3301 Real Fluids Notes

Based on the 2013 autumn lectures by Prof F T Smith

Math3301 Real Fluids   Office hours: Thurs 11am

Chapter 1: Introduction and equations of motion

(i) Pure fluid - can deform without limit when a force is applied.
- stops when the force is withdrawn (inelastic)
- includes many gases and liquids.

We treat fluids as a continuous medium.
$\Rightarrow$ we can deal with quantities such as density, velocity, pressure.
$\hookrightarrow$ can be defined at a point.

We work outside the molecular level.
The point is called a FLUID ELEMENT / FLUID PARTICLE

We can specify motion in two ways:
1. Lagrangian Method follows individual particles around.
Property $P = p(\underline{a}, t)$
initial posn of particle $\uparrow$, $\hookrightarrow$ time

Then position $\underline{r} = \underline{x}(a, t)$
velocity $\underline{u} = \frac{\partial \underline{r}}{\partial t} = \left(\frac{\partial \underline{x}}{\partial t}\right)_{\underline{a}} = \underline{q}$
acceleration $\left(\frac{\partial \underline{u}}{\partial t}\right)_{\underline{a}} = \left(\frac{\partial^2 \underline{x}}{\partial t^2}\right)_{\underline{a}}$
density $\rho(\underline{a}, t)$

Easy to apply Newton's Laws of motion, but you don't know where each fluid particle is.

we will use this
2. Eulerian method uses field quantities, functions of $\underline{x}, t$: $P = P(\underline{x}, t)$
Velocity $\underline{u} = \underline{u}(\underline{x}, t)$ is the velocity of that particle which is at position $\underline{x}$ at time t

If we can find $\underline{u}(\underline{x}, t)$, then we can get particle positions by solving $\frac{d\underline{x}}{dt} = \underline{u}(\underline{x}(t), t)$
subject to $\underline{x}(0) = \underline{a}$ (1.1)

(ii) A complication lies in deriving the acceleration in the Eulerian method, since particles move past $\underline{x}$

Use, for any $P(\underline{x}(t), t)$:

$$dP = \left(\frac{\partial P}{\partial x_i}\right)_t dx_i + \left(\frac{\partial P}{\partial t}\right)_{\underline{x}} dt \Rightarrow \frac{dP}{dt} = \frac{\partial P}{\partial x_i}\frac{dx_i}{dt} + \frac{\partial P}{\partial t}$$

$$\Rightarrow \frac{DP}{Dt} = (\underline{u} \cdot \underline{\nabla})P + \frac{\partial P}{\partial t} \qquad (1.2)$$

So acceleration $= \dfrac{D\underline{u}}{Dt} = \dfrac{\partial \underline{u}}{\partial t} + (\underline{u}.\underline{\nabla})\underline{u}$

ie. $\left(\dfrac{D\underline{u}}{Dt}\right)_i = \dfrac{\partial u_i}{\partial t} + u_j \dfrac{\partial u_i}{\partial x_j}$ ← in cartesian coordinates (1.3)

In other coordinate systems, use $(\underline{u}.\underline{\nabla})\underline{u} = \underline{\nabla}(\tfrac{1}{2}q^2) - \underline{u}\wedge\underline{\omega}$ (1.4)
where $q = |\underline{u}| = $ speed $= |\underline{q}|$ (1.5)
$\underline{\omega} = \text{curl } \underline{u} = $ vorticity

(iii) Conservation of mass

Consider "material volume" V, consisting of the same fluid particle $\forall t$.
(Not a fixed volume)

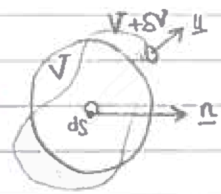
$$\frac{D}{Dt}\left[\int_V P(t)\,dV\right] = \lim_{\delta t\to 0}\frac{1}{\delta t}\left[\int_{V+\delta V} P(t+\delta t)\,dV - \int_V P(t)\,dV\right]$$

any property

$$= \frac{1}{\delta t}\left[\text{change in } P \text{ at fixed vol} + \text{effect due to change in } V\ (V+\delta V)\right]$$

lim as $\delta t \to 0$

$$= \underbrace{\frac{1}{\delta t}\left[\int_V \delta P\,dV\right]}_{*} + \underbrace{\frac{1}{\delta t}\int_S P(\underline{u}.\underline{n})\,\delta t\,\delta S}_{**}$$



$\Rightarrow$ [δV is due to ∂S moving along with $\underline{u}$
$\delta V = \text{length} \times \delta S = \underline{u}.\underline{n}\,\delta t \times \delta S$]

$$\Rightarrow \frac{D}{Dt}\left[\int_V P\,dV\right] = \underbrace{\int_V \frac{\partial P}{\partial t}\,dV}_{*} + \underbrace{\int_V \text{div}(P\underline{u})\,dV}_{**} \quad \text{(from divergence thm)}$$

$$\Rightarrow \int_V \left\{\frac{\partial P}{\partial t} + (\underline{u}.\underline{\nabla})P + P\,\text{div}\,\underline{u}\right\} dV$$

material derivative

$$\Rightarrow \frac{D}{Dt}\left[\int_V P\,dV\right] = \int_V \left\{\frac{DP}{Dt} + P\,\text{div}\,\underline{u}\right\} dV \qquad (1.6)$$

Now take $P \equiv \rho$ (density) :-

LHS $= 0$, since $\int_V \rho\,dV \equiv$ mass in V

$\Rightarrow$ RHS $= 0$. Then let $V = 0 \Rightarrow$ integrand $= 0$ $\forall \underline{x}, t$
(every point in the fluid).

$$\Rightarrow \boxed{\frac{D\rho}{Dt} + \rho \, \mathrm{div}\, \underline{u} = 0}$$ THE CONTINUITY EQUATION

ie from (1.2), $\frac{\partial \rho}{\partial t} + \mathrm{div}(\rho \underline{u}) = 0$ (1.7)

If fluid is <u>incompressible</u>, then $\boxed{\frac{D}{Dt}(\rho) = 0.}$

$\Rightarrow$ density of any fluid particle is constant.

Examples: Most liquids - water, blood, sea, even gases are incompressible provided that their speeds are considerably less than the speed of sound.

So get $\mathrm{div}\, \underline{u} = 0$. (1.8)

We can then define stream functions in 2D or axisymmetric flow of an incompressible fluid.

(a) 2D: $\underline{u} = (u, v, 0)$ in Cartesians; all are independant of z

$\Rightarrow \mathrm{div}\, \underline{u} = 0 = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}^{\nearrow 0}$ (1.9)

$\Rightarrow \exists \psi$ s.t. $\boxed{u = \frac{\partial \psi}{\partial y}}$ $\boxed{v = -\frac{\partial \psi}{\partial x}}$

streamfunction

(b) Axisymmetric: cylindrical polars  (r, θ, z)

Here $\underline{u} = (u_r, 0, u_z)$ say. All is independant of θ

$\exists \psi$ s.t. $\boxed{u_r = -\frac{1}{r}\frac{\partial \psi}{\partial z}}$, $\boxed{u_z = \frac{1}{r}\frac{\partial \psi}{\partial r}}$ (1.9b)

In spherical polars (r, θ, α); $\boxed{u_r = \frac{1}{r^2 \sin\theta}\frac{\partial \psi}{\partial \theta}}$ $\boxed{u_\theta = \frac{-1}{r\sin\theta}\frac{\partial \psi}{\partial r}}$

(iv) Motion - Look near a fixed point $\underline{x}$ and look at behaviour of motion nearby.

All at the same time t.

Use 3D's Taylor's expansion: $\underline{u}(\underline{x}+\delta\underline{x}) - \underline{u}(\underline{x}) = \delta \underline{u} = (\delta \underline{x} \cdot \underline{\nabla})\underline{u} + O(|\delta \underline{x}|^2)$

$$\Rightarrow \delta u_i \approx \overbrace{e_{ij}\,\delta x_j}^{\delta u_i^{(1)}} + \overbrace{\xi_{ij}\,\delta x_j}^{\delta u_i^{(2)}}$$

symmetric part   antisymmetric part

Here $\boxed{e_{ij} \equiv \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)}$ is a second order tensor.

$= $ rate of strain. $\Rightarrow$ pure straining motion.

And $\boxed{\xi_{ij} \equiv \left(\frac{1}{2}\frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i}\right)}$ is a second order tensor

$\Rightarrow$ pure rotation, with angular velocity $\frac{1}{2}\underline{\omega}$.

$$\delta u_i^{(2)} = \frac{1}{2}\underline{\omega} \wedge \delta \underline{x}$$

Hence motion of particle = translation [velocity $\underline{u}(\underline{x})$] (1.10a)
+ rotation (1.10b)
+ straining motion (1.10c)

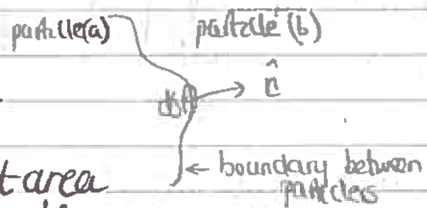
(v) Forces - There are two kinds.

(i) Long ranged body forces. eg. gravity, chemical, electromagnetic
Represented by force per unit mass, $\underline{G}(\underline{x},t)$.

(ii) Short ranged stress forces between neighbouring particles eg. friction.

Stress forces: act between neighbouring particles ($\approx$ friction).

Consider (a)(b)
Surface element dS, normal $\underline{n}$



The stress force $\underline{F}$ = force per unit area
$\Rightarrow$ force on (b) due to (a) is $\underline{F}\,dS$

Can show that $\underline{F}$ depends linearly on $\underline{n}$

$$\Rightarrow F_i(\underline{n}, \underline{x}, t) = \sigma_{ij}(\underline{x}, t)\, n_j \qquad (1.11)$$

Here σ_{ij} is a 2nd order tensor & is symmetric

Now; in fluid at rest: $\underline{F}$ is due to pressure $= \sigma_{ij} = -p\delta_{ij}$

fluid in motion: $\sigma_{ij} = p\delta_{ij} + d_{ij}$ ← derivative stress tensor (symmetric).

$$p = -\frac{1}{3}\sigma_{ii} \qquad d_{ii} = 0$$

(vi) Relation between stress & motion

motion $\equiv$ translation + straining motion (e_{ij}) + rotation (ξ_{ij})

↑ this alone contributes to stress

So d_{ij} = function of e_{ij}.

Too general! write as a power series and keep only 1 term instead.
Hence, linear relation

$\boxed{d_{ij} = A_{ijkl}\, e_{kl}}$ [Newtonian fluid]

constant of proportionality.

Tensor analysis $\Rightarrow A_{ijkl} = \mu \delta_{ik}\delta_{jl} + \mu' \delta_{il}\delta_{jk} + \lambda \delta_{ij}\delta_{kl}$

(μ, μ', λ) scalars.

But δ_{ij} is symmetric so $A_{ijkl} = A_{jikl} \Rightarrow \mu = \mu'$.

Hence $d_{ij} = 2\mu e_{ij} + \lambda \delta_{ij} e_{kk}$

$\frac{2}{3}$ as d occurs 3 times.

Also $d_{ii} = 0$ requires $\lambda = -\frac{2}{3}\mu$ (if $e_{kk} \neq 0$)

Hence, for a Newtonian fluid, $\sigma_{ij} = -p\delta_{ij} + 2\mu(e_{ij} - \frac{1}{3}\delta_{ij} e_{kk})$ (1.12)

stress tensor.

Notes: (a) $e_{kk} = \frac{1}{2}\left(\frac{\partial u_k}{\partial x_k} + \frac{\partial u_k}{\partial x_k}\right) = \frac{\partial u_k}{\partial x_k} = \operatorname{div} \underline{u}$

stress

So for an incompressible fluid, $e_{kk} = 0 \Rightarrow \sigma_{ij} = -p\delta_{ij} + 2\mu e_{ij}$

(b) Relation between p & p': see later.
For an incompressible fluid, $p = p'$.

constant = bulk velocity.

$$p - p^T = -K e_{kk} \quad (1.13)$$

same as p' — thermodynamic pressure as in $\frac{p}{\rho} = RT$

(c) Some fluids are non Newtonian, but most everyday fluids are Newtonian.

(d) Scalar μ in (1.12) is "viscosity" of the fluid.

Values:		
	Air (at normal T & p)	$\mu \simeq 0.00018$ poise (gram/cm/sec)
	Water	$\mu \simeq 0.010$
	Mercury	$\mu \simeq 0.016$
	Glycerine	$\mu \simeq 23.3$

(vii) Equations of motion – Use Newtons law of motion for a material volume V. –

material with volume V

Apply: Rate of change of momentum = body force + surface force.

$$\Rightarrow \frac{D}{Dt}\Big\{\underbrace{\int_V \rho u_i \, dV}_{\text{momentum}}\Big\} = \int_V \rho G_i \, dV + \int_S \sigma_{ij} n_j \, dS$$

$$LHS = \int_V \left[\frac{D}{Dt}(\rho u_i) + \rho u_i \operatorname{div} \underline{u}\right] dV \qquad \text{(from (1.6))}$$

$$\Rightarrow \int_V \left[\rho \frac{Du_i}{Dt} + u_i \frac{D\rho}{Dt} + \rho u_i \operatorname{div}\underline{u}\right] dV$$

$$= \int_V \rho \frac{Du_i}{Dt} dV, \text{ using (1.7) = continuity equation.}$$

$$RHS = \int_V \left[\rho G_i + \frac{\partial \sigma_{ij}}{\partial x_j}\right] dV \quad \text{from the Divergence theorem}$$

LHS = RHS for any volume V in the fluid

$\Rightarrow$ integrands must be equal everywhere

$\Rightarrow$ equation of motion is $\boxed{\rho \frac{Du_i}{Dt} = \rho G_i + \frac{\partial \sigma_{ij}}{\partial x_j}}$ (1.14)

true for any continuous medium (inc. solids)

For a Newtonian fluid, use 1.12:

$$\boxed{\rho \frac{Du_i}{Dt} = \rho G_i - \frac{\partial p}{\partial x_i} + 2\frac{\partial}{\partial x_j}\left\{\mu\left(e_{ij} - \frac{1}{3}\delta_{ij} e_{kk}\right)\right\}} \qquad (1.15)$$

THE NAVIER-STOKES' EQUATION(S).

Check: Equations are 3 from (1.15)
+ 1 from (1.7)
+ 1 from $p - p^T$ in (1.13)
+ equations of state (eg $p^T/\rho = RT$)
+ equations of heat conduction (for T)

Unknowns are $\underline{u}$, p, ρ, p^T, T
(7 each)

(a) If μ = constant, (1.15) $\rightarrow \rho \frac{Du_i}{Dt} = \rho G_i - \frac{\partial p}{\partial x_i} + \mu\left[\frac{\partial^2 u_i}{\partial x_j \partial x_j} + \frac{1}{3}\frac{\partial^2 u_j}{\partial x_i \partial x_j}\right]$ (using e_{ij})

or $\rho \frac{D\underline{u}}{Dt} = \rho \underline{G} - \nabla p + \mu\left[\nabla^2 \underline{u} + \frac{1}{3}\operatorname{grad}\operatorname{div}\underline{u}\right]$ (1.16)

NB. In other coordinate systems, use $\nabla^2 \underline{u} = \operatorname{grad}\operatorname{div}\underline{u} - \operatorname{curl}\operatorname{curl}\underline{u}$.

