3204/M204 Representation Theory Notes

Based on the 2013 autumn lectures by Mr J Nadim

MATH3204 – Representation Theory.

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Office hour – Mon 2pm, Room 603.

Recommended text: Curtis and Reiner, *Representations of finite groups and associative algebras*.
(refrence only) James and Libeck, *Group Representations and Characters*

COMPLEX REPRESENTATION THEORY.

Goal of course: To represent any finite group G as a group of invertible matrices over $\mathbb{F}$ (i.e. we want to give G the same structure).

$\boxed{\text{Definition}}$ A representation is a type of group homomorphism.

$\boxed{\text{Definition}}$ let $\mathbb{F}$ be a field. Then we define $GL_n(\mathbb{F}) = \{A \in M_n(\mathbb{F}) : \det A \neq 0\} = \{\text{invertible } n\times n \text{ matrices over } \mathbb{F}\}$.

Recall – $GL_n(\mathbb{F})$ is a group under matrix multiplication.

$\boxed{\text{Definition}}$ let $\mathbb{F}$ be a field, G be a finite group and V be an $\mathbb{F}$-vector space of dimension n (i.e. $V = \mathbb{F}^n$).

Define an $\mathbb{F}$-representation of G, ρ, as the group homomorphism $\rho : G \to GL_n(V) = \{\phi : V \to V \text{ st. } \phi \text{ is an invertible } \mathbb{F}\text{-linear map}\} = \{\phi : V\to V \text{ st. } \phi \text{ is an automorphism}\}$

If we choose a basis for V, say $B = \{e_1, \dots, e_n\}$, then $GL_n(V) \cong GL_n(\mathbb{F})$, ~~we~~

So in this way, an $\mathbb{F}$-representation of G is $\rho: G \to GL_n(\mathbb{F})$, where ρ is a group homomorphism i.e. $\rho(gh) = \rho(g)\rho(h)$ $\forall g,h \in G$.

Definition! If $\dim(V) = n$, $n \geq 1$, then n is called the degree/dimension of the representation ρ.

Examples of representations –

1. (Trivial representation) let G be any group and define $\rho : G \to GL_n(\mathbb{C})$ by $\rho(g) = I_n = \begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1\end{pmatrix}$ $\forall g \in G$.

2. (Trivial representation for cyclic groups over $\mathbb{C}$).

   Fix n and let $C_m = \langle x : x^m = 1\rangle$, then define $\rho : C_m \to GL_n(\mathbb{C})$ by $\rho(x) = I_n$. Note that we need to know where the generator of our group goes (here, x).

   Then $\rho(x^k) = \overbrace{\rho(x)\times\cdots\times\rho(x)}^{k \text{ times}} = \overbrace{I_n \times \cdots \times I_n}^{k\text{ times}} = I_n$ $\forall k \in [1,m]$.

   For instance, take $C_3 = \langle 1, x, x^2\rangle$ by $\rho(x) = \begin{pmatrix}1&0\\0&1\end{pmatrix} = I_2$ i.e. $\rho : C_3 \to GL_2(\mathbb{C})$. Then $\rho(x^3) = \rho(x)\rho(x)\rho(x) = \begin{pmatrix}1&0\\0&1\end{pmatrix}\begin{pmatrix}1&0\\0&1\end{pmatrix}\begin{pmatrix}1&0\\0&1\end{pmatrix} = I_2$.

   (this is well-defined as $x^3 = 1$, so we do expect that $\rho(x^3) = \rho(1) = I_2$.

3. (A non-trivial representation of cyclic groups)

   let $\rho : C_m \to GL_n(\mathbb{F})$ be defined by $\rho(x) = A$ for some $A \in GL_n(\mathbb{F}) \neq I_n$. For ρ to be a complex representation, A must satisfy the Group laws:

   $x^m = 1$ for C_m, so we need $\rho(x^m) = I_n$ i.e. we need A s.t. $\rho(x^m) = \rho(x)^m = A^m = I_n$.

   For instance, we can take $A = \underbrace{\begin{pmatrix}\xi & \cdots & 0\\ \vdots & \ddots & \vdots \\ 0 & \cdots & \xi\end{pmatrix}}_{n\times n}$, where $\xi = e^{\frac{2\pi i}{m}}$. Then $A^m = I_n = \begin{pmatrix}1 & \cdots & 0\\ \vdots & \ddots & \vdots\\ 0 & \cdots & 1\end{pmatrix}$

4. (Trivial representation for D_{2n}).

   $D_{2n} = \langle x, y \mid x^n = y^2 = 1,\ yx = x^{n-1}y\rangle$. Note that D_{2n} has two generators, so we need to determine where each is sent.

   Define the trivial representation as $\rho : D_{2n} \to GL_k(\mathbb{F})$ by $x \mapsto I_k$, $y \mapsto I_k$, $1 \mapsto I_k$. Note that if $k=1$, this means that $x\mapsto 1$, $y \mapsto 1$, $1\mapsto 1$.

5. (A non-trivial 1D representation of D_6)

   $D_6 = \{1, x, x^2, y, xy, x^2y\} = \langle x, y \mid x^3 = y^2 = 1, yx = x^2y\rangle$. let $\rho : D_6 \to GL_1(\mathbb{C}) = \mathbb{C}^* = \mathbb{C}\setminus\{0\}$? be defined by $x \mapsto 1$, $y\mapsto -1$, $1\mapsto 1$.

   To verify that this is a representation, we need to check the group laws: i.e. must show $\rho(x^3) = \rho(y^2) = \rho(1) = 1$ and $\rho(yx) = \rho(x^2y)$.

   check: $\rho(x^3) = \rho(x)^3 = 1\times1\times1 = 1$ ✓ $\rho(y^2) = \rho(y)\cdot\rho(y) = (-1)\times(-1) = 1$ ✓ $\rho(yx) = \rho(y)\rho(x) = (-1)\times(1) = -1$, $\rho(x^2y) = \rho(x)\rho(x)\rho(y) = 1\times1\times(-1) = -1$ ✓.

6. (A non-trivial 2D representation of $D_6/\mathbb{C}$).

   Represent D_6: let $\rho : D_6 \to GL_2(\mathbb{C})$ by $1 \mapsto \begin{pmatrix}1&0\\0&1\end{pmatrix}$, $x\mapsto\begin{pmatrix}0&-1\\1&-1\end{pmatrix}$, $y \mapsto \begin{pmatrix}0&1\\1&0\end{pmatrix}$. We need to check that ρ defines a representation:

   $\rho(x^3) = \rho(x)^3 = \begin{pmatrix}0&-1\\1&-1\end{pmatrix}\begin{pmatrix}0&-1\\1&-1\end{pmatrix}\begin{pmatrix}0&-1\\1&-1\end{pmatrix} = \begin{pmatrix}-1&1\\-1&0\end{pmatrix}\begin{pmatrix}0&-1\\1&-1\end{pmatrix} = \begin{pmatrix}1&0\\0&1\end{pmatrix} = I_2$ ✓ $\rho(y^2) = \rho(y)^2 = \begin{pmatrix}0&1\\1&0\end{pmatrix}\begin{pmatrix}0&1\\1&0\end{pmatrix} = \begin{pmatrix}1&0\\0&1\end{pmatrix} = I_2$ ✓

   $\rho(yx) = \rho(y)\rho(x) = \begin{pmatrix}0&1\\1&0\end{pmatrix}\begin{pmatrix}0&-1\\1&-1\end{pmatrix} = \begin{pmatrix}1&-1\\0&-1\end{pmatrix}$, $\rho(x^2y) = \rho(x)^2\rho(y) = \begin{pmatrix}-1&1\\-1&0\end{pmatrix}\begin{pmatrix}0&1\\1&0\end{pmatrix} = \begin{pmatrix}1&-1\\0&-1\end{pmatrix}$ $\Rightarrow$ $\rho(yx) = \rho(x^2y)$ ✓ The necessary group laws are satisfied.

7. (Another non-trivial 2D representation of D_6).

   Define $\rho : D_6 \to GL_2(\mathbb{C})$ by $1\mapsto\begin{pmatrix}1&0\\0&1\end{pmatrix}$, $x \mapsto \begin{pmatrix}w&0\\0&w^2\end{pmatrix}$, $y\mapsto\begin{pmatrix}0&1\\1&0\end{pmatrix}$ where $w = e^{\frac{2\pi i}{3}}$. check that ρ defines a representation:

   $\rho(x^3) = \begin{pmatrix}w&0\\0&w^2\end{pmatrix}^3 = \begin{pmatrix}w^3&0\\0&w^6\end{pmatrix} = \begin{pmatrix}1&0\\0&1\end{pmatrix} = I_2$ ✓ $\rho(y^2) = \begin{pmatrix}0&1\\1&0\end{pmatrix}\begin{pmatrix}0&1\\1&0\end{pmatrix} = \begin{pmatrix}1&0\\0&1\end{pmatrix} = I_2$ ✓

   $\rho(yx) = \rho(y)\rho(x) = \begin{pmatrix}0&1\\1&0\end{pmatrix}\begin{pmatrix}w&0\\0&w^2\end{pmatrix} = \begin{pmatrix}0&w^2\\w&0\end{pmatrix}$, $\rho(x^2y) = \rho(x)^2\rho(y) = \begin{pmatrix}w^2&0\\0&w^4\end{pmatrix}\begin{pmatrix}0&1\\1&0\end{pmatrix} = \begin{pmatrix}0&w^2\\w^4&0\end{pmatrix} = \begin{pmatrix}0&w^2\\w&0\end{pmatrix} = \rho(yx)$ ✓.

8. (A 3D representation of S_n)

   Recall – S_n is the group of invertible permutations on $\{1,\dots,n\}$ and $|S_n| = n!$

   Take $n=3$, so consider $S_3 = \{\overset{id}{(1)}, (1\,2), (1\,3), (2\,3), (1\,2\,3), (1\,3\,2)\}$. Recall that $S_3 \cong D_6$ by setting $(1)\mapsto 1$, $(1\,2\,3)\mapsto x$, $(1\,2)\mapsto y$.

We now consider the permutation representation in 3D (works for any n). Let $V = \mathbb{F}^3$ be a 3D vector space with the standard basis $B = \{e_1, e_2, e_3\}$.

Define the representation $\rho: S_3 \to GL_3(\mathbb{C})$ by $\rho(\sigma)\cdot e_i = e_{\sigma(i)}$ $\forall \sigma \in S_3$. We check that this actually defines a representation.

① $\rho((1))\cdot e_i = e_{(1)(i)} = e_i$ $\forall i \Rightarrow \rho((1)) = 1$ ✓ ② $\rho(\sigma\tau)\cdot e_i = e_{\sigma\tau(i)} = e_{\sigma(\tau(i))} = \rho(\sigma)\cdot e_{\tau(i)} = \rho(\sigma)\rho(\tau)e_i \Rightarrow \rho(\sigma\tau) = \rho(\sigma)\rho(\tau)$ ✓ $\forall \sigma,\tau \in S_3$.

So, returning to our example, let $\sigma = (1\ 2\ 3)$, $\tau = (1\ 2)$ be the generators of S_3. Then, $\sigma^3 = \tau^2 = (1)$ and $\tau\sigma = \sigma^2\tau$.

We want to find the representation $\rho: S_3 \to GL_3(\mathbb{C})$. Clearly, $\rho((1)) = I_3$ by definition. We need to find (1) $\rho(\sigma)$ and (2) $\rho(\tau)$.

(1) For σ, $\rho(\sigma)\cdot e_1 = e_{(1\,2\,3)(1)} = e_2$, $\rho(\sigma)\cdot e_2 = e_{(1\,2\,3)(2)} = e_3$, $\rho(\sigma)\cdot e_3 = e_{(1\,2\,3)(3)} = e_1 \Rightarrow \rho(\sigma) = \begin{pmatrix} 0&0&1\\1&0&0\\0&1&0 \end{pmatrix} = (e_2\ e_3\ e_1) \in GL_3(\mathbb{C})$.

(2) For τ, $\rho(\tau)\cdot e_1 = e_{(1\,2)(1)} = e_2$, $\rho(\tau)\cdot e_2 = e_{(1\,2)(2)} = e_1$, $\rho(\tau)\cdot e_3 = e_{(1\,2)(3)} = e_3 \Rightarrow \rho(\tau) = \begin{pmatrix} 0&1&0\\1&0&0\\0&0&1 \end{pmatrix} = (e_2\ e_1\ e_3) \in GL_3(\mathbb{C})$.

To summarise, our representation is $\rho: S_3 \to GL_3(\mathbb{C})$ with $(1) \mapsto \begin{pmatrix} 1&0&0\\0&1&0\\0&0&1 \end{pmatrix}$, $(1\ 2\ 3) \mapsto \begin{pmatrix} 0&0&1\\1&0&0\\0&1&0 \end{pmatrix}$, $(1\ 2) \mapsto \begin{pmatrix} 0&1&0\\1&0&0\\0&0&1 \end{pmatrix}$.

For checking that this is a representation, we need $\rho(\sigma^3) = \rho(\tau^2) = I_3$, and $\rho(\tau\sigma) = \rho(\sigma^2\tau)$. This is verifiable by matrix algebra.

<u>Generalised permutation representation.</u>

We can generalise a permutation representation as follows: let G be a group, and let V be a finite-dimensional vector space over $\mathbb{F}$. Then

- Choose a basis for V
- Let X be a set and choose a basis vector for each element x, say e_x.
- Define the group action $G \times X \to X$ by $(g,x) \to g \circ x$ where $1 \circ x = x$, $g \circ (h \circ x) = (g \circ h) \circ x$ $\forall x \in X$, $\forall g,h \in G$.
- Then, let $V = \bigoplus_{x \in X} \mathbb{F}e_x$
- Define $\rho: G \to GL_n(\mathbb{F})$ by $\rho(g)\cdot e_x = e_{g \circ x}$ for any group action $\circ$. (e.g. conjugation $g \circ h = ghg^{-1}$).

9. We represent D_6 over $\mathbb{R}$ by $\rho: D_6 \to GL_2(\mathbb{R})$, where $x \mapsto \begin{pmatrix} \cos 2\pi/3 & -\sin 2\pi/3 \\ \sin 2\pi/3 & \cos 2\pi/3 \end{pmatrix}$ corresponding to rotation, and $y \mapsto \begin{pmatrix} 0&1\\-1&0 \end{pmatrix}$ corresponding to reflection.

In this course, we will get a complete answer to the question:

How many complex representations are there for G, up to conjugation, where

(1) G will be always finite (2) $\mathbb{F} = \mathbb{C}, \mathbb{R}, \mathbb{F}_p$ (3) we require that $|G| \nmid \operatorname{char}(\mathbb{F})$.

Note – $\operatorname{char}(\mathbb{F}_2) = 2$ as $1+1=0$ in $\mathbb{F}_2$, $\operatorname{char}(\mathbb{F}_p) = p$ as $\overbrace{1+1+\cdots+1}^{p \text{ times}} = 0$ in $\mathbb{F}_p$, $\operatorname{char}(\mathbb{C}) = \infty$.

(4) When $\mathbb{F} = \mathbb{C}$, then every matrix is diagonalisable.

$\forall g \in G$, $\exists n$ s.t. $g^n = 1$, so $\rho(g)^n = I_m \Rightarrow$ the matrix has characteristic polynomial $x^n - 1$, and over $\mathbb{C}$, $x^n - 1 = \prod (x - \xi_i)$ where ξ_i are roots of unity.

$\Rightarrow$ the polynomial splits into distinct linear factors $\Rightarrow$ every g has a minimal polynomial $\Rightarrow \rho(g)$ is diagonalisable.

Note – $x^n - 1$ does not generally split for $\mathbb{R}$, e.g. $x^3 - 1 = (x-1)(x^2+x+1)$.

(5) The theory breaks down for infinite groups. If $\rho: \mathbb{Z} \to GL_2(\mathbb{C})$. If $n \neq 0$, then $\begin{pmatrix} 1&n\\0&1 \end{pmatrix}$ is not diagonalisable.

If $x \mapsto \begin{pmatrix} 1&n\\0&1 \end{pmatrix}$, then $m(x) = (x-1)^2$.

$\boxed{\text{Ex}}$ Define the 3D representations $\sigma, \tau: D_6 \to GL_3(\mathbb{C})$ by $\sigma(x) = \begin{pmatrix} 0&0&1\\1&0&0\\0&1&0 \end{pmatrix}$, $\sigma(y) = \begin{pmatrix} 0&1&0\\1&0&0\\0&0&1 \end{pmatrix}$ and $\tau(x) = \begin{pmatrix} 0&-1&0\\1&-1&0\\0&0&1 \end{pmatrix}$, $\tau(y) = \begin{pmatrix} 0&1&0\\1&0&0\\0&0&1 \end{pmatrix}$.

Show that σ and τ are the same representation.

<u>Soln.</u> Let $V = \mathbb{F}^3 = \operatorname{span}_{\mathbb{F}}\{e_1, e_2, e_3\}$ where $\{e_1, e_2, e_3\}$ is the canonical basis. Now define a new basis $\{\phi_1, \phi_2, \phi_3\}$ by

$\phi_1 = e_1 + \frac{1}{2}e_3$, $\phi_2 = e_2 + \frac{1}{2}e_3$, $\phi_3 = -e_1 - e_2 - \frac{1}{2}e_3$.

(1) Apply $\tau(x)$ to $\{e_1, e_2, e_3\}$. $\tau(x)(e_1) = \begin{pmatrix} 0&-1&0\\1&-1&0\\0&0&1 \end{pmatrix}\begin{pmatrix} 1\\0\\0 \end{pmatrix} = \begin{pmatrix} 0\\1\\0 \end{pmatrix} = e_2$, $\tau(x)(e_2) = \begin{pmatrix} 0&-1&0\\1&-1&0\\0&0&1 \end{pmatrix}\begin{pmatrix} 0\\1\\0 \end{pmatrix} = \begin{pmatrix} -1\\-1\\0 \end{pmatrix} = -e_1 - e_2$, $\tau(x)(e_3) = \begin{pmatrix} 0&-1&0\\1&-1&0\\0&0&1 \end{pmatrix}\begin{pmatrix} 0\\0\\1 \end{pmatrix} = \begin{pmatrix} 0\\0\\1 \end{pmatrix} = e_3$.

(2) Apply $\tau(x)$ to $\{\phi_1, \phi_2, \phi_3\}$. $\tau(x)(\phi_1) = \tau_x(e_1 + \frac{1}{2}e_3) = \tau_x(e_1) + \frac{1}{2}\tau_x(e_3) = e_2 + \frac{1}{2}e_3 = \phi_2$, $\tau(x)(\phi_2) = \cdots = \phi_3$, $\tau(x)(\phi_3) = \cdots = \phi_1$.

$\Rightarrow$ the matrix representation of $\tau(x)$ w.r.t. new basis $\{\phi_1, \phi_2, \phi_3\}$ is $\begin{pmatrix} 0&0&1\\1&0&0\\0&1&0 \end{pmatrix}$, so $\tau(x)$ on $\{\phi_1, \phi_2, \phi_3\}$ is the same as $\sigma(x)$ on $\{e_1, e_2, e_3\}$.

We check the same for y: Then

(3) Apply $\tau(y)$ to $\{e_1, e_2, e_3\}$ $\tau(y)(e_1) = \begin{pmatrix} 0&1&0\\1&0&0\\0&0&1 \end{pmatrix}\begin{pmatrix} 1\\0\\0 \end{pmatrix} = \begin{pmatrix} 0\\1\\0 \end{pmatrix} = e_2$, $\tau(y)(e_2) = \begin{pmatrix} 1\\0\\0 \end{pmatrix} = e_1$, $\tau(y)(e_3) = \begin{pmatrix} 0\\0\\1 \end{pmatrix} = e_3$.

(4) Apply $\tau(y)$ to $\{\phi_1, \phi_2, \phi_3\}$ $\tau(y)(\phi_1) = e_2 + \frac{1}{2}e_3 = \phi_2$, $\tau(y)(\phi_2) = \tau(y)(e_2 + \frac{1}{2}e_3) = e_1 + \frac{1}{2}e_3 = \phi_1$, $\tau(y)(\phi_3) = -e_2 - e_1 - \frac{1}{2}e_3 = \phi_3$.

Thus, the matrix representation of $\tau(y)$ w.r.t. new basis $\{\phi_1, \phi_2, \phi_3\}$ is $\begin{pmatrix} 0&1&0\\1&0&0\\0&0&1 \end{pmatrix}$, q.e.d.

Recall – When you choose a vector space V, you implicitly select a standard basis, $\{e_1, \ldots, e_n\}$. We change the basis with conjugating matrix P.

This is an invertible automorphism of V.

3204-02.

Definition Let $A, B \in M_n(\mathbb{F})$. Then A is conjugate to B if $\exists\, T \in GL_n(\mathbb{F})$ s.t. $B = TAT^{-1} = T^{-1}AT$.

Definition Given two representations of G, $\rho: G \to GL_n(\mathbb{F})$ and $\rho': G \to GL_n(\mathbb{F})$, of degree n, we say that ρ' is conjugate/isomorphic/equivalent to ρ if $\exists\, T \in GL_n(\mathbb{F})$ st. $\rho'(g) = T\rho(g)T^{-1}$ $\forall g \in G$.

Examples –

1. Let $G = D_8$, $\mathbb{F} = \mathbb{C}$, and define $\rho: D_8 \to GL_2(\mathbb{C})$ by $x \mapsto \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, $y \mapsto \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. Take $T = \sqrt{2}\begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix}$. We have $T^{-1} = \sqrt{2}\,\frac{1}{-i-i}\begin{pmatrix} -i & -1 \\ -i & 1 \end{pmatrix} = \frac{\sqrt{2}}{2}\begin{pmatrix} i & 1 \\ i & -1 \end{pmatrix}$.

   Then $\rho'(x) = T^{-1}\rho(x)T = \frac{1}{4}\begin{pmatrix} i & 1 \\ i & -1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} -i & -1 \\ -i & 1 \end{pmatrix} =$

2. Let $G = C_2 = \langle x \mid x^2 = 1 \rangle$, $\rho: C_2 \to GL_2(\mathbb{C})$ by $x \mapsto \begin{pmatrix} -5 & 12 \\ -2 & 5 \end{pmatrix} = A$ st. $A^2 = I_2$. Let $T = \begin{pmatrix} 2 & -3 \\ 1 & -1 \end{pmatrix}$, so $T^{-1} = \frac{1}{-2+3}\begin{pmatrix} -1 & 3 \\ -1 & 2 \end{pmatrix} = \begin{pmatrix} -1 & 3 \\ -1 & 2 \end{pmatrix}$

   Then $T^{-1}AT = \begin{pmatrix} -1 & 3 \\ -1 & 2 \end{pmatrix}\begin{pmatrix} -5 & 12 \\ -2 & 5 \end{pmatrix}\begin{pmatrix} 2 & -3 \\ 1 & -1 \end{pmatrix} = \begin{pmatrix} -1 & 3 \\ 1 & -2 \end{pmatrix}\begin{pmatrix} 2 & -3 \\ 1 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. Then let $\rho': C_2 \to GL_2(\mathbb{C})$ be given by $\rho'(x) = T^{-1}AT$.

   Check the group law: $\rho'(x^2) = \rho'(x)\rho'(x) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = I_2 \Rightarrow$ representations are equivalent.

Definition Let $\rho: G \to GL_n(\mathbb{F})$ be a representation. The kernel of the representation is $Ker(\rho) = \{g \in G \mid \rho(g) = I_n\}$.

Note – If $Ker(\rho) \trianglelefteq G$, then by the first isomorphism theorem, $G/Ker(\rho) \cong Im(\rho) \leq GL_n(\mathbb{F})$.

If $Ker(\rho) = \langle 1 \rangle$, then $G/\langle 1 \rangle \cong G \cong GL_n(\mathbb{F})$ is the trivial kernel.

Definition If $Ker(\rho) = \langle 1 \rangle$, then ρ is a faithful representation.

Note – If $\rho: G \to GL_n(\mathbb{F})$, by definition $1 \mapsto Id$, so all other elements map to matrices different from identity $\Rightarrow \rho$ is injective.

Examples of faithful / non-faithful representations –

1. Trivial representation $\rho: G \to GL_n(\mathbb{F})$ by $\rho(g) = I_n$ $\forall g \in G$ is not faithful – all elements map to I_n.
2. 2D representation of D_6 over $\mathbb{R}$ is faithful: $\rho: D_6 \to GL_2(\mathbb{R})$ by $x \mapsto \begin{pmatrix} \cos 2\pi/3 & -\sin 2\pi/3 \\ \sin 2\pi/3 & \cos 2\pi/3 \end{pmatrix}$, $y \mapsto \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$.

   Since the transformations do not fix the vertices, we cannot get an identity transformation by construction.
3. The permutation representation of S_n, $\rho: S_n \to GL_m(\mathbb{C})$ by $\rho(\sigma)\cdot e_i = e_{\sigma(i)}$ is faithful.

   To establish that it is faithful, we must show that $Ker(\rho) = \langle 1 \rangle$. Let $\sigma \in S_n$ and suppose $\rho(\sigma)e_i = e_{\sigma(i)} = e_i$ $\forall i \Leftrightarrow \sigma(i) = i$ $\forall i \Leftrightarrow \sigma = (1) \Leftrightarrow Ker(\sigma) = \langle 1 \rangle$.
4. The representation $\rho: D_{2n} \to GL_n(\mathbb{F})$ by $x \to 1$, $y \to -1$ is not faithful. In fact, $Ker(\rho) = \{1, x, \ldots, x^{n-1}\} \cong C_n$:

   Let $g \in D_{2n}$, $g = x^i y^j$ for $0 \leq i \leq n$, $0 \leq j \leq 1$. $\rho(x^i y^j) = \rho(x)^i \rho(y)^j = 1^i(-1)^j = 1 \Leftrightarrow j = 0 \Leftrightarrow g = x^i$ i.e. $g \in C_n \Leftrightarrow Ker(\rho) = C_n$.

4 October 2013
Mr. Jamil NADIM.
Maths 500.

Outline of Topics:

① Constructing all $\mathbb{C}$-representations of finite G up to conjugacy. ② Character Theory ③ Tensor products.

④ Real Representation Theory ⑤ Frobenius reciprocity.

The Roadmap:

$$\begin{tikzcd} G \arrow[r] \arrow[d, "\text{(tough)}"'] & \text{construct group ring } \mathbb{F}[G]\text{, find structure} \arrow[d] \\ \mathbb{F}\text{-rep of } G \arrow[r, leftrightarrow, "1:1"] & \text{Modules over } \mathbb{F}[G] \end{tikzcd}$$

RINGS, MODULES AND ALGEBRAS.

Definition A ~~ring~~ ring R is a set with 2 operations $(R, +, \cdot, 0, 1)$ satisfying the following axioms.

(1) $a+b = b+a$ (2) $a+(b+c) = (a+b)+c = a+b+c$ (3) $\exists\, 0 \in R$, s.t. $\forall a$, $a+0 = 0+a = a$ (4) $\forall a \in R$, $\exists\, (-a) \in R$ s.t. $a+(-a) = 0$

Axioms (1)-(4) yield that $(R, +, 0)$ is an abelian group.

(5) $\exists\, 1 \in R$ st. $a \cdot 1 = 1 \cdot a = a$ (6) $(a \cdot b) \cdot c = a \cdot (b \cdot c)$ (7) $a(b+c) = a \cdot b + a \cdot c$ (8) $(a+b) \cdot c = a \cdot c + b \cdot c$.

(9) If $a \cdot b = b \cdot a$, then R is also called a commutative ring.

Note – Rings in general are not commutative. In this course they may be non-commutative.

Examples of rings –

1. Commutative Rings: $\mathbb{Z}, \mathbb{Q}, \mathbb{R}, \mathbb{C}, \mathbb{Z}/n\mathbb{Z}, \mathbb{F}[x], \mathbb{F}_p, \mathbb{F}[x]/I$
2. Non-commutative rings: Let R be a ring, define $M_n(R) = \{(a_{ij})_{1 \leq i \leq n, 1 \leq j \leq n} : a_{ij} \in R\}$ = $n \times n$ matrices over R.

   Then for $n \geq 2$, $M_n(R)$ is not commutative, even if R is a field i.e. $M_n(\mathbb{F})$.

   $M_n(R)$ is a ring: (addition) $(\alpha + \beta)_{ij} = \alpha_{ij} + \beta_{ij}$ (multiplication) $(\alpha\beta)_{ij} = \sum_{k=1}^{n} \alpha_{ik}\beta_{kj}$ (unit) $(1)_{ij} = I_n$ (zero) $(0)_{ij} = \{0\ \forall i,j\}$.

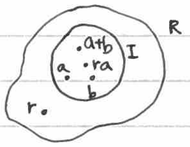
   Note – the standard basis for $M_n(R)$, $(E_{ij})_{rs} = \delta_{ir}\delta_{js}$ do not have inverses.
3. Upper triangular matrix rings $U \cdot T(R) = \left\{\begin{pmatrix} a_{11} & \cdots & a_{ij} \\ & \ddots & \\ 0 & & a_{nn} \end{pmatrix} : a_{ij} \in R\right\}$. It is non-commutative as $U \cdot T(R) \subset M_n(R)$.

$\boxed{\text{Definition}}$ Let R, S be two rings. Define product $R\times S$ as a ring by

① A set $R\times S = \{(r,s) : r\in R, s\in S\}$ with operations

② (addition) $(r_1,s_1)+(r_2,s_2) = (r_1+r_2, s_1 s_2)$ (multiplication) $(r_1,s_1)\cdot(r_2,s_2) = (r_1 r_2, s_1 s_2)$ (identity) "1" $= (1,1)$ (zero) "0" $= (0,0)$.

$\boxed{\text{Definition}}$ A subset $I \trianglelefteq R$ is called an ideal if



① $(I,+,0)$ is an additive group i.e. $\forall a,b\in I$, $a+b\in I$, and obeys

② Absorbency: $\forall a\in I$, $\forall r\in R$, $r.a\in I$

Examples of ideals —

1. If $R=\mathbb{Z}$. $0\mathbb{Z}\subseteq\mathbb{Z}$, $\mathbb{Z}\subset\mathbb{Z}$ (obvious). Also, $n\mathbb{Z}\subseteq\mathbb{Z}$ is an ideal e.g. $2\mathbb{Z} = \{2\cdot n : n\in\mathbb{Z}\}$.
2. $(p(x)) \subseteq \mathbb{F}[x]$, $(x^2-2)\subseteq\mathbb{Q}[x]$.

$\boxed{\text{Definition}}$ Let R,S be two rings, then a ring homomorphism is a map $\phi: R\to S$ s.t.

① $\phi(r_1+r_2) = \phi(r_1)+\phi(r_2)$ ② $\phi(r_1\cdot r_2) = \phi(r_1)\phi(r_2)$ } preserve $+,\cdot$.

③ $\phi(0_R) = 0_S$

Note — $\phi(1_R)=1_S$ does not always hold. For instance, in example of embedding: $\phi: M_2(\mathbb{R})\to M_3(\mathbb{R})$, $\begin{pmatrix}a&b\\c&d\end{pmatrix}\mapsto\begin{pmatrix}a&b&0\\c&d&0\\0&0&0\end{pmatrix}$, $I_2 \not\mapsto I_3$.

④ $\phi(1_R)=1_S$ if S is an integral domain or ϕ is an epimorphism (surjective).

⑤ If ϕ is bijective, then ϕ is called a ring isomorphism i.e. ϕ injective $\Leftrightarrow$ $\mathrm{Ker}(\phi)=\{0\}$ $\Leftrightarrow$ $\phi(a)=\phi(b)\Rightarrow a=b$; ϕ surjective $\Leftrightarrow$ $\mathrm{Im}(\phi)=S$ $\Leftrightarrow$ $\forall b\in S$ $\exists a\in R$ s.t. $\phi(a)=b$.

Examples of ring homomorphisms —

1. $\phi:\mathbb{Z}\to\mathbb{Z}/n\mathbb{Z}\cong\mathbb{Z}_n$. $a\mapsto a(\mathrm{mod}\, n) = \bar{a}$ is a ring homomorphism $\mathrm{Ker}(\phi)=n\mathbb{Z}$, $\mathrm{Im}(\phi)=\mathbb{Z}_n$.
   every ideal is the kernel of some ring homomorphism
   every normal subgroup is the kernel of some group homo.
2. There is no ring homomorphism $\phi:\mathbb{Z}_n\to\mathbb{Z}$.

