3301 Real Fluids Notes

Based on the 2010 autumn lectures by Dr S Timoshin

3301 REAL FLUIDS

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Moodle: MATH3301, password RealFluids2010

Structure

1. Derive eq^{n}s of motion
2. Exact solutions
3. Lubrication approximation
4. Stokes flow (very viscous flow)

Some notes on Moodle

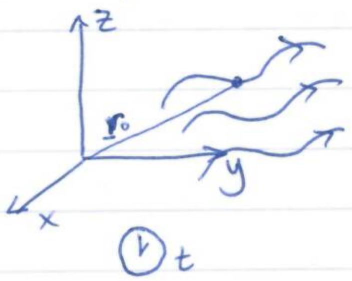
Our real fluid is viscous but incompressible

1. DERIVING EQ^{N}S OF MOTION

1. Describing a flow means finding the position of each fluid particle (or velocity) and pressure at all times.

2. Specifications of flow:

- Lagrangian specification



At $t=0$, choose fluid particle with position $\underline{r}_0$ and follow it in time.

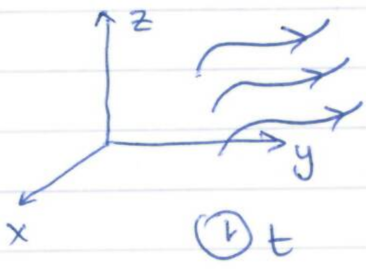
Then, e.g. position vector of chosen particle $\underline{r} = \underline{r}(\underline{r}_0, t)$
pressure $P = P(\underline{r}_0, t)$

Exercise: Given $\underline{r} = \underline{r}(\underline{r}_0, t)$,
find (i) velocity
(ii) acceleration
of a given particle.

Differentiate!: (i) $\underline{u} = \dfrac{\partial \underline{r}(\underline{r}_0, t)}{\partial t}$

(ii) $\underline{a} = \dfrac{\partial^2 \underline{r}(\underline{r}_0, t)}{\partial t^2}$

- Eulerian specification.



All quantities are f^{n}s of $(\underline{r}, t)$

e.g. velocity $\underline{u} = \underline{u}(\underline{r}, t)$

Exercise: Given velocity field $\underline{u} = \underline{u}(\underline{r}, t)$
find acceleration of an individual fluid particle.

Change to Lagrange; choose a particle with pos^n vector $\underline{r} = \underline{r}(\underline{r}_0, t)$.

Then velocity of this particle, $\underline{u} = \underline{u}(\underline{r}(\underline{r}_0, t), t)$

Accel. $\underline{a} = \left.\dfrac{\partial \underline{u}}{\partial t}\right|_{\underline{r}_0 \text{ fixed}}$

Let $\underline{r} = (x, y, z)$
$\underline{u} = (u, v, w)$

$$\underline{a} = \frac{\partial}{\partial t} \underline{u}\Big(x(\underline{r}_0, t), y(\underline{r}_0, t), z(\underline{r}_0, t), t\Big)\Big|_{\underline{r}_0 \text{ fixed}}$$

$$= \frac{\partial \underline{u}}{\partial x} \underbrace{\frac{\partial x(\underline{r}_0, t)}{\partial t}}_{u} + \frac{\partial \underline{u}}{\partial y} \underbrace{\frac{\partial y(\underline{r}_0, t)}{\partial t}}_{v} + \frac{\partial \underline{u}}{\partial z} \underbrace{\frac{\partial z(\underline{r}_0, t)}{\partial t}}_{w} + \frac{\partial \underline{u}}{\partial t}$$

$$= \frac{\partial \underline{u}}{\partial t} + u \frac{\partial \underline{u}}{\partial x} + v \frac{\partial \underline{u}}{\partial y} + w \frac{\partial \underline{u}}{\partial z}$$

$$= \frac{\partial \underline{u}}{\partial t} + (\underline{u} \cdot \nabla) \underline{u}$$

Notation: $\boxed{\dfrac{D}{Dt} = \dfrac{\partial}{\partial t} + (\underline{u} \cdot \nabla)}$ is the "material derivative"

Exercise: Given $\underline{u} = \underline{u}(\underline{r}, t)$,
find r.o.c. of density ρ in a fluid particle
rate of change

$$\frac{D\rho}{Dt} = \frac{\partial \rho}{\partial t} + (\underline{u} \cdot \nabla) \rho$$

<u>Exercise</u>: Check for any scalar $a(\underline{r},t)$, $b(\underline{r},t)$,

$$\frac{D(ab)}{Dt} = a\frac{Db}{Dt} + \frac{Da}{Dt}b$$

(obvious)

CONSERVATION OF MASS

Let $J(\underline{r},t)$ be the rate of mass production per unit volume in flow.

Take a small volume of fluid δV (small <u>material</u> volume)
i.e. same fluid particles.

Rate of change of mass in δV is

$$\frac{D(\rho\,\delta V)}{Dt} = J\,\delta V$$

$$\frac{1}{\delta V}\frac{D(\rho\,\delta V)}{Dt} = J \qquad \text{(def}^{\text{n}}\text{ of } J)$$

Product rule: $$\frac{\rho}{\delta V}\frac{D(\delta V)}{Dt} + \frac{\cancel{\delta V}}{\cancel{\delta V}}\frac{D\rho}{Dt} = J$$

Need: $$\frac{1}{\delta V}\frac{D(\delta V)}{Dt} = \frac{1}{\delta x\,\delta y\,\delta z}\frac{D(\delta x\,\delta y\,\delta z)}{Dt}$$

$$\text{(product rule)} = \frac{1}{\delta x}\underbrace{\frac{D(\delta x)}{Dt}}_{\text{What is this?!}} + \frac{1}{\delta y}\frac{D(\delta y)}{Dt} + \frac{1}{\delta z}\frac{D(\delta z)}{Dt}$$

$$\frac{D(\delta x)}{Dt} = \frac{d(x_2-x_1)}{dt} = \left(\frac{dx_2}{dt} - \frac{dx_1}{dt}\right)$$

$$= u_2 - u_1 = \delta u.$$

So $\frac{1}{\delta V}\frac{D(\delta V)}{Dt} = \frac{\delta u}{\delta x} + \frac{\delta v}{\delta y} + \frac{\delta w}{\delta z}$

If $\delta V \to 0$, $\boxed{\lim_{\delta V \to 0} \frac{1}{\delta V}\frac{D(\delta V)}{Dt} = \operatorname{div} \underline{u}}$

Back to

$$\frac{D\rho}{Dt} + \rho \frac{1}{\delta V}\frac{D(\delta V)}{Dt} = J$$

get

$$\boxed{\frac{D\rho}{Dt} + \rho \operatorname{div}(\underline{u}) = J}$$ Mass conservation.

<u>Definitions</u>: A fluid is <u>incompressible</u> if

$\frac{D(\delta V)}{Dt} = 0$ ie. $\boxed{\operatorname{div} \underline{u} = 0}$ continuity condition

Mass conservation in an incompressible fluid

is $\frac{D\rho}{Dt} = J$.

or $\frac{\partial \rho}{\partial t} + (\underline{u}\cdot\nabla)\rho = J$

or $\frac{\partial \rho}{\partial t} + u\frac{\partial \rho}{\partial x} + v\frac{\partial \rho}{\partial y} + w\frac{\partial \rho}{\partial z} = J$

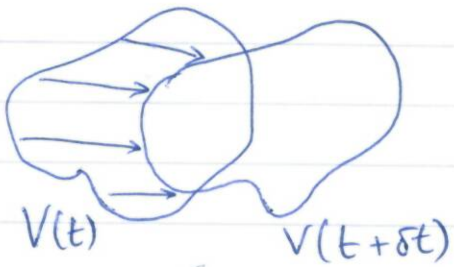
If no internal sources, $J = 0$

and $\frac{\partial \rho}{\partial t} + (\underline{u}\cdot\nabla)\rho = 0$.

A fluid is <u>homogeneous</u> if
$\rho \equiv \text{const}.$

<u>Exercise:</u> Find rate of change of mass in a finite material volume in the form

$$\frac{d}{dt}\int_{V(t)} \rho \, dV = \int_{V(t)} \left[\frac{\partial \rho}{\partial t} + (\underline{u}\cdot\nabla)\rho + \rho \operatorname{div}\underline{u} \right] dV$$



Recall: in fluid particle,

$$\frac{1}{\delta V}\frac{D(\rho \delta V)}{Dt} = \frac{D\rho}{Dt} + \rho \operatorname{div}\underline{u}$$
$$= \frac{\partial \rho}{\partial t} + (\underline{u}\cdot\nabla)\rho + \rho \operatorname{div}\underline{u}$$

Multiply by δV and take sum over individual volumes
$\Rightarrow$ the result.

Apply this to momentum.
For x-component, replace ρ with ρu. Then...

<u>Exercise:</u> Rate of change of x-momentum in finite material volume is ...

$$\frac{d}{dt}\int_{V(t)} \rho u \, dV = \int_{V(t)} \left[\frac{\partial(\rho u)}{\partial t} + (\underline{u}\cdot\nabla)(\rho u) + \rho u \,\mathrm{div}\,\underline{u} \right] dV$$

$$= \int_{V(t)} \left[u\frac{\partial \rho}{\partial t} + \rho\frac{\partial u}{\partial t} + u(\underline{u}\cdot\nabla)\rho + \rho(\underline{u}\cdot\nabla)u + \rho u \,\mathrm{div}\,\underline{u} \right] dV$$

$$= \int_{V(t)} \Big[u\overbrace{\left(\frac{\partial \rho}{\partial t} + (\underline{u}\cdot\nabla)\rho + \rho \,\mathrm{div}\,\underline{u} \right)}^{J}$$

$$+ \rho\underbrace{\left(\frac{\partial u}{\partial t} + (\underline{u}\cdot\nabla)u \right)}_{\frac{Du}{Dt}} \Big] dV$$

$\Rightarrow$ r.o.c. of x-momentum is $V(t)$

$$\frac{d}{dt}\int_{V(t)} \rho u \, dV = \int_{V(t)} \left[uJ + \rho\frac{Du}{Dt} \right] dV$$

Repeat for y, z components of momentum

e.g. $$\frac{d}{dt}\int_{V(t)} \rho v \, dV = \int_{V(t)} \left[vJ + \rho\frac{Dv}{Dt} \right] dV$$

or, in vector form

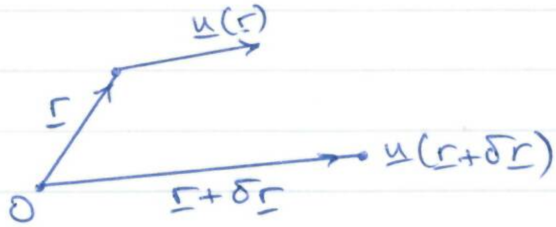
$$\frac{d}{dt}\int_{V(t)} \rho \underline{u} \, dV = \int_{V(t)} \left[\underline{u}J + \rho\frac{D\underline{u}}{Dt} \right] dV$$

Eventually I want to say,

ROC of momentum = total force.

KINEMATICS

At *fixed* time t, take 2 particles



Express $\underline{u}(\underline{r}+\delta\underline{r})$ through $\underline{u}(\underline{r})$ and derivatives of $\underline{u}(\underline{r})$.

$$\underline{r} = (x_1, x_2, x_3) \qquad \underline{u} = (u_1, u_2, u_3)$$
$$\delta\underline{r} = (\delta x_1, \delta x_2, \delta x_3)$$

Take u_1.

$$u_1(\underline{r}+\delta\underline{r}) = u_1(x_1+\delta x_1,\ x_2+\delta x_2,\ x_3+\delta x_3)$$
$$(\text{Taylor}) \equiv u_1(x_1, x_2, x_3) + \delta x_1 \frac{\partial u_1(x_1,x_2,x_3)}{\partial x_1} + \delta x_2 \frac{\partial u_1}{\partial x_2} + \delta x_3 \frac{\partial u_1}{\partial x_3} + \cdots$$

or

$$u_1(\underline{r}+\delta\underline{r}) = u_1(\underline{r}) + \frac{\partial u_1}{\partial x_i}\delta x_i \qquad \text{(in summation notation)}$$

Repeat for u_2 and u_3 to get

$$u_j(\underline{r}+\delta\underline{r}) = u_j(\underline{r}) + \frac{\partial u_j}{\partial x_i}\delta x_i \qquad j=1,2,3$$

Let $\underline{e}_1, \underline{e}_2, \underline{e}_3$ be unit vectors along x_1, x_2, x_3.

Multiply by $\underline{e}_j$ and sum over j.

$$u_j(\underline{r}+\delta\underline{r})\underline{e}_j = \underline{u}_j(\underline{r})\underline{e}_j + \underbrace{\frac{\partial u_j}{\partial x_i}\delta x_i \underline{e}_j}_{9 \text{ terms}}$$

or $\underline{u}(\underline{r}+\delta\underline{r}) = \underline{u}(\underline{r}) + \frac{\partial u_i}{\partial x_j}\delta x_j \underline{e}_i$

$$= \underline{u}(\underline{r}) + \overbrace{\frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)}^{e_{ij}} \delta x_j \underline{e}_i$$

$$+ \underbrace{\frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i}\right)}_{\xi_{ij}} \delta x_j \underline{e}_i$$

$$\Rightarrow \underline{u}(\underline{r}+\delta\underline{r}) = \overbrace{\underline{u}(\underline{r})}^{\text{translation}} + \overbrace{e_{ij}\delta x_j \underline{e}_i}^{\text{deformation}} + \underbrace{\xi_{ij}\delta x_j \underline{e}_i}_{\text{rotation}}$$

with $e_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$

$\xi_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i}\right) = -\xi_{ji}$ antisymmetric 2nd-order tensor

See handout for 'meaning' of ξ_{ij}.

So ξ_{ij} - rotation
e_{ij} - rate-of-strain tensor.

Exercise. Show that,

$$\xi_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_k} - \frac{\partial u_k}{\partial x_i}\right), \tag{1}$$

represents a rigid body rotation with angular velocity $\frac{1}{2}\boldsymbol{\omega}$ where $\boldsymbol{\omega} = \mathrm{curl}\mathbf{u}$.

Let $\mathbf{u} = (u_1, u_2, u_3)$ with the unit vectors $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$.

- First, calculate the vorticity in the flow.

  We use the alternating tensor,

$$\varepsilon_{ijk} = \begin{cases} 0 \text{ unless } i, j \text{ and } k \text{ are all different} \\ 1 \text{ if } i, j, k \text{ are in cyclic order} \\ -1 \text{ otherwise} \end{cases} \tag{2}$$

  Then

$$\boldsymbol{\omega} = \mathrm{curl}\mathbf{u} = \det\begin{pmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \\ \frac{\partial}{\partial x_1} & \frac{\partial}{\partial x_2} & \frac{\partial}{\partial x_3} \\ u_1 & u_2 & u_3 \end{pmatrix} = \varepsilon_{ijk}\mathbf{e}_i\frac{\partial u_k}{\partial x_j}. \tag{3}$$

  Since $\boldsymbol{\omega} = \omega_i \mathbf{e}_i$, we have

$$\omega_i = \varepsilon_{ijk}\frac{\partial u_k}{\partial x_j} \text{ or, changing notation, } \omega_j = \varepsilon_{jpm}\frac{\partial u_m}{\partial x_p}. \tag{4}$$

- Calculate the velocity field in rigid body rotation with angular velocity $\frac{1}{2}\boldsymbol{\omega}$.

  Let $\mathbf{r} = (r_1, r_2, r_3)$ be the position vector, $\mathbf{v} = (v_1, v_2, v_3)$ - the velocity at $\mathbf{r}$

$$\mathbf{v} = \frac{1}{2}\boldsymbol{\omega} \times \mathbf{r} = \det\begin{pmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \\ \frac{1}{2}\omega_1 & \frac{1}{2}\omega_2 & \frac{1}{2}\omega_3 \\ r_1 & r_2 & r_3 \end{pmatrix} = \varepsilon_{ijk}\mathbf{e}_i\frac{1}{2}\omega_j r_k, \tag{5}$$

  therefore

$$v_i = \frac{1}{2}\varepsilon_{ijk}\omega_j r_k = \frac{1}{2}\varepsilon_{ijk}\varepsilon_{jpm}\frac{\partial u_m}{\partial x_p}r_k, \tag{6}$$

  with summation over j, p, m, k.

  The result (6) can be written as

$$v_i = b_{ik}r_k \tag{7}$$

where

$$b_{ik} = \underbrace{\frac{1}{2}\varepsilon_{ijk}\varepsilon_{jpm}\frac{\partial u_m}{\partial x_p}}_{\text{sum over } j,p,m} = \left| \begin{array}{c} \text{note that non-zero terms can} \\ \text{only have } p = i \text{ and } m = k \text{ or} \\ p = k \text{ and } m = i \end{array} \right|$$

$$= \underbrace{\frac{1}{2}\varepsilon_{ijk}\varepsilon_{jik}\frac{\partial u_k}{\partial x_i} + \frac{1}{2}\varepsilon_{ijk}\varepsilon_{jki}\frac{\partial u_i}{\partial x_k}}_{\text{no sum over } i,k} \tag{8}$$

Since $\varepsilon_{ijk}\varepsilon_{jik} = -1$ and $\varepsilon_{ijk}\varepsilon_{jki} = 1$, we now have

$$b_{ik} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_k} - \frac{\partial u_k}{\partial x_i}\right) \tag{9}$$

We conclude that $\xi_{ij} = b_{ij}$.

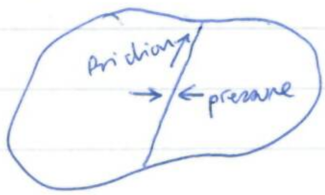
FORCES IN FLUIDS

Three classes: 1. Long-range or body forces
(like gravity)

e.g. $\underline{g}$ - gravitational acceleration
then $\underline{F} = \int_{V(t)} \rho \underline{g} \, dV$

2. Short-range or local forces
(inside the fluid)

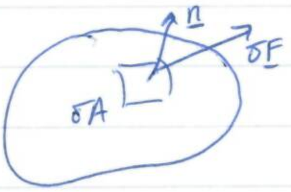
e.g. pressure and fiction.



3. Forces on boundaries between
two different fluids (water to air)

e.g. surface tension.

Short-range forces and stress tensor



Take δA - area element
$\underline{n}$ - outward unit normal to δA
$\delta \underline{F}$ - force on δA exerted by fluid outside

$\delta \underline{F} = \delta \underline{F}(\underline{r}, t, \delta A, \underline{n})$
a f^{n} of lots of things — not very convenient!!

but when $|\delta A| \ll 1$, force = stress x area

$$\delta \underline{F} = \underline{\Sigma}\, \delta A$$

$\underline{\Sigma}$ is the stress vector

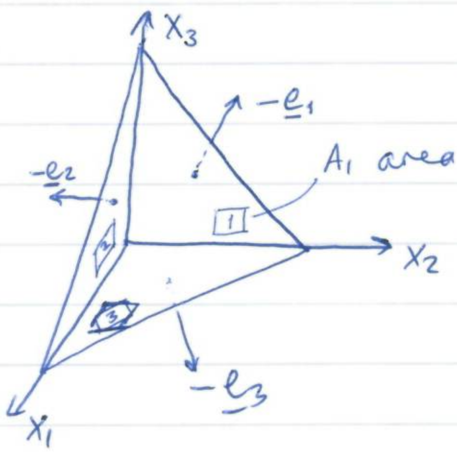
$$\underline{\Sigma} = \underline{\Sigma}(\underline{r}, t, \underline{n})$$

Want to find a quantity independent of $\underline{n}$. $\leftarrow$ not nice!

Take a tetrahedron:

Coordinates x_1, x_2, x_3

Unit vectors $\underline{e}_1, \underline{e}_2, \underline{e}_3$



Faces	1	2	3	4 (skew)
Normals	$-\underline{e}_1$	$-\underline{e}_2$	$-\underline{e}_3$	$\underline{n}$
Areas	A_1	A_2	A_3	A
Stress applied	$\underline{\Sigma}_1$	$\underline{\Sigma}_2$	$\underline{\Sigma}_3$	$\underline{\Sigma}$

Suppose $\underline{\Sigma}_1, \underline{\Sigma}_2, \underline{\Sigma}_3$ are known.
Need to find $\underline{\Sigma}$.

NII says: r.o.c. of momentum (or ma) = F.

$$\underbrace{\int_V \left(\rho \frac{D\underline{u}}{Dt} + J\underline{u} \right) dV}_{\text{r.o.c. of Momentum}} = \int_V \rho \underline{g}\, dV + \sum_{k=1}^{4} A_k \underline{\Sigma}_k \qquad \begin{pmatrix} A_4 = A \\ \underline{\Sigma}_4 = \underline{\Sigma} \end{pmatrix}$$

Let δr be typical length size of tetrahedron, and $\delta r \to 0$

$$\int_V \rho \underline{g}\, dV = O((\delta r)^3)$$

$$\int_V \left(\rho \frac{D\underline{u}}{Dt} + J\underline{u} \right) dV = O((\delta r)^3)$$

$$\sum_{k=1}^{4} A_k \underline{\Sigma}_k = O((\delta r)^2)$$

$$\Rightarrow \sum_{k=1}^{4} A_k \underline{\Sigma}_k = 0$$

Can convince myself of this by saying, for $\delta r \to 0$

$$\sum_{k=1}^{4} A_k \underline{\Sigma}_k = (\delta r)^2 \cdot \underline{F}_0 + \dots$$

$$\int_V \rho \underline{g}\, dV = (\delta r)^3 \cdot \underline{G}_0 + \dots$$

$$\int \left[\rho \frac{D\underline{u}}{Dt} + J\underline{u} \right] dV = (\delta r)^3 H_0 + \dots$$

Then by Newton:

$$(\delta r)^3 \cdot H_0 = (\delta r)^3 G_0 + (\delta r)^2 F_0 + \dots$$

Divide by $(\delta r)^2$, take limit as $\delta r \to 0$, $\Rightarrow F_0 = 0$.

$$\Rightarrow \text{(again)} \quad \boxed{\sum_{k=1}^{4} A_k \underline{\Sigma}_k = 0}$$

So for the tetrahedron,

$$A_1 \underline{\Sigma}_1 + A_2 \underline{\Sigma}_2 + A_3 \underline{\Sigma}_3 + A \underline{\Sigma} = 0$$

$$A_1 = (\underline{n} \cdot \underline{e}_1) A$$
$$A_2 = (\underline{n} \cdot \underline{e}_2) A$$
$$A_3 = (\underline{n} \cdot \underline{e}_3) A$$

$$\underline{\Sigma} = -(\underline{n} \cdot \underline{e}_1)\underline{\Sigma}_1 - (\underline{n} \cdot \underline{e}_2)\underline{\Sigma}_2 - (\underline{n} \cdot \underline{e}_3)\underline{\Sigma}_3$$ (relation for stress vector)

Notation: $\underline{\Sigma} = \begin{pmatrix} \Sigma_1 \\ \Sigma_2 \\ \Sigma_3 \end{pmatrix}$ $\underline{n} = \begin{pmatrix} n_1 \\ n_2 \\ n_3 \end{pmatrix}$

not $\underline{\Sigma}_3$

$$\underline{\Sigma}_1 = -\begin{pmatrix} \sigma_{11} \\ \sigma_{12} \\ \sigma_{13} \end{pmatrix} \quad \underline{\Sigma}_2 = -\begin{pmatrix} \sigma_{21} \\ \sigma_{22} \\ \sigma_{23} \end{pmatrix} \quad \underline{\Sigma}_3 = -\begin{pmatrix} \sigma_{31} \\ \sigma_{32} \\ \sigma_{33} \end{pmatrix}$$

$(\underline{n} \cdot \underline{e}_1) = n_1$ etc.