(b) If also, the fluid is incompressible, then we get $\rho \frac{Du}{Dt} = \rho G - \nabla p + \mu \nabla^2 u$ from (1.16) (1.17)

(1.17), $\operatorname{div} u = 0$, $\frac{D\rho}{Dt} = 0$

We now have 5 for 5 ← u, p, ρ

$\frac{Du}{Dt} = G - \nabla\left(\frac{p}{\rho}\right) + \nu \nabla^2 u$ (1.18)

$\operatorname{div} u = 0$ (1.8)

(c) If also, the density is constant, then its 4 for 4

* This applies for the rest of the course
* No general solution because the system is non linear from $\frac{\partial u}{\partial t} + (u \cdot \nabla) u$
* $\boxed{\nu = \frac{\mu}{\rho} = \text{constant}}$ = "Kinematic viscosity"

Air 0.15
Water 0.01
Mercury 0.0012
Glycerine 18.5

* There are 4 forces present: inertial, body, pressure, viscous.

(viii) Modified pressure – Often G is uniform, eg g. Then consider

$g - \nabla(p/\rho) = \nabla(g \cdot x - p/\rho)$ — x: position vector.

$\Rightarrow$ define $P = p - \rho g \cdot x - p_0$ ← constant

to get $\nabla\left(\frac{P}{\rho}\right)$. Here P is the "modified pressure".

And (1.18) $\Rightarrow \frac{Du}{Dt} = -\nabla\left(\frac{P}{\rho}\right) + \nu \nabla^2 u$ (1.19)

Here g affects only the boundary conditions.

(ix) Boundary conditions between a fluid and a solid

1. Normal component of velocity must be continuous
2. Tangential component of velocity must be continuous (no slip) *
3. Stress is continuous. **

* absent for an inviscid fluid (2301)
** this is just p, for an inviscid fluid.

FOR A VISCOUS FLUID, NEED TO IMPOSE 1, 2, 3

This is the shape of the surface (or $y = H(x,z,t)$) so

$\frac{Df}{Dt} = v - H_t - uH_x - wH_z = 0$

$\Rightarrow v = H_t + uH_x + wH_z$

(a) for a fluid/fluid interface $f(x,t) = 0$ say, 1. $\Rightarrow \boxed{\frac{Df}{Dt} = 0}$ the kinematic boundary condition.

(b) For a fluid/solid interface, $\underline{u} = 0$ at the boundary (assuming that the boundary is fixed).
3. $\Rightarrow$ stress is acting on the solid.

(c) For a "free surface", one medium is dynamically negligible.
one viscous & one non viscous fluid

$\Rightarrow \frac{DF}{Dt} = 0$ from 1.
stress $= 0$ from 3.

2. doesn't tell us anything useful.

Chapter 2: Exact Solutions of the Navier-Stokes' equations.

Incompressible fluid + gravity $\Rightarrow \rho \frac{D\underline{u}}{Dt} = \rho \underline{g} - \nabla p + \mu \nabla^2 \underline{u}$ (2.1)

or $\rho\left(\frac{\partial \underline{u}}{\partial t} + (\underline{u}\cdot\nabla)\underline{u}\right) = -\nabla p + \mu \nabla^2 \underline{u}$ (2.1')

Also have continuity: $\nabla\cdot\underline{u} = 0$. (1.8)

2.1 Unidirectional flows

Velocity vector is in just one direction, say x, so that $\underline{u} = (u, 0, 0)$ in (x,y,z) cartesian coordinates

Then (1.8) $\Rightarrow \frac{\partial u}{\partial x} = 0 \Rightarrow u = u(y,z,t)$ (2.2)

In (2.1'), $(\underline{u}\cdot\nabla)\underline{u} \rightarrow ((\underline{u}\cdot\nabla)u, 0, 0) = \left(u\frac{\partial u}{\partial x}, 0, 0\right) = 0$ → non linear effects disappear

$$\left.\begin{aligned} y \text{ momentum}: \; 0 &= -\frac{\partial p}{\partial y} + 0 \\ z \text{ momentum}: \; 0 &= -\frac{\partial p}{\partial z} + 0 \end{aligned}\right\} \Rightarrow p = p(x,t)$$

x momentum: $\underset{(y,z,t)}{\rho \frac{\partial u}{\partial t}} = -\underset{(x,t)}{\frac{\partial p}{\partial x}} + \underset{(y,z,t)}{\mu \nabla^2 u}$

$$\underbrace{(\cancel{u_{xx}} + u_{yy} + u_{zz}) - \frac{1}{\nu}u_t}_{\text{independant of } x} = \underbrace{\frac{1}{\mu}p_x}_{\text{independant of } y \,\&\, z} \qquad \boxed{\nu = \frac{\mu}{\rho}}$$

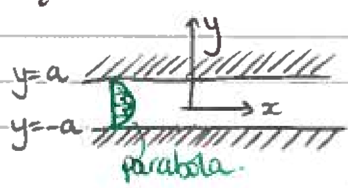
$\Rightarrow$ both sides depend on t only

Write $p_x = -G(t)$, say, the gradient of modified pressure.

Hence solve: $u_{yy} + u_{zz} - \frac{1}{\nu}u_t = -\frac{G(t)}{\mu}$ (2.3)

(a) Steady flow in a 2D channel → assume u is independant of z $\Rightarrow$ try $u = u(y)$ only.

G is constant; $\partial_t = 0$



So (2.3) $\Rightarrow u_{yy} = -\frac{G}{\mu}$ (2.4)

Need 2 boundary conditions: the 2 no slip conditions at the 2 walls.
ie. $u=0$ at $y=\pm a$

We find $u = \frac{G}{2\mu}(a^2-y^2)$ (2.5)

$\Rightarrow$ A parabolic profile $\Rightarrow$ "plane Poiseuille flow" (PPF)

NOTE: Real channels have sides (in z) & ends (in x)
We assume that these are sufficiently far away ($\gg 2a$) for
(2.4) & (2.5) to apply.

(b) Steady flow in a pipe of circular cross section (radius a)

Try an axisymmetric solution $u = u(r,\cancel{\theta},t)$

Here $(u_{yy}+u_{zz}) \Rightarrow (u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta})$ in cylindrical polars (x,r,θ)

Now (2.3) $\Rightarrow u_{rr} + \frac{1}{r}u_r = -\frac{G}{\mu}$

Only 1 BC, $u(a)=0$. But then $u = \frac{G}{4\mu}(-r^2 + A\ln r + B)$ is the sol^n

Finiteness of $u \Rightarrow A=0$.
Hence $u = \frac{G}{4\mu}(a^2-r^2)$ (2.6)

$\Rightarrow$ Another parabolic profile $\Rightarrow$ "Hagen-Poiseuille flow" (HPF)

Hence volume flow rate (flux) $Q = \int_0^a u \,.\, 2\pi r\, dr$.

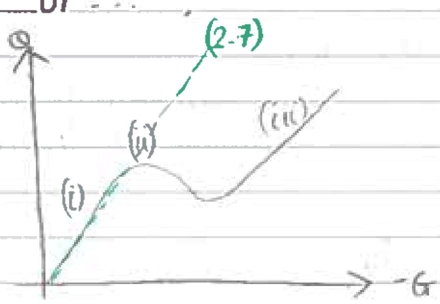
$$= \int_0^a \frac{G}{4\mu}(a^2-r^2)\,.\,2\pi r\, dr = \frac{\pi G a^4}{8\mu} \quad (2.7)$$

$\Rightarrow$ average velocity $\bar{u} \equiv \frac{Q}{\pi a^2} = \frac{Ga^2}{8\mu}$

and so $\frac{u}{\bar{u}} = 2\left(1-\frac{r^2}{a^2}\right)$ (2.6′)

↳ non dimensional form (very flexible form) showing a law in which nothing changes whatever the size of the pipe & the fluid or...

Experiments give:

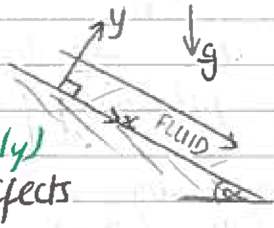


(i) agrees with our (2.7). Have unidirectional flow.

(ii) gives unsteady 3D flow ("transition")

(iii) gives turbulent flow (not laminar)

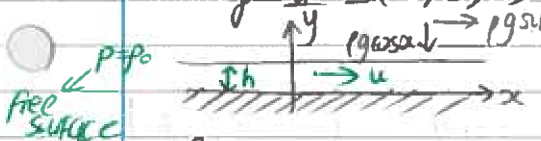
This course concentrates on laminar flow.



(c) Free-surface flows

look for 2D solution (no effects in the z-direction)
1D (x direction only)

$\Rightarrow$ try $\underline{u} = (u, 0, 0)$ and $u = u(y)$



p_0 is the atmospheric pressure

Continuity : $0 + 0 + 0 = 0$

viscous term

xM : $\rho u_t + 0 + 0 + 0 = -p_x + \rho g \sin\alpha + \mu u_{yy}$ (steady flow)

yM : $0 + 0 + 0 + 0 = -p_y - \rho g\cos\alpha + 0$

zM : $0 = 0$

$\Rightarrow p_y = -\rho g\cos\alpha \Rightarrow p = -\rho g(y-h)\cos\alpha + p_0 + g(x)$

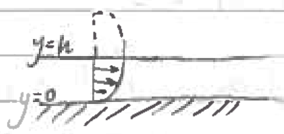
Since $p = p_0$ at $y = h$

So $xM \Rightarrow 0 = -p_x + \rho g\sin\alpha + \mu u''(y)$ $\rightarrow$ effective pressure gradient

$\Rightarrow \mu u''(y) = -\rho g\sin\alpha = -G$, say.

BCs are $\begin{cases} u = 0 \text{ at } y = 0 \text{ (no slip at floor)} \\ \mu\dfrac{du}{dy} = 0 \text{ at } y = h \text{ (zero stress).} \end{cases}$

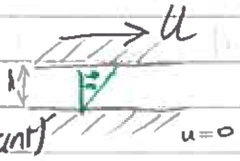
$\therefore u = \dfrac{G}{\mu} y(h - \tfrac{1}{2}y)$ (2.8)



* $\frac{1}{2}$ a PPF

* notice the "modified" pressure gradient G

(d) Couette flow, in a 2D channel with one wall moving at velocity U (constant) is $u = \dfrac{Uy}{h}$

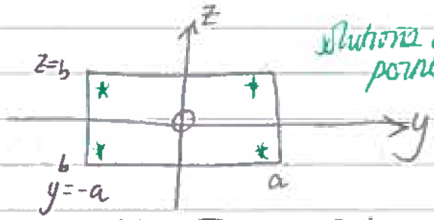


If there is also a pressure gradient, then superimpose solutions (since linear)

(e) Steady flows in pipes of general cross section

Here, solve $u_{yy} + u_{zz} = \frac{-G}{\mu}$ ... POISSON'S EQUATION. (2.9)

eg. Ellipse – see question sheet 2
Eg. Rectangle
$\Rightarrow u = 0$ at $\begin{cases} y = \pm a \\ z = \pm b. \end{cases}$



Solutions at these 4 points to be the same

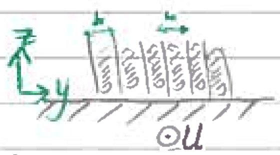
Expect symmetry in y, z $\Rightarrow$ try a double Fourier Series,
$$u = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} A_{mn} \cos\left[(2m+1)\frac{\pi y}{2a}\right] \cos\left[(2n+1)\frac{\pi z}{2b}\right]$$
This satisfies the boundary conditions and symmetries.
Need to make it satisfy (2.9)

$$\Rightarrow A_{mn} = \frac{64G(-1)^{m+n}}{\mu\pi^4} \frac{1}{(2n+1)(2m+1)} \cdot \frac{1}{B} \qquad (2.10)$$

where $B = \left(\frac{2m+1}{a}\right)^2 + \left(\frac{2n+1}{b}\right)^2$ $\quad \left(u_{yy} + u_{zz} = \frac{-G}{\mu}\right)$

(f) Model of a paint brush

Pretend the situation is thus:- parallel vertical plates, travelling horizontally with velocity U, on a plane wall, in steady 2D motion



~~Brush is going into the page~~
The wall is coming out of the page.

No pressure gradient here so solve: $u_{yy} + u_{zz} = 0$ ①
with BC's: $\begin{cases} u(y,0) = -U & [\text{moving wall}] \\ u(0,z) = u(b,z) = 0 & [\text{no slip on plates}] \\ u(y,\infty) = 0 \end{cases}$ ② ③ ④

Work with nondimensional quantity $u' = \frac{u}{U} \Rightarrow U = 1$, (replace capital U by 1)

Seek Fourier series solutions $u' = \sum_{1}^{\infty} f_n(z) \sin\left(\frac{n\pi y}{b}\right)$

① $\Rightarrow -\left(\frac{n\pi}{b}\right)^2 f_n \cancel{S} + f_n'' \cancel{S} = 0$ $\quad S \to \sin\left(\frac{n\pi y}{b}\right)$

$\Rightarrow f_n = A_n e^{-\frac{n\pi z}{b}} + B_n e^{\frac{n\pi z}{b}}$

④ $\Rightarrow B_n = 0$
③ is satisfied already.
② $\Rightarrow \sum_{1}^{\infty} A_n \sin\left(\frac{n\pi y}{b}\right) = -1$ ∴ multiply both by $\sin\left(\frac{m\pi y}{b}\right)$ & take $\int_0^b (\ )\,dy$ to find A_m's.

Find $A_m = \begin{cases} 0 \; ; & m \text{ even} \\ \frac{-4}{m\pi} \; , & m \text{ odd.} \end{cases}$

Hence $\frac{u}{U} = -\frac{4}{\pi} \sum_{n \text{ odd}} \frac{e^{-n\pi z/b}}{n} \sin\left(\frac{n\pi y}{b}\right)$

Finally, estimate thickness of layer of paint. Suppose a rear edge to the plates (an x length $\gg b$) and all of the volume flux of paint goes to that layer, of thickness h.

For one channel, volume flux $= \int_{0}^{\infty}\int_{0}^{b} u \, dy\, dz = -\frac{8Ub^2}{\pi^3} \sum_{n \text{ odd}} \frac{1}{n^3}$

$\approx -0.27\, Ub^2$

Balance this against $-Uhb$ (flux in the layer of paint)

$\Rightarrow$ mean thickness (h) of the layer is $0.27b$

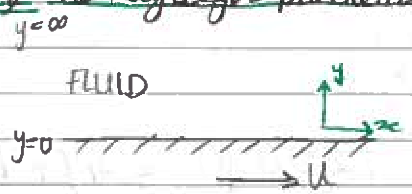
Notice: the result is independant of U (the speed at which you paint)
ν (the viscosity of the fluid)

Also, we have neglected surface tension, non-Newtonian effects, non-simple geometry.

2.2 Unsteady uni-directional flows

Use eq. (2.3): $u_{yy} + u_{zz} - \frac{1}{\nu} u_t = -\frac{G(t)}{\mu}$

(a) The Rayleigh problem



Semi infinite fluid $y>0$ all at rest for $t<0$
The rigid plane $y=0$ is set into motion with speed U at $t=0$ and maintained.
All is 2D.

Solve $u'_{yy} = \frac{1}{\nu} u'_t$ (as no z dependance & no pressure gradient is put on) (2.11)

Hence just diffusion in terms of y.

BC's: $\begin{cases} u'(0,t) = U & \text{for } t>0 \\ u'(\infty,t) = 0 \\ u'(y,0) = 0 & \text{for } y>0 \end{cases}$

(no slip)
(we expect there to be no motion far away)
(initial condition)

Set $u' \equiv \frac{u}{U}$, so that u' is dimensionless. The above stays as is except that $u'(0,t) = 1$.

Can solve by Laplace transform in t, or Fourier half-transform in y

Instead, use dimensional analysis :- solution $u' = u'(y, t, \nu)$ must be non-dimensional.

