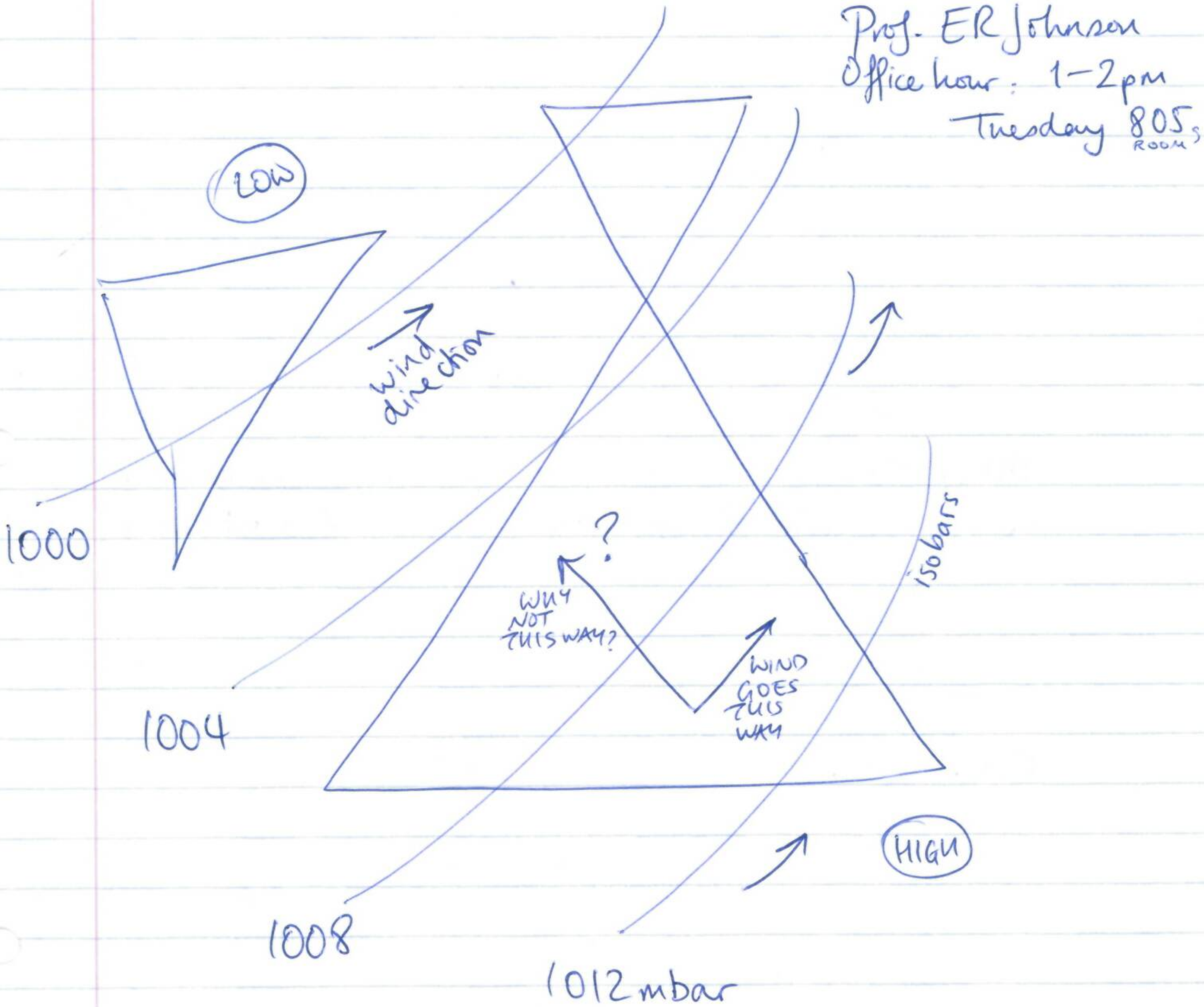
3304 Geophysical Fluid Dynamics Notes

Based on the 2011 spring lectures by Prof E R Johnson

The Author has made every effort to copy down all the content on the board during lectures. The Author accepts no responsibility what so ever for mistakes on the notes nor changes to the syllabus for the current year. The Author highly recommends that reader attends all lectures, making their own notes and to use this document as a reference only

