3203 Algebraic Topology Notes

Based on the 2014 spring lectures by Prof F E A Johnson

MATH3203 – Algebraic Topology.

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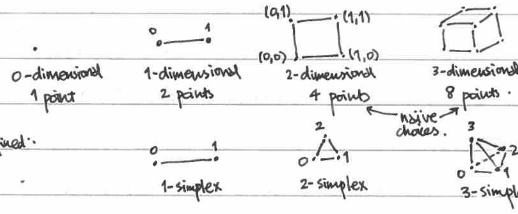
The study of topology focuses essentially on analysis (point sets / general topology). Here however, it will be considered from the perspective of algebra.
This deals with the work of Poincaré, Brouwer and Lefschetz.

We begin with an attempt to classify geometrical objects, as on the right:
this becomes increasingly complicated as we continue, so we abandon the naïve choices
and describe these objects more 'intelligently', using the idea of simplices, which are formally defined:



Definition By a simplicial complex K, we mean $K=(V_K, S_K)$ where V_K is the set of vertices of K, and S_K is a set of finite subsets of V_K s.t.

(i) if $\sigma \in S_K$ then $\sigma \neq \phi$ (ii) if $\sigma \in S_K$ and $\tau \subset \sigma$, $\tau \neq \phi$, then $\tau \in S_K$.

Using this definition, we classify our simplices again:

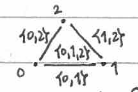
① $\{0,1\}$

Δ^1: standard 1-simplex.

$V_{\Delta^1} = \{0,1\}$, $S_{\Delta^1} = \{\{0\}, \{1\}, \{0,1\}\}$

S_{Δ^1} is the set of all non-empty subsets of V_{Δ^1}.

②

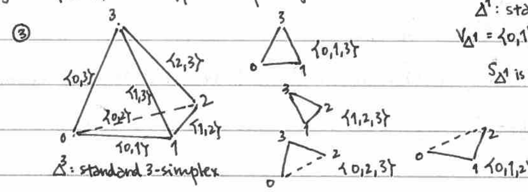


Δ^2: standard 2-simplex.

$V_{\Delta^2} = \{0,1,2\}$, $S_{\Delta^2} = \{\{0\}, \{1\}, \{2\}, \{0,1\}, \{0,2\}, \{1,2\}, \{0,1,2\}\}$

S_{Δ^2} is the set of all non-empty subsets of V_{Δ^2}.

③

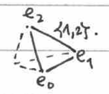


Δ^3: standard 3-simplex

interior is $\{0,1,2,3\}$. $V_{\Delta^3} = \{0,1,2,3\}$, S_{Δ^3} is the set of all non-empty subsets of V_{Δ^3}.

By analogy, Δ^n: standard n-simplex is defined by $V_{\Delta^n} = \{0,1,\dots,n\}$ with S_{Δ^n} being the set of all non-empty subsets of $\{0,1,\dots,n\}$.

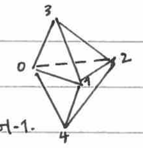
Geometric realisation of Δ^n: let $e_0, \dots, e_n$ be standard basis for $\mathbb{R}^{n+1}$. $|\Delta^n| = \{t_0 e_0 + t_1 e_1 + \dots + t_n e_n : 0 \leq t_i \leq 1, \sum_{i=0}^{n} t_i = 1\} \subset \mathbb{R}^{n+1}$.



$A=\{0,1,2\}$, then $|A| = \{t_0 e_0 + t_1 e_1 + t_2 e_2\}$. If $A=\{1,2\}$, $t_0 = 0$, $|A| = \{t_1 e_1 + t_2 e_2\}$

Examples –

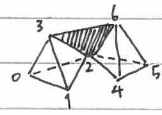
1. Consider the simplex X as on right. Then let $V_X = \{0,1,2,3,4\}$, and $S_X = \{\{0,1,3\}, \{0,2,3\}, \{1,2,3\}, \{0,1,4\}, \{1,2,4\}, \{0,2,4\}, \{0,1\}, \{0,2\}, \{0,3\}, \{0,4\}, \{1,2\}, \{1,3\}, \{1,4\}, \{2,3\}, \{2,4\}, \{0\}, \{1\}, \{2\}, \{3\}, \{4\}\}$



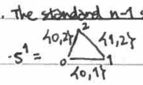
Clearly X is 2-dimensional – Here $X=(V_X, S_X)$ is the simplicial complex, V_X is the vertex set, S_X is the set of simplices. For $\sigma \in S_X$, $\dim(\sigma) = |\sigma| - 1$.

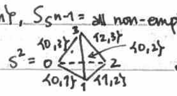
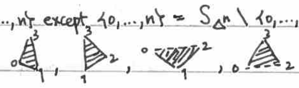
Thus, $\dim X = \max\{\dim(\sigma) : \sigma \in S_X\} = \max\{|\sigma|-1 : \sigma \in S_X\} = 2$.

2. Consider the simplex as on right. Here, $V = \{0,1,\dots,6\}$, $S = \{\{0\}, \{1\}, \{2\}, \{3\}, \{4\}, \{5\}, \{6\}, \{0,1\}, \{0,2\}, \{0,3\}, \{1,2\}, \{1,3\}, \{2,3\}, \{2,4\}, \{2,5\}, \{2,6\}, \{3,6\}, \{4,5\}, \{4,6\}, \{5,6\}, \{2,3,6\}\}$.



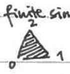
3. The standard $n-1$ sphere, S^{n-1}: $V_{S^{n-1}} = \{0,\dots,n\}$, $S_{S^{n-1}}$ = all non-empty sets of $\{0,1,\dots,n\}$ except $\{0,\dots,n\}$ = $S_{\Delta^n} \setminus \{0,\dots,n\}$

S^1 =  but not $\{0,1,2\}$.

S^2 =  and all 2-faces  but not the interior $\{0,1,2,3\}$.

Homology

Let $X=(V_X, S_X)$ be a finite simplicial complex, $\sigma \in S_X$ is an n-simplex of X when $\dim(\sigma) = n$ i.e. $|\sigma| = n+1$. Then define $C_n(X)$ to be a vector space over $\mathbb{F}_2$ with basis of n-simplices of X.

e.g. if $X = \Delta^2$ , $C_0(\Delta^2)$ is 3-dim basis $\{0\}, \{1\}, \{2\}$; $C_1(\Delta^2)$ is 3-dim basis $\{0,1\}, \{0,2\}, \{1,2\}$; $C_2(\Delta^2)$ is 1-dim basis $\{0,1,2\}$.

Moreover, $C_{-1}(\Delta^2) = 0$, $C_r(\Delta^2) = 0$ if $r \geq 3$.

In addition, we have "boundary" maps $\partial_n : C_n(X) \to C_{n-1}(X)$, which are linear. It suffices to describe $\partial_n(\sigma)$ for $\sigma \in S_X$ (i.e. σ is a basis element for $C_n(X)$.)

Over $\mathbb{F}_2$, if $\sigma = \{v_0, \dots, v_n\}$, $\partial_n(\sigma) = \sum_{r=0}^{n} \{v_0, \dots, \hat{v}_r, \dots, v_n\}$ where $\hat{v}_r$ denotes the omission of vertex v_r.

e.g. Again for $X = \Delta^2$, $C_2(\Delta^2)$ has basis $\{0,1,2\} = E_1$, $C_1(\Delta^2)$ has basis $\overset{e_1}{\{0,1\}}, \overset{e_2}{\{0,2\}}, \overset{e_3}{\{1,2\}}$, $C_0(\Delta^2)$ has basis $\overset{\varphi_1}{\{0\}}, \overset{\varphi_2}{\{1\}}, \overset{\varphi_3}{\{2\}}$. Then we examine boundary maps –

$\partial_2 : C_2(\Delta^2) \to C_1(\Delta^2)$, $\partial_2\{0,1,2\} = \overset{e_3}{\{1,2\}} + \overset{e_2}{\{0,2\}} + \overset{e_1}{\{0,1\}}$. $\partial_1 : C_1(\Delta^2) \to C_0(\Delta^2)$. For the basis elements, $\partial_1\{0,1\} = \{1\} + \{0\}$, $\partial_1\{0,2\} = \{2\} + \{0\}$, $\partial_1\{1,2\} = \{2\} + \{1\}$.

For the prior example, we attempt to calculate $\partial_1 \partial_2 \{0,1,2\} = \partial_1\{\{1,2\} + \{0,2\} + \{0,1\}\} \overset{\text{linear}}{=} \partial_1\{1,2\} + \partial_1\{0,2\} + \partial_1\{0,1\} = \{2\} + \{1\} + \{2\} + \{0\} + \{1\} + \{0\} = 2(\{0\} + \{1\} + \{2\}) = 0$ ($2 = 0$ in $\mathbb{F}_2$.)

We can generalise this to a proposition.

Proposition (Poincaré's Lemma, over $\mathbb{F}_2$)

$\partial_{n-1}\partial_n = 0$.

Proof – We prove this for basis elements, i.e. NTP: if $\sigma = \{v_0, \dots, v_n\}$ is an n-simplex, $\partial_{n-1}\partial_n(\sigma) = 0$. So we have $\partial_{n-1}\partial_n\{v_0, \dots, v_n\} = \partial_{n-1} \sum_{r=0}^{n} \{v_0, \dots, \hat{v}_r, \dots, v_n\}$

$= \sum_{r=0}^{n} \partial_{n-1}\{v_0, \dots, \hat{v}_r, \dots, v_n\} = \sum_{s<r} \{v_0, \dots, \hat{v}_s, \dots, \hat{v}_r, \dots, v_n\} + \sum_{r<s} \{v_0, \dots, \hat{v}_r, \dots, \hat{v}_s, \dots, v_n\} = \sum_{k<\ell}^{\text{(relabel)}} \{v_0, \dots, \hat{v}_k, \dots, \hat{v}_\ell, \dots, v_n\} + \sum_{k<\ell}^{\text{(relabel)}} \{v_0, \dots, \hat{v}_k, \dots, \hat{v}_\ell, \dots, v_n\}$

$= 2\sum_{k<\ell} \{v_0, \dots, \hat{v}_k, \dots, \hat{v}_\ell, \dots, v_n\} = 0$ since $2 = 0$ in $\mathbb{F}_2$ // q.e.d.

Thus far, given simplicial complex X, we have constructed (1) vector spaces $C_n(X)$ and (2) linear maps $\partial_n : C_n(X) \to C_{n-1}(X)$ s.t. $\partial_{n-1}\partial_n = 0$.

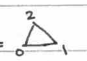
Alternatively, part two could also be expressed as $\partial_n \partial_{n+1} = 0$, using the sequence $C_{n+1}(X) \xrightarrow{\partial_{n+1}} C_n(X) \xrightarrow{\partial_n} C_{n-1}(X)$.

Proposition $Im(\partial_{n+1}) \subset Ker(\partial_n)$.

Proof – Let $x \in Im(\partial_{n+1})$. Then $\exists y \in C_{n+1}(X)$ s.t. $x = \partial_{n+1}(y)$, and $\partial_n(x) = \partial_n\partial_{n+1}(y) = 0$. Thus $x \in Ker(\partial_n)$, q.e.d.

Definition The n^{th} homology group of X (over $\mathbb{F}_2$), $H_n(X)$, is defined by $H_n(X) = Ker(\partial_n)/Im(\partial_{n+1})$. Then the n^{th} Betti number of X is $\dim H_n(X) = \dim Ker(\partial_n) - \dim Im(\partial_{n+1})$.

Example –

Consider $X = \Delta^2 =$ , $\partial_1\{0,1\} = \{1\} + \{0\}$ (e_1; v_2, v_1), $\partial_1\{0,2\} = \{2\} + \{0\}$ (e_2; v_3, v_1), $\partial_1\{1,2\} = \{2\} + \{1\}$ (e_3; v_2, v_1). Thus, $\partial_1 = \begin{pmatrix}1&1&0\\1&0&1\\0&1&1\end{pmatrix} \overset{\text{row red}}{\longleftrightarrow} \begin{pmatrix}1&0&1\\0&1&1\\0&0&0\end{pmatrix}$.

$\dim ker(\partial_1) = 1$, $\dim Im(\partial_2) = 1$ since $\partial_2 : C_2(\Delta^2) \to C_1(\Delta^2)$ is non-zero and $\dim C_2(\Delta^2) = 1$. As such, 1^{st} Betti number of $\Delta^2 = 1 - 1 = 0$.

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Having examined the case for $\mathbb{F}_2$, we now expand our consideration to arbitrary fields (such as $\mathbb{F} = \mathbb{Q}$).

Let X be a simplicial complex, $\sigma \in S_X$ be an n-simplex. Once and for all, choose an ordering on vertices of $\sigma = (v_0 < v_1 < \dots < v_n)$.

We introduce a symbol $[v_0, \dots, v_n]$ with the property that $[v_{\pi(0)}, \dots, v_{\pi(n)}] = sgn(\pi) \cdot [v_0, \dots, v_n]$ e.g. if $\sigma = \{0,1,2,3\}$, $0<1<2<3$. Then: $[1,0,2,3] = sgn(0,1)[0,1,2,3] = (-1)[0,1,2,3]$; $[2,0,1,3] = sgn\begin{pmatrix}0&1&2&3\\2&0&1&3\end{pmatrix}[0,1,2,3] = (+1)[0,1,2,3]$.

Then $[v_0, \dots, v_n]$ is called an ordered n-simplex. For arbitrary field $\mathbb{F}$, simplicial complex X, $C_n(X:\mathbb{F})$ is a vector space over $\mathbb{F}$ with ordered n-simplices as basis elements. [Notation – if $\mathbb{F}$ is understood we just write $C_n(X)$]

Analogously, we define $\partial_n : C_n(X) \to C_{n-1}(X)$ by $\partial_n[v_0, \dots, v_n] = \sum_{r=0}^{n} (-1)^k [v_0, \dots, \hat{v}_r, \dots, v_n]$.

Example – $\partial_3[v_0,v_1,v_2,v_3] = (-1)^0[v_1,v_2,v_3] + (-1)^1[v_0,v_2,v_3] + (-1)^2[v_0,v_1,v_3] + (-1)^3[v_0,v_1,v_2] = [v_1,v_2,v_3] - [v_0,v_2,v_3] + [v_0,v_1,v_3] - [v_0,v_1,v_2]$.

Remark: This agrees with previous definition as over $\mathbb{F}_2$, $+1 = -1$.

Proposition (Poincaré's lemma, general case)

$\partial_{n-1}\partial_n = 0$.

Proof – Again, it is enough to check all basis elements. Thus $\partial_{n-1}\partial_n[v_0,\dots,v_n] = \partial_{n-1}\{\sum_{r=0}^{n}(-1)^r[v_0,\dots,\hat{v}_r,\dots,v_n]\} \overset{\text{linear}}{=} \sum_{r=0}^{n}(-1)^r\partial_{n-1}[v_0,\dots,\hat{v}_r,\dots,v_n]$. Rewriting,

$= \sum_{r=0}^{n}(-1)^r\partial_{n-1}[v_0,\dots,v_{r-1},\hat{v}_r,v_{r+1},\dots,v_n] \overset{\text{cases}}{=} \sum_{r=0}^{n}(-1)^r\{\sum_{j=0}^{r-1}(-1)^j[v_0,\dots,\hat{v}_j,\dots,\hat{v}_r,v_{r+1},\dots,v_n] + \sum_{j=r+1}^{n}(-1)^{j-1}[v_0,\dots,\hat{v}_r,\dots,\hat{v}_j,\dots,v_n]\}$ (position changes due to omission of term in r^{th} position)

$= \sum_{r=0}^{n}(-1)^r\{\sum_{j<r}(-1)^j[v_0,\dots,\hat{v}_j,\dots,\hat{v}_r,\dots,v_n] + \sum_{r<j}(-1)^{j-1}[v_0,\dots,\hat{v}_r,\dots,\hat{v}_j,\dots,v_n]\} = \sum_{j<r}(-1)^{r+j}[v_0,\dots,\hat{v}_j,\dots,\hat{v}_r,\dots,v_n] - \sum_{r<j}(-1)^{r+j}[v_0,\dots,\hat{v}_r,\dots,\hat{v}_j,\dots,v_n]$

reindex $= \sum_{k<\ell}(-1)^{k+\ell}[v_0,\dots,\hat{v}_k,\dots,\hat{v}_\ell,\dots,v_n] - \sum_{k<\ell}(-1)^{k+\ell}[v_0,\dots,\hat{v}_k,\dots,\hat{v}_\ell,\dots,v_n] = 0$, q.e.d.

So, for any field $\mathbb{F}$, we have a sequence of vector spaces $C_r(X:\mathbb{F})$ and linear maps $\partial_r : C_r(X) \to C_{r-1}(X)$ s.t. $\partial_{r-1}\partial_r = 0$ [or $\partial_n\partial_{n+1} = 0$ equivalently].

Definition The n^{th} homology of X with coefficients in $\mathbb{F}$ is a quotient space defined by $H_n(X:\mathbb{F}) = \frac{Ker\,\partial_n}{Im\,\partial_{n+1}}$.

Quotient spaces.

Let V be a vector space over $\mathbb{F}$, $W \subset V$ a vector subspace. Then $x + W = \{v + w : w \in W\}$ is the coset of x mod W. ($\exists$ linear map $[\,] : V \to V/U$, $[v] = v + U$). Cosets obey the rule of equality: $v_1 + W = v_2 + W \Leftrightarrow v_1 - v_2 \in W$.

V/W is the set $V/W = \{v + W : v \in V\}$, which is a vector space over $\mathbb{F}$. This satisfies the following operations: (addition) $(v_1 + W) + (v_2 + W) = (v_1 + v_2) + W$ (scalar multiplication) $\lambda(v + W) = \lambda v + W$ (zero) $0 + W = W$ is zero

Proposition $\dim(V/W) = \dim V - \dim W$.

Proof – Define $[\,] : V \to V/W$ by $[v] = v + U$, natural map. $[\,]$ is surjective, so $Im[\,] = V/W$. $Ker[\,]$ is computable: $v \in Ker[\,] \Leftrightarrow [v] = 0$

$v + U = 0 + U \Rightarrow v - 0 \in U$, i.e. $v \in U \Rightarrow Ker[\,] = U$. $\dim Ker[\,] + \dim Im[\,] = \dim V \Rightarrow \dim W + \dim V/W = \dim V$.

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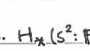
Example – from first principles. $H_*(S^2:\mathbb{F})$. $S^2 =$  standard model of S^2. $C_r = C_r(S^2:\mathbb{F})$. $C_0 = span\{[0],[1],[2],[3]\}$ (e_1, e_2, e_3, e_4) is 4-dimensional.

$C_1 = span\{[0,1],[0,2],[0,3],[1,2],[1,3],[2,3]\}$ ($E_1, E_2, E_3, E_4, E_5, E_6$), $C_2 = span\{[0,1,2],[0,1,3],[0,2,3],[1,2,3]\}$ (E_1, E_2, E_3, E_4). Then we have $0 \to C_2 \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} C_0 \to 0$.

$\partial_1(E_1) = \partial_1[0,1] = [1] - [0] = -[0] + [1] = -e_1 + e_2$. $\partial_1(E_2) = \partial_1[0,2] = -[0] + [2] = -e_1 + e_3$. $\partial_1(E_3) = \partial_1[0,3] = -[0] + [3] = -e_1 + e_4$.

$\partial_1(E_4) = \partial_1[1,2] = [2] - [1] = -e_2 + e_3$. $\partial_1(E_5) = \partial_1[1,3] = -[1] + [3] = -e_2 + e_4$. $\partial_1(E_6) = \partial_1[2,3] = [3] - [2] = -e_3 + e_4$.

Matrix of ∂_1 is $\begin{pmatrix}-1&-1&-1&0&0&0\\1&0&0&-1&-1&0\\0&1&0&1&0&-1\\0&0&1&0&1&1\end{pmatrix}$.

To calculate $Ker\,\partial_1$, $Im\,\partial_1$, we row-reduce: $\begin{pmatrix}-1&-1&-1&0&0&0\\1&0&0&-1&-1&0\\0&1&0&1&0&-1\\0&0&1&0&1&1\end{pmatrix} \overset{\text{fix bottom 3, add to row 1.}}{\longleftrightarrow} \begin{pmatrix}1&0&0&-1&-1&0\\0&1&0&1&0&-1\\0&0&1&0&1&1\\0&0&0&0&0&0\end{pmatrix}$. $Im\,\partial_1$ is 3-dimensional, $\dim Ker\,\partial_1 = 6 - 3 = 3$.

For ∂_2: $\partial_2[0,1,2] = [1,2] - [0,2] + [1,2] = E_1 - E_2 + E_4$. $\partial_2[0,1,3] = [0,1] - [0,3] + [1,3] = E_1 - E_3 + E_5$. $\partial_2[0,2,3] = [0,2] - [0,3] + [2,3] = E_2 - E_3 + E_6$.

$\partial_2[1,2,3] = [1,2] - [1,3] + [2,3] = E_4 - E_5 + E_6$. Matrix of ∂_2: $\begin{pmatrix}1&1&0&0\\-1&0&1&0\\0&-1&-1&0\\1&0&0&1\\0&1&0&-1\\0&0&1&1\end{pmatrix} \longleftrightarrow \begin{pmatrix}0&1&0&-1\\0&0&1&1\\0&0&-1&-1\\1&0&0&1\\0&1&0&-1\\0&0&1&1\end{pmatrix} \longleftrightarrow \begin{pmatrix}1&0&0&1\\0&1&0&-1\\0&0&1&1\\0&0&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix}$. $\dim Im\,\partial_2 = 3$, $\dim Ker\,\partial_2 = 1$. In fact, we can

solve to get $x_1 = -x_4$, $x_2 = x_4$, $x_3 = -x_4$, so if $x_4 = -1$, $\begin{pmatrix}1\\-1\\1\\-1\end{pmatrix}$ is a basis for the kernel, i.e. $[0,1,2] - [0,1,3] + [0,2,3] - [1,2,3]$. 

$H_*(S^2:\mathbb{F})$ is: $H_0 = C_0/Im\,\partial_1 \Rightarrow \dim H_0 = \dim C_0 - \dim Im\,\partial_1 = 4 - 3 = 1$. Thus, $H_0 \cong \mathbb{F}$. $H_1 = Ker\,\partial_1/Im\,\partial_2 \Rightarrow \dim H_1 = \dim Ker\,\partial_1 - \dim Im\,\partial_2$!

$\dim H_1 = 3 - 3 = 0 \Rightarrow H_1 = 0$. $H_2 = Ker\,\partial_2/Im\,\partial_3 = Ker\,\partial_2 \Rightarrow \dim H_2 = \dim Ker\,\partial_2 = 1 \Rightarrow H_2 \cong \mathbb{F}$ with basis $[0,1,2] - [0,1,3] + [0,2,3] - [1,2,3]$.

To summarise, the homology of S^2 is $H_k(S^2:\mathbb{F}) \equiv \begin{cases}\mathbb{F}, & k=0, k=2\\ 0, & k=1, k\geq 3\end{cases}$. We can in fact generalise this result:

Proposition $H_k(S^n:\mathbb{F}) = \begin{cases}\mathbb{F} & k=0,n\\ 0 & \text{otherwise.}\end{cases}$

Given a finite simplicial complex X, produce $C_*(X) = (0 \to C_n(X) \xrightarrow{\partial_n} C_{n-1}(X) \xrightarrow{\partial_{n-1}} C_{n-2}(X) \to \dots \to C_{r+1}(X) \xrightarrow{\partial_{r+1}} C_r(X) \xrightarrow{\partial_r} C_{r-1}(X) \to \dots \xrightarrow{\partial_2} C_1(X) \xrightarrow{\partial_1} C_0(X) \to 0)$ s.t. $\partial_r\partial_{r+1} = 0$.

By a chain complex we mean a sequence $(C_r, \partial_r)_{0\leq r}$ where C_r are vector spaces over $\mathbb{F}$, $\partial_r : C_r \to C_{r-1}$ is linear, $C_{-1} = 0$ by convention, then $\partial_r\partial_{r+1} = 0$.