But $\begin{cases} y \text{ has dimension } L \\ t \text{ has dimension } T \\ \nu \text{ has dimension } L^2/T \end{cases}$

$\Rightarrow$ the only non dimensional combination is $\frac{y^2}{\nu t}$ (or functions of this)

So try $u' = f(\eta)$, where $\eta \equiv \frac{y}{2(\nu t)^{1/2}}$, and then change variables

add for convenience $\rightarrow 2$

$(y,t) \to \eta \Rightarrow \frac{\partial}{\partial y} \to \frac{\partial \eta}{\partial y}\frac{\partial}{\partial \eta} = \frac{1}{2(\nu t)^{1/2}}\frac{\partial}{\partial \eta}$ and $\frac{\partial}{\partial t} \to \frac{\partial \eta}{\partial t}\frac{\partial}{\partial \eta} = \left(\frac{-\eta}{2t}\right)\frac{\partial}{\partial \eta}$

Hence (2.11) $\Rightarrow \frac{1}{4\nu \cancel{t}}\frac{\partial^2}{\partial \eta^2}u' = \frac{1}{\nu}\left(\frac{-\eta}{2\cancel{t}}\right)\frac{\partial}{\partial \eta}u'$

$$\Rightarrow \frac{d^2u'}{\partial \eta^2} = -2\eta\frac{du'}{d\eta}, \text{ an ODE.}$$

Solution : $\frac{du'}{d\eta} = c_1 e^{-\eta^2}$

$\therefore u' = c_2 + c_1 \int_0^{\eta} e^{-\hat{\eta}^2} d\hat{\eta}$ ← dummy variable

At $\eta = 0$, require the no slip condition $u' = 1 \Rightarrow \underline{c_2 = 1}$

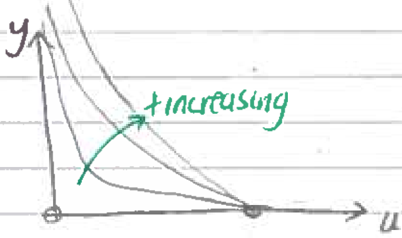
*As $\eta \to \infty$, require $u' \to 0 \Rightarrow c_2 + c_1 \underbrace{\int_0^{\infty} e^{-\hat{\eta}^2} d\hat{\eta}}_{\frac{\sqrt{\pi}}{2}} = 0$

$$\Rightarrow c_1 = -c_2 \cdot \frac{2}{\sqrt{\pi}} \cdot 1. \Rightarrow \underline{c_1 = \frac{2}{\sqrt{\pi}}}$$

*This one allows for $y \to \infty$ with $t > 0$ and $t \to 0$ with $y > 0$.

Hence $u' = \frac{u}{U} = 1 - \frac{2}{\sqrt{\pi}}\int_0^{\eta} e^{-\hat{\eta}^2} d\hat{\eta} = 1 - \underset{\text{error function}}{\mathrm{erf}(\eta)} = \underset{\text{complimentary error fn.}}{\mathrm{erfc}(\eta)}$ (2.12)

Picture :



Thickness of the layer that is moving?

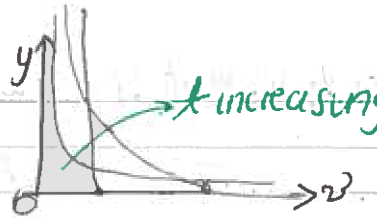
It's $O((\nu t)^{\frac{1}{2}}) = |y|$, because of the exponential in η.

(A thin fluid is easier to move the wall with because the wall moves easily against the wall, A thick fluid is more difficult to move and the wall has to move more of the thick fluid in order to move past).

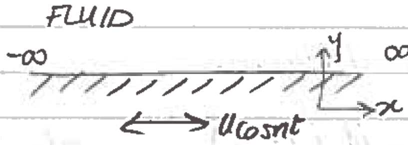
This is the Rayleigh layer.

Vorticity? $\omega = -\frac{\partial u}{\partial y} = \frac{-U}{\sqrt{\pi \nu t}} e^{-\eta^2}$

$\omega = \text{curl}\, u$



(b) Stokes' Layer



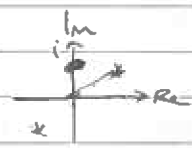
Oscillate the wall. Seek solution that persists (not transient) — can start it any way but after a long time same pattern.

Use (2.11) again since no pressure gradient $\Rightarrow$ solve $u_t = \nu u_{yy}$ (2.11)

Put $u' = \frac{u}{U} \Rightarrow u'_t = \nu u'_{yy}$ & $\begin{cases} u' = \cos nt & \text{at } y = 0 \\ u' \to 0 & \text{as } y \to \infty \end{cases}$

Put $\begin{cases} \cos nt = Re(e^{int}) \\ u' = Re(f(y) e^{int}) \end{cases}$ (as a trial)

$\Rightarrow$ solve $f(y)\, in\, \cancel{e^{int}} = \nu f''(y) \cancel{e^{int}}$



General solution is $\exp\left(\pm \left(\frac{in}{\nu}\right)^{\frac{1}{2}} y\right)$ Define $i^{\frac{1}{2}} = \frac{1+i}{\sqrt{2}}$

$\Rightarrow f(y) = Ae^{-(1+i)\left(\frac{n}{2\nu}\right)^{\frac{1}{2}} y} + \cancel{Be^{\ldots}}$ doesn't decay.

because we want the solution that decays only

BC $\Rightarrow A = 1$ Thus $u' = Re\left[e^{int - (1+i)\left(\frac{n}{2\nu}\right)^{\frac{1}{2}} y}\right]$

$\Rightarrow u = Ue^{-\left(\frac{n}{2\nu}\right)^{\frac{1}{2}} y} \cos\left\{nt - \left(\frac{n}{2\nu}\right)^{\frac{1}{2}} y\right\}$ (2.13)

ie. $u = Ue^{-y/\delta} \cos\{n(t - y/c)\}$

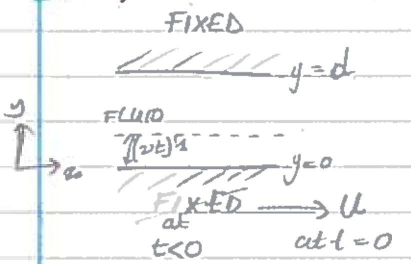
$\delta \equiv \sqrt{\frac{2\nu}{n}}$ and $c = \sqrt{2\nu n}$ — big (for persistent solution)

Velocity profile, at any given time t, is oscillatory in y but decays over distance of $O(\delta)$.

Pattern travels away from wall at speed c.

$\Rightarrow$ "Stokes' Layer": no motion outside it.

(c) Impulsive start of 1 wall parallel to another.



1. Expect early on, that a Rayleigh layer grows on $y=0$ wall, of thickness $O(\nu t)^{\frac{1}{2}}$

Unchanged for $(\nu t)^{\frac{1}{2}} \ll d$
ie $0<t \ll \frac{d^2}{\nu}$

2. For $t = O(d^2/\nu)$, the top wall has influence

3. For $t \gg d^2/\nu$, expect a steady flow, $u = \mathcal{U}\left(1-\frac{y}{d}\right)$: Couette flow

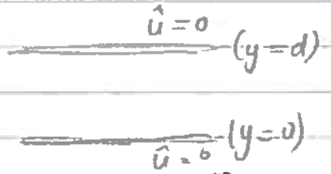
In detail, put $\frac{u}{\mathcal{U}} = \left(1-\frac{y}{d}\right) - \hat{u}(y,t)$ → deficit velocity / transient velocity (difference between the 2 velocities)

Then $\hat{u}_t = \nu \hat{u}_{yy}$ & $\begin{cases} \hat{u}(0,t) = 0 & \text{BC} \\ \hat{u}(d,t) = 0 & \text{BC} \\ \hat{u}(y,0) = 1 - \frac{y}{d} & \text{(because } u=0\text{). IC f/or } y>0) \end{cases}$

$n = 1, 2, \dots$

Seek a separable solution, $\hat{u} = f(y)g(t)$
$\Rightarrow fg' = \nu f''g \Rightarrow \underbrace{\frac{g'}{\nu g}}_{t's} = \underbrace{\frac{f''}{f}}_{y's} = \text{constant} = -k^2$, say

$g \propto e^{-k^2\nu t}$ & $f \propto \begin{cases} \sin(ky) \\ \cos(ky) \end{cases}$, BC $\Rightarrow \sin(ky)$ only (from first BC).
BC $\Rightarrow \sin(kd) = 0 \Rightarrow kd = n\pi$ (2nd BC)



Hence $\hat{u} = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi y}{d}\right) \exp\left(\frac{-n^2\pi^2\nu t}{d^2}\right)$
with the coefficients A_n to be found.

But IC $\Rightarrow 1 - \frac{y}{d} = \sum_{1}^{\infty} A_n \sin\left(\frac{n\pi y}{d}\right)$
↳ A fourier series problem - for $0<y\leq d$

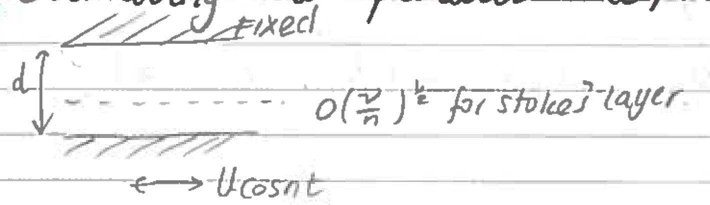
Multiply through by $\sin\left(\frac{m\pi y}{d}\right)$ and then integrate (0 to d)

$$\Rightarrow A_n = \frac{2}{d}\int_0^d \left(1-\frac{y}{d}\right)\sin\left(\frac{n\pi y}{d}\right) dy = \frac{2}{\pi n}$$

$$\Rightarrow \frac{u}{\mathcal{U}} = \left(1-\frac{y}{d}\right) - \frac{2}{\pi}\sum_{1}^{\infty} \frac{1}{n} \exp\left(\frac{-n^2\pi^2\nu t}{d^2}\right)\sin\left(\frac{n\pi y}{d}\right).$$

- For large times $\frac{\nu t}{d^2}$, get $\frac{u}{U} = \left(1-\frac{y}{d}\right) - \frac{2}{\pi}\exp\left(\frac{-\pi^2 \nu t}{d^2}\right)\sin\left(\frac{\pi y}{d}\right)$ approximately. Here this term is very small because $\pi^2 \approx 10 \Rightarrow e^{-10\frac{\nu t}{d^2}}$

(d) Oscillating wall parallel to a fixed wall.



Have $u_{yy} = \frac{1}{\nu} u_t$ & $u = \begin{cases} U\cos nt & \text{at } y=0 \\ 0 & \text{at } y=d. \end{cases}$

Try $\frac{u}{U} = \text{Real}\left(e^{int} f(y)\right) \Rightarrow f'' = \left(\frac{in}{\nu}\right) f$

$\Rightarrow f = e^{\pm \lambda y}$ with $\lambda \equiv \sqrt{\frac{n}{2\nu}}(1+i)$

BC's: $f(0) = 1$, $f(d) = 0$

So $f = A\sinh(\lambda(y-d)) + B\cosh(\lambda(y-d))$ s.t. $A\sinh(-\lambda d) = 1$.
(is not 0 when $y=d$)

$\Rightarrow \frac{u}{U} = -\text{Real}\left\{ e^{int} \frac{\sinh(\lambda(y-d))}{\sinh(\lambda d)} \right\}$. λ is complex here.

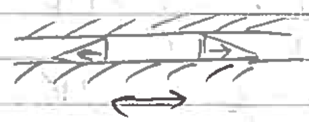
- If $|\lambda| d \gg 1$, get $-\text{Re}\left\{ e^{int} \cdot \frac{-\frac{1}{2}e^{-\lambda(y-d)}}{\frac{1}{2}e^{\lambda d}} \right\}$ $\quad \frac{nd^2}{\nu} \gg 1$

$\Rightarrow +\text{Re}\left\{ e^{int - \lambda y} \right\} \Rightarrow$ Stokes' Layer solution as we might expect.

- If $|\lambda| d \ll 1$, then $-\text{Re}\left\{ e^{int} \frac{\lambda(y-d)}{(\lambda d)} \right\}$

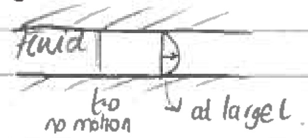
$\Rightarrow +\text{Re}\left\{ e^{int}\left(1-\frac{y}{d}\right) \right\}$

ie Couette flow for the instantaneous wall speed (quasi-steady) "as if steady" – as we might expect?



(a very thin channel compared with Stoke's layer)

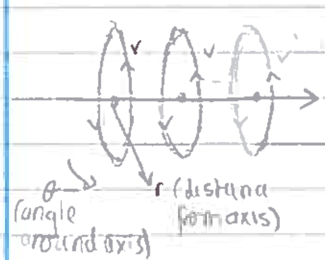
(e) A sudden or oscillating pressure gradient in a channel or pipe.
e.g channel: starting flow, constant p.g.



$u_{yy} = \frac{1}{\nu} u_t - G/\mu$

- Find the steady solution
- Then seek the deficit velocity.

2.3 With circular streamlines.



Try motion in circles, no axial or radial velocity

So in cylindrical polars (r, θ, z), $\underline{u} = (0, v, 0)$

Use Navier-Stokes equations in (r, θ, z) (see moodle)

z-momentum: gives $\frac{\partial p}{\partial z} = 0 \Rightarrow p$ independent of z

Continuity eq^n: $\frac{1}{r}\frac{\partial v}{\partial \theta} = 0$

v doesn't vary in the axial direction

r-momentum: $\frac{\rho v^2}{r} = \frac{\partial p}{\partial r} \Rightarrow \frac{\partial v}{\partial z} = 0$ and $\frac{\partial p}{\partial \theta}$ is independent of r.

since $\frac{\partial^2 p}{\partial r \partial \theta} = 0$.

above is a centrifugal (centripetal) effect.

θ momentum: $\boxed{\frac{\partial v}{\partial t} = \nu\left(\frac{\partial^2 v}{\partial r^2} + \frac{1}{r}\frac{\partial v}{\partial r} - \frac{v}{r^2}\right)}$ (2.15)

- an equation for $v(r,t)$ (since v is independent of θ, z)

[What about $\frac{1}{r}\frac{\partial p}{\partial \theta}$ in (2.15)? It is zero because in $v_t = -\frac{1}{r}\frac{p_\theta}{\rho} + \nu(v_{rr} + \ldots)$ (r,t)

$\Rightarrow p_\theta = p_\theta(r,t)$, but we said $\frac{\partial p}{\partial \theta}$ is independent of r (r,t)

$\Rightarrow p_\theta = p_\theta(t) = Q(t)$, say

$\Rightarrow p = \theta Q(t) + \text{constant} \Rightarrow Q = 0$, to make p periodic in θ.]

Note that the vorticity has only one component, $\omega = \frac{1}{r}\frac{\partial}{\partial r}(r v)$, from curl $\underline{u}$.

So (2.15) $\Rightarrow$ $\frac{\partial \omega}{\partial t} = \frac{1}{r}\frac{\partial}{\partial r}\left[r\left\{v_{rr} + \frac{1}{r}v_r - \frac{v}{r^2}\right\}\right]\nu$

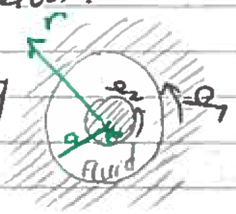
$$= \frac{\nu}{r}\left[v_{rr} + \frac{1}{r}v_r - \frac{v}{r^2} + r\left\{v_{rrr} + \frac{1}{r}v_{rr} - \frac{2v_r}{r^2} + \frac{2v}{r^3}\right\}\right]$$

$$= \nu\left[\omega_{rr} + \frac{1}{r}\omega_r\right]$$

$$\Rightarrow \frac{\partial \omega}{\partial t} = \nu \nabla^2 \omega \qquad (2.16)$$

Thus w satisfies the diffusion equation (v doesn't).

