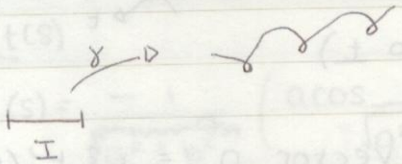
3113/M113 Differential Geometry Notes

Based on the 2012 autumn lectures by Prof R Halburd

Chapter 1 : Differential Geometry - The local theory of curves.

Definition:
A (parametrised) differentiable curve is a differentiable map
$\gamma : I \longrightarrow \mathbb{R}^3$
The set $\gamma(I) \subset \mathbb{R}^3$ is called the trace of γ



For any $t \in I$, $\gamma'(t)$ is called the velocity of γ
if it is non-zero then $\gamma'(t)$ is the tangent to γ at $\gamma(t)$

Definition:
A differentiable curve $\gamma : I \longrightarrow \mathbb{R}^3$ is said to be regular if
$\gamma'(t) \neq 0 \ \forall t \in I$.

Example:
The helix $\gamma : \mathbb{R} \longrightarrow \mathbb{R}^3$ given by $\gamma(t) = (a\cos t, a\sin t, bt)$ $a>0, b>0$
is a regular curve

Example:
The curve $\gamma : (-1,1) \longrightarrow \mathbb{R}^3$ given by $\gamma(t) = (t^3, t^2, 0)$ is not
regular at $t=0$ (and on $(-1,1)$) $(\gamma'(t) = (3t^2, 2t, 0))$

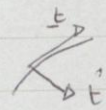
Definition:
For any curve $\gamma : I \longrightarrow \mathbb{R}^3$ and any $t \in I$ the arclength of γ
from t_0 is $s \equiv s(t) = \int_{t_0}^{t} |\dot{\gamma}(u)| \, du$ $\left(= \int \sqrt{\left(\frac{dx}{du}\right)^2 + \left(\frac{dy}{du}\right)^2 + \left(\frac{dz}{du}\right)^2} \, du \right)$

(for most purposes the choice of t_0 is not important).

Example:
$\gamma(t) = (a\cos t, a\sin t, bt)$
$\dot{\gamma}(t) = (-a\sin t, a\cos t, bt)$ $\quad |\dot{\gamma}(t)|^2 = a^2 + b^2$
So $s(t) = \int \sqrt{a^2+b^2} \, dt = \sqrt{a^2+b^2}\, t$ $\quad (t_0 = 0)$
We can reparametrise the curve
$$\tilde{\gamma}(s) = \gamma(t) = \left(a\cos \frac{s}{\sqrt{a^2+b^2}}, a\sin \frac{s}{\sqrt{a^2+b^2}}, \frac{bs}{\sqrt{a^2+b^2}} \right)$$

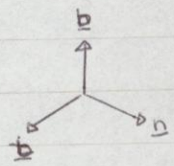
If γ is parametrised by arclength then $\gamma'(s)$ is a unit vector. (unit speed curve)

The Frenet Frame



$\underline{t}(s) = \gamma'(s)$ unit tangent vector
($\underline{t} \cdot \underline{t} = 1 \Rightarrow 2\underline{t}' \cdot \underline{t}' = 0 \Rightarrow \underline{t}'$ is orthogonal to $\underline{t}$)
$k(s) = |\underline{t}'(s)|$ is called the curvature
If $k \neq 0$, we define the principal normal vector $\underline{n} := \frac{1}{k} \underline{t}'(s)$

The unit binormal is $\underline{b} = \underline{t} \cdot \underline{n}$



$\{\underline{t}, \underline{n}, \underline{b}\}$ is called the Frenet Frame of γ

$\underline{b} = \underline{t} \times \underline{n}$
$\Rightarrow \underline{b}' = \underbrace{\underline{t}' \times \underline{n}}_{\substack{= 0 \\ \text{since} \\ \underline{t}' = k\underline{n}.}} + \underline{t} \times \underline{n}' = \underline{t} \times \underline{n}'$

$\Rightarrow \underline{b}'$ is orthogonal to $\underline{t}$
Also $\underline{b}'$ is orthogonal to $\underline{b}$
So ~~...~~ $\exists$ scalar τ (torsion) such that $\underline{b}' =$ ~~...~~ $\tau \underline{n}$

Now $\underline{n} = \underline{b} \times \underline{t}$
$\Rightarrow \underline{n}' = \underline{b}' \times \underline{t} + \underline{b} \times \underline{t}' = \tau \underline{n} \times \underline{t} + k \underline{b} \times \underline{n}$
$= -k\underline{t} - \tau \underline{b}$

Frenet Formulas

$$\left.\begin{aligned} \underline{t}' &= k\underline{n} \\ \underline{n}' &= -k\underline{t} - \tau\underline{b} \\ \underline{b}' &= \tau\underline{n} \end{aligned}\right\} *$$

The Frenet formulas can be written in a (compact) way as $F'(s) = A(s)F(s)$ **
where $\underline{F}$ is the 3×3 matrix

$$F = \begin{pmatrix} - & \underline{t} & - \\ - & \underline{n} & - \\ - & \underline{b} & - \end{pmatrix} \text{ and } A(s) = \begin{pmatrix} 0 & k & 0 \\ -k & 0 & -\tau \\ 0 & \tau & 0 \end{pmatrix}$$

Example:

$$\gamma(s) = \left(a\cos\frac{s}{\sqrt{a^2+b^2}},\ a\sin\frac{s}{\sqrt{a^2+b^2}},\ \frac{bs}{\sqrt{a^2+b^2}} \right)$$

$$\Rightarrow \gamma'(s) = \frac{1}{\sqrt{a^2+b^2}} \left(-a\sin\frac{s}{\sqrt{a^2+b^2}},\ a\cos\frac{s}{\sqrt{a^2+b^2}},\ b \right)$$

$$\underset{\underline{t}(s)}{\shortparallel}$$

$$\underline{t}'(s) = \frac{-1}{a^2+b^2} \left(a\cos\frac{s}{\sqrt{a^2+b^2}},\ a\sin\frac{s}{\sqrt{a^2+b^2}},\ 0 \right)$$

$$k(s) = |\underline{t}'| = \frac{a}{a^2+b^2} \qquad \underline{n} = \frac{\underline{t}'}{k} = -\left(\cos\frac{s}{\sqrt{a^2+b^2}},\ \sin\frac{s}{\sqrt{a^2+b^2}},\ 0 \right).$$

$$\underline{b} = \underline{t} \times \underline{n} = \frac{1}{\sqrt{a^2+b^2}} \left(b\sin\frac{s}{\sqrt{a^2+b^2}},\ -b\cos\frac{s}{\sqrt{a^2+b^2}},\ a \right)$$

$$\underline{b}' = \frac{b}{a^2+b^2} \left(\cos\frac{s}{\sqrt{a^2+b^2}},\ \sin\frac{s}{\sqrt{a^2+b^2}},\ 0 \right)$$

$$= -\frac{b}{a^2+b^2}\,\underline{n} \qquad \tau = -\frac{b}{a^2+b^2}$$

If $k \equiv 0 \Leftrightarrow \gamma$ traces out a straight line. ($\gamma'(s) = \underline{t}_0$ constant)

Theorem:

The torsion of a ~~curve vanishes completely~~ regular curve vanishes identically if and only if $\gamma(I)$ is contained in a plain.

Proof: If $\gamma(I)$ is contained in a plane P then $\underline{t}$ and $\underline{n}$ are parallel to P. So there are two choices $\underline{v}$ and $\underline{v}$ for $\underline{b}$ at each point (where $\underline{v}$ is a unit normal to P).
But $\underline{b}$ is continous so either $\underline{b} \equiv \underline{v}$ or $\underline{b} \equiv -\underline{v}$.
So $\underline{b}' \equiv 0 \Rightarrow \tau \equiv 0$.

Conversely if $\tau \equiv 0 \Rightarrow \underline{b} \equiv \underline{b}_0$ constant

$$\text{So } (\gamma \cdot \underline{b})' = (\gamma \cdot \underline{b}_0)'$$

$$= \underbrace{\underline{t} \cdot \underline{b}_0}_{\substack{=0 \\ \text{since} \\ \text{orthogonal.}}} + \gamma \cdot \underset{\substack{\shortparallel \\ 0}}{\underline{b}_0'} = 0$$

$\Rightarrow (\gamma \cdot \underline{b}_0)' = 0 \Rightarrow \gamma \cdot \underline{b}_0 = c$ constant

$(\gamma(t) = (x(t), y(t), z(t))$ equation of a plane.

Fundamental Theorem of the local theory of curves.

Given differentiable functions $k: I \to \mathbb{R}_{>0}$, $\tau: I \to \mathbb{R}$ there exists a regular curve $\gamma: I \to \mathbb{R}^3$ such that $k(s)$ and $\tau(s)$ are the curvature and torsion respectively of $\gamma(s)$ as functions of arclength.
Furthermore, γ is unique upto a rigid ~~rotation~~ motion in $\mathbb{R}^3$

Proof:
We begin by constructing an ~~[illegible]~~ orthonormal frame.
Let $(\underline{t}_0, \underline{n}_0, \underline{b}_0)$ be a right-handed system of orthonormal vectors ($\underline{b}_0 = \underline{t}_0 \times \underline{n}_0$)
Now consider the initial value problem * with $\underline{t}(s_0) = \underline{t}_0$, $\underline{n}(s_0) = \underline{n}_0$, $\underline{b}(s_0) = \underline{b}_0$ for some $s_0 \in I$. (system of 9 linear scalar ODEs).
The theory of ODEs $\Rightarrow \exists$ unique solution $(\underline{t}(s), \underline{b}(s), \underline{n}(s))$ for $s \in I$
~~[illegible]~~ We need to check that $(\underline{t}, \underline{n}, \underline{b})$ is an orthonormal frame.
Consider

$$M = \begin{pmatrix} \underline{t}\cdot\underline{t} & \underline{t}\cdot\underline{n} & \underline{t}\cdot\underline{b} \\ \underline{n}\cdot\underline{t} & \underline{n}\cdot\underline{n} & \underline{n}\cdot\underline{b} \\ \underline{b}\cdot\underline{t} & \underline{b}\cdot\underline{n} & \underline{b}\cdot\underline{b} \end{pmatrix} = \underline{F}\cdot\underline{F}^t$$

Want to show $M \equiv I$

$$\begin{aligned} \text{Now } M' &= F'F^t + F(F^t)' \\ &= AFF^t + FF^tA^t \\ &= AFF^t - FF^tA \qquad (A^t = -A) \\ &= AM - MA \end{aligned}$$

$$M' = AM - MA \quad (= [A, M]) \quad ***$$

At $s = s_0$, $\underline{t} = \underline{t}_0$, $\underline{n} = \underline{n}_0$, $\underline{b} = \underline{b}_0 \Rightarrow M(s_0) = I_{3\times3}$
There is a unique solution of *** with this initial condition.
Clearly $M \equiv I_{3\times3}$ is a solution of *** with this initial condition
$\Rightarrow M \equiv I_{3\times3}$ is the only solution.
$\therefore (\underline{t}, \underline{n}, \underline{b})$ is an orthonormal frame.
Also $\underline{b} = \underline{t} \times \underline{n}$ at $s = s_0 \Rightarrow \det(F) = \det \begin{pmatrix} - t - \\ - n - \\ - b - \end{pmatrix} = 1$ at s_0.

$\det F = \pm 1$ at each $s \in I$, but $\det F$ is continous so $\det F \equiv 1$
$\Rightarrow (\underline{t}, \underline{n}, \underline{b})$ is a right handed orthonormal frame.

Define $\gamma(s) := \int_{s_0}^{s} \underline{t}(\hat{s})\, d\hat{s} \Rightarrow \underline{t}(s) = \gamma'(s)$

So γ is a curve with curvature k and torsion τ.

Uniquness: (rigid motion = rotation + translation).

Assume we have 2 curves $\gamma : I \longrightarrow \mathbb{R}^3$ and $\tilde{\gamma} : I \longrightarrow \mathbb{R}^3$ with the same k and τ.

Some $s_0 \in I$ $(\underline{t}(s_0), \underline{n}(s_0), \underline{b}(s_0))$ and $(\hat{\underline{t}}(s_0), \hat{\underline{n}}(s_0), \hat{\underline{b}}(s_0))$ are 2 right handed orthonormal frames, so there is a rotation ρ ($\in SO_3$) such that $\hat{\underline{t}}(s_0) = \rho \cdot \underline{t}(s_0)$, $\hat{\underline{n}}(s_0) = \rho \cdot \underline{n}(s_0)$ $\hat{\underline{b}}(s_0) = \rho \cdot \underline{b}(s_0)$

Define a new frame $(\hat{\underline{t}}(s), \hat{\underline{n}}(s), \hat{\underline{b}}(s)) := (\rho^{-1} \cdot \tilde{\underline{t}}(s), \rho^{-1}\tilde{\underline{n}}(s), \rho^{-1}\underline{b}(s))$

Note ~ and ^ different

Want to show that $\hat{\underline{t}} \equiv \underline{t}$ etc.

Now $\dfrac{d}{ds}\left(|\underline{t}(s) - \hat{\underline{t}}(s)|^2 + |(\underline{n}(s) - \hat{\underline{n}}(s)|^2 + (\underline{b}(s) - \hat{\underline{b}}(s))^2 \right)$

$$= 2\left((\underline{t} - \hat{\underline{t}})\cdot(\underline{t}' - \hat{\underline{t}}') + (\underline{n} - \hat{\underline{n}})\cdot(\underline{n}' - \hat{\underline{n}}') - (\underline{b} - \hat{\underline{b}})\cdot(\underline{b}' - \hat{\underline{b}}') \right)$$

$$= 2\left(k(\underline{t} - \hat{\underline{t}})\cdot(\underline{n} - \hat{\underline{n}}) + \left[-k(\underline{n} - \hat{\underline{n}})\cdot(\underline{t} - \hat{\underline{t}}) - \tau(\underline{n} - \hat{\underline{n}}\cdot(\underline{b} - \hat{\underline{b}}) \right] + \tau(\underline{b} - \hat{\underline{b}})\cdot(\underline{n} - \underline{n}') \right) = 0.$$

$G(s) := |\underline{t}(s) - \hat{\underline{t}}(s)|^2 + |\underline{n}(s) - \hat{\underline{n}}(s)|^2 + |\underline{b}(s) - \hat{\underline{b}}(s)|^2 =$ a constant

Now at $s = s_0$

$\hat{\underline{t}}(s_0) = t(s_0)$, $\hat{\underline{n}}(s_0) = n(s_0)$ $\hat{\underline{b}}(s_0) = b(s_0)$

$\Rightarrow G(s_0) = 0 = 0 \Rightarrow G(s) \equiv 0$

$\Rightarrow \hat{t} \equiv t$ $\hat{n} \equiv n$ $\hat{b} \equiv b$.

$\hat{\underline{t}} = \underline{t} \Rightarrow \rho^{-1} \circ \tilde{\underline{t}} = \underline{t}$

$\Rightarrow \tilde{\underline{t}} = \rho \circ \underline{t} \Rightarrow \tilde{\gamma}'(s) = \rho \circ \gamma'(s)$

$\Rightarrow \tilde{\gamma}(s) = \rho \circ \gamma(s) + \underline{c}$ $\underline{c}$ · constant

Chapter 2: Surfaces

Differentiable functions $f: \mathbb{R}^m \to \mathbb{R}^n$

Definition:
Let U be an open ~~set~~ subset of $\mathbb{R}^m$ and let $f: U \to \mathbb{R}$ be a real valued function.
For any unit vector $\underline{v} \in \mathbb{R}^m$ the directional derivative of f at $\underline{x} \in U$ in the direction $\underline{v}$ is given by $\lim_{h \to 0^+} \frac{f(x+hv) - f(x)}{h}$ if this limit exists

If it exists we denote it by $D_{\underline{v}} f(\underline{x})$.

Let $\underline{e}_1, \underline{e}_2, \dots, \underline{e}_m$ be the standard basis for $\mathbb{R}^n$.
Then $D_{\underline{e}_j} f(\underline{x})$ is called the partial derivative of f wrt x_j
$\frac{\partial f}{\partial x_j} = D_{\underline{e}_j} f$

eg. $f(x,y,z)$, $\frac{\partial f}{\partial y} = \lim_{h \to 0^+} \frac{f(x,y+h,z) - f(x,y,z)}{h}$

Example:
The partial derivatives f_x and f_y for $f(x,y) = \begin{cases} \frac{xy}{x^2+y^2} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$

exist for all $(x,y) \in \mathbb{R}^2$. But f is not continuous at $(0,0)$

Definition:
Let U be an open subset of $\mathbb{R}^m$ and $f: U \to \mathbb{R}$.
We say that f is once differentiable at a point $\underline{a} = (a_1, a_2, \dots, a_m) \in U$ if $\exists$ real numbers $b_1, \dots, b_m$ such that

$$\lim_{x \to a} \frac{f(x) - f(a) - \sum_{j=1}^{m} b_j (x - a_j)}{\| x - a \|} = 0.$$

in fact $b_j = \left.\frac{\partial f}{\partial x_j}\right|_{x=a}$

From now on 'differentiable' will mean only differentiable.