Definition If $C_*(C_r, \partial_r)$ is a chain complex, we define $H_k(C_*) = \frac{Ker(\partial_k)}{Im(\partial_{k+1})}$. Special case: $H_k(C_*) = 0$, when $Ker(\partial_k) = Im(\partial_{k+1})$, it has a special name.

8203-02.

Definition Let $(\dots \to V_{n+1} \xrightarrow{f_{n+1}} V_n \xrightarrow{f_n} V_{n-1} \to \dots)$ be a sequence of vector spaces V_r and linear maps $f_n: V_n \to V_{n-1}$. The sequence is <u>exact at V_n</u> when $\boxed{\operatorname{Ker} \partial_n = \operatorname{Im} \partial_{n+1}}$

The sequence is <u>exact</u> when it is exact at each V_n.

We can consider an exact sequence of finite length: $(0 \to V_n \xrightarrow{f_n} V_{n-1} \xrightarrow{f_{n-1}} \dots \to V_{r+1} \xrightarrow{f_{r+1}} V_r \xrightarrow{f_r} V_{r-1} \to \dots \to V_1 \xrightarrow{f_1} V_0 \to 0)$; $\operatorname{Ker}(f_r) = \operatorname{Im}(f_{r+1})$ [length $n+1$].

Consider the special case $n=1$, $0 \to V_1 \xrightarrow{f} V_0 \to 0$.

Proposition $0 \to V_1 \xrightarrow{f} V_0 \to 0$ is exact $\iff$ $f: V_1 \to V_0$ is an isomorphism.

Proof – ($\Rightarrow$) $\operatorname{Im}(f) = \operatorname{Ker}(V_0 \to 0) = V_0$, so $V_0 = \operatorname{Im}(f)$, f is surjective. $\operatorname{Ker}(f) = \operatorname{Im}(0 \to V_1) = 0$, so f is injective. Thus, sequence is exact $\Rightarrow$ f is bijective i.e. isomorphism.

($\Leftarrow$) If f is an isomorphism, then f is surjective. So $V_0 = \operatorname{Im}(f) = \operatorname{Ker}(V_0 \to 0)$. So $\operatorname{Im}(V_1 \xrightarrow{f} V_0) = \operatorname{Ker}(V_0 \to 0)$. Since f is injective, $\operatorname{Ker}(f) = 0 = \operatorname{Im}(0 \to V_1)$. The sequence is exact at V_0 and V_1, so it is exact.// q.e.d.

The next special case is where $n=2$, giving <u>short exact sequences</u> of form $0 \to V_2 \xrightarrow{f_2} V_1 \xrightarrow{f_1} V_0 \to 0$.

Proposition The short exact sequence $0 \to V_2 \xrightarrow{f_2} V_1 \xrightarrow{f_1} V_0 \to 0$ is exact $\iff$ (i) f_1 is surjective, (ii) f_2 is injective and (iii) $\operatorname{Ker}(f_1) = \operatorname{Im}(f_2)$.

Proof – ($\Rightarrow$) Suppose sequence is exact. f_1 is surjective, f_2 is injective (as above). Moreover, $\operatorname{Ker} f_1 = \operatorname{Im} f_2$ by definition.

($\Leftarrow$) Suppose (i), (ii), (iii) hold. By (iii), sequence is exact at V_1. By (i), sequence is exact at V_0; and by (ii), sequence is exact at V_2.// q.e.d.

Lemma (Whitehead's Lemma).

Let $(0 \to V_n \xrightarrow{f_n} V_{n-1} \xrightarrow{f_{n-1}} \dots \xrightarrow{f_2} V_1 \xrightarrow{f_1} V_0 \to 0)$ be an exact sequence of finite dimensional vector spaces V_r and linear maps f_r. Then $\sum_{r \geq 0} \dim(V_{2r}) = \sum_{r \geq 0} \dim(V_{2r+1})$.

Proof – Let $P(n)$ be the statement of the theorem for n. $P(n)$ will be proven by induction. $P(1) = (0 \to V_1 \xrightarrow{f_1} V_0 \to 0)$. If sequence is exact, f_1 is an isomorphism, so $\dim V_0 = \dim V_1$.

$P(2) = (0 \to V_2 \xrightarrow{f_2} V_1 \xrightarrow{f_1} V_0 \to 0)$. Use the rank-nullity theorem: $\dim \operatorname{Im} f_1 + \dim \operatorname{Ker} f_1 = \dim V_1$. As sequence is exact, f_1 is surjective, so $\operatorname{Im} f_1 = V_0$. Therefore, $\dim V_0 + \dim \operatorname{Ker} f_1 = \dim V_1$. By exactness of sequence, $\operatorname{Ker} f_1 = \operatorname{Im} f_2$, so applying theorem to f_2, $\dim \operatorname{Im}(f_2) + \dim \operatorname{Ker}(f_2) = \dim V_2$. f_2 is injective so $\operatorname{Ker}(f_2) = 0$ $\dim \operatorname{Ker}(f_1) = \dim \operatorname{Im}(f_2) = \dim V_2 \Rightarrow \dim V_0 + \dim V_2 = \dim V_1$, which proves statement $P(2)$. We will prove ① $P(2) \wedge P(2n) \Rightarrow P(2n+1)$, ② $P(2) \wedge P(2n+1) \Rightarrow P(2n+2)$.

Suppose sequence $(0 \to V_{2n+1} \xrightarrow{f_{2n+1}} V_{2n} \xrightarrow{f_{2n}} V_{2n-1} \xrightarrow{f_{2n-1}} \dots \to V_1 \xrightarrow{f_1} V_0 \to 0)$ is exact. At V_{2n} we have sequence $S = (0 \to V_{2n+1} \xrightarrow{f_{2n+1}} V_{2n} \xrightarrow{f_{2n}} \operatorname{Im}(f_{2n}) \to 0)$ and $S' = (0 \to \operatorname{Ker}(f_{2n-1}) \xrightarrow{i} V_{2n-1} \xrightarrow{f_{2n-1}} \dots \to V_1 \xrightarrow{f_1} V_0 \to 0)$, where i is the inclusion of $\operatorname{Ker}(f_{2n-1}) \hookrightarrow V_{2n-1}$. S has length 2, S' has length $2n$. Both are exact.

By $P(2)$, $\dim \operatorname{Im}(f_{2n}) + \dim(V_{2n+1}) = \dim(V_{2n})$. By $P(2n)$, $\sum_{r=0}^{n-1} \dim(V_{2r}) + \dim \operatorname{Ker}(f_{2n-1}) = \sum_{r=0}^{n-1} \dim(V_{2r+1})$. But $\operatorname{Im}(f_{2n}) = \operatorname{Ker}(f_{2n-1}) \Rightarrow$ $\dim \operatorname{Ker}(f_{2n-1}) = \dim(V_{2n}) - \dim(V_{2n+1})$, so substituting, $\sum_{r=0}^{n} \dim(V_{2r}) - \dim(V_{2n+1}) = \sum_{r=0}^{n-1} \dim(V_{2r+1})$. Grouping terms, we prove $P(2n+1)$.

Thus, $P(2) \wedge P(2n) \Rightarrow P(2n+1)$. Likewise, $P(2) \wedge P(2n-1) \Rightarrow P(2n)$. Thus, building up by induction, $P(n)$ is true for all n.// q.e.d.

Suppose X is a simplicial complex, $X = X_+ \cup X_-$ where X_+, X_- are subcomplexes. $X_+ \cap X_- \subset X_- \subset X$ (and $X_+ \cap X_- \subset X_+ \subset X$). The geometrical theorem below (stated without proof) is useful:

Theorem (Mayer–Vietoris theorem – geometrical).

$\to H_{n+1}(X_+) \oplus H_{n+1}(X_-) \to H_{n+1}(X) \to \underbrace{H_n(X_+ \cap X_-) \to H_n(X_+) \oplus H_n(X_-) \to H_n(X) \to}_{\text{repeating segment}} H_{n-1}(X_+ \cap X_-) \to \dots \to H_1(X_+ \cap X_-) \to H_1(X_+) \oplus H_1(X_-) \to H_1(X) \to H_0(X_+ \cap X_-) \to H_0(X_+) \oplus H_0(X_-) \to H_0(X) \to 0$.

is an exact sequence.

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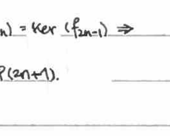
Let X be a finite simplicial complex, $X \neq \emptyset$. We want to interpret what $H_r(X : \mathbb{F})$ implies: we begin with H_0.

Proposition If $X \neq \emptyset$, $\dim H_0(X) \geq 1$.

Proof – By definition, $H_0(X) = C_0(X)/\operatorname{Im} \partial_1$ because $C_1(X) \xrightarrow{\partial_1} C_0(X) \to 0$, so $H_0(X) = \dfrac{\operatorname{Ker}(C_0(X) \to 0)}{\operatorname{Im}(\partial_1 : C_1(X) \to C_0(X))}$. Observe that $C_0(X)$ has basis $[v_1], \dots, [v_N]$ where $V_X = \{v_1, \dots, v_N\}$.

Define $\psi: C_0(X) \to \mathbb{F}$ by $\psi([v_i]) \to 1$. ψ is linear, ψ is surjective (because we hit $1 \in \mathbb{F}$). $\operatorname{Im}(\partial_1) \subset C_0(X)$. $\operatorname{Im}(\partial_1)$ is spanned by $[w] - [v]$ where $[v, w]$ is a 1-simplex (since $\partial_1 [v, w] = [w] - [v]$). Then $\psi([w] - [v]) = \psi[w] - \psi[v] = 1 - 1 = 0$, so $\psi: \operatorname{Im} \partial_1 \to \mathbb{F}$ is identically 0. Define induced map $\psi_*: C_0(X)/\operatorname{Im}(\partial_1) \to \mathbb{F}$ by $\psi_*([v] + \operatorname{Im} \partial_1) = \psi[v]$, which is well-defined. ψ_* is still linear and surjective, so $0 \to \operatorname{Ker}(\psi_*) \to H_0(X) \xrightarrow{\psi_*} \mathbb{F} \to 0$ is exact. Then $\dim H_0(X) = 1 + \dim \operatorname{Ker} \psi_* \geq 1$ q.e.d.

<u>Connectivity.</u>



Definition Let $X = (V_X, S_X)$ be a finite simplicial complex. By a <u>path</u> in X, I mean a sequence of vertices $(v_0, \dots, v_n)$ where each $v_i \in V_X$ and each $[v_i, v_{i+1}]$ is an ordered 1-simplex. We say that X is <u>connected</u> when for each $v, w \in V_X$, $\exists$ path $(v_0, \dots, v_n)$ where $v = v_0 = \dots = v_n = w$.

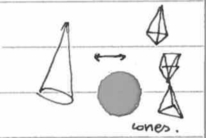
e.g. – In the diagram on the right, $(v_0, v_1, v_2, v_3, v_4)$ is a path, but (v_0, v_1, v_3, v_4) is not.

Theorem Let X be a finite non-empty simplicial complex. If X is connected, then $\dim H_0(X) = 1$.

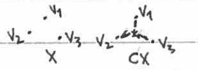
Proof – We know that $\dim H_0(X) \geq 1$, so it suffices to show that $\dim H_0(X) \leq 1$. $H_0(X) = C_0(X)/\operatorname{Im} \partial_1$. Let $v, w \in V_X$. Show that $[v] + \operatorname{Im} \partial_1 = [w] + \operatorname{Im} \partial_1$: since X is connected, let $(v_0, \dots, v_n)$ be a path from $v = v_0$ to $w = v_n$, where $[v_i, v_{i+1}]$ is a 1-simplex. Calculate $\partial_1 \left(\sum_{i=0}^{n-1} [v_i, v_{i+1}] \right) = \sum_{i=0}^{n-1} \partial_1 [v_i, v_{i+1}] = \sum_{i=0}^{n-1} [v_{i+1}] - [v_i]$ $= [v_1] - [v_0] + [v_2] - [v_1] + \dots + [v_n] - [v_{n-1}] = [v_n] - [v_0] = [w] - [v] \in \operatorname{Im} \partial_1$; so by law of equality, $[v] + \operatorname{Im} \partial_1 = [w] + \operatorname{Im} \partial_1$. Thus, as $C_0(X)$ is spanned by vertices $[w_1], \dots, [w_N]$ of X, then (?) $\alpha \in C_0(X) \Rightarrow \alpha = \sum_{i=1}^{N} \alpha_i [w_i]$. Let $\langle \alpha \rangle \in C_0(X)/\operatorname{Im}(\partial_1)$, $\langle \alpha \rangle = \alpha + \operatorname{Im} \partial_1$. Then $[w_i] - [w_j] \in \operatorname{Im} \partial_1$, so $H_0(X)$ is spanned by $[w_1] + \operatorname{Im} \partial_1$. (since $[w_i] + \operatorname{Im} \partial_1 = [w_1] + \operatorname{Im} \partial_1$ as $[w_i] - [w_1] \in \operatorname{Im} \partial_1$). $\Rightarrow$ $H_0(X)$ is spanned by a single element $[v] + \operatorname{Im} \partial_1$ where v is any vertex $\Rightarrow \dim H_0(X) \leq 1$.// q.e.d.

Definition If $X = (V_X, S_X)$ is a simplicial complex, define cone $CX = (V_{CX}, S_{CX})$ where if $* \notin V_X$ is a disjoint point, $V_{CX} = \{*\} \cup V_X$, and

$S_{CX} = S_X \cup \{\sigma \cup \{*\} : \sigma \in S_X\} \cup \{*\}$ [i.e. all original simplices + original simplices joined to $*$ + $\{*\}$ itself.].



e.g. – consider the cone on a 2-simplex X, CX, so $C\Delta^2 = \Delta^3$, $C\Delta^n = \Delta^{n+1}$ (more to follow later).



Theorem Let X be a finite simplicial complex. $H_r(CX : \mathbb{F}) = \begin{cases} \mathbb{F} & r=0 \\ 0 & r>0. \end{cases}$

Proof – Look at chain complex, $C_r = C_r(CX)$, and $C_{n+1} \xrightarrow{\partial_{n+1}} C_n \xrightarrow{\partial_n} C_{n-1}$, $C_{n+1} \underset{\partial_{n+1}}{\overset{h_n}{\leftrightarrows}} C_n \underset{\partial_n}{\overset{h_{n-1}}{\leftrightarrows}} C_{n-1}$. Define $h_n : C_n \to C_{n+1}$ by $h_n[v_0, \dots, v_n] = [*, v_0, \dots, v_n]$ if $\{v_0, \dots, v_n\} \in S_X$,

$h_n[v_0, \dots, v_n] = 0$ if $* \in \{v_0, \dots, v_n\}$. Use convention (formally) $[v_0, \dots, v_n] = 0$ if $v_i = v_j$, $i \neq j$. Formally, $h_n[v_0, \dots, v_n] = [*, v_0, \dots, v_n]$ $\forall [v_0, \dots, v_n] \in C_n$.

We want to show: $\boxed{\partial_{n+1} h_n + h_{n-1}\partial_n = \mathrm{Id}\, C_n}$ $(n-1 \geq 0)$. Here, $\partial_{n+1} h_n[v_0, \dots, v_n] = \partial_{n+1}[*, v_0, \dots, v_n] = [v_0, \dots, v_n] + \sum_{r=0}^{n} (-1)^{r+1} [*, v_0, \dots, \hat{v}_r, \dots, v_n]$. The other term

gives $h_{n-1}\partial_n[v_0, \dots, v_n] = h_{n-1}\left\{\sum_{r=0}^{n} (-1)^r [v_0, \dots, \hat{v}_r, \dots, v_n]\right\} = \sum_{r=0}^{n-1} (-1)^r [*, v_0, \dots, \hat{v}_r, \dots, v_n]$. Thus, $\partial_{n+1} h_n + h_{n-1}\partial_n = \mathrm{Id}\, C_n \Rightarrow$ let $z \in \mathrm{Ker}(\partial_n : C_n \to C_{n-1})$.

Then $\mathrm{Id}(z) = \partial_{n+1} h_n(z) + h_{n-1}\underbrace{\partial_n(z)}_{\mathrm{Ker}(\partial_n)}$. Thus, $z = \partial_{n+1} h_n(z) \in \mathrm{Im}\,\partial_{n+1}$, $\mathrm{Ker}(\partial_n) \subset \mathrm{Im}(\partial_{n+1}) \subset \mathrm{Ker}(\partial_n)$, so $\mathrm{Ker}\,\partial_n = \mathrm{Im}\,\partial_{n+1}$, so $H_n(CX : \mathbb{F}) = \frac{\mathrm{Ker}(\partial_n)}{\mathrm{Im}(\partial_{n+1})} = 0$, $n \geq 1$.

Moreover, CX is connected*, so $H_0(CX : \mathbb{F}) = \mathbb{F}$ // q.e.d.

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** **Proposition*** If K is a simplicial complex, then CK is connected.

Proof – Let $*$ be cone point, and let $v, w \in V_{CK}$. If $w = *$, then $[v, *]$ is a 1-simplex. If $v = *$, then $[w, *]$ is a 1-simplex. If $* \notin \{v, w\}$, $[v, *], [*, w]$ is a path from v to w.

Thus, we can join v to w by a path in CK.

This general theorem has some useful applications: for example, Δ^n is a cone for $n \geq 1$. In fact, $\Delta^n = C(\Delta^{n-1})$. $V_{\Delta^n} = \{0, \dots, n\}$, $V_{\Delta^{n-1}} = \{0, \dots, n-1\}$. Take cone point $* = n$. Then clearly, with cone point $*$, $C(\Delta^{n-1}) = \Delta^n$. [Note – $\Delta^1 = C(\text{point})$, $\Delta^2 = C\Delta$, $\Delta^3 = C(\Delta^2) \dots$

Corollary $H_r(\Delta^n : \mathbb{F}) = \begin{cases} \mathbb{F} & r=0 \\ 0 & r>0. \end{cases}$ [Remark – Hence, from the point of view of homology, cones behave like points].

Then, now we seek to show that $H_r(S^n : \mathbb{F}) = \begin{cases} \mathbb{F} & r = 0 \text{ or } n \\ 0 & \text{otherwise.} \end{cases}$

Definition Let $X = (V_X, S_X)$ be a simplicial complex. Then if $n \geq 0$, $X^{(n)}$ [the <u>n-skeleton</u> of X] is the simplicial complex defined as follows: Vertex set of $X^{(n)} = V_X$ = full vertex set, simplex set of $X^{(n)} = \{\sigma \in S_X, \dim \sigma \leq n\}$ [or $|\sigma| \leq n+1$]

e.g. – $S^n = (\Delta^{n+1})^{(n)}$. $S^1 = {}_0\overset{2}{\triangle}_1$ (interior absent), $\Delta^2 = {}_0\overset{2}{\blacktriangle}_1$ (interior present). Thus, $S^1 = (\Delta^2)^{(1)}$. In general, $\Delta^{n+1} = (\{0, \dots, n+1\}$, all non-empty subsets)

and $S^n = (\{0, \dots, n+1\}$, all non-empty subsets except $\{0, \dots, n+1\})$. So if σ is an r-simplex of Δ^{n+1} and $r \leq n$, then $\sigma \subset S^n$, i.e. $S^n = (\Delta^{n+1})^{(n)}$.

Theorem Let K be a simplicial complex, $K^{(n)}$ be the n-skeleton of K. $H_r(K^{(n)} : \mathbb{F}) = H_r(K : \mathbb{F})$ provided $r \leq n-1$.

Proof – For $r \leq n$, r-simplices of $K^{(n)} \equiv$ r-simplices of K. So $C_r(K^{(n)}) = C_r(K)$ for $r \leq n$. Then, we get the diagram

$$\begin{array}{ccccccc} C_{n+1}(K^{(n)}) \xrightarrow{0} & C_n(K^{(n)}) & \xrightarrow{\partial_n} & C_{n-1}(K^{(n)}) & \xrightarrow{\partial_{n-1}} & C_{n-2}(K^{(n)}) \\ & \| & & \| & & \| \\ C_{n+1}(K) \xrightarrow{\partial_{n+1}} & C_n(K) & \xrightarrow{\partial_n} & C_{n-1}(K) & \xrightarrow{\partial_{n-1}} & C_{n-2}(K) \end{array}$$

The boundary map $0 \to C_n(K^{(n)})$ is zero, but for $r \leq n$, the boundary maps ∂_r are identical.

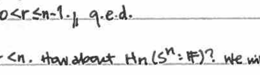
Take $r \leq n-1$ (i.e. $r+1 \leq n$). Then we have that
$$\begin{array}{ccccc} C_{r+1}(K^{(n)}) & \xrightarrow{\partial_{r+1}} & C_r(K^{(n)}) & \xrightarrow{\partial_r} & C_{r-1}(K^{(n)}) \\ \| & & \| & & \| \\ C_{r+1}(K) & \longrightarrow & C_r(K) & \longrightarrow & C_{r-1}(K) \end{array}$$
Thus, $H_r(K^{(n)}) = \mathrm{Ker}\,\partial_r / \mathrm{Im}\,\partial_{r+1} = H_r(K)$ provided $r \leq n-1$ // q.e.d.

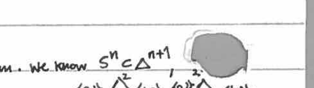
Remark – In dimension n, all that can be concluded in general is that $H_n(K^{(n)}) = \mathrm{Ker}\,\partial_n$, $H_n(K) = \mathrm{Ker}\,\partial_n / \mathrm{Im}\,\partial_{n+1}$, so we get a natural surjection $H_n(K^{(n)}) \to H_n(K)$.

Corollary For $r \leq n-1$, $n \geq 1$, $H_r(S^n : \mathbb{F}) = \begin{cases} \mathbb{F} & r=0 \\ 0 & 0 < r \leq n-1. \end{cases}$

Proof – We know $S^n = (\Delta^{n+1})^{(n)}$, so $H_r(\Delta^{n+1} : \mathbb{F}) = \begin{cases} \mathbb{F} & r=0 \\ 0 & 0<r \leq n-1. \end{cases}$ // q.e.d.

S^n is n-dimensional, so $H_r(S^n) = 0$ for $r > n$. Also, $H_r(S^n) = 0$ for $0 < r < n$ $(n \geq 1)$. How about $H_n(S^n : \mathbb{F})$? We will use the Mayer-Vietoris theorem. We know $S^n \subset \Delta^{n+1}$, also $\Delta^n \subset \Delta^{n+1}$ and in fact $\Delta^n \subset S^n \subset \Delta^{n+1}$.

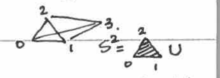




We define a simplicial complex, the witches hat: if S^{n-1} is the standard $n-1$ sphere, $C(S^{n-1})$ is the cone of S^{n-1}, which is known as the Witches Hat.

$\{0, \dots, n-1\} \subset \{0, \dots, n\} \subset \{0, \dots, n+1\}$ give inclusions of simplicial complex $\Delta^{n-1} \subset \Delta^n \subset \Delta^{n+1}$. However, we also have $\Delta^n \subset S^n \subset \Delta^{n+1}$. In fact, $S^{n-1} \subset \Delta^n \subset S^n$. $(\subset \Delta^{n+1})$.

In the inclusion $S^{n-1} \subset S^n$, the vertex n does not belong to S^{n-1}. We can use vertex n to embed $C(S^{n-1})$ inside S^n. For instance S^2 is the union ① of bottom face, the cone ${}_0\overset{2}{\blacktriangle}_1$, and ② the witches hat with 3 = cone point. The intersection of the two cones is $S^1 = {}_0\overset{2}{\triangle}_1$.

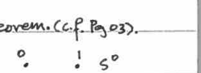


Theorem $S^n = \Delta^n \cup C(S^{n-1})$ where $\Delta^n \cap C(S^{n-1}) \cong S^{n-1}$, and $n+1$ = cone in $C(S^{n-1})$.

