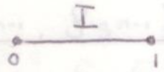
3203 Algebraic Toplogy Notes

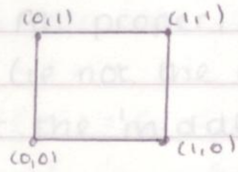
Based on the 2012 spring lectures by Prof F E A Johnson

ALGEBRAIC TOPOLOGY

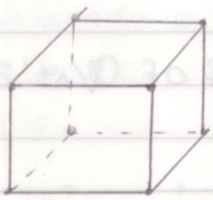
Naive view

Dim 1 

2 points

Dim 2 

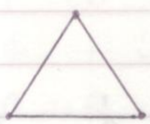
$I \times I$ 4 points.

Dim 3  $I \times I \times I$ 8 points

Dim n $\underbrace{I \times I \times \ldots \times I}_{n \text{ times}}$ 2^n points

Efficent method

 dim = 1 2 vertices 1-simplex

 dim = 2 3 vertices 2-simplex

 dim = 3 4 vertices 3-simplex

dim = n n+1 vertices

Definition: Simplicial complexes

A simplicial complex is a pair $K = (V_K, S_K)$, where

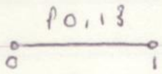
i) V_K is a set (vertex set)

ii) S_K is a set of finite subsets of V_K with the following property:

If $\sigma \in S_K$ and $\tau \subset \sigma$, $\tau \neq \emptyset$ then $\tau \in S_K$

Ex. of simplex

Example I : The 1-simplex

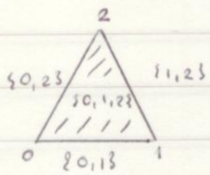


I
$\shortparallel$
Δ

$V_I = \{0,1\}$

$S_I = \{\{0\}, \{1\}, \{0,1\}\}$ ie all non-empty subsets of V_I

Example 2 : The 2-simplex Δ^2



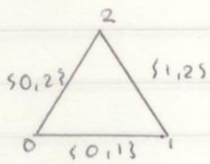
$\Delta^2 = (V_{\Delta^2}, S_{\Delta^2})$ where

$V_{\Delta^2} = \{0,1,2\}$

S_{Δ^2} = all non-empty subsets of V_{Δ^2}

Example 3: Simplicial circle S^1

$S^1(3)$



$V_{S^1} = \{0,1,2\}$

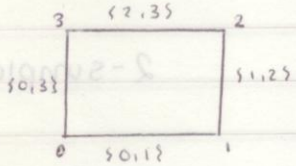
S_{S^1} = All proper, non-empty subsets of $\{0,1,2\}$

ie not $\{0,1,2\}$.

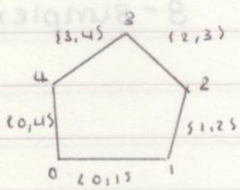
left out the middle

More complicated models for S^1

$S^1(4)$



$S^1(5)$



Example 4 : Δ^n the standard n-simplex

$\Delta^n = (V_{\Delta^n}, S_{\Delta^n})$

$V_{\Delta^n} = \{0,1,\ldots n\}$

S_{Δ^n} = All non-empty subsets of $\{0,1,\ldots n\}$

Example 5: The standard ~~spl~~ smplicial model (n-1) sphere S^{n-1}

$S^{n-1} = (V_{S^{n-1}}, \mathcal{S}_{S^{n-1}})$

$V_{S^{n-1}} = \{0, 1, \ldots, n\}$

$\mathcal{S}_{S^{n-1}}$ = All proper, non-empty subsets of $\{0, 1, \ldots, n\}$

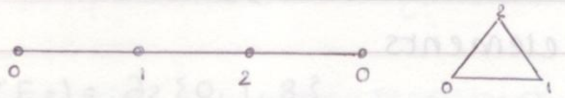
(ie not the whole of $\{0, 1, \ldots, n\}$)

Left out the 'middle' of Δ^n

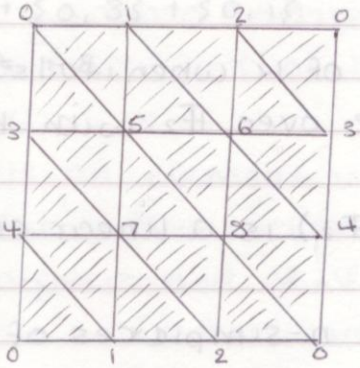
Example 6: The 2-torus T^2

Think $S^1 = \bigcirc$ $S^1 \times S^1 =$  surface of ring doughnut

S^1



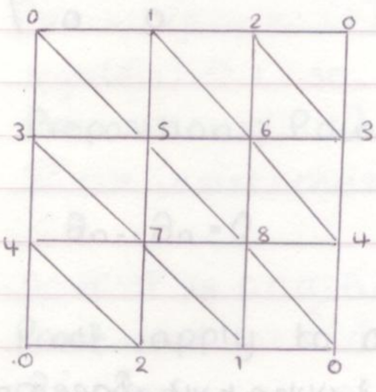
$T = S^1 \times S^1$



has 9 vertices $0, \ldots, 8$

27 1-simplexes

182 simplexes



K^2 Klien bottle.

{Simplicial Complexes} $\xrightarrow{\text{Algebraic Picture}}$ {Vector Spaces} Homology Groups

"

Geometry

{Chain Complexes} Intermidate step

Fix a field $\mathbb{F}$.

By a chain complex over $\mathbb{F}$ I mean a sequence $(C_r, \partial_r)_{0 \le r}$ where each C_r is a vector space over $\mathbb{F}$

$\partial_r : C_r \to C_{r-1}$ is a linear map such that $\partial_r \partial_{r+1} = 0$

$C_{-1} = 0$ by definition

$$\to C_3 \xrightarrow{\partial_3} C_2 \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} C_0 \longrightarrow \overset{C_{-1}}{\overset{\shortparallel}{0}}$$

$\partial_2 \partial_3 = 0 \quad \partial_1 \partial_2 = 0 \quad$ etc

We'll show how to associate to a simplicial K a chain complex $C_*(K) = (C_r(K), \partial_r)$

Simple case $\mathbb{F} = \mathbb{F}_2$ field with two elements

$K = (V_K, S_K)$

Say that $\sigma \in S_K$ is an n-simplex of K when $|\sigma| = n+1$

$C_n(K : \mathbb{F}_2)$ is the vector space over $\mathbb{F}_2$ with basis consisting of the n-simplexes of K

ie a typical element of $C_n(K : \mathbb{F}_2)$ is a linear combination

$\lambda_1 \sigma_1 + \lambda_2 \sigma_2 + \dots + \lambda_m \sigma_m$

where $\lambda_i \in \mathbb{F}_2 \quad \sigma_1, \dots, \sigma_m$ are n-simplexes of K

eg. $K = T^2$, in the model gives

$\dim C_0(T^2 : \mathbb{F}_2) = 9$

$\dim C_1(T^2 : \mathbb{F}_2) = 27$

$\dim C_2(T^2 : \mathbb{F}^3) = 18$

$\dim C_n(T^2 : \mathbb{F}^3) = 0 \quad n \geq 3$

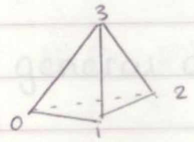
Definition of $\partial_n : C_n(K : \mathbb{F}_2) \to C_{n-1}(K : \mathbb{F}_2)$

To define a linear map only need to ~~define~~ say what it does on a basis.

A typical n-simplexes looks like $\sigma = \{v_0, v_1 \dots v_n\}$

$$\partial_n(\sigma) = \sum_{r=0}^{n} \{v_0, v_1, \dots, \hat{v}_r, \dots, v_n\}$$

Example: $K = S^2$ ($= \partial\Delta^3$ with middle left out)



$\dim C_2(S^2) = 4$

Basis elements $E_1 = \{0,1,2\}$, $E_2 = \{0,1,3\}$, $E_3 = \{0,2,3\}$

$E_4 = \{1,2,3\}$

$\dim C_1(S^2) = 6$ Basis elements $\varepsilon_1 = \{0,1\}$, $\varepsilon_2 = \{0,2\}$ $\varepsilon_3 = \{0,3\}$

$\varepsilon_4 = \{1,2\}$ $\varepsilon_5 = \{1,3\}$ $\varepsilon_6 = \{2,3\}$.

$$\partial_2(E_1) = \partial_2\{0,1,2\}$$
$$= \{1,2\} + \{0,2\} + \{0,1\}$$
$$= \varepsilon_4 + \varepsilon_2 + \varepsilon_1$$

$$\partial_2(E_2) = \partial_2\{0,1,3\}$$
$$= \{1,3\} + \{0,3\} + \{0,1\}$$
$$= \varepsilon_5 + \varepsilon_3 + \varepsilon_1$$

$$\begin{pmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 1 \\ 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}$$

complete as excercise. 6x4 matrix

Proposition: Poincaré

$$\partial_{n-1}\partial_n = 0$$

Proof: apply to an n-simplex $\{v_0, \dots, v_n\}$

$$\partial_{n-1}\partial_n\{v_0, \dots, v_n\} = \partial_{n-1}\left(\sum_{r=0}^{n} \{v_0, \dots, \hat{v}_r, \dots, v_n\}\right)$$

$$= \sum_{r=0}^{n} \partial_{n-1}\{v_0, \dots, \hat{v}_r, \dots, v_n\}.$$

$$\partial_{n-1}\{v_0, \dots, \hat{v}_r, \dots, v_n\} = \sum_{s=0}^{r-1}\{v_0, \dots, \hat{v}_s, \hat{v}_r, \dots, v_n\} + \sum_{s=r+1}^{n}\{v_0, \dots, \hat{v}_r, \hat{v}_s, \dots, v_n\}$$

$$\sum_{s=0}^{r-1}\{v_0, \dots, \hat{v}_s, \hat{v}_r, \dots, v_n\} = \sum_{s<r}\{v_0, \dots, \hat{v}_s, \hat{v}_r, \dots v_n\}.$$

$\sum_{s=r+1}^{n} \{v_0,\dots,\hat{v}_r,\hat{v}_s,\dots,v_n\} = \sum_{r<s} \{v_0,\dots,\hat{v}_r,\hat{v}_s,\dots,v_n\}$

change indices $k=r$ $l=s$

$\partial_{n-1}\partial_n(\sigma) = \sum_{k<l} \{v_0,\dots,\hat{v}_k,\hat{v}_l,\dots,v_n\} + \sum_{k<l} \{v_0,\dots \hat{v}_k,\hat{v}_l,\dots v_n\}$

So $\partial_{n-1}\partial_n(\sigma) = 2\left(\sum_{k<l}(v_0,\dots,\hat{v}_k,\hat{v}_l,\dots,v_n)\right)$

$2=0$ in $\mathbb{F}_2$

So $\partial_{n-1}\partial_n(\sigma)=0$

Poincaré's Boundary Formula in $\mathbb{F}_2$.

For a general field $\mathbb{F}$ we need to modify the definition slightly

New notation: Suppose $\{v_0,\dots,v_n\}$ is a simplex of K

Fix arbitrarily (but also fix) a specific ordering $v_0 \leq v_1 \leq \dots \leq v_n$

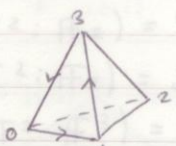
New Definition:

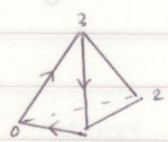
$C_n(K:\mathbb{F})$ is a vector space whose basis elements are symbols $[v_0,v_1,\dots,v_n]$ where $\{v_0,\dots,v_n\}$ is a simplex of K and $[v_0,\dots,v_n]$ is a subject to the rules

I) $[v_{\sigma(0)}, v_{\sigma(1)},\dots v_{\sigma(n)}] = \operatorname{sgn}(\sigma)[v_0,\dots,v_n]$

II) $[v_0,\dots,v_i,\dots,v_j,\dots,v_n]=0$ if $v_i=v_j$ $i\neq j$

eg S^2





$[0,1,3] = -[0,3,1]$

ie $[0,3,1] = \operatorname{sgn}(\tau)[0,1,3]$

where $\tau = \begin{pmatrix} 0 & 1 & 3 \\ 0 & 3 & 1 \end{pmatrix} = (1,3)$

$\operatorname{sgn}(\tau) = -1$

$[0,3,1] = -[0,1,3]$

$[0,1,3] = [1,3,0] = [3,0,1]$ even permutations

$[0,3,1] = [3,1,0] = [1,0,3]$ odd permutations and $[0,3,1] = -[0,1,3]$

In general case $\partial_n [v_0, \dots, v_n] = \sum_{r=0}^{n} (-1)^r [v_0, \dots \hat{v}_r, \dots, v_n]$

We'll show $\partial_{n-1}\partial_n = 0$.

11 January

$K = (V_K, S_K)$ simplicial complex
Elements of S_K are the simplices of K
$\sigma \in S_K$ is an n-simplex when $|\sigma| = n+1$
(so points are 0-simplices, $v_0 \longrightarrow v_1$ is a 1-simplex)

Def simplex simplice

Convention: "oriented simplices"

For each simplex $\{v_0, \dots, v_n\}$ of K choose (arbitrarily) some ordering $v_0 \le v_1 \le \dots \le v_n$
$[v_0, v_1, \dots, v_n]$ is an element in a vector space called $C_n(K : \mathbb{F})$
and we agree that $[v_{\sigma(0)}, \dots, v_{\sigma(n)}] = \text{sign}(\sigma)[v_0, \dots v_n]$
where σ is a permutation of $\{0, \dots, n\}$

If somehow we have repeated a vertex, $v_i = v_j$ $i \neq j$
$[v_0, \dots, v_j, \dots, v_i, \dots] = \text{sign}(i,j)[v_0, \dots v_i, \dots, v_j, \dots]$
$\text{sign}(i,j) = -1$ so,
$[v_0, \dots v_i, \dots, v_j, \dots] = -[v_0, \dots v_j, \dots v_i, \dots]$ if $v_i = v_j$
So we insist that $[v_0, \dots v_i \dots v_j, \dots] = 0$ if $v_i = v_j$, $i \neq j$

So if $\mathbb{F}$ is any field define $C_n(K, \mathbb{F})$ as the vector space whose basis elements are symbols $[v_0, \dots v_n]$ where $\{v_0 \dots v_n\}$ is an n-simplex of K

Now define $\partial_n : C_n(K : \mathbb{F}) \to C_{n-1}(K : \mathbb{F})$
("boundary map" as follows: enough to specify ∂_n on a basis)

$$\partial_n [v_0, \dots, v_n] = \sum_{i=0}^{n} (-1)^i [v_0, \dots, \hat{v}_i, \dots v_n]$$

Note: If $\mathbb{F} = \mathbb{F}_2$ this coincides with previous definition.

Proposition: Poincaré's Lemma

$$\partial_{n-1} \partial_n = 0$$

Proof: Enough to check this on a basis

$$\partial_{n-1}\partial_n [v_0, \dots, v_n] = \partial_{n-1}\left(\sum_{i=0}^{n} (-1)^i [v_0, \dots, \hat{v}_i, \dots, v_n] \right)$$

$$= \sum_{i\geq 0}^{n} (-1)^i \partial_{n-1} [v_0, \dots, \hat{v}_i, \dots, v_n]$$

$[v_0, \dots, v_{i-1}, \hat{v}_i, v_{i+1}, \dots, v_n] = [v_0, \dots, v_{i-1} ; v_{i+1}, \dots, v_n]$

$$\partial_{n-1} [v_0, \dots, v_{i-1} ; v_{i+1}, \dots, v_n] = \sum_{j=0}^{i-1} (-1)^j [v_0, \dots, \hat{v}_j, \dots, v_{i-1} ; v_{i+1}, \dots, v_n] + \sum_{j=i+1}^{n} (-1)^{j-1} [v_0, \dots, v_{i-1} ; v_{i+1}, \dots, \hat{v}_j, \dots, v_n]$$

$$\partial_{n-1} [v_0, \dots, \hat{v}_i, \dots, v_n] = \sum_{j<i} (-1)^j [v_0, \dots, \hat{v}_j, \dots, \hat{v}_i, \dots, v_n] + \sum_{i<j} (-1)^{j-1} [v_0, \dots, \hat{v}_i, \dots, \hat{v}_j, \dots, v_n]$$

$$\partial_{n-1}\partial_n [v_0, \dots, v_n] = \sum_{j<i} (-1)^{i+j} [v_0, \dots, \hat{v}_j, \dots, \hat{v}_i, \dots, v_n] + \sum_{i<j} (-1)^{i+j-1} [v_0, \dots, \hat{v}_i, \dots, \hat{v}_j, \dots, v_n]$$

$k = j \quad l = i$

$k = i \quad l = j$

$$\partial_{n-1}\partial_n [v_0, \dots, v_n] = \sum_{k<l} (-1)^{k+l} [v_0, \dots, \hat{v}_k, \dots, \hat{v}_l, \dots, v_n] + \sum_{k<l} (-1)^{k+l-1} [v_0, \dots, \hat{v}_k, \dots, \hat{v}_l, \dots, v_n]$$

$$= 0.$$

QED.

So given a field $\mathbb{F}$, simplicial complex K, we have produced "oriented chain complex"

$$C_*(K : \mathbb{F}) = (\longrightarrow C_{n+1}(K : \mathbb{F}) \xrightarrow{\partial_{n+1}} C_n(K, \mathbb{F}) \xrightarrow{\partial_n} \dots C_1(K : \mathbb{F}) \xrightarrow{\partial_1} C_0(K : \mathbb{F}) \longrightarrow 0$$

$\partial_{n-1}\partial_n = 0$.

Note: $C_{n+1} \xrightarrow{\partial_{n+1}} C_n \xrightarrow{\partial_n} C_{n-1} \quad \partial_n \partial_{n+1} = 0$

Consequence: $\mathrm{Im}(\partial_{n+1}) \subset \mathrm{Ker}(\partial_n)$

If $z = \partial_{n+1}(\omega)$ then $\partial_n(z) = \partial_n \partial_{n+1}(\omega) = 0$ so $z \in \ker \partial_n$.

Fundemental Definition: Homology

$$H_n(K:\mathbb{F}) = \mathrm{Ker}(\partial_n) / \mathrm{Im}(\partial_{n+1})$$ Noether c 1910.

$\mathrm{Ker}(\partial_n : C_n \to C_{n-1})$

$\mathrm{Im}(\partial_{n+1} : C_{n+1} \to C_n)$

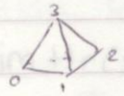
$\dim H_n = \dim \ker(\partial_n) - \dim \mathrm{Im}(\partial_{n+1})$

Pre-Noether

If $\mathbb{F} = \mathbb{Q}$

$\dim\ker(\partial_n) - \dim \mathrm{Im}(\partial_{n+1})$ is called the n^{th} Betti number of K.

Example: $H_*(S^2 : \mathbb{F})$



middle missing

$C_3 = 0$

C_2 has basis $E_1 = [0,1,2]$ $E_2 = [0,1,3]$ $E_3 = [0,2,3]$ $E_4 = [1,2,3]$

$\dim C_2 = 4$

C_1 has basis $\varepsilon_1 = [0,1]$ $\varepsilon_2 = [0,2]$ $\varepsilon_3 = [0,3]$ $\varepsilon_4 = [1,2]$ $\varepsilon_5 = [1,3]$

$\varepsilon_6 = [2,3]$ $\dim C_1 = 6$

C_0 has basis $[0]$, $[1]$, $[2]$, $[3]$

$\dim C_0 = 4$

$\partial_2[E_1] = \partial_2[0,1,2] = (-1)^0[1,2] + (-1)^1[0,2] + (-1)^2[0,1]$

$= \varepsilon_4 - \varepsilon_2 + \varepsilon_1$

$\partial_2(E_2) = \partial_2[0,1,3] = [1,3] - [0,3] + [0,1]$

$= \varepsilon_5 - \varepsilon_3 + \varepsilon_1$

$\partial_2(E_3) = \partial_2[0,2,3] = [2,3] - [0,3] + [0,2]$

$= \varepsilon_6 - \varepsilon_3 + \varepsilon_2$

$\partial_2[E_4] = \partial_2[1,2,3] = [2,3] - [1,3] + [1,2]$

$= \varepsilon_6 - \varepsilon_5 + \varepsilon_4$

Matrix of ∂_2 $\begin{pmatrix} 1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & -1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 \end{pmatrix}$

Compute $\ker \partial_2$ $\begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & -1 & -1 & 0 \\ 0 & -1 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$

$\sim \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & -1 & -1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \overset{\sim}{\text{OR}} \begin{array}{c} \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \\ x_1 \quad x_2 \quad x_3 \quad x_4 \end{array}$

eventually $\dim \operatorname{Im} \partial_2 = 3$ $\quad \operatorname{Im} \partial_3 = 0$

$\dim C_2 = 4$

$C_3 \xrightarrow{\partial_3} C_2 \xrightarrow{\partial_2} C_1$

$\overset{\shortparallel}{0}$

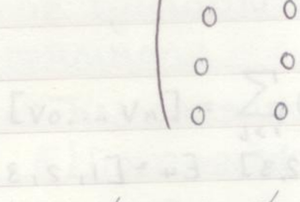
$\ker \partial_2 / \operatorname{Im} \partial_3 \cong \ker \partial_2$ which is 1 dim

Solution vector $\begin{pmatrix} -x_4 \\ x_4 \\ -x_4 \\ x_4 \end{pmatrix}$ Take $x_4 = 1$

$\dim \ker \partial_2 = 1$

Basis for $\ker \partial_2$ is $-E_1 + E_2 - E_3 + E_4$

$\dim \operatorname{Im} \partial_2 = 3 \; (= 4 - 1)$



$-E_1 + E_2 - E_3 + E_4$ Future reference 2-cycle.

$\partial_1(\varepsilon_1) = -[0] + [1]$ $\qquad \partial_1(\varepsilon_5) = -[1] + [3]$

$\partial_1(\varepsilon_2) = -[0] + [2]$ $\qquad \partial_1(\varepsilon_6) = -[2] + [3]$

$\partial_1(\varepsilon_3) = -[0] + [3]$

$\partial_1(\varepsilon_4) = -[1] + [2]$

$$\partial_1 \sim \begin{pmatrix} -1 & -1 & -1 & 0 & 0 & 0 \\ 1 & 0 & 0 & -1 & -1 & 0 \\ 0 & 1 & 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 & 1 & 1 \end{pmatrix} \rightarrow \begin{array}{c} \begin{pmatrix} 1 & 0 & 0 & -1 & -1 & 0 \\ 0 & 1 & 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \\ \begin{matrix} x_1 & x_2 & x_3 & x_4 & x_5 & x_6 \end{matrix} \end{array}$$

$\partial_1 : \mathbb{F}^6 \rightarrow \mathbb{F}^4$

$\dim \ker \partial_1 = 3$

$\dim \mathrm{Im}\, \partial_1 = 6 - 3 = 3$

$$0 \xrightarrow{\partial_3} C_2(S^2) \xrightarrow{\partial_2} C_1(S^2) \xrightarrow{\partial_1} C_0(S^2) \xrightarrow{\partial_0 = 0} 0$$

$$\begin{matrix} & & \wr\wr & & \wr\wr & & \wr\wr & & \\ 0 & \rightarrow & \mathbb{F}^4 & \longrightarrow & \mathbb{F}^6 & \longrightarrow & \mathbb{F}^4 & \longrightarrow & 0 \end{matrix}$$

$H_r = \dfrac{\ker \partial_r}{\mathrm{Im}\, \partial_{r+1}}$ $\quad \dim H_r = \dim \ker \partial_r - \dim \mathrm{Im}\, \partial_{r+1}$

$\dim H_r(S^2 : \mathbb{F}) = 0$ for $r \geq 3$ (no 3 simplices!)