Definition:
$F: U \subset \mathbb{R}^m \to \mathbb{R}^n$ $\quad (x_1, \dots, x_m)^T \in U$
$F(x) = (f_1(x), f_2(x), \dots, f_n(x))^T$ we define the differential of F to be the linear map $(DF)_x : \mathbb{R}^m \to \mathbb{R}^n$ such that

$F(x+\Delta x) = F(x) + (DF)_x(\Delta x) + R(x,\Delta x)$

where $\Delta x \in \mathbb{R}^m$ and $\lim_{\Delta x \to 0} \frac{R(x,\Delta x)}{|\Delta x|} = 0$

In matrix form ~~the~~ DF can be represented by the Jacobian matrix

$$(DF)_x = \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial f_1}{\partial x_m} \\ \frac{\partial f_2}{\partial x_1} & & & \vdots \\ \vdots & & & \\ \frac{\partial f_n}{\partial x_1} & & \cdots & \frac{\partial f_n}{\partial x_m} \end{pmatrix}$$

If $m=n$ $\quad \frac{\partial(f_1,\dots,f_n)}{\partial(x_1,\dots,x_n)} := \det \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & & \\ \frac{\partial f_n}{\partial x_1} & \cdots & \frac{\partial f_n}{\partial x_n} \end{pmatrix}$

Inverse function Theorem (multivariable)

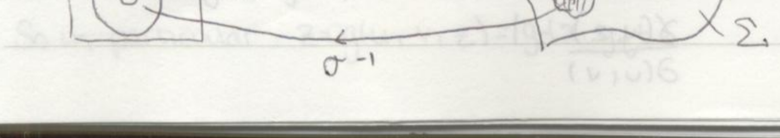
Let $f: U \subset \mathbb{R}^n \to \mathbb{R}^n$, be a smooth map and suppose that at $p \in U$, the differential DF_p is an isomorphism (ie the corresponding matrix is a non-singular ie the Jacobian is non-zero). Then there is a neighbourhood V of p in U and a nghd W of $F(p)$ in $\mathbb{R}^n$ such that the restriction of f to V, $f: V \to W$ has a smooth inverse $F^{-1}: W \to V$.

Regular Surfaces

Definition:

A non-empty subset $\Sigma \subset \mathbb{R}^3$ is called a regular surface if, for each $p \in \Sigma$ there is an open subset $U \in \mathbb{R}^2$ and an open nghd V of p in $\mathbb{R}^3$ and an onto map $\sigma: U \to V \cap \Sigma$ such that

1. σ is a smooth function (ie if $\sigma(u,v) = (x(u,v), y(u,v), z(u,v))$ then x, y, z are smooth functions)
2. σ is a homeomorphism (continuous and continous inverse) (ie show $\sigma^{-1}: V \cap \Sigma \to U$ is continuous)
3. The differential $D\sigma: \mathbb{R}^2 \to \mathbb{R}^3$ is one-to-one



(we'll see later that 3 is required to define a nice tangent plane)

$D\sigma$ is one-to-one $\Leftrightarrow \frac{\partial \sigma}{\partial u} \times \frac{\partial \sigma}{\partial v} \neq 0$

$\Leftrightarrow$ at least one of the Jacobians $\frac{\partial(x,y)}{\partial(u,v)}, \frac{\partial(y,z)}{\partial(u,v)}, \frac{\partial(z,x)}{\partial(u,v)}$ is non-zero.

$\sigma(u,v) = (x(u,v), y(u,v), z(u,v))$ $\quad \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$

Example:

The paraboloid $z = x^2+y^2$ is the image Σ of the map $\sigma : \mathbb{R}^2 \to \Sigma$ given by $\sigma(u,v) = (u, v, u^2+v^2)$

1. σ is smooth (all components are polynomials)
2. Different (u,v) give different (u, v, u^2+v^2).
   The inverse map σ^{-1} is continuous as it is the restriction of the (continuous) projection map $(x,y,z) \mapsto (x,y)$.
3. $\frac{\partial(x,y)}{\partial(u,v)} = \frac{\partial(u,v)}{\partial(u,v)} = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1 \neq 0$

$\Rightarrow \Sigma$ is a regular surface.

Theorem:

If $f : U \to \mathbb{R}$ is smooth on an open subset $U \subset \mathbb{R}^2$ then the graph of f (ie $\{x, y, f(x,y)\}$) is a regular surface.

Example:

The sphere $S^2 = \{(x,y,z) : x^2+y^2+z^2 = 1\}$ cannot be covered by a single coordinate patch.

In this example we will cover S^2 using six patches.

Let $U = \{(x,y) \in \mathbb{R}^2 : x^2+y^2 < 1\}$.

Consider the following maps:

$\sigma_j : U \to \mathbb{R}^3$, $j = 1, \ldots, 6$ given by

$\sigma_1(u,v) = (u, v, \sqrt{1-u^2-v^2})$ $\quad \frac{\partial(x,y)}{\partial(u,v)} = 1$

$\sigma_2(u,v) = (u, v, -\sqrt{1-u^2-v^2})$ $\quad$ So S^2 is a regular surface.

$\sigma_3(u,v) = (u, \sqrt{1-u^2-v^2}, v)$ $\quad \frac{\partial(x,z)}{\partial(u,v)} = 1$

$\sigma_4(u,v) = (u, -\sqrt{1-u^2-v^2}, v)$

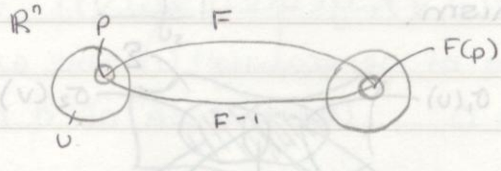
$\sigma_5(u,v) = (\sqrt{1-u^2-v^2}, u, v)$ $\quad \frac{\partial(y,z)}{\partial(u,v)} = 1$

$\sigma_6(u,v) = (-\sqrt{1-u^2-v^2}, u, v)$

Theorem:
Let $f: U \to \mathbb{R}$ be a smooth function on an open subset U of $\mathbb{R}^3$ and $a \in f(U) = \{f(\underline{x}) : \underline{x} \in U\}$.
If for all $p \in f^{-1}(a) = \{(x,y,z) \in U : f(x,y,z) = a\}$
$f_x(p), f_y(p), f_z(p)$ are not all zero, then $f^{-1}(a)$ is a regular surface in $\mathbb{R}^3$.

Example: $S^2 = \{(x,y,z) : x^2+y^2+z^2=1\}$.
Consider $f(x,y,z) = x^2+y^2+z^2$ then $S^2 = f^{-1}(1)$
$f_x = 2x, \ f_y = 2y, \ f_z = 2z$
But $f_x = f_y = f_z = 0$ iff $(x,y,z) = (0,0,0) \notin S^2$
So S^2 is a regular curve.

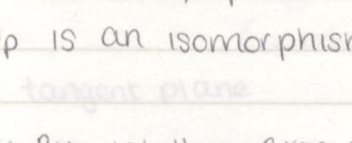
Recall: Inverse function theorem
$F: U \subset \mathbb{R}^n \to \mathbb{R}^n, \ p \in U$
$(DF)_p$ is an isomorphism



Proof: By relabelling axes if necessary, take $f_z(p) \neq 0$ and define the map $F: U \to \mathbb{R}^3$ by $F\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x \\ y \\ f(x,y,z) \end{pmatrix}$

$$DF = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ f_x & f_y & f_z \end{pmatrix} \qquad \det(DF)\big|_p = f_z(p) \neq 0$$

So by the inverse function theorem there is a locally defined smooth inverse F^{-1}.
Specifically, there exists a nghd $V \subset \mathbb{R}^3$ of p and a nghb $W \subset \mathbb{R}^3$ of $F(p)$ such that F restricted to V is invertible and maps onto W and $F^{-1}: W \to V$ is differentiable.



Writing $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = F^{-1}\begin{pmatrix} u \\ v \\ t \end{pmatrix}$ $\quad \left\{ \text{ie } \begin{pmatrix} u \\ v \\ t \end{pmatrix} = F\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x \\ y \\ f(x,y,z) \end{pmatrix} \right\}$

gives $x = u$
$y = v$
$z = g(x,y,z)$ for some differentiable function g.
So in particular $z = g(u,v,z) = g(x,y,t)$

Now on the surface $a = f(x,y,z) = t$

So on the surface $z = g(x,y,a) = h(x,y)$

$\Rightarrow$ locally the set is a smooth ~~surf~~ graph

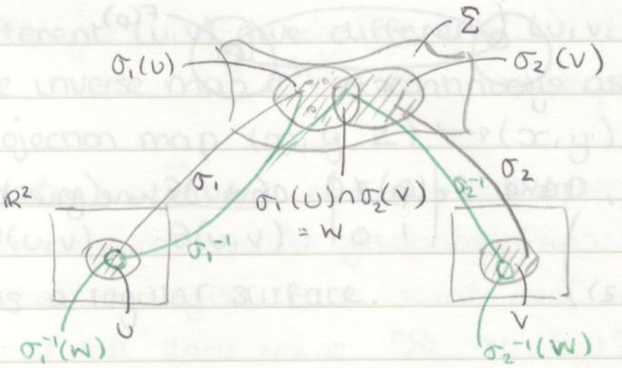
$\Rightarrow$ regular surface.

Recall: That a function $f: A \to B$ is called a diffeomorphism if it is differentiable and has a differentiable inverse $f^{-1}: B \to A$.

Theorem:

Let $\sigma_1 : U \to \Sigma$ and $\sigma_2 : V \to \Sigma$ be two parametrisations of a regular surface Σ such that $W := \sigma_1(U) \cap \sigma_2(V) \neq \emptyset$.

Then the "change of coordinates" $h := \sigma_1^{-1} \circ \sigma_2 : \sigma_2^{-1}(W) \to \sigma_1^{-1}(W)$ is a diffeomorphism.



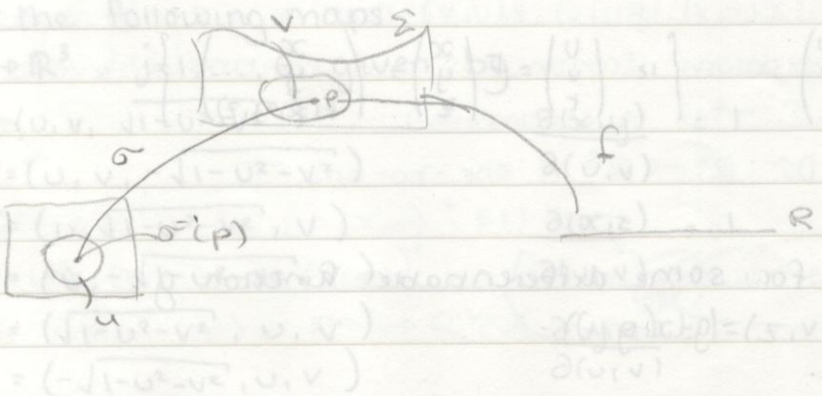
Functions on surfaces

Definition:

Let $f: V \to \mathbb{R}$ be a function on open subset V of a regular surface Σ.

Then f is said to be smooth or differentiable at $p \in V$ if, for some parametrisation $\sigma: U \to \Sigma$ with $p \in \sigma(U) \subset V$, the composition $f \circ \sigma : U \to \mathbb{R}$ is differentiable at $\sigma^{-1}(p)$

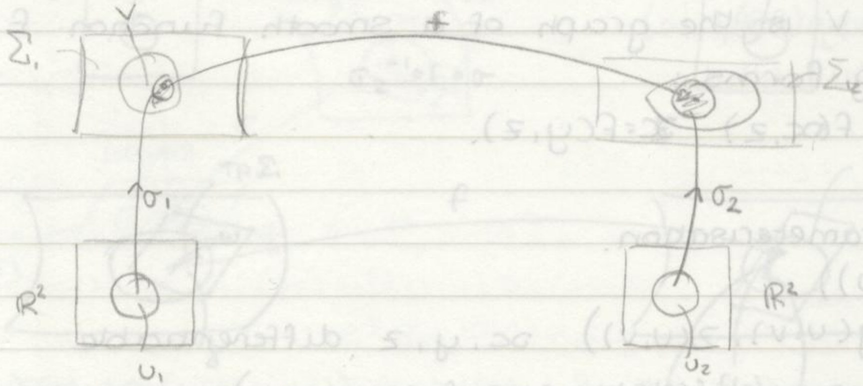
We say that f is differentiable if it is differentiable $\forall p \in V$.



(Previous thm show this definition is independent of parametrisation).

Definition:

Let Σ_1 and Σ_2 be regular surfaces. Let V be an open subset of Σ_1.
A map $f: V \to \Sigma_2$ is said to be differentiable at $p \in V$ if there are parametrisations $\sigma_1 : U_1 \to \Sigma_1$, $\sigma_2 : U_2 \to \Sigma_2$ with $p \in \sigma_1(U_1)$ and $f(\sigma_1(U_1)) \subset \sigma_2(U_2)$ such that $\sigma_2^{-1} \circ f \circ \sigma_1 : U_1 \to U_2$ is differentiable at $\sigma_1^{-1}(p)$.

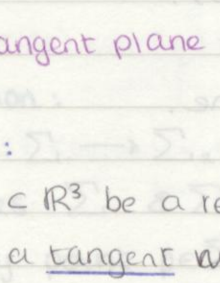


The tangent plane

Definition:

Let $\Sigma \subset \mathbb{R}^3$ be a regular surface. For any $p \in \Sigma$ a vector $v \in \mathbb{R}^3$ is called a tangent ~~normal~~ to Σ at $p \in \Sigma$ if there is a curve $\gamma : (-\varepsilon, \varepsilon) \to \Sigma$ for some $\varepsilon > 0$ such that $\gamma(0) = p$ and $\gamma'(0) = v$.
The set of all vectors tangent to Σ at p is called the tangent plane to Σ at p and is denoted by $T_p\Sigma$.

Given a parametrisation $\sigma : U \to \Sigma$, $p \in \sigma(U)$, a basis for $T_p\Sigma$ is given by $\{\sigma_u(q), \sigma_v(q)\}$ where $\sigma(q) = p$



$\dfrac{\partial \sigma(u,v)}{\partial u}$ $\qquad$ $q = (u_0, v_0)$

$\gamma(t) = \sigma(t + u_0, v_0)$

$\gamma(0) = p = \sigma(u_0, v_0)$

$\gamma'(0) = \sigma_u(u_0, v_0)$

Note that $\sigma_u(q)$ and $\sigma_v(q)$ are linearly independent since for a regular surface $\sigma_u \times \sigma_v \neq 0$.

Let $\gamma : (-\varepsilon, \varepsilon) \to \Sigma$ be a curve through $\gamma(0) = p$.

Define $(u(t), v(t)) = \sigma^{-1} \circ \gamma(t)$

Then $\gamma(t) = \sigma(u(t), v(t))$

So $\gamma'(0) = \sigma_u(q) u'(0) + \sigma_v(q) v'(0) \qquad q = \sigma^{-1}(p)$

Theorem:

Let $\Sigma \subset \mathbb{R}^3$ be a regular surface. For every $p \in \Sigma$ $\exists$ a ngbd V of p in Σ such that V is the graph of a smooth function f of one of the following forms:

$z = f(x, y)$, $y = f(x, z)$ $x = f(y, z)$.

Proof: Let σ be a parameterisation.

$\sigma : U \to \Sigma \quad (p \in \sigma(U))$

$\sigma(u, v) = (x(u,v), y(u,v), z(u,v)) \quad x, y, z$ differentiable

Without loss of generality (relable axis if necessary)

$\dfrac{\partial(x, y)}{\partial(u, v)} \neq 0$ (definition of a regular surface).

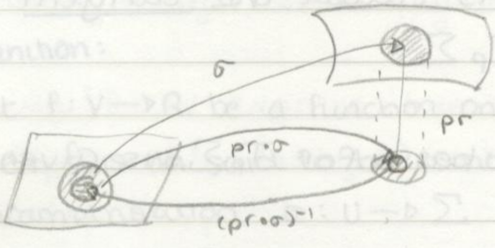
Let $pr : \mathbb{R}^3 \to \mathbb{R}^2$ be the projection onto the x-y plane.

$pr(x, y, z) = (x, y)$

So $pr \circ \sigma(u, v) = (x(u,v), y(u,v))$

$pr \circ \sigma : U \to \mathbb{R}^2$

So, since $\dfrac{\partial(x,y)}{\partial(u,v)} \neq 0$ $\exists$ a local inverse



$\sigma(u,v) = (x(u,v), y(u,v), z(u,v))$

So u and v are differentiable maps of x and y.

Also $z = z(u, v)$ is a differentiable map of u and v, so $z = z(u(x,y), v(x,y))$
$= f(x, y)$

where f is differentiable.

