}\right\} = \text{span}\left\{\frac{\partial \varphi_+}{\partial u}, \frac{\partial \varphi_-}{\partial v}\right\}$$

$$Df|_p\left(\underbrace{\begin{pmatrix}1\\0\\0\end{pmatrix}}_{\in T_p{}^1S^2}\right) = \underbrace{-\begin{pmatrix}1\\0\\0\end{pmatrix}}_{\in T_{f(p)}{}^2S^2}, \quad Df|_p\left(\begin{pmatrix}0\\1\\0\end{pmatrix}\right) = -\begin{pmatrix}0\\1\\0\end{pmatrix}$$

$Df|_p : T_p{}^1S^2 \to T_{f(p)}{}^2S^2$

$M \subset \mathbb{R}^N$ submanifold

$p_0 \in \mathbb{R}^N \setminus M$ then the squared distance to p_0,

$d : M \to \mathbb{R}$, $p \mapsto |p - p_0|^2$, is C^1.

Let φ be a local parameterisation near $p \in M$,

$\varphi(u) = p$. Since $|p-p_0|^2 = \langle p - p_0, p - p_0 \rangle$

$$Dd\Big|_p\left(\frac{\partial \varphi}{\partial u_j}\right) = \frac{d}{dt}\Big|_{t=0}\left(\langle \varphi(u+te_j) - p_0, \varphi(u+te_j) - p_0 \rangle\right)$$

$\boxed{\gamma(t) = \varphi(u + te_j)}$

$$= \frac{\partial}{\partial u_j}\left(\langle \varphi(u) - p_0, \varphi(u) - p_0 \rangle\right)$$

$$= 2 \left\langle \frac{\partial \varphi}{\partial u_j}, \varphi(u) - p_0 \right\rangle$$

If $X = \sum_{i=1}^{n} X^i \frac{\partial \varphi}{\partial u_i}$ general tangent vector.

$$Dd\Big|_p(X) = \sum_{i=1}^{n} X^i \, Dd\Big|_p\left(\frac{\partial \varphi}{\partial x^i}\right)$$

$$= \sum_{i=1}^{n} X^i \, 2 \left\langle \frac{\partial \varphi}{\partial u_i}, \varphi(u) - p_0 \right\rangle$$

$$= 2 \langle X, \varphi(u) - p_0 \rangle.$$

Def

M_i is an n_i-dim submanifold of $\mathbb{R}^{N_i}$, then

$f : M_1 \to M_2$ is said to be

i) an immersion if $Df|_p : T_pM_1 \to T_{f(p)}M_2$ is injective $\forall p \in M_1$.

ii). an embedding if $f : M_1 \to f(M_1) \subset M_2$ is both an immersion and a homeomorphism.

iii). a submersion if $Df|_p : T_pM_1 \to T_{f(p)}M_2$ is surjective $\forall p \in M_1$.

iv). a diffeomorphism if it satisfies all of the above

MATH 3113

25-10-17

The definition of a diffeo is equivalent to f and f^{-1} exist and are continuously differentiable

If $\dim M_1 > \dim M_2$ then there are no immersions.
If $\dim M_1 < \dim M_2$ then there are no submersions.
$\Rightarrow$ the dimensions of two diffeomorphic manifolds are the same.

$\gamma : (a,b) \to \mathbb{R}^2$, γ is regular $\Rightarrow$ it is an immersion
$\uparrow$ a manifold, M_1

 $\leftarrow$ immersion but not embedding

 $\leftarrow$ embedding

Now consider regular surfaces.
- 2-dim submanifold of $\mathbb{R}^3$ usually denoted Σ
- we will have local parameterisations about $p = (x,y,z) \in \Sigma$

  $U \subset \mathbb{R}$ open, $V \subset \Sigma$ open

  $\varphi : U \to V$

  coordinates on U are (u,v), I will use $\partial_u = \partial_1 = D\varphi|_p (e_1) = \dfrac{\partial \varphi}{\partial u}$,
  $\partial_v = \partial_2 = D\varphi|_p (e_2) = \dfrac{\partial \varphi}{\partial v}$, $\{\partial_u, \partial_v\}$ form a basis of $T_p \Sigma$.

<u>First Fundamental Form</u>

For $X \in T_p \Sigma \subset \mathbb{R}^3$ we can define $|X|_p^2 = |X|_{\mathbb{R}^3}^2$.
Suppose X is the tangent to a curve $\gamma : (-\varepsilon, \varepsilon) \to \mathbb{R}^3$, $\gamma(0) = p$.
at $t=0$

$$X = \gamma'(0) = \frac{d}{dt}\Big|_{t=0} \varphi(\varphi^{-1} \circ \gamma(t)) \quad \text{write } \varphi^{-1} \circ \gamma(t) = (u(t), v(t))$$

$$= D\varphi|_p \circ \frac{d}{dt}\Big|_{t=0} (u(t), v(t))$$

$\Rightarrow X = u'(0)\,\partial_u + v'(0)\,\partial_v$ using definition above.

$$|X|_p^2 = \langle X, X \rangle_{\mathbb{R}^3} = u'(0)^2 \langle \partial_u, \partial_u \rangle + 2u'(0)v'(0) \langle \partial_u, \partial_v \rangle + v'(0)^2 \langle \partial_v, \partial_v \rangle$$

$$= u'(0)^2 E + 2 u'(0) v'(0) F + v'(0)^2 G$$

where $E = \langle \partial_u, \partial_u \rangle$, $F = \langle \partial_u, \partial_v \rangle$, $G = \langle \partial_v, \partial_v \rangle$.

<u>Def</u> (1st Fundamental Form)

Σ a surface, $p \in \Sigma$.

Define the quadratic form

$I_p : T_p \Sigma \longrightarrow \mathbb{R}$, $I_p(X) = \langle X, X \rangle$

I_p is the <u>First Fundamental Form</u> on Σ.

The component functions E, F, G are called the components of the 1st fundamental form in parameterisation φ.

$X = a^1 \partial_1 + a^2 \partial_2$

$$I_p(X) = (a^1 \; a^2) \begin{pmatrix} E & F \\ F & G \end{pmatrix} \begin{pmatrix} a^1 \\ a^2 \end{pmatrix} = \sum_{i,j=1}^{2} a^i g_{ij} a^j$$

where $g_{ij} = \{ \langle \partial_i, \partial_j \rangle \}_{i,j=1,2} = \begin{pmatrix} E & F \\ F & G \end{pmatrix}$

is called the metric in local coordinates.

MATH 3113

31-10-17

$X \in T_p\Sigma$

$X = X^1\partial_1 + X^2\partial_2$

$\quad = X^1\partial_u + X^2\partial_v \qquad \left[\partial_u = \dfrac{\partial\varphi}{\partial u}\right]$

$I_p(X) = \langle X, X \rangle$

$\quad = (X^1)^2 E + 2X^1X^2F + (X^2)^2 G$

Quadratic Form

$g = \begin{pmatrix} E & F \\ F & G \end{pmatrix} \qquad I_p(X) = \sum_{i,j=1}^{2} X^i g_{ij} X^j$

$Y \in T_p\Sigma$

$\langle X, Y \rangle|_p = \frac{1}{2}\left(I_p(X) + I_p(Y) - I_p(X-Y)\right)$

$\quad = \sum_{i,j=1}^{2} X^i g_{ij} Y^j$

Can define angle:

$\measuredangle(X,Y) = \arccos\left(\dfrac{\langle X, Y\rangle|_p}{\sqrt{I_p(X) I_p(Y)}}\right)$

Also: For a curve $\gamma : I \to \Sigma$

$l(\gamma) = \int_I \sqrt{I_{\gamma(t)}(\gamma'(t))}\,dt$

Remark

Forget R^3, just define C^1 inner product functions $g_{ij}(u)$ on Σ then we could define all previous quantities.

Orientability and the Gauss map

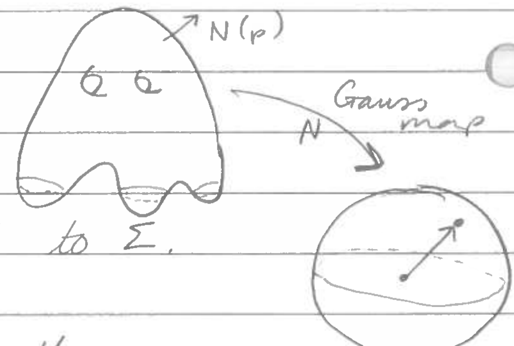
Note: from now on Σ is C^2.

Def

Let Σ be a surface, $p \in \Sigma$.

i). A unit normal at p is a vector $N(p) \in \mathbb{R}^3$ s.t.
$|N(p)| = 1$ and $(\text{span}\{N(p)\})^{\perp} = T_p\Sigma$

ii). A Gauss map for Σ is a continuously differentiable map $N : \Sigma \to S^2$ s.t. $\forall p \in \Sigma$, $N(p)$ is a unit normal to Σ.



iii). If there exists a Gauss map, then we call Σ orientable, and if we choose a Gauss map for Σ, this is called an orientation.

Remark

Locally a Gauss map always exists. Given a parameterisation φ then

$$N(p) = \frac{\left(\frac{\partial \varphi}{\partial u} \times \frac{\partial \varphi}{\partial v}\right)}{\left|\frac{\partial \varphi}{\partial u} \times \frac{\partial \varphi}{\partial v}\right|}$$

Example

The sphere: $S^2 = f^{-1}(0)$, $f(x,y,z) = x^2+y^2+z^2-1$

$N(p) = \frac{\nabla f|_p}{|\nabla f|_p|}$, $\nabla f = Df^T$, $f : \mathbb{R}^3 \to \mathbb{R}$.

Suppose $\exists X \in T_p\Sigma$ s.t. X is not $\perp$ to ∇f,

$\gamma(0) = p$, $\gamma'(0) = X$. $\gamma : I \to \Sigma$

MATH3113

31-10-17

$$0 = \frac{d}{dt}\Big|_{t=0} f(\gamma(t)) = \langle \nabla f, \gamma'(0) \rangle$$
$$= \langle \nabla f, X \rangle$$

$N(p) = p.$

We now consider $DN|_p : T_p\Sigma \to T_{N(p)}S^2$.
If fact $T_{N(p)}S^2 \equiv T_p\Sigma$ since on the sphere $N(p) = p$.

Def (Weingarten map)
Let Σ be an oriented surface. Define the map
$W|_p := -DN_p : T_p\Sigma \longrightarrow T_p\Sigma$

Prop ①
The Weingarten map is self adjoint.
$\Leftrightarrow \forall X, Y \in T_p\Sigma \quad \langle W_p(X), Y \rangle = \langle X, W_p(Y) \rangle$

Proof
Consider in basis ∂_u, ∂_v
We have $\langle N, \frac{\partial\varphi}{\partial u} \rangle = 0 = \langle N, \partial_v \rangle$.

$$\langle \partial_u, W_p(\partial_v) \rangle = \langle \frac{\partial\varphi}{\partial u}, \frac{-\partial N}{\partial v} \rangle$$
$$= +\langle \frac{\partial^2\varphi}{\partial u \partial v}, N \rangle = -\langle \frac{\partial\varphi}{\partial v}, \frac{\partial N}{\partial u} \rangle$$
$$= (\partial_v, W|_p(\partial_u) \rangle$$
$\square$

The Second Fundamental Form
We now know that $W|_p$ is self adjoint
w.r.t. $\langle \cdot, \cdot \rangle$
$\Rightarrow X \longmapsto \langle X, W_p(X) \rangle$
is a quadratic form.

Def
Let Σ be an oriented surface. The quadratic form $\mathrm{II}_p : T_p\Sigma \longrightarrow \mathbb{R}$

$$\mathrm{II}_p(X) = \langle X, W|_p(X)\rangle = -\langle X, DN_p(X)\rangle$$

is called the second fundamental form of Σ at p.
Identically as we saw earlier, if
$X = X^1\partial_1 + X^2\partial_2$ is any element of $T_p\Sigma$ ($p = \varphi(q)$, a parameter)
then $\mathrm{II}_p(X) = (X^1)^2 e(q) + 2X^1X^2 f(q) + (X^2)^2 g(q)$
where

$$e(q) = \langle \partial_u, W|_p(\partial_u)\rangle$$
$$f(q) = \langle \partial_v, W|_p(\partial_u)\rangle$$
$$g(q) = \langle \partial_v, W|_p(\partial_v)\rangle$$

$$A = \begin{pmatrix} e & f \\ f & g \end{pmatrix}, \qquad \mathrm{II}_p(X) = \sum_{\substack{i=1\\j=1}}^{2} X^i A_{ij} X^j$$

By abuse of notation, $Y = Y^1\partial_1 + Y^2\partial_2$

$$\underset{\uparrow \text{ bilinear form}}{A(X,Y)} = \sum_{\substack{i=1\\j=1}}^{2} Y^i X^j A_{ij}.$$

Corollary of (☼)

$$e(q) = \left\langle \frac{\partial^2\varphi}{\partial u^2}, N|_p \right\rangle \qquad [p = \varphi(q)]$$

$$f(q) = \left\langle \frac{\partial^2\varphi}{\partial u\partial v}, N|_p \right\rangle$$

$$g(q) = \left\langle \frac{\partial^2\varphi}{\partial v^2}, N|_p \right\rangle.$$

Theorem
Let V be a 2-dim Euclidean vector space and $T: V \longrightarrow V$ a self adjoint mapping w.r.t. scalar product on V. Then:
i) $\exists$ an orthogonal basis $\{e_1, e_2\}$ consisting of eigenvectors of T and the corresponding eigenvalues λ_1, λ_2 are real.

MATH 3113

31-10-17

ii). Define $Q(v) = \langle Tv, v \rangle$ then the eigenvalues of T are given by $\lambda_1 = \min\{Q(v) \mid v \in V, |v| = 1\}$, $\lambda_2 = \max\{Q(v) \mid v \in V, |v| = 1\}$.

Principle curvatures

Def

Let Σ be an oriented surface, $p \in \Sigma$

i). the eigenvalues $\kappa_1(p), \kappa_2(p)$ of $W|_p$ are called the principle curvatures of Σ at p.
If $\kappa_1 \neq \kappa_2$ then the corresponding eigenvectors are called the principle directions.

ii). The Gauss curvature of Σ at p is
$K := \det W|_p = \kappa_1(p)\kappa_2(p)$

iii). The mean curvature of Σ at p is
$H := \frac{\kappa_1(p) + \kappa_2(p)}{2} = \frac{1}{2} \operatorname{tr} W|_p$.

W.l.o.g. $\kappa_1 \leq \kappa_2$

$\kappa_1 = \min\{\mathrm{II}_p(X) \mid X \in T_p\Sigma, \ \mathrm{I}_p(X) = 1\}$

$\kappa_2 = \max\{\mathrm{II}_p(X) \mid X \in T_p\Sigma, \ \mathrm{I}_p(X) = 1\}$.

$g : \mathbb{R}^2 \longrightarrow \mathbb{R}$

define the hessian as

$$\operatorname{Hess}(g) := \begin{pmatrix} \frac{\partial^2 g}{\partial u^2} & \frac{\partial^2 g}{\partial u \partial v} \\ \frac{\partial^2 g}{\partial u \partial v} & \frac{\partial^2 g}{\partial v^2} \end{pmatrix}$$

Prop

Let Σ be an oriented surface, $p \in \Sigma$. Then $\exists$ a neighbourhood U of 0 in $T_p\Sigma$, V of p in Σ and a twice differentiable function $g : U \to \mathbb{R}$ st.
$V = \{p + X + g(X)N(p) \mid X \in U\}$ where g satisfies

$g(0)=0$ and $Dg|_0 = 0$.

Then:

$$X^T \cdot \mathrm{Hess}(g)|_0 \cdot X = \mathrm{II}_p(X)$$

for $X \in T_p\Sigma$.

(Locally Σ is a graph and $A = \mathrm{Hess}(g)|_0$.)

Proof (part of) w.l.o.g. $p=0$.

Suppose we have a graph function, then locally we have a parameterisation

$$\varphi(X) = (X^1 e_1 + X^2 e_2 + g(X) e_3)$$

$$\partial_1 = e_1 + \frac{\partial g}{\partial x_1} e_3$$

$$\partial_2 = e_2 + \frac{\partial g}{\partial x_2} e_3$$

$$N = \frac{-(Dg - e_3)}{\sqrt{1+|Dg|^2}} \quad \text{where } Dg = \frac{\partial g}{\partial x_1} e_1 + \frac{\partial g}{\partial x_2} e_2$$

$$\langle N, \partial_i \rangle = 0 \quad i=1,2.$$

A is the matrix with entries

$$A_{ij} = \left\langle \frac{\partial^2 \varphi}{\partial x^i \partial x^j}, N \right\rangle$$

$$= \left\langle \frac{\partial^2 g}{\partial x^i \partial x^j} e_3, \frac{-(Dg - e_3)}{\sqrt{1+|Dg|^2}} \right\rangle$$

$$= \frac{\left(\frac{\partial^2 g}{\partial x^i \partial x^j}\right)}{\sqrt{1+|Dg|^2}}$$

But at $p = \varphi(0)$, $Dg = 0$

$$\Rightarrow A_{ij} = \frac{\partial^2 g}{\partial x^i \partial x^j} = \mathrm{Hess}(g) \longleftarrow \text{graph, not metric}$$

$\square$

MATH 3113

31-10-17

Remarks / Def

i). If $\kappa_1|_p = \kappa_2|_p$ then p is called an umbillic point. ("curvature like on S^2")

ii). If $\kappa_1 = \kappa_2 = 0$ then this is called a flat point.

iii). If $K = \kappa_1\kappa_2 > 0$ and Σ is entirely on one side of $T_p\Sigma$ then p is an elliptical point.

iv). If $K = \kappa_1\kappa_2 < 0$ p is called a hyperbolic point.

v). If $K = 0$ but one $\kappa_i \neq 0$ then this is a parabolic point.

vi). If $H = 0$ $\forall p \in \Sigma$ then Σ is called a minimal surface.