In components: $\Sigma_1 = \sigma_{11} n_1 + \sigma_{21} n_2 + \sigma_{31} n_3$

$\Sigma_2 = \ldots$

$\Sigma_3 = \ldots$

or $\boxed{\Sigma_i = \sigma_{ji} n_j}$

Q1 2005

$\Rightarrow$ the stress vector $\underline{\Sigma}$ is a linear function of the so-called <u>stress tensor</u> σ_{ij}

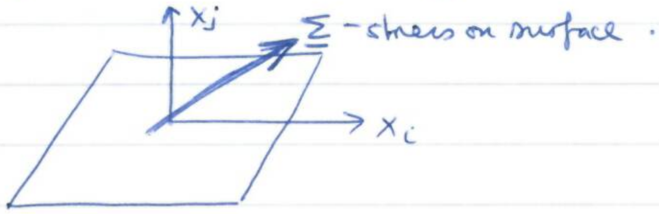
Q1 2008

Can prove that $\sigma_{ij} = \sigma_{ji}$ $\Rightarrow$ $\boxed{\Sigma_i = \sigma_{ij} n_j}$

stress tensor

Physical meaning of σ_{ij}

Take a surface normal to the j-axis.



Take i-component of $\underline{\Sigma}$:

$$\Sigma_i = \sum_{k=1}^{3} \sigma_{ik} n_k = \sigma_{ij} \quad \text{since } n_i = 0, \; n_j = 1$$

$$\left(\begin{matrix} n_1 = n_2 = n_i = 0 \\ n_3 = n_j = 1 \end{matrix} \right)$$

$\Rightarrow$ σ_{ij} is the i-component of stress (force per unit area) exerted on the surface normal to j-axis.

CONSTITUTIVE RELATION

Constitutive relation is the relation between forces and motion in fluid (similar to force-displacement relation in Hooke's Law).

$$\underline{F} = m\underline{a}$$

Gravity   Surface force $\rightarrow$ stress tensor.

__Simplest model:__ σ_{ij} is a 'constant', same in all reference frames

or $\sigma_{ij} = -p\delta_{ij}$

ie. $\|\sigma\| = \begin{pmatrix} -p & 0 & 0 \\ 0 & -p & 0 \\ 0 & 0 & -p \end{pmatrix} \leftarrow$ inviscid incompressible fluids from second year.

__More general, Newtonian viscous fluid:__

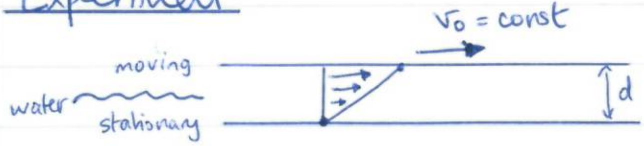
Q1 2006 $$\sigma_{ij} = -p\delta_{ij} + 2\mu e_{ij}$$

Q1 2010

μ — viscosity coefficient

therefore σ_{ij}, the stress tensor = linear function of rate-of-strain tensor.

__Experiment__



v_0 is constant if force (shear force) is constant

Force $\sim \mu \frac{v_0}{d}$

__Equations of motion —__
__NAVIER STOKES EQUATIONS__

For any continuous medium,

Q1 2005

Q1 2008

$$\text{Newton: } \int_{V(t)} \rho \left[\frac{\partial \underline{u}}{\partial t} + (\underline{u}\cdot\nabla)\underline{u} \right] dV = \int_{V(t)} \rho \underline{g}\, dV + \int_{A} \underline{\Sigma}\, dA$$

(drop J).

Take i-component:

$$\int \rho \left[\frac{\partial u_i}{\partial t} + (\underline{u}\cdot\nabla)u_i \right] = \int_{V(t)} \rho g_i \, dV + \underbrace{\int_A \Sigma_i \, dA}$$

use $\Sigma_i = \sigma_{ij} n_j$

Then $$\int_A \Sigma_i \, dA = \int_A \sigma_{ij} n_j \, dA$$

$$= \int_{V(t)} \frac{\partial \sigma_{ij}}{\partial x_j} dV \quad \text{by the divergence theorem}$$

Get $$\int_{V(t)} \rho \left[\frac{\partial u_i}{\partial t} + (\underline{u}\cdot\nabla)u_i \right] dV = \int_{V(t)} \left[\rho g_i + \overbrace{\frac{\partial \sigma_{ij}}{\partial x_j}} \right] dV$$

$$\Rightarrow \boxed{\rho \left[\frac{\partial u_i}{\partial t} + u_j \frac{\partial u_i}{\partial x_j} \right] = \rho g_i + \frac{\partial \sigma_{ij}}{\partial x_j}}$$

CAUCHY'S EQN. OF MOTION FOR ANY CONTINUOUS MEDIUM

Navier-Stokes: Use $\sigma_{ij} = -p\delta_{ij} + 2\mu e_{ij}$ (more general viscous fluid form)

Q1 2006

Q1 2010

$$\frac{\partial \sigma_{ij}}{\partial x_j} = -\underbrace{\frac{\partial p}{\partial x_j}\delta_{ij}}_{= \frac{\partial p}{\partial x_i}} + 2\mu \frac{\partial e_{ij}}{\partial x_j}$$

Meanwhile, what is ~~Simplify~~: $$\frac{\partial e_{ij}}{\partial x_j} = \frac{\partial}{\partial x_j} \frac{1}{2} \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right)$$

$$= \frac{1}{2}\frac{\partial^2 u_i}{\partial x_j \partial x_j} + \frac{1}{2}\frac{\partial}{\partial x_i}\left(\frac{\partial u_j}{\partial x_j}\right)$$

$$= \frac{1}{2}\nabla^2 u_i \qquad \underbrace{}_{\text{div } \underline{u} = 0}$$

because incompressible.

Sub $\frac{\partial \sigma_{ij}}{\partial x_j}$ into Cauchy:

$$\rho\left[\frac{\partial u_i}{\partial t} + u_j \frac{\partial u_i}{\partial x_j}\right] = \rho g_i - \frac{\partial p}{\partial x_i} + \mu \nabla^2 u_i$$

In vector form:

momentum → $$\boxed{\rho\left[\frac{\partial \underline{u}}{\partial t} + (\underline{u}\cdot\nabla)\underline{u}\right] = \rho \underline{g} - \nabla p + \mu \nabla^2 \underline{u}}$$

add continuity → $\boxed{\text{div } \underline{u} = 0}$

NAVIER-STOKES

And we used:

- incompressibility
- fluid is Newtonian.

<u>Exercise</u>: Evaluate viscosity of coffee (instant Kenco).
Ignore terms we don't like!
Pretend to be physicists.

$$\rho \frac{\partial u}{\partial t} = \mu \frac{\partial^2 u}{\partial x^2}$$

$$\rho \frac{\cancel{u}}{t} = \mu \frac{\cancel{u}}{x^2} \rightarrow \frac{\rho x^2}{t} = \mu \text{ - viscosity coeff.}$$

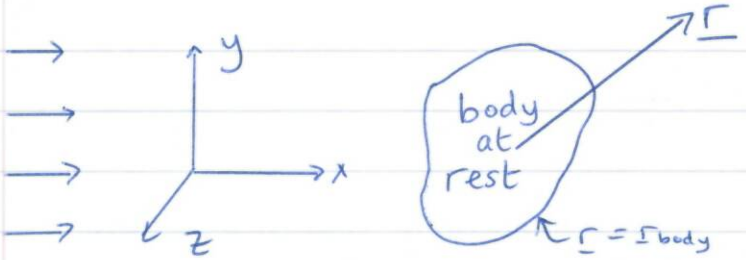
$$\frac{\mu}{\rho} = \nu \text{ - kinematic viscosity}$$

$$\nu = \frac{x^2}{t} = \frac{(0.1m)^2}{60sec} = \frac{10^{-2}}{60} = 1.6 \times 10^{-4}\ m^2/sec$$

A little table. In old units.

	μ (g/cm.s = poise)	ν (cm^2/s)
air room temp	0.00018	0.145
water	0.0114	0.0114
glycerine	23.3	18.5

Boundary conditions: Typically,

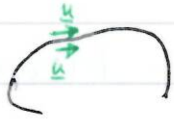


At infinity, $\underline{u} \to u_\infty \cdot \underline{i} \iff \begin{cases} u \to u_\infty \\ v \to 0 \\ w \to 0 \end{cases}$

as $r \to \infty$.

$$p \to p_\infty \text{ as } r \to \infty.$$

On the body, $\underline{u} = \underline{u}_{body}$ at $\underline{r} = \underline{r}_{body}$

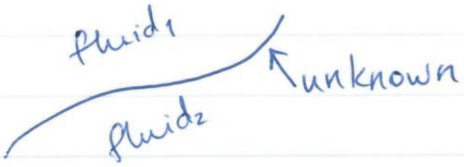


If body is at rest, $\underline{u} = 0$ at $\underline{r} = \underline{r}_{body}$. (no slip conditions)

Different from inviscid, where $\underline{u}_\perp = \underline{u}_{body\perp}$.

In viscous fluid, both normal and tangential velocities are specified on $\underline{r}_{body}$.

Interface between 2 fluids:

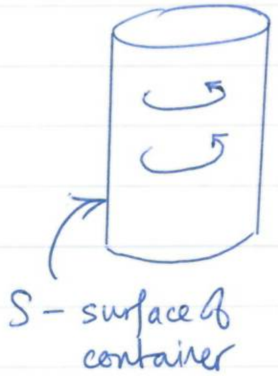


Conditions:
(1) Kinematic condition
(2) Continuity of velocity <u>vector</u>
(3) Continuity of stress <u>vector</u>

Q1
2007

ENERGY EQUATION

<u>Example</u>: Sealed container filled without gaps



All boundaries are at rest.

Begin with i-component in Cauchy's eqn.

$$\rho\left[\frac{\partial u_i}{\partial t} + u_j \frac{\partial u_i}{\partial x_j}\right] = \frac{\partial \sigma_{ij}}{\partial x_j}$$

this is Cauchy eqn's minus the g term.

(have ignored $\underline{g}$, can put it back in as exercise)

Multiply by u_i, sum for $i=1,2,3$:

$$\rho\left[\underbrace{u_i \frac{\partial u_i}{\partial t}}_{②} + \underbrace{u_i u_j \frac{\partial u_i}{\partial u_j}}_{③}\right] = \underbrace{u_i \frac{\partial \sigma_{ij}}{\partial x_j}}_{①}$$

Manipulate terms to get divergence terms:

① $$u_i \frac{\partial \sigma_{ij}}{\partial x_j} = \frac{\partial}{\partial x_j}\left[u_i \sigma_{ij}\right] - \sigma_{ij}\frac{\partial u_i}{\partial x_j}$$

② $$u_i \frac{\partial u_i}{\partial t} = \frac{1}{2}\frac{\partial}{\partial t}(u_i)^2$$

③ $u_i u_j \frac{\partial u_i}{\partial x_j} = \frac{1}{2}\frac{\partial}{\partial x_j}\left[u_i u_j u_i\right] - \frac{1}{2} u_i u_i \underbrace{\frac{\partial u_j}{\partial x_j}}_{= \operatorname{div} \underline{u} = 0}$

$$= \frac{1}{2}\frac{\partial}{\partial x_j}\left[u_i u_i u_j\right]$$

Throw this back into Cauchy to get:

$$\underbrace{\frac{\partial}{\partial t}\left[\frac{\rho u_i u_i}{2}\right]}_{②} + \underbrace{\frac{\rho}{2}\frac{\partial}{\partial x_j}\left[u_i u_i u_j\right]}_{③}$$

$$= \underbrace{\frac{\partial}{\partial x_j}\left(u_i \sigma_{ij}\right) - \sigma_{ij}\frac{\partial u_i}{\partial x_j}}_{①}$$

Integrate over container V

$$\frac{d}{dt}\int_V \rho \frac{u_i u_i}{2}\, dV + I_1 = I_2 - \int_V \sigma_{ij}\frac{\partial u_i}{\partial x_j}\, dV$$

where $I_1 = \frac{\rho}{2}\int_V \frac{\partial}{\partial x_j}\left(u_i u_i u_j\right) dV$ div. thm

$$= \frac{\rho}{2}\int_S \underbrace{\underset{0}{u_i}\,\underset{0}{u_i}\,\underset{0}{u_j}}_{\text{no slip}}\, n_j\, dS = 0$$

$$I_2 = \int_V \frac{\partial}{\partial x_j}\left(\sigma_{ij} u_i\right) dV = \int_S \sigma_{ij} \underset{\substack{\parallel \\ 0}}{u_i} n_j\, dS = 0.$$

Also $\int_V \rho \frac{u_i u_i}{2}\, dV = \int_V \frac{\rho}{2} |\underline{u}|^2\, dV = K$

kinetic energy inside container

Cauchy's energy equation:

$$\frac{dK}{dt} = -\int_V \sigma_{ij} \frac{\partial u_i}{\partial x_j}\, dV$$

For Newtonian fluid, $\sigma_{ij} = -p\delta_{ij} + 2\mu e_{ij}$

"Physicists: For an incompressible fluid, does pressure do any work?"
"Mmm... yes"
"But how because pressure can't compress it"
"Mmm... no."

Proving mathematically:

$$-\frac{dK}{dt} = \int_V \sigma_{ij} \frac{\partial u_i}{\partial x_j}\, dV = \cancel{-\int_V p\, \delta_{ij} \underbrace{\frac{\partial u_i}{\partial x_j}}_{= \frac{\partial u_i}{\partial x_i} = \operatorname{div} \underline{u} = 0}\, dV} \rightarrow 0$$

$\sigma_{ij} = -p\delta_{ij} + 2\mu e_{ij}$

$$+ \underbrace{2\mu \int_V e_{ij} \frac{\partial u_i}{\partial x_j}\, dV}$$

$$= \mu \int_V e_{ij} \frac{\partial u_i}{\partial x_j}\, dV + \mu \int_V e_{ij} \frac{\partial u_i}{\partial x_j}\, dV$$

$$= \mu \int_V e_{ij} \frac{\partial u_i}{\partial x_j}\, dV + \mu \int_V e_{ji} \frac{\partial u_j}{\partial x_i}\, dV \qquad \text{(relabelling subscripts)}$$

$$= \mu \int_V e_{ij} \frac{\partial u_i}{\partial x_j}\, dV + \mu \int_V e_{ij} \frac{\partial u_j}{\partial x_i}\, dV \qquad \text{(by symmetry of } e\text{)}$$

$$= \mu \int_V e_{ij} \underbrace{\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)}_{=2e_{ij}} dV = 2\mu \int_V e_{ij} e_{ij}\, dV$$

Back to Cauchy...

$$\frac{dK}{dt} = - \int_V 2\mu e_{ij} e_{ij} \, dV$$

or $$\frac{dK}{dt} = - \int_V \Phi(\underline{r}, t) \, dV$$

where $\Phi(\underline{r}, t) = 2\mu e_{ij} e_{ij} = 2\mu \sum_{i=1}^{3} \sum_{j=1}^{3} (e_{ij})^2$

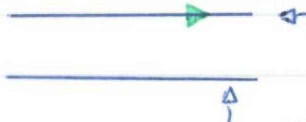
$\underline{\Phi}$ - dissipation f^{n}, positive definite.

$\Rightarrow$ K - monotonically decreasing in time

2 EXACT NAVIER-STOKES SOLUTIONS

3 Steps:
(1) Physical (wordy) description
(2) Maths formulation : eq^{n}s and b.c.'s
(3) Solve, analyse.

1 **Example**: The Couette flow in a channel.



moving to the right with speed $u_0 = \text{const}$

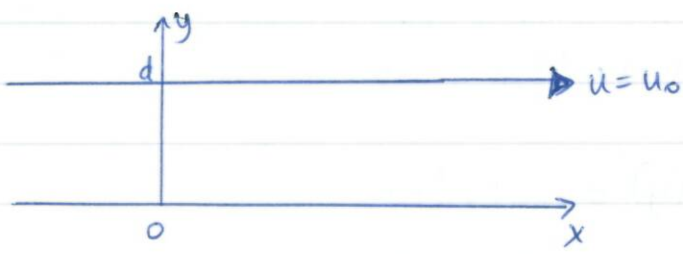
plate at rest.

What's the flow?

Q2 2007 similar

Assume infinitely long plates.
steady flow (ignore transients) ← dependence on t
velocity field is indpt of coord along the channel

Navier-Stokes equations in 2D:



N-S:
$$\rho\left[\frac{\partial \underline{u}}{\partial t} + (\underline{u}\cdot\nabla)\underline{u}\right] = \cancel{\rho \underline{g}}^{\,0} - \nabla p + \mu \nabla^2 \underline{u}$$

and $\operatorname{div} \underline{u} = 0$.

NAVIER STOKES
$$\rho(u_t + u u_x + v u_y) = -p_x + \mu(u_{xx} + u_{yy}) \qquad (1)$$
$$\rho(v_t + u v_x + v v_y) = -p_y + \mu(v_{xx} + v_{yy}) \qquad (2)$$

CONT.
$$u_x + v_y = 0 \qquad (3)$$

STEADY FLOW
Steady flow: $u_t = v_t = 0 \Rightarrow u = u(x,y)$
$v = v(x,y)$.

SETUP

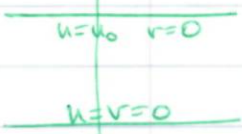
x-independent $\Rightarrow u_x = 0 \quad \Rightarrow u = u(y)$.

From ③: $v_y = 0 \quad \Rightarrow v = v(x)$

but $v_x = 0 \Rightarrow v = \text{const}$.

B.C.s

Boundary conditions for ①-③:



No slip on the walls i.e. at $y=0 \quad u=v=0$
$y=d \quad u=u_0, \; v=0$.

From b.c.'s, $v \equiv 0$.

$\Rightarrow$ ① becomes $0 = -p_x + \mu u_{yy}$
② becomes $0 = -p_y$
$\Rightarrow p = p(x,t)$

u is a f^n of $y \Rightarrow u_{yy} = u''(y)$

$\Downarrow$

$$u''(y) = \frac{1}{\mu} p_x(x,t)$$

$p_x(x,t)$ can only be constant. $\because u''(y)$ is a f^n of y !!

Assume no extra pressure gradient (on top of sliding wall as the source of motion). $p_x \equiv 0$.

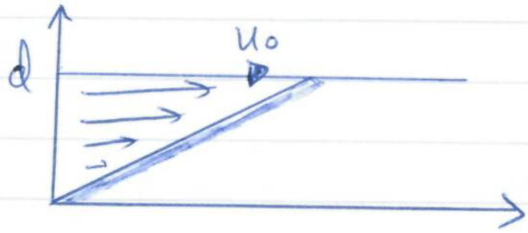
$\Rightarrow$ we have $u''(y) = 0$ [what's left of N-S]

B.C.s

b.c.'s $u|_{y=0} = 0$
$u|_{y=d} = u_0 \quad \Rightarrow u(y) = \frac{u_0}{d} y$

LINEAR PROFILE

COUETTE FLOW



Exercise: compute 'wall shear' (tangential stress) on bottom wall.

x-component of stress exerted on surface normal to y-axis → $\sigma_{xy} = 2\mu \frac{1}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)$

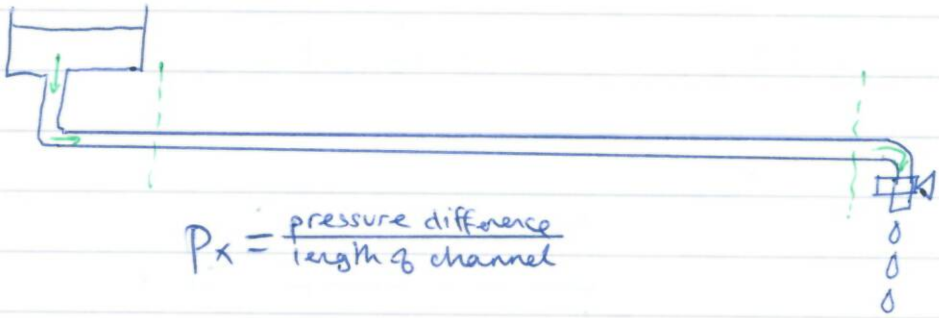
↑ v const.

At $y=0$, $\sigma_{xy} = \mu u'(0) = \mu \frac{u_0}{d}$

Solution: $u(y) = \frac{u_0}{d} y$.

$\boxed{2}$ Example: Infinitely long channel, walls at rest, 2D steady flow due to a constant pressure gradient.

Q2 2010



$p_x = \frac{\text{pressure difference}}{\text{length of channel}}$

Retrace the Couette steps

$$u''(y) = \frac{1}{\mu} p_x(x,t)$$

now $p_x = -G = \text{const}.$

B.C.s { b.c.'s: $u|_{y=0} = 0$
$u|_{y=d} = 0$

↑channel width

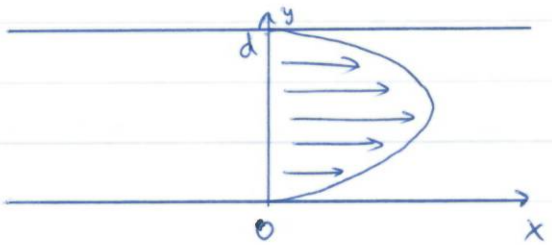
$$u''(y) = \frac{-G}{\mu}$$

General solution: $u(y) = -\frac{G}{2\mu}y^2 + C_1 y + C_2$

B.C.s Find $C_{1,2}$ from b.c's: $u(0)=0 \Rightarrow C_2 = 0$
$u(d)=0 \Rightarrow C_1 = \frac{G}{2\mu}d$

$\Rightarrow$ answer $u(y) = \frac{G}{2\mu}(yd - y^2)$ ← POISEUILLE FLOW

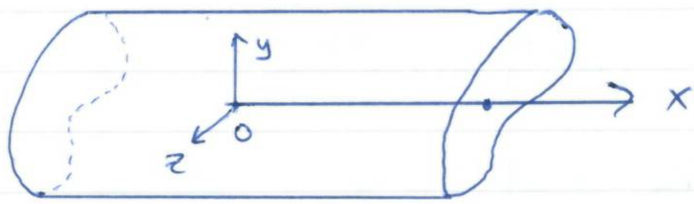
Parabolic velocity profile:



$\boxed{3}$ General 3D flow invariant to x-translations

Q1 2005

Q1 2008



$$u = u(y,z,t)$$
$$v = v(y,z,t)$$
$$w = w(y,z,t).$$

Continuity: $u_x + v_y + w_z = 0$
$\Rightarrow v_y + w_z = 0$ ④

$\nabla\cdot\underline{u} = 0$

NAV. STOKES

Navier-Stokes

x-momentum: $\rho[u_t + \cancel{uu_x} + vu_y + wu_z] = -p_x + \mu[\cancel{u_{xx}} + u_{yy} + u_{zz}]$

$$\rho[u_t + vu_y + wu_z] = -p_x + \mu[u_{yy} + u_{zz}] \quad (5)$$

y-momentum: $\rho[v_t + \cancel{uv_x} + vv_y + wv_z] = -p_y + \mu[\cancel{v_{xx}} + v_{yy} + v_{zz}]$

$$\rho[v_t + vv_y + wv_z] = -p_y + \mu[v_{yy} + v_{zz}] \quad (6)$$

z-momentum: In the cross-section, y-mom. = z-mom.

$$\rho[w_t + vw_y + ww_z] = -p_z + \mu[w_{yy} + w_{zz}] \quad (7)$$

Further assumption: flow is parallel to x-axis

"parallel" or "unidirectional" (even if in opposite dir^{n}s)

$\Rightarrow v = 0,\ w = 0.$

(4) - satisfied.

