3203 Algebraic Topology Notes

Based on the 2018 spring lectures by Prof F E A Johnson

MATH 3203

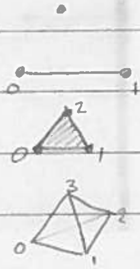
08-01-18 **Algebraic Topology**
(Not to be confused with General
Topology or Point Set Topology)

Prof F Johnson
100% exam.

Poincaré c 1885
Brouwer c 1910
Lefschetz 1925 – 1960 } Algebraic Topology

[Frechet
Hausdorff } General and Point Set Topology]

Book:
A Combinatorial Introduction to Topology – M Henle (Dover)



0-dim
1-dim building block
2-simplex
3-simplex

 I × I
Naive!
 I × I × I

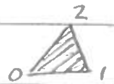
Def^n
By a simplicial complex K we mean a pair
$K = (V_K, S_K)$ where
i) V_K is a set (vertex set)
ii). S_K is a set of finite non-empty subsets of V_K
such that
(a). $\{v\} \in S_K$ for each $v \in V_K$
(b) If $\sigma \in S_K$ and $\tau \subset \sigma$, $\tau \neq \emptyset$, then $\tau \in S_K$

V_K is the vertex set of K
S_K is the simplex set of K

Examples

(Δ^1 or I)

$$\Delta^1 = (\underset{\substack{\uparrow \\ \text{vertex set}}}{\{0,1\}}, \underset{\substack{\uparrow \\ \text{simplex set}}}{\{\{0\}, \{1\}, \{0,1\}\}})$$



$\Delta^2 = (\{0,1,2\}, S_{\Delta^2})$

$S_{\Delta^2} = \{\{0\}, \{1\}, \{2\}, \{0,1\}, \{0,2\}, \{1,2\}, \{0,1,2\}\}.$

Def^n

Let $K = (V_K, S_K)$

If $\sigma \in S_K$ define $\dim(\sigma) = |\sigma| - 1$

So $\dim\{v\} = 0$

$\dim\{v,w\} = 1$

$\dim\{u,v,w\} = 2$

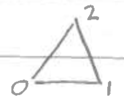
The standard n-simplex Δ^n

$\Delta^n = (V_{\Delta^n}, S_{\Delta^n})$

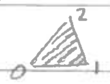
$V_{\Delta^n} = \{0, \dots, n\}$

$S_{\Delta^n} = \{\sigma \in \{0, \dots, n\} : \sigma \neq \emptyset\}$

The basic circle S^1:



$= S^1(3)$ $\Delta^2 =$



middle missing!

$V_{S^1} = \{0,1,2\}$

$S_{S^1} = \{\{0\}, \{1\}, \{2\}, \{0,1\}, \{0,2\}, \{1,2\}\}$

$S^1(4) =$

$V_{S^1(4)} = \{0,1,2,3\}$

$S_{S^1(4)} = \{\{0\}, \{1\}, \{2\}, \{3\}, \{0,1\}, \{1,2\}, \{2,3\}, \{0,3\}\}$

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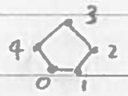
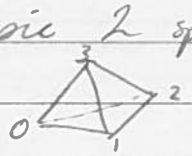
$S'(5) =$ 

$S'(n)$ = circle with n subdivision points.

Basic $\overset{S^n}{n\text{-sphere}}$ = boundary of basic $(n+1)$-simplex

$V_{S^n} = \{0, \dots, n+1\}$
$\mathcal{S}_{S^n} = \{\alpha \subset \{0, \dots, n+1\} : \alpha \neq \emptyset, |\alpha| < n+2\}$

So basic 2 sphere, S^2:
$S^2 =$  middle missing

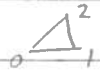
Vertex set = $\{0, 1, 2, 3\}$
Simplex set = $\{\{0\}, \{1\}, \{2\}, \{3\}, \{0,1\}, \{0,2\}, \{0,3\},$
$\{1,2\}, \{1,3\}, \{2,3\}, \{0,1,2\}, \{0,1,3\},$
$\{0,2,3\}, \{1,2,3\}\}$

4 vertices
6 1-simplices
4 2-simplices.

Let $K = (V_K, \mathcal{S}_K)$ be a simplicial complex.
Pick your favourite field (eg $\mathbb{Q}$)
For each $n \geq 0$ (integer) we will construct
i) a vector space $C_n(K)$ $(= C_n(K; \mathbb{F}))$ over $\mathbb{F}$
ii) linear maps $\partial_n : C_n(K) \to C_{n-1}(K)$ $(n \geq 1)$.

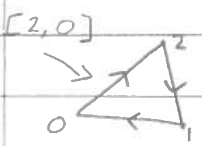
Construct $C_n(K)$:
If $\sigma = \{v_0, \dots, v_n\} \in \mathcal{S}_K$ (n-simplex) choose (arbitrarily)
some order $v_0 < v_1 < \dots < v_n$.
The basis vectors for $C_n(K)$ are "symbols"
$[v_0, v_1, \dots, v_n] \in C_n(K)$ such that $[v_{\sigma(0)}, \dots, v_{\sigma(n)}] = \text{sign}(\sigma) \cdot [v_0, \dots, v_n]$
plus the obvious rule $[v_0, \dots, v_n] = 0$ if $v_r = v_s$, $r \neq s$.

[I.e. if we are silly enough to repeat a vertex then we get zero!]

Take $K = S'(3) =$ 

$C_0(K)$ is 3-dimensional with basis $[0], [1], [2]$

$C_1(K)$ is 3-dimensional with basis $[0,1], [0,2], [1,2]$



$-[0,2] = [2,0]$

$[0,2]$

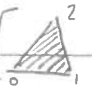
Boundary maps

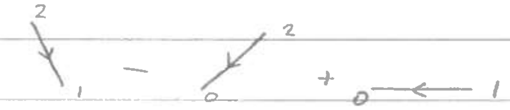
$\partial_n : C_n(K) \longrightarrow C_{n-1}(K)$

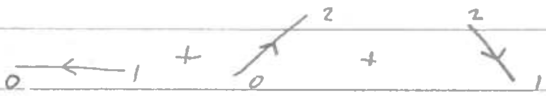
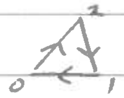
linear, defined by specifying on basis elements.

$$\partial_n[v_0,\dots,v_n] = \sum_{r=0}^{n} (-1)^r [v_0,\dots,\hat{v}_r,\dots,v_n]$$

↖ hat means omit v_r

$\partial\Big[$  $\Big] = [1,2] - [0,2] + [0,1]$

$=$ 

$=$  $=$ 

Proposition (Poincaré's Lemma)

$\partial_{n-1}\partial_n = 0$

$C_n \xrightarrow{\partial_n} C_{n-1} \xrightarrow{\partial_{n-1}} C_{n-2}$

Proof

Enough to show $\partial_{n-1}\partial_n = 0$ on each basis element.

$$\partial_{n-1}\partial_n[v_0,\dots,v_n] = \partial_{n-1}\left(\sum_{r=0}^{n} (-1)^r [v_0,\dots,\hat{v}_r,\dots,v_n]\right)$$

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$$= \sum_{r=0}^{n} (-1)^r \partial_{n-1} [v_0, \dots, \hat{v}_r, \dots, v_n]$$

v_r is "missing"

$$\partial_{n-1} [v_0, \dots, \hat{v}_r, \dots, v_n] = \sum_{s=0}^{r-1} (-1)^s [v_0, \dots, \hat{v}_s, \dots, \hat{v}_r, \dots, v_n]$$

$$+ \sum_{s=r+1}^{n} (-1)^{s-1} [v_0, \dots, \hat{v}_r, \dots, \hat{v}_s, \dots, v_n]$$

since v_r is missing

$$\text{So } \partial_{n-1} \partial_n [v_0, \dots, v_n] = \sum_{s<r} (-1)^{r+s} [v_0, \dots, \hat{v}_s, \dots, \hat{v}_r, \dots, v_n]$$

$$+ \sum_{r<s} (-1)^{r+s-1} [v_0, \dots, \hat{v}_r, \dots, \hat{v}_s, \dots, v_n]$$

reindex $k=s, l=r$ then $k=r, l=s$.

$$= \sum_{k<l} (-1)^{k+l} \underset{A}{[v_0, \dots, \hat{v}_k, \dots, \hat{v}_l, \dots, v_n]}$$

$$+ \sum_{k<l} (-1)^{k+l-1} \underset{B}{[v_0, \dots, \hat{v}_k, \dots, \hat{v}_l, \dots, v_n]}$$

But $B = -A \Rightarrow \partial_{n-1} \partial_n [v_0, \dots, v_n] = 0$ □

K simplicial complex

$C_{n+1}(K) \xrightarrow{\partial_{n+1}} C_n(K) \xrightarrow{\partial_n} C_{n-1}(K)$ $\quad \partial_n \cdot \partial_{n+1} = 0$

i.e.

<u>Prop</u>

$Im(\partial_{n+1}) \subset Ker(\partial_n)$

$Ker(\partial_n) \subset C_n(K)$ vector subspace called the set of n-cycles.

$Im(\partial_{n+1}) \subset Ker(\partial_n)$ $(\subset C_n(K))$

vector subspace "n-boundaries".

$U \subset V$

Take quotient U/V

$Ker(\partial_n) / Im(\partial_{n+1})$.

Def^n

$K = (V_K, S_K)$ simplicial complex (fix field $\mathbb{F}$)

Define $H_n(K) = \mathrm{Ker}(\partial_n) / \mathrm{Im}(\partial_{n+1})$ ← n^{th} homology group with coefficients in $\mathbb{F}$

(or write $H_n(K;\mathbb{F})$ if you want to stress $\mathbb{F}$).

$H_n(K;\mathbb{F})$ = n^{th} homology of K with coeffs in $\mathbb{F}$.

1st objective: Find out how to compute these groups EASILY.

Recall def^n:

U vector subspace of V.

$V/U = \{x + U : x \in V\}$

ie. set of cosets of U in V.

Rule of Equality:

$x + U = x' + U \iff x' - x \in U$.

V/U is naturally a vector space

Addition: $x + U + y + U = x + y + U$

Scalar multiplication: $\lambda(x+U) = \lambda x + U$

Zero: $0 + U = U$

Observe we have a cannonical linear mapping

$\natural : V \longrightarrow V/U$

$\natural(x) = x + U$

obviously surjective so $\mathrm{Im}(\natural) = V/U$

$\natural(x) = 0$ iff $x + U = U \iff x + U = 0 + U \iff x = x - 0 \in U$

So $\mathrm{Ker}(\natural) = U$

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Prop

Let $U \subset V$ be a vector subspace

If $\dim(V)$ is finite then

$\dim(V/U) = \dim(V) - \dim(U)$.

Proof

Kernel-Rank Thm:

$\dim \operatorname{Ker} q + \dim \operatorname{Im} q = \dim V$

$\Rightarrow \dim U + \dim V/U = \dim V$

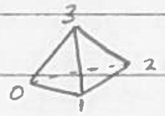
$\square$

How to compute homology the hard way

ie from 1st principles.

$H_n(K:\mathbb{F}) = \operatorname{Ker} \partial_n / \operatorname{Im} \partial_{n+1}$

$H_*(S^2;\mathbb{F}) = H_*(S^2)$

S^2 = 

middle missing.

$C_0(S^2)$ is 4-dimensional, basis: $\overset{e_1}{[0]}, \overset{e_2}{[1]}, \overset{e_3}{[2]}, \overset{e_4}{[3]}$.

$C_1(S^2)$ is 6-dim, basis: $\overset{\varphi_1}{[0,1]}, \overset{\varphi_2}{[0,2]}, \overset{\varphi_3}{[0,3]}, \overset{\varphi_4}{[1,2]}, \overset{\varphi_5}{[1,3]}, \overset{\varphi_6}{[2,3]}$

$C_2(S^2)$ is 4-dim, basis: $\overset{E_1}{[0,1,2]}, \overset{E_2}{[0,1,3]}, \overset{E_3}{[0,2,3]}, \overset{E_4}{[1,2,3]}$

$C_2 \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} C_0$

$\partial_1(\varphi_1) = \partial_1[0,1] = [1] - [0] = -e_1 + e_2$

$\partial_1(\varphi_2) = [2] - [0] = -e_1 + e_3$

$\partial_1(\varphi_3) = [3] - [0] = -e_1 + e_4$

$\partial_1(\varphi_4) = [2] - [1] = -e_2 + e_3$

$\partial_1(\varphi_5) = [3] - [1] = -e_2 + e_4$

$\partial_1(\varphi_6) = [3] - [2] = -e_3 + e_4$

Matrix of ∂_1 is

$$\begin{pmatrix} -1 & -1 & -1 & 0 & 0 & 0 \\ 1 & 0 & 0 & -1 & -1 & 0 \\ 0 & 1 & 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 & 1 & 1 \end{pmatrix}$$

row reduce!

$$= \begin{pmatrix} 0 & 0 & -1 & 0 & -1 & -1 \\ 1 & 0 & 0 & -1 & -1 & 0 \\ 0 & 1 & 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & -1 & -1 & 0 \\ 0 & 1 & 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$\dim \operatorname{Ker} \partial_1 = 3$

$\dim \operatorname{Im} \partial_1 = 3$

$\partial_2(E_1) = [1,2] - [0,2] + [0,1] = \varphi_4 - \varphi_2 + \varphi_1$

$\partial_2(E_2) = [1,3] - [0,3] + [0,1] = \varphi_5 - \varphi_3 + \varphi_1$

$\partial_2(E_3) = [2,3] - [0,3] + [0,2] = \varphi_6 - \varphi_3 + \varphi_2$

$\partial_2(E_4) = [2,3] - [1,3] + [1,2] = \varphi_6 - \varphi_5 + \varphi_4$

Matrix of ∂_2 is

$$\begin{pmatrix} 1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & -1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 1 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

$\dim \operatorname{Ker} \partial_2 = 1$

$\dim \operatorname{Im} \partial_2 = 3$

$\dim \operatorname{Ker} \partial_1 = 3$

$\Rightarrow \operatorname{Im} \partial_2 = \operatorname{Ker} \partial_1$

so $H_1(S^1) = \operatorname{Ker} \partial_1 / \operatorname{Im} \partial_2 = 0$

$0 = C_3 \xrightarrow{\partial_3} C_2 \xrightarrow{\partial_2} C_1$

$H_2(S^2) = \operatorname{Ker} \partial_2 / 0 \cong \operatorname{Ker} \partial_2$

$\dim H_2(S^2) = 1$

$C_1 \xrightarrow{\partial_1} C_0 \longrightarrow 0 = C_{-1}$

$H_0(S^2) = C_0 / \operatorname{Im} \partial_1$

$\dim H_0(S^2) = 4 - 3 = 1$

$$\boxed{H_*(S^2 ; \mathbb{F}) = \begin{cases} \mathbb{F} & k = 0 \\ 0 & k = 1 \\ \mathbb{F} & k = 2 \\ 0 & k \geqslant 3 \end{cases}}$$

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$$\begin{tikzcd} \{\text{Simplicial complexes}\} \arrow[rr, "(H_n)"] & & \{\text{Vector spaces}\} \end{tikzcd}$$

and simplicial maps ... and linear maps

$$\begin{tikzcd} X \arrow[dr] & & H_n(C_*) = \dfrac{\mathrm{Ker}(\partial_n)}{\mathrm{Im}(\partial_{n+1})} \\ & (C_*(X), \partial) \arrow[ur] & \end{tikzcd}$$

Homology groups

$C_*(X) = (C_n(X), n \geq 0)$, $\partial = (\partial_n : C_n(X) \to C_{n-1}(X))$

First example

$$H_k(S^2 : \mathbb{F}) = \begin{cases} \mathbb{F} , & k=0 \\ 0 , & k=1 \\ \mathbb{F} , & k=2 \\ 0 , & k \geq 3 \end{cases}$$

[Coming soon:
$H_0(X : \mathbb{F}) = \mathbb{F}$ if X is connected.]

Functoriality:

Suppose $f : X \to Y$ is a simplicial map.
We construct a linear map
$H_n(f) : H_n(X) \longrightarrow H_n(Y)$
in a consistent way.

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[rr, bend right, "g \circ f"'] & Y \arrow[r, "g"] & Z \end{tikzcd}$$

$$\begin{tikzcd} H_n(X) \arrow[r, "H_n(f)"] \arrow[rr, bend right, "H_n(g \circ f)"'] & H_n(Y) \arrow[r, "H_n(g)"] & H_n(Z) \end{tikzcd}$$

1. $H_n(g \circ f) = H_n(g) \circ H_n(f)$
2. $H_n(Id) = Id_{H_n(X)}$

Functor

Simplicial maps

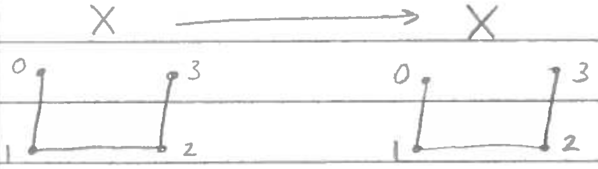
$X = (V_X, S_X)$, $Y = (V_Y, S_Y)$ simplicial complexes

By a simplicial mapping $f: X \to Y$ I mean a mapping of sets $f: V_X \to V_Y$ in such a way that

$f(\sigma) \in S_Y$ if $\sigma \in S_X$

$\sigma \in S_X \Rightarrow f(\sigma) \in S_Y$

simplices $\longmapsto$ simplices.



e.g. $f(0) = 3$, $f(3) = 0$, $f(1) = 2$, $f(2) = 1$

defines a simplicial map $f: X \to X$.

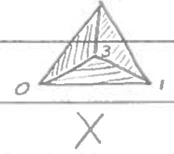
whereas $g: \{0,1,2,3\} \longrightarrow \{0,1,2,3\}$

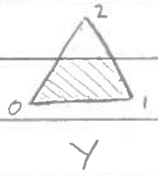
$g(0) = 1$, $g(1) = 2$, $g(2) = 3$, $g(3) = 0$

is not simplicial because $\{2,3\} \in S_X$ but

$g(\{2,3\}) = \{3,0\} \notin S_X$

Squash map





Formally:

$X = (\{0,1,2,3\}, S_X)$

$S_X = \{\{0\}, \{1\}, \{2\}, \{3\}, \{0,1\}, \{0,2\}, \{0,3\}, \{1,2\}, \{1,3\}, \{2,3\},$
$\{0,2,3\}, \{0,1,3\}, \{1,2,3\}\}$

$Y = \Delta^2$

Consider $f: X \to Y$, $f(0) = 0$, $f(1) = 1$, $f(2) = 2$, $f(3) = 0$

$f(\{0,2,3\}) = \{0,2\}$, $f(\{0,1,3\}) = \{0,1\}$, $f(\{1,2,3\}) = \{0,1,2\}$.

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Exercises

1) Let $f : X \to Y$, $g : Y \to Z$ be simplicial mappings.
Show that $g \circ f : X \to Z$ is also simplicial

2) $Id : X \to X$ is simplicial.

Given simplicial $f : X \to Y$ we want to produce (for each n) a linear map

$H_n(f) : H_n(X : \mathbb{F}) \longrightarrow H_n(Y : \mathbb{F})$ in such a way that

1). $H_n(Id) = Id_{H_n}$

2). If $g : Y \to Z$ is also simplicial then

$H_n(g \cdot f) = H_n(g) \circ H_n(f)$.

Intermediate step: Chain complexes and chain mappings
(Purely algebraic).
Fix a field $\mathbb{F}$.
By a chain complex over $\mathbb{F}$ I mean a collection
$(C_*, \partial) = (C_n, \partial_n)_{n \geq 0}$ where each C_n is a vector space over $\mathbb{F}$ and $\partial_n : C_n \longrightarrow C_{n-1}$ is a linear map such that
$\partial_{n-1} \partial_n = 0$

$$\to C_{n+1} \xrightarrow{\partial_{n+1}} C_n \xrightarrow{\partial_n} C_{n-1} \xrightarrow{\partial_{n-1}} \dots \xrightarrow{\partial_3} C_2 \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} C_0 \xrightarrow{\partial_0} \overset{C_{-1}}{\overset{\shortparallel}{0}}$$

Convention:

$C_{-1} = 0$, $\partial_0 \equiv 0 : C_0 \to C_{-1}$.

Chain mappings:
Suppose that (C_*, ∂), (C_*', ∂') are chain complexes over $\mathbb{F}$
By a chain mapping $f : (C_*, \partial) \to (C_*', \partial')$ I mean a collection $f = (f_n)$ where each f_n is a linear map $f_n : C_n \to C_n'$ in such a way that the following diagram commutes:

$$\begin{tikzcd} C_n \arrow[r, "\partial_n"] \arrow[d, "f_n"'] & C_{n-1} \arrow[d, "f_{n-1}"] \\ C_n' \arrow[r, "\partial_n'"'] & C_{n-1}' \end{tikzcd}$$

$f_{n-1} \circ \partial_n = \partial_n' \circ f_n$

$$\begin{tikzcd}
{} \arrow[r] & C_{n+1} \arrow[r, "\partial_{n+1}"] \arrow[d, "f_{n+1}"] & C_n \arrow[r, "\partial_n"] \arrow[d, "f_n"] & C_{n-1} \arrow[r, "\partial_{n-1}"] \arrow[d, "f_{n-1}"] & \cdots \arrow[r, "\partial_3"] & C_2 \arrow[r, "\partial_2"] \arrow[d, "f_2"] & C_1 \arrow[r, "\partial_1"] \arrow[d, "f_1"] & C_0 \arrow[r, "\partial_0"] \arrow[d, "f_0"] & 0 \\
{} \arrow[r] & C_{n+1}' \arrow[r, "\partial_{n+1}'"] & C_n' \arrow[r, "\partial_n'"] & C_{n-1}' \arrow[r, "\partial_{n-1}'"] & \cdots \arrow[r, "\partial_3'"] & C_2' \arrow[r, "\partial_2'"] & C_1' \arrow[r, "\partial_1'"] & C_0' \arrow[r, "\partial_0'"] & 0
\end{tikzcd}$$

Given a simplicial mapping $f : X \to Y$ I need to produce a chain mapping

$$C_*(f) : (C_*(X), \partial) \longrightarrow (C_*(Y), \partial)$$

Remember that $C_n(X)$ is the vector space with basis the symbols $[v_0, \dots, v_n]$ where $\{v_0, \dots, v_n\} \in S_X$ such that $[v_{\sigma(0)}, \dots, v_{\sigma(n)}] = \operatorname{sign}(\sigma)[v_0, \dots, v_n]$ and $[v_0, \dots, v_r, \dots, v_s, \dots, v_n] = 0$ if $r = s$.

To define $C_n(f) : C_n(X) \longrightarrow C_n(Y)$; it is enough to specify what it does on a basis.

Def^n

$$C_n(f)[v_0, \dots, v_n] = [f(v_0), \dots, f(v_n)]$$

Prop

$C_*(f) = (C_n(f))_{n \geq 0}$ defines a chain mapping $C_*(f) : C_*(X) \to C_*(Y)$.

Proof

$$\partial_n' C_n(f)[v_0, \dots, v_n] = \partial_n'[f(v_0), \dots, f(v_n)]$$
$$= \sum_{r=0}^{n} (-1)^r [f(v_0), \dots, \widehat{f(v_r)}, \dots, f(v_n)]$$

$$C_{n-1}(f)[v_0, \dots, \hat{v}_r, \dots, v_n] = [f(v_0), \dots, \widehat{f(v_r)}, \dots, f(v_n)]$$

$$\Rightarrow \partial_n' C_n(f)[v_0, \dots, v_n] = C_{n-1}(f)\left(\sum_{r=0}^{n} (-1)^r [v_0, \dots, \hat{v}_r, \dots, v_n]\right)$$

$$\Rightarrow \partial_n' \circ C_n(f) = C_{n-1}(f) \circ \partial_n$$

i.e. it is a chain mapping $\square$

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$$\begin{tikzcd} \left\{ \begin{array}{l} \text{Simplicial complexes} \\ \text{\& simplicial maps} \end{array} \right\} \arrow[d, "C_* \checkmark"] & \left\{ \begin{array}{l} \text{Vector spaces} \\ \text{\& linear maps} \end{array} \right\} \\ \left\{ \begin{array}{l} \text{Chain complexes} \\ \text{\& chain maps} \end{array} \right\} \arrow[ur, dashed, "H_*"'] & \end{tikzcd}$$

Pick a field $\mathbb{F}$

Last time:

If X is a simplicial complex

? $C_*(X, \mathbb{F}) = C_x(X), \partial$

$C_n(X)$ vector space on symbols $[v_0, \dots, v_n]$, $\{v_0, \dots, v_n\} \in S_x$

$\partial : C_n \to C_{n-1}$, $\partial[v_0, \dots, v_n] = \sum (-1)^r [v_0, \dots, \hat{v}_r, \dots, v_n]$

$f : X \to Y$

$C_n(f) : C_n(X) \to C_n(Y)$

$C_n(f)[v_0, \dots, v_n] = [f(v_0), \dots, f(v_n)]$

$\partial_n^Y C_n(f) = C_{n-1}(f) \partial_n^X$

$X \xrightarrow{f} Y \xrightarrow{g} Z$, f, g simplicial maps

$C_n(f \circ g) = C_n(f) \circ C_n(g)$

Today:

Now investigate $H_n : \left\{ \begin{array}{l} \text{Chain complexes} \\ \text{\& chain mappings} \end{array} \right\} \longrightarrow \left\{ \begin{array}{l} \text{Vector spaces} \\ \text{\& linear mappings} \end{array} \right\}$

Def

If $C_* : (C_n, \partial_n)_{n \geq 0}$ is a chain complex, then

$$H_n(C_*) = \frac{\operatorname{Ker} \partial_n}{\operatorname{Im} \partial_{n+1}} = n^{\text{th}} \text{ homology group of } C_*$$

In the liturature often see $Z_n(C_*) = \operatorname{Ker} \partial_n$ n-cycles

$B_n(C_*) = \mathrm{Im}(\partial_{n+1})$ n-boundaries

$H_n(C_*) = Z_n(C_*) / B_n(C_*)$.

Suppose $f = (f_n)_{n\geq 0} : (C_*, \partial) \longrightarrow (D_*, \delta)$ is a chain mapping, i.e.

$$\begin{tikzcd} C_{n+1} \arrow[r, "\partial_{n+1}"] \arrow[d, "f_{n+1}"'] & C_n \arrow[r, "\partial_n"] \arrow[d, "f_n"'] & C_{n-1} \arrow[d, "f_{n-1}"'] \\ D_{n+1} \arrow[r, "\delta_{n+1}"] & D_n \arrow[r, "\delta_n"] & D_{n-1} \end{tikzcd}$$

commutes for each n.

Need to define linear map $H_n(f) : H_n(C) \longrightarrow H_n(D)$.

$H_n(C) = Z_n(C) / B_n(C)$, $H_n(D) = Z_n(D) / B_n(D)$.

Elements of $H_n(C)$ have form $z + \mathrm{Im}(\partial_{n+1})$, $z \in Z_n(C)$

i.e. $\partial_n(z) = 0$

Elements of $H_n(D)$ have form $z' + \mathrm{Im}(\delta_{n+1})$, $z' \in Z_n(D)$

$\delta_n(z') = 0$.

Def^n

$H_n(f)(z + \mathrm{Im}(\partial^C_{n+1})) = f_n(z) + \mathrm{Im}(\delta^D_{n+1})$

This is a meaningful def^n.

If $z \in \mathrm{Ker}(\partial^C_n)$, $\delta^D_n f_n(z) = f_{n-1}(\partial^C_n(z))$

$= f_{n-1}(0) = 0$

$\Rightarrow f_n(z) \in \mathrm{Ker}\, \delta^D_n$.