Example:

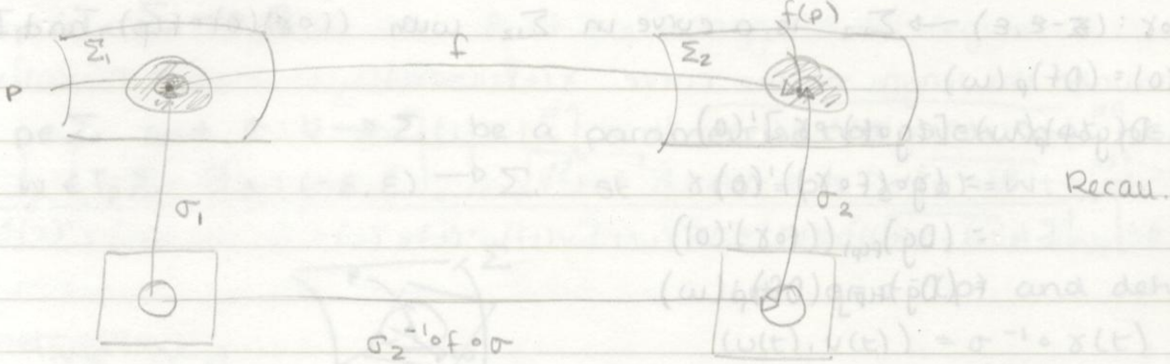
The cone $z = \sqrt{x^2 + y^2}$ is not a regular surface

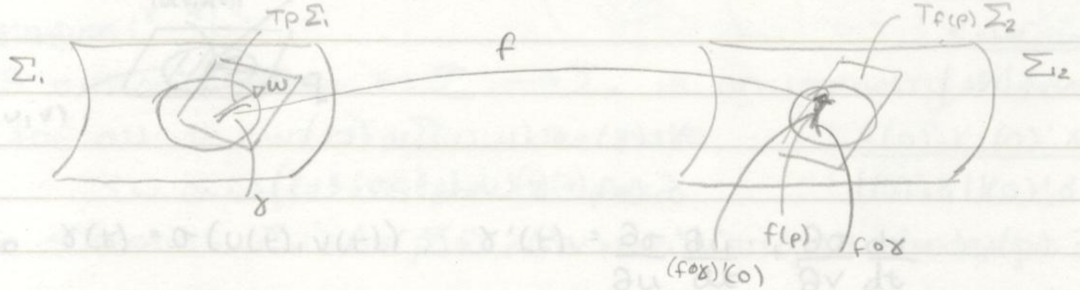
If it were it could be written as a graph with respect to one of the coordinate planes in a ngbd of $(0, 0, 0)$

1. $z = f(x, y) = \sqrt{x^2 + y^2}$ but this is not differentiable

2. $x = f(y,z)$ or $y = f(x,z)$ but it could not be single valued.

Recall.





Definition:

Let $f : \Sigma_1 \to \Sigma_2$ be a differentiable function between the regular surfaces Σ_1 & Σ_2.

For any point $p \in \Sigma_1$ and any vector $w \in T_p\Sigma_1$, let $\gamma : (-\varepsilon, \varepsilon) \to \Sigma_1$ be a curve such that $\gamma(0) = p$ and $\gamma'(0) = w$.

Then the map $(Df)_p : T_p\Sigma_1 \to T_{f(p)}\Sigma_2$ is called the differential of f at p and is given by

$$(Df)_p(w) = (f \circ \gamma)'(0).$$

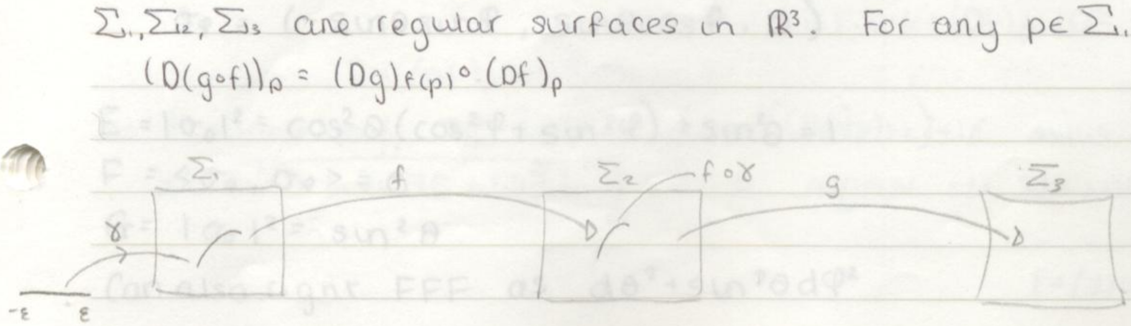
Lemma:

The differential $(Df)_p$, defined above, is independent of choice of ~~gra~~ γ.

Theorem: The Chain Rule

Let $f : \Sigma_1 \to \Sigma_2$, $g : \Sigma_2 \to \Sigma_3$ be two differentiable maps where $\Sigma_1, \Sigma_2, \Sigma_3$ are regular surfaces in $\mathbb{R}^3$. For any $p \in \Sigma_1$

$$(D(g \circ f))_p = (Dg)_{f(p)} \circ (Df)_p$$



Proof: compositions are well defined

Choose $w \in T_p\Sigma_1$ and let $\gamma : (-\varepsilon, \varepsilon) \to \Sigma_1$, be such that $\gamma(0) = p$ & $\gamma'(0) = w$

Then $f \circ \gamma : (-\varepsilon, \varepsilon) \to \Sigma_2$ is a curve in Σ_2 with $(f \circ \gamma)(0) = f(p)$ and $(f \circ \gamma)'(0) = (Df)_p(w)$

Hence $D(g \circ f)_p(w) = [(g \circ f) \circ \gamma]'(0)$

$$= (g \circ (f \circ \gamma))'(0)$$

$$= (Dg)_{f(p)}((f \circ \gamma)'(0))$$

$$= (Dg)_{f(p)} \circ (Df)_p(w)$$

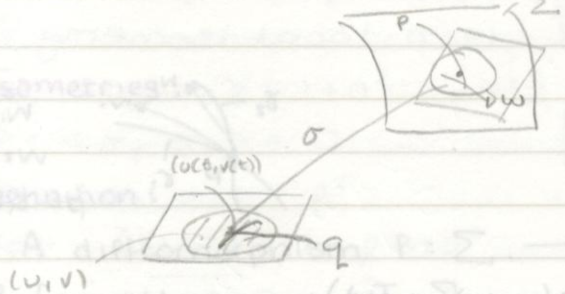
Definition:

The FFF is the function $I_p(w) = \langle w, w\rangle = |w|^2 \quad \forall w \in T_p\Sigma$

$I_p : T_p\Sigma \to \mathbb{R}$.

Let $p \in \Sigma$ and $\sigma : U \to \Sigma$ be a parametrisation such that $p \in \sigma(U)$

For $w \in T_p\Sigma$ $\exists \gamma : (-\varepsilon, \varepsilon) \to \Sigma$ st $\gamma(0) = p$, $\gamma'(0) = w$.



Let $q = \sigma^{-1}(p)$ and define $(u(t), v(t)) = \sigma^{-1} \circ \gamma(t)$

So $\gamma(t) = \sigma(u(t), v(t))$ $\gamma'(t) = \frac{\partial\sigma}{\partial u}\frac{du}{dt} + \frac{\partial \sigma}{\partial v}\frac{dv}{dt}$

$w = \gamma'(0) = \sigma_u(q)u'(0) + \sigma_v(q)v'(0)$ $\quad (q = (u(0), v(0)))$

$$\begin{aligned} I_p(w) = \langle w, w\rangle &= \langle \sigma_u u'(0) + \sigma_v v'(0), \sigma_u u'(0) + \sigma_v v'(0)\rangle \\ &= \langle \sigma_u, \sigma_u\rangle u'(0)^2 + 2\langle \sigma_u, \sigma_v\rangle u'(0)v'(0) + \langle \sigma_v, \sigma_v\rangle v'(0)^2 \\ &= Eu'^2 + 2Fu'v' + Gv'^2 \quad * \end{aligned}$$

where $E = \langle \sigma_u, \sigma_u\rangle$, $F = \langle \sigma_u, \sigma_v\rangle$, $G = \langle \sigma_v, \sigma_v\rangle$

E, F, G are called the components of the FFF

Motivated by $*$ we also call $E\,du^2 + 2F\,du\,dv + G\,dv^2$ (formal exp) the FFF

Example

Consider the part of the unit sphere covered by

$\sigma(\theta, \varphi) = (\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta)$ $\quad (0 < \theta < \pi,\ 0 < \varphi < 2\pi)$

$\Rightarrow \sigma_\theta = (\cos\theta\cos\varphi, \cos\theta\sin\varphi, -\sin\theta)$

$\sigma_\varphi = (-\sin\theta\sin\varphi, \sin\theta\cos\varphi, 0)$

$E = |\sigma_\theta|^2 = \cos^2\theta(\cos^2\varphi + \sin^2\varphi) + \sin^2\theta = 1$

$F = \langle \sigma_\theta, \sigma_\varphi\rangle = 0$

$G = |\sigma_\varphi|^2 = \sin^2\theta$

Can also right FFF as $d\theta^2 + \sin^2\theta\, d\varphi^2$

Properties that can be calculated using the FFF are called intrinsic.

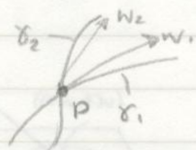
Example:

To calculate the length of the curve $\gamma(t) = (x(t), y(t), z(t)) = \sigma(u(t), v(t))$

$$s = \int_{t_0}^{t_1} \sqrt{x'(t)^2 + y'(t)^2 + z'(t)^2}\, dt = \int_{t_0}^{t_1} |\gamma'(t)|^2 dt$$

$$= \int_{t_0}^{t_1} \sqrt{E(u(t), v(t))u'(t)^2 + 2F(u(t), v(t))u'(t)v'(t) + G(u(t), v(t))v'(t)^2}\, dt$$

Example:



$w_1 = \gamma_1'(0)$

$w_2 = \gamma_2'(0)$

$p = \gamma_1(0) = \gamma_2(0)$

Angle between curves

$$\cos\theta = \frac{\langle w_1, w_2 \rangle}{|w_1||w_2|}$$

$$= \frac{\langle \gamma_1'(0), \gamma_2'(0) \rangle}{|\gamma_1'(0)||\gamma_2'(0)|} \qquad \gamma_1(t) = \sigma(u_1(t), v_1(t))$$

$$\gamma_2(t) = \sigma(u_2(t), v_2(t))$$

$$\langle \gamma_1', \gamma_2' \rangle = E(q)u_1'u_2' + F(q)(u_1'v_2' + u_2'v_1') + G(q)v_1'v_2'$$

Example:

Area: $\sigma: U \to \Sigma$.

The area of $\sigma(U)$ is $\iint_U |\sigma_u \times \sigma_v|\, du\, dv = A(R)$

Use the identity $|\sigma_u \times \sigma_v|^2 + \langle \sigma_u, \sigma_v \rangle^2 = |\sigma_u|^2 |\sigma_v|^2$

So

$$A(R) = \iint_U \sqrt{EG - F^2}\, du\, dv$$

Example:

The helicoid is the image of $\mathbb{R}^2$ under the mapping $\sigma(u,v) = (v\cos u, v\sin u, au)$

$a > 0$

$x = v\cos u$, $y = v\sin u$ $z = au$

We can also calculate the FFF ~~form~~ from

$$dx^2 + dy^2 + dz^2 = (-v\sin u\, du + \cos u\, dv)^2 + (v\cos u\, du + \sin u\, dv)^2$$
$$+ a^2 du^2$$

$$= (v^2 + a^2) du^2 + dv^2$$

So $E = v^2 + a^2$, $F = 0$, $G = 1$

The image of the curve $\gamma(t) = (\cos t, \sin t, at)$ $0 \leq t \leq 2\pi$ lies on the helicoid. To calculate its length ie $s = \int_0^{2\pi} \sqrt{E\dot{u}^2 + 2F\dot{u}\dot{v} + G\dot{v}^2}\, dt$

where $u(t) = t$ $v(t) = 1$

$$S = \int_0^{2\pi} \sqrt{(v^2+a^2)}\Big|_{v=1} \cdot 1 \, dt = \int_0^{2\pi} \sqrt{a^2+1}\, dt = 2\pi\sqrt{a^2+1}$$

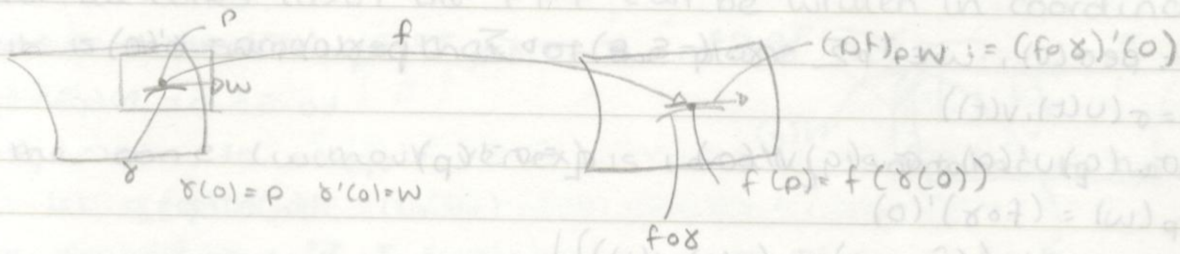
Also we can find the area A of the image of the region $U := \{(u,v) : 0 < u < 2\pi,\ 0 < v < 1\}$.

$$A = \int_0^{2\pi}\int_0^1 \sqrt{EG-F^2}\, du\, dv = \int_0^{2\pi}\int_0^1 \sqrt{v^2+a^2}\, du\, dv = \pi\left(v\sqrt{a^2+v^2} + \sinh^{-1}(v/a)\right)\Big|_{v=1}$$
$$= \pi\left(\sqrt{a^2+1} + \sinh^{-1}(1/a)\right).$$

Isometries:

Definition:

A diffeomorphism $f : \Sigma_1 \to \Sigma_2$ is an isometry if for all $p \in \Sigma_1$ & for all $w_1, w_2 \in T_p\Sigma_1$, we have
$$\langle w_1, w_2 \rangle = \langle (Df)_p w_1, (Df)_p w_2 \rangle$$
The surfaces Σ_1 and Σ_2 are said to be isometric.



Clearly if f is an isometry then $I_p(w) = I_{f(p)}((Df)_p w)$

Now suppose that $I_p(w) = I_{f(p)}((Df)_p w) \quad \forall w \in T_p\Sigma$

$$2\langle w_1, w_2\rangle = \langle w_1+w_2, w_1+w_2\rangle - \langle w_1, w_1\rangle - \langle w_2, w_2\rangle$$
$$= I_p(w_1+w_2) - I_p(w_1) - I_p(w_2)$$
$$= I_{f(p)}(\underbrace{(Df)_p w_1 + (Df)_p w_2}_{\substack{\text{linearity} \\ (Df)_p(w_1+w_2)}}) - I_{f(p)}((Df)_p w_1) - I_{f(p)}((Df)_p w_2)$$
$$= \langle (Df)_p w_1 + (Df)_p w_2, (Df)_p w_1 + (Df)_p w_2\rangle - \langle (Df)_p w_1, (Df)_p w_1\rangle - \langle (Df)_p w_2, (Df)_p w_2\rangle$$
$$= 2\langle (Df)_p w_1, (Df)_p w_2\rangle$$

So f isometry $\Leftrightarrow I_p(w) = I_{f(p)}((Df)_p w)$

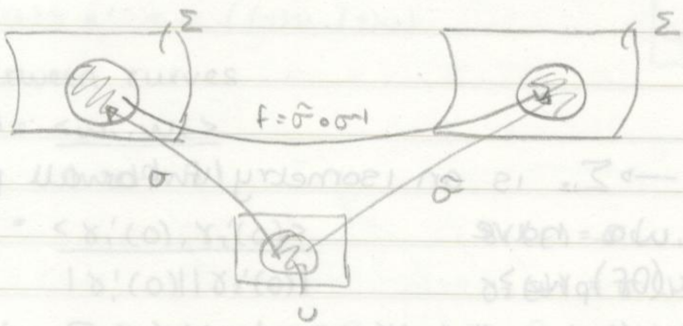
Definition:

A function $f: V \to \Sigma_2$ of a nghd V of a point $p \in \Sigma_1$ is called a local isometry if $\exists$ a nghd $\hat{V}$ of $f(p)$ st $f: V \to \hat{V}$ is an isometry.

Theorem:

Let $\sigma: U \to \Sigma$ and $\tilde{\sigma}: U \to \tilde{\Sigma}$ be parametrisations of the regular surfaces $\Sigma, \tilde{\Sigma}$ such that $E = \tilde{E}$, $F = \tilde{F}$, $G = \tilde{G}$

Then the map $f := \tilde{\sigma} \circ \sigma^{-1} : \sigma(U) \to \tilde{\Sigma}$ is a local isometry.



Proof: $p \in \sigma(U)$, $w \in T_p\Sigma$ $\quad \gamma: (-\varepsilon, \varepsilon) \to \Sigma \quad p = \gamma(0),\ w = \gamma'(0)$

$\gamma(t) = \sigma(u(t), v(t))$

$w = \sigma_u(q)u'(0) + \sigma_v(q)v'(0) \qquad q = \sigma^{-1}(p)$

$$(Df)_p(w) = (f \circ \gamma)'(0)$$
$$= \frac{d}{dt}\Big(\underbrace{(\tilde{\sigma} \circ \sigma^{-1})}_{f} \circ \underbrace{\sigma(u(t), v(t))}_{\gamma(t)}\Big)\Big|_{t=0}$$
$$= \frac{d}{dt}\big(\tilde{\sigma} \circ (\sigma^{-1} \circ \sigma(u(t), v(t))\big)\Big|_{t=0}$$
$$= \frac{d}{dt}\big(\tilde{\sigma}(u(t), v(t))\big)\Big|_{t=0}$$
$$= \tilde{\sigma}_u u'(0) + \tilde{\sigma}_v v'(0).$$

So $I_{f(p)}((Df)_p(w)) = |\tilde{\sigma}_u u'(0) + \tilde{\sigma}_v v'(0)|^2$
$$= \langle \tilde{\sigma}_u, \tilde{\sigma}_u \rangle (u')^2 + \langle \tilde{\sigma}_u, \tilde{\sigma}_v \rangle u'v' + \langle \tilde{\sigma}_u, \tilde{\sigma}_v \rangle (v')^2$$
$$= \tilde{E}(u')^2 + 2\tilde{F}u'v' + \tilde{G}(v')^2$$
$$= E(u')^2 + 2Fu'v' + G(v')^2$$
$$= |\sigma_u u' + \sigma_v v'|^2 = I_p(w) \Rightarrow \text{local isometry}$$

Example: Consider the cone $z = a\rho$ in polar coordinates (without vertex), where a is a constant.