(5) becomes $\Rightarrow \rho u_t = -p_x + \mu(u_{yy} + u_{zz})$

(6) $\Rightarrow p_y = 0$

(7) $\Rightarrow p_z = 0$

Hence $p = p(x,t)$

$u = u(y,z,t)$

and

MOMENTUM EQ^{N}

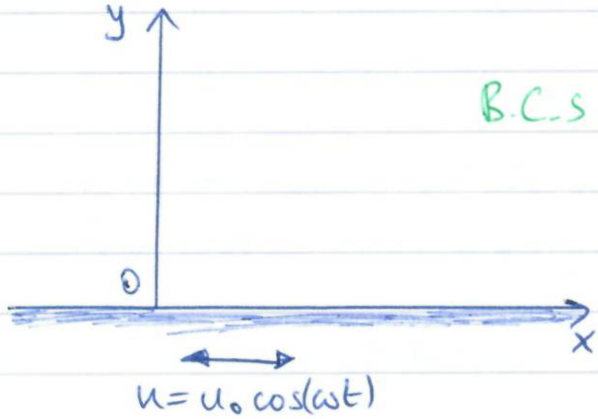
$$\boxed{u_t = -\frac{1}{\rho} p_x(x,t) + \nu(u_{yy} + u_{zz})} \qquad u = u(y,z,t). \quad (*)$$

Conclude: $p_x = f^n$ of t only. (RHS has no x dependence)

Need to specify p_x for every case.

$\boxed{4}$ <u>Example</u>: Stokes flow due to an oscillating flat plate
2D, t-periodic.

Q2 2008 similar



B.C.s $\begin{cases} \text{Wall velocity} \\ u|_{y=0} = u_0 \cos(\omega t) \\ v|_{y=0} = 0 \end{cases}$

Infinite plate, region $y \geq 0$.

Assume parallel flow, then solve ⊛.

mom eqn $\left\{ \right.$

mom. eq^n: $u_t = -\frac{1}{\rho} p_x(x,t) + \nu(u_{yy} + u_{zz})$

do this to find p_x → far from the plate, fluid is at rest,

B.C.

$u \to 0$ as $y \to \infty$. Assume that u_{yy} and $u_{zz} \to 0$ then, though this is not always true.

$\Rightarrow$ from ⊛ $\underline{p_x = 0}$.

EQNS TO SOLVE

Drop u_{zz} (2D), solve: $\begin{cases} u_t = \nu u_{yy} \\ u|_{y=0} = u_0 \cos(\omega t) \\ u|_{y \to \infty} \to 0. \end{cases}$

Find t-periodic solution (using separation of vars).

Separable solution [but not $u = Y(y)\cos(\omega t)$]

Write $u(y,t) = e^{i\omega t} f(y)$ and take real part in the answer.

Replace $\cos\omega t$ with $e^{i\omega t}$

$$u_t = i\omega e^{i\omega t} f(y)$$
$$u_{yy} = e^{i\omega t} f''(y)$$

Plug in: $i\omega f = \nu f''$

$$f'' = \frac{i\omega}{\nu} f$$

$$f = C_1 e^{\left[\frac{i\omega}{\nu}\right]^{1/2} y} + C_2 e^{-\left[\frac{i\omega}{\nu}\right]^{1/2} y}$$

Choose branch of $(\ )^{1/2}$ s.t. $\mathrm{Re}\left[\left(\frac{i\omega}{\nu}\right)^{1/2}\right] > 0$.

As $y \to \infty$, $e^{\left(\frac{i\omega}{\nu}\right)^{1/2} y} \to \infty \Rightarrow C_1 = 0$.

At $y = 0$, $f(0) = u_0 \Rightarrow C_2 = u_0$

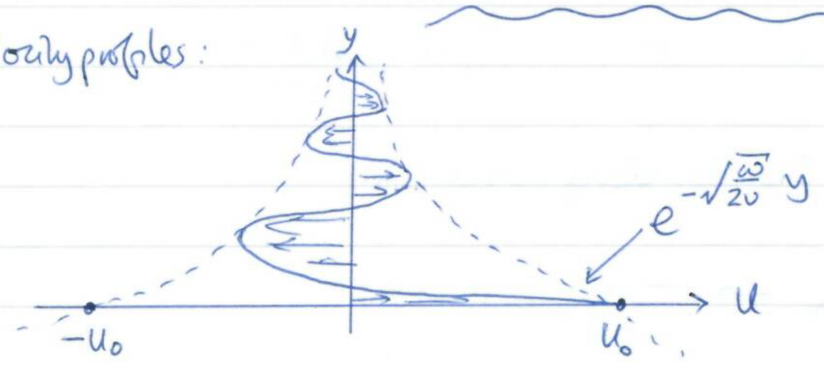
$$\Rightarrow f(y) = u_0 e^{-\left(\frac{i\omega}{\nu}\right)^{1/2} y}$$

$$\Rightarrow \text{Answer: } u(y,t) = \mathrm{Re}\left[u_0 e^{i\omega t - \left(\frac{i\omega}{\nu}\right)^{1/2} y}\right]$$

$$= \mathrm{Re}\left[u_0 e^{i\omega t - \frac{1+i}{\sqrt{2}}\sqrt{\frac{\omega}{\nu}} y}\right]$$

$$= u_0 e^{-\sqrt{\frac{\omega}{2\nu}} y} \cos\left(\omega t - \sqrt{\frac{\omega}{2\nu}} y\right).$$

Velocity profiles:



Fluid is disturbed in a layer adjacent to the wall.

Q: How thick is this layer in terms of frequency and viscosity?

A: $y = O\left(\sqrt{\frac{\nu}{\omega}}\right)$

$\boxed{5}$ Example: Fixed plate, flow due to an oscillating pressure gradient.

$$\frac{\partial p}{\partial x} = -G \sin(\omega t)$$

in region $y \geq 0$.

For parallel flow, need to solve

$u_t = -\frac{1}{\rho} p_x(x,t) + \nu(u_{yy} + u_{zz})$

$$\begin{cases} \dfrac{\partial u(y,t)}{\partial t} = \dfrac{G}{\rho} \sin(\omega t) + \nu \dfrac{\partial^2 u(y,t)}{\partial y^2} \\ u\big|_{y=0} = 0 \end{cases}$$

MOM EQN

Find b.c. as $y \to \infty$ from eq^n

$y \to \infty$, $u \to$ f^n of time

$u_{yy} \to 0$

and $\dfrac{\partial u}{\partial t} = \dfrac{G}{\rho} \sin(\omega t) + \cdots$

$$u = -\frac{G}{\rho\omega} \cos(\omega t) - \cdots$$

hence 2nd boundary condition:

$$u\big|_{y\to\infty} \to -\frac{G}{\rho\omega} \cos(\omega t).$$

Reduce to Stokes flow by

$$u = -\frac{G}{\rho\omega}\cos(\omega t) + U(y,t)$$

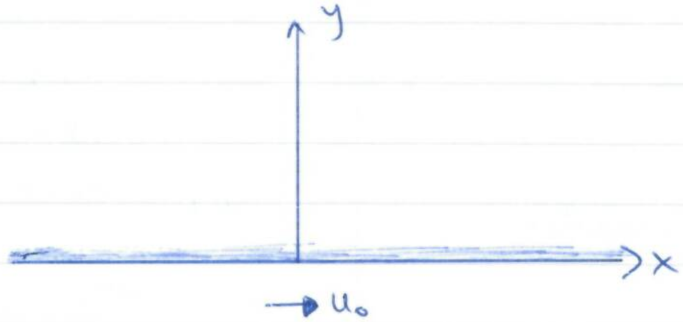
[CHECK and retrace to find out where Dr Timoshin cheated!]

"important" Example: Rayleigh problem for impulsively started plate flow (similarity or self-similar solutions).

$\boxed{6}$

2D flow in $y=0$
Solid plate at $y=0$
Initially fluid and plate are at rest
At $t \geq 0$, plate moves along x-axis with const. speed u_0.

Find the flow in $y \geq 0$.



Assume parallel flow

$\Rightarrow \quad v = 0 \qquad u = u(y,t) \qquad p = p(x,t).$

and momentum eq^{n} reduces to

MOM EQN
$u_t = -\frac{1}{\rho}p_x(x,t) + \nu(u_{yy}+u_{zz})$

$$\Longrightarrow \quad \frac{\partial u(y,t)}{\partial t} = -\frac{1}{\rho}\frac{\partial p(x,t)}{\partial x} + \nu\frac{\partial^2 u(y,t)}{\partial y^2}$$

$\Rightarrow$ from here, $\dfrac{\partial p}{\partial x} = \dfrac{\partial p}{\partial x}(t)$ (no x dep. on LHS) b.c. As $y \to \infty$, assume fluid is at rest at each $t \geq 0$.

Find, $u \to 0$, $\quad 0 = -\frac{1}{\rho}\frac{dp}{dx} + 0$

$$\to \frac{\partial p}{\partial x} = 0.$$

Hence, solve $\frac{\partial u}{\partial t} = \nu \frac{\partial^2 u}{\partial y^2}$ in $y > 0$, $t \geq 0$

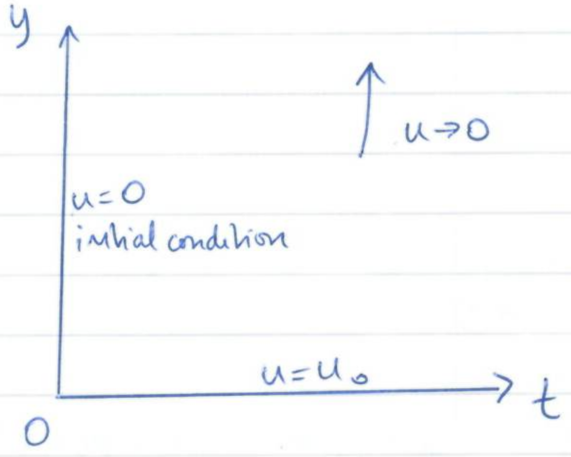
with initial condition at $t=0$, $u=0$ $\forall y \geq 0$.

Boundary conditions: at wall, no slip:

$u = U_0$ at $y=0$, $t>0$

at infinity

$u \to 0$ as $y \to \infty$, $t > 0$.



There is no length scale

There is no time scale

Change (y,t)-variables trying not to change the formulation (affine group properties)

$$\left.\begin{aligned} y &= b\bar{y} \\ t &= a\bar{t} \end{aligned}\right\} \text{some constants } a, b.$$

Equation: $\frac{\partial}{\partial t} = \frac{1}{a}\frac{\partial}{\partial \bar{t}}$, $\frac{\partial^2}{\partial y^2} = \frac{1}{b^2}\frac{\partial^2}{\partial \bar{y}^2}$

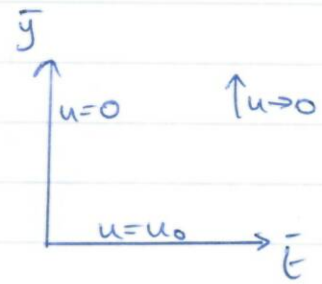
$\frac{\partial u}{\partial t} = \nu \frac{\partial^2 u}{\partial y^2} \Rightarrow \frac{1}{a}\frac{\partial u}{\partial \bar{t}} = \frac{\nu}{b^2}\frac{\partial^2 u}{\partial \bar{y}^2}$

Choose $a = b^2$, eq^n remains unaltered

$$\frac{\partial u}{\partial \bar{t}} = \nu \frac{\partial^2 u}{\partial \bar{y}^2}$$

Initial/boundary conditions:

$\bar{t} = 0 \ \forall \ \bar{y} \geq 0, \quad \bar{u} = 0$
$\bar{y} = 0 \ \forall \ \bar{t} \geq 0, \quad u = u_0$
$\bar{y} \to \infty \ \forall \ \bar{t} \geq 0 \quad u \to 0$



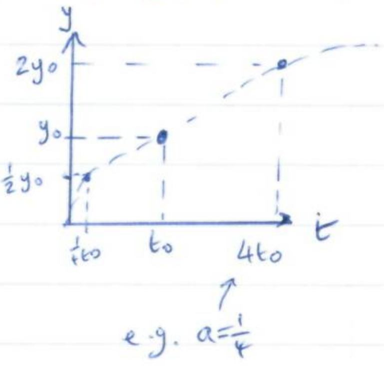
Conclude: if $a = b^2$, formulation in (t, y) is the same as in $(\bar{t}, \bar{y})$ variables.

Assuming uniqueness of solution, claim $u(y,t) = u(\bar{y}, \bar{t})$

$$\Rightarrow \underbrace{u(y,t) = u\left(\frac{y}{\sqrt{a}}, \frac{t}{a}\right)}$$

↑ HOLDS FOR ANY $a > 0$.

Pick a point (t_0, y_0).
If $u(t_0, y_0)$ known, then u is known at $\frac{t_0}{a}, \frac{y_0}{\sqrt{a}}$.



or $u = $ const. on the line $\begin{cases} t = \frac{t_0}{a} \\ y = y_0/\sqrt{a} \end{cases}$ param'd by a.

or $\frac{y}{\sqrt{t}} = \frac{y_0}{\sqrt{t_0}}$

Finally, $\underbrace{u(y,t) = u\left(\frac{y}{\sqrt{t}}\right)}$ ← similarity property

Rewrite formulation in similarity variable

$$\eta = \frac{y}{\sqrt{t}}$$

Eq 1: $\frac{\partial u}{\partial t} = \frac{\partial}{\partial t} u(\eta) = u'(\eta) \frac{\partial \eta}{\partial t}$

$(t,y) \to (t,\eta) \Rightarrow \quad = u'(\eta) \cdot \left[- \frac{y}{2t^{3/2}} \right] = -\frac{\eta}{2t} u'(\eta)$

$$\frac{\partial u}{\partial y} = \frac{\partial u(\eta)}{\partial y} = u'(\eta) \frac{\partial \eta}{\partial y} = \frac{1}{\sqrt{t}} u'(\eta)$$

$$\frac{\partial^2 u}{\partial y^2} = \frac{\partial}{\partial y} \frac{1}{\sqrt{t}} u'(\eta) = \frac{1}{t} u''(\eta)$$

$\frac{\partial u}{\partial t} = \nu \frac{\partial^2 u}{\partial y^2} \Rightarrow$ into eq 1: $\quad -\frac{\eta}{2t} u'(\eta) = \frac{\nu}{t} u''(\eta)$

$$\Rightarrow \underbrace{-\frac{\eta}{2} u'(\eta) = \nu u''(\eta)}$$

get an ODE instead of PDE.

$$-\frac{\eta}{2}u'(\eta) = \nu u''(\eta)$$

Practical method

Observe lack of time and space scales, suspect similarity
$\Rightarrow$ attempt solution of the form

$$u = t^{\alpha} f(\eta) \qquad , \eta = \frac{y}{t^{\beta}}$$

α, β consts to be found.

Sub into eq^{n} and initial/bdry conditions

Eq^{n}: $$\frac{\partial u}{\partial t} = \nu \frac{\partial^2 u}{\partial y^2}$$

- $$\frac{\partial u}{\partial t} = \frac{\partial}{\partial t}\left(t^{\alpha} f(\eta)\right) = \alpha t^{\alpha-1} f(\eta) + t^{\alpha} f'(\eta)\frac{\partial \eta}{\partial t}$$
$$= \alpha t^{\alpha-1} f(\eta) + t^{\alpha} f(\eta)(-\beta)\frac{y}{t^{\beta+1}}$$
$$= \alpha t^{\alpha-1} f(\eta) - \beta t^{\alpha-1}\eta f'(\eta)$$
$$= t^{\alpha-1}\left[\alpha f(\eta) - \beta \eta f'(\eta)\right]$$

(RHS should have t and η, not y !)

- $$\frac{\partial u}{\partial y} = \frac{\partial}{\partial y}\left(t^{\alpha} f(\eta)\right) = t^{\alpha} f'(\eta)\frac{\partial \eta}{\partial y}$$
$$= t^{\alpha} f'(\eta)\frac{1}{t^{\beta}}$$
$$= t^{\alpha-\beta} f'(\eta)$$

- $$\frac{\partial^2 u}{\partial y^2} = \frac{\partial}{\partial y}\left(t^{\alpha-\beta} f'(\eta)\right) = t^{\alpha-2\beta} f''(\eta)$$

$$\Rightarrow t^{\alpha-1}\underbrace{\left[\alpha f - \beta \eta f'\right]}_{\text{f}^{\text{n}}\text{ of }\eta} = \nu t^{\alpha-2\beta}\underbrace{f''}_{\text{f}^{\text{n}}\text{ of }\eta}$$

t cancels out if $\alpha - 1 = \alpha - 2\beta \Rightarrow \beta = \frac{1}{2}$

$$\Downarrow$$

$$\eta = \frac{y}{\sqrt{t}}$$

$$\Rightarrow \alpha f - \beta\eta f' = \nu f''$$

$$\Rightarrow \alpha f - \tfrac{1}{2}\eta f' = \nu f'' \quad \text{.......} (*)$$

<u>Use condition at wall:</u> $u = u_0$ at $y = 0$ for $t > 0$.

In terms of $\eta = \frac{y}{\sqrt{t}}$, $u = u_0$ at $\eta = 0$ $t > 0$.

$$\Rightarrow t^{\alpha} f(0) = u_0 \qquad \text{(from first line)}.$$

How can a f^n of time be equal to a constant?
Only if $\alpha = 0$.

$$\left.\begin{array}{l} \alpha = 0 \\ \Downarrow \\ f(0) = u_0 . \end{array}\right\} \text{two pieces of info from 1 b.c.}$$

$$\Rightarrow -\tfrac{1}{2}\eta f' = \nu f'' \qquad \text{(from } (*)\text{)}$$

<u>Second condition</u>: $u = 0$ at $y > 0$, $t = 0$.

In (t, η), $\eta = \frac{y}{\sqrt{t}} \to \infty$ as $t \to 0$ $\forall y > 0$.

Initial condition becomes $u = f(\eta) \to 0$ as $\eta \to \infty$.

<u>Third condition</u>: $y \to \infty$. $u \to 0$ as $y \to \infty$, $t > 0$.

Since $\eta = \frac{y}{\sqrt{t}}$, $y \to \infty$, $t > 0$ is same as $\eta \to \infty$

$$\Rightarrow u = f(\eta) \to 0 \text{ as } \eta \to \infty .$$

Observe that the second and third conditions are the same in η-variable.

Final self-similar formulation:

$$\begin{cases} -\frac{1}{2}\eta f' = \nu f'' \\ f(0) = u_0 \\ f(\eta) \to 0, \quad \eta \to \infty \end{cases}$$

Changed from PDE to ODE.

Solve:
$$f' = Ce^{-\eta^2/4\nu}$$
$$f = K + C\int_0^\eta e^{-s^2/4\nu}\,ds$$

$$f(0) = u_0 \Rightarrow K = u_0$$

$$f(\infty) = 0 \Rightarrow u_0 + C\int_0^\infty e^{-s^2/4\nu}\,ds = 0$$

$$\Rightarrow u_0 + C\frac{1}{2}\sqrt{\pi 4\nu} = 0$$

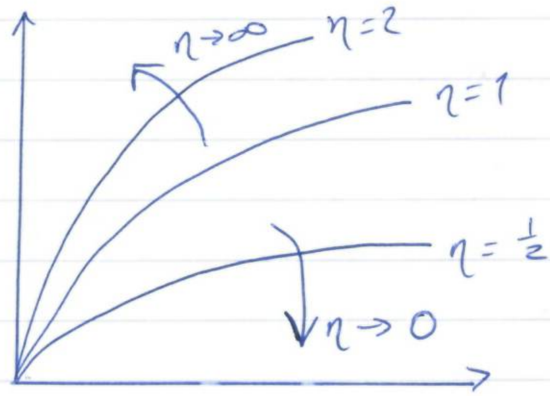
$$\left(\because \int_0^\infty e^{-as^2}\,ds = \sqrt{\frac{\pi}{a}}, \quad a = 1/4\nu\right)$$

$$\Rightarrow C = -\frac{u_0}{\sqrt{4\nu}}$$

$$\Rightarrow u(y,t) = u_0\left[1 - \frac{1}{\sqrt{\pi\nu}}\int_0^\eta e^{-s^2/4\nu}\,ds\right]$$

$\eta = y/\sqrt{t}$, can rewrite ↑ using error f^n.

Discussion: 1) 3 conditions for PDE
turned into 2 conditions for ODE ?!



$\eta = y/\sqrt{t}$

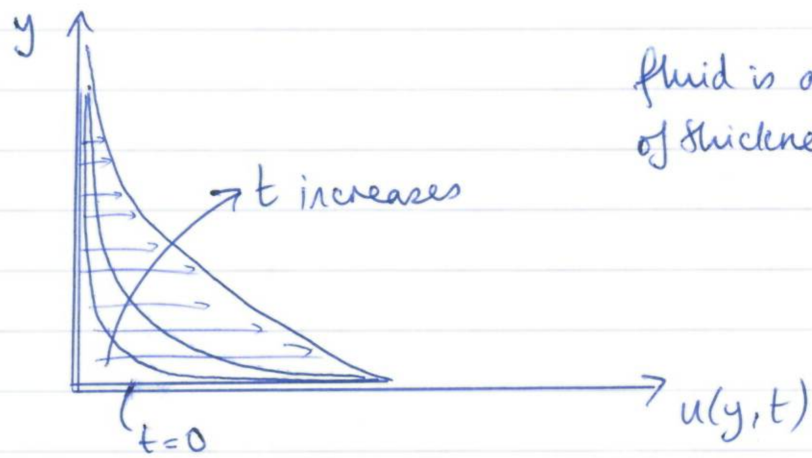
It's because

$\eta \to 0 \equiv y \to 0$

but $\eta \to \infty \equiv y \to \infty$ or $t \to 0$.