GEOPHYSICAL FLUID DYNAMICS

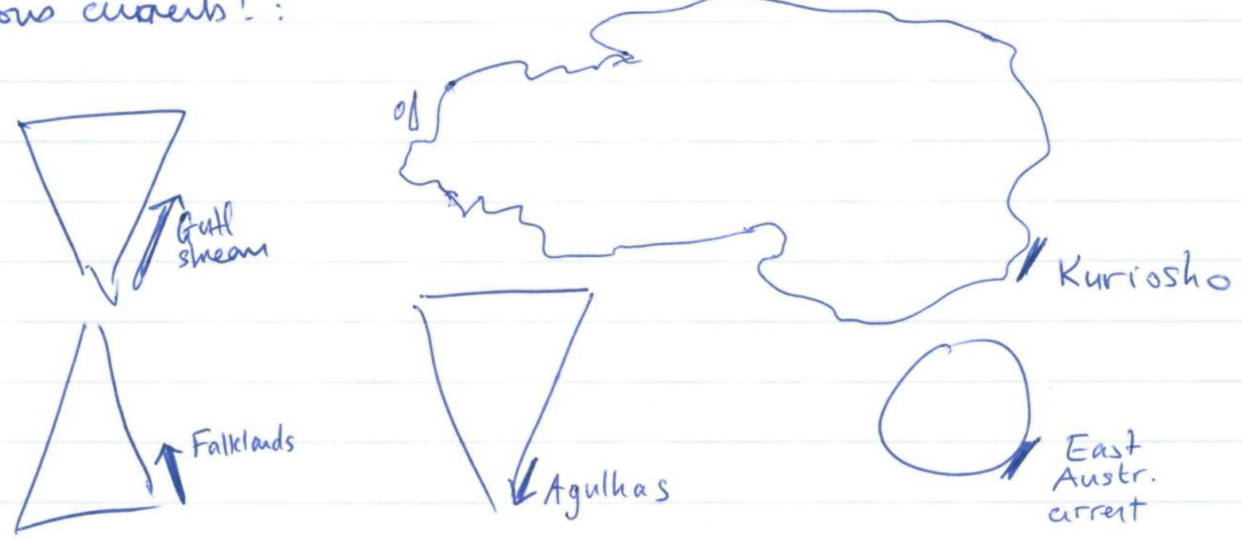
Prof. ER Johnson
Office hour: 1–2pm
Tuesday 805 ROOM



Pressure gradient is balanced by Coriolis force, "geostrophic balance"

Will look at large scale motion of the atmosphere, and the ocean.

Famous currents! :



All of the intense currents are 'western boundary currents', i.e. on the right of continents. We'll find out why.

Things to know

Euler eq^n: $\quad \dfrac{D\underline{u}}{Dt} = \dfrac{\partial \underline{u}}{\partial t} + (\underline{u}\cdot\nabla)\underline{u}$

Mom^m eq^n $\quad = -\dfrac{1}{\rho}\nabla p + \underline{F}$ ← external forces per unit mass

Cons. of mass: $\quad \dfrac{\partial \rho}{\partial t} + \nabla\cdot(\rho\underline{u}) = 0$

Assume inviscid flow on the whole.

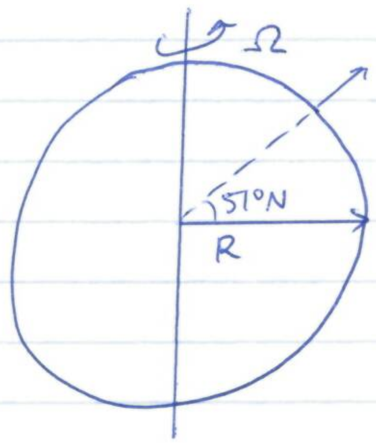
else mom^m eq^n becomes $\quad \dfrac{D\underline{u}}{Dt} = -\dfrac{1}{\rho}\nabla p + \underline{F} + \nu\nabla^2\underline{u}$

Limits valid for Earth-sized flows

1. Geostrophy
2. Small deviations from geostrophy (linear)
3. Solve these wavelike linear PDEs (almost) all constant coeffs, i.e. $e^{ikx+ily+imz-i\omega t}$

May use Fourier Transforms and maybe Stationary Phase.