Also $f_n(\mathrm{Im}\, \partial^C_{n+1}) \subset \mathrm{Im}(\delta^D_{n+1})$:

Suppose $b \in \mathrm{Im}(\partial^C_{n+1})$, $b = \partial^C_{n+1}(\omega)$

$f_n \partial^C_{n+1}(\omega) = \delta^D_{n+1} f_{n+1}(\omega) \in \mathrm{Im}(\partial^D_{n+1})$

Clear that each $H_n(f)$ is linear

(everything involved is linear i.e. ∂^C_n, δ^D_n, f_n)

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Functoriality conditions

If $$\begin{tikzcd} C_* \arrow[r, "f"] \arrow[rr, bend right, "g\circ f"'] & D_* \arrow[r, "g"] & E_* \end{tikzcd}$$ are chain maps

Then $g \circ f$ is also a chain mapping.

$$\begin{tikzcd}
C_{n+1} \arrow[r, "\partial^C_{n+1}"] \arrow[d, "f_{n+1}"] & C_n \arrow[r, "\partial^C_n"] \arrow[d, "f_n"] & C_{n-1} \arrow[d, "f_{n-1}"] \\
D_{n+1} \arrow[r, "\partial^D_{n+1}"] \arrow[d, "g_{n+1}"] & D_n \arrow[r, "\partial^D_n"] \arrow[d, "g_n"] & D_{n-1} \arrow[d, "g_{n-1}"] \\
E_{n+1} \arrow[r, "\partial^E_{n+1}"] & E_n \arrow[r, "\partial^E_n"] & E_{n-1}
\end{tikzcd}$$

By hypothesis all small squares commute.
Rub out middle line.

$$\begin{tikzcd}
C_{n+1} \arrow[r, "\partial^C_{n+1}"] \arrow[d, "g_{n+1}\circ f_{n+1}"] & C_n \arrow[r, "\partial^C_n"] \arrow[d, "g_n \circ f_n"] & C_{n-1} \arrow[d, "g_{n-1}\circ f_{n-1}"] \\
E_{n+1} \arrow[r, "\partial^E_{n+1}"] & E_n \arrow[r, "\partial^E_n"] & E_{n-1}
\end{tikzcd}$$ commutes

$H_n(f)(z + \mathrm{Im}\, \partial^C_{n+1}) = f_n(z) + \mathrm{Im}\, \partial^D_{n+1}$ (Def^n)
$H_n(g)(\omega + \mathrm{Im}\, \partial^D_{n+1}) = g_n(\omega) + \mathrm{Im}\, \partial^E_{n+1}$ (Def^n)

$$\begin{aligned} H_n(g) \circ H_n(f) &= (g_n \circ f_n)(z) + \mathrm{Im}\, \partial^E_{n+1} \\ &= H_n(g\circ f)(z + \mathrm{Im}\, \partial^C_{n+1}) \end{aligned}$$

1). So we've shown $f : C_* \to D_*$, $g : D_* \to E_*$ are chain mappings then $H_n(g \circ f) = H_n(g) \circ H_n(f)$
2). If $1d : C_* \to C_*$, clear that $H_n(1d) = 1d_{H_n(C_*)}$

$$\begin{tikzcd}
\left\{\begin{array}{l}\text{Simp. complexes} \\ \text{\& simp. maps}\end{array}\right\} \arrow[rr, "H_n\ (= H_n \circ C_*)"] \arrow[dr, "C_*"'] & & \left\{\begin{array}{l}\text{Vector spaces} \\ \text{\& linear maps}\end{array}\right\} \\
& \left\{\begin{array}{l}\text{Chain complexes} \\ \text{\& chain maps}\end{array}\right\} \arrow[ur, "H_n"'] &
\end{tikzcd}$$

One such diagram for each $n \geq 0$.

$H_n(X; \mathbb{F}) = H_n(C_*(X : \mathbb{F}))$ Def^n

Prop

If X is a non-empty simplicial complex then $H_0(X : \mathbb{F}) \neq 0$.

Proof

Define $*$ to be "simplicial point".

$V_* = \{*\}$, $S_* = \{\{*\}\}$ i.e. just a single point.

$$C_n(* : \mathbb{F}) = \begin{cases} \mathbb{F} & n = 0 \\ 0 & n \neq 0 \end{cases}$$

$$\underset{C_1}{\overset{}{0}} \xrightarrow{\partial_1} C_0(* : \mathbb{F}) \xrightarrow{\partial_0} \underset{C_{-1}}{0}$$

$$H_0(* : \mathbb{F}) = \frac{\operatorname{Ker}(C_0(* : \mathbb{F}) \to 0)}{\operatorname{Im}(0 \to C_0(* : \mathbb{F}))} \cong C_0(*, \mathbb{F}) \equiv \mathbb{F}$$

$\square$

Now consider the simplicial map

$c : X \longrightarrow *$, $c(v) = * \quad \forall v \in V_X$

c induces $H_0(c) : H_0(X) \longrightarrow H_0(*)$

Claim: $H_0(c)$ is surjective.

As $X \neq \emptyset$, choose $v \in V_X$.

Consider $i : * \longrightarrow X$, $i(*) = v$

$c \circ i : * \longrightarrow *$ is the identity.

so $H_0(c) \circ H_0(i) = H_0(c \circ i) = Id_{H_0(*)} = Id$

so $H_0(c)$ is surjective, $H_0(i)$ is injective

$\Rightarrow H_0(X : \mathbb{F}) \neq 0$.

$\square$

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Connectivity

$X = (V_X, S_X)$ simplicial complex.

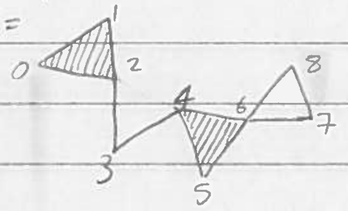
Let $v, w \in V_X$ (vertices of X), $v \neq w$.

By a path in X from v to w I mean a finite sequence of vertices $(v_0, \dots, v_N)$ $v_i \in V_X$ such that (i) $v_0 = v$, (ii) $v_N = w$, (iii) $\{v_{i-1}, v_i\}$ is a 1-simplex in X.

We say that X is connected when for any vertices $v, w \in V_X$, $v \neq w$, $\exists$ a path in X from v to w.

Examples

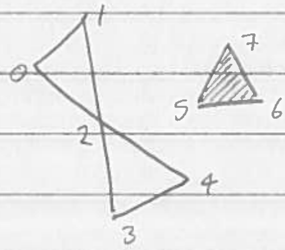
$X =$



X is connected.

e.g. $(0, 2, 3, 4, 5)$ is a path from 0 to 5, but $(0, 2, 4, 5)$ is not a path.

$Y =$



Y is not connected. No path from 2 to 5.

Theorem

Let X be a non-empty simplicial complex. If X is connected then $H_0(X:\mathbb{F}) \cong \mathbb{F}$.

Proof

We know $H_0(X:\mathbb{F}) \neq 0$ so it suffices to show that $\dim H_0(X:\mathbb{F}) \leq 1$. Also assume X is finite.

List the vertices of X thus $v_0, v_1, \dots, v_N$ (arbitrarily)

Then $[v_0], [v_1], \dots, [v_n]$ forms a basis for $C_0(X:\mathbb{F})$.

Elementary basis change:

$[v_0], [v_1]-[v_0], [v_2]-[v_0], \ldots, [v_N]-[v_0]$ is still a basis for $C_0(X)$.

First claim that if $w \in V_X$, $w \neq v$ then

$[w]-[v] \in \operatorname{Im} \partial_1$.

Since X is connected, I can choose a path

$(w_0, w_1, \ldots, w_n)$ in X from $w_0 = v$ to $w_n = w$.

$[w_i, w_{i+1}]$ is a 1-simplex for $0 \leq i \leq n-1$.

$\partial_1 [w_i, w_{i+1}] = [w_{i+1}] - [w_i]$ (by def^n).

$$[w_n]-[w_0] = \sum_{i=0}^{n-1} [w_{i+1}]-[w_i] = \sum_{i=0}^{n-1} \partial_1 [w_i, w_{i+1}]$$

$$= \partial_1 \left(\sum_{i=0}^{n-1} [w_i, w_{i+1}] \right) \in \operatorname{Im} \partial_1$$

But $w_0 = v$, $w_n = w$, so

$[w]-[v] \in \operatorname{Im} \partial_1$ as claimed.

In the above basis $\{[v_0]\} \cup \{[v_r]-[v_0]\}_{1 \leq r \leq N}$.

Each $[v_r]-[v_0] \in \operatorname{Im} \partial_1$

so $H_0(X:\mathbb{F}) = C_0(X:\mathbb{F}) / \operatorname{Im} \partial_1$

so $H_0(X:\mathbb{F})$ is represented by $|V_0| + \operatorname{Im}(\partial_1)$

so $\dim H_0(X:\mathbb{F}) \leq 1$. $\square$

ie. we've proved:

Thm

If $X \neq \emptyset$ is connected then $\dim H_0(X;\mathbb{F}) = 1$

In general, we'll see that $\dim H_0(X:\mathbb{F}) =$ no. of connected components of X.

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If $*$ is a single point, then

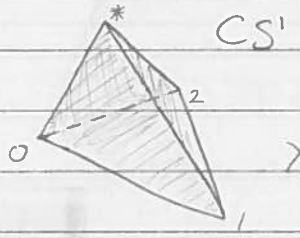
$$H_r(*, \mathbb{F}) = \begin{cases} \mathbb{F}, & r=0 \\ 0, & r \neq 0. \end{cases}$$

Cones

Let $X = (V_X, S_X)$ be a simplicial complex.

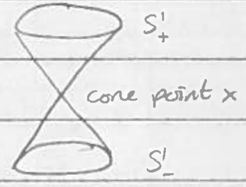
Define a new complex CX = cone on X

by choosing a disjoint point $*$, $* \notin V_X$



$X = S'$ (middle missing)

: cone on two circles →



$C[S'_+ \sqcup S'_-]$

Definition

$X = (V_X, S_X)$ simplicial complex, $* \notin V_X$

$CX = (V_{CX}, S_{CX})$ where

$V_{CX} = \{*\} \sqcup V_X$

$$S_{CX} = \underset{\substack{\uparrow \\ \text{original simplices}}}{S_X} \cup \underset{\substack{\uparrow \\ \text{cone point}}}{\{\{*\}\}} \cup \{\{*\} \underset{\substack{\uparrow \\ \text{all simplices joined to the cone point}}}{\cup} \sigma : \sigma \in S_X\}$$

Theorem

Let X be a simplicial complex.

$$H_r(CX : \mathbb{F}) \cong \begin{cases} \mathbb{F}, & r=0 \\ 0, & r \neq 0 \end{cases}$$

i.e. CX has the homology of a point.

Proof

$C_*(CX)$ is the chain complex of CX.

Define $h_n : C_n(CX) \longrightarrow C_{n+1}(CX)$ linear map

by $h_n[v_0, \dots, v_n] = [*, v_0, \dots, v_n]$.

[Note if $v_r = *$ then $h[v_0, \dots, v_n] = 0$ due to repetition of vertex.]

$$\partial_{n+1} h_n[v_0,\dots,v_n] = \partial_{n+1}[*, v_0,\dots,v_n]$$
$$= [v_0,\dots,v_n] + \sum_{r=0}^{n} (-1)^{r+1}[*, v_0,\dots,\hat{v}_r,\dots,v_n]$$
$$= [v_0,\dots,v_n] + h_{n-1}\Big(\sum_{r=0}^{n}(-1)^{r+1}[v_0,\dots,\hat{v}_r,\dots,v_n]\Big)$$
$$= [v_0,\dots,v_n] - h_{n-1}\partial_n[v_0,\dots,v_n]$$

and so $\partial_{n+1}h_n + h_{n-1}\partial_n = Id$

Suppose $n \geq 1$ and let $z \in Z_n(CX)$

then $\partial_n(z) = 0$.

So $Id(z) = \partial_{n+1}h_n(z) + h_{n-1}(0) = \partial_{n+1}h_n(z) \in B_n(CX)$

ie. $Z_n(CX) = B_n(CX)$ for $n \geq 1$

So $H_n(CX : \mathbb{F}) = 0$ for $n \geq 1$

(Check CX is connected). □

17-01-18 Theorem

Let X be any simplicial complex and CX = cone on X

Then $H_r(CX; \mathbb{F}) = \begin{cases} \mathbb{F}, & r=0 \\ 0, & r \neq 0. \end{cases}$

ie. CX behaves like a point as regards homology.

Proof

CX is connected:

$V_{CX} = V_X \sqcup \{*\}$

Let $v, w \in V_{CX}$, $v \neq w$.

Either $* \in \{v, w\}$ or $v \in V_X$ and $w \in V_X$.

Suppose $w = \overset{v \neq *}{*}$, then $[v, *] \in S_{CX}$ so I've joined v to w

Likewise if $v = *$, $w \neq *$.

If $v, w \in V_X$, then $[v, *] \in S_{CX}$, $[*, w] \in S_{CX}$.

So again I've joined v to w.

So CX is connected.

So $H_0(CX : \mathbb{F}) \cong \mathbb{F}$.

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Now suppose $n \geq 1$.

Need to show $H_n(CX : F) = 0$. Write $C_n = C_n(CX ; \mathbb{F})$.

For each $n \geq 0$ define $h_m : C_m \to C_{m+1}$, defined on basis $h_m[v_0, \dots, v_m] = [*, v_0, \dots, v_m]$

$$\begin{tikzcd}
C_{n+1} \arrow[r, "\partial_{n+1}"] & C_n \arrow[r, "\partial_n"] \arrow[d, "\mathrm{Id}"'] \arrow[dl, "h_n"'] & C_{n-1} \arrow[dl, "h_{n-1}"'] \\
C_{n+1} \arrow[r, "\partial_{n+1}"] & C_n \arrow[r, "\partial_n"] & C_{n-1}
\end{tikzcd}$$

Claim: $h_{n-1}\partial_n + \partial_{n+1}h_n = \mathrm{Id}$. "Chain contraction."

It is enough to check this on the standard basis of C_n.

$[v_0, \dots, v_n] \in C_n \quad (= C_n(CX ; \mathbb{F}))$

$$\partial_n [v_0, \dots, v_n] = \sum_{r=0}^{n} (-1)^r [v_0, \dots, \hat{v_r}, \dots, v_n]$$

$$h_{n-1}\partial_n [v_0, \dots, v_n] = \sum_{r=0}^{n} (-1)^r [*, v_0, \dots, \hat{v_r}, \dots, v_n]$$

$$h_n [v_0, \dots, v_n] = [*, v_0, \dots, v_n]$$

$$\partial_{n+1}h_n [v_0, \dots, v_n] = [v_0, \dots, v_n] + \sum_{r=0}^{n} (-1)^{r+1} [*, v_0, \dots, \hat{v_r}, \dots, v_n]$$

$$= [v_0, \dots, v_n] - \sum_{r=0}^{n} (-1)^r [*, v_0, \dots, \hat{v_r}, \dots, v_n]$$

$\Rightarrow [\partial_{n+1}h_n + h_{n-1}\partial_n][v_0, \dots, v_n]$

$$= [v_0, \dots, v_n] - \sum_{r=0}^{n} (-1)^r [*, v_0, \dots, \hat{v_r}, \dots, v_n] + \sum_{r=0}^{n} (-1)^r [*, v_0, \dots, \hat{v_r}, \dots, v_n]$$

$$= [v_0, \dots, v_n]$$

$\Rightarrow \partial_{n+1}h_n + h_{n-1}\partial_n = \mathrm{Id}$ as claimed.

Let $z \in Z_n(CX)$, $n \geq 1$

$\partial_n(z) = 0$

$z = \partial_{n+1}h_n(z) + h_{n-1}\partial_n(z)$

$\Rightarrow z = \partial_{n+1}h_n(z) \in B_n(CX)$

i.e. $Z_n(CX) \subseteq B_n(CX)$

But $B_n(CX) \subset Z_n(CX)$

$\Rightarrow B_n(CX) = Z_n(CX)$

$\Rightarrow H_n(CX;\mathbb{F}) = Z_n(CX)/B_n(CX) = 0$

$\square$

Example

$\Delta^{n+1} = C\Delta^n \qquad (n \geqslant 0)$

Proof

$V_{\Delta^{n+1}} = \{0, \dots, n+1\}, \quad V_{\Delta^n} = \{0, \dots, n\}$

Put $* = n+1$. $\square$

Corollary

$$H_r(\Delta^n;\mathbb{F}) = \begin{cases} \mathbb{F} & r = 0 \\ 0 & r \neq 0 \end{cases}$$

$\forall n \geqslant 1$.

$\Delta^1 = C\Delta^0, \quad \Delta^0 = \{0\}$ single point.

$\updownarrow$

here taking cone point to be 1

$\Delta^2 =$ 

$= C($$)$ cone point $= 2$.

Skeleta

Let $X = (V_X, S_X)$ be a simplicial complex. Let $n \geqslant 0$.
Define $X^{(n)} = (V_X, \{\sigma \in S_X : |\sigma| \leqslant n+1\}$ i.e. st. $\dim(\sigma) \leqslant n)$
$X^{(n)}$ is called the n-skeleton of X.

In English, to get $X^{(n)}$, throw away all simplices in X of dimension $> n$.

Theorem

$$H_r(X^n;\mathbb{F}) = \begin{cases} H_r(X;\mathbb{F}) & r < n \\ Z_n(X;\mathbb{F}) & r = n \\ 0 & r > n \end{cases}$$

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Proof

By def^n of $X^{(n)}$,

$C_r(X^{(n)};\mathbb{F}) \equiv C_r(X;\mathbb{F}) \quad r \leq n.$

$\partial_r : C_r(X^{(n)}) \to C_{r-1}(X^{(n)}) \equiv \partial_r : C_r(X) \to C_{r-1}(X)$ provided $r \leq n$.

$$\begin{tikzcd}
0 \arrow[r] & C_n(X^{(n)}) \arrow[r, "\partial_n"] \arrow[d, phantom, "\equiv" rotate=90] & C_{n-1}(X^{(n)}) \arrow[d, phantom, "\equiv" rotate=90] \\
C_{n+1}(X) \arrow[r, "\partial_{n+1}"] & C_n(X) \arrow[r, "\partial_n"] & C_{n-1}(X)
\end{tikzcd}$$

For $r < n$:

$$H_r(X^{(n)};\mathbb{F}) \equiv H_r(X;\mathbb{F})$$
$$\overset{\shortparallel}{\frac{\operatorname{Ker}\partial_r}{\operatorname{Im}\partial_{r+1}}} \equiv \overset{\shortparallel}{\frac{\operatorname{Ker}\partial_r}{\operatorname{Im}\partial_{r+1}}}$$

$H_n(X^{(n)};\mathbb{F}) = \operatorname{Ker}\partial_n$

$H_n(X:\mathbb{F}) = \operatorname{Ker}\partial_n / \operatorname{Im}\partial_{n+1}$

$\square$

[Natural surjection:
$H_n(X^{(n)}) \to H_n(X)$
Not an isomorphism in general]

Example

Take the standard model of S^n

$V_{S^n} = \{0, \dots, n+1\}$

$\mathcal{S}_{S^n} = \sigma \subset \{0, \dots, n+1\}$, $\sigma \neq \emptyset$ and $\dim(\sigma) \leq n$ ie. $|\sigma| \leq n+1$

Prop

$S^n = (\Delta^{n+1})^{(n)} \quad (n \geq 0)$

Corollary

Let $n \geq 1$.

$$H_r(S^n:\mathbb{F}) = \begin{cases} \mathbb{F} & r = 0 \\ 0 & 1 \leq r \leq n-1 \\ ? & r = n \\ 0 & r > n \end{cases}$$

Proof

For $r < n$

$H_r(S^n ; \mathbb{F}) \cong H_r(\Delta^{n+1} ; \mathbb{F})$. $\square$

But so far we don't know the value $H_n(S^n ; \mathbb{F})$ in general

We did show that $H_2(S^2 ; \mathbb{F}) = \mathbb{F}$

22-01-18 Recall: $H_r(CX ; \mathbb{F}) = \begin{cases} \mathbb{F}, & r = 0 \\ 0, & r \neq 0. \end{cases}$

$$\Downarrow$$

$$H_r(\Delta^n : \mathbb{F}) = \begin{cases} \mathbb{F}, & r = 0 \\ 0, & r \neq 0. \end{cases}$$

S^n standard model of n-sphere.

$S^n = (\Delta^{n+1})^{(n)}$, $n \geqslant 2$

$$H_r(S^n : \mathbb{F}) = \begin{cases} \mathbb{F}, & r = 0 \\ 0, & 0 < r < n \\ ?, & r = n \\ 0, & r > n. \end{cases}$$

$H_2(S^2 : \mathbb{F}) = \mathbb{F}$.

Exact sequences [Hurewicz c1936]

$$U \xrightarrow{S} V \xrightarrow{T} W$$

U, V, W vector spaces

S, T linear maps

Say the sequence is exact at V when $\operatorname{Ker} T = \operatorname{Im} S$.

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Types of exact sequences

(i) $V \xrightarrow{T} W \longrightarrow 0$

exact at $W \Leftrightarrow T$ is surjective

$\mathrm{Ker}(W \to 0) = W = \mathrm{Im}\,T$

(ii) $0 \longrightarrow U \xrightarrow{S} V$

exact at $U \Leftrightarrow S$ is injective

$\mathrm{Im}(0 \to U) = 0 = \mathrm{Ker}\,S$

(iii) Very short exact sequence

$0 \to U \xrightarrow{T} V \longrightarrow 0$

exact (at U and V) $\Leftrightarrow T$ is an isomorphism.

(T is both injective and surjective).

(iv) Short exact sequence (SES)

$0 \longrightarrow U \xrightarrow{S} V \xrightarrow{T} W \longrightarrow 0$

exact (at U, V, W) $\Leftrightarrow$ S injective, T surjective and $\mathrm{Ker}\,T = \mathrm{Im}\,S$.

If U, V, W are ∞-dimensional (For us NEVER)
then this is all we can say.
However if V is finite dimensional then so are U, W
and in this case $\dim V = \dim U + \dim W$
(by Kernel-Rank Thm).

If $0 \longrightarrow U \xrightarrow{S} V \xrightarrow{T} W \longrightarrow 0$ is exact, then
T surjective so $\dim W = \dim \mathrm{Im}(T)$
S injective so $\dim U = \dim \mathrm{Im}(S) = \dim \mathrm{Ker}(T)$
$\dim(V) = \dim \mathrm{Ker}(T) + \dim \mathrm{Im}(T) = \dim U + \dim W$.

In general given a sequence of vector spaces and linear maps

$$\dots \to V_{n+1} \xrightarrow{T_{n+1}} V_n \xrightarrow{T_n} V_{n-1} \xrightarrow{T_{n-1}} \dots$$

then the sequence is is exact at V_n

$\iff \mathrm{Ker}(T_n) = \mathrm{Im}(T_{n+1})$

Such a sequence is exact when exact at each V_n.

Main Theorem of the Course:

Mayer-Vietoris Thm

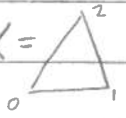
Let X be a simplicial complex and suppose we can write $X = X_+ \cup X_-$ where X_+ and X_- are subcomplexes of X, then there exists a long exact sequence of the following form: ($\mathbb{F}$ = coefficients)

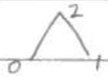
$$H_{n+1}(X) \to H_n(X_+ \cap X_-) \to H_n(X_+) \oplus H_n(X_-) \to H_n(X) \to H_{n-1}(X_+ \cap X_-) \to \dots$$

and finishes like

$$\dots \to H_1(X_+ \cap X_-) \to H_1(X_+) \oplus H_1(X_-) \to H_1(X) \to H_0(X_+ \cap X_-) \to H_0(X_+) \oplus H_0(X_-) \to H_0(X) \to 0.$$

Example

$H_*(S^1 : \mathbb{F})$, $X =$  $(= S^1(3))$

$X_- =$ , $X_+ =$ 

X_+ is the cone on two points, $\{0, 1\}$

X_- is the cone on a single point, $\{0\}$

$X_+ \cap X_- =$ 

Use the Mayer-Vietoris Thm

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X is connected.

$$\begin{tikzcd}
H_1(X_+)\oplus H_1(X_-) \arrow[r] \arrow[d, equal] & H_1(S^1) \arrow[r] & H_0(X_+\cap X_-) \arrow[r] & H_0(X_+)\oplus H_0(X_-) \arrow[r] & H_0(X) \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & H_1(S^1) \arrow[r] & \mathbb{F}\oplus\mathbb{F} \arrow[r] & \mathbb{F}\oplus\mathbb{F} \arrow[r] & \mathbb{F} \arrow[r] & 0
\end{tikzcd}$$

$$C_n(X_+\cap X_-) = \begin{cases} \mathbb{F}\oplus\mathbb{F} & n=0 \\ 0 & n\neq 0 \end{cases} \Rightarrow H_0(X_+\cap X_-) \cong \mathbb{F}\oplus\mathbb{F}.$$

So we have

$$\begin{tikzcd}
0 \arrow[r] & H_1(S^1) \arrow[r, "J"] & \mathbb{F}^2 \arrow[r, "S"] \arrow[d] & \mathbb{F}^2 \arrow[r, "T"] & \mathbb{F} \arrow[r] & 0 \\
 & & \operatorname{Im} S = \operatorname{Ker} T \arrow[ur] & & &
\end{tikzcd}$$

Split into two exact sequences:

$$0 \to H_1(S^1) \xrightarrow{J} \mathbb{F}^2 \xrightarrow{S} \operatorname{Im}(S) \to 0$$

$$0 \to \operatorname{Ker}(T) \to \mathbb{F}^2 \xrightarrow{T} \mathbb{F} \to 0$$

$\dim \operatorname{Ker} T = 1$ so

$\dim \operatorname{Im} S = 1$

$$0 \to H_1(S^1) \to \mathbb{F}^2 \to \underset{\operatorname{Im}(S)}{\overset{}{\underset{\|}{\mathbb{F}}}} \to 0$$

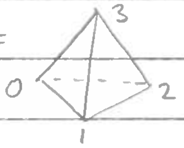
So $\dim H_1(S^1) + 1 = 2$

$\dim H_1(S^1) = 1$

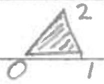
To summarise:

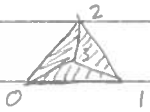
$$H_k(S^1 : \mathbb{F}) = \begin{cases} \mathbb{F} & k=0 \quad S^1 \text{ connected} \\ \mathbb{F} & k=1 \quad \text{see above calculation} \\ 0 & k>1 \quad (1\text{-dimensional}) \end{cases}$$

$H_*(S^2 : \mathbb{F})$

$X =$ 

All 2-dim faces included but middle is missing

$X_- =$  = bottom face

$X_+ =$  $= C(\partial X_-)$

$X_+ \cap X_- = S^1$

Use Mayer-Vietoris Thm.

$$H_2(X_+) \oplus H_2(X_-) \to H_2(S^2) \to H_1(X_+ \cap X_-) \to H_1(X_+) \oplus H_1(X_-)$$

$$\begin{array}{ccccc} \parallel & & \parallel & & \parallel \\ 0 & & H_1(S^1) & & 0 \\ \text{(cones)} & & \parallel & & \text{(cones)} \\ & & \mathbb{F} & & \end{array}$$

$\Rightarrow \quad 0 \longrightarrow H_2(S^2) \longrightarrow \mathbb{F} \longrightarrow 0$

So $\dim H_2(S^2) = 1 \quad \Rightarrow H_2(S^2) \cong \mathbb{F}$

$$\Rightarrow H_k(S^2 : \mathbb{F}) = \begin{cases} \mathbb{F} & k=0 \quad S^2 \text{ connected} \\ 0 & k=1 \quad S^2 = (\Delta^3)^{(2)} \\ \mathbb{F} & k=2 \quad \text{see above} \\ 0 & k>2 \end{cases}$$

$H_*(S^n : \mathbb{F}) \quad n \geqslant 2$

$S^n = (\Delta^{n+1})^{(n)}$

vertex set $= \{0, \dots, n+1\}$

simplex set $= \{\alpha \subset \{0, \dots, n+1\} : \alpha \neq \emptyset, \alpha \neq \{0, \dots, n+1\}\}$

Decompose as follows:

$X_- = \Delta^n \subset S^n$, vertex set $= \{0, \dots, n\}$, simplex set $= \{\alpha \subset \{0, \dots, n\} : \alpha \neq \emptyset\}$

Notice that X_- contains a copy of S^{n-1}

i.e. the boundary of Δ^n.