$\boxed{\text{Definition}}$ Let R be a ring. Then a left module M over R is an abelian group $(M,+,\cdot)$ endowed with a map which is a left ring action,
$\varphi: R\times M\to M$
$(r,m)\mapsto r.m$ s.t. the following axioms are satisfied:

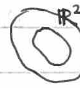
① $r(m+n) = rm+rn$ ② $1\cdot m = m$ ③ $r\cdot(sm) = (rs)m = rsm$ ④ $(r+s)m = rm+sm$ $\forall r,s\in R$, $\forall m,n\in M$.

Remark — Think of a module over R as a vector space over R, but there is no basis theorem in this degree of generality.

$\boxed{\text{Definition}}$ The (external) direct sum of modules, $M\oplus N$ is defined as a module, for M,N being two modules over R, by

① $M\oplus N = M\times N$ as sets, ② $(m_1,n_1)+(m_2,n_2) = (m_1+m_2, n_1+n_2)$ ③ $\lambda(m,n) = (\lambda m, \lambda n)$ ④ $\underline{0} = (0,0)$

$\boxed{\text{Definition}}$ Let $N\subseteq M$ be a subset, then N is called a submodule of M, written $N\leq M$, if


$\mathbb{R}^2\subseteq\mathbb{R}^3$.

① $0\in N$, $N\neq\emptyset$ ② $\forall n_1,n_2\in N$, $n_1+n_2\in N$ ③ $\forall r\in R$, $\forall n\in N$, $r\cdot n\in N$

Examples of modules —

1. Any vector space over a field $\mathbb{F}$ is a left module.
2. $\mathbb{R}^n$ is an $\mathbb{R}$-module $\varphi: \mathbb{R}\times\mathbb{R}^n\to\mathbb{R}^n$
3. $\mathbb{Q}$ is a $\mathbb{Z}$-module. $\varphi:\mathbb{Z}\times\mathbb{Q}\to\mathbb{Q}$, $(z,q)\mapsto z.q$
4. Any abelian group $A = \mathbb{Z}^r\times\mathbb{Z}_{n_1}\times\cdots\times\mathbb{Z}_{n_q} \overset{\text{Chinese Remainder Thm}}{\cong} \underset{(r\text{ is rank})}{\mathbb{Z}^r}\times\mathbb{Z}_{k_1}\times\cdots\times\mathbb{Z}_{k_s}$ (Smith Normal Form) with $k_1 | k_2 | \cdots | k_s$
   is a $\mathbb{Z}$-module. $\varphi:\mathbb{Z}\times\mathbb{Z}_3\to\mathbb{Z}_3 = A = \{0,1,2\}$.
5. Let R be a ring, then the principal ideal generated by $a\in R$ is $(a) = \{r\cdot a : r\in R\} = Ra$.
   Then by defining R-action, it is a R-submodule $Ra\subseteq R$, e.g. $(2) = 2\mathbb{Z}\subseteq\mathbb{Z}$ is a $\mathbb{Z}$-submodule.
6. Let $I\trianglelefteq R$ be an ideal, then I is an R-submodule of R.
7. $M_n(\mathbb{R})$ is an $\mathbb{R}$-module and an $M_n(\mathbb{R})$-module [Think of $M_n(\mathbb{R})$ as vectors]
8. The quaternions $\mathbb{H}$ is a real dimensional vector space $=\mathbb{R}$-module. $\mathbb{H} = \{a\cdot 1 + b\cdot i + c\cdot j + d\cdot k : \{1,i,j,k\}\text{ basis}, a,b,c,d\in\mathbb{R}\}$.
   is an $\mathbb{R}$-module of dimension 4 and $\mathbb{C}$-module of dimension 2. $i^2=j^2=k^2=1$, $ij=k=-ji$
   If $z=a+bi$, $h = a\cdot 1+b\cdot i+c j+d\cdot k = a\cdot 1+bi+(c+di)j = \mathbb{C}\oplus\mathbb{C}j \Rightarrow$ basis $\{1,j\}$ over $\mathbb{C}$.
   Check $jz = \bar{z}j$, so not commutative.

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$\boxed{\text{Definition}}$ An R-module M is called finitely generated if $\exists\{m_1,\dots,m_k\}$ s.t. $\forall m\in M$, $m=\sum_{i=1}^{k}\lambda_i m_i$, $\forall\lambda_i\in R$.

Examples of finitely generated modules —

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1. Any vector space V of finite dimension over $\mathbb{F}$ $\equiv$ modules over "division rings"

2. $M_n(\mathbb{F})$ is finitely generated over $\mathbb{F}$ by E_{ij}.

3. Any abelian group $A = \mathbb{Z}^r \times \mathbb{Z}_{q_1} \times \dots \times \mathbb{Z}_{q_s}$ as a finitely generated $\mathbb{Z}$-module.

4. $\mathbb{F}[x]$ as an $\mathbb{F}$-module is not finitely generated.

5. $\mathbb{Q}$ as a $\mathbb{Z}$-module is not finitely generated. $\varphi: \mathbb{Z} \times \mathbb{Q} \to \mathbb{Q}$. Suppose $\{q_1, q_2, \dots, q_s\}$ is a generated set of $\mathbb{Q}$. Choose n s.t. n is coprime to all denominators.
Then $\frac{1}{n}$ cannot be expressed as a linear combination of $\{q_1, \dots, q_s\}$.

Note – Modules in this course will be finitely generated.

$\boxed{\text{Definition}}$ Let N be an R-submodule of M. Then M/N is the <u>quotient module</u>, where $M/N = \{m + N : m \in M\}$.

M/N is a R-module.

Proof – (+) Since $(m_1 + N) + (m_2 + N) = (m_1 + m_2) + N$. We can add cosets, so we have an abelian group

(scalar mult). Under R-action, $\lambda(m+N) = \lambda m + N$ for $\lambda \in R$. To determine if action is well-defined, we need rule of equality for cosets, i.e.

$\boxed{x + N = y + N \iff x - y \in N}$. Thus, action is well defined: $x+N = y+N \iff x - y \in N \iff \lambda(x-y) \in N$ by absorbency of submodule.

$\iff \lambda x - \lambda y \in N \iff \lambda x + N = \lambda y + N$ // q.e.d.

Example of quotient module –

Let $I \lhd R$ be an ideal of R, R/I is an additive group which becomes an R-module via the action $r.(a + I) = ra + I$.

For instance, for $2\mathbb{Z} \subset \mathbb{Z}$, $\mathbb{Z}/2\mathbb{Z} = \{\bar{0}, \bar{1}\} = \{0 + 2\mathbb{Z}, 1 + 2\mathbb{Z}\}$. We define $\mathbb{Z}$-action $\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \to \mathbb{Z}/2\mathbb{Z}$, $\mathbb{Z} \times \{\bar{0}, \bar{1}\} \mapsto \{\bar{0}, \bar{1}\}$.

$\boxed{\text{Definition}}$ Let N_1, N_2 be R-submodules of M. Then we define their sum to be $N_1 + N_2 = \{n_1 + n_2 : n_1 \in N_1, n_2 \in N_2\}$.

If $N_1 \cap N_2 = \{0\}$, then the sum is the <u>(internal) direct sum</u>, $N_1 \oplus N_2$.

$\boxed{\text{Definition}}$ Let $N \leq M$ be an R-submodule. Then, we say N is a <u>direct summand</u> of M if $\exists N'$ s.t. $N \oplus N' = M$.

$\boxed{\text{Definition}}$ Let I be an ideal / R-submodule of R. If I is a left and right ideal simultaneously, then I is called a <u>2-sided ideal</u>. [i.e. $R \cdot I \subseteq I$, $I \cdot R \subseteq I$].

$\boxed{\text{Definition}}$ A ring R is called a <u>simple ring</u> if its only 2-sided ideals are $\{0\}$ and R.

$\boxed{\text{Proposition}}$ Let R be a ring, I an ideal $I \lhd R$. Then the 2-sided ideals of $M_n(R)$ are of the form $M_n(I)$, where I has to be 2-sided.

Proof – See Ex 2.

Consequences –

(I) If R is a field, then the only 2-sided ideals of $\mathbb{F}$ are $\{0\}$ and $\mathbb{F}$. $\therefore$ The only 2-sided ideals of $M_n(\mathbb{F})$ are $M_n(\mathbb{F})$ and $\{0\}$.

$\Rightarrow M_n(\mathbb{F})$ is a simple ring.

(II) If $R = D$ is a division ring, then $M_n(D)$ is also a simple ring.

(III) Does $M_n(\mathbb{F})$ have non-trivial left ideals? Yes. Take $c_j = \left\{ \begin{pmatrix} 0 & a_{1j} & 0 \\ & \vdots & \\ & a_{nj} & \end{pmatrix} \middle| \, a_{ij} \in \mathbb{F} \right\}$. Then $M_n(\mathbb{F}) \times c_j \to c_j$, $A \times \begin{pmatrix} 0 & a_{1j} & 0 \\ & \vdots & \\ & a_{nj} & \end{pmatrix} \mapsto$

Trivia – $M_2(\mathbb{F}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, b, c, d \in \mathbb{F} \right\}$. Left ideals are $c_1 = \left\{ \begin{pmatrix} a & 0 \\ c & 0 \end{pmatrix} \middle| \, a, c \in \mathbb{F} \right\} \lhd M_2(\mathbb{F})$, $c_2 = \left\{ \begin{pmatrix} 0 & b \\ 0 & d \end{pmatrix} \middle| \, b, d \in \mathbb{F} \right\} \lhd M_2(\mathbb{F})$

$\begin{pmatrix} a & 0 \\ c & 0 \end{pmatrix} + \begin{pmatrix} a' & 0 \\ c' & 0 \end{pmatrix} \in c_1$, $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \begin{pmatrix} a' & 0 \\ c' & 0 \end{pmatrix} \in c_1$.

Do not confuse ideals with subrings. $S = \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} \subset \begin{pmatrix} a & b \\ c & d \end{pmatrix} = M_2(\mathbb{F})$. S is a subring, but not a left ideal: $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \begin{pmatrix} a' & 0 \\ 0 & 0 \end{pmatrix} \notin S$.

$\therefore$ every ideal is a subring, but not every subring is an ideal.

$\therefore M_n(\mathbb{F}) \cdot c_j \subseteq c_j$ $\therefore$ c_j absorbs $M_n(\mathbb{F})$-action and is a left $M_n(\mathbb{F})$ module. Corresponding right $M_n(\mathbb{F})$-modules are r_i.

$\boxed{\text{Definition}}$ An (left) R-module M is called <u>simple</u> if its only (left) submodules are 0 and M.

$\boxed{\text{Definition}}$ An R-module M is called <u>semisimple</u> if $M = \bigoplus_{i \in I} M_i$ where each M_i is simple for every $i \in I$, where I could be a finite or infinite indexing set.

i.e. $M_i \cap M_j = \{0\}$ $\forall i \neq j$.

Examples of simple R-modules –

1. $\mathbb{F}$ as an $\mathbb{F}$-vector space is a simple module, as it is 1-dimensional.

2. $\mathbb{F}^n$ as an $\mathbb{F}$-module is semisimple. We choose a basis for $\mathbb{F}^n$, say $\{e_1, \dots, e_n\}$. Then $\mathbb{F}^n = \mathbb{F}e_1 \oplus \dots \oplus \mathbb{F}e_n = \bigoplus_{i=1}^{n} \mathbb{F}e_i$.

3. c_j are simple $M_n(\mathbb{F})$-modules ($M_n(\mathbb{F}) \times c_j \to c_j$). $\Rightarrow$ 4. $M_n(\mathbb{F})$ is semisimple as an $M_n(\mathbb{F})$-module. $\Rightarrow M_n(\mathbb{F}) = \bigoplus_{j=1}^{n} c_j$, e.g. $\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & 0 \\ c & 0 \end{pmatrix} + \begin{pmatrix} 0 & b \\ 0 & d \end{pmatrix}$.

5. c_j is not simple as an $\mathbb{F}$-module. $\mathbb{F} \times c_j \to c_j$. c_j is semi-simple as $\mathbb{F}$-module, since we identify c_j with $\mathbb{F}^n = \oplus \mathbb{F}e_i$.
$c_j = \begin{pmatrix} 0 & a_{1j} & 0 \\ & \vdots & \\ & a_{nj} & \end{pmatrix} \mapsto \begin{pmatrix} a_{1j} \\ \vdots \\ a_{nj} \end{pmatrix} = a_{1j} e_1 + \dots + a_{nj} e_n \cong \mathbb{F}^n$, which is semisimple over $\mathbb{F}$.

6. $\mathbb{Z}$ as a $\mathbb{Z}$-module is not simple nor semisimple. Not semisimple: has ideals/submodules such as $2\mathbb{Z} \subset \mathbb{Z}$.

It is not simple as $\mathbb{Z} = \bigoplus M_i$

Beware! Some modules are neither simple nor semisimple. e.g. $\mathbb{Z}/4\mathbb{Z}$ as a $\mathbb{Z}$-module, $\mathbb{Z}/2\mathbb{Z} \subseteq \mathbb{Z}/4\mathbb{Z}$ but $\mathbb{Z}/4\mathbb{Z} \not\cong \mathbb{Z}/2\mathbb{Z} \oplus$?.

Distinguish between rings and modules over rings. For instance,

- $M_n(\mathbb{F})$ is a simple ring, with only two 2-sided ideals. No action here. However,
- $M_n(\mathbb{F})$ over $M_n(\mathbb{F})$ is a semi-simple module, with scalar action.

$\boxed{\text{Definition}}$ Let M and N be left R-modules, then an R-module homomorphism $\varphi: M \to N$ satisfies

(1) $\varphi(0)=0$ (2) $\varphi(m_1+m_2) = \varphi(m_1)+\varphi(m_2)$ (3) $\varphi(\lambda m) = \lambda\varphi(m)$ $\forall m, m_1, m_2 \in M$, $\lambda \in R$.

$\boxed{\text{Definition}}$ The kernel, $\mathrm{Ker}(\varphi) = \{m \in M : \varphi(m) = 0\}$. If $\mathrm{Ker}(\varphi) = \{0\}$, φ is injective. (Ker measures φ's injectivity).

The image, $\mathrm{Im}(\varphi) = \{\varphi(m) \in N : m \in M\}$. If $\mathrm{Im}(\varphi) = N$, φ is surjective.

$\boxed{\text{Proposition}}$ $\mathrm{Ker}(\varphi) \leq M$ and $\mathrm{Im}(\varphi) \leq N$ are R-submodules.

Proof — (1) $\varphi(0) = 0 \Rightarrow 0 \in \mathrm{Ker}(\varphi)$, $0 \in \mathrm{Im}(\varphi)$

(2) Let $a, b \in \mathrm{Ker}(\varphi) \Rightarrow \varphi(a) = 0$, $\varphi(b) = 0$. Then $\varphi(a+b) \overset{\text{[module homomorphism]}}{=} \varphi(a)+\varphi(b) = 0+0 = 0 \Rightarrow a+b \in \mathrm{Ker}(\varphi)$.

Let $\alpha, \beta \in \mathrm{Im}(\varphi) \Rightarrow \exists\, a, b \in M$ s.t. $\varphi(a) = \alpha$, $\varphi(b) = \beta$. $\varphi(a+b) = \varphi(a)+\varphi(b) = \alpha+\beta \Rightarrow \alpha+\beta \in \mathrm{Im}(\varphi)$.

(3) Let $\lambda \in R$, $\varphi(\lambda a) = \lambda \cdot \varphi(a) = \lambda \cdot 0 = 0 \Rightarrow \lambda a \in \mathrm{Ker}(\varphi)$. $\varphi(\lambda a) = \lambda\varphi(a) = \lambda\alpha \in \mathrm{Im}(\varphi)$ $\therefore$ $\mathrm{Im}(\varphi) \leq N$ // q.e.d.

$\boxed{\text{Lemma}}$ (Schur's Lemma, Version 1)

Let M and N be two non-zero simple left R-modules. Then $\varphi: M \to N$ is either

① $\varphi = \underline{0}$ map i.e. $\underline{0}: M \to N$, $m \mapsto 0$ s.t. $\underline{0}(m) = 0$ or ② φ is an isomorphism (i.e. φ invertible, $M \cong N$).

Proof — Suffices to prove that if $\varphi \neq 0$, then φ is an isomorphism (NTP: module homomorphism ✓, injective, surjective).

M (simple over R) $\cong$ N (simple over R)

(Injectivity) Since $\mathrm{Ker}(\varphi) \leq M$ is an R-submodule, M is simple module $\Rightarrow$ $\mathrm{Ker}(\varphi) = \{0\}$ or M. Since $\varphi \neq 0$, $\mathrm{Ker}(\varphi) \neq M$.

$\Rightarrow \mathrm{Ker}(\varphi) = \{0\} \Rightarrow \varphi$ is injective.

(Surjectivity). $\mathrm{Im}(\varphi) \leq N$ is an R-submodule. By simplicity of N, $\mathrm{Im}(\varphi) = \{0\}$ or N. Since $\varphi \neq 0$, $\mathrm{Im}(\varphi) \neq 0 \Rightarrow \mathrm{Im}(\varphi) = N \Rightarrow \varphi$ is surjective // q.e.d.

$\boxed{\text{Definition}}$ Let R be a ring, M be an R-module. We define the endomorphism ring, $\mathrm{End}_R(M) = \mathrm{Hom}_R(M, M) = \{\alpha: M \to M \mid \alpha \text{ is an } R\text{-module homomorphism}\}$.

$\boxed{\text{Proposition}}$ $\mathrm{End}_R(M)$ is naturally a ring.

Proof — $\overset{(+)}{\text{Let}}$ $\alpha, \beta \in \mathrm{End}_R(M)$; $(\alpha+\beta)(m) = \alpha(m)+\beta(m)$ for $m \in M$. $^{(\times)}$ Define composition of maps, $(\alpha \circ \beta)(m) = \alpha(\beta(m))$.

(0) Zero map $\underline{0}(m) = 0$ (1) Identity map $\mathrm{Id}(m) = m$ $\forall m \in M$ // q.e.d. [check ring axioms].

Note — $\mathrm{Hom}_R(M, N)$ is an abelian group (add maps) which becomes an R-module $R \times \mathrm{Hom}(M,N) \to \mathrm{Hom}(M,N)$ if R is a commutative ring.

$\therefore$ $\mathrm{End}_R(M)$ is an R-module if R is commutative $R \times \mathrm{End}_R(M) \longrightarrow \mathrm{End}_R(M)$.

$\boxed{\text{Definition}}$ A ring D is called a division ring if $\forall x \in D$, $x \neq 0$, $\exists\, y$ s.t. $xy = 1$. [Nomenclature: these are also called skew-fields]

Note — Here, every element has an inverse (except 0). Unlike fields however, it is not necessarily commutative.

Examples —

1. Any field $\mathbb{F}$ is a division ring. 2. $\mathbb{Z}$ is a ring, but not division ring. $\{\pm 1\} \in \mathbb{Z}$ are only elements with inverses.

3. $M_n(\mathbb{F})$ is not a division ring — E_{ij} has no inverse. 4. Quaternions $\mathbb{H} = \{a \cdot 1 + b \cdot i + c \cdot j + d \cdot k\}$ is a division ring (non-commutative). $ij = k = -ji$.

Let $\alpha \in \mathbb{H}$, $\alpha = a \cdot 1 + b \cdot i + c \cdot j + d \cdot k$. Let $\alpha \neq 0$. Define $\bar{\alpha} = a \cdot 1 - b \cdot i - c \cdot j - d \cdot k$. Then let $N(\alpha) = a^2+b^2+c^2+d^2$. $\therefore$ $\alpha^{-1} = \frac{\bar{\alpha}}{N(\alpha)}$, $\forall \alpha$, $\exists \alpha^{-1}$.

5. $\left(\frac{-1, 3}{\mathbb{Q}}\right) = \{a1 + bi + cj + dk,\ i^2 = -1,\ j^2 = 3,\ ij = k = -ji\}$ is a division ring $=$ $\mathbb{Q}$-vector space with basis $\{1, i, j, k\}$.

$\boxed{\text{Lemma}}$ (Schur's Lemma, Version 2).

Let M be a simple R-module, then $\mathrm{End}_R(M)$ is a division ring.

Proof — Let $\alpha \in \mathrm{End}_R(M)$. Suppose $\alpha \neq \underline{0}$, since M is simple, by Schur's Lemma V1, $\alpha: M \to M \Rightarrow \alpha$ is an isomorphism

$\Rightarrow \exists\, \alpha^{-1}: M \to M$ s.t. $\alpha \circ \alpha^{-1} = \mathrm{Id}$ and $\alpha^{-1} \circ \alpha = \mathrm{Id}$. $\therefore$ $\forall \alpha \in \mathrm{End}_R(M)$, an inverse exists $\Rightarrow$ $\mathrm{End}_R(M)$ is a division ring // q.e.d.

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$\mathrm{End}_R(M)$ is a tool for measuring the simplicity of M. As an application,

1. Let $\mathbb{F}$ be a field, and consider the $\mathbb{F}$-module $\mathbb{F}$. Let $\varphi_\lambda: \mathbb{F} \to \mathbb{F}$ be $\mathbb{F}$-linear module homomorphisms defined by $x \mapsto \lambda x$ for some $\lambda \in \mathbb{F}$.

   $\mathrm{End}_{\mathbb{F}}(\mathbb{F}) = \{f: \mathbb{F} \to \mathbb{F} : f \text{ is } \mathbb{F}\text{-linear}\} \cong \mathbb{F}$ = division ring $\Rightarrow$ $\mathbb{F}$ is a simple $\mathbb{F}$-module.

2. Let $M = \mathbb{F}^2$ over $\mathbb{F}$. Compute $\mathrm{End}_{\mathbb{F}}(\mathbb{F}^2) \cong \{f: \mathbb{F}^2 \to \mathbb{F}^2 : f \text{ is } \mathbb{F}\text{-linear}\}. \cong M_2(\mathbb{F})$, which is not a division ring since E_{ij} have no inverse.

   Clearly, $\mathbb{F}^2$ is not simple because $\mathbb{F} \leq \mathbb{F}^2$ is a non-trivial submodule.

3. Recall: $c_j = \{\sum_{k=1}^{n} c_k E_{kj} : c_k \in \mathbb{F}\}$ is an $M_n(\mathbb{F})$ module. $M_n(\mathbb{F}) \times c_j \to c_j$.

$\boxed{\text{Proposition}}$ c_j is a simple $M_n(\mathbb{F})$-module.

Proof – ① compute $\mathrm{End}_{M_n(\mathbb{F})}(c_j) = \{f: c_j \to c_j \mid f \text{ is } M_n(\mathbb{F})\text{-linear}\} = \left\{ \begin{matrix} f(x+y) = f(x) + f(y) & x, y \in c_j \\ f(Ax) = Af(x) & A \in M_n(\mathbb{F}) \end{matrix} \right\}$, which we hope is isomorphic to $\mathbb{F}$ = division ring.

② Choose canonical basis for $\mathbb{F}^n$, say $\{e_1, \dots, e_n\}$. s.t. $\mathbb{F}^n \cong \mathbb{F}e_1 \oplus \cdots \oplus \mathbb{F}e_n$.

③ Define f linear maps to identify c_j with $\mathbb{F}^n$, $f: c_j \to \mathbb{F}^n$, $\begin{pmatrix} 0 & a_{1j} & 0 \\ & \vdots & \\ & a_{nj} & \end{pmatrix} \mapsto \begin{pmatrix} a_{1j} \\ \vdots \\ a_{nj} \end{pmatrix} = \sum_{i=1}^{n} a_{ij} e_i$

④ Use $f(Ax) = Af(x)$. Since f is a linear map, $\exists$ matrix $\Phi = (\varphi_{ij})$ s.t. $f_\lambda: \mathbb{F}^n \to \mathbb{F}^n$, $\begin{pmatrix} \varphi_{11} & \cdots & \varphi_{1n} \\ \vdots & & \vdots \\ \varphi_{n1} & \cdots & \varphi_{nn} \end{pmatrix} \begin{pmatrix} a_{1j} \\ \vdots \\ a_{nj} \end{pmatrix} = \begin{pmatrix} \lambda a_{1j} \\ \vdots \\ \lambda a_{nj} \end{pmatrix}$ for some $\lambda \in \mathbb{F}$.

$\Rightarrow \Phi A = A \Phi$ using $f(Ax) = Af(x)$.

$f_\lambda \mapsto \lambda$

⑤ $\mathrm{End}_{M_n(\mathbb{F})}(c_j) = \{f: c_j \to c_j \mid f \text{ commutes with all } A \in M_n(\mathbb{F})\} = \{B \in M_n(\mathbb{F}) : AB = BA\} = \{\mathrm{diag}(\lambda, \dots, \lambda) : \lambda \in \mathbb{F}\} \cong \mathbb{F}$, division ring.

$\therefore$ c_j is a simple $M_n(\mathbb{F})$-module. $\Rightarrow M_n(\mathbb{F}) = \bigoplus_{j=1}^{n} c_j$ is semisimple as an $M_n(\mathbb{F})$-module, q.e.d.

The converse of Schur's lemma V2 does not hold in general. $\mathrm{End}_R(M)$ = division ring $\not\Rightarrow$ M is simple.

4. $\mathbb{Q}$ is a $\mathbb{Z}$-module. $\mathbb{Z} \times \mathbb{Q} \to \mathbb{Q}$. It is obviously not simple, as $\mathbb{Z} \leq \mathbb{Q}$ is a non-trivial submodule. $\therefore$ $\mathbb{Q}$ is not simple, but $\mathrm{End}_{\mathbb{Z}}(\mathbb{Q}) \cong \mathbb{Q}$ which is a division ring.

   $\mathrm{End}_{\mathbb{Z}}(\mathbb{Q}) = \{f: \mathbb{Q} \to \mathbb{Q} \mid f \text{ is } \mathbb{Z}\text{-linear}\}$ (module homomorphisms) $= \left\{ f: \mathbb{Q} \to \mathbb{Q} \,\middle|\, \begin{matrix} f(m_1 + m_2) = f(m_1) + f(m_2) \\ f(zm) = zf(m) \quad z \in \mathbb{Z} \end{matrix} \right\}$. Let $f \in \mathrm{End}_{\mathbb{Z}}(\mathbb{Q})$ be an element.

   Then $f(n) = f(n \cdot 1) = nf(1)$, $n \in \mathbb{Z}$; $f(\tfrac{1}{n}) = \tfrac{1}{n} f(1) \because f(1) = f(\tfrac{n}{n}) = nf(\tfrac{1}{n})$ $\Rightarrow f(\tfrac{m}{n}) = mf(\tfrac{1}{n}) = \tfrac{m}{n} f(1)$.

   $\therefore \forall q \in \mathbb{Q}$, $f(q) = qf(1) \Rightarrow f: \mathbb{Q} \to \mathbb{Q}$ is solely determined by $f(1)$. $\Rightarrow$ $\mathrm{End}_{\mathbb{Z}}(\mathbb{Q}) \cong \mathbb{Q}$, where $f \mapsto f(1)$ defines ring isomorphism.

   $\Rightarrow \mathbb{Z} \leq \mathbb{Q}$ is not simple but $\mathrm{End}_{\mathbb{Z}}(\mathbb{Q})$ is division ring.

   Remark – There is a more powerful version of Schur's lemma, V3 which applies to $\mathbb{F}[G]$ modules. Then $\mathrm{End}_{\mathbb{F}[G]}(M)$ = division ring $\Rightarrow$ M is simple.

$\boxed{\text{Definition}}$ $\boxed{\text{Proposition}}$ Let M be a simple R-module. Then $M = Rm$ for some $m \neq 0$, $m \in M$. Such a module is called a cyclic module.

Proof – Clearly $R \cdot m \leq M$, since $m \neq 0$ and M is simple, $R \cdot m \neq \{0\} \Rightarrow R \cdot m = M$, q.e.d.

Conversely, if $\forall m \in M$, we have $M = Rm \Rightarrow M$ is simple. If $M = Rm$, $M \cong R/I$ where $I = \{r \in R : rm = 0\} \trianglelefteq R$.

Example – Classifying all simple $\mathbb{Z}$-modules –

Let M be a simple $\mathbb{Z}$-module. Define $\varphi: \mathbb{Z} \to M$, $n \mapsto n \cdot m$ for $n \in \mathbb{Z}$. Since $\mathrm{Ker}(\varphi) \leq \mathbb{Z}$, $\mathrm{Ker}(\varphi) = n\mathbb{Z}$. Thus, by 1st isomorphism theorem,

φ induces a map $\varphi^*: \frac{\mathbb{Z}}{\mathrm{Ker}\varphi} \to \mathrm{Im}\varphi \leq M$ (= M since M is simple) $\Rightarrow$ $\boxed{\varphi^*: \mathbb{Z}/n\mathbb{Z} \xrightarrow{\cong} M}$

Case 1: If $n = n_1 n_2$, $(n_1, n_2) = 1$ coprime. Then $M \cong \mathbb{Z}/n\mathbb{Z} \cong \mathbb{Z}/n_1\mathbb{Z} \times \mathbb{Z}/n_2\mathbb{Z}$. Is M simple in this case? No, $\mathbb{Z}/n_1\mathbb{Z} \leq \mathbb{Z}/n\mathbb{Z}$.

Case 2: If $n = p^k$, p is prime, $k > 1$. Then $M \cong \mathbb{Z}/p^k\mathbb{Z}$ is not simple because $\mathbb{Z}/p\mathbb{Z} \leq \mathbb{Z}/p^k\mathbb{Z}$.

Case 3: If $n = p$, then $M \cong \mathbb{Z}/p\mathbb{Z}$, which is a simple $\mathbb{Z}$-module as $\{0\}$ (or x) and $\mathbb{Z}/p\mathbb{Z}$ are the only submodules.

Thus, semisimple $\mathbb{Z}$-modules look like $M = \mathbb{Z}/p_1 \oplus \mathbb{Z}/p_2 \oplus \cdots \oplus \mathbb{Z}/p_n\mathbb{Z}$ (could continue on, but would not be finitely generated).

For instance, $\mathbb{Z}/2\mathbb{Z}$ is simple, $\mathbb{Z}/6\mathbb{Z} \cong \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/3\mathbb{Z}$ is semisimple, $\mathbb{Z}/4\mathbb{Z}$ is neither.

$\boxed{\text{Theorem}}$ (Classification Theorem of Semisimple Modules)

Let M be an R-module, then the following are equivalent:

(1) M is semisimple $[M = \bigoplus_{i \in I} M_i,\ M_i \text{ simple}]$; (2) $\forall N \leq M$ $\exists N' \leq M$ s.t. $M = N \oplus N'$

Example – $M = \mathbb{F}^n$, then $\forall N = \mathbb{F}^a$, $\exists N' = \mathbb{F}^b$ s.t. $\mathbb{F}^n \cong \mathbb{F}^a \oplus \mathbb{F}^b$, $a + b = n$.

Proof – (1) $\Rightarrow$ (2): Suppose M is semisimple, $M = \bigoplus_{i \in I} M_i$, I is a finite indexing set. Let $J \subseteq I$, $J \neq I$ be a non-empty maximal subset $J = \emptyset$ which exists by Zorn's lemma.

Let $N \leq M$ be a non-zero submodule. Define $M^* = N \oplus \bigoplus_{i \in J} M_i$ and show $M = M^*$. $\Rightarrow$ Then you can take $N = \bigoplus_{i \in I/J} M_i$.

Let $i \in I/J$, consider $N \cap M_i$. If $N \cap M_i = \{0\}$, contradicts maximality of J, can add it. $\therefore N \cap M_i = M_i \Rightarrow M_i \subset N \subset M \Rightarrow M \neq M^*$ contradiction

$\therefore M = M^*$, q.e.d.

(2) $\Rightarrow$ (1): To do this we need to first introduce a vital lemma, which we will not yet prove until later:

"Any non-zero finitely generated module contains a simple submodule, which is also non-zero".

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Suppose $M = N \oplus N' \Rightarrow$ both N and N' are semisimple. Let $M_0 \leq M$, where M_0 = sum of all simple submodules.

Show that $M_0 = M$: Suppose $M_0 \neq M$, let $W \leq M_i$, so $W \neq \{0\} \Rightarrow M = W \oplus M_0$. $\Rightarrow W$ contains a simple submodule

by the vital lemma, which contradicts the definition of M_0. $\Rightarrow M = M_0 = \bigoplus_{i \in I} M_i$. The sum is finite as M is finitely generated // q.e.d.