Proof – Apply definitions, $V_{S^n} = \{0, \dots, n, n+1\}$, $V_{S^{n-1}} = \{0, \dots, n\}$. S_{S^n} = all non-empty subsets of $\{0, \dots, n+1\}$ except the whole set. Let $\sigma \in S_{S^n}$. Then either (i) $n+1 \notin \sigma$ or (ii) $n+1 \in \sigma$. If (i), $\sigma \subset \Delta^n$. If (ii), $\sigma \subset C(S^{n-1})$, the witches hat. $S^n = \Delta^n \cup C(S^{n-1})$, and $\Delta^n \cap C(S^{n-1}) = S^{n-1}$ // q.e.d.

Hence, we have decomposed S^n into two parts: $S^n = X_+ \cup X_-$ where $X_- = \Delta^n$, $X_+ = C(S^{n-1})$, the witches hat. So we can apply the Mayer-Vietoris theorem. (c.f. Pg 03).

Theorem $H_n(S^n : \mathbb{F}) \cong \mathbb{F}$ for all $n \geq 1$, and $H_0(S^0 : \mathbb{F}) \cong \mathbb{F} \oplus \mathbb{F}$.



Proof – Show that $H_r(S^0 : \mathbb{F}) = \begin{cases} \mathbb{F} \oplus \mathbb{F} & r=0 \\ 0 & r>0. \end{cases}$ $S^0 = (\{0,1\}, S_{S^0} = \{\{0\}, \{1\}\})$ (no 1-simplexes). $C_r(S^0) = 0$, $r \geq 1$, $C_0(S^0) = \mathbb{F} \oplus \mathbb{F}$.

$0 \to \mathbb{F} \oplus \mathbb{F} \to 0$, $H_0 = \mathrm{Ker}(\mathbb{F} \oplus \mathbb{F} \to 0) / \mathrm{Im}(0 \to \mathbb{F} \oplus \mathbb{F}) = \mathbb{F} \oplus \mathbb{F}$. Then, we show that $H_r(S^1 : \mathbb{F}) = \begin{cases} \mathbb{F} & r = 0,1 \\ 0 & r \geq 2. \end{cases}$ $S^1 = {}_0\overset{2}{\triangle}_1$ is connected so $H_0(S^1) \cong \mathbb{F}$.

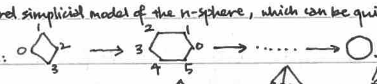
Decompose $S^1 = \Delta^1 \cup C(S^0)$, $X^+ = \Delta^1$, $X^- = C(S^0)$, $X_+ \cap X_- = S^0$. Now use Mayer-Vietoris theorem: $\underbrace{H_1(X_+)}_{0} \oplus \underbrace{H_1(X_-)}_{0} \text{(cones)} \to H_1(S^1) \to \underbrace{H_0(\overbrace{X_+ \cap X_-}^{S^0})}_{\mathbb{F} \oplus \mathbb{F}}$

$\to \underbrace{H_0(X_+)}_{\mathbb{F}} \oplus \underbrace{H_0(X_-)}_{\mathbb{F}} \text{(cones, connected)} \to \underbrace{H_0(S^1)}_{\mathbb{F} \text{ (connected)}} \to 0$. Our exact sequence is $0 \to \underline{H_1(S^1)} \to \overline{\mathbb{F} \oplus \mathbb{F}} \to \underline{\mathbb{F} \oplus \mathbb{F}} \to \mathbb{F} \to 0$. Then use Whitehead's lemma: we get the relation that $\dim H_1(S^1) + 2 = 2 + 1 \Rightarrow \dim H_1(S^1) = 1 \Rightarrow H_1(S^1) \cong \mathbb{F}$, indeed. Finally, we compute the general homology: NTP: $H_n(S^n : \mathbb{F}) \cong \mathbb{F}$, $n \geq 1$.

Let this proposition be $P(n)$. From above $P(1)$ is proven to be true. Suppose $n \geq 2$, $P(n-1)$ proven. Consider $P(n)$: $S^n = \Delta^n \cup C(S^{n-1}) = X^- \cup X^+$ respectively, so $X_+ = C(S^{n-1})$, $X_- = \Delta^n$, $X_+ \cap X_- = S^{n-1}$. Then Mayer-Vietoris theorem gives $\underset{0 \text{ (cones)}}{H_n(X_+) \oplus H_n(X_-)} \to H_n(S^n) \to H_{n-1}(S^{n-1}) \to \underset{0 \text{ (cones)}}{H_{n-1}(X_+) \oplus H_{n-1}(X_-)}$ as exact.

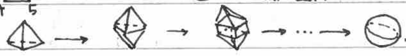
Hence we get $0 \to H_n(S^n) \to H_{n-1}(S^{n-1}) \to 0$ is a (very short) exact sequence, which is an isomorphism. $H_n(S^n) \cong H_{n-1}(S^{n-1}) \cong \mathbb{F}$ // q.e.d.

Remark – S^n is known as the standard simplicial model of the n-sphere, which can be quite a crude approximation: e.g. $_0\triangle^2_1$ approximates ◯. We could better approximate the circle by adding points:



. We still need to show that with these extra points (subdivisions), homologies stay the same.

Likewise for approximations to 2-spheres:



29 January 2014
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Invariance of homology under subdivision.

Simplicial maps: Let $X = (V_X, S_X)$, $Y = (V_Y, S_Y)$ be simplicial complexes. By a simplicial map $f: X \to Y$ we mean a mapping $f: V_X \to V_Y$ with the property that if $\sigma \in S_X$, then $f(\sigma) \in S_Y$.

Example – If $X = (V_X, S_X)$ is a simplicial complex, then $Id_{V_X}: V_X \to V_X$ defines a simplicial mapping $Id_X: X \to X$. (identity map).

Proposition If $f: X \to Y$, $g: Y \to Z$ are simplicial mappings, then $g \circ f: X \to Z$ is also simplicial.

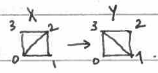
Proof – obvious.

Remark – Algebraic topology produces a "coherent" algebraic picture of geometry. $\underset{\text{simplicial complex}}{X} \longmapsto \underset{\text{vector space.}}{H_n(X:\mathbb{F})}$ such that if $f: X \to Y$ is a simplicial map, then we also get a linear map $H_n(f): H_n(X:\mathbb{F}) \to H_n(Y:\mathbb{F})$, and "coherence" implies that $\boxed{H_n(Id_X) = Id_{H_n(X)}}$ (I). Also, if $f: X \to Y$, $g: Y \to Z$ are simplicial, $H_n(X) \xrightarrow{H_n(f)} H_n(Y) \xrightarrow{H_n(g)} H_n(Z)$ s.t. $\boxed{H_n(g \circ f) = H_n(g) \circ H_n(f)}$ (II)

The two properties (I) and (II) define H_n as a functor. (E. Noether, S. Eilenberg)

Examples of simplicial mappings –

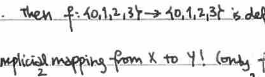
(1) Triangulate the square in two ways:



. Then $f: \{0,1,2,3\} \to \{0,1,2,3\}$ is defined by $f(0)=1$, $f(1)=0$, $f(2)=3$, $f(3)=2$. f is a simplicial isomorphism, as $f^2 = Id$.

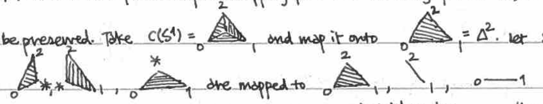
However, $Id: \{0,1,2,3\} \to \{0,1,2,3\}$ does NOT define a simplicial mapping from X to Y! (only for $X \to X$, or $Y \to Y$). Note here that $S_X \neq S_Y$ so they are not the same complex!

(2) Squash map: dimension need not be preserved. Take $C(S^1) =$



$= \Delta^2$. Let $Sq: \{0,1,2,*\} \to \{0,1,2\}$ be $Sq(0)=0$, $Sq(1)=1$, $Sq(2)=2$, $Sq(*)=1$.

Observe how the 2-simplexes map:



respectively. Note that the dimensions have been reduced.

To turn homology into a functor, we need to recall definitions. This is a two-stage process: $\underset{\text{simplicial complex}}{X} \xrightarrow{H_n} \underset{\text{vector space}}{H_n(X:\mathbb{F})}$, $X \xrightarrow{C_*} \underset{\text{chain complex}}{C_*(X)} \xrightarrow{H_n} H_n(X:\mathbb{F})$. Here, $H_n(X:\mathbb{F}) = H_n(C_*(X:\mathbb{F}))$.

Recall – By a chain complex over $\mathbb{F}$, we mean a collection $(C_r, \partial_r)_{r \geq 0}$ where

(i) each C_r is a vector space over $\mathbb{F}$, and (ii) $\partial_r: C_r \to C_{r-1}$ is linear, and (iii) $\partial_r \partial_{r+1} = 0$ for all r, with the convention that $C_{-1} = 0$.

Definition Let $C_* = (C_r, \partial_r)_{r \geq 0}$, $C'_* = (C'_r, \partial'_r)_{r \geq 0}$ be chain complexes over $\mathbb{F}$. By a chain mapping $f: C_* \to C'_*$ we mean a collection $f = (f_r)_{r \geq 0}$ where $f_r: C_r \to C'_r$ is linear and such that each of the following diagrams commute: [i.e. $f_{r-1}\partial_r = \partial'_r f_r$, $f_r \partial_{r+1} = \partial'_{r+1} f_{r+1}$].

$$\begin{array}{ccccc} C_{r+1} & \xrightarrow{\partial_{r+1}} & C_r & \xrightarrow{\partial_r} & C_{r-1} \\ \downarrow f_{r+1} & & \downarrow f_r & & \downarrow f_{r-1} \\ C'_{r+1} & \xrightarrow{\partial'_{r+1}} & C'_r & \xrightarrow{\partial'_r} & C'_{r-1} \end{array}$$

Composition of chain mappings – $C_* \xrightarrow{f} C'_* \xrightarrow{g} C''_*$, where $(g \circ f)_r = g_r \circ f_r$. i.e.

$$\begin{array}{ccccc} C_{r-1} & \xrightarrow{\partial_{r+1}} & C_r & \xrightarrow{\partial_r} & C_{r+1} \\ \downarrow f_{r-1} & & \downarrow f_r & & \downarrow f_{r+1} \\ C'_{r-1} & \longrightarrow & C'_r & \longrightarrow & C'_{r+1} \\ \downarrow g_{r-1} & & \downarrow g_r & & \downarrow g_{r+1} \\ C''_{r-1} & \xrightarrow{\partial''_{r+1}} & C''_r & \xrightarrow{\partial''_r} & C''_{r+1} \end{array}$$

If $C_* = (C_r, \partial_r)$, then $Id_{C_*} = (Id_{C_r})_{r \geq 0}$ is a chain mapping.

If $f: X \to Y$ is a simplicial map, $C_n(f): C_n(X) \to C_n(Y)$ is defined by $C_n(f)[v_0, \ldots, v_n] = [f(v_0), \ldots, f(v_n)]$, $C_*(f) = (C_r(f))$, also if $g: Y \to Z$, $C_n(g \circ f) = C_n(g) \circ C_n(f)$, $C_*(g \circ f) = C_*(g) \circ C_*(f)$ and $C_*(Id) = Id_{C_*}$. We have constructed chain complexes from simplicial complexes.

3 February 2014
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Recall that we are turning homologies to functors. $\{\text{simplicial complexes and simplicial maps}\} \overset{H_n}{\dashrightarrow} \{\text{vector spaces and linear maps}\}$, via $\xrightarrow{\text{functor } C_*} \{\text{chain complexes and chain maps}\} \xrightarrow{H_n}$ and $H_n(X:\mathbb{F}) = H_n(C_*(X:\mathbb{F}))$. Schild LT / Roberts G08.

$C_n(g \circ f): C_n(X) \to C_n(Z)$

$C_n(X)$ is the vector space on the oriented simplices, and we defined $C_n(f): C_n(X) \to C_n(Y)$. Moreover, if $X \xrightarrow{f} Y \xrightarrow{g} Z$, then we have $C_n(g \circ f)[v_0, \ldots, v_n] = [g \circ f(v_0), \ldots, g \circ f(v_n)]$.

$= C_n(g)[f(v_0), \ldots, f(v_n)] = C_n(g) \circ C_n(f)$. Thus $C_n(g \circ f) = C_n(g) \circ C_n(f)$. Likewise, $C_n(Id_X)[v_0, \ldots, v_n] = [v_0, \ldots, v_n] = Id(C_n(X))$ $\therefore$ $(C_n(f))_{n \geq 0} = C_*(f)$ and C_* is a functor.

Finally, we show that H_n is a functor: let $f: C_* \to D_*$ be a chain mapping over $\mathbb{F}$ i.e. $C_* = (C_r, \partial_r)_{r \geq 0}$, $D_* = (D_r, \delta_r)_{r \geq 0}$. $f = (f_r)_{r \geq 0}$ for $f_r: C_r \to D_r$.

These fit into the commutative diagram on the right. Also, $H_n(C_*) = Ker(\partial_n)/Im(\partial_{n+1})$ and $H_n(D_*) = Ker(\delta_n)/Im(\delta_{n+1})$ by definition.

$$\begin{array}{ccccc} C_{n+1} & \xrightarrow{\partial_{n+1}} & C_n & \xrightarrow{\partial_n} & C_{n-1} \\ \downarrow f_{n+1} & & \downarrow f_n & & \downarrow f_{n-1} \\ D_{n+1} & \xrightarrow{\delta_{n+1}} & D_n & \xrightarrow{\delta_n} & D_{n-1} \end{array}$$

Then we want to define $H_n(f): H_n(C_*) \to H_n(D_*)$. Note that elements of $H_n(C_*)$ look like $z + Im\,\partial_{n+1}$ [for $z \in Ker\,\partial_n$] and elements of $H_n(D_*)$ look like $w + Im\,\delta_{n+1}$ [for $w \in Ker\,\delta_n$].

Define $H_n(f): H_n(C_*) \to H_n(D_*)$ by $H_n(f)(z + Im\,\partial_{n+1}) = f_n(z) + Im\,\delta_{n+1}$. Check that this is well-defined: Suppose $z + Im\,\partial_{n+1} = z' + Im\,\partial_{n+1} \Rightarrow z - z' \in Im(\partial_{n+1}) \Rightarrow z - z' = \partial_{n+1}(\xi)$ for some ξ.

Apply f_n: $f_n(z) - f_n(z') = f_n(\partial_{n+1}(\xi)) \overset{\text{diagram commutes}}{=} \delta_{n+1} f_{n+1}(\xi) \in Im(\delta_{n+1}) \Rightarrow f_n(z) - f_n(z') \in Im(\delta_{n+1}) \overset{\text{law of equality}}{\Longrightarrow} f_n(z) + Im(\delta_{n+1}) = f_n(z') + Im(\delta_{n+1})$; thus the map is well-defined. $H_n(f)$ is linear as f_n is.

(II) $H_n(g \circ f) = H_n(g) \circ H_n(f)$: Let $C_* \xrightarrow{f} D_* \xrightarrow{g} E_*$. $H_n(g \circ f)(z + Im\,\partial^C_{n+1}) = (g_n \circ f_n)(z) + Im\,\partial^E_{n+1} = H_n(g)[f_n(z) + Im\,\partial^D_{n+1}] = H_n(g)\,H_n(f)[z + Im\,\partial^C_{n+1}]$, so indeed $H_n(g \circ f) = H_n(g) \circ H_n(f)$.

Likewise, (I) $H_n(Id) = Id_{H_n}$. Thus, H_n is a functor.

Mayer-Vietoris Theorem, revisited.

Let $X = X_+ \cup X_-$, where X is a simplicial complex and X_+, X_- are subcomplexes. They intersect at another subcomplex, $X_+ \cap X_-$. We have the following commutative diagram:

$$\begin{array}{ccc} & X_+ & \\ {}^{i_+}\nearrow & & \searrow^{j_+} \\ X_+ \cap X_- & & X_+ \cup X_- \\ {}_{i_-}\searrow & & \nearrow_{j_-} \\ & X_- & \end{array}$$

Here, the maps i_+, j_+, i_-, j_- are inclusion maps. They are trivially simplicial maps.

Thus, we can apply the functor H_n to get another commutative diagram (as shown overleaf).

Now, we have $H_n(X_+\cap X_-) \xrightarrow{i_*} H_n(X_+)\oplus H_n(X_-) \xrightarrow{j_*} H_n(X_+\cup X_-) = H_n(X)$ where we define $j_*: H_n(X_+)\oplus H_n(X_-) \to H_n(X_+\cup X_-)$, $\binom{z_+}{z_-} \mapsto H_n(j_+)(z_+) \oplus H_n(j_-)(z_-)$

$$\begin{array}{ccccc} & & H_n(X_+) & & \\ & \overset{H_n(i_+)}{\nearrow} & & \overset{H_n(j_+)}{\searrow} & \\ H_n(X_+\cap X_-) & & & & H_n(X) \\ & \underset{H_n(i_-)}{\searrow} & & \underset{H_n(j_-)}{\nearrow} & \\ & & H_n(X_-) & & \end{array}$$

and $i_*: H_n(X_+\cap X_-) \longrightarrow H_n(X_+)\oplus H_n(X_-)$, $w \longmapsto \begin{pmatrix} H_n(i_+)(w) \\ -H_n(i_-)(w)\end{pmatrix}$ (using column convention for direct sums). From the Mayer-Vietoris theorem, the above sequence is exact (verify!).

The difficult part remains however, which is to show the existence of the "boundary" map ∂ as shown to the right.

$$H_{n+1}(X_+\cap X_-) \xrightarrow{i_*} H_{n+1}(X_+)\oplus H_{n+1}(X_-) \xrightarrow{j_*} H_{n+1}(X) \overset{\partial}{\dashrightarrow} H_n(X_+\cap X_-) \xrightarrow{i_*} H_n(X_+)\oplus H_n(X_-) \xrightarrow{j_*} H_n(X) \overset{\partial}{\dashrightarrow} H_{n-1}(X_+\cap X_-) \xrightarrow{i_*} H_{n-1}(X_+)\oplus H_{n-1}(X_-) \xrightarrow{j_*} H_{n-1}(X)$$

The Mayer-Vietoris sequence is functorial w.r.t. decomposition. Suppose $X = X_+\cup X_-$, $Y = Y_+\cup Y_-$, and f is a simplicial map $f: X\to Y$ s.t. $f(X_+)\subset Y_+$, $f(X_-)\subset Y_-$.

Then the following sequence commutes:

$$\begin{array}{ccccccccc} H_{n+1}(X) & \xrightarrow{\partial^X} & H_n(X_+\cap X_-) & \xrightarrow{i_*} & H_n(X_+)\oplus H_n(X_-) & \xrightarrow{j_*} & H_n(X) & \xrightarrow{\partial^X} & H_{n-1}(X_+\cap X_-) \\ \downarrow H_{n+1}(f) & & \downarrow H_n(f) & & \downarrow \begin{pmatrix} H_n(f_+) & 0 \\ 0 & H_n(f_-)\end{pmatrix} & & \downarrow H_n(f) & & \downarrow H_{n-1}(f) \\ H_{n+1}(Y) & \xrightarrow[\partial^Y]{} & H_n(Y_+\cap Y_-) & \xrightarrow[i'_*]{} & H_n(Y_+)\oplus H_n(Y_-) & \xrightarrow[j'_*]{} & H_n(Y) & \xrightarrow[\partial^Y]{} & H_{n-1}(Y_+\cap Y_-) \end{array}$$

Note that H_n is invariant under isomorphism:

i.e. If we have $f: X \xrightarrow{\simeq} Y$, $g: Y\xrightarrow{\simeq} X$ s.t. $g\circ f = 1_X$, $f\circ g = 1_Y$, then $H(g)\circ H(f) = \mathrm{Id}$ and $H(f)\circ H(g) = \mathrm{Id}$. This is a straightforward consequence of functoriality.

Moreover, homology is invariant under subdivision (this is an even stronger statement).

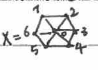
Let X be a finite simplicial complex. Then,

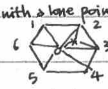
Definition A simplex $\sigma\in S_X$ is called <u>maximal</u> (or principal) if σ is not properly contained in any other simplex.

A finite simplicial complex X can be regarded as a finite union $X = \Delta_1\cup\Delta_2\cup\cdots\cup\Delta_n$, where each Δ_i is a maximal simplicial complex (and dimension of each Δ_i can differ).

e.g. Take $X =$ , then X has 5 maximal simplices: $\{0,1,5\}$, $\{0,2,3\}$, $\{0,3,4\}$, $\{1,2\}$, $\{5,4\}$.

The notion of the subdivision at a maximal simplex is easy to describe. We first consider an example:

Example – Take $X =$  with all interiors included. Then the simplex $\sigma = (0,2,3) =$  is maximal. We subdivide it with a lone point $*$: 

subdivision of $\sigma = C(\partial\sigma)$ i.e. cone on the boundary. Thus $X \longrightarrow Sd_\sigma(X)$, the subdivision of X at σ.  $Sd_\sigma(X)$.

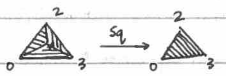
Definition If $X = \Delta_1\cup\cdots\cup\Delta_n$ where the Δ_i are the maximal simplices, and $\sigma = \Delta_i$ for some particular i, then we define the <u>subdivision of X at σ</u> as $Sd_\sigma(X) = \left(\bigcup_{r=1}^{i-1}\Delta_r\right)\cup C(\partial\Delta_i)\cup\left(\bigcup_{r=i+1}^{n}\Delta_r\right)$

i.e. we replace Δ_i by $C(\partial\Delta_i)$.

Theorem There exists a simplicial map (squash map) $Sq: Sd_\sigma(X)\to X$ $(\sigma=\Delta_i)$ s.t. (1) $Sq = \mathrm{Id}$ on Δ_j if $j\neq i$ and $Sq = \mathrm{Id}$ on $\partial\Delta_i$, and (2) $H_n(Sq): H_n(Sd_\sigma(X)) \xrightarrow{\simeq} H_n(X)$ $\forall n$.

Proof – (1) Choose a vertex v in Δ_i (so $v\in\partial\Delta_i$). Define $Sq: C(\partial\Delta_i)\to\Delta_i$ by $Sq(w) = w$ if $w\in\Delta_i$, $Sq(*) = v$ if $*$ is the cone point. Clearly $Sq|_{\partial\Delta_i} = \mathrm{Id}$. Now simply extend this map to get

$Sq: Sd_\sigma(X)\to X$ by $Sq = \mathrm{Id}$ on Δ_j $(j\neq i)$ // q.e.d.

e.g. – In previous example, let $v=3$ and map the point $*$ to $v=3$.  Note again that the choice of v is not unique.

(2) $X = \Delta_1\cup\cdots\cup\Delta_{i-1}\cup\Delta_i\cup\Delta_{i+1}\cup\cdots\cup\Delta_n$. Put $X_+ = \Delta_i$ and $X_- = \Delta_1\cup\cdots\cup\Delta_{i-1}\cup\Delta_{i+1}\cup\cdots\cup\Delta_n$ so that $X_+\cap X_-\subset\partial\Delta_i$. Write $X' = Sd_\sigma(X)$.

Then $X'_+ = C(\partial\Delta_i)$ and $X'_- = X_-$. Then $X'_+\cap X'_- = X_+\cap X_-$. Now we consider the following diagram:

$$\begin{array}{ccccccccc} H_n(X'_+\cap X'_-) & \longrightarrow & H_n(X'_+)\oplus H_n(X'_-) & \longrightarrow & H_n(X') & \longrightarrow & H_{n-1}(X'_+\cap X'_-) & \longrightarrow & H_{n-1}(X'_+)\oplus H_{n-1}(X'_-) \\ ①\downarrow Sq^* = \mathrm{Id} & & ②\downarrow\begin{pmatrix} Sq^* & 0 \\ 0 & 1\end{pmatrix} & & ③\downarrow Sq^* & & ④\downarrow Sq^* = \mathrm{Id} & & ⑤\downarrow\begin{pmatrix} Sq^* & 0 \\ 0 & 1\end{pmatrix} \\ H_n(X_+\cap X_-) & \longrightarrow & H_n(X_+)\oplus H_n(X_-) & \longrightarrow & H_n(X) & \longrightarrow & H_{n-1}(X_+\cap X_-) & \longrightarrow & H_{n-1}(X_+)\oplus H_{n-1}(X_-) \end{array}$$

This is a commutative diagram with exact rows.