$\dim H_2(S^2 : \mathbb{F}) = \dim \ker \partial_2 - \dim \mathrm{Im}\, \partial_3 = 1 - 0 = 1$ $\quad$ 2nd Betti number

So $H_2(S^2 : \mathbb{F}) \cong \mathbb{F}$

$\dim H_1 = \dim \ker \partial_1 - \dim \mathrm{Im}\, \partial_2$

$= 3 - 3 = 0$

$H_1(S^2 : \mathbb{F}) = 0$

$\dim H_0 = \dim \ker \partial_0 - \dim \mathrm{Im}\, \partial_1$

$= 4 - 3 = 1$

So we've shown:

Theorem:

$$H_r(S^2 : \mathbb{F}) \cong \begin{cases} \mathbb{F} & r = 0 \\ 0 & r = 1 \\ \mathbb{F} & r = 2 \\ 0 & r \geq 3. \end{cases}$$

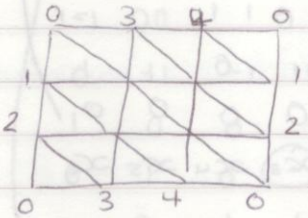
We'll show (coming soon!)

$$H_r(S^n : \mathbb{F}) \cong \begin{cases} \mathbb{F} & r = 0, n \\ 0 & \text{otherwise} \end{cases}$$

We need some technology to do this easily.

Bare hand calculations can get quite big quite quickly.

Consider T^2



$\dim C_2 = 18$
$\dim C_1 = 27$
$\dim C_0 = 9$

$$0 \longrightarrow \mathbb{F}^{18} \xrightarrow{27\times 18} \mathbb{F}^{27} \xrightarrow{9\times 27} \mathbb{F}^{9} \longrightarrow 0$$

Quotient constructions for vector spaces

V vector space
$W \subset V$ vector subspace
$x \in V$ $\quad x + W = \{x + w \mid w \in W\}$
$V/W = \{x + W \mid x \in V\}$

Rule of equality: $x + W = x' + W \iff x - x' \in W$

Recall $H \subset G$ $\quad G/H = \{gH \dots\}$ $\quad g_1 H = g_2 H \iff g_2^{-1} g_1 \in H$.

Let $C_* = (\xrightarrow{\partial_{n+1}} C_n \xrightarrow{\partial_n} C_{n-1} \xrightarrow{\partial_{n-1}} \dots \longrightarrow C_1 \xrightarrow{\partial_1} C_0 \xrightarrow{\partial_0 = 0} 0)$

$\partial_n \partial_{n+1} = 0$ and we define

$$\boxed{H_n(C) = \ker \partial_n / \operatorname{Im} \partial_{n+1}}$$ Homology of chain complexes.

$$\{\text{Simplicial Complexes}\} \longrightarrow \{\text{Vector spaces}\}$$
$$\{\text{Simplicial Complexes}\} \xrightarrow{C_*} \{\text{Chain Complexes}\} \xrightarrow{H_n \ (n = 0, 1, \dots)} \{\text{Vector spaces}\}$$

Exact seq.

$H_n(K : \mathbb{F}) = H_n(C_*(K : \mathbb{F}))$

Advantage of homology $\dim H_n \ll \dim C_n$.

Exact sequences

Definition: Let $\dots V_{n+2} \xrightarrow{T_{n+2}} V_{n+1} \xrightarrow{T_{n+1}} V_n \xrightarrow{T_n} V_{n-1} \xrightarrow{T_{n-1}} V_{n-2} \dots$ be a sequence of vector spaces and linear maps. Say that the sequence is exact at V_n when $\operatorname{Ker}(T_n) = \operatorname{Im}(T_{n+1})$

Say that the sequence is exact when it is exact at each V_r
ie $\operatorname{Ker} T_r = \operatorname{Im} T_{r+1}$ for all r.
We shall see lots of exact sequences.

Two important special cases.

1. $0 \longrightarrow V \xrightarrow{T} W \longrightarrow 0$ ("very short" exact sequence).

Proposition:

This sequence is exact if and only if T is an isomorphism.

Proof: suppose sequence is exact. So $\ker(T) = \operatorname{Im}(0 \to V) = 0$
So $\ker T = 0$, so T injective
Also $\ker(W \to 0) = \operatorname{Im}(V \xrightarrow{T} W)$
$W = \operatorname{Im}(T)$
so T is also surjective.
Hence T is bijective, hence an isomorphism.
Argument is reversible. If T is an isomorphism, then T:
is surjective so $\operatorname{Im}(T) = W = \ker(W \to 0)$
is injective so $\ker(T) = 0 = \operatorname{Im}(0 \to V)$ QED.

2. $0 \longrightarrow U \xrightarrow{S} V \xrightarrow{T} W \longrightarrow 0$
An exact sequence of this form is called a short exact sequence (SES)

Proposition:

Such a sequence is exact if and only if
i) S is ~~surj~~ injective
ii) T is surjective
iii) $\ker(T) = \operatorname{Im}(S)$

Proof: $\Rightarrow$ Exact at U
$\ker(S) = \operatorname{Im}(0 \longrightarrow U) = 0$
So $\ker(S) = 0$, so S injective
Exact at W, $\underset{\overset{\parallel}{W}}{\ker(W \to 0)} = \operatorname{Im}(T)$

So $Im(T) = W$. T is surjective
$\Leftarrow$ Arguments are reversible Complete

Consider a SES, U, V, W finite dimensional
$$0 \longrightarrow U \xrightarrow{S} V \xrightarrow{T} W \longrightarrow 0$$
Then

Whiteheads Lemma.

Proposition:

$\dim V = \dim(U) + \dim(W)$

Proof: Ker - Rank Thm for T.
$\dim(V) = \dim \ker T + \dim Im T$ — S injective
$= \dim Im S + \dim Im T$ — T surjective.
$= \dim U + \dim W$ QED.

Whiteheads Lemma:

Let $0 \longrightarrow V_n \xrightarrow{T_n} V_{n-1} \xrightarrow{T_{n-1}} \cdots \longrightarrow V_1 \xrightarrow{T_1} V_0 \longrightarrow 0$
be an exact sequence of finite dimensional vector spaces and linear maps.
Then $\sum_{r \geq 0} \dim(V_{2r}) = \sum_{r \geq 0} \dim(V_{2r+1})$

Proof: Let $P(n)$ be the statement that $0 \to V_n \to \cdots \to V_1 \to V_0 \to 0$ is exact then $\sum_{r\geq 0} \dim V_{2r} = \sum_{r \geq 0} \dim V_{2r+1}$

First note that $P(1)$ is true
$0 \to V_1 \xrightarrow{T_1} V_0 \to 0$ exact $\Rightarrow T_1$ is isomorphism $\Rightarrow \dim V_0 = \dim V_1$
P_2 is also true
If $0 \to V_2 \xrightarrow{T_2} V_1 \xrightarrow{T_2} V_0 \to 0$ is exact then $\dim V_0 + \dim V_2 = \dim V_1$
To complete proof we must show that $P(2n) \Rightarrow P(2n+1)$ and $P(2n+1) \Rightarrow P(2n+2)$

$P(2n) \Rightarrow P(2n+1)$
Take the exact sequence
$0 \to V_{2n+1} \to V_{2n} \to V_{2n-1} \to \cdots \to V_1 \to V_0 \to 0$
split here

Define $V_{2n}' = \ker(T_{2n-1}) = Im(T_{2n})$

We now have the exact sequences

$$0 \longrightarrow V'_{2n} \hookrightarrow V_{2n-1} \xrightarrow{T_{2n-1}} \dots \longrightarrow V_1 \xrightarrow{T_1} V_0 \longrightarrow 0$$

and also

$$0 \longrightarrow V_{2n+1} \xrightarrow{T_{2n+1}} V_{2n} \xrightarrow{T_{2n}} V'_{2n} \longrightarrow 0$$

By hypothesis $P(2n)$ we get

$$\dim V'_{2n} + \sum_{r=0}^{n-1} \dim V_{2r} = \sum_{r=0}^{n-1} \dim (V_{2r+1})$$

But $\dim V'_{2n} = \dim V_{2n} - \dim V_{2n+1}$

So now $\displaystyle\sum_{r=0}^{n} \dim V_{2r} - \dim V_{2n+1} = \sum_{r=0}^{n-1} \dim (V_{2r+1})$

So $\displaystyle\sum_{r=0}^{n} \dim (V_{2r}) = \sum_{r=0}^{n} \dim (V_{2r+1})$ QED.

Proof that $P(2n+1) \Rightarrow P(2n+2)$ is identical except for slight change of indexing. Complete!

Main Technique in this course:

Mayer - Vietoris Theorem:

Suppose $X = X_+ \cup X_-$ $X_+ \cap X_- \neq \emptyset$

(Think of glueing X_+ to X_- along the intersection $X_+ \cap X_-$

Then:

$$H_n(X_+ \cap X_-) \longrightarrow H_n(X_+) \oplus H_n(X_-) \longrightarrow H_n(X) \longrightarrow H_{n-1}(X_+ \cap X_-) \longrightarrow$$
$$H_{n-1}(X_+) \oplus H_{n-1}(X_-) \longrightarrow H_{n-1}(X) \longrightarrow H_{n-2}(X_+ \cap X_-) \longrightarrow \dots$$

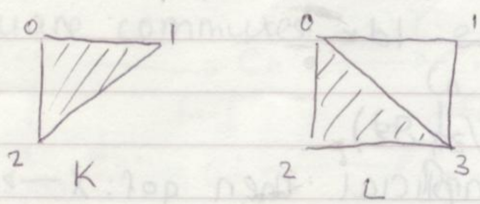
Whitehead's Theorem + MV Thm gives relations between $\dim H_r(X)$, $\dim H_r(X_+)$ $\dim H_r(X_-)$ $\dim H_r(X_+ \cap X_-)$

Definition:

$K = (V_K, S_K)$, $L = (V_L, S_L)$ simplicial complexes

By a simplicial mapping $f: K \longrightarrow L$ we mean a mapping $f: V_K \longrightarrow V_L$ such that $\forall \sigma \in S_K$ $f(\sigma) \in S_L$

ie f takes simplicies to simplicies.

eg.  $V_K = \{0, 1, 2\}$

$V_L = \{0, 1, 2, 3\}$.

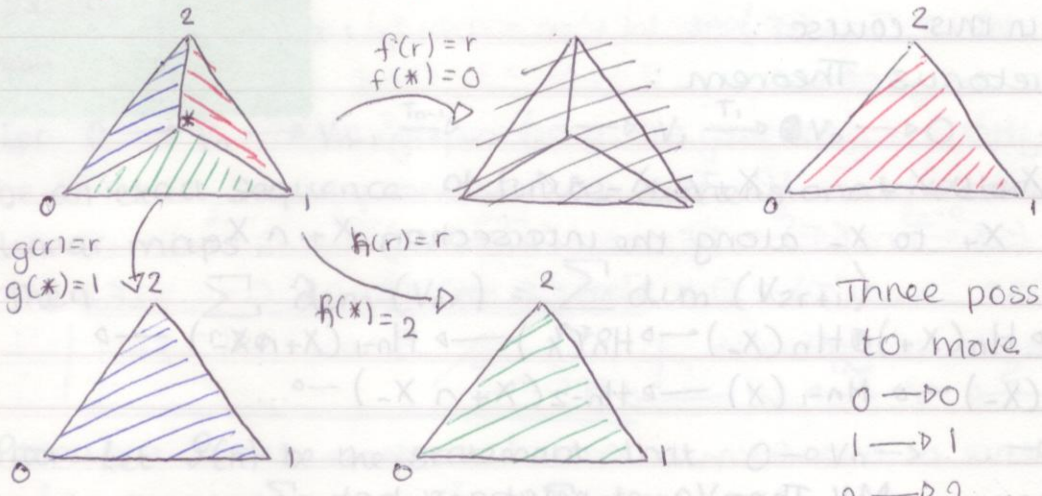
1) $g: V_K \longrightarrow V_L$: $g(0)=0$, $g(1)=1$, $g(2)=2$
This is not a simplicial map.

2) $f: V_K \longrightarrow V_L$ $\left.\begin{array}{l} f(0)=2 \\ f(1)=0 \\ f(2)=3 \end{array}\right\}$ simplicial mapping

3) $h: V_K \longrightarrow V_L$ $\left.\begin{array}{l} h(0)=0 \\ h(1)=1 \\ h(2)=3 \end{array}\right\}$ Not simplicial.

4) If I were to change def^n of L and fill in the blank 2-simplex then h would be simplicial.

Example: "squash map"



Three possible choices for to move $*$
$0 \longrightarrow 0$
$1 \longrightarrow 1$
$2 \longrightarrow 2$

Notice in these squash maps dimensions of simplices can be lowered.
So with f the 1-simplex $[0, *]$ gets squashed to 0.

Obvious properties of simplicial maps:

I) If $K=(V_K, S_K)$ is a simplicial complex then $Id_{V_K}: K \longrightarrow K$ is simplicial (write it normally as Id_K.

II) If $X=(V_X, S_X)$, $Y=(V_Y, S_Y)$, $Z=(V_Z, S_Z)$ and $f: X \longrightarrow Y$, $g: Y \longrightarrow Z$ are simplicial then $g \circ f: X \longrightarrow Z$

is also simplicial.

Simplicial complexes and simplicial maps form a category.

$K \longmapsto H_n(K, \mathbb{F})$

$$\left\{\begin{matrix}\text{Simplicial complexes}\\ \text{\& simplicial maps}\end{matrix}\right\} \xrightarrow{H_n} \left\{\begin{matrix}\text{Vector spaces}\\ \text{and linear maps}\end{matrix}\right\}$$

$$f \longmapsto H_n(f)$$

$$\{f : X \to X\} \longmapsto \{H_n(f) : H_n(X) \to H_n(Y)\}.$$

This is what I will now define.

H_n functor.

$$\begin{tikzcd} \left\{\begin{matrix}\text{Simplicial complexes}\\ \text{and simplicial maps}\end{matrix}\right\} \arrow[rr] \arrow[dr, "C_*"'] & & \left\{\begin{matrix}\text{Vector spaces}\\ \text{and linear maps}\end{matrix}\right\} \\ & \left\{\begin{matrix}\text{Chain complexes}\\ \text{and chain maps}\end{matrix}\right\} \arrow[ur, "H_n"'] & \end{tikzcd}$$

Chain mappings: (ie transformations of chain complexes)

$$\begin{tikzcd} C_* = (\cdots \arrow[r] & C_{n+1} \arrow[r, "\partial_{n+1}"] \arrow[d, "f_{n+1}"] & C_n \arrow[r, "\partial_n"] \arrow[d, "f_n"] & C_{n-1} \arrow[r] \arrow[d, "f_{n-1}"] & \cdots) \quad \partial\partial = 0 \\ D_* = (\cdots \arrow[r] & D_{n+1} \arrow[r, "\delta_{n+1}"] & D_n \arrow[r, "\delta_n"] & D_{n-1} \arrow[r] & \cdots) \quad \delta\delta = 0 \end{tikzcd}$$

Definition:

Let $C_* = (C_n, \partial_n)$, $D_* = (D_n, \delta_n)$ be chain complexes

By a chain mapping $f : C_* \to D_*$

I mean a collection of linear maps

~~for~~ $f = (f_n)$ $f_n : C_n \to D_n$ such that for each n the following square commutes

$$\begin{tikzcd} C_n \arrow[r, "\partial_n"] \arrow[d, "f_n"'] & C_{n-1} \arrow[d, "f_{n-1}"] \\ D_n \arrow[r, "\delta_n"] & D_{n-1} \end{tikzcd}$$

ie $\delta_n \circ f_n = f_{n-1} \circ \partial_n$

Given a simplicial mapping $f: X \to Y$ I need to produce a chain mapping $C_*(f): C_*(X) \to C_*(Y)$

~~Recall~~

Recall that $C_n(X)$ is a vector space whose n-simplices are the "oriented n-simplicies" of X

ie symbols $[v_0, \ldots, v_n]$ where $\{v_0, \ldots, v_n\} \in S_X$

To define $C_n(f): C_n(X) \to C_n(Y)$ it is enough to define it on a basis.

So define

$$C_n(f)[v_0, \ldots, v_n] = [f(v_0), \ldots, f(v_n)]$$

(ie do obvious)

Claim that:

Proposition:

$C_*(f) = (C_n(f))_n$ is a chain mapping.

Proof: $C_*(X) = (C_n(X), \partial_n^X)$

$C_*(Y) = (C_n(Y), \partial_n^Y)$

I need to show following commutes

$$\begin{array}{ccc} C_n(X) & \xrightarrow{\partial_n^X} & C_{n-1}(X) \\ {\scriptstyle C_n(f)}\downarrow & & \downarrow{\scriptstyle C_{n-1}(f)} \\ C_n(Y) & \xrightarrow{\partial_n^Y} & C_{n-1}(Y) \end{array}$$

$$\partial_n^Y C_n(f)[v_0 \ldots v_n] = \partial_n^Y \{[f(v_0) \ldots f(v_n)]\}$$

$$= \sum_{r=0}^{n} (-1)^r [f(v_0) \ldots f(\hat{v_r}), \ldots f(v_n)]$$

$$C_{n-1}(f)\partial_n^X [v_0, \ldots v_n] = C_{n-1}(f)\left(\sum_{r=0}^{n} (-1)^r [v_0, \ldots \hat{v_r}, \ldots v_n]\right)$$

$$= \sum_{r=0}^{n} (-1)^r C_{n-1}(f)[v_0 \ldots \hat{v_r} \ldots v_n]$$

$$= \sum_{r=0}^{n} (-1)^r [f(v_0) \ldots f(\hat{v_r}) \ldots f(v_n)]$$

{Simplicial complexes} $\longrightarrow$ {Chain Complexes} $\longrightarrow$ {Vector spaces}

$X \mapsto C_*(X)$ $\quad f \mapsto C_*(f)$ $\quad C_* \xrightarrow{H_n(C_*)}$ $\quad C_*(f) \mapsto$

$$H_n(C_*) = \frac{\operatorname{Ker}\partial_n}{\operatorname{Im}\partial_{n+1}} = \mathbb{Z}_n(C)\big/ B_n(C)$$

$\mathbb{Z}_n(C) = \operatorname{Ker}(\partial_n)$ $\qquad B_n = \operatorname{Im}\partial_{n+1}$

n-cycles $\qquad$ n-boundaries

$H_n(C_*)$ is quotient space in which the zero element is represented by B_n.

$(z + B_n) + (z' + B_n) = z + z' + B_n$

Given a chain mapping $f = (f_r) : \underbrace{(C_r, \partial_r)}_{C_*} \longrightarrow \underbrace{(C_r', \partial_r')}_{C'_*}$

I need to construct a linear map

$H_n(f) : H_n(C_*) \longrightarrow H_n(C_*')$

Definition:

Define $H_n(f) : H_n(C_*) \longrightarrow H_n(C_*')$ by

$H_n(f)[z + B_n(C)] = f_n(z) + B_n(C')$

Need to check that:

Proposition: $H_n(f)$ is a well defined mapping $H_n(C) \longrightarrow H_n(C')$

Proof: Must show that the form of $H_n(f)$ does not depend on the way we represent cosets.

ie suppose that $z_1 + B_n = z_2 + B_n$ have to show that $f(z_1) + B_n(C') = f_n(z_2) + B_n(C')$ for every $z_1, z_2 \in \mathbb{Z}_n(C)$

$$\begin{tikzcd}
C_{n+1} \arrow[r, "\partial_{n+1}"] \arrow[d, "f_{n+1}"'] & C_n \arrow[r, "\partial_n"] \arrow[d, "f_n"'] & C_{n-1} \arrow[d, "f_{n-1}"] \\
C'_{n+1} \arrow[r, "\partial'_{n+1}"] & C'_n \arrow[r, "\partial'_n"] & C'_{n-1}
\end{tikzcd}$$

commutes.

If $z \in Ker(\partial_n) = Z_n(C)$ then $\partial_n(z) = 0$

so $f_{n-1}\partial_n(z) = 0 = \partial'_n f_n(z)$ so $f_n(z) \in Z_n(C')$

$f_n(Z_n(C)) \subset Z_n(C')$

Similarly if $b \in B_n(C)$.

Write $b = \partial_{n+1}(\beta)$ $\beta \in C_{n+1}$

$f_n(b) = f_n\partial_{n+1}(\beta) = \partial'_{n+1}f_{n+1}(\beta)$

so $f_n(b) \in B_n(C') = Im\,\partial'_{n+1}$

so $f_n(B_n(C)) \subset B_n(C')$.

Suppose $z_1 + B_n(C) = z_2 + B_n(C)$ $z_i \in Z_n(C)$

so $z_1 - z_2 \in B_n(C)$ (Rule of equalities for cosets)

so $f_n(z_1 - z_2) \in B_n(C')$ so $f_n(z_1) - f_n(z_2) \in B_n(C')$

so $f_n(z_1) + B_n(C') = f_n(z_1) + B_n(C')$ as required.

f_n induces mapping $H_n(f) : Z_n(C)/B_n(C) \longrightarrow Z_n(C')/B_n(C')$

Proposition:

$H_n(f) : H_n(C) \longrightarrow H_n(C')$ is linear

Proof: Each $f_r : C_r \longrightarrow C'_r$ is linear.

$$\begin{aligned} H_n(f)(z_1 + z_2 + B_n) &= f(z_1 + z_2) + B_n(C') \\ &= f(z_1) + f(z_2) + B_n(C') \\ &= H_n(f)(z_1) + H_n(f)(z_2) \end{aligned}$$

and likewise with scalar multiplication.

So now we have constructed machine.

$$\begin{tikzcd} \left\{\begin{array}{l}\text{Simplicial complexes}\\ \text{and simplicial maps}\end{array}\right\} \arrow[rr, "H_n"] \arrow[dr, dashed] & & \left\{\begin{array}{l}\text{Vector Spaces}\\ \text{and linear maps}\end{array}\right\} \\ & \left\{\begin{array}{l}\text{Chain complexes}\\ \text{and chain maps}\end{array}\right\} \arrow[ur, dashed] & \end{tikzcd}$$

$\left\{\begin{array}{l} X \longmapsto H_n(X) \\ f \longmapsto H_n(f) \end{array}\right\}$ is an algebraic representation of geometry.

The picture is consistent in the following sense

$X \xrightarrow{f} Y \xrightarrow{g} Z$

simplicial maps

$H_n(g \circ f) = H_n(g) \circ H_n(f)$

$H_n(X) \xrightarrow[H_n(f)]{} H_n(Y) \xrightarrow[H_n(g)]{} H_n(Z)$

and also $X \xrightarrow{Id} X$ $\quad H_n(Id_X) = Id_{H_n(X)}$

$H_n(X) \xrightarrow[H_n(Id)]{} H_n(X)$

Composites $\rightarrow$ composites Identities $\rightarrow$ identities	This sort of thing is called a covariant functor.

Proof: Very easy, follow def^{n}s.

Proof of covariance: Need to show $C_n(g \circ f) = C_n(g) \circ C_n(f)$

$C_n(g \circ f)[v_0 \ldots v_n] = [g(f(v_0)), \ldots, g(f(v_n))]$

$= C_*(g)[f(v_0), \ldots, f(v_n)]$

$= C_*(g)(C_*(f)[v_0, \ldots v_n])$

$C_*(g \circ f) = C_*(g)\, C_*(f)$

Also $C_n(Id) = Id$.