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$X_+ =$ cone on $\partial\Delta^n \subset S^n$
taking $n+1$ to be the cone point.
In detail:
vertex set $= \{0, \dots, n+1\}$
simplex set $= \{ \alpha \in \{0, \dots, n+1\}$ s.t. $\alpha \neq \emptyset,\ \alpha \neq \{0, \dots, n+1\},\ \alpha \neq \{0, \dots, n\} \}$

So $S^n = X_+ \cup X_-$, $S^{n-1} = X_+ \cap X_-$
Now use MV Thm:

$$\begin{tikzcd}
H_n(X_+) \oplus H_n(X_-) \arrow[r] \arrow[d, equal] & H_n(S^n) \arrow[r] & H_{n-1}(S^{n-1}) \arrow[r] & H_{n-1}(X_+) \oplus H_{n-1}(X_-) \arrow[d, equal] \\
0 \arrow[r] & H_n(S^n) \arrow[r, "\cong"] & H_{n-1}(S^{n-1}) \arrow[r] & 0 \\
\text{(cones)} & & & \text{(cones)}
\end{tikzcd}$$

$\Rightarrow H_n(S^n) \cong H_{n-1}(S^{n-1})$
Inductive hypothesis $H_{n-1}(S^{n-1}) \cong \mathbb{F}$
Conclusion: $H_n(S^n) \cong \mathbb{F}$

Summary: $H_k(S^n : \mathbb{F}) = \begin{cases} \mathbb{F} & k = 0, \\ \mathbb{F} & k = n, \\ 0 & \text{otherwise.} \end{cases}$ $(n \geq 1)$

Whitehead's Trick
Suppose $0 \longrightarrow V_n \xrightarrow{T_n} V_{n-1} \longrightarrow \dots \longrightarrow V_1 \xrightarrow{T_1} V_0 \longrightarrow 0$
is an exact sequence of finite dimensional vector spaces and linear maps.
Then $\sum_{r \geq 0} \dim V_{2r} = \sum_{r \geq 0} \dim V_{2r+1}$

Proof
Let $\mathbb{P}(2n)$ be the statement that if
$0 \longrightarrow V_{2n} \longrightarrow V_{2n-1} \rightarrow \dots \longrightarrow V_1 \rightarrow V_0 \longrightarrow 0$
is exact then $\sum_{r=0}^{n} \dim(V_{2r}) = \sum_{r=0}^{n} \dim(V_{2r+1})$.

Let $\mathbb{P}(2n+1)$ be the statement that if

$0 \to V_{2n+1} \to V_{2n} \to \dots \to V_1 \to V_0 \to 0$ is exact

then $\sum_{r=0}^{n} \dim(V_{2r}) = \sum_{r=0}^{n} \dim(V_{2r+1})$.

We know:

$0 \to V_1 \xrightarrow{T_1} V_0 \to 0$ exact, then $\dim V_1 = \dim V_0$

This is $\mathcal{P}(1)$.

Also, $0 \to V_2 \xrightarrow{T_2} V_1 \xrightarrow{T_1} V_0 \to 0$ exact, then

$\dim V_0 + \dim V_2 = \dim V_1$. This is $\mathcal{P}(2)$.

It suffices to show that $\mathcal{P}(2n) \Rightarrow \mathcal{P}(2n+1)$ and $\mathcal{P}(2n+1) \Rightarrow \mathcal{P}(2n+2)$.

Proofs are very similar. $\mathcal{P}(2n+1) \Rightarrow \mathcal{P}(2n+2)$ left as exercise.

$\mathcal{P}(2n) \Rightarrow \mathcal{P}(2n+1)$:

We'll start with $0 \to V_{2n+1} \xrightarrow{T_{2n+1}} V_{2n} \xrightarrow{T_{2n}} V_{2n-1} \to \dots \to V_1 \xrightarrow{T_1} V_0 \to 0$

Split the exact sequence ⟶

$0 \to \operatorname{Ker} T_{2n-1} \to V_{2n-1} \xrightarrow{T_{2n-1}} \dots \to V_1 \xrightarrow{T_1} V_0 \to 0$

$0 \to V_{2n+1} \xrightarrow{T_{2n+1}} V_{2n} \xrightarrow{T_{2n}} \operatorname{Im} T_{2n} \to 0$

Both exact.

By hypothesis $\mathcal{P}(2n)$:

$$\dim(\operatorname{Ker} T_{2n-1}) + \sum_{r=0}^{n-1} \dim V_{2r} = \sum_{r=0}^{n-1} \dim V_{2r+1}$$

Ker-Rank Thm tells us

$$\dim(V_{2n+1}) + \dim(\operatorname{Im} T_{2n}) = \dim(V_{2n})$$

So $\dim(V_{2n+1}) + \dim(\operatorname{Ker} T_{2n-1}) = \dim(V_{2n})$.

So $\dim V_{2n+1} + \dim(\operatorname{Ker} T_{2n-1}) + \sum_{r=0}^{n-1} \dim V_{2r} = \dim V_{2n+1} + \sum_{r=0}^{n-1} \dim V_{2r+1}$

$$\dim(V_{2n}) + \sum_{r=0}^{n-1} \dim(V_{2r}) = \sum_{r=0}^{n-1} \dim(V_{2r+1})$$

So $\mathcal{P}(2n) \Rightarrow \mathcal{P}(2n+1)$ □

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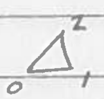
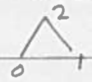
Let's redo $H_* S^1$ using Whitehead's trick.

$S^1 =$ 

, $X_- =$ 

, $X_+ =$ 

$X_+ \cap X_- =$ 

$$\begin{array}{ccccccccccc} H_1(X_+)\oplus H_1(X_-) & \to & H_1(S^1) & \to & H_0(X_+\cap X_-) & \to & H_0(X_+)\oplus H_0(X_-) & \to & H_0(S^1) & \to & 0 \\ \| & & \| & & \| & & \| & & \| & & \\ 0 & & ? & & \mathbb{F}\oplus\mathbb{F} & & \mathbb{F}\oplus\mathbb{F} & & \mathbb{F} & & \\ \text{cones} & & & & \text{2 connected components} & & & & \text{connected} & & \end{array}$$

$\Rightarrow 0 \to H_1(S^1) \to \mathbb{F}^2 \to \mathbb{F}^2 \to \mathbb{F} \to 0$

Use Whitehead:

$\dim \mathbb{F} + \dim \mathbb{F}^2 = \dim \mathbb{F}^2 + \dim H_1(S^1)$

$\Rightarrow 1 + 2 = 2 + \dim H_1(S^1)$

$\Rightarrow \dim H_1(S^1) = 1 \Rightarrow H_1(S^1) = \mathbb{F}$

$$\Rightarrow H_k(S^1;\mathbb{F}) = \begin{cases} \mathbb{F}, & k=0,1 \\ 0, & k>1. \end{cases}$$

$$H_n : \begin{Bmatrix} \text{Simp. complexes} \\ \text{\& simp. maps} \end{Bmatrix} \longrightarrow \begin{Bmatrix} \text{Vector spaces} \\ \text{\& linear maps} \end{Bmatrix}$$

Functor $\quad X \xrightarrow{f} Y \xrightarrow{g} Z$

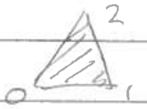
$H_n(g\circ f) = H_n(g)\circ H_n(g)$

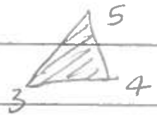
$H_n(Id) = Id.$

$f: X \to Y$ simplicial map

We say f is a simplicial isomorphism when $f: V_X \to V_Y$ is bijective and $f_*: S_X \to S_Y$ is also bijective.

$f_*(\sigma) = f(\sigma).$



$\cong$


$f(0) = 4$, $f(1) = 5$, $f(2) = 3$ defines a simplicial isomorphism.

If $f : X \to Y$ is a simplicial isomorphism then $f^{-1} : Y \to X$ is also a simplicial isomorphism

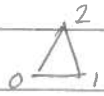
Obvious prop:
If $f : X \to Y$ is a simplicial isomorphism then $H_n(f) : H_n(X) \to H_n(Y)$ is a linear isomorphism.

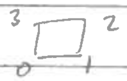
Proof
Consider $f^{-1} : Y \to X$, $f \circ f^{-1} = Id_Y$, $f^{-1} \circ f = Id_X$.
Apply H_n.

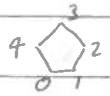
$$H_n(f \circ f^{-1}) = H_n(f) H_n(f^{-1})$$
$$\overset{\shortparallel}{H_n(Id)} = Id \qquad \Rightarrow H_n(f) H_n(f^{-1}) = Id$$

Also $H_n(f^{-1}) H_n(f) = Id$
$\Rightarrow H_n(f)^{-1} = H_n(f^{-1})$. $\square$

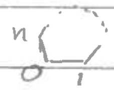
Stronger property
Homology is invariant under subdivision



$= S^1(3) \longrightarrow$


$= S^1(4) \longrightarrow$


$= S^1(5)$

$\cdots \longrightarrow S^1(n) =$


$H_*(S^1(3)) \cong H_*(S^1(n))$

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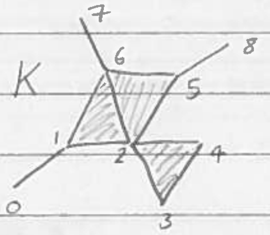
Homology is invariant under subdivision (to be shown in next few lectures)

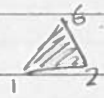
First: Subdivision of a principle simplex

Assume that X is a finite dimensional simplicial complex.

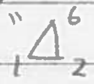
i.e. $\exists N : \forall \sigma \in S_X \quad \dim(\sigma) \leq N$.

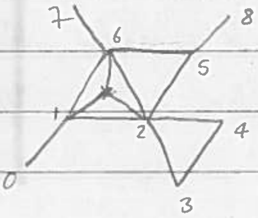
In a finite dimensional complex, every simplex is contained in at least one maximal simplex (or principle simplex), i.e. σ is maximal $\Leftrightarrow$ $\sigma \subset \tau$, $\tau \in S_X \Rightarrow \sigma = \tau$.



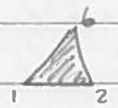
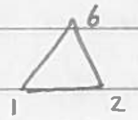
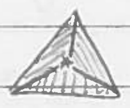
Here  is principle.

$Sd_\sigma(K)$ = subdivision of K at σ

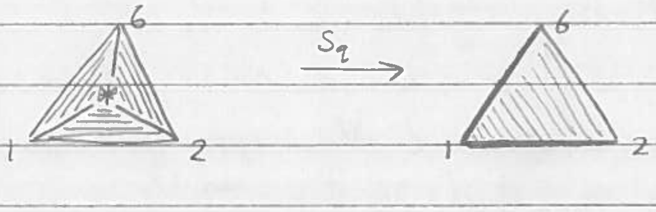




Formally one replaces σ (leaving all its faces in place) and substitutes $C(\partial\sigma)$

$\sigma =$ , $\partial\sigma =$ , $C(\partial\sigma) =$ 

To show $H_*(Sd_\sigma(K)) \cong H_*(K)$ we construct a (squash) mapping $Sq : Sd_\sigma(K) \longrightarrow K$ and show $H_*(Sq) : H_*(Sd_\sigma(K)) \xrightarrow{\cong} H_*(K)$ is an isomorphism.



$\{1, 2, *\} \longmapsto \{1, 2\}$

$\{1, 6, *\} \longmapsto \{1, 6\}$

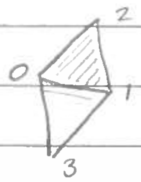
$\{*\} \longmapsto \{1\}$

$\{2, 6, *\} \longmapsto \{1, 2, 6\}$

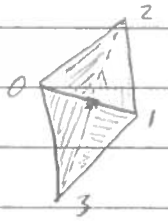
Sq is the identity wherever it makes sense.

$Sq(*)$ is a vertex in $\partial\sigma$.

To subdivide a non principle simplex is more complicated.



$\sigma = [0,1]$



Where do we send $*$?
Coming soon!

To show that $H_*(S_q)$ is an isomorphism, need to consider MV sequence in slightly more detail.
$X = X_+ \cup X_-$
X_+, X_- are subcomplexes

Definition
Let K be a simplicial complex, $K = (V_K, S_K)$.
By a subcomplex $L \subset K$ I mean that $V_L \subset V_K$ and $S_L \subset S_K$.

If X_+, X_- are subcomplexes of X, I write $X = X_+ \cup X_-$ when $V_X = V_{X_+} \cup V_{X_-}$ and $S_X = S_{X_+} \cup S_{X_-}$.

If X_+, X_- are subcomplexes of X then $X_+ \cap X_-$ is the complex $(V_{X_+} \cap V_{X_-}, S_{X_+} \cap S_{X_-})$.

Some elementary linear algebra:
V vector space. Let $\{e_1, \dots, e_m, e_{m+1}, \dots, e_{m+n}\}$ be a spanning set.
Put $V_+ = \text{span}\{e_1, \dots, e_m\}$ and $V_- = \text{span}\{e_{m+1}, \dots, e_{m+n}\}$.
Get a linear mapping $\text{Add}: V_+ \oplus V_- \longrightarrow V$,
$\text{Add}\begin{pmatrix} v_+ \\ v_- \end{pmatrix} = v_+ + v_-$.

Add is surjective because $\{e_1, \dots, e_{m+n}\}$ spans V.

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In general Add is not injective.

$\mathrm{Ker}(\mathrm{Add}) = \left\{ \begin{pmatrix} v_+ \\ v_- \end{pmatrix} : v_+ + v_- = 0 \right\}$

i.e. $\begin{cases} v_- = -v_+ \in V_+ \\ v_+ = -v_- \in V_- \end{cases}$

$\Rightarrow V_+ \in V_+ \cap V_-$ and $V_- = -V_+$.

So get an exact sequence

$$0 \longrightarrow V_+ \cap V_- \xrightarrow{i} V_+ \oplus V_- \xrightarrow{\mathrm{Add}} V \longrightarrow 0$$

$i(v) = \begin{pmatrix} v \\ -v \end{pmatrix}$.

$\dim(V_+) + \dim(V_-) = \dim(V) + \dim(V_+ \cap V_-)$ (algebra 2)

<u>Geometric part of MV sequence.</u>

$X = X_+ \cup X_-$, X, X_+, X_- simplicial complexes

Look at vector spaces

$C_n(X)$, $C_n(X_+)$, $C_n(X_-)$, $C_n(X_+ \cap X_-)$ and so

get an exact sequence

$$\begin{tikzcd}
0 \arrow[r] & C_{n+1}(X_+ \cap X_-) \arrow[r, "i"] \arrow[d, "\partial_{n+1}"] & C_{n+1}(X_+) \oplus C_{n+1}(X_-) \arrow[r] \arrow[d, "\begin{pmatrix} \partial^+_{n+1} & 0 \\ 0 & \partial^-_{n+1} \end{pmatrix}"] & C_{n+1}(X) \arrow[r] \arrow[d, "\partial_{n+1}"] & 0 & \forall n \\
0 \arrow[r] & C_n(X_+ \cap X_-) \arrow[r] \arrow[d, "\partial_n"] & C_n(X_+) \oplus C_n(X_-) \arrow[r] \arrow[d, "\begin{pmatrix} \partial_n^+ & 0 \\ 0 & \partial_n^- \end{pmatrix}"] & C_n(X) \arrow[r] \arrow[d, "\partial_n"] & 0 \\
0 \arrow[r] & C_{n-1}(X_+ \cap X_-) \arrow[r] & C_{n-1}(X_+) \oplus C_{n-1}(X_-) \arrow[r] & C_{n-1}(X) \arrow[r] & 0
\end{tikzcd}$$

(everything commutes)

These sequences are compatible with the boundary maps

<u>Definition</u>

By an exact sequence of (algebraic) chain complexes

$$0 \to A_* \xrightarrow{i} B_* \xrightarrow{p} C_* \longrightarrow 0$$

I mean that (i) $A_* = (A_n, \partial_n^A)$, $B_* = (B_n, \partial_n^B)$, $C_* = (C_n, \partial_n^C)$ are all chain complexes,

(ii) i, p are chain mappings (i.e. they commute with boundary maps)

(iii) For each n, $0 \longrightarrow A_n \xrightarrow{i_n} B_n \xrightarrow{p_n} C_n \longrightarrow 0$ is exact.

$$\begin{tikzcd}
0 \arrow[r] & A_{n+1} \arrow[r, "i_{n+1}"] \arrow[d, "\partial^A_{n+1}"] & B_{n+1} \arrow[r, "p_{n+1}"] \arrow[d, "\partial^B_{n+1}"] & C_{n+1} \arrow[r] \arrow[d, "\partial^C_{n+1}"] & 0 \\
0 \arrow[r] & A_n \arrow[r, "i_n"] & B_n \arrow[r, "p_n"] & C_n \arrow[r] & 0
\end{tikzcd}$$

So the geometric part of MV Thm is:
if $X = X_+ \cup X_-$ (union of complexes)
then $\exists$ exact sequence of chain complexes
$0 \longrightarrow C_*(X_+ \cap X_-) \xrightarrow{i} C_*(X_+) \oplus C_*(X_-) \xrightarrow{Add} C_*(X) \rightarrow 0$

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$X = X_+ \cup X_-$ get an exact sequence of chain complexes.
$0 \longrightarrow C_*(X_+ \cap X_-) \rightarrow C_*(X_+) \oplus C_*(X_-) \longrightarrow C_*(X) \rightarrow 0$

differential here is $\begin{pmatrix} \partial_+ & 0 \\ 0 & \partial_- \end{pmatrix}$

Geometric aspect (done!) of Mayer Vietoris.

The algebraic part of MV:
Given an exact sequence of chain complexes
$0 \rightarrow A_* \xrightarrow{i} B_* \xrightarrow{p} C_* \rightarrow 0$
i induces $H(i) : H_*(A) \rightarrow H_*(B)$
p induces $H(p) : H_*(B) \rightarrow H_*(C)$.

$$H_{n+2}(C) \xrightarrow{\delta} H_{n+1}(A) \xrightarrow{H(i)} H_{n+1}(B) \xrightarrow{H(p)} H_{n+1}(C) \xrightarrow{\delta} H_n(A) \xrightarrow{H(i)} H_n(B) \xrightarrow{H(p)} H_n(C) \xrightarrow{\delta} H_{n-1}(A)$$

$\exists$ linear maps (connecting homomorphisms or boundary maps)
$\delta : H_{n+1}(C) \rightarrow H_n(A)$ such that the whole sequence is exact.
[Long exact sequence in homology (H. Cartan, S. Eilenberg c1944).]

If $X = X_+ \cup X_-$
$0 \rightarrow C_*(X_+ \cap X_-) \rightarrow C_*(X_+) \oplus C_*(X_-) \rightarrow C_*(X) \rightarrow 0$
we get a LES:

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Let $Y = Y_+ \cup Y_-$

$$\begin{tikzcd}[ampersand replacement=\&]
\cdots \arrow[r] \& H_{n+2}(X) \arrow[r, "\delta"] \arrow[dd, "f_*"] \& H_{n+1}(X_+ \cap X_-) \arrow[r, "i_*"] \arrow[dd, bend right, "f_*"] \& H_{n+1}(X_+) \oplus H_{n+1}(X_-) \arrow[r, "Add_*"] \arrow[dd, "\begin{pmatrix} f_* \& 0 \\ 0 \& f_* \end{pmatrix}"] \& H_{n+1}(X) \arrow[r, "\delta"] \arrow[dd, "f_*"] \& H_n(X_+ \cap X_-) \arrow[r, "i_*"] \arrow[dd, "f_*"] \& {} \& {} \\
\& {} \arrow[r, "i_*"] \& H_n(X_+) \oplus H_n(X_-) \arrow[r, "Add_*"] \arrow[drrrr, dashed, "\begin{pmatrix} f_* \& 0 \\ 0 \& f_* \end{pmatrix}"] \& H_n(X) \arrow[r, "\delta"] \arrow[drrrr, dashed, "f_*"] \& \cdots \& \& \& \\
\cdots \arrow[r] \& H_{n+2}(Y) \arrow[r] \& H_{n+1}(Y_+ \cap Y_-) \arrow[r] \& H_{n+1}(Y_+) \oplus H_{n+1}(Y_-) \arrow[r] \& H_{n+1}(Y) \arrow[r] \& H_n(Y_+ \cap Y_-) \arrow[r] \& H_n(Y_+) \oplus H_n(Y_-) \arrow[r] \& H_n(Y) \arrow[r] \& \cdots
\end{tikzcd}$$

Let $f : X \to Y$, $f(X_+) \subset Y_+$, $f(X_-) \subset Y_-$

so $f(X_+ \cap X_-) \subset Y_+ \cap Y_-$

Naturality: the diagram above commutes.

We'll show that if $X' = Sd_\sigma(X)$ where σ is a principle simplex then

$Sq : H_*(X') \xrightarrow{Sq_*} H_*(X)$ where Sq is the squash map.

X is a simplicial complex and $\sigma \in S_X$ is a principle simplex. Assume X is finite ~~[illegible]~~ (not necessary but easier).

List the principle simplicies of X starting with $\sigma = \sigma_0, \sigma_1, \dots, \sigma_N$.

Write X_+ for the subcomplex generated by σ, ie. σ and all its faces.

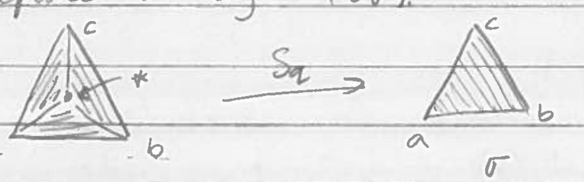
Write X_- for the subcomplex of X generated by $\sigma_1, \dots, \sigma_N$ (ie. $\sigma_1, \dots, \sigma_N$ and all their faces).

So $X = X_+ \cap X_-$ and $X_+ \cap X_- \subset \partial\sigma$.

Now form subdivision $Sd_\sigma(X) = X'$.

$X' = X_+' \cup X_-'$ where $X_-' \equiv X_-$ and $X_+' = C(\partial_\sigma)$

ie. replace σ by $C(\partial_\sigma)$.



map $* \to$ either a, b, or c

ie. suppose $Sq(*) = a$ and

$Sq|_{\partial_\sigma} = Id_{\partial_\sigma}$.

Now extend Sq to $Sq : X' \longrightarrow X$

$Sq|_{X_+} = Sq$, $Sq|_{X_-} = Id$ $\quad (X'_- \equiv X_-)$

$$\begin{tikzcd}
H_n(X'_+ \cap X'_-) \arrow[r] \arrow[d, "\cong"', "Id"] & H_n(X'_+) \oplus H_n(X'_-) \arrow[r] \arrow[d, "\cong"', "\begin{pmatrix} Sq & 0 \\ 0 & Id \end{pmatrix}"] & H_n(X') \arrow[r] \arrow[d, "Sq"] & H_{n-1}(X'_+ \cap X'_-) \arrow[r] \arrow[d, "\cong"', "Id"] & H_{n-1}(X'_+) \oplus H_{n-1}(X'_-) \arrow[d, "\cong"', "\begin{pmatrix} Sq & 0 \\ 0 & Id \end{pmatrix}"] \\
H_n(X_+ \cap X_-) \arrow[r] & H_n(X_+) \oplus H_n(X_-) \arrow[r] & H_n(X) \arrow[r] & H_{n-1}(X_+ \cap X_-) \arrow[r] & H_{n-1}(X_+) \oplus H_{n-1}(X_-)
\end{tikzcd}$$

$X'_+ \cap X'_- \subset \partial\sigma$ and $Sq|_{\partial\sigma} = Id$, $Sq : X'_- \longrightarrow X_- = Id$

Assume X is connected (for simplicity)

Then X' is also connected

Prop

$Sq : H_n(X') \xrightarrow{\simeq} H_n(X)$ is an isomorphism $\forall n \geq 0$.

Proof

For $n=0$, both connected so okay.

Suppose $n=1$.

Then $Sq : H_0(X'_+) \oplus H_0(X'_-) \xrightarrow{\simeq} H_0(X_+) \oplus H_0(X_-)$ $\quad \begin{pmatrix} Sq \checkmark & 0 \\ 0 & Id \end{pmatrix}$

$Sq : H_1(X'_+) \oplus H_1(X'_-) \xrightarrow{\simeq} H_1(X_+) \oplus H_1(X_-)$ because

$H_1(X'_+) = 0$ since X'_+ is a cone, $\quad \begin{pmatrix} 0 & 0 \\ 0 & Id \end{pmatrix} \checkmark$

$H_1(X_+) = 0$ since $X_+ = \langle \sigma \rangle$ is a simplex

When $n \geq 2$.

$H_n(X'_+) = 0 = H_{n-1}(X'_+)$ cone.

$H_n(X_+) = 0 = H_{n-1}(X_+)$ simplex.

So in each case $n \geq 1$, we have a commutative diagram as above with exact rows and where outer maps are isomorphisms.

Now appeal to Five Lemma to conclude that $Sq : H_n(X') \xrightarrow{\simeq} H_n(X)$ is an isomorphism. □

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Five Lemma

Given a commutative diagram of abelian groups and homomorphisms with exact rows, and f_0, f_1, f_3, f_4 are isomorphisms. Then f_2 is also an isomorphism.

$$\begin{tikzcd}
A_0 \arrow[r, "\alpha_0"] \arrow[d, "f_0"] & A_1 \arrow[r, "\alpha_1"] \arrow[d, "f_1"] & A_2 \arrow[r, "\alpha_2"] \arrow[d, "f_2"] & A_3 \arrow[r, "\alpha_3"] \arrow[d, "f_3"] & A_4 \arrow[d, "f_4"] \\
B_0 \arrow[r, "\beta_0"] & B_1 \arrow[r, "\beta_1"] & B_2 \arrow[r, "\beta_2"] & B_3 \arrow[r, "\beta_3"] & B_4
\end{tikzcd}$$

Proof (diagram chasing).

1). f_2 is injective:

Suppose $x \in A_2$ has $f_2(x) = 0$.

WTS: $x = 0$.

So $\beta_2(f_2(x)) = 0$

So $f_3(\alpha_2(x)) = 0$ (commutativity)

f_3 injective so $\alpha_2(x) = 0$

$\Rightarrow x \in \mathrm{Ker}\,\alpha_2 = \mathrm{Im}\,\alpha_1$

Choose $y \in A_1$ s.t. $\alpha_1(y) = x$

So $f_2(\alpha_1(y)) = f_2(x) = 0$

$\Rightarrow \beta_1(f_1(y)) = 0$ (commutativity)

$\Rightarrow f_1(y) \in \mathrm{Ker}\,\beta_1 = \mathrm{Im}\,\beta_0$

Choose $z \in \beta_0$ s.t. $\beta_0(z) = f_1(y)$

f_0 is surjective so choose $w \in A_0$ s.t. $f_0(w) = z$

$\beta_0(f_0(w)) = \beta_0(z) = f_1(y)$

$\beta_0(f_0(w)) = f_1(\alpha_0(w))$

So $f_1(\alpha_0(w)) = f_1(y) \Rightarrow \alpha_0(w) = y$ since f_1 injective

$\Rightarrow \alpha_1\alpha_0(w) = \alpha_1(y) = x$

But $\alpha_1\alpha_0 \equiv 0$ so $x = 0$. So f_2 is injective.