If $a=0$ then this is the plane $z=0$.

Use the parametrisation $\sigma(\rho,\theta) = (\rho\cos\theta, \rho\sin\theta, a\rho)$

$\sigma_\rho = (\cos\theta, \sin\theta, a)$

$\sigma_\theta = (-\rho\sin\theta, \rho\cos\theta, 0)$

$E = \langle \sigma_\rho, \sigma_\rho \rangle = 1 + a^2$

$F = \langle \sigma_\rho, \sigma_\theta \rangle = 0$

$G = \langle \sigma_\theta, \sigma_\theta \rangle = \rho^2$

FFF $(1+a^2)\,d\rho^2 + \rho^2 d\theta^2$

Let $\hat{\rho} = \sqrt{1+a^2}\,\rho \qquad \hat{\theta} = \dfrac{\theta}{\sqrt{1+a^2}}$

$\Rightarrow$ $d\hat{\rho}^2 + \hat{\rho}^2 d\hat{\theta}^2$

So for all cones $(a>0)$ the FFF can be written in coordinates st it is the same as that of the plane.

So the cone (without vertex) is locally isometric to the plane.

Curvature & the 2^{nd} Fundemental Form.

Definition:

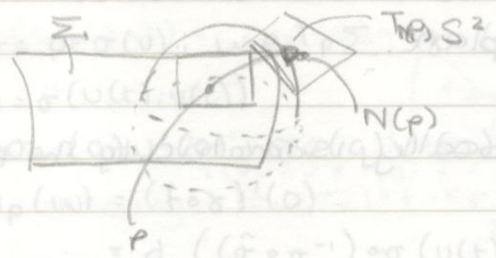
An orientation on a surface Σ is a continuous map $N : \Sigma \to \mathbb{R}^3$ such that $\forall p \in \Sigma$, $N(p)$ is a unit normal to $T_p\Sigma$.
If Σ admits an orientation, it is called orientable.

Mobius strip is non-orientable!
Small enough pieces are orientable.

In a single coordinate ~~maps~~ patch $\sigma : U \to \sigma(U) \subset \Sigma$, we have

$N(p) = \pm \dfrac{\sigma_u \times \sigma_v}{|\sigma_u \times \sigma_v|}$ (2 choices of orientation)
(with the +, we call this the standard orientation)

If we identify unit vectors with the unit sphere S^2 (in the obvious way) we have $N : \Sigma \to S^2$
Natural to consider $(DN)_p : T_p\Sigma \to T_{N(p)}S^2$



$T_{N(p)}S^2$ = space of vectors tangent to S^2 at $N(p)$
= space of vectors perp. to $N(p)$
$= T_p\Sigma$

So $(DN)_p : T_p\Sigma \to T_p\Sigma$

Unless stated otherwise, all surfaces will be assumed orientable from now on.

Self-adjoint maps

Definition:

A linear map $A : V \to V$ is self adjoint if $\langle Av, w\rangle = \langle v, Aw\rangle$ $\forall v, w \in V$ (V inner product space)

For each self-adjoint map $A : V \to V$ $\exists$ a symmetric bilinear map $B(v,w) = \langle Av, w\rangle$

If $\{e_1, e_2\}$ is an orthonormal basis for V, then the matrix $b_{ij} = \langle Ae_i, e_j\rangle$ is symmetric

For each symmetric bilinear form B on V there is a quadratic form

$Q(v) = B(v, v)$

Q determines B uniquely by $B(u,v) = \frac{1}{2}(Q(u+v) - Q(u) - Q(v))$.

Theorem:

Let $A: V \to V$ be a self-adjoint linear map on the real 2-dimensional inner product space V. Then the unit eigen vectors e_1 and e_2 of A form an ~~anor~~ orthonormal basis of V. The corresponding eigenvalues λ_1 and λ_2 are the max and min of $Q(v) = \langle Av, v \rangle$ on the unit circle of V.

The Second Fundemental Form.

$p \to N(p)$ is called the <u>Gauss map</u>

Theorem:

The differential $(DN)_p : T_p\Sigma \to T_p\Sigma$ of the Gauss map is self-adjoint.

Proof: Let $q = \sigma^{-1}(p) =: (u_0, v_0)$

Since $\{\sigma_u(q), \sigma_v(q)\}$ is a basis for $T_p\Sigma$, it suffices to show that $\langle (DN)_p\sigma_u(q), \sigma_v(q)\rangle = \langle \sigma_u(q), (DN)_p\sigma_v(q)\rangle$

Let $\gamma(t) = \sigma(u_0 + t, v_0)$.

Then $p = \gamma(0)$ and $\gamma'(0) = \sigma_u(q)$

So $(DN)_p\sigma_u(q) = (N\circ\gamma)'(0)$

$$= \frac{d}{dt}\left(N\circ\sigma(u_0+t, v_0)\right)\Big|_{t=0}$$

$$= (N\circ\sigma)_u(q)$$

$$= \tilde{N}_u(q) \quad \text{where } \tilde{N} = N\circ\sigma$$

So we want to show $\langle \tilde{N}_u, \sigma_v\rangle = \langle \sigma_u, \tilde{N}_v\rangle$

$\langle \underset{\in T_p\Sigma}{\sigma_u}, \underset{\perp T_p\Sigma}{\tilde{N}} \rangle = 0$

So diff. wrt v : $\langle \sigma_{uv}, \tilde{N}\rangle + \langle \sigma_u, \tilde{N}_v\rangle = 0$

Also $\langle \sigma_v, \tilde{N}\rangle = 0 \Rightarrow \langle \sigma_{uv}, \tilde{N}\rangle + \langle \sigma_v, \tilde{N}_u\rangle = 0$

Definition:

The quadratic form $\mathrm{II}_p(w) := -\langle (DN)_p w, w\rangle \quad \forall w \in T_p\Sigma$ is called the second Fundemental Form.

- The eigenvalues k_1 and k_2 of $-(DN)_p$ are called the principle curvatures of Σ at p.
- $K = k_1 k_2 = \det((DN)_p)$ is called the Gauss curvature
- $H = \frac{1}{2}(k_1+k_2)$ is called the mean curvature
  $= -\frac{1}{2}\operatorname{Tr}((DN)_p)$

For any $w \in T_p\Sigma$, $\exists \gamma$ st $\gamma(0)=p$, $\gamma'(0)=w$

$\gamma(t) = \sigma(u(t), v(t))$.

$w = \gamma'(0) = \sigma_u(q)u'(0) + \sigma_v(q)v'(0)$

$$\begin{aligned}
\mathrm{II}_p(w) &:= -\langle (DN)_p w, w\rangle \\
&= -\langle (DN)_p(\sigma_u u' + \sigma_v v'), \sigma_u u' + \sigma_v v'\rangle \\
&= -\langle u'(DN)_p(\sigma_u) + v'(DN)_p(\sigma_v), u'\sigma_u + v'\sigma_v\rangle \\
&= -(u')^2\langle (DN)_p\sigma_u, \sigma_u\rangle - u'v'(\langle (DN)_p\sigma_u, \sigma_v\rangle + \langle (DN)_p\sigma_v, \sigma_u\rangle) \\
&\quad - (v')^2\langle (DN)_p\sigma_v, \sigma_v\rangle \\
&= e(u,v)(u')^2 + 2f(u,v)u'v' + g(u,v)(v')^2
\end{aligned}$$

where $e(u,v) = -\langle (DN)_p\sigma_u, \sigma_u\rangle$

$f(u,v) = -\langle (DN)_p\sigma_u, \sigma_v\rangle = -\langle (DN)_p\sigma_v, \sigma_u\rangle$ self adjoint

$g(u,v) = -\langle (DN)_p\sigma_v, \sigma_v\rangle$

Also $(DN)_p\sigma_u = \tilde{N}_u$, $(DN)_p\sigma_v = \tilde{N}_v$ where $\tilde{N} = N\circ\sigma$

So $e = -\langle \tilde{N}_u, \sigma_u\rangle$

$f = -\langle \tilde{N}_u, \sigma_v\rangle = -\langle \tilde{N}_v, \sigma_u\rangle$

$g = -\langle \tilde{N}_v, \sigma_v\rangle$

Also note that $\langle \sigma_u, \tilde{N}\rangle = 0$

$\Rightarrow \langle \sigma_{uu}, \tilde{N}\rangle + \langle \sigma_u, \tilde{N}_u\rangle = 0 \quad \Rightarrow e = \langle \tilde{N}, \sigma_{uu}\rangle$

$f = \langle \tilde{N}, \sigma_{uv}\rangle$

$g = \langle \tilde{N}, \sigma_{vv}\rangle$

The Second Fundamental Form is often expressed as

$$e\,du^2 + 2f\,du\,dv + g\,dv^2$$

Relationships between first and second fundamental forms (FFs).

Recall that $\hat{N}$ is a unit vector

So $\hat{N}_u$ & $\hat{N}_v$ are orthogonal to $\hat{N}$ & hence in $T_p\Sigma$.

So ~~there~~ $\exists$ scalars such that

$$\left.\begin{aligned} 1 \quad \hat{N}_u &= a_{11}\sigma_u + a_{21}\sigma_v \\ 2 \quad \hat{N}_v &= a_{12}\sigma_u + a_{22}\sigma_v \end{aligned}\right\} *$$

For any $w = \alpha\sigma_u + \beta\sigma_v$ we have

$$\begin{aligned} (DN)_p w &= \alpha (DN)_p(\sigma_u) + \beta (DN)_p(\sigma_v) \\ &= \alpha \hat{N}_u + \beta \hat{N}_v \\ &= (a_{11}\alpha + a_{12}\beta)\sigma_u + (a_{21}\alpha + a_{22}\beta)\sigma_v \quad \text{using } * \end{aligned}$$

So $(DN)_p$ acts on the coeffs as

$$\begin{pmatrix}\alpha \\ \beta\end{pmatrix} \longrightarrow \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22}\end{pmatrix}\begin{pmatrix}\alpha \\ \beta\end{pmatrix}$$

So Gauss curvature $K = \det(a_{ij}) = a_{11}a_{22} - a_{12}a_{21}$

mean curvature $H = \frac{1}{2}\mathrm{Tr}(a_{ij}) = \frac{1}{2}(a_{11} + a_{22})$

Take inner product of $*_1$ with σ_u:

$$\langle \hat{N}_u, \sigma_u \rangle = a_{11}\langle \sigma_u, \sigma_u\rangle + a_{21}\langle \sigma_v, \sigma_u\rangle$$

$$-e = a_{11}E + a_{21}F$$

Inner product of $*_1$ with σ_v:

$$-f = a_{11}F + a_{21}G$$

$\langle *_2, \sigma_u\rangle$: $-f = a_{12}E + a_{22}F$

$\langle *_2, \sigma_v\rangle$: $-g = a_{12}F + a_{22}G$.

These equations can be written as:

$$\begin{pmatrix} E & F \\ F & G\end{pmatrix}\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22}\end{pmatrix} = -\begin{pmatrix} e & f \\ f & g\end{pmatrix}$$

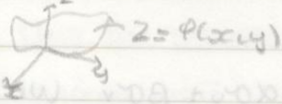
$$\Rightarrow \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = - \begin{pmatrix} E & F \\ F & G \end{pmatrix}^{-1} \begin{pmatrix} e & f \\ f & g \end{pmatrix}$$

$$= \frac{1}{EG - F^2} \Bigg($$

So $K = a_{11} a_{22} - a_{12} a_{22} = \dfrac{eg - f^2}{EG - F^2}$

$H = \frac{1}{2}(a_{11} + a_{22}) = \frac{1}{2} \dfrac{eG - 2fF + gE}{EG - F^2}$

Example: $z = \varphi(x, y)$



$\sigma(u, v) = (u, v, \varphi(u, v))$

$\sigma_u = (1, 0, \varphi_u)$

$\sigma_v = (0, 1, \varphi_v)$

$$\left.\begin{array}{l} E = \langle \sigma_u, \sigma_u \rangle = 1 + \varphi_u^2 \\ F = \langle \sigma_u, \sigma_v = \varphi_u \varphi_v \\ G = \langle \sigma_v, \sigma_v \rangle \; 1 + \varphi_v^2 \end{array}\right\} \text{Components of 1st FF}$$

$\sigma_u \times \sigma_v = (-\varphi_u, -\varphi_v, 1)$

Choose $\tilde{N} = + \dfrac{\sigma_u \times \sigma_v}{|\sigma_u \times \sigma_v|} = \dfrac{(-\varphi_u, -\varphi_v, 1)}{\sqrt{1 + \varphi_u^2 + \varphi_v^2}}$

$\sigma_{uu} = (0, 0, \varphi_{uu}) \quad \sigma_{uv} = (0, 0, \varphi_{uv}) \quad \sigma_{vv} = (0, 0, \varphi_{vv})$

$$\left.\begin{array}{l} e = -\langle \tilde{N}_u, \sigma_u \rangle = \langle \tilde{N}, \sigma_{uu} \rangle = \dfrac{\varphi_{uu}}{\sqrt{1 + \varphi_u^2 + \varphi_v^2}} \\[2ex] f = \langle \tilde{N}, \sigma_{uv} \rangle = \dfrac{\varphi_{uv}}{\sqrt{1 + \varphi_u^2 + \varphi_v^2}} \\[2ex] g = \langle \tilde{N}, \sigma_{vv} \rangle = \dfrac{\varphi_{vv}}{\sqrt{1 + \varphi_u^2 + \varphi_v^2}} \end{array}\right\} \text{Components of 2nd FF.}$$

$K = \dfrac{eg - f^2}{EG - F^2} = \dfrac{\varphi_{uu}\varphi_{vv} - \varphi_{uv}^2}{1 + \varphi_u^2 + \varphi_v^2}$

$EG - F^2 = (1 + \varphi_u^2)(1 + \varphi_v^2) - \varphi_v^2 \varphi_u^2$

$= 1 + \varphi_u^2 + \varphi_v^2$

$H = \dfrac{(1 + \varphi_u^2)\varphi_{uv} - 2\varphi_u \varphi_v \varphi_{uv} - (1 + \varphi_v^2)\varphi_{uu}}{2(1 + \varphi_u^2 + \varphi_v^2)^{3/2}}$

Let's look at how $\{\sigma_u, \sigma_v, \hat{N}\}$ varies by considering derivatives.

$$[(\sigma_u)_v =] \quad \sigma_{uu} = \Gamma_{11}^{1}\sigma_u + \Gamma_{11}^{2}\sigma_v + \lambda \hat{N} \quad 1$$
$$\sigma_{uv} = \Gamma_{12}^{1}\sigma_u + \Gamma_{12}^{2}\sigma_v + \mu \hat{N} \quad 2$$
$$\sigma_{vv} = \Gamma_{22}^{1}\sigma_u + \Gamma_{22}^{2}\sigma_v + \nu \hat{N} \quad 3$$

The scalar functions Γ_{ij}^{k} are called Christoffel symbols.

$\langle 1, \hat{N}\rangle$: $\lambda = \langle \sigma_{uu}, \hat{N}\rangle = e$

Similarly $\mu = f$ and $\nu = g$.

$\langle 1, \sigma_u\rangle$: $\Gamma_{11}^{1}\langle \sigma_u, \sigma_u\rangle + \Gamma_{11}^{2}\langle \sigma_v, \sigma_u\rangle = \langle \sigma_{uu}, \sigma_u\rangle$

$\Rightarrow \Gamma_{11}^{1}E + \Gamma_{11}^{2}F = \frac{1}{2}E_u$ $\qquad E = \langle \sigma_u, \sigma_u\rangle$, $E_u = 2\langle \sigma_u, \sigma_{uu}\rangle$

$\langle 1, \sigma_v\rangle$: $\Gamma_{11}^{1}F + \Gamma_{11}^{2}G = \langle \sigma_{uu}, \sigma_v\rangle = F_u - \frac{1}{2}E_v$ $\qquad F_u = \langle \sigma_{uu}, \sigma_v\rangle + \underbrace{\langle \sigma_u, \sigma_{uv}\rangle}_{=\frac{1}{2}E_v}$

Similarly we get four more equations from:

$\langle 2, \sigma_u\rangle$ $\langle 2, \sigma_v\rangle$ $\langle 3, \sigma_u\rangle$ $\langle 3, \sigma_v\rangle$

in total we get 6 equations

$$\begin{pmatrix} E & F \\ F & G \end{pmatrix}\begin{pmatrix} \Gamma_{11}^{1} & \Gamma_{12}^{1} & \Gamma_{22}^{1} \\ \Gamma_{11}^{2} & \Gamma_{12}^{2} & \Gamma_{22}^{2} \end{pmatrix} = \begin{pmatrix} \frac{1}{2}E_u & \frac{1}{2}E_v & F_v - \frac{1}{2}G_u \\ F_u - \frac{1}{2}E_v & \frac{1}{2}G_u & \frac{1}{2}G_v \end{pmatrix}$$

So all Christoffel symbols are determined by FFF only.