Definition Let $N \leq M$. Then N is called a <u>maximal submodule</u> if $\forall K \leq M$ s.t. $N \leq K \leq M$, $K = N$ or $K = M$.

Example – Let R be an R-module. Then any maximal ideal $I \triangleleft R$ is a maximal R-module. For instance, $\mathbb{Z}$ is a $\mathbb{Z}$-module, then maximal ideal of $\mathbb{Z}$ is $p\mathbb{Z}$.

Facts about maximal submodules:

1. Submodule $N \leq M$ is maximal $\Rightarrow M/N$ is simple. e.g. $p\mathbb{Z} \subseteq \mathbb{Z} \Leftrightarrow \mathbb{Z}/p\mathbb{Z}$ is a simple $\mathbb{Z}$-module and maximal ideal $\Leftrightarrow$ quotient is a field.
2. Any proper submodule of a finitely-generated module is contained in a maximal submodule. This fails if M is not finitely generated, as we need axiom of choice.

Lemma (Vital lemma)

Any non-zero finitely generated module contains a simple submodule, which is also non-zero.

Proof – Let M be such that $\forall N \leq M$, $\exists N'$ s.t. $M = N \oplus N'$. Take $v \in M$, $v \neq 0$ and consider the homomorphism $\varphi: R \to \underset{(\text{cyclic})}{R \cdot v} \leq M$ by $\lambda \mapsto \lambda v$ $\forall \lambda \in R$.

Then φ is surjective $\Rightarrow \operatorname{Im}(\varphi) = R \cdot v$, and we know $\operatorname{Ker}(\varphi) \leq R$. By fact 2 about maximal submodules, $\operatorname{Ker}(\varphi) \leq I$ for some maximal ideal $I \triangleleft R$. By definition, Iv is a maximal submodule of R. By construction, $M = Iv \oplus M'$ as a direct sum, $m = x + y$ $\forall m \in M$ uniquely.

Intersect with Rv: $M \cap Rv = Iv \cap Rv \oplus M' \cap Rv = Iv \oplus M' \cap Rv$. Then $M' \cap Rv \cong Rv/Iv \cong R/I$, which is simple by Fact 1 // q.e.d.

Proposition Every submodule and every quotient module of a semi-simple module is semisimple.

Proof – Let M be semisimple and let $N \leq M$. Let $W \leq N \Rightarrow W \leq M \Rightarrow$ by simplicity of M, $\exists W'$ s.t. $M = W \oplus W'$.

Intersect with $N \Rightarrow M \cap N = W \cap N \oplus W' \cap N$. Now, $M \cap N = N$, $W \cap N = W$, so $N = W \oplus W' \cap M$ $\therefore N$ is semisimple by Characterisation Theorem of semisimple modules, so $N = \bigoplus M_i$ // q.e.d.

Now assume $N = W \oplus W'$. Let $N \leq W \leq M \Rightarrow W/N \leq M/N$ since $M = W \oplus W' \Rightarrow M \cap N = W \cap N \oplus W' \cap N$, $M/N \cong W/N \oplus \frac{W'}{W' \cap N}$

$\Rightarrow M/N$ is semi-simple by the characterisation theorem.

ALGEBRAS.

Definition By an <u>algebra</u> A over $\mathbb{F}$ we mean a ring with a vector space structure satisfying

(1) $\underset{(\text{in ring})}{a+b} = \underset{(\text{as vectors})}{\underline{a}+\underline{b}}$ (2) $(\lambda a)b = \lambda(ab) = a(\lambda b)$ $\forall a, b \in A$, $\forall \lambda \in \mathbb{F}$.

We call it an <u>$\mathbb{F}$-algebra</u>. By an $\mathbb{F}$-algebra we mean a ring which is automatically a vector space over $\mathbb{F}$.

More generally, an R-algebra is a module over a ring R [i.e. we can look at all modules over a ring A]

[We can classify all semi-simple algebras over $\mathbb{F}(G) = \bigoplus M_{n_i}(D_i)$]

Examples of algebras –

1. $\mathbb{F}$ is an $\mathbb{F}$-algebra. 2. $\mathbb{F}[x]$ is an $\mathbb{F}$-algebra. 3. $\mathbb{F}[x]/I$ is an $\mathbb{F}$-algebra.

4. $M_n(\mathbb{F})$ is an $\mathbb{F}$-algebra. The scalars are scalar diagonal matrices, $\begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix}$ 5. Any ring is a $\mathbb{Z}$-algebra.

6. $\operatorname{End}_R(M)$ is an R-algebra if R is commutative, due to the map $\varphi: R \to \operatorname{End}_R(M)$.

7. $\mathbb{H}$ is an $\mathbb{R}$-algebra but not a $\mathbb{C}$-algebra: $\mathbb{H} = \operatorname{span}_{\mathbb{R}}\{1, j, k, 1\}$, $\mathbb{H}$ has $jz = \bar{z}j$ in $\mathbb{C}$, so it is not a $\mathbb{C}$-algebra.

Definition An algebra A is <u>finite-dimensional</u> if it is finite dimensional as a vector space over $\mathbb{F}$.

It is <u>semisimple</u> if every finitely-generated module over A is semisimple.

Example – $\mathbb{F}$ is an $\mathbb{F}$-algebra, but $\mathbb{F}$ is semisimple because every finitely generated module over $\mathbb{F}$ is isomorphic to $\mathbb{F}^n$.

(Characterisation of semisimple algebras)

Proposition A is semisimple $\Leftrightarrow$ A viewed as an A-module is semisimple.

Proof – ($\Rightarrow$) Trivial by definition.

($\Leftarrow$) Suppose A is a semi-simple A-module, and let $M \neq \{0\}$ be an A-module. Choose a generating set $\{m_1, \dots, m_k\} \subset M$.

Let $\varphi: \underset{[A \oplus \cdots \oplus A]}{A^r} \to M$ be a homomorphism of A-modules by $\underset{(r)}{(a_1, \dots, a_r)} \mapsto \overset{(\text{direct sum})}{\sum_i a_i m_i}$. Since A is semi-simple, we can write $A = S_1 \oplus \cdots \oplus S_t$ where S_i are simple A-modules

$\therefore A^r$ is semi-simple, because $A^r = \overbrace{(S_1 \oplus \cdots \oplus S_t) \oplus \cdots \oplus (S_1 \oplus \cdots \oplus S_t)}$. Since φ is surjective, as the m_i generate M over A.

By 1st isomorphism theorem, $\operatorname{Im}(\varphi) = M \cong A^r/\operatorname{Ker}(\varphi)$ is semisimple as it is a quotient of semi-simple modules $\Rightarrow A$ is semisimple by definition // q.e.d.

Examples and Consequences —

1. D is a division algebra $\Rightarrow M_n(D)$ is semisimple as a D-algebra.

2. If $A = \mathbb{Z}/p^2\mathbb{Z}$ is an $\mathbb{F}_p$-algebra, then A is not semisimple because $\mathbb{Z}/p^2\mathbb{Z}$ has no complement.

$\boxed{\text{Proposition}}$ Let A be a semisimple algebra over $\mathbb{F}$ s.t. $A = A_1 \oplus \cdots \oplus A_t$ is a sum of simple submodules. Then any simple A-module is isomorphic to an A_i for some i.

Proof — Let S be a simple module over A, then show $S \cong A_i$ for some i. Take $s \in S$, $s \neq 0$. Consider the linear map $\varphi: A \longrightarrow a \cdot s \leq S$ (where $A = \bigoplus_{i=1}^{r} A_i$) by $a \mapsto a \cdot s$

$\varphi \neq 0$ because $s \neq 0$, so restrict φ. Let $\varphi_i = \varphi|_{A_i} : A_i \to A_i s \leq S$. $\operatorname{Im}(\varphi) \leq S$, but S is simple, so $A_i s = S$. Since $\varphi_i \neq 0 \ \forall i$, $\varphi_i : A_i \to S$ is always an isomorphism by Schur's lemma $\forall i$ fixed.

$\boxed{\text{Proposition}}$ Let A be a semisimple algebra and let $\{S_i\}_r$ be a collection of simple A-modules. Let M be an A-module. Then M is semisimple i.e. $M = S_1^{n_1} \oplus \cdots \oplus S_r^{n_r}$ and the decomposition is unique $\therefore$ if $M = T_1^{m_1} \oplus \cdots \oplus T_s^{m_s}$, then $r = s$, $T_i \cong S_i \ \forall i$.

$\boxed{\text{Definition}}$ A <u>division algebra</u> D is a division ring with a vector space structure, i.e. every non-zero element has an inverse: $\forall x \in D$, $\exists x^{-1}$ s.t. $x \cdot x^{-1} = 1$.

Examples of division algebras —

1. $\mathbb{F}$ over $\mathbb{F}$ 2. $\mathbb{H}$ over $\mathbb{R}$ 3. $M_n(\mathbb{F})$ is not

4. $D_1 \times D_2$ is not a division algebra, as $(d_1, 0)^{-1}$ does not exist.

$\boxed{\text{Theorem}}$ (Frobenius Theorem on algebras)

If D is a finite dimensional division algebra over $\mathbb{R}$, then:

(1) $D \cong \mathbb{R}$ (dim 1 over $\mathbb{R}$) or (2) $D \cong \mathbb{C}$ (dim 2 over $\mathbb{R}$) or (3) $D \cong \mathbb{H}$ (dim 4 over $\mathbb{R}$) [useful for $\mathbb{R}[G]$].

Proof — Beyond the scope of this course; omitted.

Let D be a finite-dimensional $\mathbb{F}$-algebra. Then $\forall n$, $M_n(D)$ is an $\mathbb{F}$-algebra and has dimension $n^2 \dim(D)$. For instance, $D = \mathbb{H}$, then $\dim_{\mathbb{R}}(M_n(\mathbb{H})) = n^2 \dim(\mathbb{H}) = 4n^2$.

$\boxed{\text{Proposition}}$ Let D be a division algebra and $n \geq 1$, with $M_n(D)$ defined as usual. Then

(1) Any simple $M_n(D)$ module is isomorphic to $C_j \cong D^n$ (2) $M_n(D) \cong \bigoplus C_j = \bigoplus D^n$, $\therefore M_n(D)$ is semisimple.

Proof — Easy, previously seen.

$\boxed{\text{Definition}}$ A field $\mathbb{F}$ is <u>algebraically closed</u> if every polynomial $f(x) \in \mathbb{F}[x]$ of $\deg(f) \geq 1$ has a root in $\mathbb{F}$. We denote this by $\overline{\mathbb{F}}$.

Examples of algebraically closed fields —

1. $\mathbb{C} = \overline{\mathbb{R}}, \overline{\mathbb{Q}}, \overline{\mathbb{F}_p}$. 2. $\mathbb{R}$ is not closed as $x^2+1 \in \mathbb{R}[x]$ but has no root in $\mathbb{R}$ (so we close $\mathbb{R}$ with $\mathbb{C}$).

$\boxed{\text{Theorem}}$ (Burnside's theorem)

Let S be a simple module over A, i.e. an algebra over $\overline{\mathbb{F}}$. Then $\operatorname{End}_A(S) \cong \overline{\mathbb{F}}$.

Proof — Let $\varphi \in \operatorname{End}_A(S)$ s.t. $\varphi \neq 0$. S is an $\overline{\mathbb{F}}$-vector space, φ is a linear map. Let $ch_\varphi(x) \in \overline{\mathbb{F}}[x]$ be the characteristic polynomial of φ. Then, $(\varphi - \lambda \mathrm{Id})v = 0 \Rightarrow$

since $\overline{\mathbb{F}}$ is algebraically closed, then $ch_\varphi(x)$ has a root / eigenvalue in $\overline{\mathbb{F}}$, λ. By definition, $\varphi - \lambda \mathrm{Id}_S \in \operatorname{End}_A(\varphi)$ is not invertible as an $\overline{\mathbb{F}}$-linear map.

By Schur's lemma, φ is invertible since $\varphi = \lambda \mathrm{Id}_S$, because $\varphi - \lambda \mathrm{Id}_S = \underline{0}$. Define $\Phi: \operatorname{End}_A(S) \to \overline{\mathbb{F}}$ as an isomorphism by $\varphi_A \mapsto x$.

$\boxed{\text{Definition}}$ Let A be an algebra. We define the <u>opposite algebra</u>, A^{op}, by

(1) $A^{op} = A$ as a set, (2) $+$ in $A^{op} = +$ in A i.e. $(A, +, \circ) \cong (A^{op}, +, \circ)$ (3) $\cdot = *$ is different in A^{op}, as $a * b = b \cdot a$ in A^{op}.

Obviously, $(A^{op}, +, *)$ is an algebra.

$\boxed{\text{Proposition}}$ 1. A is a division algebra $\Leftrightarrow A^{op}$ is a division algebra.

2. $(A^{op})^{op} = A$, $((a * b)^{op})^{op} = (ba)^{op} = ab$.

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$\boxed{\text{Lemma}}$ If B is an algebra, then $M_n(B)^{op} \cong M_n(B)^{op}$

Proof — Define $\psi: M_n(B)^{op} \to M_n(B^{op})$ by $X \mapsto X^T$. It is clear that ψ is bijective, since $\operatorname{Ker}(\psi) = 0$, $\operatorname{Im}(\psi)$ is all matrices.

$\psi(X * Y) = (YX)^T = X^T Y^T = \psi(X)\psi(Y) \Rightarrow \psi$ is a bijective morphism of algebras.

$\boxed{\text{Lemma}}$ Let S be a simple A-module, where A is an algebra. Then $\forall n$, $\operatorname{End}_A(S) \cong M_n(\operatorname{End}(S))$.

Proof — See Exercise 3.

Example — $\operatorname{End}_{\mathbb{F}}(\mathbb{F}^n) \cong M_n(\mathbb{F})$ because $\mathbb{F}$ is a simple algebra over $\mathbb{F}$.

Lemma If $U_1, U_2 \leq M$ are submodules of M s.t. $U_1 \cup U_2 = \{0\}$, then $\text{End}(U_1 \oplus U_2) \cong \text{End}(U_1) \oplus \text{End}(U_2)$.

Proof – see Ex. 3.

Theorem (Wedderburn Decomposition Theorem)

An algebra A is semisimple over $\mathbb{F}$ $\Leftrightarrow$ $A \cong M_{n_1}(D_1) \times \cdots \times M_{n_k}(D_k)$ where D_i are division algebras over $\mathbb{F}$. (same as $\oplus$ by finiteness.)

Proof – ($\Leftarrow$) Trivial, since A is a direct sum of simple modules $\Rightarrow$ semisimple by definition.

($\Rightarrow$). Suppose A is semisimple. Then $A = S_1^{n_1} \oplus \cdots \oplus S_k^{n_k}$ where S_i are simple modules of dimension n_i. $A^{op} = \text{End}_A(A) = \text{End}_A(S_1^{n_1} \oplus \cdots \oplus S_k^{n_k})$.

Since $S_i \cap S_j = \{0\}$, $A^{op} = \text{End}_A(S_1^{n_1}) \oplus \cdots \oplus \text{End}_A(S_k^{n_k}) = M_{n_1}(\text{End}_A(S_1)) \oplus \cdots \oplus M_{n_k}(\text{End}_A(S_k))$. Then taking opposite rings,

$A = (A^{op})^{op} = [M_{n_1}(\text{End}_A(S_1)) \oplus \cdots \oplus M_{n_k}(\text{End}_A(S_k))]^{op} = M_{n_1}(\text{End}_A(S_1))^{op} \oplus \cdots \oplus M_{n_k}(\text{End}_A(S_k))^{op} = M_{n_1}(\text{End}_A(S_1)^{op}) \oplus \cdots \oplus M_{n_k}(\text{End}_A(S_k)^{op})$.

Since S_i are simple, by Shur's lemma V2, $\text{End}(S_i)$ and $\text{End}(S_i^{op})$ are division rings, so let $D_i = \text{End}(S_i^{op})$

$\therefore A \equiv M_{n_1}(D_1) \oplus \cdots \oplus M_{n_k}(D_k)$ for some division rings D_i // q.e.d.

Consequences –

If $\mathbb{F}$ is algebraically closed (hint: $\mathbb{C} = \mathbb{F}$), then $A \equiv M_{n_1}(\mathbb{C}) \oplus \cdots \oplus M_{n_k}(\mathbb{C})$, if A is a semisimple algebra. If A is a simple algebra, then $\boxed{A \equiv M_{n_i}(\mathbb{C})}$ as rings.

Group ring / Group algebra.

Definition By a group ring/group algebra $\mathbb{F}[G] = \{\sum_{g \in G} \lambda_g g : g \in G, \lambda_g \in \mathbb{F}\}$ i.e. formal linear combinations of group elements as a basis with $\mathbb{F}$ coefficients.

Proposition $\mathbb{F}[G]$ is a ring.

Proof – (addition) $\sum_{g \in G} \lambda_g g + \sum_{g \in G} \mu_g g = \sum_{g \in G} (\underset{\in \mathbb{F}}{\lambda_g + \mu_g}) \underset{\in G}{g}$ ✓ (multiplication) $(\sum_{g \in G} \lambda_g g)(\sum_{h \in G} \mu_h h) = \sum_{g,h} (\lambda_g \mu_h) gh = \sum_{g,h} \lambda_g \mu_{h^{-1}g}\, g$ ✓

(zero) Since all $g \neq 0$ in G, $0 = \sum \lambda_g g$ where $\lambda_g = 0$ $\forall g$. ✓ (unit) $1 = 1_{\mathbb{F}} \cdot 1_G$ ✓ // q.e.d.

Note – The set $(\mathbb{F}[G], +, 0, \cdot, 1)$ is a ring.

Ex Show that $\mathbb{F}[G]$ is an algebra over $\mathbb{F}$ with basis elements the group elements $\{1, g, \cdots, g_n\}$ and scalar multiplication from $\mathbb{F}$.

Soln. $\lambda_g \cdot g = g \cdot \lambda_g$ $\forall g$, $\mathbb{F}[G]$ is an $\mathbb{F}$-algebra. $\lambda \cdot (\sum \lambda_g g) = \sum \lambda \lambda_g g$

Note – The algebra $\mathbb{F}[G]$ is not commutative unless G is commutative.

Ex Show that $\mathbb{F}[C_2] = \{a + bx : a, b \in \mathbb{F},\ 1, x \in C_2 = \langle x \mid x^2 = 1 \rangle\}$ is a group ring.

Soln. (+) $(2+6x) + (3+9x) = 5 - 3x$ (say). ($\times$) $(2+x) \cdot (3-4x) = 6 - 5x - 4x^2 = 6 - 5x - 4 = 2 - 5x \in \mathbb{F}[C_2]$.

Note – Is $(1+x)$ invertible? G is commutative $\Rightarrow$ $\mathbb{F}[C_2]$ is a commutative ring $\Rightarrow$ $(a+bx) \cdot (1+x) = 1$ $\Rightarrow$ no $a+bx$ exists, $\mathbb{F}[C_2]$ is not a field.

Lemma If $|G| > 1$, then $\mathbb{F}[G]$ is not a division ring/algebra.

Proof – $|G| = 1 \Rightarrow \mathbb{F}[G] = \mathbb{F}[1] = \mathbb{F}$, which is a division ring. So suppose $|G| > 1$, G is finite $\Rightarrow$ $\exists n$ st. $g^n = 1$ $\forall g$, then we get the expression:

$(1-g)(1+g+\cdots+g^{n-1}) = 1 - g^n = 1 - 1 = 0 \Rightarrow 1-g, 1+\cdots+g^{n-1}$ are non-zero divisors of zero, which cannot exist in an integral domain // q.e.d.

Definition By an $\mathbb{F}[G]$-module V, I will always mean a finitely generated module.

Example –

- $\mathbb{F}[G]$ as an $\mathbb{F}[G]$-module acting on its basis elements (group elements) by left-multiplication $\varphi : \underset{\text{ring}}{\mathbb{F}[G]} \times \underset{\text{v.s.}}{\mathbb{F}[G]} \to \underset{\text{v.s.}}{\mathbb{F}[G]}$

Definition Let V, W be two $\mathbb{F}[G]$-modules. Then a $\mathbb{F}[G]$-module homomorphism is a map $\varphi : V \to W$ that satisfies

(1) $\varphi(v + v') = \varphi(v) + \varphi(v')$, and (2) $\varphi(\lambda v) = \lambda \varphi(v)$ $\forall \lambda \in \mathbb{F}[G]$, $\forall v, v' \in V$. Moreover,

(3) $\varphi(gv) = g\varphi(v)$ $\forall g \in G$ $\forall v \in V$.

As before, $\text{Ker}(\varphi) \leq V$ and $\text{Im}(\varphi) \leq W$ are $\mathbb{F}[G]$-submodules.

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Theorem (Correspondence Theorem)

Let G be a finite group, V a finite dimensional vector space over $\mathbb{F}$, $\rho : G \to GL(V)$ be an $\mathbb{F}$-representation. Then $\exists$ 1-1 correspondence (bijection) between representations G and finitely generated left $\mathbb{F}[G]$ modules. [i.e. $\{\rho : G \to GL(V)\} \overset{1:1}{\longleftrightarrow} \{\text{modules over } \mathbb{F}[G]\}$]

Proof – ($\Leftarrow$) Let V be an $\mathbb{F}[G]$-module $\Rightarrow$ V is an $\mathbb{F}$-vector space. Now $\forall g \in G$, define an automorphism (linear map) $\psi : V \to V$, $v \mapsto g \cdot v$

$\forall v \in V = \text{span} \overbrace{\{b_1, \ldots, b_n\}}^{B.}$. Write the map ψ as a matrix, $\therefore [\psi]_B = \rho(g)$. We check that $\rho(g)$ defines a representation:

(1) $\rho(g)(\lambda v + w) = g \cdot (\lambda v + w) = \lambda g \cdot v + g \cdot w = \lambda \rho(g) \cdot v + \rho(g) \cdot w$ (2) Check $\rho(g)$ is a homomorphism: factorise
$$\begin{tikzcd} V \arrow[r, "\rho(h)"] \arrow[rr, bend left, "\rho(gh)"] & V \arrow[r, "\rho(g)"] & V \end{tikzcd}$$

$\rho(gh) \cdot v = g(h \cdot v) = \rho(g) \cdot (hv) = \rho(g)\rho(h)v$, which is a composition of linear maps $\equiv$ matrix multiplication.

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(3) $\rho(1) = \begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix}$ corresponds to identity map $Id: V \to V$. (4) $\rho(g) \in GL_n(\mathbb{F})$ is invertible, $1 = g g^{-1}$ in $\mathbb{F}[G]$ and in G, factorise

$$\begin{tikzcd} V \arrow[r, "\rho(g)"] \arrow[rr, bend right, "\rho(1)"'] & V \arrow[r, "\rho(g^{-1})"] & V \end{tikzcd}.$$

$\rho(g^{-1})\rho(g)\cdot v = \rho(g^{-1}) gv = g^{-1} g v = v = Id(v)$ $\therefore$ $\mathbb{F}[G]$ modules give $\mathbb{F}$-map of G.

($\Rightarrow$) let $\rho: G \to GL(V) = GL_n(V)$ for some $B = \{b_1, \dots, b_n\} = V = \mathbb{F}^n$. then associate to ρ an $\mathbb{F}[G]$ module constructed from $V = \mathbb{F}^n$ by keeping the same addition, and defining scalar multiplication on it by letting $\alpha = \sum \lambda_g g \in \mathbb{F}[G]$, $\underset{\text{modules}}{\alpha \cdot v} = (\sum \lambda_g g)\cdot v = \sum \lambda_g \cdot (gv) = \sum \overset{\mathbb{F}}{\underset{\cup}{\lambda_g}} \overset{\mathbb{F}^n}{\underset{\cup}{\rho(g)}} v$ $\therefore$ $\mathbb{F}^n$ becomes an $\mathbb{F}[G]$-module. q.e.d.

Examples —

1. let $G = D_8 = \langle x, y \mid x^4 = y^2 = 1, yx = x^3 y \rangle$ and $\mathbb{F} = \mathbb{R}$. Define $\rho: D_8 \to GL_2(\mathbb{R})$, $x \mapsto \overset{\text{reflection}}{\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}}$, $y \mapsto \overset{\text{rotation}}{\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}}$. let $V = \mathbb{R}^2 = \mathrm{span}_{\mathbb{R}}\{e_1, e_2\}$. Apply matrices: $\rho(x)\cdot e_1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \end{pmatrix} = -e_2$, $\rho(x)\cdot e_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} = e_1$. $\rho(y)\cdot e_1 =$ $\rho(y)\cdot e_2 =$. This is the module information, which defines the structure of $V = \mathbb{R}^2$ as an $\mathbb{R}[D_8]$ module.

2. let $G = S_n$, define $\rho: S_n \to GL_n(\mathbb{C})$, which is the permutation representation on $V = \mathbb{C}^n$, $\rho(\sigma)(e_i) = e_{\sigma(i)}$ is the module definition. This gives a matrix/representation from the $\mathbb{C}[S_n]$-module $V = \mathbb{C}^n$. For instance, if $n=4$, $\rho: S_4 \to GL_4(\mathbb{C})$. let $V = \mathbb{C}^4 = \{e_1, \dots, e_4\}$. let $\sigma = (1\ 2) = S_4$. Then $\left.\begin{matrix}\rho(\sigma)\cdot e_1 = e_{\sigma(1)} = e_2, & \rho(\sigma)\cdot e_2 = e_1, & \rho(\sigma)\cdot e_3 = e_3, & \rho(\sigma)\cdot e_4 = e_4\end{matrix}\right\} \longleftrightarrow \rho(\sigma) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$.

$\boxed{\text{Definition}}$ let G be a finite group, V be a finite dimensional vector space over $\mathbb{F}$. An $\mathbb{F}$-representation (matrix) $\rho: G \to GL(V)$ is called irreducible $\equiv$ simple module of $V \neq \{0\}$, and the only invariant subspaces of V under ρ are $\{0\}$ and V. The representation is reducible (semisimple) if $\exists W \leq V$, $W \neq \{0\}$ s.t. $\rho(g)\cdot W \subseteq W$ $\forall g \in G$.

i.e. W is $\rho(g)$-stable / invariant subspace of V.

Hence, by correspondence theorem, $\rho: G \to GL(V)$ is irreducible $\longleftrightarrow$ V is a simple $\mathbb{F}[G]$ module.

How do we recognise reducible representations?

$\boxed{\text{Definition}}$ An $\mathbb{F}$-representation $\rho: G \to GL_n(\mathbb{F})$ is called reducible if $\exists T \in GL_n(\mathbb{F})$ s.t. $\forall g \in G$, we have equivalent matrices of the form $\rho'(g) = T^{-1}\rho(g) T = \left(\begin{array}{c|c} X_g & Y_g \\ \hline 0 & Z_g \end{array}\right)$ where X_g is a $\dim W \times \dim W$ matrix.

Examples of irreducible / reducible maps —

1. $\rho: D_8 \to GL_2(\mathbb{R})$ is irreducible, $x \mapsto \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, $y \mapsto \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. suppose ρ is reducible, then $\exists W \subseteq V = \mathbb{R}^2$ s.t. W is a $\mathbb{R}[D_8]$-invariant submodule of V, where $\dim_{\mathbb{R}}(W) = 1$. suppose $W = \mathrm{span}_{\mathbb{R}} \overset{\text{one-dimensional}}{\{\lambda v_1 + \mu v_2\}} \subseteq V$. Apply $\rho(g)$ to W: let $w = \lambda v_1 + \mu v_2$, $\rho(x)\cdot w = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} \lambda \\ \mu \end{pmatrix} = \begin{pmatrix} \mu \\ -\lambda \end{pmatrix} = \mu v_1 - \lambda v_2 = \underset{\mathbb{R},\ \text{scalar multiple of the subspace.}}{\overset{}{\alpha}} \cdot (\lambda v_1 + \mu v_2)$.
$\rho(y)\cdot w = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} \lambda \\ \mu \end{pmatrix} = \begin{pmatrix} \lambda \\ -\mu \end{pmatrix} = \lambda v_1 - \mu v_2 = \beta(\lambda v_1 + \mu v_2)$. Solve system to get $\lambda = 0$ or $\alpha = \beta$. Then we see that:
If $\lambda = 0$, $w = \mu v_2 = \mathrm{span}\{v_2\}$ and this is not stable by $\rho(x)$ since $\rho(x)\cdot v_2 = v_1 \notin W$. Thus $\alpha = \beta$, but if $\mu \neq 0$, $w = \lambda v_1 = \mathrm{span}(v_1)$, which is not stable.
$\Rightarrow W = \{0\}$ since $\alpha = \beta = 0$, $\therefore$ ρ is irreducible. We will instead compute $\mathrm{End}_{\mathbb{R}[D_8]}(\rho) =$ division ring (in the future).

2. If $\mathbb{F} = \mathbb{F}_2$, then $W = \mathrm{span}_{\mathbb{R}}\{v_1 + v_2\} \subseteq \mathbb{R}^2$ is $\rho(D_8)$-stable. $\rho(x)\cdot w = \rho(x)(v_1 + v_2) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \in W$ in $\mathbb{F}_2$. $\rho(y)\cdot w = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \in W$. Hence, ρ is reducible over $\mathbb{F}_2$ into direct sum of 1-dimensional representations.

Note — Reducibility depends on field chosen, as seen — $\rho: D_8 \to GL_2(\mathbb{R})$ is irreducible with above representation, but $\rho: D_8 \to GL_2(\mathbb{F}_2)$ can.

3. Let $G = C_3$ and define the permutation representation $\rho: C_3 \to GL_3(\mathbb{R})$, $V = \mathbb{R}^3 = \mathrm{span}\{1, x, x^2\}$. $e_i \mapsto g\cdot e_i$ for $g \in G$, $C_3 = \langle x \mid x^3 = 1 \rangle$. $\rho(x)\cdot e_1 = x \cdot 1 = x = e_2$, $\rho(x)\cdot e_2 = x \cdot x = x^2 = e_3$, $\rho(x)\cdot e_3 = x \cdot x^2 = x^3 = 1 = e_1$ $\Rightarrow$ $\rho(x) = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$. $\rho(x)$ is a reducible representation i.e. $\exists$ $\mathbb{R}[C_3]$-invariant subspaces.
i.e. V is a semisimple $\mathbb{R}[C_3]$-module = direct sum of simple modules. Let $W = \mathrm{span}_{\mathbb{R}}\{w\} = \mathrm{span}\{e_1 + e_2 + e_3\} = \mathrm{span}\left\{\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}\right\}$. W is an $\mathbb{R}[C_3]$-invariant subspace.
i.e. $\rho(1)\cdot\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$, $\rho(x)\cdot\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$, $\rho(x^2)\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$, $\rho(g)\cdot w \subseteq \lambda \cdot w$. choose new basis $B' = \mathrm{span}\{w, e_2, e_3\}$. Apply $\rho(x)\cdot w = w$, $\rho(x)\cdot e_2 = e_3$, $\rho(x)\cdot e_3 = w - e_2 - e_3$.
Thus, $\rho'(x) = \left(\begin{array}{c|cc} 1 & 0 & 1 \\ \hline 0 & 0 & -1 \\ 0 & 1 & -1 \end{array}\right) \sim \left(\begin{array}{c|c} X_g & Y_g \\ \hline 0 & Z_g \end{array}\right)$. $\therefore$ $\exists T \in GL_n(\mathbb{R})$ s.t. $\rho'(x) = T^{-1}\rho(x) T$, $T = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & -1 \\ 0 & 1 & -1 \end{pmatrix}$.

Projections.

If $V = U \oplus W$, then we can construct a special automorphism of V that depends on the expression $V = U \oplus W$.

$\boxed{\text{Proposition}}$ suppose $V = U \oplus W$, define $\pi: V \to U \leq V$, $v = u + w \mapsto u$ $\forall u \in U, w \in W$. Then π is an endomorphism of V and $\mathrm{Ker}(\pi) = W$, $\mathrm{Im}(\pi) = U$ and $\pi^2 = \pi$.

$\boxed{\text{Definition}}$ An endomorphism π of a vector space V which satisfies $\pi^2 = \pi$ is called a projection.

$\boxed{\text{Proposition}}$ Suppose π is a projection of V, then $V = \mathrm{Ker}(\pi) \oplus \mathrm{Im}(\pi)$.