Also if $C_* \xrightarrow{f} C'_* \xrightarrow{g} C''_*$

$H_n(g \circ f)(z + B_n(C)) = g(f(z)) + B_n(C'')$

$= H_n(g)\{f(z) + B_n(C')\}$

$= H_n(g) H_n(f)\{z + B_n(C)\}$

$H_n(g \circ f) = H_n(g) H_n(f)$

$H_n(Id) = Id$ also QED

We know how to compute a homology. Now we learn how to use it.

Interpretation of H_0:

Simplest non-empty simplicial complex is a point

$* = (V_*, S_*) \quad V_* = \{*\} \quad S_* = \{\{*\}\}$

Proposition:

$$H_n(*, \mathbb{F}) = \begin{cases} \mathbb{F} & n = 0 \\ 0 & n \neq 0 \end{cases}$$

Proof: $C_0(*)$ is 1 dimensionally spanned by $\{*\}$
$C_n(*) = 0$ for $n > 0$ (no higher dimension simplicies)

$0 \xrightarrow{\partial_1} C_0(*) \xrightarrow{\partial_0} 0$

$H_0(*) = \operatorname{Ker}\partial_0 / \operatorname{Im}\partial_1 = C_0(*) \cong \mathbb{F}$

$H_n(*) = 0 \quad n > 0.$ QED.

Corollary:

If X is a non-empty simplicial complex then $H_0(X : \mathbb{F}) \neq 0$.

Proof: Choose $* \in (V_X)$ (a vertex)
Let $r : X \longrightarrow *$ be the constant mapping $r(v) = *$
r is obviously simplicial
Let $i : * \longrightarrow X$ be $i(*) = *$ $\qquad r \circ i = \mathrm{Id}_*$
Apply ~~the~~ H_0

$$\begin{tikzcd} & H_0(X) \arrow[dr, "H_0(r)"] & \\ H_0(*) \arrow[ur, "H_0(i)"] \arrow[rr, "\mathrm{Id}"] & & H_0(*) \end{tikzcd}$$

$H_0(r) \circ H_0(i) = \mathrm{Id}_{H_0(*)} = \mathrm{Id}_{\mathbb{F}}$ $\qquad H_0(*) \cong \mathbb{F}$

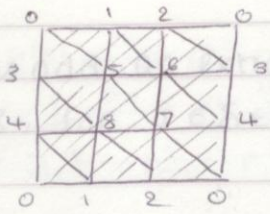
So $H_0(r)$ is surjective
$\quad H_0(i)$ is injective $\Big\}$ So $H_0(X) \neq 0$ QED.

Obvious Question: What is $\dim H_0(X)$?

If X is "connected" $\dim H_0(X) = 1$
In general $\dim H_0(X) =$ number of connected components.

Connectivity:

Let $K = (V_K, \mathcal{S}_K)$ be a simplicial complex
Let $v, w \in V_K$, $v \neq w$. By a path m on K from v to w, I mean a sequence of 1-simplicies $\{v_r, v_{r+1}\}_{0 \leq r \leq m}$ in K such that $v_0 = v$, $v_{m+1} = w$.

eg: $K = T^2$ 

So $\{0,7\}$ is a path $0 \longrightarrow 7$

$(\{0,5\},\{5,7\})$ is a path $0 \to 7$

$(\{0,3\},\{3,4\},\{4,8\},\{8,7\})$ is a path $0 \longrightarrow 7$.

Say that K is connected when given any $v, w \in V_K$, $v \neq w$ there exists a path $v \longrightarrow w$ in K.

Clearly T^2 is connected
Δ^n is connected
S^n is connected $n \geq 1$
$S^0 = \underset{0}{\bullet} \quad \underset{1}{\bullet}$ is not connected.

$H_0(X:\mathbb{F}) = \mathbb{F}$

Proposition:

If X is connected then $\dim H_0(X:\mathbb{F}) = 1$

Proof: $H_0(X:\mathbb{F}) = C_0(X) / \mathrm{Im}\,\partial_1$ $\quad C_1 \xrightarrow{\partial_1} C_0 \xrightarrow{\partial_0 = 0} 0 \quad \ker(\partial_0) = C_0$

The set $\{[w] : w \in V_X\}$ is a basis for $C_0(X)$

Choose elementary vector $v \in V_X$

Then $\{[v]\} \cup \{[w]-[v] : w \in V_X \; w \neq v\}$ is also a basis for $C_0(X)$ (elementary basis change)

But each $[w]-[v] \in \mathrm{Im}\,\partial_1$

Let $(\{v_r, v_{r+1}\})_{0 \leq r \leq m}$ be from $v \longrightarrow w$, $v_0 = v$, $v_{m+1} = w$

$\partial[v_r, v_{r+1}] = v_{r+1} - v_r \in \mathrm{Im}\,\partial_1$

So $[w]-[v] = \sum_{r=0}^{m} [v_{r+1}] - [v_r] \in \mathrm{Im}(\partial_1)$

$\{[v]\} \dot\cup \underbrace{\{[w]-[v]\}}_{\mathrm{Im}(\partial_1)}$

So $\underbrace{C_0/\mathrm{Im}(\partial_1)}_{H_0}$ is at most 1-dimensional generated by $[v]$

But I've shown that $H_0(X) \neq 0$ so $\dim H_0(X) \geq 1$

So $1 \geq \dim H_0(X) \geq 1$ so $H_0(X:\mathbb{F}) \cong \mathbb{F}$ $\quad$ QED.

We've shown that in a connected complex, if v is an arbitrary vertex then $[v]$ generates $H_0(X)$

Later on we'll use the following:

Corollary:

Let X be a connected simplicial complex and $f: X \to X$ a simplicial map, then $H_0(f) = \mathrm{Id};\ H_0(X:\mathbb{F}) \to H_0(X:\mathbb{F})$

Proof: $[f(v)] - [v] \in \mathrm{Im}(\partial_1)$

$f(v) + \mathrm{Im}(\partial_1) = [v] + \mathrm{Im}(\partial_1)$

So $H_0(f) = \mathrm{Id}$, $[v]$ generates H_0 QED.

A general finite complex X is a disjoint union.

$X = X_1 \sqcup X_2 \sqcup \dots \sqcup X_m$

where each X_i is a maximal connected ~~simplex~~ simplical complex

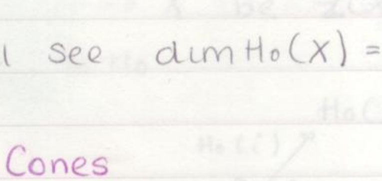
We'll see $\dim H_0(X) = m$ (follows from MVT).

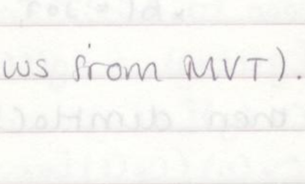
Cones

Let K be a simplicial complex. Let $*$ be a point, $* \notin V_K$

We'll construct a new complex CK called the cone on K ($*$ will be the conepoint).

Example: Take K to be 3 disjoint 1-simplexes.

$K =$ 

CK 

Definition:

If, $K = (V_K, S_K)$ is a simplicial complex.

Define $CK = (V_K \cup \{*\}, \{\{*\}\} \cup S_K \cup \{\sigma \cup \{*\} : \sigma \in S_K\})$ where $* \notin V_K$

in english you add an extra vertex. The extra simplicies are $\{\{*\}\}$ and $\sigma \cup \{*\}$, for $\sigma \in S_K$.

Homology of cone.

Theorem:

Let K be a simplicial complex.
Then $H_r(CK : \mathbb{F}) \cong \begin{cases} \mathbb{F} & r = 0 \\ 0 & r \neq 0. \end{cases}$

Proof: CK is connected (even if K isn't).
If $v, w \in V_K$, I have a path $v \longrightarrow * \longrightarrow w$
$v \in V_K$ so also have path $v \longrightarrow *$.
Put $X = CK$. Define $h_r : C_r(X : \mathbb{F}) \longrightarrow C_{r+1}(X : \mathbb{F})$ (linear).
by $h_r[v_0, \dots, v_r] = [*, v_0, \dots, v_r]$ $*$ is cone point of $X = CK$.
(note if $* = v_i$ some i, $[*, v_0 \dots v_r] = 0$ repeated vertex).

$$\begin{tikzcd}
C_{r+1}(X) \arrow[r, "\partial_{r+1}"] & C_r(X) \arrow[r, "\partial_r"] \arrow[d, "Id"] \arrow[dl, "h_r"'] & C_{r-1}(X) \arrow[dl, "h_{r-1}"] \\
C_{r+1}(X) \arrow[r, "\partial_{r+1}"'] & C_r(X) \arrow[r, "\partial_r"'] & C_{r-1}(X)
\end{tikzcd} \qquad r \geq 1.$$

Claim: $Id = \partial_{r+1} h_r + h_{r-1} \partial_r$

$$\begin{aligned}
\partial_{r+1} h_r [v_0, \dots, v_r] &= \partial_{r+1}[*, v_0, \dots v_r] \\
&= [v_0, \dots v_r] + \sum (-1)^{n+1} [*, v_0, \dots \hat{v}_n, \dots v_m] \\
&= [v_0, \dots, v_r] + \sum (-1)^{n+1} h_{r-1} [v_0, \dots \hat{v}_n, \dots, v_r] \\
&= [v_0, \dots, v_r] + h_{r-1}\left(\sum (-1)^{n+1} [v_0, \dots \hat{v}_n \dots v_r]\right) \\
&= [v_0 \dots v_r] - h_{r-1} \partial_r [v_0 \dots v_r].
\end{aligned}$$

$(\partial_{r+1} h_r + h_{r-1} \partial_r)[v_0 \dots v_r] = [v_0 \dots v_r]$

$Id = \partial_{r+1} h_r + h_{r-1} \partial_r$

Suppose $z \in Z_r(X)$ $\partial_r(z) = 0$, $z = \partial_{r+1} h_r(z) \neq 0$
So $\partial_r(z) = 0 \Rightarrow z \in \mathrm{Im}\, \partial_{r+1}$ $z \in Z_r(X) \Rightarrow z \in B_n(X)$
ie. $H_r(X : \mathbb{F}) = 0$ for $r \geq 1$

$H_r(X : \mathbb{F}) = Z_r / B_r = 0$ $r \geq 1$

X connected, $H_0(X : \mathbb{F}) = \mathbb{F}$

Example: Δ^n is a cone

$\Delta^n = C(\Delta^{n-1})$ where we take $\{*\} = \{n\}$

$V_{\Delta^n} = \{0, \dots, n\}$ $\quad V_{\Delta^{n-1}} = \{0, \dots, n-1\}$

S_{Δ^n} = all nonempty subsets of $\{0, \dots, n\}$ $\quad S_{\Delta^{n-1}}$ = all non empty subsets of $\{0, \dots, n-1\}$.

If $A \subset \{0, \dots, n\}$ $A \neq \emptyset$

Then either i) $n \notin A$ or ii) $n \in A$

If i) $A \subset \{0, \dots, n-1\}$

If ii) either $A = \{n\}$ or $A = A' \cup \{n\}$ where A' is a nonempty subset of $\{0, \dots, n-1\}$

So $C_{\Delta^{n-1}} \cong \Delta^n$ $\quad * = n$.

Corollary:

$$H_r(\Delta^n : \mathbb{F}) = \begin{cases} \mathbb{F} & r = 0 \\ 0 & r > 0 \end{cases}$$

Proof: Δ^n is a cone.

Cones have homology of points.

n-skeleton of a simplicial complex:

$K = (V_K, S_K)$ $\quad n \geq 0$

Define $K^{(n)} = (V_K, \{\sigma \subset V_K, \sigma \neq \emptyset, |\sigma| \leq n+1\})$

$= (V_K$, simplices of dim $\leq n)$

Example $S^n = (\Delta^{n+1})^{(n)}$

Look at definition.

Theorem:

$H_r(K^{(n)} : \mathbb{F}) \cong H_r(K : \mathbb{F})$ for $r < n$

$H_r(K^{(n)} : \mathbb{F}) = H_r(K : \mathbb{F})$

Proof: Look at definition

$C_r(K^{(n)}) \cong C_r(K)$ for $r \leq n$

$0 \longrightarrow C_n(K^{(n)}) \xrightarrow{\partial_n} C_{n-1}(K^{(n)}) \longrightarrow 0$ $\qquad K^{(n)}$ has no n+1 simplicies.

$C_{n+1}(K) \xrightarrow{\partial_{n+1}} \overset{\shortparallel}{C_n}(K) \xrightarrow{\partial_n} \overset{\shortparallel}{C_{n-1}}(K) \longrightarrow 0$ $\qquad C_r(K^{(n)}) = C_r(K) \quad r \leq n$

and $\partial_r = \partial_r^k$ for $r \leq n$.

For $r<n$

$$H_r(K^{(n)};\mathbb{F}) = \frac{\ker \partial_r^{kn}}{\operatorname{Im} \partial_{r+1}^{kn}} = \frac{\ker \partial_r^k}{\ker \partial_{r+1}^k} = H_r(K;\mathbb{F}).$$

For $r=n$ $\quad H_n(K^{(n)}) \longrightarrow\!\!\!\!\rightarrow H_n(K)$ $\qquad (H_n(K^{(n)}) = \ker(\partial n)$

$H_n(K) = \ker(\partial n) / \operatorname{Im} \partial_{n+1}$

$$Z_n(K) \longrightarrow\!\!\!\!\rightarrow Z_n(K)/B_n(K) \quad \text{cononical surjection.}$$

Corollary :

For $n \geq 1$

$$H_r(S^n:\mathbb{F}) = \begin{cases} \mathbb{F} & r=0 \\ 0 & r > n \\ ? & r=n \\ 0 & 1 \leq r < n \end{cases}$$

Proof : $H_r(S^n:\mathbb{F}) = H_r(\Delta^n:\mathbb{F})$ for $r<n$

Still have to determine $H_n(S^n:\mathbb{F})$

To compute $H_n(S^n:\mathbb{F})$ we will use Mayer-Vietons sequence. "Glueing theorem"

Mayer-Vietoris Theorem :

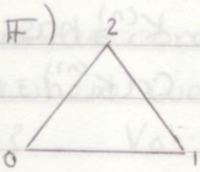
Let X be a finite simplicial complex written as a union $X = X_+ \cup X_-$, where X_+, X_- are subcomplexes of X. Then MV Theorem says (geometric form of MV) $\exists$ long exact sequence.

$$\longrightarrow H_{n+1}(X_+) \oplus H_{n+1}(X_-) \longrightarrow H_{n+1}(X) \xrightarrow{\delta} H_n(X_+ \cap X_-) \longrightarrow H_n(X_+) \oplus H_n(X_-)$$
$$\longrightarrow H_n(X) \longrightarrow H_{n-1}(X_+ \cap X_-) \longrightarrow \dots \longrightarrow H_1(X) \longrightarrow H_0(X_+ \cap X_-)$$
$$\longrightarrow H_0(X_+) \oplus H_0(X_-) \longrightarrow H_0(X) \longrightarrow 0$$

There is a corresponding purely algebraic form which we will need eventually.

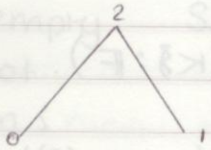
Example: $H_*(S^1 : \mathbb{F})$

Standard model (middle missing)



$$H_r(S^1 : \mathbb{F}) = \begin{cases} \mathbb{F} & r = 0 \\ \mathbb{F} & r = 1 \quad ? \\ 0 & r > 1 \ (\dim = 1) \end{cases}$$

$X = S^1$ $\quad X_+$



(= cone on two disjoint points 0, 1

X_-

$= \Delta^1$ so also a cone

$X_+ \cap X_- = \,:\,$ two disjoint points.

$$H_1(X_+) \oplus H_1(X_-) \longrightarrow H_1(S^1) \to H_0(X_+ \cap X_-) \longrightarrow H_0(X_+) \oplus H_0(X_-) \to H_0(S^1) \to 0$$

$$\begin{array}{ccccccccc} \| & & & & \| & & \| & & \| \\ 0 & \longrightarrow & H_1(S^1) & \to & \underset{\dim \overset{!!}{=} 2}{\mathbb{F} \oplus \mathbb{F}} & \longrightarrow & \underset{\dim = 2}{\mathbb{F} \oplus \mathbb{F}} & \longrightarrow & \mathbb{F} \to 0 \end{array}$$

X_+ X_- cones

(exact sequence

X_+, X_- both connected

S^1 connected

Use Whiteheads lemma

$1 + 2 = 2 + \dim H_1(S^1)$ so $\dim H_1(S^1) = 1$ $\quad H_1(S^1) \cong \mathbb{F}$.

Example: $H_*(S^2)$

So far we know

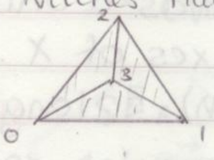
$$H_r(S^2 : \mathbb{F}) = \begin{cases} \mathbb{F} & r = 0 \\ 0 & r = 1 \\ ? & r = 2 \\ 0 & r > 2 \end{cases}$$

$S^2 =$

(middle 3 simplex missing).

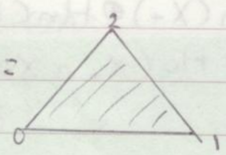
X_+ = "Witches Hat" take out bottom 2 simplex

=



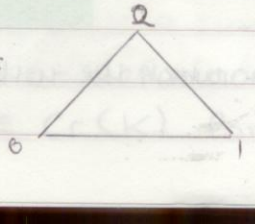
$= C(S^1)$ 3 is cone point.

$X_- =$



= bottom face.

$X_+ \cap X_- =$



$= S^1$ standard model!

$$H_2(X_+)\oplus H_2(X_-) \longrightarrow H_2(S^2) \longrightarrow H_1(X_+\cap X_-) \longrightarrow H_1(X_+)\oplus H_1(X_-) \longrightarrow$$

$$\underset{X_+\ X_-\text{ cones}}{0} \longrightarrow H_2(S^2) \longrightarrow H_1(S^1) \longrightarrow \underset{X_+\ X_-\text{ cones}}{0}$$

$H_2(S^2) \cong H_1(S^1)$ by exactness

$H_2(S^2 : \mathbb{F}) \cong \mathbb{F}$.

$$H_r(S^2:\mathbb{F}) = \begin{cases} \mathbb{F} & r=0 \\ 0 & r=1 \\ \mathbb{F} & r=2 \\ 0 & r>2 \end{cases}$$

We will show by induction that $H_n(S^n : \mathbb{F}) = \mathbb{F}$ for $n \geq 1$

Proved already for $n=1,2$.

So we'll get

$$H_r(S^n:\mathbb{F}) = \begin{cases} \mathbb{F} & r=0 \\ 0 & 1\leq r\leq n-1 \\ \mathbb{F} & r=n \\ 0 & n<r \end{cases}$$

Canonical decomposition of S^n $(n\geq 2)$

$V_{S^n} = \{0,\ldots,n+1\}$, $S_{S^n} = \{\sigma \subset \{0,\ldots,n+1\}, \sigma \neq \emptyset$ and $|\sigma| \leq n+1\}$.

(Beware $|\{0,\ldots,n+1\}| = n+2$).

$V_{\Delta^n} = \{0,\ldots,n\}$, $S_{\Delta^n} = \{$all nonempty subsets of $\{0,\ldots,n\}\}$.

So we've got $\Delta^n \subseteq S^n$ (I'll take $X_- = \Delta^n$)

$S^{n-1} \subset \Delta^n \subseteq S^n$ (I'll take $X_+\cap X_- = S^{n-1}$)

We'll take $X_+ =$ Cone on S^n where we take $n+1$ to be cone point.

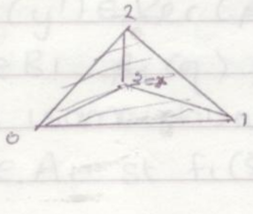
Definition:

$X_+ = (V', S')$ $V' = \{0,\ldots,n+1\}$

$S' = \{\sigma \subset \{0,\ldots,n+1\};\ \sigma \neq \emptyset,\ \sigma \neq \{0,\ldots,n\}\ \sigma \neq \{0,\ldots,n+1\}\}$.

$X_+ \subset S^n$, $X_+ = C(S^{n-1})$ taking $n+1$ as cone point

e.g. $n=2$ X_+



$X_+ = C(S^1)$

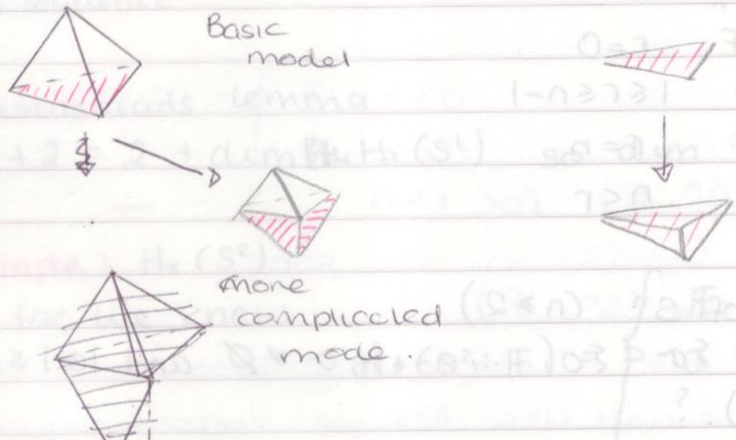
So we get MV sequence $(n \geq 2)$

$$\begin{array}{ccccccc} H_n(X_+) \oplus H_n(X_-) & \longrightarrow & H_n(S^n) & \longrightarrow & H_{n-1}(X_+ \cap X_-) & \longrightarrow & H_{n-1}(X_+) \oplus H_{n-1}(X_-) \\ \| & & \| & & \| & & \| \\ 0 & \longrightarrow & H_n(S^n) & \longrightarrow & H_{n-1}(S^{n-1}) & \longrightarrow & 0 \end{array}$$

By exactness $H_n(S^n) \cong H_{n-1}(S^{n-1})$ so by induction

$$H_r(S^n : \mathbb{F}) = \begin{cases} \mathbb{F} & r = 0 \\ 0 & 0 < r < n \\ \mathbb{F} & r = n \\ 0 & n < r. \end{cases}$$

2-simplex



Need to define what I mean by "subdivision"
Need to show ~~F~~ if X' is a subdivision of X then
$H_*(X' : \mathbb{F}) \cong H_*(X : \mathbb{F})$

Five Lemma:

Let
$$\begin{array}{ccccccccc} A_0 & \xrightarrow{\alpha_0} & A_1 & \xrightarrow{\alpha_1} & A_2 & \xrightarrow{\alpha_2} & A_3 & \xrightarrow{\alpha_3} & A_4 \\ \downarrow f_0 & & \downarrow f_1 & & \downarrow f_2 & & \downarrow f_3 & & \downarrow f_4. \\ B_0 & \xrightarrow{\beta_0} & B_1 & \xrightarrow{\beta_1} & B_2 & \xrightarrow{\beta_2} & B_3 & \xrightarrow{\beta_3} & B_4 \end{array}$$

~~Thm~~ i) Both rows are exact
ii) f_0, f_1, f_3, f_4 are isomorphisms
Then f_2 is an isomorphism.

Five lemma.

Proof: (by "Diagram chasing")

Need to prove f_2 is a) injective ~~surject~~

b) surjective.

a) injectivity

Suppose $x \in A_2$ satisfies $f_2(x) = 0$

Got to show $x = 0$.