Since they are identical, the maps ① and ④ are obviously isomorphisms. We will show that ② and ⑤ are also isomorphisms: Clearly, $X'_+ = C(\partial\Delta_i)$ is a cone, and $X_+ = \Delta_i$ is a simplex, so it is also a cone. Now, $H_r(X'_+) = H_r(X_+) = 0$ $\forall r\geq 1$. Hence, $Sq: H_r(X'_+)\xrightarrow{\simeq} H_r(X_+)$ is an isomorphism $\forall r\geq 1$. Then consider $r=0$. Here, Sq takes a generating vertex to a generating vertex: $H_0(X'_+)\cong\mathbb{F}\to\mathbb{F}\cong H_0(X'_+)$. Using that, the maps ② and ⑤ are clearly isomorphisms.

Thus, from the Five lemma (Exercises Sheet 2), ①,②,④,⑤ are isomorphisms $\Rightarrow$ ③: $H_n(Sq): H_n(Sd_\sigma(X))\longrightarrow H_n(X)$ is an isomorphism // q.e.d.

10 February 2014
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Schafd LT / Roberts G06.

Lemma (Five lemma).

$$\begin{array}{ccccccccc} \overset{y}{A_0} & \xrightarrow{\alpha_0} & \overset{z}{A_1} & \xrightarrow{\alpha_1} & \overset{x}{A_2} & \xrightarrow{\alpha_2} & A_3 & \xrightarrow{\alpha_3} & A_4 \\ \downarrow f_0 & & \downarrow f_1 & & \downarrow f_2 & & \downarrow f_3 & & \downarrow f_4 \\ \underset{w}{B_0} & \xrightarrow[\beta_0]{} & B_1 & \xrightarrow[\beta_1]{} & B_2 & \xrightarrow[\beta_2]{} & B_3 & \xrightarrow[\beta_3]{} & B_4 \end{array}$$

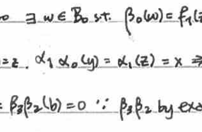
Suppose given a commutative diagram of vector spaces and linear maps as follows, in which both rows are exact:

then if f_0, f_1, f_3, f_4 are isomorphisms, then so is f_2.

Proof – (a) NTP: f_2 injective. Let $x\in A_2$ be such that $f_2(x) = 0$. NTP: $x=0$. $\beta_2 f_2(x) = 0$. By commutativity, $f_3\alpha_2(x) = 0$. f_3 is isomorphism, injective $\Rightarrow$ $\alpha_2(x) = 0$. $x\in\mathrm{Ker}\,\alpha_2$ $\Rightarrow$ by exactness, $x\in\mathrm{Im}\,\alpha_1 \Rightarrow \exists z\in A_1$ s.t. $\alpha_1(z) = x$. Then $f_2\alpha_1(z) = f_2(x) = \beta_1 f_1(z)$. So $f_1(z)\in\mathrm{Ker}(\beta_1) = \mathrm{Im}(\beta_0)$, so $\exists w\in B_0$ s.t. $\beta_0(w) = f_1(z)$. f_0 is an isomorphism so it is surjective, hence $\exists y\in A_0$ s.t. $f_0(y) = w$. $\beta_0 f_0(y) = \beta_0(w) = f_1(z)$. Also, $f_1(z) = f_1\alpha_0(y)$. f_1 injective so $\alpha_0(y) = z$. $\alpha_1\alpha_0(y) = \alpha_1(z) = x \Rightarrow \alpha_1\alpha_0 = 0 \Rightarrow x = 0$ // q.e.d.

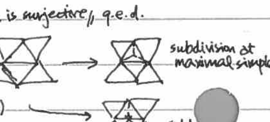
(b) NTP: f_2 surjective. Let $b\in B_2$. Need to find $a\in A_2$ s.t. $f_2(a) = b$. f_3 is surjective, so $\exists x\in A_3$ s.t. $f_3(x) = \beta_2(b)$. $f_4\alpha_3(x) = \beta_3 f_3(x) = \beta_3\beta_2(b) = 0$ ∵ $\beta_3\beta_2$ by exactness. But f_4 injective, so $\alpha_3(x) = 0 \Rightarrow x\in\mathrm{Ker}(\alpha_3) = \mathrm{Im}(\alpha_2)$. Choose $y\in A_2$ s.t. $\alpha_2(y) = x$. Then $f_3\alpha_2(y) = \beta_2 f_2(y)$. Consider $z = b - f_2(y)$. Then $\beta_2(z) = \beta_2(b) - \beta_2 f_2(y)$ $= \beta_2(b) - f_3\alpha_2(y) = \beta_2(b) - f_3(x) = 0$ because $\beta_2(b) = f_3(x)$. So $z\in\mathrm{Ker}(\beta_2) = \mathrm{Im}(\beta_1)$. Choose $w\in B_1$ s.t. $\beta_1(w) = z$. f_1 is an isomorphism, so it is surjective. Choose $t\in A_1$ s.t. $f_1(t) = w$. Therefore $\beta_1 f_1(t) = z = f_2\alpha_1(t) = b - f_2(y) \Rightarrow b = f_2(y + \alpha_1(t))$. Put $a = y + \alpha_1(t)\in A_2$. Then $f_2(a) = b$, and f_2 is surjective // q.e.d.

So now, we have shown that if K' is a subdivision of K at a maximal simplex, then $H_*(K')\cong H_*(K)$. What happens at a non-maximal simplex?



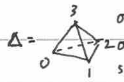
Definition Let $X = (V_X, S_X)$, $Y = (V_Y, S_Y)$ be simplicial complexes, with $V_X\cap V_Y = \phi$ (i.e. disjoint). The join of simplices X and Y is $X * Y = (V_X\cup V_Y, S_{X*Y})$ where $S_{X*Y} = \{S_X\cup S_Y\cup\{\sigma\cup\tau : \sigma\in S_X, \tau\in S_Y\}\}$.

Remark – A cone is a special case of a join, where X is a single point. Then $X * Y = C(Y)$, the cone on Y.



Note – $\Delta^1 * \Delta^1 \cong \Delta^3$ and moreover in general, $\Delta^m * \Delta^n \cong \Delta^{m+n+1}$, $S^m * S^n \cong S^{m+n+1}$.

If σ is a non-maximal simplex in X and Δ is a maximal simplex in X s.t. $\sigma \subset \Delta$, then $\exists$ unique simplex $\sigma' \subset \Delta$, $\sigma \cap \sigma' = \phi$ and $\sigma * \sigma' = \Delta$. σ' is called the <u>opposite face</u> of σ.

$\Delta =$  $\sigma = \{0,3\}$, $\sigma' = \{1,2\}$ s.t. $\sigma * \sigma' = \Delta$.

Observe that join has the following properties:

(I) $X * Y = Y * Z$ (commutativity) (II) $X*(Y*Z) = (X*Y)*Z$. (III) $(CX)*Y$ is a cone, since $CX = \{\text{point}\} * X$, so $(CX)*Y = \{\text{point}\} * (X*Y) = C(X*Y)$.

<u>Subdivision at a non-maximal simplex.</u>

Let X be a finite simplicial complex, σ a non-maximal simplex. List the maximal simplices of X: thus, $\Delta_1, \Delta_2, \dots, \Delta_m, \Delta_{m+1}, \dots, \Delta_N$ so that $\sigma \subset \Delta_i$, $1 \leq i \leq m$, $\sigma \not\subset \Delta_j$, $m+1 \leq j \leq n$.

Let X_+ be the subcomplex of X determined by $\Delta_1, \dots, \Delta_m$ with all faces of $\Delta_1, \dots, \Delta_m$. $X_+ = \Delta_1 \cup \dots \cup \Delta_m$. Likewise, $X_- = \Delta_{m+1} \cup \dots \cup \Delta_N$. Write $\Delta_i = \sigma * \delta_i$, $1 \leq i \leq m$ where δ_i is the opposite face in Δ_i. $X_+ = \sigma * (\bigsqcup_{i=1}^m \delta_i)$. In particular, X_+ is a cone, because σ is a cone, and a cone joined to anything is a cone.

Definition $Sd_\sigma(X) = X'_+ \cup X'_-$ where $X'_+ = C(\partial\sigma) * (\bigsqcup_{i=1}^m \delta_i)$, $X'_- = X_-$ and $X'_+ \cap X'_- = X_+ \cap X_-$. Claim: $H_*(Sd_\sigma(X)) \cong H_*(X)$. Get a generalised squash map $Sq: X'_+ \cup X'_- \to X_+ \cup X_-$.
Then $Sq|_{X'_+} = sq * (Id \bigsqcup_{i=1}^m \delta_i)$ where $sq: C(\partial\sigma) \to \sigma$ is a standard squash map, $Sq|_{X'_-} = Id_{X'_-} = Id_{X_-}$. Again, use Mayer-Vietoris Thm and Five Lemma:

Here, $Sq: H_k(X'_+) \to H_k(X_+)$ is an isomorphism as X'_+, X_+ are cones.

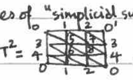
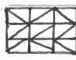
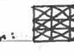
By Five Lemma, Sq is an isomorphism.

$$\begin{array}{ccccccccc} H_n(X'_+ \cap X'_-) & \longrightarrow & H_n(X'_+) \oplus H_n(X'_-) & \longrightarrow & H_n(X'_+ \cup X'_-) & \longrightarrow & H_{n-1}(X'_+ \cap X'_-) & \longrightarrow & H_{n-1}(X'_+) \oplus H_{n-1}(X'_-) \\ \downarrow Id & & \downarrow \begin{pmatrix} sq & 0 \\ 0 & Id \end{pmatrix} & & \downarrow Sq & & \downarrow Id & & \downarrow \begin{pmatrix} sq & 0 \\ 0 & 1 \end{pmatrix} \\ H_n(X_+ \cap X_-) & \longrightarrow & H_n(X_+) \oplus H_n(X_-) & \longrightarrow & H_n(X_+ \cup X_-) & \longrightarrow & H_{n-1}(X_+ \cap X_-) & \longrightarrow & H_{n-1}(X_+) \oplus H_{n-1}(X_-) \end{array}$$

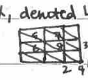
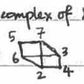
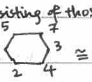
Definition Let X, X' be simplicial complexes. We say that X, X' are <u>combinatorially equivalent</u> when $\exists$ sequence $X_0, X_1, \dots, X_N$ of complexes, s.t. $X_0 = X$, $X_N = X'$ and for each i, $0 \leq i \leq N-1$,

either $X_{i+1} = Sd_\sigma(X_i)$ or $X_i = Sd_{\sigma'}^{-1}(X_{i+1})$ for some σ, σ'. We write $X \sim X'$.

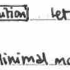
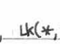
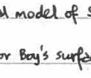
We look at two examples of "simplicial surfaces".

- T^2, the torus. $T^2 =$ , with all 2-simplices included. Or, we can triangulate.  $= \hat{T}^2$. Clearly, T^2 and $\hat{T}^2$ are not the same. However, we can subdivide them further to get an isomorphism: . Hence, $T^2 \not\cong \hat{T}^2$, but they share a common subdivision so as to become isomorphic. i.e. $T^2, \hat{T}^2$ are combinatorially equivalent.
Hence, $H_*(T^2 : \mathbb{F}) \cong H_*(\hat{T}^2 : \mathbb{F})$.

Definition Let v be a vertex in X. The link of v in X, denoted $Lk(v, X)$ is the subcomplex of X consisting of those simplices σ in X s.t. $\{v\} * \sigma$ is a simplex in X.

e.g. – Using our model of T^2 earlier, , then $Lk(\{8\}, X) =$  $=$  $\cong S^1(6)$ [Define: $S^1(n)$ = circle with n (≥ 3) subdivision points].

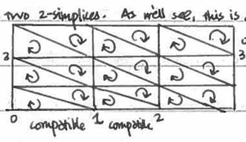
Definition Let Σ be a simplicial complex. We say Σ is a <u>combinatorial surface</u> when $\forall$ vertices $v \in \Sigma$, $\exists n \geq 3$ s.t. $Lk(v, \Sigma) \cong S^1(n)$.

- Minimal model of S^2. . Then $Lk(\{0\}, S^2) \cong S^1(3)$ . If we take a non-minimal model, e.g. , $Lk(*, X) =$  $= S^1(4)$.
- $\mathbb{RP}^2$: (or Boy's surface) . Then $Lk(3, \mathbb{RP}^2) =$  $\cong S^1(5)$.

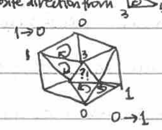
A common feature of the two surfaces is that * every 1-simplex lies in exactly two 2-simplices. As we'll see, this is a general property of surfaces.

12 February 2014. Prof FEA JOHNSON. Chemistry LT.

We introduce an orientation for our model of the torus:

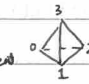
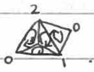
T_2: 

orienting adjacent 2-simplices in opposite directions gives a compatible arrangement.



However, for $\mathbb{RP}^2$, we get a clash in directions — as seen on right. So it is not orientable.

Definition A simplicial surface Σ is said to be <u>orientable</u> when it is possible to choose orientations of the 2-simplices in such a way that every 1-simplex receives opposite orientations from the 2-simplices to which it belongs.

e.g. S^2 is orientable. We represent  by .

After reading week, we will prove the orientation theorem.

Theorem (Orientation theorem).

Let Σ be a connected surface, and let $\mathbb{F}$ be a field.

(i) If $1+1=0$, then $H_2(\Sigma : \mathbb{F}) \cong \mathbb{F}$ (e.g. $\mathbb{F} = \mathbb{F}_2$), and (ii) If $1+1 \neq 0$, then $H_2(\Sigma : \mathbb{F}) = \begin{cases} \mathbb{F} & \text{if } \Sigma \text{ is orientable} \\ 0 & \text{if } \Sigma \text{ is non-orientable} \end{cases}$ (e.g. $\mathbb{F} = \mathbb{Q}, \mathbb{R}, \mathbb{F}_3, \mathbb{F}_5$ etc)

Proof – To be provided later.

<u>Euler characteristics</u>

Definition Let $K = (V_K, S_K)$ be a finite simplicial complex. Let $\nu_K(n)$ be the number of n-simplices of K. Then the <u>geometric Euler characteristic</u> is given by $\chi_{geom}(K) = \sum_{n \geq 0} (-1)^n \nu_K(n)$.

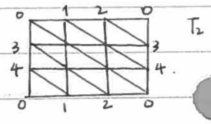
The <u>homological Euler characteristic</u> is given by $\chi_{hom}(K) = \sum_{n \geq 0} (-1)^n \dim H_n(K : \mathbb{F})$.

Theorem Let K be a finite simplicial complex, $\mathbb{F}$ a field. Then $\chi_{hom}(K) = \chi_{geom}(K)$.

Proof – Let $C_*(K : \mathbb{F})$ be chain complex of K with coefficients in $\mathbb{F}$. Then $\dim C_n(K : \mathbb{F}) = \nu_K(n)$ = no. of n-simplices of K. Consider $C_{n+1}(K) \xrightarrow{\partial_{n+1}} C_n(K) \xrightarrow{\partial_n} C_{n-1}(K)$, and also put $C_n = C_n(K)$, $Z_n = \mathrm{Ker}(\partial_n)$, $B_n = \mathrm{Im}(\partial_{n+1})$, so $H_n = Z_n / B_n$ s.t. $\dim H_n = \dim Z_n - \dim B_n \Rightarrow \dim H_n + \dim B_n = \dim Z_n$. $(*)$ Moreover, $0 \to Z_n \to C_n \xrightarrow{\partial_n} B_{n-1} \to 0$ is exact, so $\dim C_n = \dim Z_n + \dim B_{n-1} \Rightarrow \dim C_n - \dim B_{n-1} = \dim Z_n$. $(**)$ Then $(*) = (**)$ s.t. $\dim H_n + \dim B_n = \dim C_n - \dim B_{n-1}$ and then, we get $(-1)^n \dim H_n + (-1)^n \dim B_n = (-1)^n \dim C_n + (-1)^{n-1} \dim B_{n-1}$. Summing over all n, $\sum_n (-1)^n \dim H_n + \sum_n (-1)^n \dim B_n = \sum_n (-1)^n \dim C_n + \sum_n (-1)^{n-1} \dim B_{n-1}$.

Then B_n, B_{n-1} terms are equal, so $\chi_{hom}(K) = \sum_n (-1)^n \dim H_n = \sum_n (-1)^n \dim C_n = \chi_{geom}(K)$.

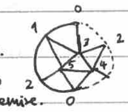
Examples – • Calculate $H_*(T^2)$: We triangulate T^2 as on right: $\chi_{geom}(T^2) = \nu(0) - \nu(1) + \nu(2) = 9 - 27 + 18 = 0$. Thus, $\chi_{hom}(T^2) = 0 \Rightarrow$



$\dim H_0(T^2:\mathbb{F}) - \dim H_1(T^2:\mathbb{F}) + \dim H_2(T^2:\mathbb{F}) = 0$. $\dim H_0 = 1$ as T^2 is connected. $\dim H_2 = 1$ as T^2 is orientable. Hence,

$1 + \dim H_1(T^2:\mathbb{F}) + 1 = 0 \Rightarrow \dim H_1(T^2:\mathbb{F}^2) = 2$. Hence, $H_k(T^2:\mathbb{F}) = \begin{cases} \mathbb{F} & k=0 \\ \mathbb{F}\oplus\mathbb{F} & k=1 \\ \mathbb{F} & k=2 \end{cases}$ for any field $\mathbb{F}$.

• Calculate $H_*(\mathbb{RP}^2)$: Two cases, if $1+1=0$ or $\neq 0$. Then $\chi_{geom}(\mathbb{RP}^2) = \nu(0) - \nu(1) + \nu(2) = 6 - 15 + 10 = 1 \Rightarrow \chi_{hom}(\mathbb{RP}^2) = 1$ over any $\mathbb{F}$.

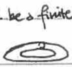


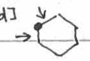
If $1+1 \neq 0$, $h_r = \dim H_r$, then $h_0 - h_1 + h_2 = 1$. $h_0 = 1$ (connected), $h_2 = 0$ (not orientable) $\Rightarrow -h_1 = 0 \Rightarrow h_1 = 0$. $H_k(\mathbb{RP}^2:\mathbb{F}) = \begin{cases} \mathbb{F} & k=0 \\ 0 & \text{otherwise}. \end{cases}$

If $1+1=0$, $h_0 - h_1 + h_2 = 1$. $h_0 = 1$ (connected) $h_2 = 1$ (always true), so $2 - h_1 = 1 \Rightarrow h_1 = 1$. $H_k(\mathbb{RP}^2:\mathbb{F}) = \begin{cases} \mathbb{F} & k = 0,1,2 \\ 0 & \text{otherwise}. \end{cases}$

24 February 2014
Prof FEA JOHNSON.
Schild LT / Roberts G06.

Let Σ be a finite simplicial complex. We say that Σ is a surface when $\forall$ vertex $v \in \Sigma$, $Lk_\Sigma(v) \cong S^1(n)$ for some $n \geq 3$. The idea is that the horizon is a circle. [e.g. $S^1(5) =$ ⬠].

 this is a surface, however...  this is only "almost" a surface – it has one "bad point".

First observe that in $S^1(n)$, every vertex belongs to exactly two 1-simplices. [good]  but not more e.g. [bad] 

Corollary In a surface, every 1-simplex lies in exactly two 2-simplices.

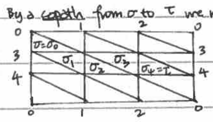
Proof – Let $e = \{v, w\}$ be a 1-simplex. Consider $Lk_\Sigma(v) \cong S^1(n)$. Now $w \in Lk_\Sigma(v)$ as w is joined to v. So w lies in exactly two 1-simplices, $e_1, e_2 \subset Lk_\Sigma(v)$.

Then the joins $e_i * v$ are 2-simplices containing $e = \{v, w\}$, $i = 1, 2$ // q.e.d.

Definition Two 2-simplices σ, τ are said to be adjacent when $\sigma \cap \tau$ is a 1-simplex. i.e. σ, τ are the 2-simplices containing $\sigma \cap \tau$.

Definition Let Σ be a surface, and let σ, τ be 2-simplices in Σ. By a copath from σ to τ we mean a sequence of 2-simplices $\sigma_0, \sigma_1, \dots, \sigma_N$ such that $\sigma = \sigma_0$, $\tau = \sigma_N$ and σ_i is adjacent to σ_{i+1} $\forall 0 \leq i \leq N-1$.

e.g. for T_2, we have the standard triangulation

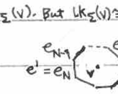


Theorem Let Σ be a connected surface, and let σ, τ be distinct 2-simplices in Σ. Then there exists a copath from σ to τ.

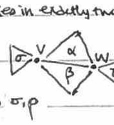
Proof – A priori we have three cases: (0) $\sigma \cap \tau$ is a 1-simplex, (1) $\sigma \cap \tau$ is a vertex or (2) $\sigma \cap \tau = \phi$. Case (0) is easily resolved by setting $\sigma_0 = \sigma$, $\sigma_1 = \tau$ by adjacency.

For case (1), let $\sigma \cap \tau = \{v\}$. Then consider $Lk_\Sigma(v) \cong S^1(n)$. Since $v \in \sigma$, $\sigma = e * v$ where $e \subset Lk_\Sigma(v)$. Likewise $v \in \tau \Rightarrow \tau = e' * v$ where $e' \subset Lk_\Sigma(v)$. But $Lk_\Sigma(v) \cong S^1(n)$

Let $e = e_0$, $e' = e_N$, then we can clearly go from e to e' along adjacent edges $e = e_0, e_1, \dots, e_N = e'$. Put $\sigma_0 = e_0 * v = \sigma$, $\sigma_1 = e_1 * v, \dots,$ then



$\sigma_N = e_N * v = \tau$, so this is a copath. Finally, we prove (2) using induction on the minimum length between σ and τ. Let v be a vertex of σ, w be a vertex of τ. Then consider minimum length n of a path from v to w. Then for $P(1)$, our induction base, $n = 1 \Rightarrow \{v, w\}$ is an edge, a 1-simplex. Then $\{v, w\}$ lies in exactly two 2-simplices α and β. $v \in \alpha \cap \sigma$, so by case (1), there is a copath from σ to $\alpha = \sigma_M$. $w \in \beta \cap \tau$, so by (1) $\exists$ copath $\beta = \sigma_{M+1}, \dots, \sigma_N = \tau$



Since α, β are adjacent, there is a copath $\sigma = \sigma_0, \sigma_1, \dots, \sigma_{N-1}, \sigma_N = \tau$ from σ to τ. Then for inductive step, suppose proven for $P(m)$ i.e. simplices σ, ρ joined by a path of length $\leq m$. Suppose $\exists$ path of length $m+1$ from σ to τ. The path is of the form $v_0, v_1, \dots, v_{m+1}$ where v_i are vertices, $v_0 \in \sigma$, $v_{m+1} \in \tau$. where $\{v_i, v_{i+1}\}$ is a 1-simplex. Take a 2-simplex ρ which contains v_m. By induction, $\exists$ copath $\sigma = \sigma_0, \dots, \sigma_K = \rho$, and ρ and τ are joined by a path of length 1.

So $\exists$ copath $\rho = \sigma'$, $\sigma_{K+1}, \dots, \sigma_N = \tau$. Then $\sigma = \sigma_0, \dots, \sigma_N = \tau$ is a copath // q.e.d.

Let $\sigma = \{a, b, c\}$ be a 2-simplex. We can orient σ in 2 different ways. $[\sigma] = [a,b,c] = [b,c,a] = [c,a,b]$. The opposite orientation is $-[\sigma] = [a,c,b] = [c,b,a] = [b,a,c]$.