2) Velocity profiles.

$u = u(y/\sqrt{t})$



fluid is affected in layer of thickness $y = O(\sqrt{t})$

3) What's the inviscid limit of the sol^{n}? (i.e. $\nu \to 0$)

4) At $t = 0^+$, vorticity = $\begin{cases} 0 & \text{in } y > 0 \\ \infty & \text{at } y = 0 \end{cases}$

At $t > 0$ vorticity diffuses from wall into the bulk

Example: Couette flow started from rest.

Q2 2005 (similar)

Channel, width d.
2D flow
Initially, fluid at rest

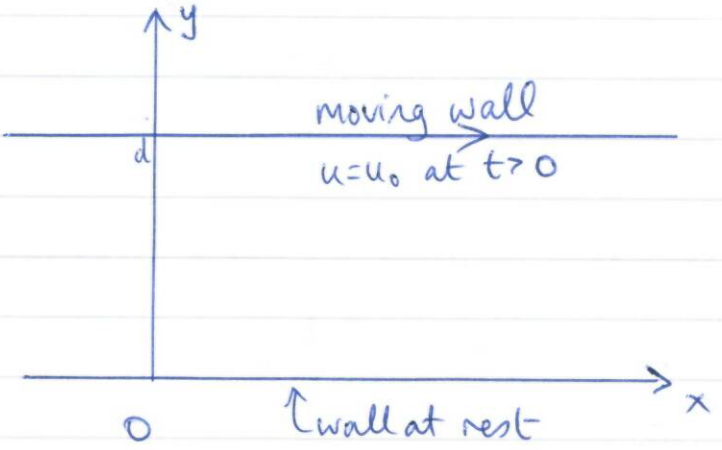
Q2 2010 (similar but completely different)

At $t \geq 0$, upper wall moves with speed u_0.
Bottom wall remains at rest.
Assume parallel flow $\quad v \equiv 0 \quad u = u(y,t)$

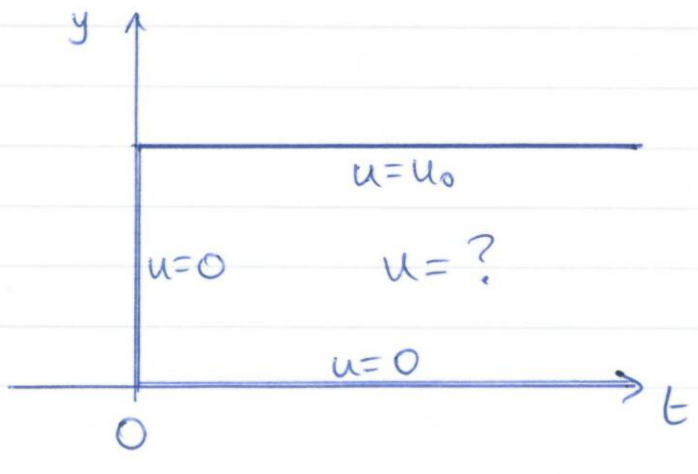
Momentum: $$\frac{\partial u}{\partial t} = -\frac{1}{\rho}\frac{\partial p(x,t)}{\partial x} + \nu \frac{\partial^2 u}{\partial y^2}$$

Take $\frac{\partial p}{\partial x} = 0$

$$\Rightarrow \frac{\partial u}{\partial t} = \nu \frac{\partial^2 u}{\partial y^2}.$$



PHYSICAL PLANE



(t-y) PLANE

Methods of solving: (1) Computer : actual computer £500
software
(unless good hacker) £500.
= £1000. Press Enter.

Cheaper ways: (2) Green's function

(3) Laplace transform in time.

(4) Separable solution.

Use (4). Need to deal first with the nonzero condition at $y=d$.

Substitution: $u(y,t) = \underbrace{\frac{u_0}{d} y}_{}$ $+ f(y,t)$.

Couette flow, or limit of u as $t \to \infty$ and satisfies condition at $y=d$.

Sub into eq^n and initial/boundary condition:

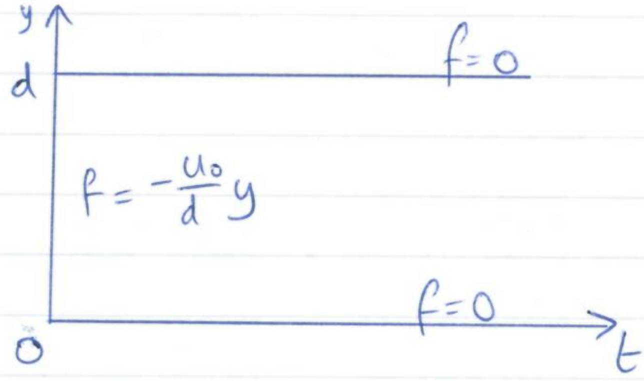
$$\frac{\partial u}{\partial t} = \frac{\partial f}{\partial t}, \quad \frac{\partial^2 u}{\partial y^2} = \frac{\partial^2 f}{\partial y^2}$$

- eq^n: $\dfrac{\partial f}{\partial t} = \nu \dfrac{\partial^2 f}{\partial y^2}$

- b.c.'s: $t>0$ $y=0$, $f=0$
  $t>0$ $y=d$, $f=0$.

• Initial: $t=0, \quad 0 \le y \le d, \quad f = -\frac{u_0}{d} y$

(t,y)-plane for $f(y,t)$:



suitable for separation of variables.

Write $f = \sum_n T_n(t) Y_n(y)$, each pair T_n, Y_n satisfies eq^n.

$$T'Y = \nu T Y''$$

à la Methods 3.

$$\frac{T'}{T} = \nu \frac{Y''}{Y} = \text{const}$$

Since $Y(0)=0 \quad Y(d)=0 \quad \Rightarrow \quad Y_n(y) = \sin\left(\frac{n\pi y}{d}\right)$.

$$\Rightarrow \frac{T'}{T} = -\nu\left(\frac{n\pi}{d}\right)^2 \quad \Rightarrow \quad T = Ce^{-\nu\left(\frac{n\pi}{d}\right)^2 t}$$

n is an integer.

Full sol^n: $f(y,t) = \sum_n C_n \sin\left(\frac{n\pi y}{d}\right) e^{-\nu\left(\frac{n\pi}{d}\right)^2 t}$

Find C_n from initial condition:

$$t=0, \quad f = -\frac{u_0}{d} y$$

$$\sum_{n=1}^{\infty} C_n \sin\left(\frac{n\pi}{d}\right) y = -\frac{u_0}{d} y$$

So, need Fourier coefficients for RHS.

Since $\sin\left(\frac{n\pi y}{d}\right)$ are orthogonal in $[0,d]$,

$$C_n \int_0^d \sin^2\left(\frac{n\pi y}{d}\right) dy = -\frac{u_0}{d}\int_0^d y \sin\left(\frac{n\pi y}{d}\right) dy$$

and $\int_0^d \sin\left(\frac{n\pi y}{d}\right) \sin\left(\frac{m\pi y}{d}\right) dy = 0 \qquad m \neq n.$

$$\sin^2\theta = \frac{1}{2}\left[1-\cos 2\theta\right]$$

$$C_n \frac{1}{2} d - C_n \frac{1}{2} \underbrace{\int_0^d \cos\left[\frac{2\pi n y}{d}\right] dy}_{=0} = -\frac{u_0}{d}\underbrace{\int_0^d y \sin\left(\frac{n\pi y}{d}\right) dy}_{\text{by parts.}}$$

find (check!), $C_n = \frac{2u_0}{n\pi}(-1)^n$

Answer: $$\underbrace{u(y,t) = \frac{u_0}{d} y + \sum_{n=1}^{\infty} \frac{2u_0}{n\pi}(-1)^n \sin\left(\frac{n\pi}{d} y\right) e^{-\nu\left(\frac{n\pi}{d}\right)^2 t}}$$

This answer is good for large t analysis, but the series converges really slowly, so bad for small t.

Flow at large t:

$$u(y,t) = \underbrace{\frac{u_0}{d} y - \frac{2u_0}{\pi}\sin\left(\frac{\pi y}{d}\right) e^{-\nu\left(\frac{\pi}{d}\right)^2 t}}_{n=1} + O\Big(\underbrace{e^{-\nu\left(\frac{\pi}{d}\right)^2 4t}}_{n\geq 2.}\Big)$$

Conclude: flow approaches steady state exponentially, rate of decay as exponential is:

$$t = O\left[\left(\frac{d}{\pi}\right)^2 \frac{1}{\nu} \right]$$

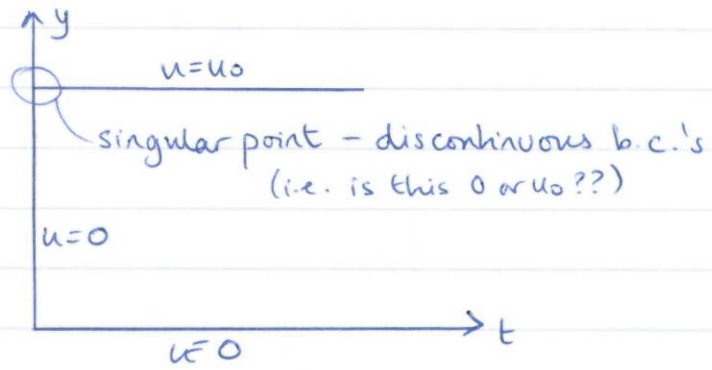
e.g. time scale is small if ν is large.

<u>Exercise:</u> Find approximate solution for small times.

Q2 2005 In the separable solution, convergence is poor in $\frac{\partial u}{\partial t}$, $\frac{\partial u}{\partial y}$ as $y \to d$, $t \to 0$ since, roughly,

$$u \sim \sum_n \frac{(-1)^n}{n} \sin(ny)\, e^{-n^2 t}$$

This comes from



Use perturbations method as $t \to 0$.

Formulation: $$\frac{\partial u}{\partial t} = \nu \frac{\partial^2 u}{\partial y^2}$$

$$u\big|_{\substack{y=0\\ t\geq 0}} = 0$$

$$u\big|_{\substack{y=d\\ t\geq 0}} = u_0$$

$$u\big|_{\substack{t=0\\ 0\leq y\leq d}} = 0.$$

Define an artificial small parameter ε.
Write $t = \varepsilon T$. Assume $\varepsilon \to 0$, $T = O(1)$

Sub into eq^n: $\frac{\partial}{\partial t} = \frac{1}{\varepsilon}\frac{\partial}{\partial T}$

$$\frac{1}{\varepsilon}\frac{\partial u}{\partial T} = \nu \frac{\partial^2 u}{\partial y^2}$$

$$\frac{\partial u}{\partial T} = \varepsilon \nu \frac{\partial^2 u}{\partial y^2}$$

Try a series solution in ε:

$$u = u_0 + \varepsilon u_1 + \varepsilon^2 u_2 + \cdots$$

$$\Rightarrow \frac{\partial u_0}{\partial T} + \varepsilon \frac{\partial u_1}{T} + \cdots = \varepsilon \nu \frac{\partial^2 u_0}{\partial y^2} + \varepsilon^2 \nu \frac{\partial^2 u_1}{\partial y^2} + \cdots$$

Equate powers of ε: $\frac{\partial u_0}{\partial T} = 0$, $u_0 = u_0(y)$.

But from initial condition,

$u\big|_{\substack{t=0 \\ 0 \le y \le d}} = 0$ find $u_0 = 0$.

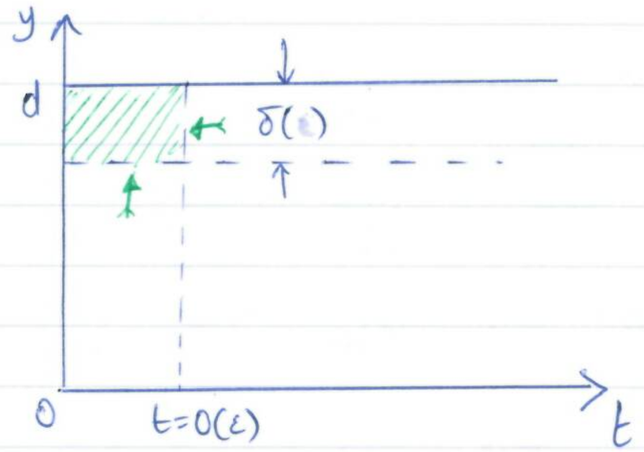
Next term in ε: $\frac{\partial u_1}{\partial T} = \nu \frac{\partial^2 u_0}{\partial y^2} = 0$ by initial condition

$u_1 = u_1(y) \equiv 0$ from initial condition.

Conclude: $u = u(y, T, \varepsilon) = 0 + \varepsilon \cdot 0 + \varepsilon^2 0 + \cdots$

$\Rightarrow$ $u \ll \varepsilon^n \quad \forall n > 0$ (exponentially smaller)

Look near upper wall for non-trivial solutions.



Want to be in region $t=O(\varepsilon)$ and $d-y=O(\delta)$ with $\delta\to 0$ as $\varepsilon\to 0$. Write $y=d-\delta(\varepsilon)Y$.

Eqn:
$$\frac{\partial}{\partial t} = \frac{1}{\varepsilon}\frac{\partial}{\partial T}$$
$$\frac{\partial}{\partial y} = \frac{1}{\delta}\frac{\partial}{\partial Y}, \qquad \frac{\partial^2}{\partial y^2} = \frac{1}{\delta^2}\frac{\partial^2}{\partial Y^2}$$
$$\frac{1}{\varepsilon}\frac{\partial u}{\partial T} = \frac{1}{\delta^2}\nu\frac{\partial^2 u}{\partial Y^2}$$

Terms are in balance if $\varepsilon=\delta^2$, ie. $\delta=\sqrt{\varepsilon}$.

$$\frac{\partial u}{\partial T} = \nu\frac{\partial^2 u}{\partial Y^2} \quad \text{... new eqn for small time } t\text{, close to upper wall.} \qquad \cdots(1)$$

Initial condition was $u\big|_{\substack{t=0\\ 0\le y\le d}} = 0$.

now $u\big|_{\substack{T=0\\ 0\le Y\le d/\sqrt{\varepsilon}}} = 0$

Take limit as $\varepsilon\to 0$:
$$u\big|_{\substack{T=0\\ Y\ge 0}} = 0 \qquad \cdots(2)$$

B.c. at top wall was $u\big|_{\substack{y=d\\t\geq 0}} = u_0$

now $u\big|_{\substack{Y=0\\T\geq 0}} = u_0$ . . . (3)

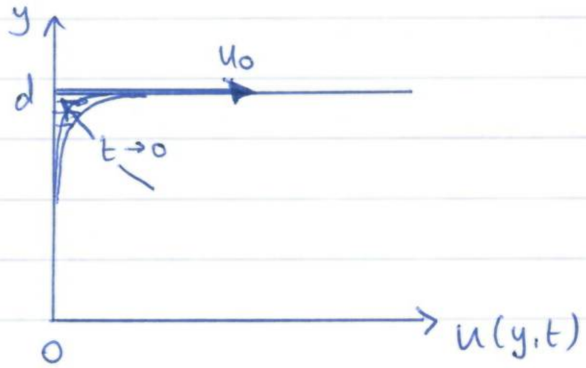
B.c. at bottom wall was $u\big|_{\substack{y=0\\t\geq 0}} = 0$

now $u\big|_{\substack{Y=d/\sqrt{\varepsilon}\\T\geq 0}} = 0$

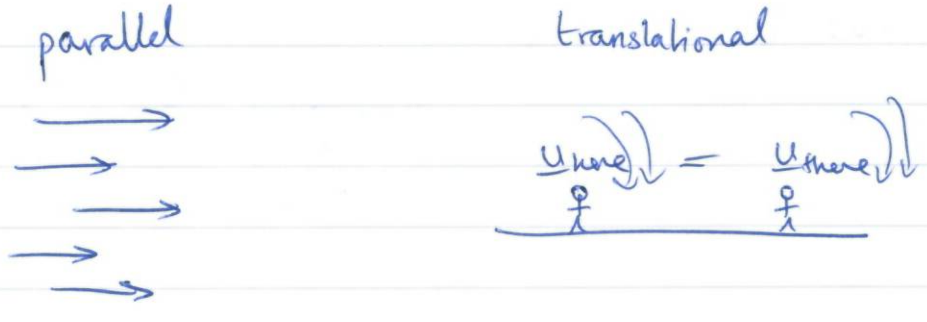
take $\lim \varepsilon \to 0$: $u\big|_{\substack{Y\to\infty\\T\geq 0}} = 0$ . . . (4).

Then (1) – (4) is the Rayleigh flow for impulsive plate in $\frac{1}{2}$-infinite region.

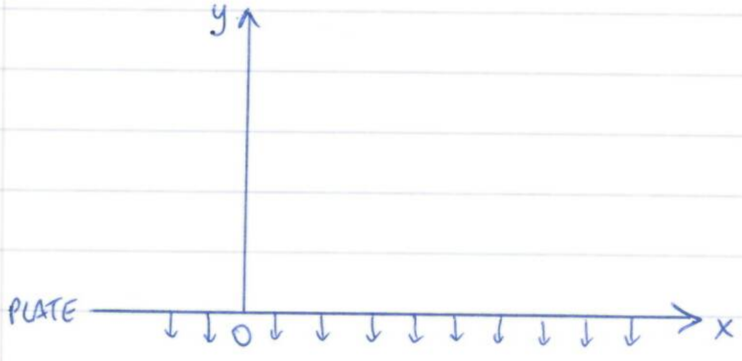
Velocity profiles in channel flow at small times:



Example: Translationary invariant but not a parallel flow



2D Steady flow past an infinite plate with suction



At wall, velocity in y is constant, $-v_w$

$$\Rightarrow \left.\begin{matrix} u=0 \\ v=-v_w \end{matrix}\right\} \text{ at } y=0 .$$

Far from wall, $u \to U_0$ as $y \to \infty$.

Flow is <u>not</u> parallel. But expect $\underline{u}=(u,v)$ to be independent of x [invariance to translations along x-axis].

hence $v = v(y)$, continuity $u_x + v_y = 0$

$$u = u(y) \Rightarrow v_y = 0 \Rightarrow v = \text{const}.$$

From wall condition, $v(y) \equiv -v_w$

y-momentum: $\cancel{u v_x} + \cancel{v v_y} = -\frac{1}{\rho} p_y + \cancel{\nu \nabla^2 v}$

$$p_y = 0, \quad p = p(x).$$

x-momentum: $\underset{0}{\underset{\shortparallel}{u u_x}} + v u_y = -\frac{1}{\rho} p'(x) + \nu \big(\underset{0}{\underset{\shortparallel}{u_{xx}}} + u_{yy} \big)$

and $u = u(y)$ $\qquad v = -v_w$

$$\Rightarrow \quad -v_w u'(y) = -\frac{1}{\rho} p'(x) + \nu u''(y)$$

Conclude $p'(x) = \text{const}.$

b.c.'s: at $y=0$ (no slip), strictly vertical suction
$$u|_{y=0} = 0.$$

(Q: What changes in solution if suction velocity has a horizontal component, e.g. $u|_{y=0} = u_w = \text{const}$?)

at ∞: Want to have non-trivial horizontal flow,
$$u \to u_0 \text{ as } y \to \infty$$

Then, from momentum,
if $u \to u_0$, then
$$u' \to 0$$
$$u'' \to 0$$
$$p'(x) \equiv 0.$$

Hence, solve
$$\begin{cases} -v_w u'(y) = \nu u''(y) \\ u(0) = 0 \\ u(\infty) = u_0 \end{cases}$$

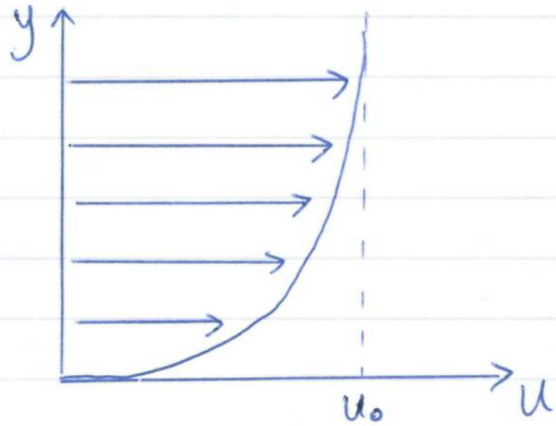
$$u' = C_1 e^{-\frac{v_w}{\nu} y}$$

$$u = C_2 e^{-\frac{v_w}{\nu} y} + C_3$$

$u(0) \Rightarrow C_2 + C_3 = 0.$
$u(\infty) = U_0 \Rightarrow C_3 = u_0, \; C_2 = -u_0.$

Ans:
$$\begin{cases} u(y) = u_0 \left[1 - e^{-\frac{v_w}{\nu} y}\right] \\ v(y) = -v_w \end{cases}$$

Horizontal velocity profiles:

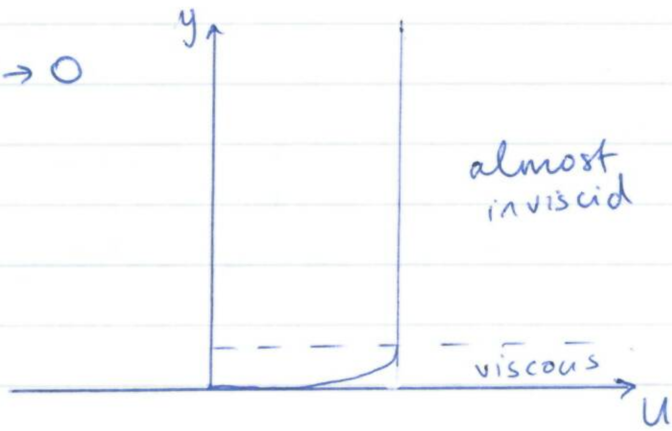


boundary-layer profile
'thickness' of boundary layer is

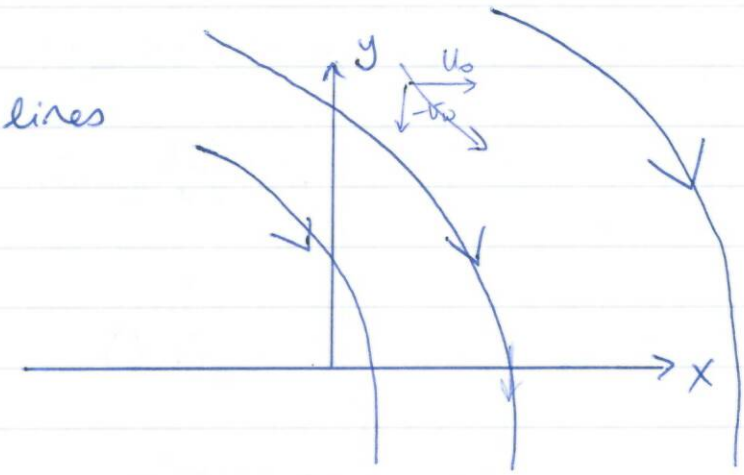
$$y = O\left(\frac{\nu}{v_w}\right)$$

i.e. small if $v_w \to \infty$
large if $v_w \to 0$.