$f_2(x) = 0 \Rightarrow \beta_2 f_2(x) = 0$

$\Rightarrow f_3 \alpha_2(x) = 0$

But f_3 is isomorphism so $\alpha_2(x) = 0$

ie $x \in \ker \alpha_2 = \operatorname{Im} \alpha_1$ (exactness)

Choose $y \in A_1$ st $\alpha_1(y) = x$

$f_2 \alpha_1(y) = 0$ $(= f_2(x))$

So $\beta_1 f_1(y) = 0$, so $f_1(y) \in \ker \beta_1 = \operatorname{Im} \beta_0$

Choose $z \in B_0$ st $\beta_0(z) = f_1(y)$.

But f_0 is isomorphism, hence surjective.

Choose $w \in A_0$ st $f_0(w) = z$.

So $\beta_0 f_0(w) = \beta_0(z) = f_1(y)$

$f_1 \alpha_0(w) = f_1(y)$

But f_1 is isomorphism hence injective, hence

$\alpha_0(w) = y$

so $\alpha_1 \alpha_0(w) = = \alpha_1(y) = x$

But $\alpha_1 \alpha_0 \equiv 0$ as top row is exact.

Hence $x = 0$

b) Surjectivity.

Let $x \in B_2$. I have to find $y \in A_2$ st $f_2(y) = x$.

Put $z \equiv \beta_2(x)$, f_3 is surjective so find $w \in A_3$ st $f_3(w) = z = \beta_2(x)$

$\beta_3 f_3(w) = \beta_3 \beta_2(x) = 0$ as $\beta_3 \beta_2 \equiv 0$ by exactness.

So $f_4 \alpha_3(w) = 0$

But f_4 isomorphism hence injective so $\alpha_3(w) = 0$

so $w \in \ker(\alpha_3) = \operatorname{Im}(\alpha_2)$

Choose $y' \in A_2$ such that $\alpha_2(y') = w$.

$\beta_2 f_2(y') = f_3 \alpha_2(y') = f_3(w) = \beta_2(x)$

So $x - f_2(y') \in \ker(\beta_2) = \operatorname{Im}(\beta_1)$

Choose $\eta \in B_1$ $\beta_1(\eta) = x - f_2(y')$

But f_1 is isomorphism hence surjective

Choose $\xi \in A_1$ st $f_1(\xi) = \eta$

So $\beta_1 f_1(\xi) = x - f_2(y')$

$f_2 \alpha_1(\xi) = x - f_2(y')$

So $f_2(\alpha_1(\xi) + y') = x$ Put $y = \alpha_1(\xi) + y'$

$f_2(y) = x$ f_2 surjective. QED

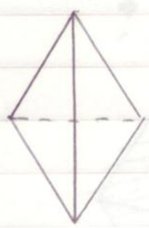
Suppose $X = X_+ \cup X_-$ $X' = X'_+ \cup X'_-$

$$\begin{tikzcd} H_n(X_+ \cap X_-) \arrow[r] \arrow[d] & H_n(X_+) \oplus H_n(X_-) \arrow[r] \arrow[d] & H_n(X) \arrow[r] \arrow[d] & H_{n-1}(X_+ \cap X_-) \arrow[r] \arrow[d] & H_{n-1}(X_+) \oplus H_{n-1}(X_-) \arrow[d] \\ H_n(X'_+ \cap X'_-) \arrow[r] & H_n(X'_+) \oplus H_n(X'_-) \arrow[r] & H_n(X') \arrow[r] & H_{n-1}(X'_+ \cap X'_-) \arrow[r] & H_{n-1}(X'_+) \oplus H_{n-1}(X'_-) \end{tikzcd}$$

Homology is invariant under subdivision.

(a "round" 3-sphere : $S^2 = \{x \in \mathbb{R}^3 : x_1^2 + x_2^2 + x_3^2 = 1\}$)

Tetrahedron



4 vertices

6 edges

4 2-faces

Model III

62 vertices

180 edges

120 2-simplices

Principal/Maximal Simplex:

Let K be a finite simplicial complex. $K = (V_K, S_K)$ we say that a simplex $\sigma \in S_K$ is maximal/principle when given:

$$\tau \in S_K \quad \sigma \subset \tau \Longrightarrow \sigma = \tau$$

ie the biggest simplex

Subdivision of a principle simplex:

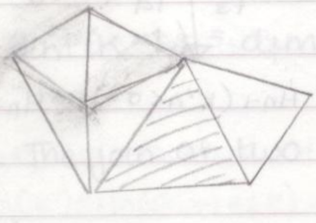
K finite complex, σ principal simplex in K,

$$\partial\sigma = \{\sigma \in S_K : \tau \in \sigma, \tau \neq \sigma\}$$

sub division of σ is the complex obtained by removing σ and replacing it by introducing a new cone point $*$.

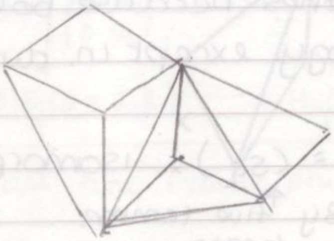
Example:

K

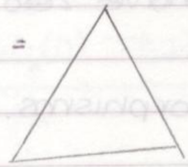


$\langle \sigma \rangle$ = whole of shaded complex.

$sd = (K)$



$\partial \sigma =$



like $X_- \simeq \sigma$

$X = X_+ \cup X_-$

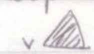
Let σ be a principle simplex of K. Write $K = K' \cup \langle \sigma \rangle$ where $\langle \sigma \rangle$ is the subcomplex defined by σ and K' consists of all the simplicies except of σ.

Definition $sd\sigma$:

$sd\sigma(K) = K' \cup C(\partial\sigma)$

Squash map:

Define a squash map ~~by~~ $sq: sd\sigma(K) \longrightarrow K$ by.

(ie  $*$ $s_2 \longrightarrow v$ )

Choose vertex $v \in \partial\sigma$ $\quad sd\sigma(K) = K' \cup C(\partial\sigma)$

$K = K' \cup \langle \sigma \rangle \quad sq: K' \longrightarrow K'$ is the identity

$sq: C(\partial\sigma) \longrightarrow \langle \sigma \rangle$ obtained by $* \longmapsto v$.

Theorem:

If σ is a principal simplex of K, then

$H_*(sd\sigma(K); \mathbb{F}) \cong H_*(K; \mathbb{F})$

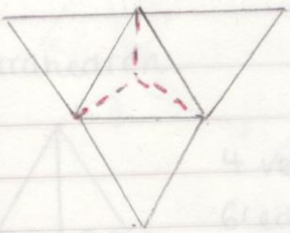
is an isomorphism.

Proof: By MV sequence and five lemma

$$\begin{array}{ccccccccc}
H_n(K'\cap\langle\sigma\rangle) & \to & H_n(K')\oplus H_n(C(\partial\sigma)) & \to & H_n(Sd\sigma(K)) & \to & H_{n-1}(K\cap\langle\sigma\rangle) & \to & H_{n-1}(K')\oplus H_{n-1}(C(\partial\sigma)) \\
f_0 \downarrow \mathrm{Id} & & f_1 \downarrow \begin{pmatrix} \mathrm{Id} & 0 \\ 0 & sq \end{pmatrix} & & f_2 \downarrow sq_* & & f_3 \downarrow \mathrm{Id} & & f_4 \downarrow \begin{pmatrix} \mathrm{Id} & 0 \\ 0 & sq \end{pmatrix} \\
H_n(K'\cap\langle\sigma\rangle) & \to & H_n(K')\oplus H_n(\langle\sigma\rangle) & \to & H_n(K) & \to & H_{n-1}(K'\cap\langle\sigma\rangle) & \to & H_{n-1}(K')\oplus H_{n-1}\langle\sigma\rangle
\end{array}$$

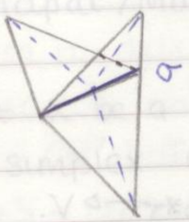
$f_0, f_3 = \mathrm{Id}$ obviously isomorphisms.
Also $sq : H_r(C(\partial\sigma)) \to H_*(\langle\sigma\rangle)$ are cones because both $C(\partial\sigma)$ & $\langle\sigma\rangle$ are cones so have zero homology except in dim 0 where isomorphism.
So f_1, f_4 are isomorphisms. So $f_2 = (sq)_*$ isomorphism
By five lemma

Top row = MV sequence for $sd\sigma(K) = K' \cup C(\partial\sigma)$
Bottom = MV sequence for $K = K' \cup \langle\sigma\rangle$



Subdividing a principle simplex only disturbs that simplex.

Whereas if we subdivide a non principle simplex σ, we have to disturb all principle simplexes which contain σ.



Joins, Links, Stars ★

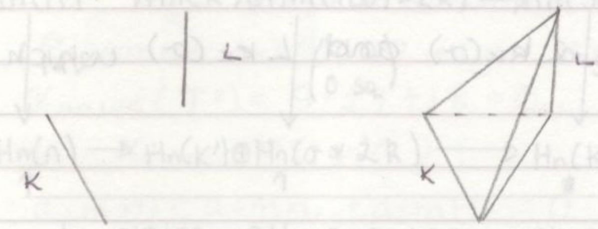
Definition:

Let K, L be simplicial complexes such that $K \cap L = \emptyset$
Then the join $K * L$ is defined formally thus:
$V_{K*L} = V_K \cup V_L \quad (= V_K \sqcup V_L)$
S_{K*L} is the following collection of finite subsets of $V_K \sqcup V_L$
$S_{K*L} = S_K \cup S_L \cup \{\sigma \cup \tau : \sigma \in S_K, \tau \in S_L\}$.
The idea is to join every simplex σ in K to each simplex τ in L

by means of $\sigma \cup \tau$

Exercise: $\dim(K * L) = \dim K + \dim L + 1$.

Example: The join of two disjoint 1-simplicies.



$\Delta^1 * \Delta^1 \cong \Delta^3$

Exercise: $\Delta^n * \Delta^m = \Delta^{n+m-1}$

Special case: $L = \{\text{point}\} = \{pt\}$

Then $K * \{pt\} \cong CK$ cone on K, where $\{pt\}$ = (disjoint) cone point.

~~Tedious but easy to show~~

~~1) $K*(L*M) \cong (K*L)*M$~~

~~2)~~

Tedious but easy to show

1) $K * (L * M) \cong (K * L) * M$

2) $L * K \cong K * L$.

Corollary:

If K, L are simplicial complexes then $(CK) * L \cong C(K * L)$

So join of L with a cone is a cone!

Proof: Write $CK = \{pt\} * K$

Then $CK * L = (\{pt\} * K) * L = \{pt\} * K * L$

$\cong C(K * L)$ Q.E.D.

Subdivision at a non-principle simplex

K finite simplicial complex

σ is a simplex of K and σ is non-maximal.

Definition:

Define $\mathcal{L}k_K(\sigma) = \{\tau \in S_K : \sigma \cap \tau = \emptyset \text{ and } \sigma \cup \tau \in S_K\}$.

I'll use $Lk_K(\sigma)$ to mean the subcomplex of K, whose simplicies are $\mathcal{L}k_K(\sigma)$.
There is a pedantic distinction between $\mathcal{L}k_K(\sigma)$ and $Lk_K(\sigma)$ which we'll ignore.

Proposition:

$\sigma * Lk_K(\sigma)$ is a subcomplex of K

Proof: Tautologous.
Not just that but:

Proposition: The principal simplicies of $\sigma * Lk_K(\sigma)$ are the principle simplicies of K which contain σ.
(So when I subdivide σ these are the only simplicies I must distort).

JFP Hudson Piecewise Linear Topology.

So let σ be a non principal simplex of K. I can decompose K into a union: $K = K' \cup (\sigma * Lk_K(\sigma))$
where K' consists of the principal simplicies that don't contain σ and $\sigma * \mathcal{L}k_K(\sigma)$ consists of principal simplicies which do contain σ.
I'll write $\cap$ for $K' \cap (\sigma * Lk_K(\sigma))$.

Definition:

$Sd_\sigma(K) = K' \cup (C(\partial\sigma) * Lk_K(\sigma))$
Subdivision of K at non principle simplex σ
Replacing σ by $C(\partial\sigma)$.

Choose a squash map $Sq : C(\partial\sigma) \longrightarrow \sigma$ as before.
Now extend this, squash map by identity on every other simplex.
$Sq : Sd_\sigma(K) \longrightarrow K$.

Theorem:

$Sq : Sd\sigma(K) \to K$ induces an isomorphism $Sq_* : H_*(Sd\sigma(K)) \cong H(K)$.

Proof:

$$\begin{tikzcd}
H_n(\cap) \arrow[r] \arrow[d, "id \checkmark"] & H_n(K') \oplus H_n(C(\partial\sigma) * LK) \arrow[r] \arrow[d, "\begin{pmatrix} id & 0 \\ 0 & Sq \end{pmatrix}"] & H_n(Sd\sigma(K)) \arrow[r] \arrow[d, "Sq_*"] & H_{n-1}(\cap) \arrow[r] \arrow[d, "id \checkmark"] & H_{n-1}(K') \oplus H_{n-1}(C(\partial\sigma) * LK) \arrow[d, "\begin{pmatrix} id & 0 \\ 0 & Sq \end{pmatrix}"] \\
H_n(\cap) \arrow[r] & H_n(K') \oplus H_n(\sigma * LK) \arrow[r] & H_n(K) \arrow[r] & H_{n-1}(\cap) \arrow[r] & H_{n-1}(K') \oplus H_{n-1}(\sigma * LK).
\end{tikzcd}$$

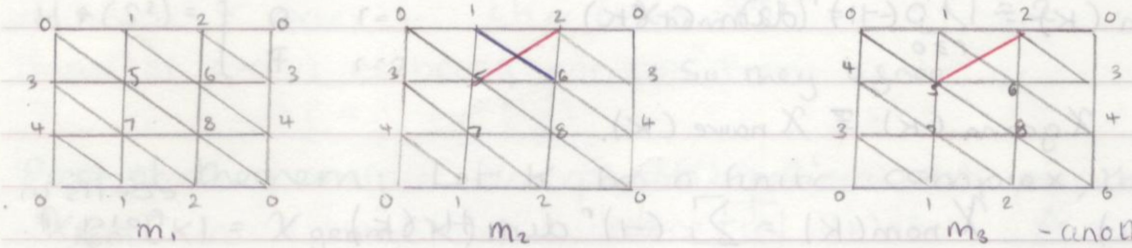
I claim all are isomorphisms.

Obvious id ~~are~~ is isomorphism.
So it ~~remains~~ suffices to show $S : H_*(C(\partial\sigma) * LK) \xrightarrow{\cong} H_*(\sigma * LK)$ is an isomorphism.
However $C(\partial\sigma) * LK$ is a cone and because σ is a cone then $\sigma * LK$ is also a cone.
So $Sq : H_*(C(\partial\sigma) * LK) \to H_*(\sigma * LK)$ is isomorphism
So by 5 Lemma
$Sq_* : H_*(Sd\sigma(K)) \xrightarrow{\cong} H_*(K)$ is an isomorphism QED.

Corollary:

H_* is invarient under subdivision.

One model of T^2



m_1 m_2 m_3 - another model.

m_2 is a subdivision of m_1 and m_2 is a sudivision of m_3.

$m_1 \not\cong m_3$. However $H_*(m_1) \cong H_*(m_2)$.

Combinatorial Equivalence:
Let K, K' be simplicial complexes. Say that K and K' are combinatorially equivalent ($K \sim K'$) when $\exists$ sequence

$(K_r)_{0 \le r \le N}$ of simplicial complexes K_r such that

i) $K_0 = K$

ii) $K_N = K'$

iii) for each r, $1 \le r \le N$ either K_r is a subdivison of K_{r-1} at a simplex σ, ~~on~~ or K_{r-1} is is a subdivison of K_r at a simplex σ.

Corollary:

If $K \simeq K'$ then $H_*(K) \cong H_*(K')$.

Euler Characteristic: "computing H_* in low dimensions"

Euler Chara.

Naive definition: K finite simplicial complex.

Write ~~∀~~ ν_r = no. of r-simplicies in K.

$$\chi_{naive}(K) = \sum_{r \ge 0} (-1)^r \nu_r$$

Rather better way for us:

Definition: (geometric)

K finite complex

$C_*(K)$ = oriented chain complex

$C_r(K)$ = vector space with basis $[v_0, \dots, v_r]$ the r-simplices of K.

$$\chi_{geom}(K) = \sum_{r \ge 0} (-1)^r \dim C_r(K).$$

Obviousbly $\chi_{geom}(K) = \chi_{naive}(K)$.

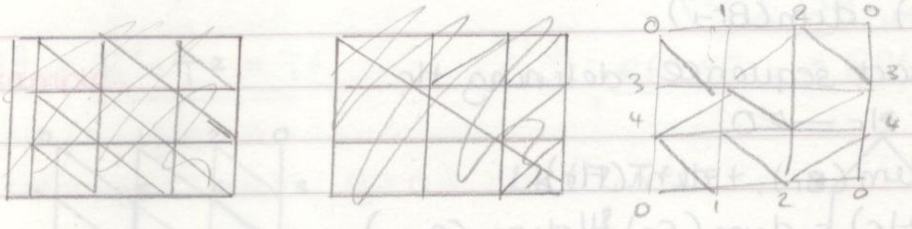
(homological) $\chi_{hom}(K) = \sum_{r \ge 0} (-1)^r \dim H_r(K)$ coeffs in field $\mathbb{F}$.

We will show:

Theorem:

$$\chi_{hom}(K) = \chi_{geom}(K) \left(= \chi_{naive}(K)\right)$$

What does χ tell us?



9 vertices $v_0 = 9$, 27 edges $v_1 = 27$ 18 2-simplicies $v_2 = 18$

$\chi_{naive}(T^2) = 9 - 27 + 18 = 0.$

So $\chi_{hom}(T^2) = 0$

$\dim H_0 - \dim H_1 + \dim H_2 = 0$

But $\dim H_0 = 1$

$\dim H_1 = 1 + \dim H_2$

So we only need to compute $\dim H_2$.

In fact $H_2(T^2 : \mathbb{F}) \cong \mathbb{F}$

So we get $H_1(T^2 : \mathbb{F}) \cong \mathbb{F}^2$

$$H_r(T^2) = \begin{cases} \mathbb{F} & r = 0 \\ \mathbb{F} \oplus \mathbb{F} & r = 1 \\ \mathbb{F} & r = 2 \\ 0 & r > 2 \end{cases}$$

Example: S^2

$v_0 = 4$ $v_2 = 4$

$v_1 = 6$

$\chi_{naive}(S^2) = 4 - 6 + 4 = 2.$

$$H_*(S^2) = \begin{cases} \mathbb{F} & r = 0 \\ 0 & r = 1 \\ \mathbb{F} & r = 2 \end{cases} \qquad \chi_{hom}(S^2) = 1 - 0 + 1 = 2$$

So they agree.

Proof of theorem: Let K be a finite complex, then $\chi_{hom}(K) = \chi_{geom}(K)$.

~~Let K be a finite complex~~

$\partial_r : C_r(K) \to C_{r-1}(K)$

Put $Z_r = \operatorname{Ker}(\partial_r)$, $B_r = \operatorname{Im}(\partial_{r+1})$ $\Big\}$ so $H_r = H_r(K) = Z_r / B_r$.

Get 2 exact sequences

(I) $0 \to Z_r \to C_r \to B_{r-1} \to 0$

$\operatorname{Im}(\partial_r) \cong C_r / \operatorname{Ker}(\partial_r)$

So $\dim(C_r) = \dim(Z_r) + \dim(B_{r-1})$

$\dim(Z_r) = \dim(C_r) - \dim(B_{r-1})$.

Also get canonical exact sequence defining H_r.

$0 \to B_r \to Z_r \to H_r \to 0$

and so $\dim(Z_r) = \dim(B_r) + \dim(H_r)$

So $\dim(B_r) + \dim(H_r) = \dim(C_r) - \dim(B_{r-1})$

So take alternating sums

$$\sum_r (-1)^r \dim(B_r) + \sum_r (-1)^r \dim(H_r) = \sum_r (-1)^r \dim(C_r) + \sum_r (-1)^r \dim(B_{r-1})$$

$$= \sum_r (-1)^r \dim(C_n) + \sum_{r-1} (-1)^{r-1} (B_{r-1})$$

Clearly $\sum_r (-1)^r \dim B_r = \sum_s (-1)^s \dim B_s$

$$= \sum_{r-1=s} (-1)^{r-1} \dim(B_{r-1})$$

Hence $\sum_r (-1)^r \dim H_r = \sum_r (-1)^r \dim C_r$.

So $\chi_{hom}(K) \cong \chi_{geom}(K)$ QED.

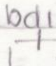
Corollary:

The Euler characteristic $\chi(K)$ of a complex K is invariant under subdivision.

Proof: $\chi(K) = \sum_r (-1)^r \dim H_r(K)$ and H_r is invarient under subdivision QED.

Definition:

$S'(n)$ is the circle with n-subdivision points

eg. $S'(3)$  $S'(4)$  etc.

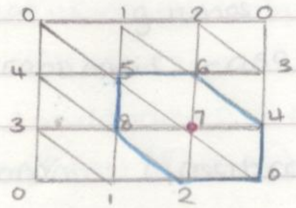
In $S'(n)$ any vertex belongs to exactly two edges.

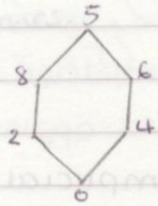
Definition:

A (simplicial) surface, Σ, is a simplicial complex in which for each vertex v $LK_\Sigma(v) \cong S'(n)$ for some $n \geq 3$.

eg. $S'(2)$  is not simplicial (cell complex).

Example. T^2

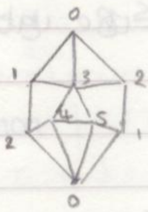


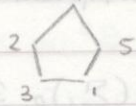
$LK(7, T^2) =$ 

$\cong$

$S'(6)$

Example: $\mathbb{R}P^2$



$LK(0, \mathbb{R}P^2) =$ 

Definition:

The star $St_\Sigma(v) = \{v\} * LK_\Sigma(\sigma)$

= cone on LK with $\{v\}$ the cone point.

If K is a simplicial complex, get "genuine" topological space ($|K|$) by replacing a formal n-simplex by a "geometric" n-simplex.

$|\Delta^n| = \{ \sum_{i=1}^{n} t_i e_i \quad 0 \leq t_i \leq 1 \quad 0 \leq \sum t_i \leq 1 \}$

$e_1 \ldots e_n$ standard basis in $\mathbb{R}^n$

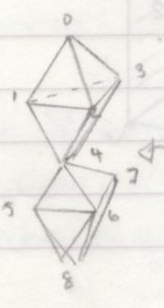
Generalisation: A simplicial n-manifold M is a simplicial complex in which $\forall$ vertex v, $LK_M(v) \sim S^{n-1}$

Think of $LK(v)$ as being "horizon" from v

In a simplicial surface Σ, $St_\Sigma(v) \sim \Delta^2 \sim$ 2-disc

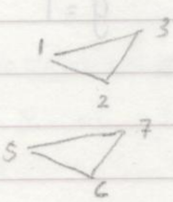
In particular a simplicial surface in 2-dim.