Examples: (a) Steady flow between 2 cylinders, rotating



$$(2.15) \Rightarrow v_{rr} + \frac{v}{r} - \frac{v}{r^2} = 0$$

for $v(r)$ s.t. $v = \begin{cases} a\Omega_1 & \text{at } r=a \\ b\Omega_2 & \text{at } r=b. \end{cases}$

Math1401.

Try $v \propto r^n \Rightarrow$ need $n(n-1)r^{n-2} + \frac{nr^{n-1}}{r} - \frac{r^n}{r^2} = 0$

$$n(n-1)r^{n-2} + nr^{n-2} + r^{n-2} = 0$$

$\Rightarrow n(n-1) + n - 1 = 0 \Rightarrow n = 1, -1$

So $v = Ar^{-1} + Br$

Then BC's $\Rightarrow \begin{cases} a\Omega_1 = Aa^{-1} + Ba & \text{(condition at } r=a) \\ b\Omega_2 = Ab^{-1} + Bb & \text{(condition at } r=b) \end{cases}$

$$\Rightarrow \begin{aligned} A &= (\Omega_1 - \Omega_2)/\left(\tfrac{1}{a^2} - \tfrac{1}{b^2}\right) \\ B &= (b^2\Omega_2 - a^2\Omega_1)/(b^2 - a^2) \end{aligned}$$

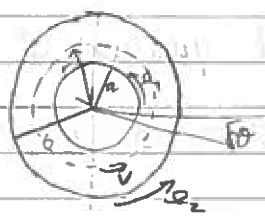
Hence v

Hence $w \overset{\text{defn}}{=} \underset{\text{earlier}}{\frac{1}{r}\frac{\partial}{\partial r}}(rv) = \frac{1}{r}\overset{\text{steady flow}}{\frac{d}{dr}}(A + Br^2) = 2B$, constant.

satisfies the heat equation (diffusion)

Steady flow ⇒ satisfies Laplaces equation.

Recap



Found $\begin{cases} v = Ar^{-1} + Br \\ w = 2B \end{cases}$

Couple exerted across any cylindrical surface of radius r is due to shear stress $\sigma_{r\theta}$ acting on element of length $r\,d\theta$

$$\longrightarrow \text{couple} = \int_{\theta=0}^{2\pi} (r\,d\theta)\,\sigma_{r\theta} \times r = 2\pi r^2 \sigma_{r\theta}$$

since $\sigma_{r\theta}$ is independant of θ.

In fact, $\sigma_{r\theta} = 2\mu e_{r\theta} = \mu\left(\frac{dv}{dr} - \frac{v}{r}\right)$ (quoted result)

$$\Rightarrow \text{couple} = 2\pi r^2\mu\left(\frac{dv}{dr} - \frac{v}{r}\right) = 2\pi r^2\mu\left(-Ar^{-2} + \cancel{B} - \frac{A}{r^2} - \cancel{B}\right) = -4\pi\mu A$$

independent of r.

Special cases:-

(i) $\Omega_1 = 0$, $a \to 0 \Rightarrow v = \Omega_2 r \Rightarrow$ solid-body rotation inside a cylinder.

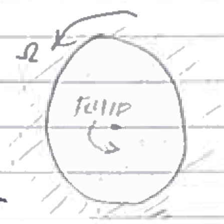
Has $A = 0$; couple $\xrightarrow{}$ moment, torque. $= 0$. (no force required to keep it moving.

(ii) $\Omega_2 \to 0$, $b \to \infty \Rightarrow v = \Omega_1 \frac{a^2}{r} \Rightarrow$ flow outside rotating cylinder.

Has $B = 0$, $A = \Omega_1 a^2$; couple $\neq 0$. "potential vortex"

(b) Unsteady flow.

eg. starting or stopping a steady flow, as in (a).



Take the case of stopping the solid-body rotation, How long does the flow go on for

Equation is (2.15): $v_t = \nu\left(v_{rr} + \frac{1}{r}v_r - \frac{v}{r^2}\right)$

BC's $v(a,t) = 0$, $v(0,t) = 0$ $\forall t > 0$ → (or solution is to be finite)

IC: $v(r,0) = \Omega r$

Try $v(r,t) = f(r)\,g(t)\exp\left(-\lambda^2 \frac{\nu t}{a^2}\right)$ ← we find $g(t)$ is $e^{-\lambda^2 \nu t / a^2}$

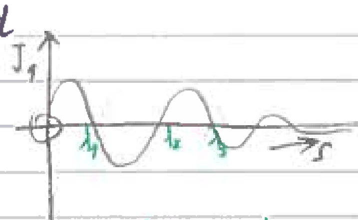
$\Rightarrow f \cdot \cancel{\exp} \cdot -\frac{\lambda^2 \cancel{\nu}}{a^2} = \cancel{\nu}\left(f'' + \frac{f'}{r} - \frac{f}{r^2}\right)\cancel{\exp}$

$\Rightarrow f'' + f'/r + \left(\frac{\lambda^2}{a^2} - \frac{1}{r^2}\right)f = 0$

Put $r = as \Rightarrow f_{ss} + \frac{f_s}{s} + \left(\lambda^2 - \frac{1}{s^2}\right)f = 0$

— Bessel equation of the first kind

The solution is $f = AJ_1(\lambda s) \cancel{+BY_1(\lambda s)}$



because we want solution to be finite at the origin

$BC\big|_{r=a} \Rightarrow J_1(\lambda) = 0 \Rightarrow \lambda = \lambda_1, \lambda_2, \lambda_3, \ldots$

So $v = \sum_{n=1}^{\infty} A_n J_1\left(\lambda_n \frac{r}{a}\right)\exp\left(-\lambda_n^2 \frac{\nu t}{a^2}\right)$

IC $\Rightarrow \sum_{n=1}^{\infty} A_n J_1\left(\lambda_n \frac{r}{a}\right) = \Omega r$ at $t = 0$

Use orthogonality of these functions: $\int_0^1 s J_1(\lambda_m s) J_1(\lambda_n s)\,ds = \begin{cases} 0 & m \neq n \\ \mu_n & m = n \end{cases}$

just a constant

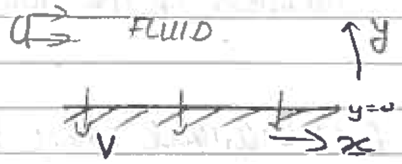
& $\mu_n \equiv \frac{1}{2}\left[J_1'(\lambda_n)\right]^2 \Rightarrow A_n = \frac{2}{(J_1'(\lambda_n))^2}\int_0^1 \Omega a s^2 J_1(\lambda_n s)\,ds = \dots$

$$= \frac{-2\Omega a}{\lambda_n J_0(\lambda_n)}$$

Here $\lambda_1 \simeq 3.83$, $\lambda_2 = 7.02$ $\Rightarrow$ a fast decaying exponential term

$\Rightarrow$ at time $\frac{\nu t}{a^2} = 1$ we have $\sim \exp(\overset{-(3.83)^2}{-13})$ approx in v, ie very small.

2.4 Flow past a porous plate



Rigid boundary with suction at $y=0$
Suction speed V, into the plate
Also have a longitudinal velocity (x component of velocity) U in the far field.

Seek a solution independant of z (a two dimensional solution)
Also independant of x (because assuming infinite plate & flow).

$\Rightarrow \underline{u} = (u(y,t), v(y,t), 0)$ $\quad p = p(y,t)$ ← no axial (x-wise) pressure gradient (longitudinal)

Continuity, $\hookrightarrow \frac{\partial v}{\partial y} = 0$

x Momentum: $\frac{\partial u}{\partial t} + 0 + v\frac{\partial u}{\partial y} + 0 = -\frac{1}{\rho}0 + \nu\frac{\partial^2 u}{\partial y^2}$

y Momentum: $\frac{\partial v}{\partial t} + 0 + v\frac{\partial v}{\partial y} + 0 = -\frac{1}{\rho}\frac{\partial p}{\partial y} + \nu\frac{\partial^2 v}{\partial y^2}$

z Momentum: $0 = 0$

BCs: $\begin{cases} u = 0 \quad v = -V \quad \text{at } y = 0 \quad \text{(no slip condition)} \\ u \to U \quad \text{as } y \to \infty \end{cases}$

Hence $v(y,t) = -V$ throughout

Then x-Momentum $\Rightarrow \frac{\partial u}{\partial t} - V\frac{\partial u}{\partial y} = \nu\frac{\partial^2 u}{\partial y^2}$

y-Momentum $\Rightarrow \frac{\partial p}{\partial y} = 0 \Rightarrow p = p(t)$ only

So we are only left with solving for $u(y,t)$.
Seek a steady state, $u = u(y)$ only $\Rightarrow \nu\frac{\partial^2 u}{\partial y^2} = -V\frac{\partial u}{\partial y}$

ie $\nu u'' = -Vu'$ ← no longer partial as dependent on y only

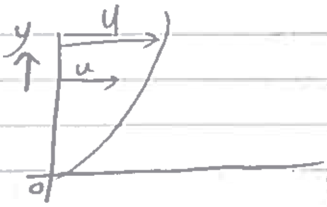
$\therefore u' = c_1 e^{-yV/\nu}$

$\therefore u = C_1 e^{-yV/\nu} + C_2$, where C_1, C_2 are constants to be determined

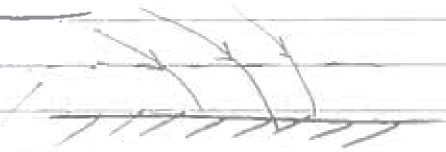
BCs ⇒ $C_2 = U$, $C_1 + C_2 = 0$

Hence $u = U\left[1 - e^{-yV/\nu}\right]$ longitudinal velocity profile.

ie



⇒ Streamlines are



order

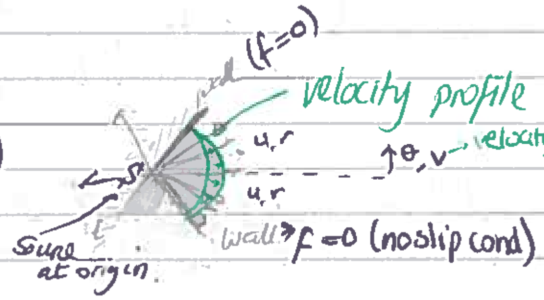
Exponential of yV/ν ⇒ "thickness" of the flow $= O(\nu/V)$

$(\underline{u}\cdot\underline{\nabla})u$.

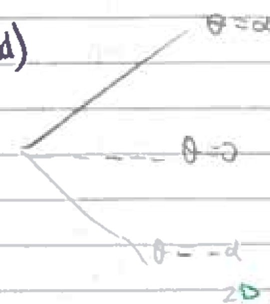
Also, this is the first example where the inertia terms are non zero.

2.5 Other Solutions

(a) Radial flow (Jeffrey-Hamel flow)



Can we get a solution between 2 diverging walls?



only need to know for lecture version

will be given in the exam

Try $u \propto \frac{1}{r}$, motivated by potential flow

Say $u = \frac{f(\theta)}{r}$, $v = 0$, steady flow 2D.

from N-S in polars (as z indep. as 2D)

Continuity: $\frac{1}{r}\partial_r(ru) + 0 + 0 = 0$ ① $= \text{div}\,\underline{u}$ $\left(= U_r + \frac{U}{r} + \frac{1}{r}V_\theta + W_z\right)$

r Momentum: $u\frac{\partial u}{\partial r} + 0 + 0 = -\frac{1}{\rho}\frac{\partial p}{\partial r} + \nu\left\{\partial_r^2 + \frac{1}{r}\partial_r - \frac{1}{r^2} + \frac{1}{r^2}\partial_\theta^2\right\}u$ ②

θ momentum: $0 + 0 + 0 = -\frac{1}{\rho r}\frac{\partial p}{\partial \theta} + \frac{2\nu}{r^2}\left\{\frac{\partial u}{\partial \theta}\right\}$ ③

z Momentum: $0 = 0$

① confirms $u = \frac{f(\theta)}{r}$

Eliminate p between ②, ③ by taking ∂_θ(②) $-$ ∂_r(③)

$$\Rightarrow \partial_\theta(uu_r) = -\frac{1}{\rho}p_{r\theta} + \frac{1}{\rho}p_{\theta r} + \nu\left\{u_{rr\theta} + \frac{1}{r}u_{r\theta} - \frac{1}{r^2}u_\theta + \frac{1}{r^2}u_{\theta\theta\theta} - 2\partial_r\left[\frac{f'(\theta)}{r^2}\right]\right\}$$

$$\Rightarrow \partial_\theta\left(\frac{-f^2}{r^3}\right) = \nu\partial_\theta\left\{\cancel{\frac{2f}{r^3}} \cancel{-\frac{f}{r^3}} \cancel{-\frac{f}{r^3}} + \frac{f''}{r^3} + \frac{4f}{r^3}\right\}$$

⇒ ode for $f(\theta)$ $\quad -2ff' = \nu(f''' + 4f')$

ie. $\nu f''' + 2ff' + 4\nu f' = 0$ (4)

Normalise: $\frac{\theta}{\alpha} \equiv \eta$, $\frac{f}{f_{max}} \equiv F$

Expect/guess that maximum value of velocity is at $\theta = 0$
ie. $f_{max} = f(0)$

$$\Rightarrow \nu F''' \cdot \frac{f_{max}}{\alpha^3} + 2FF' \frac{f_{max}^2}{\alpha} + 4\nu F' \frac{f_{max}}{\alpha} = 0$$

$\times \frac{\alpha^3}{\nu f_{max}}$

$$\Rightarrow F''' + 2(\alpha Re) FF' + 4\alpha^2 F' = 0 \quad (5)$$

Here $Re = \alpha \frac{U r}{\nu} = \frac{\alpha f_{max}}{\nu}$ This is a Reynold's number. $\to f_{max}$
↳ a measure of the strength of the flow relative to the viscous forces

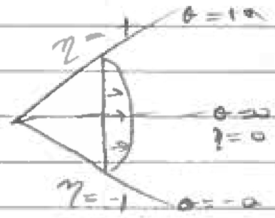
Two parameters which will affect the solution: α, Re

BCs are $F(\pm 1) = 0$ (no slip condition at $\pm\alpha$)
$F(0) = 1$.

Integrate (5) wrt $\eta \Rightarrow F'' + (\alpha Re)F^2 + 4\alpha^2 F = c_1$

Integrate wrt $F \Rightarrow \frac{1}{2}F'^2 + (\alpha Re)\frac{F^3}{3} + 2\alpha^2 F^2 = c_1 F + c_2$

At $\eta = 0$, $F = 1$ & $F' = 0$ (for max)



$\Rightarrow$ one condition on c_1, c_2

dummy variable for F

Hence find $$\eta = \int_F^1 \frac{d\tilde{F}}{(1-\tilde{F})^{1/2}\{\frac{2}{3}(\alpha Re)(\tilde{F}^2 + \tilde{F}) + 4\alpha^2 \tilde{F} + c_3\}^{1/2}}$$

this term gets ignored in next step because Re term is very large in comparison

Impose $\eta = 1$ at $F = 0$ to determine c_3.