If $\nu \to 0$

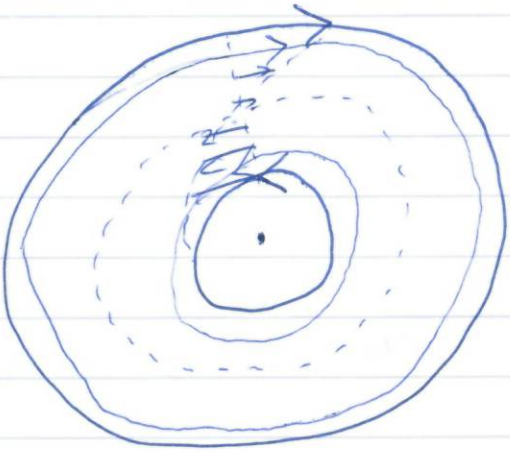


Streamlines

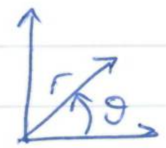


Flows with circular streamlines

(in effect, parallel in cylindrical coordinates)



Use cylindrical polars (r, θ, z).



Assume $u_z \equiv 0$
$\frac{\partial}{\partial z} \equiv 0$

no longer using $\frac{\partial}{\partial x} u = u_x$ notation.

2D flow. Let u_r, u_θ be velocity components.

Circular streamlines $\Rightarrow u_r \equiv 0$.

Continuity (from printout, A.35, 4th eq^n)

CONTINUITY $$\frac{1}{r}\frac{\partial}{\partial r}(\underbrace{r u_r}_{0}) + \frac{1}{r}\frac{\partial u_\theta}{\partial \theta} + \cancel{\frac{\partial u_z}{\partial z}} = 0$$

$$\Rightarrow \frac{\partial u_\theta}{\partial \theta} = 0 \Rightarrow u_\theta = u_\theta(r,t).$$

azimuthal velocity

MOM.

z-momentum: $0 = 0$.

r-momentum: 1st eq^n in (A.35)

Close to $F = \frac{m u_\theta^2}{r}$

$$-\frac{u_\theta^2}{r} = -\frac{1}{\rho}\frac{\partial p}{\partial r}$$

i.e. $\frac{\partial p}{\partial r} = \rho \frac{u_\theta^2(r,t)}{r}$

Integrate: $p = p_0(\theta,t) + \int_{r_0}^{r} \rho \frac{u_\theta^2(s,t)}{s}\, ds$

MOM. $\Big\{$ θ-momentum: 2^{nd} eq^{n} in (A.35)

$$\frac{\partial u_\theta}{\partial t} = -\frac{1}{\rho r}\underbrace{\frac{\partial p}{\partial \theta}}_{= \frac{\partial p_0}{\partial \theta}} + \nu\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u_\theta}{\partial r}\right) - \frac{u_\theta}{r^2}\right]$$

$u_\theta = u_\theta(r,t)$

$$\frac{\partial p_0(\theta,t)}{\partial \theta} = F(r,t)$$

$$p_0(\theta,t) = \theta F(r,t) + C(r,t)$$

$p_0(\theta + 2\pi, t) = p_0(\theta, t)$ — periodicity

$\Rightarrow F = 0 \Rightarrow \frac{\partial p_0}{\partial \theta} = 0$

General eq^{n} for flow with circular streamlines:

$u_z = 0$

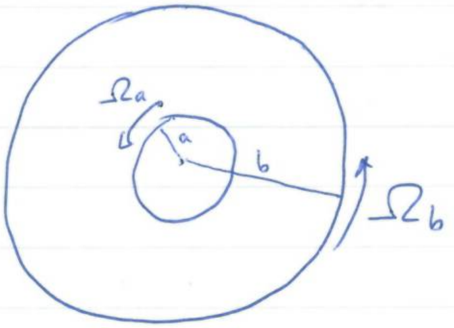
$u_r = 0$

$u_\theta = u_\theta(r,t)$

$$\frac{\partial u_\theta}{\partial t} = \nu\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u_\theta}{\partial r}\right) - \frac{u_\theta}{r^2}\right]$$

Q3 similar 2006 **Example:** Steady flow between coaxial rotating cylinders.

Q2 similar 2009



$\Omega_{a,b}$ angular velocities

$u_\theta = u_\theta(r)$

usually $\frac{\partial u_\theta}{\partial t}$, but $= u_\theta(r)$

from bottom of previous page

Eq^n: $\frac{1}{r}\frac{d}{dr}\left(r\frac{du_\theta}{dr}\right) - \frac{u_\theta}{r^2} = 0.$

no-slip: $u_\theta(a) = a\Omega_a$ $\quad (v = r\omega)$

$u_\theta(b) = b\Omega_b$

$$\frac{d^2u_\theta}{dr^2} + \frac{1}{r}\frac{du_\theta}{dr} - \frac{u_\theta}{r^2} = 0$$

$$\Rightarrow \quad r^2\frac{d^2u_\theta}{dr^2} + r\frac{du_\theta}{dr} - u_\theta = 0$$

Try $u = r^\alpha$

$$\Rightarrow \quad \alpha(\alpha-1) + \alpha - 1 = 0$$

$$(\alpha+1)(\alpha-1) = 0$$

$$\alpha = \pm 1.$$

General sol^n: $u_\theta = C_1 r + \frac{C_2}{r}$

Find C_1, C_2 from b.c.'s.

Answer: $u_\theta = \frac{\Omega_a a^2 - \Omega_b b^2}{a^2-b^2}r + b^2a^2\frac{\Omega_b - \Omega_a}{a^2-b^2}\frac{1}{r}.$

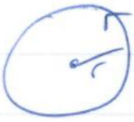
<u>Analysis</u>: Let $\Omega_b = \Omega_a$,
then
$$u_\theta = \Omega_a r \qquad \text{— rigid body rotation.}$$

Trick to kill first term:
Let $b \to \infty$, $\Omega_b \to 0$. $\left(\frac{\text{finite}}{\infty} = 0\right)$

then $u_\theta \sim b^2 a^2 \cdot \dfrac{-\Omega_a}{-b^2} \cdot \dfrac{1}{r} = \underbrace{\dfrac{a^2 \Omega_a}{r}}_{\text{line vortex}}$

Q3 2010?

Circulation $\Gamma = \oint \underline{u} \cdot d\underline{r}$



$$= \int u_\theta \, r \, d\theta$$

$$= \int \frac{a^2 \Omega_a}{r} r \, d\theta$$

$$= a^2 \Omega_a 2\pi$$

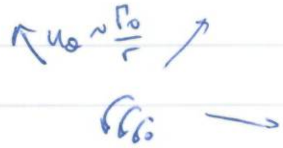
<u>Vortex dissipation</u>

Q3 2010

At time $t = 0$, line vortex is immersed in viscous fluid.

Solve $\dfrac{\partial u_\theta}{\partial t} = \nu \left[\dfrac{1}{r} \dfrac{\partial}{\partial r}\left(r \dfrac{\partial u_\theta}{\partial r} \right) - \dfrac{u_\theta}{r^2} \right]$

with initial conditions $u_\theta \big|_{t=0} = \dfrac{\Gamma_0}{r}$, Γ_0 const.



B.c.'s at $t>0$, $|u_\theta| < \infty$ at $r=0$.

$u_\theta \sim \frac{\Gamma_0}{r}$ as $r \to \infty$.

Consider $\Gamma(r,t) = r u_\theta(r,t)$

b.c.'s and i.c.'s:

$$\left.\begin{aligned} \Gamma\big|_{t=0} &= \Gamma_0 \\ \Gamma\big|_{\substack{r\to\infty \\ t>0}} &\to \Gamma_0 \\ \Gamma\big|_{\substack{r=0 \\ t>0}} &= 0 \end{aligned}\right\}$$

no length or time scales
$\Downarrow$
expect self-similar solutions.

In θ-momentum: write $u_\theta = \frac{\Gamma(r,t)}{r}$

get $$\frac{\partial \Gamma}{\partial t} = \nu\left[\frac{\partial^2 \Gamma}{\partial r^2} - \frac{1}{r}\frac{\partial \Gamma}{\partial r}\right]$$

"I'm writing all sorts of random symbols here and nobody is complaining!"

General form $\Gamma(r,t) = t^\alpha f\left(\frac{r}{t^\beta}\right)$ "self-similar"

From initial condition: $\Gamma_0 = t^\alpha f(\infty)$

$\Rightarrow \alpha = 0, \quad f(\infty) = \Gamma_0$

$\Rightarrow \Gamma(r,t) = f\left(\frac{r}{t^\beta}\right), \qquad \eta = \frac{r}{t^\beta}$

As $r \to \infty$, finite time

$\Gamma_0 = f(\infty)$

as $r \to 0$, $\Gamma \to 0 \quad \Rightarrow f(0) = 0$.

$$\Rightarrow \quad \Gamma = f(\eta).$$

$$\frac{\partial \Gamma}{\partial t} = f'(\eta)\frac{\partial \eta}{\partial t} = f'(\eta)(-\beta)\frac{r}{t^{\beta+1}}$$

$$= -\beta \frac{\eta}{t} f'(\eta)$$

$$\frac{\partial \Gamma}{\partial r} = f'(\eta)\frac{\partial \eta}{\partial r} = \frac{1}{t^{\beta}} f'(\eta)$$

$$\frac{\partial^2 \Gamma}{\partial r^2} = \frac{1}{t^{2\beta}} f''(\eta)$$

Sub into eq^n

$$-\beta\frac{\eta}{t} f' = \frac{\nu}{t^{2\beta}}\left[f'' - \frac{f'}{\eta}\right]$$

$\beta = \frac{1}{2} \qquad \eta = \frac{r}{\sqrt{t}}$ $\leftarrow$ Pick β s.t. t cancels out

$$\Rightarrow -\frac{\eta}{2\nu} f' = f'' - \frac{1}{\eta} f'$$

$$f'' = \left(\frac{1}{\eta} - \frac{\eta}{2\nu}\right) f'$$

integrate

$$\ln f' = \ln \eta - \frac{\eta^2}{4\nu} + \ln C$$

$$f' = C\eta e^{-\eta^2/4\nu}$$

integrate

$$f(\eta) = C_0 + C_1 e^{-\eta^2/4\nu}$$

$f(\infty) = \Gamma_0 \Rightarrow C_0 = \Gamma_0$

$f(0) = 0 \Rightarrow C_1 = -\Gamma_0$

Recall $\eta = \frac{r}{\sqrt{t}}$

Answer:

$$\boxed{\begin{aligned} \Gamma(r,t) &= \Gamma_0\left(1 - e^{-\frac{r^2}{4\nu t}}\right) \\ u_\theta(r,t) &= \frac{\Gamma_0}{r}\left(1 - e^{-\frac{r^2}{4\nu t}}\right) \end{aligned}}$$

Near $r=0$, $u_\theta(r,t) = \frac{\Gamma_0}{r}\left[1 - \left(1 - \frac{r^2}{4\nu t} + O(r^4)\right)\right]$

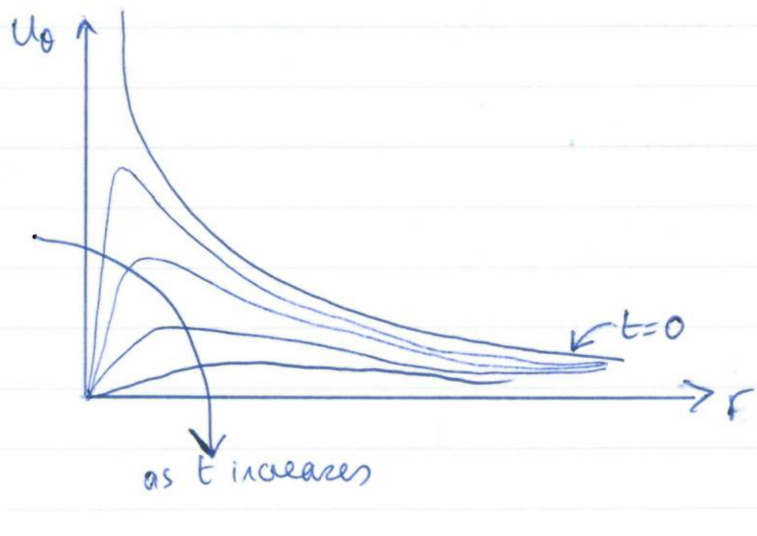
$$= \frac{\Gamma_0}{4\nu t} r + O(r^3)$$

$\Rightarrow$ $u_\theta = O(r)$ as $r \to 0$.

The term $u_\theta \sim \frac{r}{t}$ is rigid body rotation with angular velocity $\approx \frac{1}{t}$ (decreasing with time).

As $r \to \infty$ for finite t,

$u_\theta \approx \frac{\Gamma_0}{r}$ dissipation is not felt at large r and moderate t.

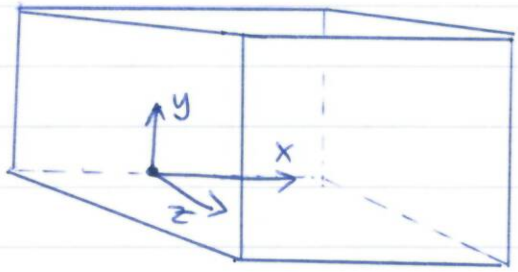


Flows in pipes

Parallel 3D flows

Parallel flow

$\Rightarrow v = w = 0$ along y, z.



$$\frac{\partial p}{\partial y} = \frac{\partial p}{\partial z} = 0.$$

$u = u(z, y, t)$, and

$$\frac{\partial u}{\partial t} = -\frac{1}{\rho}\frac{\partial p(x,t)}{\partial x} + \nu\left(\frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right)$$

$$\frac{\partial p(x,t)}{\partial x} = \begin{cases} 0 \\ \text{or const.} \\ \text{or function of } t \end{cases}$$

<u>Example</u> Steady Poiseuille flow in a round pipe, radius a

$$\frac{\partial u}{\partial t} \equiv 0 \qquad \frac{\partial p}{\partial x} = -G$$

In cylindrical polars

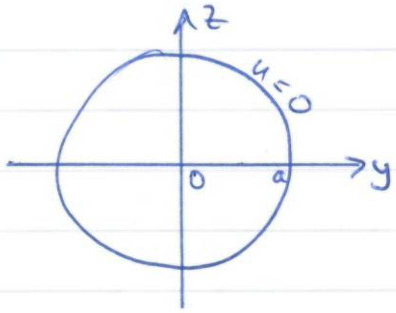
$y = r\cos\theta$

$z = r\sin\theta$

$$\frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}$$

$\Rightarrow$ solving $\quad 0 = \dfrac{G}{\rho} + \nu\left[\dfrac{1}{r}\dfrac{\partial}{\partial r}\left(r\dfrac{\partial u}{\partial r}\right) + \dfrac{1}{r^2}\dfrac{\partial^2 u}{\partial \theta^2}\right]$ in $r \le a$

No slip at wall $u=0$ at $r=a$.



Look for solution with $u = u(r)$.

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{du}{dr}\right) = -\frac{G}{\mu} \quad , \mu = \rho\nu.$$

DO NOT EXPAND THIS!

$$\frac{d}{dr}\left(r\frac{du}{dr}\right) = -\frac{Gr}{\mu}$$

$$r\frac{du}{dr} = -\frac{G}{2\mu}r^2 + C_1$$

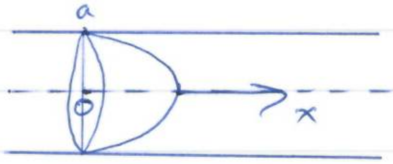
$$\frac{du}{dr} = -\frac{G}{2\mu}r + \frac{C_1}{r}$$

$$u = -\frac{G}{4\mu}r^2 + C_1 \ln r + C_2$$

Use $|u(0)| < \infty \Rightarrow C_1 = 0$
Use $u(a) = 0 \Rightarrow C_2 = Ga^2/4\mu$

$\Rightarrow$ Poiseuille flow in a pipe.

$$u(r) = \frac{G}{4\mu}(a^2 - r^2)$$

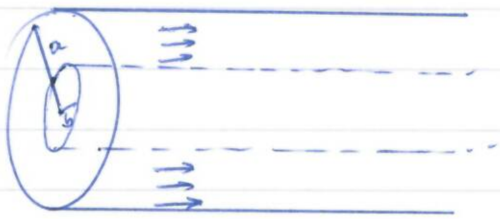


Q1
2005

(also Sheet 2)

Q1
2008

Note: 1)



Flow between two coaxial pipes $b \le r \le a$

$$u(r) = -\frac{G}{4\mu} r^2 + C_1 \ln r + C_2$$

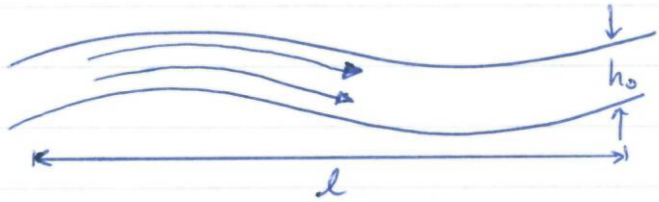
Find $C_{1,2}$ from $u(a) = u(b) = 0$.

2) Elliptic pipe

$$u(y,z) = C_0 + C_1 y^2 + C_2 z^2$$

LUBRICATION THEORY

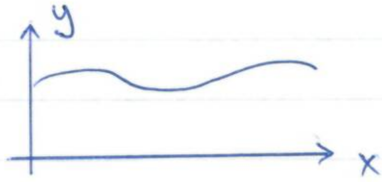
Flow in thin layers with narrow gaps, cracks:



$$\frac{h_0}{\ell} \ll 1$$

1st assumption: let u_0 be typical speed along the channel

Try to simplify the N.S. equations



In 2D: Continuity $\quad \dfrac{\partial u}{\partial x} + \dfrac{\partial v}{\partial y} = 0$

$$\frac{\partial u}{\partial x} \sim \frac{u_0}{\ell}$$

is this an OK approximation?

e.g. $u = u_0 \sin\left(\frac{x}{\ell}\right)$

$$\frac{\partial u}{\partial x} = \frac{u_0}{\ell}\cos\left(\frac{x}{\ell}\right)$$

Keep balance of terms in continuity,

$$\frac{\partial v}{\partial y} \sim \frac{u_0}{\ell}$$

$$\frac{v}{h_0} \sim \frac{u_0}{\ell} \Rightarrow v \sim u_0 \frac{h_0}{\ell}.$$

Momentum in x:

$$\underbrace{\rho\left(u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}\right)}_{\frac{\rho u_0^2}{\ell}} = \underbrace{-\frac{\partial p}{\partial x}}_{\frac{p}{\ell}} + \mu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)$$

$$\frac{u_0}{\ell^2} \ll \frac{u_0}{h_0^2}$$

so ignore this term

$$u\frac{\partial u}{\partial x} \sim u_0 \frac{u_0}{l} = \frac{u_0^2}{l}$$

$$v\frac{\partial u}{\partial y} \sim u_0 \frac{h_0}{l}\frac{u_0}{h_0} = \frac{u_0^2}{l}$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right) \sim \frac{1}{l}\left(\frac{u_0}{l}\right) = \frac{u_0}{l^2}$$

$$\frac{\partial^2 u}{\partial y^2} = \frac{u_0}{h_0^2}$$

Want no convective (non-linear) terms.

$$\frac{\rho u_0^2}{l} \ll \mu \frac{u_0}{h_0^2}$$

$$\frac{\rho u_0 h_0^2}{\mu l} \ll 1 \quad , \quad \frac{h_0}{l}\cdot\frac{\rho u_0 h_0}{\mu} \ll 1$$

Let $\alpha = \frac{h_0}{l}$ – typical angle.

$\frac{\rho u_0 h_0}{\mu} = Re$ – Reynolds number

Q4 2006
Q4 2007
Q4 2008
Q4 2009
Q4 2010

Require $\boxed{\alpha \cdot Re \ll 1}$ ⟵ 2nd assumption

$\boxed{\alpha \ll 1}$ ⟵ 1st assumption

Get $0 = -\frac{\partial p}{\partial x} + \mu \frac{\partial^2 u}{\partial y^2}$

this works for a nearly dead giraffe.

Provided $p \sim l \cdot \frac{\mu u_0}{h_0^2}$

Momentum in y

$$\underbrace{\rho\left(\underbrace{u\frac{\partial v}{\partial x}}_{\frac{\rho u_0^2 h_0}{l^2}} + \underbrace{v\frac{\partial v}{\partial y}}_{\frac{\rho (u_0 h_0)^2}{l^2}\cdot\frac{1}{h_0}}\right)}_{1} = \underbrace{-\frac{\partial p}{\partial y}}_{\underbrace{\frac{l\mu u_0}{h_0^3}}_{2}} + \mu\left(\cancel{\frac{\partial^2 v}{\partial x^2}} + \underbrace{\frac{\partial^2 v}{\partial y^2}}_{\underbrace{\frac{\mu u_0 h_0}{l}\cdot\frac{1}{h_0^2} = \mu\frac{u_0}{h_0 l}}_{3}}\right)$$

$\because$ much smaller than next term

$$\frac{1}{2} = \frac{\rho u_0^2 h_0}{l^2}\cdot\frac{h_0^3}{l\mu u_0} = \frac{\rho u_0 h_0}{\mu}\cdot\frac{h_0^3}{l^3} = Re\cdot\alpha^3$$

$$= (Re\cdot\alpha)\cdot\alpha^2 \ll 1$$

$$\frac{3}{2} = \frac{\cancel{\mu u_0}}{h_0 l}\cdot\frac{h_0^3}{l\cancel{\mu u_0}} = \frac{h_0^2}{l^2} = \alpha^2 \ll 1$$

These ratios mean that 2 is the biggest, most important part of the eq^n. Drop 1 and 3.

$$\frac{\partial p}{\partial y} = 0$$

And we get lubrication equations in 2D:

p4 2005

but not for

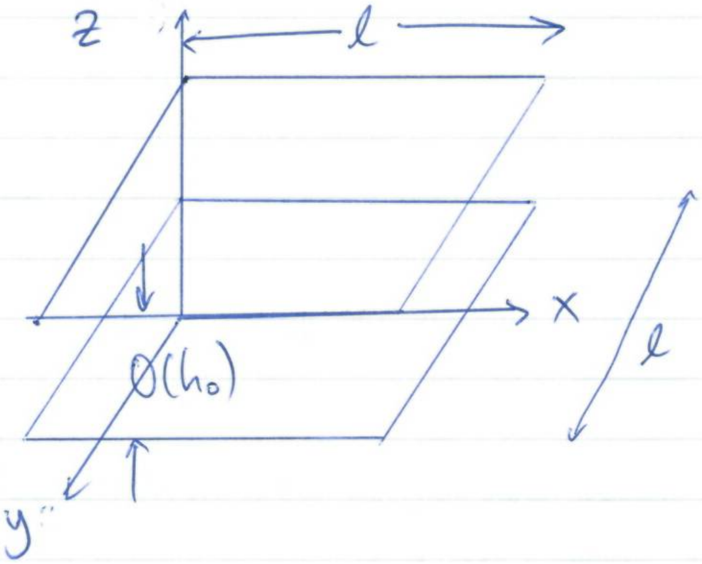
See ape' 2 fluid flows

$$\boxed{\begin{aligned} 0 &= -\frac{\partial p}{\partial x} + \mu \frac{\partial^2 u}{\partial y^2} \\ 0 &= \frac{\partial p}{\partial y} \\ \frac{\partial u}{\partial x} &+ \frac{\partial v}{\partial y} = 0 \end{aligned}}$$

LUBRICATION EQ^{N}S IN 2D

Looks like parallel channel flow, but the channel is not parallel!