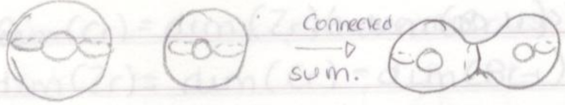
Example: "Simplicial" surface



$LK(v)$ is a circle

except for where $LK =$

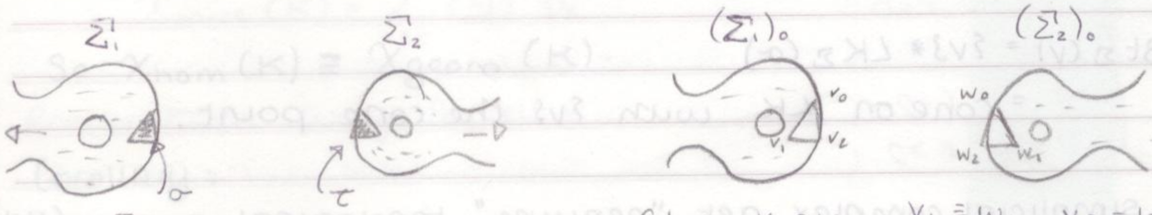




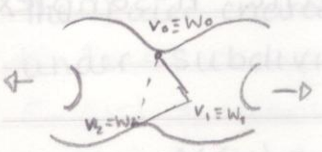
Σ_1 $\quad\Sigma_2$ $\quad\Sigma_1 \# \Sigma_2$

Definition: (Formal).

Let Σ_1, Σ_2 be simplicial surfaces
Let $(\Sigma_1)_0$ denote the complex obtained by removing the interior of 2-simplex σ.
Let $(\Sigma_2)_0$ denot the complex obtained by removing the interior of 2-simplex τ.
$\partial(\Sigma_1)_0$ = boundary of $(\Sigma_1)_0 \cong S'(3)$
$\partial(\Sigma_2)_0$ = boundary of $(\Sigma_2)_0 \cong S'(3)$
$\Sigma_1 \# \Sigma_2 = (\Sigma_1)_0 \cup (\Sigma_2)_0$
$\partial(\Sigma_1)_0 \equiv \partial(\Sigma_2)_0$ glue boundaries together.



Glue $v_0 \equiv w_0$; $v_1 \equiv w_1$ $\quad v_2 \equiv w_2$.



$\Sigma_1 \# \Sigma_2$.

Standard models for simplicial surfaces.

+ List $\quad S^2, \quad T^2, \quad T^2 \# T^2, \ldots \quad \underbrace{T^2 \# T^2 \# \ldots \# T^2}_{g \text{ times}}$

$g=0 \quad g=1 \quad g=2 \qquad\qquad g = \text{genus}$

$\Sigma^0_+ \quad \Sigma^1_+ \quad \Sigma^2_+ \qquad \Sigma^g_+$

− List $\quad \Sigma^0_- \quad \Sigma^1_- \quad \Sigma^2_- \qquad \Sigma^g_-$

$\mathbb{R}P^2 \quad \mathbb{R}P^2 \# \mathbb{R}P^2 \quad \mathbb{R}P^2 \# \mathbb{R}P^2 \# \mathbb{R}P^2 \qquad \underbrace{\mathbb{R}P^2 \# \ldots \# \mathbb{R}P^2}_{g+1}$

$g=0 \quad g=1 \quad g=2$

Definition:

$\Sigma_+^g = \underbrace{T^2 \# \dots \# T^2}_{g \text{ times}}$ $\Sigma_-^g = \underbrace{\mathbb{R}P^2 \# \dots \# \mathbb{R}P^2}_{g+1 \text{ times}}$

(exceptional case $g=0$).

Theorem: Classification Theorem

If Σ is a finite simplicial ~~com~~ surface then $\Sigma \sim \Sigma_s^g$ for exactly $g \geq 0$ and one $s = \pm$

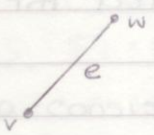
Definition:

A surface Σ is a simplicial complex in which $LK(v, \Sigma) \sim S^1$ for each vertex $v \in \Sigma$

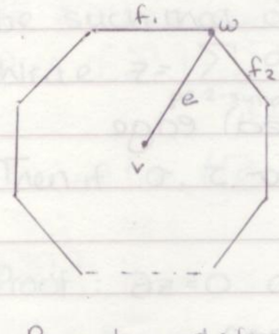
Proposition:

If Σ is a surface and e is a 1-simplex ('edge') in Σ, then e belongs to exactly two 2-simplicies.

Proof:



So $w \in LK(v, \Sigma) \cong S^1(n)$



$LK(v, \Sigma) \cong S^1(n)$

$w \in S^1(n)$ belongs to exactly two 1-simplicies f_1, f_2 as shown.

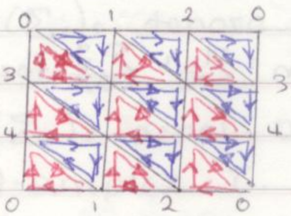
But by definition of link, both f_1, f_2 are joinable in Σ to v. So draw it.

$v * f_1$, $v * f_2$ both 2-simplicies QED.

Orientability:

Definition: (informal).

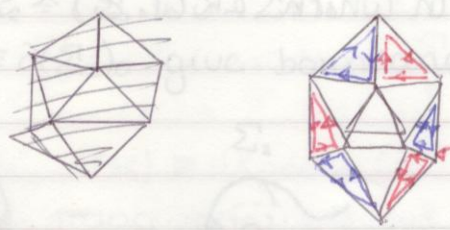
Say that surface Σ is consistently orientated when it is possible to orient the 2-simplicies of Σ in such a way that each 1-simplex e recieves oppisite orientation from the 2-simplices it belongs to.

eg. 

T^2 is orientable

S^2 is also orientable (see previous notes).

$\mathbb{R}P^2$ is non-orientable:

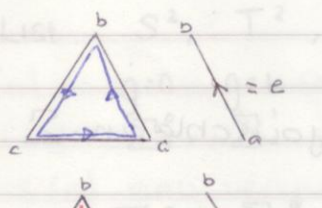

directions here are the same.

Corollary: (of drawing)

If Σ is a surface and Σ contains a punctared $\mathbb{R}P^2$ then Σ is not orientable.

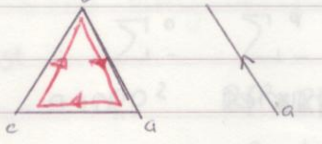
Punctare $\mathbb{R}P^2 = \mathbb{R}P^2_0 = \mathbb{R}P^2 - \{2\text{-simplex}\}$.

More formally suppose Σ is a surface, e is a (directed) edge in Σ and σ is an (oriented) 2-simplex. $e \subset \sigma$.



$[\sigma, e] = +1$

"Intersection numbers."



$[\sigma, e] = -1$

Now suppose that Σ is a (finite) surface and let $z \in C_2(\Sigma : \mathbb{F})$ ($\mathbb{F}$ some field). So $z = \sum_{\sigma \in 2\text{ simplices of } \Sigma} a_\sigma \sigma$ and assume each σ is locally orientated

Now ∂z is a linear in the 1-simplices of Σ (edges).
Let e be some edge.
e lies exactly in two 2-simplicies σ, τ.
What is coefficent of e in expression for ∂z.
Coeff of e in ∂z is $\pm (a_\sigma [\sigma, e] + a_\tau [\tau, e])$.

Definition: (Formal).

A surface Σ is consistently orientated iff it is possible to orient the 2-simplicies in such a way that for each edge e in Σ,
$[\sigma, e] + [\tau, e] = 0$ (σ, τ being the 2-simplicies which contain e)
ie $[\tau, e] = -[\sigma, e]$.

Theorem:

Let Σ be a consistently orientated surface and let $z \in C_2(\Sigma, \mathbb{F})$
If e is an edge and σ, τ are the 2-simplicies which contain e
then, coeff of e in $\partial z = \pm (a_\sigma - a_\tau)$.

Corollary:

Let Σ be a consistently orientated surface and let $z \in C_2(\Sigma, \mathbb{F})$
be such that $\partial z = 0$
Write $z = \sum_{p \in \text{2-simplices of } \Sigma} a_p p$.

Then if σ, τ interact on an edge then $a_\sigma = a_\tau$

Proof: $\partial z = 0$ and coeff of $e = \pm(a_\sigma - a_\tau) = 0$

We'll now generalise this to:

Theorem:

Let Σ be a finite, connect, consistently orientated surface.
and $z \in C_2(\Sigma : \mathbb{F})$ st $\partial z = 0$
Write $z = \sum_{\sigma \in \text{2-simplices}} a_\sigma \sigma$, then $\sigma \mapsto a_\sigma$ is constant

Copathy stuff

Definition:

Let σ, τ be 2-simplices in a surface Σ. By a copath from σ to τ I mean a collection $(\sigma_r)_{0 \le r \le N}$ of 2-simplicies such that

i) $\sigma_0 = \sigma$ ii) $\sigma_N = \tau$

iii) $\sigma_i \cap \sigma_{i+1}$ is an edge for $0 \le i \le N-1$

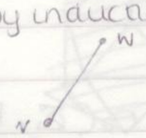
Proposition:

If Σ is a connected surface and σ, τ are 2-simplicies $(\sigma \ne \tau)$ then $\exists$ a copath from σ to τ.

Proof: Can join any vertex in σ to any vertex in τ (Def connected)

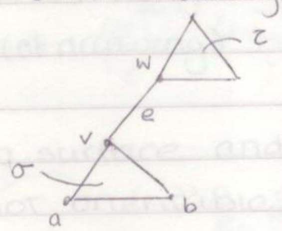
Let m = smallest path length of a vertex in σ to a vertex in τ.

Proof by induction on m.

$m=1$  $v \in \sigma$, $w \in \tau$.

If $\{v, w\} = \sigma \cap \tau$ nothing to prove. $N=1$ $\sigma_0 = \sigma$, $\sigma_1 = \tau$.

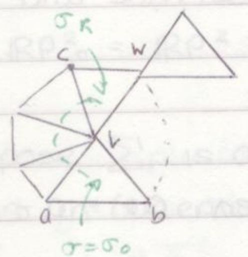
Otherwise



Note that $w \in LK(v, \Sigma) \sim S^1$

$\{a, b\} \subset LK(v, \Sigma)$.

~~So I've got~~



So I've got a copath $\sigma_0 \ldots \sigma_K$ where

$\{c, v\} \subset \sigma_K$

$\{c, v\} \subset LK(w, \Sigma)$.

Now consider $LK(w, \Sigma)$

Choose a copath $\sigma_K, \sigma_{K+1}, \ldots \sigma_N = \tau$ such that $w \in \sigma_{K+r}$

So $\sigma_0, \ldots \sigma_N$ is a copath from σ to τ QED $(m=1)$

Suppose proved ~~from~~ for $m-1$. Let v, w be vertices in σ, τ respectively seperated by path length m.

$v = v_0, v_1, \ldots, v_m = w$. By induction I get copath $\sigma_0, \ldots, \sigma_p$ where

σ_φ contains v_{m-1}.
By case $m=1$ above $\exists$ copath $\sigma_\varphi, \sigma_{\varphi+1} \ldots \sigma_N = \tau$ and so
$\sigma_0 = \sigma_0, \sigma_1, \ldots, \sigma_N = \tau$ is a copath QED.

Corollary:

Let Σ be a finite connected surface which is consistently orientated and let $z = \sum_{\sigma \in 2\text{-simp. of } \Sigma} a_\sigma \sigma \in C_2(\Sigma : \mathbb{F})$.

If $\partial z = 0$ then $\sigma \mapsto a_\sigma$ is constant

Proof: $\sigma \mapsto a_\sigma$ remains constant as we cross an edge.
Hence it remains constant on any copath.
But there is a copath joining any two 2-simplicies
Hence $\sigma \mapsto a_\sigma$ is constant. QED.

Corollary:

Let Σ be a finite, connected, oriented surface then
$H_2(\Sigma : \mathbb{F}) \cong \mathbb{F}$.

Proof: $C_3(\Sigma : \mathbb{F}) = 0$ (2-dim)
So $H_2(\Sigma : \mathbb{F}) = \ker(\partial_2 : C_2(\Sigma : \mathbb{F}) \to C_1(\Sigma : \mathbb{F}))$
If $z \in \ker(\partial_2)$ then we've just shown that $z = a\left(\sum_{\sigma \in 2\text{simplexes of } \Sigma} \sigma\right)$
Put $[\Sigma] = \sum_{\sigma \in 2\text{-simplices}} \sigma$

So we've got $\ker(\partial_2) = \{a[\Sigma] : a \in \mathbb{F}\} \cong \mathbb{F}$ QED.

$[\Sigma]$ is called the fundamental class (unique up to ± 1)
(I've actually shown $H_2(\Sigma : \mathbb{Z}) \cong \mathbb{Z}$ provided Σ connected, oriented)

What about non-orientable surfaces?

If I take $\mathbb{F} = \mathbb{F}_2$ same argument shows that for adjacent simplicies $a_\sigma = \pm a_\tau$
In $\mathbb{F}_2$ $+1 = -1$ so $a_\sigma = a_\tau$
and same proof gives.

Theorem:

If Σ is any finite connected surface then $H_2(\Sigma : \mathbb{F}_2) = \mathbb{F}_2$.

However if $2 \neq 0$ in $\mathbb{F}$, then $H_2(\mathbb{R}P^2 : \mathbb{F}) = 0$
More generally if Σ contains a punctured $\mathbb{R}P^2$
$H_2(\Sigma : \mathbb{F}) = 0$

To summarise

Theorem

Let Σ be a finite connected surface.
i) If Σ is orientable then $H_2(\Sigma : \mathbb{F}) \cong \mathbb{F}$
ii) Regardless of orientability $H_2(\Sigma : \mathbb{F}_2) \cong \mathbb{F}_2$
iii) If Σ contains a punctured $\mathbb{R}P^2$ and $2 \neq 0$ in $\mathbb{F}$
then $H_2(\Sigma : \mathbb{F}) = 0$

Note: iii) remains true for arbitrary non-orientable surfaces but we still need to prove it.

Example:

$$H_r(T^2 : \mathbb{F}) = \begin{cases} \mathbb{F} & r = 0 \\ \mathbb{F} \oplus \mathbb{F} & r = 1 \\ \mathbb{F} & r = 2 \\ 0 & r \geq 3 \end{cases}$$

Proof: $\chi(T^2) = 0$ $(9 - 27 + 18)$ so $\sum_{r=0}^{2} (-1)^r \dim H_r = 0$
$H_0(T^2) \cong \mathbb{F}$ connected
$H_2(T^2) \cong \mathbb{F}$ orientable
$1 - \dim H_1 + 1 = 0$
So $\dim H_1 = 2$, $H_1(T^2 : \mathbb{F}) \cong \mathbb{F} \oplus \mathbb{F}$.

Example: $H_*(\mathbb{R}P^2 : \mathbb{F})$ (usual to consider $\mathbb{F} = \mathbb{Q}$, $\mathbb{F} = \mathbb{F}_2$)

$$H_r(\mathbb{R}P^2 : \mathbb{Q}) = \begin{cases} \mathbb{Q} & r = 0 \\ 0 & r \geq 1 \end{cases} \qquad H_r(\mathbb{R}P^2 : \mathbb{F}_2) = \begin{cases} \mathbb{F}_2 & r = 0 \\ \mathbb{F}_2 & r = 1 \\ \mathbb{F}_2 & r = 2 \\ 0 & r \geq 3 \end{cases}$$

Proof: $\chi(\mathbb{RP}^2) = 1$

6 vertices, 10 2-simplicies, 15 1-simplicies.

$H_2(\mathbb{RP}^2 : \mathbb{Q}) = 0$

So $H_1(\mathbb{RP}^2 : \mathbb{Q}) = 0$ because $1 \mp 0 + 0 = 1$

$H_2(\mathbb{RP}^2 : \mathbb{F}_2) = \mathbb{F}_2$

So $H_1(\mathbb{RP}^2 : \mathbb{F}_2) = \mathbb{F}_2$

$1 - 1 + 1 = 1$

$H_0 \cong \mathbb{F}$.

Σ, Σ' surfaces

$\Sigma_0 = \Sigma - \{2\text{-simplex}\}$

$\partial \Sigma_0 = S'(3) = \triangle$

$\Sigma_0' = \Sigma' - \{2\text{-simplex}\}$.

$\partial \Sigma_0' = S'(3) = \triangle$

$\Sigma \# \Sigma' = \Sigma_0 \cup \Sigma_0'$
$\qquad \partial\Sigma_0 = \partial\Sigma_0'$

"It can be shown that" up to combinatorial equivalence $\Sigma \# \Sigma'$ is independent of the particular 2-simplicies removed.

Exercise: $\Sigma \# \Sigma'$ is a surface

Proposition:

$\chi(\Sigma \# \Sigma') = \chi(\Sigma) + \chi(\Sigma') - 2$.

Proof: When you form $\Sigma \# \Sigma'$ you are losing.

i) two 2-simplicies

ii) three 1-simplicies ← when you glue

iii) three 0-simplicies

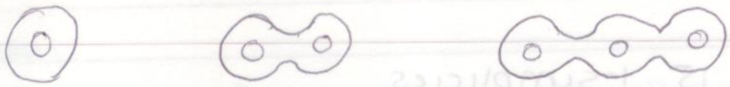
$\chi(\Sigma \# \Sigma') = \chi(\Sigma) + \chi(\Sigma') - 3 - (-3) - 2$

$\qquad = \chi(\Sigma) + \chi(\Sigma') - 2$. QED.

2-families

$\Sigma_+^0 = S^2$, $\Sigma_+^1 = T^2$, $\Sigma_+^g = \underbrace{T^2 \# \dots \# T^2}_{g \text{ times.}}$

Σ_+^g = orientable surface of genus g.



Σ^1_+ Σ^2_+ Σ^g_+

Proposition:

$\chi(\Sigma^g_+) = 2 - 2g$

Proof: $g=0$ $\Sigma^0_+ = S^2$, $\chi(S^2) = 2$
$g=1$ $\Sigma^1_+ = T^2$ $\chi(T^2) = 0$
ok for $g = 0,1$
suppose proved that $\chi(\Sigma^{g-1}_+) = 2 - 2(g-1)$
Then $\chi(\Sigma^g_+) = \chi(\Sigma^{g-1}_+ \# \Sigma^1_+)$
$= \chi(\Sigma^{g-1}_+) + \chi(T^2) - 2$
$= 2 - 2(g-1) + 0 - 2$
$= 2 - 2g.$ QED.

- Family.

$\mathbb{R}P^2$, $\mathbb{R}P^2 \# \mathbb{R}P^2$, ..., $\mathbb{R}P^2 \# \dots \# \mathbb{R}P^2$
Σ^0_- Σ^1_- Σ^g_-

Klien bottle coming soon!

Proposition:

$\chi(\Sigma^g_-) = 1 - g$

Proof: For $g=0$ $\chi(\mathbb{R}P^2) = 1$
Suppose $g \geq 1$ so proved for $g-1$
$\chi(\Sigma^g_-) = \chi(\Sigma^{g-1}_- \# \mathbb{R}P^2)$
$= \chi(\Sigma^{g-1}_-) + \chi(\mathbb{R}P^2) - 2$
$= 1 - (g-1) + 1 - 2$
$= 1 - g$. QED.

Complete homology of Σ_+^g, Σ_-^g

Proposition:

Let Σ, Σ' be orientable surfaces. Then $\Sigma \# \Sigma'$ is orientable.

Proof: Take a consistent orientation of Σ and remove a 2-simplex. This gives Σ_0.
Now remove a 2-simplex from Σ'. This gives Σ_0'.
When you glue: $\partial\Sigma_0 \cong \partial\Sigma_0'$ ensure that orientation on Σ' is chosen so boundary edges recieve oppisite directions from Σ, Σ'
This gives an orientation of $\Sigma \# \Sigma'$ QED.

Obvious observation: Σ_+^g, Σ_-^g are all connected.
$H_0(\Sigma_+^g : \mathbb{F}) \cong \mathbb{F}$, $H_0(\Sigma_-^g : \mathbb{F}) \cong \mathbb{F}$ for any field $\mathbb{F}$.

Theorem:

For any field $\mathbb{F}$.

$$H_r(\Sigma_+^g : \mathbb{F}) = \begin{cases} \mathbb{F} & r=0 \\ \mathbb{F}^{2g} & r=1 \\ \mathbb{F} & r=2 \\ 0 & r\geq 3 \end{cases} \qquad \mathbb{F}^{2g} = \underbrace{\mathbb{F}\oplus\cdots\oplus\mathbb{F}}_{2g}$$

Proof: $H_0(\Sigma_+^g) = \mathbb{F}$ connected
$H_2(\Sigma_+^g) = \mathbb{F}$ orientable ($\Sigma_+^g = T^2 \# \cdots \# T^2$, T^2 orientable).
$\chi(\Sigma_+^g) = 2-2g$
$\chi(\Sigma_+^g) = \dim H_0 - \dim H_1 + \dim H_2$
$= 1 - \dim H_1 + 1$
So $\dim H_1(\Sigma_+^g) = 2g$ QED.

Theorem:

$$H_r(\Sigma_-^g : \mathbb{Q}) = \begin{cases} \mathbb{Q} & r=0 \\ \mathbb{Q}^g & r=1 \\ 0 & r=2 \\ 0 & r\geq 3 \end{cases}$$

(can replace $\mathbb{Q}$ by any field in which $2 \neq 0$)

Proof: $H_0(\Sigma_-^g : \mathbb{Q}) = \mathbb{Q}$ connected

$H_2(\Sigma_-^g : \mathbb{Q}) = 0$ because Σ_-^g contains a punctuated $\mathbb{RP}^2$

$$\begin{aligned}\chi(\Sigma_-^g) &= 1-g \\ &= \dim H_0 - \dim H_1 + \dim H_2 \\ &= 1 - \dim H_1 + 0\end{aligned}$$

So $\dim H_1(\Sigma_-^g, \mathbb{Q}) = g$ QED.

Theorem:

$$H_r(\Sigma_-^g : \mathbb{F}^2) \cong \begin{cases} \mathbb{F} & r=0 \\ \mathbb{F}^{g+1} & r=1 \\ \mathbb{F} & r=2 \\ 0 & r\geq 3 \end{cases}$$

Proof: $H_0(\Sigma_-^g : \mathbb{F}_2) \cong \mathbb{F}_2$ connected

$H_2(\Sigma_-^g : \mathbb{F}_2) \cong \mathbb{F}_2$ connected

$$\begin{aligned}\chi(\Sigma_-^g) &= 1-g \\ &= \dim H_0 - \dim H_1 + \dim H_2 \\ &= 1 - \dim H_1 + 1\end{aligned}$$

$\dim H_1 = g+1.$ QED.

From the homology calculations we see that,

$\Sigma_+^g \sim \Sigma_+^h \Rightarrow g=h$

$\Sigma_-^g \sim \Sigma_-^h \Rightarrow g=h$

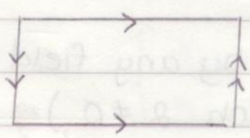
and $\Sigma_+^g \not\sim \Sigma_-^h$ (calculate $H_2(- : \mathbb{Q})$).

The classification Theorem for surfaces says:

"Any finite connected surface Σ is combinatorial equivalent to exactly one of Σ_+^g, Σ_-^h for some $g \geq 0$, $h \geq 0$"

Σ finite connected surface

Qu 1. What does Σ look like if Σ contains a subcomplex $\sim$ Möb (= Mobius Band).

Möb =


Proposition: 1

$\partial \text{Möb} \sim S^1$

Proposition: 2

$\mathbb{RP}^2 - 2\text{-simplex} \simeq \text{Möb}$. (Exercise)

Corollary: 3

$\text{Möb} \cup D^2 \sim \mathbb{RP}^2$ $\quad D^2 = 2$ disc ie some triangulation of Δ^2.
$\partial \text{Möb} = \partial D^2$

~~Answer~~ Answer Qu1: Decompose Σ ~~to~~ as:
$\Sigma = \text{Möb} \cup C$ $\quad C =$ complement of Möb
$\partial \text{Möb} = \partial C$
where C is a subcomplex $\partial C = S^1$
Form $\mathbb{RP}^2 = \text{Möb} \underset{\partial = \partial}{\cup} D_1^2$ $\qquad \Sigma' = C \underset{\partial=\partial}{\cup} D_2^2$

where D_1, D_2 are (different) 2 discs.