ROTATION



$\Omega = 1$ rev / day
$= 2\pi / 86400$ sec
$\approx 10^{-4}\ s^{-1}$

$R \approx 6400$ km
$= 6.4 \times 10^6$ m

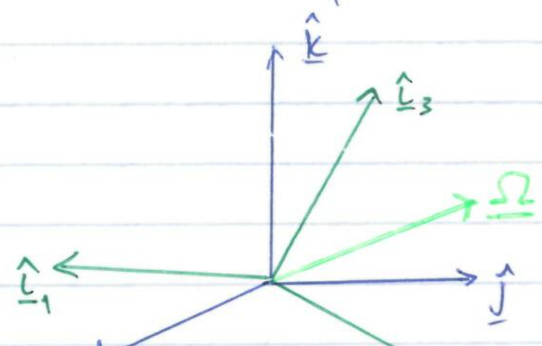
So Earth's surface moves at 640 m/s compared to the centre

Typical ocean current $\approx$ 4 to 10 knots
$\approx$ 2 to 5 m/s

Typical wind speed $\approx$ strong wind 25 knots
$\approx$ 12 m/s

jet stream 100 m/s

A frame fixed in London is not an inertial frame, so we need suitable eq^{n}s.



Let's have an inertial frame I with unit vectors $\hat{\underline{i}}, \hat{\underline{j}}, \hat{\underline{k}}$

Let R be a frame with unit vectors $\hat{\underline{i}}_1, \hat{\underline{i}}_2, \hat{\underline{i}}_3$ rotating at ang. vel. $\underline{\Omega}$ relative to the frame.

$$\left(\frac{d\hat{\underline{\imath}}}{dt}\right)_I = 0$$

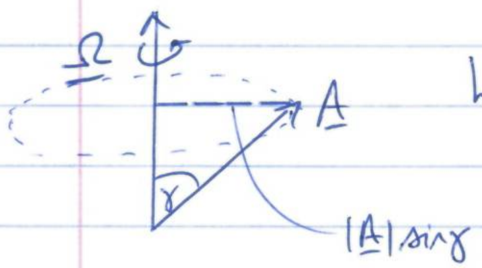
$$\left(\frac{d\hat{\underline{\imath}}_1}{dt}\right)_I \neq 0 \qquad \text{obviously since } \hat{\underline{\imath}}_i \text{ is rotating}$$

Choose any vector $\underline{A}$, fixed in the rotating frame R.

ie. $$\left(\frac{d\underline{A}}{dt}\right)_R = 0 .$$

What is $\left(\frac{d\underline{A}}{dt}\right)_I$?

By defⁿ, $$\left(\frac{d\underline{A}}{dt}\right)_I = \lim_{\delta t \to 0} \frac{\underline{A}(t+\delta t) - \underline{A}(t)}{\delta t}$$



Let the angle between $\underline{\Omega}$ and $\underline{A}$ be γ.