(ii) f_2 is surjective: NTF: $y \in A_2$ s.t. $f_2(y) = x$, $x \in B_2$

f_3 is surjective so choose $w \in A_3$: $f_3(w) = \beta_2(x)$

$\beta_3(f_3(w)) = \beta_3(\beta_2(x)) = 0$ $(\beta_3\beta_2 \equiv 0)$

$f_4(\alpha_3(w)) = \beta_3(f_3(w)) = 0.$

f_4 is injective so $\alpha_3(w) = 0$.

$w \in \operatorname{Ker} \alpha_3 = \operatorname{Im} \alpha_2$

Choose $y' \in A_2$ s.t. $\alpha_2(y') = w$

$\beta_2(f_2(y')) = f_3(\alpha_2(y')) = f_3(w) = \beta_2(x)$

So $\beta_2(x - f_2(y')) = 0$

So $x - f_2(y') \in \operatorname{Ker} \beta_2 = \operatorname{Im} \beta_1$

Choose $z \in B_1$ s.t. $\beta_1(z) = x - f_2(y')$

f_1 is surjective, so choose $u \in A_1$ s.t. $f_1(u) = z$.

$\beta_1(f_1(u)) = \beta_1(z) = x - f_2(y')$

$f_2(\alpha_1(u)) = x - f_2(y')$

$\Rightarrow x = f_2(y' + \alpha_1(u))$

Put $y = y' + \alpha_1(u)$.

So f_2 is surjective.

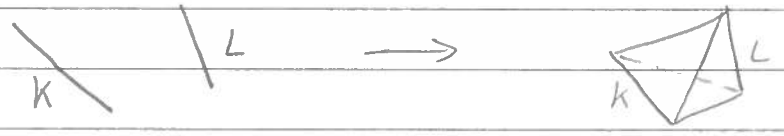
□

So now we've proved that $H_*(Sd_\sigma(X)) \cong H_*(X)$ provided σ is a principle simplex.

Joins, links and stars:

Suppose K, L are simplicial complexes and $K \cap L = \emptyset$.

Define the abstract join $K * L$ as follows;



$\Delta^1 * \Delta^1 = \Delta^3$ $[S^m * S^n = S^{m+n+1}]$

$V_{K*L} = V_K \sqcup V_L$ $(K \cap L = \emptyset)$

$S_{K*L} = S_K \sqcup S_L \sqcup \{\sigma \cup \tau : \sigma \in S_K, \tau \in S_L\}$

i.e. we join everything in K to everything in L.

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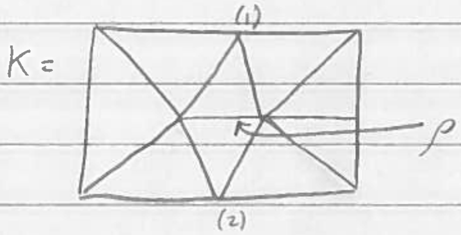
Special case: K = single point = pt.
$K * L = \{pt\} * L = C(L)$ (cone on L).

Obvious fact
Suppose K, L, M are complexes
st. $K \cap L = \emptyset$, $K \cap M = \emptyset$, $L \cap M = \emptyset$.
Then $(K * L) * M \equiv K * (L * M)$.

Proof
Look at definition. $\square$

Corollary
If $K \cong \Delta^m$, then $K * L$ is a cone.
$K * L = C(\Delta^{m-1} * L)$

Proof
Write $\Delta^m = \{pt\} * \Delta^{m-1}$
$$\Delta^m * L = \{pt\} * (\Delta^{m-1} * L)$$
$$= C(\Delta^{m-1} * L) \quad \square$$



(non principle simplex in K).
$Lk(\rho, K) = (1) \sqcup (2)$

Let $\sigma \in S_K$. Say that σ is joinable to ρ in K
when $\rho \cap \sigma = \emptyset$ and $\rho \cup \sigma \in S_K$
Define $Lk(\rho, K)$ by link of ρ in K.
$Lk(\rho, K)$ consists of all simplices σ of K st. σ is joinable to ρ in K.

Tautologically: $\rho * Lk(\rho, K)$ imbeds as a subcomplex of K.
This subcomplex is called the Star of ρ in K.
[In above example $Star(\rho, K) =$ ]

Prop
$\rho * Lk(\rho, K)$ is a cone (so it has trivial homology)

Proof
ρ is a simplex.
□

Def
Subdivision at a non-principle simplex.
Let ρ be a non-principle simplex of X.
Let X_+ be subcomplex of X generated by the principle simplices which contain ρ.
Then $X_+ = \rho * Lk(\rho, K)$
Let X_- be subcomplex of X generated by the principle simplices which do not contain ρ.
$X_+ \cap X_- = \partial\rho * Lk(\rho, K)$.
Then $Sd_\rho(x) = X_+' \cup X_-$ where $X_+' = C(\partial\rho) * Lk(\rho, K)$.

Prop
$H_* Sd_\rho(x) \cong H_*(X)$.

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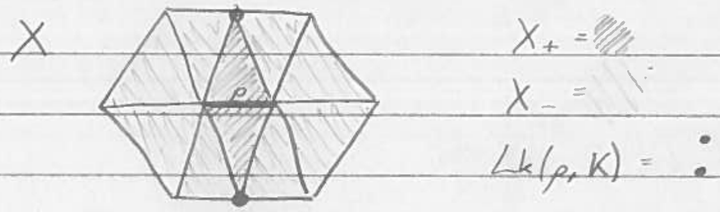
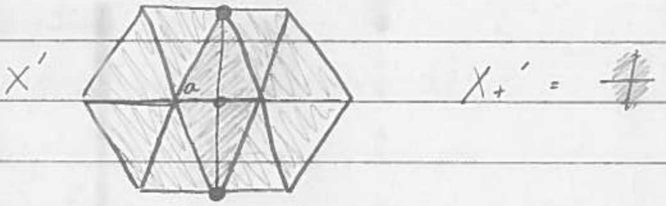
X simplicial complex (finite)
ρ a non-principle simplex
$X = X_+ \cup X_-$ where $X_+ = \rho * Lk(\rho, K)$
$X_+ \cap X_- \subset \partial\rho * Lk(\rho, K)$

Define $Sd_\rho(X) = X_+' \cup X_-$
$X_+' = C(\partial\rho) * Lk(\rho, K)$





Define $S_q : X' \to X$
$S_q|_{X_-} = Id$, $S_q|_{X_+}$ maps cone point to a vertex in $\partial\rho$

Map cone point to a for example.

$$\begin{tikzcd}
H_n(X_+\cap X_-) \arrow[r] \arrow[d, "Id"] & H_n(X_+')\oplus H_n(X_-) \arrow[r] \arrow[d, "\begin{pmatrix} S_q & 0 \\ 0 & Id \end{pmatrix}"] & H_n(X') \arrow[r] \arrow[d, "S_q"] & H_{n-1}(X_+\cap X_-) \arrow[r] \arrow[d, "Id"] & H_{n-1}(X_+')\oplus H_{n-1}(X_-) \arrow[d, "\begin{pmatrix} S_q & 0 \\ 0 & Id \end{pmatrix}"] \\
H_n(X_+\cap X_-) \arrow[r] & H_n(X_+)\oplus H_n(X_-) \arrow[r] & H_n(X) \arrow[r] & H_{n-1}(X_+\cap X_-) \arrow[r] & H_{n-1}(X_+)\oplus H_{n-1}(X_-)
\end{tikzcd}$$

⓪ ① ② ③ ④

At ⓪ and ③ we obviously have isomorphisms (=Id)
At ① and ④ both $X_+ = \rho * Lk(\rho, K)$ and $X_- = C(\partial\rho) * Lk(\rho, K)$ are cones, so both are isomorphisms.
$\Rightarrow$ We have an isomorphism at ② by the Five-Lemma
ie. $S_q : \underset{H_n(Sd_\rho(X))}{\overset{\shortparallel}{H_n(X')}} \xrightarrow{\cong} H_n(X)$.

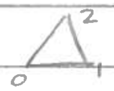
This finishes the proof of homology invarience under subdivision.

Definition

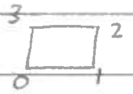
Let X, Y be simplicial complexes.
Say that X, Y are combinatorially equivalent $(X \sim Y)$
when $\exists$ finite sequence of complexes $X_0, X_1, \dots, X_N$
s.t. $X_0 \cong X$, $X_N \cong Y$ (simplicially isomorphic)
and for each $i \geqslant 1$ either
$X_i = Sd_\rho(X_{i-1})$ for some ρ
or $X_{i-1} = Sd_\rho(X_i)$ for some ρ.

Corollary

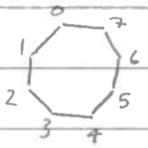
If $X \sim Y$, then $H_*(X) \cong H_*(Y)$.

Basic circle: $S^1 =$ 

$= S^1(3)$ (no middle)

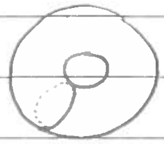
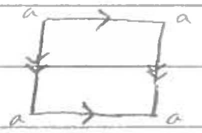
Subdivide: $S^1(4) =$ 

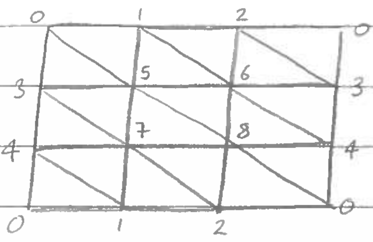
Then $H_*(S^1(4)) \cong H_*(S^1(3))$



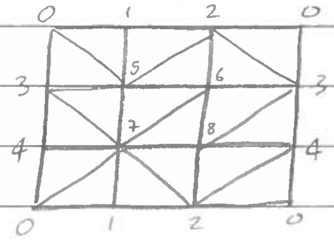
$= S^1(8)$, $H_*(S^1(8)) \cong H_*(S^1(3))$

$H_*(S^1(10^{10})) \cong H_*(S^1(3))$

$T^2 = S^1 \times S^1$ 2-torus  = 

$X(1) =$ 

A simplicial model of $T^2 = S^1 \times S^1$

$X(2) =$ 

Also a simplicial model of T^2.

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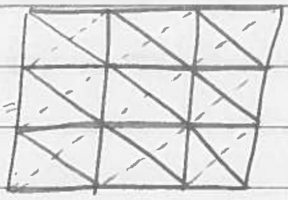
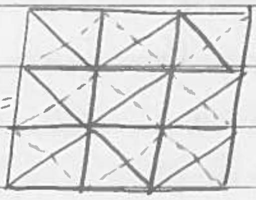
$X(1) \not\cong X(2)$ However $X(1) \sim X(2)$

$X(1)' =$



$X(2)' =$



However $X(1)' =$ subdivision of $X(1)$

simp. iso. $\wr\wr$

$X(2)' =$ subdivision of $X(2)$

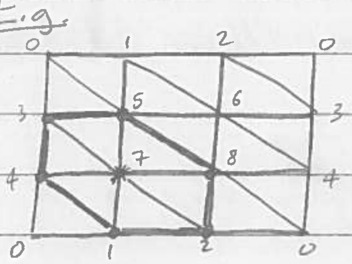
$\Rightarrow H_*(X(1)) \cong H_*(X(2))$.

Definition

Let X be a simplicial complex.

Say that X is a simplicial surface when $\forall$ vertex v of X $\exists n$ s.t. $Lk(v, X) \cong S^1(n)$

E.g.

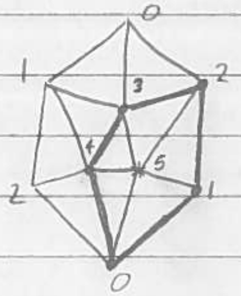


$= X$, $v = 7$

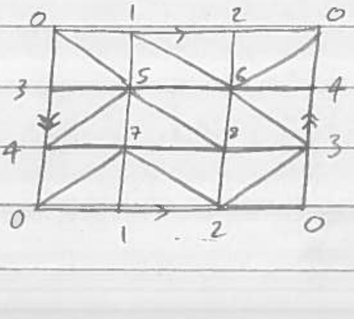
$Lk(7, X) \cong S^1(6)$

Example

$\mathbb{RP}(2)$



$Lk(5, \mathbb{RP}(2)) \cong S^1(5)$



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Definition

A simplicial surface is a complex $K = (V_K, S_K)$ such that $\forall v \in V_K$ $\exists n$ s.t. $Lk(v, K) \cong S'(n)$, where $S'(n)$ = circle with n vertices.

Generalisation

By a simplicial n-manifold $X = (V_X, S_X)$ we mean a simplicial complex in which $\forall v \in V_K$, $Lk(v, X) \sim S^{n-1}$.
A surface is a simplicial 2-manifold.

Simplicial 1-manifolds : 

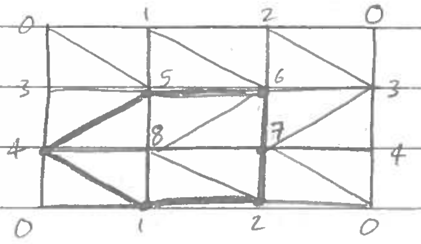
Example (tetrahedron)

Basic S^2 is a surface.
$Lk(v, S^2) \cong S'(3)$.

If Σ is a surface, ρ a simplex, then $Sd_\rho(\Sigma)$ is still a surface.

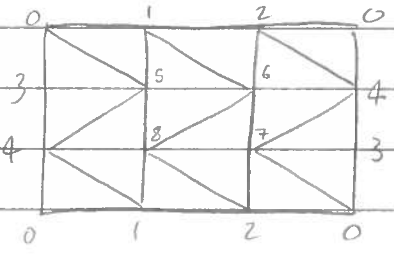
In the dodecahedron Σ, $Lk(v, \Sigma) \cong S'(5)$.

T^2 standard model of 2-torus. 



$Lk(8, T^2) \cong S'(6)$

Klein bottle, K^2 



$Lk(8, K^2) \cong S'(6)$

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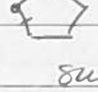
Orientability of Surfaces

Prop

If Σ is a surface and ρ is a 1-simplex, then ρ is contained in exactly two 2-simplices.

Proof

In $S^1(n)$ every vertex belongs to exactly two 1-simplices.



Suppose Σ is a surface, v a vertex.

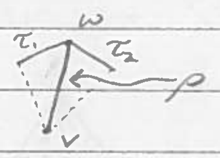
$Lk(v, \Sigma) \cong S^1(n)$.

Let ρ be a 1-simplex in Σ. Let v be a vertex, $v \in \rho$.

Let $w \in \rho$ be the other vertex; $\rho = \{v, w\}$.

$Lk(v, \Sigma) \cong S^1(n)$

$w \in Lk(v, \Sigma)$. Let $\tau_1, \tau_2 \subset Lk(v, \Sigma)$ be two 1-simplices, $w \in \tau_1$, $w \in \tau_2$.



Then $v * \tau_1$ is a 2-simplex,

$v * \tau_2$ " " " ".

These are the only 2-simplices which contain ρ.

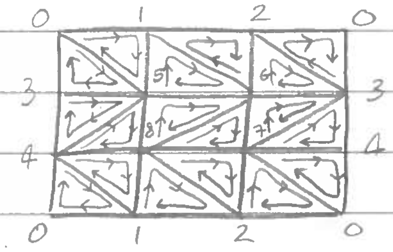
$\square$

Definition

Let Σ be a connected surface. Say that Σ is <u>orientable</u> when it is possible to orient each 2-simplex in such a way that every 1-simplex recieves opposite orientations from the 2-simplices to which it belongs.

Example

T^2 is orientable.

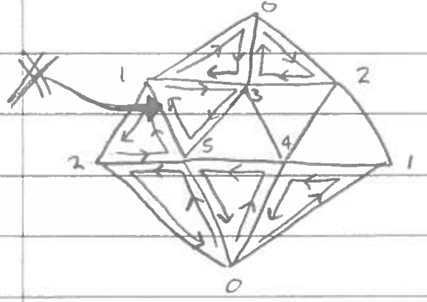


Exercise

Show K^2 is non-orientable

Example

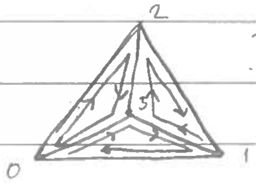
$\mathbb{R}P^2$ is nonorientable



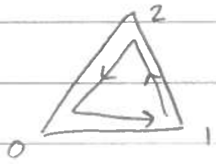
(Boy's Surface)

Example

S^2 is orientable



Top view



bottom view

Theorem (Orientation Thm)

Let Σ be a finite connected surface and let $\mathbb{F}$ be a field.

(i) If char $\mathbb{F} \neq 2$ (e.g. $\mathbb{Q}$, $\mathbb{F}_3, \ldots$) then Σ is orientable iff $H_2(\Sigma; \mathbb{F}) \cong \mathbb{F}$, and Σ is non-orientable iff $H_2(\Sigma; \mathbb{F}) = 0$.

(ii) If char $\mathbb{F} = 2$ then $H_2(\Sigma; \mathbb{F}) \cong \mathbb{F}$ regardless of whether Σ is orientable or not.

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How to compute homology of surfaces quickly.

Euler Characteristic (Primitive def^n)

Let $K = (V_K, S_K)$ be a finite simplicial complex.

For each $r \geq 0$ put $S_K(r) = \{ \sigma \in S_K : \dim(\sigma) = r, \text{ i.e. } |\sigma| = r+1 \}$.

$$\chi_{geom}(K) = \sum_{r \geq 0} (-1)^r |S_K(r)|$$

[In English this is the alternating sum of r-simplices.]

Observe that $|S_K(r)| = \dim_{\mathbb{F}}(C_r(K;\mathbb{F}))$ for any field $\mathbb{F}$.

So we think of $\chi_{geom}(K)$ as

$$\chi_{geom}(K) = \sum_{r \geq 0} (-1)^r \dim_{\mathbb{F}}(C_r(K:\mathbb{F}))$$

We also have homological Euler characteristic

$$\chi_{hom}(K) = \sum_{r \geq 0} (-1)^r \dim_{\mathbb{F}}(H_r(K:\mathbb{F})).$$

Theorem

If K is a finite simplicial complex then $\chi_{hom}(K) = \chi_{geom}(K)$.

Proof

Write $C_r = C_r(K:\mathbb{F})$ (fix $\mathbb{F}$)

$\partial_r : C_r \to C_{r-1}$

$Z_r = \operatorname{Ker} \partial_r$, $B_{r-1} = \operatorname{Im} \partial_r$

$B_r = \operatorname{Im} \partial_{r+1}$

$H_r = Z_r / B_r$ (write $H_r = H_r(K:\mathbb{F})$)

We have two exact sequences

(I): $0 \to Z_r \to C_r \to B_{r-1} \to 0$

(II): $0 \to B_r \to Z_r \to H_r \to 0$

From (I), $\dim C_r = \dim Z_r + \dim B_{r-1}$
From (II), $\dim Z_r = \dim H_r + \dim B_r$

So $\dim C_r = \dim H_r + \dim B_r + \dim B_{r-1}$
Take alternating sums, $-\infty < r < \infty$.

$$\sum_r (-1)^r \dim C_r = \sum_r (-1)^r \dim H_r + \underbrace{\sum_r (-1)^r \dim B_r}_{B} + \sum_r (-1)^r \dim B_{r-1}$$

$$\sum_r (-1)^r \dim B_{r-1} = (-1) \sum_r (-1)^{r-1} B_{r-1} = -B$$

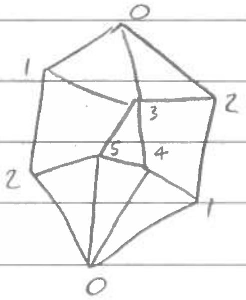
$$\text{So} \quad \sum_r (-1)^r \dim C_r = \sum_r (-1)^r \dim H_r$$

So $\chi_{geom}(K) = \chi_{hom}(K)$.

□

How to compute homology of surfaces assuming the orientation thm.

$H_*(\mathbb{RP}^2; \mathbb{Q})$



$\chi_{geom}(\mathbb{RP}^2) = 6 - 15 + 10$
$\Rightarrow \chi_{geom}(\mathbb{RP}^2) = 1$

$H_*(\mathbb{RP}^2; \mathbb{Q})$
$h_0 = \dim H_0(\mathbb{RP}^2, \mathbb{Q}) = 1$ (connected)
$h_1 = \dim H_1(\mathbb{RP}^2, \mathbb{Q}) = ?$
$h_2 = \dim H_2(\mathbb{RP}^2, \mathbb{Q}) = 0$ (non-orientable)

$$\chi_{hom}(\mathbb{RP}^2) = 1 = h_0 - h_1 + h_2$$
$$= 1 - h_1 + 0 \Rightarrow h_1 = 0.$$

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$$\Rightarrow H_r(\mathbb{RP}^2, \mathbb{Q}) = \begin{cases} \mathbb{Q} & , r=0 \\ 0 & , r \geq 1 \end{cases}$$

$H_*(\mathbb{RP}^2, \mathbb{F}_2)$ ($\mathbb{F}_2$ field with two elements).

$\dim H_0 = h_0 = 1$ connected

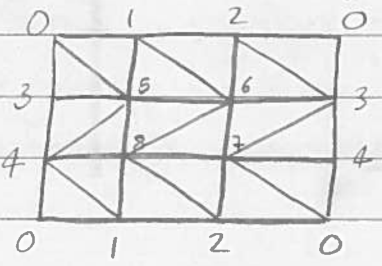
$\dim H_1 = h_1 = ?$

$\dim H_2 = h_2 = 1$ ($\mathbb{F}_2$ has char $= 2$)

$$\chi_{hom}(\mathbb{RP}^2) = 1 = 1 - h_1 + 1 \Rightarrow h_1 = 1$$

$$\Rightarrow H_k(\mathbb{RP}^2, \mathbb{F}_2) = \begin{cases} \mathbb{F}_2 & , k = 0, 1, 2 \\ 0 & , k \geq 3. \end{cases}$$

$H_*(T^2 : \mathbb{F})$ $\mathbb{F}$ any field, T^2 orientable.



9 vertices, 27 1-simplices

18 2-simplices

$$\chi(T^2) = 9 - 27 + 18 = 0.$$

Write $h_i = \dim H_i(T^2 : \mathbb{F})$

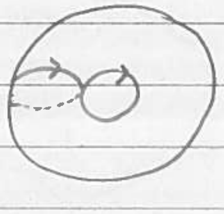
$h_0 = 1$ (connected)

$h_1 = ?$

$h_2 = 1$ (orientable)

$h_0 - h_1 + h_2 = 0 \Rightarrow h_1 = h_0 + h_2 = 2$

$$\Rightarrow H_k(T^2 : \mathbb{F}) = \begin{cases} \mathbb{F} & k = 0 \\ \mathbb{F} \oplus \mathbb{F} & k = 1 \\ \mathbb{F} & k = 2 \\ 0 & k \geq 3 \end{cases}$$



Σ surface.

$H_0(\Sigma : \mathbb{F})$ tells you whether Σ is connected.

$H_1(\Sigma : \mathbb{F})$ tells you how "big" Σ is.

$H_2(\Sigma : \mathbb{F})$ tells you whether Σ is orientable.

Connected sum

Definition

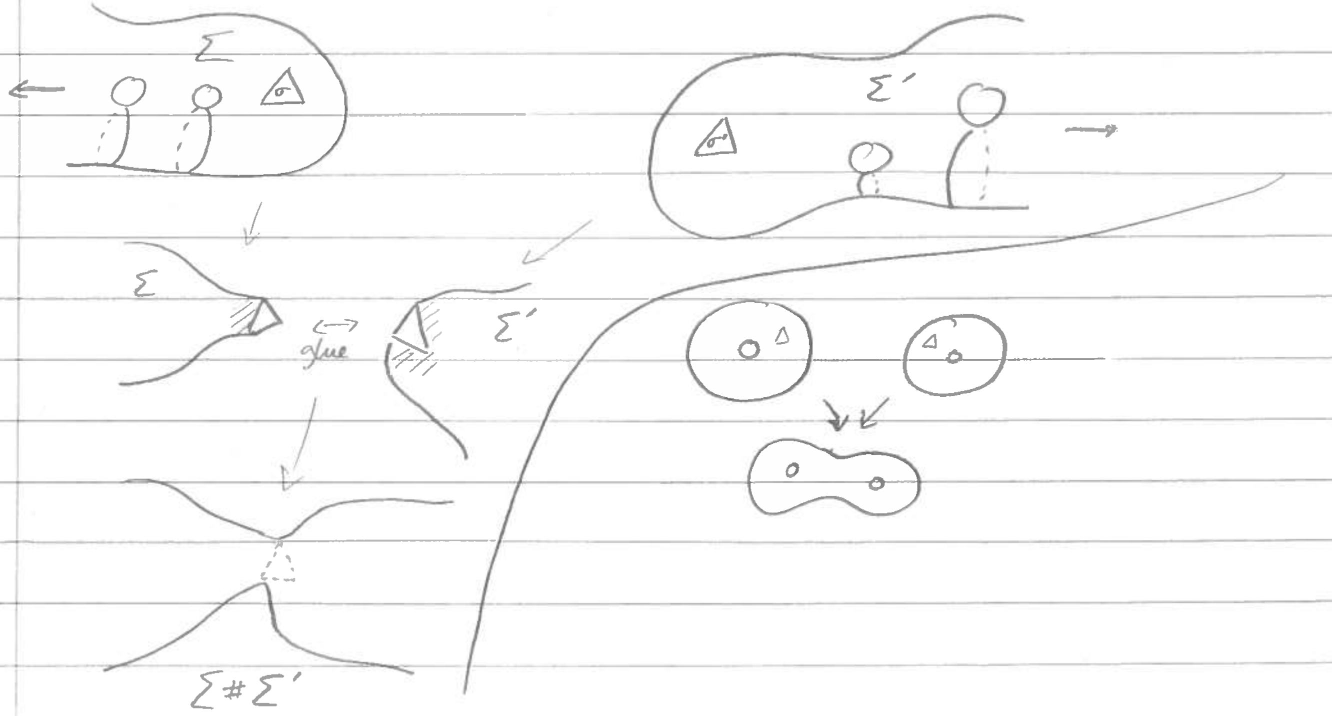
Let Σ be a simplicial surface.
Let Σ_0 be the complex obtained by removing exactly one 2-simplex σ.
Σ_0 is a "bounded surface", it has boundary $\partial\Sigma_0 = \partial\sigma$.

Definition

Suppose Σ, Σ' are simplicial surfaces, $\Sigma \cap \Sigma' = \emptyset$.
To form the connected sum
$\Sigma \# \Sigma'$, we remove a 2-simplex σ from Σ, σ' from Σ' and glue the boundaries.

$$\Sigma \# \Sigma = \Sigma_0 \underset{\partial\Sigma_0 = \partial\Sigma'_0}{\cup} \Sigma'_0 .$$

Picture:



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If Σ is a finite connected surface, then Σ is combinatorially equivalent to exactly one of the following:

$S^2 \quad T^2 \quad T^2 \# T^2 \qquad \underbrace{T^2 \# \dots \# T^2}_{g}$

$\mathbb{RP}^2 \quad \underset{\underset{\mathbb{RP}^2 \# \mathbb{RP}^2}{\shortparallel}}{K^2} \quad \dots \qquad \underbrace{\mathbb{RP}^2 \# \dots \# \mathbb{RP}^2}_{g}$

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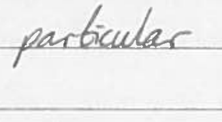
Σ, Σ' connected surfaces, $\Sigma \# \Sigma' = \Sigma_0 \underset{\partial = \partial'}{\cup} \Sigma_0'$

Σ_0, Σ_0' are Σ, Σ' with one 2-simplex removed.
Up to combinatorial equivalence the result is independent of the particular simplices removed.