Proposition: 4

$\Sigma = \mathbb{RP}^2 \# \Sigma'$

Proof: Reverse steps and look at definition of connected sum
$\mathbb{RP}^2 \# \Sigma' = \mathbb{RP}^2 - \{2\text{disc}\} \underset{\partial=\partial}{\cup} \Sigma' - \{2 \text{ disc}\}$

$= \text{Möb} \underset{\partial=\partial}{\cup} C = \Sigma$ QED.

Qu 2: What is the relationship between $H_*(\Sigma : \mathbb{F}_2)$ and $H_*(\Sigma' : \mathbb{F}_2)$?

Ans: $\chi(\Sigma) = \chi(\mathbb{RP}^2 \# \Sigma')$
$= \chi(\mathbb{RP}^2) + \chi(\Sigma') - 2$
$= \chi(\Sigma') - 1$
Σ, Σ' both connected surfaces so
$H_2(\Sigma : \mathbb{F}_2) \cong H_2(\Sigma' : \mathbb{F}_2) \cong \mathbb{F}_2$.

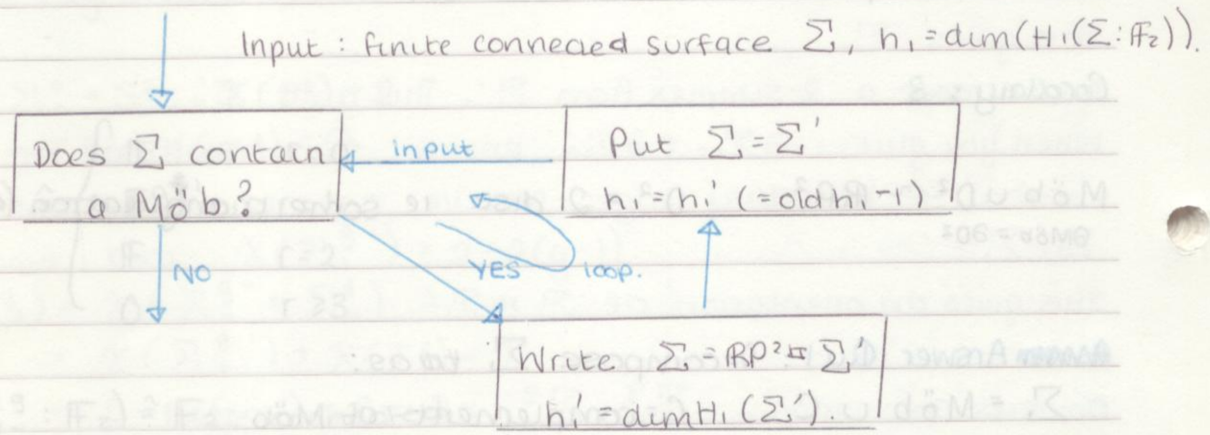
$H_0(\Sigma : \mathbb{F}_2) \cong H_0(\Sigma' : \mathbb{F}_2) \cong \mathbb{F}_2$

$\chi(\Sigma) = 2 - \dim H_1(\Sigma)$

$\chi(\Sigma') = 2 - \dim H_1(\Sigma')$

So $\quad -\dim H_1(\Sigma : \mathbb{F}) = -\dim(H_1(\Sigma' : \mathbb{F}_2)) - 1$

So if Σ contains a Möb then $\dim H_1(\Sigma : \mathbb{F}_2) \neq 0$ and this process reduces the dimension by 1.



Note already you can only go round loop finitely many times, controlled by h_1.

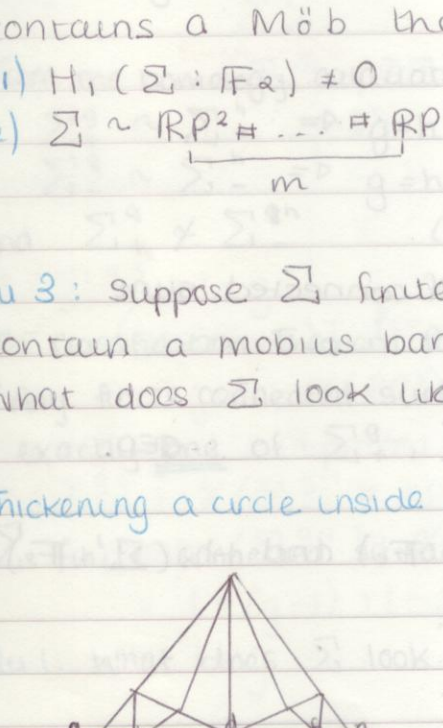
First conclusion: If Σ is a finite connected surface which contains a Möb then

1) $H_1(\Sigma : \mathbb{F}_2) \neq 0$

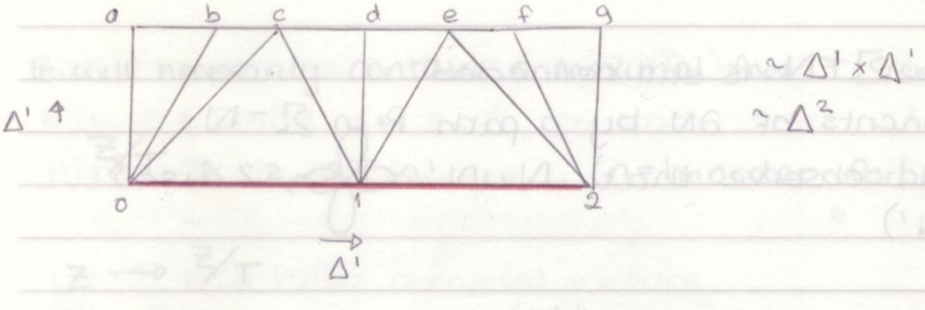
2) $\Sigma \sim \underbrace{\mathbb{RP}^2 \# \dots \# \mathbb{RP}^2}_{m} \# \Sigma'$ where Σ' does not contain a Möb.

Qu 3: Suppose Σ finite connected surface which does not ~~have~~ contain a mobius band and that $H_1(\Sigma : \mathbb{F}_2) \neq 0$. What does Σ look like?

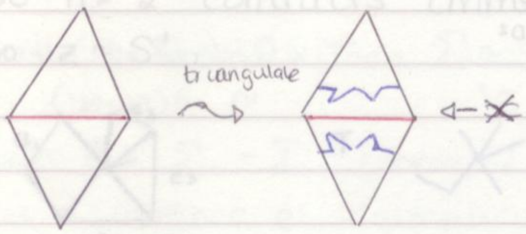
Thickening a circle inside a surface:



Preform 1st and 2nd barycentre subdivision



In a surface every 1-simplex — belongs to exactly two 2-simplicies. So I need to double up.

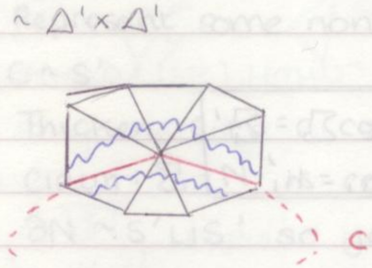


Proposition:

If e is a simplex inside a surface Σ, then I can triangulate Σ' so that it has a subcomplex X, such that $X \sim \Delta^1 \times \Delta^1$ and collopses onto e.

Extension: Let C be a finite simplicial complex $C \cong S^1(n)$ and $C \subset \Sigma$ where Σ surface.

(Cannocial Nbd) Then Σ can be triangulated so that C has a neighbourhood N which collapses onto C and such that locally $N \sim \Delta^1 \times \Delta^1$



A circle has exactly two distinct thickenings.

i) cylinder ii) mobius band

Next Step: Σ finite connected surface, Σ contains no Möb $H_1(\Sigma : \mathbb{F}_2) \neq 0$

i) I want to produce a circle C inside Σ such that C represents a non-trival element of H_1

ii) Thicken C to N $\sim$ cylinder

iii) Remove N and show $\Sigma - N$ is still connected
iv) Join the two components of ∂N by a path P in $\Sigma - N$
v) Thicken P to N' and observe that $N \cup N' \sim T^2 - S^2$ disc$\}$
vi) Put $X = \Sigma - (N \cup N')$

$\partial X \sim S^1$

$\partial(N \cup N') \sim S^1$

So $\Sigma = X \cup (N \cup N')$

Put $\Sigma' = X \cup D^2$ $\quad$ $T^2 = (N \cup N') \cup D^2$

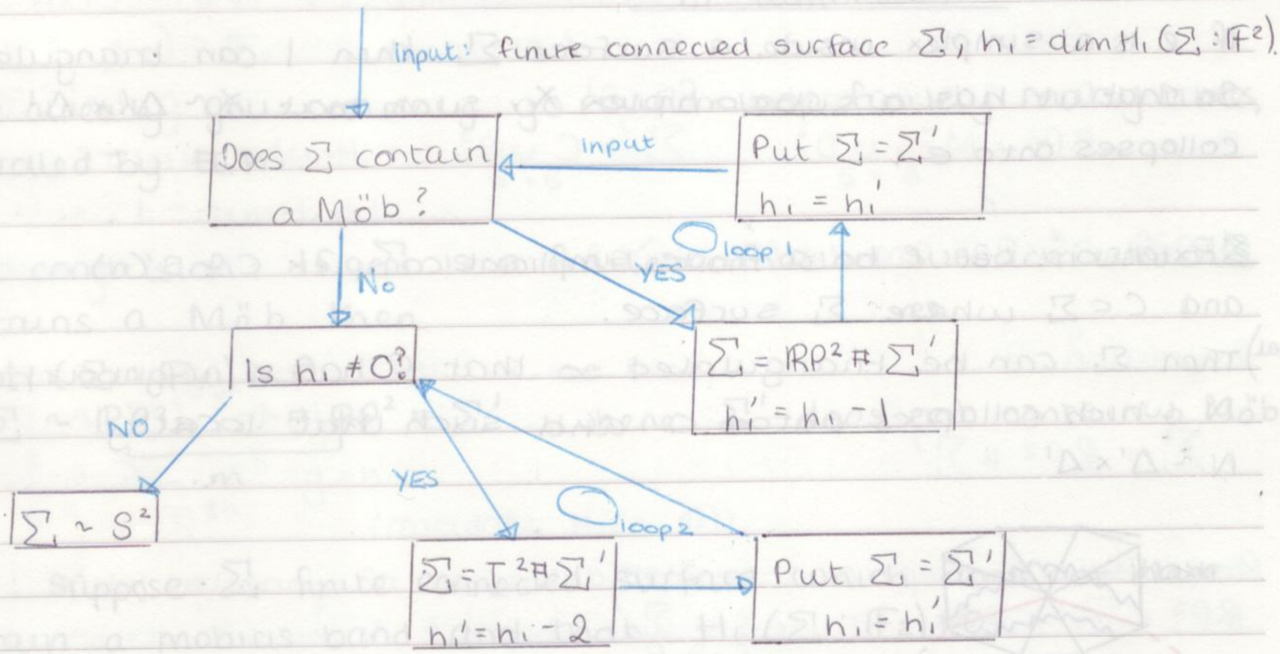
$\partial X = \partial D^2$ $\quad$ $\partial(N \cup N') = \partial D^2$

and what I've got is $\Sigma \sim \Sigma' \# T^2$

Flow chart

Proposition:

$$\dim H_1(\Sigma') = \dim H_1(\Sigma) - 2.$$

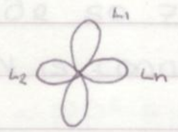


Details for loop 2: Let Σ be a finite connected surface (Σ contains no Möb). and $H_1(\Sigma : F_2) \neq 0$.
Elements of H_1 are represented by collections of edges.
$C_1(\Sigma : F_2)$ spanned by edges.
Choose smallest collection, z in Σ, which represents a non-zero element of H_1

i) z contains no "free edges" (otherwise $\partial z \neq 0$ and we want $\partial z = 0$)
ii) z is connected (otherwise throw some of it away).

z is a finite 1-complex.

It will necessarily contain a maximal free T. (subcomplex with no loops)

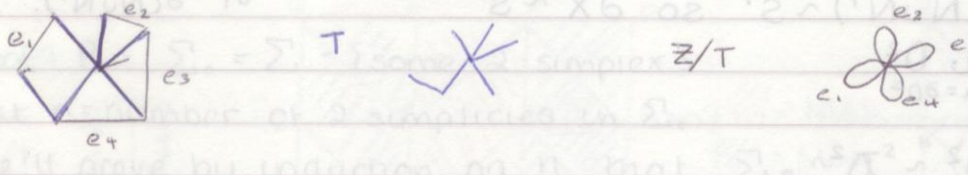
$Z/T \simeq$  ($n=1$ in order for z to be minimal).

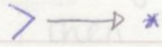
$Z \to Z/T$

Otherwise put $z_i = h^{-1}(L_i)$ and each z_i represents a non zero element of H_1

So $n \geq 2$ contracts immediatly.

So $z \sim S^1$



$h \equiv$ 

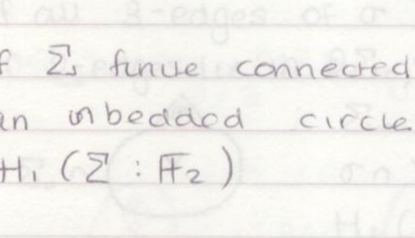
Proposition:

If Σ finite connected surface $H_1(\Sigma : \mathbb{F}_2) \neq 0$ then Σ contains an imbedded circle repeating some non-zero element of $H_1(\Sigma : \mathbb{F}_2)$

~~Thicken~~

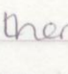
What's involved in loop 2.

1. Represent some non-zero element $z \in H_1(\Sigma : \mathbb{F}_2)$ by an imbedded $C \sim S^1$
2. Thicken C to a canonical nbd N, then $N \sim$ cylinder $(S^1 \times I)$
3. Claim: $\Sigma - N$ is connected
4. $\partial N \sim S^1 \sqcup S^1$ so join one boundary component of ∂N to ~~other~~ other by a path P, $\Sigma - N$.

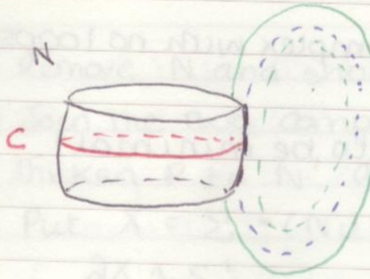


Put $C' = P \cup \gamma$

$C' \sim S^1$

 $= C'$

5. Thicken C' out to another cylinder N'



6. $N \cup N' \sim T^2 - \{2\text{-disc}\}$

7. Now write $\Sigma_1 = (N \cup N') \cup X$

Observe that $\partial(N \cup N') \sim S^1$ so $\partial X \sim S^1$

X intersects $N \cup N'$ in $\partial(N \cup N')$.

Define $\Sigma_1' = X \cup_{\partial X = \partial D^2} D^2$

$T = (N \cup N') \cup_{\partial(N \cup N') = \partial D^2} D^2 \sim T^2$

Then $\Sigma_1 \sim \Sigma_1' \# T^2$

Proposition:

$\dim H_1(\Sigma_1') = \dim H_1(\Sigma_1) - 2$

Proof: $\chi(\Sigma_1) = \chi(\Sigma_1') + \chi(T^2) - 2$
$= \chi(\Sigma_1') - 2 \quad (\chi(T^2) = 0)$

Σ_1' is connected so $\dim H_0(\Sigma_1') = 1$

With $\mathbb{F}_2$ coeffs, $\dim H_2(\Sigma_1') = 1$

$\dim H_0(\Sigma_1) - \dim H_1(\Sigma_1) + \dim H_2(\Sigma_1) = \dim H_0(\Sigma_1') - \dim H_1(\Sigma_1') + \dim H_2(\Sigma_1') - 2$

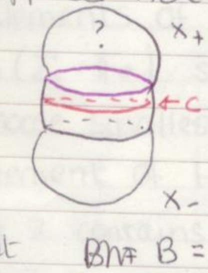
$2 - h_1 = 2 - 2 - h_1'$

$h_1' = h_1 - 2$.

So we still need to show

3. $\Sigma_1 - N$ is connected

suppose not, $\Sigma_1 - N = X_+ \cup X_-$



where X_+ intersects N in top boundary component

X_- " bottom boundary component

Put ~~BnT~~ B = (= ½ of N)

$X_+ \cup B$ is 2 complex contained in Σ_1 and $\partial(X_+ \cup B) = C$

But C represents $z \neq 0$, $z \in H_1(\Sigma : \mathbb{F}_2)$
But $C \in \operatorname{Im} \partial_2$ so $z = 0$, contradiction.
Hence $\Sigma - N$ is connected.

Let Σ be a finite connected surface.

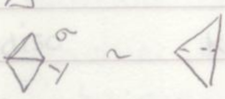
Theorem:

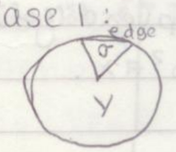
if $H_1(\Sigma : \mathbb{F}_2) = 0$. Then $\Sigma \sim S^2$.

Proof: Put $\Sigma_0 = \Sigma - \{\text{some 2-simplex}\}$
Put n = number of 2 simplicies in Σ_0. $H_1(\Sigma_0) = 0$.
We'll prove by induction on n that $\Sigma_0 \sim \Delta^2$ (clear that Σ_0 connected).
if $n = 1$, then $\Sigma_0 \cong \Delta^2$ (this case is empty, therefore true).
Suppose proved for $< n$ 2-simplicies.
Let σ be a 2-simplex of Σ_0 such that some edge of σ lies in $\partial\Sigma_0$
If all 3-edges of σ line in $\partial\Sigma_0$, then $\Sigma_0 = \sigma \cong \Delta^2$ 
If 2-edges lie in $\partial\Sigma_0$;

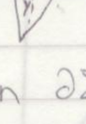
$\Sigma_0 \sim$ 

$\Sigma_0 = \sigma \cup Y$
$\sigma \cap Y$ single edge.
$H_1(Y) \cong H_1(\Sigma_0) = 0$.

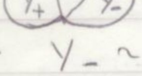
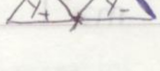
Collapse: Y has 1 less 2-simplex than Σ_0.
So $Y \sim \Delta^2$
$\Sigma_0 \sim$  $\cong \Delta^2$

If only one edge of σ lies in $\partial\Sigma_0$, there are two cases.
Case I:


opposite vertex not in $\partial\Sigma_0$.

$\Sigma_0 = \sigma \cup Y$. Collapse $H_1(Y) = 0$
$Y \sim \Delta^2$, $\sigma \cup Y \sim \Delta^2$  $Y =$

Case II: Opposite edge is in $\partial\Sigma_0$.

Σ_0  $\sim$ 

$Y_+ \sim \Delta^2$ $Y_- \sim \Delta^2$

$\sim$ 

Proposition 2:

$\mathbb{R}P^2 - \{2\text{-simplex}\} \sim$ Möb.

Proof:  $\sim$ Möb  $\sim \mathbb{R}P^2 - \{2\text{-simplex}\}$ 

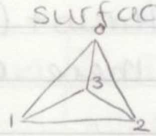
-absent

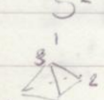
To Summarise:

Input finite, connected surface Σ.

1) Don't go around Loop 1 or Loop 2 and get $\Sigma \sim S^2$
2) Don't go around Loop 1 but go around Loop 2 n times
Then $\Sigma \sim S^2 \# \underbrace{T^2 \# \ldots \# T^2}_{n}$
3) Go around Loop 1 m times but don't go around Loop 2
$\Sigma \sim S^2 \# \underbrace{\mathbb{R}P^2 \# \ldots \# \mathbb{R}P^2}_{m}$
4). Go around Loop 1 m times and Loop 2 n times
$\Sigma \sim S^2 \# \underbrace{\mathbb{R}P^2 \# \ldots \# \mathbb{R}P^2}_{m} \# \underbrace{T^2 \# \ldots \# T^2}_{n}$

Simplifications:

1) $S^2 \# X \sim X$ for any surface X.
$S^2 - \{2\text{ simplex}\}$  $S^2 \# X$ is simply the subdivision of X at the 2-simplex you remove to form $S^2 \# X$.

 (Remove bottom face)

After simplification it looks like we get 4 cases:

1) $\Sigma \sim S^2$
2) $\Sigma \sim \underbrace{T^2 \# \ldots \# T^2}_{n}$ $n \geq 1$
3) $\Sigma \sim \underbrace{\mathbb{R}P^2 \# \ldots \# \mathbb{R}P^2}_{m}$ $m \geq 1$
4) $\Sigma \sim \underbrace{\mathbb{R}P^2 \# \ldots \# \mathbb{R}P^2}_{m} \# \underbrace{T^2 \# \ldots \# T^2}_{n}$ $n, m \geq 1$.

We get rid of mixed case using:

Theorem:

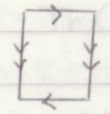
$\mathbb{R}P^2 \# T^2 \sim \mathbb{R}P^2 \# \mathbb{R}P^2 \# \mathbb{R}P^2$.

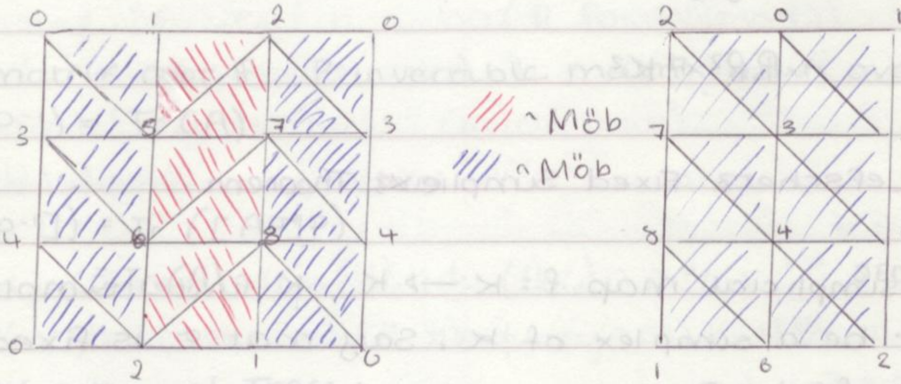
First prove:

Proposition:

$K^2 \sim \mathbb{R}P^2 \# \mathbb{R}P^2$ (Klien bottle).

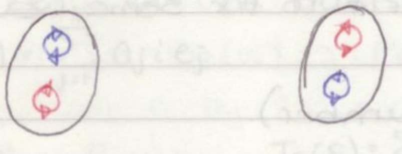
Proof:

$K^2 =$ 



$K^2 \sim$ Möb $\underset{\partial \equiv \partial}{\cup}$ Möb

QED.

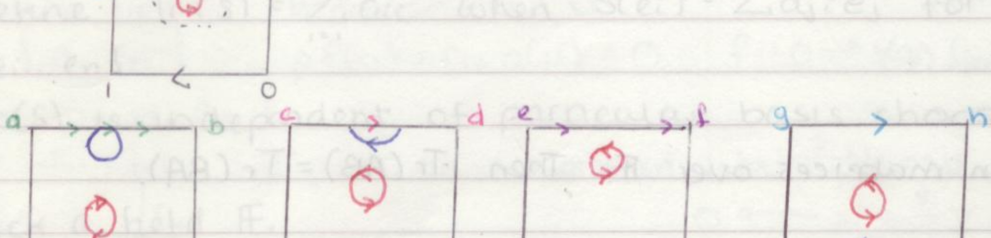


$T^2 - 2$-disc $\qquad K^2 - 2$-disc.