Examples

Consider local graphs g st. $g(0) = 0$, $Dg(0) = 0$, then

i). 0 is a flat point of the graph, $g = x^4 + y^4$

ii). 0 is umbillic for graph of $g = x^2 + y^2$ (also elliptic) 

iii). 0 is hyperbolic for $g = x^2 - y^2$ 

Remark Change of orientation.

Suppose we take $-N$ instead of N.

$I \longmapsto I$ change of normal, $N \longmapsto -N$,

$I\!I \longmapsto -I\!I$, $K \longmapsto K$, $H \longmapsto -H$

Calculations in Local Coordinates

Suppose at $p = \varphi(q)$, φ a local parameterisation, we have $Y, X \in T_p\Sigma$ where $Y = Y^1\partial_1 + Y^2\partial_2$ and $X = X^1\partial_1 + X^2\partial_2$. Define $\tilde{W}$ as W in these coordinates.
(by abuse of notation)

Then, writing X as $\begin{pmatrix} X^1 \\ X^2 \end{pmatrix}$ we have

$$X^T A Y = \langle X, W|_p(Y)\rangle = X^T g \cdot \tilde{W} Y$$

$$\Rightarrow \tilde{W} = g^{-1}A$$

[Recall: for $X, Y \in T_p\Sigma$, $\langle X, Y\rangle = \sum_{i,j=1}^{2} X^i g_{ij} Y^j = X^T g Y = g(X,Y)$

g = inner product on $\mathbb{R}^3$ restricted to $T_p\Sigma$ and written wrt basis $\partial_u \partial_v$]

01-11-17 Last time:

$N : \Sigma \longrightarrow S^2$, $p \in \Sigma$

$W_p : T_p\Sigma \to T_p\Sigma$

$W_p := -DN|_p$

$\mathrm{II}_p : T_p\Sigma \longrightarrow \mathbb{R}$ (quadratic form)

$\mathrm{II}_p(X) = \langle X, W_p(X) \rangle$

$X = X^1\partial_1 + X^2\partial_2$, $\tilde{X} = \begin{pmatrix} X^1 \\ X^2 \end{pmatrix}$, $A = \begin{pmatrix} e & f \\ f & g \end{pmatrix}$

$\mathrm{II}_p(x) = \tilde{X}^T A \tilde{X} = \sum_{i,j} X^i A_{ij} X^j$

We also have a bilinear form

$\left[Y = Y^1\partial_1 + Y^2\partial_2 \right.$, $\left. \tilde{Y} = \begin{pmatrix} Y^1 \\ Y^2 \end{pmatrix} \right]$

$A(X,Y) = \sum_{i,j=1}^{2} Y^i A_{ij} X^j = \tilde{Y}^T A \tilde{X}$

principle curvatures $\varkappa_1(p), \varkappa_2(p)$ eigenvalues of W_p

$H_p = \frac{1}{2}(\varkappa_1(p) + \varkappa_2(p))$ mean curvature

$= \frac{1}{2} \operatorname{tr} DW$ (metric trace)

$K_p = \varkappa_1(p)\varkappa_2(p) = \det W_p$ Gauss curvature.

Now calculate in local coordinates.

Write everything in basis $\partial_u = \partial_1$, $\partial_v = \partial_2$.

Write $\tilde{W} : T_p\Sigma \to T_p\Sigma$ for W in these coordinates

(this is a matrix mapping)

$\tilde{W} = g^{-1} \cdot A$ $\qquad g = \begin{pmatrix} E & F \\ F & G \end{pmatrix}$, $A = \begin{pmatrix} e & f \\ f & g \end{pmatrix}$

WARNING: these g's are different!!!

sometimes write $\tilde{g} = \begin{pmatrix} E & F \\ F & G \end{pmatrix}$

$\tilde{g}^{-1} = \frac{1}{EG - F^2} \begin{pmatrix} G & -F \\ -F & E \end{pmatrix}$

MATH 3113

01-11-17

In local coordinates : $\varphi : U \to V$ a local parameterisation

$\varphi(q) = p \in V$

i). tangent vectors $\partial_u = \frac{\partial \varphi}{\partial u}\Big|_q$, $\partial_v = \frac{\partial \varphi}{\partial v}\Big|_q$

ii). $E = \langle \partial_u, \partial_u \rangle$, $F = \langle \partial_u, \partial_v \rangle$, $G = \langle \partial_v, \partial_v \rangle$

entries of g. Entries of A: $e = \langle \frac{\partial^2 \varphi}{\partial u^2}, N \rangle$, f, g etc.

iii). Gauss map $N_p = \dfrac{\partial_u \times \partial_v}{|\partial_u \times \partial_v|}$

iv). The entries of $\tilde{\omega} = \tilde{g}^{-1} \circ A$

v). $K_p = \det \tilde{\omega} = \dfrac{\det A}{\det \tilde{g}} = \dfrac{eg - f^2}{EG - F^2}$ (Gauss curvature)

vi). Mean curvature:

$$H = \frac{1}{2} \operatorname{tr} \tilde{\omega} \qquad \operatorname{tr} A^T B = \sum_{i,j=1}^{2} A_{ij} B_{ij}$$

$$= \frac{1}{2} \cdot \frac{Ge - 2Ff + Eg}{EG - F^2}$$

vii). Principle curvatures are eigenvalues of $\tilde{\omega}$

$\Leftrightarrow$ solutions of $x^2 - \operatorname{tr} \tilde{\omega} x + \det \tilde{\omega} = \det(\tilde{\omega} - x I)$

$x_1 = H - \sqrt{H^2 - K}$, $x_2 = H + \sqrt{H^2 - K}$

viii). Principle directions are given by $\tilde{e}_1, \tilde{e}_2$ where

$\tilde{\omega} \tilde{e}_i = x_i \tilde{e}_i$, $i = 1, 2$.

Example

Cylinder. $C = \{ (x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 = 1 \}$

$\varphi : (-\pi, \pi) \times \mathbb{R} \longrightarrow \mathbb{R}^3$

$(u, v) \longmapsto (\cos u, \sin u, v)$.

i). $\partial_u = \begin{pmatrix} \sin u \\ \cos u \\ 0 \end{pmatrix}$, $\partial_v = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

ii). $\tilde{g} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

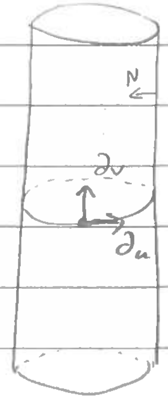
iii). $N_p = \dfrac{\partial_u \times \partial_v}{|\partial_u \times \partial_v|} = -\begin{pmatrix} \cos u \\ \sin u \\ 0 \end{pmatrix}$

iv). $e = \left\langle \frac{\partial^2 \varphi}{\partial u \partial u}, N \right\rangle = \left\langle \begin{pmatrix} -\cos u \\ -\sin u \\ 0 \end{pmatrix}, \begin{pmatrix} -\cos u \\ -\sin u \\ 0 \end{pmatrix} \right\rangle = 1$

$f = \langle 0, N \rangle = 0$

$g = \langle 0, N \rangle = 0$

$$\Rightarrow A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$



v). $K = \frac{eg - f^2}{EG - F^2} = 0$

vi). $H = \frac{1}{2} \cdot \frac{eG - 2fF + Eg}{EG - F^2} = \frac{1}{2}$

vii). $\kappa_1 = H - \sqrt{H^2 - K} = \frac{1}{4} - \sqrt{1/4} = 0$

$\kappa_2 = \frac{1}{2} + \sqrt{1/4} = 1$

viii). We observe that

$\tilde{\omega} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ already in diagonal form.

So in basis ∂_u, ∂_v the principle directions are e_1, e_2

i.e. ∂_u, ∂_v are principle directions.

Recall:

Parallelogram $P(a,b)$ defined by $a, b \in \mathbb{R}^2$,

$\text{area}(P(a,b)) = \sqrt{\det \begin{pmatrix} \langle a,a \rangle & \langle b,a \rangle \\ \langle a,b \rangle & \langle b,b \rangle \end{pmatrix}}$

Def (Surface Element)

Let Σ be a surface, $\varphi : U \to V \subset \Sigma$ a local parameterisation,

then: i). the surface element w.r.t. φ is

$dS = \sqrt{\det \tilde{g}}\, du\, dv = \sqrt{EG - F^2}\, du\, dv = |\partial_u \times \partial_v|\, du\, dv$

ii). If $W \subset V$ is open and $\tilde{W} = \varphi^{-1}(W)$ then define

$\text{area}(W) = \int_{\tilde{W}} dS = \int_{\tilde{W}} \sqrt{EG - F^2}\, du\, dv.$

MATH 3113

01-11-17

In homework we parameterised the upper half sphere

$\varphi: (0,\pi)\times(0,\pi)$ $\qquad \tilde{g} = \begin{pmatrix} 1 & 0 \\ 0 & \sin^2\theta \end{pmatrix}$

calculate area of sphere

$$\text{area}(S^2) = 2\int_0^{\pi}\int_0^{\pi} \sqrt{\sin^2\theta}\, d\theta\, d\alpha$$

$$= 2\pi \int_0^{\pi} \sin\theta\, d\theta = 2\pi \cos\theta \Big|_0^{\pi} = 4\pi$$

14-11-17

<u>Isometry</u>

<u>Def</u>

Let Σ_1, Σ_2 be surfaces with first fundamental forms I_1, I_2.

(i) a diffeomorphism $\phi: \Sigma_1 \longrightarrow \Sigma_2$ is called an isometry if $I_{1,p}(X) = I_{2,\phi(p)}(D\phi|_p(X))$.
If such an isometry exists, Σ_1, Σ_2 are isometric.

(ii) Let $V_1 \subset \Sigma_1$ be open, $p \in V_1$, then a map $\phi_1: V_1 \longrightarrow \Sigma_2$ is called a local isometry if $\exists$ an open set $V_2 \subset \Sigma_2$ s.t. $\phi: V_1 \rightarrow V_2$ is an isometry.
Similarly Σ_1, Σ_2 are locally isometric if $\forall p \in \Sigma_1$, $\exists$ a local isometry $\phi^1: V_1^{\ni p} \longrightarrow \Sigma_2$ and $\forall q \in \Sigma_2$ $\exists$ a local isometry $\phi^2: V_2^{\ni q} \rightarrow \Sigma_1$.

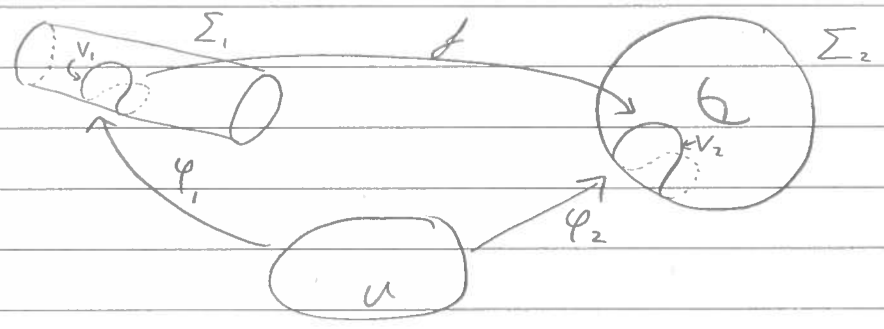
(iii) A surface is called flat if $\forall p \in \Sigma$ $\exists$ a local isometry $\phi: V_1^{\ni p} \longrightarrow \mathbb{R}^2$.

<u>Prop</u>

helpful for hw6 q'6

Let Σ_1, Σ_2 be surfaces and $f: \Sigma_1 \rightarrow \Sigma_2$ a continuously differentiable bijective local isometry $\forall p \in \Sigma_1$. Then f is an isometry.

Theorem
Let $\varphi_j : U \longrightarrow V_j \subset \Sigma_j$ be local parameterisations of surfaces Σ_1, Σ_2, and $f := \varphi_2 \circ \varphi_1^{-1} : V_1 \longrightarrow V_2$.
Then the map f is an isometry iff
$E_1 = E_2$, $F_1 = F_2$, $G_1 = G_2$.
(E_i, F_i, G_i from first fundamental form).



Proof:
$p \in V_1$, $X \in T_p \Sigma_1$
Let γ be a curve representing X.
$\gamma(t) = \varphi_1(u(t), v(t))$
$X = u'(0)\partial_{1,u} + v'(0)\partial_{1,v}$
[$\Leftarrow$]

$$Df|_p(X) = \frac{d}{dt}\Big|_{t=0} f \circ \gamma = \frac{d}{dt}\Big|_{t=0} f \circ \varphi_1(u(t), v(t))$$
$$= \frac{d}{dt}\Big|_{t=0} \varphi_2(u(t), v(t)) = u'(0)\partial_{2,u} + v'(0)\partial_{2,v}$$

$$\begin{aligned} I_{1,p}(X) &= \langle u'(0)\partial_{1,u} + v'(0)\partial_{1,v}, u'(0)\partial_{1,u} + v'(0)\partial_{1,v} \rangle \\ &= (u'(0))^2 E_1 + 2v'(0)u'(0)F_1 + (v'(0))^2 G_1 \\ &= (u'(0))^2 E_2 + 2v'(0)u'(0)F_2 + (v'(0))^2 G_2 \\ &= \langle u'(0)\partial_{2,u} + v'(0)\partial_{2,v}, u'(0)\partial_{2,u} + v'(0)\partial_{2,v} \rangle \\ &= I_{2,f(p)}(Df(X)) \end{aligned}$$

[$\Rightarrow$]
Suppose f is an isometry.
Choose $X = \partial_{1,u}$
$$E_1 = I_{1,p}(\partial_{1,u}) = I_{2,f(p)}(Df(\partial_{1,u})) = I_{2,f(p)}(\partial_{2,u}) = E_2$$

14-11-17

$G_1 = G_2$ follows identically $(X = \partial_{1,v})$

$F_1 = F_2$ follows from polarisation identity

$$F_1 = \langle \partial_{1,u}, \partial_{1,v} \rangle = \frac{1}{2}\left(I_1(\partial_{1,u} + \partial_{1,v}) - I_1(\partial_{1,u}) - I_1(\partial_{1,v})\right).$$

Given $V_1 \subset \Sigma_1$ and $V_2 \subset \Sigma_2$ and a diffeo $f: V_1 \to V_2$, then if $\varphi: U \to V_1$ is a parameterisation, then $f \circ \varphi : U \to V_2$ is a parameterisation of V_2.

So f is an isometry $\Leftrightarrow$ $E_1 = E_2$, $F_1 = F_2$, $G_1 = G_2$ in these coordinates. $\square$

Examples

(i) In hw we calculated E, F, G for half sphere for $\varphi = $ (trig. functions), $(\theta, \alpha) \in (-\pi/2, \pi/2)^2$

$E = 1$, $F = 0$, $G = \sin^2\theta$

This is not an isometry.

(ii) Previously we calculated the 1st fundamental form for a cylinder; $E = 1$, $F = 0$, $G = 1$.

This is a local isometry

$\varphi: \mathbb{R}^2 \to C$, $(x, y) \mapsto (\cos x, \sin x, y)$

(iii) Consider $K_\sigma = \{(r, \omega, z) \in \mathbb{R}^3 \mid z = ar^2\}$

(r, ω) are polar coordinates on $\mathbb{R}^2$.

Local parameterisation:

$\varphi(r, \omega) = (r\cos\omega, r\sin\omega, ar)$

on $\mathbb{R}_{>0} \times (-\pi, \pi)$

In these coordinates, 1st fundamental form is given by $\tilde{g} = \begin{pmatrix} 1 + a^2 & 0 \\ 0 & r^2 \end{pmatrix}$

Let $\hat{r} = \sqrt{1+a^2}\, r$, $\hat{\omega} = \omega / \sqrt{1+a^2}$

Consider new parameterisation:

$$\hat{\varphi}(\hat{r}, \hat{\omega}) = \left(\frac{\hat{r}}{\sqrt{1+a^2}} \cos(\sqrt{1+a^2}\,\hat{\omega}), \frac{\hat{r}}{\sqrt{1+a^2}} \sin(\sqrt{1+a^2}\,\hat{\omega}), \frac{a\hat{r}}{\sqrt{1+a^2}}\right)$$

Calculate $\hat{E}, \hat{F}, \hat{G}$ w.r.t. $\hat{\varphi}$,
then $\hat{E} = 1$, $\hat{F} = 0$, $\hat{G} = \hat{r}^2$.
These are independent of a (the 'gradient' of cone)
(away from 0).
This is an isometry.