$\boxed{\text{Theorem}}$ (Maschke's version1 theorem) [important, but proof omitted] $\mathrm{char}(\mathbb{F}) = n \Rightarrow \overbrace{1 + 1 + \cdots + 1}^{n} = 0$, n minimal.

Suppose G is a finite group, $\mathbb{F}$ a field s.t. $\underbrace{\mathrm{char}(\mathbb{F})}_{} \nmid G$. Let V be an $\mathbb{F}[G]$ module, then for any $U \leq V$ where U is an $\mathbb{F}[G]$-submodule.

$\exists W \leq V$, an $\mathbb{F}[G]$-submodule s.t. $V \cong U \oplus W$.

Remark — In our course, this tells us that any $\mathbb{F}[G]$-module V with the above conditions is semisimple. i.e. $V = S_1 \oplus \cdots \oplus S_n$ where each S_i is a simple $\mathbb{F}[G]$-submodule.

$\boxed{\text{Definition}}$ Let A,B,C be R-modules. A sequence of homomorphisms $A \xrightarrow{\varphi} B \xrightarrow{\psi} C$ is ~~exact~~ exact at B if $\ker(\psi) = \operatorname{Im}(\varphi)$.

A sequence $A_{n+1} \xrightarrow{\varphi_{n+1}} A_n \xrightarrow{\varphi_n} A_{n-1} \xrightarrow{\varphi_{n-1}} \cdots$ is exact everywhere if $\forall n$ $\ker(\varphi_n) = \operatorname{Im}(\varphi_{n+1})$. i.e. exact at every A_i.

Example — let A, C be R-modules, then $0 \to A \xrightarrow{i} A \oplus C \xrightarrow{p} C \to 0$ is an exact sequence. Here, the mappings are $\underset{\text{inclusion}}{i(a) = (a,0)}$, $\underset{\text{projection}}{p(a,c) = c}$

claim: $\ker(p) = \operatorname{Im}(i)$. $p(a,c) = 0 \iff c = 0$ i.e. $(a,0) \in \ker(p)$. $\iff$ $(a,0) = i(a)$ $\Rightarrow$ $(a,0) \in \operatorname{Im}(i)$ // q.e.d.

$\therefore$ sequence is exact at $A \oplus C$.

claim: $0 \to A \xrightarrow{i} A \oplus C \xrightarrow{p} C \to 0$ is exact at every module if i injective, p surjective $\iff \ker(p) = \operatorname{Im}(i)$

$\boxed{\text{Definition}}$ A <u>short exact sequence</u> $0 \to A \xrightarrow{i} B \xrightarrow{p} C \to 0$ means that:

(1) $\ker(i) = \operatorname{Im}(0 \to A) = 0$ is injective (2) $\ker(p) = \operatorname{Im}(i)$ (3) $\ker(C \to 0) = C = \operatorname{Im}(p)$, p is surjective.

$\boxed{\text{Definition}}$ The ~~trivial~~ short <u>exact sequence</u> is a sequence s.t. if A, C are R-modules, then $0 \to A \xrightarrow{i} A \oplus C \xrightarrow{p} C \to 0$ is always exact, $i(a) = (a,0)$, $p(a,c) = c$,

$\ker(p) = \{(a,c) : c = 0\} = \{(a,0) : a \in A\} = \operatorname{Im}(i)$.

$\boxed{\text{Definition}}$ A short exact sequence $0 \to A \to B \to C \to 0$ splits if $\exists$ isomorphism of R-modules $\psi : A \oplus C \to B$ s.t. the following diagram commutes:

$$\begin{array}{ccccccccc} 0 & \to & A & \xrightarrow{i} & A \oplus C & \xrightarrow{p} & C & \to & 0 \\ & & \downarrow \mathrm{Id} & & \cong \downarrow \psi & & \downarrow \mathrm{Id} & & \\ 0 & \to & A & \to & B & \to & C & \to & 0. \end{array}$$

Note — Short exact sequences usually do not split!

Examples —

1. Consider the $\mathbb{Z}$-module short exact sequence:
$$0 \to \mathbb{Z} \xrightarrow{\times 2} \mathbb{Z} \xrightarrow{[\,]} \mathbb{Z}/2\mathbb{Z} \to 0.$$
$$z \mapsto 2z, \quad n \mapsto [n] \bmod 2.$$
This is exact, as $\ker([\,]) = \operatorname{Im}(\times 2)$

check: $\ker(\mathbb{Z}/2\mathbb{Z} \xrightarrow{?} 0) = \operatorname{Im}(\mathbb{Z} \xrightarrow{[\,]} \mathbb{Z}/2\mathbb{Z})$. suppose sequence splits, $\mathbb{Z} \cong \mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$. This is a contradiction, as RHS contains element of finite order $\because 1+1=0$, whereas LHS has no element of finite order.

2. $0 \to \mathbb{Z}/2\mathbb{Z} \xrightarrow{\times 2} \mathbb{Z}/4\mathbb{Z} \xrightarrow{(\text{mod } 2)} \mathbb{Z}/2\mathbb{Z} \to 0.$ Then
$$\begin{array}{ccccc} \mathbb{Z}/2\mathbb{Z} & \xrightarrow{\times 2} & \mathbb{Z}/4\mathbb{Z} & \xrightarrow{(\text{mod } 2)} & \mathbb{Z}/2\mathbb{Z} \\ 0 & \mapsto & 0 & \mapsto & 0 \\ 1 & \mapsto & 2 & \mapsto & 0 \\ & & 1 & \mapsto & 1 \\ & & 3 & \mapsto & 1 \end{array}$$

This is exact as $\ker((\text{mod } 2)) = \operatorname{Im}(\times 2)$. However, this does not split since we observe that $\mathbb{Z}/4\mathbb{Z} \ncong \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$.

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3. The SES $0 \to \mathbb{Z} \to \mathbb{Z} \oplus \mathbb{Z}/n\mathbb{Z} \to \mathbb{Z}/n\mathbb{Z} \to 0$ is split exact by definition

4. The SES $0 \to \mathbb{Z}/2\mathbb{Z} \to \mathbb{Z}/6\mathbb{Z} \to \mathbb{Z}/3\mathbb{Z} \to 0$ is split exact by the fact that $\mathbb{Z}/6\mathbb{Z} \cong \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/3\mathbb{Z}$.

i.e. $\exists \psi : \mathbb{Z}/6\mathbb{Z} \to \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/3\mathbb{Z}$ which is an isomorphism.

$\boxed{\text{Theorem}}$ (Basis Theorem for Vector Spaces)

Let $\mathbb{F}$ be a field. Then every short exact sequence of vector spaces over $\mathbb{F}$ splits.

Proof — 0. Let $0 \to A \xrightarrow{i} B \underset{s}{\overset{p}{\rightleftarrows}} C \to 0$ be a SES of $\mathbb{F}$-vector spaces. 1. Let $\{e_1, \ldots, e_n\}$ be a basis for C (assume C is finite). 2. Choose $b_i \in B$ st. $p(b_i) = e_i$, for since p is surjective, we can hit all e_i. 3. Define $s : C \to B$ by $s(e_i) = b_i$. 4. check $p \circ s = \mathrm{Id}_C$ $\therefore p \circ s(e_i) = p(b_i) = e_i$

5. Construct a splitting map $\psi : A \oplus C \to B$, by $\psi(a,c) = i(a) + s(c) \in B$. Claim: ψ is an isomorphism. ψ is obviously $\mathbb{F}$-linear because i and s are both $\mathbb{F}$-linear. 6. For surjectivity, let $b \in B$, define $\alpha = b - s \circ p(b)$. Apply $p(\alpha) = p(b) - p(s \circ p(b)) = p(b) - \mathrm{Id}_C\, p(b) = p(b) - p(b) = 0$.

$\Rightarrow \alpha \in \ker(p) = \operatorname{Im}(i)$ by exactness. $\therefore \alpha = i(a)$ for some $a \in A$. Plug in and rearrange: $b = \alpha + s \circ p(b) = i(\alpha) + s(p(b)) = \psi(a, p(b))$ by definition of ψ.

$\therefore \psi$ is surjective as every b is hit. 7. For injectivity, suppose $\psi(a,c) = 0 \Rightarrow i(a) + s(c) = 0 \Rightarrow s(c) = -i(a) \overset{\text{linear}}{=} i(-a) \in \operatorname{Im}(i) = \ker(p)$.

Hence, $p(s(c)) = p(i(-a)) = 0 \Rightarrow (p \circ s)(c) = \mathrm{Id}_C(c) = 0 \Rightarrow c = 0$ $\therefore i(a) + s(0) = 0 \Rightarrow i(a) = 0 \Rightarrow \alpha = 0$ since i is injective. $\Rightarrow (a,c) = (0,0)$.

$\therefore \psi$ is injective. Then ψ is an isomorphism that splits our SES // q.e.d.

$\boxed{\text{Theorem}}$ (Maschke's theorem, Version 2: Modern Form).

Let G be a finite group, let $\mathbb{F}$ be a field in which $|G| \neq 0$ (e.g. If $G = C_3$, don't want $\mathbb{F} = \mathbb{F}_3$ as $|G| = 3 \equiv 0$ in $\mathbb{F}_3$). If $\mathcal{E} = (0 \to A \to B \to C \to 0)$ is a SES of $\mathbb{F}[G]$ modules, then $\mathcal{E}$ splits. i.e. $\exists$ an $\mathbb{F}[G]$ module isomorphism $\psi : A \oplus C \to B$ s.t. for
$$\begin{array}{ccccccccc} 0 & \to & A & \to & A \oplus C & \to & C & \to & 0 \\ & & \downarrow \mathrm{Id}_A & & \downarrow \exists \psi & & \downarrow \mathrm{Id}_C & & \\ 0 & \to & A & \xrightarrow{i} & B & \xrightarrow{p} & C & \to & 0, \end{array}$$
the diagram commutes.

Proof — We need a homomorphism of $\mathbb{F}[G]$-modules, $\psi : A \oplus C \to B$ st. $\psi(\lambda(a,c)) = \lambda \psi(a,c)$ $\forall \lambda \in \mathbb{F}$, and $\psi(g(a,c)) = g\psi(a,c)$ $\forall g \in G$.

To start, let A, B, C be vector spaces over $\mathbb{F}$. (forget about G in $\mathbb{F}[G]$ for the moment). $0 \to A \xrightarrow{i} B \underset{\sigma}{\overset{p}{\rightleftarrows}} C \to 0$. From the basis theorem, $\exists$ $\mathbb{F}$-linear map $\sigma : C \to B$ s.t. $p \circ \sigma = \mathrm{Id}_C$. Need to produce a linear map s s.t. $p \circ s = \mathrm{Id}_C$ and s commutes with $g \in G$ as well.

Maschke's trick: $\forall g \in G$, define $\sigma_g : C \to B$ by $\sigma_g(c) = g^{-1}\sigma(gc)$. Then $\sigma_g(hc) = g^{-1}\sigma(ghc) = hh^{-1}g^{-1}\sigma(ghc) = h(gh)^{-1}\sigma(ghc) = h\sigma_{gh}(c)$ for $h \in G$.

Put $\hat{s} : C \to B$. then $\hat{s}(c) = \sum_{g \in G} \sigma_g(c)$, the sum over all group elements. Then $\hat{s}(hc) = \sum_{g \in G} \sigma_g(hc) = \sum_{g \in G} h\sigma_{gh}(c) \overset{\text{reindex}}{=} h\left(\sum_{\gamma \in G} \sigma_\gamma(c)\right)$

by linearity, $= h\hat{s}(c)$. Now $\hat{s} : C \to B$ is $\mathbb{F}[G]$-linear. However, $p \circ \hat{s}(c) = \sum_g p\sigma_g(c)$ where $p\sigma_g(c) = p(g^{-1}\sigma(gc)) = g^{-1}p\sigma(gc)$ since p is $\mathbb{F}[G]$-linear.

$\therefore p\sigma(gc) = gc$, so $p \circ \sigma_g(c) = g^{-1} \circ gc = c \Rightarrow p \circ \sigma_g = \mathrm{Id}$. So apply to arbitrary element: $p \circ \hat{s}(c) = |G|c$. If $|G| \neq 0$ in $\mathbb{F}$, we can invert:

$\therefore$ Define splitting map $s(c) = \frac{1}{|G|}\hat{s}(c)$ where s is $\mathbb{F}[G]$ linear. Define $\psi : A \oplus C \to B$ by $\psi(a,c) = i(a) + s(c)$. By basis theorem, ψ is $\mathbb{F}$-isomorphism and $\mathbb{F}[G]$-linear
q.e.d.

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Examples that falsify Maschke's Theorem – If $|G| \neq 0$ in $\mathbb{F}$, then every $\mathbb{F}[G]$-submodule $U \leq V$ has a complement $W \leq V$ s.t. $V = U \oplus W$. $(B = A \oplus C)$

1. If G is infinite, Maschke fails. Define $\rho: \mathbb{Z}_{C_\infty} \to GL_2(\mathbb{C})$ by $n \mapsto \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}$ for $n \neq 0$. This is a representation: $\rho(n+m) = \begin{pmatrix} 1 & n+m \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & m \\ 0 & 1 \end{pmatrix} = \rho(n) \cdot \rho(m)$)

Take a 1-dimensional $\mathbb{C}[C_\infty]$-submodule $U = \text{span}\{e_1\} = \lambda\begin{pmatrix} 1 \\ 0 \end{pmatrix}$. Then U is $\mathbb{C}[C_\infty]$-invariant. $\begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \in U$ is an eigensubspace.

But U has no complement since matrix is not diagonalisable because $m(x) = (x-1)^2$. $\mathbb{C}^2 \equiv U \oplus ?$

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Recap of course thus far:

① Representations $\rho: G \to GL_n(\mathbb{C})$, $g \mapsto A$. ② Modules over rings M over R maps, submodules ③ simple/semisimple modules $M = \bigoplus_{i \in I} S_i$.

④ Group ring $\mathbb{F}[G]$: modules over $\mathbb{F}[G] \overset{1:1}{\longleftrightarrow} \rho$ representations of G ⑤ Schur's lemma V1, V2 (simple modules). ⑥ Maschke's theorem using SES.

More examples where Maschke's theorem fails –

2. If $|G| = 0$ in $\mathbb{F}$ i.e. $\text{char}(\mathbb{F}) \mid |G|$. then $\rho: C_p \to GL_2(\mathbb{F}_p) \cong \text{Aut}(C_p \times C_p)$, $C_p = \langle x \mid x^p = 1 \rangle$. Then $x \mapsto \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ s.t. $x^j \mapsto \begin{pmatrix} 1 & j \\ 0 & 1 \end{pmatrix}$, $j \in \{0, 1, \dots, p-1\}$.

ρ defines a representation since $\rho(x)^p = \rho(x^p) = \rho(1)$. Let $U = \text{span}\{e_1\} \leq V$ be an $\mathbb{F}_p[C_p]$-submodule of $V = \mathbb{F}_p^2$. Then U is C_p-invariant:

$\rho(x^j)(e_1) = \begin{pmatrix} 1 & j \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} = e_1 \in U$. But $\nexists\, W \leq V$ s.t. $\mathbb{F}_p^2 \cong U \oplus W$. If there was such a W, then $\begin{pmatrix} 1 & j \\ 0 & 1 \end{pmatrix}$ would be diagonalisable. However,

$m_p(\alpha) = (\alpha - 1)^2$, so there is only one eigenvalue $\Rightarrow$ no complement for U.

Hint: $\left.\begin{array}{l} \rho \text{ reducible completely} \\ V \text{ over } \mathbb{F}[G] \text{ semisimple} \end{array}\right\} \Rightarrow \rho(g)$ diagonalisable $\forall g$ $\begin{pmatrix} 1 & j \\ 0 & 1 \end{pmatrix} \cong \begin{pmatrix} u^* & 0 \\ 0 & ? \end{pmatrix}$.

3. If $|G| = 0$ in $\mathbb{F}$, let $\rho: D_8 \to GL_2(\mathbb{F}_2)$. $|D_8| = 8 = 0$ in $\mathbb{F}_2$. $x \mapsto \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, $y \mapsto \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. Then let $U = \text{span}(e_1 + e_2) = \text{span}\left\{\begin{pmatrix} 1 \\ 1 \end{pmatrix}\right\}$. Then

$U \leq V = \mathbb{F}_2^2$ is a D_8-invariant submodule. $\rho(x) \cdot u = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \underbrace{\begin{pmatrix} 1 \\ 1 \end{pmatrix}}_{\in U}$. Let $\underset{2D}{V} = \underset{1D}{U} \oplus \underset{1D}{W}$, let $W = \text{span}\{\lambda e_1 + \mu e_2 : \lambda, \mu \text{ not both zero}\}$.

If $\lambda = 0, \mu \neq 0 \Rightarrow W = \text{span}(e_2)$ and if $\lambda \neq 0, \mu = 0 \Rightarrow W = \text{span}(e_1)$. However, both are not D_8-invariant by x. So $\nexists\, W$ s.t. $V = \mathbb{F}_2^2 = U \oplus W$.

Thus, $V = \mathbb{F}_2^2$ is not a simple $\mathbb{F}_2[D_8]$-module.

The point of Maschke's theorem is to give us an algorithm to decompose V as an $\mathbb{F}[G]$-module into $\bigoplus \mathbb{F}[G]$-modules: Here we have an example where Maschke works:

Use S.E.S. proof / splitting map $s: A \oplus C \to B$. Define $\rho: S_3 \to GL_3(\mathbb{C})$ by $\sigma \cdot e_i = e_{\sigma(i)}$ $\forall \sigma \in S_3$. $\sigma = (1\,2\,3) = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$, $\tau = (1\,2) = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$.

Let $p: B \to C$ be the projection map as $\mathbb{F}$-modules: $\begin{array}{l} e_1 \mapsto 0 \\ e_2 \mapsto 0 \\ e_3 \mapsto e_1 + e_2 + e_3 \end{array}$. By basis theorem, $B \cong A \oplus C$ as $\mathbb{F}$-modules, and also as $\mathbb{F}[G]$-modules using splitting map.

Use $s(v) = \frac{1}{\underset{6}{\underset{\|}{|G|}}} \sum_{g \in S_3} g^{-1} \sigma(g \cdot v)$ to perform an averaging process. Then we get $V = \mathbb{C}^3 = U \oplus W = \text{Im}(s) \oplus \text{Ker}(s) = \text{span}(e_1 - e_2, e_3 - e_2)$ [dim 2].

By Maschke, $S_3 \cong D_6$ with $\rho: S_3 \mapsto GL_3(\mathbb{C})$ is reducible i.e. $V = \mathbb{C}^3$ is a semi-simple $\mathbb{C}[S_3]$-module $= \underset{(\text{simple})}{1 \text{ dim submodule}} \underset{(\text{or } \oplus)}{\oplus}$ 2 dim submodule. (simple? even if not, can be further broken down).

$\boxed{\text{Lemma}}$ Let V and W be R-modules s.t. $\text{Hom}_R(V, W) = \text{Hom}_R(W, V) = 0$. Then $\text{End}_R(V \oplus W) \cong \text{End}_R(V) \times \text{End}_R(W)$.

Proof – $\text{Hom}_R(V, W)$ = ring matrices of the form $\begin{pmatrix} \alpha_{VV} & \alpha_{VW} \\ \alpha_{WV} & \alpha_{WW} \end{pmatrix}$, where $\left.\begin{array}{ll} \alpha_{VV}: V \to V & \alpha_{VW}: V \to W \\ \alpha_{WV}: W \to V & \alpha_{WW}: W \to W \end{array}\right\}$ are R-linear module homomorphisms.

Since $\text{Hom}_R(V, W) = \text{Hom}_R(W, V) = 0$ $\therefore$ $\text{End}_R(V \oplus W) \overset{\cong}{\longmapsto} \text{End}_R(V) \times \text{End}_R(W)$ defines an isomorphism $\alpha \mapsto (\alpha_{VV}, \alpha_{WV})$ // q.e.d.

Schur's lemma revisited for $\mathbb{F}[G]$-modules $\leftrightarrow \rho$ reps of G.

(V1). If M, N are simple non-zero modules, then $\varphi: M \to N$ is either 0 or homomorphism.

(V2) If M is a simple $\overset{R}{\overset{\|}{\mathbb{F}[G]}}$-module, then $\text{End}_{\mathbb{F}[G]}(M)$ is a division ring. i.e. $\varphi \in \text{End}_{\mathbb{F}[G]}(M)$ then $\varphi = \lambda \cdot \text{Id}$ for λ a non-zero eigenvalue since $\mathbb{F}$ is algebraically closed.

Note – In (V1) we do not require $\mathbb{F}$ to be algebraically closed.

(V3) [Strong form]. If V is a finitely-generated $\mathbb{F}[G]$-module s.t. $\text{char}(\mathbb{F}) \nmid |G|$, then V is simple $\Leftrightarrow$ $\text{End}_{\mathbb{F}[G]}(V)$ is a division ring

Proof – ($\Rightarrow$) (V2). ($\Leftarrow$). Suppose $V \cong S_1^{n_1} \oplus \dots \oplus S_m^{n_m}$ is semi-simple. Then $\text{End}_{\mathbb{F}[G]}(V) = \text{End}_{\mathbb{F}[G]}(S_1^{n_1} \oplus \dots \oplus S_m^{n_m})$, n_i are dimension of S_i, $S_i \neq S_j$ if $i \neq j$.

$= \text{End}_{\mathbb{F}[G]}(S_1^{n_1}) \oplus \dots \oplus \text{End}_{\mathbb{F}[G]}(S_m^{n_m})$ [consult HW2 or previous lemma]. $= M_{n_1}(\text{End}_{\mathbb{F}[G]}(S_1)) \oplus \dots \oplus M_{n_m}(\text{End}_{\mathbb{F}[G]}(S_n))$.

$= \bigoplus_{i=1}^{m} M_{n_i}(\text{End}_{\mathbb{F}[G]}(S_i))$. Assume $\text{End}_{\mathbb{F}[G]}(V)$ is a division ring, then $\bigoplus_{i=1}^{m} M_{n_i}(\text{End}_{\mathbb{F}[G]}(S_i)) \underset{(V2)}{\overset{\text{Schur}}{=}} \bigoplus_{i=1}^{m} M_{n_i}(D_i)$ ← division rings.

The RHS is a division ring if $\exists$ unique r s.t. $n_i = \begin{cases} 1 & i = r \\ 0 & i \neq r \end{cases}$. $\Rightarrow V \equiv S_r$ simple. $\text{End}_{\mathbb{F}[G]}(V) = \bigoplus M_{n_i}(D_i)$.

Schur's lemma (V3) is a tool for detecting when a representation is irreducible/reducible i.e. when V as an $\mathbb{F}[G]$-module is simple/semi-simple.

Some elegant examples of Schur (V3) –

1. Let $\rho: D_8 \to GL_2(\mathbb{C})$. $x \mapsto \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, $y \mapsto \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. Is ρ irreducible (= simple as a $\mathbb{C}[D_8]$-module)? Compute $\text{End}_{\mathbb{C}[D_8]}(\rho)$ = All complex 2×2 matrices that commute with all $g \in D_8$.

i.e. $A \in GL_2(\mathbb{C})$ s.t. $A\rho(g) = \rho(g)A$ $\forall g \in D_8$. Only need to do $A\rho(x) = \rho(x)A$, $A\rho(y) = \rho(y)A$ (conjugates the generators). Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$.

$A\rho(x) = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} -b & a \\ -d & c \end{pmatrix}$, $\rho(x) \cdot A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} c & d \\ -a & -b \end{pmatrix} \Rightarrow a = d, c = -b$, simply by comparing terms. Then $A = \begin{pmatrix} a & b \\ -b & a \end{pmatrix}$. Then we observe that

$A\rho(y) = \begin{pmatrix} a & b \\ -b & a \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} a & -b \\ -b & -a \end{pmatrix}$, $\rho(y) \cdot A = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} a & b \\ -b & a \end{pmatrix} = \begin{pmatrix} a & b \\ b & -a \end{pmatrix} \Rightarrow a = a, b = -b \Rightarrow 2b = 0, b = 0. \Rightarrow A = \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}$. $\therefore \text{End}_{\mathbb{C}[D_8]}(\rho) = \left\{\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} \middle| a \in \mathbb{C}\right\} \cong \mathbb{C}$.

by $A \mapsto a$, which is a division ring. Therefore, ρ is an irreducible 2-dimensional representation of $D_8 \Rightarrow V = \mathbb{C}^2$ is simple as $\mathbb{C}[D_8]$-module.

2. Let $\sigma: D_6 \to GL_3(\mathbb{C})$. $\sigma(x) = (1\,2\,3) = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$, $\sigma(y) = (1\,2) = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$. Is σ irreducible? Compute $\text{End}_{\mathbb{C}[D_6]}(\rho) \Rightarrow$ all $M_3(\mathbb{C})$ that commutes with generators.

$\mathrm{End}_{\mathbb{C}[D_6]}(\sigma) = \{A \in GL_3(\mathbb{C}) \mid A\sigma(x) = \sigma(x)A,\ A\sigma(y) = \sigma(y)A\}$. Let $A = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & k \end{pmatrix}$. $A\sigma(x) = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & k \end{pmatrix}\begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} b & c & a \\ e & f & d \\ h & k & g \end{pmatrix}$, $\sigma(x)\cdot A = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & k \end{pmatrix} = \begin{pmatrix} g & h & k \\ a & b & c \\ d & e & f \end{pmatrix}$

Then $a = k = e$, $b = g = f$, $c = h = d \Rightarrow A = \begin{pmatrix} a & b & c \\ c & a & b \\ b & c & a \end{pmatrix}$. Then $A\sigma(y) = \begin{pmatrix} a & b & c \\ c & a & b \\ b & c & a \end{pmatrix}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} b & a & c \\ a & c & b \\ c & b & a \end{pmatrix}$, $\sigma(y)\cdot A = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} a & b & c \\ c & a & b \\ b & c & a \end{pmatrix} = \begin{pmatrix} c & a & b \\ a & b & c \\ b & c & a \end{pmatrix}$.

As such, $a = a$, $b = c \Rightarrow A = \begin{pmatrix} a & b & b \\ b & a & b \\ b & b & a \end{pmatrix}$. $\therefore \mathrm{End}_{\mathbb{C}[D_6]}(\sigma) = \left\{\begin{pmatrix} a & b & b \\ b & a & b \\ b & b & a \end{pmatrix} \middle| a, b \in \mathbb{C}\right\} \cong \mathbb{C}^2$ by $A \mapsto (a,b)$. $\mathbb{C}^2$ is not a division ring.
↓
e.g. $(1,0)^{-1}$ does not exist.

$\therefore \rho: D_6 \to GL_3(\mathbb{C})$, $g \mapsto A$ is not irreducible $\Rightarrow$ it is reducible $\Rightarrow$ $V = \mathbb{C}^3$ is a semisimple $\mathbb{C}[D_6]$-module.

$\boxed{\text{Definition}}$ Let $\rho_1: G \to GL(U)$, $g \mapsto A$ and $\rho_2: G \to GL(W)$, $g \mapsto B$ be two representations of G. Then define the <u>direct sum of representations</u> $(\rho_1 \oplus \rho_2)(g) = \rho_1(g) \oplus \rho_2(g)$ be the map $\rho_1 \oplus \rho_2$ with representation space $U \oplus W$. Let $\mathrm{span}\{u_1, \dots, u_m\} = U$ and $\mathrm{span}\{w_1, \dots, w_n\} = W$. Then with respect to these bases $\rho_1: G \to GL_m(\mathbb{F})$ and $\rho_2: G \to GL_n(\mathbb{F})$, $\therefore \rho_1 \oplus \rho_2$ w.r.t. $\{(u_1, 0), \dots, (u_m, 0), (0, w_1), \dots, (0, w_n)\}$ gives a representation.

$\rho_1 \oplus \rho_2: G \to GL_{m+n}(\mathbb{F})$, $g \mapsto \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$ is a block diagonal matrix.

Observe that from our last example, $\sigma: D_6 \to GL_3(\mathbb{C})$ in the last example was a direct sum of 2 simple (1-dim & 2-dim) $\mathbb{F}[G]$-modules.

$T\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}T^{-1} = \begin{pmatrix} 0 & 0 & \boxed{1} \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$ was conjugated, so we could not easily identify A and B.

We now classify all representations of finite abelian groups over $\mathbb{C}$:

Let G be a finite abelian group, $C_{n_1} \times \dots \times C_{n_s}$. Let V be a simple $\mathbb{C}[G]$-module. Since G is abelian, $g\cdot(xv) = gxv = xg\cdot v$ $\forall x, g \in G$, $\forall v \in V$.

Let $\varphi_x \in \mathrm{End}_{\mathbb{F}[G]}(V)$, $\varphi_x: V \to V$, $v \mapsto xv$ for a fixed $x \in G$. By Schur (v2), $\mathrm{End}_{\mathbb{C}[G]}(V)$ is a division ring as V is simple, so every map has an inverse

$\therefore \varphi_x = \lambda_x \mathrm{Id}$ for $\lambda_x \neq 0$, $\lambda_x \in \mathbb{C}$. $\therefore$ by substitution, our map is $\lambda_x \mathrm{Id}: V \to V$, $v \mapsto \lambda_x \cdot v$ $\Rightarrow$ scalar multiple of $v \Rightarrow \lambda_x \in \mathbb{C}$.

Then V is simple $\Rightarrow \dim V = 1 \Rightarrow$ every irreducible $\mathbb{C}[G]$-module is one-dimensional: every irreducible representation of G is 1-dimensional.

Example of all irreducible reps of G over $\mathbb{C}$. — 1. Let $\rho_{\lambda_i}: C_n \to GL_1(\mathbb{C})$ $x \mapsto \lambda_i^k$, $\lambda_i^k = e^{\frac{2\pi i}{n}}$, i.e. $x \mapsto \xi_i$, $\qquad \mathbb{C}[C_n] \cong \mathbb{C}[x]/(x^n - 1) = \mathbb{C}[x]/(x-1) \times \dots \times \mathbb{C}[x]/(x - \xi_{n-1})$.

2. $\rho: C_2 \times C_2 \to GL_1(\mathbb{C})$ $\quad \rho_1: \begin{matrix} x \mapsto 1 \\ y \mapsto 1 \end{matrix}$ $\quad \rho_2: \begin{matrix} x \mapsto 1 \\ y \mapsto -1 \end{matrix}$ $\quad \rho_3: \begin{matrix} x \mapsto -1 \\ y \mapsto 1 \end{matrix}$ $\quad \rho_4: \begin{matrix} x \mapsto -1 \\ y \mapsto -1 \end{matrix}$.

repeat (a) Let $G = C_n = \langle x \mid x^n = 1 \rangle$. We can define $|C_n| = n$ irreducible 1D representations by $\rho_{\lambda_k}: C_n \to GL_1(\mathbb{C})$, $x \mapsto \lambda_k \in \mathbb{C}$, $\lambda_k = e^{\frac{2\pi i k}{n}}$. $\qquad$ 1 November 2013

$\mathbb{C}[C_n] \cong \mathbb{C}[x]/(x^n - 1) \cong \mathbb{C}[x]/(x - \lambda_0) \times \dots \times \mathbb{C}[x]/(x - \lambda_n) \overset{1D}{\cong} \mathbb{C} \times \dots \times \mathbb{C}$ $\Rightarrow \rho_0 \times \dots \times \rho_k$ are representations. $\qquad$ Mr Jamil NADIM

Maths 500.

repeat (b) Let $G = C_2 \times C_2$. There are $|G| = 4$ irreducible representations of $C_2 \times C_2$. $\rho_i: C_2 \times C_2 \to GL_1(\mathbb{C})$ by $\begin{matrix} x \mapsto 1 \\ y \mapsto 1 \end{matrix}$ or $\begin{matrix} x \mapsto 1 \\ y \mapsto -1 \end{matrix}$ or $\begin{matrix} x \mapsto -1 \\ y \mapsto 1 \end{matrix}$ or $\begin{matrix} x \mapsto 1 \\ y \mapsto 1 \end{matrix}$

So $\mathbb{C}[C_2 \times C_2] \cong U_1 \oplus U_2 \oplus U_3 \oplus U_4 \cong \underset{\rho_1}{\mathbb{C}} \times \underset{\rho_2}{\mathbb{C}} \times \underset{\rho_3}{\mathbb{C}} \times \underset{\rho_4}{\mathbb{C}}$, where the U_i are all submodules of dimension 1.