Then $\underline{A}$ moves into the page, ie. in a direction $\hat{\underline{n}}$, $\perp^r$ to $\underline{A}$ and $\underline{\Omega}$ s.t. $[\underline{\Omega}, \underline{A}, \hat{\underline{n}}]$ is a right-handed system, an amount $\underbrace{|\underline{A}|\sin\gamma}_{\text{arm length}} \times \underbrace{|\underline{\Omega}|\,\delta t}_{\text{increase in angle}}$

ie. $$\left(\frac{d\underline{A}}{dt}\right)_I = \lim_{\delta t \to 0} \frac{|\underline{\Omega}||\underline{A}|\sin\gamma\,\hat{\underline{n}}\,\cancel{\delta t}}{\cancel{\delta t}} = \underline{\Omega} \times \underline{A}$$

Now consider a variable vector $\underline{B}(t)$. In the rotating frame R,

let $$\underline{B}(t) = B_1(t)\hat{\underline{\imath}}_1 + B_2(t)\hat{\underline{\imath}}_2 + B_3(t)\hat{\underline{\imath}}_3$$
$$= B_j(t)\hat{\underline{\imath}}_j$$

Then $\left(\frac{d\underline{B}}{dt}\right)_R = \frac{d}{dt}\left(B_j \hat{\underline{i}}_j\right)_R = \frac{dB_j}{dt}\hat{\underline{i}}_j + B_j \cancel{\frac{d\hat{\underline{i}}_j}{dt}}^{\,0}$ $\therefore \hat{\underline{i}}_j$ is const in rot. frame

and $\left(\frac{d\underline{B}}{dt}\right)_I = \frac{d}{dt}\left(B_j \hat{\underline{i}}_j\right)_I = \frac{dB_j}{dt}\hat{\underline{i}}_j + B_j\left(\frac{d\hat{\underline{i}}_j}{dt}\right)_I$

$$= \left(\frac{d\underline{B}}{dt}\right)_R + B_j\left(\underline{\Omega} \times \hat{\underline{i}}_j\right)$$

$$= \left(\frac{d\underline{B}}{dt}\right)_R + \underline{\Omega} \times \left(B_j \hat{\underline{i}}_j\right)$$

$$= \left(\frac{d\underline{B}}{dt}\right)_R + \underline{\Omega} \times \underline{B}$$

e.g. if $\underline{B} = \underline{r}$, $\left(\frac{d\underline{r}}{dt}\right)_I = \left(\frac{d\underline{r}}{dt}\right)_R + \underline{\Omega} \times \underline{r}$

i.e. $\underline{u}_I = \underline{u}_R + \underline{\Omega} \times \underline{r}$

<u>Acceleration</u>: $\left(\frac{d^2\underline{r}}{dt^2}\right)_I = \left(\frac{d\underline{u}}{dt}\right)_I = \left(\frac{d\underline{u}_I}{dt}\right)_R + \underline{\Omega} \times \underline{u}_I$

$$\left(\frac{d\underline{u}_I}{dt}\right)_I = \left[\frac{d}{dt}\left(\underline{u}_R + \underline{\Omega} \times \underline{r}\right)\right]_R + \underline{\Omega} \times \left(\underline{u}_R + \underline{\Omega} \times \underline{r}\right)$$

$$= \left(\frac{d\underline{u}_R}{dt}\right) + \underline{\Omega} \times \underbrace{\left(\frac{d\underline{r}}{dt}\right)_R}_{\underline{u}_R} + \underline{\Omega} \times \underline{u}_R + \underline{\Omega} \times \left(\underline{\Omega} \times \underline{r}\right)$$

$$= \underbrace{\left(\frac{d\underline{u}_R}{dt}\right)_R}_{\text{Accel in rotating frame}} + \underbrace{2\underline{\Omega} \times \underline{u}_R}_{\text{Coriolis}} + \underbrace{\underline{\Omega} \times \left(\underline{\Omega} \times \underline{r}\right)}_{\text{Centripetal acceleration}}$$

Notice Newton's laws apply in an inertial frame to give,

$$\left(\frac{d\underline{u}_I}{dt}\right)_I = \frac{\underline{F}}{m}$$

or relative to a rotating frame,

$$\left(\frac{d\underline{u}_R}{dt}\right)_R + 2\underline{\Omega}\times\underline{u}_R + \underline{\Omega}\times(\underline{\Omega}\times\underline{r}) = \underline{F}/m$$