Remarks
1). We can use the above theorem to prove that the given f is an isometry.
2). It may be hard to find such an isometry.
3). To prove there is no isometry we look for quantities which "are invarient under isometries".

Def
A vector field X of class C^k on a surface patch $V \subset \Sigma$ is a k-times continuously differentiable function
$X : V \longrightarrow \mathbb{R}^3$ s.t. $\forall p \in V$ $X|_p \in T_p\Sigma$.
We denote the set of such vector fields (v.f.s) is denoted $\mathfrak{X}^k(V)$.
Given the parameterisation $\varphi : U \longrightarrow V$ we will mostly consider X as $X = X^u(p)\partial_u + X^v(p)\partial_v$.

$X \in \mathfrak{X}^k(V)$

Given $f \in C^l(V)$, $X \in \mathfrak{X}^k(V)$ we may consider $fX \in \mathfrak{X}^m(V)$ where $(fX)(p) := f(p)X(p)$ and $m = \min\{l, k\}$.

We may also define $Xf \in C^r(V)$
$$(Xf)(p) := Df|_p(X) = \frac{d}{dt}\Big|_{t=0} f(\gamma) \quad (\gamma'(0) = X)$$

and $r = \min\{k, l-1\}$

MATH 3113

14-11-17

<u>Derivatives of v.f.s. w.r.t. v.f.s:</u>

In $\mathbb{R}^3$ ✓ : We may define

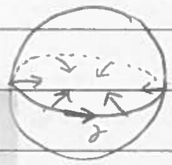
$$(\nabla_X^{\mathbb{R}^3} Y)(p) := DY|_p (X(p))$$

$$:= \frac{d}{dt}\Big|_{t=0} Y(\gamma(t)) \quad \text{(where } \gamma \text{ as above)}$$

We want a derivative of v.f.s on Σ.

Problem: If we use $\nabla^{\mathbb{R}^3}$ then $\nabla_X^{\mathbb{R}^3} Y|_p$ need not be in $T_p\Sigma$.

e.g. consider the equator circle contained in S^2.



$$\nabla_{\dot\gamma}^{\mathbb{R}^3} \dot\gamma = \ddot\gamma = \kappa n \notin T_p S^2$$

<u>Def</u>

Let Σ be an oriented surface, $X, Y \in \mathfrak{X}^k(\Sigma)$. Then the covarient derivative of Y w.r.t. X is the v.f $\nabla_X Y \in \mathfrak{X}^{k-1}(\Sigma)$ defined by

$$(\nabla_X Y)(p) = (\nabla_X^{\mathbb{R}^3} Y)(p) - \langle (\nabla_X^{\mathbb{R}^3} Y)(p), N(p) \rangle N(p)$$

$$= ((\nabla_X^{\mathbb{R}^3} Y)(p))^T \longleftarrow \text{orthog. proj. to } T_p\Sigma$$

<u>Lemma</u>

Let $X, Y, X_1, X_2, Y_1, Y_2, Z \in \mathfrak{X}^k(\Sigma)$, $f \in C^k(\Sigma)$, then:

(i) $\nabla_{X_1+X_2} Y = \nabla_{X_1} Y + \nabla_{X_2} Y$

$\nabla_{fX} Y = f \nabla_X Y$

(ii) $\nabla_X (Y_1 + Y_2) = \nabla_X Y_1 + \nabla_X Y_2$

$\nabla_X (fY) = X(f) Y + f \nabla_X Y$

(iii) $X(\langle Y, Z \rangle) = \langle \nabla_X Y, Z \rangle + \langle Y, \nabla_X Z \rangle$

Proof

The map $(X,Y) \longmapsto \nabla_X^{\mathbb{R}^3} Y$ is linear in the first variable and "a derivative in the second".

$\Rightarrow$ (i) for $\nabla^{\mathbb{R}^3}$ and $D(Y_1+Y_2)(X) = DY_1(X) + DY_2(X)$,

$D(fY)(X) = Df(X)Y + fDY(X)$.

$\Rightarrow$ (i) follows since projection is a linear mapping.

(ii) follows " " " " " "

(iii) :

$$X(\langle Y, Z\rangle) = D(\langle Y, Z\rangle)(X)$$
$$= \langle DY(X), Z\rangle + \langle Y, DZ(X)\rangle$$
$$= \langle \nabla_X^{\mathbb{R}^3} Y, Z\rangle + \langle Y, \nabla_X^{\mathbb{R}^3} Z\rangle$$

Recall now $X, Y, Z \in \mathfrak{X}^k(\Sigma)$

$\langle \nabla_X^{\mathbb{R}^3} Y + \lambda N, Z\rangle = \langle \nabla_X^{\mathbb{R}^3} Y, Z\rangle$

Choose $\lambda = -\langle N(p), \nabla_X^{\mathbb{R}^3} Y\rangle$

$$X(\langle Y, Z\rangle) = \langle \nabla_X Y, Z\rangle + \langle Y, \nabla_X Z\rangle$$

$\square$

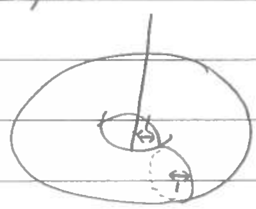
[Technically:

$\nabla : \mathfrak{X}^k(\Sigma) \times \mathfrak{X}^k(\Sigma) \to \mathfrak{X}^{k-1}(\Sigma)$]

Example : The Torus

We consider the param. $\varphi : U = (0,\pi)^2 \longrightarrow V$

$(u,v) \longmapsto ((2+\cos u)\cos v, (2+\cos u)\sin v, \sin u)$



$$\partial_u = \frac{\partial \varphi}{\partial u} = \begin{pmatrix} -\sin u \cos v \\ -\sin u \sin v \\ \cos u \end{pmatrix}$$

$$\partial_v = \frac{\partial \varphi}{\partial v} = (2+\cos u)\begin{pmatrix} -\sin v \\ \cos v \\ 0 \end{pmatrix}$$

Want to calculate $\nabla_{\partial_u} \partial_u$ etc.

$$\nabla_{\partial_u}^{\mathbb{R}^3} \partial_u = (D\partial_u)(\partial_u) = (D\partial_u)(D\varphi(e_1))$$
$$= D(\partial_u \circ \varphi)(e_1) = \frac{\partial^2 \varphi}{\partial u^2}$$

$$= \begin{pmatrix} -\cos u \cos v \\ -\cos u \sin v \\ -\sin u \end{pmatrix}$$

MATH 3113

14-11-17

Identically:

$$\nabla^{\mathbb{R}^3}_{\partial u} \partial v = -\sin u \begin{pmatrix} -\sin v \\ \cos v \\ 0 \end{pmatrix}$$

$$\nabla^{\mathbb{R}^3}_{\partial v} \partial v = (2+\cos u)\begin{pmatrix} -\cos v \\ -\sin v \\ 0 \end{pmatrix}$$

$$N = \frac{\partial u \times \partial v}{|\partial u \times \partial v|} = -\begin{pmatrix} \cos u \cos v \\ \cos u \sin v \\ \sin u \end{pmatrix}$$

So $\langle \nabla^{\mathbb{R}^3}_{\partial u} \partial u, N \rangle = 1$

$\langle \nabla^{\mathbb{R}^3}_{\partial u} \partial v, N \rangle = 0$

$\langle \nabla^{\mathbb{R}^3}_{\partial v} \partial v, N \rangle = (2+\cos u)\cos u$

$$\nabla_{\partial u} \partial u = 0$$

$$\nabla_{\partial v} \partial u = \nabla_{\partial u} \partial v = \frac{-\sin u}{2+\cos u} \partial v$$

$$\nabla_{\partial v} \partial v = -(2+\cos u)\begin{pmatrix} \cos v - \cos^2 u \cos v \\ \sin v - \cos^2 u \sin v \\ -\cos u \sin u \end{pmatrix}$$

$$= (2+\cos u)\begin{pmatrix} -\cos v \sin^2 u \\ -\sin v \sin^2 u \\ \cos u \sin u \end{pmatrix} = (2+\cos u)\sin u \, \partial u$$

Def

Let $\varphi: U \to V \subset \Sigma$ be a local param. Then the Christoffel symbols (of the second kind) of Σ w.r.t. φ are the functions $\Gamma^k_{ij} : U \to \mathbb{R}$ defined via $\nabla_{\partial_i} \partial_j = \sum_k \Gamma^k_{ij} \partial_k$ where $i, j, k \in \{u, v\}$

The Christoffel symbols of the first kind are defined by $\Gamma_{ij,k} = \langle \nabla_{\partial_i} \partial_j, \partial_k \rangle = \sum_l \Gamma^l_{ij} \langle \partial_l, \partial_k \rangle$.

15-11-17

Recap: $f : \Sigma_1 \to \Sigma_2$ diffeo
then f is an isometry if $\forall p \in \Sigma_1$ $\forall X \in T_p\Sigma_1$
$I_{1,p}(X) = I_{2,f(p)}(Df(X))$.

Suppose we have a diffeo $f : \Sigma_1 \to \Sigma_2$, $\varphi : U \to V$ is a local parameterisation of Σ_1, then $f \circ \varphi$ is a parameterisation, and f is an isometry $\Leftrightarrow E_1 = E_2$, $F_1 = F_2$, $G_1 = G_2$ where E_i, F_i, G_i are calculated w.r.t. φ_i, $\varphi_1 = \varphi$, $\varphi_2 = f \circ \varphi$.

X is a C^k vector field on Σ if $\forall p \in \Sigma$, $X(p) \in T_p\Sigma$.
$\mathfrak{X}^k(\Sigma)$ = set of all C^k vector fields.
Given $f \in C^\infty(\Sigma)$, $X \in \mathfrak{X}^\infty(\Sigma)$, γ a curve representing X, i.e. $\gamma(0) = p$, $\gamma'(p) = X(p)$, then
$Xf = X(f)$, $X(f)|_p = \frac{d}{dt}\Big|_{t=0} f(\gamma) = Df(X)$.

fX - multiplication.

$\nabla_X^{\mathbb{R}^3} Y = DY(X)$ v.fs X, Y defined on $\mathbb{R}^3$
$= \frac{d}{dt}\Big|_{t=0} Y(\gamma(t))$ $\leftarrow$ useful definition!!

Covarient Derivative:
$\nabla_X Y = \nabla_X^{\mathbb{R}^3} Y - \langle \nabla_X^{\mathbb{R}^3} Y, N \rangle N$

$$\nabla_{\partial_u}^{\mathbb{R}^3} \partial_u \Big|_{p=\varphi(u,v)} = \frac{d}{dt}\Big|_{t=0} \partial_u(\varphi(u+t,v)) , \quad \partial_u = \frac{\partial \varphi}{\partial u}$$

$$= \frac{d}{dt}\Big|_{t=0} \frac{\partial \varphi}{\partial u}(u+t, v) = \frac{\partial^2 \varphi}{\partial u^2}$$

$$\nabla_{\partial_v}^{\mathbb{R}^3} \partial_u = \frac{\partial^2 \varphi}{\partial u \partial v} = \nabla_{\partial_u}^{\mathbb{R}^3} \partial_v$$

$\Rightarrow \forall i,j,k \quad \Gamma_{ij}^k = \Gamma_{ji}^k$ (using next definition).

MATH 3113

15-11-17

Def: Christoffel symbols.
Let $\varphi: U \to V$ be a local parameterisation.
Then the Christoffel symbols (of the second kind) are defined by $\nabla_{\partial_i} \partial_j = \sum_{k \in \{u,v\}} \Gamma_{ij}^k \partial_k$
where from now on, $i, j, k, l \in \{u, v\}$.
The Christoffel symbols (of the first kind) are defined by $\Gamma_{ij,k} = \langle \nabla_{\partial_i} \partial_j, \partial_k \rangle = \sum_{l \in \{u,v\}} \Gamma_{ij}^l \langle \partial_l, \partial_k \rangle$

$$= \sum_{l \in \{u,v\}} \Gamma_{ij}^l \tilde{g}_{lk}$$

Suppose $Y = Y^1 \partial_1 + Y^2 \partial_2$.
Calculate $X = X^1 \partial_1 + X^2 \partial_2$, $\nabla_X Y$.
$\nabla_X Y = X^1 \nabla_{\partial_1} Y + X^2 \nabla_{\partial_2} Y$, $\nabla_X (Y^1 + Y^2) = \nabla_X Y^1 + \nabla_X Y^2$,
$\nabla_X f Y = X(f) Y + f \nabla_X Y$
Calculate

$$\nabla_{\partial_1} Y = \nabla_{\partial_1}(Y^1 \partial_1 + Y^2 \partial_2)$$
$$= \partial_1 (Y^1) \partial_1 + Y^1 \nabla_{\partial_1} \partial_1 + \partial_1 (Y^2) \partial_2 + Y^2 \nabla_{\partial_1} \partial_2$$
$$= \left[\frac{\partial Y^1}{\partial u^1} + Y^1 \Gamma_{11}^1 + Y^2 \Gamma_{12}^1 \right] \partial_1$$
$$+ \left[\frac{\partial Y^2}{\partial u^1} + Y^1 \Gamma_{11}^2 + Y^2 \Gamma_{12}^2 \right] \partial_2$$

Recall: (Torus example)

$$\nabla_{\partial_u} \partial_u = 0 \quad ①, \quad \nabla_{\partial_u} \partial_v = \frac{-\sin u}{2 + \cos u} \partial_v \quad ②, \quad \nabla_{\partial_v} \partial_v = (2 + \cos u) \sin u \, \partial_u \quad ③$$

① $\Rightarrow \Gamma_{uu}^u = 0 = \Gamma_{uu}^v$

② $\Rightarrow \Gamma_{uv}^u = 0$, $\Gamma_{uv}^v = \dfrac{-\sin u}{2 + \cos u}$

③ $\Rightarrow \Gamma_{vv}^u = (2 + \cos u) \sin u$, $\Gamma_{vv}^v = 0$

Prop

Let i, j, k denote either u or v, $\Gamma_{ij,k}$ denote Christoffel symbols of the first kind. Then if $\tilde{g}_{ij}$ is the matrix associated to 1st fundamental form w.r.t. some local param. then

$$\Gamma_{ij,k} = \frac{1}{2}\left(\frac{\partial \tilde{g}_{ik}}{\partial u^j} + \frac{\partial \tilde{g}_{jk}}{\partial u^i} - \frac{\partial \tilde{g}_{ij}}{\partial u^k}\right)$$

$$X(\langle Y, Z\rangle) = \langle \nabla_X Y, Z\rangle + \langle Y, \nabla_X Z\rangle \quad (*)$$

Proof

Let $X = \partial_i$, $Y = \partial_j$, $Z = \partial_k$

$$(*) \Rightarrow \frac{\partial}{\partial u^i}\tilde{g}_{jk} = \partial_i(\langle \partial_j, \partial_k\rangle) = \langle \nabla_{\partial_i}\partial_j, \partial_k\rangle + \langle \partial_j, \nabla_{\partial_i}\partial_k\rangle$$
$$= \Gamma_{ij,k} + \Gamma_{ik,j}$$

$$\Rightarrow \frac{\partial}{\partial u^i}\tilde{g}_{jk} + \frac{\partial}{\partial u^j}\tilde{g}_{ik} - \frac{\partial}{\partial u^k}\tilde{g}_{ij} = \left(\Gamma_{ij,k} + \cancel{\Gamma_{ik,j}}\right) + \left(\Gamma_{ij,k} + \cancel{\Gamma_{jk,i}}\right) - \left(\cancel{\Gamma_{kj,i}} + \cancel{\Gamma_{ik,j}}\right)$$

$$= 2\Gamma_{ij,k}$$

□

MATH 3113

21-11-17

$$\nabla_X^{\mathbb{R}^3} Y\big|_p = \frac{d}{dt}\Big|_{t=0} Y(\gamma(t)) \qquad \gamma \text{ a curve, } \gamma(0)=p,\ \gamma'(0)=X$$

$X, Y \in \mathfrak{X}^k(\Sigma)$

$\nabla_X^{\mathbb{R}^3} Y\big|_p \notin T_p\Sigma$

$$\nabla_X Y = \nabla_X^{\mathbb{R}^3} Y - \langle \nabla_X^{\mathbb{R}^3} Y, N\rangle N$$

<u>Christoffel Symbols:</u>

$$\nabla_{\partial_i} \partial_j = \sum_k \Gamma_{ij}^{\ k} \partial_k \qquad \text{(2nd kind)}$$

$$\langle \nabla_{\partial_i} \partial_j, \partial_k \rangle = \Gamma_{ij,k} \qquad \text{(1st kind)}$$

$$= \sum_{l=1}^{2} \langle \Gamma_{ij}^{\ l} \partial_l, \partial_k \rangle = \sum_{l=1}^{2} \Gamma_{ij}^{\ l} \tilde{g}_{lk} \qquad \text{since } \tilde{g}_{lk} = \langle \partial_l, \partial_k \rangle$$

We may write any $\nabla_X Y$ in terms of these symbols.

e.g. $X = \partial_u$, $Y = Y^u \partial_u + Y^v \partial_v$

$$\nabla_{\partial_u}(Y^u \partial_u + Y^v \partial_v) = \partial_u(Y^u)\partial_u + Y^u \nabla_{\partial_u}\partial_u + \partial_u(Y^v)\partial_v + Y^v \nabla_{\partial_u}\partial_v$$

$$= \frac{\partial Y^u}{\partial u}\partial_u + \frac{\partial Y^v}{\partial u}\partial_v + \sum_k \left[Y^u \Gamma_{uu}^{\ k} + Y^v \Gamma_{uv}^{\ k} \right] \partial_k$$

<u>Lemma</u>

$$\Gamma_{ij,k} = \frac{1}{2}\left(\frac{\partial}{\partial_j}\tilde{g}_{ik} + \frac{\partial}{\partial_i}\tilde{g}_{jk} - \frac{\partial}{\partial_k}\tilde{g}_{ik} \right)$$

"C.S. of 1st kind depend only on 1st fundamental form."