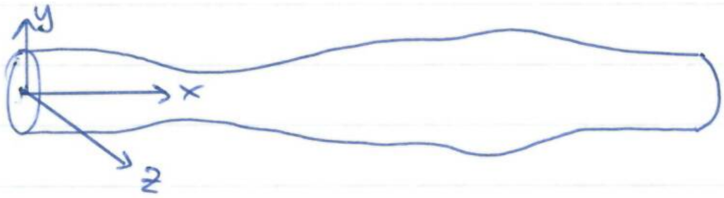
Exercise: write out eq^{n}s of lubrication approximation in a 3D 'narrow cushion' (?!)



$$\boxed{\begin{aligned} 0 &= -\frac{\partial p}{\partial x} + \mu \frac{\partial^2 u}{\partial z^2} \\ 0 &= -\frac{\partial p}{\partial y} + \mu \frac{\partial^2 v}{\partial z^2} \\ 0 &= \frac{\partial p}{\partial z} \\ \frac{\partial u}{\partial x} &+ \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 . \end{aligned}}$$

LUBRICATION EQNS IN 3D

Example: 3D flow in a long, thin pipe



$$\text{Continuity eq}^{\underline{n}}: \quad \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$$

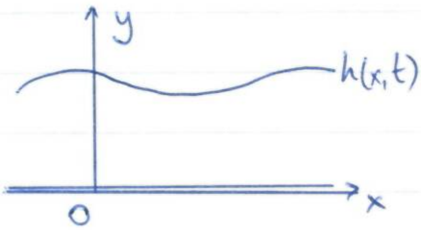
$$\text{Pressure gradients:} \quad \frac{\partial p}{\partial y} = 0$$

$$\frac{\partial p}{\partial z} = 0$$

$$\rho \frac{D\underline{u}}{Dt} = -\nabla p + \mu \nabla^2 \underline{u} : \quad 0 = -\frac{\partial p}{\partial x} + \mu\left(\frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right)$$

~~Q4 2005~~ Example: Reynold's lubrication equation for 2D steady; channel

Q4 2006 similar



$$0 \le y \le h(x,t)$$

$$\text{At } y = h, \quad \left.\begin{aligned} u &= U(x,t) \\ v &= V(x,t) \end{aligned}\right\} \text{ given f}^{\underline{n}}\text{s.}$$

Q4 2009 similar

~~Q4 2010 similar~~

Note: time scale $t \sim \frac{\text{length}}{\text{speed}} = \frac{l}{u_0}$

$$\frac{\partial u}{\partial t} = \frac{u_0}{(l/u_0)} = \frac{u_0^2}{l} \sim u\frac{\partial u}{\partial x} \sim v\frac{\partial u}{\partial y}$$

$$\Rightarrow \frac{\partial u}{\partial t} \sim u\frac{\partial u}{\partial x} \sim v\frac{\partial u}{\partial y} \quad \text{— all neglected}$$

i.e. for time-dependent flow eq$^{\underline{n}}$s of motion are the same.

Solving: $\begin{cases} \dfrac{\partial^2 u}{\partial y^2} = \dfrac{1}{\mu}\dfrac{\partial p}{\partial x} \\ \dfrac{\partial p}{\partial y} = 0 \\ \dfrac{\partial u}{\partial x} + \dfrac{\partial v}{\partial y} = 0 \end{cases}$ $\begin{cases} u = v = 0 \text{ at } y = 0 \\ u = U \\ v = V \text{ at } y = h \end{cases}$

$$\frac{\partial p}{\partial y} = 0 \Rightarrow p = p(x,t)$$

$$\frac{\partial^2 u}{\partial y^2} = \frac{1}{\mu}\frac{\partial p(x,t)}{\partial x}$$

integrate twice: $u = \frac{1}{2\mu}\frac{\partial p(x,t)}{\partial x} y^2 + C_1(x,t) y + C_2(x,t)$

At $y=0$, $u=0$ $\Rightarrow C_2 \equiv 0$
At $y=h$, $u = U(x,t)$. Find C_1.

$$U = \frac{1}{2\mu}\frac{\partial p}{\partial x} h^2 + C_1 h,$$

$$C_1 = -\frac{1}{2\mu}\frac{\partial p}{\partial x} h + \frac{U}{h}$$

Find v:

$$\frac{\partial v}{\partial y} = -\frac{\partial u}{\partial x} = -\frac{\partial}{\partial x}\left[\frac{1}{2\mu}\frac{\partial p}{\partial x} y^2 + \left(-\frac{1}{2\mu}\frac{\partial p}{\partial x} h + \frac{U}{h}\right) y\right]$$

$$= -\frac{1}{2\mu}\frac{\partial^2 p}{\partial x^2} y^2 + \frac{\partial}{\partial x}\left(\frac{1}{2\mu}\frac{\partial p}{\partial x} h - \frac{U}{h}\right) y$$

Integrate in y:

$$v = -\frac{1}{6\mu}\frac{\partial^2 p}{\partial x^2} y^3 + \frac{\partial}{\partial x}\left(\frac{1}{2\mu}\frac{\partial p}{\partial x} h - \frac{U}{h}\right)\frac{y^2}{2} + C_2(x,t)$$

At $y=0$, $v=0 \Rightarrow C_2 \equiv 0$.
At $y=h$, $v=V(x,t)$.

$$V = -\frac{1}{6\mu}\frac{\partial^2 p}{\partial x^2} h^3 + \frac{\partial}{\partial x}\left(\frac{1}{2\mu}\frac{\partial p}{\partial x} h - \frac{U}{h}\right)\frac{h^2}{2}$$

product rule

Simplify

$$= -\frac{1}{6\mu}\frac{\partial^2 p}{\partial x^2} h^3 + \frac{h^2}{2}\frac{1}{2\mu}\left(\frac{\partial^2 p}{\partial x^2} h + \frac{\partial p}{\partial x}\frac{\partial h}{\partial x}\right) - \frac{h^2}{2}\frac{\partial}{\partial x}\left(\frac{U}{h}\right)$$

$$= \left(\frac{1}{4\mu} - \frac{1}{6\mu}\right)\frac{\partial^2 p}{\partial x^2} h^3 + \frac{1}{4\mu}\frac{\partial p}{\partial x} h^2 \frac{\partial h}{\partial x} - \frac{h^2}{2}\frac{\partial}{\partial x}\left(\frac{U}{h}\right)$$

$$= \frac{1}{12\mu}\frac{\partial^2 p}{\partial x^2} h^3 + \frac{1}{12\mu}\frac{\partial p}{\partial x} \cdot 3h^2 \frac{\partial h}{\partial x} - \frac{h^2}{2}\frac{\partial}{\partial x}\left(\frac{U}{h}\right)$$

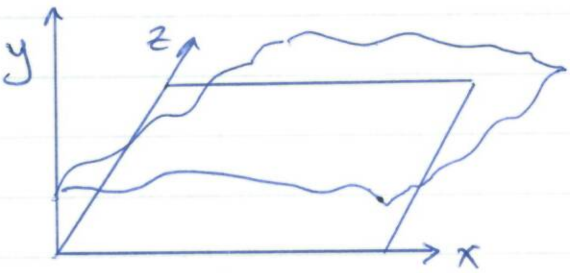
$$= \frac{1}{12\mu}\frac{\partial}{\partial x}\left(h^3 \frac{\partial p}{\partial x}\right) - \frac{h^2}{2}\frac{\partial}{\partial x}\left(\frac{U}{h}\right)$$

$\Rightarrow$ our equation becomes

$$\frac{1}{12\mu}\frac{\partial}{\partial x}\left(h^3\frac{\partial p}{\partial x}\right) = V + \frac{h^2}{2}\frac{\partial}{\partial x}\left(\frac{U}{h}\right)$$

Given h, V and U, need to find p.

For a 3D cushion:



$u=v=w=0$ at $y=0$

$$\left.\begin{aligned} u &= U(x,z,t) \\ v &= V(x,z,t) \\ w &= W(x,z,t) \end{aligned}\right\} y=h.$$

$y = h(x, z, t)$ – given shape with given velocities.

Show

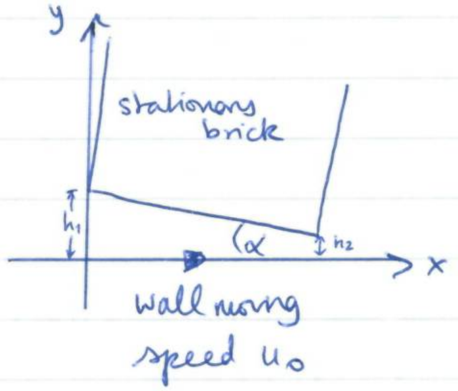
$$\frac{1}{12\mu}\frac{\partial}{\partial x}\left(h^3 \frac{\partial p}{\partial x}\right) + \frac{1}{12\mu}\frac{\partial}{\partial z}\left(h^3\frac{\partial p}{\partial z}\right)$$

$$= V + \frac{h^2}{2}\frac{\partial}{\partial x}\left(\frac{U}{h}\right) + \frac{h^2}{2}\frac{\partial}{\partial z}\left(\frac{W}{h}\right).$$

Left as exercise.

<u>Example:</u> <u>Slider bearing</u>

In the frame of the brick:



Lubrication approximation for flow <u>inside</u> the gap:

$$\alpha \ll 1$$
$$Re \cdot \alpha \ll 1 \qquad \left(\text{recall } Re = \frac{\rho u_0 h_1}{\mu}\right)$$

eq^{n}s are

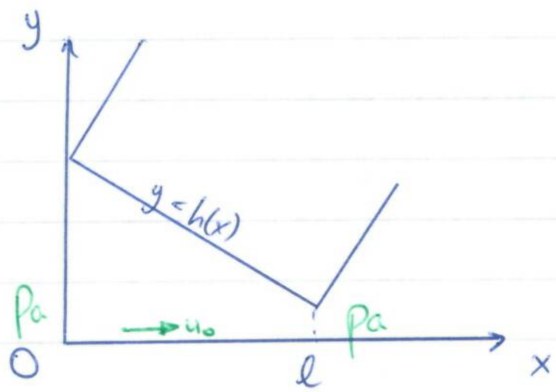
$$\frac{\partial^2 u}{\partial y^2} = \frac{1}{\mu}\frac{\partial p}{\partial x} \qquad (1)$$

$$\frac{\partial p}{\partial y} = 0 \qquad (2)$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0. \qquad (3)$$

standard lubrication eq^{n}s

b.c.s :



$u = u_0,\ v = 0$ at $y = 0$
$u = 0\ ,\ v = 0$ at $y = h(x)$
$\underbrace{\qquad\qquad}$
for $0 \leq x \leq l$

What's going on outside $0 \leq x \leq l$? Not much, so we can assume $p = p_a$ at $x = 0$ and $x = l$.

(2) $\dfrac{\partial p}{\partial y} = 0 \qquad p = p(x)$

(1): $\dfrac{\partial^2 u}{\partial y^2} = \dfrac{1}{\mu} p'(x) \quad \Rightarrow \quad u = \dfrac{1}{2\mu} p' y^2 + C_1 y + C_2 \qquad\qquad p' = p_x$

b.c.: at $y = 0$, $u = u_0 \Rightarrow C_2 = u_0$
at $y = h(x)$, $u = 0$

$$\Rightarrow \quad 0 = \frac{1}{2\mu} p' h^2 + C_1 h + u_0$$

$$\Rightarrow C_1 = -\frac{1}{2\mu} p' h - \frac{u_0}{h}$$

$$\Rightarrow \quad u = \frac{1}{2\mu} p' y^2 + y\left(-\frac{1}{2\mu} p' h - \frac{u_0}{h}\right) + u_0$$

$$= \underbrace{\frac{1}{2\mu}\left(y^2 - yh\right) p'}_{\text{this resembles Poiseville flow}} + \underbrace{u_0\left(1 - \frac{y}{h}\right)}_{\text{this resembles Couette flow}}$$

Now find v from continuity

(3) $$\frac{\partial v}{\partial y} = -\frac{\partial u}{\partial x} = -\frac{d}{dx}\left[\frac{1}{2\mu}(y^2-yh)p' + u_0\left(1-\frac{y}{h}\right)\right]$$

$$= -\frac{1}{2\mu}y^2p'' + \frac{1}{2\mu}y(h'p'+hp'') - u_0 y\frac{h'}{h_2}$$

Integrate with $v=0$ at $y=0$

$$\Rightarrow \quad v = -\frac{1}{6\mu}y^3p'' + \frac{1}{4\mu}y^2(h'p'+hp'') - \frac{u_0}{2}y^2\frac{h'}{h_2}$$

b.c.: Use $v=0$ at $y=h(x)$

$$0 = -\frac{h^3}{6\mu}p'' + \frac{h^2}{4\mu}(h'p'+hp'') - \frac{u_0}{2}h'$$

$$= \frac{1}{12\mu}h^3p'' + \frac{1}{4\mu}\underbrace{h^2h'p'}_{\frac{1}{3}(h^3)'} - \frac{u_0}{2}h'$$

$$\Rightarrow \quad 0 = \frac{1}{12\mu}(h^3p')' - \frac{u_0}{2}h' \qquad \leftarrow \text{eq}^{\text{n}} \text{ for the pressure.}$$

Now solve for $p(x)$
using $p(0) = p(\ell) = p_a = \text{const}$.

$$\frac{1}{12\mu}h^3\frac{dp}{dx} - \frac{u_0}{2}h = D_1 \qquad \leftarrow \text{(integrating once)}$$

$$\frac{1}{12\mu}\frac{dp}{dx} = \frac{D_1 + \frac{u_0}{2}h}{h^3}$$

If $h(x)$ is a linear function, change

$$\frac{d}{dx} = h'(x)\frac{d}{dh} = -\tan\alpha \frac{d}{dh} \approx -\alpha\frac{d}{dh} \qquad (\because \alpha \text{ small})$$

Gives us $\quad -\dfrac{\alpha}{12\mu}\dfrac{dp}{dh} = \dfrac{D_1}{h^3} + \dfrac{u_0}{2h^2}$

$$\Rightarrow -\frac{\alpha}{12\mu}p = -\frac{D_1}{2h^2} - \frac{u_0}{2h} + D_2 \qquad (\text{integrating once})$$

$$\Rightarrow p = \frac{12\mu}{\alpha}\left[\frac{D_1}{2h^2} + \frac{u_0}{2h} - D_2\right]$$

Two unknowns, two conditions.

Use $p = p_a$ at $h = h_1$, $h = h_2$.

Find D_1, D_2

to get, finally, $\quad p = p_a + \dfrac{6\mu}{\alpha}u_0 \dfrac{\overbrace{(h_1 - h(x))(h(x) - h_2)}^{>0}}{h^2(x)\,(h_1 + h_2)}$

Total normal force on the block

$$= \int_0^l (p - p_a)\,dx = \frac{6\mu u_0}{\alpha^2}\left[\ln\left(\frac{h_1}{h_2}\right) - \frac{2(h_1 - h_2)}{h_1 + h_2}\right]$$

Total tangential force

$$= \int_0^l \left(-\mu\frac{\partial u}{\partial y}\right)_{y=h} dx = \frac{2\mu u_0}{\alpha}\left[\frac{3(h_1 - h_2)}{h_1 + h_2} - \ln\left(\frac{h_1}{h_2}\right)\right]$$

$$\text{Friction coefficient} = \frac{\text{Tangential force}}{\text{Normal force}} := C_f$$

$$= \alpha F\left(\frac{h_1}{h_2}\right)$$

Important: $C_f = O(\alpha)$ for given geometry

For solid-on-solid friction, $C_f = O(1)$

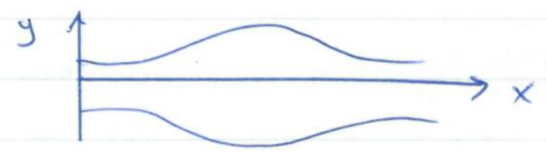
But C_f does not depend on u_0 nor μ !!

So why do people buy expensive engine oils?
Because if u_0 - large
temp - large
μ - small
$Re - \frac{\rho u_0 h}{\mu}$ - rises.

Exercise 'Unsteady lubricating flow'

Q4 2010

Flow between two walls at $y = \pm h(x,t)$



boundaries change in time.

eq^{n}s:

$$\left.\begin{aligned} \frac{\partial^2 u}{\partial y^2} &= \frac{1}{\mu}\frac{\partial p}{\partial x} \\ \frac{\partial p}{\partial y} &= 0 \\ \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} &= 0 \end{aligned}\right\} \text{ no time dependence!}$$

Boundary conditions: no-slip:

$$\left.\begin{aligned} u &= 0 \\ v &= \pm \frac{\partial h}{\partial t} \end{aligned}\right\} \text{ at } y = \pm h(x,t)$$

these apply to a 'rubber'/flexible wall

Q: what is the boundary (in 2D) given by $y=0$?
A: It's the x-axis, either stationary or sliding.
Need more conditions

Solve eq^{n}s:

$$u = \frac{1}{2\mu}\frac{\partial p}{\partial x} y^2 + C_1(x,t) + \underbrace{C_2(x,t)}_{\text{0 by symmetry}} y$$

$$\frac{\partial p}{\partial y} = 0 \Rightarrow p = p(x,t)$$

$$\frac{\partial v}{\partial y} = -\frac{\partial}{\partial x}\left[\frac{1}{2\mu}\frac{\partial p}{\partial x} y^2 + C_1\right]$$

$$= -\frac{1}{2\mu}\frac{\partial^2 p}{\partial x^2} y^2 - \frac{\partial C_1}{\partial x}$$

$$\Rightarrow v = -\frac{1}{6\mu}\frac{\partial^2 p}{\partial x^2} y^3 - \frac{\partial C_1}{\partial x} y + \overbrace{C_3(x,t)}^{\text{0 by symmetry}}$$

$$u|_{y=\pm h} = 0 \rightarrow C_1 = -\frac{1}{2\mu}\frac{\partial p}{\partial x} h^2$$

$$v|_{y=h} = v|_{y=-h} = \frac{\partial h}{\partial t}$$

$$\Rightarrow \frac{\partial h}{\partial t} = -\frac{1}{6\mu}\frac{\partial^2 p}{\partial x^2} h^3 + \left(\frac{1}{2\mu}\frac{\partial^2 p}{\partial x^2} h^2 + \frac{1}{\mu}\frac{\partial p}{\partial x} h \frac{\partial h}{\partial x}\right) h$$

$$\frac{\partial h}{\partial t} = -\frac{1}{6\mu}\frac{\partial^2 p}{\partial x^2}h^3 + \left(\frac{1}{2\mu}\frac{\partial^2 p}{\partial x^2}h^2 + \frac{1}{\mu}\frac{\partial p}{\partial x}h\frac{\partial h}{\partial x}\right)h$$

$$= \frac{1}{3\mu}\frac{\partial^2 p}{\partial x^2}h^3 + \frac{1}{\mu}\frac{\partial p}{\partial x}h^2\frac{\partial h}{\partial x}$$

$$\frac{\partial h}{\partial t} = \frac{1}{3\mu}\frac{\partial}{\partial x}\left(h^3\frac{\partial p}{\partial x}\right)$$ ... eq^n for $p(x,t)$ given $h(x,t)$: t-parameter.

<u>Example from past exam paper</u>

$h = a + b\cos(\omega t)\cos(\alpha x)$

$|x| \leq \pi/2\alpha \qquad (a,b) > 0, \quad a > b.$

Also $p = p_0 = \text{const.}$ at $x = \pm\pi/2\alpha$.

Find $p = p(x,t)$

Use $\frac{\partial h}{\partial t} = \frac{1}{3\mu}\frac{\partial}{\partial x}\left(h^3\frac{\partial p}{\partial x}\right)$

$\frac{\partial h}{\partial t} = -\omega b\sin(\omega t)\cos(\alpha x)$

$\Rightarrow -\omega b\sin(\omega t)\cos(\alpha x)3\mu = \frac{\partial}{\partial x}\left(h^3\frac{\partial p}{\partial x}\right)$

$\rightarrow -\frac{\omega}{\alpha}b\sin(\omega t)\sin(\alpha x)3\mu + C_1(t) = h^3\frac{\partial p}{\partial x}$

Let $\frac{3\mu b\omega}{\alpha} = A$.

$$\Rightarrow \quad \frac{\partial p}{\partial x} = -\frac{A\sin(\omega t)\sin(\alpha x)}{[a+b\cos(\omega t)\cos(\alpha x)]^3} + \frac{C_1(t)}{[a+b\cos(\omega t)\cos(\alpha x)]^3}$$

$C_1(t) = 0$ by symmetry, but can argue it more rigorously:

$$p = -A\sin(\omega t)\int \frac{\sin(\alpha x)\,dx}{[a+b\cos(\omega t)\cos(\alpha x)]^3} + C_1(t)\int \frac{dx}{[a+b\cos(\omega t)\cos(\alpha x)]^3}$$

$$= -\frac{A\sin(\omega t)}{-\alpha b\cos\omega t}\left(-\frac{1}{2}\right)\frac{1}{[a+b\cos(\omega t)\cos(\alpha x)]^2} + C_1(t)\int_{-\frac{\pi}{2\alpha}}^{x}\frac{ds}{[a+b\cos(\omega t)\cos(\alpha s)]^3} + C_2(t)$$

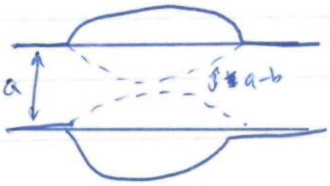
- Say $p(x=-\frac{\pi}{2\alpha}) = p_0$.

$$\Rightarrow \quad C_2(t) = \frac{A\sin(\omega t)}{\alpha b\cos(\omega t)}\cdot\frac{1}{2}\frac{1}{a^2} + p_0$$

- Next, $p(x=\frac{\pi}{2\alpha}) = p_0$.