Generalisation:
If X, X' are connected simplicial n-manifolds.
$X \# X' = X_0 \underset{\partial = \partial'}{\cup} X_0'$ where X_0, X_0' are X, X' with one n-simplex removed.

Prop
If Σ, Σ' are connected surfaces, then
$\chi(\Sigma \# \Sigma') = \chi(\Sigma) + \chi(\Sigma') - 2$

Proof



$\chi(\Sigma_0) = \chi(\Sigma) - 1$, $\chi(\Sigma_0') = \chi(\Sigma') - 1$ (removing a 2 simplex in each case).
$\chi(\Sigma \# \Sigma') = \chi(\Sigma_0) + \chi(\Sigma_0')$

↖ visual proof

Chain complex proof:

$\Sigma \# \Sigma' = \Sigma_0 \underset{\partial = \partial'}{\cup} \Sigma_0'$ $(\partial \sim S^1)$

$\Sigma_0 \cap \Sigma_0' = \partial \sim S^1$

We get an exact sequence of chain complexes.

$$0 \to C_*(\partial) \to C_*(\Sigma_0) \oplus C_*(\Sigma_0') \to C_*(\Sigma \# \Sigma') \to 0$$

In each dimension we have an exact sequence

$$0 \to C_n(\partial) \to C_n(\Sigma_0) \oplus C_n(\Sigma_0') \to C_n(\Sigma \# \Sigma') \to 0$$

$\dim C_n(\Sigma \# \Sigma') + \dim C_n(\partial) = \dim C_n(\Sigma_0) + \dim C_n(\Sigma_0')$

Take alternating sums:

$\chi(\Sigma \# \Sigma') + \chi(\partial) = \chi(\Sigma_0) + \chi(\Sigma_0')$

$\partial = \Sigma_0 \cap \Sigma_0' \sim S^1$, $\chi(\partial) = 0$

$\Rightarrow \chi(\Sigma \# \Sigma') = \chi(\Sigma_0) + \chi(\Sigma_0')$

$= \chi(\Sigma) + \chi(\Sigma') - 2$

Here we've used the following

$0 \to A_* \to B_* \to C_* \to 0$ exact sequence of chain complexes, then $\chi(B_*) = \chi(A_*) + \chi(C_*)$

[Same formula true in even dimensions]

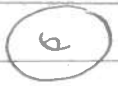
$\square$

$\chi(S^2) = 2$

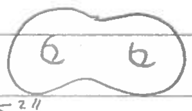
orientable surface of genus 0

$\Sigma_+^0 =$

$= \Sigma_+^1$ (orientable, genus 1)



$\chi(T^2) = 0$

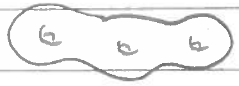


$\Sigma_+^{2''}$

$\chi(T^2 \# T^2) = -2$

$[= \chi(T^2) + \chi(T^2) - 2]$

$, \Sigma_+^3$



$\chi(T^2 \# T^2 \# T^2) = -4$

<u>Def</u>

For $g \geq 2$, $\Sigma_+^g = \underbrace{T^2 \# \ldots \# T^2}_{g}$

<u>Prop</u> $\chi(\Sigma_+^g) = 2 - 2g$

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Proof

True for $g = 0, 1, 2$.

Suppose proved for g.

$$\Sigma_+^{g+1} = \Sigma_+^g \# T^2$$

$$\chi(\Sigma_+^{g+1}) = \chi(\Sigma_+^g) + \chi(T^2) - 2$$
$$= 2 - 2g \quad - 2 \quad = 2 - 2(g+1)$$

□

Corollary

If $\mathbb{F}$ is any field, then

$$H_k(\Sigma_+^g : \mathbb{F}) = \begin{cases} \mathbb{F} & , k = 0 \\ \mathbb{F}^{2g} & , k = 1 \\ \mathbb{F} & , k = 2 \\ 0 & , k \geq 3 \end{cases}$$

Proof

Put $h_i = \dim H_i(\Sigma_+^g)$

$\chi(\Sigma_+^g) = h_0 - h_1 + h_2 = 2 - 2g$

$h_0 = 1$ connected

$h_2 = 1$ orientable

$\Rightarrow 2 - h_1 = 2 - 2g \Rightarrow h_1 = 2g$.

□

Corollary

$\Sigma_+^g \sim \Sigma_+^h \iff g = h$

Proof

They are distinguished by homology.

□

$\mathbb{RP}^2 = \Sigma_-^0$ (non-orientable surface of genus 0)

[S^2 double covers $\mathbb{RP}^2$]

$\mathbb{RP}^2 \# \mathbb{RP}^2 = \Sigma_-^1$ (non orientable, genus 1)

[T^2 double covers Σ_-^1]

$\underbrace{\mathbb{RP}^2 \# \ldots \# \mathbb{RP}^2}_{g+1} = \Sigma_-^g$ (non orientable, genus g)

[$\underbrace{T^2 \# \ldots \# T^2}_{g}$ double covers Σ_-^g]

Already seen $\chi(\mathbb{RP}^2) = 1$

Prop

$\chi(\Sigma_-^g) = 1 - g$

Proof

True for $g = 0$.

Suppose true for g.

$$\Sigma_-^{g+1} = \underbrace{\mathbb{RP}^2 \# \ldots \# \mathbb{RP}^2}_{g+2} = \Sigma_-^g \# \mathbb{RP}^2$$

$$\chi(\Sigma_-^{g+1}) = \chi(\Sigma_-^g) + \chi(\mathbb{RP}^2) - 2$$
$$= 1 - g + 1 - 2 = 1 - (g+1).$$

□

$H_*(\Sigma_-^g : \mathbb{F})$ depends on $\mathbb{F}$.

(I) 2 invertible in $\mathbb{F}$, e.g. $\mathbb{Q}$, $\mathbb{F}_p$ (p odd)

(II) $2 = 0$ in $\mathbb{F}$.

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Prop

$H_*(\Sigma_-^g : \mathbb{F})$ 2 invertible in $\mathbb{F}$

$$H_k(\Sigma_-^g : \mathbb{F}) = \begin{cases} \mathbb{F}, & k=0 \\ \mathbb{F}^g, & k=1 \\ 0, & k=2 \\ 0, & k \geq 3 \end{cases}$$

($\mathbb{F}^0 = 0$ by convention)

Proof

$\chi = h_0 - h_1 + h_2$, $h_0 = 1$ connected, $h_2 = 0$ non-orientable

$= 1 - g$

$\Rightarrow 1 - h_1 = 1 - g \Rightarrow h_1 = g.$ $\square$

Prop

If $2 = 0$ in $\mathbb{F}$, then

$$H_k(\Sigma_-^g ; \mathbb{F}) = \begin{cases} \mathbb{F}, & k=0 \\ \mathbb{F}^{g+1}, & k=1 \\ \mathbb{F}, & k=2 \\ 0, & k \geq 3 \end{cases}$$

Proof

$h_1 = 1$ (connected), $h_2 = 1$ (orientation thm $2=0$ in $\mathbb{F}$)

$\Rightarrow 1 - h_1 + 1 = 1 - g$

$\Rightarrow h_1 = 2 - 1 + g = g + 1.$ $\square$

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Orientation Thm

X finite connected surface, then

(i) $H_2(X:\mathbb{F}_2) \cong \mathbb{F}_2$ if $1+1=0$ in $\mathbb{F}_2$

(ii) $H_2(X:\mathbb{F}) \cong \mathbb{F}$ if X is orientable

(iii) $H_2(X:\mathbb{F}) = 0$ if X is not orientable and $1+1 = 2 \neq 0$.

In particular if $\mathbb{F}$ is a field of $char(\mathbb{F}) \neq 2$, then $H_2(X:\mathbb{F}) \neq 0 \Leftrightarrow$ X orientable.

Recall that X is connected when $\forall\, v, w \in V_X$ $\exists$ sequence $(v_r)_{0 \le r \le N}$, $v_r \in V_X$ and $v = v_0, \dots, v_N = w$, $\{v_r, v_{r+1}\}$ is a 1-simplex for $0 \le r \le N-1$.

$(v_r)_{0 \le r \le N}$ is a path from v to w.

Definition

Let X be a simplicial surface.

Let σ, ρ be 2-simplices of X. By a copath from σ to ρ I mean a sequence of 2-simplices $(\sigma_r)_{0 \le r \le N}$ s.t. $\sigma = \sigma_0, \dots, \sigma_N = \rho$ s.t. $\sigma_r \cap \sigma_{r+1}$ is a 1-simplex for $0 \le r \le N-1$.

Theorem

If X is a connected simplicial surface and σ, ρ are 2-simplices in X, then $\exists$ copath from σ to ρ $(\sigma \neq \rho)$.

Proof

Let σ, ρ be 2-simplices of X, $\sigma \neq \rho$. Then a priori there are 3 possibilities.

(i) $\sigma \cap \rho$ is a 1-simplex.

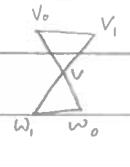
(ii) $\sigma \cap \rho$ is a 0-simplex.

(iii) $\sigma \cap \rho = \phi$.

If (i) then (σ, ρ) is a copath, so okay.

(ii) Suppose $\sigma \cap \rho = \{v\}$, $v \in V_X$.

So $\sigma = \{v_0, v_1, v\}$, $\rho = \{w_0, w_1, v\}$.



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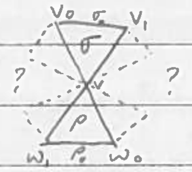
Observe that

$\{v_0, v_1\} \subset Lk(v, X)$, $\{w_0, w_1\} \subset Lk(v, X)$

The points v_0, v_1, w_0, w_1 are pairwise distinct, otherwise back to case 1.

The hypothesis on X (surface) means $Lk(v, X) \sim S^1(n)$.



Now join v_1 to w_0 by a path in $Lk(v, X)$ $(\sim S^1(n))$.

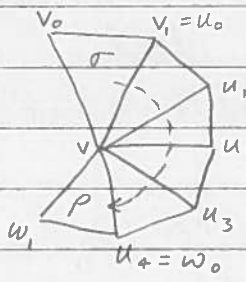
I have a sequence $(u_r)_{0 \le r \le M}$, $u_r \in Lk(v, X)$,

$\{u_r, u_{r+1}\} \subset Lk(v, X)$, $u_0 = v_1$, $u_M = w_0$.

Define $\sigma_0 = \sigma$, $\sigma_1 = \{\overset{v_1}{\overset{\shortparallel}{u_0}}, u_1, v\} \in S_X$, $\sigma_2 = \{u_1, u_2, v\}, \ldots$,

$\sigma_M = \{u_{M-1}, \overset{w_0}{\overset{\shortparallel}{u_M}}, v\}$, $\sigma_{M+1} = \{w_0, w_1, v\}$.

Then $(\sigma_r)_{0 \le r \le M+1}$ is a copath from σ to ρ.



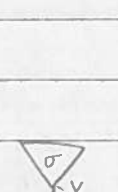
(iii) $\sigma \cap \rho = \emptyset$

Let ν = length of a shortest path from a vertex of σ to a vertex of ρ.

If $\nu = 0$, back to case 1 or 2.

Induction base: $\nu = 1$.

So $\exists$ vertex $v \in \sigma$, vertex $w \in \rho$.



$\{v, w\}$ is a 1-simplex of X.

Let τ be a 2-simplex st. $\{v, w\} \subset \tau$.

By cases 1 and 2, $\exists$ a copath from σ to τ $(v \in \sigma \cap \tau)$ and $\exists$ copath from τ to ρ $(w \in \tau \cap \rho)$.

Hence $\exists$ copath from σ to ρ (concatenate the copaths).

This proves the induction base.

Induction step:

Suppose that ν = length of a shortest path from σ to ρ, $\nu \geqslant 2$.

Suppose statement is true for $\nu - 1$.

Let $(v_0, \ldots, v_\nu)$ be a shortest path from σ to ρ, $v_0 \in \sigma$, $v_\nu \in \rho$.

Let τ be a 2-simplex s.t. $v_{\nu-1} \in \tau$.

By induction$^{\text{hyp.}}$ $\exists$ copath from σ to τ.

By induction base $\exists$ copath from τ to ρ.

So $\exists$ copath from σ to ρ. $\square$

Proof (of orientation thm)

Let X be a finite connected simplicial surface.

Enumerate the 2-simplices of X $\sigma_0, \ldots, \sigma_N$ and give each σ_i an arbitrary, but fixed, local orientation.

To start with, let $\mathbb{F}$ be some field.

A typical element $\alpha \in C_2(X:\mathbb{F}_2)$ looks like

$$\alpha = \sum_{r=0}^{N} a_r \sigma_r$$

Question: When does $\alpha \in Z_2(X:\mathbb{F}) = \operatorname{Ker}(\partial_2 : C_2 \to C_1)$?

Enumerate the 1-simplices of X, and give each (arbitrarily) a local (fixed) orientation, $\tau_0, \ldots, \tau_M$.

$$\partial_2(\alpha) = \sum_{s=0}^{M} b_s \tau_s$$

Fix s.

Let σ_+, σ_- be the two 2-simplices such that $\tau_s = \sigma_+ \cap \sigma_-$.

$\partial_2(\sigma_+) = \pm\tau_s$ + other terms not involving τ_s

$\partial_2(\sigma_-) = \pm\tau_s$ + " " " " ".

$\partial_2(\alpha) = (\pm a_{\sigma_+} \pm a_{\sigma_-})\tau_2$ + other terms not involving τ_s.

Incidence number:

σ n-simplex $\tau \subset \sigma$ $(n-1)$-simplex

$\sigma = [v_0, \ldots, v_n]$, $\underset{\nwarrow \text{incidence no.}}{[\sigma, \tau]} = (-1)^r$ where $\tau = [v_0, \ldots, v_{r-1}, v_{r+1}, \ldots, v_n]$

$\partial_2(\alpha) = (a_{\sigma_+}[\sigma_+, \tau_s] + a_{\sigma_-}[\sigma_-, \tau_s])\tau_s$ + terms not involving τ_s

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If $\partial_2(\alpha) = 0$ then $a_{\sigma_+}[\sigma_+, \tau_s] + a_{\sigma_-}[\sigma_-, \tau_s] = 0$

$\alpha = \sum_{r=0}^{N} a_r \sigma_r$, $\partial_2(\alpha) = 0$

if $\sigma_r \cap \sigma_t$ is a 1-simplex then $a_t = \pm a_r$.

Claim

If $\partial_2(\alpha) = 0$ then $a_r = \pm a_0$ $\forall r$.

Take a copath from σ_0 to σ_r $(\sigma_0 = \rho_0, \rho_1, \dots, \rho_N = \sigma_r)$

$a_{\rho_1} = \pm a_{\rho_0}$, $a_{\rho_2} = \pm a_{\rho_1}$, ..., $a_{\rho_N} = \pm a_{\rho_{N-1}}$

(consecutive 2-simplices intersect in a 1-simplex).

so $a_r = \pm a_0$.

If $\partial_2(\alpha) = 0$, then $\alpha = a_0 (\sigma_0 + \sum_{r=1}^{n} c(r) \sigma_r)$, $a_0 \in \mathbb{F}$,

where $c : \{0, \dots, n\} \longrightarrow \{\pm 1\}$, $c(0) = +1$.

If $c : \{0, \dots, n\} \longrightarrow \{\pm 1\}$, define $\xi(c) \in C_2(X : \mathbb{F})$ by

$\xi(c) = \sum_{r=0}^{N} c(r) \sigma_r$.

Preliminary conclusion:

X is a finite connected surface, $\sigma_0, \dots, \sigma_N$ are 2-simplices,

$\alpha = \sum_{r=0}^{N} a_r \sigma_r \in C_2(X : \mathbb{F})$

If $\partial_2(\alpha) = 0$ then $\alpha = a_0 \xi(c)$ for some function

$c : \{0, \dots, N\} \longrightarrow \{\pm 1\}$ $(a_0 \in \mathbb{F})$.

Question: When is $\partial_2(\xi(c)) = 0$?

Choose a 1-simplex τ.

Look at the coefficient of τ in $\partial_2(\xi(c))$.

If σ_+, σ_- are the two 2-simplices which contain τ

then the coefficient of τ in $\partial_2(\xi(c)) = [c(\sigma_+)\sigma_+, \tau] + [c(\sigma_-)\sigma_-, \tau]$

$= (\pm 1) + (\pm 1)$

$\Rightarrow$ possible values are $+2, 0, -2$.

If $2 = 0$ in $\mathbb{F}$ then $\partial_2(\xi(c)) = 0$ $\forall c$.

If $2 \neq 0$ in $\mathbb{F}$ then either

(i) $\exists\ c : \{0, \dots, N\} \to \{\pm 1\}$ s.t. $[c(\sigma_r)\sigma_r, \tau] + [c(\sigma_s)\sigma_s, \tau] = 0$
whenever $\tau = \sigma_r \cap \sigma_s$.
This is precisely the definition of orientability.

or (ii) $\forall c : \{0, \dots, N\} \to \{\pm 1\}$ $\exists\ r, s$ $(r \neq s)$ s.t.
$[c(\sigma_r)\sigma_r, \tau] + [c(\sigma_s)\sigma_s, \tau] = \pm 2$, $\sigma_r \cap \sigma_s$ is a 1-simplex.

Intermediate conclusion:

ⓐ If $2 = 0$ in $\mathbb{F}$, then $H_2(X : \mathbb{F}) \neq 0$.
(any $\xi(c) \in Z_2(X : \mathbb{F})$)

ⓑ If $2 \neq 0$ then $H_2(X : \mathbb{F}) \neq 0 \iff X$ orientable.
(In particular $H_2(X : \mathbb{F}) = 0$ if X nonorientable).

(Final stretch!)
Assume $2 \neq 0$ in $\mathbb{F}$, X orientable.
Then we know $H_2(X : \mathbb{F}) \neq 0$.
In particular $\exists\ c : \{0, \dots, N\} \to \{\pm 1\}$ s.t. $\partial_2(\xi(c)) \neq 0$.

Q: How big is $H_2(X : \mathbb{F})$ when X connected, orientable, $2 \neq 0$ in $\mathbb{F}$?
Every $\alpha \in Z_2(X : \mathbb{F})$ has form $\alpha = a\,\xi(c)$, $a \in \mathbb{F}$, and
$c : \{0, \dots, N\} \to \{\pm 1\}$ is some function s.t. $\partial_2(\xi(c)) = 0$.

So suppose I have two functions $c : \{0, \dots, N\} \to \{\pm 1\}$,
$d : \{0, \dots, N\} \to \{\pm 1\}$ s.t. $\partial_2(\xi(c)) = 0$ and $\partial_2(\xi(d)) = 0$, $d \neq c$.
Wlog. assume $\exists\ r$ s.t. $c(r) = +1$, $d(r) = -1$.
Suppose $\sigma_s \cap \sigma_r$ is a 1-simplex τ, then
$[\sigma_s, \tau] + [\sigma_r, \tau] = 0$.
Then $[d(s)\sigma_s, \tau] + [d(r)\sigma_r, \tau] = 0$
$\Rightarrow d(r) = -c(r)$, $d(s) = -c(s)$.
So for adjacent 2-simplices σ_r, σ_s, changing $c(r)$ to $-c(r) = d(r)$
necessarily changes $c(s)$ to $-c(s) = d(s)$.
Now go along a copath, $d(w) = -c(w)$ $\forall w \in \{0, \dots, N\}$.

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Final conclusion:

If X connected, orientable surface,
$\sigma_0, \dots, \sigma_N$ are the 2-simplices, then
$\exists c : \{0, \dots, N\} \longrightarrow \{\pm 1\}$ such that
$\alpha \in Z_2(X : \mathbb{F}) \Rightarrow \alpha = a_0 \zeta(c)$ st. $\zeta(c)$ generates $Z(X : \mathbb{F})$.
$\zeta(c)$ is a "fundamental class".
$-\zeta(c) = \zeta(-c)$ is the only other fundamental class.
So $H_2(X : \mathbb{F}) \cong \mathbb{F}$.

If $1+1=0$ in $\mathbb{F}$, then $\exists!$ function $c : \{0, \dots, N\} \rightarrow \{\pm 1\} \subset \mathbb{F}$ since $-1 = 1$.
So if $1+1=0$ in $\mathbb{F}$, then $H_2(X : \mathbb{F}) \cong \mathbb{F}$.

$\square$

21-02-18

Lefschetz Fixed Simplex Theorem (coming soon)

R commutative ring. $A \in M_n(R)$ ($n \times n$ matrices over R).
$A = (a_{ij})_{\substack{1 \le i \le n \\ 1 \le j \le n}}$, $T_r(A) := \sum_{i=1}^{n} a_{ii}$ (trace of A)

Proposition

Let $A, B \in M_n(R)$. Then $T_r(AB) = T_r(BA)$.

Proof

$AB = (c_{ik})$, $c_{ik} = \sum_{j=1}^{n} a_{ij} b_{jk}$

$c_{ii} = \sum_{j=1}^{n} a_{ij} b_{ji}$.

$$T_r(AB) = \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij} b_{ji} = \sum_{j=1}^{n} \sum_{i=1}^{n} a_{ij} b_{ji}$$

$$= \sum_{j=1}^{n} \sum_{i=1}^{n} b_{ji} a_{ij} \quad \text{(since } R \text{ commutative)}$$

$$= T_r(BA). \quad \square$$

Warning: $\mathrm{Tr}\,AB \neq \mathrm{Tr}\,A\,\mathrm{Tr}\,B$

Corollary

If $A, P \in M_n(\mathbb{R})$ and P is invertible, then
$\mathrm{Tr}(PAP^{-1}) = \mathrm{Tr}(A)$.

Proof

$\mathrm{Tr}(PAP^{-1}) = \mathrm{Tr}(P(AP^{-1})) = \mathrm{Tr}((AP^{-1})P) = \mathrm{Tr}(A)$.

□

Trace of a linear map.

Let $\mathbb{F}$ be a field.

$T: V \to V$ be an $\mathbb{F}$-linear map where $\dim_{\mathbb{F}}(V)$ is finite.

Take a basis $\mathcal{E} = \{e_1, \dots, e_n\}$ for V.

Represent T as a matrix.

$$T(e_i) = \sum_{j=1}^{n} a_{ji} e_j$$

$$\mathrm{Tr}(T)_{\mathcal{E}} = \sum_{i=1}^{n} a_{ii}$$

This is Trace referred to a specific basis $\mathcal{E}$.

Suppose we change basis, $\phi = \{\varphi_1, \dots, \varphi_n\}$.

$$T(\varphi_i) = \sum_{j=1}^{n} \tilde{a}_{ji} \varphi_j$$

$$\mathrm{Tr}(T)_{\phi} = \sum_{i=1}^{n} \tilde{a}_{ii}$$

$A = (a_{ji})$, $\tilde{A} = (\tilde{a}_{ji})$

Then $\tilde{A} = PAP^{-1}$ where P is the matrix of change of basis

$$\underset{\tilde{A}}{M(T)^{\phi}_{\phi}} = \underset{P}{M(Id)^{\phi}_{\mathcal{E}}}\,\underset{A}{M(T)^{\mathcal{E}}_{\mathcal{E}}}\,\underset{P^{-1}}{M(Id)^{\mathcal{E}}_{\phi}}$$

So $\mathrm{Tr}(T)_{\phi} = \mathrm{Tr}(\tilde{A}) = \mathrm{Tr}(PAP^{-1}) = \mathrm{Tr}(A) = \mathrm{Tr}(T)_{\mathcal{E}}$

So $\mathrm{Tr}(T)$ is an absolute invariant of linear maps $T: V \to V$,
V f.d $/\mathbb{F}$.

21-02-18

Lefschetz number of a simplicial map Fix a field $\mathbb{F}$.

Let $K = (V_K, S_K)$ finite simplicial complex.

$f: K \to K$ simplicial map.

$\forall n \geq 0$ we have

$C_n(f) : C_n(K:\mathbb{F}) \to C_n(K:\mathbb{F})$, the induced map on n-chains.

Define the geometric Lefschetz number $\lambda_{geom}(f)$ (relative to $\mathbb{F}$, fixed in advace).

$$\lambda_{geom}(f) = \sum_{r \geq 0} (-1)^r Tr(C_r(f))$$

The homological Lefschetz number $\lambda_{hom}(f)$ is defined

$\lambda_{hom}(f) = \sum_{r \geq 0} (-1)^r Tr(H_r(f))$ where $H_r(f): H_r(K:\mathbb{F}) \to H_r(K:\mathbb{F})$ is the induced map on homology.

We'll show:

(I) with $\mathbb{F}$ fixed, $\lambda_{hom}(f) = \lambda_{geom}(f)$ $(= \lambda(f))$

(II) (Lefschetz' Thm)

$f: K \to K$ simplicial map, K finite complex.

If $\lambda(f) \neq 0$ $\exists$ simplex σ of K st. $f(\sigma) = \sigma$.

Proof (of I)

We first prove that Tr is additive on exact sequences.

$$\begin{tikzcd}
0 \arrow[r] & U \arrow[r, "i"] \arrow[d, "T_U"] & V \arrow[r, "p"] \arrow[d, "T_V"] \arrow[rd, phantom, "\circlearrowleft"] & W \arrow[r] \arrow[d, "T_W"] & 0 \\
0 \arrow[r] & U \arrow[r, "i"] & V \arrow[r, "p"] & W \arrow[r] & 0
\end{tikzcd}$$

Prop: Given such a commutative diagram of vector spaces and linear maps over $\mathbb{F}$, U, V, W finite dimensional over $\mathbb{F}$. Then $Tr(T_V) = Tr(T_U) + Tr(T_W)$

Proof: Essentially the Kernel-Rank Thm.

Let $\{e_1, \ldots, e_k\}$ be a basis for U.

Let $\{\varphi_1, \ldots, \varphi_q\}$ be a basis for W.

For each $1 \leq r \leq q$, choose $e_{k+r} \in V$ st. $p(e_{k+r}) = \varphi_r$ (p surjective).

Then $\{e_1, \dots, e_k, e_{k+1}, \dots, e_{k+q}\}$ is a basis for V.

We want to express matrix of T_V in terms of matrices of T_u, T_W.

Write $T_u(e_i) = \sum_{j=1}^{k} a_{ji} e_j$, $A = (a_{ji})$ ($k \times k$ matrix)

Write $T_W(\varphi_r) = \sum_{s=1}^{q} d_{sr} \varphi_s$, $D = (d_{sr})$ ($q \times q$ matrix)

Then $\exists$ $k \times q$ matrix $B = (b_{j\,k+i})$ $1 \leq j \leq k$, $1 \leq i \leq q$, such that matrix of T_V w.r.t. $\{e_1, \dots, e_{k+q}\}$ is $\begin{pmatrix} A & B \\ O & D \end{pmatrix}$

$$T_V(e_{k+i}) = \sum_{j=1}^{k} b_{j\,k+i} e_j + \sum_{s=1}^{q} d'_{k+s,k+i} e_{k+s}$$

Commutativity of G shows that $d'_{k+s,k+i} = d_{si}$.

$$\mathrm{Tr} \begin{pmatrix} A & B \\ O & D \end{pmatrix} = \mathrm{Tr}(A) + \mathrm{Tr}(D)$$

$$\Rightarrow \mathrm{Tr}(T_V) = \mathrm{Tr}(T_u) + \mathrm{Tr}(T_W)$$

$\square$

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$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r, "i"] \arrow[d, "f_A"] & B \arrow[r, "p"] \arrow[d, "f_B"] & C \arrow[r] \arrow[d, "f_C"] & 0 \\
0 \arrow[r] & A \arrow[r, "i"] & B \arrow[r, "p"] & C \arrow[r] & 0
\end{tikzcd}$$

A, B, C f.d / $\mathbb{F}$

Diagram commutes

Rows exact.