At large Reynolds numbers, Re: $F = 1 - \epsilon \hat{F}$ say, with $\epsilon \ll 1$

$$\Rightarrow \eta \doteq \int_{\hat{F}}^{0} \frac{-\epsilon\, d\hat{F}}{(\epsilon \hat{F})^{1/2}\{\frac{2}{3}(\alpha Re)\cdot 2 + c_3\}^{1/2}}$$

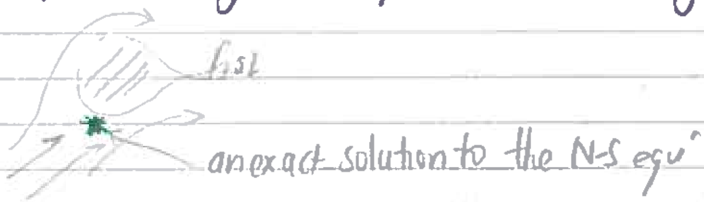
$$= \frac{2\epsilon^{1/2}\hat{F}^{1/2}}{\{\cdots\}^{1/2}}$$

goes from 0 to 1

so $\eta \sim O(1)$ (order 1)

$\Rightarrow c_3 = -\frac{4}{3}\alpha Re$ to leading order

(b) At a front stagnation point on a body



Chapter 3 - General Results

3.1 Vorticity

How does vorticity ($\underline{\omega} = \text{curl}\, \underline{u}$) spread in a flow?

(A) Consider 2D flow: N/S equations are $u_x + v_y = 0 \Rightarrow u = \psi_y,\ v = -\psi_x$

xM: $u_t + uu_x + vu_y = -p_x/\rho + \nu\nabla^2 u$ (3.1)

yM: $v_t + uv_x + vv_y = -p_y/\rho + \nu\nabla^2 v$

Eliminate p by cross differentiation $\Rightarrow (u_{yt} - v_{xt})\ \partial_y(uu_x + vu_y) - \partial_x(uv_x + vv_y)$
$= 0 + \nu\nabla^2(u_y - v_x)$

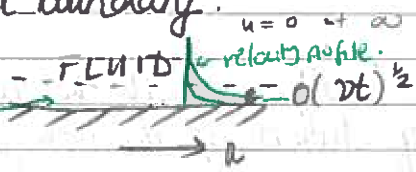
$$\Rightarrow \frac{D}{Dt}(u_y - v_x) = \nu\nabla^2(u_y - v_x)$$

But $u_y - v_x = \nabla^2\psi = -\omega$, since $\underline{\omega} = \text{curl}\,\underline{u} = (0, 0, \underbrace{-\nabla^2\psi}_{\omega})$ in 2D

So $$\frac{D\omega}{Dt} = \nu\nabla^2\omega \qquad (3.2)$$

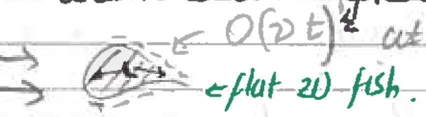
If inviscid, ω is conserved for each fluid particle.
Hence if $\omega = 0$ at $t = 0$ $\forall x, y, t$ (everywhere), then ω
$\Rightarrow$ potential flow ($\nabla^2\psi = 0$) (3.3)

If viscous, a similar argument almost works.
So if $\omega = 0$ at $t = 0$ then (3.2) $\Rightarrow \omega = 0\ \forall t > 0$
But this isn't true at any solid boundary.
eg. Rayleigh problem shows



$\Rightarrow$ vorticity ω ($\propto u_y$ here)
is infinite at $y = 0$ at $t = 0$
$\Rightarrow$ the boundary conditions alter the vorticity.

(B) At what rate does spreading of ω take place?



What about at $t = O(1)$?
Ratio of terms in (3.2) is $\dfrac{\frac{D\omega}{Dt}}{(\nu\nabla^2\omega)} \Rightarrow$ orders $\dfrac{|\underline{u}.\nabla\omega|}{|\nu\nabla^2\omega|}$

Estimate $|\underline{u}| = U$, $|\underline{\nabla}| = \frac{1}{L}$, $|\nu| = \nu$

$\Rightarrow$ we get $\dfrac{UL^{-1}|\omega|}{\nu L^{-2}|\omega|}$

$\Rightarrow$ ratio is $\dfrac{UL}{\nu}$

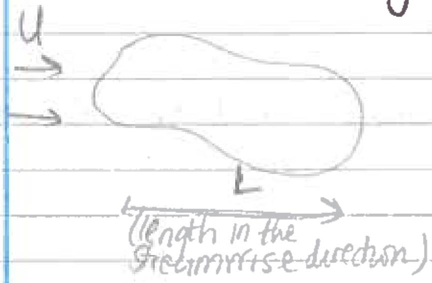
(3.4) $\boxed{\dfrac{UL}{\nu} = \text{Reynolds number}}$ (3.5)

3.2 Reynolds number

(a) Dynamical similarity

Effects of altering ρ, μ, U, L are what??

- For a 2D solid body with length scale L in a stream of speed U:



Equ^n's are
$$\begin{cases} \frac{\partial \underline{u}}{\partial t} + (\underline{u}\cdot\nabla)\underline{u} = -\nabla p + \nu\nabla^2\underline{u} & (3.6) \\ \nabla\cdot\underline{u} = 0 \end{cases}$$

Make variables dimensionless : Set $\frac{\underline{u}}{U} \equiv \hat{\underline{u}}$, $\frac{p}{\rho U^2} \equiv \hat{p}$, $\frac{tU}{L} \equiv \hat{t}$, $\frac{x_i}{L} \equiv \hat{x}_i$

so we can decide if a body is large or small (relative size)

and substitute into (3.6) $\Rightarrow$ $\frac{\partial\hat{\underline{u}}}{\partial\hat{t}}\cdot\frac{U^2}{L} + (\hat{\underline{u}}\cdot\hat{\nabla})\hat{\underline{u}}\cdot\frac{U^2}{L} = -\frac{\hat{\nabla}\hat{p}}{\rho}\frac{\rho U^2}{L} + \nu\hat{\nabla}^2\hat{\underline{u}}\frac{U}{L^2}$ (3.7)

$\hat{\nabla} = (\partial_{\hat{x}}, \partial_{\hat{y}}, \partial_{\hat{z}})$

i.e. $\frac{\partial\hat{\underline{u}}}{\partial\hat{t}} + (\hat{\underline{u}}\cdot\hat{\nabla})\hat{\underline{u}} = -\hat{\nabla}\hat{p} + \frac{1}{R}\hat{\nabla}^2\hat{\underline{u}}$ (3.8)

Here $R \equiv \frac{UL}{\nu}$ is dimensionless, the Reynolds no. again.

The solution now depends on
(i) $\hat{x}_i$, $\hat{t}$
(ii) R
(iii) non dimensional geometry & initial cond

- Any two flows with the same (i), (ii), (iii) are dynamically similar.
- Every solution of the non-dimensional problem (for given R) gives us a triply infinite family of solutions, because the same R comes from varying U, L, ρ, μ s.t. $\frac{UL\rho}{\mu} = R$
- Modelling experimentally :

scale model, with L decreased by α ($\approx \frac{1}{10}$). wing into a wind tunnel

R must be the same $\Rightarrow \frac{UL}{\nu}$ must be the same.

$\Rightarrow$ must increase U by $\frac{1}{\alpha}$ as ν will stay the same because we want to use air (cheap)

$\Rightarrow$ v. high speeds & noisy!

- Another: tiny particle in blood or water

$\Rightarrow$ increase L by β

Do this by altering the fluid to make ν increase by β.

$\Rightarrow$ Same $\frac{UL}{\nu}$ eg. use oil (has an increased velocity)

(b) Effects of R

$$\frac{D\hat{u}}{Dt} = -\hat{\nabla}\hat{p} + \frac{1}{R}\hat{\nabla}^2\hat{u} \rightarrow \frac{\text{inertia forces}}{\text{viscous forces}} = O(R)$$

all are $O(1)$

see chapt.

Hence if $R \ll 1$ then expect to solve $0 = -\hat{\nabla}\hat{p} + \frac{1}{R}\hat{\nabla}^2\hat{u}$

If $R \gg 1$ then expect to solve $\frac{D\hat{u}}{Dt} = -\hat{\nabla}\hat{p}$, except in some thin layers (Rayleigh, Stokes)

→ 4th yr course (Boundary layers)

3.3 Other numbers

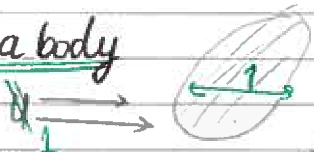
- Strouhal/Stokes/Womersley number $= a\sqrt{\frac{\Omega}{\nu}}$

- Froude number $= \frac{U}{\sqrt{gh}}$

- Mach number $= \frac{U}{c_s}$ ← speed of sound

not examinable

3.4 Drag on a body



General result: $D = Q$ → vol of velocity deficit

drag

$$Q = \int_{\text{far field}} (1-u)\, dA$$

$O(\frac{1}{r})$ [r/10]

Comes from integrating the Navier-stokes eq^n over the space involved.

Chapter 4: Flow at low Reynolds numbers   ie Slow flow / Stokes flow.

$R \ll 1 \Rightarrow$ neglect the inertia terms $\frac{Du}{Dt}$

$\Rightarrow$ Stokes equation instead of Navier–Stokes

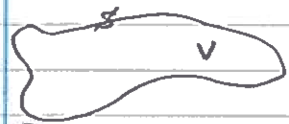
$\hookrightarrow \boxed{\nabla p = \mu \nabla^2 \underline{u}}$ (4.1)

↳ linear equation because inertia terms have been eliminated

4.1 Two theorems

COULD BE ASKED TO PROVE THIS

(i) Uniqueness. Volume V, surface S.
If $(\underline{u}, p)$ and $(\underline{u}', p')$ are two solutions, each satisfying (4.1) and $\operatorname{div} \underline{u} = 0$, and if $\underline{u} = \underline{u}'$ on the boundary S, then $\underline{u} = \underline{u}'$ in V.



Proof: Let e_{ij}, e'_{ij} be the rate-of-strain tensors

Then $\int_V (e_{ij} - e'_{ij}) e_{ij}\, dV = \int_V \left\{ \frac{1}{2}\frac{\partial}{\partial x_j}(u_i - u'_i) + \frac{1}{2}\frac{\partial}{\partial x_i}(u_j - u'_j) \right\} e_{ij}\, dV$

added this in to make working easier

$= \int_V \frac{\partial}{\partial x_j}(u_i - u'_i)\, e_{ij}\, dV$ (since $e_{ij} = e_{ji}$)

By parts $\Rightarrow \underbrace{\int_S n_j (u_i - u'_i) e_{ij}\, dS}_{=0} - \int_V (u_i - u'_i) \frac{\partial}{\partial x_j} e_{ij}\, dV$

(using the Divergence thm)

(because $u_i = u'_i$ on the surface)

$\Rightarrow -\int_V (u_i - u'_i) \underbrace{\frac{\partial}{\partial x_j} e_{ij}}\, dV$ (defn in Chap 1)

↳ viscous term in the N-S eqns (except for a factor μ)

$\Rightarrow -\frac{1}{2\mu}\int_V (u_i - u'_i) \frac{\partial p}{\partial x_i}\, dV$ (using the fact that each equation satisfies the stoke equations (4.1))

$\therefore \int_V (e_{ij} - e'_{ij}) e_{ij}\, dV = -\frac{1}{2\mu}\int_S (u_i - u'_i) p n_i\, dS$ from (4.1)

↳ div them

↳ using divergence theorem and $\operatorname{div} \underline{u} = 0$.

$\therefore \int_V (e_{ij} - e'_{ij}) e_{ij}\, dV = 0$ (because $u_i = u'_i$ on S).

Similarly, $\int_V (e'_{ij} - e_{ij}) e'_{ij}\, dV = 0$, Hence $\int_V (e_{ij} - e'_{ij})^2\, dV = 0$

$\Rightarrow e_{ij} = e'_{ij}$ in V.

Integration $\Rightarrow$ $\underline{u} = \underline{u}'$ in V $\quad \left(\int e_{ij} = \int e_{ij}' \right)$

(ii) Minimum dissipation $\Rightarrow$ IT IS DOUBTFUL THAT YOU WOULD BE ASKED TO PROVE THIS

Given unique solution (just shown) of (4.1) & div $\underline{u} = 0$ in V (& $\underline{u}$ given on S)
suppose there is another motion in which $\nabla p' \neq \mu \nabla^2 \underline{u}'$, div $\underline{u}' = 0$
and satisfies again, $\underline{u}' = \underline{u}$ on S.
Then $\int (e_{ij} - e_{ij}') \, e_{ij} \, dV = 0$ (†) (ie everything until step ⊕ holds)

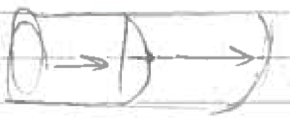
Rate of dissipation of energy in flow $(\underline{u}', p')$ is : $\Phi' = 2\mu \int e_{ij}' e_{ij}' \, dV$

$$\Phi' = 2\mu \int_V \left\{ (e_{ij}' - e_{ij})(e_{ij}' - e_{ij}) + \underbrace{2(e_{ij}' - e_{ij}) e_{ij}}_{=0 \text{ from } (†)} + e_{ij} e_{ij} \right\} dV \quad \text{completing the sq.}$$

$$\Phi' \geq 2\mu \int_V e_{ij} e_{ij} \, dV$$

$$\Phi' \geq \Phi$$

ie. $\boxed{\text{the rate of dissipation of energy in Stokes flow} \leq \text{the rate of dissipation of energy in any other flow.}}$

eg HPF  is a very efficient way to transport fluid

4.2 Stokes flow when a stream function exists

Eqn of motion is $\nabla p = \mu \nabla^2 \underline{u}$ (4.1).

Take curl $\Rightarrow$ $\nabla^2 \underline{\omega} = 0$ (since $\underline{\omega} = \text{curl}\, \underline{u}$) (4.2)

div $\Rightarrow$ $\nabla^2 p = 0$ (since div $\underline{u} = 0$) (4.3)

Also, div $\underline{u} = 0$ $\Rightarrow$ $\exists$ vector potential $\underline{A}$ s.t. $\underline{u} = \nabla \times \underline{A}$.
& $\nabla \cdot \underline{A} = 0$

Then $\underline{\omega} = \nabla \times (\nabla \times \underline{A}) = -\nabla^2 \underline{A}$ (vector triple product)
& so (4.2) $\Rightarrow$ $\nabla^4 \underline{A} = 0$ (4.4)

This simplifies in two cases :

(i) 2D flow Here $\underline{A} = \psi(x, y)\, \underline{k}$ $\therefore$ $\underline{u} = \text{curl}\, A$
$= (\psi_y, -\psi_x, 0)$

Then $\underline{\omega} = \text{curl}\, \underline{u} = (0, 0, -\nabla^2 \psi)$

$\nabla^2\omega = \nabla^2(-\nabla^2\psi \underline{k}) = -\nabla^4\psi\underline{k}$

$\Rightarrow$ solve the scalar equation $\nabla^4\psi = 0$ (4.5)

In cartesians, $(\partial_x^2 + \partial_y^2)^2\psi = 0$

[Check: $p_x = \mu\nabla^2 u$ & $p_y = \mu\nabla^2 v \Rightarrow 0 = \mu\nabla^2(u_y - v_x)$
$\Rightarrow 0 = \nabla^2(\psi_{yy} + \psi_{xx})$
$\Rightarrow \nabla^4\psi = 0.$]

In plane polars, $\left(\partial_r^2 + \frac{1}{r}\partial_r + \frac{1}{r^2}\partial_\theta^2\right)^2\psi(r,\theta) = 0$

(ii) Axisymmetric flow.

In general orthogonal coordinates (q_1, q_2, ϕ), use $(ds)^2 = h_1^2(dq_1) + h_2^2(dq_2)^2 + h_3^2(d\phi)^2$
(hs will vary from coordinate system to coordinate system)

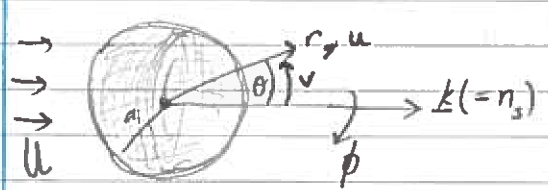
Here $\underline{A} = (0, 0, \frac{1}{h_3}\psi(q_1, q_2))$ ← no dependance on ϕ in axisymmetric cases.