Then $[a,b]$ has incidence $+1$ in $[a,b,c]$ and -1 in $[a,c,b]$. $\partial[a,b,c] = [b,c] - [a,c] \oplus [a,b]$ ($+1$), $\partial[a,c,b] = [c,b] \ominus [a,b] + [a,c]$ (-1)

Definition Let Σ be a connected surface. Σ is said to be orientable when one can assign orientations to 2-simplices in such a way that if $\sigma \cap \tau$ is a 1-simplex, then the incidence number of $\sigma \cap \tau$ in σ is opposite to the incidence number of $\sigma \cap \tau$ in τ.

Let Σ be a finite connected surface. We seek to compute $H_2(\Sigma:\mathbb{F})$ where $\mathbb{F}$ is some field. Consider the sequence $0 \to C_2(\Sigma:\mathbb{F}) \xrightarrow{\partial_2} C_1(\Sigma:\mathbb{F}) \xrightarrow{\partial_1} C_0(\Sigma:\mathbb{F}) \to 0$. There are no 3-simplices, so $H_2(\Sigma:\mathbb{F}) = \mathrm{Ker}(\partial_2 : C_2 \to C_1)$. List the 2-simplices of Σ $\sigma_1, \dots, \sigma_N$. $[\sigma_i]$ will denote σ_i with a particular orientation. An element of $C_2(\Sigma:\mathbb{F})$ looks like $z = \sum_{i=1}^{N} a_i[\sigma_i]$, $a_i \in \mathbb{F}$ and an element of $H_2(\Sigma:\mathbb{F})$ is such an expression in which $\partial_2(z) = 0$.

Proposition If $\partial_2(z) = 0$ and some $a_i = 0$, then every $a_i = 0$ and $z = 0$.

Proof – Suppose that $[\sigma_i] = [\sigma]$ and $a_i = 0$, and let $[\tau]$ be any other 2-simplex. Say $[\tau] = [\sigma_j]$, $j \neq i$. Have to show $a_j = 0$. Choose copath $\sigma = \rho_1, \dots, \rho_K = \tau$. Reindex such that the remaining simplices are $\rho_{K+1}, \dots, \rho_N$ so that $z = \sum_{j=1}^{N} b_j[\rho_j]$ such that the b_j terms are reindexed a_i terms. By hypothesis, take $b_1 = 0$. Then since ρ_1, ρ_2 are adjacent, then the coefficient of $\rho_1 \cap \rho_2$ in the expression $\partial_2(z) = 0$ is simply $b_1 \pm b_2 = 0$ as coefficient of every 1-simplex in $\partial_2(z)$ must be 0 as $\partial_2(z) = 0$. $b_1 = 0 \Rightarrow b_2 = 0$.

We iterate this process. Likewise, ρ_2, ρ_3 are adjacent so $b_3 = 0, \dots, b_K = 0 \Rightarrow$ coefficient of $[\tau] = 0$ // q.e.d.

Hence, if $z = \sum_{i=1}^{N} a_i[\sigma_i] \in H_2(\Sigma:\mathbb{F})$, then $z \neq 0 \Rightarrow$ every $a_i \neq 0$. We can refine this statement further...

Examples – (1) Let $\mathbb{F} = \mathbb{F}_2 = \{0, 1\}$. Then for finite connected surface Σ, we list the (unoriented) simplices $\sigma_1, \dots, \sigma_N$. $[\Sigma] = \sum_{i=1}^{N} [\sigma_i]$, then $\partial_2[\Sigma] = 0$. Here, we need not worry about signs in calculating ∂_2. If e is any 1-simplex, $e = \sigma \cap \tau$. σ, τ are 2-simplices and coefficient of $[e]$ in $\partial_2([\Sigma])$ = coefficient of σ + coefficient of $\tau = 1 + 1 = 0$. This is true for every 1-simplex e, so $\partial_2(\Sigma) = 0$. $[\Sigma] = \sum_{i=1}^{N} [\sigma_i]$ is called the mod 2 Fundamental class of Σ.

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Corollary If Σ is a finite connected surface, $\mathbb{F}=\mathbb{F}_2=\{0,1\}$. Then $H_2(\Sigma:\mathbb{F}_2)\cong\mathbb{F}_2$ with a single non-zero element $[\Sigma]$ mod 2 fundamental class.

Now let Σ be a finite connected orientable surface, and let $\mathbb{F}$ be any field in which $1+1\neq 0$. As Σ is orientable, we assign definite orientations to the 2-simplices $[\sigma_1],\dots,[\sigma_N]$ s.t. if $e=\sigma_i\cap\sigma_j$ $(i\neq j)$ is a 1-simplex, then incidence number of $[e]$ in $[\sigma_i]$ = $-$incidence number of $[e]$ in $[\sigma_j]$ [$(e,i)=-(e,j)$]. Define $[\Sigma]=\sum_{i=1}^{N}[\sigma_i]$. In the expression of $\partial_2([\Sigma])$, the coefficient of $[e]$ is $(e,i)+(e,j)=0$ where $[\sigma_i],[\sigma_j]$ are such that $e=\sigma_i\cap e_j$. This is true for every e, so $\partial_2[\Sigma]=0$. Clearly $[\Sigma]\neq 0$.

Corollary If Σ is a finite oriented surface and $\mathbb{F}$ is a field in which $1+1=0$, then $H_2(\Sigma:\mathbb{F})\neq 0$ and contains $[\Sigma]$. (fundamental class over $\mathbb{F}$).

Theorem Let Σ be a finite connected oriented surface ($\mathbb{F}$ some field). Then $H_2(\Sigma:\mathbb{F})\cong\mathbb{F}$ generated by $[\Sigma]$.

Proof – List the 2-simplices $\sigma_1,\dots,\sigma_N$ and assume they are compatibly oriented so that if $\sigma_i\cap\sigma_j$ is a 1-simplex, then $(e,i)+(e,j)=0$. Let $z=\sum_{i=1}^{N}a_i[\sigma_i]\in\operatorname{Ker}\partial_2=H_2$.

Choose a particular 1-simplex e. Then $e=\sigma_i\cap\sigma_j$. The coefficient of $[e]$ in expression $\partial_2(z)=0$ must be 0. But also, coefficient of $e=a_i(e,i)+a_j(e,j)$, so $a_i(e,i)=-a_j(e,j)$

But $(e,i)=-(e,j)$ so $a_i=a_j$. Then claim that $\forall i,k$, $a_i=a_k$. Choose a copath from σ_i to σ_k. Then $\sigma_i=\rho_0,\dots,\rho_p=\sigma_k$. Let b_j be the coefficient of ρ_j (reindexing a_i terms as before). Coefficients remain constant – $a_i=b_0=b_1=\dots=b_p=a_k$. Let λ = constant value of a_i as $i=1,\dots,N$. $z=\lambda[\Sigma]$ $(\lambda\in\mathbb{F})$. So $H_2(\Sigma:\mathbb{F})\cong\mathbb{F}$ generated by $[\Sigma]$ (generated by $[\Sigma]$ = sum of consistently oriented 2-simplices). // q.e.d.

So far, we have shown let if Σ is a finite connected surface, (1) $H_2(\Sigma:\mathbb{F}_2)\cong\mathbb{F}_2$. (2) $H_2(\Sigma:\mathbb{F})\cong\mathbb{F}$ if Σ is orientable, $1+1\neq 0$. It remains only to show $H_2(\Sigma:\mathbb{F})=0$ if Σ is non-orientable.

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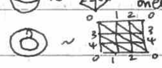
To show the last part, let Σ be a finite connected non-orientable surface, $1+1\neq 0$. List the 2-simplices $[\sigma_1],\dots,[\sigma_N]$ taken with arbitrary (but fixed) local orientations. Then let us define function $\rho:\{1,\dots,N\}\to\{\pm1\}$, and $\Sigma(\rho)\in C_2(\Sigma:\mathbb{F})$ by $\Sigma(\rho)=\sum_{i=1}^{N}\rho(i)[\sigma_i]$. There are 2^N such ρ terms. Regardless of ρ chosen, $\partial_2[\Sigma(\rho)]\neq 0$ because Σ is nonorientable. Now suppose $\alpha\in\operatorname{Ker}(\partial_2:C_2\to C_1)$, $\partial_2(\alpha)=0$. We claim $\alpha=0$. $\alpha=\sum_{i=1}^{N}a_i[\sigma_i]$ $a_i\in\mathbb{F}$. Suppose that σ_i,σ_j are adjacent and $\sigma_i\cap\sigma_j=e$. In the formal expression for $\partial_2(\alpha)$, coefficient of e is $(\pm1)a_i+(\pm1)a_j$. Since $\partial_2(\alpha)=0$, then coefficient of e is 0, so $a_j=(\pm1)a_i$. This is true for any adjacent 2-simplices σ_i,σ_j. For each $k\geq 2$, choose a copath from σ_1 to σ_k. Going along copath, we get $a_k=(\pm1)a_1$. So $\alpha=a_1\left(\sum_{i=1}^{N}\rho(i)[\sigma_i]\right)$ for some $\rho:\{1,\dots,N\}\to\{\pm1\}$. So by linearity, $\partial_2(\alpha)=a_1\,\partial_2(\Sigma(\rho))=0$. However, $\partial_2(\Sigma(\rho))\neq 0$, so $a_1=0\Rightarrow\alpha=0\left(\sum_{i=1}^{N}\rho(i)[\sigma_i]\right)=0$. Hence $\partial_2(\alpha)=0\Rightarrow\alpha=0$.

Hence, $H_2(\Sigma:\mathbb{F})=0$ for Σ non-orientable // q.e.d.

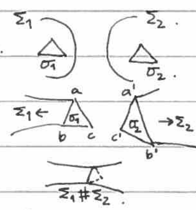
Standard examples of surfaces.

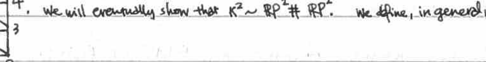
(0) S^2:  orientable. $H_K(S^2:\mathbb{F})=\begin{cases}\mathbb{F} & k=0\\ 0 & k=1\\ \mathbb{F} & k=2.\end{cases}$ S^2 is the orientable surface of genus 0.

(1) T^2:  $H_K(T^2:\mathbb{F})=\begin{cases}\mathbb{F} & k=0\\ \mathbb{F}\oplus\mathbb{F} & k=1\\ \mathbb{F} & k=2\\ 0 & k\geq 3.\end{cases}$ H_0 implies T^2 is connected, H_2 implies that T^2 is orientable.

Connected sum.

Suppose Σ_1,Σ_2 are simplicial surfaces. Let σ_1 be a 2-simplex in Σ_1, σ_2 be a 2-simplex in Σ_2. Remove the interiors of σ_1,σ_2 and then "glue" the boundaries together i.e. identify $a=a'$, $b=b'$, $c=c'$. Then the resulting simplex is the connected sum of Σ_1 and Σ_2, denoted $\Sigma_1\#\Sigma_2$.



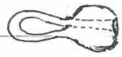
For instance, figure on the right is $T^2\#T^2$.  This is described as Σ_2^+.

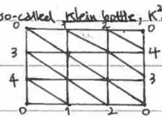
Looking back, S^2 is also called Σ_0^+, T^2 is also called Σ_1^+.

Definition The orientable surface of genus g is defined as $\Sigma_g^+=\overbrace{T^2\#\cdots\#T^2}^{g}$ For instance if $g=3$, . If $g=5$, 

Remark – This gives an infinite family Σ_g^+ $(g\geq 0)$.

For non-orientable surfaces, our basic building block is $\mathbb{RP}^2$. We call this surface Σ_0^-, and it is triangulated as such . Then, building from it, we define $\Sigma_1^-=\mathbb{RP}^2\#\mathbb{RP}^2$.

This has another description, the so-called Klein bottle, K^2. We can describe K^2 as follows: We describe T^2 by , then K^2 is identified  .

So we can triangulate it by . We will eventually show that $K^2\sim\mathbb{RP}^2\#\mathbb{RP}^2$. We define, in general, $\Sigma_g^-=\overbrace{\mathbb{RP}^2\#\cdots\#\mathbb{RP}^2}^{g+1\text{ copies}}$. With all this background, we can state a theorem.

Theorem (Classification theorem)

Let Σ be a finite connected surface. Then Σ is combinatorially equivalent to exactly one of Σ_g^+ or Σ_g^- $(g\geq 0)$.

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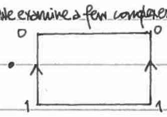
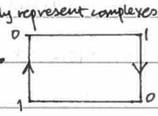
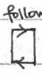
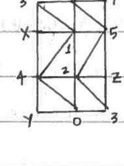
We examine a few complexes. On a rectangle, if we identify edges to schematically represent complexes

 we get a cylinder  or  we get a Möbius band. (denoted Möb).

The boundary of the cylinder is given by $\partial(\text{cylinder})=S^1\sqcup S^1$, whereas the boundary of the cylinder is given by $\partial(\text{Möb})\cong S^1$. Hence, the cylinder $\not\sim$ Möb!

Earlier, we claimed that $K^2\sim\mathbb{RP}^2\#\mathbb{RP}^2$. Consider $\mathbb{RP}^2\setminus\{3,4,5\}$, which is the plane with the central 2-simplex excluded.  Then we make the following subdivision:

 introducing points X, Y, Z. Then we also triangulate the Möbius band as follows:

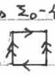
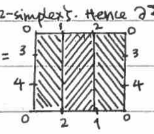
Match each of the existing edges from $\mathbb{RP}^2$ to this triangulation. Möb =  =  As such, we see that the two complexes are combinatorially equivalent.

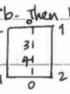
Theorem $\mathbb{RP}^2$ – {2-simplex} $\sim$ Möb.

Corollary If Σ is a surface that contains Möb, then Σ is non-orientable.

Proof – Trying to orient $\mathbb{RP}^2$, we get a contradiction without considering at least one 2-simplex.

Theorem $K^2 \sim \mathbb{RP}^2 \# \mathbb{RP}^2$.

Proof – Recall definition of #. Let Σ, Σ' be triangulated surfaces. Define Σ_0 to be $\Sigma - \{2\text{-simplex}\}$, likewise Σ_0' as $\Sigma_0 - \{2\text{-simplex}\}$. Hence $\partial\Sigma_0 \cong S^1$, $\partial\Sigma_0' \cong S^1$. Then $\Sigma \# \Sigma' = \Sigma_0 \underset{\partial=\partial}{\cup} \Sigma_0'$ i.e. boundaries are glued together. We have just seen that $\mathbb{RP}^2_0 = $ Möb. Then $K^2 =$  $=$  with colourings ▨ and ▧.

Clearly, ▨ is a Möbius band, ▨ = Möb = $\mathbb{RP}^2_0$. For ▧, we join the common edge 

Hence, ▧ is also Möb $= \mathbb{RP}^2_0$. So $K^2 = \text{Möb} \underset{\partial=\partial}{\cup} \text{Möb} = \mathbb{RP}^2_0 \underset{\partial=\partial}{\cup} \mathbb{RP}^2_0 = \mathbb{RP}^2 \# \mathbb{RP}^2$ // q.e.d.

Overall, we get a family of standard surfaces:

- Orientable: $\underset{\Sigma_0^+}{S^2}, \underset{\Sigma_1^+}{T^2}, \underset{\Sigma_2^+}{T^2 \# T^2}, \ldots\ldots, \underset{\Sigma_g^+}{\underbrace{T^2 \# \cdots \# T^2}_{g}}$
- Non-orientable: $\underset{\Sigma_0^-}{\mathbb{RP}^2}, \underset{\Sigma_1^- = K^2}{\mathbb{RP}^2 \# \mathbb{RP}^2}, \ldots\ldots, \underset{\Sigma_g^-}{\underbrace{\mathbb{RP}^2 \# \cdots \# \mathbb{RP}^2}_{g+1}}$

The surfaces Σ_g^+ and Σ_g^- correspond (c.f. Fundamental groups: Σ_g^+ is a "double cover" of Σ_g^-).

Proposition If Σ, Σ' are finite simplicial surfaces, then $\chi(\Sigma \# \Sigma') = \chi(\Sigma) + \chi(\Sigma') - 2$

Proof – Consider Σ, Σ': Σ  Σ' then $\Sigma \# \Sigma' = \Sigma_0 \underset{\partial=\partial}{\cup} \Sigma_0'$ and $\chi(\Sigma_0) = \chi(\Sigma) - 1$ (since we have lost a 2-simplex). Likewise $\chi(\Sigma_0') = \chi(\Sigma') - 1$.

Let $\nu(r) = |r\text{-simplices of } \Sigma_0|$, $\nu'(r) = |r\text{-simplices of } \Sigma_0'|$. Then $\chi(\Sigma \# \Sigma') = \sum_{r=0}^{2} (-1)^r \nu_r + \sum_{r=0}^{2} (-1)^r \nu_r' \underbrace{- 3 + 3}_{\text{losing 3 0-simplices and 3 1-simplices}} = \chi(\Sigma_0) + \chi(\Sigma_0') = \chi(\Sigma) + \chi(\Sigma') - 2$ //.

Theorem For any field $\mathbb{F}$, $H_r(\Sigma_g^+ : \mathbb{F}) \cong \begin{cases} \mathbb{F} & r=0 \\ \mathbb{F}^{2g} & r=1 \\ \mathbb{F} & r=2 \end{cases}$ where $\mathbb{F}^{2g} = \overbrace{\mathbb{F} \oplus \cdots \oplus \mathbb{F}}^{2g \text{ copies}}$ where $\mathbb{F}^0 = \{0\}$ by convention.

Proof – Already seen this is true for $g = 0, 1$. Suppose true for some $g \geq 1$, then $\Sigma_{g+1}^+ = T^2 \# \Sigma_g^+$. Then $\chi(\Sigma_{g+1}^+) = \chi(T^2) + \chi(\Sigma_g^+) - 2$, then by induction hypothesis we know that $\chi(\Sigma_g^+) = 2 - 2g$. Since $\chi(T^2) = 0$, $\chi(\Sigma_{g+1}^+) = 0 + 2 - 2g - 2 = 2g = 2 - 2(g+1)$. Since Σ_{g+1}^+ is orientable, $H_2(\Sigma_g^+ : \mathbb{F})$ is 1-dimensional. Since it is connected, $H_0(\Sigma_g^+ : \mathbb{F})$ is 1-dimensional. Hence, $\chi(\Sigma_{g+1}^+) = 2 - 2(g+1) = 2 - \dim H_1(\Sigma_g^+ : \mathbb{F}) \Rightarrow \dim H_1(\Sigma_g^+ : \mathbb{F}) = 2(g+1)$. Hence, proven by induction // q.e.d.

Theorem Let $\mathbb{F}_2 = \{0,1\}$ be the field with 2 elements. Then $H_r(\Sigma_g^- : \mathbb{F}_2) \cong \begin{cases} \mathbb{F}_2 & r=0 \\ \mathbb{F}_2^{g+1} & r=1 \\ \mathbb{F}_2 & r=2 \end{cases}$

Proof – This is true for $g = 0$, as $H_r(\mathbb{RP}_2 : \mathbb{F}_2) = \begin{cases} \mathbb{F}_2 & r=0 \\ \mathbb{F}_2 & r=1 \\ \mathbb{F}_2 & r=2 \end{cases}$. Then suppose this is true for $g \geq 0$. Then $\Sigma_{g+1}^- = \mathbb{RP}^2 \# \Sigma_g^-$. Then $\chi(\Sigma_{g+1}^-) = \chi(\mathbb{RP}^2) + \chi(\Sigma_g^-) - 2$.

Since $\chi(\mathbb{RP}^2) = 1$, by induction hypothesis, $\chi(\Sigma_{g+1}^-) = \chi(\Sigma_g^-) - 1 = [1 - (g+1) + 1] - 1 = 1 - (g+1)$. Since Σ_{g+1}^- is connected, $H_0(\Sigma_{g+1}^- : \mathbb{F}_2) \cong H_2(\Sigma_{g+1}^- : \mathbb{F}_2) \cong \mathbb{F}$.

Hence, $\chi(\Sigma_{g+1}^-) = 2 - \dim H_1(\Sigma_{g+1}^- : \mathbb{F}) = 2 - (g+2) \Rightarrow \dim H_1(\Sigma_{g+1}^- : \mathbb{F}) = g + 2 = (g+1) + 1$. Hence, proven by induction // q.e.d.

We now know that $\chi(\Sigma_g^-) = 1 - g$ [which is half of $2 - 2g$]. Since χ is field-independent, we can calculate rational homologies quite easily.

Corollary Let $\mathbb{F}$ be a field in which $1 + 1 \neq 0$. (e.g. $\mathbb{F} = \mathbb{Q}, \mathbb{R}, \mathbb{F}_3, \ldots$), then $H_r(\Sigma_g^- : \mathbb{F}) = \begin{cases} \mathbb{F} & r=0 \\ \mathbb{F}^g & r=1 \\ 0 & r=2 \end{cases}$.

Proof – $H_2(\Sigma_g^- : \mathbb{F}) = 0$ as Σ_g^- is non-orientable, and since $\chi(\Sigma_g^-)$ is conserved at $1 - g$, $\dim H_0(\Sigma_g^- : \mathbb{F}) = 1 \Rightarrow \dim H_1(\Sigma_g^- : \mathbb{F}) = g$ // q.e.d.

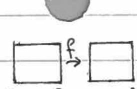
Theorem Let $g, h \geq 0$ be integers, $s, t \in \{\pm 1\}$. Then $\Sigma_g^s \sim \Sigma_h^t \Leftrightarrow g = h$ and $s = t$. In particular, if $g \neq h$ or $s \neq t$, then $H_*(\Sigma_g^s, \mathbb{Q}) \not\cong H_*(\Sigma_h^t, \mathbb{Q})$.

Proof – compute $H_*(-, \mathbb{Q})$, which distinguishes between the standard spaces.

Question: What happens for $\Sigma_g^+ \# \Sigma_h^-$? We will eventually see that $T^2 \# \mathbb{RP}^2 \sim \mathbb{RP}^2 \# \mathbb{RP}^2 \# \mathbb{RP}^2$. Analogously, we infer that $\Sigma_g^+ \# \Sigma_h^- \sim \Sigma_{2g+h}^-$. This will be proven later.

Fixed "Point" Theorems.

Recall the Intermediate Value Theorem: If $f : [-1,1] \to [-1,1]$ is continuous, then $\exists x \in [-1,1]$ s.t. $f(x) = x$. We then consider similar theorems in higher dimensions.



Here, for continuous $f : [-1,1]^2 \to [-1,1]^2$, $\exists \underline{x} \in [-1,1]^2$ s.t. $f(\underline{x}) = \underline{x}$. Likewise for three dimensions and more, $\exists \underline{x} \in [-1,1]^n$, $f(\underline{x}) = \underline{x}$. First proved by Brouwer. $f : [-1,1]^2 \to [-1,1]^2$

Theorem (Brouwer's Fixed Point Theorem) ~1908

Let $f : [-1,1]^n \to [-1,1]^n$ be a continuous mapping. Then $\exists \underline{x} \in [-1,1]^n$ s.t. $f(\underline{x}) = \underline{x}$.

Theorem (Lefschetz's Fixed Simplex Theorem).

Let X be a simplicial complex, $f : X \to X$ be a simplicial mapping. $H_r(f) : H_r(X : \mathbb{Q}) \to H_r(X : \mathbb{Q})$ induces linear map. Define Lefschetz's number $\lambda(f) = \sum_{r \geq 0} (-1)^r \operatorname{Tr}(H_r(f))$. Then if $\lambda(f) \neq 0$, then $\exists$ simplex σ in X s.t. $f(\sigma) = \sigma$.

Remark – If $H_r(f) = \begin{pmatrix} a_{11} & \cdots & a_{1N} \\ \vdots & \ddots & \vdots \\ a_{N1} & \cdots & a_{NN} \end{pmatrix}$, then the trace is $\operatorname{Tr}(H_r(f)) = \sum_{r=1}^{N} a_{rr}$.

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Definition Let $A = (a_{ji})_{\substack{1 \leq i \leq n \\ 1 \leq j \leq n}}$, $a_{ji} \in \mathbb{F}$. Then the trace of A is $\operatorname{Tr}(A) = \sum_{i=1}^{n} a_{ji} \in \mathbb{F}$.