Then $Tr(f_B) = Tr(f_A) + Tr(f_C)$

$f : K \to K$ simplicial map

K finite complex.

$$\begin{tikzcd}
0 \arrow[r] & B_r(K) \arrow[r] \arrow[d, "B_r(f)"] & Z_r(K) \arrow[r] \arrow[d, "Z_r(f)"] & H_r(K) \arrow[r] \arrow[d, "H_r(f)"] & 0 \\
0 \arrow[r] & B_r(K) \arrow[r] & Z_r(K) \arrow[r] & H_r(K) \arrow[r] & 0
\end{tikzcd}$$

$$C_{r+1}(K) \xrightarrow{\partial_{r+1}} C_r(K) \xrightarrow{\partial_r} C_{r-1}(K)$$

$B_r(K) = Im\, \partial_{r+1}$

$Z_r(K) = Ker\, \partial_r$

$\Rightarrow Tr\, Z_r(f) = Tr\, B_r(f) + Tr\, H_r(f)$

$$\begin{tikzcd}
0 \arrow[r] & Z_r(K) \arrow[r] \arrow[d, "Z_r(f)"] & C_r(K) \arrow[r] \arrow[d, "C_r(f)"] & B_{r-1}(K) \arrow[r] \arrow[d, "B_{r-1}(f)"] & 0 \\
0 \arrow[r] & Z_r(K) \arrow[r] & C_r(K) \arrow[r] & B_{r-1}(K) \arrow[r] & 0
\end{tikzcd}$$

$\Rightarrow Tr(C_r(f)) = Tr(Z_r(f)) + Tr(B_{r-1}(f))$

$\Rightarrow Tr(C_r(f)) = Tr(H_r(f)) + Tr(B_r(f)) + Tr(B_{r-1}(f))$

Take alternating sums:

$$\sum_r (-1)^r Tr(C_r(f)) = \sum_r (-1)^r Tr(H_r(f)) + \sum_r (-1)^r Tr(B_r(f)) + \sum_r (-1)^r Tr(B_{r-1}(f))$$

$$\Rightarrow \lambda_{geom}(f) = \lambda_{hom}(f) + \sum_r (-1)^r Tr(B_r(f)) - \sum_{r-1} (-1)^{r-1} Tr(B_{r-1}(f)).$$

So $\lambda_{geom}(f) = \lambda_{hom}(f)$.

Note if we take $f = Id : K \to K$
we get $\chi_{geom}(K) = \chi_{hom}(K)$.
Beware: χ is independent of $\mathbb{F}$,
however $\lambda(f)$ depends on $\mathbb{F}$.
Usually take $\mathbb{F} = \mathbb{Q}$.

Theorem (Lefschetz Fixed Simplex Thm).
Let $f : K \to K$ be a simplicial map where K is a finite simplicial complex. Pick a field (advice : $\mathbb{F} = \mathbb{Q}$) If $\lambda(f) \neq 0$ then $\exists$ a simplex σ of K st. $f(\sigma) = \sigma$. (as sets, not necessarily as oriented simplices).

Proof
Work with $\lambda_{geom}(f)$

$\lambda_{geom}(f) = \sum_r (-1)^r T_r(C_r(f))$

$C_r(f) : C_r(K : \mathbb{F}) \to C_r(K : \mathbb{F})$.

This is quite a special sort of linear map

$$C_r(f)(\text{basis element}) = \begin{cases} \pm \text{ some basis element} \\ 0 \end{cases}$$

Expressed as a matrix, in each row / column, $C_r(f)$ has at most one non-zero element ± 1.
If we enumerate the r-simplices of K $\sigma_1, \dots, \sigma_N$, $C_r(f)$ is $N \times N$ matrix, $C_r(f)$ has a non-zero entry (± 1) in $(i,i)^{th}$ position $\Leftrightarrow f(\sigma_i) = \pm \sigma_i$.

If f fixes no r-simplex, then diagonal of $C_r(f) \equiv 0$
$\Rightarrow T_r(C_r(f)) = 0$
Suppose f fixes no simplex whatsoever, then
$T_r(C_r(f)) = 0 \quad \forall r$
So if f fixes no simplex whatsoever, then $\lambda_{geom}(f) = \sum_r (-1)^r T_r(C_r(f)) = 0$

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In the contrapositive, if $\lambda_{geom}(f) \neq 0$, then $\exists$ simplex σ of K s.t. $f(\sigma) = \sigma$

$\square$

Corollary (Brower Fixed Simplex Thm).
Let D be a finite simplicial complex s.t. $D \sim \Delta^n$,
(Think of D as a triangulated disc.)
Then if $f: D \to D$ is any simplicial map, then $\exists$ a simplex σ of D s.t. $f(\sigma) = \sigma$.

Proof
Calculate $\lambda_{hom}(f)$.

$f: D \to D$, $D \sim \Delta^n$, $H_r(D:\mathbb{F}) = \begin{cases} \mathbb{F}, & r=0 \\ 0, & r \neq 0 \end{cases}$

So $\lambda_{hom}(f) = Tr(H_0(f): H_0(D:\mathbb{F}) \to H_0(D:\mathbb{F}))$

Observe that if v, w are vertices in D,

$v - w \in Im\, \partial_1 (: C_1(D;\mathbb{F}) \to C_0(D:\mathbb{F}))$

(see proof that $H_0(\text{connected}:\mathbb{F}) \cong \mathbb{F}$)

So if v is a vertex of D,

$f(v) - v \in Im\, \partial_1$

$H_0(f)[v] = [v]$ $\quad$ $H_0(f) = Id$

$\Rightarrow Tr\, H_0(f) = 1$

so $\lambda_{hom}(f) = 1$ so $\lambda_{geom}(f) = 1$

so f fixes a simplex.

$\square$

[Brower Fixed Point Thm (c 1908)
$D = \{x \in \mathbb{R}^n : \|x\| \leq 1\}$, $f: D \to D$ a continuous map.
Then $\exists\, x \in D$ s.t. $f(x) = x$.]

28-02-18

Products

Problem: $\Delta^m \times \Delta^n$ is not a simplex.

[$I = [0,1]$, $I^2 =$  , $I^3 =$  , $I^{n+1} = I^n \times I$]

Posets

Definition

A partially ordered set (poset) $(A, \leq)$ where

(i). A is a set

(ii) $\leq \subset A \times A$ $a \leq b \Leftrightarrow (a,b) \in \leq$

$\wedge =$ and satisfies $a \leq b \wedge b \leq c \Rightarrow a \leq c$ (transitive) and $a \leq b \wedge b \leq a \Rightarrow a = b$, and $a \leq a$ (reflexive).

Special case: totally ordered set:
in which case $\forall a,b \in A$ either $a \leq b$ or $b \leq a$.
e.g. $(\mathbb{N}, \leq)$ is totally ordered
e.g. $(\{0,\dots,n\}, \leq)$ is totally ordered

Product of posets:
$(A, \leq_A)$, $(B, \leq_B)$
Define $\preccurlyeq$ on $A \times B$ by
$(a,b) \preccurlyeq (a',b') \Leftrightarrow (a \leq_A a') \wedge (b \leq_B b')$

Beware: if $(A, \leq)$, $(B, \leq)$ then $(A \times B, \leq)$ isn't unless either A or B is a point.

$A = \{0,1\}$, $0 \leq 1$

$A \times A$

$$\begin{tikzcd}
(1,0) \arrow[d] & (1,1) \arrow[l] \arrow[d] \arrow[dl] \\
(0,0) & (0,1) \arrow[l]
\end{tikzcd}$$

This triangulates $\Delta^1 \times \Delta^1$.

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Simplicial realisation of a poset

Let $(A, \leq)$ be a poset.

Define a simplicial complex

$N(A, \leq)$ (the "nerve of $(A, \leq)$")

Vertex set $= A$.

Simplices are the finite totally ordered subsets.

(In previous example, $N(A, \leq) =$ triangulated square)

How to triangulate $\Delta^m \times \Delta^n$:

Take $A = \{0, \dots, m\}$, $B = \{0, \dots, n\}$,

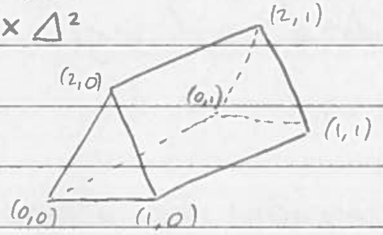
both with standard ordering.

$N(A, \leq) = \Delta^m$, $N(B, \leq) = \Delta^n$

$N(A \times B, \leq)$ triangulates $\Delta^m \times \Delta^n$.

Example

$\Delta^1 \times \Delta^2$



The maximal totally ordered subsets are

$(0,0) \leq (1,0) \leq (2,0) \leq (2,1)$

$(0,0) \leq (1,0) \leq (1,1) \leq (2,1)$

$(0,0) \leq (0,1) \leq (1,1) \leq (2,1)$

Triangulated by three 3-simplices.

Definition

By an ordered simplicial complex $(K, \leq)$ I mean that

(i) $K = (V_K, S_K)$ is a simplicial complex

(ii) $\leq$ is a partial ordering on V_K in such a way that

(iii) each $\sigma \in S_K$ is totally ordered.

Prop

Any finite simplicial complex K imbeds as a subcomplex of Δ^n (where $n = |V_K|-1$)

Proof

Index V_K as $\{v_0, \dots, v_n\}$.
Map $V_K \to \{0, \dots, n\}$, $v_r \mapsto r$.
Every simplex of K maps to a simplex of Δ^n
(Every nonempty subset of Δ^n is a simplex).

$\square$

Consequently:

Prop

Every finite simplex K admits the structure of an ordered simplicial complex $(K, \leq)$

Proof

Imbed K in Δ^n and take induced ordering. $\square$

Def

Let $(K, \leq)$, $(L, \leq)$ be ordered simplicial complexes.
Define $(K \times L, \leq)$ as follows:
Take $V_{K\times L} = V_K \times V_L$ and take the product ordering on $V_{K \times L}$.
Define the simplices of $K \times L$ to be the totally ordered subsets of $\sigma \times \tau$ where σ ranges through S_K and τ ranges through S_L.

eg. try $S^1(3) \times S^1(3)$.

So $H_*(K \times L ; \mathbb{F})$ is defined for any two simplicial complexes K and L.
How do we compute $H_*(K \times L : \mathbb{F})$? ans: Künneth Thm

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We'll show:

Thm

Let K, L be finite simplicial complexes, then
$\chi(K \times L) = \chi(K)\chi(L)$

Prop

Let $(A, \leq)$ be a poset and suppose $(A, \leq)$ has an absolute minimum (i.e. suppose $\exists\, a \in A$ s.t. $\forall\, b \in A\ a \leq b$)
Then $N(A, \leq)$ is a cone.

Proof

Let a be the absolute minimum.
Define $A' = A - \{a\}$.
So $(A', \leq)$ is also a poset, and
(i) $Lk(a, N(A)) = N(A')$
(ii) $N(A) = C(N(A'))$ by taking a as the cone point.
□

Corollary

$$H_k(\Delta^m \times \Delta^n : \mathbb{F}) = \begin{cases} \mathbb{F} &, k=0 \\ 0 &, k \neq 0 \end{cases}$$

Proof

$\Delta^m = N(\{0, \dots, m\})$, $\Delta^n = N(\{0, \dots, n\})$
and $(0,0)$ is an absolute minimum.
□

We'll first prove:

Thm

Let K be a finite complex. Then
$\chi(K \times \Delta^n) = \chi(K)$.

Prop

Let K, L be simplicial complexes s.t.

$V_K \cap V_L = \phi$

Then $H_n(K \cup L) \cong H_n(K) \oplus H_n(L)$ for each n.

Proof

$K \cup L = K \cup L$ with $K \cap L = \phi$

$H_n(K \cap L) = 0 \ \forall\ n$.

$$\underset{\overset{\shortparallel}{0}}{H_n(K\cap L)} \to H_n(K) \oplus H_n(L) \xrightarrow{\cong} H_n(K \cup L) \to \underset{\overset{\shortparallel}{0}}{H_{n-1}(K \cap L)}$$

$\square$

Prop

Let $K = \{1, \dots, m\}$ be 0-dim complex with m vertices.

Then $\chi(K \times \Delta^n) = m$.

Proof (by induction on m)

$m=1$, nothing to prove.

Suppose proved for $m-1$.

Let $K_+ = \{1, \dots, m-1\}$, $K_- = \{m\}$

$K = K_+ \cup K_-$, $K_+ \cap K_- = \emptyset$

Put $X = K \times \Delta^n$, $X_+ = K_+ \times \Delta^n$, $X_- = K_- \times \Delta^n$

$X_+ \cap X_- = \phi$

$$0 \to \underset{\overset{\shortparallel}{0}}{C_*(X_+ \cap X_-)} \to C_*(X_+) \oplus C_*(X_-) \to C_*(X) \to 0$$

$\Rightarrow \chi(X_+) + \chi(X_-) = \chi(X)$

$\Rightarrow \chi(\{1, \dots, m-1\} \times \Delta^n) + \chi(\{m\} \times \Delta^n) = \chi(X)$

$\Rightarrow (m-1) + 1 = m$

$\square$

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X, Y finite complexes

$\chi(X \times Y) = \chi(X)\chi(Y)$

Suppose $X = X_+ \cup X_-$, X_+, X_- subcomplexes of X , X finite.

Additivity of χ:

$\chi(X) + \chi(X_+ \cap X_-) = \chi(X_+) + \chi(X_-)$

Proof

Have an exact sequence of chain complexes

$0 \to C_*(X_+ \cap X_-) \to C_*(X_+) \oplus C_*(X_-) \to C_*(X) \to 0$

For each n, $\dim C_n(X) + \dim C_n(X_+ \cap X_-) = \dim C_n(X_+) + \dim C_n(X_-)$

Take alternating sums:

$\chi_{geom}(X) + \chi_{geom}(X_+ \cap X_-) = \chi_{geom}(X_+) + \chi_{geom}(X_-)$.

Corollary

If $X = X_+ \cup X_-$, $X_+ \cap X_- = \phi$

Then $\chi(X) = \chi(X_+) + \chi(X_-)$.

Corollary

Let Y be a finite complex, let $\{1, \dots, m\}$ be a 0-dim complex (m distinct points). Form $\{1, \dots, m\} \times Y$.

Then $\chi(\{1, \dots, m\} \times Y) = m\,\chi(Y)$.

Proof (by induction on m)

For $m=1$, nothing to prove.

Suppose proved for $m-1$.

Define $Z_+ = \{1, \dots, m-1\} \times Y$, $Z_- = \{m\} \times Y$

$Z_+ \cap Z_- = \phi$

$$\begin{aligned}\chi(\{1, \dots, m\} \times Y) &= \chi(Z_+) + \chi(Z_-) \\ &= (m-1)\chi(Y) + \chi(Y) \\ &= m\,\chi(Y).\end{aligned}$$

□

We also showed that

$\Delta^n \times \Delta^m$ is triangulable as a cone,

so $\chi(\Delta^n \times \Delta^m) = 1 = \chi(\Delta^m)$.

Proposition

Let X be a finite complex.

Then $\chi(X \times \Delta^m) = \chi(X) \quad \forall m \geqslant 0$.

Proof

Fix n and let $P(n,k)$ be the statement

$P(n,k)$: $\chi(X \times \Delta^m) = \chi(X)$ when X is a finite complex of dimension $\leq n$ and X has exactly k simplices of dimension n.

$P(n)$: $\chi(X \times \Delta^m) = \chi(X)$ whenever X is a finite complex of dimension $\leq n$.

Observe that $P(n+1, 0) \equiv P(n)$.

$P(0,k)$ is true for all k.

So if X is a 0-dimensional complex having exactly k points $\{v_1, \dots, v_k\}$ then $\chi(X \times \Delta^m) = k\,\chi(\Delta^m) = k = \chi(X)$.

We've already shown this.

$P(0) = \bigwedge_{k=0}^{\infty} P(0,k)$, so $P(0)$ is true.

We'll show that $P(n-1) \wedge P(n,k) \Rightarrow P(n, k+1)$.

Let X be a finite complex of dimension n, which has exactly $(k+1)$ simplices of dimension n, $\sigma_1, \dots, \sigma_{k+1}$. Write $X = X_+ \cup X_-$ where

$X_+ = X^{(n-1)} \cup \sigma_1 \cup \dots \cup \sigma_k$, $X_- = \sigma_{k+1}$ (subcomplex determined by σ_{k+1})

Then $X_+ \cap X_- \subset X^{(n-1)}$

$\dim(X_+ \cap X_-) \leq n-1$.

$\chi(X) + \chi(X_+ \cap X_-) = \chi(X_+) + \overset{=1}{\chi(\sigma_{k+1})}$

Now take product with Δ^m:

$X \times \Delta^m = X_+ \times \Delta^m \cup X_- \times \Delta^m$

$(X_+ \times \Delta^m) \cap (X_- \times \Delta^m) = (X_+ \cap X_-) \times \Delta^m$

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$$\chi(X \times \Delta^m) + \chi((X_+ \cap X_-) \times \Delta^m) = \chi(X_+ \times \Delta^m) + \chi(\sigma_{k+1} \times \Delta^m)$$

$\chi((X_+ \cap X_-) \times \Delta^m) = \chi(X_+ \cap X_-)$ by $P(n-1)$.
$\chi(X_+ \times \Delta^m) = \chi(X)$ by $P(n,k)$.
$\chi(\sigma_{k+1} \times \Delta^m) = 1 = \chi(\sigma_{k+1})$

So
$$\begin{array}{ccccccc} \chi(X \times \Delta^m) & + & \chi((X_+ \cap X_-) \times \Delta^m) & = & \chi(X_+ \times \Delta^m) & + & \chi(\sigma_{k+1} \times \Delta^m) \\ & & \| & & \| & & \| \\ \chi(X) & + & \chi(X_+ \cap X_-) & = & \chi(X_+) & + & \chi(\sigma_{k+1}) \end{array}$$

Hence $\chi(X \times \Delta^m) = \chi(X)$ and $P(n-1) \wedge P(n,k) \Rightarrow P(n,k+1)$
Now consider
$$P(0,1) \Rightarrow P(0,2) \Rightarrow \dots \Rightarrow P(0,k) \Rightarrow P(0,k+1) \Rightarrow \dots \Rightarrow P(0)$$

$$P(0) = P(1,0) \Rightarrow P(1,1) \Rightarrow P(1,2) \Rightarrow \dots \Rightarrow P(1,k+1) \Rightarrow \dots \Rightarrow P(1)$$
$$P(1) = P(2,0) \Rightarrow P(2,1) \Rightarrow P(2,2) \Rightarrow \dots \Rightarrow P(2)$$

So $P(0) \wedge P(n) \Rightarrow P(n+1)$, so $P(n)$ true for all n $\square$

Theorem
Let X, Y be finite complexes, then
$\chi(X \times Y) = \chi(X)\chi(Y)$.

Proof
Fix X and define statements $Q(n,k)$, $Q(n)$ as follows:
$Q(n,k)$: $\chi(X \times Y) = \chi(X)\chi(Y)$ whenever Y is a finite complex of dim $\leq n$ and has exactly k simplices of dimension n, $\sigma_1, \dots, \sigma_k$
$Q(n)$: $\chi(X \times Y) = \chi(X)\chi(Y)$ whenever Y is a finite complex of dimension $\leq n$.
Observe: $Q(n) \equiv Q(n+1, 0)$.
By last proposition, $Q(n,1)$ is true for all n.

We need to show $Q(0,k)$ is true for all k.

$Y = \{v_1, \dots, v_k\}$, $\chi(Y) = k$

$X \times Y = \bigsqcup_{i=1}^{k} X \times \{v_i\}$

? $\chi(X \times Y) = k\,\chi(X) =$

So $Q(0)$ is true so $Q(1,0)$ is true

We'll show that $Q(n-1) \wedge Q(n,k) \Rightarrow Q(n,k+1)$

This will be enough by the same type of double induction argument we had before

Assume $Q(n-1)$ is true and $Q(n,k)$ is true.

Let Y be a finite complex of dimension $\leq n$ having exactly $(k+1)$ simplices of dimension n, $\sigma_1, \dots, \sigma_k, \sigma_{k+1}$

Write $Y = Y_+ \cup Y_-$ where $Y_+ = Y^{(n-1)} \cup \sigma_1 \cup \dots \cup \sigma_k$,

$Y_- = \sigma_{k+1}$. $Y_+ \cap Y_- \subset Y^{(n-1)}$ so $\dim(Y_+ \cap Y_-) \leq n-1$.

$\chi(X \times Y) + \chi(X \times (Y_+ \cap Y_-)) = \chi(X \times Y_+) + \chi(X \times \sigma_{k+1})$ $\quad \sigma_{k+1} \cong \Delta^n$

|| $Q(n-1)$ || $Q(n,k)$ || by previous lemma

$\chi(X \times Y) + \chi(X)\chi(Y_+ \cap Y_-) = \chi(X)\chi(Y_+) + \chi(X)$

Then $\chi(X \times Y) = \chi(X)\left(\chi(Y_+) + \chi(Y_-) - \chi(Y_+ \cap Y_-)\right)$

$= \chi(X)\,\chi(Y)$ □

In terms of General Topology:

S^4 is compact, connected, locally like $\mathbb{R}^n$, and simply connected.

$S^2 \times S^2$ " " " "

$\chi(S^4) = 2$, $\chi(S^2 \times S^2) = 2 \times 2 = 4$

$\mathbb{F}$ field. Künneth Theorem:

If X, Y are finite complexes, then

$H_n(X \times Y : \mathbb{F}) = \bigoplus_{k=0}^{n} H_k(X : \mathbb{F}) \otimes H_{n-k}(Y : \mathbb{F})$.

($\dim V \otimes W = \dim V \dim W$).

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$H_*(S^3 \times S^1)$

$H_0 = H_0(S^3) \otimes H_0(S^1) = \mathbb{F}$

$H_1 = \underset{\overset{\shortparallel}{0}}{H_1(S^3)} \otimes H_0(S^1) \oplus H_0(S^3) \otimes \underset{\overset{\shortparallel}{\mathbb{F}}}{H_1(S^1)} = \mathbb{F}$

$H_2 = H_2(S^3) \otimes H_0(S^1) \oplus H_1(S^3) \otimes H_1(S^1) \oplus H_0(S^3) \otimes H_2(S^1) = 0$

$H_3 = H_3(S^3) \otimes H_0(S^1) \oplus \underbrace{H_2(S^3) \otimes H_1(S^1) \oplus H_1(S^3) \otimes H_2(S^1) \oplus H_0(S^3) \otimes H_3(S^1)}_{=0}$

$= \mathbb{F}$

$H_4 = H_4(S^3) \otimes H_0(S^1) \oplus H_3(S^3) \otimes H_1(S^1) \oplus 0$

$= \mathbb{F}$

$$\Rightarrow H_k(S^3 \times S^1) = \begin{cases} \mathbb{F}, & k=0 \\ \mathbb{F}, & k=1 \\ 0, & k=2 \\ \mathbb{F}, & k=3 \\ \mathbb{F}, & k=4 \end{cases}$$

<u>Definition</u>

Let X, Y be simplicial complexes

Let $p: X \to Y$ be a simplicial map.

Let d be an integer ≥ 1.

We say that p is a covering map of degree d when for each simplex σ of Y $\exists$ exactly d simplices $\sigma_1, \dots, \sigma_d$ of X $\sigma_i \cap \sigma_j = \emptyset$ $i \neq j$ and $p: \sigma_i \xrightarrow{\cong} \sigma$ for each i.

<u>Proposition</u>

Let $p: X \to Y$ be a simplicial covering of degree d where X, Y are finite complexes.

Then $\chi(X) = d\chi(Y)$.

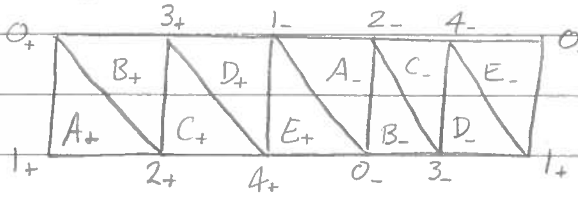
<u>Proof</u>

Calculate χ_{geom} in each case.

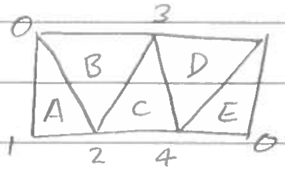
□

Example

The cylinder is a double covering of the Möbius band.



= Cylinder



= Möb

$C_2 \times \text{Cylinder} \longrightarrow \text{Cylinder}$

$\tau \cdot v_+ = v_- \, , \quad \tau \cdot v_- = v_+$

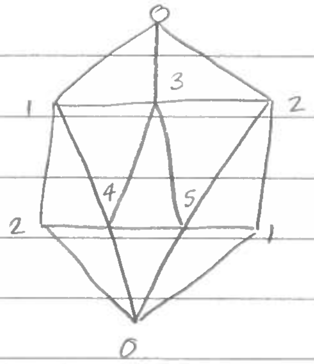
$\text{Cylinder} / C_2 = \text{Möbius}$

$\chi(\text{Cylinder}) = 2\chi(\text{Möb})$ (but both $= 0$)

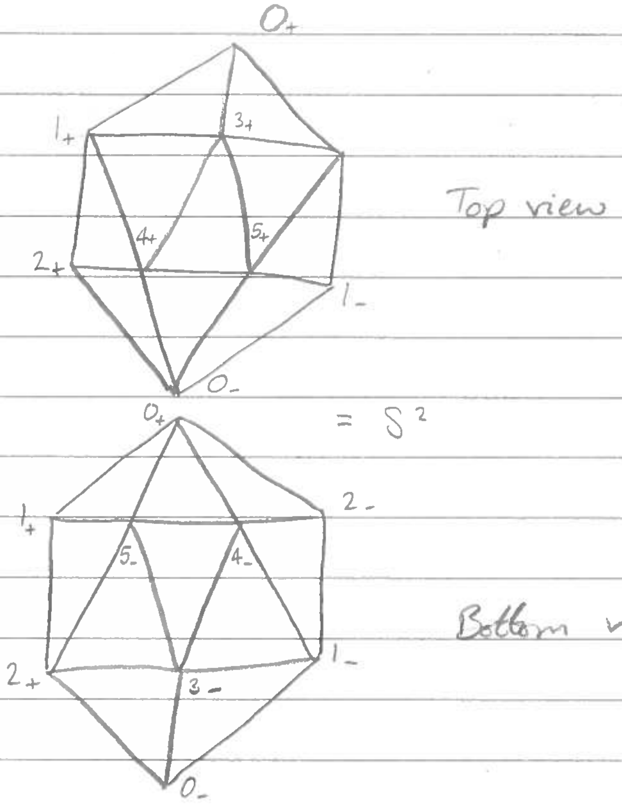
Example

S^2 is a double cover of $\mathbb{RP}(2)$.