In particle dynamics, it's traditional to rearrange this as

$$m\left(\frac{d\underline{u}_R}{dt}\right)_R = \underline{F} - \underbrace{2m\underline{\Omega}\times\underline{u}_R}_{\substack{\text{CORIOLIS FORCE} \\ \text{(to the right in} \\ \text{Northern Hemisphere)}}} - \underbrace{m\underline{\Omega}\times(\underline{\Omega}\times\underline{r})}_{\text{centrifugal}}$$

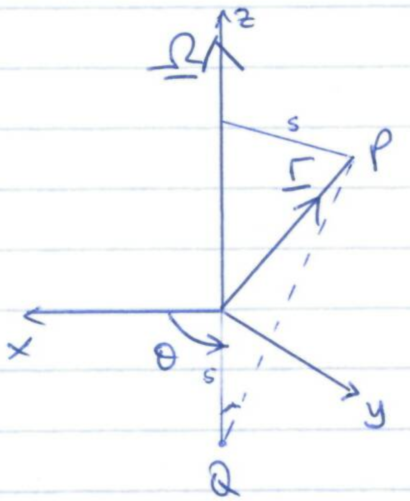
Thus our equations of motion for a constant density fluid are, <u>relative</u> to a <u>rotating</u> frame,

$$\boxed{\begin{gathered} \nabla\cdot\underline{u} = 0 \qquad (*) \\ \left(\frac{D\underline{u}_R}{Dt}\right)_R + \underbrace{2\underline{\Omega}\times\underline{u}_R + \underline{\Omega}\times(\underline{\Omega}\times\underline{r})}_{\substack{\text{new: Coriolis and centripetal} \\ \text{acceleration due to} \\ \text{rotating frame}}} = -\frac{1}{\rho}\nabla p + \underbrace{\underline{F}}_{\substack{\text{external} \\ \text{forces per} \\ \text{unit mass.}}} \end{gathered}}$$

momentum eqn with extra terms but I'm not sure how this follows from above.

The centripetal acceleration can be expressed as a potential and so absorbed into the pressure (exactly as done for gravity through the defn of hydrostatic pressure in a non-rotating fluid).
To show this,
Introduce (temporarily) cylindrical polar coords

(s, θ, z) with O_z along $\underline{\Omega}$:



$\underline{r} = \overrightarrow{OP}$

Q is projection of P in xy plane

Now, $\underline{\Omega} \times \underline{r} = \Omega s \hat{\underline{\theta}}$

$(= \Omega r \sin\chi \, \hat{\underline{\theta}})$

$$\underline{\Omega} \times (\underline{\Omega} \times \underline{r}) = |\underline{\Omega}||\Omega s \hat{\underline{\theta}}| \cdot 1 \cdot (-\hat{\underline{s}})$$
$$= \Omega^2 s\,(-\hat{\underline{s}}) \qquad \ldots (1)$$

<u>Centripetal</u>: towards axis of rotation

Now, for any function G_c, we have in these coordinates,

$$\nabla G_c = \frac{\partial G_c}{\partial s}\hat{\underline{s}} + \frac{1}{s}\frac{\partial G_c}{\partial \theta}\hat{\underline{\theta}} + \frac{\partial G_c}{\partial z}\hat{\underline{z}} \qquad \ldots (2)$$

(1) cf (2): $\quad \dfrac{\partial G_c}{\partial \theta} = 0 \quad \dfrac{\partial G_c}{\partial z} = 0$ but $\dfrac{\partial G_c}{\partial s} = -\Omega^2 s$

ie we can take $G_c = -\frac{1}{2}\Omega^2 s^2$

$$= -\frac{1}{2}|\underline{\Omega} \times \underline{r}|^2$$

ie. the centripetal acceleration is derivable from a potential,

$$\underline{\Omega} \times (\underline{\Omega} \times \underline{r}) = \nabla G_c$$

where $G_c = -\frac{1}{2}|\underline{\Omega} \times \underline{r}|^2$.