This leads to $D^4\psi = 0$ (4.6).

Eg. In spherical (r,θ,ϕ) polars, $h_1 = 1$, $h_2 = r$, $h_3 = r\sin\theta$
and $D^2 = \partial_r^2 + \frac{\sin\theta}{r^2}\partial_\theta\left(\frac{1}{\sin\theta}\partial_\theta\right)$

Eg. In cylindrical polars (r, z, ϕ), $h_1 = 1$, $h_2 = 1$, $h_3 = r$.
and $D^2 = \partial_r^2 - \frac{1}{r}\partial_r + \partial_z^2$

4.3 Flow past a sphere (slow/stokes flow & steady).



BCs: $\underline{u} = 0$ on $r = a$ (no slip)
$\underline{u} \to U\underline{k}$ as $r \to \infty$

Seek $\underline{u}, p$ for $r \geq a$
Axisymmetric $\Rightarrow \psi(r,\theta)$ satisfies $D^4\psi = 0$ (assumption) with $u = \frac{1}{r^2\sin\theta}\frac{\partial\psi}{\partial\theta}$, $v = \frac{-1}{r\sin\theta}\frac{\partial\psi}{\partial r}$.
↳ can only look for a stream function because it is axisymmetric. Can't do this for full 3D problem.

So the BC's are 1. on $r = a$, $\psi =$ constant (WLOG, set to zero.) & $\frac{\partial\psi}{\partial r} = 0$

2. As $r \to \infty$, $\psi \sim \frac{1}{2}Ur^2\sin^2\theta$ (since $u \to U\cos\theta$, $v \to -U\sin\theta$ as $r \to \infty$).

Try seperable solution $\psi = f(r)\sin^2\theta$ (and then appeal to uniqueness!).

$$\Rightarrow D^2\psi = f''\sin^2\theta + \frac{\sin\theta}{r^2}\partial_\theta\left(\frac{1}{\sin\theta}f\cdot 2\sin\theta\cos\theta\right)$$

$$= f''\sin^2\theta + \frac{\sin\theta}{r^2}f\cdot -2\sin\theta$$

$$= \underbrace{\left(f'' - \frac{2f}{r^2}\right)}_{F}\sin^2\theta$$

Repeat $\Rightarrow D^4\psi = \left(F'' - \frac{2F}{r^2}\right)\sin^2\theta$, where $F \equiv f'' - \frac{2f}{r^2}$

$\Rightarrow$ Solve $F'' - \frac{2F}{r^2} = 0.$ (ODE)

Try $F \propto r^n$: $n(n-1)r^{n-2} - \frac{2r^n}{r^2} = 0 \Rightarrow n^2 - n - 2 = 0$
$\Rightarrow n = 2, -1$

So $F = C_1 r^2 + C_2 r^{-1}$

Then solve $f'' - \frac{2f}{r^2} = C_1 r^2 + C_2 r^{-1}$ for $f(r)$.

$\Rightarrow f = Ar^4 + Br^2 + Cr + Dr^{-1}$ is the general solution.

So find A, B, C, D.

Boundary conditions: $\begin{cases} r \to \infty, f \sim \frac{1}{2}Ur^2 \Rightarrow A = 0, B = \frac{1}{2}U \\ r = a, f = f' = 0 \Rightarrow \frac{1}{2}Ua^2 + Ca + Da^{-1} = 0 \end{cases}$

$$Ua + C - Da^{-2} = 0$$

Hence $C = -\frac{3}{4}Ua$ $D = \frac{1}{4}Ua^3$

Hence $\psi = \frac{1}{4}Ua^2\left(\frac{2r^2}{a^2} - \frac{3r}{a} + \frac{a}{r}\right)\sin^2\theta$ (4.7)

$\Rightarrow u, v, \omega, p_n$ etc. can be obtained

$$u = U\cos\theta\left(1 - \frac{3a}{2r} + \frac{a^3}{2r^3}\right)$$
$$v = -U\sin\theta\left(1 - \frac{3a}{4r} - \frac{a^3}{4r^3}\right)$$ (4.8)

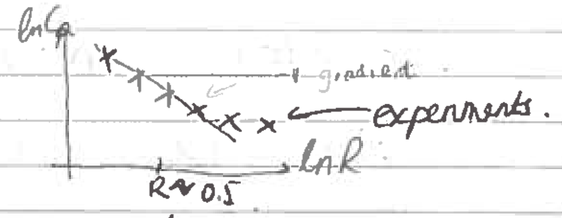
Vorticity ω (one component) (only ϕ) $= -\frac{D^2\psi}{r^2\sin\theta} = -\frac{3}{2}\frac{Ua\sin\theta}{r^2}$

– Symmetric about $\theta = \frac{\pi}{2}$ – vorticity as ~~far~~ much ahead as behind the sphere

Pressure $p = p_0 - \frac{3}{2}\mu\frac{Ua}{r^2}\cos\theta$.

Drag on sphere $\bar{D} = 6\pi\mu U a$ (long proof) (4.9)

$\Rightarrow$ Drag coefficient $(C_D) \equiv \frac{\bar{D}}{\frac{1}{2}\rho U^2 \pi a^2} = \frac{3\pi}{2R}$



Next, check on the neglect of inertia forces from (4.8)

$\underline{u} = U\underline{k}\,(1 + O(a/r))$ as $r \to \infty$

multiplication

$\Rightarrow \begin{cases} \text{inertial forces } (\underline{u}\cdot\nabla)\underline{u} = O\left(U * \frac{Ua}{r^2}\right) \\ \text{viscous forces } \nu\nabla^2\underline{u} = O\left(\nu * \frac{Ua}{r^3}\right) \end{cases}$ from the correction term differentiated $\to O(a/r)$.

$\frac{\text{Inertia}}{\text{viscous}} = O\left(\frac{U^2 a}{r^2} * \frac{r^3}{\nu U a}\right) = O\left(\frac{Ur}{\nu}\right)$ $\qquad R \equiv \frac{Ua}{\nu}$ (Reynold's no.)

$= O\left(\frac{R\nu}{a} * \frac{r}{\nu}\right)$

$= O\left(\frac{Rr}{a}\right)$

Hence inertia comes back into play at large distances s.t. $\frac{r}{a} = O\left(\frac{1}{R}\right) \gg 1$.

There $\underline{u} = U\underline{k}$ + small effects.

4.4 The outer flow

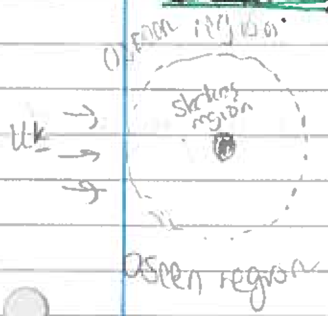
For $r = O\left(\frac{a}{R}\right)$, write $\underline{u} = U\underline{k} + \underline{u}'$ st $|\underline{u}'| \ll U$ and approximate on that basis.

& Navier-Stokes equations become $(\underline{U}\cdot\nabla)\underline{u}' = -\frac{\nabla p}{\rho} + \nu\nabla^2\underline{u}'$

$\Rightarrow (\underline{U}\cdot\nabla)\underline{u} = -\frac{\nabla p'}{\rho} + \nu\nabla^2\underline{u}$ ← can get rid of prime because you are just adding to the constant. $\quad (\underline{U} = U\underline{k})$ (x direction) (4.10)

↓

- Oseen's Equation (still linear equation).



BCs are; $\underline{u} \to \underline{U}$ as $r \to \infty$

& solution of (4.10) must match (4.8) - Stokes solution as $\frac{r}{a} \to O\left(\frac{1}{R}\right)$.

To solve :- take curl $\to (\underline{U}\cdot\nabla)\underline{\omega} = \nu\nabla^2\underline{\omega}$

i.e. $U\frac{\partial \underline{\omega}}{\partial x} = \nu\nabla^2\underline{\omega}$ (*) to solve.

In Cylindrical polars (σ, x, ϕ) 

$$\begin{cases} \omega = \left(0, 0, -\frac{1}{\sigma} D^2\psi\right) \\ \nabla^2 \omega = \left(0, 0, -\frac{1}{\sigma} D^4\psi\right) \end{cases}$$

$$(*) \Rightarrow -\frac{U}{\sigma}\frac{\partial}{\partial x}\left(D^2\psi\right) = -\frac{\nu}{\sigma} D^4\psi$$

$$\Rightarrow \left(D^2 - k\frac{\partial}{\partial x}\right) D^2\psi = 0 \qquad (4.11)$$

where $k \equiv \frac{U}{\nu}$

Put $D^2\psi = e^{kx/2}\chi \Rightarrow \left(D^2 - \frac{k^2}{4}\right)\chi = 0 \quad (**)$ for this equation

$\left[\text{Recall } D^2 \equiv \partial_r^2 - \frac{1}{r}\partial_r + \partial_x^2\right]$

Return to spherical polars (r, θ, ϕ): BC's are $\chi \xrightarrow{①} 0$ as $r \to \infty$

② $\cancel{e^{\frac{kx}{2}}}\chi \to (D^2\psi)_{Stokes}$ + smaller terms, as $r \to O(a)$.

② $\chi \simeq \frac{3Ua}{2r}\sin^2\theta$, as $r \to O(a)$

Seek separable solution again, $\chi = g(r)\sin^2\theta$, wrt $(**)$

$$\Rightarrow g'' - \frac{2g}{r^2} - \frac{k^2 g}{4} = 0 \qquad \text{Solve this!}$$

Put $g = e^{-\frac{kr}{2}}G(r)$ $\qquad g = A\left(1 + \frac{2}{kr}\right)e^{-\frac{kr}{2}} + B\left(1 - \frac{2}{kr}\right)e^{\frac{kr}{2}}$

Then BCs $\Rightarrow B = 0$ & $A = \frac{3}{4}Uak$

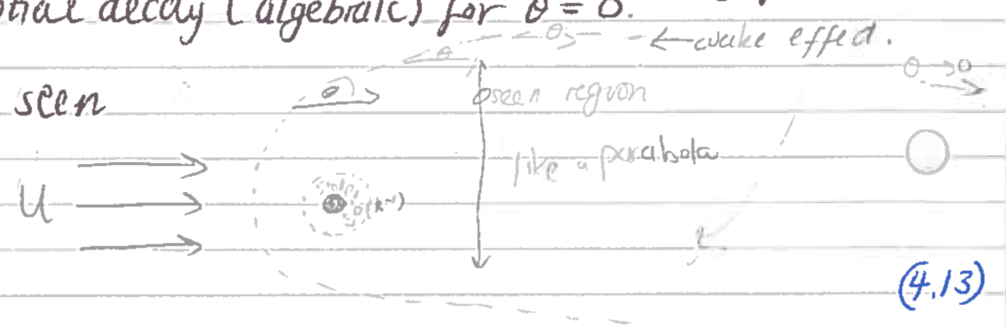
$$\therefore D^2\psi = \frac{3}{4}kUa\left(1 + \frac{2}{kr}\right)e^{-\frac{kr}{2}(1-\cos\theta)} \cdot \sin^2\theta. \qquad (4.12)$$

$\Rightarrow$ vorticity (where the viscous effects are concentrating)

$$\omega = -\frac{D^2\psi}{r\sin\theta} = -\frac{3}{4}kUa\left(\frac{1}{r} + \frac{2}{kr^2}\right)e^{-\frac{kr}{2}(1-\cos\theta)} \cdot \sin\theta$$

$\Rightarrow$ behaviour as $r \to \infty$ depends on θ: exponential decay for $\theta \neq 0$ but no exponential decay (algebraic) for $\theta = 0$.

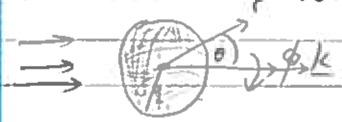
$\Rightarrow$ wake effect is seen



(4.13)

4.5 Non Dimensional thinking (only have one parameter R to worry about).

Reconsider Sections 4.3 & 4.4 in terms of non dimensional quantities



From (3.8) in a clearer notation

$$(\underline{u}\cdot\underline{\nabla})\underline{u} = -\underline{\nabla}p + \underbrace{\frac{1}{R}\nabla^2\underline{u}}_{\text{non dimensional Navier Stokes equations}} \qquad (4.14)$$

and BCs : $\begin{cases} \underline{u} \to \underline{k}, & r \to \infty \\ \underline{u} = 0, & r = 1. \end{cases}$

Want solution for $R \ll 1$.

- Inner region ($r \sim 1$) has approximations $\begin{cases} \underline{u} = \underline{u}_0 + (\text{smaller terms}) \\ p = \frac{p_0}{R} + (\text{smaller terms}) \end{cases}$ (4.15)

diverge (but need to cancel)

$$(4.14) \Rightarrow \underbrace{O(1)}_{\text{small}} = -\underline{\nabla}p_0 + \underbrace{\frac{1}{R}\nabla^2\underline{u}_0}_{\text{huge}}$$

make this $-\frac{\underline{\nabla}p_0}{R}$ to make equation balance

$\Rightarrow$ solve $0 = -\underline{\nabla}p_0 + \nabla^2\underline{u}_0$, at leading order.

$\Rightarrow$ work exactly as in 4.3 but with (a, U) replaced by $(1, 1)$

↳ find that p_0 decays like r^{-2} at large r, (and other findings for drag, vorticity etc).

- Outer region ($r = R^{-1}\hat{r}$) (Oseen) has $\begin{cases} \underline{u} = \underline{k} + R\hat{\underline{u}}_0 + R^2\hat{\underline{u}}_1 \ldots \\ p = R\hat{p}_0 + \ldots \end{cases}$ (4.16)

$[\underline{\nabla} = R\hat{\underline{\nabla}}]$ [illegible] $O(1)$.

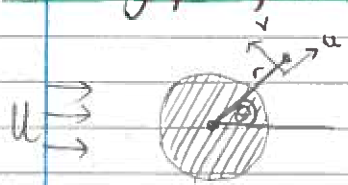
Now substituting into (4.14) $\Rightarrow$ $\hat{\underline{\nabla}}\cdot\hat{\underline{u}}_0 = 0$

$$(\underline{k}\cdot\hat{\underline{\nabla}})\hat{\underline{u}}_0 \cdot \cancel{R^2} = -\hat{\underline{\nabla}}\hat{p}_0 \cancel{R^2} + \frac{1}{\cancel{R}}\hat{\nabla}^2\hat{\underline{u}}_0 \cancel{R^2 R}$$

$\Rightarrow$ Oseen's equation

$\Rightarrow \begin{cases} \text{work as in §4.4 with } (a, U) \to (1, 1) \\ \text{match to (4.15) as } \hat{r} \to 0^+ \end{cases}$

4.6 Steady flow past a circular cylinder.



Plane polars (r, θ) in 2 dimensions.

$$\Rightarrow \boxed{\nabla^2 \equiv \partial_r^2 + \frac{1}{r}\partial_r + \frac{1}{r}\partial_\theta^2}$$ learn this!

We are to solve $\nabla^4\psi = 0$.