3. $\oplus$ example: define $\rho: C_n \to GL_n(\mathbb{C})$. $x \mapsto \begin{pmatrix} \lambda_0 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix}$ = direct sum of all the simple representations of dimension 1.

$\boxed{\text{Definition}}$ The <u>regular representation</u> corresponds to a module $\mathbb{C}[G]$ as a $\mathbb{C}[G]$-module (by correspondence theorem). Since $\mathbb{C}[G] = V$ is a $\mathbb{C}$-algebra with basis $B = G = \{1 = g_1, \dots, g_n\}$ define the regular representation $\rho_{reg}: G \to GL(\mathbb{C}[G]) = GL(|G|)$ by $g \mapsto \rho_g$ where ρ_g is a matrix obtained by left g-action on the basis B $g \circ g_i = g_i \to$ matrix $\rho_{reg}(G)$. Point: ρ_{reg} is always reducible.

Examples of regular representations —

1. Let $G = C_3 = \{1, x, x^2\}$. Label $\begin{matrix} g_1 = 1 \\ g_2 = x \\ g_3 = x^2 \end{matrix}$. $\rho_{reg}: C_3 \to GL_3(\mathbb{C}) = GL(\mathbb{C}[C_3])$. $\rho_x(g_1) = \rho_x(1) = x1 = x = g_2$. $\rho_x(g_2) = \rho_x(x) = x \cdot x = x^2 = g_3$ $\rho_x(g_3) = \rho_x(x^2) = x \cdot x^2 = x^3 = 1 = g_1$.
$\rho: x = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} = \rho_{reg}(x)$.
Fact: $\mathbb{C}[G] \equiv \rho_{reg}$ is reducible = semisimple and we decompose using Wedderburn and Maschke. $\mathbb{C}[G] \cong U_1 \oplus \dots \oplus U_r \cong S_1^{n_1} \oplus \dots \oplus S_r^{n_r}$, where U_i are simple $\mathbb{C}[G]$-submodules of dim $\neq n$. $\cong M_{n_1}(\mathbb{C}) \times \dots \times M_{n_r}(\mathbb{C}) \cong \rho_1 \times \dots \times \rho_r$ = all irreducible reps of G. By an old proposition, any simple $\mathbb{C}[G]$-submodule is $\cong S_i$ for some i.

2. $C_3 = \langle x \mid x^3 = 1 \rangle$. Decompose $\mathbb{C}[C_3]$: let $u_1 = 1 + x + x^2 \in \mathbb{C}[C_3]$, $U_1 = \mathrm{span}\{u_1\}$ then $x \cdot u_1 = x + x^2 + 1 = 1 \cdot u_1 \Rightarrow U_1 \leq \mathbb{C}[C_3]$ (as it is $\mathbb{C}[C_3]$-invariant), where U_1 is a 1D submodule. (and we said every submodule is giving us a representation). Let $u_2 = 1 + w^2 x + wx$, $x \cdot u_2 = w \cdot u_2 \Rightarrow U_2 = \mathrm{span}\{u_2\} \leq \mathbb{C}[C_3]$. $\therefore U_2$ is a 1D submodule giving us a representation $\rho: C_3 \to GL_1(\mathbb{C})$ by $x \mapsto w$. Let $u_3 = 1 + wx + w^2 x$. Repeat as above. Then $\mathbb{C}[C_3] \cong U_1 \oplus U_2 \oplus U_3 \cong \mathbb{C} \oplus \mathbb{C} \oplus \mathbb{C}$.

$\boxed{\text{Corollary}}$ If all irreducible representations are of dimension 1, then G is abelian.

Proof — By Maschke and Wedderburn, we can decompose $\mathbb{C}[G] \cong U_1 \oplus U_2 \oplus \dots \oplus U_r$ where all the U_i are 1D by assumption. Choose a basis: $\{u_1, \dots, u_r\}$ and write matrix action on the basis. $\rho(g) = \begin{pmatrix} \lambda_0 & & 0 \\ & \ddots & \\ 0 & & \lambda_r \end{pmatrix}$, which is diagonal. $G/\ker(\rho) \cong \mathrm{Im}(\rho) \in \{\text{diagonal matrices}\}$ $\therefore G$ is abelian, q.e.d.

Remark — A consequence of this is that if $\exists$ an irreducible/simple representation of dim ≥ 2, then G is noncommutative i.e. non-abelian i.e. $\exists \rho: G \to GL_n(\mathbb{C})$ $n \geq 2$ (e.g. D_6, Q_{4m} etc).

$\boxed{\text{Theorem}}$ $\mathbb{C}[G]$ is a semisimple algebra $\Leftrightarrow$ $\mathbb{C}[G]$ viewed as a $\mathbb{C}[G]$-module is semisimple.

Proof — Suppose $\mathbb{C}[G]$ is semisimple, where the S_i are non-pairwise isomorphic simple submodules of dim n: $\mathbb{C}[G]^{op} = \mathrm{End}_{\mathbb{C}[G]}(\mathbb{C}[G]) = \mathrm{End}(S_1^{n_1} \oplus \dots \oplus S_r^{n_r}) = \mathrm{End}(S_1^{n_1}) \oplus \dots \oplus \mathrm{End}(S_r^{n_r}) = M_{n_1}(\mathrm{End}(S_1)) \oplus \dots \oplus M_{n_r}(\mathrm{End}(S_r))$. Claim: $\mathrm{End}(S_i) \cong \mathbb{C}$. This is because $\mathrm{End}(S_i)$ is a division ring (by Schur's v2 + Burnside's thm), but the only division ring over $\mathbb{C}$ is $\mathbb{C}$, so it must be $\mathbb{C}$, so $= M_{n_1}(\mathbb{C}) \oplus \dots \oplus M_{n_r}(\mathbb{C}) = \mathbb{C}[G]^{op}$. Take opposites again: then we get that overall,

$\mathbb{C}[G] = (\mathbb{C}[G]^{op})^{op} = (M_{n_1}(\mathbb{C}) \oplus \dots \oplus M_{n_r}(\mathbb{C}))^{op} = M_{n_1}(\mathbb{C})^{op} \oplus \dots \oplus M_{n_r}(\mathbb{C}) = M_{n_1}(\mathbb{C}^{op}) \oplus \dots \oplus M_{n_r}(\mathbb{C}^{op}) = M_{n_1}(\mathbb{C}) \oplus \dots \oplus M_{n_r}(\mathbb{C}) = \rho_1 \oplus \dots \oplus \rho_r$, q.e.d.

Definition The values $\dim_{\mathbb{C}}(S_i) = n_i$ are the degrees of the irreducible representations of G.

Corollary order of group, $|G| = n_1^2 + \dots + n_r^2$.

Proof – $|G| = \dim_{\mathbb{C}}(\mathbb{C}[G]) = \dim_{\mathbb{C}}(\overbrace{\bigoplus_{i=1}^{r} M_{n_i}(\mathbb{C})}^{\text{weddeburn}}) = \sum_{i=1}^{r} \dim(M_{n_i}(\mathbb{C})) = n_1^2 + \dots + n_r^2$ where $\dim_{\mathbb{C}}(M_{n_i}(\mathbb{C})) = n_i^2$ //.

thus, we obtain a generalised approach to decomposing $\mathbb{C}[G]$ and classifying representations for G.

Note – This only provides us with information on the type of representations (and their degrees), but not the matrices themselves.

1. Decompose $\mathbb{C}[G]$, then $\mathbb{C}[G] = M_{n_1}(\mathbb{C}) \times \dots \times M_{n_r}(\mathbb{C})$.
2. Note that $|G| = n_1^2 + \dots + n_r^2$
3. We can always take $n_1 = 1$ as we always have the trivial representation.
4. Each $n_i \mid |G|$ exactly.
5. We stop at r = number of conjugacy classes of $G = \#\{x^G\} = \#\{gxg^{-1} : g \in G\}$.

We examine several examples of group rings, as follows.

1. $\mathbb{C}[C_2]$. We begin by finding the conjugacy classes of $C_2 = \{1, x\}$ $x^2 = 1$. These are $\{1\}$ and $\{x\}$, so $r = 2$. $|G| = |C_2| = 2 = n_1^2 + n_2^2$ and we may take $n_1 = 1$. So, $2 = 1 + n_2^2$.

   Thus, we must have $n_2 = 1$. So there are 2 representations: $\mathbb{C}[C_2] = M_1(\mathbb{C}) \times M_1(\mathbb{C}) = \underset{\rho_1 \left(\begin{smallmatrix} x \mapsto 1 \\ 1 \mapsto 1 \end{smallmatrix}\right)}{\mathbb{C}} \times \underset{\rho_{-1} \left(\begin{smallmatrix} x \mapsto -1 \\ 1 \mapsto -1 \end{smallmatrix}\right)}{\mathbb{C}}$.

2. $\mathbb{C}[C_3]$. The conjugacy classes of $C_3 = \{1, x, x^2\}$ $x^3 = 1$ are $\langle 1 \rangle, \langle x \rangle, \langle x^2 \rangle \Rightarrow r = 3$. Then we have $|G| = |C_3| = 3 = n_1^2 + n_2^2 + n_3^2 = 1^2 + n_2^2 + n_3^2$.

   This again forces us to take $n_2 = n_3 = 1$. As such, we have $n_1 = n_2 = n_3 = 1$ and therefore,

   $\mathbb{C}[C_3] = M_1(\mathbb{C}) \times M_1(\mathbb{C}) \times M_1(\mathbb{C}) = \mathbb{C} \times \mathbb{C} \times \mathbb{C}$ which corresponds to $\rho_1 \times \rho_\omega \times \rho_{\omega^2}$ where $\omega = e^{\frac{2\pi i}{3}}$ is the cube root of unity.

   $\Rightarrow C_3$ has only 3 irreducible 1D representations.

3. $\mathbb{C}[D_6]$. The conjugacy classes of D_6 are $\langle 1 \rangle, \langle x \rangle, \langle y \rangle \Rightarrow r = 3$. $|G| = |D_6| = 6 = n_1^2 + n_2^2 + n_3^2 = 1 + n_2^2 + n_3^2 \Rightarrow n_1 = n_2 = 1$, $n_3 = 2$ WLOG. Then,

   $\mathbb{C}(D_6) = M_1(\mathbb{C}) \times M_1(\mathbb{C}) \times M_2(\mathbb{C}) \cong \mathbb{C} \times \mathbb{C} \times M_2(\mathbb{C}) \cong \rho_1 \times \rho_2 \times \rho_3$.

   $\left(\begin{smallmatrix} x \mapsto 1 \\ y \mapsto 1 \end{smallmatrix}\right) \left(\begin{smallmatrix} x \mapsto 1 \\ y \mapsto -1 \end{smallmatrix}\right) \left(x \mapsto \left(\begin{smallmatrix} \omega & 0 \\ 0 & \omega^2 \end{smallmatrix}\right), y \mapsto \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)\right)$.

   Thus, we see that D_6 has 2 simple 1D representations, and 1 simple (i.e. irreducible) 2D representation only.

   Note – The matrix $\left(\begin{smallmatrix} \omega & 0 \\ 0 & \omega^2 \end{smallmatrix}\right)$ is irreducible.

   We can always bring it to the standard form by finding T s.t. $T\left(\begin{smallmatrix} \omega & 0 \\ 0 & \omega^2 \end{smallmatrix}\right)T^{-1}$ = standard model.

4. Wedderburn Decomposition of $\mathbb{C}[Q_8]$. $Q_8 = \{1, x, x^2, x^3, y, xy, x^2y, x^3y\}$ $x^2 = y^2$, $x^4 = 1$. We have conjugacy classes: $(x)^G = \{gxg^{-1} : g \in G\}$. 11 November 2013. Mr Jamil NADIM. Gordon St (25) D103.

   These are $\{1\}, \{x^2\}, \{x, x^3\}, \{y, x^2y\}, \{xy, x^3y\} \Rightarrow r = 5$. Solve $|Q_8| = 8 = n_1^2 + n_2^2 + n_3^2 + n_4^2 + n_5^2 = 1^2 + 1^2 + 1^2 + 1^2 + 2^2$ is only solution up to order. $\mathbb{C}[Q_8] = M_1(\mathbb{C}) \times M_1(\mathbb{C}) \times M_1(\mathbb{C}) \times M_1(\mathbb{C}) \times M_2(\mathbb{C}) \equiv \mathbb{C}^{(4)} \times M_2(\mathbb{C})$. $\Rightarrow$ 4 irreducible 1D reps: $\rho_1, \rho_2, \rho_3, \rho_4 : Q_8 \to \mathbb{C}$; 1 unique 2D reps: $\rho_5 : Q_8 \to GL_2(\mathbb{C})$.

5. Take $G = D_{10}$. $= \langle x, y \mid x^5 = y^2 = 1, yx = x^4y \rangle$. Conjugacy classes are $\{1\}, \{x, x^4\}, \{x^2, x^3\}, \{y, xy, x^2y, x^3y, x^4y\} \Rightarrow r = 4$. Solve $|D_{10}| = 10 = n_1^2 + n_2^2 + n_3^2 + n_4^2 = 1^2 + 1^2 + 2^2 + 2^2$.

   Then $\mathbb{C}[D_{10}] \cong M_1(\mathbb{C})^{(2)} \times M_2(\mathbb{C})^{(2)} = \mathbb{C}^{(2)} \times M_2(\mathbb{C})^{(2)} \Rightarrow$ 2 irreducible 1D reps, 2 irreducible 2D reps only.

6. Take $G = A_4 = \{\sigma \in S_4 : \operatorname{sgn}(\sigma) = +1\}$ = even permutations on 4 items. $A_4 = \langle s, t, x \mid s^2 = t^2 = (st)^2 = 1, x^3 = 1, xsx^{-1} = st, xtx^{-1} = s, x\,st\,x^{-1} = t \rangle$ where we have

   $s = \left(\begin{smallmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \end{smallmatrix}\right)$, $t = \left(\begin{smallmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 1 & 2 \end{smallmatrix}\right)$, $st = \left(\begin{smallmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 2 & 1 \end{smallmatrix}\right)$, $x = \left(\begin{smallmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 1 & 4 \end{smallmatrix}\right)$. $A_4 = \{1, s, t, st, x, xs, xt, xst, x^2, x^2s, x^2t, x^2st\}$. We then compute conjugacy classes:

   $\{1\}, \{s, t, st\}, \{x, xs, xt, xst\}, \{x^2, x^2s, x^2t, x^2st\} \Rightarrow r = 4$. Solve $|A_4| = 12 = n_1^2 + n_2^2 + n_3^2 + n_4^2 = 1^2 + 1^2 + 1^2 + 3^2$ up to order $\Rightarrow$ 3 irreducible 1D reps. 1 irreducible 3D rep only.

Definition An $\mathbb{F}[G]$-module M is completely reducible when $M = \bigoplus_{i=1}^{r} M_i = M_1 \oplus \dots \oplus M_r$, where each M_i is a simple $\mathbb{F}[G]$-submodule.

Lemma (Maschke's corollary)

If G finite, $|G| \neq 0$ in $\mathbb{F}$, then every $\mathbb{F}[G]$-module M is completely reducible (semisimple) into $\mathbb{F}[G]$-submodules. $\therefore$ each M_i is simple and of any dimension.

Proof – If M is simple, nothing to prove. So suppose M is semisimple. Then choose an $\mathbb{F}[G]$ submodule of each possible dimension, say M. Proof by induction.

By induction hypothesis, any $\mathbb{F}[G]$-module of dimension $< \dim_{\mathbb{F}}(M)$ is completely ~~neutral~~ reducible. Form a SES: $0 \to M_1 \to M \xrightarrow{[\,]} M/M_1 \to 0$, $[x] \mapsto x + M_1$ which splits by Maschke.

since $M_1 \neq \{0\}$, $\dim(M_1) < \dim M \Rightarrow \dim_{\mathbb{F}}(M/N) < \dim_{\mathbb{F}}(M)$ $\therefore$ By induction hypothesis, $M/M_1 \equiv M_2 \oplus \dots \oplus M_r$ // $M \equiv M_1 \oplus M_2 \oplus \dots \oplus M_r$

$\mathbb{C}[G] = M_{n_1}(\mathbb{C}) \oplus \dots \oplus M_{n_r}(\mathbb{C})$.

We seek a more rapid approach to test conjugacy classes. The conjugacy classes of x is $x^G = \{gxg^{-1} : g \in G\}$. Also, note that conjugacy classes are disjoint.

Definition The centre of a group ring $Z(\mathbb{C}[G]) = \{z \in \mathbb{C}[G] \mid zx = xz \ \forall x \in \mathbb{C}[G]\}$ (i.e. the set of elements that commute with all others).

$\boxed{\text{Lemma}}$ $\dim_{\mathbb{C}}(Z(\mathbb{C}[G])) = r$.

Proof – By Wedderburn and Maschke, $\mathbb{C}[G] \cong M_{n_1}(\mathbb{C}) \times \cdots \times M_{n_r}(\mathbb{C})$, so $Z(\mathbb{C}[G]) \cong Z(M_{n_1}(\mathbb{C})) \times \cdots \times Z(M_{n_r}(\mathbb{C})) \cong \overbrace{\mathbb{C} \times \mathbb{C} \times \cdots \times \mathbb{C}}^{r \text{ copies.}}$

since $Z(M_{n_1}(\mathbb{C})) = \left\{\begin{pmatrix} * & & 0 \\ & \ddots & \\ 0 & & * \end{pmatrix} : * \in \mathbb{C}\right\} \cong \mathbb{C} \Rightarrow \therefore \dim_{\mathbb{C}}(Z(\mathbb{C}[G])) = r$ where $\mathbb{C}[G]$ is a $\mathbb{C}$-algebra, $Z(\mathbb{C}[G])$ is a $\mathbb{C}$-subalgebra.

$\boxed{\text{Theorem}}$ If G is finite, in $\mathbb{C}[G] = M_{n_1}(\mathbb{C}) \times \cdots \times M_{n_r}(\mathbb{C})$, where r = no. of conjugacy classes.

Proof – let $z = \sum_{g \in G} \lambda_g g \in Z(\mathbb{C}[G]) = \{z \in \mathbb{C}[G] \mid xz = zx \ \forall x \in \mathbb{C}[G]\}$ and conjugate $\forall h \in G$: $hzh^{-1} = z = \sum \lambda_g g$ (since $z \in Z(\mathbb{C}[G])$).

Also by definition, $hzh^{-1} = h(\sum \lambda_g g)h^{-1} = \sum \lambda_g hgh^{-1} \overset{\text{reindex}}{=} \sum \lambda_{hgh^{-1}} g \Rightarrow$ coefficients are constant on conjugacy classes. $\Rightarrow \therefore r$ in n_r refers to the number of conjugacy classes. Basis for $Z(\mathbb{C}[G])$ consists of linear combinations: $\sum_{g_i \in K_i} g_i$ where K_i is a conjugacy class. $\therefore \dim_{\mathbb{C}}(Z(\mathbb{C}[G])) = r$, q.e.d.

Examples of applications –

1. $G = D_6 = \langle x, y \mid x^3 = y^2 = 1,\ yx = x^2 y\rangle$. Conjugacy classes are $\overset{K_1}{\{1\}}, \overset{K_2}{\{x, x^2\}}, \overset{K_3}{\{y, xy, x^2y\}} \Rightarrow r = 3$. $6 = 1^2 + 1^2 + 2^2$, $\mathbb{C}[D_6] = \mathbb{C} \times \mathbb{C} \times M_2(\mathbb{C})$, and a basis for $Z(\mathbb{C}[D_6])$ is $\{1,\ x + x^2,\ y + xy + x^2y\}$ and $\dim Z(\mathbb{C}[D_6]) = 3$.

We now examine some formulae to compute conjugacy classes: We first recall some definitions – ① the conjugacy classes of x: $x^G = \{gxg^{-1} : g \in G\}$. ② The centraliser of x is $C_G(x) = \{g \in G : gx = xg\}$. Then the size of $|x^G| = \frac{|G|}{|C_G(x)|}$ by the orbit-stabiliser theorem. ③ The Class Equation is $G = Z(G) + \sum_{x_i \notin Z(G)} x_i^G$.

Conjugacy classes are as follows –

1. For all cyclic groups, $C_n = \langle x \mid x^n = 1\rangle$. $\forall x^j \in C_n$, $x^i x^j x^{-i} = x^{i+j-i} = x^j \Rightarrow$ each x^j is its own conjugacy class. $C_n = \{1, x, x^2, \ldots, x^{n-1}\}$. $1^{C_n} = \{1\}$, $x^{C_n} = \{x\}$, $(x^2)^{C_n} = \{x^2\}$ etc.
$\Rightarrow n$ conjugacy classes $\{1\}, \{x\}, \ldots, \{x^{n-1}\} \Rightarrow n$ irreducible 1D representations, $\mathbb{C}[C_n] = \underbrace{\overset{\rho_1}{\mathbb{C}} \times \overset{\rho_2}{\mathbb{C}} \times \cdots \times \overset{\rho_n}{\mathbb{C}}}_{n}$.
2. Finite abelian groups $C_{n_1} \times \cdots \times C_{n_r}$ works the same way: $C_2 \times C_2 = \{1, x, y, xy\}$, $x^{C_2 \times C_2} = \{x\}$
3. Dihedral groups: $D_{2n} = \langle x, y \mid x^n = y^2 = 1,\ yx = x^{n-1}y\rangle$. Case 1 – If n is odd (e.g. D_6, D_{10}, D_{14}). There are $\frac{n+3}{2}$ conjugacy classes, which are $\{1\}, \{x^1, x^{-1}\}, \{x^2, x^{-2}\}, \ldots \{x^{(\frac{n-1}{2})}, x^{-(\frac{n-1}{2})}\}$ and $\{y, xy, x^2y, \ldots, x^{n-1}y\}$. For instance, $D_{14} \Rightarrow 14 = 2 \times 7 \Rightarrow \frac{7+3}{2} = 5$ conjugacy classes, which are $\{1\}, \{x, x^6\}, \{x^2, x^5\}, \{x^3, x^4\}, \{y, xy, x^2y, \ldots, x^6y\}$.
Case 2 –
If n is even (e.g. D_8, D_{12}), then there exist $m+3$ conjugacy classes, where $m = \frac{n}{2}$. These are $\{1\}, \{x^m\}, \underbrace{\{x^i, x^{-i}\} \text{ for } i = 1, 2, \ldots, m-1}_{\text{several sets, possibly.}}$, and $\{x^{2i}y : 0 \leq i \leq m-1\}$, $\{x^{2i+1}y : 0 \leq i \leq m-1\}$
E.g. $D_8 \Rightarrow 8 = 4 \cdot 2 \Rightarrow m + 3 = 5$ conjugacy classes, which are $\{1\}, \{x^2\}, \{x, x^3\}, \{y, x^2y\}, \{xy, x^3y\}$.
4. ~~Likewise~~ S_n conjugacy classes: # conjugacy classes = # permutation sets of same shape and size. e.g. $S_3 = \{Id, (1\,2), (1\,3), (2\,3), (1\,2\,3), (1\,3\,2)\}$. Then, the conjugacy classes are $\{Id\}, \{(1\,2), (1\,3), (2\,3)\}, \{(1\,2\,3), (1\,3\,2)\} \cong \{1\}, \{y, xy, x^2y\}, \{x, x^2\}$, which is congruent as $S_3 \cong D_6$. Likewise, we have the structure of S_4 – for $|S_4| = 24$, we have $\{1\}$, $\{(1\,2), (1\,4), \ldots\}$ transpositions, $\{(1\,2)(3\,4), \ldots\}$ products of transpositions, $\{(1\,2\,3), (1\,3\,2), \ldots\}$ 3-cycles, and $\{(1\,2\,3\,4), \ldots\}$ 4-cycles.
This is consistent with the partition of n, $p(n)$ e.g. for 5, $5 = 0+5, 1+4, 2+3, 1+1+3, 1+1+1+2, 1+1+1+1, 1+2+2 \Rightarrow S_4$ has $p(4) = 5$ conjugacy classes, S_5 has 7.
$\mathbb{C}[S_5] = n_1^2 + \cdots + n_7^2$.

Tensor Products.

These give us a way to break down structures such as $\mathbb{C}[G \times H]$ to the form $\bigoplus M_{n_i}(\mathbb{C})$. This works as $\mathbb{C}[G \times H] \cong \mathbb{C}[G] \otimes \mathbb{C}[H]$ for a tensor product $\otimes$.

Beware! These are not like $V \times W$, direct products of vector spaces... nor like direct sums $\oplus$. We know that $V \times W \cong V \oplus W$ for finite V, W. What about infinite vector spaces? We know that $\bigoplus_{i=1}^{\infty} V_i \subset \prod_{i=1}^{\infty} V_i$ is a proper subset, but $\bigoplus_{i=1}^{\infty} V_i \neq \prod_{i=1}^{\infty} V_i$. For instance, $\bigoplus_{i=1}^{\infty} \mathbb{R}^{(i)} \neq \prod_{i=1}^{\infty} \mathbb{R}$, since $(1, 0, 0, \ldots) \in \bigoplus_{i=1}^{\infty} \mathbb{R}^{(i)}$. However, $(1, 1, 1, \ldots) \notin \bigoplus_{i=1}^{\infty} \mathbb{R}^{(i)}$. Both vectors belong in $\prod_{i=1}^{\infty} \mathbb{R} = \mathbb{R} \times \mathbb{R} \times \cdots$

We care about finite vector spaces: recall $\dim_{\mathbb{F}}(V \oplus W) = \dim_{\mathbb{F}}(V \times W) = \dim(V) + \dim(W)$

Consider tensor products $V \otimes_{\mathbb{F}} W$ over fields $\mathbb{F}$. The idea is to construct a vector space $V \otimes_{\mathbb{F}} W$ whose elements look like $\sum_{i,j} v_i \otimes w_j$ where i, j are arbitrary. 15 November 2013. Mr Jamil NADIM Maths 500.

So for example: $\sum_{i=1}^{k} v_i \otimes w_i = v_1 \otimes w_1 + v_2 \otimes w_2 + \cdots + v_k \otimes w_k$, k arbitrary. Contrast this with $V \oplus W \ni (v, w) = (v, 0) + (0, w)$. We want $- \otimes_{\mathbb{F}} -$ to obey:

(1) $v \otimes (w + w') = v \otimes w + v \otimes w'$ (2) $(v + v') \otimes w = v \otimes w + v' \otimes w$ and (3) (Key Rule) $\lambda(v \otimes w) = \lambda v \otimes w = v \otimes \lambda w$ $\forall \lambda \in \mathbb{F}$ $\forall v, v' \in V$ $\forall w, w' \in W$.

Generally, if $v = \sum \lambda_i v_i$ and $w = \sum \mu_j w_j$, then $v \otimes w = \sum_{i,j} \lambda_i \mu_j (v_i \otimes w_j)$. For example – $v = 2v_1 - v_2$, $w = w_1 + 3w_2$. Then $v \otimes w = (2v_1 \otimes w_1) + (2v_1 \otimes 3w_2) - (v_2 \otimes w_1) - (v_2 \otimes 3w_2)$.

If given a basis $\{e_i\}_{1 \leq i \leq m}$ for V, $\{f_j\}_{1 \leq j \leq n}$ for W, then we want $\{e_i \otimes f_j\}$ to be a basis for $V \otimes_{\mathbb{F}} W$. However, this is not always possible to realise, although it is possible for vector spaces over $\mathbb{F}$. In this case, $\dim_{\mathbb{F}}(V \otimes W) = \dim_{\mathbb{F}}(V) \times \dim_{\mathbb{F}}(W)$ Example: $\dim_{\mathbb{R}}(\mathbb{R}^n \otimes \mathbb{R}^m) = \dim_{\mathbb{R}}(\mathbb{R}^n) \times \dim(\mathbb{R}^m) = nm$. "loads of elements in $V \otimes W$".

Can we make such a space exist? Yes. (Used heavily by physicists: for elasticity, electromagnetic fields, stress/strain).

$\boxed{\text{Definition}}$ Given vector spaces U, V, W over $\mathbb{F}$, a bilinear map $f : V \times W \to U$ satisfies

(1) $f(v_1 + v_2, w) = f(v_1, w) + f(v_2, w)$ (2) $f(v, w_1 + w_2) = f(v, w_1) + f(v, w_2)$ (3) $f(\lambda v, w) = f(v, \lambda w) = \lambda f(v, w)$ $\lambda \in \mathbb{F}$, $v_i \in V$, $w_j \in W$

Note – f is not linear! since $f(\lambda v, \lambda w) = \lambda f(v, \lambda w) = \lambda^2 f(v,w)$.

Examples –

1. The dot product $v \cdot w$ 2. The inner product $f = \langle , \rangle : V \times V \to \mathbb{R}$, $\langle v, w \rangle = v \cdot Aw$, $\forall A \in M_n(\mathbb{R})$. [Note: Dot product is just taking $A = I_n$].

3. Triple product $f : \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}^3$, $v \times w$. 4. $\begin{matrix} R \times M \to M \\ (\lambda, m) \mapsto \lambda m \end{matrix}$ scalar multiplication on modules is bilinear. 5. $R \times R \to R$ multiplication is linear.

<u>Definition</u> Let V, W be vector ~~sp~~ spaces over $\mathbb{F}$. By a <u>tensor product</u> $(V \otimes_{\mathbb{F}} W, - \otimes_{\mathbb{F}} -)$ we mean (0) $V \otimes_{\mathbb{F}} W$ is a vector space,

(1) a bilinear map $- \otimes_{\mathbb{F}} - : V \times W \to V \otimes W$ (2) given any bilinear map $f : V \times W \to U$ ~~st~~ $\exists$ a unique linear map $\tilde{f} : V \otimes W \to U$ making the diagram commute [i.e. $\tilde{f} \circ (- \otimes_{\mathbb{F}} -) = f$].

$$\begin{tikzcd} V \times W \arrow[r, "f", "\text{bilinear}"'] \arrow[dr, "\text{bilinear } - \otimes_{\mathbb{F}} -"'] & U \\ & V \otimes W \arrow[u, dashed, "\tilde{f} \text{ linear.}"'] \end{tikzcd}$$

$\therefore$ Every bilinear map f can be factored through $- \otimes -$ (this is called the <u>universal property</u> of $\otimes$). $\Rightarrow$ $V \otimes W$ turn bilinear maps into linear maps.

~~Note~~ Note –
1. To show two spaces are isomorphic using $- \otimes -$, just define linear maps $\tilde{f}$ and show universal property definition works.

2. To calculate, use bilinearity (key rule). $\lambda(v \otimes_{\mathbb{F}} w) = \lambda v \otimes_{\mathbb{F}} w = v \otimes_{\mathbb{F}} \lambda w$, $\lambda \in \mathbb{F}$.

We generalise this to modules over R = commutative rings: $V \otimes_{\mathbb{F}} W$. If rings are non-commutative, we encounter trouble. $\underset{\text{right mod}}{\lambda v} \otimes \underset{\text{left mod.}}{w}$

Examples of $\mathbb{Z}$-modules tensored together – Use Key Rule: $\lambda v \otimes w = v \otimes \lambda w = \lambda(v \otimes w)$.

1. $\mathbb{Z}/2\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/3\mathbb{Z} \cong \{0\}$ as $\mathbb{Z}$-modules. Let $\mathbb{Z}/2\mathbb{Z} = \{e, a\}$ s.t. $a + a = 2a = e$, $\mathbb{Z}/3\mathbb{Z} = \{e, b, 2b\}$ s.t. $3b = e$. [Later let $e=0$, $a=1$, $b=1$].

Note that elements of $\mathbb{Z}/2\mathbb{Z} \otimes \mathbb{Z}/3\mathbb{Z}$ look like $\lambda_1(e \otimes e) + \lambda_2(e \otimes b) + \lambda_3(e \otimes 2b) + \lambda_4(a \otimes e) + \lambda_5(a \otimes b) + \lambda_6(a \otimes 2b)$. So we only have 6 $\overbrace{\text{simple tensors}}^{\text{cannot be broken.}}$ to consider

Claim: All these simple tensors collapse to $e \otimes e = 0 \otimes 0 = 0$. ① $e \otimes e$ is itself ② $e \otimes b = 3e \otimes_{\mathbb{Z}} b \overset{3 \in \mathbb{Z}, \text{ by key rule, shift vectors.}}{=} e \otimes_{\mathbb{Z}} 3b = e \otimes e$ ③ $e \otimes 2b = 3e \otimes 2b = e \otimes 6b = e \otimes e$.