Equations 1, 2, 3 & * have three compatability conditions

$$(\hat{N}_u)_v = (\hat{N}_v)_u \quad 6$$
$$(\sigma_{uu})_v = (\sigma_{uv})_u \quad 7$$
$$(\sigma_{vv})_u = (\sigma_{uv})_v \quad 8$$

$$\begin{pmatrix} \hat{N}_u = a_{11}\sigma_u + a_{21}\sigma_v & 4 \\ \hat{N}_v = a_{12}\sigma_u + a_{22}\sigma_v & 5 \end{pmatrix}$$

Consider 6. From 1

$(\sigma_{uu})_v = (\Gamma_{11}^{1})_v\sigma_u + \Gamma_{11}^{1}\sigma_{uv} + (\Gamma_{11}^{2})_v\sigma_v + \Gamma_{11}^{2}\sigma_{vv} + e_v\hat{N} + e\hat{N}_v$

(using equations 1–5 to eliminate σ_{uv}, $\hat{N}_v$ ~~etc~~ etc.)

$$(\sigma_{uu})_v = ((\Gamma_{11}^{1})_v + \Gamma_{11}^{1}\Gamma_{12}^{1} + \Gamma_{11}^{2}\Gamma_{22}^{1} + e\,a_{12})\sigma_u + ((\Gamma_{11}^{2})_v + \Gamma_{11}^{1}\Gamma_{12}^{2} + \Gamma_{11}^{2}\Gamma_{22}^{2} + e\,a_{22})\sigma_v + (\Gamma_{11}^{1}f + \Gamma_{11}^{2}g + e_v)\hat{N} \quad 9$$

Similarly

$$(\sigma_{uv})_u = ((\Gamma_{12}^{1})_u + \Gamma_{12}^{1}\Gamma_{11}^{1} + \Gamma_{12}^{2}\Gamma_{12}^{1} + f a_{11})\sigma_u + ((\Gamma_{12}^{2})_u + \Gamma_{12}^{1}\Gamma_{11}^{2} + \Gamma_{12}^{2}\Gamma_{12}^{2} + f a_{21})\sigma_v + (\dots)\hat{N} \quad 10$$

Equations 9 and 1 gives 3 ~~scalar~~ scalar equations (coeffs of σ_u, σ_v, $\hat{N}$). The coeffs of σ_v give:

$$(\Gamma_{11}^{1})_v - (\Gamma_{12}^{2})_u + \Gamma_{11}^{1}\Gamma_{12}^{2} + \Gamma_{11}^{2}\Gamma_{21}^{2} - \Gamma_{12}^{1}\Gamma_{11}^{2} - \Gamma_{12}^{2}\Gamma_{12}^{2} = f a_{21} - e a_{22} \quad 11$$

Using our previous expressions for a_{21} and a_{22} and the fact that $K = \frac{eg-f^2}{EG-F^2}$ we find that RHS of 11 is EK.

$\Rightarrow$ The Gauss Curvature depends on the FFF (and its derivatives) only

Theorem: Theorema Egresium

The Gauss curvature is uniquely determined by the FFF and is therefore preserved by isometries.

The typical notion in maths is to group together things which we consider as more or less equivalent. This is what we are doing with respect to isometries

We have three vector equations (6, 7, 8) each giving 3 scalar equations but only 3 of these 9 scalar equations are independent. Together with, 11 = EK - called the Gauss equation we also have:

$$e_v - f_u = e\Gamma_{12}^{1} + f(\Gamma_{12}^{2} - \Gamma_{11}^{1}) - g\Gamma_{11}^{2} \quad 12$$

$$f_v - g_u = e\Gamma_{22}^{1} + f(\Gamma_{22}^{2} - \Gamma_{12}^{1}) - g\Gamma_{12}^{2} \quad 13$$

12 & 13 are called the Mainardi - Codazzi equations

What makes the Gauss Curvature so important when considering abstract geometry is that you only need FFF, you don't need to consider embeddings.

Theorem: Bonnet

Let E, F, G, e, f, g be differentiable functions on some on some open set $V \subset \mathbb{R}^2$ on which E, G and $EG - F^2$ are all strictly positive. If these functions sastify the Mainardi-Codazzi equations (where the Christoffel symbols are defined in terms of E, F, G as before) then for each $q \in V$ $\exists$ a neighbourhood $U \subset V$ of q and a diffeomorphism $\sigma : U \to \sigma(U)$ st the regular surface $\sigma(U)$ has

$E\,du^2 + 2F\,du\,dv + G\,dv^2$, and

$e\,du^2 + 2f\,du\,dv + g\,dv^2$

as its FFF and 2FF respectively.

Further more if U is connected and if $\hat{\sigma} : U \to \hat{\sigma}(U)$ is another diffeomorphism satisfying the same conditions, then there is a rigid body motion R st $\hat{\sigma} = R \circ \sigma$.

The Intrinsic Geometry of Surfaces

For stuff that is defined from FFF only, we will look at
- characterisation of surfaces through curvature
- geodesics - natural analogs of straight lines, surfaces
- covariarient derivatives.

Covariant Derivative

Definition:
Let V be an open set in Σ. A differential (tangential) vector field on V is a smooth function $w: V \to \mathbb{R}^3$ st $\forall p \in V$ $w(p) \in T_p\Sigma$.

Let $\gamma: I \to \Sigma$ be a curve on Σ. Any vector field w restricted to $\gamma(I)$ could be written locally as
$w(\gamma(t)) = a(t)\sigma_u(u(t),v(t)) + b(t)\sigma_v(u(t),v(t))$
where $\gamma(t) = \sigma(u(t),v(t))$
So $\frac{dw}{dt} = a'\sigma_u + a(\sigma_{uu}u' + \sigma_{uv}v') + b'\sigma_v + b(\sigma_{vu}u' + \sigma_{vv}v')$.

So using the expressions for σ_{uu}, σ_{uv} etc in terms of Γ's, e, f, g we have:

$$\frac{dw}{dt} = \overbrace{(a' + a(\Gamma_{11}^1 u' + \Gamma_{12}^1 v') + b(\Gamma_{12}^1 u' + \Gamma_{22}^1 v'))}^{A}\sigma_u$$
$$+ \underbrace{(b' + \Gamma_{11}^2 au' + \Gamma_{12}^2 av' + \Gamma_{21}^2 bu' + \Gamma_{22}^2 bv')}_{B}\sigma_v$$
$$+ (eau' + f(av' + bu') + gbv')\hat{N}$$

$u, v \leftrightarrow$ determine γ
$a, b \leftrightarrow$ " w

The covariant derivative of w in the direction γ' is the projection of $\frac{dw}{dt}$ onto the tangent plane $T_p\Sigma$

$$\nabla_{\gamma'} w = (\overset{A}{\dots})\sigma_u + (\overset{B}{\dots})\sigma_v$$

To emphasise dependence on t, we sometimes write $\frac{\nabla_{\gamma'} w}{dt}$.

Definition:
A smooth vector field w along a curve $\gamma: I \to \Sigma$ is said to be <u>parallel</u> if $\nabla_{\gamma'} w = 0$ $\forall t \in I$

(To define $\nabla_{\gamma'} w$ we need only be defined on $\gamma(I)$, not necessarily on an open set in Σ.)

Theorem:

Let w_1 and w_2 be parallel vector fields along a smooth curve $\gamma: I \to \Sigma$. Then $\langle w_1, w_2 \rangle_p$ is a constant. In particular $|w_1|, |w_2|$ and the angle between them is constant.

Proof: Note: if w is a parallel vector field then $\frac{dw}{dt}\big|_p$ is propotional to $\hat{N}$ and therefore orthogonal to $T_p\Sigma$. In particular $\langle w, \frac{dw}{dt} \rangle = 0$

Consider $\frac{d}{dt} \langle w_1, w_2 \rangle = \langle \underset{\substack{\uparrow \\ \text{orthogonal} \\ \text{to } T_p\Sigma}}{w_1'}, \underset{\substack{\uparrow \\ T_p\Sigma}}{w_2} \rangle + \langle w_1, w_2' \rangle = 0$

Theorem:

Let $\gamma: I \to \Sigma$ be a curve in Σ and choose $w_0 \in$ ~~$T_p\Sigma$~~ $T_{\gamma(t_0)}\Sigma$ for some $t_0 \in I$.

Then there is a unique parallel vector field $w(t)$ along $\gamma(t)$ with $w(t_0) = w_0$.

Definition:

The vector field defined above is called the <u>parallel transport</u> of w_0 along γ.

Geodesics

Definition:

A non-constant parametric curve $\gamma: I \to \Sigma$ is said to be <u>geodesic</u> if γ' is parallel along γ

ie $\underline{\nabla_{\gamma'}\gamma' = 0}$

ie Geodesics are curves such that γ'' is orthogonal to $T_p\Sigma$ at each point.

Parallel vector field $\Rightarrow$ length is constant

γ geodesic $\Rightarrow |\gamma'|$ is constant

ie γ is constant speed, so we can always reparametrise so that it is unit speed (parametrised by arclength)

$1 = |\gamma'|^2 = \langle u'\sigma_u + v'\sigma_v, u'\sigma_u + v'\sigma_v \rangle \qquad \gamma(t) = \sigma(u(t), v(t))$

$\qquad = E(u')^2 + 2Fu'v' + G(v')^2$

$\nabla_{\gamma'}\gamma' = 0 \iff \gamma'' =$ orthogonal to $T_p\Sigma$

$\gamma(t) = \sigma(u,v)$, $w = a\sigma_u + b\sigma_v$, $\gamma' = u'\sigma_u + v'\sigma_v$

ie plug $a = u'$, $b = v'$ in our formula for $\nabla_{\gamma'} w = 0$

$$\begin{cases} u'' + \Gamma_{11}^{1}(u')^2 + 2\Gamma_{12}^{1}u'v' + \Gamma_{22}^{1}(v')^2 = 0 \\ v'' + \Gamma_{11}^{2}(u')^2 + 2\Gamma_{12}^{2}u'v' + \Gamma_{22}^{2}(v')^2 = 0 \end{cases}$$

Corollary:

Choose $p \in \Sigma$ and $w \in T_p\Sigma$ then $\exists$ a unique geodesic γ on Σ which passes through p and has tangent vector w there.

Theorem:

An alternate form of the geodesic ~~form~~ equations is

$$\frac{d}{dt}(Eu' + Fv') = \frac{1}{2}(E_u u'^2 + 2F_u u'v' + G_u v'^2) \qquad 1$$

$$\frac{d}{dt}(Fu' + Gv') = \frac{1}{2}(E_v u'^2 + 2F_v u'v' + G_v v'^2) \qquad 2$$

Proof: γ'' is orthogonal to $T_p\Sigma$, $\gamma = \sigma(u,v)$

$\gamma' = u'\sigma_u + v'\sigma_v$

So $0 = \gamma''\cdot\sigma_u = \left(\frac{d}{dt}(u'\sigma_u + v'\sigma_v)\right)\cdot\sigma_u$

$$= \frac{d}{dt}\big((u'\sigma_u + v'\sigma_v)\cdot\sigma_u\big) - (u'\sigma_u + v'\sigma_v)\cdot\frac{d\sigma_u}{dt}$$

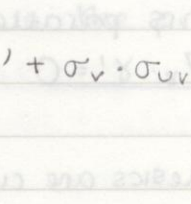
$$= \frac{d}{dt}(u'E + v'F) - (u'\sigma_u + v'\sigma_v)\cdot(\sigma_{uu}u' + \sigma_{uv}v')$$

$$\Rightarrow \frac{d}{dt}(Eu' + Fv') = (\sigma_u\cdot\sigma_{uu})(u')^2 + (\sigma_u\cdot\sigma_{uv} + \sigma_v\cdot\sigma_{uu})u'v' + \sigma_v\cdot\sigma_{uv}v'$$

$$= \frac{1}{2}E_u(u')^2 + F_u u'v' + \frac{1}{2}G_u(v')^2$$

Same for other equation starting with $\gamma''\cdot\sigma_v = 0$.

Example: Geodesics of rotationally symmetric surfaces



$\sigma(f(v), g(v))$

$\sigma(u,v) = (f(v)\cos u, f(v)\sin u, g(v))$

$\Rightarrow \sigma_u = (-f(v)\sin u, f(v)\cos u, 0)$

$\sigma_v = (f'(v)\cos u, f'(v)\sin u, g'(v))$

$E = |\sigma_u|^2 = f(v)^2 \quad F = \sigma_u\cdot\sigma_v = 0 \quad G = (f')^2 + (g')^2$

Take the curve $(f(v), g(v))$ to be unit speed (ie v is arclength for this curve)

Then $G = (f')^2 + (g')^2 = 1$

FFF $f(v)^2 du^2 + dv^2$

1 becomes $\frac{d}{dt}\left(f(v)^2 u'\right) = 0$ 3

2 becomes $v'' = f(v) f'(v)(u')^2$ 4

Consider the case when $u(t) \equiv u_0$ a constant
(curves on the surface contained in a plane also containing the z-axis)

Then 3 is satisfied identically and 4 $\Rightarrow v'' = 0 \Rightarrow v(t) = \alpha t + \beta$

$(u(t), v(t)) = (u_0, \alpha t + \beta)$

So any such curve with constant speed is a geodesic.

Next consider $v(t) = v_0$ is a constant.

Then 3 becomes $u'' = 0$

and 4 becomes $0 = \underset{\neq 0}{f(v_0)} f'(v_0) \underset{R}{u'^2}$

$\neq 0$ since otherwise image of γ is a point.

$\Rightarrow f'(v_0) = 0$

So geodesics in planes perp to the z-axis occur at values of v for which $f(v)$ is stationary
(eg local max/min of distance from z-axis)

In general note that

$\langle \sigma_u, \gamma' \rangle = \langle \sigma_u, \sigma_u u' + \sigma_v v' \rangle$

$= Eu' + Fv'$

$= f(v)^2 u'$

3 $\Leftrightarrow \langle \sigma_u, \gamma' \rangle$ is a constant.

ie $|\sigma_u||\gamma'| \cos\theta = c$

angel between them

Also $|\sigma_u| = \sqrt{E} = f(v)$ = distance to z-axis, which we call r.

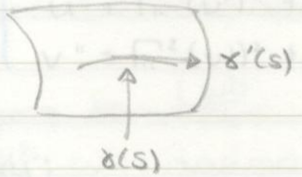
If γ is parametrised by arclength, then $|\gamma'| = 1$

So we have Clairaut's relation: $r \cos\theta = \text{const}$

We can show 4 is automatically satisfied

Geodesic and normal curvatures

Regular surface Σ, curve parametrised by arclength in Σ



$\gamma'(s)$ is a unit tangent vector
$N(\gamma(s))$ is the unit normal to the surface (and $\perp$ to $\gamma'(s)$)

Obviously $\{\gamma'(s), N(\gamma(s)), N(\gamma(s)) \times \gamma'(s)\}$ is an orthonormal basis for $\mathbb{R}^3$

Also $\gamma''(s)$ is orthogonal to $\gamma'(s)$
$\Rightarrow \exists$ scalars k_g and k_n st $\gamma''(s) = k_n N + k_g N \times \gamma'$

k_n normal curvature
k_g geodesic curvature

$k_g = 0 \Leftrightarrow \gamma$ is a geodesic.

'Usual' curvature of curve (chapter 1)
$k = |\gamma''(s)| = \sqrt{k_g^2 + k_n^2}$

Normal curvature : $k_n = \langle N, \gamma'' \rangle$
Now $\langle N \circ \gamma(s), \gamma'(s) \rangle = 0$
$\langle (N\circ\gamma)'(s), \gamma'(s) \rangle + \underbrace{\langle (N\circ\gamma)(s), \gamma''(s) \rangle}_{k_n} = 0$

$\Rightarrow k_n = -\langle (N\circ\gamma)'(s), \gamma'(s) \rangle$
$\quad = -\langle (DN)(\gamma'(s)), \gamma'(s) \rangle$

Let $w \in T_p\Sigma$
$\gamma(0) = p$, $\gamma'(0) = w$
$\Rightarrow k_n = -\langle (DN)w, w \rangle$
$\quad = \mathrm{II}_p(w)$

This gives another way of thinking about the 2nd FF

Recall : $-(DN)_p : T_p\Sigma \longrightarrow T_p\Sigma$ is self adjoint
$\Rightarrow$ orthogonal e'vectors e_1, e_2 (principal directions E'valules k_1, k_2 are called the principle curvatures)
$[K = k_1 k_2 \quad H = \frac{1}{2}(k_1 + k_2)]$

Any vector $w \in T_p\Sigma$ of unit length, can be written as

$$w = e_1\cos\varphi + e_2\sin\varphi$$

for some φ.

$$\begin{aligned} k_n(p) &= \mathrm{II}_p(w) \\ &= -\langle (DN)_p w, w\rangle \\ &= -\langle (DN)_p(e_1\cos\varphi + e_2\sin\varphi), e_1\cos\varphi + e_2\sin\varphi\rangle \\ &= \langle k_1 e_1\cos\varphi + k_2 e_2\sin\varphi, e_1\cos\varphi + e_2\sin\varphi\rangle \\ &= k_1(\cos\varphi)^2 + k_2(\sin\varphi)^2 \end{aligned}$$

Euler's formula:

$$k_n(p) = \mathrm{II}_p(w) = k_1(\cos\varphi)^2 + k_2(\sin\varphi)^2$$

In particular if k_1, k_2 have the same sign, then $k_n(p)$ has the same sign for all curves through p (unit speed).