BC : $\begin{cases} \psi = \psi_r = 0 & \text{at } r = a \\ \psi \sim Ur\sin\theta & \text{as } r \to \infty \end{cases}$

Recall $u = \frac{\psi_\theta}{r}$, $v = -\psi_r$

Try $\psi = f(r)\sin\theta$

$$\Rightarrow \nabla^2\psi = f''s + \frac{f'}{r}s - \frac{f}{r^2}s. \qquad s = \sin\theta$$

$$= F(r)\sin\theta \text{, say. where } F = \left(f'' + \frac{f'}{r} - \frac{f}{r^2}\right) \quad \text{— ①.}$$

$$\therefore \nabla^4\psi = 0 \Rightarrow F'' + \frac{F'}{r} - \frac{F}{r^2} = 0 \quad \text{eq. for } F(r)$$

Hence $F = C_1 r + C_2 r^{-1}$

Then solve ① for $f(r)$ ie. $f'' + \frac{f'}{r} - \frac{f}{r^2} = C_1 r + C_2 r^{-1}$

$$\Rightarrow f = Ar^3 + Br\ln r + Cr + Dr^{-1} \tag{4.17}$$

Then BC's at ∞. $\Rightarrow A = B = 0,\ C = U$

But BC's at $r = a \Rightarrow 2$ conditions to satisfy on D $(\psi = \psi_r = 0$ at $r = a)$, which cannot be satisfied as done!!

Resolution: recognise that there is an oseen region further out, between Stokes region and the stream U. So make (4.17) satisfy the BCs at $r = a$ & hope the oseen region helps ~~to~~ us satisfy the BC at ∞.

$$\Rightarrow f(a) = f'(a) = 0$$

$$\Rightarrow \psi = \hat{E}\left(\frac{2r}{a}\ln\left(\frac{r}{a}\right) - \frac{r}{a} + \frac{a}{r}\right)\sin\theta \tag{4.18}$$

(We omitted the A term because it diverges so fast as $r \to \infty$)

$$\Rightarrow \begin{cases} u = \hat{E}\left(2\ln\left(\frac{r}{a}\right) - 1 + \frac{a^2}{r^2}\right)\cos\theta \\ v = -\frac{\hat{E}}{a}\left(2\ln\left(\frac{r}{a}\right) + 1 - \frac{a^2}{r^2}\right)\sin\theta \end{cases}$$

uu_r

At large $\frac{r}{a}$, inertia $\sim \frac{\hat{E}^2}{a^2}\frac{1}{r}\ln\left(\frac{r}{a}\right)$

viscous $\sim \frac{\nu\hat{E}}{ar^2}$ ← νu_{rr}

By def^n of oseen region

$\Rightarrow$ ratio $\frac{\text{inertia}}{\text{viscous}} = \frac{\hat{E}r}{\nu a}\ln\left(\frac{r}{a}\right)$. This ratio must become $O(1)$ when $\frac{r}{a} \to O(R^{-1})$.

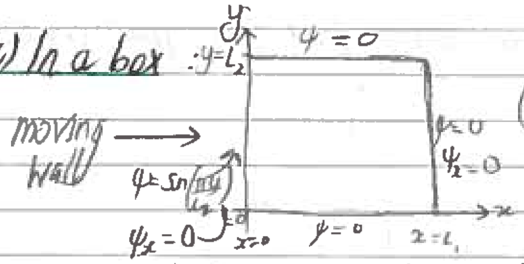
ie. in the oseen region.

$$\Rightarrow \frac{\hat{F}}{U_a} = O\left(\frac{-1}{\ln R}\right)$$

4.7 Small scale flows (micro- & nano-), physiology etc.

Most have $R \ll 1$.

(i) In a box



(in 2D)

... and periodicity. Solve $\nabla^4 \psi = 0$ inside.

Expect/try $\psi = \sin(\alpha y) f(x)$ with $\alpha = \frac{\pi}{L_2}$

$$\Rightarrow \nabla^2 \psi = (f'' - \alpha^2 f)\sin \alpha y \equiv F \sin \alpha y, \text{ say}$$

$$\nabla^4 \psi = (F'' - \alpha^2 F)\sin \alpha y.$$

$\therefore$ Solve $F'' - \alpha^2 F = 0 \Rightarrow F = \hat{A}\cosh \alpha x + \hat{B}\sinh \alpha x$.

and then solve $f'' - \alpha^2 f = \cancel{F}\ \hat{A}\cosh \alpha x + \hat{B}\sinh \alpha x$

$$\Rightarrow f(x) = C\cosh \alpha x + D\sinh \alpha x + A x\cosh \alpha x + B x \sinh \alpha x.$$

BCs: $f(0) = 1$ (from $\psi = \sin\left(\frac{\pi y}{L_2}\right)$) $\Rightarrow C = 1$

$f'(0) = 0 \Rightarrow \alpha D + A = 0$

$f(L_1) = 0 \Rightarrow C\cosh \alpha L_1 + D\sinh \alpha L_1 + AL_1 \cosh + BL_1 \sinh$

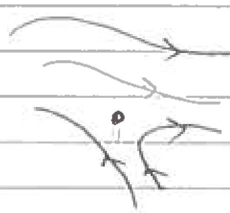
$f'(L_1) = 0 \Rightarrow C\alpha \sinh + D\alpha \cosh + (4 \text{ others}) = 0.$

$\Rightarrow A, B, C, D \Rightarrow$ solution $\Rightarrow$ test case for computations/experiments

eg. $B = \dfrac{-\alpha \sinh^2(\alpha L_1)}{\sinh^2(\alpha L_1) - \alpha^2 L_1^2}$

(ii) Flow near a stagnation point

$\nabla^4 \psi = 0$ again.



(guess) one of the simplest solution of Laplace's equation.

Take $\nabla^2 \psi = x^2 - y^2$ as a solution.

Integrate to find an ψ.

e.g. $\psi = \psi_1(x) + \psi_2(y)$

$\Rightarrow \psi_1'' + \psi_2'' = x^2 - y^2 \Rightarrow \psi_1'' = x^2, \quad \psi_2'' = -y^2$

$\Rightarrow \psi = \frac{1}{12}(x^4 - y^4) + \underbrace{(\text{terms of integration})}_{0}$

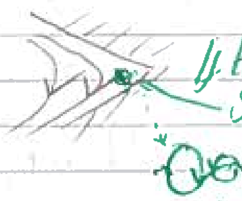
0 because we are working at a stagnation point.

Streamlines are $x^4 - y^4 = c$

Similar to potential flow but curves are flatter.

potential flow would have $x^2 - y^2$

(iii) Flow in a corner. (Sheet 4) Q2.



If the corner angle is too small, the solution is complex - get eddies.

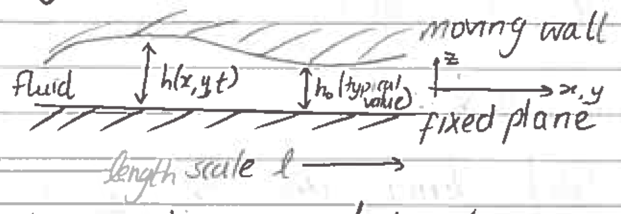
the eddies counter rotate.

Chapter 5: Lubrication Theory

$\Rightarrow$ angles involved are small.

Flow in channels and pipes of slowly varying cross section
Inertia forces is negligable again. $\Rightarrow$ small Reynolds number.

5.1 The theory



The gap is slowly varying $\Rightarrow$ take $h_0 \ll \ell$
Here $h_0 = O(h)$
Velocity of upper wall is, say $(U, V, W)(x,y)$ & typical (U, V) scale is
(U_0, V_0) with $V_0 \sim U_0$

Aim: determine $\underline{u}, p$.

Consider continuity, ie $u_x + v_y + w_z = 0$ (5.1)
$\Rightarrow$ orders are $\frac{U_0}{\ell}$, $\frac{U_0}{\ell}$, $\frac{W_0}{h_0}$
u scale, x scale, y scale

we expect that
$W_0 = O\left(\frac{U_0 h_0}{\ell}\right) \ll U_0$ since $\frac{h_0}{\ell}$ is very small (5.2) (5.1')
ie $\frac{h_0}{\ell} \ll 1$ (5.2)

Consider the Navier Stokes' equations in component form:

① $u_t + (\underline{u}\cdot\nabla)u = -\frac{p_x}{\rho} + \nu(\cancel{u_{xx}} + \cancel{u_{yy}} + u_{zz})$

② $v_t + (\underline{u}\cdot\nabla)v \quad -\frac{p_y}{\rho} + \nu(\cancel{v_{xx}} + \cancel{v_{yy}} + v_{zz})$

③ $w_t + (\underline{u}\cdot\nabla)v = -\frac{p_z}{\rho} + \nu(\cancel{w_{xx}} + \cancel{w_{yy}} + w_{zz})$

The z derivatives are dominant over the x & y derivatives
ie (the change in z direction is much greater than in the x & y direction)

pressure forces (p_z) are much greater

In Lubrication theory, knock out all secondary terms (inertia forces are negligible)

That's how Lubrication theory works. Let us check on its validity below.

In ①, LHS $= O\left(\frac{U_0^2}{\ell}\right)$, RHS $= O\left(\frac{\nu U_0}{h_0^2}\right)$ ④ since $|\partial_z| \gg \left|\begin{matrix}\partial_x \\ \partial_y\end{matrix}\right|$ $[h_0 \ll \ell]$
and since p adjusts to the largest term in the equation

So RHS is larger provided $\frac{U_0^2}{\ell} \ll \frac{\nu U_0}{h_0^2}$, ie $\left(\frac{U_0 h_0}{\nu}\right)\left(\frac{h_0}{\ell}\right) \ll 1$

ie $R\left(\frac{h_0}{\ell}\right) \ll 1$ [Sometimes written as $\alpha R \ll 1$]. $\alpha = \frac{h_0}{\ell}$ (5.3)
$\hookrightarrow$ the typical angle in the geometry.

$\alpha \ll 1$ is our first assumption.
$\alpha R \ll 1$ is our second assumption

Take $|t| \sim \frac{l}{U_0}$ $\left(\frac{\text{distance}}{\text{speed}}\right)$

we have justified (1) & (2).

In (3), consider $\left|\frac{p_z}{\rho}\right|$, know (4) $\Rightarrow \left|\frac{p_x}{\rho}\right| \sim \nu|u_{zz}|$

$$\Rightarrow |p| \sim \frac{\mu l U_0}{h_0^2} \Longrightarrow \left|\frac{p_z}{\rho}\right| = O\left(\frac{\mu l U_0}{h_0^2} \cdot \frac{1}{\rho h_0}\right) = O\left(\frac{\nu l U_0}{h_0^3}\right)$$

Hence $\left|\frac{p_z}{\rho}\right| \gg |\nu w_{zz}|$ provided $\frac{\nu l U_0}{h_0^3} \gg \frac{\nu W_0}{h_0^2}$

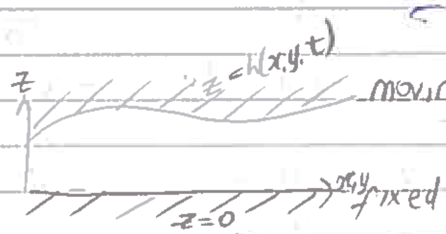
ie $\frac{l U_0}{h_0} \gg \frac{U_0 h_0}{l}$ $\left(W_0 = \frac{U_0 h_0}{l}\right)$

ie $\left(\frac{h_0}{l}\right)^2 \ll 1$ which is true (our original assumption).
Hence (3) is justified

$$\Rightarrow p_z = 0 \rightarrow \underline{p = p(x,y,t)}$$

We have then:

$$\begin{cases} u_x + v_y + w_z = 0 & \text{(continuity)} \\ u_{zz} = \frac{1}{\mu} p_x & (x \text{ mom}^{m}) \\ v_{zz} = \frac{1}{\mu} p_y & (y \text{ mom}^{n}) \\ p = p(x,y,t) & \end{cases} \quad (5.4)$$



$$\text{So } u = \underbrace{\frac{p_x}{2\mu} z(z-h)}_{PPF} + \underbrace{\frac{Uz}{h}}_{CF} \quad \text{with } \left.\begin{array}{ll} u=0, & z=0 \\ u=U, & z=h(x,y,t) \end{array}\right\} \text{bcs.} \quad \leftarrow (5.5)$$

$$\& \; v = \frac{p_y}{2\mu} z(z-h) + \frac{Vz}{h}$$

Then substitute (5.5) into the continuity equation $u_x + v_y + w_z = 0$, solve for w (remembering $h(x,y)$ & U). & apply the 2 boundary conditions on w.

$$\Rightarrow \boxed{(h^3 p_x)_x + (h^3 p_y)_y = 6\mu\left[h(U_x + V_y) - Uh_x - Vh_y + 2W\right]}$$

REYNOLDS LUBRICATION EQUATION [RLE] will show this next lec

5.2 Derivation of the RLE (Reynolds Lubrication Equation)

Given $\begin{cases} u_{zz} = \frac{1}{\mu} p_x \\ v_{zz} = \frac{1}{\mu} p_y \end{cases}$ where $p = p(x,y,t)$, $h = h(x,y,t)$

(lubrication approximation), we can integrate to get

$$u = \underbrace{\frac{Uz}{h}}_{\text{linear term (Couette flow)}} + \underbrace{\frac{p_x}{2\mu} z(z-h)}_{\text{PPF}}$$

$$v = \frac{Vz}{h} + \frac{p_y}{2\mu} z(z-h) \qquad (5.5)$$

Also have continuity: $u_x + v_y + w_z = 0$ (5.6)

Integrate (5.6) wrt z from 0 to h

$$\Rightarrow \int_0^h u_x \, dz + \int_0^h v_y \, dz + \left[w \right]_0^h = 0$$

$$\Rightarrow \int_0^h u_x \, dz + \int_0^h v_y \, dz + W = 0 \quad \leftarrow \text{from boundary conditions}$$

$$\Rightarrow -W = \int_0^h u_x \, dz + \int_0^h v_y \, dz$$

$$= \int_0^h \left\{ \left(\frac{U}{h}\right)_x z + \frac{p_{xx}}{2\mu} z^2 - \frac{(p_x h)_x z}{2\mu} \right\} dz + \text{similar for } v_y \text{ integral}$$

$$= \left(\frac{U}{h}\right)_x \frac{h^2}{2} + \frac{p_{xx}}{2\mu}\frac{h^3}{3} - \frac{(p_x h)_x}{2\mu} \cdot \frac{h^2}{2} + \text{"} \quad \text{"}$$

$$\Rightarrow -W = \frac{1}{2} h^2 \left(\frac{U_x}{h} - \frac{U h_x}{h^2} \right) - \frac{1}{12\mu} \partial_x \left(h^3 p_x \right) + \text{"} \quad \text{"}$$

$$= \frac{1}{2\mu} \left\{ \frac{h^3}{3} p'' - \frac{h^2}{2} \left(p'' h + p' h' \right) \right\} + \text{"} \quad \text{"}$$

$$\Rightarrow \frac{1}{2\mu} \left\{ -\frac{1}{6} h^3 p'' - \frac{h^2}{2} h' p' \right\} + \text{"} \quad \text{"}$$

$$\Rightarrow \frac{1}{12\mu} \left\{ h^3 p'' + 3h^2 h' p' \right\} + \text{"} \quad \text{"}$$

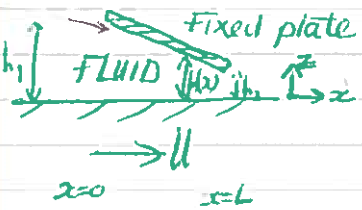
$$\Rightarrow -W = \frac{1}{2} h^2 \left(\frac{U_x}{h} - \frac{U h_x}{h^2} \right) - \frac{1}{12\mu} \partial_x \left(h^3 p_x \right) + \frac{1}{2} h^2 \left(\frac{V_y}{h} - \frac{V h_y}{h^2} \right) - \frac{1}{12\mu} \partial_y \left(h^3 p_y \right)$$