Write $= \underset{\text{"top"}}{D_+} \cup \underset{\text{"bottom"}}{D_-} \, , \quad D_+ \cong D_-$



$\mathbb{RP}(2)$



Let C_2 act on S^2 by

$v_+ \longmapsto v_- \, , \quad v_- \longmapsto v_+$

Then $S^2 / C^2 \cong \mathbb{RP}(2)$.

$\chi(S^2) = 2\chi(\mathbb{RP}(2))$

$2 = 2 \times 1$.

05-03-18 Finite connected surfaces come in two families

$$\begin{tikzcd}
\text{Orientable:} & S^2 \arrow[d] & T^2 \arrow[d] & T^2 \# T^2 \arrow[d] & T^2 \# T^2 \# T^2 \arrow[d] & \cdots \\
\text{Non-orientable:} & \mathbb{R}P(2) & \mathbb{R}P(2) \# \mathbb{R}P(2) \arrow[d, phantom, "\shortparallel"] & \mathbb{R}P(2) \# \mathbb{R}P(2) \# \mathbb{R}P(2) & \mathbb{R}P(2) \# \mathbb{R}P(2) \# \mathbb{R}P(2) \# \mathbb{R}P(2) & \\
 & & K^2 & & &
\end{tikzcd}$$

There is a double covering in each case.

$$\begin{tikzcd}
\Sigma^g_+ = \underbrace{T^2 \# \ldots \# T^2}_{g} \arrow[d, "\text{double cover}"] \\
\Sigma^g_- = \underbrace{\mathbb{R}P(2) \# \ldots \# \mathbb{R}P(2)}_{g+1}
\end{tikzcd}$$

$\chi_+ = 2 - 2g = 2(1-g)$

$\chi_- = 1 - g$

07-03-18 <u>Classification of Surfaces</u> [All homology will be with $\mathbb{F}_2$ coeffs, $H_k(\Sigma) = H_k(\Sigma, \mathbb{F}_2)$]

We proceed via steps I - V

I). Let Σ be a finite connected surface. If Σ contains a copy of Möb, then $\Sigma \sim \mathbb{R}P(2) \# \Sigma'$ and $\dim H_1(\Sigma') = \dim H_1(\Sigma) - 1$

II). If Σ is a finite connected surface such that

a). Σ contains no copy of Möb, and

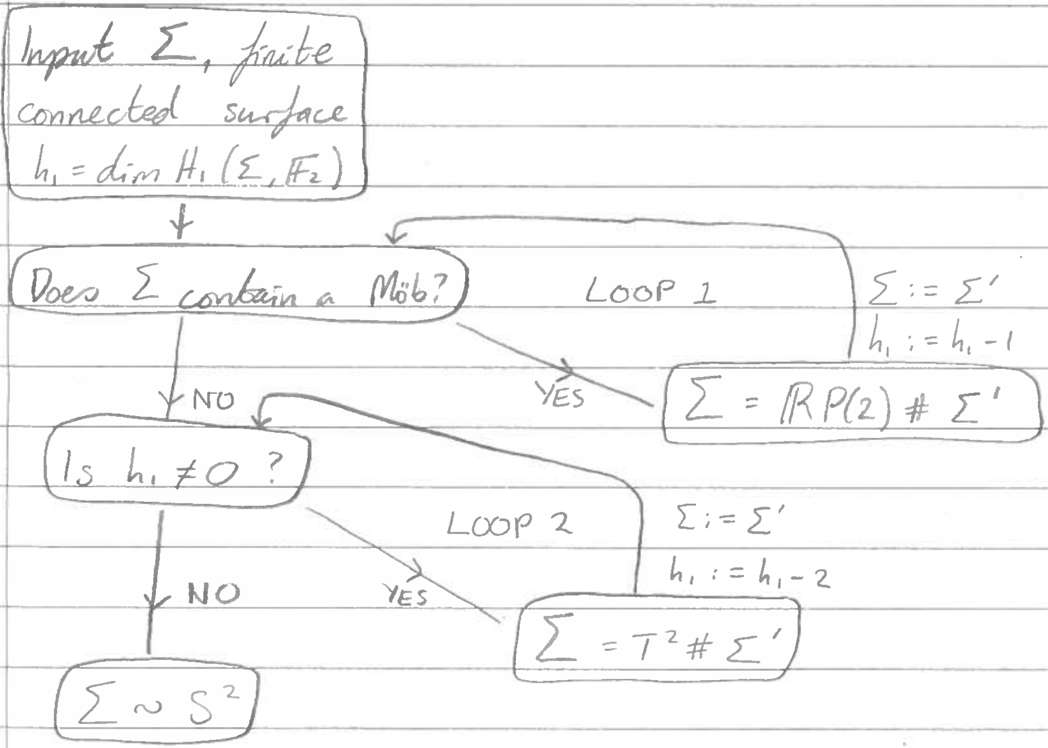
b). $H_1(\Sigma) \neq 0$ then Σ is again a connected sum

$\Sigma = T^2 \# \Sigma'$ and $\dim H_1(\Sigma') = \dim H_1(\Sigma) - 2$

III). If Σ is a finite connected surface such that $H_1(\Sigma) = 0$ then $\Sigma \sim S^2$.

IV). If Σ is a finite connected surface, then $\Sigma \# S^2 \sim \Sigma$

V). $\mathbb{R}P(2) \# T^2 \sim \mathbb{R}P(2) \# \mathbb{R}P(2) \# \mathbb{R}P(2)$



h_1 diminishes by 1 every time we go around loop 1 so eventually you drop out.

In loop 2 h_1 diminishes by 2, so eventually you doop out.

Input Σ. If we traverse LOOP 1 n times, then

$$\Sigma \sim \underbrace{\mathbb{R}P(2) \# \dots \# \mathbb{R}P(2)}_{n \text{ times}} \# \Sigma'$$

where Σ' contains no Möb.

If Σ' goes around LOOP 2 m times then

$$\Sigma' \sim \underbrace{T^2 \# \dots \# T^2}_{m \text{ times}} \# S^2 \quad \text{so} \quad \Sigma' \sim \underbrace{T^2 \# \dots \# T^2}_{m \text{ times}}$$

$$\Rightarrow \Sigma \sim \underbrace{\mathbb{R}P(2) \# \dots \# \mathbb{R}P(2)}_{n \text{ times}} \# \underbrace{T^2 \# \dots \# T^2}_{m \text{ times}} \qquad (n, m \neq 0)$$

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Special cases

(i) $n - m = 0 \Rightarrow \Sigma \sim S^2$

(ii) $n = 0,\ m \neq 0 \Rightarrow \Sigma \sim \underbrace{T^2 \# \dots \# T^2}_{m \text{ copies}}$

(iii) $n \neq 0,\ m = 0 \Rightarrow \Sigma \sim \underbrace{\mathbb{R}P(2) \# \dots \# \mathbb{R}P(2)}_{n \text{ copies}}$

However if $n \neq 0, m \neq 0$, then

$$\Sigma \sim \underbrace{\mathbb{R}P(2) \# \dots \# \mathbb{R}P(2)}_{n} \# \underbrace{T^2 \# \dots \# T^2}_{m}$$

$$\Rightarrow \Sigma \sim \underbrace{\mathbb{R}P(2) \# \dots \# \mathbb{R}P(2)}_{n+2m \text{ copies}}$$

Conclusion

A finite connected surface Σ is combinatorially equivalent to exactly one of

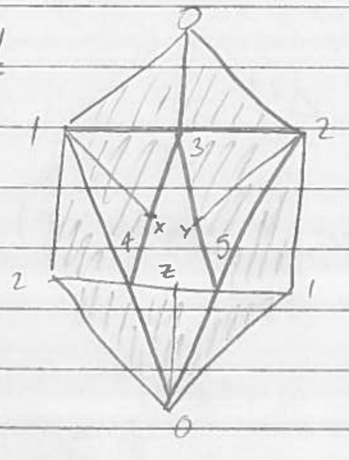
(i) S^2, (ii) T^2, (iii) $\underbrace{T^2 \# \dots \# T^2}_{g}$

(iv) $\mathbb{R}P(2)$, (v) $\underbrace{\mathbb{R}P(2) \# \dots \# \mathbb{R}P(2)}_{g+1}$.

Prop

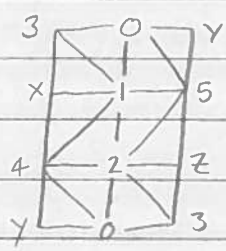
$\mathbb{R}P(2) - \text{Disc} \sim \text{Möb}$

Proof



$= \mathbb{R}P(2) - \text{Disc}$

$\sim$



$= \text{Möb}$

□

$\Sigma = Möb \cup_{S^1} \Sigma_0'$

$\partial Möb = S^1$, $\partial \Sigma_0' = S^1$

Define

$\Sigma' = \Sigma_0' \cup_{S^1} D^2$

$\dim H_1 \Sigma' = \dim H_1 \Sigma - 1$

12-03-18 $[\mathbb{F} = \mathbb{F}_2]$

I). Σ finite connected surface, if Σ contains a copy of Möb then $\Sigma \sim \mathbb{R}P(2) \# \Sigma'$ where Σ' is a finite connected surface and $\dim H_1(\Sigma') = \dim H_1(\Sigma) - 1$.

Proof

Triangulate Σ in such a way that Möb consists of a finite number of 2-simplices.
Enumerate 2-simplices of Σ, $\sigma_1, \dots, \sigma_m, \sigma_{m+1}, \dots, \sigma_n$ s.t. $\sigma_1, \dots, \sigma_m$ triangulate Möb.
Let Σ_0' be the subcomplex determined by $\sigma_{m+1}, \dots, \sigma_n$.
Then $Möb \cap \Sigma_0'$ = boundary of Möb $\sim S^1(n)$.
Put $\Sigma' = \Sigma_0' \cup C(\partial Möb)$.
Then Σ' is a finite connected surface and $\Sigma = \mathbb{R}P(2) \# \Sigma'$.
(Note that $C(\partial Möb) \sim C(S^1) \sim \Delta^2$)
$\mathbb{R}P(2) \sim Möb \cup C(\partial Möb)$
Now compute $h_1' = \dim H_1(\Sigma')$ [where $h_1 = \dim H_1(\Sigma)$]

$\chi(\Sigma) = \chi(\mathbb{R}P(2)) + \chi(\Sigma') - 2$

$2 - h_1 = 1 + 2 - h_1' - 2$

$-h_1 = -h_1' - 1$ $\qquad h_1 = h_1' + 1 \qquad h_1' = h_1 - 1$

$\square$

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II). Let Σ be a finite connected surface which contains no Möb. If $h_1 = \dim H_1(\Sigma) \neq 0$, then $\Sigma \sim T^2 \# \Sigma'$ where $h_1' = \dim H_1(\Sigma')$ satisfies $h_1' = h_1 - 2$

Recall that $H_k(\text{Cylinder}) \cong H_k(\text{Möb}) = \begin{cases} \mathbb{F} & k=0 \\ \mathbb{F} & k=1 \\ 0 & k \geq 2 \end{cases}$

Def^n^

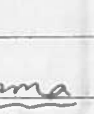
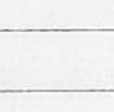
Let Cyl be a cylinder $\subset \Sigma$ (our finite connected surface)
Let $i : \text{Cyl} \hookrightarrow \Sigma$. Say that Cyl is essential if $i_* : H_1(\text{Cyl}) \longrightarrow H_1(\Sigma)$ is injective
Likewise if Möb $\hookrightarrow \Sigma$, say that Möb is essential iff $i_* : H_1(\text{Möb}) \rightarrow H_1(\Sigma)$ is injective.

Lemma

Let Σ be a finite connected surface.
If $H_1(\Sigma ; \mathbb{F}_2) \neq 0$ then Σ contains either
(i) an essential cylinder, or
(ii) an essential Möb.

Proof

Let $\xi \in H_1(\Sigma : \mathbb{F}_2) = Z_1(\Sigma) / B_1(\Sigma)$
Represent ξ by a finite linear combination (with $\mathbb{F}_2$ coefficients) of 1-simplices, with $\partial = 0$ because $\xi \in Z_1 / B_1$

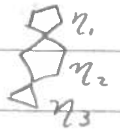



So represent ξ by a subgraph in which for every 1-simplex in ξ, $\partial \xi \subset \xi$ (i.e. no free edges)
(otherwise $\partial \xi \neq 0$)

Now search through all $\xi \in H_1(\Sigma) = Z_1/B_1$, $\xi \neq 0$, and choose one (call it η) represented by the fewest no. of 1-simplices.

Claim that η is an imbedded circle.

If not, η has a singularity

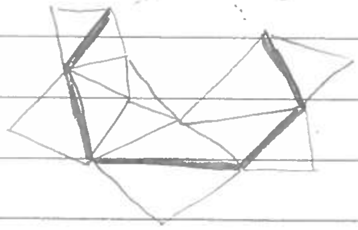


$\eta = \eta_1 + \eta_2 + \ldots + \eta_m$

Choose the smallest.

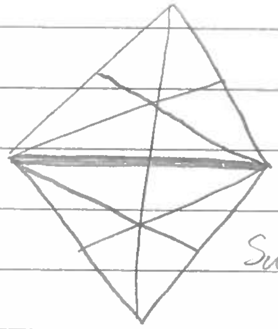
So now we have $\eta \subset \Sigma$, $\eta \sim S^1(n)$ for some n.

Draw it: Locally represented thus:

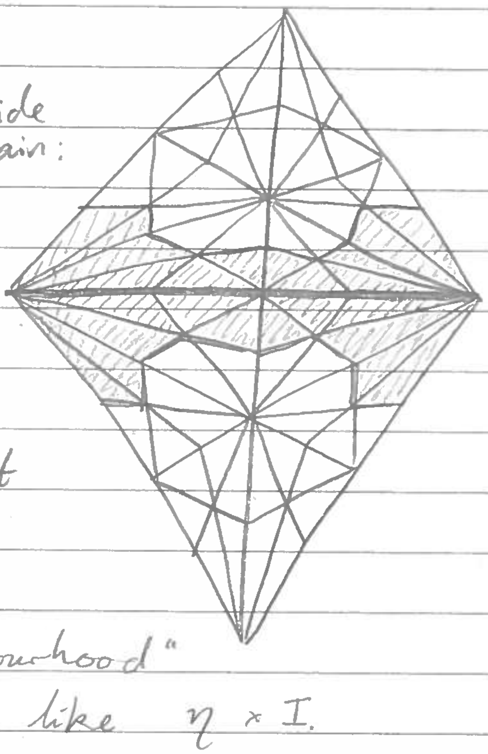


Each 1-simplex — of η is contained in exactly two 2-simplices of Σ.

Subdivide everything:



Subdivide again:



This is called "thickening"

Take the 2-simplices in the 2nd subdivision which intersect η somewhere.

We now have a "neighbourhood" of η which locally looks like $\eta \times I$.

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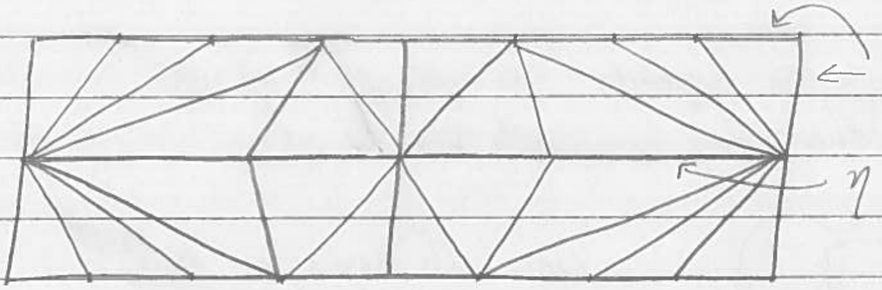


So given a finite connected surface Σ with $H_1(\Sigma : \mathbb{F}_2) \neq 0$, then $\exists$ imbedded circle $\eta \subset \Sigma$ which represents a nontrivial element of $H_1(\Sigma : \mathbb{F}_2)$.

Now thicken η to a neighbourhood N which is locally a product $\eta \times I$.

Ask whether ∂N is connected ($N \sim$ Möb) or disconnected ($N \sim$ Cyl).

In either case $N \subset \Sigma$ is essential.

$$\begin{tikzcd} N \arrow[dr, "\text{collapse}"'] & & \Sigma \\ & \eta \arrow[ur] & \end{tikzcd} \qquad H_1(N) \cong H_1(\eta) \longrightarrow H_1(\Sigma)$$

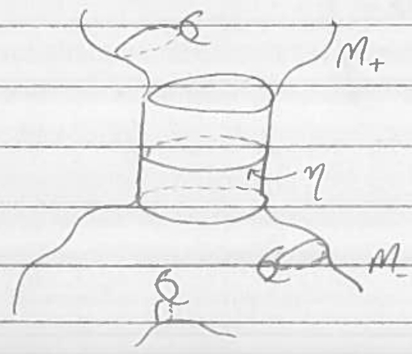
□

II).(b) Let Σ be a finite connected surface which contains no Möb. If $H_1(\Sigma ; \mathbb{F}_2) \neq 0$ then Σ contains an essential cylinder C ($\equiv N$ nbd of η).

Moreover $\Sigma - C$ is connected.

Why? $\Sigma - C = M_+ \sqcup M_-$

then either M_+ or M_- will bound the homology class of η which C represents.



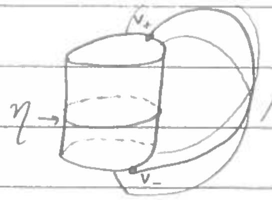
So $\eta \sim \partial M_-$

so η represents 0 in $H_1(\Sigma : \mathbb{F}_2)$

contradiction

So $\Sigma - C$ is connected.

Take v_+, v_- in opposite components of ∂C
and join them by a path μ in $\Sigma - C$.



Now thicken μ to C'

Because Σ contains no Möb then $C \cup C'$ is a punctured torus.

Put $\Sigma_0' = \Sigma - (C \cup C')$

$\partial(C \cup C') \sim S^1$, $\partial \Sigma' \sim S^1$

Put $\Sigma' = (\Sigma - C \cup C') \underset{\partial = \partial}{\cup} C(\partial(C \cup C'))$

then Σ' is a finite connected surface

$\Sigma \sim T^2 \# \Sigma'$

Now put $h_1 = \dim H_1(\Sigma)$, $h_1' = \dim H_1(\Sigma')$

$$\chi(\Sigma) = \chi(T^2) + \chi(\Sigma') - 2$$
$$= \chi(\Sigma') - 2$$
$$2 - h_1 = 2 - h_1' - 2 = - h_1'$$
$$\Rightarrow h_1' = h_1 - 2$$

$\square$

III). Let Σ be a finite connected surface
If $H_1(\Sigma : \mathbb{F}_2) = 0$ then $\Sigma \sim S^2$

[Poincaré conjecture 2004 G Perelman
Survey paper by John Milnor]

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Σ finite connected surface, $h_1 = \dim H_1(\Sigma ; \mathbb{F}_2) = 0$

Define $\Sigma_0 = \Sigma - (2\text{-disc})$ this is a "surface with boundary".
$= \Sigma - D^2$

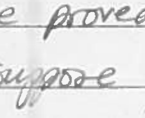
Σ_0 is connected and $H_1(\Sigma_0 : \mathbb{F}_2) = 0$ $(H_2(\Sigma_0 : \mathbb{F}_2) = 0$ obvious$)$

$$\begin{tikzcd}
H_2(\Sigma_0)\oplus H_2(D^2) \arrow[r] \arrow[d, equal] & H_2(\Sigma) \arrow[r] \arrow[d, equal] & H_1(\partial\Sigma_0) \arrow[r] \arrow[d, equal] & H_1(\Sigma_0)\oplus H_1(D^2) \arrow[r] \arrow[d, Rightarrow] & H_1(\Sigma) \arrow[d, equal] \\
0 & \mathbb{F} \arrow[r, "\cong"] & \mathbb{F} & H_1(\Sigma_0) = 0 & 0
\end{tikzcd}$$

We prove $\Sigma_0 \sim D^2 = \Delta^2$ by induction on no. $= n$ of 2 simplices of Σ_0.

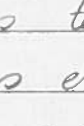
If $n = 1$, nothing to prove.
If $n = 2$, $\Theta \sim D^2$

Suppose proved when no. of 2-simplies of $\Sigma_0 < n$ and suppose Σ_0 has n 2-simplices.
Choose a 2-simplex σ of Σ_0 s.t. at least one face of σ is in $\partial\Sigma_0$.
There are three possible configurations:
(A) σ has two 1-faces in $\partial\Sigma_0$
(B) σ has exactly one 1-face τ in $\partial\Sigma_0$ and the opposite vertex to τ is in the interior
(C) σ has exactly one 1-face τ in $\partial\Sigma_0$ and the opposite vertex to τ is also in $\partial\Sigma_0$.

(A) $\Sigma_0 =$ 

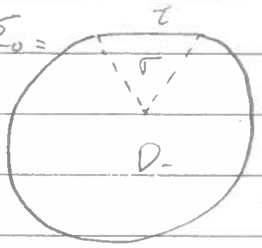
$\Sigma_0 = \sigma \cup D_-$
and by induction $D_- \sim D^2$

Σ_0 collapses onto D_-, $\Sigma_0 \searrow D_- \Rightarrow H_*(\Sigma_0) = H_*(D_-)$

$\Rightarrow \Sigma_0 \sim$ 

$\sim D^2$

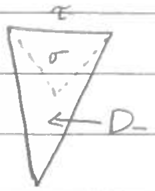
(B) $\Sigma_0 =$



Again $\Sigma_0 \searrow D_-$

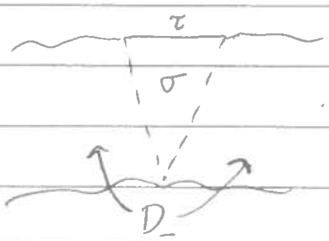
$\Rightarrow H_*(D_-) \cong H_*(\Sigma_0)$

$\sim$

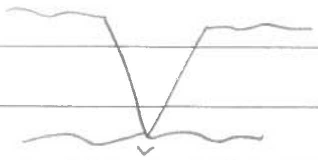


$\Sigma_0 \sim D^2 = \Delta^2$

(C)



$D_- = \Sigma_0 - \sigma$

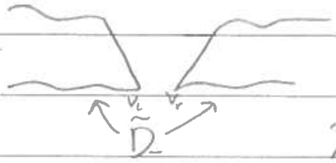


$\Sigma_0 \searrow D_-$

so $H_*(D_-) = H_*(\Sigma_0)$

$\chi(D_-) = \chi(\Sigma_0) = 1$

Let $\tilde{D}_-$ be obtained from D_- by duplicating v



$\chi(\tilde{D}_-) = \chi(D_-) + 1 = 2$

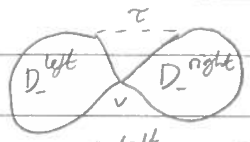
If $H_0(\tilde{D}_- : \mathbb{F}_2) = \mathbb{F}_2$ $\quad 1 - \tilde{h}_1 + \tilde{h}_2 = 2$

So $H_2(\tilde{D}_- : \mathbb{F}_2) \neq 0$ Impossible

$H_2(\tilde{D}_-) \cong H_2(D_-) = 0$

$\tilde{D}_-$ disconnected

So $D_- =$



By induction, $D_-^{left} \sim D^2$, $D_-^{right} \sim D^2$

$\Rightarrow \Sigma_0 \sim$

$\sim \Delta^2$

$\square$

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If Σ_0 is a finite connected "bounded" surface with boundary $\sim S^1$, and $H_1(\Sigma_0 : \mathbb{F}_2) = 0$
then $\Sigma_0 \sim D^2$

Corollary (Step IV)
Σ finite connected surface with $H_1(\Sigma : \mathbb{F}_2) = 0$
then $\Sigma \sim S^2$.

Proof
Put $\Sigma_0 = S^2 - \Delta^2$
Δ^2 some 2-simplex

$$\underset{\overset{\shortparallel}{0}}{H_2(\Sigma_0) \oplus H_2(\Delta^2)} \rightarrow \underset{\overset{\shortparallel}{\mathbb{F}_2}}{H_2(\Sigma)} \rightarrow \underset{\overset{\shortparallel}{S_1(S^1) = \mathbb{F}_2}}{H_1(\Sigma_0 \cap \Delta^2)} \rightarrow \underset{\overset{\shortparallel}{H_1(\Sigma_0)}}{H_1(\Sigma_0) \oplus H_1(\Delta^2)} \rightarrow \underset{\overset{\shortparallel}{0}}{H_1(\Sigma)}$$

By Whitehead's trick, $H_1(\Sigma_0) = 0$.
$\Rightarrow \Sigma_0 \sim D^2$
$\Rightarrow \Sigma \sim \underset{\overset{\shortparallel}{\Sigma_0}}{D^2} \cup_{\partial = \partial} \Delta^2 \sim S^2$

□

So if Σ is a finite connected surface, then we have one of the following:
$\Sigma \sim S^2$
$\Sigma \sim \underbrace{\mathbb{R}P(2) \# \dots \# \mathbb{R}P(2)}_{n}$ loop 1 n times, loop 2 0 times
$\Sigma \sim \underbrace{T^2 \# \dots \# T^2}_{m}$ loop 1 0 times, loop 2 m times

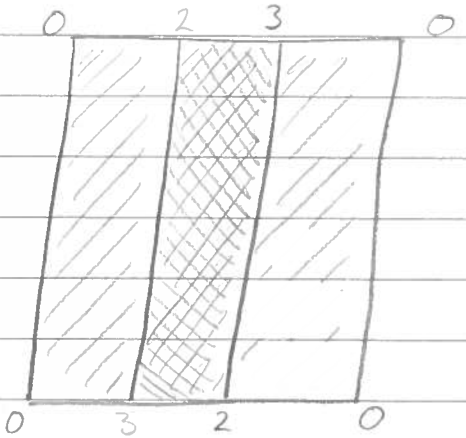
$\Sigma \sim \underbrace{\mathbb{R}P(2) \# \dots \# \mathbb{R}P(2)}_{n} \# \underbrace{T^2 \# \dots \# T^2}_{m}$ loop 1 n times, loop 2 m times.