$k_n(p) = \langle \gamma''(0), (N\circ\gamma)(0)\rangle$

frenet frame $\gamma''(0) = \underline{t}'(0)$

$= k\hat{\underline{n}}$

curvature principle normal vector

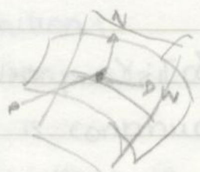
So $k_n(p) = \langle k\hat{\underline{n}}, N\rangle$

$= k\langle \hat{\underline{n}}, N\rangle$

principle normal normal to surface

Definition:

For any $p \in \Sigma$ & $w \in T_p\Sigma$, let P_w be the plane through p parallel to w & $N(p)$



The intersection $\Sigma \cap P_w$ is called the normal section of Σ at p in direction w

Normal section lies in P_w

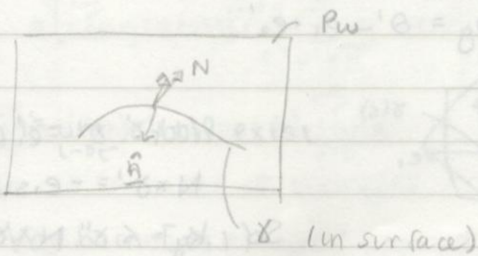
So $\hat{\underline{n}}$ lies in P_w

Principle normal to normal section

N (normal to surface, ie orientation)

also lies in P_w (by construction)

so $N = \pm\hat{\underline{n}}$



$k_n = k\langle \hat{n}, N\rangle = \pm k.$
(+ sign when curve turns in direction of N)

From euler's formula, if $k_1 > 0$, $k_2 > 0 \Rightarrow k_n(p) > 0$ (for all ω)

If $k_1 < 0$, $k_2 < 0 \Rightarrow k_n(p) < 0$

If $k_1 > 0$, $k_2 < 0$

Definition:
$K(p) = k_1 k_2 > 0$ p is called an elliptic point.
$K(p) < 0 \Rightarrow k_1, k_2$ have different signs, p is called a hyperbolic point.
$K(p) = 0$ and $(DN)_p = 0 \Rightarrow$ p is a planar point.

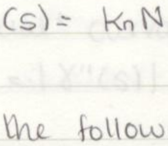
$\gamma''(s) = k_n N + k_g N \times \gamma'$ γ unit speed.

In the following e_1, e_2 will be an orthonormal basis for $T_p\Sigma$.

eg $e_1 = \dfrac{\sigma_u}{\sqrt{E}}$ $e_2 = \dfrac{E\sigma_v - F\sigma_u}{\sqrt{E(EG-F^2)}}$

Will say $N = e_1 \times e_2$

Lemma:
Let Σ be an oriented regular surface with orientation N.
Let e_1, e_2 be smooth functions st at each $p \in \Sigma$, $\{e_1, e_2\}$ is an orthonormal basis for $T_{\gamma(s)}\Sigma$ and $N = e_1 \times e_2$. (γ curve through p).
Let θ be a smooth function st $\gamma' = e_1\cos\theta + e_2\sin\theta$
Then $k_g = \theta' - e_1 \cdot e_2'$



Proof: $\gamma'' = e_1'\cos\theta + e_2'\sin\theta + (-e_1\sin\theta + e_2\cos\theta)\theta'$
$N \times \gamma' = -e_1\sin\theta + e_2\cos\theta$
So $k_g = \langle \gamma'', N \times \gamma'\rangle$
$= \langle e_1'\cos\theta + e_2\sin\theta + \theta'(-e_1\sin\theta + e_2\cos), -e_1\sin\theta + e_2\cos\theta\rangle$

Use $\langle e_1, e_1\rangle = 1$, $\langle e_1, e_2\rangle = 0 \Rightarrow \langle e_1, e_1'\rangle = 0$, $\langle e_1', e_2\rangle + \langle e_1, e_2'\rangle = 0$ etc.

$\Rightarrow K_g = \langle e_1', e_2 \rangle \cos^2\theta - \langle e_2', e_1 \rangle \sin^2\theta + \theta'(\sin^2\theta + \cos^2\theta)\rangle$
$= \theta' - e_1 \cdot e_2'$

Lemma

Using the same notation as above

$$(e_1)_u \cdot (e_2)_v - (e_1)_v \cdot (e_2)_u = \frac{eg - f^2}{\sqrt{EG - F^2}}$$

Proof: $\{e_1, e_2, \hat{N}\}$ is an orthonormal basis for $\mathbb{R}^3$

$e_1 (e_1)_u = 0$ and $e_2 (e_2)_v = 0$ etc

So $\exists$ scalars a, b, c, d st

$(e_1)_u = a e_2 + c\hat{N}$

$(e_1)_v = b e_2 + d\hat{N}$

Noting $e_1 (e_2)_u = -(e_1)_u e_2$ we have

$(e_2)_u = -a e_1 + \hat{c}\hat{N}$ for some $\hat{c}, \hat{d}$

$(e_2)_v = -b e_1 + \hat{d}\hat{N}$

So $(e_1)_u (e_2)_v - (e_1)_v (e_2)_u = c\hat{d} - \hat{c}d$

$= (\hat{N} \cdot (e_1)_u)(\hat{N} \cdot (e_2)_v) - (\hat{N} \cdot (e_2)_u)(\hat{N} (e_1)_v)$

$= (\hat{N}_u \cdot e_1)(\hat{N}_v \cdot e_2) - (\hat{N} \cdot e_2)(\hat{N}_v \cdot e_1)$

Use the identity

$(A \times B) \cdot (C \times D) = (A \cdot C)(B \cdot D) - (A \cdot D)(B \cdot C)$ to get

$(e_1)_u \cdot (e_2)_v - (e_1)_v (e_2)_u = (\hat{N}_u \times \hat{N}_v) \cdot (e_1 \times e_2)$

$= (\hat{N}_u \times \hat{N}_v) \cdot \hat{N}$

$= \frac{eg - f^2}{\sqrt{EG - F^2}}$ using equation 4.12

Definition:

A map $\gamma : [0,1] \to \Sigma$ is a parametrised piecewise regular curve if γ is continuous and $\exists$ $t_0, t_1, \ldots, t_{n+1} \in [0,1]$ where $0 = t_0 < t_1 < \ldots < t_n < t_{n+1} = 1$ such that the restriction of γ to $[t_j, t_{j+1}]$ is a regular curve (called a regular arc). ~~It follows that γ'~~

It follows that $\gamma'(t_j^+) := \lim_{t \to t^+} \gamma'(t)$ and $\gamma'(t_j^-) := \lim_{t \to t^-} \gamma'(t)$ exist.

Furthermore, γ is called simple if $\gamma(a) \neq \gamma(b)$ $\forall a, b$ in $[0,1]$

It is closed $\gamma(0) = \gamma(1)$.

The points $\gamma(t_0), \ldots, \gamma(t_{n+1})$ are called vertices.

Define the exterior angle $\alpha \in [-\pi, \pi]$ at $\gamma(t_j)$ as follows:
$|\alpha_j|$ is the smallest determination of the angle from $\gamma'(t_j^-)$ to $\gamma'(t_j^+)$
If $|\alpha_j| \neq 0$ or π, then $\gamma'(t_j^-) \times \gamma'(t_j^-)$ is non-zero
If it points in the same direction as N then we define α_j to be positive (otherwise it is negative)

For $|\alpha_j| = \pi$ look in online notes

Theorem: Turning Tangents Theorem
With above notation,
$$\sum_{j=0}^{n} [\theta(s_{j+1}^-) - \theta(s_j^+)] + \sum_{j=0}^{n} \alpha_j = 2\pi$$
t=s arclength.

Definition:
A region R of an oriented surface is called simple if it is homeomorphic to the disc (ie bounded and has no holes). and its boundary ∂R is the trace of a simple closed ~~pei~~ piecewise regular curve $\gamma: I \to \Sigma$

~~look~~ Look up stuff about orientation in notes

Gauss - Bonnet Theorem (Local)

$U \subset \mathbb{R}^2$ homeomorphic to open disc, $\sigma: U \to \Sigma$.
Let $R \subset \sigma(U)$ be a simple regular region of Σ with bounding curve $\gamma: I \to \Sigma$ parametrised by arclength
Let $\gamma(s_0), \dots, \gamma(s_n)$ and $\alpha_0, \dots, \alpha_n$ be the vertices and exterior angles respectively. Then
$$\sum_{j=0}^{n} \int_{s_j}^{s_{j+1}} \underbrace{k_g}_{\text{geodesic curvature}}(s)\, ds + \iint_R \underbrace{K}_{\text{Gauss curvature}}\, dA + \sum_{j=0}^{n} \alpha_j = 2\pi$$

Proof: intergrating | gives
$$\sum_{j=0}^{n} \int_{s_j}^{s_{j+1}} k_g \, ds = \sum_{j=0}^{n} \int_{s}^{s_{j+1}} \theta' ds - \sum \int e_1 \cdot e_2' ds$$
$$= \sum_{j=0}^{n} (\theta(s_{j+1}^-) - \theta(s_j^+)) - \sum \int e_1 \cdot e_2' \, ds$$

Using the turning tangent's theorem, the proof is done if we can show that
$$\sum_{j=0}^{n} \int_{s_j}^{s_{j+1}} e_1 \cdot e_2' \, ds = \iint_R K \, dA$$

$$\sum_{j=0}^{n}\int_{s_j}^{s_{j+1}} e_1\cdot e_2'\,ds = \sum_{j=0}^{n}\int_{s_j}^{s_{j+1}} e_1\cdot\left((e_2)_u u' + (e_2)_v v'\right) ds$$

$$= \sum_{j=0}^{n}\int_{s_j}^{s_{j+1}}\left([e_1\cdot(e_2)_u]u' + [e_1\cdot(e_2)_v]v'\right) ds$$

use green's thm $\int P\,du + Q\,dv = \iint_R \left(\frac{\partial Q}{\partial u} - \frac{\partial P}{\partial v}\right) du\,dv$

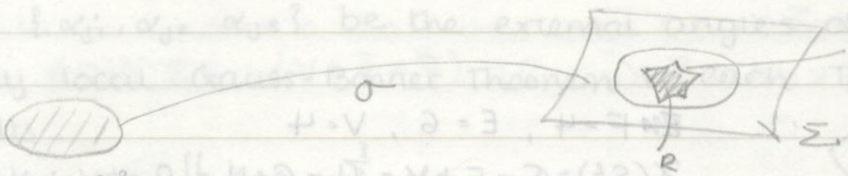
$$= \iint_R \left([e_1\cdot(e_2)_v]_u - [e_1\cdot(e_2)_u]_v\right) du\,dv$$

$$= \iint_R \left[(e_1)_u\cdot(e_2)_v - (e_1)_v\cdot(e_2)_u\right] du\,dv$$

$$= \iint_U \frac{eg-f^2}{\sqrt{EG-F^2}}\,du\,dv$$

$dA = \sqrt{EG-F^2}\,du\,dv$

$$= \iint_U \frac{eg-f^2}{\sqrt{EG-F^2}}\,\frac{dA}{\sqrt{EG-F^2}} = \iint_U K\,dA.$$



$U \subset \mathbb{R}^2$.

Definition:

A region $R \subset \Sigma$ is said to be regular if it is compact and its boundary ∂R is the finite union of non-intersecting, piecewise regular curves.

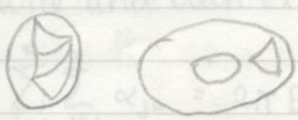
R compact $\Rightarrow \partial R = \emptyset$.

A simple region with only 3 vertices is called a triangle.

Definition:

A triangulation of a regular region $R \subset \Sigma$ is a finite family $\mathcal{T}$ of triangles $T_1, \dots, T_n$ such that

1. $\cup T_i = R$
2. For $i \neq j$ $T_i \cap T_j$ is either empty, a single vertex or a single edge



Given a triangulation, we define

F = number of faces (number of triangles

E = number of edges

V = number of vertices.

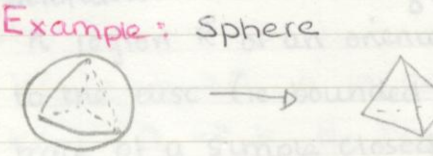
The Euler characteristic of $\mathcal{T}$ is $\chi = F - E + V$

The following facts will be assumed:

1. Every regular region of a regular surface admits a triangulation.
2. Euler characteristic is independent of triangulation.
3. Let Σ be an oriented surface and $\{\sigma_\alpha\}$ be a parametrisation compatible with this orientation. Then $\exists$ a triangulation $\mathcal{T}$ of R st each $T \in \mathcal{T}$ is contained in the image of some parametrisation $\sigma_\alpha(U_\alpha)$.

   Furthermore if the boundary of every triangle in $\mathcal{T}$ is positively oriented, then adjacent triangles determine opposite orientation on the common edge.

Example: Sphere



$F = 4$, $E = 6$, $V = 4$

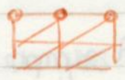
$\chi(S^2) = F - E + V = 4 - 6 + 4 = 2.$

Example: Disc, D.

$\chi(D) = F - E + V$

$= 1 - 3 + 3 = 1$

Example:



share, 2 vertices

$F = 18$, $E = 27$, $V = 9$

$\chi(T^2) = 18 - 27 + 9 = 0.$

$$\sum \int_{c_i} k_g \, ds + \iint K \, dA + \sum \alpha_i = 2\pi \chi(R).$$

(Global) Gauss - Bonnet Theorem:

Let $R \subset \Sigma$ be a regular region of an oriented surface and let $C_1, \dots, C_p$ be simple closed regular ~~surfaces~~ curves which form the boundary ∂R of R. Suppose that each C_i is positively oriented and let $\alpha_1, \dots, \alpha_n$ be the set of external angles of the curves $C_1, \dots, C_p$.

Then $\sum \int_{C_i} k_g(s)\, ds + \iint_R K\, dA + \sum \alpha_i = 2\pi\, \chi(R)$

where s is the arclength of C_i and $\int_{C_i} k_g(s)\, ds$ is the sum of intergrals over the regular arcs of C_i.

Proof: Consider a triangulation of R of the form above, $\mathcal{T} = \{T_i\}$

Let $\{\alpha_{j1}, \alpha_{j2}, \alpha_{j3}\}$ be the external angles of T_j.

Apply local Gauss-Bonnet Theorem to each T_j and sum the results.

$$\int_{\partial T_i} k_g(s)\, ds + \iint_{T_i} K\, dA + \sum_{j=1}^{3} \alpha_{ij} = 2\pi$$

$$\sum \int_{C_i} k_g(s)\, ds + \iint_R K\, dA + \sum_{i=1}^{F} \sum_{j=1}^{3} \alpha_{ij} = 2\pi F$$

because the common edges have ~~oppisite~~ opposite orientations, so $\int k_g\, ds$ terms cancel between adjacent Δs.

In terms of the interior angles, $\varphi_{jk} = \pi - \alpha_{jk}$, we have

$$\sum_{j=1}^{F} \sum_{k=1}^{3} \alpha_{jk} = 3\pi F - \sum_{j=1}^{F} \sum_{k=1}^{3} \varphi_{jk}$$

Let E_e = the number of external edges in $\mathcal{T}$. (ie in ∂R)

E_i = the number of internal edges in $\mathcal{T}$.

V_e = number of external vertices

V_i = number of internal vertices.

Since the C_i are closed $E_e = V_e$.

$3F = 2E_i + E_e$

(for each Δ, if I count the 3 edges, we have counted each interior twice and each exterior edge once).

$$\sum_{j=1}^{F} \sum_{k=1}^{3} \alpha_{jk} = 2\pi E_i + \pi E_e - \sum_{j=1}^{F} \sum_{k=1}^{3} \varphi_{jk} \quad *$$

$V_e = V_{ec} + V_{et}$

V_{et}: exterior vertices from triangulation only.

V_{ec}: # exterior vertices at 'corners' of bounding curves C_i

The sum of the internal angles ~~a~~ at each interior point is 2π.
The sum of the interior angles at each exterior vertex that is not at a vertex of one of the C_is. is π.
The sum of the internal angles at each exterior vertex that is ~~a~~ at a vertex of a C_i is $\pi - \alpha_j$.

So $\sum\sum \varphi_{jk} = 2\pi V_i + \pi V_{et} + \sum_{l=1}^{n} (\pi - \alpha_l)$

$= 2\pi V_i + \pi V_{et} + \pi V_{ee} - \sum_{l=1}^{n} \alpha_l$.

So * becomes, $\sum\sum \alpha_{jk} = 2\pi E_i + \pi E_e - \pi V_e - 2\pi V_i + \sum_{l=1}^{n} \alpha_l$

subtract $\pi(V_e - E_e) = 0$ from RHS.

So $\sum\sum \alpha_{ij} = 2\pi(\underbrace{E_i + E_e}_{F} \underbrace{- V_e - V_i}_{-V}) + \sum e_l$

$= 2\pi(E - V) + \sum \alpha_l$

So * $\sum \int k_g(s)ds + \iint_R K dA + \sum \alpha_l = 2\pi(F - E + V) = 2\pi \chi(R)$.