$\Rightarrow$ RLE, i.e. $\boxed{(h^3 p_x)_x + (h^3 p_y)_y = 6\mu \left[h(U_x + V_y) - (U h_x + V h_y) + 2W \right]}$ (5.7)

Sometimes we use (5.7) for a given problem, sometimes need to start from scratch

5.3 Example: slider bearing

pressure is larger than p_0 ~~outside~~ inside the gap



Angle α (small), axes fixed in the plate, pressure p is atmospheric (p_0) outside the gap

Could use (5.7), but instead start from scratch

Assume Lubrication theory: $\boxed{\begin{matrix}\alpha \ll 1 \\ \alpha R \ll 1\end{matrix}}$ where $R \equiv \dfrac{Uh}{\nu}$

$$\Rightarrow u_{zz} = \mu^{-1} p_x(x, \cancel{t}) \quad \text{steady.} \tag{5.8}$$

BC's are $\begin{cases} u = 0 & \text{at } z = h \\ u = U & \text{at } z = 0 \end{cases}$

$$(5.8) \Rightarrow u = \frac{p_x}{2\mu} z^2 + Az + B$$

$$\& \text{ BC's} \Rightarrow \frac{p_x}{2\mu} h^2 + Ah + B = 0 \;\&\; B = U$$

$$\text{Hence } A = -\frac{B}{h} - \frac{p_x}{2\mu} h^2$$

$$\text{Hence } u = -\frac{p'}{2\mu} z(h-z) + U\left(1 - \frac{z}{h}\right) \tag{5.9}$$

The continuity gives w $[u_x + w_z = 0]$

Or use mass flux: $Q \equiv \int_0^h u \, dz$ must be constant

$$\Rightarrow Q = \frac{-p'}{2\mu}\left(\frac{h^2}{2} \cdot h - \frac{h^3}{3}\right) + U\left(h - \frac{1}{2}h\right)$$

$$\therefore Q = \frac{-p' h^3}{12\mu} + \frac{Uh}{2}$$

$$\Rightarrow p' = 6\mu\left(\frac{U}{h^2} - \frac{2Q}{h^3}\right)$$

$$\Rightarrow \frac{dp}{dh} = 6\mu\left(\frac{U}{h^2} - \frac{2Q}{h^3}\right) \cdot \frac{-1}{\alpha} \tag{5.10}$$

$$p - p_0 = \frac{-6\mu}{\alpha}\left(-\frac{U}{h} + \frac{U}{h_1} + \frac{Q}{h^2} - \frac{Q}{h_1^2}\right) \text{ using } p = p_0 \text{ at } h = h_1.$$

Then impose $p = p_0$ at $h = h_2$ ($\equiv h_1 - \alpha L$) (because α is small)

$$\Rightarrow 0 = \left(\frac{-U}{h_2} + \frac{U}{h_1} + \frac{Q}{h_2^2} - \frac{Q}{h_1^2} \right)$$

$$\Rightarrow Q = \frac{U h_1 h_2}{(h_1 + h_2)}$$

$$\text{So } p - p_0 = \frac{6\mu U}{\alpha} \frac{(h_1 - h)(h - h_2)}{h^2 (h_1 + h_2)} \quad (> 0 \text{ in the gap}) \qquad (5.11)$$

$\Rightarrow u, v$, drag etc (we can now get these)

E.g. the total normal force on the plate (block)

$$= \int_0^L (p - p_0)\, dx = \frac{6\mu U}{\alpha^2} \left[\ln\left(\frac{h_1}{h_2}\right) - \frac{2(h_1 - h_2)}{(h_1 + h_2)} \right]$$

& the total tangential force is $\int_0^L -\mu \left(\frac{\partial u}{\partial z}\right)_{z=h} dx$

$$= \frac{2\mu U}{\alpha} \left[\frac{3(h_1 - h_2)}{(h_1 + h_2)} - \ln \frac{h_1}{h_2} \right]$$

$$\Rightarrow \frac{\text{Tangential force on plate}}{\text{Normal force on plate}} = \alpha \cdot \text{function}\left(\frac{h_1}{h_2}\right)$$

$\Rightarrow$ the effective coefficient of friction (small in this case)

friction effect would be order 1 without the fluid between the two solids.

we have high pressures & low drag

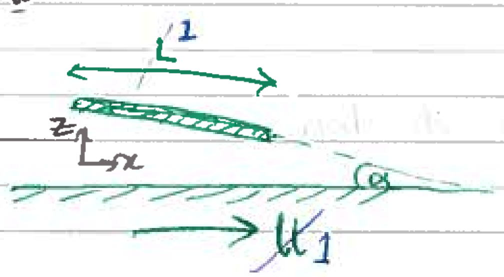
This ratio is equivalent to the coefficient of friction

$\Rightarrow$ the effect of lubrication is to reduce friction by a factor α (e.g. 1% of the original compared with solid-solid friction).

5.4 Non dimensional thinking

Use (3.8) - the non dimensional Navier Stokes equations:

$$\begin{cases} (\underline{u} \cdot \nabla)\underline{u} = -\nabla p + R^{-1} \nabla^2 \underline{u} \\ \nabla \cdot \underline{u} = 0 \end{cases} \qquad (5.13)$$



Apply this to case of 5.3 $\Rightarrow$

Put $\begin{cases} z = \alpha \bar{z} \\ x = \bar{x} \end{cases} \Rightarrow \bar{x}, \bar{z}$ are $O(1)$

And $\begin{cases} u = \bar{u} \\ w = \alpha \bar{w} \end{cases}$ $\Rightarrow$ our expectation is $\bar{u}, \bar{w}$ are $O(1)$ from continuity equation

Put $p = \frac{\lambda}{R}\bar{p}$ (unknown λ)

Substitute into (5.13) $\Rightarrow \begin{cases} \bar{u}\bar{u}_{\bar{x}} + \bar{w}\bar{u}_{\bar{z}}\frac{\alpha}{\alpha} = -\bar{p}_{\bar{x}} \cdot \frac{\lambda}{Re} + \frac{1}{Re}\left(\frac{\bar{u}_{\bar{z}\bar{z}}}{\alpha^2} + \bar{u}_{\bar{x}\bar{x}}\right) \\ \bar{u}\bar{w}_{\bar{x}}\alpha + \bar{w}\bar{w}_{\bar{z}}\alpha \\ \quad = -\bar{p}_{\bar{z}}\frac{\lambda}{R\alpha} + \frac{1}{Re}\left(\bar{w}_{\bar{z}\bar{z}}\frac{\alpha}{\alpha^2} + \bar{w}_{\bar{x}\bar{x}}\alpha\right) \end{cases}$

this term is dominant because it is divided by a v. small (α^2).

and the continuity equation $\Rightarrow \bar{u}_{\bar{x}} + \bar{w}_{\bar{z}} = 0$

Hence $O\left(\frac{1}{R\alpha^2}\right) \Rightarrow 0 = -\bar{p}_{\bar{x}}\frac{\lambda}{Re} + \frac{1}{Re}\frac{\bar{u}_{\bar{z}\bar{z}}}{\alpha^2}$ provided $\frac{1}{R\alpha^2} \gg 1$ | x Mom

Hence $0 = -\bar{p}_{\bar{x}} + \bar{u}_{\bar{z}\bar{z}}$ & $\lambda = \frac{1}{\alpha^2}$

Also $\bar{p}_{\bar{z}} = 0$ at $O\left(\frac{\lambda}{R\alpha}\right)$ | z Mom

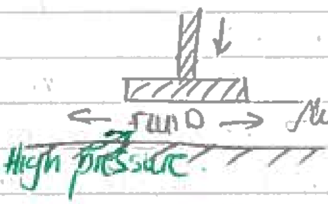
Also $\bar{u}_{\bar{x}} + \bar{w}_{\bar{z}} = 0$ | continuity

$(R\alpha^2 \ll 1)$ where $R_e = \frac{R}{\alpha}$ in chapter 5.

what we had in chapter 3

5.5 Other cases

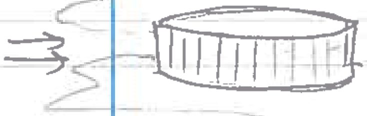
Thrust bearing (Sheet 5)



Journal bearing

5.6 Hele-Shaw flow

Flow between parallel plates, past 2D obstacles.



Here, $h = $ constant, $U = V = W = 0$.

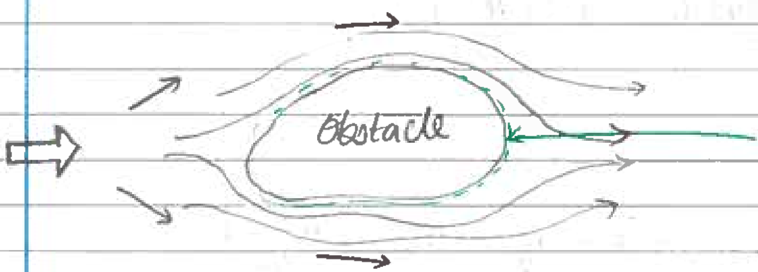
So RLE $\Rightarrow (h^3 p_x)_x + (h^3 p_y)_y = 6\mu[0]$ $\Rightarrow \nabla^2 p = 0$ (in terms of x & y).

with $\begin{cases} u = \dfrac{p_x}{2\mu} z(z-h) \\ v = \dfrac{p_y}{2\mu} z(z-h) \end{cases}$ (5.15)

$\Rightarrow \dfrac{u}{v}$ is independant of z

$\Rightarrow$ direction of the velocity vector is independant of z

Hence the birds eye view of this is



Get potential flow past the obstacle? Yes.

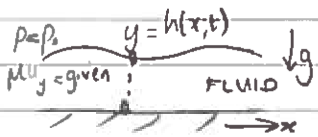
$\hookrightarrow$ Set $\begin{cases} \bar{u} \equiv \text{mean } u = -\dfrac{h^2}{12\mu} p_x \\ \bar{v} \equiv -\dfrac{h^2}{12\mu} p_y. \end{cases}$ (5.16)

& put $\phi = -\dfrac{h^2 p}{12\mu}$, to obtain $\begin{cases} \bar{\underline{u}} = \nabla\phi \\ \nabla^2\phi = 0 \end{cases}$ (5.17)

$\Rightarrow$ streamlines are those of potential flow. flow that satisfies laplaces equations

We can visualise these flows.

5.7 Final Problem



(Hand out)

Find an equation for $h(x,t)$.

At $y = h$, $p = p_0$ & $\mu \dfrac{\partial u}{\partial y} = \tau(x,t)$ (given)

& $v = h_t + u h_x$ (5)

YM: $\dfrac{\partial p}{\partial y} = -\rho g$ (2) $\Rightarrow p = -\rho g y + G(x,t)$

But (6) $\Rightarrow p_0 = -\rho g h + C_1$

So $p = -\rho g(y-h) + p_0$.

xM: (1) $\Rightarrow u_{yy} = \frac{1}{\mu} p_x = \mu^{-1}\rho g h_x$

$\Rightarrow u = \mu^{-1}\rho g h_x y^2 + C_2 y + C_3 \quad (*)$.

Then $y=0 \Rightarrow C_3 = 0$

$y=h \Rightarrow \mu^{-1}\rho g h_x y\big|_h + C_2 = \mu^{-1}\tau$

$\Rightarrow C_2 = \mu^{-1}\{\tau - \rho g h_x h\} \quad (7)$.

Then (3) $\Rightarrow -v_y = \frac{\mu^{-1}\rho g y^2}{2} h_{xx} + C_{2x} y$ (differentiating $*$).

$\Rightarrow v = \frac{\mu^{-1}\rho g y^3}{6} h_{xx} + C_{2x}\frac{y^2}{2} + C_4$

But $\overset{\text{(no slip)}}{y=0} \Rightarrow C_4 = 0$

$y = h \Rightarrow -\frac{\mu^{-1}\rho g}{6} h^3 h_{xx} - C_{2x}\frac{h^2}{2} = h_t + u\big|_{top} h_x$

$= h_t + \{\frac{\mu^{-1}\rho g h_x h^2}{2} + C_2 h\} h_x$.

$\Rightarrow h_t + \{\frac{\mu^{-1}\rho g h^2 h_x}{2} + C_2 h\} h_x + \frac{\mu^{-1}\rho g}{6} h^3 h_{xx} + C_{2x}\frac{h^2}{2} = 0$

Now use $C_2 = \mu^{-1}(\tau - \rho g h_x h)$

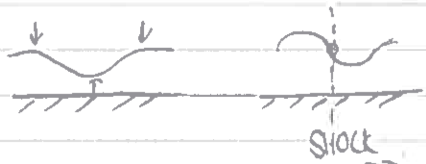
$\Rightarrow h_t + \{\frac{\rho g}{2\mu} h^2 h_x h_x + \frac{h h_x}{\mu}(\tau - \rho g h h_x)\} + \frac{\rho g h^3}{6\mu} h_{xx} + \frac{h^2}{2\mu}(\tau - \rho g h h_x)_x = 0$

$\Rightarrow h_t + \frac{1}{2\mu}(\tau h^2)_x - \frac{\rho g}{3\mu}(h^3 h_x)_x = 0$

Ex. $h_t + \frac{\tau_0}{2\mu} 2 h h_x = 0$

Inviscid burgers equation.

$h_t + h h_x = 0$



$\Rightarrow$ breaking waves.

From Moodle

A film is on a horizontal solid surface. A constant atmospheric pressure, p_0, and a specified shear stress, $\mu \partial u/\partial y = \tau(x,t)$ are applied to the free surface at $y = h(x,t)$.

Derive an equation for $h(x,t)$ in lubrication approximation.

Governing equations:

$$\mu\frac{\partial^2 u}{\partial y^2} = \frac{\partial p}{\partial x}, \qquad (1)$$

$$\frac{\partial p}{\partial y} = -\rho g, \qquad (2)$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0. \qquad (3)$$

Boundary conditions:

$$u = v = 0 \text{ at } y = 0 \text{ (no-slip at the wall)}, \qquad (4)$$

$$v|_{y=h(x,t)} = \frac{\partial h}{\partial t} + u|_{y=h(x,t)}\frac{\partial h}{\partial x} \text{ (kinematic condition)}, \qquad (5)$$

$$p|_{y=h(x,t)} = p_0, \quad \mu\frac{\partial u}{\partial y}\bigg|_{y=h(x,t)} = \tau(x,t) \text{ (given stresses at the free surface)}. \qquad (6)$$

Answer:

$$\frac{\partial h}{\partial t} + \frac{1}{2\mu}\frac{\partial}{\partial x}\left(\tau h^2\right) - \frac{\rho g}{3\mu}\frac{\partial}{\partial x}\left(h^3\frac{\partial h}{\partial x}\right) = 0.$$

Exercise. Consider the case $\tau(x,t) = \tau_0 = const$ and negligible gravity, $g = 0$. Verify the general solution of the resulting equation,

$$\frac{\partial h}{\partial t} + \frac{\tau_0}{\mu}h\frac{\partial h}{\partial x} = 0,$$

in the implicit form

$$h = h_0\left(x - \frac{\tau_0}{\mu}h(x,t)\,t\right),$$

where h_0 is an arbitrary function.

Consider the (t,x)-plane and observe that h remains constant on straight lines (i.e. the characteristics of the governing equation are straight lines). Interpreting h_0 as an initial shape of the film surface at time $t = 0$, show that the film with a negative gradient in the initial shape will overturn in a finite time.

Exercise. Consider the flow with $\tau = \tau_0 = const$ and non-zero gravity. The film surface is perturbed slightly from it's equilibrium state,

$$h = h_0 + \varepsilon h_1(x,t) + O\left(\varepsilon^2\right),$$

where $h_0 = const, \varepsilon$ is a small parameter and $h_1(x,t)$ is the shape of the perturbation.

Derive an equation for h_1.