We'll show $\mathbb{R}P(2) \# T^2 \sim \mathbb{R}P(2) \# \mathbb{R}P(2) \# \mathbb{R}P(2)$
$\Rightarrow \Sigma \sim \underbrace{\mathbb{R}P(2) \# \dots \# \mathbb{R}P(2)}_{n+2m}$

Prop

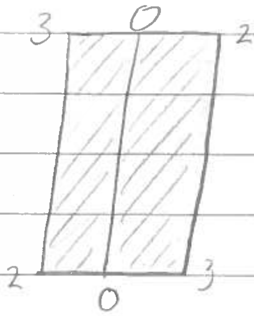
$\mathbb{R}P(2) \# \mathbb{R}P(2) \sim K^2$ klein bottle

Proof



▦ = Möb_1

▨ = Möb_2



$K^2 = \text{Möb}_1 \underset{\partial=\partial}{\cup} \text{Möb}_2$

$\mathbb{R}P(2)_1 = \text{Möb}_1 \underset{\partial}{\cup} D^2$, $\mathbb{R}P(2)_2 = \text{Möb}_2 \underset{\partial}{\cup} D^2$

$\Rightarrow K^2 \sim \mathbb{R}P(2)_1 \# \mathbb{R}P(2)_2$

$\square$

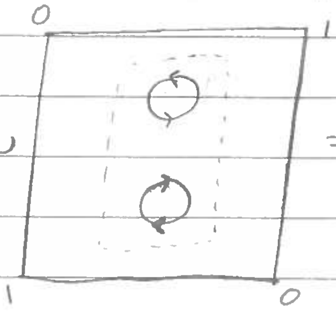
Prop

$\mathbb{R}P(2) \# T^2 \sim \mathbb{R}P(2) \# K^2$ $(\sim \mathbb{R}P(2) \# \mathbb{R}P(2) \# \mathbb{R}P(2))$

Proof

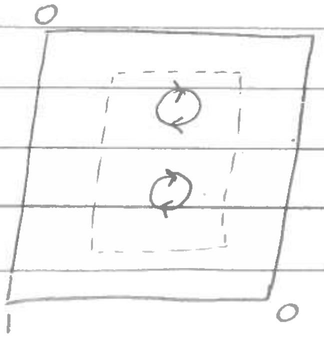
Write $\mathbb{R}P(2) = D^2 \underset{\partial=\partial}{\cup} \text{Möb}$ (D^2 will now become invisible)

$D^2 \cup$



$= \mathbb{R}P(2) \# T^2$

$D^2 \cup$



$= \mathbb{R}P(2) \# K^2$

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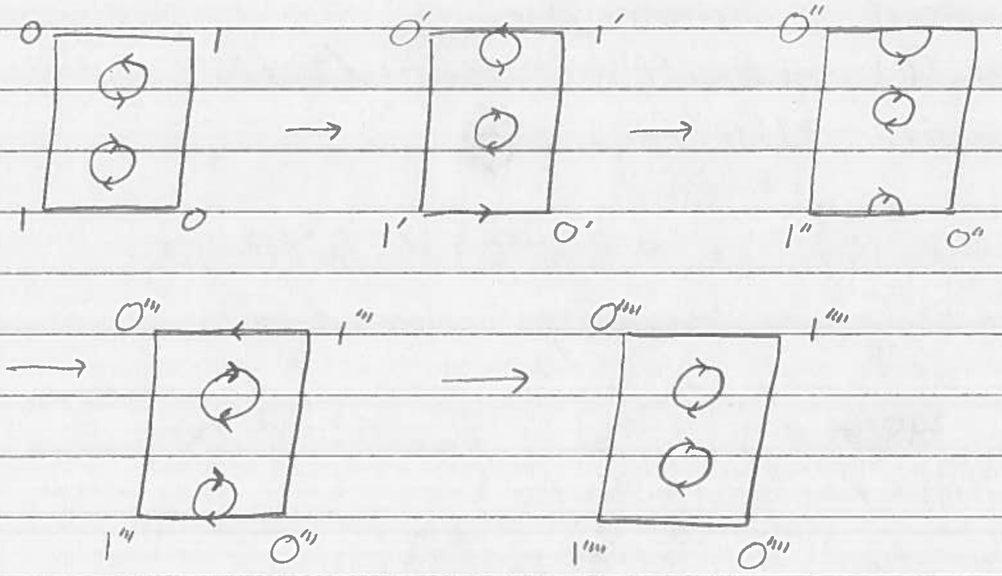

□

$\Sigma^g_+ = \underbrace{T^2 \# \dots \# T^2}_{g}$

⤵ double covers

$\Sigma^g_- = \underbrace{\mathbb{R}P(2) \# \dots \# \mathbb{R}P(2)}_{g+1}$

19-03-18 Algebraic MV-Thm

$0 \to A_* \xrightarrow{i} B_* \xrightarrow{p} C_* \to 0$

exact sequence of chain complexes

1). Then exist homomorphisms

$\delta : H_n(C) \to H_{n-1}(A)$ (connecting homomorphisms)

s.t. the sequence

$H_{n+1}(C) \xrightarrow{\delta} H_n(A) \xrightarrow{i_*} H_n(B) \xrightarrow{p_*} H_n(C) \xrightarrow{\delta} H_{n-1}(A) \xrightarrow{i_*} H_{n-1}(B) \xrightarrow{p_*} \dots$

is exact for all n.

2) δ are "natural" in an obvious way.

$$\begin{tikzcd}
0 \arrow[r] & A_{n+1} \arrow[r, "i_{n+1}"] \arrow[d, "\partial^A_{n+1}"] & B_{n+1} \arrow[r, "p_{n+1}"] \arrow[d, "\partial^B_{n+1}"] & C_{n+1} \arrow[r] \arrow[d, "\partial^C_{n+1}"] & 0 \\
0 \arrow[r] & A_n \arrow[r, "i_n"] \arrow[d, "\partial^A_n"] & B_n \arrow[r, "p_n"] \arrow[d, "\partial^B_n"] & C_n \arrow[r] \arrow[d, "\partial^C_n"] & 0 \\
0 \arrow[r] & A_{n-1} \arrow[r, "i_{n-1}"] \arrow[d, "\partial^A_{n-1}"] & B_{n-1} \arrow[r, "p_{n-1}"] \arrow[d, "\partial^B_{n-1}"] & C_{n-1} \arrow[r] \arrow[d, "\partial^C_{n-1}"] & 0 \\
0 \arrow[r] & A_{n-2} \arrow[r, "i_{n-2}"] & B_{n-2} \arrow[r, "p_{n-2}"] & C_{n-2} \arrow[r] & 0
\end{tikzcd}$$

"Snake Lemma"

We'll construct a homomorphism

$$\delta : Z_n(C) \longrightarrow H_{n-1}(A) = Z_n(A) / B_n(A).$$

$$\overset{\shortparallel}{\mathrm{Ker}\,\partial_n^C} \longrightarrow \left(\frac{\mathrm{Ker}\,\partial_n^A}{\mathrm{Im}\,\partial_{n+1}^A} \right)$$

Let $z \in Z_n(C)$, i.e. $z \in C_n(C)$ st. $\partial_n^C(z) = 0$.

$p_n : B_n \to C_n$ is surjective (exactness)

Choose $b \in B_n$ s.t. $p_n(b) = z$

Consider $\partial_n^B(b)$ st. $p_{n-1}\partial_n^B(b) = \partial_n^C p_n(b) = \partial_n^C(z) = c$

so $\partial_n^B(b) \in \mathrm{Ker}(p_{n-1}) = \mathrm{Im}(i_{n-1})$

so $\exists\, a \in A_{n-1}$ st. $i_{n-1}(a) = \partial_n^B(b)$

Def

Let $z \in Z_n(C)$. By a choice for z I mean a pair (b, a) s.t. $\begin{cases} b \in B_n \text{ s.t. } p_n(b) = z \\ a \in A_n \text{ s.t. } i_{n-1}(a) = \partial_n(b) \end{cases}$

Prop

If (b, a) is a choice for $z \in Z_n(C)$, then $a \in Z_{n-1}(A) = \mathrm{Ker}(\partial_{n-1}^A : A_{n-1} \to A_{n-2})$.

Proof

$i_{n-2}\,\partial_{n-1}^A(a) = \partial_{n-1}^B(i_{n-1}(a)) = \partial_{n-1}^B\,\partial_n^B(b) = 0$, $\partial\partial = 0$

But i_{n-2} is injective, so $\partial_{n-1}^A(a) = 0$

$z \in Z_n(C)$. Make a choice (b, a) for z, so $a \in Z_{n-1}(A)$. $z \longmapsto a$ Have to make this into a mapping. (b, a) is not unique, so have to ensure end result doesn't depend on choice.

Special case : $z = 0$

Make a choice (b, a) for $z = 0$.

Claim $a \in B_{n-1}(A) = \mathrm{Im}(\partial_n^A : A_n \to A_{n-1})$.

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Since $p_n(b) = 0$ $(= z)$

Then $b \in \mathrm{Ker}(p_n) = \mathrm{Im}(i_n)$

Choose $\alpha \in A_n$ s.t. $i_n(\alpha) = b$

$i_{n-1}\partial_n^A(\alpha) = \partial_n^B i_n(\alpha) = \partial_n^B(b)$

But $i_{n-1}(a) = \partial_n^B(b)$

i_{n-1} is injective so $a = \partial_n^A(\alpha) \in B_{n-1}(A)$.

In general suppose (b, a), (b', a') are both choices for z. Then

$(b-b', a-a')$ is a choice for 0.

So $a - a' \in B_{n-1}(A)$

$z \rightsquigarrow a \in Z_{n-1}(A)$, $z \rightsquigarrow a' \in Z_{n-1}(A)$

and $a - a' \in B_{n-1}(A)$

So $[a] = [a'] \in H_{n-1}(A)$

We've now constructed a mapping

$\delta : Z_n(C) \longrightarrow H_{n-1}(A)$

by $\delta(z) = [a] \in Z_{n-1}(A) / B_{n-1}(A)$

whenever (b, a) is a choice for z for some b.

It is easy to show that this is a group homomorphism.

What we really want is a homomorphism

$\delta : H_n(C) \longrightarrow H_{n-1}(A)$

$\overset{\shortparallel}{Z_n(C)/B_n(C)}$ Need to show $\delta(B_n(C)) = 0$

So suppose $z \in B_n(C)$

Choose $\zeta \in C_{n+1}$ s.t. $\partial_{n+1}^C(\zeta) = z$

p_{n+1} is surjective so choose $\beta \in B_{n+1}$ st. $p_{n+1}(\beta) = \zeta$

$\partial_n^C p_{n+1}(\beta) = \partial_n^C(\zeta) = z$ so $p_n \partial_{n+1}^B(\beta) = z$

and $\partial_n^B \partial_{n+1}^B(\beta) = 0$ so $i_{n-1}(0) = \partial_n^B \partial_{n+1}^B(\beta)$

so $(\partial_{n+1}^B(\beta), 0)$ is a choice for $z \in B_n(C)$

i.e. $\delta(z) = 0$ so $\delta(B_n(C)) = 0$

So δ induces $\delta : H_n(C) \to H_{n-1}(A)$ (connecting homo)

$\shortparallel$

$Z_n(C)/B_n(C)$

Now need to show three things:

(I) $H_n(A) \xrightarrow{i_*} H_n(B) \xrightarrow{p_*} H_n(C)$ is exact

(II) $H_n(B) \xrightarrow{p_*} H_n(C) \xrightarrow{\delta} H_{n-1}(A)$ is exact

(III) $H_{n+1}(C) \xrightarrow{\delta} H_n(A) \xrightarrow{i_*} H_n(B)$ is exact.

Proof of I

$p_* i_* = 0$ because $p_n i_n = 0$

NTS: if $\beta \in Z_n(B)$ is such that

$p_*[\beta] = 0$ then $\exists \alpha \in Z_n(A)$ s.t. $i_*[\alpha] = [\beta]$

$\equiv$ $p_n(\beta) \in Im(\partial^C_{n+1} : C_{n+1} \to C_n)$

So $\exists \zeta \in C_{n+1}$ s.t. $p_n(\beta) = \partial^C_{n+1}(\zeta)$.

p_{n+1} is surjective so choose $\eta \in B_{n+1}$

s.t. $p_{n+1}(\eta) = \zeta$

$\partial^C_{n+1} p_{n+1}(\eta) = \partial^C_{n+1}(\zeta)$

so $p_n \partial^B_{n+1}(\eta) = \partial^C_{n+1}(\zeta)$

$p_n(\beta) = p_n \partial^B_{n+1}(\eta)$

$\beta - \partial^B_{n+1}(\eta) \in Ker(p_n) = Im(i_n)$

so $\exists \alpha \in A_n$ s.t. $i_n(\alpha) = \beta - \partial^B_{n+1}(\eta)$ [note $\partial^B_n(\beta) = 0$ since $\beta \in Z_n(B)$]

$\partial^B_n i_n(\alpha) = \partial^B_n(\beta) - \partial^B_n \partial^B_{n+1}(\eta) = 0$ $[\partial\partial = 0]$

So $i_{n-1}\partial^A_n(\alpha) = \partial^B_n i_n(\alpha) = 0$ i_{n-1} is injective so

$\partial^A_n(\alpha) = 0$, $\alpha \in Z_n(A)$, & $i_*[\alpha] = [\beta]$

$\square$

Proof of (II)

First observe that $\delta p_* = 0$.

Let $b \in Z_n(B)$, so $p_n(b) \in Z_n(C)$

We have to evaluate δ on $[p_n(b)]$

Make a choice for $p_n(b)$. There is an obvious choice,

$(b, 0)$. $(\partial^B_n(b) = 0)$ So $\delta p_*[b] = 0$.

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Next we have to show that if $z \in Z_n(C)$ satisfies $\delta[z] = 0$ then $\exists b \in Z_n(B)$ s.t. $p_*[b] = [z]$.

We know that $\exists$ choice (b,a) for z s.t.

$a \in B_{n-1}(A) = Im(\partial_n^A)$. $b \in B_n$ s.t. $p_n(b) = z$

and $i_{n-1}(a) = \partial_n^B(b)$

Choose $\alpha \in A_n$ s.t. $\partial_n^A(\alpha) = a$ $[p_n i_n = 0]$

and consider $b - i_n(\alpha)$

Then $p_n(b - i_n(\alpha)) = z$

so now $\partial_n^B(b - i_n(\alpha)) = \partial_n^B(b) - \partial_n^B(i_n(\alpha))$

$= \partial_n^B(b) - i_{n-1}\partial_n^A(\alpha)$

$= \partial_n^B(b) - i_{n-1}(a) = 0$

So $(b - i_n(\alpha), 0)$ is a choice for $p_n(b) = z$

So $p_*[b] = [z]$. □

Proof of III

Let $z \in Z_{n+1}(C)$ and make a choice (b,a) for z.

Can assume $a \in Z_n(A)$.

$p_{n+1}(b) = z$, $\partial_n^B(b) = i_n(a)$

$i_*[a] = [\partial_{n+1}^B(b)] = 0$ $\partial_{n+1}^B(b) \in B_n(B)$

$i_* \circ \delta = 0$

NTS: if $a \in Z_n$ satisfies $i_*[a] = 0$ then

$\exists z \in Z_{n+1}(C) : \delta[z] = [a]$.

? $i_*[a] = 0$ means $i_n(a) = \partial_{n+1}^B(b)$ for some $b \in B_{n+1}$

Put $z = p_{n+1}(b)$ and calculate $\partial_{n+1}^C(z)$

$\partial_{n+1} p_{n+1}(b) - p_n \partial_{n+1}^B(b) = p_n i_n(a) = 0$

so $\partial_{n+1}(z) = 0$ and now look at def^n of δ

$z \in Z_{n+1}(C)$ and (b,a) is a choice for z

$\Rightarrow \delta[z] = [a]$. □

We've established the existence of long exact sequences.

"Naturality of δ"

Suppose $(0 \to A_* \xrightarrow{i} B_* \xrightarrow{\rho} C_* \to 0)$

and $(0 \to A'_* \xrightarrow{i'} B'_* \xrightarrow{\rho'} C'_* \to 0)$

are both short exact sequences of chain complexes.

Suppose we also have chain maps

$\alpha : A_* \to A'_*$, $\beta : B_* \to B'_*$, $\gamma : C_* \to C'_*$

in such a way that for each n the following diagram commutes:

$$\begin{tikzcd}
0 \arrow[r] & A_n \arrow[r, "i_n"] \arrow[d, "\alpha_n"] & B_n \arrow[r, "\rho_n"] \arrow[d, "\beta_n"] & C_n \arrow[r] \arrow[d, "\gamma_n"] & 0 \\
0 \arrow[r] & A_n' \arrow[r, "i_n'"] & B_n' \arrow[r, "\rho_n'"] & C_n' \arrow[r] & 0
\end{tikzcd}$$

$$\begin{tikzcd}
H_{n+1}(B) \arrow[r, "\rho_*"] \arrow[d, "\beta_*"] & H_{n+1}(C) \arrow[r, dashed, "\delta"] \arrow[d, "\gamma_*"] & H_n(A) \arrow[r, "i_*"] \arrow[d, "\alpha_*"] & H_n(B) \arrow[r, "\rho_*"] \arrow[d, "\beta_*"] & H_n(C) \arrow[r, dashed, "\delta"] \arrow[d, "\gamma_*"] & H_{n-1}(A) \arrow[r, "i_*"] \arrow[d, "\alpha_*"] & H_{n-1}(B) \arrow[r] \arrow[d, "\beta_*"] & {} \\
H_{n+1}(B') \arrow[r, "\rho'_*"] & H_{n+1}(C') \arrow[r, dashed, "\delta'"] & H_n(A') \arrow[r, "i'_*"] & H_n(B') \arrow[r, "\rho'_*"] & H_n(C') \arrow[r, dashed, "\delta'"] & H_{n-1}(A') \arrow[r, "i'_*"] & H_{n-1}(B') \arrow[r] & {}
\end{tikzcd}$$

All this commutes. "Naturality" says that when we add in the connecting homomorphisms then everything commutes.

In particular

$$\begin{tikzcd}
H_n(C) \arrow[r, "\delta"] \arrow[d, "\gamma_*"] & H_{n-1}(A) \arrow[d, "\alpha_*"] \\
H_n(C') \arrow[r, "\delta'"] & H_{n-1}(A')
\end{tikzcd}$$

commutes

MATH3203

21-03-18

Possible problem class: room 500, 1st wed of Term 3, 11-1

Trivial case

$$0 \to A_* \xrightarrow{\binom{1}{0}} A_* \oplus C_* \xrightarrow{(0,1)} C_* \to 0$$

exact sequence of chain complexes

? Def^n of $A_* \oplus C_*$ $\quad A_n \oplus C_n \quad \begin{pmatrix} \partial_n^A & 0 \\ 0 & \partial_n^C \end{pmatrix}$

Prop

In the trivial case

$\delta : H_n(C_*) \to H_{n-1}(A_*)$ is zero.

Proof

If $z \in Z_n(C)$ then $\binom{0}{z}$ is already a choice for z, because $(0,1)\binom{0}{z} = z$ and $\partial\binom{0}{z} = \binom{0}{\partial z}$ and $\partial z = 0$

so $\partial[z] = 0$

$\square$

Corollary

In a trivial exact sequence of chain complexes, the long exact sequence becomes

$$0 \to H_n(A) \to H_n(A_* \oplus C_*) \to H_n(C_*) \to 0$$

Proof

$$\underset{\substack{\| \\ 0}}{H_{n+1}(C) \xrightarrow{\delta} H_n(A)} \to H_n(A_* \oplus C_*) \to \underset{\substack{\| \\ 0}}{H_n(C_*) \xrightarrow{\delta} H_n(A_*)}$$

$$\Downarrow$$

$$0 \to H_n(A) \to H_n(A_* \oplus C_*) \to H_n(C_*) \to 0$$

$\square$

Split exact sequences

$\mathcal{E} = (0 \to A \xrightarrow{i} B \xrightarrow{p} C \to 0)$ exact sequence of abelian groups

Say that $\mathcal{E}$ splits when $\exists$ commutative diagram

$$\begin{tikzcd}
0 \arrow[r] \arrow[d, dotted] & A \arrow[r, "i"] \arrow[d, "\mathrm{Id}"] & B \arrow[r, "p"] \arrow[d, "\varphi"] & C \arrow[r] \arrow[d, "\mathrm{Id}"] & 0 \arrow[d, dotted] \\
0 \arrow[r] & A \arrow[r, "\binom{1}{0}"] & A \oplus C \arrow[r, "{(0,1)}"] & C \arrow[r] & 0
\end{tikzcd}$$

$\binom{1}{0}(a) = \binom{a}{0}$, $(0,1)\binom{a}{c} = c$

injection $\quad$ projection

It follows that φ is an isomorphism.

Say that
$$\begin{tikzcd} \mathcal{E} = (0 \arrow[r] & A \arrow[r, "i"] & B \arrow[l, dashed, bend left] \arrow[r, "p"] & C \arrow[r] & 0) \end{tikzcd}$$
splits on the left when $\exists$ homomorphism $r : B \to A$ s.t. $r \circ i = Id_A$

Prop
If $\mathcal{E}$ splits on the left then $\mathcal{E}$ splits.

Proof
Let $r : B \to A$ be the homomorphism s.t. $r \circ i = Id_A$.
Define $\varphi : B \to A \oplus C$ by $\varphi(b) = \begin{pmatrix} r(b) \\ p(b) \end{pmatrix}$

Then
$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r, "i"] \arrow[d, "Id"] & B \arrow[r, "p"] \arrow[d, "\varphi"] & C \arrow[r] \arrow[d, "Id"] & 0 \\ 0 \arrow[r] & A \arrow[r, "\binom{1}{0}"] & A \oplus C \arrow[r, "{(0,1)}"] & C \arrow[r] & 0 \end{tikzcd}$$
commutes

so $\mathcal{E}$ splits.

$\square$

[Converse also true]

A right splitting is a homomorphism $s : C \to B$
s.t. $p \circ s = Id_C$.

Exercise
$\mathcal{E}$ splits $\Leftrightarrow$ $\mathcal{E}$ splits on the left $\Leftrightarrow$ $\mathcal{E}$ splits on the right.

Corollary
If $0 \longrightarrow A_* \xrightarrow{\binom{1}{0} = i} A_* \oplus C_* \xrightarrow{(0,1) = p} C_* \longrightarrow 0$
is trivial exact sequence of chain complexes, then
$H_n(A_* \oplus C_*) \cong H_n(A_*) \oplus H_n(C_*)$

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Proof

The LES looks like

$$0 \to H_n(A) \xrightarrow{i_*} H_n(A_* \oplus C_*) \xrightarrow{p_*} H_n(C_*) \to 0 \quad \text{exact}$$

This sequence splits on the left.

Define $r: A_* \oplus C_* \to A_*$, $r\binom{a}{c} = a$, $r \circ i = Id_{A_*}$

$H_n(r) \circ H_n(i) = Id_{H_n(A)}$

$\overset{\|}{r_*} \circ i_* = Id$

$\square$

Geometric MV Thm

$X = X_+ \cup X_-$, X_+, X_- subcomplexes of X

For each n get an exact sequence of vector spaces:

$$0 \to C_n(X_+ \cap X_-) \to C_n(X_+) \oplus C_n(X_-) \to C_n(X) \to 0$$

and boundary map of $C_*(X_+) \oplus C_*(X_-) = \begin{pmatrix} \partial^{X_+} & 0 \\ 0 & \partial^{X_-} \end{pmatrix}$

i.e. for the middle term, we get a trivial exact sequence of chain complexes

$$0 \to C_*(X_+) \to C_*(X_+) \oplus C_*(X_-) \to C_*(X_-) \to 0$$

so $H_n(C_*(X_+) \oplus C_*(X_-)) \cong H_n(C_*(X_+)) \oplus H_n(C_*(X_-))$

$= H_n(X_+) \oplus H_n(X_-)$

$$0 \to C_*(X_+ \cap X_-) \to C_*(X_+) \oplus C_*(X_-) \to C_*(X) \to 0$$

If we apply LES we get

$$H_{n+1}(X) \to H_n(X_+ \cap X_-) \to \underset{\substack{\cong \\ H_n(X_+) \oplus H_n(X_-)}}{H_n(C_*(X_+) \oplus C_*(X_-))} \to H_n(X) \to H_{n-1}(X_+ \cap X_-)$$

Finally we get

$$\begin{tikzcd}
H_{n+1}(X) \arrow[r, "\delta"] \arrow[d, "f_*"] \arrow[dr, phantom, "\circlearrowright"] & H_n(X_+ \cap X_-) \arrow[r] \arrow[d, "f_*"] & H_n(X_+) \oplus H_n(X_-) \arrow[r] \arrow[d, "\begin{pmatrix} f_* & 0 \\ 0 & f_* \end{pmatrix}"] & H_n(X) \arrow[r, "\delta"] \arrow[d, "f_*"] \arrow[dr, phantom, "\circlearrowright"] & H_{n-1}(X_+ \cap X_-) \arrow[r] \arrow[d, "f_*"] & \ldots \\
H_{n+1}(Y) \arrow[r, "\delta"] & H_n(Y_+ \cap Y_-) \arrow[r] & H_n(Y_+) \oplus H_n(Y_-) \arrow[r] & H_n(Y) \arrow[r, "\delta"] & H_{n-1}(Y_+ \cap Y_-) \arrow[r] & \ldots
\end{tikzcd}$$

Supposing $Y = Y_+ \cup Y_-$, $f: X \to Y$, $f(X_+) \subset Y_+$, $f(X_-) \subset Y_-$

$\circlearrowright$ commutes by naturality.

Course finished.

$$H_k(S^n\times S^1 : \mathbb{F}) = \begin{cases} \mathbb{F}, & k=0,1,n,n+1 \\ 0, & \text{otherwise} \end{cases} \qquad n \geq 2.$$

$\underline{n=2}$

$S^2 = D_+ \cup D_-$ $\qquad D_+\cap D_- = S^1$

D_+, D_- are cones

$$\underset{\underset{0}{\parallel}}{H_3(D_+\times S^1)\oplus H_3(D_-\times S^1)} \to H_3(S^2\times S^1) \xrightarrow{\cong} H_2(S^1\times S^1) \to \underset{\underset{0}{\parallel}}{H_2(D_+\times S^1)} \oplus \underset{\underset{0}{\parallel}}{H_2(D_-\times S^1)}$$

and $H_2(S^1\times S^1) \cong \mathbb{F} \Rightarrow H_3(S^2\times S^1) \cong \mathbb{F}$

Now write $S^1 = D_+\cup D_-$

$$\underset{\underset{0}{\parallel}}{H_1(S^2\times D_+)\oplus H_1(S^2\times D_-)} \to H_1(S^2\times S^1) \to \underset{\underset{\underset{\mathbb{F}\oplus\mathbb{F}}{\parallel}}{H_0(S^2)\oplus H_0(S^2)}}{\underset{\parallel}{H_0(S^2\times S^0)}} \to \overset{\overset{\mathbb{F}\oplus\mathbb{F}}{\parallel}}{H_0(S^2\times D_+)\oplus H_0(S^2\times D_-)}$$
$$\to \underset{\underset{\mathbb{F}}{\parallel}}{H_0(S^2\times S^1)} \to 0$$

$\Rightarrow 1+2 = 2 + \dim H_1(S^2\times S^1)$ by Whitehead's Trick

$\Rightarrow \dim H_1(S^2\times S^1) = 1$, $H_1(S^2\times S^1) \cong \mathbb{F}$

$H_0(S^2\times S^1) = \mathbb{F}$ connected

$H_1(S^2\times S^1) = \mathbb{F}$

$H_2(S^2\times S^1) = ?$ write $\dim H_2 = h_2$

$H_3(S^2\times S^1) = \mathbb{F}$

$\chi(S^2\times S^1) = \chi(S^2)\chi(S^1) = 0$

$\Rightarrow h_0 - h_1 + h_2 - h_3 = 0 \Rightarrow 1-1+h_2-1 = 0 \Rightarrow h_2 = 1$

$\Rightarrow H_2 = \mathbb{F}$

$S^3\times S^1$

$S^3 = D_+\cup D_-$

$$\underset{\underset{0}{\parallel}}{H_4(D_+\times S^1)\oplus H_4(D_-\times S^1)} \to H_4(S^3\times S^1) \xrightarrow{\cong} \underset{\underset{\mathbb{F}}{\parallel}}{H_3(S^2\times S^1)} \to \underset{\underset{0}{\parallel}}{H_3(D_+\times S^1)\oplus H_3(D_-\times S^1)}$$

$H_4(S^3\times S^1) \cong \mathbb{F}$

$$\underset{\underset{0}{\parallel}}{H_3(D_+\times S^1)\oplus H_3(D_-\times S^1)} \to H_3(S^3\times S^1) \xrightarrow{\cong} H_2(S^2\times S^1) \to \underset{\underset{0}{\parallel}}{H_2(D_+\times S^1)\oplus H_2(D_-\times S^1)}$$

$\Rightarrow H_3(S^3\times S^1) \cong H_2(S^2\times S^1) \cong \mathbb{F}$ $\qquad$ Now decompose $S^1 = D_+\cup D_-$