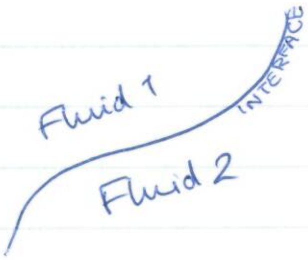
$$\Rightarrow \quad p_0 = \frac{-A\sin(\omega t)}{\alpha b\cos(\omega t)}\frac{1}{2a^2} + C_1(t)\overbrace{\int_{-\frac{\pi}{2\alpha}}^{\pi/2\alpha}\left(\text{positive def } f^n\right)ds}^{>0} + C_2(t).$$

$$\Rightarrow \quad 0 = C_1(t)$$



Q: does lubrication approx. remain valid for $a-b$ small?

2-FLUID FLOWS



For N-S eq^{n}s, interfacial conditions are:

(1) Continuity in velocity <u>vector</u>
(2) Continuity of stresses
(3) Kinematic condition

<u>Recall: (3) Kinematic condition:</u>

Let $F(x, y, z, t) = 0$ be the eq^n of a fluid boundary.
Statement: if a fluid particle is on the boundary at time t, it remains on the boundary at time $t+dt$.
Change to Lagrange, $\underline{r} = \underline{r}(\underline{r}_0, t)$.

At time $t+dt$, $F\big(x(t+dt, \underline{r}_0), y(\ ), z(\ ), t+dt\big) = 0$.

$$\frac{\partial x}{\partial t} = u, \quad \frac{\partial y}{\partial t} = v \quad \frac{\partial z}{\partial t} = w$$

$$\boxed{\begin{gathered} u\frac{\partial F}{\partial x} + v\frac{\partial F}{\partial y} + w\frac{\partial F}{\partial z} + \frac{\partial F}{\partial t} = 0 \\ \text{or} \quad \underline{u}\cdot\nabla F + \frac{\partial F}{\partial t} = 0 \quad \text{i.e. } \frac{DF}{Dt} = 0 \end{gathered}}$$

F = eq^n of bdy

In 2D, if $y = h(x,t)$,
write $F(x, y, t) = y - h(x,t)$

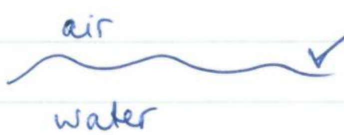
$$-u\frac{\partial h}{\partial x} + v \cdot 1 + \left(-\frac{\partial h}{\partial t}\right) = 0$$

or $$\boxed{v = \frac{\partial h}{\partial t} + u\frac{\partial h}{\partial x}}$$

Variants

(i) 2-fluids with surface tension:
continuity in velocity
kinematic condition
continuity in tangential stress.
jump in normal stress ~ curvature of surface

(ii) Free-surface approximation (air/water)

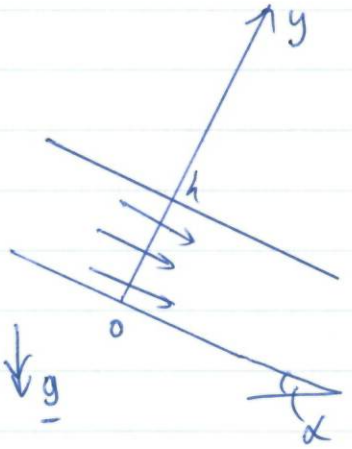


kinematic condition
+ given normal & tangential stress.

Shape of interface and velocity at interface found from solution

Exercise (exact N-S sol$^{\underline{n}}$)

Liquid layer on a sloping wall in free-surface approximation



Try parallel flow:

$$\begin{cases} \bullet\ \rho\left[\dfrac{\partial \underline{u}}{\partial t} + (\underline{u}\cdot\nabla)\underline{u}\right] = -\nabla p + \rho \underline{g} + \mu \nabla^2 \underline{u} \\ \bullet\ \operatorname{div} \underline{u} = 0. \end{cases}$$

Steady $\Rightarrow \dfrac{\partial}{\partial t} = 0$

Parallel $\Rightarrow v = 0, \quad u = u(y)$

momentum in y: $\quad 0 = -\frac{\partial p}{\partial y} - \rho g \cos\alpha + \mu \nabla^2 \underline{v} \overset{0}{\parallel}$

$$\Rightarrow \quad \underbrace{\frac{\partial p}{\partial y} = -\rho g \cos\alpha}$$

momentum in x: $\quad \underline{0 = -\frac{\partial p}{\partial x} + \rho g \sin\alpha + \mu \frac{d^2 u}{dy^2}}$

b.c. at $\underline{y=0}$, no slip, $\underline{u=v=0}$.

At $\underline{y=h}$, free surface conditions: $\underline{p = p_a = \text{const}}$ – normal stress
tangential stress $= 0$.

$\quad \hookrightarrow \underline{\mu \frac{du}{dy} = 0} \qquad$ (check!)

Let's look at pressure:

$$\frac{\partial p}{\partial y} = -\rho g \cos\alpha$$

$$p = -\rho g \cos\alpha (y-h) + p_a$$

but $\frac{\partial p}{\partial x} \equiv 0$ for $h = \text{const}$, gives

$$\begin{cases} \mu \frac{d^2 u}{dy^2} = -\rho g \sin\alpha \\ u(0) = 0, \quad \frac{du}{dy}(h) = 0 \end{cases}$$

$$u = -\frac{\rho g \sin\alpha}{2\mu} y^2 + C_1 y + C_2$$

$u(0) = 0 \Rightarrow C_2 = 0$

$\frac{du}{dy}(h) = 0 \Rightarrow C_1 = \frac{\rho g \sin\alpha h}{\mu}$

$$\Rightarrow \quad \underbrace{u = \frac{\rho g \sin\alpha}{\mu}\left(yh - \frac{y^2}{2}\right)}$$

Note: $u(h)$ – speed on free surface – is not known in advance.

For practice, can do same problem, but

$$\mu \frac{du}{dy}(h) = \tau_0 \quad \text{(given const)}.$$

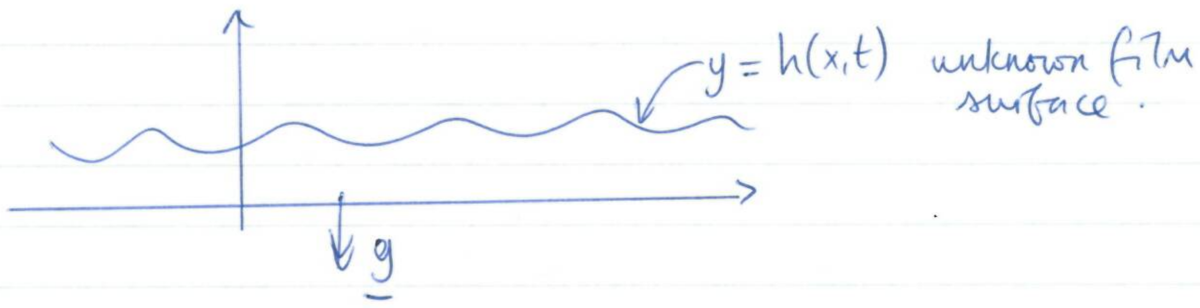
Sketch $u(y)$ for $\tau_0 < 0$.

BACK TO LUBRICATION THEORY

Q4 2005 **Exercise:** Liquid film on a horizontal wall with gravity and given pressure and tangential stress on free surface.

Q4 2007 similar

Q4 2008



Lubrication eq^{n}s:

$$\frac{\partial^2 u}{\partial y^2} = \frac{1}{\mu}\frac{\partial p}{\partial x} \qquad \text{horizontal} \qquad \cdots(1)$$

$$0 = -\frac{\partial p}{\partial y} \underbrace{- \rho g}_{\text{new}} + 0 \qquad \text{vertical momentum} \qquad \cdots(2)$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \qquad \text{continuity} \qquad \cdots(3)$$

b.c.s: At wall, no-slip

i.e. $y = 0 \Rightarrow u = v = 0.$ $\cdots(4)$

On free surface,

kinematic condition: $v|_{y=h} = \dfrac{\partial h}{\partial t} + u|_{y=h}\dfrac{\partial h}{\partial x}$ $\cdots(5)$

given stresses:

$$\left.\begin{aligned} p|_{y=h} &= p_a(x,t) && \text{normal} \\ \mu \frac{\partial u}{\partial y}\Big|_{y=h} &= \tau(x,t) && \text{tangential} \end{aligned}\right\} (6)$$

p_a, τ – known ☺

u, v, p, h – unknown ☹

Solve. (2) $\Rightarrow \dfrac{\partial p}{\partial y} = -\rho g$

$$\Rightarrow p = -\rho g y + C_1(x,t)$$

b.c. $p|_{y=h} = p_a(x,t) \Rightarrow$

$$\Rightarrow \underline{p = -\rho g (y-h) + p_a}$$

$$(1) \Rightarrow \frac{\partial^2 u}{\partial y^2} = \frac{1}{\mu}\frac{\partial p}{\partial x}$$

$$\Rightarrow \frac{\partial^2 u}{\partial y^2} = \frac{1}{\mu}\frac{\partial}{\partial x}\left(\rho g h + p_a\right)$$

$$\Rightarrow u = \frac{1}{2\mu}\frac{\partial}{\partial x}\left(\rho g h + p_a\right)y^2 + C_2 y + C_3$$

$$(4) \Rightarrow C_3 = 0$$

In (6), $\mu \dfrac{\partial u}{\partial y}\Big|_{y=h} = \tau$

$$\Rightarrow \mu \frac{1}{2\mu}\frac{\partial}{\partial x}\left(\rho g h + p_a\right) 2h + \mu C_2 = \tau$$

$$\Rightarrow C_2 = -\frac{h}{\mu}\frac{\partial}{\partial x}\left(\rho g h + p_a\right) + \frac{\tau}{\mu}$$

$$\Rightarrow \underline{u = \frac{1}{2\mu}\underbrace{\frac{\partial}{\partial x}\left(\rho g h + p_a\right)}_{\text{call this } G} y^2 + \frac{1}{\mu}\left[\tau - h\frac{\partial}{\partial x}\left(\rho g h + p_a\right)\right] y}$$

Use (3), ie. $\dfrac{\partial v}{\partial y} = -\dfrac{\partial u}{\partial x}$, find v and apply $v|_{y=0} = 0$.

and apply kinematic condition (5).

continues on sheet

Having called $\frac{\partial}{\partial x}(\rho g h + p_a) = G$,

$$u = \frac{1}{2\mu} G y^2 + \frac{1}{\mu}[\tau - hG] y$$

Now, $\frac{\partial v}{\partial y} = -\frac{\partial u}{\partial x} = -\frac{1}{2\mu}\frac{\partial G}{\partial x} y^2 + \frac{1}{\mu}\left[\frac{\partial (hG)}{\partial x} - \frac{\partial \tau}{\partial x}\right] y$

Integrate to find v, use $v|_{y=0} = 0$,

$$\Rightarrow \quad v = -\frac{1}{6\mu}\frac{\partial G}{\partial x} y^3 + \frac{1}{\mu}\left[\frac{\partial (hG)}{\partial x} - \frac{\partial \tau}{\partial x}\right]\frac{y^2}{2}$$

Use 5 $\Rightarrow$

$$-\frac{1}{6\mu}\frac{\partial G}{\partial x} h^3 + \frac{1}{\mu}\left[\frac{\partial (hG)}{\partial x} - \frac{\partial \tau}{\partial x}\right]\frac{h^2}{2}$$

$$= \frac{\partial h}{\partial t} + \left[\frac{1}{2\mu} G h^2 + \frac{1}{\mu}(\tau - hG) h\right]\frac{\partial h}{\partial x}$$

$$-\frac{1}{6\mu}\frac{\partial G}{\partial x} h^3 + \frac{\partial G}{\partial x}\frac{h^3}{2\mu} + G\frac{\partial h}{\partial x}\frac{h^2}{2\mu} - \frac{\partial \tau}{\partial x}\frac{h^2}{2\mu}$$

$$= \frac{\partial h}{\partial t} + \frac{1}{2\mu} G h^2 \frac{\partial h}{\partial x} + \frac{\tau h}{\mu}\frac{\partial h}{\partial x} - \frac{G h^2}{\mu}\frac{\partial h}{\partial x}$$

$$\frac{\partial h}{\partial t} + \frac{1}{2\mu}\frac{\partial}{\partial x}\left(\tau h^2\right) = \frac{\rho g}{3\mu}\frac{\partial}{\partial x}\left(\underbrace{h^3 \frac{\partial h}{\partial x}}\right) + \frac{1}{3\mu}\frac{\partial}{\partial x}\left(\frac{\partial p_a}{\partial x} h^3\right)$$

VERY NONLINEAR!!

this is an eq^n for $h(x,t)$ for given $\tau(x,t)$ and $p_a(x,t)$.

It's 1st order in time
2nd order in X.

This can't be solved by hand, but we can look at some cases:

<u>Simple solution:</u> Let $\tau = \text{const}$ and $p_a = \text{const}$.
Then $h = h_0 = \text{const}$. possible sol^n?

<u>Exercise:</u> <u>Stability of uniform film.</u>

Let $\tau = \tau_0 = \text{const}$, $p = p_a = \text{const}$.
Assume small disturbance to $h = h_0 = \text{const}$.

$$h(x,t) = h_0 + \varepsilon h_1(x,t) + O(\varepsilon^2)$$

Sub into eq^n, at top of page ignore terms of $O(\varepsilon^2)$.

$$\Rightarrow \frac{\partial}{\partial t}\left[h_0 + \varepsilon h_1\right] + \frac{1}{2\mu}\frac{\partial}{\partial x}\left[\tau h_0^2 + 2\varepsilon h_0 h_1\right]$$

$$= \frac{\rho g}{3\mu}\frac{\partial}{\partial x}\left[\left(h_0^3 + \varepsilon 3 h_0^2 h_1\right)\left(\varepsilon \frac{\partial h_1}{\partial x}\right)\right] + 0$$

(h_0=const)

$$\Rightarrow \varepsilon \frac{\partial h_1}{\partial t} + \varepsilon \underbrace{\frac{2\tau_0 h_0}{2\mu}}_{A} \frac{\partial h_1}{\partial x} = \varepsilon \underbrace{\frac{\rho g h_0^3}{3\mu}}_{B} \frac{\partial^2 h_1}{\partial x^2}$$

Get eq^n for perturbation in film shape

$$\frac{\partial h_1}{\partial t} + A\frac{\partial h_1}{\partial x} = B\frac{\partial^2 h_1}{\partial x^2} \qquad A, B \sim \text{const}$$

a LINEAR EQ^N ☺

Before solving a linear eq^n sometimes people like to do...

<u>Wave disturbances of const. wavelengths.</u>

Let $h_1(x,t) = \underset{\substack{\text{const.}\\\text{amplitude}}}{H_0}\, e^{i(\overset{\text{wave n}^o}{k}x - \overset{\text{frequency}}{\omega} t)}$ + complex conjugate

Sub into linearised eq^n: (at top of this page)

$$-i\omega H_0 e^{i(kx-\omega t)} + A\cancel{H_0}(ik)e^{i(kx-\omega t)} = B\cancel{H_0}(ik)^2 e^{i(kx-\omega t)}$$

Get the so-called dispersion relation

a dispersion relation is any relation $F(k,\omega)=0$

$$-i\omega + Aik = B(ik)^2$$

$$\Rightarrow \quad \omega = Ak - iBk^2$$

Assume k real
ω complex:

$$h_1(x,t) = H_0 e^{i(kx-\omega t)} + \text{complex conjugate}$$

$$= H_0 e^{i(kx-Akt)} e^{-Bk^2 t} + \text{complex conjugate}$$

$$= H_0 \underbrace{e^{ik(x-At)}}_{\substack{\text{travelling wave,}\\\text{speed } A}} \underbrace{e^{-Bk^2 t}} + \text{complex conjugate}$$

If $B>0$ $(g>0)$, any wave will decay exponentially with $t \Rightarrow$ stable flow

If $B<0$ $(g<0)$, then $h_1 \sim e^{|B|k^2 t}$
$\Rightarrow$ exponential growth instability!

Exercise (fixed frequency)

Liquid film, const $\tau=\tau_0$, $p=p_a$
$h(x,t) = h_0 = $ const - undisturbed flow

At $x=0$, $h = h_0 + \varepsilon H_0 \cos(\omega t)$.
What happens at $x>0$? $\varepsilon \ll 1$.

Write $h(x,t) = h_0 + \varepsilon h_1(x,t) + O(\varepsilon^2)$
get linearised eq^n for h_1 as before.

$$\begin{cases} \dfrac{\partial h_1}{\partial t} + A\dfrac{\partial h_1}{\partial x} = B\dfrac{\partial^2 h_1}{\partial x^2} \\ \text{At } x=0, \quad h_1 = H_0\cos(\omega t) \\ \text{Solve for } x>0. \end{cases}$$

Write $h_1 = H_0 e^{i\omega t} f(x)$, take real part in answer.

Sub into eq^n: $i\omega f + Af' = Bf''$.

One initial condition: $f(0)=1$.

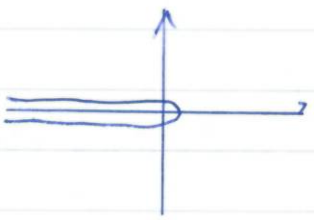
Write $f = e^{\lambda x}$, find λ
$$i\omega + A\lambda = B\lambda^2$$

$$B\lambda^2 - A\lambda - i\omega = 0$$

$$\lambda = \frac{A \pm \sqrt{A^2 + 4i\omega B}}{2B}$$

Choose $\sqrt{z}$ s.t. $\sqrt{1} = 1$ and $|\arg(z)| < \pi$.

then $\mathrm{Re}\sqrt{A^2 + 4i\omega B} > A$



or $\mathrm{Re}\sqrt{1 + \frac{4i\omega B}{A^2}} > 1$

(obviously true)

$\Rightarrow$ $\mathrm{Re}(\lambda_+) > 0 \Rightarrow e^{\lambda x} \to \infty$ as $x \to \infty \Rightarrow$ exclude.

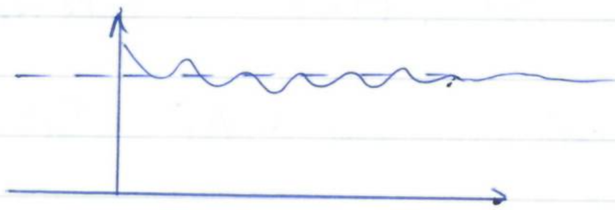
$\mathrm{Re}(\lambda_-) < 0$

$\Rightarrow$ $f(x) = e^{\lambda_- x}$

$h_1(x,t) = \mathrm{Re}\left[H_0 e^{i\omega t} e^{\lambda_- x} \right]$

where $\lambda_- = \dfrac{A - \sqrt{A^2 + 4i\omega B}}{2B}$

Since $\mathrm{Re}(\lambda_-) < 0$, $h_1(x,t)$ decays exponentially with x.



(F. transforms won't be on exam)

Exercise Linear (small) perturbation of arbitrary initial shape.

Let $h(x,t) = h_0 + \varepsilon h_1(x,t) + O(\varepsilon^2)$

Then $\dfrac{\partial h_1}{\partial t} + A\dfrac{\partial h_1}{\partial x} = B\dfrac{\partial^2 h_1}{\partial x^2}$

Let $h_1 = f(x)$ at $t=0$.

Use Fourier transform:

$$h_1(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ikx}\,\bar{h}_1(k,t)\,dk$$

then $$\bar{h}_1(k,t) = \int_{-\infty}^{\infty} e^{-ikx}\,h_1(x,t)\,dx$$

Plan: 1) Find eq^n for $\bar{h}_1(k,t)$
2) Find initial condition on $\bar{h}_1$ at $t=0$
3) Solve for $\bar{h}_1$
4) Use inverse transform to find $h_1(x,t)$

Note: $$\frac{\partial h_1}{\partial t} = \frac{1}{2\pi}\int e^{ikx}\frac{\partial \bar{h}_1}{\partial t}\,dk$$
$$\frac{\partial h_1}{\partial x} = \frac{1}{2\pi}\int e^{ikx}\,ik\bar{h}_1\,dk$$
$$\frac{\partial^2 h_1}{\partial x^2} = \frac{1}{2\pi}\int e^{ikx}(ik)^2\bar{h}_1\,dk$$

Sub into eq^n, get
$$\frac{\partial \bar{h}_1}{\partial t} + Aik\bar{h}_1 = B(ik)^2\bar{h}_1$$
$$\frac{\partial \bar{h}_1}{\partial t} = \left[-Bk^2 - Aik\right]\bar{h}_1$$

PDE reduced to ODE !!

$$\bar{h}_1(k,t) = \bar{f}(k)\, e^{-(Bk^2 + Aik)t}$$

Initial condition: if $h_1(x,t) = f(x)$ at $t=0$,
then Fourier transform of $h_1(x,0)$ is
known $= \bar{f}(k) = \int_{-\infty}^{\infty} e^{-ikx} f(x)\,dx$.
This gives initial condition.

Answer: $$h_1(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \bar{f}(k)\, e^{ikx-(Bk^2-Aik)t}\, dk$$

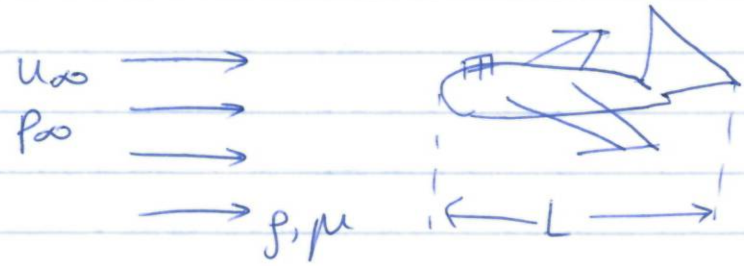
Fourier transform of i.c.

"integral of simple harmonic waves"

Reynold's number

$$Re = \frac{\rho u l}{\mu} = \frac{ul}{\nu}$$

How many parameters?



$$\begin{cases} \rho \dfrac{D\underline{u}}{Dt} = -\nabla p + \mu \nabla^2 \underline{u} \\ \operatorname{div} \underline{u} = 0 \end{cases}$$

$\underline{u} = 0$ on solid walls

$$\left.\begin{array}{l} \underline{u} \to u_\infty \underline{i} \\ p \to p_\infty \end{array}\right\} \text{as } \underline{r} \to \infty$$

($\underline{i}$ is along the flow)

5 parameters

↳ every time 1 changes, need to recalculate
N-S. Urgh. Solution is . . .