Proposition If $A, B \in M_n(\mathbb{F})$, then $\operatorname{Tr}(AB) = \operatorname{Tr}(BA)$.

Proof – Let $A = (a_{ji})$, $B = (b_{k\ell})$. Then $(AB)_{j\ell} = \sum_{i=1}^{n} a_{ji} b_{i\ell}$, so $(AB)_{jj} = \sum_{i=1}^{n} a_{ji} b_{ij}$, and $\operatorname{Tr}(AB) = \sum_{j=1}^{n} (AB)_{jj} = \sum_{j=1}^{n} \sum_{i=1}^{n} a_{ji} b_{ij}$. Then $(BA)_{j\ell} = \sum_{i=1}^{n} b_{ji} a_{i\ell}$, so $(BA)_{jj} = \sum_{i=1}^{n} b_{ji} a_{ij}$.

Then $\operatorname{Tr}(BA) = \sum_{j=1}^{n} (BA)_{jj} = \sum_{j=1}^{n} \sum_{i=1}^{n} b_{ji} a_{ij}$. $\mathbb{F}$ is commutative, so $b_{ji} a_{ij} = a_{ij} b_{ji}$. Then $\operatorname{Tr}(BA) = \sum_{j=1}^{n} \sum_{i=1}^{n} a_{ij} b_{ji} = \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ji} b_{ij}$ (interchange order of Σ) $= \operatorname{Tr}(AB)$ // q.e.d.

Remark – Beware that $\operatorname{Tr}(AB) \neq \operatorname{Tr}(A)\operatorname{Tr}(B)$. To see this, let $A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $A^2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. Then $\operatorname{Tr}(A^2) = 2$ but $\operatorname{Tr}(A)^2 = 0$.

Corollary Let $A, P \in M_n(\mathbb{F})$ with P invertible, then $\operatorname{Tr}(PAP^{-1}) = \operatorname{Tr}(A)$.

Proof – $\operatorname{Tr}(PAP^{-1}) = \operatorname{Tr}(P(AP^{-1})) = \operatorname{Tr}((AP^{-1})P) = \operatorname{Tr}(AP^{-1}P) = \operatorname{Tr}(A)$ // q.e.d.

Remark – If $c_A(t)$ is the characteristic polynomial of A, then $c_A(t) = \pm \det(A - tI)$, then $\operatorname{Tr}(A) = \pm$ coefficient of t.

Trace of a linear map.

Let V be a finite dimensional vector space over $\mathbb{F}$, $S: V \to V$ a linear map. Then let $\mathcal{E} = \{e_1, \dots, e_n\}$ be a basis for V. Write $S(e_i) = \sum_{j=1}^{n} e_j a_{ji}$, then $M^{\mathcal{E}}_{\mathcal{E}}(S) = A$. Let $\Phi = \{\varphi_1, \dots, \varphi_n\}$ also be a basis. Then $S(\varphi_i) = \sum_{j=1}^{n} \varphi_j b_{ji}$. Then $M^{\Phi}_{\Phi}(S) = B = (b_{ji})$. Then $M^{\Phi}_{\Phi}(S) = M^{\Phi}_{\mathcal{E}}(\mathrm{Id})\, M^{\mathcal{E}}_{\mathcal{E}}(S)\, M^{\mathcal{E}}_{\Phi}(\mathrm{Id}) \Rightarrow B = PAP^{-1}$ where $P = M^{\Phi}_{\mathcal{E}}(\mathrm{Id})$, the matrix of the change of basis. Then $\mathrm{Tr}(B) = \mathrm{Tr}(A)$. Hence, the trace of a linear map is independent of the chosen basis. If $S: V \to V$ is linear, $\dim V < +\infty$, then $\mathrm{Tr}(S) \in \mathbb{F}$ is defined as follows: for a basis $\mathcal{E} = \{e_1, \dots\}$ for V, write $S(e_i) = \sum_{j=1}^{n} e_j a_{ji}$, $A = (a_{ji})$, and define $\mathrm{Tr}(S) = \mathrm{Tr}(A)$.

Lefschetz numbers.

Let K be a finite simplicial complex, $f: K \to K$ a simplicial map. Then we get induced maps $C_r(f): C_r(K:\mathbb{F}) \to C_r(K:\mathbb{F})$. We can take $\mathrm{Tr}(C_r(f)) \in \mathbb{F}$. Define $\lambda_{geom}(f) = \sum_{r \geq 0} (-1)^r \mathrm{Tr}(C_r(f))$ to be the geometric Lefschetz number. We can also get induced maps $H_r(f): H_r(K:\mathbb{F}) \to H_r(K:\mathbb{F})$, so $\mathrm{Tr}(H_r(f)) \in \mathbb{F}$. Then $\lambda_{hom}(f) = \sum_{r \geq 0} (-1)^r \mathrm{Tr}(H_r(f))$ is the homological Lefschetz number.

Theorem With the above notation, $\lambda_{geom}(f) = \lambda_{hom}(f)$.

We first need to prove additivity of Tr on exact sequences. Take an exact sequence of $\mathbb{F}$ linear maps, $\dim V$ finite. Assume diagram commutes, Q, R, S linear.

$$\begin{array}{ccccccccc} 0 & \to & U & \xrightarrow{i} & V & \xrightarrow{p} & W & \to & 0 \\ & & \downarrow Q & & \downarrow S & & \downarrow R & & \\ 0 & \to & U & \xrightarrow{i} & V & \xrightarrow{p} & W & \to & 0 \end{array}$$

Then claim $\boxed{\mathrm{Tr}(S) = \mathrm{Tr}(Q) + \mathrm{Tr}(R)}$.

Proof – observe $U = \mathrm{Ker}(p)$. Let $\{e_i\}_{1 \leq i \leq k}$ be a basis for U, $\{\varphi_j\}_{1 \leq j \leq m}$ be a basis for W. p is surjective so chose $\Phi_j \in V$, s.t. $p(\Phi_j) = \varphi_j$. Let $\{e_i\}_{1 \leq i \leq k} \cup \{\Phi_j\}_{1 \leq j \leq m}$ be a basis for V. We can write S as a matrix in block form. $S(e_i) = \sum_{j=1}^{k} e_j \alpha_{ji} + \sum_{k=1}^{m} \Phi_k \gamma_{ki}$, $S(\Phi_i) = \sum_{j=1}^{k} e_j \beta_{ji} + \sum_{k=1}^{m} \Phi_k \delta_{ki}$, so $S \sim \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$. Claim each $\gamma_{ki} = 0$.

$pS(e_i) = \sum_j p(e_j)\alpha_{ji} + \sum_k \varphi_k \gamma_{ki} = 0 + \sum_k \varphi_k \gamma_{ki}$, so $Rp(e_i) = pS(e_i)$ and each $p(e_i) = 0$. Hence $S \sim \begin{pmatrix} \alpha & \beta \\ 0 & \delta \end{pmatrix}$, so $\mathrm{Tr}(S) = \mathrm{Tr}(\alpha) + \mathrm{Tr}(\delta)$. But $Q(e_i) = \sum_{j=1}^{n} e_j \alpha_{ji}$ so $\mathrm{Tr}(\alpha) = \mathrm{Tr}(Q)$. $R(\varphi_i) = \sum \varphi_j \nu_{ji}$ for some $\nu_{ji} \in \mathbb{F}$. Then $Rp(\Phi_i) = \sum \varphi_j \nu_{ji}$, $pS(\Phi_i) = \sum \varphi_j \nu_{ji}$ by commutativity. Then $pS(\Phi_i) = p(\sum e_k \beta_{ji}) + \sum_j p(\Phi_j)\delta_{ji} = 0 + \sum_j \varphi_j \delta_{ji}$, so $\nu_{ji} = \delta_{ji}$, hence $\mathrm{Tr}(R) = \mathrm{Tr}(\delta)$. Hence, $\mathrm{Tr}(S) = \mathrm{Tr}(Q) + \mathrm{Tr}(R)$ // q.e.d.

Proof (thm) – $\partial_r: C_r \to C_r$. $B_r = \mathrm{Im}(\partial_{r+1})$, $Z_r = \mathrm{Ker}(\partial_r)$. $B_r \subset Z_r$, $H_r = B_r / Z_r$. Then we get the commutative diagram

$$\begin{array}{ccccccccc} 0 & \to & B_r & \to & Z_r & \to & H_r & \to & 0 \\ & & \downarrow B_r(f) & & \downarrow Z_r(f) & & \downarrow H_r(f) & & \\ 0 & \to & B_r & \to & Z_r & \to & H_r & \to & 0 \end{array}$$

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Since both rows are exact, we conclude that $\mathrm{Tr}(Z_r(f)) = \mathrm{Tr}(B_r(f)) + \mathrm{Tr}(H_r(f))$. Moreover, we also have the diagram as on right: this gives us also the conclusion $\mathrm{Tr}(C_r(f)) = \mathrm{Tr}(Z_r(f)) + \mathrm{Tr}(B_{r-1}(f))$.

$$\begin{array}{ccccccccc} 0 & \to & Z_r & \to & C_r & \xrightarrow{\partial} & B_{r-1} & \to & 0 \\ & & \downarrow Z_r(f) & & \downarrow C_r(f) & & \downarrow B_{r-1}(f) & & \\ 0 & \to & Z_r & \to & C_r & \xrightarrow{\partial} & B_{r-1} & \to & 0 \end{array}$$

Then $\mathrm{Tr}(Z_r(f)) = \mathrm{Tr}(C_r(f)) - \mathrm{Tr}(B_{r-1}(f)) = \mathrm{Tr}(B_r(f)) + \mathrm{Tr}(H_r(f)) \Rightarrow \mathrm{Tr}(C_r(f)) = \mathrm{Tr}(H_r(f)) + \mathrm{Tr}(B_r(f)) + \mathrm{Tr}(B_{r-1}(f))$.

i.e. $(-1)^r \mathrm{Tr}\, C_r(f) = (-1)^r \mathrm{Tr}\, H_r(f) + (-1)^r \mathrm{Tr}\, B_r(f) + (-1)^r \mathrm{Tr}\, B_{r-1}(f)$. Take alternating sum: $\lambda_{geom}(f) = \lambda_{hom}(f) + \sum_r (-1)^r \mathrm{Tr}\, B_r(f) + \sum_r (-1)^r \mathrm{Tr}\, B_{r-1}(f)$.

$\Rightarrow \lambda_{geom}(f) = \lambda_{hom}(f) + \sum_r (-1)^r \mathrm{Tr}\, B_r(f) - \sum_{(s=r-1)} (-1)^s B_s(f) = \lambda_{hom}(f)$ // q.e.d.

What information does $\lambda_{geom}(f)$ provide? Let $f: K \to K$, $C_r(f): C_r(K) \to C_r(K)$, $C_r(f)[v_0, \dots, v_r] = [f(v_0), \dots, f(v_r)]$.

The matrix of $C_r(f)$ is square. In each column we get at most one non-zero entry which is either $+1$ or -1. On the diagonal of $C_r(f)$, we get non-zero entries only when that particular r-simplex gets mapped isomorphically to itself up to sign. If no r-simplex is mapped to itself, then diagonal of $C_r(f) \equiv 0$, so $\mathrm{Tr}(C_r(f)) = 0$ (very weak statement). So if no simplex of K is mapped to itself, then $\sum (-1)^r \mathrm{Tr}\, C_r(f) = 0$ (even weaker statement).

Theorem (Lefschetz's theorem).

If $f: K \to K$ is simplicial, K finite complex, then $\lambda(f) \neq 0 \Rightarrow f$ maps at least one simplex of K to itself isomorphically, up to sign.

Proof – As we have seen, if f maps no simplex to itself, then $\lambda_{geom}(f) = 0$ // q.e.d.

Note – $\lambda_{geom}(f)$ retains enough information to prove the theorem, but $\lambda_{hom}(f)$ is useful in applications.

Corollary (Brouwer's Fixed Simplex Theorem).

Let K be a finite complex s.t. $K \sim \Delta^n$ (i.e. combinatorially equivalent). Let $f: \Delta^k \to \Delta^k$ be simplicial. Then $\exists$ simplex σ of K s.t. $f(\sigma) = \sigma$ (ignoring sign).

Proof – Calculate $\lambda_{hom}(f)$. $H_r(K:\mathbb{F}) = \begin{cases} \mathbb{F} & k = 0 \\ 0 & \text{otherwise.} \end{cases}$ So, $\lambda_{hom}(f) = \mathrm{Tr}\, H_0(f)$. Since K is connected, v, w are vertices of K, then $[v] = [w]$ in $H_0(K:\mathbb{F})$

so $[v] = [f(v)]$, so $H_0(f) = \mathrm{Id}$, $\mathrm{Tr}\, H_0(f) = 1$. So f has a fixed simplex by Lefschetz's theorem.

Proposition Let K be a finite complex. Then $\chi(K) = \lambda(\mathrm{Id}_K)$.

Proof – $\mathrm{Id}: K \to K$ induces $\mathrm{Id}_r: C_r(K:\mathbb{F}) \to C_r(K:\mathbb{F})$. So $\mathrm{Tr}(\mathrm{Id}_r) = \dim C_r(K:\mathbb{F})$. So $\sum_r (-1)^r \mathrm{Tr}(\mathrm{Id}_r) + \sum_r (-1)^r \dim(C_k:\mathbb{F}) = \lambda(K)$ // q.e.d.

In general, $\lambda(f)$ depends on the field $\mathbb{F}$ we are working over. For a sensible choice, if possible take $\mathbb{F} = \mathbb{Q}$.

Corollary Let $K \sim \mathbb{RP}^2$, $f: K \to K$ a simplicial map. Then $\exists$ simplex σ of K s.t. $f(\sigma) = \sigma$ up to sign.

Proof – Take $H_k(K:\mathbb{Q}) = \begin{cases} \mathbb{Q} & k = 0 \\ 0 & k > 0. \end{cases}$ Now apply same proof as for Brouwer's Theorem // q.e.d.

So far, we have dealt entirely with simplicial complexes and simplicial maps. As such, it is a finite theory and is completely computible in principle.

Take $f = f_1, \dots, f_k: \mathbb{R}^n \to \mathbb{R}$ smooth, $X(f) = \{\underline{x} \in \mathbb{R}^n\ \forall i\ \ f_i(x) = 0\}$. With a bit of luck, $X(f)$ will be a manifold of dimension $n-k$. i.e. $\forall x \in X(f)$, $\exists$ neighbourhood V in $X(f)$, such that $X \equiv \mathbb{R}^{n-k}$ is diffeomorphic. Each such $X(f)$ can be triangulated as a simplicial complex i.e. $\exists$ simplicial complex K with maximal simplices $\sigma_1, \dots, \sigma_N$ let σ_i be a complex set in $\mathbb{R}^{n-k}$. Then $|K| = \bigcup_{i=1}^{N} |\sigma_i|$, and $h: |K| \to X(f)$ is a homeomorphism (JHC Whitehead, C^1 triangulation Lemma, 1940).

Let K be a (finite) simplicial complex, $\mathcal{L}: |K| \xrightarrow{\simeq} X(f)$. Since $H_*(K)$ defined for K, we can expect to define it in the same way for $X(f)$. However, it was later shown in 1915 that topological invariance holds

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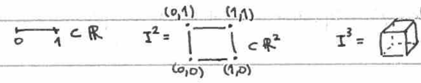
Topological invariance of H_*:

The Brouwer Fixed Point theorem gives that if $D^n = \{x \in \mathbb{R}^n : \|x\| \leq 1\}$, $\|x\| = \sqrt{x_1^2 + \dots + x_n^2}$, $f: D^n \to D^n$ continuous. $\exists\, \underline{x} \in D^n$, $f(\underline{x}) = \underline{x}$. Brouwer's proof not effective.

Lefschetz's Fixed Point theorem. $f: X \to X$ continuous, $\lambda_{hom}(f) \neq 0$, $\exists\, x \in X$ s.t. $f(x) = x$. sidenote: Alexander's proof: $X \xrightarrow{f} X$ continuous, triangulation: $h \uparrow$, $|K| \xrightarrow{|F|} |K|$, $\uparrow h$, $F: K \to K$ of his simplicial approximation thm. $h \circ |F|$ as close as you like to $f \circ h$.



Products. 

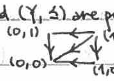
$I^n \sim 2^n$.

The n-simplex is the efficient way to describe a portion of space. It is not, however, the intuitive way. It is possible to describe homology in cubes e.g. cubical homology (Hilton-Wylie). c. 1950.

It is computationally inefficient to use cubes, but they are natural for products, $I^m \times I^n \cong I^{m+n}$. Contrastingly however, $\Delta^m \times \Delta^n$ is not a simplex. It instead yields a prism.

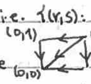
For instance, consider $\Delta^2 \times I$, , which is not a simplex. To triangulate $\Delta^m \times \Delta^n$, we introduce some background–

By a partially ordered set (poset) we mean a pair $(X, \leq)$ where $\leq$ is a relation on X satisfying (i) $\forall x$, $x \leq x$, (ii) $\forall x, y$, $(x \leq y) \wedge (y \leq x) \Leftrightarrow x = y$. and (iii) $\forall x, y, z$ $(x \leq y) \wedge (y \leq z) \Leftrightarrow x \leq z$. We do not assume (iv) $\forall x, y \in X$ either $x \leq y$ or $y \leq x$ (which would make it a total ordering).

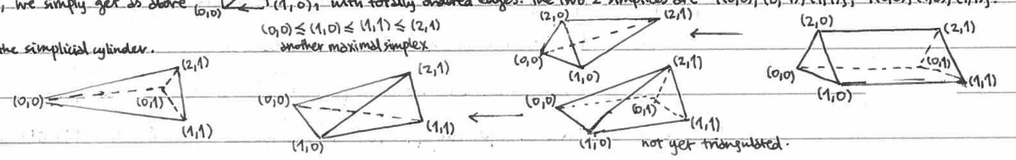
Product of posets: suppose $(X, \leq)$ and $(Y, \leq)$ are posets. Then define a new partial ordering on $X \times Y$ by $(x, y) \leq (x', y') \Leftrightarrow (x \leq x') \wedge (y \leq y')$. For instance, let us take $X = Y = (0 \leftarrow 1)$. Then $X \times Y =$ . This is not totally ordered, as there is no comparison between $(0,1)$ and $(1,0)$. Δ^n has vertex set $\{0, 1, \dots, n\}$.

We can impose standard ordering on vertices $0 \leftarrow 1 \leftarrow \dots \leftarrow n-1 \leftarrow n$. So $\exists$ natural partial ordering on vertices of $\Delta^m \times \Delta^n$, the standard triangulation of $\Delta^m \times \Delta^n$ is as follows:

- As vertex set take $\{0, 1, \dots, m\} \times \{0, 1, \dots, n\}$ i.e. $\{(r, s) : 0 \leq r \leq m,\ 0 \leq s \leq n\}$.
- As simplices take the totally ordered subsets.

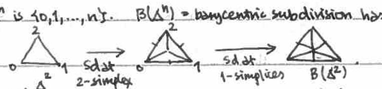
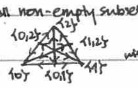
In the case of $\Delta^1 \times \Delta^1$, we simply get as above , with totally ordered edges. The two 2-simplices are $\{(0,0), (0,1), (1,1)\}$, $\{(0,0), (1,0), (1,1)\}$.

For $\Delta^2 \times \Delta^1$, we get the simplicial cylinder.



How to triangulate a cube: To every simplex Δ^n, we can associate its barycentric subdivision – i.e. subdivide every simplex from original Δ^n exactly once. Simpler description:

- vertex set of Δ^n is $\{0, 1, \dots, n\}$. $B(\Delta^n)$ = barycentric subdivision has
  - vertex set: all non-empty subsets of $\{0, \dots, n\}$
  - simplices are totally ordered subsets ordered by (reverse) inclusion.

For Δ^2, we get  Then we get  with central vertex $\{0,1,2\}$. Now take $C(B(\Delta^n))$, which yields a natural triangulation of I^{n+1}, an $(n+1)$-cube. This is because $I^{n+1} = \underbrace{[0,1] \times \dots \times [0,1]}_{n+1}$ with vertices $(v_0, \dots, v_n)$, $v_i \in \{0,1\}$. Think of a vertex of I^{n+1} as a function $v: \{0, 1, \dots, n\} \to \{0,1\}$. To each subset X of $\{0, \dots, n\}$ associate $\chi_X : \{0, \dots, n\} \to \{0,1\}$, $\chi_X(v) = \begin{cases} 1 & v \in X \\ 0 & v \notin X \end{cases}$, $\chi_\phi : \{0, \dots, n\} \to \{0,1\}$, $\chi_\phi(v) = 0\ \forall v$.

(i.e. imagine this is a cone to invisible vertex $(0,0,\dots,0)$ with barycentre at $(1,1,\dots,1)$)

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We have shown that $\Delta^m \times \Delta^n$ can be triangulated using natural order in Δ^m, Δ^n. Suppose X, Y are simplicial complexes. We can describe $X \times Y$ in terms of $\sigma \times \tau$, where σ is a simplex of X, τ a simplex of Y. To triangulate $X \times Y$, need to introduce into X, Y a partial ordering in such a way that each $\sigma \in S_X$, $\tau \in S_Y$ is totally ordered.

The easiest "get out" is to replace X by $B(X)$, Y by $B(Y)$ – the barycentric subdivisions of X, Y. Then simplices in $B(X), B(Y)$ are naturally totally ordered, so can take product.

Note that obviously, $X \sim B(X)$, $Y \sim B(Y)$. Also, I^n triangulated as $C(B(\Delta^{n-1}))$. Then $I^m \times I^n \cong I^{m+n}$. $\Delta^n \sim C(\Delta^{n-1}) \sim C(B(\Delta^{n-1})) \sim I^n$. So $\boxed{H_*(I^n) \cong H_*(\Delta^n) \cong H_*(pt)}$

Obvious problem: Given X, Y can we express $H_*(X \times Y : \mathbb{F})$ in terms of $H_*(X : \mathbb{F})$, $H_*(Y : \mathbb{F})$? Answer: Yes. We state a result, and then prove several specific examples of it.

This theorem was first formalised by Künneth (c. 1930).

Theorem (Künneth Theorem).

$$H_k(X \times Y : \mathbb{F}) \cong \bigoplus_{r=0}^{k} H_r(X : \mathbb{F}) \otimes H_{k-r}(Y : \mathbb{F}); \quad \text{or numerically,} \quad \dim H_k(X \times Y : \mathbb{F}) = \sum_{r=0}^{k} \dim H_r(X : \mathbb{F}) \dim H_{k-r}(Y : \mathbb{F}).$$

We will not prove this, but we will prove two consequences of it:

Corollary (I) $H_k(X \times \Delta^n : \mathbb{F}) \cong H_k(X : \mathbb{F})$

(II) $\chi(X \times Y) = \chi(X)\chi(Y)$.

Proof – (I) By induction. Define maps $i: X \to X \times \Delta^n$, $v \mapsto (v, 0)$; $\pi: X \times \Delta^n \to X$, $\pi(v, r) = v$. i is an inclusion, π a projection. Then we have

Theorem If X is a finite simplicial complex, then
(a) $i_*: H_*(X : \mathbb{F}) \xrightarrow{\simeq} H_*(X \times \Delta^n : \mathbb{F})$ is an isomorphism
(b) $\pi_*: H_*(X \times \Delta^n : \mathbb{F}) \xrightarrow{\simeq} H_*(X : \mathbb{F})$ is an isomorphism.

Proof – (a) Let $P(m)$ be the statement "$i_*: H_*(X : \mathbb{F}) \to H_*(X \times \Delta^n : \mathbb{F})$ is an isomorphism of $\dim(X) \leq m$". Let $P(m, k)$ be the statement "$i_*: H_*(X : \mathbb{F}) \to H_*(X \times \Delta^n : \mathbb{F})$ is an isomorphism if $\dim(X) \leq m$ and X has exactly k m-simplices. Note that • $P(m) = \bigwedge_{k \geq 0} P(m, k)$ and • $P(m+1, 0) \equiv P(m)$.