<u>Lemma</u>

"C.S. of 2nd kind depend only on 1st fundamental form"

$$\Gamma_{ij}^{\ k} = \sum_{l=1}^{2} \underbrace{(\tilde{g}^{-1})^{kl}}_{\text{inverse of } \tilde{g}} \Gamma_{ij,l} = \frac{1}{\det \tilde{g}}\left(\Gamma_{ij,k}\, \tilde{g}_{\bar{k}\bar{k}} - \Gamma_{ij,\bar{k}}\, \tilde{g}_{k\bar{k}} \right)$$

where $\bar{k} \neq k$ i.e. $k=u \Rightarrow \bar{k}=v$

$k=v \Rightarrow \bar{k}=u$

Proof

Consider $\sum_{l=1}^{2} \Gamma_{ij,l} (\tilde{g}^{-1})^{lk} = \sum_{\substack{l=1\\m=1}}^{2} \Gamma_{ij,m} \tilde{g}_{m,l} (\tilde{g}^{-1})^{lk}$

$$= \sum_{m=1}^{2} \Gamma_{ijm} \underbrace{\sum_{l=1}^{2} \tilde{g}_{m,l} (\tilde{g}^{-1})^{lk}}_{= (\tilde{g} \circ \tilde{g}^{-1})_m^k = (Id)_m^k = \delta_m^k}$$

$$= \sum_{m=1}^{2} \Gamma_{ij}^m \delta_m^k = \Gamma_{ij}^k$$

(See notes for second part). $\square$

Corollary

The covarient derivative is invarient under isometries. That is, if $f : \Sigma \to \hat{\Sigma}$ is an isometry and Γ_{ij}^k, $\hat{\Gamma}_{ij}^k$ are Christoffel Symbols of Σ, $\hat{\Sigma}$ resp. w.r.t. parameterisations φ, $f \circ \varphi$, then
$\Gamma_{ij}^k(p) = \hat{\Gamma}_{ij}^k(f(p))$.

Remark

Given such an isometry $f : \Sigma \to \hat{\Sigma}$ we transform $X \in T_p\Sigma$ to a vector in $T_{f(p)}\hat{\Sigma}$ by $f_* X|_{f(p)} := Df(X)|_p \in T_{f(p)}\hat{\Sigma}$.
f_* is called the "push forward".

Corollary above $\iff f_*(\nabla_X Y) = \hat{\nabla}_{f_* X} f_* Y$

We can also define the "pull back":
Suppose we have a function $\Omega : \hat{\Sigma} \to \mathbb{R}$ then we define $f^*\Omega : \Sigma \to \mathbb{R}$ by $f^*\Omega(p) := \Omega(f(p))$. We will say a function, e.g. Gauss curvature, is preserved under an isometry, f, if $f^*(\hat{K}) = K$

Gauss in $\hat{\Sigma}$ Gauss in Σ

MATH 3113

21-11-17

Theorema Egregium - coming soon!
"From now on, everything is at least 3 times differentiable"

Lemma
Let φ be a local parameterisation of an oriented surface with Gauss map N. Then if $X \in \mathfrak{X}^k(\Sigma)$ then

(i) $\dfrac{\partial^2 \varphi}{\partial u^i \partial u^j} = \nabla_{\partial_i} \partial_j + \underset{=\left(\begin{smallmatrix} e & f \\ f & g \end{smallmatrix}\right)}{A_{ij}} N$ ← Weingarten formula

(ii) $\nabla^{\mathbb{R}^3}_{\partial_i} X = \nabla_{\partial_i} X + \langle W(\partial_i), X \rangle N$

Proof
(ii) By def: $\nabla^{\mathbb{R}^3}_{\partial_i} X = \nabla_{\partial_i} X + \langle \nabla^{\mathbb{R}^3}_{\partial_i} X, N \rangle N$
But $\langle X, N \rangle = 0$
$\Rightarrow 0 = \partial_i \langle X, N \rangle = \langle \nabla^{\mathbb{R}^3}_{\partial_i} X, N \rangle + \langle X, \underbrace{\nabla^{\mathbb{R}^3}_{\partial_i} N}_{DN(\partial_i)} \rangle$

$= \langle \nabla^{\mathbb{R}^3}_{\partial_i} X, N \rangle - \langle W(\partial_i), X \rangle$
$\Rightarrow$ (ii)

(i) write $X = \partial_j$

$\square$

$$\frac{\partial}{\partial u^i}\left(\frac{\partial^2 \varphi}{\partial u^j \partial u^l}\right) = \nabla^{\mathbb{R}^3}_{\partial_i} \frac{\partial^2 \varphi}{\partial u^j \partial u^l}$$

$$= \nabla^{\mathbb{R}^3}_{\partial_i} (\nabla_{\partial_j} \partial_l + A_{jl} N)$$

$$= \nabla_{\partial_i} \nabla_{\partial_j} \partial_l + \langle W(\partial_i), \nabla_{\partial_j} \partial_l \rangle N - A_{jl} W(\partial_i) + ?N$$

$$0 = \frac{\partial^3 \varphi}{\partial u^i \partial u^j \partial u^l} - \frac{\partial^3 \varphi}{\partial u^j \partial u^i \partial u^l}$$

$$= \nabla_{\partial_i} \nabla_{\partial_j} \partial_l - \nabla_{\partial_j} \nabla_{\partial_i} \partial_l - A_{jl} W(\partial_i) + A_{il} W(\partial_j) + ?N$$

$$\Rightarrow \nabla_{\partial_i} \nabla_{\partial_j} \partial_l - \nabla_{\partial_j} \nabla_{\partial_i} \partial_l = A_{jl} W(\partial_i) - A_{il} W(\partial_j)$$

(tangential part)

Def
Riemann curvature tensor
Let $i,j,k,l \in \{u,v\}$ as before, then define
$R_{klij} := \langle \nabla_{\partial_i} \nabla_{\partial_j} \partial_l - \nabla_{\partial_j} \nabla_{\partial_i} \partial_l \, , \, \partial_k \rangle$

Gauss Equation
$R_{klij} = A_{jl} A_{ik} - A_{il} A_{jk}.$

$$R_{uvuv} = A_{vv} A_{uu} - A_{uv} A_{vu}$$
$$= ge - f^2 = K \det \tilde{g} \ (= \det A)$$

Theorema Egregium
Gauss curvature (K) is invariant under isometries,
i.e. it is an intrinsic quantity.

Proof
$$K = \frac{R_{uvuv}}{\det \tilde{g}}$$
If the Riemann curvature tensor is intrinsic, then so is the Gauss curvature, but Riemann curvature tensor is defined only in terms of 1st fundamental form and covarient derivative, which are intrinsic. $\square$

Example
You cannot (isometrically) flatten a sphere.
$\varphi := r(\cos\theta\cos\varphi, \sin\theta\cos\varphi, \sin\varphi)$
$$\tilde{g} = \begin{pmatrix} r^2 & 0 \\ 0 & r^2\cos^2\theta \end{pmatrix} \qquad A = r\begin{pmatrix} 1 & 0 \\ 0 & \cos^2\theta \end{pmatrix}$$
$$K = \frac{\det A}{\det \tilde{g}} = \frac{1}{r^2} \neq 0$$
Since the Gauss curvature on an open subset is not equal

MATH 3113

21-11-17

to zero, then due to the Theorema Egregian, there can be no isometry to a subset of $\mathbb{R}^2$.

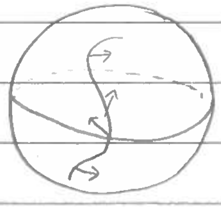
Non Examinable

The normal part of the commutation of the derivatives earlier gives the Codazzi-Mainardi eqns., namely
$$\sum_k \left(\Gamma^k_{ji} A_{ik} - \Gamma^k_{il} A_{jk}\right) - \frac{\partial}{\partial u^i} A_{jl} - \frac{\partial}{\partial u^j} A_{il} = 0$$

Theorem (Bonnet)
Let $U \subset \mathbb{R}^2$ be open and suppose $\tilde{g}, A : U \longrightarrow \mathbb{R}^{2\times 2}$ are continuously differentiable s.t.
(i) $\forall q \in U$, $\tilde{g}, A$ are symmetric
(ii) $\forall q \in U$, $\tilde{g}$ is positive definite
(iii) $\tilde{g}, A$ satisfy the Gauss equation and Codazzi-Mainardi equations.
Then $\forall q \in U$ $\exists$ an open neighbourhood $\tilde{U} \ni q$ and a parameterisation $\varphi : \tilde{U} \longrightarrow \varphi(\tilde{U}) \subset \mathbb{R}^3$ s.t. $\Sigma =$ union of images of the $\tilde{U}$'s $\subset \mathbb{R}^3$ is a manifold with 1st and 2nd fundamental forms given by $\tilde{g}$ and A respectively.
Moreover if U is connected and $\tilde{\varphi}$ is another such parameterisation, then φ and $\tilde{\varphi}$ differ only by a Euclidean Isometry.

22-11-17

Curves on Surfaces - Chapter 6



Def
Let Σ be a surface, $\gamma : I \to \Sigma$ a regular parameterised curve.
A vector field along γ (of class C^k) is a (k-times differentiable) map $X : I \to \mathbb{R}^3$ s.t. $X(t) \in T_{\gamma(t)}\Sigma \quad \forall t \in I$. We may write $X \in \mathfrak{X}^k(\gamma)$.
We may define
$$\frac{dX}{dt} = \nabla^{\mathbb{R}^3}_{\gamma'(t)} X \Big|_{\gamma(t)} = \frac{d}{dr}\Big|_{t=r} X(\gamma(r))$$
Want: derivative also in $\mathfrak{X}^{k-1}(\gamma)$.
So we define
$$\frac{\nabla}{dt} X = \nabla^{\mathbb{R}^3}_{\gamma'} X - \langle \nabla^{\mathbb{R}^3}_{\gamma'} X, N \rangle N$$
$$\left(= \nabla_{\gamma'} X \right) \leftarrow \text{if } X \text{ existed outside of } \gamma.$$

If there is an extension of γ' to a v.f. $\tilde{Y}$ near γ and extension of X to $\tilde{X}$ in a neighbourhood of γ then
$$\frac{\nabla}{dt} X = \nabla_{\tilde{Y}} \tilde{X} \Big|_{\gamma(t)}$$

Def (Parallel vector fields)
Let γ be a curve on a surface Σ,
(i) the map $\frac{\nabla}{dt} : \mathfrak{X}^k(\gamma) \to \mathfrak{X}^{k-1}(\gamma)$ is called the covariant derivative along γ.
(ii) if $\frac{\nabla}{dt} X = 0 \quad \forall t \in I$, then X is said to be parallel along γ.

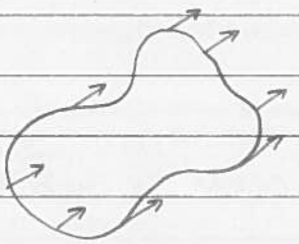
MATH 3113

22-11-17

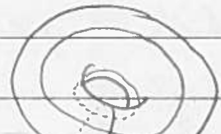
Examples

(i) $\mathbb{R}^2$: For any regular curve γ in $\mathbb{R}^2$ we have that $\frac{\nabla}{dt} X(t) = \frac{\partial X^1}{\partial t} e_1 + \frac{\partial X^2}{\partial t} e_2$ where $X = X^1 e_1 + X^2 e_2$

X parallel $\iff \frac{\partial X^1}{\partial t} \equiv 0 \equiv \frac{\partial X^2}{\partial t}$, X is a constant vector field.



(ii)



$\varphi : (0, 2\pi)^2 \to V$

$(u,v) \mapsto ((2+\cos u)\cos v, (2+\cos u)\sin v, \sin u)$

$$\partial_u = \begin{pmatrix} -\sin u \cos v \\ -\sin u \sin v \\ \cos u \end{pmatrix}, \quad \partial_v = \begin{pmatrix} -(2+\cos u)\sin v \\ (2+\cos u)\cos v \\ 0 \end{pmatrix}$$

$\nabla_{\partial_u} \partial_u = 0 \qquad \nabla_{\partial_u} \partial_v = \frac{-\sin u}{2+\cos u} \partial_v$

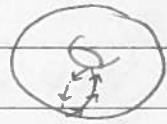
$\nabla_{\partial_v} \partial_v = (2+\cos u)\sin u \, \partial_u$

Consider curves $\gamma(t) = \varphi(t, \pi)$, $\tilde{\gamma}(t) = \varphi(r, t)$ for some fixed $r \in (0, 2\pi)$

Q: Is γ' parallel along γ?
Is $\tilde{\gamma}'$ parallel along $\tilde{\gamma}$?

$\frac{\nabla}{dt} \gamma' = \nabla_{\gamma'} \gamma' = \nabla_{\partial_u} \partial_u \Big|_{\gamma(t)} = 0 \Rightarrow \gamma'$ is parallel along γ.

vector fields extend to V



$\frac{\nabla}{dt} \tilde{\gamma}' = \nabla_{\tilde{\gamma}'} \tilde{\gamma}' = \nabla_{\partial_v} \partial_v \Big|_{\gamma(t)} = (2+\cos u)\sin u \, \partial_u$

$= 0 \iff u = \pi$

Two properties

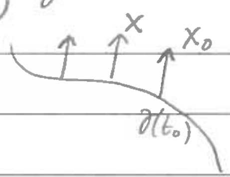
$X \in \mathfrak{X}^k(\gamma)$, $f \in C^\infty(\Sigma)$

(i) $\dfrac{\nabla}{dt} fX = \dfrac{d}{dt}(f(\gamma))X + f\dfrac{\nabla}{dt}X$

(ii) $\dfrac{d}{dt}\langle X, Y\rangle = \langle \dfrac{\nabla}{dt}X, Y\rangle + \langle X, \dfrac{\nabla}{dt}Y\rangle$

Proposition

Let $\gamma : I \to \Sigma$ be a parameterised curve in surface Σ, let $t_0 \in I$, $X_0 \in T_{\gamma(t_0)}\Sigma$.
Then $\exists$ a unique v.f. X along γ which is parallel and satisfies $X(t_0) = X_0$



Proof

local param. $\varphi : U \to V$, $\operatorname{tr}\gamma \subset V$

First assume that $\operatorname{tr}\gamma$ is in a single coordinate patch.
In local coordinates $\gamma = \varphi(u(t), v(t))$ ← local parameterisation.
Write a vector field along γ as $X = X^u(t)\partial_u + X^v(t)\partial_v$,
$\gamma'(t) = u'(t)\partial_u + v'(t)\partial_v$. Write $u'(t) = \gamma_u'(t)$, $v'(t) = \gamma_v'(t)$.
Then $\dfrac{\nabla}{dt}\partial_j = \nabla_{\gamma'}\partial_j = \displaystyle\sum_{i=1}^{2} \gamma_i' \nabla_{\partial_i}\partial_j$

$$= \sum_{\substack{i=1\\k=1}}^{2} \gamma_i' \Gamma_{ij}^k \partial_k$$

We now may calculate $\dfrac{\nabla}{dt}X$

$$\frac{\nabla}{dt}X = \sum_{j=1}^{2} \nabla_{\gamma'}(X^j\partial_j) = \sum_{j=1}^{2}\left[(X^j)'\partial_j + X^j\nabla_{\gamma'}\partial_j\right]$$

$$= \sum_{j=1}^{2}\left[(X^k)' + \sum_{i,j=1}^{2}\gamma_i'(X^j)'\Gamma_{ij}^k\right]\partial_k$$

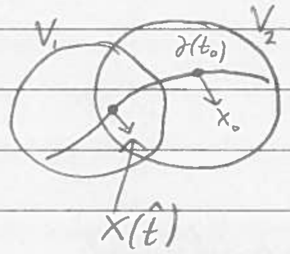
So X is parallel $\iff (X^k)'(t) + \displaystyle\sum_{i,j=1}^{2}\gamma_i'(X^j)'\Gamma_{ij}^k = 0^{(*)}$, $k = u, v$.