Applications:

For a compact connected surface Σ, the quantity $g: \frac{2 - \chi(\Sigma)}{2}$ is called the genus
("# ~~num~~ of holes")

Theorem:
Let $\Sigma \subset \mathbb{R}^3$ be a compact surface. Then $\chi(\Sigma)$ takes the values $2, 0, -2, -4, \dots$ (i.e. $g(\Sigma) = 0, 1, 2, 3, \dots$)
Furthermore if $\hat{\Sigma} \subset \mathbb{R}^3$ is a second compact connected surface st $\chi(\hat{\Sigma}) = \chi(\Sigma)$ then Σ is homeomorphic to $\hat{\Sigma}$
(continuous map $\varphi: \Sigma \to \hat{\Sigma}$, φ^{-1} continuous).

Also assuming Jordan curve lemma.

Corollary:
Local Gauss-Bonnet Thm is true, even if we drop the condition that $R \subset \sigma(U)$

Corollary:
Let Σ be an orientable compact surface. ~~then~~
Then $\iint_\Sigma K dA = 2\pi \chi(\Sigma)$

Corollary:
Any compact surface with positive Gauss curvature is homeomorphic to the sphere.

Proof: $K>0 \Rightarrow \chi(\Sigma) = \frac{1}{2\pi}\iint K\,dA > 0$

Since $\chi(\Sigma) \in \{2, 0, -2, \dots\}$.
$\Rightarrow \chi(\Sigma) = 2$, but $\chi(S^2) = 2$
$\Rightarrow \Sigma$ homeomorphic to S^2

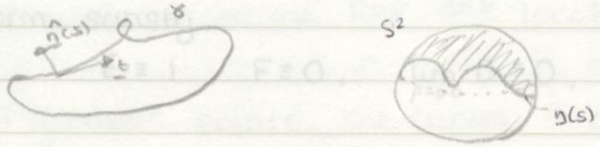
Corollary:
Let Σ be an orientable surface with $K \leq 0$. Then 2 geodesics cannot meet twice in such a way that they form the boundary of a simple region R of Σ

Proof:

Gauss Bonnet thm
$$\iint_R K\,dA + \alpha_1 + \alpha_2 = 2\pi \quad \left(\int k_g\,ds = 0\right\}, k_g = 0).$$

By uniqueness of geodesics, $\alpha_1 < \pi$, $\alpha_2 < \pi$
But $\alpha_1 + \alpha_2 + \iint K\,dt < \pi + \pi + 0 < 2\pi$ contradiction.

Jacobi's Theorem:
Let $\gamma : I \rightarrow \mathbb{R}^3$ be a closed regular curve with non-zero curvature
Assume the curve $\underline{\hat{n}}(I) \subset S^2$ traced by the principle normal is simple
Then $\underline{n}(I)$ divides the sphere into 2 regions of equal area.



Proof: Let $\hat{s}$ be the arclength of n and $\hat{k_g}$ be the geodesic curvature of $\underline{n}$ as a function of $\hat{s}$.
$\hat{k_g} = \ddot{n}\cdot(\dot{n}\times n)$ Taking N = "outer normal" $= n$ "$\cdot$" $= \frac{d}{d\hat{s}}$
$\int k_g\,d\hat{s} + \iint K\,dA = 2\pi$
" ^ (area)

$\gamma'' = k_n N + k_g \gamma \times N$
Frenet: $\frac{d\underline{t}}{ds} = k\underline{n}$ $\quad \frac{dn}{ds} = -k\underline{t} - \tau b$ $\quad \frac{db}{ds} = \tau n$

$\dot{\underline{n}} = \frac{d\underline{n}}{d\hat{s}} = \frac{d\underline{n}}{ds}\frac{ds}{d\hat{s}} = -(\kappa\underline{t} + \tau\underline{b})\frac{ds}{d\hat{s}}$

$\hat{s}$ arclength for $\underline{n} \Leftrightarrow |\dot{\underline{n}}| = 1 \quad \Rightarrow \left|\frac{ds}{d\hat{s}}\right| = \frac{1}{\sqrt{\kappa^2 + \tau^2}}$ *

$\ddot{\underline{n}} = -(\kappa\underline{t} + \tau\underline{b})\frac{d^2 s}{d\hat{s}^2} - \left(\frac{ds}{d\hat{s}}\right)^2\left((\kappa_s\underline{t} + \tau_s\underline{b}) + \underbrace{(\kappa^2 + \tau^2)\underline{n}}_{(\kappa\, dt/ds + \tau\, db/ds)}\right)$

$\hat{\kappa}_g = (n \times \dot{n}) \cdot \ddot{n} = \frac{ds}{d\hat{s}}(\kappa\underline{b} - \tau\underline{t}) \cdot \ddot{\underline{n}}$

$= 0 + \left(\frac{ds}{d\hat{s}}\right)^3(-\tau\kappa_s + \kappa\tau_s) + 0$

$= -\frac{\kappa\tau_s - \tau\kappa_s}{\kappa^2 + \tau^2}\frac{ds}{d\hat{s}}$ from *

$= -\frac{d}{ds}\tan^{-1}\left(\frac{\tau}{\kappa}\right)\frac{ds}{d\hat{s}}$

GB Thm ** $\underbrace{-\oint \frac{d}{ds}\tan^{-1}(\tau/\kappa)\frac{ds}{d\hat{s}}\, d\hat{s}}_{=0} + \iint_R \underbrace{\overset{1}{K}\, dA}_{\text{area of region}} = 2\pi$

$\Rightarrow$ Area of region is $2\pi = \frac{1}{2}$ total surface area.

The exponential map and geodesic polar coordinates.

Recall: Given $p \in \Sigma$ and $w \in T_p\Sigma$ there is a unique geodesic $\gamma : (-\varepsilon, \varepsilon) \to \Sigma$ with $\gamma(0) = p$, $\gamma'(0) = w$

To keep track write $\gamma(t) \equiv \gamma(t; p, w)$.

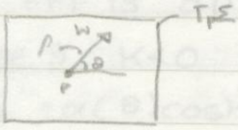
Definition:

For any $p \in \Sigma$ and sufficiently small w ($|w|$ small), we define $\exp_p$ by
$$\exp_p(w) := \gamma(1, p, w)$$

Recall that geodesics are constant speed and here $|\gamma'(0)| = |w|$, so $\exp_p(w)$ results in the point in Σ obtained by moving a distance $|w|$ along the geodesic through p in direction w.

Given a point $p \in \Sigma$ $\exists\ \varepsilon > 0$ such that $\exp_p$ is a diffeomorphism from $B_\varepsilon(0) \subset T_p\Sigma$ onto its image in Σ.

If we describe $w \in T_p\Sigma$ using cartesian coordinates, we get coordinates on a neighbourhood of p in Σ called <u>geodesic normal coordinates</u>.
If we use polar coordinates, we get <u>geodesic polar coordinates</u>



So (ρ, θ)-polars on $T_p\Sigma$ give a parametrisation $\exp_p(\underbrace{\rho, \theta}_{w})$

Theorem:

Let $\sigma : U \setminus l \to V \setminus L \subset \Sigma$ be a parametrisation by geodesic polar coordinates (ρ, θ). Then the coeffs $E(\rho, \theta)$, $F(\rho, \theta)$, $G(\rho, \theta)$ of the first fundemental form satisfy,
$$E \equiv 1, \quad F \equiv 0, \quad \lim_{\rho \to 0} G \equiv 0, \quad \lim_{\rho \to 0} (\sqrt{G})_\rho \equiv 1$$

Proof: parametrisation: $\sigma(\rho, \theta) = \exp_p(\rho, \theta)$

Consider the curve $\gamma(\rho) = \sigma(\rho, \theta_0) = \exp_p(\rho, \theta_0)$ $\quad \theta_0$ const (not in direction of l)

ρ = distance we move along the geodesic γ, so ρ = arclength of γ.

So $1 = |\gamma'(\rho)|^2 = |\sigma_\rho(\rho, \theta)|^2 = E(\rho, \theta) \Rightarrow E \equiv 1$

Geodesic equations
$$\frac{d}{dt}(E\dot\rho + F\dot\theta) = \frac{1}{2}\left(E_\rho \dot\rho^2 + 2F_\rho \dot\rho\dot\theta + G_\rho \dot\theta^2\right)$$
$$\frac{d}{dt}(F\dot\rho + G\dot\theta) = \frac{1}{2}\left(E_\theta \dot\rho^2 + 2F_\theta \dot\rho\dot\theta + G_\theta \dot\theta^2\right)$$

We know that $\theta \equiv \theta_0$ is a geodesic and $E \equiv 1$

$$\frac{d}{dt}(1\dot{\rho}) = 0 \iff \ddot{\rho} = 0$$

$$\frac{d}{dt}(F\dot{\rho}) = 0 \iff F_\rho = 0 \iff \frac{\partial F(\rho, \theta)}{\partial \rho} = 0$$

$$\left((F_\rho \dot{\rho} + \underbrace{\cancel{F_\theta \dot{\theta}}}_{0})\dot{\rho} + \underbrace{\cancel{F\ddot{\rho}}}_{0} = 0 \implies F_\rho = 0. \right)$$

$\sigma(\rho, \theta) = \exp_p(\rho, \theta)$ is a diffeomorphism (derivatives are continuous)

$\implies \frac{\partial}{\partial \theta} \exp_p(\rho, \theta)$ is continuous.

But $\exp_p(0, \theta) = p$ const

So $\lim_{\rho \to 0} \frac{\partial}{\partial \theta} \exp_p(\rho, \theta) = \frac{\partial}{\partial \theta} \text{const} = 0$

$$\lim_{\rho \to 0} F(\rho, \theta) = \lim_{\rho \to 0} \langle \sigma_\rho, \sigma_\theta \rangle = \lim_{\rho \to 0} \langle \frac{\partial}{\partial \rho} \exp_p(\rho, \theta), \underbrace{\frac{\partial}{\partial \theta} \exp_p(\rho, \theta)}_{0} \rangle$$

$$= 0$$

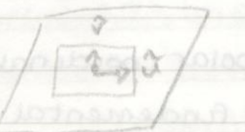
But F doesn't depend on $\rho \implies F \equiv 0$.

in terms of the geodesic normal coordinates

$\hat{u} = \rho\cos\theta$, $\hat{v} = \rho\sin\theta$ (usual cartesian coords on $B_\varepsilon(0)$)

We have $\sqrt{\underbrace{E}_{1} G - \underbrace{F^2}_{0}} = \sqrt{\hat{E}\hat{G} - \hat{F}^2}\ \frac{\partial(\hat{u}, \hat{v})}{\partial(\rho, \theta)}$

$$\sqrt{G} = \sqrt{\hat{E}\hat{G} - \hat{F}^2}\ \rho$$



$\hat{E} = 1$, $\hat{G} = 1$, $\hat{F} = 0$ at p.

$$\implies \sqrt{G} = \rho\sqrt{(1 \cdot 1 - 0^2) + o(\rho)}$$

$\sqrt{G} \to 0$ as $\rho \to 0$

$(\sqrt{G})_\rho \to 1$

FFF is just $d\rho^2 + G\,d\theta^2$

Theorem (Mindins)

Any 2 regular surfaces with the same constant Gauss curvature are locally isometric

Proof: Recall (HW problem)

if $F \equiv 0$ $\quad K = \frac{1}{2\sqrt{EG}} \left(\left(\frac{E_v}{\sqrt{EG}} \right)_v + \left(\frac{G_u}{\sqrt{EG}} \right)_u \right)$

In geodesic polar coordinates $E = 1, F = 0$

$$\Rightarrow K = -\frac{(\sqrt{G})_{\rho\rho}}{\sqrt{G}}$$

$\Leftrightarrow (\sqrt{G})_{\rho\rho} + K\sqrt{G} = 0$ (2^{nd} order const coeff "ODE" for $\sqrt{G}$) $*$

Case 1: $K = 0 \Rightarrow (\sqrt{G})_{\rho\rho} = 0 \Rightarrow (\sqrt{G})_\rho = \alpha(\theta)$

Also $(\sqrt{G})_\rho \to 1$ as $\rho \to 0$ so $\alpha(\theta) \equiv 1$.

$\Rightarrow (\sqrt{G})_\rho = 1 \Rightarrow \sqrt{G} = \rho + \beta(\theta)$

$\sqrt{G} \to 0$

$\beta(\theta) = \lim_{\rho \to 0} (\sqrt{G} - \rho) = 0 - 0 = 0 \quad \Rightarrow \sqrt{G} = \rho$

$\Rightarrow$ FFF is $d\rho^2 + \rho^2 d\theta^2$

Case 2: $K > 0$ (look for solutions of form $\sqrt{G} = e^{r\rho}$)

$r^2 + K = 0 \Rightarrow \sqrt{G} = \alpha(\theta)\cos\sqrt{K}\rho + \beta(\theta)\sin\sqrt{K}\rho$

$\lim \rho \to 0, \sqrt{G} \to 0$

$0 = \alpha(\theta) \Rightarrow \sqrt{G} = \beta(\theta)\sin\sqrt{K}\rho$

$\Rightarrow (\sqrt{G})_\rho = \sqrt{K}\beta(\theta)\cos\sqrt{K}\rho$

$\rho \to 0 \quad 1 = \sqrt{K}\beta(\theta) \quad \Rightarrow \beta(\theta) = 1/\sqrt{K}$

$\Rightarrow \sqrt{G} = \frac{1}{\sqrt{K}}\sin\sqrt{K}\rho$

So FFF is $d\rho^2 + \frac{1}{K}(\sin\sqrt{K}\rho)^2 d\theta^2$

Case 3: $K < 0$

$\sqrt{G} = \alpha(\theta)\cosh\sqrt{-K}\rho + \beta(\theta)\sinh\sqrt{-K}\rho$

$\therefore$ FFF is $d\rho^2 + \frac{1}{(-K)}(\sinh^2\sqrt{-K}\rho)d\theta^2$

So any 2 surfaces with the same constant K have the same FFFs (each answer obtained above is unique.)

2 surfaces, same FFF $\Leftrightarrow$ locally isometric.

In geodesic polars, the curves θ = const are geodesics, the curves ρ = const are called "geodesic circles" (but they are not geodesics)

Theorem:

Let L be the arclength of the geodesic circle $\rho = r$ centred at $p \in \Sigma$

Then $K(p) = \lim_{r \to 0} \frac{3}{\pi}\frac{2\pi r - L}{r^3}$

where $K(p)$ is the Gauss curvature at p.

Proof: Work in geodesic polars (ρ, θ) centred at p.

$* \Rightarrow (\sqrt{G})_{\rho\rho\rho} + K_\rho \sqrt{G} + K(\sqrt{G})_\rho = 0$ $**$

Take limit $\rho \to 0$, $\sqrt{G} \to 0$, $(\sqrt{G})_\rho \to 1$

$* \Rightarrow (\sqrt{G})_{\rho\rho} = -K\sqrt{G} \to 0$

$** \Rightarrow (\sqrt{G})_{\rho\rho\rho} = -K_\rho \sqrt{G} - K(\sqrt{G})_\rho \to -K$

Taylor series $\sqrt{G}$

$$\sqrt{G}(\rho,\theta) = \sqrt{G}(0,\theta) + \rho(\sqrt{G})_\rho(0,\theta) + \frac{\rho^2}{2}(\sqrt{G})_{\rho\rho}(0,\theta) + \frac{\rho^3}{6}(\sqrt{G})_{\rho\rho\rho} + o(\rho^3)$$

where $o(\rho^3)$ represents some function $g(\rho,\theta)$ st $\frac{g(\rho,\theta)}{\rho^3} \to 0$ as $\rho \to 0$.

So $\sqrt{G} = \rho - \frac{\rho^3}{6}K + o(\rho^3)$

Recall the length of a curve

$$s = \int_{t_0}^{t_1} \sqrt{\dot{x}^2 + \dot{y}^2 + \dot{z}^2}\, dt = \int_{t_0}^{t_1} \sqrt{\underset{1}{E}(u')^2 + 2\underset{0}{F}u'v' + G(v')^2}\, dt$$

In our case

$$L = \lim_{\varepsilon \to 0} \int_\varepsilon^{2\pi - \varepsilon} \sqrt{G(r,\theta)}\, d\theta \quad (\rho = r \text{ cons})$$

$$\Rightarrow L = 2\pi r - \frac{r^3}{3}\pi K + o(r^3)$$

$$\Rightarrow K(p) = \lim_{r \to 0} \frac{3}{\pi} \frac{2\pi r - L}{r^3}$$

Lemma: Wirtinger's inequality

For any differentiable function $F : [0,\pi] \to \mathbb{R}$ with $F(0) = 0$, $F(\pi) = 0$

$$\int_0^\pi F(t)^2 dt \le \int_0^\pi \left(\frac{dF}{dt}\right)^2 dt.$$

where the equality holds if and only if $\exists$ const c, st $F(t) = c\sin t$ $\forall t \in [0,\pi]$.