So it suffices to show that: $P(0)$ is true (induction base) and $P(m, k) \Rightarrow P(m, k+1)$ (induction step). Take X to be a 0-dim complex, $X = \{x_1, \dots, x_N\}$ with no simplices of dim ≥ 1.

Then $X \times \Delta^m = \bigsqcup_{i=1}^{N} (\{x_i\} \times \Delta^m)$. Claim: $H_0(X \times \Delta^n) \cong \mathbb{F}^N$, where N is the number of points, $H_k(X \times \Delta^n) = 0\ \forall k \geq 1$. Whereas $H_0(X) \cong \mathbb{F}$, $H_k(X) = 0$ $k \geq 1$. $X = X' \sqcup \{x_N\}$, $X' = \{x_1, \dots, x_{n-1}\}$ $X' \cap \{x_N\} = \phi$, so

$$\begin{array}{ccccccc} 0 \to & H_0(X') \oplus H_0(\{x_N\}) & \to & H_0(X) & \to 0 \\ & \downarrow \begin{pmatrix} i & 0 \\ 0 & i \end{pmatrix} & & \downarrow i & \\ 0 \to & H_0(X' \times \Delta^n) \oplus H_0(\{x_N\} \times \Delta^n) & \longrightarrow & H_0(X \times \Delta^n) & \to 0 \end{array}$$

Inductively, $\begin{pmatrix} i & 0 \\ 0 & i \end{pmatrix}$ is an isomorphism. For $k \geq 1$,

$$\begin{array}{ccccccc} 0 \to & H_k(X') \oplus H_k(\{x_N\}) & \to & H_k(X) & \to 0 \\ & \cong \downarrow & & \downarrow & \\ 0 \to & H_k(X' \times \Delta^n) \oplus H_k(\{x_N\} \times \Delta^n) & \to & H_k(X \times \Delta^n) & \to 0 \end{array}$$

Everything is 0, so isomorphic. Thus $P(0)$ is true.

We then need to show $P(m,k) \Rightarrow P(m,k+1)$. Suppose $\dim X \leq m$ and X has exactly $k+1$ simplices of dimension n, $\sigma_1, \dots, \sigma_{k+1}$. Then $X = (X^{(m-1)} \cup \sigma_1 \cup \dots \cup \sigma_k) \cup \sigma_{k+1}$,

$X' = X^{(m-1)} \cup \sigma_1 \cup \dots \cup \sigma_k$. $X = X' \cup \sigma_{k+1}$, $\sigma_{k+1} \cong \Delta^m$ with $\dim(X' \cap \sigma_{k+1}) \leq m-1$. This gives us:

$$\begin{tikzcd}
H_r(X'\cap\sigma_{k+1}) \arrow[r] \arrow[d, "(\mathrm{I})\ i_*"] & H_r(X') \oplus H_r(\sigma_{k+1}) \arrow[r] \arrow[d, "(\mathrm{II})\ \begin{pmatrix} i_* & 0 \\ 0 & i_* \end{pmatrix}"] & H_r(X) \arrow[r] \arrow[d, "(\mathrm{III})\ i_*"] & H_{r-1}(X'\cap\sigma_{k+1}) \arrow[r] \arrow[d, "(\mathrm{IV})\ i_*"] & H_{r-1}(X') \oplus H_{r-1}(\sigma_{k+1}) \arrow[d, "(\mathrm{V})\ \begin{pmatrix} i & 0 \\ 0 & i \end{pmatrix}"] \\
H_r(X'\cap\sigma_{k+1} \times \Delta^n) \arrow[r] & H_r(X'\times\Delta^n) \oplus H_r(\sigma_{k+1}\times\Delta^n) \arrow[r] & H_r(X\times\Delta^n) \arrow[r] & H_{r-1}(X'\cap\sigma_{k+1}\times\Delta^n) \arrow[r] & H_{r-1}(X'\times\Delta^n) \oplus H_{r-1}(\sigma_{k+1}\times\Delta^n)
\end{tikzcd}$$

(I) and (IV) are isomorphisms, because $\dim(X' \cap \sigma_{k+1}) \leq m-1$.

(II) and (V) are isomorphisms by induction hypothesis (for X') and fact that $H_*(\sigma_{k+1} \times \Delta^n) \cong H_*(\Delta^m \times \Delta^n) \cong H_*(I^{m+n}) \cong H_*(pt)$.

By Five Lemma, (III) is an isomorphism, q.e.d. So for any finite complex X, $H_*(X) \cong H_*(X \times \Delta^n)$.

Remark – $H_*(X \times I) \cong H_*(X)$ – "homotopy invariance".

(II) For this, we require addition and multiplication properties of χ:
(addition) If $X = X_+ \cup X_-$, then $\chi(X) + \chi(X_+ \cap X_-) = \chi(X_+) + \chi(X_-)$.
(multiplication) $\chi(X \times Y) = \chi(X)\chi(Y)$. [target].

Additive property: get exact sequence of chain complexes $0 \to C_*(X_+ \cap X_-) \to C_*(X_+) \oplus C_*(X_-) \to C_*(X) \to 0$. For each r, we get exact sequence $0 \to C_r(X_+ \cap X_-) \to C_r(X_+) \oplus C_r(X_-) \to C_r(X) \to 0$.

Thus, $\dim C_r(X) + \dim C_r(X_+ \cap X_-) = \dim C_r(X_+) + \dim C_r(X_-)$. Take alternating sum: $\sum_r (-1)^r \dim C_r(X) + \sum_r (-1)^r \dim C_r(X_+ \cap X_-) = \sum_r (-1)^r \dim C_r(X_+) + \sum_r (-1)^r \dim C_r(X_-)$

$\Rightarrow \chi(X) + \chi(X_+ \cap X_-) = \chi(X_+) + \chi(X_-)$.

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We have previously shown that $H_*(X \times \Delta^n) \cong H_*(X)$ for finite X $\Rightarrow$ $\chi(X \times \Delta^n) = \chi(X)$. Moreover, $\chi(X)\chi(Y)$, the proof of which we will continue below:

For X, let $Q(n,m)$ be the statement that $\chi(X \times Y) = \chi(X)\chi(Y)$, when Y is a complex with $\dim Y \leq n$, in which Y has exactly m simplices of dimension n. Observe that $Q(n,1)$ is true:

$\chi(X \times \Delta^n) = \chi(X) = \chi(X)\chi(\Delta^n)$ as $\chi(\Delta^n) = 1$. Then put $Q(n) = \bigwedge_{m=0}^{\infty} Q(n,m)$. Observe that $Q(n,0) \equiv Q(n-1)$. Hence it suffices to prove that $Q(n,m) \Rightarrow Q(n,m+1)$

assuming that $Q(n-1)$ is true. Assume that $Q(n,m)$ and $Q(n-1)$ are true. Then let Y be a complex of dim $\leq n$ with exactly $m+1$ n-simplices. Then $Y = Y^{(n-1)} \cup \sigma_1 \cup \dots \sigma_m \cup \sigma_{m+1}$

where $\sigma_i \sim \Delta^n$. Define $Y' = Y^{(n-1)} \cup \sigma_1 \cup \dots \cup \sigma_m$, so $I = Y' \cap \sigma_{m+1} \subset Y^{(n-1)}$. Then $Y = Y' \underset{I}{\cup} \sigma_{m+1}$, so $X \times Y = (X \times Y') \underset{X \times I}{\cup} (X \times \sigma_{m+1})$. By addition property,

$\chi(X \times Y) + \chi(X \times I) = \chi(X \times Y') + \chi(X \times \sigma_{m+1})$. By hypotheses given, $\chi(X \times Y) + \overbrace{\chi(X)\chi(I)}^{Q(n-1)} = \overbrace{\chi(X)\chi(Y')}^{Q(n,m)} + \chi(X)\chi(\overset{\sigma_{m+1} \cong \Delta^n}{\sigma_{m+1}})$ — ①. However, we also

have $\chi(Y) + \chi(I) = \chi(Y') + \chi(\sigma_{m+1})$. Multiplying through by $\chi(X)$ gives $\chi(X)\chi(Y) + \chi(X)\chi(I) = \chi(X)\chi(Y') + \chi(X)\chi(\sigma_{m+1})$ — ②. Comparing ①, ②

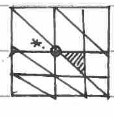
we can cancel like terms to give $\chi(X \times Y) = \chi(X)\chi(Y)$, q.e.d.

The product formula for χ is definitely weaker than the Künneth Theorem, which states that $H_n(X \times Y) \cong \bigoplus_{r=0}^{k} H_r(X) \otimes H_{n-r}(Y)$. Even so, χ can still distinguish between interesting spaces: e.g. Take $S^n = \{\underline{x} \in \mathbb{R}^{n+1} : \sum_{i=1}^{n+1} x_i^2 = 1\}$. Then compare S^4 and $S^2 \times S^2$. From the point of view of analysis, S^4 and $S^2 \times S^2$ are virtually identical – compact, metricable, connected, locally $\mathbb{R}^4$. However, $\chi(S^4) = 2$ but $\chi(S^2 \times S^2) = \chi(S^2)\chi(S^2) = 2 \times 2 = 4$. Hence, they are not identical.

Classification Theorem

Recall –
Let Σ be a finite connected simplicial surface. Then if (i) Σ is orientable, then $\Sigma \sim S^2$ or $\Sigma \sim T^2$ or $\Sigma \sim \overbrace{T^2 \# \cdots \# T^2}^{g}$ where $g \geq 2$. If (ii) Σ is non-orientable, then $\Sigma \sim \mathbb{RP}^2$ or $\Sigma \sim \overbrace{\mathbb{RP}^2 \# \cdots \# \mathbb{RP}^2}^{g}$ for some $g \geq 2$.

We now move in to prove this theorem, after providing a small definition. Consider T^2 with indicated point $*$. $Lk(*) =$  $\cong S^1$ is a circle.

T^2. 

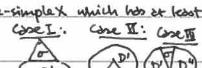
Now remove the shaded triangle, then $Lk(*) =$  is an arc. (subdivided Δ^1).

$\boxed{\text{Definition}}$ A bounded surface X is a simplicial complex in which $Lk_X(v) \sim S^1$ (circle) or $Lk_X(v) \sim \Delta^1$ (arc).

$\boxed{\text{Proposition}}$ Let D be a finite connected bounded surface in which (i) $\partial D \sim S^1$, (ii) $H_1(D : \mathbb{F}_2) = 0$. Then $D \sim \Delta^2$.

Proof – By induction on number of 2-simplices, N. If $N=1$, nothing to prove. Suppose true for $N-1$ and let D have N 2-simplices. Let σ be a 2-simplex which has at least one edge in ∂D. If all three edges are in ∂D, then $D = \Delta^2$ so we have nothing to prove. So assume σ has 1 or 2 edges in ∂D.

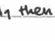
Case I: Case II: Case III



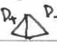
$D = D' \cup \sigma$ $D = D' \cup \sigma \cup D''$

We split cases: Case I, Case II with opposite vertex of edge in boundary lying in interior, Case III: where $D = D' \cup \sigma \cup D''$.

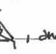
No other cases exist as $H_1(D : \mathbb{F}_2) = 0$. In Case I, observe that $H_1(D' : \mathbb{F}_2) = 0$ by Mayer-Vietoris sequence: $\sigma \cap D' \cong \Delta_1$ then we have

$H_2(D) \to \overset{0}{H_1(D' \cap \sigma)} \to \overset{H_1(D') \oplus 0}{H_1(D') \oplus H_1(\sigma)} \to \overset{0}{H_1(D)}$, so $H_1(D) = 0$. D' has only $N-1$ 2-simplices, so $D' \sim \Delta^2$. $\Rightarrow$ $D \sim$  $\sim \Delta^2$ by subdivision.

Case II is almost the same, where here $I = D' \cap \sigma$ is a subdivided interval $\Rightarrow H_k(I) = 0$ $\forall k \geq 1$. So we focus on the final result, Case III:

Write $D_+ = D'$, $D_- = \sigma \cup D''$. Then $D_+ \cap D_-$ is a single edge. Then $\overset{0}{H_1(D_+ \cap D_-)} \to H_1(D_+) \oplus H_1(D_-) \to \overset{0}{H_1(D)}$, so $H_1(D_+) \cong H_1(D_-) \cong 0$. By induction hypothesis, $D_+ \sim \Delta^2$, $D_- \sim \Delta^2$. Then $D =$  $\sim \Delta^2$ by subdivision, q.e.d.

$\boxed{\text{Theorem}}$ Let Σ be a finite connected surface s.t. $H_1(\Sigma : \mathbb{F}_2) = 0$. Then $\Sigma \sim S^2$.

Proof – Let σ be a 2-simplex of Σ and write $D = \Sigma - \sigma$. Then D is a bounded surface with $\partial D = \partial\sigma = S^1(3)$. Claim $H_1(D : \mathbb{F}_2) = 0$. $\Sigma = D \underset{I}{\cup} \sigma$, $I = D \cap \sigma \cong S^1$. Then we get the exact sequence $\overset{0}{H_2(D)} \oplus \overset{0}{H_2(\sigma)} \to H_2(\Sigma) \to H_1(S^1) \to H_1(D) \oplus H_1(\sigma) \to \overset{0}{H_1(\Sigma)}$. The proof that $H_2(\Sigma : \mathbb{F}_2)$ also shows that $H_2(D) = 0$, as we can never cancel ∂D in expression for $H_2(D)$. Then $0 \to \mathbb{F}_2 \to \mathbb{F}_2 \to H_1(D) \oplus 0 \to 0$, and by Whitehead's Lemma, $H_1(D) = 0$. Hence $D \sim \Delta^2 \sim$ , and we have $\Sigma = D \cup \sigma \sim S^2$. We identify σ with bottom 2-simplex .

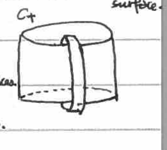
Remark – This has an analogue in dimension 3: If X is finite simplicial 3-manifold in which every loop is contractible within X, then $X \sim S^3$. This is the famous Poincaré conjecture

An even more general analogue is this: If X is any compact manifold in which every loop is contractible and $H_*(X) \cong H_*(S^n)$. Then $X \sim S^n$. (n-dim Poincaré conjecture)

This was solved for dimension ≥ 5 by 1961., dimension 4 by 1976. (Friedman), dimension 3 by 2003 (Perelmann).

Proof for classification Theorem –

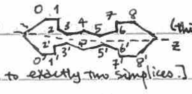
STEP 0: $H_1(\Sigma) = 0 \Rightarrow \Sigma \sim S^2$. STEP 1: let Σ be a finite connected surface which contains no Möbius band, then $H_1(\Sigma : \mathbb{F}_2) \neq 0 \Rightarrow \Sigma \sim \Sigma' \# T^2$ where Σ' is a connected surface.



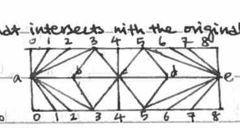
the boundary of $\{T^2 - \text{2-simplex}\}$ is connected. let this be C_+, then $\partial C_+ \sim S^1$. Proof of STEP 1 – $H_1(\Sigma) \neq 0$. since we work over $\mathbb{F}_2$, an element of $C_1(\Sigma : \mathbb{F}_2)$ is simply a collection of 1-simplices. Take a non-zero element $z \in H_1(\Sigma : \mathbb{F}_2)$, which is represented by the smallest possible number of 1-simplices. $z \in C_1(\Sigma : \mathbb{F}_2)$, $\partial z = 0$.

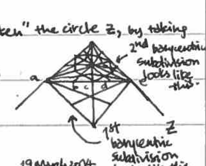
Then z is an imbedded S^1 contained in Σ: z is a connected 1-complex, which cannot have any free edges (i.e.) otherwise $\partial z \neq 0$. [one circle as z is minimal]

It must be a collection of circles [as something like is not minimal]. so $z \sim$ $= S^1(n)$, $n \geq 3$. (stage 2) then we seek to "thicken" the circle z, by taking the second barycentric subdivision. Then we follow the external outline of every small 2-simplex that intersects with the original (z) edge, i.e. in this case z ——— $\mapsto$



(thickened edge)...... then,





[Note – since edge lies in a surface, it belongs to exactly two simplices.] Locally, this thickens out to a strip (as seen on triangulation on right). Complete thickening C must have disconnected boundary, otherwise Σ contains a Möbius band,

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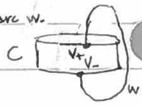
which would produce a contradiction. (stage 3) Now consider $\Sigma - C$. $\partial(\Sigma - C) = S^1 \sqcup S^1$ disconnected, but $\Sigma - C$ is connected. otherwise, if $\Sigma - C = X_+ \cup X_-$, where $X_+ \cap X_- = \phi$, then

$[z'] = [z] \in H_1(\Sigma : \mathbb{F}_2)$, so $z - z'$ is bounded by z, z', $z' = \partial(X_+ \cup C)$

so $[z'] = 0$ in $H_1[\Sigma_1 : \mathbb{F}_2)$, which is a contradiction as $z \neq 0$ in H_1. Choose v_+, v_- on different components of ∂C and join them by an arc w.



Then thicken w out, such that $C_+ = C \cup$ thickening of w. claim $C_+ = T^2 - 1$ disc $=$. We have written $\Sigma = (\Sigma - C_+) \cup C_+$,

$\partial(\Sigma - C_+) = S^1 = \partial C_+$. Put $\Sigma' = (\Sigma - C_+) \underset{\partial = \partial}{\cup} \Delta^2$, $T^2 = C_+ \underset{\partial = \partial}{\cup} \Delta^2$. Hence, $\Sigma \sim \Sigma' \# T^2$, q.e.d.

Thus far our flowchart looks like this: Σ is a finite connected surface, contains no Möb. ⟶ $\boxed{\text{Is } H_1(\Sigma : \mathbb{F}_2) = 0?}$ —YES→ $\boxed{\Sigma \sim S^2}$; —NO→ $\boxed{\Sigma \sim \Sigma' \# T^2}$ —LOOP 2→ set $\Sigma = \Sigma'$.

let $\Sigma = \Sigma' \# T^2$. $\chi(\Sigma) = \chi(\Sigma') + \chi(T^2) - 2 = \chi(\Sigma') - 2$. But $H_0 \Sigma \cong H_0 \Sigma' = \mathbb{F}_2$, $H_2 \Sigma \cong H_2 \Sigma' = \mathbb{F}_2$. Hence $\dim H_1(\Sigma' : \mathbb{F}_2) = \dim H_1(\Sigma : \mathbb{F}_2) - 2$. So we can only have finitely many iterations of the loop, picking up a copy of T_2 each time. Hence, we have:

Corollary: If Σ is a finite connected surface which contains no Möb, then (i) $\Sigma \sim S^2$ OR (ii) $\Sigma \sim T^2$ (loop once) OR (iii) $\Sigma \sim \overbrace{T^2 \# \cdots \# T^2}^{g}$ (go around loop g times).

$\boxed{\text{Corollary}}$ let Σ be a finite connected surface. then Σ is orientable $\Leftrightarrow$ Σ contains no Möb.

Σ is a finite connected surface. ⟶ $\boxed{\Sigma \text{ contains Möb?}}$ —YES→ $\boxed{\Sigma = \Sigma' \# \mathbb{R}P^2}$ —LOOP 1→ identify $\Sigma = \Sigma'$; —NO→ (connects to above flowchart)

Proof – ($\Rightarrow$) Trivial ($\Leftarrow$) By above, $\Sigma \sim S^2, T^2$ or $\overbrace{T^2 \# \cdots \# T^2}^{g}$

$h_1 = h_1 - 1$ where $h_1 = \dim_{\mathbb{F}_2}(H_1(\Sigma : \mathbb{F}_2))$.

(cont'd): Suppose Σ does contain Möb. $\Sigma' = (\Sigma - \text{Möb}) \underset{\partial = \partial}{\cup} \text{Möb}$. Put $\Sigma' = (\Sigma - \text{Möb}) \underset{\partial = \partial}{\cup} \Delta^2$, $\mathbb{R}P^2 = \text{Möb} \underset{\partial = \partial}{\cup} \Delta^2$, so $\Sigma = \Sigma' \# \mathbb{R}P^2$.

Explanation of LOOP 1: If Σ contains a Möb, $\Sigma = (\Sigma - \text{Möb}) \underset{\partial = \partial}{\cup} (\text{Möb})$. Define $\Sigma' = (\Sigma - \text{Möb}) \underset{\partial = \partial}{\cup} D^2$ and $\mathbb{R}P^2 = (\text{Möb}) \underset{\partial = \partial}{\cup} D^2$ so $\Sigma \sim \Sigma' \# \mathbb{R}P^2$.

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Moreover, $\chi(\Sigma) = \chi(\Sigma' \# \mathbb{R}P^2) = \chi(\Sigma') + \chi(\mathbb{R}P^2) - 2 = \chi(\Sigma') + 1 - 2 = \chi(\Sigma') - 1$. $\chi(\Sigma) = h_0 - h_1 + h_2 = 2 - h_1$. $\chi(\Sigma') = h_0' - h_1' + h_2' = 2 - h_1'$

$\Rightarrow 2 - h_1 = (2 - h_1') - 1 \Rightarrow h_1 = h_1' + 1$, $h_1' = h_1 - 1$, so we can only go around LOOP 1 finitely many times.

$\boxed{\text{Proposition}}$ If Σ is a finite connected surface, then $\Sigma \sim S^2 \# \Sigma$.

Proof – $S^2 \# \Sigma$ is simply a subdivision of Σ at a principal simplex, q.e.d.

(cont'd): Consider the individual cases: (I) $\Sigma \sim S^2$ $\left(\begin{matrix}\text{0 times LOOP 1} \\ \text{0 times LOOP 2}\end{matrix}\right)$. (II) $\Sigma \sim S^2 \# \overbrace{\mathbb{R}P^2 \# \cdots \# \mathbb{R}P^2}^{b} \sim \overbrace{\mathbb{R}P^2 \# \cdots \# \mathbb{R}P^2}^{b}$ $\left(\begin{matrix}\text{b times around LOOP 1} \\ \text{0 times around LOOP 2}\end{matrix}\right)$.

(III) $\Sigma \sim S^2 \# \overbrace{T^2 \# \cdots \# T^2}^{a} \sim \overbrace{T^2 \# \cdots \# T^2}^{a}$ $\left(\begin{matrix}\text{0 times LOOP 1} \\ \text{a times LOOP 2}\end{matrix}\right)$. (IV) seemingly general case: $\Sigma \sim S^2 \# \overbrace{T^2 \# \cdots \# T^2}^{a} \# \overbrace{\mathbb{R}P^2 \# \cdots \# \mathbb{R}P^2}^{b}$ $\left(\begin{matrix}\text{b times LOOP 1} \\ \text{a times LOOP 2}\end{matrix}\right)$.

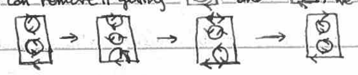
where $a, b > 0$ for all cases. For case (IV), we use the following theorem to simplify working – such that case (IV) reduces to case (II).

$\boxed{\text{Theorem}}$ $T^2 \# \mathbb{R}P^2 \sim \mathbb{R}P^2 \# \mathbb{R}P^2 \# \mathbb{R}P^2$

Remark – Accepting this theorem gives the classification theorem (given on previous page – cross-refer), as $\overbrace{T^2 \# \cdots \# T^2}^{a} \# \overbrace{\mathbb{R}P^2 \# \cdots \# \mathbb{R}P^2}^{b} \sim \overbrace{\mathbb{R}P^2 \# \cdots \# \mathbb{R}P^2}^{2a+b}$

Proof – consider $D^2 \underset{\partial = \partial}{\cup}$ Möb. Then puncture Möb as such . Then take a direction and follow it through to get the schematic representation then $D^2 \underset{\partial = \partial}{\cup}$ $= \mathbb{R}P^2 \# T^2$. then $\mathbb{R}P^2 \# K^2 \sim \mathbb{R}P^2 \# \mathbb{R}P^2 \# \mathbb{R}P^2$; and taking same Möbius band, $\mathbb{R}P^2 \# K^2$ has a schematic representation as on right. Since disc D^2 is only imaginary, we can remove it getting A and B. We can deform them into each other:

A = more reference line down to get tangent →



Algebraic Mayer-Vietoris Theorem.