This is a system of linear 1st order ODEs, so by

MATH 3113

22-11-17

Picard-Lindlof theorem, given X_1, X_2 at t_0 $\exists!$ solution to $(*)$ $\forall t \in I$.



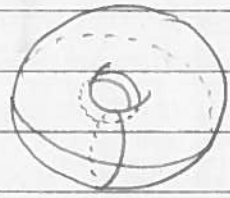
$\gamma(\hat{t}) \in V_1 \cap V_2$

$\varphi_i : U_i \to V_i$

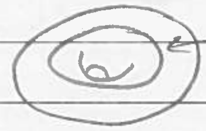
By uniqueness, and patching our vector field X, we have existence of a unique vector field along all of γ, even if it crosses to other coordinate patches. $\square$

Def

Let $\gamma : I \to \Sigma$ be a regular parameterised curve, γ is a geodesic if γ' is parallel along γ, ie. $\frac{D}{dt}\gamma' = 0$ $\forall t \in I$.



Geodesics on T^2



not a geodesic!

28-11-17 Last time:

$\frac{D}{dt} X$ $\quad$ X is a v.f. along γ

$$\frac{D}{dt} X\Big|_t = \left(\nabla^{\mathbb{R}^3}_{\gamma'}(X)\right)^T \leftarrow \text{tangent}$$

$$= \left(\frac{d}{dr}\Big|_{r=0} X(\gamma(t+r))\right)^T$$

$$= \nabla^{\mathbb{R}^3}_{\gamma'} X - \langle \nabla^{\mathbb{R}^3}_{\gamma'} X, N \rangle N$$

X is parallel along γ if $\frac{D}{dt} X = 0$

Def

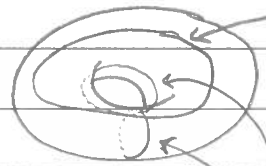
γ is a geodesic iff γ' is parallel along γ,
iff $\frac{\nabla}{dt}\gamma' = 0$.

Examples

$\mathbb{R}^2$

← straight lines parameterised at constant speed.

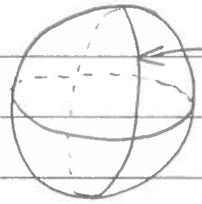
T^2



not geodesic.

geodesic

S^2



great circles are geodesic (see hw 8)

Fact

If γ is a geodesic, $|\gamma'|^2 = \text{const}$

Proof

$$\frac{d}{dt}|\gamma'|^2 = 2\langle \underbrace{\frac{\nabla}{dt}\gamma'}_{=0}, \gamma' \rangle = 0 \qquad \square$$

Fact

"Geodesics are invarient under isometries"

Why? Christoffel symbols are invarient under isometries + eqn below...

(Suppose we have $f: \Sigma_1 \to \Sigma_2$ an isometry, γ_1 is a geodesic in Σ_1, then $f \circ \gamma_1 = \gamma_2$ is a geodesic in Σ_2)

MATH 3113

28-11-17

<u>Prop</u> (Local existence and uniqueness of geodesics).
Let Σ be a smooth surface, $p \in \Sigma$, $X \in T_p\Sigma$, then $\exists\, \varepsilon > 0$ and a smooth geodesic $\gamma : (-\varepsilon, \varepsilon) \to \Sigma$ s.t. $\gamma(0) = p$, $\gamma'(0) = X$.

<u>Proof</u>
Take $\varphi : U \to V \subset \Sigma$ s.t. $p \in V$.
Suppose $\gamma(t) = \varphi(\tilde{\gamma}(t)) = \varphi(\tilde{\gamma}_1(t)e_1 + \tilde{\gamma}_2(t)e_2)$

Recall if $X = X^1\partial_1 + X^2\partial_2$ then

$$\frac{\nabla}{dt} X = \sum_{i=1}^{2}\left(\frac{\partial}{\partial t} X^i + \sum_{k,j=1}^{2} \frac{\partial \tilde{\gamma}_k}{\partial t} X^j \Gamma^i_{jk}\right)\partial_i$$

Applying to $\gamma' = \frac{\partial \tilde{\gamma}_1}{\partial t}\partial_1 + \frac{\partial \tilde{\gamma}_2}{\partial t}\partial_2$

$$\frac{\nabla}{dt}\gamma' = 0 \;,\quad \gamma'(0) = X \;,\quad \gamma(0) = p$$

$$\Updownarrow \qquad\qquad \Updownarrow \qquad\qquad \Updownarrow$$

$$\frac{\partial^2 \tilde{\gamma}_i}{\partial t^2} + \sum_{j,k=1}^{2} \frac{\partial \tilde{\gamma}_j}{\partial t}\frac{\partial \tilde{\gamma}_k}{\partial t}\Gamma^i_{jk} = 0 \quad \text{for } i = 1, 2. \qquad \tilde{\gamma}(0) = \varphi^{-1}(p) \qquad \frac{\partial \tilde{\gamma}_i}{\partial t}(0) = X^i$$

This is a non-linear 2nd order system of ODEs, so we may apply Picard-Lindlof (Thm I.19) to obtain the existence of a unique solution on the above 2nd order ODE for some $\varepsilon > 0$. $\square$

<u>Example</u>
Geodesics on cylinders.
C - cylinder
$V \subset C$ parameterised by $\varphi(\theta, z) = (\cos\theta, \sin\theta, z)$

$\varphi : \underbrace{(-\pi, \pi) \times \mathbb{R}}_{U} \to \mathbb{R}^3$ (φ is an isometry)

$V = \varphi(U)$

$p = (1, 0, 0) = \varphi(0, 0)$

$T_p\Sigma = \text{span}\left\{ \underbrace{\begin{pmatrix}0\\1\\0\end{pmatrix}}_{\partial_\theta},\ \underbrace{\begin{pmatrix}0\\0\\1\end{pmatrix}}_{\partial_z} \right\}$

Pick $V = V^\theta \partial_\theta + V^z \partial_z$

$\gamma(t) = \varphi(V^\theta t, V^z t) \subset C$

$\gamma'(t) = V^\theta \partial_\theta + V^z \partial_z = (V^\theta(-\sin(V^\theta t)), V^\theta \cos(V^\theta t), V^z)$

$$\nabla^{\mathbb{R}^3}_{\gamma'} \gamma' = \gamma''(t) = -(V^\theta)^2 \underbrace{(\cos(V^\theta t), \sin(V^\theta t), 0)}_{= N}$$

$$= -(V^\theta)^2 N$$

$$\Rightarrow \frac{\nabla}{\partial t} \gamma' = 0$$

<u>Prop</u>

Writing $\gamma = \varphi(u(t), v(t))$ and writing

$E = E(u(t), v(t))$, $F = F(u(t), v(t))$, $G = G(u(t), v(t))$

then the geodesic equations are equivalent to

$$\frac{d}{dt}(u'E + v'F) = \frac{1}{2}\left((u')^2 \frac{\partial E}{\partial u} + 2u'v' \frac{\partial F}{\partial u} + (v')^2 \frac{\partial G}{\partial u}\right)$$

$$\frac{d}{dt}(u'F + v'G) = \frac{1}{2}\left((u')^2 \frac{\partial E}{\partial v} + 2u'v' \frac{\partial F}{\partial v} + (v')^2 \frac{\partial G}{\partial v}\right)$$

Geodesics and shortest connections

Given two points $p, q \in \Sigma$ write $\gamma : p \rightsquigarrow q$ if γ is a curve in Σ starting at p and ending at q. Suppose Σ is path-connected, then we may define the distance from p to q by

$$\mathrm{dist}(p,q) := \inf\{\, l(\gamma) \mid \gamma : p \rightsquigarrow q \,\}$$

<u>Exercise:</u>

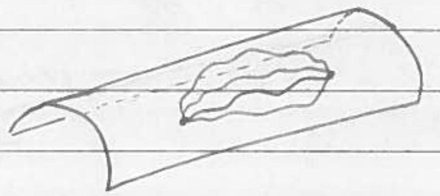
Σ equipped with this metric is a metric space.

Want to study how length in a family of curves changes.

MATH 3113

28-11-17

Suppose γ is parameterised by arc length, $\gamma : p \rightsquigarrow q$ of length L
then define a variation of γ to be a mapping (C^2)
$h : [0, L] \times (-\varepsilon, \varepsilon) \longrightarrow \Sigma$ s.t. $h(s, 0) = \gamma(s)$
$h(0, t) = p$, $h(L, t) = q$.



We define the length of the variation $L(t) : (-\varepsilon, \varepsilon) \longrightarrow \mathbb{R}$

$$L(t) = \int_0 \left| \frac{\partial L}{\partial s}(s, t) \right| ds$$

$$= \int_0^L |\gamma_t'(s)| \, ds$$

where $\gamma_t(s) = h(s, t)$.

Prop (First Variation of arc length)
The function $L : (-\varepsilon, \varepsilon) \longrightarrow \mathbb{R}$ is continuously differentiable for ε small enough
and at $t = 0$ its derivative is given by

$$L'(0) = -\int_0^L \left\langle \frac{\nabla}{ds} \gamma'(s), \left. \frac{d}{dt} \right|_{t=0} \gamma_t(s) \right\rangle ds$$

"variation of arc length"

Proof
Since $|\gamma'(s)| = 1$ then for ε small enough γ_t is a regular curve.
We observe that since $\dfrac{\partial^2 h}{\partial s \partial t} = \dfrac{\partial^2 h}{\partial t \partial s}$ we have that

$$\nabla_{\frac{\partial h}{\partial s}} \frac{\partial h}{\partial t} = \nabla_{\frac{\partial h}{\partial t}} \frac{\partial h}{\partial s}$$ where we consider $\dfrac{\partial h}{\partial s}$, $\dfrac{\partial h}{\partial t}$ as

vector fields along γ_t. So now

$$L'(t) = \frac{1}{2} \int_0^L \left| \frac{\partial h}{\partial s}(s, t) \right|^{-1} \frac{d}{dt} \left(\left\langle \frac{\partial h}{\partial s}, \frac{\partial h}{\partial s} \right\rangle \right) ds$$

$$= \frac{1}{2} \int_0^L \left| \frac{\partial h}{\partial s} \right|^{-1}(s, t) \left\langle \frac{\nabla}{\partial t} \frac{\partial h}{\partial s}, \frac{\partial h}{\partial s} \right\rangle ds$$

$$= \frac{1}{2} \int_0^L \left| \frac{\partial h}{\partial s} \right|^{-1}(s, t) \left\langle \frac{\nabla}{\partial s} \frac{\partial h}{\partial t}, \frac{\partial h}{\partial s} \right\rangle ds$$

We now evaluate at $t=0$:

$$L'(0) = \int_0^L - \left\langle \frac{\partial h}{\partial t}, \frac{\nabla}{\partial s}\frac{\partial h}{\partial s} \right\rangle + \frac{\partial}{\partial s}\left\langle \frac{\partial h}{\partial t}, \frac{\partial h}{\partial s}\right\rangle ds$$

$$= -\int_0^L \left\langle \frac{\partial h}{\partial t}, \frac{\nabla}{\partial s}\frac{\partial h}{\partial s}\right\rangle ds + \cancel{\left\langle \frac{\partial h}{\partial t}, \frac{\partial h}{\partial s}\right\rangle}\Big|_{s=0}^{s=L}$$

$$= -\int_0^L \left\langle \frac{\partial}{dt}\Big|_{t=0} \gamma_t(s), \frac{\nabla}{ds}\gamma' \right\rangle ds \quad \text{as required.}$$

$\square$

Suppose γ minimises the length between p and q.
Then $L'(0) = 0$ for any variation.
e.g. we could pick a variation s.t.

$$\frac{d}{dt}\Big|_{t=0} \gamma_t(s) = \frac{\nabla}{ds}\gamma' \qquad \text{(some details required)}$$

But now $L'(0) = -\int_0^L \left|\frac{\nabla}{ds}\gamma'\right|^2 ds = 0$

$\Leftrightarrow \frac{\nabla}{ds}\gamma' \equiv 0 \Rightarrow \gamma$ is a geodesic.

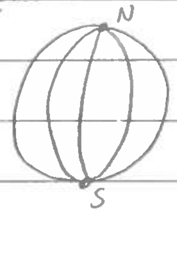
non-examinable

[Similarly if we had a minimiser of area
(i.e. fix the manifold outside $V \subset \Sigma$ and vary as in hw 7)

$$\frac{d}{dt}\,\text{Area}(\Sigma_t) = -2\int aH\, dS$$]

__Theorem__

Let Σ be a complete path connected surface then
(i) if $p, q \in \Sigma$ and $\gamma : p \rightsquigarrow q$ and satisfies $L(\gamma) = \text{dist}(p,q)$ then γ is a geodesic
(ii) furthermore, $\forall\, p, q \in \Sigma\ \exists$ a minimising geodesic $\gamma : p \rightsquigarrow q$ s.t. $L(\gamma) = \text{dist}(p,q)$.



← shows non-uniqueness ($p=N$, $q=S$)

MATH 3113

28-11-17

Geodesic and Normal Curvature

Suppose $\gamma : I \to \Sigma$ is a curve parameterised by arc length.
From now on, Σ will be orientable.

Def (Normal curvature)
Let Σ be an orientable surface, γ regular, parameterised by unit speed. Then
$$\kappa_n(s) = \langle \gamma''(s), N(\gamma(s)) \rangle.$$

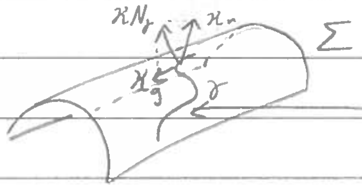
Prop
We have that the normal curvature satisfies
$$\kappa_n(s) = \langle \ddot{\gamma}(s), N(\gamma(s)) \rangle = \mathrm{II}_{\gamma(s)}(\dot{\gamma})$$

Proof
$N(\gamma(s)) \perp \dot{\gamma}(s) \ \forall s.$
$$0 = \frac{d}{ds} \langle N(\gamma(s)), \dot{\gamma}(s) \rangle$$
$$= \langle N(\gamma(s)), \ddot{\gamma}(s) \rangle + \langle \dot{\gamma}(s), DN(\dot{\gamma}(s)) \rangle$$
$$= \langle N(\gamma(s)), \ddot{\gamma}(s) \rangle - \langle \dot{\gamma}(s), W(\dot{\gamma}(s)) \rangle$$
$$\iff \langle N(\gamma(s)), \ddot{\gamma}(s) \rangle = \mathrm{II}_{\gamma(s)}(\dot{\gamma}(s)) \quad \square$$

Def (Geodesic curvature)
The function $\kappa_g(s) := \langle \frac{D}{ds} \dot{\gamma}(s), N(\gamma(s)) \times \dot{\gamma}(s) \rangle$

and of course
$$\kappa^2(s) = \kappa_g^2(s) + \kappa_n^2(s).$$

29-11-17



parameterised by arc length

$\varkappa_n = \langle N, \ddot{\gamma} \rangle$

$\varkappa_g = \langle \ddot{\gamma}, N \times \dot{\gamma} \rangle$

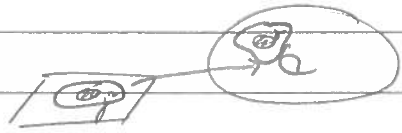
Gauss - Bonnet Theorem

Local Gauss Bonnet

Contained in an open set V s.t. $\exists$ param. $\varphi : \mathcal{U} \to V$

Def (Local closed curves)

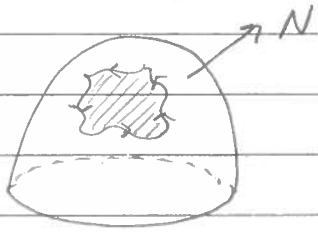
(i) Parameterised curve $\gamma : [a,b] \to \Sigma$ is a simple closed curve in Σ if $\exists$ a local parameterisation $\varphi : \mathcal{U} \to V$ and a simple closed curve $\tilde{\gamma} : [a,b] \to \mathcal{U}$ s.t. $\gamma = \varphi \circ \tilde{\gamma}$. We will write $\text{int}(\gamma) := \varphi(\text{int}(\tilde{\gamma}))$



(ii) A continuous parameterised curve $\gamma : [a,b] \to \Sigma$ is a chain if $\exists$ a plane chain $\tilde{\gamma} : [a,b] \to \mathcal{U}$ s.t. $\gamma = \varphi \circ \tilde{\gamma}$.

Def

Let $\gamma : [a,b] \to \Sigma$ be a closed chain in an (oriented) surface Σ and let N be a Gauss map. We say that γ is positively oriented (w.r.t. N) iff $N \times \gamma'$ points into $\text{int}(\gamma)$. (i.e. the int γ is on the "left side" of the curve.)