④ $a \otimes e = a \otimes 2e = 2a \otimes e = e \otimes e$ ⑤ $a \otimes b = 3a \otimes b = a \otimes 3b = a \otimes e = e \otimes e$. ⑥ $a \otimes 2b = 2a \otimes b = e \otimes b = e \otimes e$. $\Rightarrow$ all 6 products are $e \otimes e$.

$\mathbb{Z}/2\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/3\mathbb{Z} \cong \sum 0 \otimes 0 = \{0\}$.

2. $\mathbb{Z}/2\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z} \cong \mathbb{Z}/2\mathbb{Z}$. Let $\mathbb{Z}/2\mathbb{Z} = \{e, a\}$ $2a = e$, $\mathbb{Z} = \{0, \pm 1, \pm 2, \cdots\}$. Claim: We will only obtain 2 elements in tensor space. Our possibilities are as such:
$e \otimes$ even no. $a \otimes$ even no.
$e \otimes$ odd no. $a \otimes$ odd no. $\cdot\ e \otimes \text{even} = 0 \otimes \text{even} = 0 \cdot 0 \otimes \text{even} = 0 \otimes 0 \cdot \text{even} = 0 \otimes 0$. $\cdot\ e \otimes \text{odd} = 0 \cdot 0 \otimes \text{odd} = 0 \otimes 0 \cdot \text{odd} = 0 \otimes 0$.

$\cdot\ a \otimes$ even e.g. $a \otimes 2n = 2a \otimes n = e \otimes n = 0 \otimes n = 0 \otimes 0$ from above. $\cdot\ a \otimes$ odd e.g. $a \otimes 9 = a \otimes 3 \cdot 3 = 3a \otimes 3 = a \otimes 3$, cannot simplify.
$= 3a \otimes 1 = a \otimes 1$

Thus we get two elements, $\{0 \otimes 0, a \otimes 1\} \cong \mathbb{Z}/2\mathbb{Z}$.

3. $\mathbb{Z}^2 \otimes_{\mathbb{Z}} \mathbb{Z}^2 \cong \mathbb{Z}^4$ [to be seen later].

Formal properties of tensor products, using Universal Property –

1. $\begin{matrix} R \otimes V \cong V \\ \lambda \otimes v \mapsto \lambda v \end{matrix}$ 2. $\begin{matrix} V \otimes W \cong W \otimes V \\ (v \otimes w) \mapsto (w \otimes v) \end{matrix}$ 3. $\begin{matrix} U \otimes (V \otimes W) \cong (U \otimes V) \otimes W \\ u \otimes (v \otimes w) \mapsto (u \otimes v) \otimes w \end{matrix}$ 4. $\begin{matrix} U \otimes (V \oplus W) \cong U \otimes V \oplus U \otimes W \\ u \otimes (v, w) \mapsto (u \otimes v, u \otimes w). \end{matrix}$

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Refined definition of Tensor Products: Let U, V, W be R-modules. Then a tensor product $(V \otimes W, - \otimes -)$ (0) is an R-module, $V \otimes_R W$,
(1) ~~s.t.~~ with a bilinear map $- \otimes - : V \times W \to V \otimes W$ s.t. (2) $\forall$ bilinear map $f : V \times W \to U$, $\exists$ unique linear map $\tilde{f} : V \otimes W \to U$ making the diagram commute (as above).

Idea: consider the following diagram:

$$\begin{tikzcd} V \times W \arrow[r, "\text{bilinear } f"] \arrow[dr, "- \otimes - ?"'] & U \\ & T \arrow[u, "\tilde{f} \text{ linear}"'] \end{tikzcd}$$

This is beautiful, as instead of constructing bilinear maps $f : V \times W \to U$, we can equivalently construct linear maps $\tilde{f} : V \otimes W \to U$ from our constructed space $V \otimes W$.

Last time, we showed that $\mathbb{Z}/2\mathbb{Z} \otimes \mathbb{Z}/3\mathbb{Z} \cong \{0\}$ as $\mathbb{Z}$-modules. We calculated using spanning sets e.g. $\{e, a\}$, but we did not use universal property.

$$\begin{tikzcd} V \times W \arrow[r, "f"] \arrow[dr, "- \otimes -"'] & U \\ & V \otimes W \cong \{0\} \arrow[u, "\exists \tilde{f}"'] \end{tikzcd}$$

Question: What does it mean to say $V \otimes_R W \cong \{0\}$? $\Rightarrow$ every bilinear map out of $V \times W$ is a zero map $\Leftrightarrow$ Every linear map out of $V \otimes W$ is a zero map

$\Rightarrow$ every simple tensor $v \otimes w = \overset{0}{\cancel{0}}$ is zero in the image of $f(v, w)$. $V \otimes W = \sum \lambda_i (v_i \otimes w_i) = \{0\}$.

We now examine proofs that tensor spaces are isomorphic: from above. 1. $f : \begin{matrix} R \otimes V \cong V \\ \lambda \otimes v \mapsto \lambda v \end{matrix}$. Why do we need universal property? Danger – the map $\sum \lambda_i \otimes v_i \mapsto \sum \lambda_i v_i$ is not well-defined.

This is because the symbols $\lambda_i \otimes v_i$ do not form a basis. They are however a generating / spanning set (not LI). Let $\begin{matrix} - \otimes - : R \times V \to R \otimes V \\ (\lambda, v) \mapsto \lambda \otimes v \end{matrix}$ be a bilinear map. Let the function $\begin{matrix} f : R \times V \to V \\ (\lambda, v) \mapsto \lambda v \end{matrix}$ be bilinear. Let $\tilde{f} : R \otimes V \to V$ be linear, given by $\lambda \otimes v \mapsto \lambda v$. [check commutativity: $\tilde{f} \circ - \otimes - (v, v) = f(v, v)$].

$$\begin{tikzcd} R \times V \arrow[r, "f"] \arrow[dr, "- \otimes -"'] & V \\ & R \otimes V \arrow[u, dashed, "\exists \text{ unique linear map, isomorphism}"'] \end{tikzcd}$$

To show isomorphism, we need an inverse $\tilde{f}^{-1} : V \to R \otimes V$, $v \mapsto 1 \otimes v$. Then $\tilde{f} \circ \tilde{f}^{-1}(v) = \tilde{f}(1 \otimes v) = \mathrm{Id}(v)$, $\tilde{f}^{-1} \circ \tilde{f}(\lambda \otimes v) = \tilde{f}^{-1}(\lambda v) = 1 \otimes_R \lambda v = \lambda \otimes v = \mathrm{Id}(\lambda \otimes v)$

$\therefore$ $\tilde{f}$ is an isomorphism // q.e.d.

<u>Theorem</u> Let $\mathbb{F}$ be a field, then $M_m(\mathbb{F}) \otimes M_n(\mathbb{F}) \cong M_{mn}(\mathbb{F})$.

Proof – Let $f : M_m(\mathbb{F}) \times M_n(\mathbb{F}) \to M_{mn}(\mathbb{F})$ be the "secret" bilinear map given by $(A, B) \mapsto \begin{pmatrix} a_{11}B & \cdots & a_{1m}B \\ \vdots & \ddots & \vdots \\ a_{m1}B & \cdots & a_{mm}B \end{pmatrix}$. Then take $\begin{matrix} - \otimes - : M_m(\mathbb{F}) \times M_n(\mathbb{F}) \to M_m(\mathbb{F}) \otimes M_n(\mathbb{F}) \\ (A, B) \mapsto A \otimes B. \end{matrix}$
We work around this indirectly.

NTS: $\exists\ \tilde{f}$ linear which is an isomorphism. Let U, V be finite dimensional $\mathbb{F}$-vector spaces. Let $S : U \to U$, $T : V \to V$ be $\mathbb{F}$-linear maps (endomorphisms). Then, define $\begin{matrix} S \hat{\otimes} T : U \otimes V \to U \otimes V \\ u \otimes v \mapsto S(u) \otimes T(v), \end{matrix}$ which is linear. Then we have $\begin{matrix} \mathrm{End}(U) \otimes \mathrm{End}(V) \to \mathrm{End}(U \otimes V) \\ S \otimes T \mapsto S \hat{\otimes} T \end{matrix}$ which gives the isomorphism: let $\dim(U) = m$, $\dim(V) = n$.

Then choose bases, $\mathrm{End}(U) \cong M_m(\mathbb{F})$, $\mathrm{End}(V) \cong M_n(\mathbb{F})$, and $\mathrm{End}(U \otimes V) \cong M_{mn}(\mathbb{F})$ // q.e.d.

Examples – 1. $I_m \otimes I_n \cong I_{mn}$, 2. $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \otimes I_2 = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} a & 0 & b & 0 \\ 0 & a & 0 & b \\ c & 0 & d & 0 \\ 0 & c & 0 & d \end{pmatrix}$. as $M_2(\mathbb{F}) \times M_2(\mathbb{F}) \mapsto M_{2\times 2}(\mathbb{F}) = M_4(\mathbb{F})$

Tensor Products of algebras.

Let A, B be algebras over $\mathbb{F}$. Then $A \otimes_{\mathbb{F}} B$ becomes an $\mathbb{F}$-algebra by defining $\sum (a\otimes b)\cdot(a'\otimes b') = \sum aa' \otimes bb'$ with unit $1 \otimes 1$.

Example – $\mathbb{C}$ is a $\mathbb{C}$-algebra, so $\mathbb{C} \otimes_{\mathbb{C}} \mathbb{C}$ is a $\mathbb{C}$-algebra. Likewise, group ring $\mathbb{C}[G]$ is a $\mathbb{C}$-algebra, so let V, W be $\mathbb{C}[G]$ modules, then $V \otimes_{\mathbb{C}[G]} W$ is an algebra by defining $g(v \otimes w) = gv \otimes gw$.

Further Wedderburn Decomposition.

By Wedderburn and Maschke, we know that $\mathbb{C}[G] \cong \prod_{i=1}^{r} M_{n_i}(\mathbb{C})$. We want $\boxed{\mathbb{C}[G\times H] \cong \mathbb{C}[G] \otimes_{\mathbb{C}} \mathbb{C}[H], \text{ with isomorphism } (g,h) \mapsto g \otimes h}$.

Examples –

1. We know that $\mathbb{C}[D_6 \times D_6]$ breaks down. $\mathbb{C}[D_6] \cong \mathbb{C} \times \mathbb{C} \times M_2(\mathbb{C})$ $\because 6 = 1^2 + 1^2 + 2^2$. We use the theorem $M_m(\mathbb{F}) \times M_n(\mathbb{F}) \cong M_{mn}(\mathbb{F})$. Think of $\mathbb{C}$ as $M_1(\mathbb{C})$.

Then by isomorphism, $\mathbb{C}[D_6 \times D_6] \cong \mathbb{C}[D_6] \otimes_{\mathbb{C}} \mathbb{C}[D_6] \cong [\mathbb{C} \times \mathbb{C} \times M_2(\mathbb{C})] \otimes [\mathbb{C} \times \mathbb{C} \times M_2(\mathbb{C})] \cong (\mathbb{C}\otimes\mathbb{C}) \times (\mathbb{C}\otimes\mathbb{C}) \times (\mathbb{C} \otimes M_2(\mathbb{C})) \times (\mathbb{C}\otimes\mathbb{C}) \times (\mathbb{C}\otimes\mathbb{C}) \times (\mathbb{C} \otimes M_2(\mathbb{C})) \times (M_2(\mathbb{C}) \otimes \mathbb{C}) \times (M_2(\mathbb{C}) \otimes \mathbb{C}) \times (M_2(\mathbb{C}) \otimes M_2(\mathbb{C}))$

$\cong \mathbb{C}^{(4)} \times M_2(\mathbb{C})^{(4)} \times M_4(\mathbb{C})$ $\because$ $\mathbb{C} \otimes \mathbb{C} = M_1(\mathbb{C}) \otimes M_1(\mathbb{C}) = M_1(\mathbb{C})$, $\mathbb{C} \otimes M_2(\mathbb{C}) = M_1(\mathbb{C}) \otimes M_2(\mathbb{C}) = M_2(\mathbb{C})$ ...

$\Rightarrow D_6 \times D_6$ has 4 irreducible distinct 1D representations, 4 irreducible distinct 2D representations and 1 irreducible 4D representation.

We verify that both sides have the same $\mathbb{C}$-dimension: $\mathbb{C}[D_6 \times D_6] \cong \mathbb{C}[D_6] \times \mathbb{C}[D_6] \cong \mathbb{C}^{(4)} \times M_2(\mathbb{C})^{(4)} \times M_4(\mathbb{C})$ $\Rightarrow$ $6 \times 6 = 4 + 4(2\times 2) + 4\times 4 = 36$.

2. $D_6^* = \langle x, y \mid x^3 = y^4 = 1, yx = x^2 y \rangle = C_3 \rtimes C_4$ has 12 elements $\{1, x, x^2, y, xy, x^2y, y^2, xy^2, x^2y^2, y^3, xy^3, x^2y^3\}$. We then seek $\mathbb{C}[D_6^*]$. We need $r =$ no. of classes $= 6$.

These are $\{1\}, \{x, x^2\}, \{y, xy, x^2y\}, \{y^2\}, \{y^2x, y^2x^2\}, \{y^3, xy^3, x^2y^3\}$. Then $|D_6^*| = 12 = n_1^2 + n_2^2 + n_3^2 + n_4^2 + n_5^2 + n_6^2 = 1^2 + 1^2 + 1^2 + 1^2 + 2^2 + 2^2$.

i.e. $\mathbb{C}[D_6^*] \cong \mathbb{C}^{(4)} \times M_2(\mathbb{C})^{(2)}$, isotypic decomposition. Then $\mathbb{C}[D_6^* \times D_6^*] \cong \mathbb{C}[D_6^*] \otimes_{\mathbb{C}} \mathbb{C}[D_6^*]$, $(g,h) \mapsto g \otimes h$. $\cong [\mathbb{C}^{(4)} \times M_2(\mathbb{C})^{(2)}] \otimes [\mathbb{C}^{(4)} \times M_2(\mathbb{C})^{(2)}]$

$\cong \mathbb{C}^{(16)} \times M_2(\mathbb{C})^{(16)} \times M_4(\mathbb{C})^{(4)}$. Again we verify, $12 \times 12 = 144$. $16 + 4\cdot 16 + 16 \cdot 4 = 16 + 64 + 64 = 144$.

$\therefore$ $D_6^* \times D_6^*$ has 16 distinct 1D representations, 16 distinct simple 2D representations and 4 distinct simple 4-dim reps.

Thus, we always know for any G how many representations $\rho: G \to GL_n(\mathbb{C})$ there are. Now, we attempt to actually construct these matrices.

Induced representations. = constructing $\begin{array}{l} \rho: G \to GL_n(\mathbb{C}) \\ g \mapsto ? \end{array}$

Idea: To construct all $\mathbb{C}$-representations of G, starting from cyclic group representations (since we know these).

Construction: Let G be a finite group, $H \subset G$ a subgroup. Let V be a $\mathbb{C}[H]$-module (a representation of H). Then define the $\mathbb{C}[G]$-module (targeted representation of G that we want) by $\mathrm{Ind}_H^G(V) = \mathbb{C}[G] \otimes_{\mathbb{C}[H]} V$, where we will think of V as a $\mathbb{C}[G]$-module, where G acts trivially, i.e. $g.V = V$.

Then consider submodule $Y = \{ gh \otimes v - g \otimes hv : g \in G, h \in H, v \in V\}$ s.t. we get a $\mathbb{C}$-vector space $\mathbb{C}[G] \otimes_{\mathbb{C}} V/Y \cong \mathbb{C}[G/H] \otimes_{\mathbb{C}} V \cong \mathrm{Ind}_H^G(V)$. Then (right mod, left mod; $\otimes_{\mathbb{C}}$: thinking of elements as scalars)

1. Define the set of left cosets $G/H = \{gH : g \in G\}$ where $g_1 H = g_2 H \Leftrightarrow g_2^{-1} g_1 \in H$.

2. Let $\{g_1, \dots, g_n\}$ be a complete set of coset representatives, $G = \bigcup_{i=1}^{n} g_i H$.

3. Define the quotient ~~space~~ module $\mathbb{C}[G/H]$, which is a $\mathbb{C}$-vector space with basis $\{g_1, \dots, g_n\}$ = set of coset representatives.

4. Choose a basis for $\mathbb{C}[G/H] \otimes_{\mathbb{C}[H]} V$, and we need to define how g acts on the tensors $g \cdot (g_i \otimes v) = ?$ We use group relations of G and key rule of $\otimes$. ($\otimes_{\mathbb{C}[H]}$: thinking of elements as in $\mathbb{C}[H]$.)

Examples –

1. Let us find a representation of $D_6 = \langle x, y \mid x^3 = y^2 = 1, yx = x^2 y \rangle$. Let $H = C_3 = \{1, x, x^2\}$ $x^3 = 1$ be the subgroup. V is a $\mathbb{C}[C_3]$-module, on which x acts as $w = e^{2\pi i/3}$, i.e. $\rho: C_3 \to GL_1(\mathbb{C})$, $x \mapsto w$.

1. $G/H = D_6/C_3 \cong C_2$ since $C_3 \trianglelefteq D_6$ normal. $\because |D_6/C_3| = |D_6|/|C_3| = 2 \Rightarrow$ index 2 = normal.
2. Let $\{1, y\} = Q$ be the set of coset representatives. Then, i.e. $D_6 = C_3 \cdot 1 \sqcup C_3 \cdot y$

3. Define quotient module $\mathbb{C}[D_6/C_3] \cong \mathbb{C}[Q]$ with basis $\{1, y\}$. (Q: need not be a group generally)
4. Let $\{1 \otimes 1, y \otimes 1\}$ be a basis for $\mathbb{C}[Q] \otimes V$. V is a 1D rep, so $V \cong \mathbb{C}$ as modules. i.e. $\{1\}$ is a basis for V.

Now find x and y actions on the basis: use relations for D_6. x-action – $x(\underbrace{1 \otimes 1}_{e_1}) = x \cdot 1 \otimes 1 = 1 \cdot x \otimes_{\mathbb{C}[C_3]} 1 = 1 \otimes w \cdot 1 = (\underbrace{1 \otimes 1}_{e_1}) w$. ($x \sim w \in \mathbb{C}$) Then we have

$x(\underbrace{y \otimes 1}_{e_2}) = xy \otimes 1 \overset{D_6}{=} yx^2 \otimes 1 = y \otimes x^2 \cdot 1 = y \otimes w^2 \cdot 1 = (\underbrace{y \otimes 1}_{e_2}) w^2$. Therefore, $\mathrm{Ind}_{C_3}^{D_6}(V) = \rho: D_6 \to GL_2(\mathbb{C})$ sends $x \mapsto \begin{pmatrix} w & 0 \\ 0 & w^2 \end{pmatrix}$ (first element of basis; second element of basis). Likewise, we evaluate y-action.

$y(\underbrace{1 \otimes 1}_{e_1}) = \underbrace{y \otimes 1}_{e_2}$, $y(\underbrace{y \otimes 1}_{e_2}) = y^2 \otimes 1 = \underbrace{1 \otimes 1}_{e_1}$, so $y \mapsto \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

Note – we said earlier that in $\mathbb{Z}$-modules, $\lambda v \otimes w = v \otimes \lambda w$, but this applies only in commutative group rings! i.e. $xy \otimes 1 \neq y \otimes x$!

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Using Wedderburn Decomposition theorem, $\mathbb{C}[D_6] \cong \mathbb{C} \times \mathbb{C} \times M_2(\mathbb{C})$. $6 = 1^2 + 1^2 + 2^2 \Rightarrow r = 3$ conjugacy classes. We have just induced a 2D representative of D_6, but is it • irreducible rep $\leftrightarrow M_2(\mathbb{C})$, or • direct sum $\rho = \underset{\mathbb{C}}{\rho'} \oplus \underset{\mathbb{C}}{\rho''}$ of 1-dim representatives?

To figure this out, use Schur's lemma V3. Compute $\mathrm{End}_{\mathbb{C}[C_6]}(\mathrm{Ind}_{C_3}^{D_6}(V)) = \{A \in GL_2(\mathbb{C}) : A\rho(g) = \rho(g)A \ \forall g \in D_6\}$. Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, find $A\rho(x) = \rho(x)A$, $A\rho(y) = \rho(y)A$

$A\rho(y) = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} b & a \\ d & c \end{pmatrix}$, $\rho(y)A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} c & d \\ a & b \end{pmatrix} \Rightarrow a = d, b = c \Rightarrow A = \begin{pmatrix} a & b \\ b & a \end{pmatrix}$. Then $A\rho(x) = \begin{pmatrix} a & b \\ b & a \end{pmatrix}\begin{pmatrix} w & 0 \\ 0 & w^2 \end{pmatrix} = \begin{pmatrix} aw & bw^2 \\ bw & aw^2 \end{pmatrix}$, $\rho(x)A = \begin{pmatrix} aw & bw \\ bw^2 & aw^2 \end{pmatrix}$

$\Rightarrow a = a$, $bw^2 - bw = 0 \Rightarrow b = 0$ since $w^2 - w \neq 0$. $\Rightarrow A = \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} \Rightarrow \mathrm{End}_{\mathbb{C}[D_6]}(\mathrm{Ind}_{C_3}^{D_6}(V)) \cong \mathbb{C}$, a division ring $\Rightarrow$ representation is simple $\Rightarrow$ we have that $\rho: D_6 \to GL_2(\mathbb{C})$ is simple, i.e. irreducible, where ρ is our constructed induced representation.

Note – We can induce from any subgroup. In particular, if we induce from $\{1\} \trianglelefteq G$, we get the regular representation of G: $\mathrm{Ind}_{\{1\}}^{G}(V) = \rho_{reg}: G \to GL_{|G|}(\mathbb{C})$. This is always reducible!

2. Let $G = D_6 = \langle x, y \mid x^3 = y^2 = 1,\ yx = x^2 y \rangle$. Let $H = C_2 = \langle y \mid y^2 = 1 \rangle \leq D_6$ be a subgroup. However, $C_2 \ntrianglelefteq D_6$ since $gC_2 \neq C_2 g\ \forall g \in D_6$. Let V be the 1-dimensional trivial $\mathbb{C}[C_2]$-module.
$y \cdot 1 = 1$
$\rho_v : C_2 \to \mathbb{C}$ by $y \mapsto 1$. $\left|\frac{D_6}{C_2}\right| = 3 \Rightarrow$ take Q (quotient) $= \{1, x, x^2\}$ to be coset representatives. Construct $\mathrm{Ind}_{C_2}^{D_6}(V)$ (or ρ_V) $= \mathbb{C}[D_6] \otimes_{\mathbb{C}[C_2]} V \cong \mathbb{C}[Q] \otimes_{\mathbb{C}} V$; which is a $\mathbb{C}[D_6]$ mod.

As a basis, we take $\{1 \otimes 1, x \otimes 1, x^2 \otimes 1\}$. Then x-action: $x(1 \otimes 1) = x \cdot 1 \otimes 1 = x \otimes 1$. $x(x \otimes 1) = x^2 \otimes 1$. $x(x^2 \otimes 1) = x^3 \otimes 1 = 1 \otimes 1 \Rightarrow x \mapsto \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$, $\rho: D_6 \to GL_3(\mathbb{C})$.
y acts trivially on right factor.
y-action: $y(1 \otimes 1) = y \cdot 1 \otimes 1 = 1 \cdot y \otimes 1 = 1 \otimes y \cdot 1 = 1 \otimes 1$. $y(x \otimes 1) = yx \otimes 1 = x^2 y \otimes 1 = x^2 \otimes y \cdot 1 = x^2 \otimes 1$. $y(x^2 \otimes 1) = yx^2 \otimes 1 = xy \otimes 1 = x \otimes y \cdot 1 = x \otimes 1$.

Then $y \mapsto \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$. This representation $\rho : D_6 \to GL_3(\mathbb{C})$ is reducible for two reasons: ① $\mathbb{C}[D_6] \cong \mathbb{C} \times \mathbb{C} \times M_2(\mathbb{C})$, $3 > 2$. ② $\mathrm{End}_{\mathbb{C}[D_6]}(\rho) = \left\{\begin{pmatrix} a & b & b \\ b & a & b \\ b & b & a \end{pmatrix}\right\} \cong \mathbb{C}^2$ not division ring

$\Rightarrow$ not simple representation.

3. Let $G = Q_8 = \langle x, y \mid x^4 = 1,\ x^2 = y^2,\ xy = yx^3 \rangle$. Take $H = C_4 = \{1, x, x^2, x^3 \mid x^4 = 1\} \trianglelefteq Q_8$ ($\because$ every subgroup of Q_8 is normal). Let V be the $\mathbb{C}[C_4]$-module where x acts as i.
$\leftrightarrow$ $\rho_v : C_4 \to \mathbb{C}$, $x \mapsto i$ $\therefore V \cong \mathbb{C} \Rightarrow$ basis $\{1\}$ for V. $|Q_8 / C_4| = \frac{8}{4} = 2$, so take $Q = \{1, y\}$ to be coset reps. Construct $\mathbb{C}[Q_8]$-module, $\mathrm{Ind}_{C_4}^{Q_8}(V) = \mathbb{C}[Q_8] \otimes_{\mathbb{C}[C_4]} V \cong \mathbb{C}[Q] \otimes_{\mathbb{C}} V$.

This has basis $\{1 \otimes 1, y \otimes 1\}$. x-action: $x(1 \otimes 1) = x \cdot 1 \otimes 1 = 1 \cdot x \otimes 1 = 1 \otimes x \cdot 1 = (1 \otimes i) = (1 \otimes 1) \cdot i$. $x(y \otimes 1) = xy \otimes 1 = yx^3 \otimes 1 = y \otimes x^3 \cdot 1 = (y \otimes 1) \cdot (-i)$. $x \sim \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$.

y-action: $y(1 \otimes 1) = y \otimes 1$, $y(y \otimes 1) = y^2 \otimes 1 = x^2 \otimes 1 = 1 \cdot x^2 \otimes 1 = 1 \otimes x^2 \cdot 1 = 1 \otimes (-1) = (1 \otimes 1)(-1)$, so $y \sim \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$.

Check: $\mathrm{End}_{\mathbb{C}[Q_8]}(\rho)$. Recall $r = 5$ for $\mathbb{C}[Q_8]$, and we have just constructed a $M_2(\mathbb{C})$-representation. Is it the simple one, or the direct sum of 2 1D-reps?

Find $A \in GL_2(\mathbb{C})$ s.t. $A\rho(x) = \rho(x)A$, $A\rho(y) = \rho(y)A$. Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $A\rho(x) = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} = \begin{pmatrix} ai & -bi \\ ci & -di \end{pmatrix}$, $\rho(x) \cdot A = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} ai & bi \\ -ci & -di \end{pmatrix} \Rightarrow b = 0, c = 0$.

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Then $A = \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}$. $A\rho(y) = \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -a \\ d & 0 \end{pmatrix}$, $\rho(y)A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} = \begin{pmatrix} 0 & -d \\ a & 0 \end{pmatrix} \Rightarrow a = d$, so $\mathrm{End}_{\mathbb{C}[Q_8]}(\rho) = \left\{\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}\right\} \cong \mathbb{C}$, division ring

$\Rightarrow \rho$ is a (the) simple representation. And we can write a full list of all representations of Q_8. $\mathbb{C}[Q_8] = \underset{\substack{x \mapsto 1 \\ y \mapsto 1 \\ \text{trivial}}}{\mathbb{C}} \times \underset{\substack{x \mapsto 1 \\ y \mapsto -1}}{\mathbb{C}} \times \underset{\substack{x \mapsto -1 \\ y \mapsto 1}}{\mathbb{C}} \times \underset{\substack{x \mapsto -1 \\ y \mapsto -1}}{\mathbb{C}} \times \underset{\rho}{M_2(\mathbb{C})}$.

Note – In $\otimes$ key rule, we said $\lambda(x \otimes y) = \lambda x \otimes y = x \otimes \lambda y$ $\lambda \in \mathbb{R}$. However, in $V \otimes W$, V and W were both left modules. However, in the induced reps "definition": $\mathrm{Ind}_H^G(V) = \underset{\text{right}}{\mathbb{C}[G]} \otimes_{\mathbb{C}[H]} \underset{\text{left}}{V}$.

$\boxed{\text{Theorem}}$ Let V be a $\mathbb{C}[H]$-module, and let $|G/H| = n$ so that $\{g_1, \ldots, g_n\}$ is a complete set of coset representatives for coset list $g_1 H, \ldots, g_n H$. Then $\mathrm{Ind}_H^G(V) = \mathbb{C}[G] \otimes_{\mathbb{C}[H]} V \cong \bigoplus_{i=1}^n g_i \otimes_{\mathbb{C}} V$ as $\mathbb{C}$-modules, where $g_i \otimes V = \{g_i \otimes v : g_i \text{ coset rep}, v \in V\} \subseteq \mathbb{C}[G] \otimes_{\mathbb{C}[H]} V$, and each $g_i \otimes V \cong V$ as $\mathbb{C}$-modules.

Moreover, if V is free (i.e. has a basis), then $\dim \mathrm{Ind}_H^G(V) = |G/H| \times \mathrm{rk}_{\mathbb{C}}(V)$. If $x \in G$, then $x \cdot (g_i \otimes V) = g_j \otimes V$ where $xg_i = g_j h$ for some $h \in H$.

$\therefore$ the submodules $g_i \otimes_{\mathbb{C}} V$ are permuted by the action of G.

Proof – ${}_{\mathbb{C}[H]}\mathbb{C}[G] = \bigoplus_{i=1}^n g_i \mathbb{C}[H] \cong \mathbb{C}[H]^n$, and H has a permutation action on the basis of $\mathbb{C}[G]$ with n orbits $g_1 H, \ldots, g_n H$. Each orbit spans a left $\mathbb{C}[H]$-module $\mathbb{C}[g_i H]$ of $\mathbb{C}[G]$. $\therefore \mathrm{Ind}_H^G(V) = \mathbb{C}[G] \otimes_{\mathbb{C}[H]} V = \bigoplus_{i=1}^n g_i \mathbb{C}[H] \otimes V = \bigoplus_{i=1}^n (\mathbb{C}[g_i H] \otimes V) = \bigoplus_{i=1}^n g_i \otimes_{\mathbb{C}[H]} V$ and as $\mathbb{C}$-modules, $g_i\, \mathbb{C}[H] \otimes_{\mathbb{C}[H]} V = \mathbb{C}[H] \otimes_{\mathbb{C}[H]} V \cong V$. Next, g-action permutes the basis. Let $x \in G$ s.t. $xg_i = g_j h$ for some $h \in H$. Then $x(g_i \otimes v) = xg_i \otimes v = g_j h \otimes v = g_j \otimes hv \in g_j \otimes V$ and equality follows from x^{-1}: $x^{-1} g_j \otimes v = g_i \otimes V$. q.e.d.

CHARACTER THEORY.

Thus far, the theory is sound for $\mathbb{C}$-reps, but consider the following Wedderburn decomposition: $\mathbb{C}[C_2 \times C_2] \cong \mathbb{C} \times \mathbb{C} \times \mathbb{C} \times \mathbb{C} \cong \mathbb{C}[C_4]$, but $C_2 \times C_2 \ncong C_4$. (group rings isomorphic $\not\Rightarrow$ groups isomorphic.) So we need some kind of invariant to distinguish between group rings. We use character tables (denoted χ-tables).

$\boxed{\text{Theorem}}$ If 2 χ-tables are isomorphic, then $\mathbb{C}[G] \cong \mathbb{C}[H]$, but $\mathbb{C}[G] \cong \mathbb{C}[H] \not\Rightarrow$ χ-tables are the same.

Remark – (Beyond scope of course:) If $\mathbb{Z}[G] \cong \mathbb{Z}[H]$ then $G \cong H$.

$\boxed{\text{Definition}}$ Let $A = (a_{ij})$ be an $n \times n$ matrix over $\mathbb{F}$, then the trace of A, $\mathrm{Tr}(A) = \sum_{i=1}^n a_{ii}$.

Recall – Property: $\mathrm{Tr}(\lambda A) = \lambda \mathrm{Tr}(A)$.

$\boxed{\text{Proposition}}$ Let $A, B \in M_n(\mathbb{F})$. Then $\mathrm{Tr}(AB) = \mathrm{Tr}(BA)$. [Although $AB \neq BA$ in general.]