$T^2 \#$ Möb 

irrelevant.

gradually change line of identification.



$= K^2 \#$ Möb.

Finally we have:

Proposition:

$\mathbb{R}P^2 \# T^2 \sim \mathbb{R}P^2 \# K^2$ so $\mathbb{R}P^2 \# T^2 \sim \mathbb{R}P^2 \# \mathbb{R}P^2 \# \mathbb{R}P^2$

Proof: Shown, $\text{Möb} \# T^2 \sim \text{Möb} \# K^2$

so $\mathbb{R}P^2 \# T^2 = D^2 \underset{\partial = \partial}{\cup} (\text{Mob} \# T^2) - \{2\text{-disc}\}$

$= D^2 \underset{\partial=\partial}{\cup} \{(\text{Möb} \# K^2 - 2\text{disc}\}$

$= \mathbb{R}P^2 \# K^2$ QED.

Lefschetz Fixed Simplices Theorem

Given simplicial map $f: K \to K$, K finite simplicial complex.
Let σ be a simplex of K. Say that σ is fixed under f when $f(\sigma) = \sigma$.
Ignore orientation on σ.

Lefschetz gives a sufficient condition for f to fix some simplex.

Lefschetz number (generalisation of Euler number).

Definition:

Let $\mathbb{F}$ be a field and let $A = (a_{ij})_{\substack{1 \le i \le n \\ 1 \le j \le m}}$ be an $n \times n$ matrix $/\mathbb{F}$.
Define $Tr(A) = \sum_{i=1}^{n} a_{ii} = a_{11} + a_{22} + \dots + a_{nn}$.

Proposition:

Let A, B be $n \times n$ matrices over $\mathbb{F}$. Then $Tr(AB) = Tr(BA)$.

Proof: $A = (a_{ij})$, $B = (b_{jk})$

$(AB)_{ii} = \sum_{j=1}^{n} a_{ij} b_{ji}$

$\sum_{i=1}^{n} (AB)_{ii} = \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij} b_{ji}$ Interchange order of summation.

$\sum_{i=1}^{n}(AB)_{ii} = \sum_{j=1}^{n}\sum_{i=1}^{n} a_{ij}b_{ji}$ $\quad a_{ij}b_{ji} = b_{ji}a_{ij}$ $\mathbb{F}$ field so commutative

$\sum_{i=1}^{n}(AB)_{ii} = \sum_{j=1}^{n}\sum_{i=1}^{n} b_{ji}a_{ij} = \sum_{j=1}^{n}(BA)_{jj}$

$Tr(AB) = Tr(BA)$ QED.

$Tr(AB) \cancel{=} Tr(A)Tr(B)$ ⚠ eg $A = B = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$.

Proposition:

If A is $n \times n$ matrix over $\mathbb{F}$. P invertible matrix $n \times n$ over $\mathbb{F}$.
Then $Tr(PAP^{-1}) = Tr(A)$

Proof: $Tr(P[AP^{-1}]) = Tr([AP^{-1}]P)$
$= Tr(A)$ QED.

Coordinate free definition of Trace:

Let $S: V \to V$ be linear map, V finitely dimensional vs $/\mathbb{F}$.
Let $\{e_1, \dots, e_n\}$ be basis for V.
$S(e_i) = \sum_{j=1}^{n} a_{ji}e_j$

I'd like to define $Tr(S) = \sum_{i=1}^{n} a_{ii}$
If I take another basis for V $\{\varepsilon_1 \dots \varepsilon_n\}$ can also write
$S(\varepsilon_i) = \sum_{j=1}^{n} b_{ji}\varepsilon_j$ $\quad B = (b_{ij})$
Then $B = PAP^{-1}$, P matrix of change of basis
So $Tr(B) = Tr(PAP^{-1}) = Tr(A)$
Define $Tr(S) = \sum_{i=1}^{n} a_{ii}$ when $S(e_i) = \sum a_{ji}e_j$ for some basis $\{e_1 \dots e_n\}$
$Tr(S)$ is independent of particular basis choosen.

Pick a field $\mathbb{F}$.
$f: K \to K$ simplicial map, K finite simplicial complex.
$H_i(f): H_i(K:\mathbb{F}) \to H_i(K:\mathbb{F})$
induced map on homology.

Definition:

$\lambda_{hom}(f) = \sum_{i\geq 0} (-1)^i \, Tr(H_i(f))$. Homological Lefschetz number (coeffs in $\mathbb{F}$)

Obviously $\lambda_{hom}(f) \in \mathbb{F}$.

Lefshetz Fixed Simplex Theorem:

Let $f: K \to K$ be simplicial map, where K finite simplicial complex. (Choose field $\mathbb{F}$)
If $\lambda_{hom}(f) \neq 0$ then f fixes a simplex.

X finite simplicial complex $f: X \to X$ simplicial ~~complex~~ map.
Fix a field $\mathbb{F}$.
$\lambda_{hom}(f) = \sum_{k\geq 0} (-1)^k \, Tr(H_k(f))$ homological Lefschetz number.
where $H_k(f): H_k(X:\mathbb{F}) \to H_k(X:\mathbb{F})$ is induced map on homology.
$\lambda_{hom}(f) \in \mathbb{F}$.

Alternative definition:

For each k we have induced on k-chains, $C_k(f): C_k(X:\mathbb{F}) \to C_k(X:\mathbb{F})$.
Def: $\lambda_{geom}(f) = \sum_k (-1)^k \, Tr(C_k(f))$.

Note: $\chi(X) = \lambda(Id_X)$ Exercise
So λ is generalisation of χ.

Proposition:

$\lambda_{geom}(f) = \lambda_{hom}(f)$.

Proof: Suppose given exact sequence of finite dimensional vs / $\mathbb{F}$.

$$0 \to U \xrightarrow{i} V \xrightarrow{\varphi} W \to 0$$

$U = \ker(\varphi)$.

Suppose next given a linear map $f: V \to V$ which "preserves exact sequence" ie:

$$\begin{array}{ccccccccc} 0 & \to & U & \xrightarrow{i} & V & \xrightarrow{\varphi} & W & \to & 0 \\ & & \downarrow{f_U} & & \downarrow{f} & & \downarrow{f_W} & & \\ 0 & \to & U & \xrightarrow{i} & V & \xrightarrow{\varphi} & W & \to & 0 \end{array}$$

Commutes.

~~Previous~~:

Proposition:

Given a commutative diagram as above $Tr(f) = Tr(f_u) + Tr(f_V)$ (Additivity of trace).

Proof: choose a basis $\varphi_1 \dots \varphi_m$ for W. Because p is surjective, choose $\epsilon_1 \dots \epsilon_m \in V$ st $p(\epsilon_i) = \varphi_i$

Put $W' = \mathrm{span}_{\mathbb{F}}\{\epsilon_1 \dots \epsilon_m\}$. $\dim(W') \le m$.

$p: W' \longrightarrow W$ is surjective so $\dim(W') \ge \dim(W) = m$

So $\dim(W') = m$ and $\{\epsilon_1 \dots \epsilon_m\}$ is a basis for W'.

Claim: $V = U \dot{+} W'$ (internal direct sum)

$\dim V = \dim U + \dim W$ by exactness, (Kernal Rank)

so $\dim V = \dim U + \dim W'$

So $\dim(U \cap W') = 0$

$\dim(U + W') + \dim(U \cap W') = \dim(U) + \dim(W')$.

So every $v \in V$ can be expanded uniquely as $v = u + w$, $u \in U$, $w \in W$

Let $f: V \to V$ be a linear map

$f(u+w) = f(u) + f(w)$

$f(u) = f_{11}(u) + f_{21}(u)$ $\quad f_{11}(u) \in U \quad f_{21}(u) \in W'$

$f(w) = f_{12}(w) + f_{22}(w)$ $\quad f_{12}(w) \in U \quad f_{22}(w) \in W'$.

So represent f by 2×2 matrix of linear maps

$f = \begin{pmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{pmatrix}$ $\quad f_{11}: U \to U \quad f_{12}: W' \to U$

$\quad f_{21}: U \to W' \quad f_{22}: W' \to W'$

$$\begin{array}{ccccccccc} 0 & \longrightarrow & U & \hookrightarrow & V & \xrightarrow{p} & W & \longrightarrow & 0 \\ & & \downarrow f_u & & \downarrow f & & \downarrow f_W & & \\ 0 & \longrightarrow & U & \hookrightarrow & V & \longrightarrow & W & \longrightarrow & 0 \end{array}$$

$f = \begin{pmatrix} f_u & f_{12} \\ 0 & f_{22} \end{pmatrix}$ Because $u \in U$

$p f(u) = f_W p(u) = 0 \quad f: 0 \to \mathrm{Ker}(p) = 0$.

How about f_{22}?

$$\begin{array}{ccc} W' & \xrightarrow{p} & W \\ \downarrow f_{22} & & \downarrow f_W \\ W' & \xrightarrow{p} & W \end{array}$$

p is an isomorphism $W' \xrightarrow{\cong} W$.

$f_W = p f_{22} p^{-1}$

So $Tr(f_W) = Tr(f_{22})$

$Tr(f) = Tr\begin{pmatrix} f_u & f_{12} \\ 0 & f_{22} \end{pmatrix} = Tr(f_u) + Tr(f_{22}) = Tr(f_u) + Tr(f_W)$. QED.

Theorem:

$\lambda_{geom}(f) = \lambda_{hom}(f)$ where $f : X \to X$ simplicial map, X finite simplicial complex.

Proof: $\lambda_{geom}(f) = \sum_k (-1)^k \operatorname{Tr}(C_k(f))$.

$$\begin{array}{ccccccccc} 0 & \to & Z_k(X) & \hookrightarrow & C_k(X) & \xrightarrow{\partial_k} & B_{k-1}(X) & \to & 0 \\ & & \downarrow Z_k(f) & & \downarrow C_k(f) & & \downarrow B_{k-1}(f) & & \\ 0 & \to & Z_k(X) & \hookrightarrow & C_k(X) & \xrightarrow{\partial_k} & B_{k-1}(X) & \to & 0 \end{array}$$

$Z_k = \operatorname{Ker} \partial_k$
$B_{k-1} = \operatorname{Im}(\partial_k)$

Commutative, rows exact.

So $\operatorname{Tr}(C_k(f)) = \operatorname{Tr}(Z_k(f)) + \operatorname{Tr}(B_{k-1}(f))$.

$$\begin{array}{ccccccccc} 0 & \to & B_k(X) & \hookrightarrow & Z_k(X) & \xrightarrow{\pi} & H_k(X) & \to & 0 \\ & & \downarrow B_k(f) & & \downarrow Z_k(f) & & \downarrow H_k(f) & & \\ 0 & \to & B_k(X) & \hookrightarrow & Z_k(X) & \xrightarrow{\pi} & H_k(X) & \to & 0 \end{array}$$

Commutative rows exact.

so $\operatorname{Tr}(Z_k(f)) = \operatorname{Tr}(H_k(f)) + \operatorname{Tr}(B_k(f))$

So $\operatorname{Tr}(C_k(f)) = \operatorname{Tr}(H_k(f)) + \operatorname{Tr}(B_k(f)) + \operatorname{Tr}(B_{k-1}(f))$.

Take alternating sum.

$$\sum_k (-1)^k \operatorname{Tr}(C_k(f)) = \sum_k (-1)^k \operatorname{Tr}(H_k(f)) + \sum_k (-1)^k \left(\operatorname{Tr}(B_k(f)) + \operatorname{Tr}(B_{k-1}(f))\right)$$

But $\sum_k (-1)^k \left(\operatorname{Tr}(B_k(f)) + \operatorname{Tr}(B_{k-1}(f))\right) = 0$

So $\lambda_{geom}(f) = \lambda_{hom}(f) + 0$ QED.

Theorem: Lefschetz Fixed Simplex Theorem.

Let $f : X \to X$ be simplicial map, X finite simplicial complex ($\mathbb{F}$ some field)
If $\lambda(f) \neq 0$ then f forces some simplex (ignore orientation)
ie $\exists$ simplex σ in X ; $f(\sigma) = \sigma$.

Proof: Observe that $C_k(f) : C_k(X) \to C_k(X)$. has the following (atypical) property; $C_k(X)$ has a basis which consists of the oriented k-simplexes of X $\sigma_1 \dots \sigma_N$ (N may be huge).

For each i

$$C_k(f) = \begin{cases} 0 \\ \text{or} \\ \pm \text{ some othe } \sigma_\gamma \end{cases}$$

In any column of matrix $\exists$ at most one non-zero entry ± 1
(maybe all enteries in column are 0)
So looking at the diagonal of matrix of $C_k(f)$, a non-zero entry on

diagonal corresponds to a k-simplex σ_i such that $f(\sigma_i) = \pm\sigma_i$ taking orientation into account. (or $f(\sigma_i) = \sigma_i$ ignoring orientation)
So we get: If there is no k-simplex fixed by f then $Tr(C_k(f)) = 0$
So if no simplex of whatever dimension is fixed by f then $Tr(C_k(f)) = 0$ for all k.
So if no simplex of X is fixed then $\lambda(f) = \sum_k (-1)^k Tr\, C_k(f) = 0$
Take contrapositive:
If $\lambda(f) \neq 0$ some simplex of X is fixed by f. QED.

Brower Fixed Simplex Theorem:
Let D be a "combinatorial disc" (ie D finite simplicial complex $D \sim \Delta^n$ for some $n \geq 1$).
Let $f: D \to D$ be a simplicial map
Then $\exists$ simplex σ in D st $f(\sigma) = \sigma$ (up to orientation).

Proof: $H_k(D : \mathbb{F}) = \begin{cases} \mathbb{F} & k=0 \\ 0 & k \neq 0 \end{cases}$

So $\lambda(f_*) = Tr(H_0(f))$ $\quad H_0(f) : \underset{\mathbb{F}}{\underset{\shortparallel}{H_0(D)}} \to \underset{\mathbb{F}}{\underset{\shortparallel}{H_0(D)}}$

But if X is connected complex, $g: X \to X$ $H_0(g) = Id : H_0(X) \to H_0(X)$.
So in this case $\lambda(f) = 1 \neq 0$. So f fixes a simplex QED.

Corollary:
Let $X \sim \mathbb{R}P^2$, X finite simplicial complex and let $f: X \to X$ be simplicial map. Then f fixes a simplex.

Proof: $H_r(X : \mathbb{Q}) \cong \begin{cases} \mathbb{Q} & r=0 \\ 0 & r \neq 0. \end{cases}$

So $\lambda(f) = Tr\, H_0(f) = 1 \neq 0$ QED.

Euler characteristic (Again)!

Euler char extra

Want to show (※)
I If $X = X_+ \cup X_-$ then $\chi(X) + \chi(X_+ \cap X_-) = \chi(X_+) + \chi(X_-)$ (Additivity)
II $\chi(X \times Y) = \chi(X)\chi(Y)$
SNAG: I need to say what $X \times Y$ actually means.
eg $S^2 \times S^2$ $\chi(S^2 \times S^2) = \chi(S^2)\chi(S^2) = 4$ $\chi(S^4) = 2$.

Recall Internal and External Direct Sums.

Suppose W is a vector space and $V_1, V_2 \subset W$ are vector subspaces.
Say that W is the "sum" of V_1, V_2 when $\forall w \in W\ \exists v_1 \in V_1\ \exists v_2 \in V\ \ w = v_1 + v_2$

The external direct sum $V_1 \oplus V_2$ is the vector space $\left\{ \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} : v_i \in V_i \right\}$
$\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} + \begin{pmatrix} v_1' \\ v_2' \end{pmatrix} = \begin{pmatrix} v_1 + v_1' \\ v_2 + v_2' \end{pmatrix}$ $\quad \lambda \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} \lambda v_1 \\ \lambda v_2 \end{pmatrix}$.

Clearly $\dim(V_1 \oplus V_2) = \dim V_1 + \dim V_2$.

Relation to the sum: Suppose $V_1, V_2 \subset W$. Get a linear map $\alpha : V_1 \oplus V_2 \to W$
$\alpha \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = v_1 + v_2$.

Proposition:
W is "the sum" of V_1, V_2 if and only if $\alpha : V_1 \oplus V_2 \to W$ is surjective

$\operatorname{Ker}(\alpha)$? $\alpha \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = v_1 + v_2$
$v_1 + v_2 = 0 \Leftrightarrow v_2 = -v_1$
$v_1 \in V_1\ \ v_2 \in V_2$ so $v_2 = -v_1$ means $v_1 \in V_1 \cap V_2\ \ v_2 \in V_1 \cap V_2$.

Standard Notation: '$W = V_1 + V_2$' means $\forall w \in W\ \exists v_1 \in V_1, v_2 \in V_2 : w = v_1 + w_2$
or better $\alpha : V_1 \oplus V_2 \to W$ is surjective.
$0 \to V_1 \cap V_2 \xrightarrow{i} V_1 \oplus V_2 \xrightarrow{\alpha} V_1 + V_2 \to 0$
is exact when $i(v) = \begin{pmatrix} v \\ -v \end{pmatrix}$.
So $\dim(V_1 \oplus V_2) = \dim(V_1 + V_2) + \dim(V_1 \cap V_2)$.

Definition: $V_1 + V_2$ is called the internal direct sum of V_1, V_2 if and only if $V_1 \cap V_2 = 0$
Then write $V_1 \dot{+} V_2$ $(\cong V_1 \oplus V_2)$.

Reference to MV Theorem:

Suppose $X = X_+ \cup X_-$, X finite simplicial complex, X_+, X_- subcomplexes
ie every simplex σ of X ie either a simplex of X_+ or of X_-
$C_k(X_+) \oplus C_k(X_-) \xrightarrow{\alpha} C_k(X) \to 0$
To say $X = X_+ \cup X_-$ means that for each k α is surjective.

Proposition:
If $X = X_+ \cup X_-$ then we get an exact sequence for each k.
$0 \to C_k(X_+ \cap X_-) \to C_k(X_+) \oplus C_k(X_-) \to C_k(X) \to 0$

So $\dim C_k(X) + \dim C_k(X_+ \cap X_-) = \dim C_k(X_+) + \dim C_k(X_-)$

Take alternating sum.

$$\sum_k (-1)^k \dim C_k(X) + \sum_k (-1)^k \dim C_k(X_+ \cap X_-) = \sum_k (-1)^k \dim C_k(X_+) + \sum_k (-1)^k \dim C_k(X_-)$$

$$\chi(X) + \chi(X_+ \cap X_-) = \chi(X_+) + \chi(X_-)$$

~~II [illegible]~~

II $\chi(X \times Y) = \chi(X)\chi(Y)$ Multiplicianty formula.

I need to say what I mean by $X \times Y$.

For a cubical description of space products are not a problem

$I^m \times I^n \cong I^{m+n}$

Snag for simplicial homology is that $\Delta^m \times \Delta^n$ is not actually a simplex (its a prism).

So we need to triangulate $\Delta^m \times \Delta^n$

Start from Posets $(X, \leq)$.

Set X with a relation $\leq$ on X such that

$\forall x \in X \quad x \leq x$

$\forall x, y, z \in X, \quad x \leq y,\ y \leq z \Rightarrow x \leq z$

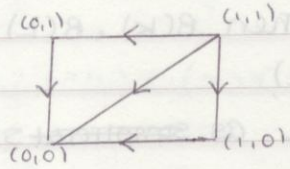
$x \leq y$ and $y \leq x \Rightarrow x = y$.

A totally ordered set is a Poset which satisfies

$\forall x, y \in X$ either $x \leq y$ or $y \leq x$.

$\{0,1\}$ has total order $0 \leftarrow 1$

$\{0,1\} \times \{0,1\}$ has partial order.



Product of Posets:

$(X, \leq_1), (Y, \leq_2)$

$(x_1, y_1) \leq (x_2, y_2)$ iff $x_1 \leq_1 x_2,\ y_1 \leq_2 y_2$.

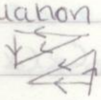
Definition:

Simplicial complex associated to a poset $(X, \leq)$. Take X finite

$S(X, \leq)$ Vector set is X

Take simplices to be totally orderded subsets.

Example: Gives triangulation of $I \times I$.

Maximal simplices 

Suppose $K = S(X, \leq)$ $L = S(Y, \leq)$.

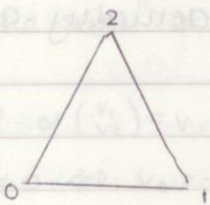
Define $K \times L = S(X \times Y, \leq)$.

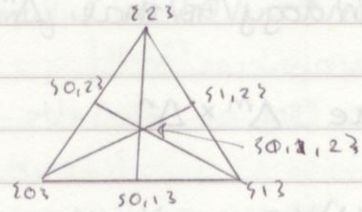
So I can triangulate $K \times L$ provided I can describe K, L as simplicial complexes associated to posets.

(Every finite simplicial complex can be so described)

Slightly more naturally if K finite simplicial complex and $B(K)$ is its barycentric subdivision.

Then $B(K) \cong S(X, \leq)$ for some $X, \leq$. which I'll describe.

Δ^2 

$B(\Delta^2)$ 

I've taken the non empty subsets of $\{0,1,2\}$ partially ordered by inclusion.

If $K = (V_K, S_K)$

$B(K)$ = barycentric subdivision obtained as follows.

Vertex set of $B(K) = \bigcup_{\sigma \in S_K} \{$non empty subsets of $\sigma\}$.

There is a natural partial ordering by inclusion.

The associated simplicial complex is $B(K)$.

K, L simplicial complexes.

$\downarrow$

$B(K)\ B(L)$ Then $B(K), B(L)$ well defined.

$K \sim B(K)$ $L \sim B(L)$

Triangulate $K \times L$ as ~~simplicial~~ subcomplex of $B(K) \times B(L)$.

Here I will write $\Delta^n = \{\{0,1, \dots n\}$ non empty subsets$\}$.

Take $X(n)$ = all subsets of $\{1, \dots, n\}$ partially ordered by inclusion

$Y(n)$ = all non-empty subsets of $\{1, \dots, n\}$. partially ordered by inclusion

Proposition:

$X(n) \cong C(Y(n))$ Cone on $Y(n)$

Proof: Cone point is $\emptyset$.

$$\begin{array}{ccc} \{1\} & \subset & \{1,2\} \\ \cup & \diagup & \cup \\ \emptyset & \subset & \{2\} \end{array}$$

$\{1\} \subset \{1,2\} \subset \{2\}$

$X(2)$ = cone on $Y(2)$.

$X(2)$ $\qquad$ $Y(2)$.

Proposition:

$Y(N) = B(\Delta^{n-1})$ = Barycentric subdivision of Δ^{n-1}

Corollary:

$X(n)$ is a subdivision of Δ^n (not $n-1$).

Proof: $\Delta^n = C(\Delta^{n-1})$

$$\wr$$

$$C(B(\Delta^{n-1})) \cong X(n)$$

QED.

$X(n) \sim \Delta^n$.

Proposition:

$X(n) \cong I \times \ldots \times I$ with triangulation obtained from the product poset on $\{0,1\}$.

Proof: Vertices of $X(n)$ are subsets of $\{1, \ldots, n\}$

For each subset $A \subset \{1 \ldots n\}$ define a point $P_A \in \underbrace{\{0,1\} \times \ldots \times \{0,1\}}_{n}$

Coordinates of $P_A = (x_1, \ldots x_n)$

where $x_i = \begin{cases} 1 & \text{if } i \in A \\ 0 & \text{if } i \notin A. \end{cases}$

$P_\emptyset = (0, \ldots 0)$

$\vdots$

$P_{\{1, \ldots n\}} = (1, \ldots 1)$.