MATH 3113

29-11-17

Let's work on V ($\exists$ param $\varphi: U \to V$).
Suppose $\{e_1, e_2\}$ is an orthonormal frame on V.
($e_1, e_2 \in \mathfrak{X}^\infty(V)$ vector fields, $|e_1| = 1$, $|e_2| = 1$, $\langle e_1, e_2 \rangle = 0$, $e_1 \times e_2 = N$)
For a local curve $\gamma: I \to V$ define an angular function $\alpha: I \to \mathbb{R}$ st.

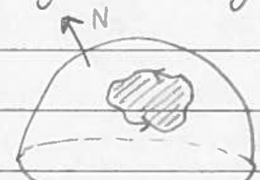
$$\dot\gamma = \frac{\gamma'(t)}{|\gamma'(t)|} = \cos(\alpha(t))\, e_1 + \sin(\alpha(t))\, e_2.$$

VII.3 <u>Theorem</u> (Turning tangents)
Suppose γ a simple closed curve parameterised by arc-length as above. Then $\exists$ angular function $\alpha: I \to V$ for γ and

$$\int_I \dot\alpha = \pm 2\pi$$

with $+$ if γ is positively parameterised.

<u>Theorem</u> (Local Gauss Bonnet - smooth curves)
Let $\gamma: [0, l] \to \Sigma$ be a regular simple positively oriented closed curve, parameterised by arc length.
Then $\displaystyle\int_\gamma \kappa_g \, ds + \int_{int(\gamma)} K \, dS = 2\pi$



<u>Lemma</u>
Let Σ, γ be as in VII.3.
Let $e_1, e_2 \in \mathfrak{X}^\infty(\Sigma)$ an oriented frame on V (coord. nbhd)
and let α be an angular function for γ w.r.t. $\{e_1, e_2\}$.
Then $\kappa_g = \dot\alpha - \langle e_1, \frac{\nabla}{ds} e_2 \rangle$.

<u>Proof</u>
We have $\dot\gamma = \cos\alpha\, e_1 + \sin\alpha\, e_2$.

$$\frac{\nabla}{ds}\dot\gamma = \cos\alpha \frac{\nabla}{ds} e_1 + \sin\alpha \frac{\nabla}{ds} e_2 - \dot\alpha \sin\alpha\, e_1 + \dot\alpha \cos\alpha\, e_2$$

$$N \times \dot\gamma = \cos\alpha\, e_2 - \sin\alpha\, e_1 \qquad (N = e_1 \times e_2)$$

$$\varkappa_g = \left\langle \frac{\nabla}{ds}\dot{\gamma}, N \times \dot{\gamma} \right\rangle$$

$\dot{\gamma}^T \leftarrow$ tangent

$\frac{\nabla}{ds} e_1 \perp e_1$ since $|e_1|^2 = 1$

$\frac{\nabla}{ds} e_2 \perp e_2$ similarly

$$\left\langle \frac{\nabla}{ds} e_1, e_2 \right\rangle = -\left\langle e_1, \frac{\nabla}{ds} e_2 \right\rangle$$

$$\Rightarrow \varkappa_g = \dot{\alpha} + \cos^2\alpha \left\langle \frac{\nabla}{ds} e_1, e_2 \right\rangle - \sin^2\alpha \left\langle \frac{\nabla}{ds} e_2, e_1 \right\rangle$$

$$= \dot{\alpha} - \left\langle \frac{\nabla}{ds} e_1, e_2 \right\rangle \qquad \square$$

Lemma

Let $\{e_1, e_2\}$ be as above. By abuse of notation we write $N(u,v) = N(\varphi(u,v))$ and $\frac{\partial N}{\partial u} = DN\left(\frac{\partial \varphi}{\partial u}\right)$

similar for $\frac{\partial N}{\partial v}$. Then

(i) $\frac{\partial N}{\partial u} \times \frac{\partial N}{\partial v} = K \cdot \partial_u \times \partial_v$

(ii) $\frac{\partial N}{\partial u} \times \frac{\partial N}{\partial v} = \left(\left\langle \frac{\partial e_1}{\partial u}, \frac{\partial e_2}{\partial v} \right\rangle - \left\langle \frac{\partial e_1}{\partial v}, \frac{\partial e_2}{\partial u} \right\rangle \right) N$

Proof

(i) For any vector $X \in \mathbb{R}^3$,

$$\left\langle \frac{\partial N}{\partial u} \times \frac{\partial N}{\partial v}, X \right\rangle = \det \begin{pmatrix} | & | & | \\ \frac{\partial N}{\partial u} & \frac{\partial N}{\partial v} & X \\ | & | & | \end{pmatrix}$$

det doesn't see orthogonal changes of basis so w.l.o.g. take $e_3 = N$

$$\Rightarrow \left\langle \frac{\partial N}{\partial u} \times \frac{\partial N}{\partial v}, X \right\rangle = \det \begin{pmatrix} | & | & 0 \\ \frac{\partial N}{\partial u} & \frac{\partial N}{\partial v} & 0 \\ 0 & 0 & X_3 \end{pmatrix}$$

$$= \det \begin{pmatrix} \sum_{i=1}^{2} \tilde{W}^{ui} \partial_i & -\sum_{j=1}^{2} \tilde{W}^{vj} \partial_j & 0 \\ | & | & 0 \\ 0 & 0 & X_3 \end{pmatrix}$$

$\left(\text{since } DN\left(\frac{\partial \varphi}{\partial u}\right) = -W\left(\frac{\partial^2 \varphi}{\partial u^2}\right) = -\sum_{i=1}^{2} \tilde{W}^{ui} \partial_i \right)$

MATH 3113

29-11-17

$$= \det\left(\begin{pmatrix} \tilde{W} & & 0 \\ & & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} \frac{\partial \varphi}{\partial u} & \frac{\partial \varphi}{\partial v} & 0 \\ & & 0 \\ 0 & 0 & X_3 \end{pmatrix}\right)$$

$= K < \partial_u \times \partial_v , X >$ as required. $\square$

05-12-17

Theorem (Gauss-Bonnet)

Let $\gamma : [0, l] \to \Sigma$ be a regular, simple, closed, smooth curve, parameterised by arc length and positively oriented. Then

$$\int_\gamma \kappa_g \, ds + \int_{\text{int}_\gamma} K \, dS = 2\pi$$

Lemma 1

Let Σ be an orientable surface, $\varphi : U \to V$ a parameterisation of Σ, suppose $tr\,\gamma \subset V$ and we have an oriented frame $\{e_1, e_2\}$ on V $(e_1 \times e_2 = N)$, then $\kappa_g = \dot{\alpha} - < e_1 , \frac{\nabla}{ds} e_2 >$ where γ is parameterised by arc length and α is an angular function of γ w.r.t. $\{e_1, e_2\}$.

As we are in one coordinate patch we abuse notation and write $N : U \to \mathbb{R}^3$ $N = N(\varphi(u,v))$.

So we may write $\frac{\partial N}{\partial u} = DN\left(\frac{\partial \varphi}{\partial u}\right)$.

Similarly we write $e_1 : U \to \mathbb{R}^3$, $e_2 : U \to \mathbb{R}^3$

Lemma 2

Let $\{e_1, e_2\}$ be an oriented orthonormal frame on V. Then

(i) $\frac{\partial N}{\partial u} \times \frac{\partial N}{\partial v} = K \partial_u \times \partial_v$

(ii) $\frac{\partial N}{\partial u} \times \frac{\partial N}{\partial v} = \left(< \frac{\partial e_1}{\partial u} , \frac{\partial e_2}{\partial v} > - < \frac{\partial e_1}{\partial v} , \frac{\partial e_2}{\partial u} > \right) N$

Proof

(ii) $N = e_1 \times e_2$

$|e_1|^2 = 1$

$\Rightarrow \frac{\partial e_1}{\partial u} = \langle e_2, \frac{\partial e_1}{\partial u} \rangle e_2 + \langle N, \frac{\partial e_1}{\partial u} \rangle N$

and we see similar for $\frac{\partial e_1}{\partial v}, \frac{\partial e_2}{\partial u}, \frac{\partial e_2}{\partial v}$

$$\frac{\partial N}{\partial u} \times \frac{\partial N}{\partial v} = \left(\frac{\partial e_1}{\partial u} \times e_2 + e_1 \times \frac{\partial e_2}{\partial u}\right) \times \left(\frac{\partial e_1}{\partial v} \times e_2 + e_1 \times \frac{\partial e_2}{\partial v}\right)$$

$e_2 \times e_2 = 0$

$$= \left(\langle N, \frac{\partial e_1}{\partial u} \rangle N \times e_2 + e_1 \times N \langle N, \frac{\partial e_2}{\partial u} \rangle\right)$$

$$\times \left(\langle N, \frac{\partial e_1}{\partial v} \rangle N \times e_2 + e_1 \times N \langle N, \frac{\partial e_2}{\partial v} \rangle\right)$$

$$= \left(\langle N, \frac{\partial e_1}{\partial u} \rangle e_1 + \langle N, \frac{\partial e_2}{\partial u} \rangle e_2\right)$$

$$\times \left(\langle N, \frac{\partial e_1}{\partial v} \rangle e_1 + \langle N, \frac{\partial e_2}{\partial v} \rangle e_2\right)$$

$\leftarrow$ minuses cancel

$$= \left(\langle N, \frac{\partial e_1}{\partial u} \rangle \langle N, \frac{\partial e_2}{\partial v} \rangle - \langle N, \frac{\partial e_2}{\partial u} \rangle \langle N, \frac{\partial e_1}{\partial v} \rangle\right) N$$

$\langle e_1, e_2 \rangle = 0 \Rightarrow \langle \frac{\partial e_1}{\partial u}, e_2 \rangle = - \langle e_1, \frac{\partial e_2}{\partial u} \rangle$

$$\langle \frac{\partial e_1}{\partial u}, \frac{\partial e_2}{\partial v} \rangle = \langle ? e_2 + \langle N, \frac{\partial e_1}{\partial u} \rangle N, \; ? e_1 + \langle N, \frac{\partial e_2}{\partial v} \rangle N \rangle$$

$$= \langle N, \frac{\partial e_1}{\partial u} \rangle \langle N, \frac{\partial e_2}{\partial v} \rangle$$

$\langle \frac{\partial e_1}{\partial v}, \frac{\partial e_2}{\partial u} \rangle$ follows similarly

$$\Rightarrow \frac{\partial N}{\partial u} \times \frac{\partial N}{\partial v} = \left(\langle \frac{\partial e_1}{\partial u}, \frac{\partial e_2}{\partial v} \rangle - \langle \frac{\partial e_1}{\partial v}, \frac{\partial e_2}{\partial u} \rangle\right) N$$

$\square$

MATH 3113

05-12-17

Proof (of Gauss-Bonnet (local smooth curves))

We require Green's Thm:
Suppose $\tilde{\gamma} : [0, l] \to \mathbb{R}^2$ is a smooth, simple, regular, closed curve s.t. $V = \text{int}\,\tilde{\gamma}$ and $P, Q : V_\varepsilon \to \mathbb{R}$, and $\tilde{\gamma}(s) = (u(s), v(s))$. Then

$$\int_0^l (\dot{u}P + \dot{v}Q)\, ds = \int_V \left(\frac{\partial Q}{\partial v} - \frac{\partial P}{\partial u} \right) du\, dv.$$

We consider $\displaystyle \mathcal{I} = \int_0^l \langle e_1(\gamma(s)), \frac{\nabla}{ds} e_2(\gamma(s)) \rangle ds$

Let γ be our arc length parameterised curve on Σ. Choose $\tilde{\gamma}$ such that $\underset{\varphi(u(s), v(s))}{\overset{\shortparallel}{\varphi(\tilde{\gamma}(s))}} = \gamma(s)$

Since $e_1 \perp N$, then

$$\langle e_1(\gamma(s)), \frac{\nabla}{ds} e_2(\gamma(s)) \rangle = \langle e_1(\gamma(s)), \frac{d}{ds}(e_2(\gamma(s))) \rangle$$

Chain rule:

$$\frac{d}{ds}(e_2(\gamma(s))) = De_2(\dot{u}\partial_u + \dot{v}\partial_v) = \dot{u}\frac{\partial e_2}{\partial u} + \dot{v}\frac{\partial e_2}{\partial v}$$

$$\begin{aligned}
\mathcal{I} &= \int_0^l \left(\langle e_1, \frac{\partial e_2}{\partial u} \rangle \dot{u} + \langle e_1, \frac{\partial e_2}{\partial v} \rangle \dot{v} \right) ds \\
&= \int_{V = \text{int}\,\tilde{\gamma}} \frac{\partial}{\partial u}\left(\langle e_1, \frac{\partial e_2}{\partial v} \rangle \right) - \frac{\partial}{\partial v}\left(\langle e_1, \frac{\partial e_2}{\partial u} \rangle \right) du\, dv \\
&= \int_V \langle \frac{\partial e_1}{\partial u}, \frac{\partial e_2}{\partial v} \rangle - \langle \frac{\partial e_1}{\partial v}, \frac{\partial e_2}{\partial u} \rangle du\, dv \quad \text{since } \frac{\partial^2 e_2}{\partial u \partial v} = \frac{\partial^2 e_2}{\partial v \partial u} \\
&= \int_V \langle \frac{\partial N}{\partial u} \times \frac{\partial N}{\partial v}, N \rangle \quad \text{by Lemma 2(ii)} \\
&= \int_V K |\partial_u \times \partial_v|\, du\, dv \quad \text{by Lemma 2(i)} \\
&= \int_{\varphi(V)} K\, dS
\end{aligned}$$

Also $\mathcal{I} = \int_0^l (\dot\alpha(s) - \kappa_g(s))\, ds$

$$= 2\pi - \int_\gamma \kappa_g(s)\, ds$$

$$\Rightarrow \int_{\mathrm{int}(\gamma)} K\, dS + \int_\gamma \kappa_g(s)\, ds = 2\pi \qquad \square$$

Remark

Orientation is important!

Suppose γ as above, $\hat\gamma(s) := \gamma(l-s)$

$$\kappa_g^{\hat\gamma}(s) = -\kappa_g^{\gamma}(l-s)$$

$$\int_{\hat\gamma} \kappa_g^{\hat\gamma} = \int_0^l \kappa_g^{\hat\gamma}\, ds = -\int_0^l \kappa_g^{\gamma}(l-s)\, ds$$

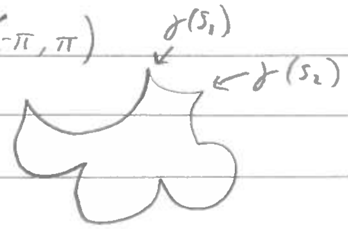
$$= \int_l^0 \kappa_g^{\gamma}\, ds = -\int_0^l \kappa_g^{\gamma}\, ds$$

$\Rightarrow$ positively oriented changes sign of 2^{nd} integral in Gauss Bonnet (K doesn't change)

We may define exactly as we did in week 2 external angles for a chain.

So if we have an oriented frame $\{e_1, e_2\}$ on V s.t. $\dot\gamma = \cos\alpha\, e_1 + \sin\alpha\, e_2$ for some $\alpha : I \to \mathbb{R}$, where s_i is a non-differentiability point, then we may define the exterior angle

$$\theta_i = -\lim_{s \nearrow s_i} \alpha + \lim_{s \searrow s_i} \alpha \in (-\pi, \pi)$$



MATH 3113

05-12-17

From now on the $\gamma(s_i) \in \Sigma$ will be called vertices and the image of the intervals $\gamma([s_i, s_{i+1}])$ will be called edges

If γ is a chain which is closed and simple then it is called a generalised polygon and int γ exists.

Theorem (Local Gauss-Bonnet for chains)
Let $P \subset \Sigma$ be a generalised polygon with exterior angles $\theta_1, \dots, \theta_n$.
Then
$$\int_{\partial P} \kappa_g \, dl + \int_P K \, dS + \sum_{i=1}^{n} \theta_i = 2\pi.$$

Proof
Exercise.

Examples
1) In $\mathbb{R}^2$ we could consider polygons

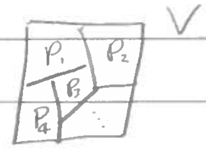
$$\left.\begin{array}{l} \kappa_g = 0 \\ K \equiv 0 \end{array}\right\} \Rightarrow \sum_{i=1}^{5} \theta_i = 2\pi$$

2). Let P be a geodesic triangle in S^2

$$K \equiv 1, \quad \kappa_g = 0 \Rightarrow \sum_{i=1}^{3} \theta_i = 2\pi - \int_P K \, dS$$
$$= 2\pi - \text{area}(P)$$

θ_i are exterior angles.

Def (Polygonal Covers, Euler Characteristic)
Let Σ be a surface, $V \subset \Sigma$ be compact.
A polygonal cover is a set $P = \{P_1, \dots, P_m\}$ where
(i) each P_i is a generalised polygon
i.e. $\exists$ an injective chain γ_i in Σ s.t. $P_i = \text{int}\,\gamma_i$.
(ii) $V = \bigcup_{i=1}^{m} \overline{P_i}$
(iii) The intersections $\overline{P_i} \cap \overline{P_j}$ for $i \neq j$ are either empty or intersect along one entire edge or intersect at one vertex.