Proof: Let $G(t) = F(t)/\sin t$ $\forall t \in (0,\pi)$

Then $\dot{F} = \dot{G}\sin t + G\cos t$

$\dot{F}^2 = \dot{G}^2 \sin^2 t + 2\dot{G}G\sin t\cos t + G^2\cos^2 t$ #

Using intergration by parts

$$\int_0^\pi \underbrace{(2G\dot{G})}_{\frac{dG^2}{dt}} \sin t\cos t\, dt = G^2 \sin t \cancel{\cos t}\Big|_0^\pi - \int_0^\pi G^2(\cos^2 t - \sin^2 t)\, dt$$

$$= \int_0^\pi F(t)^2 dt - \int_0^\pi G^2 \cos^2 t\, dt$$

Intergrating # gives

$\int_0^\pi \dot{F}^2 dt = \int_0^\pi \dot{G}^2 \sin^2 t \, dt + \int_0^\pi F^2 dt \geq \int_0^\pi F^2 dt$

where equality holds iff $\int_0^\pi \dot{G}^2 \sin^2 t \, dt = 0$

$\Leftrightarrow \dot{G} = 0 \Leftrightarrow G = c$ is a const $\Leftrightarrow F = c\sin t$.

Theorem: Isoperimetric inequality

Let C be a simple closed curve with length L and let A be the area of the interior of C.

Then $A \leq \frac{L^2}{4\pi}$ where equality holds if and only if C is a circle.

Proof: Let $\tilde{\gamma} : [0, L] \to \mathbb{R}^2$ be parametrised by arclength, tracing C in the +ve direction.

Define $\gamma(t) = \tilde{\gamma}(s) - \tilde{\gamma}(0)$, where $t = \frac{\pi s}{L}$

Express γ in polar coordinates

$\gamma(t) = (x(t), y(t)) = (\rho(t)\cos\theta(t), \rho(t)\sin\theta(t))$

$\dot{x} = \dot{\rho}\cos\theta - \rho\dot{\theta}\sin\theta \qquad \dot{y} = \dot{\rho}\sin\theta + \rho\dot{\theta}\cos\theta$

Recall Greens thm

$$\iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy = \int_0^\pi (P\dot{x} + Q\dot{y})\,dt$$

Choose $P = -y/2$, $Q = x/2$

$$\Rightarrow A = \iint_R dx\,dy = \frac{1}{2}\int_0^\pi (x\dot{y} - y\dot{x})\,dt$$

$$= \frac{1}{2}\int_0^\pi \rho^2\dot{\theta}\,dt$$

Also from det^n of $t = \pi s/L$

$$\frac{L^2}{\pi^2} = \left(\frac{ds}{dt}\right)^2 = \dot{x}^2 + \dot{y}^2 = \dot{\rho}^2 + \rho^2\dot{\theta}^2$$

Hence $\frac{L^2}{4\pi} - A = \frac{1}{4}\int_0^\pi (\dot{\rho}^2 + \rho^2\dot{\theta}^2 - 2\rho^2\dot{\theta})\,dt$

$$= \underbrace{\frac{1}{4}\int_0^\pi \rho^2(\dot{\theta}-1)\,dt}_{I_1} + \underbrace{\frac{1}{4}\int_0^\pi (\dot{\rho}^2 - \rho^2)\,dt}_{I_2}$$

$I_1 \geq 0$ and $I_1 = 0 \Leftrightarrow \dot{\theta} = 1 \Leftrightarrow \theta = t + c$

$I_2 \geq 0$ by Wirtingers lemma $F = \rho$ $\rho(0) = \rho(\pi) = 0$

with equality $\Leftrightarrow \rho(t) = c\sin t$ - eliminate $t \Rightarrow$ polar eq^n of circle.

The Rigidity of the Sphere

Recall: $p \in \Sigma$ is umbillical if and only if the principle curvatures are equal: $k_1(p) = k_2(p)$

Σ is called totally umbilical $\Leftrightarrow$ p is umbilical, $\forall p \in \Sigma$.

Theorem:
The only totally umbilical connected surfaces are the open subsets of planes and spheres

$(k = k_1 = k_2)$
$-(DN)_p e_1 = k e_1$
$-(DN)_p e_2 = k e_2$
$\Rightarrow -(DN)_p v = kv$

proof: Since Σ is totally umbillical $\exists$ smooth function $f(u,v)$ st $(DN) = f(u,v)\, id$ $\quad (f(u,v) = -k)$.

identity

So in particular $(DN)\sigma_u = f\sigma_u$ and $(DN)\sigma_v = f\sigma_v$.

But $(DN)\sigma_u = \tilde{N}_u$ etc.

So $\tilde{N}_u = f\sigma_u$ and $\tilde{N}_v = f\sigma_v$ $\quad *$

So $(f\sigma_u)_v = (\tilde{N}_u)_v = (f\sigma_v)_u$

$\Leftrightarrow f_v\sigma_u + \cancel{f\sigma_{uv}} = f_u\sigma_v + \cancel{f\sigma_{vu}}$

$\quad f_v\sigma_u = f_u\sigma_v$

$\Rightarrow f_u = f_v = 0$ since σ_u, σ_v are linearly independent

$\Rightarrow f = \text{const}$

Case 1 $f = 0$

Then $*$ $\Rightarrow \tilde{N}_u = \tilde{N}_v = 0 \Rightarrow \tilde{N} = \text{const}$

$\Rightarrow \Sigma$ is a plane

Case 2 $f \neq 0$

$\sigma - \frac{1}{f}\tilde{N} = \underline{c}$

$\Rightarrow |\sigma(u,v) - \underline{c}| = |\frac{1}{f}\tilde{N}| = \frac{1}{|f|}$ $\Rightarrow \sigma(u,v)$ lies on the sphere centred at $\underline{c}$ with radius $1/|f|$.

Corollary:
The only totally umbilical surfaces which are closed subsets of $\mathbb{R}^3$ are spheres and planes.

Lemma:
Any non-umbilical point of a regular surface has a nghd which is the image of a parametrisation for which $F = f = 0$.

Theorem: (Hilbert).

Let Σ be an oriented surface with principal curvatures $k_1 \le k_2$.
Suppose that the following conditions hold at some point $p \in \Sigma$.

1. $K(p) > 0$
2. k_1 has a local minimum at p
3. k_2 has a local maximum at p

Then p is an umbilical point.

Proof: Assume that p is not umbilical.
Therefore there exists parametrisations st $F = f = 0$

$\Rightarrow K = \dfrac{eg}{EG} \quad H = \dfrac{1}{2}\left(\dfrac{e}{E} + \dfrac{g}{G}\right)$

$\Rightarrow k_1 = \dfrac{e}{E},\ k_2 = \dfrac{g}{G}$ (wlog changing $u \leftrightarrow v$ if necessary).

$$\begin{pmatrix} \Gamma_{11}^1 & \Gamma_{12}^1 & \Gamma_{22}^1 \\ \Gamma_{11}^2 & \Gamma_{12}^2 & \Gamma_{22}^2 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} E^{-1} & 0 \\ 0 & G^{-1} \end{pmatrix}\begin{pmatrix} E_u & E_v & -G_u \\ -E_v & G_u & G_v \end{pmatrix}$$

$$= \frac{1}{2}\begin{pmatrix} E_u/E & E_v/E & -G_u/E \\ -E_v/G & G_u/G & G_v/G \end{pmatrix}$$

Mainardi-Codazzi equations.

$e_v - \cancel{f_u} = e\Gamma_{12}^1 + \cancel{f} - g\Gamma_{11}^2$

$\cancel{f_v} - g_u = e\Gamma_{22}^1 + \cancel{f} - g\Gamma_{12}^2$

$\Rightarrow e_v = \dfrac{1}{2}\left(e\dfrac{E_v}{E} + g\dfrac{E_v}{G}\right) = \dfrac{E_v}{2}(k_1 + k_2)$ 1

$g_u = \dfrac{1}{2}\left(e\dfrac{G_u}{E} + g\dfrac{G_u}{G}\right) = \dfrac{G_u}{2}(k_1 + k_2)$ 2

$(k_1)_v = \dfrac{e_v}{E} - \dfrac{e}{E}\dfrac{E_v}{E}$

$= \dfrac{e_v}{E} - k_1\dfrac{E_v}{E} = \dfrac{E_v}{E}\left(\dfrac{k_1}{2} + \dfrac{k_2}{2} - k_1\right)$

Using 1 to eliminate e_v: $E_v = \dfrac{2E(k_1)_v}{k_2 - k_1}$

Similarly $G_u = \dfrac{2G(k_2)_u}{k_1 - k_2}$

Gauss equation: $EK = (\Gamma_{11}^2)_v - (\Gamma_{12}^2)_u + \Gamma_{11}^1\Gamma_{12}^2 + \Gamma_{11}^2\Gamma_{22}^2 - \Gamma_{12}^1\Gamma_{11}^1 - \Gamma_{12}^2\Gamma_{12}^2$ 3

Now $\Gamma_{11}^2 = -\dfrac{1}{2}\dfrac{E_v}{G} = \dfrac{E}{G}\dfrac{(k_1)_v}{k_1 - k_2}$

So $(\Gamma_{11}^{\ 2})_v = \frac{E}{G}\frac{(k_1)_{vv}}{k_1-k_2} + \left(\frac{E}{G}\frac{1}{k_1-k_2}\right)_v (k_1)_v$

Similarly $(\Gamma_{12}^{\ 2})_u = \frac{(k_2)_{uu}}{k_1-k_2} + \left(\frac{1}{k_1-k_2}\right)_u (k_2)_u$

Assumption 1 $\Rightarrow$ LHS of 3 is >0 at p

2 $\Rightarrow$ $(k_1)_v(p)=0$ and $(k_1)_{vv}(p) \geq 0$

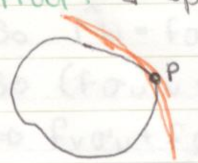
3 $\Rightarrow$ $(k_2)_u(p)=0$ and $(k_2)_{uu}(p) \leq 0$

$\Rightarrow$ RHS of 3 ≤ 0 contradiction!

Lemma:

A regular compact surfaces $\Sigma \subset \mathbb{R}^3$ has at least one eliptic ~~curve~~ point.
(ie $\exists\, p \in \Sigma$ st $K(p)>0$.

Proof: $\exists$ sphere S centred at O of maximum radius st $\Sigma \cap S \neq \emptyset$



Choose $p \in \Sigma \cap S$, Σ and S are tangent at p

Choose $w \in T_p\Sigma$

Let $\gamma_1 : (-\varepsilon, \varepsilon) \to \Sigma$ and $\gamma_2 : (-\varepsilon,\varepsilon) \to S$ be unit speed parametrisations of the normal sections of Σ and S in direction w. Let $N(p)$ be the unit normal to S (and Σ) at p pointing towards O.

$N(p) = \frac{p}{|p|}$

Since S is the largest sphere intersecting Σ we must have

$\langle \gamma_1(s), N(p)\rangle \geq \langle \gamma_2(s), N(p)\rangle$

$\Leftrightarrow \langle \gamma_1(s)-p, N(p)\rangle \geq \langle \gamma_2(s)-p, N(p)\rangle$

$\Leftrightarrow \langle \gamma_1(0) + s\gamma_1'(0) + \frac{s^2}{2}\gamma_1''(0) + O(s^2) - p, N(p)\rangle \geq \langle \gamma_2(0) + s\gamma_2'(0) + \dots - p, N(p)\rangle$

Now $\gamma_1(0) = \gamma_2(0) = p$, $\gamma_1'(0) = \gamma_2'(0) = w \in W_pS \perp N$

So $\frac{s^2}{2}\langle \gamma_1''(0), N\rangle \geq \frac{s^2}{2}\langle \gamma_2''(0), N\rangle + O(s^3)$

$\Rightarrow \langle \gamma_1''(0), N\rangle \geq \langle \gamma_2''(0), N\rangle \quad (s \to 0)$

$\Rightarrow$ normal curvature of $\gamma_1 \geq$ normal curvature of $\gamma_2 = 1/R$ ← radius of sphere
(great circle on sphere)

$\Rightarrow$ each principal curvature of Σ at $p \geq 1/R$

So $K(p) = k_1(p)k_2(p) \geq \frac{1}{K^2} > 0$

Theorem: (Liebmann)

Let Σ be a compact connected regular surface with constant Gauss curvature. Then Σ is a sphere.

Proof: Since K is constant, the previous lemma shows that $K>0$. Label principle curvature st $k_1(q) \leq k_2(q)$ $\forall q \in \Sigma$.

Since Σ is compact k_2 must have a maximum at some point $p \in \Sigma$. Also $k_1(q) = K/k_2(q)$ so k_1 has a minimum at p.

$\therefore$ Hilbert's Theorem $\Rightarrow$ p is umbilical.

For any $q \in \Sigma$

$k_2(q) \leq k_2(p)$ since max of k_1 at p

$= k_1(p)$ p is umbilical.

$\leq k_1(q)$ since min of k_1 is at p

So $k_2(q) \leq k_1(q)$ but by defn $k_1(q) \leq k_2(q)$

$\Rightarrow k_1(q) = k_2(q)$ $\forall q \in \Sigma$ $\Rightarrow \Sigma$ is totally umbilical. $\boxtimes$

Theorem: Rigidity of spheres

Let S be a sphere of radius $R>0$ and let Σ be a connected surface. If Σ is ~~a~~ locally isometric to S the Σ is a sphere of radius R.

Proof: Isometries preserve K. So Σ has constant Gauss curvature $K = \frac{1}{R^2}$. By Liebmann's Thm $\Rightarrow$ sphere.

Revision Notes

- Gives important formula.

Example questions:

1a i) Verify that $(s-\tan^{-1}s, \log(s^2+1), 1)$ is parametrised by arclength.

ii) Let $\underline{t}, \underline{n}, \underline{b}$ be $\mathbb{R}^3$ valued functions of $s \in I$ solving
$\underline{t}' = \kappa \underline{n}$ $\quad \underline{n}' = -\kappa \underline{t} - \underline{b}$ $\quad \underline{b}' = \tau \underline{n}$
for $s \in I$, where

Define
$$M(s) = \begin{pmatrix} t(s)t(s) & \underline{t}\cdot\underline{n} & \underline{t}\cdot\underline{b} \\ \underline{n}\cdot\underline{t} & \underline{n}\cdot\underline{n} & \underline{n}\cdot\underline{b} \\ \underline{b}\cdot\underline{t} & \underline{b}\cdot\underline{n} & \underline{b}\cdot\underline{b} \end{pmatrix} \qquad M' = AM - MA \quad *$$

Suppose that for some value s_0 of s, the vectors $\underline{t}(s_0)$, $\underline{n}(s_0)$, $\underline{b}(s_0)$ form a right handed orthonormal frame.
Show that $(\underline{t}(s), \underline{n}(s), \underline{b}(s))$ remains right handed orthonormal frame.

We have initial value problem $*$ with $M(s_0) = I$
Existence and uniqueness says there is exactly one soln of the problem.
Clearly $M \equiv I$ is a solution of this initial value problem
$\therefore$ It is the only solution.
So $M \equiv I \Rightarrow \{\underline{t}, \underline{n}, \underline{b}\}$ is orthonormal.
Initially right handed continuity shows it can't suddenly become left handed.

2. Let σ be a parametrisation of a regular surface Σ with orientation N and let $\tilde{N} = N \circ \sigma$.

i) Show that $(DN)_p \sigma_u(u_0, v_0) = \tilde{N}_u(u_0, v_0)$ where $p = \sigma(u_0, v_0)$

Let $\gamma(t) = \sigma(u_0 + t, v_0)$
So $\gamma(0) = p = \sigma(u_0, v_0)$
$\gamma'(0) = \sigma_u(u_0, v_0)$

So $(DN)_p \sigma_u(u_0, v_0) = (N \circ \gamma)'(0) = \frac{d}{dt}((N \circ \gamma)(u_0 + t, v_0))\Big|_{t=0} = (N \circ \sigma)_u(u_0, v_0)$

ii) Show that $\tilde{N}_u(u,v) = a(u,v)\sigma_u(u,v) + b\sigma_v$ 1
$\tilde{N}_v(u,v) = c(u,v)\sigma_u(u,v) + d\sigma_v$ 2

with $\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \dots$ (explicity given in paper).

$\tilde{N}$ is a unit (ie constant length) vector, so $\tilde{N}_u$ and $\tilde{N}_v$ are orthogonal to $\tilde{N}$ and hence in $T_p\Sigma$. Furthermore $\{\sigma_u, \sigma_v\}$ is a basis for $T_p\Sigma$, so $\exists$ a, b, c, d st 1 and 2 are true.

$\langle \sigma_u, 1 \rangle \Rightarrow \langle \tilde{N}_u, \sigma_u \rangle = a\underbrace{\langle \sigma_u, \sigma_u \rangle}_{E} + b\underbrace{\langle \sigma_u, \sigma_v \rangle}_{F}$

$\langle \tilde{N}_u, \sigma_u \rangle = \langle (DN)_p\sigma_u, \sigma_u \rangle$ from i
$= -e$

$-e = aE + bF.$

...

$-f = aF + bG$
$-f = cE + dF$
$-g = cF + dG$
$\Rightarrow \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} E & F \\ F & G \end{pmatrix} = -\begin{pmatrix} e & f \\ f & g \end{pmatrix}$ ...

iii) A surface $\hat{\Sigma}$ is said to be parallel to Σ if it has a parametrisation $\hat{\sigma}(u,v) = \sigma(u,v) + \alpha\tilde{N}(u,v)$. where α is a constant.
Show that $\hat{\sigma}_u \times \hat{\sigma}_v = (1 - 2H\alpha + K\alpha^2)\sigma_u \times \sigma_v$.

$\hat{\sigma}_u = \sigma_u + \alpha\tilde{N}_u$
$\hat{\sigma}_v = \sigma_v + \alpha\tilde{N}_v$