Proof $(AB)_{ii} = \sum_{j=1}^n a_{ij} b_{ji}$, so $\mathrm{Tr}(AB) = \sum_{i=1}^n \sum_{j=1}^n a_{ij} b_{ji} = \sum_{j=1}^n \sum_{i=1}^n b_{ji} a_{ij} = \sum_{j=1}^n (BA)_{jj} = \mathrm{Tr}(BA)$ q.e.d.

$\boxed{\text{Corollary}}$ If A, B are conjugate (similar), then $\mathrm{Tr}(A) = \mathrm{Tr}(B)$.

Proof – Suppose $\exists T \in GL_n(\mathbb{F})$ s.t. $B = TAT^{-1}$, then $\mathrm{Tr}(B) = \mathrm{Tr}(TAT^{-1}) = \mathrm{Tr}(ATT^{-1}) = \mathrm{Tr}(A)$ q.e.d.

$\boxed{\text{Definition}}$ Let G be a finite group, $\mathbb{F} = \mathbb{C}$. Let V be a finite dimensional $\mathbb{C}[G]$-module corresponding to $\rho: G \to GL_n(\mathbb{C})$. Then the <u>character afforded</u> by ρ is the mapping $\chi_\rho : G \to \mathbb{C}$ defined by $\chi_\rho(g) = \mathrm{Tr}(\rho(g))$, $\forall g \in G$. Every rep ρ has a character χ_ρ.

Examples –

1. Let $G = C_3 = \{x \mid x^3 = 1\}$. G has 3 conjugacy classes $\langle 1 \rangle, \langle x \rangle, \langle x^2 \rangle$. $\therefore r = 3$, $\mathbb{C}[C_3] = \underset{\rho_1}{\mathbb{C}} \times \underset{\rho_2}{\mathbb{C}} \times \underset{\rho_3}{\mathbb{C}}$, $\rho_1 : C_3 \to \mathbb{C}$, $x \mapsto (1)$; $\rho_2 : C_3 \to \mathbb{C}$, $x \mapsto (\omega)$; $\rho_3 : C_3 \to \mathbb{C}$, $x \mapsto (\omega^2)$

Then we get the following χ-table as shown on right –

	$\langle 1 \rangle$	$\langle x \rangle$	$\langle x^2 \rangle$
χ_1	1	1	1
χ_2	1	ω	ω^2
χ_3	1	ω^2	ω

Note: $\chi_{reg} = \chi_1 + \chi_2 + \chi_3 = \begin{cases} 3 & g = 1 \\ 0 & g \neq 1 \end{cases}$ is still a function.

$\boxed{\text{Definition}}$ If ρ is irreducible, then $\chi_\rho : G \to \mathbb{C}$ is called an <u>irreducible character</u>. If ρ is a 1-dim rep $\rho: G \to GL_1(\mathbb{C})$, then $\chi_\rho : G \to \mathbb{C}$ is called a <u>linear character</u>.

The degree n of a representation $\rho: G \to GL_n(\mathbb{C})$ is also the degree of the character $\chi_\rho: G \to \mathbb{C}$, $\deg(\chi_\rho) = [V:\mathbb{C}] = n$. To find degree of χ_ρ, compute $Tr(\rho(1)) = Tr\begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix} = n$.

Note that χ_ρ is not a homomorphism in general, $\chi_\rho: G \to \mathbb{C}$, $\chi_\rho(gh) \neq \chi_\rho(g)\chi_\rho(h)$. It is a homomorphism if $\rho: G \to GL_1(\mathbb{C}) = \mathbb{C}$ is a 1-dim rep,

since $\chi_\rho(gh) = Tr(\rho(gh)) = Tr(\rho(g)\rho(h)) = Tr(\rho(g))\, Tr(\rho(h)) = \chi_\rho(g)\chi_\rho(h)$.

Recall the definition: two representations $\underset{G \to GL_n(\mathbb{C})}{\rho}$ and $\underset{G \to GL_n(\mathbb{C})}{\sigma}$ were equivalent/conjugate if $\exists T \in GL_n(\mathbb{C})$ s.t. $\sigma(g) = T^{-1}\rho(g)T \ \forall g \in G$ i.e. $\exists$ a change of basis matrix for $V = \mathbb{C}^n$.

$\boxed{\text{Proposition}}$ If ρ and σ are equivalent, then $\chi_\rho = \chi_\sigma$.

Proof – let $\sigma(g) = T^{-1}\rho(g)T \ \forall g \in G$. So $\chi_\sigma(g) = Tr(\sigma(g)) = Tr(T^{-1}\rho(g)T) = Tr(\rho(g)T \cdot T^{-1}) = Tr(\rho(g)) = \chi_\rho(g)$, q.e.d.

Thus, characters are also independent of changes in basis. $\Rightarrow$ Practical point: If $\chi_\rho \neq \chi_\sigma$, then ρ and σ are not equivalent.

$\boxed{\text{Proposition}}$ Characters are constant on conjugacy classes. (i.e. $\chi(\rho)$ is constant on $(g)^G = \{xgx^{-1} : x \in G\}$).

Proof – Suppose $g = xhx^{-1}$ for some $x, h \in G$. Then $g \in (h)^G$. Then $\rho(g) = \rho(xhx^{-1}) \overset{\rho \text{ is group hom}}{=} \rho(x)\rho(h)\rho(x^{-1})$. Then $\chi_\rho(g) = Tr(\rho(g)) = Tr(\rho(x)\rho(h)\rho(x^{-1})) = Tr(\rho(h)\rho(x)^{-1}\rho(x))$

$= Tr(\rho(h)) = \chi_\rho(h)$ [Point: We don't need to find the matrices for all elements in a conjugacy class and then trace them to find χ_ρ.]

Example –

1. $D_6 = \langle x, y \mid x^3 = y^2 = 1, yx = x^2y \rangle = \langle 1 \rangle \sqcup \langle x, x^2 \rangle \sqcup \langle y, xy, x^2y \rangle$ as conjugacy classes. $\mathbb{C}[D_6] = \mathbb{C} \times \mathbb{C} \times M_2(\mathbb{C})$. Try a 2-dim rep:

Let $\rho: D_6 \to GL_2(\mathbb{C})$, $x \mapsto \begin{pmatrix} \omega & 0 \\ 0 & \omega^2 \end{pmatrix}$, $y \mapsto \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. Then $\chi_\rho(1) = Tr(\rho(1)) = Tr\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = 2 \Rightarrow$ degree of character is 2. Check $\chi_\rho(x) = \chi_\rho(x^2)$

$\chi_\rho(x) = Tr(\rho(x)) = Tr\begin{pmatrix} \omega & 0 \\ 0 & \omega^2 \end{pmatrix} = \omega + \omega^2$, $\chi_\rho(x^2) = Tr(\rho(x^2)) = Tr\begin{pmatrix} \omega^2 & 0 \\ 0 & \omega^4 \end{pmatrix} = \omega^2 + \omega^4 = \omega + \omega^2$. (indeed these are the same, and $= -1$).

Likewise, $\{y, xy, x^2y\}$ will share a value of χ. $\chi_\rho(y) = Tr(\rho(y)) = Tr\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = 0$, $\chi_\rho(xy) = Tr(\rho(x)\rho(y)) = Tr\left(\begin{pmatrix} \omega & 0 \\ 0 & \omega^2 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right) = 0$ etc.

$\boxed{\text{Theorem}}$ Let G be finite of order m, $|G| = m$. Let $\rho: G \to GL_n(\mathbb{C})$ be a rep of $G \leftrightarrow V$ be a finite dimensional $\mathbb{C}[G]$-module. Let $\chi_\rho: G \to \mathbb{C}$ be the character afforded by ρ.

Then $\forall g \in G$, we have:

(1) $\rho(g)$ is diagonalisable, (2) $\chi_\rho(g)$ is a sum of roots of unity, (3) $\chi_\rho(g^{-1}) = \overline{\chi_\rho(g)}$. (conjugated). (4) $|\chi_\rho(g)| \leq n$.

Example – $\sigma: D_6 \to GL_2(\mathbb{C})$, $|D_6| = 6$. $x \mapsto \begin{pmatrix} * & * \\ * & * \end{pmatrix}$, $y \mapsto \begin{pmatrix} * & * \\ * & * \end{pmatrix}$ not diagonal. Then (1): $\exists$ new basis and $T \in GL_n(\mathbb{C})$ s.t. $x \mapsto \begin{pmatrix} \omega & 0 \\ 0 & \omega^2 \end{pmatrix}$ is diagonal. Likewise we can modify y. (2):

It is clear that $\chi_\rho(x) = \omega + \omega^2$ is a sum of roots of unity. (3): $\chi_\rho(x^{-1}) = \overline{\chi_\rho(x)} = \overline{\omega + \omega^2} = \omega + \omega^2$. (4): $|\chi_\rho(x)| = |\omega + \omega^2| \leq |\omega| + |\omega^2| = 1 + 1 = 2$.

Since $|D_6| = 6$, $\forall g$, $g^6 = 1 \Rightarrow g^6 - 1 = 0$. Look at minimal polynomial for matrix $\rho(g)$, since $\rho(g)^6 - Id = 0 \Rightarrow x^6 - 1 = 0 \Rightarrow (x^3 - 1)(x^3 + 1) = 0$

$\therefore$ either $x^3 - 1 = 0$ or $x^3 + 1 = 0 \Rightarrow x = 1, \omega, \omega^2$

Proof – Since $|G| = m$, $\forall g \in G$, $g^m = 1$ $\therefore$ minimal polynomial div $g^m - 1 = 0 \iff \rho(g)^m - I_n = 0$. $\therefore$ diagonalisable over $\mathbb{C} \iff \exists$ a basis for $V = \mathbb{C}^n$ and minimal polynomial is a product of roots of unity. $\chi_\rho(g) = Tr\begin{pmatrix} \omega_1 & & 0 \\ & \ddots & \\ 0 & & \omega_n \end{pmatrix} = \omega_1 + \dots + \omega_n$. Then $\chi_\rho(g^{-1}) = Tr\begin{pmatrix} \omega_1^{-1} & & 0 \\ & \ddots & \\ 0 & & \omega_n^{-1} \end{pmatrix} = Tr\begin{pmatrix} \bar{\omega}_1 & & 0 \\ & \ddots & \\ 0 & & \bar{\omega}_n \end{pmatrix} = \overline{\chi_\rho(g)}$. Then (4):

$|\chi_\rho(g)| = |\omega_1 + \dots + \omega_n| \leq |\omega_1| + \dots + |\omega_n| = 1 + \dots + 1 = n$, q.e.d.

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If g and g^{-1} are in the same conjugacy class, then $\chi_\rho(g) \in \mathbb{R}$. Reason: If $g^{-1} \in (g)^G$, then $g = xg^{-1}x^{-1}$ for some $x \in G$. $\therefore \chi_\rho(g) = \chi_\rho(g^{-1}) = \overline{\chi_\rho(g)} \Rightarrow \chi_\rho(g) \in \mathbb{R}$. Example: We cannot try cyclic groups as each element is its own conjugacy class: $C_3 = \langle 1 \rangle \sqcup \langle x \rangle \sqcup \langle x^2 \rangle$. We try $D_6 = \langle x, y \mid x^2 = y^2 = 1, yx = x^2y \rangle$.

Let $\rho: D_6 \to GL_2(\mathbb{C})$, $x \mapsto \begin{pmatrix} \omega & 0 \\ 0 & \omega^2 \end{pmatrix}$, $y \mapsto \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. $\chi_\rho(x) = \omega + \omega^2 = \cos\left(\frac{2\pi}{3}\right) \in \mathbb{R}$, $\chi_\rho(x) = \chi_\rho(x^2)$. Another example – take $G = S_n$. Then permutations in same conjugacy class have same shape i.e. inverses $\in (g)^G$. $\chi_\rho(g) \in \mathbb{R} \ \forall g \in S_n$. Beware that this does not hold for A_n.

$\boxed{\text{Proposition}}$ Let $\rho: G \to GL_n(\mathbb{C})$, $\chi_\rho: G \to G$ its character. $|G| = m$. Then if $\forall g$, $|\chi_\rho(g)| = \chi_\rho(1) = Tr(I_n) = n \iff \rho(g) = \omega I_n$ for some root of unity.

Proof – ($\Leftarrow$) Suppose $\rho(g) = \omega I_n$ for some root ω of $g^m - 1 = 0$. $\chi_\rho(g) = Tr(\rho(g)) = Tr\begin{pmatrix} \omega & & 0 \\ & \ddots & \\ 0 & & \omega \end{pmatrix} = n\omega$. Now $|\chi_\rho(g)| = |n\omega| = |n||\omega| = n = \chi_\rho(1)$, $|\omega| = 1$.

($\Rightarrow$) Suppose $|\chi_\rho(g)| = \chi_\rho(1) = n$. Then w.r.t. some basis, $\rho(g) = \begin{pmatrix} \omega_1 & & 0 \\ & \ddots & \\ 0 & & \omega_s \end{pmatrix}$ for some roots of unity. $\chi_\rho(g) = Tr\begin{pmatrix} \omega_1 & & 0 \\ & \ddots & \\ 0 & & \omega_s \end{pmatrix} = \omega_1 + \dots + \omega_s$.

$n = |\chi_\rho(g)| = |\omega_1 + \dots + \omega_s| \leq |\omega_1| + \dots + |\omega_s|$ with equality if all roots lie on a straight line $\Rightarrow \omega_1 = \dots = \omega_s = \omega$ for some $\omega \Rightarrow \rho(g) = \begin{pmatrix} \omega & & 0 \\ & \ddots & \\ 0 & & \omega \end{pmatrix}$, q.e.d.

$\boxed{\text{Definition}}$ Let $\rho: G \to GL_n(\mathbb{C})$. Let $\chi_\rho: G \to \mathbb{C}$ be character. Then the <u>kernel of the character</u> is the set $Ker(\chi_\rho) = \{g \in G : \chi_\rho(g) = \chi_\rho(1) = n\}$.

$\boxed{\text{Proposition}}$ $Ker(\chi_\rho) = Ker(\rho)$, where $Ker(\rho) = \{g \in G : \rho(g) = I_n\}$.

Proof – $Ker(\rho) \subseteq Ker(\chi_\rho)$. Let $g \in Ker(\chi_\rho) \Rightarrow \chi_\rho(g) = \chi_\rho(1) = n$. By previous propn, $|\chi_\rho(g)| = \chi_\rho(1) = n \Rightarrow \rho(g) = \begin{pmatrix} \omega & & 0 \\ & \ddots & \\ 0 & & \omega \end{pmatrix} = |\chi_\rho(g)| = |n\omega| = |n||\omega|$ $\therefore g = \begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix}$ $\therefore g \in Ker(\rho)$, q.e.d.

$\boxed{\text{Definition}}$ If $Ker(\chi_\rho) = \{1\}$, then χ_ρ is called a <u>faithful character</u>. (analogous – "injective").

Examples –

$\rho_1: x \mapsto 1, y \mapsto 1$ $\rho_2: x \mapsto 1, y \mapsto -1$ $\rho_3: x \mapsto \begin{pmatrix} \omega & 0 \\ 0 & \omega^2 \end{pmatrix}, y \mapsto \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

	$\langle 1 \rangle$	$\langle x, x^2 \rangle$	$\langle y, xy, x^2y \rangle$
χ_1	1	1	1
χ_2	1	1	-1
χ_3	2	$\omega + \omega^2 = -1$	0
χ_{reg}	6	0	0

χ_{reg} corresponds to $\rho_{reg}: G \to GL_{|G|}(\mathbb{C}) = GL_6(\mathbb{C})$.

1. $D_6 = \langle x, y \mid x^3 = y^2 = 1, yx = x^2y \rangle = \langle 1 \rangle \sqcup \langle x, x^2 \rangle \sqcup \langle y, xy, x^2y \rangle$. $\mathbb{C}[D_6] = \mathbb{C} \times \mathbb{C} \times M_2(\mathbb{C})$

$Ker(\chi_1) = D_6$, $Ker(\chi_2) = \langle x \mid x^3 = 1 \rangle = C_3$, $Ker(\chi_3) = \{1\}$ hence faithful! $\because Ker(\chi_3) = \{g \in G : \chi_\rho(g) = \chi_\rho(1) = 2^{(n)}\}$

$Ker(\chi_{reg}) = n_1\chi_1 + n_2\chi_2 + n_3\chi_3 = \chi_1 + \chi_2 + 2\chi_3$ [sum of dimensions $\times$ characters] clearly then, χ_{reg} is faithful.

Now define another 2-dim rep of D_6: $\sigma: D_6 \to GL_2(\mathbb{C})$, $x \mapsto \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}$, $y \mapsto \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. $\chi_\sigma(1) = 2$, $\chi_\sigma(x) = \chi_\sigma(x^2) = -1$, $\chi_\sigma(y) = \chi_\sigma(xy) = \chi_\sigma(x^2y) = 0 \Rightarrow \chi_\sigma = \chi_3$ equivalent

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$\Rightarrow$ we can find $T \in GL_2(\mathbb{C})$ s.t. $\sigma(g) = T\rho_3(g)T^{-1}$.

The regular character

Recall the regular representation $\rho_{reg}: G \to GL(\mathbb{C}[G]) = GL_{|G|}(\mathbb{C})$ given by $g \mapsto \rho_g$ matrix. We obtained ρ_g by treating $\mathbb{C}[G]$ as a $\mathbb{C}$-vector space with basis $\{1=g_1, \dots, g_n\}$ and g acts on $\mathbb{C}^n$ like $g \cdot g_i = g_j$, then write matrix ρ_g w.r.t. basis. $\rho_{g_i} = \begin{pmatrix} 0 & & 1 \\ & \ddots & \\ 1 & & 0 \end{pmatrix}$ $\forall g_i \neq 1$, $\rho_{g_1} = \begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix}$. All elements of G except for $1=g_1$ go to permutation matrices, with zeroes on diagonal since if $g \cdot g_i = g_i \Rightarrow g=1 \Rightarrow g_i = 1$. $\therefore \chi_{reg}$ is faithful and decomposable. $\Rightarrow \chi_{reg}(g) = \begin{cases} |G| & g=1 \\ 0 & g \neq 1 \end{cases}$. Also, $\mathbb{C}[G]$ as a $\mathbb{C}[G]$-module gives $\mathbb{C}[G] = S_1^{n_1} \oplus \dots \oplus S_r^{n_r}$.

Example – $\rho_{reg}: D_6 \to GL_6(\mathbb{C})$, $\rho_{reg}(1) \mapsto I_6$, $\rho_{reg}(x) = ?$ $\mathbb{C}[D_6] = \text{span}\{\overset{g_1}{1}, \overset{g_2}{x}, \overset{g_3}{x^2}, \overset{g_4}{y}, \overset{g_5}{xy}, \overset{g_6}{x^2y}\}$. $xg_1 = x \cdot 1 = x = g_2$, $xg_2 = x \cdot x = x^2 = g_3$, $xg_3 = x \cdot x^2 = 1 = g_1$, $xg_4 = xy = g_5$, $xg_5 = g_6$, $xg_6 = x^3y = y = g_4$. Then $\rho(x) = \begin{pmatrix} 0&0&1&0&0&0 \\ 1&0&0&0&0&0 \\ 0&1&0&0&0&0 \\ 0&0&0&0&0&1 \\ 0&0&0&1&0&0 \\ 0&0&0&0&1&0 \end{pmatrix} = \rho_{reg}(x)$. Then $\chi_{reg}(x) = \mathrm{Tr}(\rho_{reg}(x)) = 0$.

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We use χ to avoid computing $\mathrm{End}_{\mathbb{C}[D_6]}(\rho)$. Exercise: find $\rho_{reg}(x^2)$, $\rho_{reg}(y)$ in above example.

Proposition Recall that $\mathbb{C}[G] \cong \bigoplus_{i=1}^{r} M_{n_i}(\mathbb{C}) \cong S_1^{n_1} \oplus \dots \oplus S_r^{n_r}$ where S_i are pairwise nonisomorphic $\mathbb{C}[G]$-simple modules corresponding to representations $\rho_1, \dots, \rho_r: G \to GL_{n_i}(\mathbb{C})$ s.t. $\dim_{\mathbb{C}}(S_i) = n_i$. $\therefore$ Each $M_{n_i}(\mathbb{C}) \cong S_i^{n_i} = \overbrace{S_i \oplus \dots \oplus S_i}^{n_i}$ as $\mathbb{C}[G]$-modules. So the regular $\mathbb{C}[G]$-module $\underset{\bigoplus_{i=1}^{r} S_i^{n_i}}{\mathbb{C}[G]} \overset{1:1}{\longleftrightarrow} \underset{\rho_{reg}: G \to GL_{|G|}(\mathbb{C})}{\bigoplus_{i=1}^{r} n_i \rho_i(g)}$.

Take trace, we can rewrite $\boxed{\chi_{reg} = \sum_{i=1}^{r} n_i \chi_i}$, and applied to a group element g. Then $\chi_{reg}(g) = n_1\chi_1(g) + \dots n_r\chi_r(g)$.

Example – Take $G = D_6$. Then $\mathbb{C}[D_6] \cong \underset{\rho_1}{\mathbb{C}} \times \underset{\rho_2}{\mathbb{C}} \times \underset{\rho_3}{M_2(\mathbb{C})}$. Then we get the following table:

$\chi_{reg}(1) = 1 \cdot 1 + 1 \cdot 1 + 2 \cdot 2 = 6$, $\chi_{reg}(x) = 1 \cdot 1 + 1 \cdot 1 + 2 \cdot (-1) = 0$, $\chi_{reg}(y) = 1 \cdot 1 + 1 \cdot (-1) + 2 \cdot 0 = 0$.

conjugacy classes

characters applied to g	$\langle 1 \rangle$	$\langle x \rangle$	$\langle y \rangle$	
χ_1	1	1	1	dim 1
χ_2	1	1	-1	dim 1
χ_3	2	-1	0	dim 2
χ_{reg}	6	0	0	

Proposition Two irreducible representations of G are equivalent $\Leftrightarrow$ their characters are equal.

Proof – ($\Rightarrow$) Suppose $\rho, \sigma: G \to GL_n(\mathbb{C})$ are equivalent, $\exists T \in GL_n(\mathbb{C})$ s.t. $\sigma(g) = T\rho(g)T^{-1}$ $\forall g$ $\therefore \chi_\sigma(g) = \mathrm{Tr}(\sigma(g)) = \mathrm{Tr}(T\rho(g)T^{-1}) = \mathrm{Tr}(\rho(g)TT^{-1}) = \mathrm{Tr}(\rho(g)) = \chi_\rho(g)$ // q.e.d.

($\Leftarrow$) NTP: if $\chi_U = \chi_V$, then $U \cong V$ as $\mathbb{C}[G]$-modules. Let $U = S_1^{a_1} \oplus \dots \oplus S_r^{a_r}$ and $V = S_1^{b_1} \oplus \dots \oplus S_r^{b_r}$ be two semisimple $\mathbb{C}[G]$-modules. Take trace of corresponding representations: $\chi_U = a_1\chi_1 + \dots + a_r\chi_r$, $\chi_V = b_1\chi_1 + \dots + b_r\chi_r$. Since χ_i are unique irreducible characters, $\chi_U = \chi_V \Rightarrow a_i = b_i \ \forall i \Rightarrow U \cong V$ as $\mathbb{C}[G]$-modules // q.e.d.

Tool – the invariance of χ_i is used to detect when representations are equivalent / not equivalent.

Example – Take $Q_8 = \langle x, y \mid x^4 = 1, x^2 = y^2, y^{-1}xy = x^{-1} \rangle$, with representations $\rho_1: Q_8 \to GL_2(\mathbb{C})$, $x \mapsto \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$, $y \mapsto \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$; $\rho_2: Q_8 \to GL_2(\mathbb{C})$, $x \mapsto \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$, $y \mapsto \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$; $\rho_3: Q_8 \to GL_2(\mathbb{C})$, $x \mapsto \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$, $y \mapsto \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$.

We know that $\mathbb{C}[Q_8] \cong \mathbb{C} \times \mathbb{C} \times \mathbb{C} \times \mathbb{C} \times M_2(\mathbb{C})$ with conjugacy classes $\{1\}, \{x^2\}, \{x, x^3\}, \{y, x^2y\}, \{xy, x^3y\}$. We plot our character table:

	$\langle 1 \rangle$	$\langle x^2 \rangle$	$\langle x \rangle$	$\langle y \rangle$	$\langle xy \rangle$
χ_1	2	-2	0	0	0
χ_2	2	-2	0	0	0
χ_3	2	2	0	0	-2

$\rho_1(x^2) = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}^2 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$, $\rho_2(x^2) = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}^2 = \begin{pmatrix} i^2 & 0 \\ 0 & i^2 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$ ... etc. Then from character table, $\chi_1 = \chi_2 \Leftrightarrow \rho_1 \cong \rho_2$, hence $\exists T$ s.t. $\rho_2 = T\rho_1T^{-1}$ (equivalent representation). However, $\chi_1 \neq \chi_3 \Rightarrow \rho_1 \ncong \rho_3$, so ρ_3 is different. We can compute $\mathrm{End}_{\mathbb{C}[Q_8]}(\rho_1) \cong \mathbb{C}$ irreducible.

However, ρ_3 is reducible (semisimple as a $\mathbb{C}[G]$-module). Let $B = \{e_1, e_2\}$ be a basis for $\mathbb{C}$. $\rho_3(x) \cdot e_1 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \end{pmatrix} = -1\begin{pmatrix} 1 \\ 0 \end{pmatrix} = -e_1$. This is stable by ρ_3. Then $\rho_3(y) \cdot e_2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \end{pmatrix} = -e_2$, this is stable too, so $\underset{(2\text{-dim})}{\rho_3} \cong \underset{(1\text{-dim})}{\rho'} \oplus \underset{(1\text{-dim})}{\rho''}$ where $\rho': Q_8 \to \mathbb{C}$, $x \mapsto -1, y \mapsto 1$ and $\rho'': Q_8 \to \mathbb{C}$, $x \mapsto 1, y \mapsto -1$. This gives our representation.

Nilpotency and Idempotency

Definition An element $a \in R$ is called nilpotent if $\exists n \in \mathbb{N}$ s.t. $a^n = 0$.

Proposition If R is an integral domain (i.e. $ab = 0 \Rightarrow a = 0$ or $b = 0$), then the only nilpotent element is trivial (which is 0).

Proof – Suppose a nilpotent, then $a^n = 0 \Rightarrow a(a^{n-1}) = 0$. If $a \neq 0$, then $a^{n-1} \neq 0 \Rightarrow a$ is a zero divisor $\Rightarrow$ contradicts integral domain // q.e.d.

Example – 1. Any $\mathbb{F}_p$ is an integral domain: $\mathbb{F}_5 = \{\bar{0}, \bar{1}, \dots, \bar{4}\}$. $\bar{0}$ is only nilpotent. However 2. $\mathbb{Z}_9$ is not an integral domain because $3 \times 3 = 9 \equiv 0$. But 3 is nilpotent $\because 3^2 = 0$.

3. If $R = M_2(\mathbb{F})$, $a = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ is nilpotent $\because a^2 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$. Since $a^2 = 0$ but $a \neq 0$, $M_2(\mathbb{F})$ is not an integral domain.

We want to find another decomposition of $\mathbb{C}[G] \cong \bigoplus_{i=1}^{r} \mathbb{C}[G]e_i$.

Definition An element $e \in R$ is called idempotent if $e^2 = e$.

Examples –

1. Let $R = \mathbb{Z}_6$. Then $3^2 \equiv 9 \equiv 3$, so 3 is idempotent. 2. If $R = M_2(\mathbb{F})$, then $a = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, $b = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ are idempotent.

Proposition If R is an integral domain, then the only idempotent elements are trivial.

Proof – Let $e^2 = e$ be idempotent, $e^2 - e = 0$, $e(e-1) = 0 \Rightarrow e - 1 = 0 \Rightarrow e = 1$ // q.e.d.

Convention: We do not consider 0 to be idempotent, although $0^2 = 0$.

Definition The centre of ring R is $Z(R) = \{z \in R \text{ s.t. } rz = zr \ \forall r \in R\}$.

Definition An element $e \in R$ is called a central idempotent if $e^2 = e$ and $e \in Z(R)$.

Definition A set of idempotents $\{e_1, \dots, e_s\}$ are called orthogonal where $\forall i \neq j$, $e_ie_j = 0$.

The point is, we want to write the unit generator $1 \in R$ as a sum of idempotents.

Example – let $R = R_1 \times R_2 \times R_3$ be a product of 3 subrings. Then we can decompose $1 = (1,1,1)$ into a sum of orthogonal idempotents: $1 = (1,1,1) = (1,0,0) + (0,1,0) + (0,0,1) = e_1 + e_2 + e_3$

$\boxed{\text{Theorem}}$ A ring R can be written as a product of subrings $R = R_1 \times \dots \times R_r \iff 1 = e_1 + \dots + e_r$ is a sum of orthogonal central idempotents. In this case, each $R_i \cong Re_i = \{\Sigma re_i : r \in R\}$.

$\boxed{\text{Theorem}}$ A ring R is semisimple $\iff$ every left ideal $I \triangleleft R$ is of the form $I = Re_i$ where e_i is an idempotent element. [sketch of proof – $1 = \Sigma e_i \Rightarrow e_i = 1 - \sum_{i \neq j} e_j \dots$]

Proofs – omitted, not relevant.

Apply to $\mathbb{C}[G]$: We know that $\mathbb{C}[G] \cong \bigoplus_{i=1}^{r} M_{n_i}(\mathbb{C}) \cong \bigoplus_{i=1}^{r} \mathbb{C}[G]\,\varepsilon_i$ with $(0,\dots,0,I_{n_i},0,\dots,0) \xmapsto{\rho_i} \varepsilon_i$ where $\{\varepsilon_1,\dots,\varepsilon_r\}$ are a set of orthogonal idempotents.

Key point – let $\rho_1,\dots,\rho_r : \mathbb{C}[G] \to GL_{n_i}(\mathbb{C})$ be the representations of the group ring (irreducible). $\rho_i(\Sigma a_g g) = \Sigma a_g \rho_i(g)$ $\forall g \in G$. Then $\rho_i(\varepsilon_i) = I_{n_i}$ and $\rho_i(\varepsilon_j) = 0$ if $i \neq j$.

$\Rightarrow \chi_i(\varepsilon_i) = \mathrm{Tr}(\rho_i(\varepsilon_i)) = \mathrm{Tr}(I_{n_i}) = \mathrm{Tr}\begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix} = n_i = \deg(\rho_i)$. $\chi_i(\varepsilon_j) = \mathrm{Tr}(0) = 0$.

(Idempotent Formula)

$\boxed{\text{Theorem}}$ Let G be finite, $|G| \neq 0$ in $\mathbb{F}$. Let $\rho_1,\dots,\rho_r : G \to GL_{n_i}(\mathbb{C})$ be a set of irreducible unique representations of G with corresponding irreducible characters $\{\chi_1,\dots,\chi_r\}$ where $\chi_i(g) = \mathrm{Tr}(\rho_i(g))$. Then $\{\varepsilon_1,\dots,\varepsilon_r\}$ is a set of central orthogonal idempotents $\varepsilon_i^2 = \varepsilon_i$ given by the formula $\boxed{\varepsilon_i = \frac{n_i}{|G|}\sum_{h \in G} \chi_i(h^{-1})h}$ where each ε_i is associated with a factor of $\mathbb{C}[G]$ s.t. $\mathbb{C}[G] = \bigoplus_{i=1}^{r} \mathbb{C}[G]\,\varepsilon_i$.