Denote • $f(P)$ = number of faces (polygons) $= m$
• $e(P)$ = number of edges
• $v(P)$ = number of vertices

The Euler Characteristic of P is defined to be
$\chi(P) = f(P) - e(P) + v(P)$

Remark
We may always cut our polygonal covers into smaller subdivisons – a refinement of the polygonal cover.

Prop
Let V be a compact subset of a surface Σ, and suppose ∂V is piecewise smooth.
(i) $\exists$ a finite triangulation of V (polygonal cover such that all polygons are triangles)
(ii) The Euler characteristic of two covers of V are the same.
(iii) If there is a homeomorphism $h: V \to V'$ then $\chi(V) = \chi(V')$.

MATH 3113

05-12-17

Examples

1). $S =$ a square $\quad P =$ just the square

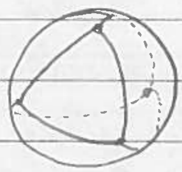
$v(P) = 4, \quad e(P) = 4, \quad f(P) = 1$

$\Rightarrow \chi(P) = 1 - 4 + 4 = 1$

$\tilde{P}$ is  $v(\tilde{P}) = 5, \quad e(\tilde{P}) = 8, \quad f(\tilde{P}) = 4$

$\Rightarrow \chi(\tilde{P}) = 4 - 8 + 5 = 1$

2). S^2

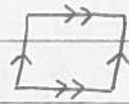
P is $\rightarrow$  $\longrightarrow$ 

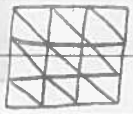
$v(P) = 4$

$e(P) = 6$

$f(P) = 4$ $\qquad \Rightarrow \chi(S^2) = \chi(P) = 4 - 6 + 4 = 2$

3). T^2

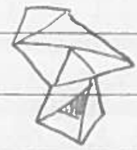


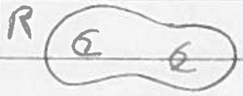
$P \rightarrow$  $v(P) = 9, \quad e(P) = 27, \quad f(P) = 18$

$\chi(P) = 18 - 27 + 9 = 0$

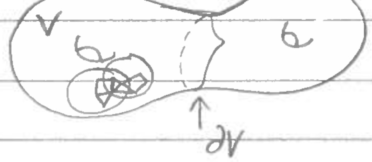
4). Consider a triangulation of some compact $V \subset \mathbb{R}^2$. Remove a face to obtain $\tilde{V}$.

Then $\chi(\tilde{V}) = \chi(V) - 1$



[ $\chi(R) = -2$]

06-12-17



Plan: Cut a region into sufficiently small triangles, apply local G-B to obtain a global G-B.

Theorem (global Gauss-Bonnet)
Let Σ be an orientable surface, $V \subset \Sigma$ be compact and have a piecewise smooth boundary with exterior angles $\theta_1, \dots, \theta_n$. Then

$$\int_V K \, dS + \int_{\partial V} \kappa_g \, dl + \sum_{i=1}^{n} \theta_i = 2\pi \chi(V)$$

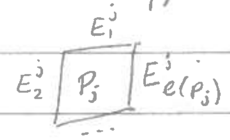
Proof
Take a polygonal cover $\mathcal{P} = \{P_1, \dots, P_m\}$ s.t. each P_i is contained in one coordinate neighbourhood

Remark
On a polygonal cover, each interior edge is positively parameterised in opposite directions for each of the two faces adjoining it.

Apply local Gauss-Bonnet for each P_i.

$$\sum_{j=1}^{m} 2\pi = \sum_{j=1}^{m} \int_{\partial P_j} \kappa_g^j \, dl + \sum_{j=1}^{m} \int_{P_j} K \, dS + \sum_{j=1}^{m} \sum_{i=1}^{v(P_j)} \theta_i^j$$

where $\theta_i^j \quad 1 \le j \le v(P_j)$ are the external angles of P_j



$$\sum_{j=1}^{m} \int_{P_j} K \, dS = \int_V K \, dS$$

$$\sum_{j=1}^{m} \int_{\partial P_j} \kappa_g^j \, dl = \sum_{j=1}^{m} \sum_{i=1}^{e(P_j)} \int_{E_i^j} \kappa_g \, dl$$

But due to our remark, along any interior edge κ_g is integrated

MATH3113

06 - 12 - 17

in both directions.

$$\Rightarrow \sum_{j=1}^{m} \int_{\partial P_j} \kappa_g^{\,j}\, dl = \int_{\partial V} \kappa_g\, dl$$

Define interior angles as $\phi_i^{\,j} = \pi - \theta_i^{\,j}$

$$\sum_{i=1}^{v(P_j)} \theta_i^{\,j} = \sum_{i=1}^{v(P_j)} (\pi - \phi_i^{\,j})$$

$$= \pi v(P_j) - \sum_{i=1}^{v(P_j)} \phi_i^{\,j}$$

number of edges = number of vertices

$$= \pi e(P_j) - \sum_{i=1}^{v(P_j)} \phi_i^{\,j}$$

n = number of boundary edges

$$\sum_{j=1}^{m} \pi e(P_j) = 2\pi e(\mathcal{P}) - n\pi$$

interior angles at boundary vertices

$$\sum_{j=1}^{m} \sum_{i=1}^{v(P_j)} \phi_i^{\,j} = 2\pi(v(\mathcal{P}) - n) + \sum_{k=1}^{n} \phi_k$$

exterior boundary angles

$$= 2\pi v(\mathcal{P}) - \pi n - \sum_{k=1}^{n} \theta_k$$

$$\Rightarrow \sum_{j=1}^{m} \sum_{i=1}^{v(P_j)} \theta_i^{\,j} = 2\pi e(\mathcal{P}) - \cancel{n\pi} - 2\pi v(\mathcal{P}) + \cancel{\pi n} + \sum_{k=1}^{n} \theta_k$$

$m = f(\mathcal{P})$

$$\Rightarrow 2\pi f(\mathcal{P}) = 2\pi e(\mathcal{P}) - 2\pi v(\mathcal{P}) + \sum_{k=1}^{n} \theta_k + \int_V K\, dS + \int_{\partial V} \kappa_g\, dl$$

$$\Rightarrow \int_V K\, dS + \int_{\partial V} \kappa_g\, dl + \sum_{i=1}^{n} \theta_i = 2\pi \chi(\mathcal{P}) = 2\pi \chi(V)$$

$\square$

<u>Corollary</u> G-B for compact surfaces.

Suppose Σ is compact and orientable, then

$$\int_\Sigma K\, dS = 2\pi \chi(\Sigma)$$

<u>Corollary</u>

Suppose V is homeomorphic to a disc, ∂V piecewise smooth with boundary angles $\theta_1, \dots, \theta_n$, then

$$\int_V K\, dS + \int_{\partial V} \kappa_g\, dl + \sum_{i=1}^{n} \theta_i = 2\pi \qquad [\chi(\text{disc}) = 1]$$

Classification of compact surfaces

If Σ is compact, orientable, then the surface is entirely determind up to homeomorphism by its genus, g. For such a surface, $g = \frac{1}{2}(2 - \chi(\Sigma))$

g integer ≥ 0 $\Rightarrow \chi(\Sigma) \in \{2, 0, -2, -4, -6, \dots\}$

Corollary

Suppose Σ is compact and $K > 0$ everywhere.
Then Σ is homeomorphic to a sphere.
Furthermore any two simple closed geodesics intersect.

Proof

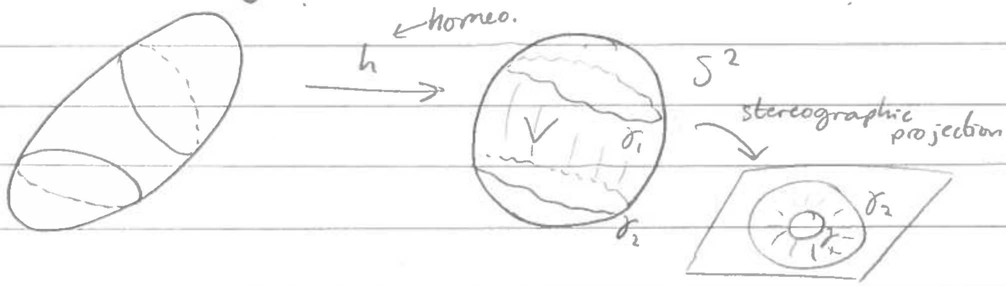
$$\int K \, dS > 0$$

$$\int K \, dS = 2\pi \chi(\Sigma) \Rightarrow \int K \, dS = 4\pi \text{ and } g = 0, \ \chi(\Sigma) = 2$$

$\Rightarrow \Sigma$ is homeomorphic to a sphere.

Suppose we have two simple closed geodesics γ_1, γ_2 which are disjoint.
There is a reigion bounded by γ_1 and γ_2 which is homeomorphic to a cylinder.

$\chi(S^2) = 2$
$\chi(\text{cube}) = 2$



$\chi(\text{Cylinder}) = 0$
since cylinder is cube − 2 faces ()

$$\Rightarrow \int_V K = 0 \quad ※ \quad \square$$

MATH 3113

06-12-17

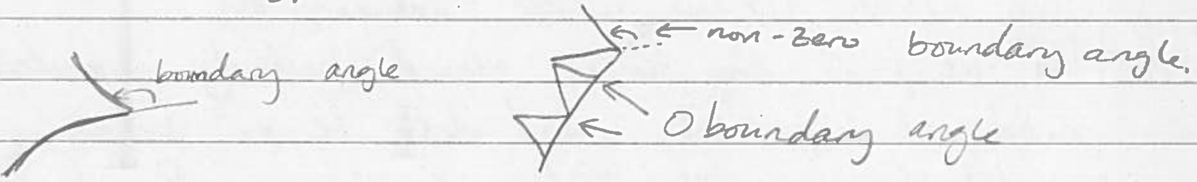
Suppose $K<0$ and Σ is homeomorphic to a cylinder.
Then $\exists$ at most one simple closed geodesic.
May assume any closed simple curve on surface homeomorphic to a cylinder either bounds a disc or goes once around the cylinder.

12-12-17

Last time:

Theorem G-B (general)
Suppose Σ is orientable, $V \subset \Sigma$ is compact with piecewise smooth boundary ∂V and external angles $\theta_1, \dots, \theta_n$, then

$$\int_V K\,dS + \int_{\partial V} \varkappa_g\,dl + \sum_{i=1}^{n} \theta_i = 2\pi \chi(V)$$



Corollary G-B with smooth boundary
Suppose V, Σ as above but ∂V is smooth, then

$$\int_V K\,dS + \int_{\partial V} \varkappa_g\,dl = 2\pi\chi(V)$$

Corollary G-B without boundary
Suppose Σ is compact (as metric space), then

$$\int_\Sigma K\,dS = 2\pi\chi(\Sigma)$$

Corollary
Suppose $V \subset \Sigma$ ← orientable is homeomorphic to a disc, ∂V is smooth, then

$$\int_V K\,dS + \int_{\partial V} \varkappa_g\,dl = 2\pi$$

Application: Hairy Ball Thm

You cannot comb a hairy ball:

If $X \in \mathfrak{X}^\infty(S^2)$ then $\exists p \in S^2$ s.t. $X(p) = 0$.

Def

p_0 is an isolated stationary point of a vector field $X \in \mathfrak{X}(\Sigma)$ (Σ orientable surface) if $X(p_0) = 0$ and $\exists$ an open set $W \subset \Sigma$ s.t. $p_0 \in W$ and $X(q) \neq 0 \; \forall q \in W, q \neq p_0$.

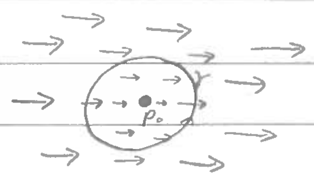
We define the index of an isolated stationary point p_0 as follows:

Suppose $\varphi : U \to V$ is a parameterisation of Σ and wlog $W \subset V$ (W from isolated stationary pt).

Then let γ is any simple closed positively parameterised curve s.t. $p_0 \in \operatorname{int} \gamma$ and let $\{e_1, e_2\}$ be an oriented frame on V. Define $\beta : W \setminus p_0 \to \mathbb{R}$ to be a function s.t. $\tilde{X} := \frac{X}{|X|} = \cos\beta e_1 + \sin\beta e_2$.

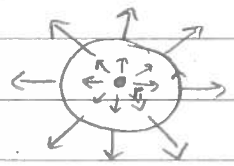
Then the index of p_0 is defined by

$$\mu_X(p_0) = \frac{1}{2\pi} \int_\gamma \dot{\beta} \, dl.$$



$\mu_X(p_0) = 0$

$\mu_X(p_2) = 1$



$\mu_X(p_1) = 1$

MATH 3113

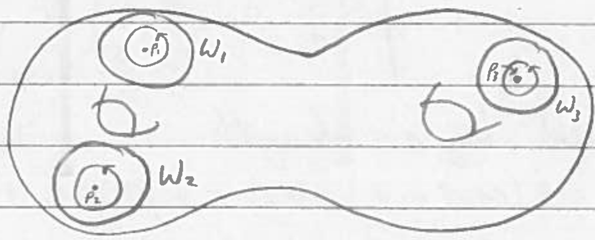
12-12-17

Poincaré Hopf Theorem

Suppose Σ is compact oriented and $X \in \mathfrak{X}^\infty(\Sigma)$ has n isolated stationary points $p_1, \dots, p_n$.
Then (the index is well-defined and)
$$\sum_{i=1}^{n} \mu_x(p_i) = \chi(\Sigma)$$

Proof of Hairy Ball Thm
Suppose $\Sigma = S^2$ then $\sum_{i=1}^{n} \mu_x(p_i) = 2$
$\Rightarrow \exists$ at least one stationary point. $\square$

Proof
Take local disjoint neighbourhoods W_i of p_i and curves γ_i as above.



On W_i define $\{e_1, e_2\}$ an oriented frame and $Q = \bigcup_{i=1}^{n} W_i$

On $\Sigma \setminus Q$ we define oriented frame $f_1 = \tilde{X} = \frac{X}{|X|}$, ← well-defined ($|X| \neq 0$ on $\Sigma \setminus Q$)
$f_2 = N \times f_1$.
Recall: for any positively oriented local curve γ
$$\int_{\text{int}\gamma} K \, dS = \int_\gamma \langle e_1, \frac{\nabla}{ds} e_2 \rangle \, dl.$$

From this we have that
$$\sum_{i=1}^{n} \int_{\text{int}\gamma_i} K \, dS = \sum_{i=1}^{n} \int_{\gamma_i} \langle e_1, \frac{\nabla}{ds} e_2 \rangle \, dl \quad ① \checkmark$$

$$\int_{\Sigma \setminus Q} K \, dS = -\sum_{i=1}^{n} \int_{\gamma_i} \langle f_1, \frac{\nabla}{ds} f_2 \rangle \, dl \quad ②$$

② : Observe that on any triangle T_j
$$\int_{T_j} K \, dS = \int_{\partial T_j} \langle f_1, \frac{\nabla}{ds} f_2 \rangle \, dl$$

Take $T_1, \dots, T_m$ a triangulation of $\Sigma \setminus Q$ and sum

$$\Rightarrow \int_{\Sigma \setminus Q} K\, dS = \sum_{j=1}^{m} \int_{T_j} K\, dS$$

$$= \sum_{j=1}^{m} \int_{\partial T_j} \langle f_1, \frac{\nabla}{ds} f_2 \rangle\, dl$$

$$= -\sum_{i=1}^{n} \int_{\gamma_i} \langle f_1, \frac{\nabla}{ds} f_2 \rangle\, dl$$

due to orientation

Gauss - Bonnet

$$\Rightarrow 2\pi \chi(\Sigma) = \int_{\Sigma \setminus Q} K\, dS + \sum_{i=1}^{n} \int_{\text{int}\,\gamma_i} K\, dS$$

$$= \sum_{i=1}^{n} \int_{\gamma_i} \langle e_1, \frac{\nabla}{ds} e_2 \rangle - \langle f_1, \frac{\nabla}{ds} f_2 \rangle\, dl$$

By definition we have

$$f_1 = \cos\beta\, e_1 + \sin\beta\, e_2$$
$$f_2 = -\sin\beta\, e_1 + \cos\beta\, e_2$$

$$\left[\langle \frac{\nabla}{ds} e_i, e_i \rangle = 0, \quad \langle \frac{\nabla}{ds} e_1, e_2 \rangle = -\langle e_1, \frac{\nabla}{ds} e_2 \rangle \right]$$

W.l.o.g. γ_i parameterised by arc length

$$\langle f_1, \frac{\nabla}{ds} f_2 \rangle = \langle f_1, -\dot\beta \underbrace{(\cos\beta\, e_1 + \sin\beta\, e_2)}_{f_1} - \sin\beta \frac{\nabla}{ds} e_1 + \cos\beta \frac{\nabla}{ds} e_2 \rangle$$

$$= -\dot\beta - \sin^2\beta \langle \frac{\nabla}{ds} e_1, e_2 \rangle + \cos^2\beta \langle \frac{\nabla}{ds} e_2, e_1 \rangle$$

$$= -\dot\beta + \langle e_1, \frac{\nabla}{ds} e_2 \rangle$$

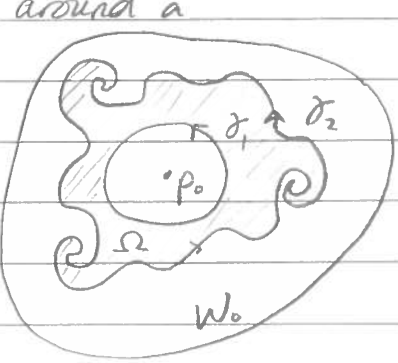
$$\Rightarrow 2\pi \chi(\Sigma) = \sum_{i=1}^{n} \int_{\gamma_i} \dot\beta\, dl$$

Well-defined:

Suppose γ_1, γ_2 are two disjoint positively parameterised curves around a stationary point p_0 of a v.f. $X \in \mathfrak{X}^\infty(\Sigma)$.