$P_i : X(n) \longrightarrow \underbrace{I \times \ldots \times I}_{n}$ is a simplicial isomorphism

Corollary:

$\Delta^m \times \Delta^n \sim \Delta^{m+n}$

Proof: $\Delta^m \sim \underbrace{I \times \ldots \times I}_{m}$ $\qquad$ $\Delta^n \sim \underbrace{I \times \ldots \times I}_{n}$

$\Delta^m \times \Delta^n = \underbrace{I \times \ldots \ldots \times I}_{m+n}$.

$\chi(X \times Y) = \chi(X)\,\chi(Y)$

Let $\mathcal{P}(d,k)$ be the statement $\chi(X \times \Delta^n) = \chi(X) (= \chi(X)\chi(\Delta^n))$.
when X is a finite complex of dimension d with exactly k d-simplicies.

$\mathcal{P}(d)$ is the statement that $\chi(X \times \Delta^n) = \chi(X)$ for X finite of dimension d.

$\mathcal{P}(d) = \bigwedge_{i=1}^{\infty} \mathcal{P}(d,k) = \mathcal{P}(d+1) = 0.$

Want to prove each $\mathcal{P}(d)$ is true.

Induction Base $\mathcal{P}(0)$

Induction Step. $\mathcal{P}(d-1) \wedge \mathcal{P}(d,k) \Rightarrow \mathcal{P}(d,k+1)$.

First need to establish $\mathcal{P}(0)$:

$\chi(X_+ \cup X_-) + \chi(X_+ \cap X_-) = \chi(X_+) + \chi(X_-)$

Special case that $X_+ \cap X_- = \emptyset$

so $X = X_+ \sqcup X_-$ then $\chi(X) = \chi(X_+) + \chi(X_-)$

So if $X = X_1 \sqcup X_2 \sqcup \dots \sqcup X_k$ and $X_1 \cong X_2 \cong \dots \cong X_k$ then $\chi(X) = k\,\chi(X_1)$.

Proof of $\mathcal{P}(0)$: Let X be a finite complex of dimension 0.

$X = \{v_1\} \sqcup \{v_2\} \sqcup \dots \sqcup \{v_k\}$ ($v_1 \dots v_k$ distinct points).

$X \times \Delta^n = \{v_1\} \times \Delta^n \sqcup \dots \sqcup \{v_k\} \times \Delta^n$ $\quad (\{v_i\} \times \Delta^n \cong \{v_j\} \times \Delta^n)$

$\chi(X \times \Delta^n) = \sum_{i=1}^{k} \chi(\{v_i\} \times \Delta^n) = k\,\chi(\Delta^n)\ (= k).$

But $\chi(X) = k$.

So $\chi(X \times \Delta^n) = \chi(X)\,\chi(\Delta^n)$ when $\dim X = 0$ $\quad$ QED $\mathcal{P}(0)$.

Proof of induction step: $\mathcal{P}(d-1) \wedge \mathcal{P}(d,k) \Rightarrow \mathcal{P}(d,k+1)$

Suppose $\dim(X) = d$ and X has exactly $k+1$ simplicies of dim d.

Write $X = X_- \cup \Delta^d$ (where X_- has exactly k d-simplicies)

and $X_- \cap \Delta^d \subset \partial\Delta^d$, $\dim(X_- \cap \Delta^d) \leq d-1$.

Consider $X \times \Delta^n = (X_- \cup \Delta^d) \times \Delta^n$

$$= (X_- \times \Delta^n) \cup (\Delta^d \times \Delta^n)$$
$$(X_- \cap \Delta^d) \times \Delta^n$$

$\chi(X \times \Delta^n) = \chi(X_- \times \Delta^n) + \chi(\Delta^d \times \Delta^n) - \chi((X_- \cap \Delta^d) \times \Delta^n)$

$\chi(X_-) + 1 - \chi(X_- \cap \Delta^d)$

Apply induction hypothesis

$\chi(X_- \times \Delta^n) = \chi(X_-)\,\chi(\Delta^n) = \chi(X_-)$ $\mathcal{P}(d,k)$

$\chi(\Delta^d \times \Delta^n) = \chi(\Delta^{d+n}) = 1$

$\chi((X_- \cap \Delta^d) \times \Delta^n) = \chi(X_- \cap \Delta^d)\chi(\Delta^n) = \chi(X_- \cap \Delta^d)$ $P(d-1)$

But $X = X_- \cup \Delta^d$

$\chi(X) = \chi(X_-) + \chi(\Delta^d) - \chi(X_- \cap \Delta^d)$

$\chi(X) = \chi(X_-) + 1 - \chi(X_- \cap \Delta^d)$ $**$

Comparing $*$ and $**$ we see that $\chi(X \times \Delta^n) = \chi(X)$

True for all finite complexes X.

Fix a finite complex X and consider the following ~~complex~~ statements.

- $Q(d,k)$: $\chi(X \times Y) = \chi(X)\chi(Y)$ where Y is a finite complex of dimension d having exactly k simplices of dimension d.
- $Q(d)$: $\chi(X \times Y) = \chi(X)\chi(Y)$ for all finite complexes of dim $\leq d$.

Induction Base : $Q(0)$

Induction Step : $Q(d-1) \wedge Q(d,k) \Rightarrow Q(d,k+1)$

Proof of $Q(0)$: Let $Y = \{v_1\} \sqcup \dots \sqcup \{v_k\}$ $\chi(Y) = k$.

$X \times Y = X \times \{v_1\} \sqcup \dots \sqcup X \times \{v_k\}$

$\chi(X \times Y) = \sum_{i=1}^{k} \chi(X \times \{v_i\}) = k\,\chi(X) = \chi(X)\chi(Y)$.

So $Q(0)$ is true.

Proof of induction step: So suppose $Q(d-1) \wedge Q(d,k)$

Let Y be a d-dim complex with exactly $k+1$ simplicies of dim d

Write $Y = Y_- \underset{Z}{\cup} \Delta^d$ $Z = Y_- \cap \Delta^d$ $Z \subseteq \partial\Delta^d$ $\dim(Z) \leq d-1$.

Take product with X:

$X \times Y = (X \times Y_-) \underset{X \times Z}{\cup} (X \times \Delta^d)$ so

$\chi(X \times Y) = \chi(X \times Y_-) + \chi(X \times \Delta^d) - \chi(X \times Z)$

$= \underbrace{\chi(X)\chi(Y_-)}_{Q(d,k)} + \underbrace{\chi(X)}_{\text{Previous Result}} - \underbrace{\chi(X)\chi(Z)}_{Q(d-1)}$ $***$

But $\chi(Y) = \chi(Y_-) + \chi(\Delta^d) - \chi(Z)$.

So $\chi(X)\chi(Y) = \chi(X)\chi(Y_-) + \chi(X) - \chi(X)\chi(Z)$ $****$

Comparing $***$ and $****$ we get $\chi(X \times Y) = \chi(X)\chi(Y)$.

For all finite complexes X, Y.

The internal logic of the proof is based on the following observation:

If X has dimension d we can write

$X = X^{(d-1)} \cup (D_1 \cup \dots \cup D_{k+1})$ where $D_1 \dots D_{k+1}$ are the d-simplices of X

$D_i \cong \Delta^d$.

So $X_- = X^{(d-1)} \cup (D_1 \cup \dots \cup D_k)$

$X \cong X_- \cup \Delta^d \quad X_- \cap \Delta^d \subset X^{(d-1)}$

$P(d) \equiv P(d+1, 0)$.

$P(0) \equiv P(1,0) \Rightarrow P(1,1) \Rightarrow P(1,2) \Rightarrow \dots \Rightarrow P(1,k) \Rightarrow \dots \Rightarrow P(1)$

$P(1) \equiv P(2,0) \Rightarrow P(2,1) \Rightarrow P(2,2) \Rightarrow \dots \Rightarrow P(2,k) \Rightarrow \Rightarrow P(2)$.

$P(2) \equiv P(3,0) \Rightarrow P(3,1) \dots$

$\chi(S^3 \times S^5) = \chi(S^3)\chi(S^5) = 0$

$\chi(S^4 \times S^4) = \chi(S^4)\chi(S^4) = 2 \times 2 = 4$

$S^3 \times S^5 \not\simeq S^4 \times S^4$.

$H_n(X \times Y : \mathbb{F}) = \bigoplus_{r=0}^{n} H_r(X : \mathbb{F}) \otimes H_{n-r}(Y : \mathbb{F})$

$\mathbb{F}$ field Künnett Thm.

Mayer-Vietoris Theorem

Not examinable

Geometric Form:

$X = X_+ \cup X_-$ $\exists$ long sequence in homology.

$H_n(X_+) \oplus H_n(X_-) \to H_n(X) \xrightarrow[\partial]{} H_{n-1}(X_+ \cap X_-) \to X_{n-1}(X_+) \oplus X_{n-1}(X_-) \to H_{n-1}(X) \xrightarrow[\partial]{} H_{n-2}(X_+ \cap X_-) \dots$

Difficult bit.

Algebraic Form:

Given an exact sequence of chain complexes

$0 \to A_* \xrightarrow{i} B_* \xrightarrow{\varphi} C_* \to 0$

Then $\exists$ long exact sequence

$H_{n+1}(B) \xrightarrow[\varphi_*]{} H_{n+1}(C) \xrightarrow[\partial]{} H_n(A) \xrightarrow[i_*]{} H_n(B) \xrightarrow[\varphi_*]{} H_n(C) \xrightarrow[\partial]{} H_{n-1}(A) \xrightarrow[i_*]{} \dots$

difficult bit.

Algebraic Form $\Rightarrow$ Geometric Form.

Given ~~X~~ $X = X_+ \cup X_-$ $\exists$ exact sequence of chain complexes

$0 \to C_*(X_+ \cap X_-) \to C_*(X_+) \oplus C_*(X_-) \to C_*(X) \to 0$ $\xRightarrow{\text{now}}$ apply algebraic form.

$\begin{pmatrix} x_+ \\ x_- \end{pmatrix} \longmapsto x_+ + x_-$

$z \longmapsto \begin{pmatrix} z \\ -z \end{pmatrix}$

Given following commutative diagram:

$$\begin{tikzcd}
0 \arrow[r] & A_{n+2} \arrow[r, "i"] \arrow[d, "\partial"] & B_{n+2} \arrow[r, "p"] \arrow[d, "\partial"] & C_{n+2} \arrow[r] \arrow[d, "\partial"] & 0 \\
0 \arrow[r] & A_{n+1} \arrow[r, "i_{n+1}"] \arrow[d, "\partial^A_{n+1}"] & B_{n+1} \arrow[r, "p_{n+1}"] \arrow[d, "\partial^B_{n+1}"] & C_{n+1} \arrow[r] \arrow[d, "\partial^C_{n+1}"] & 0 \\
0 \arrow[r] & A_n \arrow[r, "i_n"] \arrow[d, "\partial^A_n"] & B_n \arrow[r, "p_n"] \arrow[d, "\partial^B_n"] & C_n \arrow[r] \arrow[d, "\partial^C_n"] & 0 \\
0 \arrow[r] & A_{n-1} \arrow[r, "i_{n-1}"] & B_{n-1} \arrow[r, "p_{n-1}"] & C_{n-1} \arrow[r] & 0
\end{tikzcd}$$

Rows are exact.

Got obvious maps induced on homology.

$$H_n(A) \xrightarrow[\underset{H_n(i)}{\shortparallel}]{i_*} H_n(B) \xrightarrow{p_*} H_n(C)$$

Proposition:

This sequence is exact for each n.

Proof: First observe that $p \circ i = 0$. (exactness)

So $p_* \circ i_* = 0$.

So $\mathrm{Im}(i_*) \subset \mathrm{Ker}(p_*)$

Let $[z] \in \mathrm{Ker}(p_*)$ $[z] \in H_n(B)$ and $p_*[z] = 0$.

$[z] = z + \mathrm{Im}\, \partial^B_{n+1}$ where $z \in B_n$ and $\partial^B_n(z) = 0$.

$p_*[z] = 0$ means $p_n[z] \in \mathrm{Im}\, \partial^C_{n+1}$

So I'm given $z \in B_n$: $\partial^B_n(z) = 0$ and $p_n(z) = \partial^C_{n+1}(w)$ for some $w \in C_{n+1}$

$p_{n+1} : B_{n+1} \to C_{n+1}$ is surjective so choose $y \in B_{n+1}$

$p_{n+1}(y) = w$.

$\partial^C_{n+1} p_{n+1}(y) = \partial^C_{n+1}(w) = p_n(z)$ so $p_n \partial^B_{n+1}(y) = p_n(z)$

So $z - \partial^B_{n+1}(y) \in \mathrm{Ker}(p_n) = \mathrm{Im}(i_n)$.

So choose $\alpha \in A_n$: $i_n(\alpha) = z - \partial^B_{n+1}(y)$

Claim that $\alpha \in Z_n(A) = \mathrm{Ker}(\partial^A_n)$ Why?

$\partial^B_n i_n(\alpha) = \partial^B_n(z) - \partial_n \partial_{n+1}(y) = 0 - 0$.

$i_{n-1} \partial^A_n(\alpha) = 0$ But i_{n-1} is injective (by exactness)

So $\partial^A_n(\alpha) = 0$ $\alpha \in Z_n(A)$

$i_n(\alpha) = z - \partial_{n+1}(y)$ $[\alpha] \in H_n(A)$

Take homology classes

$i_*([\alpha]) = [z - \partial^B_{n+1}(y)] = [z]$ ($[\partial(\xi)] = 0$

ie $[z] \in \mathrm{Im}(i_*)$

$H_n = Z_n / \mathrm{Im}\, \partial$ QED.

~~So given short exact sequence of chain complexes~~

0

So given a short exact sequence of chain complexes.

$$0 \longrightarrow A_* \xrightarrow{i} B_* \xrightarrow{p} C_* \longrightarrow 0$$

I get an exact sequence

$$H_{n+1}(A) \xrightarrow{i_*} H_{n+1}(B) \xrightarrow{p_*} H_{n+1}(C)$$

∂ = connecting homomorphism.

$$H_n(A) \xrightarrow{i_*} H_n(B) \xrightarrow{p_*} H_n(C)$$

$$H_{n-1}(A) \xrightarrow{i_*} H_{n-1}(B) \xrightarrow{p_*} H_{n-1}(B).$$

The difficulty is to construct a homomorphism $\partial : H_{n+1}(C) \longrightarrow H_n(A)$.

"Snake Lemma"

First stage: Construct homomorphism

$\partial : Z_{n+1}(C) \longrightarrow H_n(A)$

$\quad \ker(\partial^C_{n+1})$.

Choose $z \in Z_{n+1}(C)$: $\partial^C_{n+1}(z) = 0$.

p_{n+1} is surjective so choose $b \in B_{n+1}$: $p_{n+1}(b) = z$.

consider $\partial^B_{n+1}(b)$.

Claim : $\partial^B_{n+1}(b) \in \ker(p_n)$.

$p_n \partial^B_{n+1}(b) = \partial^C_{n+1} p_n(b) = \partial^C_{n+1}(z) = 0$

So $\partial^B_{n+1}(b) \in \operatorname{Im}(i_n)$. Choose $a \in A_n$ st $i_n(a) = \partial^B_{n+1}(b)$

Claim that $a \in Z_n(A) = \ker(\partial^A_n)$.

$\partial_n i_n(a) = \partial_n \partial_{n+1}(b) = 0$

$i_{n-1} \partial_n(a) = 0$ But i_{n-1} injective so $\partial_n(a) = 0$.

To summerise: given $z \in Z_{n+1}(C)$ I've produced (via a single choice $b \in B_{n+1}$) an element $a \in Z_n(A)$ and hence a homologly class $[a] \in H_n(A)$.

Claim: $z \longmapsto [a]$ is a well defined (linear) map and is independent of choice of b.

$z \longmapsto [a]$ is a mapping

$z \rightsquigarrow a$ is not a mapping (dependent on b).

So suppose $b' \in B_{n+1}$, $a' \in A_n$ $\quad i_n(a') = \partial_{n+1}(b')$ $\quad p_{n+1}(b') = z$.

Claim: $[a] = [a']$

$p_{n+1}(b - b') = p_{n+1}(b) - p_{n+1}(b') = z - z = 0$

So $\exists \alpha \in A_{n+1}$ $\quad i_{n+1}(\alpha) = b - b'$

$i_n \partial_{n+1}(\alpha) = \partial_{n+1} i_{n+1}(\alpha) = \partial_{n+1}(b - b') = \partial_{n+1}(b) - \partial_{n+1}(b')$

$\qquad\qquad\qquad\qquad\qquad\qquad = i_n(a) - i_n(a')$

$i_n(a) = i_n(a') + \partial_{n+1}(\alpha)$ ~~But a~~

But i_n injective so $a = a' + \partial_{n+1}(\alpha)$ so $[a] = [a']$ QED.

So far $\partial : Z_{n+1}(C) \longrightarrow H_n(A)$ well defined (connecting homomorphism).

$\partial[z] = [i_n^{-1} \partial_{n+1} p_{n+1}^{-1}(z)]$ $i_n, p_{n+1}, \partial_{n+1}$ linear $\Rightarrow \partial$ is linear.

Suppose $z \in B_{n+1}(C)$ ie $\exists\ \hat{z} \in C_{n+2} : \partial_{n+1}(\hat{z}) = z$.

Claim: $\partial[z] = 0$.

Choose $\hat{b} \in B_{n+2}$, $p_{n+2}(\hat{b}) = \hat{z}$

So now $p_{n+1}\partial_{n+2}(\hat{b}) = z$

So in constructing $\partial[z]$ I can take any b to be $b = \partial_{n+2}(\hat{b})$.

So then $i_n(a) = \partial_{n+1}(b) = \partial_{n+1}\partial_{n+2}(\hat{b}) = 0$.

i_n injective so $a = 0$ QED.

So now $\partial : Z_{n+1}(C) \longrightarrow H_n(A)$ that if $z \in B_{n+1}(C)$ $\partial[z] = 0$.

So ∂ induces a homomorphism $\partial : H_{n+1}(C) \longrightarrow H_n(A)$

$H_{n+1}(C) = Z_{n+1}(C)/B_{n+1}(C)$.

$\partial(z + \operatorname{Im}\partial^B_{n+2}) = \partial[z]$.

So now:

$$H_{n+1}(B) \xrightarrow{p_*} H_{n+1}(C) \xrightarrow{\partial} H_n(A) \xrightarrow{i_*} H_n(B).$$

Claim: this sequence is exact.

$\partial[z] = [i_n^{-1} \partial^B_{n+1} p_{n+1}^{-1}(z)]$.

Four conditions to check:

1. $\partial p_* = 0$
2. $\operatorname{Ker}(\partial) \subset \operatorname{Im}(p_*)$
3. $i_* \partial = 0$
4. $\operatorname{Ker}(i_*) \subset \operatorname{Im}\partial$.

1). $\partial p_*[b] = [i_n^{-1} \partial^B_{n+1} p_{n+1}^{-1} p_{n+1}(b)]$. $b \in Z_{n+1}(B)$

$= [i_n^{-1} \partial^B_{n+1}(b)]$ $\partial_{n+1}(b) = 0$

$= 0$

3). $i_*[i_n^{-1} \partial_{n+1} p_{n+1}^{-1}(z)] = [\partial_{n+1}(?)] = 0$

2). Suppose $[z] \in H_{n+1}(C)$ is such that $\partial[z] = 0$

So $\exists\ a \in A_n$ $\exists b \in B_{n+1}$ $i_n(a) = \partial_{n+1}(b)$ $p_{n+1}(b) \in z$. and $a \in B_n(A)$

ie $\exists\ \alpha \in A_{n+1}$ $a = \partial_{n+1}(\alpha)$.

So $\partial_{n+1} i_{n+1}(\alpha) = \partial_{n+1}(b)$

$i_n \partial_{n+1}(\alpha) = \partial_{n+1} i_{n+1}(\alpha) = \partial_{n+1}(b)$

put $b' = b - i_{n+1}(\alpha)$

$p_{n+1}(b') = p_{n+1}(b) - p_{n+1} i_{n+1}(\alpha)$

$= p_{n+1}(b) = z.$

and now $\partial_{n+1}(b') = \partial_{n+1}(b) - \partial_{n+1} i_{n+1}(\alpha) = 0$

So $b' \in Z_{n+1}(B)$ and $p_{n+1}(b') = z$.

So $[z] \in Im(p_{n+1})_*$ QED.

4) Suppose $[a] \in H_n(A)$ is such that $i_*[a] = 0$.

Got to show $\exists [z] \in H_{n+1}(C) : \partial[z] = [a]$

So I have $a \in Z_n(A)$ ie $\partial_n(a) = 0$ is such that $i_*[a] = 0$

ie $\exists b \in B_{n+1}$ st $i_n(a) = \partial_{n+1}(b)$

Put $z = p_{n+1}(b) \in C_{n+1}$

Claim: $\partial_{n+1}(z) = 0$.

$\partial_{n+1}(z) = \partial_{n+1} p_{n+1}(b)$

$= p_n \partial_{n+1}(b) = p_n i_n(a) = 0.$

Now consider $\partial[z]$. I have $b \in B_{n+1}$, $p_{n+1}(b) = z$ and $i_n(a) = \partial_{n+1}(b)$

So $[a] = \partial[z]$.

QED.

This completes the proof.

So given an exact sequence of chain complexes

$$0 \longrightarrow A_* \xrightarrow{i} B_* \xrightarrow{p} C_* \longrightarrow 0$$

we get a long exact sequence.

$$H_{n+1}(B) \xrightarrow{p_*} H_{n+1}(C) \xrightarrow{\partial} H_n(A) \xrightarrow{i_*} H_n(B) \xrightarrow{p_*} H_n(C) \xrightarrow{\partial} H_{n-1}(A)$$

This the Algebraic MV Theorem.

Back to geometric Form.

Special Case: $X = X_+ \sqcup X_-$

ie $X_+ \cap X_- = \emptyset$ $C_*(\emptyset) \equiv 0$ $H_*(\emptyset) \equiv 0$

$$0 \longrightarrow H_n(C_*(X_+) \oplus C_*(X_-)) \xrightarrow{\cong} H_n(X) \longrightarrow 0.$$

To complete proof in Geometric case I need to show:

Addendum:

Suppose $B_* = A_* \oplus C_*$ direct sum of chain complexes

$$\partial_n^B = \begin{pmatrix} \partial_n^A & 0 \\ 0 & \partial_n^C \end{pmatrix}$$

Then $Z_n(B) = Z_n(A) \oplus Z_n(C)$

$B_n(B) = B_n(A) \oplus B_n(C)$.

So $H_n(B) = \dfrac{(Z_n(A) \oplus Z_n(C))}{B_n(A) \oplus B_n(C)}$

$$\cong \frac{Z_n(A)}{B_n(A)} \oplus \frac{Z_n(C)}{B_n(C)}$$

$$\cong H_n(A) \oplus H_n(\cancel{B}C).$$

So if $X = X_+ \sqcup X_-$

$H_n(X) \cong H_n(X_+) \oplus H_n(X_-)$

So in general an arbitrary finite simplicial complex X is a disjoint union

$X = X_1 \sqcup X_2 \sqcup \ldots \sqcup X_m$

where $X_1 \ldots X_m$ maximal connected subcomplexes of X

$H_*(X) = H_*(X_1) \oplus H_*(X_2) \oplus \ldots \oplus H_*(X_m)$.