$\boxed{\text{Theorem}}$ let $0 \to A_* \xrightarrow{i} B_* \xrightarrow{p} C_* \to 0$ be an exact sequence of chain complexes. Then $\exists$ homomorphisms $\delta : H_{n+1}(C) \to H_n(A)$ such that the following sequence is exact for all n:

$$H_{n+1}(B) \xrightarrow[(p_*)]{H_{n+1}(p)} H_{n+1}(C) \xrightarrow{\delta} H_n(A) \xrightarrow[(i_*)]{H_n(i)} H_n(B) \xrightarrow{H_n(p)} H_n(C) \xrightarrow{\delta} H_{n-1}(A) \xrightarrow[(i_*)]{H_{n-1}(i)} H_{n-1}(B).$$

Proof – (not examinable) Consider the commutative diagram as on right: Hypothesis – each such diagram is a commutative diagram of linear maps in which rows are exact, $\partial_n^K \partial_{n+1}^K = 0$ for $K = A, B, C$. We begin by constructing maps δ (snake lemma). Start with diagram:

$$\begin{tikzcd}
0 \arrow[r] & A_{n+1} \arrow[r, "i_{n+1}"] \arrow[d, "\partial_{n+1}^A"] & B_{n+1} \arrow[r, "p_{n+1}"] \arrow[d, "\partial_{n+1}^B"] & C_{n+1} \arrow[r] \arrow[d, "\partial_n^C"] & 0 \\
0 \arrow[r] & A_n \arrow[r, "i_n"] \arrow[d, "\partial_n^A"] & B_n \arrow[r, "p_n"] \arrow[d, "\partial_n^B"] & C_n \arrow[r] \arrow[d, "\partial_n^C"] & 0 \\
0 \arrow[r] & A_{n-1} \arrow[r, "i_{n-1}"] & B_{n-1} \arrow[r, "p_{n-1}"] & C_{n-1} \arrow[r] & 0
\end{tikzcd}$$

$H_{n+1}(C) = Z_{n+1}(C) / \mathrm{Im}(\partial_{n+2}^c)$ where $Z_{n+1}(C) = \{z \in C_{n+1} ; \partial_{n+1}^C(z) = 0\}$. $H_n(A) = Z_n(A) / \mathrm{Im}(\partial_{n+1}^A)$. $Z_n(A) = \{a \in A_n : \partial_n^A(a) = 0\}$.

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then observe the diagram on the right: the idea is to define $\delta(z) = i_n^{-1} \partial^B_{n+1} p^{-1}_{n+1}(z)$. We first define $\delta: Z_{n+1}(C) \to H_n(A) = \frac{Z_n(A)}{\text{Im}\, \partial^A_{n+1}}$

$$\begin{array}{ccccccccc} & & & & B_{n+2} & \xrightarrow{p_{n+2}} & C_{n+2} & \to & 0 \\ & & & & \downarrow \partial & & \downarrow & & \\ 0 & \to & A_{n+1} & \xrightarrow{i_{n+1}} & B_{n+1} & \xrightarrow{p_{n+1}} & C_{n+1} & \to & 0 \\ & & \downarrow \partial & & \downarrow \partial^B_{n+1} & & \downarrow & & \\ 0 & \to & A_n & \xrightarrow{i_n} & B_n & \xrightarrow{p_n} & C_n & \to & 0 \\ & & \downarrow \partial^A_n & & \downarrow \partial & & & & \\ 0 & \to & A_{n-1} & \xrightarrow{i_{n-1}} & B_{n-1} & & & & \end{array}$$

Take $z \in Z_{n+1}(C)$ so $\partial_{n+1}(z) = 0$. By exactness, $p_{n+1}: B_{n+1} \to C_{n+1}$ surjective. so choose $b \in B_{n+1}: p_{n+1}(b) = z$.

then $\partial_c \circ p_{n+1}(b) = \partial^c(z) = 0$, but $\partial^c \circ p_{n+1}(b) = p_n \partial^B_{n+1}(b) = 0$, so $\partial^B_{n+1}(b) \in \text{Ker}(p_n) = \text{Im}(i_n)$. so choose $a \in A_n$,

such that $i_n(a) = \partial^B_{n+1}(b)$. The idea is to associate $z \rightsquigarrow a$. Calculate $\partial^A_n(a)$: $i_{n-1}\partial^A_n(a) = \partial^B_n i_n(a) \Rightarrow$

$i_{n-1}\partial^A_n(a) = \partial^B_n \partial^B_{n+1}(b) = 0$. Since i_{n-1} is injective, $\partial^A_n(a) = 0$. so $a \in Z_n(A)$. We would like to say $z \mapsto a$ is a mapping, but it isn't! However, we do know:

Proposition The above defines a mapping $\delta: Z_{n+1}(C) \to H_n(A)$, $\delta(z) = [a] = [i_n^{-1} \partial_{n+1} p^{-1}_{n+1}(z)]$, $[a] \in H_n(A)$.

Proof – Apparently $[a]$ depends upon a specific choice of $b \in B_{n+1}$ s.t. $p_{n+1}(b) = z$. We must show that if we take another choice of b, then homology class of $[a]$ does not change.

i.e. we have $b, b' \in B_{n+1}$ both satisfying $p_{n+1}(b) = z = p_{n+1}(b')$. We have $a, a' \in A_n$ such that $i_n(a) = \partial^B_{n+1}(b)$, $i_n(a') = \partial^B_{n+1}(b')$. so $i_n(a-a') = \partial^B_{n+1}(b-b')$ and also

$p_{n+1}(b-b') = p_{n+1}(b) - p_{n+1}(b') = z - z = 0$. then $b - b' \in \text{Ker}(p_{n+1}) = \text{Im}(i_{n+1})$, so $\exists\, \alpha \in A_{n+1}$ s.t. $i_{n+1}(\alpha) = b - b'$. so now, $\partial^B_{n+1} \circ i_{n+1}(\alpha) = \partial^B_{n+1}(b-b')$. so,

$i_n \partial^A_{n+1}(\alpha) = i_n(a-a')$. But i_n is injective (exactness), so $a - a' = \partial^A_{n+1}(\alpha) \Rightarrow [a] = [a'] \in H_n(A)$. i.e. $[a]$ is independent of particular choice of b i.e.

$\delta: Z_{n+1}(C) \to H_n(A)$, $\delta(z) = [i_n^{-1} \partial^B_{n+1} p^{-1}_{n+1}(z)]$ is well-defined // q.e.d.

(cont'd) – It is clear that $\delta: Z_{n+1}(C) \to H_n(A)$ is linear because $p_{n+1}, \partial^B_{n+1}, i_n$ are all linear. We really want to define $\delta: H_{n+1}(C) \to H_n(A)$, $H_{n+1}(C) = \frac{Z_{n+1}(C)}{\text{Im}(\partial^c_{n+2})}$.

Proposition If $z \in \text{Im}(\partial^c_{n+2})$, then $\delta(z) = 0$.

Proof – Write $z = \partial^c_{n+2}(\zeta)$, $\zeta \in C_{n+2}$. $p_{n+2}: B_{n+2} \to C_{n+2}$ is surjective so choose $\beta \in B_{n+2}$, $p_{n+2}(\beta) = \zeta$. $p_{n+1}\partial^B_{n+2}(\beta) = \partial^c_{n+2} p_{n+2}(\beta) = \partial^c_{n+2}(\zeta) = z$. so in defining

$a: \partial(z) = [a]$, we can choose $b = \partial^B_{n+2}(\beta)$, so $i_n(a) = \partial^B_{n+1}(b) = \partial^B_{n+1}\partial^B_{n+2}(\beta) = 0$. But i_n is injective, so $a = 0$, $[a] = 0$ i.e. $z \in \text{Im}(\partial^c_{n+2})$ then $\partial(z) = 0$ // q.e.d.

(cont'd) – Final definition: $\delta_*: H_{n+1}(C) \to H_n(A)$, $\delta_*[z] = [i_n^{-1} \partial^B_{n+1} p^{-1}_{n+1}(z)]$. this is well-defined as $[z] = z + \text{Im}\,\partial^c_{n+2}$ and $\delta(\text{Im}\,\partial^c_{n+2}) = 0$. δ_* is linear.

so given an exact sequence of chain complexes, $0 \to A_* \xrightarrow{i} B_* \xrightarrow{p} C_* \to 0$, we have produced a sequence of linear maps $\searrow H_{n+1}(B) \xrightarrow{p_*} H_{n+1}(C) \xrightarrow{\delta_*} H_n(A) \xrightarrow{i_*} H_n(B) \nearrow$.

We still have to prove that the sequence itself is exact.

Remark – the construction of δ_* is natural in the following sense – suppose

$$\begin{array}{ccccccccc} 0 & \to & A_* & \xrightarrow{i} & B_* & \xrightarrow{p} & C_* & \to & 0 \\ & & \downarrow f_A & & \downarrow f_B & & \downarrow f_C & & \\ 0 & \to & A'_* & \xrightarrow{i'} & B'_* & \to & C'_* & \to & 0 \end{array}$$

is a diagram, then it commutes.

Likewise we have

$$\begin{array}{ccccccccccc} \to & H_{n+1}(B) & \to & H_{n+1}(C) & \xrightarrow{\delta_*} & H_n(A) & \to & H_n(B) & \to & H_n(C) & \to \\ & \downarrow H_{n+1}(f_B) & & \downarrow H_{n+1}(f_C) & & \downarrow H_n(f_A) & & \downarrow H_n(f_B) & & \downarrow H_n(f_C) & \\ \to & H_{n+1}(B') & \to & H_{n+1}(C') & \to & H_n(A') & \to & H_n(B') & \to & H_n(C') & \to \end{array}$$

26 March 2014
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Let $0 \to A_* \xrightarrow{i} B_* \xrightarrow{p} C_* \to 0$ be exact, constructed, $\delta: H_{n+1}(C) \to H_n(A)$. We get a sequence as follows:

$H_{n+1}(B) \xrightarrow{p} \underset{(\text{I})}{H_{n+1}(C)} \xrightarrow{\delta} \underset{(\text{II})}{H_n(A)} \xrightarrow{i} \underset{(\text{III})}{H_n(B)} \xrightarrow{p} H_n(C)$. We then claim that this sequence is exact, and we demonstrate this by parts.

Exactness at (I): (a) $\delta p = 0$ (b) $\text{Ker}(\delta) \subset \text{Im}(p)$. For (a), let $[b] \in H_{n+1}(B)$ so $b \in B_{n+1}$ and $\partial b = 0$. $\delta p[b] = [a]$ when $a \in Z_n(A)$ satisfies $i_n(a) = \partial(?)$ when

$p(?) = p(b)$. So we can take $? = b$, and $i_n(a) = \partial b = 0$. i_n is injective, so $a = 0$ //. For (b), let $[z] \in H_{n+1}(C)$ be such that $\delta[z] = 0$ so $z \in Z_{n+1}(C)$ i.e. $\partial z = 0$.

$\delta[z] = 0$ means that $i_n(\partial\alpha) = \partial b$ where $p(b) = z$, $\partial i_{n+1}\alpha = \partial b$ so $\partial(b - i_{n+1}(\alpha)) = 0 \Rightarrow b - i_{n+1}(\alpha) \in Z_{n+1}(B)$ and $p(b - i_{n+1}(\alpha)) = p(b) - pi(\alpha) = p(b) - 0$; so

$p(b - i_{n+1}(\alpha)) = z \Rightarrow p[b - i_{n+1}(\alpha)] = [z]$ i.e. $\delta[z] = 0 \Rightarrow [z] \in \text{Im}(p)$ // so q.e.d (a), q.e.d. (b) $\Rightarrow$ q.e.d. (I) exact.

For exactness at (II), (a) $i\delta = 0$ (b) $\text{Ker}(i) \subset \text{Im}(\delta)$. For (a), let $[z] \in H_{n+1}(C)$ (so $z \in C_{n+1}$, $\partial z = 0$) $i\delta[z] = i[i^{-1}\partial p^{-1}(z)] = [\partial p^{-1}(z)] = 0 \because [\partial(?)] = 0$ //.

For (b), suppose $[a] \in H_n(A)$ satisfies $i[a] = [0] = [\partial ?]$, so $a \in A_n$ and $\partial a = 0$, and $i(a) = \partial b$. ($b \in B_{n+1}$). Put $z = p(b) \in C_{n+1}$. $\partial(z) = \partial p(b) = p\partial b = pi(a) = 0$ because

$pi = 0$. Thus, we have $z \in Z_{n+1}(C)$ and $i(a) = \partial b$ where $p(b) = z$. By definition of δ, we see that $\delta[z] = [a]$, so $i[a] = 0 \Rightarrow [a] \in \text{Im}(\delta)$ // $\Rightarrow$ q.e.d. (II) exact.

For exactness at (III), (a) $p \circ i = 0$, (b) $\text{Ker}(p) \subset \text{Im}(i)$. (a) is trivial because $A_n \xrightarrow{i_n} B_n \xrightarrow{p_n} C_n$ is already 0 as $p_n \circ i_n = 0$. For (b), suppose that $[b] \in H_n(B)$ and $p[b] = 0$

so $b \in B_n$ and $\partial b = 0$. then $p_n(b) = \partial(?)$ for $? \in C_{n+1}$. Put $z = ?$ i.e. $p_n(b) = \partial(z)$ where $z \in C_{n+1}$. p_{n+1} is surjective, so choose

$$\begin{array}{ccccc} B_{n+1} & \xrightarrow{p_{n+1}} & C_{n+1} & & z \\ \downarrow & & \downarrow & & \downarrow \\ B_n & \xrightarrow{p_n} & C_n & & \\ b & \longrightarrow & & & \end{array}$$

$\beta \in B_{n+1}$ s.t. $p_{n+1}(\beta) = z$, $p_n(b - \partial\beta) = p_n(b) - p_n\partial(\beta) = p_n(b) - \partial p_{n+1}(\beta) = \partial z - \partial z = 0$. So $\exists\, a \in A_n$ such that $i_n(a) = b - \partial\beta$.

Then $\partial i_n(a) = \partial b - \partial\partial\beta = 0 - 0$. So $i_{n-1}(\partial a) = 0$, but i_{n-1} is injective so $\partial a = 0 \Rightarrow a \in Z_n(A)$ and $i_n(a) = b - \partial\beta$. So $i[a] = [b - \partial\beta] = [b]$, so

$p[b] = 0 \Rightarrow \exists\, [a] \in H_n(A) \Rightarrow i[a] = [b]$. q.e.d. (b), q.e.d. (III) exact. $\Rightarrow$ // q.e.d.

How does this all relate to the geometric form of the theorem?

Definition If $A_* = (A_r, \partial^A_r)$, $C_* = (C_r, \partial^c_r)$, then define the direct sum of chain complexes $A_* \oplus C_* = \left(A_r \oplus C_r, \begin{pmatrix} \partial^A_r & 0 \\ 0 & \partial^c_r \end{pmatrix}\right)$. i.e. $\partial: A_{n+1} \oplus C_{n+1} \to A_n \oplus C_n$, $\partial\begin{pmatrix} a \\ c \end{pmatrix} = \begin{pmatrix} \partial^A_{n+1}(a) & 0 \\ 0 & \partial^c_{n+1}(\end{pmatrix}$

Corollary $H_n(A_* \oplus C_*) = H_n(A) \oplus H_n(C)$.

Proof – We have an exact sequence of chain complexes $0 \to A_* \xrightarrow{i^A} A_* \oplus C_* \xrightarrow{p^c} C_* \to 0$. However the boundary map $\delta: H_{n+1}(C_*) \to H_n(A)$ is necessarily equal

to 0 for this exact sequence. $A_{n+1} \xrightarrow{i^A} A_{n+1} \oplus C_{n+1} \xrightarrow{p^c} C_{n+1}$. choose $b = \begin{pmatrix} 0 \\ z \end{pmatrix}$ to hit $A_n \oplus C_n$. $\partial b = \begin{pmatrix} 0 \\ \partial z \end{pmatrix}$. If $\partial z = 0$, $\partial b = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$. Then $i^A(0) = \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \partial b$

where the maps are... $\rightsquigarrow$

$$\begin{array}{ccccc} A_{n+1} & \xrightarrow{i^A} & A_{n+1} \oplus C_{n+1} & \xrightarrow{p^c} & C_{n+1} \\ \downarrow \partial^A & & \downarrow \begin{pmatrix} \partial^A & 0 \\ 0 & \partial^c \end{pmatrix} & & \downarrow \partial^c \\ A_n & \longrightarrow & A_n \oplus C_n & \longrightarrow & C_n \end{array}$$

$\rightsquigarrow$ $i^A: A_n \to A_n \oplus C_n$, $i^A(a) = \begin{pmatrix} a \\ 0 \end{pmatrix}$; $p^c: A_n \oplus C_n \to C_n$, $p^c\begin{pmatrix} \alpha \\ \gamma \end{pmatrix} = \gamma$. So long exact sequences splits into SES:

$H_{n+1}(A) \xrightarrow{i^A} H_{n+1}(A \oplus C) \xrightarrow{p^c} H_{n+1}(C) \xrightarrow{\delta = 0} H_n(A) \xrightarrow{i^A} H_n(A \oplus C) \xrightarrow{p^c} H_n(C) \xrightarrow{\delta = 0}$ (with $\searrow 0 \nearrow$ under each $\delta = 0$; the part from 0 to 0 braced as "short exact sequence."). So for each n we have an exact sequence $0 \to H_n(A) \xrightarrow{i^A} H_n(A \oplus C) \xrightarrow{p^c} H_n(C)$

However, we can interchange the roles of A, C in order to get the alternative sequence $0 \to H_n(C) \xrightarrow{i^c} H_n(A \oplus C) \xrightarrow{p^A} H_n(A)$ where $p^A: A_n \oplus C_n \to A_n$, $p^A\begin{pmatrix} a \\ c \end{pmatrix} = a$, $i^c: C_n \to A_n \oplus$, $i^c(\gamma) = ($

Then $p^A i^A = \text{Id}_A$, $p^c i^c = \text{Id}_C$. $p^A i^A(a) = p^A\begin{pmatrix} a \\ 0 \end{pmatrix} = a$, $p^c i^c(\gamma) = p^c\begin{pmatrix} 0 \\ \gamma \end{pmatrix} = \gamma$. So we have the commutative diagram –

$$
\begin{array}{ccccccccc}
 & & [a] \longmapsto \begin{pmatrix}[a]\\0\end{pmatrix} & & \begin{pmatrix}[a]\\ [\gamma]\end{pmatrix} \longrightarrow [\gamma] & & & & \\
0 & \longrightarrow & H_n(A) & \longrightarrow & H_n(A) \oplus H_n(C) & \longrightarrow & H_n(C) & \longrightarrow & 0 \\
 & & \downarrow id & & \downarrow \varphi & & \downarrow id & & \\
0 & \longrightarrow & H_n(A) & \longrightarrow & H_n(A) \oplus H_n(C) & \longrightarrow & H_n(C) & \longrightarrow & 0
\end{array}
$$

$\varphi : \begin{pmatrix}[a]\\ [\gamma]\end{pmatrix} = i_A(a) + i_C(\gamma)$. By Five lemma, $\varphi: H_n(A) \oplus H_n(C) \xrightarrow{\cong} H_n(A \oplus C)$ is an isomorphism, q.e.d.

In the geometric case, $X = X_+ \cup X_-$, $\begin{pmatrix}a\\b\end{pmatrix} \mapsto a+b$. Then $a \longmapsto \begin{pmatrix}a\\-a\end{pmatrix}$.

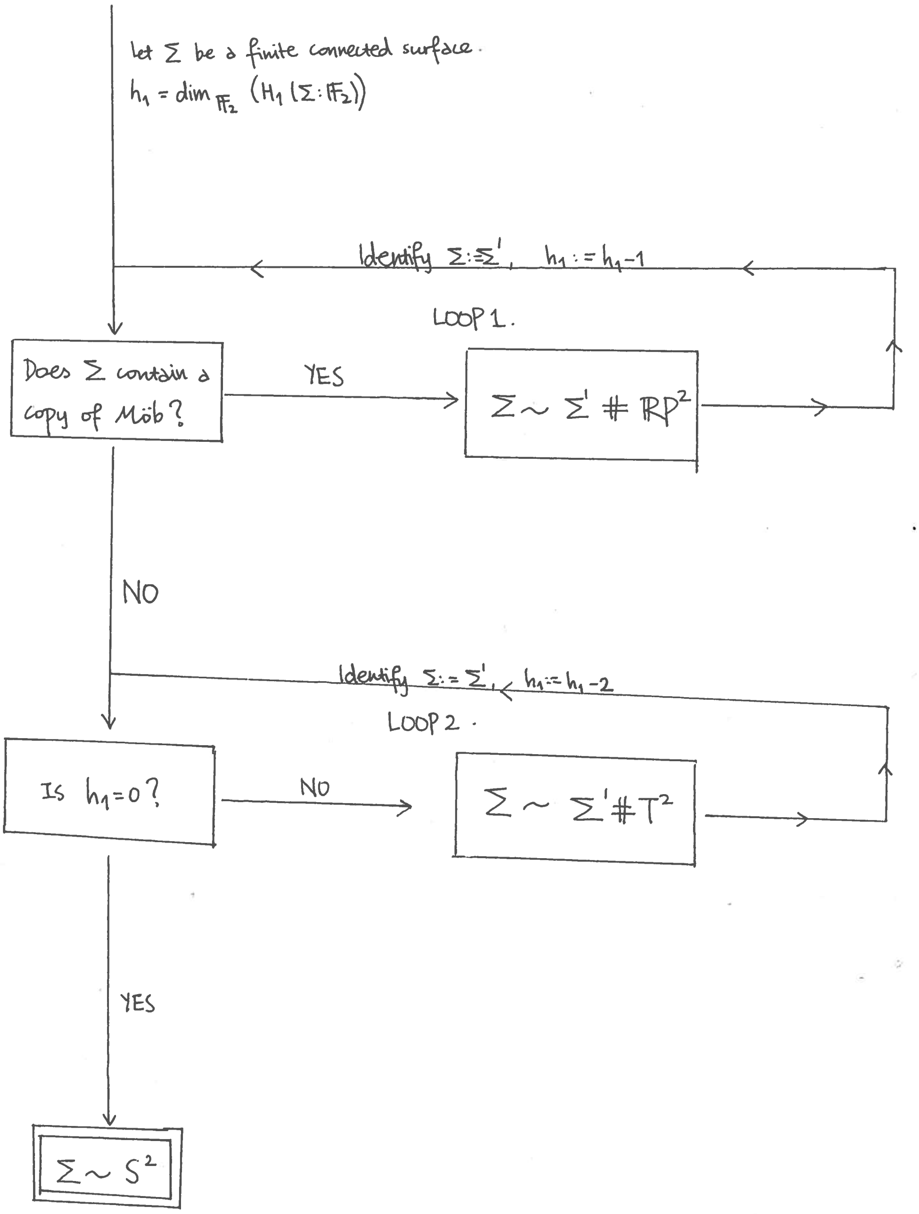
$$0 \longrightarrow C_*(X_+ \cap X_-) \longrightarrow C_*(X_+) \oplus C_*(X_-) \longrightarrow C_*(X) \longrightarrow 0;$$ and $$H_*(C_*(X_+) \oplus C_*(X_-)) \cong H_*(C_*(X_+)) \oplus H_*(C_*(X_-)).$$

END OF SYLLABUS.

END OF COURSE.

Revision class – Apr 28 (Mon) 1200–1300 Roberts G08.

203-16.



Let Σ be a finite connected surface.
h₁ = dim_F₂ (H₁(Σ; F₂))
Identify Σ := Σ', h₁ := h₁ − 1
LOOP 1.
Does Σ contain a copy of Möb?
YES
Σ ∼ Σ' # RP²
NO
Identify Σ := Σ', h₁ := h₁ − 2
LOOP 2.
Is h₁ = 0?
NO
Σ ∼ Σ' # T²
YES
Σ ∼ S²