Proof – Let $\varepsilon_i \in \mathbb{C}[G]$, $\therefore \varepsilon_i = \sum_{g \in G} a_g g \in \mathbb{C}[G]$, $\varepsilon_i h^{-1} = \sum_{g \in G} a_g g h^{-1}$. Apply both definitions of χ_{reg} to $\varepsilon_i h^{-1}$. First, recall $\chi_{reg}(g) = \begin{cases} |G| & \text{if } g = 1 \\ 0 & \text{if } g \neq 1 \end{cases}$. Then we get

$\chi_{reg}(\varepsilon_i h^{-1}) = \mathrm{Tr}[\rho_{reg}(\varepsilon_i h^{-1})] = \mathrm{Tr}[\rho_{reg}(\sum_g a_g g h^{-1})] = \mathrm{Tr}[\rho_{reg}(a_h h h^{-1} + 0)] = \mathrm{Tr}[\rho_{reg}(a_h \cdot 1)] = a_h |G|$. Then, recall second definition – $\chi_{reg} = \sum_{j=1}^{r} n_j \chi_j$

where χ_j are irreducible characters. $\chi_{reg}(\varepsilon_i h^{-1}) = \sum_{j=1}^{r} n_j \chi_j(\varepsilon_i h^{-1})$. Recall key point: $\chi_j(\varepsilon_i) = 0$ if $i \neq j$, $\chi_j(\varepsilon_j) = \mathrm{Tr}(S_j \xrightarrow{h^{-1}} S_j) = \mathrm{Tr}(I_{n_j}) = n_j$ $\Rightarrow$ we get that

$\chi_{reg}(\varepsilon_i h^{-1}) = \overset{j=i}{n_i \chi_i(h^{-1})} + \overset{j \neq i}{0}$ $\therefore a_h |G| = n_i \chi_i(h^{-1}) \Rightarrow a_h = \frac{n_i}{|G|} \cdot \chi_i(h^{-1}) \Rightarrow$ with the coefficients of ε_i, $\varepsilon_i = \sum_{g \in G} a_g g = \sum_{h \in G} a_h h = \frac{n_i}{|G|}\sum_{h \in G} \chi_i(h^{-1})h$ // q.e.d.

Example – D_6, we use the character table from before with same ρ_1, ρ_2, ρ_3. We can find $\mathbb{C}[D_6] \cong \mathbb{C}[G](\varepsilon_1 \oplus \varepsilon_2 \oplus \varepsilon_3)$ with $\varepsilon_i^2 = \varepsilon_i$, $\varepsilon_i \cdot \varepsilon_j = 0$ $\forall i \neq j$. Then $1 \mapsto \varepsilon_1 + \varepsilon_2 + \varepsilon_3$.

	$\langle 1 \rangle$	$\langle x \rangle$	$\langle y \rangle$
χ_1	1	1	1
χ_2	1	1	-1
χ_3	2	-1	0

$\varepsilon_1 = \frac{n_1}{|D_6|}\sum_{h \in D_6} \chi_1(h^{-1}) \cdot h = \frac{1}{6}[(\chi_1(1^{-1}) \cdot 1 + \chi_1(x^{-1}) \cdot x + \chi_1(x^{-2})x^2 + \chi_1(y^{-1})y + \chi_1((xy)^{-1})xy + \chi_1((x^2y)^{-1})x^2y]$

$\Rightarrow \varepsilon_1 = \frac{1}{6}[\chi_1(1) \cdot 1 + \chi_1(x^2)x + \chi_1(x)x^2 + \chi_1(y)y + \chi_1(xy)xy + \chi_1(x^2y)x^2y] = \frac{1}{6}(1 \cdot 1 + 1 \cdot x + 1 \cdot x^2 + 1 \cdot y + 1 \cdot xy + 1 \cdot x^2y)$ using χ_1 table. $\varepsilon_1 = \frac{1}{6}(1 + x + x^2 + y + xy + x^2y)$.

Check $\varepsilon_1^2 = \varepsilon_1$:

Repeat for $\varepsilon_2, \varepsilon_3$. We get $\varepsilon_2 = \frac{1}{6}(1 + x + x^2 - y - xy - x^2y)$, $\varepsilon_3 = \frac{1}{3}(2 - x - x^2)$. [$\varepsilon_3 = \frac{2}{6}(2 \cdot 1 + 1 \cdot x + 1 \cdot x^2 + 0(xy) + 0(x^2y) + 0y$].

Then $1 = \varepsilon_1 + \varepsilon_2 + \varepsilon_3$.

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A beautiful application of this is decomposing $\mathbb{C}[G]$-modules into $\mathbb{C}[G]$-submodules using ε_i where we need to know irreducible χ_i.

Recipe to find submodules –

1. Choose basis for $V \cong \mathbb{C}^n$, say $\{v_1,\dots,v_n\}$.
2. Compute the set $\varepsilon_i \cdot V = \{\varepsilon_i v_i : 1 \leq i \leq n\} = \{\frac{n_i}{|G|}\Sigma \chi_i(h^{-1}) \cdot h\, v_i\}$. Ignore coefficient,
3. Let $V_i = \mathrm{span}_{\mathbb{C}}\{\varepsilon_i V\}$, then 4. $V = \bigoplus V_i$ $\therefore$ we have decomposed $\mathbb{C}[G]$-module V into submodules.

Example – continue from above: $D_6 \cong S_3 = G = \{(1)\} \sqcup \{\overset{x}{(1\ 2\ 3)}, \overset{x^2}{(1\ 3\ 2)}\} \sqcup \{\overset{y}{(1\ 2)}, \overset{xy}{(2\ 3)}, \overset{x^2y}{(1\ 3)}\}$. We take $V = \mathbb{C}^3$ to be the $\mathbb{C}[S_3]$-module (permutation module) given by action $\sigma \cdot v_i = v_{\sigma(i)}$ $\forall \sigma \in S_3$ $\forall v_i \in V$. $\rho : S_3 \to GL_3(\mathbb{C})$, $(1\,2\,3) \mapsto \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$, $(1\,2) \mapsto \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$. Let $\{v_1, v_2, v_3\}$ be a standard basis for $V = \mathbb{C}^3$. Compute $\varepsilon_1 \cdot V \Rightarrow$

$\varepsilon_1 v_1 = \frac{1}{6}(v_{(1)(1)} + v_{(1\,2\,3)(1)} + v_{(1\,3\,2)(1)} + v_{(1\,2)(1)} + v_{(2\,3)(1)} + v_{(1\,3)(1)}) = \frac{1}{6}(v_1 + v_2 + v_3 + v_2 + v_1 + v_3) = \frac{1}{3}(v_1 + v_2 + v_3)$. Likewise, we have

$\varepsilon_1 v_2 = \frac{1}{6}(v_2 + v_3 + v_1 + v_1 + v_2 + v_2) = \frac{1}{3}(v_1 + v_2 + v_3)$, $\varepsilon_1 v_3 = \frac{1}{3}(v_1 + v_2 + v_3)$, so $V_1 = \mathrm{span}_{\mathbb{C}}\{\varepsilon_1 V\} = \mathrm{span}\{v_1 + v_2 + v_3\} \subseteq V$. Next we do $\varepsilon_2 \cdot V$, and we $\varepsilon_2 = \frac{1}{6}(1 + (1\,2\,3) + (1\,3\,2) - (1\,2) - (1\,3) - (2\,3))$

realise $\varepsilon_2 \cdot v_1 = \varepsilon_2 v_2 = \varepsilon_2 \cdot v_3 = 0$, so $V_2 = \mathrm{span}\{\varepsilon_2 V\} = 0$. $\varepsilon_3 = \frac{1}{3}(2(1) - (1\,2\,3) - (1\,3\,2))$. $\varepsilon_3 \cdot v_1 = \frac{1}{3}(2v_{(1)(1)} - v_{(1\,2\,3)(1)} - v_{(1\,3\,2)(1)}) = \frac{1}{3}(2v_1 - v_2 - v_3)$.

$\varepsilon_3 \cdot v_2 = \frac{1}{3}(2 v_{(1)(2)} - v_{(1\,2\,3)(2)} - v_{(1\,3\,2)(2)}) = \frac{1}{3}(2v_2 - v_3 - v_1)$. $\varepsilon_3 \cdot v_3 = \frac{1}{3}(2v_{(1)(3)} - v_{(1\,2\,3)(3)} - v_{(1\,3\,2)(3)}) = \frac{1}{3}(2v_3 - v_1 - v_2)$. $V_3 = \mathrm{span}\{\varepsilon_3 \cdot V\}$. If we ignore coefficients and label $f_i = \varepsilon_3 \cdot v_i$, then $f_1 + f_2 + f_3 = 0 \Rightarrow$ not LI. Then $V_3 = \mathrm{span}\{\varepsilon_3 v_i\} = \mathrm{span}\{2v_1 - v_2 - v_3, 2v_2 - v_1 - v_3\}$. Then we have

$V = V_1 \oplus V_3 = \mathrm{span}\{v_1 + v_2 + v_3\} \oplus \mathrm{span}\{2v_1 - v_2 - v_3, 2v_2 - v_1 - v_3\}$.

Class Functions

Recall that characters are $\chi : G \to \mathbb{C}$ are functions which are constant on conjugacy classes of G. So if $g = xhx^{-1}$, $h \in (g)^G$, then $\chi(g) = \chi(h)$.

$\boxed{\text{Definition}}$ A class function $\varphi : G \to \mathbb{C}$ is a function that is constant on conjugacy classes of G. (i.e. $\varphi(g) = \varphi(h)$ if g, h in same conjugacy class).

Examples – clearly, characters of representations.

Let $\mathcal{C} = \{\varphi : G \to \mathbb{C}$ where φ is a class function$\}$. This is a $\mathbb{C}$-vector space of dimension r = number of conjugacy classes. Then $\{\chi_1,\dots,\chi_r\}$ (irreducible characters of G) form a basis for $\mathcal{C}$.

$\boxed{\text{Theorem}}$ Let G be finite, then $\mathcal{C} = \mathrm{span}_{\mathbb{C}}\{\chi_1,\dots,\chi_r\}$ is a basis for the space of class functions, s.t. any $\varphi \in \mathcal{C}$ can be uniquely expressed as $\varphi = \sum_{i=1}^{r} \lambda_i \chi_i$, $\lambda_i \in \mathbb{C}$ and χ_i are irreducible characters of G.

Proof – We know $\mathbb{C}[G] = \bigoplus_{i=1}^{r} M_{n_i}(\mathbb{C}) \leftrightarrow \{\chi_1,\dots,\chi_r\}$ irreducible characters and r is number of conjugacy classes. Let $\{K_1,\dots,K_r\}$ be conjugacy classes of G.

Define the class functions $\chi_i : G \to \mathbb{C}$, $\chi_i(g) = \begin{cases} 1 & \text{if } g \in K \\ 0 & \text{if } g \notin K \end{cases}$. Then $\{\chi_1,\dots,\chi_r\}$ form a basis if we establish LI. Let $0 = \sum_{j=1}^{r} \lambda_j \chi_j \in \mathcal{C}$, must show $\lambda_j = 0$ $\forall j$.

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Use idempotents $\varepsilon_i \sim \chi_i$. $0 = \sum_{j=1}^{r} \lambda_j \chi_j(\varepsilon_i)$ where $\chi_j(\varepsilon_i) = \deg(\chi_i(1)) = \deg(\rho_i)$ if $i=j$, $\chi_j(\varepsilon_i) = 0$ if $i \neq j$. Then $0 = \sum_{j=1}^{r} \lambda_j \chi_j(\varepsilon_i) = \lambda_j \deg(\rho_i)$. Since $\deg(\rho_i) \neq 0$, $\lambda_j = 0 \ \forall j$ q.e.d.

Positive Definite Hermitian Forms

A positive definite Hermitian form is provided by the complex inner product space $(V, \langle -,- \rangle)$, $\langle , \rangle : V \times V \to \mathbb{C}$ satisfying:

(1) $\langle v, w \rangle = \overline{\langle w, v \rangle}$ (conjugate symmetry) (2) $\langle \lambda_1 v_1 + \lambda_2 v_2, w \rangle = \lambda_1 \langle v_1, w \rangle + \lambda_2 \langle v_2, w \rangle$ (linearity in first argument – beware, not second! $\langle v, \lambda w_1 + w_2 \rangle = \bar{\lambda} \langle v, w_1 \rangle + \langle v, w_2 \rangle$)

(3) $\langle v, v \rangle \geq 0$ and $\langle v, v \rangle = 0 \Leftrightarrow v = 0$.

Example of complex inner product –

let $V = \mathbb{C}^n$, $\langle , \rangle : \mathbb{C}^n \times \mathbb{C}^n \to \mathbb{C}$, $\langle v, w \rangle = \bar{v}^T A w$ where A is any positive Hermitian matrix $a_{ij} = \overline{a_{ji}}$ e.g. $A = \begin{pmatrix} 1 & 2-3i \\ 2+3i & 4 \end{pmatrix}$, $\bar{A}^T = A$.

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Definition Let φ, ψ be two class functions (characters), then their inner product is the complex number $\langle \varphi, \psi \rangle = \frac{1}{|G|} \sum_{g \in G} \varphi(g) \overline{\psi(g)}$ for $\varphi : G \to \mathbb{C}$, $\psi : G \to \mathbb{C}$. [$\because \varphi, \psi$ as complex vectors].

Example – The set $\{\chi_1, \dots, \chi_r\}$ of irreducible characters $\leftrightarrow \{\rho_1, \dots, \rho_r\}$ form an orthonormal basis for the space $\mathcal{C}$ of class functions, if we choose r conjugacy class representatives $\{g_1, \dots, g_r\}$ for G. $\therefore \langle \chi_i, \chi_j \rangle = \delta_{ij} = \begin{cases} 1 & i=j \\ 0 & i \neq j \end{cases}$

Properties (1), (2) are easy to check, so we just check (3) positive definiteness – $\langle \varphi, \varphi \rangle = \frac{1}{|G|} \sum_g \varphi(g) \overline{\varphi(g)} = \frac{1}{|G|} \sum_g |\varphi(g)|^2 \geq 0$, with equality iff $\varphi(g) = 0 \ \forall g$.

Example of inner product of class functions. Let $G = C_3$, φ, ψ be class functions $C_3 \to \mathbb{C}$. Character table is given on right. Then, by definition,

	$\{1\}$	$\{x\}$	$\{x^2\}$
$\varphi(g)$	1	1	1
$\psi(g)$	2	i	-1

$\langle \varphi, \psi \rangle = \frac{1}{|G|} \sum_{g \in G} \varphi(g) \overline{\psi(g)} = \frac{1}{3} (\varphi(1)\overline{\psi(1)} + \varphi(x)\overline{\psi(x)} + \varphi(x^2)\overline{\psi(x^2)}) = \frac{1}{3}(1 \cdot \bar{2} + 1 \cdot \bar{i} + 1 \cdot \overline{-1}) = \frac{1}{3}(1-i)$. Then $\langle \psi, \varphi \rangle = \overline{\langle \varphi, \psi \rangle} = \frac{1}{3}(1+i)$.

$\langle \varphi, \varphi \rangle = \frac{1}{3} (\varphi(1)\overline{\varphi(1)} + \varphi(x)\overline{\varphi(x)} + \varphi(x^2)\overline{\varphi(x^2)}) = \frac{1}{3}(1+1+1) = 1$. $\Rightarrow \varphi$ is a basis element and an irreducible character. $\langle \psi, \psi \rangle = \frac{1}{3}(2 \cdot \bar{2} + i \cdot \bar{i} + (-1)\overline{(-1)}) = \frac{1}{3}(4+1+1) = 2$ $\Rightarrow \therefore \psi$ is a reducible character $\therefore \psi = \sum n_i \chi_i$.

We can use this to check if $\rho : G \to \mathbb{C}$ is simple, without computing $\mathrm{End}_{\mathbb{C}[C_3]}(\rho)$.

Proposition Let G have conjugacy classes $\{g_1, \dots, g_r\}$. Let χ and ψ be two characters of G. Then we have:

(1) $\langle \chi, \psi \rangle = \langle \psi, \chi \rangle = \frac{1}{|G|} \sum_{g \in G} \chi(g) \overline{\psi(g)}$ and (2) $\langle \chi, \psi \rangle = \sum_{g_i \in G} \frac{\chi(g_i)\overline{\psi(g_i)}}{|C_G(g_i)|}$ where $C_G(g_i) = \{x \in G : x g_i = g_i x\}$ (3) $\langle \chi, \psi \rangle \in \mathbb{R}$

Proof – (1) Since $\chi(g^{-1}) = \overline{\chi(g)}$ if $g^{-1} \in (g)^G$, then $\langle \chi, \psi \rangle = \frac{1}{|G|} \sum_g \chi(g)\overline{\psi(g)} = \frac{1}{|G|} \sum_g \chi(g) \psi(g^{-1}) = \frac{1}{|G|} \sum_g \chi(g^{-1}) \psi(g) = \frac{1}{|G|} \sum_g \psi(g) \chi(g^{-1}) = \frac{1}{|G|} \sum \psi(g) \overline{\chi(g)} = \langle \psi, \chi \rangle$ q.e.d.

(2) Consider $\sum_g \chi(g)\overline{\psi(g)} = \sum_{g_i \in G} |(g_i)^G| \chi(g_i) \overline{\psi(g_i)}$ where g_i are conjugacy class reps. $\therefore \frac{1}{|G|} \sum_{g_i \in G} \chi(g)\overline{\psi(g)} = \langle \chi, \psi \rangle = \sum_{g_i \in G} \frac{|(g_i)^G|}{|G|} \chi(g_i)\overline{\psi(g_i)} = \sum_{i=1}^{r} \frac{\chi(g_i)\overline{\psi(g_i)}}{|C_G(g_i)|}$

(3) $\langle \chi, \psi \rangle = \overline{\langle \psi, \chi \rangle}$ (conjugate symmetric) $= \langle \psi, \chi \rangle$ (by definition) $\Rightarrow \langle \chi, \psi \rangle \in \mathbb{R}$ q.e.d.

Example – Let $G = A_4 = \{\sigma \in S_4 : \mathrm{sgn}(\sigma) = +1\}$. $\mathbb{C}[A_4] \cong \mathbb{C} \times \mathbb{C} \times \mathbb{C} \times M_3(\mathbb{C})$, $r = 4$ conjugacy classes. Define $g_1 = (1)$, $g_2 = (1\,2)(3\,4)$, $g_3 = (1\,2\,3)$, $g_4 = (1\,3\,2)$.

$\lvert C_{A_4}(g_i)\rvert$	12	4	3	3
	g_1	g_2	g_3	g_4
χ	1	1	ω	ω^2
ψ	4	0	ω^2	ω

Define two characters $\chi, \psi : A_4 \to \mathbb{C}$. Then we have $\langle \chi, \psi \rangle = \frac{1 \cdot \bar{4}}{12} + \frac{1 \cdot \bar{0}}{4} + \frac{\omega \cdot \overline{\omega^2}}{3} + \frac{\omega^2 \cdot \bar{\omega}}{3} = \frac{4}{12} + \frac{\omega^2}{3} + \frac{\omega}{3} = \frac{1}{3} - \frac{1}{3} = 0$.

Then χ, ψ are orthogonal. $\langle \chi, \chi \rangle = \frac{1 \cdot \bar{1}}{12} + \frac{1 \cdot \bar{1}}{4} + \frac{\omega \cdot \bar{\omega}}{3} + \frac{\omega^2 \cdot \overline{\omega^2}}{3} = \frac{1}{12} + \frac{1}{4} + \frac{1}{3} + \frac{1}{3} = 1 \Rightarrow \chi$ is irreducible, corresponds to a simple character. $\langle \psi, \psi \rangle = \frac{4 \cdot \bar{4}}{12} + \frac{\omega^2 \cdot \overline{\omega^2}}{3} + \frac{\omega \cdot \bar{\omega}}{3} = \frac{16}{12} + \frac{1}{3} + \frac{1}{3} = 2 \Rightarrow$ reducible.

Motivation: We know characters are constant on conjugacy classes represented by $\{g_1, \dots, g_r\}$ and they form a basis for $\mathcal{C} = \mathrm{span}\{\chi_1, \dots, \chi_r\}$, so we can arrange our characters into an $r \times r$ matrix with entries $\chi_i(g_j)$ where $1 \leq i, j \leq r$.

	g_1	$\cdots$	g_j	$\cdots$	g_r
χ_1	$\chi_1(g_1)$	$\cdots$	$\chi_1(g_j)$	$\cdots$	$\chi_1(g_r)$
$\vdots$	$\vdots$	$\ddots$	$\vdots$		$\vdots$
χ_i	$\chi_i(g_1)$	$\cdots$	$\chi_i(g_j)$	$\cdots$	$\chi_i(g_r)$
$\vdots$	$\vdots$		$\vdots$	$\ddots$	$\vdots$
χ_r	$\chi_r(g_1)$	$\cdots$	$\chi_r(g_j)$	$\cdots$	$\chi_r(g_r)$

Definition The character table of G is the $r \times r$ invertible matrix (for basis χ_i) given with the $(i,j)^{th}$-entry $\chi_i(g_j)$:

Example – if $G = D_6$, we have the following table

	$\{1\}$	$\langle x \rangle$	$\langle y \rangle$
χ_1	1	1	1
χ_2	1	1	1
χ_3	2	-1	0

Row and Column Orthogonality Relations

Used to reconstruct $(\chi_i(g_j))_{ij}$ = character table, find $|G|$, $|(g_i)^G|$, $|C_G(g_i)|$. Used to induce characters etc –

(1) Row orthogonality relation – for two fixed characters, run through the row of conjugacy classes. Since $\chi_1, \dots, \chi_r$ form an orthonormal basis for $\mathcal{C}$, by irreducibility theorem, $\langle \chi_i, \chi_j \rangle = \delta_{ij}$ so for conjugacy class representatives $\{g_1 = (1), \dots, g_r\}$, we have $\sum_{k=1}^{r} \frac{\chi_i(g_k)\overline{\chi_j(g_k)}}{|C_G(g_k)|} = \delta_{ij} = \langle \chi_i, \chi_j \rangle$.

(2) Column orthogonality relation (more useful). Fix two conjugacy classes, run through all characters on table. $\sum_{k=1}^{r} \chi_k(g_i)\overline{\chi_k(g_j)} = \delta_{ij} |C_G(g_j)| = \begin{cases} |C_G(g_j)| & \text{if } g_i \text{ is conjugate to } g_j \text{ (same column)} \\ 0 & \text{otherwise (different column)} \end{cases}$

Proof – Define the class function $\psi_j(g_i) = \delta_{ij}$. Then since $\{\chi_1, \dots, \chi_r\}$ form basis for $\mathcal{C}$, $\psi_j \in \mathcal{C} \Rightarrow \psi_j = \sum_{k=1}^{r} \lambda_k \chi_k$, $\lambda_k \in \mathbb{C}$. Using Irreducibility Theorem, $\langle \chi_i, \chi_k \rangle = \delta_{ik}$.

Then $\lambda_k = \langle \psi_j, \chi_k \rangle = \frac{1}{|G|} \sum_{g \in G} \psi_j(g) \overline{\chi_k(g)}$ [not yet running through $(g_i)^G$]. Now $\psi_j(g) = 1$ if g is conjugate to g_j, $\psi_j(g) = 0$ otherwise. We also have $|(g_j)^G|$ elements which are conjugate to g_j. Thus, $\lambda_k = \frac{1}{|G|} \cdot |(g_j)^G| \overline{\chi_k(g_j)} = \frac{\overline{\chi_k(g_j)}}{|C_G(g_j)|}$. $\therefore \delta_{ij} = \psi_j(g_i) = \sum_{k=1}^{r} \lambda_k \chi_k(g_i) = \sum_{k=1}^{r} \frac{\overline{\chi_k(g_j)}}{|C_G(g_j)|} \cdot \chi_k(g_i) \Rightarrow \sum_{k=1}^{r} \chi_k(g_i)\overline{\chi_k(g_j)} = \delta_{ij} |C_G(g_j)|$ q.e.d.

Example – 1. $G = D_6 = \langle 1 \rangle \sqcup \langle x \rangle \sqcup \langle y \rangle$. $|(1)^{D_6}| = 1$, $|(x)^{D_6}| = 2$, $|(y)^{D_6}| = 3$. $|C_{D_6}(1)| = 6$, $|C_{D_6}(x)| = 3$, $|C_{D_6}(y)| = 2$. Then

	$\langle 1 \rangle$	$\langle x \rangle$	$\langle y \rangle$
χ_1	1	1	1
χ_2	1	1	-1
χ_3	a?	b?	c?

How can we fill in the last row?

By orthogonality, examine first column: $i = 1, j = 1$. $\sum_{k=1}^{r} \chi_k(g_i)\overline{\chi_k(g_j)} = \delta_{ij} |C_G(g_j)|$. Fix i, j, here $g_i = g_j = 1 \Rightarrow \sum_{k=1}^{3} \chi_k(1) \cdot \overline{\chi_k(1)} = \delta_{11} \cdot |C_G(1)| = 1 \cdot 6 = 6$.

$\Rightarrow$ by table: $\chi_1(1) \cdot \overline{\chi_1(1)} + \chi_2(1) \cdot \overline{\chi_2(1)} + \chi_3(1) \cdot \overline{\chi_3(1)} = 6 \Rightarrow 1 \cdot \bar{1} + 1 \cdot \bar{1} + a \cdot \bar{a} = 6 \Rightarrow 2 + a^2 = 6 \Rightarrow a = 2$. (we reject -2, because first column is a trace of Id).

Then, we see that $a=2$. Since this is identity column, this produces a 2D representation. Likewise for column 2, fix $i=j$, $g_2=x$ s.t. $1\cdot\bar{1}+1\cdot\bar{1}+b\cdot\bar{b}=|C_{D_6}(x)|=3 \Rightarrow b^2=1$.

$b=\pm1$? This is ambiguous, so we try another method – first and second column. $i=1, j=2 \Rightarrow \delta_{ij}=0$, so we can ignore centraliser s.t.

$\sum_{k=1}^{3} \chi_k(\underset{g_1}{1}) \cdot \overline{\chi_k(\underset{g_2}{x})} = 0 \Rightarrow 1\cdot\bar{1}+1\cdot\bar{1}+2\cdot\bar{b}=0 \Rightarrow 2\bar{b}=-2 \Rightarrow \bar{b}=-1 \Rightarrow b=-1$. For third column, $1\cdot\bar{1}+1\cdot\overline{(-1)}+2\bar{c}=0\cdot|C_3(\omega)| \Rightarrow 2\bar{c}=0 \Rightarrow c=0$.

2. A group of order 12 with 4 conjugacy classes $\{g_1,g_2,g_3,g_4\}$ and centralisers $|C_G(g_1)|=12$, $|C_G(g_2)|=4$, $|C_G(g_3)|=3$, $|C_G(g_4)|=3$.

	g_1	g_2	g_3	g_4
χ_1	1	1	1	1
χ_2	1	1	ω	ω^2
χ_3	1	1	ω^2	ω
χ_4	?	?	?	?

It has partial χ-table as on right. We seek final row of table. First column with itself gives $\delta_{11}|C_G(1)|=12 \Rightarrow 1\cdot\bar{1}+1\cdot\bar{1}+1\cdot\bar{1}+\chi_4(1)\cdot\overline{\chi_4(1)}=12$

$\Rightarrow |\chi_4(1)|^2=9 \Rightarrow \chi_4(1)=3 \Rightarrow$ (reject -3) 3-dimensional rep. Then $i=1, j=2 \Rightarrow 3+3\overline{\chi_4(g_2)}=0 \Rightarrow \chi_4(g_2)=-1$. Then $i=1, j=3$

$\Rightarrow 1\cdot\bar{1}+1\cdot\bar{\omega}+1\cdot\overline{\omega^2}+3\cdot\overline{\chi_4(g_3)} = 1+\omega^2+\omega+3\overline{\chi_4(g_3)}=0 \Rightarrow 3\overline{\chi_4(g_3)}=0 \Rightarrow \chi_4(g_3)=0$. $i=1, j=4 \Rightarrow 1\cdot\bar{1}+1\cdot\overline{\omega^2}+1\cdot\bar{\omega}+3\overline{\chi_4(g_4)}=0 \Rightarrow \chi_4(g_4)=0$

Last row is $\{3,-1,0,0\}$.

Summary on characters.

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We know that $\mathbb{C}[G] \cong S_1^{n_1} \oplus \dots \oplus S_r^{n_r}$ where each S_i is a simple $\mathbb{C}[G]$-submodule of dimension n_r. For each S_i we have irreducible characters $\{\chi_1,\dots,\chi_r\}$ where $r=\#$ conjugacy classes of G. Then ① $\langle\chi_i,\chi_j\rangle=\delta_{ij}$ and ② if ψ is any character of G, then $\psi=d_1\chi_1+\dots+d_r\chi_r$ for some non-negative integers d_i where $d_i=\langle\psi,\chi_i\rangle$. ③ $\langle\psi,\psi\rangle=\sum_{i=1}^{r} d_i^2$.

	$\langle 1\rangle$	$\langle(1\,2\,3)\rangle$	$\langle(1\,2)\rangle$
χ_1	1	1	1
χ_2	1	1	-1
χ_3	2	0	-1

Example – $D_6 \cong S_3 = \langle 1\rangle \sqcup \langle(1\,2\,3)\rangle \sqcup \langle(1\,2)\rangle$. Then the character table is as on right: $|\langle 1\rangle|=1$, $|\langle 1\,2\,3\rangle|=2$, $|\langle 1\,2\rangle|=3$.

$|C_{D_6}(g_i)|=\frac{|G|}{|g^G|}$, $C_{S_3}(1)=\frac{6}{1}=6$, $C_{S_3}((1\,2\,3))=\frac{6}{2}=3$ $C_{S_3}((1\,2))=\frac{6}{3}=2$. Let $\rho: S_3 \to GL_3(\mathbb{C})$ be the permutation representation

$(1\,2\,3) \mapsto \begin{pmatrix} 0&0&1\\1&0&0\\0&1&0 \end{pmatrix}$, $(1\,2) \mapsto \begin{pmatrix} 0&1&0\\1&0&0\\0&0&1 \end{pmatrix}$. $\psi_\rho(1)=3$, $\psi_\rho((1\,2\,3))=0$, $\psi_\rho((1\,2))=1$. By ②, ψ is a linear combination of irreducible

characters: $\psi_\rho=d_1\chi_1+d_2\chi_2+d_3\chi_3$. $\langle\psi_\rho,\chi_1\rangle=d_1=\sum_{i=1}^{3}\psi_\rho(g_i)\overline{\chi_1(g_i)}\cdot\frac{1}{|C_{S_3}(g_i)|}=\frac{3\cdot\bar{1}}{6}+\frac{1\cdot\bar{1}}{2}+\frac{0}{3}=1$. $\langle\psi_\rho,\chi_2\rangle=0=d_2$. $\langle\psi_\rho,\chi_3\rangle=1=d_3$.

$\therefore \psi_\rho=1\cdot\chi_1+1\cdot\chi_3$.

$\boxed{\text{Theorem}}$ (Frobenius Reciprocity Theorem).

$\chi\downarrow H: H \longrightarrow \mathbb{C}$ $\forall h\in H$

Let $H\leq G$, then if $\chi: G\to\mathbb{C}$ is a character of G, then $\chi\downarrow H$ is called the <u>restricted character</u> of H, $h \mapsto \chi(h)$ i.e. we ignore elements of G that are not in H [For example, $\rho: G_6 \to GL_2(\mathbb{C})$, $\chi: D_6 \to \mathbb{C}$. Let $H=C_3$, $\chi(1)=2$ $\chi(x)=-1$ $\chi(x^2)=-1$ [take] / $\chi(y)=0$ $\chi(xy)=0$ $\chi(x^2y)=0$ [ignore].].

If $\varphi_1,\dots,\varphi_s$ are irreducible characters of H, then $(\chi\downarrow H)=d_1\varphi_1+\dots+d_s\varphi_s$. s.t. $\sum_{i=1}^{3} d_i^2 \leq [G:H]$. For each $\mathbb{C}[H]$-module V, the induced $\mathbb{C}[G]$-module $\mathrm{Ind}_H^G(V)$ has character defined using the character ψ of H written as $(\psi\uparrow G)(g)=\frac{1}{|H|}\sum_{y\in G}\dot{\psi}(y^{-1}gy)$. where $\dot{\psi}(g)=\begin{cases}\psi(g) & g\in H\\ 0 & g\notin H.\end{cases}$ $\dot{\psi}: G\to\mathbb{C}$

Then $\langle\psi\uparrow G,\chi\rangle_G=\langle\psi,\chi\downarrow H\rangle_H$ where ψ is a character of H and χ is a character of G.

END OF SYLLABUS.

The Frobenius Reciprocity theorem is important as it allows us to write characters for G and H.

$$\begin{array}{c} \\ \chi_1 \\ \vdots \\ \chi_r \end{array} \begin{array}{c} \psi_1 \dots \psi_s \\ \left(\begin{array}{c} \quad \\ \quad \\ \quad \end{array} \right) \end{array}$$

rows are restrictions,
columns are inductions.
[refer to James, Liebeck for more info]

END OF COURSE.

A problem class will be convened on Friday, 25 April 2014.