Non-dimensional variables

$$\underline{r} = L\,\underline{\tilde{r}}$$

$$\underline{u} = u_\infty\,\underline{\tilde{u}}$$

$$t = \frac{L}{u_\infty}\,\tilde{t}$$

$$p = p_\infty + \rho u_\infty^2\,\tilde{p}$$

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + \underline{u}\cdot\nabla = \frac{u_\infty}{L}\frac{\partial}{\partial \tilde{t}} + \left(u_\infty\,\underline{u}\cdot\frac{1}{L}\tilde{\nabla}\right)$$

$$= \frac{u_\infty}{L}\left(\frac{\partial}{\partial \tilde{t}} + (\underline{\tilde{u}}\cdot\tilde{\nabla})\right)$$

where $\tilde{\nabla} = \frac{\partial}{\partial \tilde{x}}\underline{i} + \frac{\partial}{\partial \tilde{y}}\underline{j} + \frac{\partial}{\partial \tilde{z}}\underline{k}$

$$\nabla^2 = \frac{1}{L^2}\tilde{\nabla}^2 .$$

Sub in N-S, get

$$\rho\frac{u_\infty}{L}\left[\frac{\partial}{\partial \tilde{t}} + (\underline{\tilde{u}}\cdot\tilde{\nabla})\right]u_\infty\,\underline{\tilde{u}} = -\frac{\rho u_\infty^2}{L}\tilde{\nabla}\tilde{p} + \mu\frac{u_\infty}{L^2}\tilde{\nabla}^2\tilde{u}$$

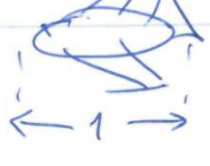
$$\Rightarrow \frac{\partial \tilde{u}}{\partial \tilde{t}} + (\tilde{u}\cdot\tilde{\nabla})\,\tilde{u} = -\tilde{\nabla}\tilde{p} + \frac{\mu}{\rho u_\infty L}\tilde{\nabla}^2\tilde{u} \quad \ldots (1)$$

Continuity: was $\operatorname{div}\underline{u} = 0$ i.e. $\nabla\cdot\underline{u} = 0 \Rightarrow \tilde{\nabla}\cdot\underline{\tilde{u}} = 0$ .. (2)

B.C.s: as $\tilde{r}\to\infty$, $\underline{\tilde{u}}\to\underline{i}$, $\tilde{p}\to 0$ - - - - (3)

On solid walls $\underline{\tilde{u}} = 0$ . . . (4)

(1) to (4) is non-dimensional formulation.



It has 1 parameter $Re = \frac{\rho u_\infty L}{\mu}$

s.t. $$\frac{\tilde{D}\tilde{u}}{\tilde{D}\tilde{t}} = -\tilde{\nabla}\tilde{p} + \frac{1}{Re}\tilde{\nabla}^2\tilde{\underline{u}}$$

<u>Flow pattern for various Re</u>

Small $Re = \frac{\rho u_\infty L}{\mu}$ $\Rightarrow$ large viscosity
or slow motion
or tiny objects

STOKES FLOWS

Stokes flows have large μ and/or small u,
small length scale
(formally, small Re.)

Exercise 2D-flow, non-dimensional

$$\begin{cases} \dfrac{\partial \tilde{u}}{\partial \tilde{t}} + (\tilde{u}\cdot\tilde{\nabla})\tilde{u} = -\tilde{\nabla}\tilde{p} + \dfrac{1}{Re}\tilde{\nabla}^2\tilde{u} \\ \tilde{\nabla}\cdot\tilde{u} = 0 \end{cases}$$ (Nav-Stok)

Take $Re \to 0$.
LHS as small

write $\tilde{u} = \tilde{u}_0 + Re\,\tilde{u}_1 + \cdots$
for $Re \ll 1$,
if $\tilde{p} \ll \frac{1}{Re}$ then in the limit, $\tilde{\nabla}^2\tilde{u}_0 = 0$
$\tilde{\nabla}\cdot\tilde{u}_0 = 0$

and there are no sol^{n}s in general.

Must have $\tilde{p} = \frac{1}{Re}\tilde{P}_0 + \cdots$ as $Re \to 0$

and $\begin{cases} 0 = -\tilde{\nabla}\tilde{p}_0 + \tilde{\nabla}^2\tilde{u}_0 \\ 0 = \operatorname{div} u_0 \end{cases}$

pressure-viscosity balance w/o inertia

Let's forget all these zero expansions and in 2D, hence

$$\begin{cases} 0 = -\nabla p + \frac{1}{Re}\nabla^2 \underline{u} \\ \mathrm{div}\,\underline{u} = 0 \end{cases}$$

(non-dimensional) $\underline{u} = (u, v)$

Having p in eqⁿs is not convenient

Q5 2005

Q5 2008

Q5 2009

In scalar form,

$$\begin{cases} 0 = -\frac{\partial p}{\partial x} + \frac{1}{Re}\nabla^2 u & \cdots (1) \\ 0 = -\frac{\partial p}{\partial y} + \frac{1}{Re}\nabla^2 v & \cdots (2) \\ 0 = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \end{cases}$$

Define streamfunction ψ s.t. $u = \frac{\partial \psi}{\partial y}$, $v = -\frac{\partial \psi}{\partial x}$

Continuity – satisfied

Take $\frac{\partial}{\partial y}(1) - \frac{\partial}{\partial x}(2)$:

$$0 = \overbrace{-\frac{\partial}{\partial y}\left(\frac{\partial p}{\partial x}\right) + \frac{\partial}{\partial x}\left(\frac{\partial p}{\partial y}\right)}^{=0}$$

$$+ \frac{1}{Re}\left[\frac{\partial}{\partial y}\left(\nabla^2 u\right) - \frac{\partial}{\partial x}\left(\nabla^2 v\right)\right]$$

Note, $\frac{\partial u}{\partial y} - \frac{\partial v}{\partial x} = \omega$ vorticity (sometimes $-\omega$)

Then get $0 = 0 + \frac{1}{Re}\nabla^2 \omega$

Next, $\omega = \frac{\partial u}{\partial y} - \frac{\partial v}{\partial x} = \frac{\partial^2 \psi}{\partial y^2} + \frac{\partial^2 \psi}{\partial x^2} = \nabla^2 \psi.$

$\Rightarrow$ Stokes eqⁿ $\boxed{\nabla^2(\nabla^2\psi) = 0}$ biharmonic eqⁿ

often written $\nabla^4\psi = 0$
but be careful,
$\nabla^4 = \nabla^2(\nabla^2)$
$\neq \frac{\partial^4}{\partial x^4} + \frac{\partial^4}{\partial y^4}$

Task: solve for ψ.

Exercise: Same in 3D, in dimensional form.
Ignore inertia

Q5 2006

$$\begin{cases} 0 = -\nabla p + \mu \nabla^2 \underline{u} \\ \operatorname{div} \underline{u} = 0 \end{cases}$$

Q5 2007

Q5 2010

Since $\operatorname{div}\underline{u} = 0$, can define vector potential $\underline{A}$
(instead of ψ) s.t. $\begin{cases} \underline{u} = \operatorname{curl} \underline{A} = \nabla \times \underline{A} \\ \operatorname{div} \underline{A} = 0 \end{cases}$ for uniqueness

Continuity satisfied and
$0 = -\nabla p + \mu \nabla^2 \underline{u}$

Take curl (to get rid of p)
$\Rightarrow \operatorname{curl}(\nabla^2 \underline{u}) = 0$
$\rightarrow \nabla^2(\operatorname{curl} \underline{u}) = 0$

or $\nabla^2 \underline{\Omega} = 0$ where $\underline{\Omega} = \operatorname{curl}\underline{u}$ = vorticity
$= \operatorname{curl}(\operatorname{curl} \underline{A})$
$= \nabla \times (\nabla \times \underline{A})$
$= \operatorname{grad}(\underbrace{\operatorname{div} \underline{A}}_{=0}) - \nabla^2 \underline{A}$
$= -\nabla^2 \underline{A}$

$\underline{\Omega}$

End up with Stokes eq^n $\quad \underline{\nabla^2(\nabla^2 \underline{A}) = 0}$.
Solve for $\underline{A}$.

<u>Special cases of Stokes eq^n</u>

Q5 2007 1. 2D flow
Define $\underline{A} = (0, 0, \psi(x,y))$
z-component of $\underline{A}$

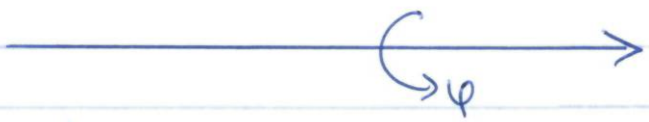
$$\underline{\Omega} = -\nabla^2 \underline{A} = (0, 0, -\nabla^2\psi)$$

Stokes eq^n is $\nabla^2(\nabla^2)(0, 0, -\nabla^2\psi) = 0$
reduces to $\nabla^2(\nabla^2\psi) = 0$.

In Cartesians, $\quad \nabla^2 = \dfrac{\partial^2}{\partial x^2} + \dfrac{\partial^2}{\partial y^2}$

In Polars: $\quad \nabla^2 = \dfrac{1}{r}\dfrac{\partial}{\partial r}\left(r\dfrac{\partial}{\partial r}\right) + \dfrac{1}{r^2}\dfrac{\partial^2}{\partial \theta^2}$

2. Axisymmetric 3D flow



(φ, q_1, q_2) – orthogonal system $\quad$ (e.g. (φ, r, θ))
$$ds^2 = h_1^2(dq_1)^2 + h_2^2(dq_2)^2 + h_3^2(d\varphi)^2$$

Define $\underline{A} = \left(0, 0, \dfrac{\psi(q_1, q_2)}{h_3}\right)$

defined by continuity eq^n

SWINE FLU

Then $$\underline{u} = \nabla \times \underline{A} = \frac{1}{h_1 h_2 h_3} \begin{vmatrix} h_1 \underline{n}_1 & h_2 \underline{n}_2 & h_3 \underline{n}_3 \\ \partial/\partial q_1 & \partial/\partial q_2 & \partial/\partial \varphi \\ \underbrace{h_1 A_1}_{=0} & \underbrace{h_2 A_2}_{=0} & h_3 A_3 \end{vmatrix}$$

$$\Rightarrow \left(\frac{1}{h_2 h_3} \frac{\partial \psi}{\partial q_2} , \; -\frac{1}{h_3 h_1} \frac{\partial \psi}{\partial q_1} , \; 0 \right)$$

$$\underline{\Omega} = \text{curl } \underline{u}$$
$$= -\frac{D^2 \psi}{h_3} \underline{n}_3$$

where $$D^2 = \frac{h_3}{h_1 h_2} \left[\frac{\partial}{\partial q_1} \left(\frac{h_2}{h_1 h_3} \frac{\partial}{\partial q_1} \right) + \frac{\partial}{\partial q_2} \left(\frac{h_1}{h_2 h_3} \frac{\partial}{\partial q_2} \right) \right]$$

The Stokes eq^n in 3D is $\nabla^2 \underline{\Omega} = 0$.
Repeat the steps for $\underline{\Omega}$.

get $$\boxed{D^2 (D^2 \psi) = 0}$$

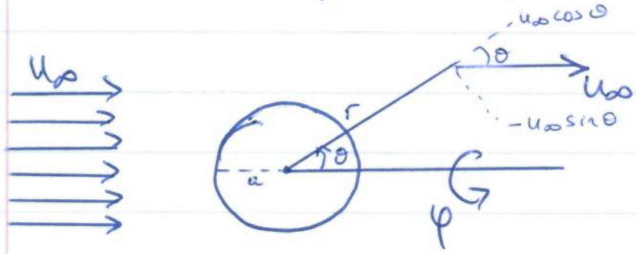
<u>Special cases</u>

(1) Spherical polars (r, θ, φ)
$h_1 = 1, \quad h_2 = r, \quad h_3 = r \sin\theta$
$$D^2 = \frac{\partial^2}{\partial r^2} + \frac{\sin\theta}{r^2} \frac{\partial}{\partial \theta} \left(\frac{1}{\sin\theta} \frac{\partial}{\partial \theta} \right)$$

(2) Cylindrical polars (r, φ, z)
$h_1 = 1 \quad h_2 = 1 \quad h_3 = r$
$$D^2 = \frac{\partial^2}{\partial r^2} - \frac{1}{r} \frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2}$$

Q5
2008

Stokes flow past a sphere



in (r, θ, φ)

Need to solve $D^2(D^2\psi) = 0$

Look at b.c.s. Need velocity components relative to ψ.
Use continuity eq^n.

i.e. $$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 u_r\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left(u_\theta \sin\theta\right) = 0$$

Rewrite in divergence form

$$\frac{\partial}{\partial r}\left(\sin\theta\, r^2 u_r\right) + \frac{\partial}{\partial\theta}\left(r u_\theta \sin\theta\right) = 0$$

Define ψ by

$$r^2 \sin\theta\, u_r = \frac{\partial\psi}{\partial\theta} \qquad u_r = \frac{1}{r^2\sin\theta}\frac{\partial\psi}{\partial\theta}$$

$$u_\theta r\sin\theta = -\frac{\partial\psi}{\partial r} \qquad u_\theta = -\frac{1}{r\sin\theta}\frac{\partial\psi}{\partial r}$$

As $r \to \infty$ $\begin{cases} u_r \to U_\infty \cos\theta \\ u_\theta \to -U_\infty \sin\theta \end{cases}$

or $\begin{cases} \dfrac{\partial\psi}{\partial\theta} \sim U_\infty r^2 \sin\theta\cos\theta \\ \dfrac{\partial\psi}{\partial r} \sim U_\infty r \sin^2\theta \end{cases}$

$$\Rightarrow \boxed{\psi \sim \frac{1}{2} U_\infty r^2 \sin^2\theta \quad \text{as } r \to \infty}$$

At solid wall,

$$u_r = u_\theta = 0$$

$$\Rightarrow \boxed{\frac{\partial \psi}{\partial r} = 0, \quad \psi = 0 \quad \text{at } r = a}$$

Looking for separable sol^n

Try $\psi = f(r) \sin^2\theta$

$$D^2\psi = f'' \sin^2\theta + \frac{\sin\theta}{r^2} f(r) \frac{\partial}{\partial\theta}\left(2 \frac{\cancel{\sin\theta} \cos\theta}{\cancel{\sin\theta}}\right)$$

$$= \left(f'' - \frac{2}{r^2} f\right) \sin^2\theta = F(r) \sin^2\theta$$

$$D^2(D^2\psi) = D^2\left[F \sin^2\theta\right] = \left(F'' - \frac{2}{r^2}F\right) \sin^2\theta$$

Need $D^2(D^2\psi) = 0 \Rightarrow F'' - \frac{2}{r^2} F = 0$

$$\Rightarrow F = r^\lambda$$

then
$$\lambda(\lambda - 1) - 2 = 0$$
$$(\lambda - 2)(\lambda + 1) = 0$$
$$\lambda = \{2, -1\}$$

$$\rightarrow F = C_1 r^2 + C_2 r^{-1}$$

Find f from

$$f'' - \frac{2}{r^2} f = F$$

$$f'' - \frac{2}{r^2} f = C_1 r^2 + C_2 r^{-1}$$

$$f(r) = \overbrace{D_1 r^4 + D_2 r}^{\text{from inspection of } C_1 r^2 + C_2 r^{-1}} + D_3 r^2 + D_4 r^{-1}$$

At $r=a$, $f(a) = 0$, $f'(a) = 0$

As $r \to \infty$ $f(r) \sim \frac{U_\infty}{2} r^2$

Find D_1 to D_4.

<u>As $r \to \infty$</u> $D_1 = 0$ (r^4 goes too fast)

$D_3 = \frac{U_\infty}{2}$

$$f(r) = D_2 r + \frac{U_\infty}{2} r^2 + D_4 r^{-1}$$

<u>At $r=a$</u> Use $f(a)=0$, $f'(a)=0$ to find D_2, D_4.

Exercise: Same as sphere but in 2D.

Cylinder, radius a, u_∞ flow field

Solve $\nabla^2(\nabla^2\psi) = 0$

Cylindrical polars: $\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}$

b.c.s: $\left.\begin{array}{l} u \to u_\infty \\ v \to 0 \end{array}\right\}$ as $r \to \infty$.

$u = v = 0$ at $r = a$ but u, v are Cartesian,

$u = \frac{\partial \psi}{\partial y}$, $v = -\frac{\partial \psi}{\partial x}$

make const. so this is true.

Find b.c.s for ψ: at $r = a$, $\frac{\partial \psi}{\partial r} = 0$, $\psi = 0$

as $r \to \infty$, $\psi \sim u_\infty y$
$\sim u_\infty r \sin\theta$

Attempt separable ψ: $\underline{\psi = \sin\theta \cdot f(r)}$

$\nabla^2\psi = \left[f'' + \frac{1}{r}f' - \frac{1}{r^2}f\right]\sin\theta = F(r)\sin\theta$

$\nabla^2(\nabla^2\psi) = \left[F'' + \frac{1}{r}F' - \frac{1}{r^2}F\right]\sin\theta$.

$\nabla^2(\nabla^2\psi) = 0 \Rightarrow \underline{F'' + \frac{1}{r}F' - \frac{1}{r^2}F = 0}$

Try $F = r^\lambda$: $\lambda(\lambda-1) + \lambda - 1 = 0$
$(\lambda+1)(\lambda-1) = 0$

$F = D_1 r + D_2 r^{-1}$

$$\Rightarrow \quad f'' + \frac{1}{r}f' - \frac{1}{r^2}f = D_1 r + D_2 r^{-1}$$

from homogeneous

General solution: $f(r) = C_1 r^3 + C_2 r \ln r + C_3 r + C_4 r^{-1}$

$\because$ r cancels.

b.c.s: $\psi \sim u_\infty r \sin\theta + \cdots, \quad r \to \infty$

$\Rightarrow f(r) \sim u_\infty r + \cdots, \quad r \to \infty$

If this is used then $C_1 = 0$, $C_2 = 0$, $C_3 = u_\infty$

($\because$ of size)

$$\Rightarrow \quad f(r) = u_\infty r + C_4 r^{-1}$$

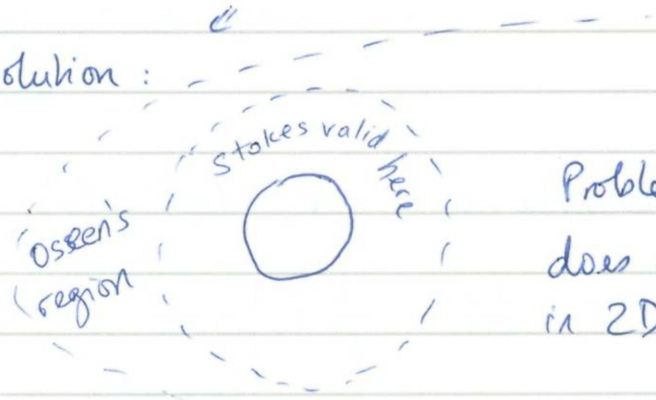
On the sphere, $\psi = \frac{\partial \psi}{\partial r} = 0$

$$\Longrightarrow f(a) = 0$$
$$f'(a) = 0$$

..... not possible

STOKES PARADOX

Resolution :



Problem is, the Laplacian does not decay at infinity in 2D ($\because \ln r$ is fundamental sol^{n})

True solution in stokes region is:

$$\psi = \sin\theta \left[r \ln r \, C_2 + C_3 r + C_4 r^{-1} \right]$$

3 unknowns

2 conditions to find C_{2-4} are $f(a) = f'(a) = 0$

(no slip)

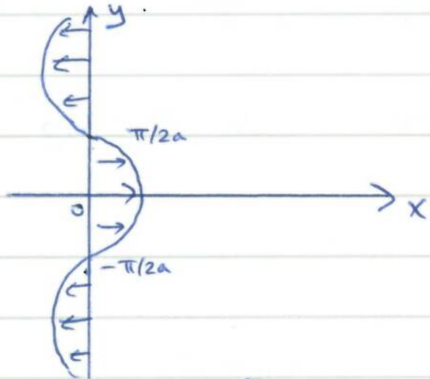
The condition as $r \to \infty$ is <u>not</u> imposed.
Need to consider region of size

$$\frac{r}{a} = O(1/Re) \gg 1$$

impose condition $\psi \to u_\infty r \sin\theta$ in Oseen region

Q5 2007 vaguely similar

<u>Exercise</u>: 2D permeable wall



Flow in $x \geq 0$

At $x=0$, $\begin{cases} u = u_0 \cos(ay) \\ v = 0 \end{cases}$

Q3 2009 similar

Find ψ by solving $\nabla^2(\nabla^2\psi) = 0$

Q5 2010 similar

[For practice, try $u = u_0 \cos(ay)$, $v = v_0 \cos(ay)$ at $x=0$]

Write conditions on ψ (instead of u, v)

At $x=0$, $v=0 \Rightarrow \frac{\partial\psi}{\partial x} = 0$

$u = u_0\cos(ay) \Rightarrow \frac{\partial\psi}{\partial y} = u_0 \cos(ay)$

$\Bigg\} \; \psi = \frac{u_0}{a}\sin(ay)$

Solving for $\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$

$$\nabla^2(\nabla^2\psi) = 0.$$

Seek separable sol^n: $\psi = \sin(ay) f(x)$

$$\nabla^2\psi = (f'' - a^2 f)\sin(ay) = F(x)\sin(ay)$$

$$\nabla^2(\nabla^2\psi) = (F'' - a^2 F)\sin(ay) = 0$$

$$F'' - a^2 F = 0 \Rightarrow F = D_1 e^{ax} + D_2 e^{-ax}$$
$$\Rightarrow f'' - a^2 f = D_1 e^{ax} + D_2 e^{-ax}$$

General sol$^{\underline{n}}$: $f(x) = C_1 e^{ax} + C_2 x e^{ax} + C_3 e^{-ax} + C_4 x e^{-ax}$

$$\left.\frac{\partial \psi}{\partial x}\right|_{x=0} = 0 \Rightarrow f'(0) = 0$$

$$\psi\Big|_{x=0} = \frac{u_0}{a}\sin(ay) \Rightarrow f(0) = \frac{u_0}{a}$$

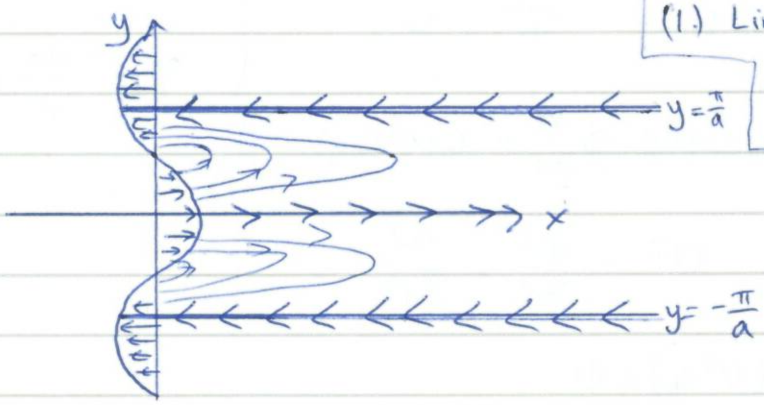
Require $|\psi| <$ exp as $x \to \infty$ $\Rightarrow C_1 = C_2 = 0$,

$$f(0) = \frac{u_0}{a} \Rightarrow C_3 = \frac{u_0}{a}$$

$$f'(0) = 0 \Rightarrow -aC_3 + C_4 = 0 \Rightarrow C_4 = u_0$$

$$\Rightarrow \psi = u_0\left[\frac{1}{a} + x\right] e^{-ax} \sin(ay).$$

Sketch streamlines.



(1) Lines $\psi = 0$
$\Rightarrow ay = n\pi$

(2) Stag. pts
NONE

(3) Guesswork