As before

$$\int_{\Omega} K\, dS = \int_{\gamma_2} \langle f_1, \frac{\nabla}{ds} f_2 \rangle\, dl - \int_{\gamma_1} \langle f_1, \frac{\nabla}{ds} f_2 \rangle\, dl$$



MATH 3113

12-12-17

$$\Rightarrow \int_{\Omega} K\, dS = \int_{\gamma_2} -\dot{\beta}_2 + \langle e_1, \tfrac{\nabla}{ds} e_2 \rangle\, dl \;-\; \int_{\gamma_1} -\dot{\beta}_1 + \langle e_1, \tfrac{\nabla}{ds} e_2 \rangle\, dl$$

Recall from "lemma for G-B",

$$\kappa_g = \dot{\alpha} - \langle e_1, \tfrac{\nabla}{ds} e_2 \rangle$$

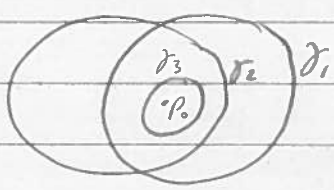
$$\Rightarrow \int_{\Omega} K\, dS = \int_{\gamma_2} -\dot{\beta}_2 + \dot{\alpha}_2 - \kappa_{g_2}\, dl \;-\; \int_{\gamma_1} -\dot{\beta}_1 + \dot{\alpha}_1 - \kappa_{g_1}\, dl$$

angular function w.r.t. γ_2 etc $\quad [\beta_i = \beta(\gamma_i)]$

$\int \dot{\alpha}_i$ cancel and so

$$\Rightarrow \underbrace{\int_{\Omega} K\, dS + \underbrace{\int_{\gamma_2} \kappa_{g_2}\, dl - \int_{\gamma_1} \kappa_{g_1}\, dl}_{= \int_{\partial\Omega} \kappa_g\, dl}}_{= 2\pi \chi(\Omega)} = \int_{\gamma_2} \dot{\beta}_2\, dl - \int_{\gamma_1} \dot{\beta}_1\, dl$$

Ω is an annulus $\Rightarrow \chi(\Omega) = 0$



This also holds for these γ_i's. $\square$

Hyperbolic Plane

Claim: $\exists\, \varphi : B_1(0) \to \mathbb{R}^3$ s.t. $\tilde{g}_{ij} = f\delta_{ij}$, $f = \dfrac{1}{(1-u^2-v^2)^2}$

Calculate Γ_{ij}^k

$$\Gamma_{ij}^k = \frac{1}{2f}\left(\frac{\partial f}{\partial u^j}\delta_{ik} + \frac{\partial f}{\partial u^i}\delta_{jk} - \frac{\partial f}{\partial u^k}\delta_{ij} \right)$$

Claim: φ(y-axis) is a geodesic.

$\tilde{\gamma}(t) = \varphi(0,t)$

$$l(\tilde{\gamma}|_{[0,t]}) = \int_0^t |\tilde{\gamma}'(y)|\, dy = \int_0^t \frac{1}{1-y^2}\, dy = \tanh^{-1}(t)$$

Prove that $\gamma(s) = \varphi(0, \tanh s)$ is a geodesic.

If $\gamma = \varphi(u(s), v(s))$

$$u'' + (u')^2 \Gamma_{uu}^{u} + 2u'v' \Gamma_{uv}^{u} + (v')^2 \Gamma_{vv}^{u} = 0$$
$$v'' + (u')^2 \Gamma_{uu}^{v} + 2u'v' \Gamma_{uv}^{v} + (v')^2 \Gamma_{vv}^{v} = 0$$

$u(s) = 0$, $v(s) = \tanh s$

$v'(s) = 1/\cosh^2(s)$ $\quad$ $v''(s) = \dfrac{-2\sinh s}{\cosh^3 s}$

So WTS $\begin{cases} (v')^2 \Gamma_{vv}^{u} = 0 & ① \\ v'' + (v')^2 \Gamma_{vv}^{v} = 0 & ② \end{cases}$

$\Gamma_{vv}^{u} = \dfrac{-1}{2f} \dfrac{\partial f}{\partial u} = 0$ on v-axis

$\Gamma_{vv}^{v} = \dfrac{1}{2f} \dfrac{\partial f}{\partial v}$

$\dfrac{\partial f}{\partial u} = \dfrac{4u}{(1-u^2-v^2)^3}$, $\dfrac{\partial f}{\partial v} = \dfrac{4v}{(1-u^2-v^2)^3}$

$\Rightarrow$ ① ✓

② : $\dfrac{-2\sinh s}{\cosh^3 s} + \dfrac{1}{\cosh^4 s} \cdot \dfrac{2\tanh s}{(1-\tanh^2 s)}$

$$= \frac{-2\sinh s}{\cosh^3 s} + \frac{2\sinh s}{\cosh^3 s} = 0 \quad ✓$$

MATH 3113

13-12-17

In $\mathbb{R}^3$ we may take derivatives of v.f.'s w.r.t. vf:

$$\nabla_X^{\mathbb{R}} Y\Big|_p = \frac{d}{dt}\Big|_{t=0} Y(\gamma(t)) \quad , \gamma(0)=p \quad , \gamma'(0)=X.$$

$$= \sum_{\substack{i=1\\ j=1}}^{3} X^i \frac{\partial Y^j}{\partial x^i} e_j \qquad , \quad X = \sum_{i=1}^{3} X^i e_i$$

$\Sigma \subset \mathbb{R}^3$, X, Y vfs on Σ then

$$\nabla_X Y = \left(\nabla_X^{\mathbb{R}^3} Y\right)^T \leftarrow \text{orthogonal projection}$$

$$= \nabla_X^{\mathbb{R}^3} Y - \langle \nabla_X^{\mathbb{R}^3} Y, N \rangle N$$

Let $Y = Y^1\partial_1 + Y^2\partial_2$

$$\nabla_{\partial_i} Y = \partial_i(Y^1)\partial_1 + Y^1\nabla_{\partial_i}\partial_1 + \partial_i(Y^2)\partial_2 + Y^2\nabla_{\partial_i}\partial_2$$

$$= \frac{\partial Y^1}{\partial u^i}\partial_1 + Y^1 \sum_{\substack{j=1\\ k=1}}^{2} \Gamma_{1j}^{\,k}\partial_k + \frac{\partial Y^2}{\partial u^i}\partial_2 + Y^2 \sum_{\substack{j=1\\ k=1}}^{2} \Gamma_{2j}^{\,k}\partial_k$$

$$= \sum_{j=1}^{2}\left[\frac{\partial Y^j}{\partial u^i} + \sum_{k=1}^{2} Y^k \Gamma_{ki}^{\,j}\right]\partial_j$$

$\varphi : B_1(0) \to$ "$\mathbb{R}^3$" s.t. $\tilde{g}_{ij} = f\,\delta_{ij}$

$$f = \frac{1}{(1-u^2-v^2)^2}$$

We showed  is a geodesic

Calculate K at $(0,0)$.

Problem: we can't calculate II (2nd fund. form).

$$K = \frac{R_{uvuv}}{\det\tilde{g}} = \frac{\langle \nabla_{\partial_u}(\nabla_{\partial_v}\partial_v) - \nabla_{\partial_v}(\nabla_{\partial_u}\partial_v), \partial_u \rangle}{\det\tilde{g}}$$

$$\langle \partial_u, \partial_v \rangle = 0$$

$$\Gamma_{ij}^{\,k} = \frac{1}{2f}\left(\frac{\partial f}{\partial u^i}\delta_{jk} + \frac{\partial f}{\partial u^j}\delta_{ik} - \frac{\partial f}{\partial u^k}\delta_{ij}\right)$$

$$\nabla_{\partial_u}\partial_u = \Gamma_{uu}^{\,u}\partial_u + \Gamma_{uu}^{\,v}\partial_v$$

$$\nabla_{\partial_v}\partial_v = \Gamma_{vv}^{\,u}\partial_u + \Gamma_{vv}^{\,v}\partial_u$$

$$\nabla_{\partial_u}\partial_v = \Gamma_{vu}^{\,u}\partial_u + \Gamma_{vu}^{\,v}\partial_v = \nabla_{\partial_v}\partial_u$$

$$\langle \nabla_{\partial_u}(\Gamma_{vv}^{u}\partial_u + \Gamma_{vv}^{v}\partial_v) - \nabla_{\partial_v}(\Gamma_{vu}^{u}\partial_u + \Gamma_{vu}^{v}\partial_v), \partial_u \rangle$$

can drop some terms since $\langle \partial_v, \partial_u \rangle = 0$

$$\nabla_{\partial_k}(\nabla_{\partial_i}\partial_j) = \nabla_{\partial_k}\left(\sum_{l=1}^{2}\Gamma_{ij}^{l}\partial_l\right)$$

Liebnitz rule

$$= \sum_{l=1}^{2}\frac{\partial \Gamma_{ij}^{l}}{\partial u^k}\partial_l + \sum_{\substack{l=1\\m=1}}^{2}\Gamma_{ij}^{l}\Gamma_{lk}^{m}\partial_m$$

$$= \sum_{l=1}^{2}\left[\frac{\partial \Gamma_{ij}^{l}}{\partial u^k} + \sum_{n=1}^{2}\Gamma_{ij}^{n}\Gamma_{nk}^{l}\right]\partial_l$$

$$K \det \tilde{g} = \langle \nabla_{\partial_u}(\nabla_{\partial_v}\partial_v) - \nabla_{\partial_v}(\nabla_{\partial_u}\partial_v), \partial_u \rangle$$

$$= \langle \partial_u, \partial_u \rangle \left[\frac{\partial \Gamma_{vv}^{u}}{\partial u} + \sum_{n=1}^{2}\Gamma_{vv}^{n}\Gamma_{nu}^{u} - \frac{\partial \Gamma_{uv}^{u}}{\partial v} - \sum_{m=1}^{2}\Gamma_{uv}^{m}\Gamma_{mv}^{u}\right]$$

$$\frac{\partial f}{\partial u} = \frac{4u}{(1-u^2-v^2)^3}, \quad \frac{\partial f}{\partial v} = \frac{4v}{(1-u^2-v^2)^3}$$

$\Rightarrow$ at $(0,0)$, $\Gamma_{ij}^{k} = 0$

$$\Rightarrow K \det \tilde{g} = \langle \partial_u, \partial_u \rangle \left[\frac{\partial \Gamma_{vv}^{u}}{\partial u} - \frac{\partial \Gamma_{uv}^{u}}{\partial v}\right]$$

$$\Gamma_{vv}^{u} = \frac{1}{2f}\left(-\frac{\partial f}{\partial u}\right)$$

$$\Gamma_{uv}^{u} = \frac{1}{2f}\left(\frac{\partial f}{\partial v}\right) \qquad \tilde{g}_{ij} = f\,\delta_{ij}$$

$$K|_{(0,0)} \det \tilde{g} = -f\left[\frac{\partial}{\partial u}\left(\frac{\partial f}{\partial u}\Big/2f\right) + \frac{\partial}{\partial v}\left(\frac{\partial f}{\partial v}\Big/2f\right)\right]$$

$$\frac{\partial}{\partial u}\left(\frac{f_u}{2f}\right)\Bigg|_{(0,0)} = \frac{2f f_{uu} - 2f_u^2}{4f^2}\Bigg|_{(0,0)} \qquad f_{uu} = \frac{4(1-u^2-v^2)^3 - 4u(1-u^2-v^2)^2(3(-2u))}{(1-u^2-v^2)^3}$$

$$= \frac{2f \cdot 4}{4f^2}\Bigg|_{(0,0)}$$

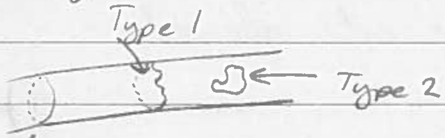
$$\Rightarrow K|_{(0,0)} \det \tilde{g} = -f\left(\frac{2}{f} + \frac{2}{f}\right)\Bigg|_{(0,0)} = -4$$

$$\det \tilde{g}|_{(0,0)} = f^2|_{(0,0)} = 1 \quad \Rightarrow K = -4.$$

MATH 3113

13-12-17

Suppose Σ a surface st. $K < 0$ everywhere and Σ homeomorphic to a cylinder. Show there is at most one closed injective geodesic.



Suppose $\exists$ 2 such geodesics.

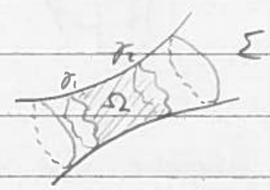
Aim: G-B.

Suppose γ "Type 2" geodesic curve

G-B (homeo. to a disc)

$\Rightarrow \int_{int_\gamma} K = 2\pi$ ※ since $K<0$.

$\Rightarrow$ both curves wind once around Σ.



Do they intersect?

Suppose not.

$\Rightarrow \gamma_1 \cap \gamma_2 = \emptyset$ and they both wind once around Σ

$\Rightarrow \gamma_1, \gamma_2$ bound a region Ω (compact)

G-B on Ω:

$$\int_\Omega K\, dS = 2\pi \chi(\Omega) = 0 \quad ※$$

$\Rightarrow \gamma_1 \cap \gamma_1 \neq \emptyset$

Suppose γ_1, γ_2 distinct.

Suppose they intersect at 1 point

$\Rightarrow \gamma_1, \gamma_2$ must intersect tangentially at p.

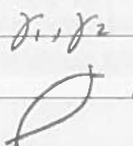
$\Rightarrow \gamma_1(p) = \gamma_2(p) \Rightarrow \dot{\gamma}_1(p) = \dot{\gamma}_2(p)$

$\Rightarrow$ locally $\gamma_1 = \gamma_2$ — due to existence and uniqueness of geodesic.

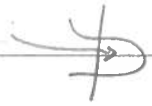
$\Rightarrow \gamma_1 = \gamma_2$ everywhere

$\Rightarrow \gamma_1, \gamma_2$ intersect in at least 2 points

$\Rightarrow \gamma_1, \gamma_2$ bound some reigion

 not allowed due to hw9

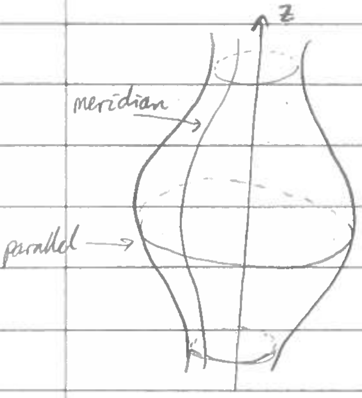
 not allowed due to injectivity

We will always get  each time we

needs some topological stuff!

push one curve over the other giving reigions which are not allowed.

□



$\varphi : (-\pi,\pi)\times \mathbb{R} \longrightarrow \mathbb{R}^3$

$\varphi(u,v) = (f(v)\cos u,\ f(v)\sin u,\ g(v))$

where $(f(v), g(v))$ parameterise the profile curve.

What are the geodesics?

Suppose we have a geodesic param. by arc length $\varphi(u(t), v(t))$.

$$\tilde{g} = \begin{pmatrix} f^2 & 0 \\ 0 & f'^2+g'^2 \end{pmatrix}$$

$$u'' + \frac{2f(v)f'(v)u'v'}{f^2(v)} = 0,$$

$$v'' - \frac{f(v)f'(v)}{(f'(v))^2+(g'(v))^2}(u')^2 + \frac{f'(v)f''(v)+g'(v)g''(v)}{(f'(v))^2+(g'(v))^2}(v')^2 = 0$$

meridians are given by $u = \text{const}$

parallels " " " $v = \text{const}$

integrating 1st eqn

$$\frac{(u'f^2)'}{f^2} = 0$$

$\Rightarrow u'f^2 = \text{const}$ on geodesic

$\Rightarrow r\cos\theta = \text{const}$ on geodesic

$$\Rightarrow v'' + \frac{f'f''+g'g''}{f'^2 g'^2}v'^2 = 0$$

$$= \frac{(\overbrace{v'(f'^2+g'^2)}^{\text{const.}})'}{f'^2+g'^2}$$

when $f' = 0$