Thus we can write, dropping subscript 'R' for rotating, as from now on, all velocities are measured relative to rotating axes (unless otherwise stated):

(*) becomes

$$\frac{D\underline{u}}{Dt} + 2\underline{\Omega}\times\underline{u} + \nabla G_c = -\frac{1}{\rho}\nabla p + \underline{F} \quad (**)$$

The only external force we will consider is gravity, for which the force per unit mass, i.e. acceleration, is

$$\underline{F} = -g\hat{\underline{z}} \qquad (\hat{\underline{z}} \text{ upwards 'local vertical'})$$

Then $\underline{F} = -\nabla G_g$ where $G_g = gz$ $\Rightarrow \frac{D\underline{u}}{Dt} + 2\underline{\Omega}\times\underline{u} + \nabla G_c = -\frac{1}{\rho}\nabla p - \nabla G_g$

Now we can proceed exactly as in the derivation of hydro<u>static</u> pressure, expressing p as the deviation from the pressure when $\underline{u}=0$. (no motion rel. to rotating frame i.e. $\underline{u}_I = \underline{\Omega}\times\underline{r}$ i.e. solid body rotation)

$\underline{u}_R = 0$ in old notation

i.e. write $p = p_e + p_*$ ← dynamic pressure

↑ equilibrium pressure ($\underline{u}=0$)

Putting $\underline{u}=0$,

$$\nabla G_c = -\frac{1}{\rho}\nabla p_e - \nabla G_g \quad (\dagger)$$

i.e. $\nabla(p_e + \rho G_c + \rho G_g) = 0$

i.e. $p_e = p_0 - \rho G_c - \rho G_g$ } plugging in def^{n}s of G_c, G_g.

$$= \underbrace{p_0 + \rho g z}_{\text{'traditional' hydrostatic}} - \frac{1}{2}\rho\Omega^2 s^2 \quad (\square)$$

<u>Note:</u>

(1) If $\Omega = 0$, this is the usual hydrostatic pressure $p_e = p_0 + \rho g z$

(2) The surfaces of constant pressure are the paraboloid

(I) $\rho g z = \frac{1}{2}\rho\Omega^2 s^2 - p_0$

$z = \frac{\Omega^2 s^2}{2g} - \frac{p_0}{\rho g}$

$$\Rightarrow z = \frac{\Omega^2}{2g} s^2 + \text{const} \quad \cdots\cdots (\otimes)$$

$$= \frac{\Omega^2}{2g}(x^2 + y^2) + \text{const}$$

The free surface is a surface of constant pressure and so is a paraboloid.

Now, measure pressure as the deviation from p_e, i.e. write $p = p_e + p_D$

$$(**) \Rightarrow \quad \frac{D\underline{u}}{Dt} + 2\underline{\Omega}\times\underline{u} + \cancel{\nabla G_c} = -\frac{1}{\rho}\nabla(p_e + p_D) - \nabla G_g$$

$$\text{from}(\dagger) \rightarrow \quad = \cancel{\nabla G_c} + \cancel{\nabla G_g} - \frac{1}{\rho}\nabla p_D - \cancel{\nabla G_g}$$

$$\Rightarrow \quad \frac{D\underline{u}}{Dt} + 2\underline{\Omega}\times\underline{u} = -\frac{1}{\rho}\nabla p_D$$

Summary

(i) $\frac{D\underline{u}}{Dt} + 2\underline{\Omega}\times\underline{u} = -\frac{1}{\rho}\nabla p_D$

(ii) $\nabla\cdot\underline{u} = 0$

(iii) $p = p_e + p_D$

$$= p_0 + \rho g z - \frac{1}{2}\rho|\underline{\Omega}\times\underline{r}|^2 + p_D \quad \leftarrow \text{from (I)}$$

Recall $\frac{D\underline{u}}{Dt} = \frac{\partial \underline{u}}{\partial t} + (\underline{u}\cdot\nabla)\underline{u}$.

For <u>steady</u> flows (or 'almost' steady) we get $\frac{\partial \underline{u}}{\partial t} \approx 0$

For <u>slow</u> flows $|\underline{u}| \ll 1$, $|\underline{u}|^2 \ll |\underline{u}|$

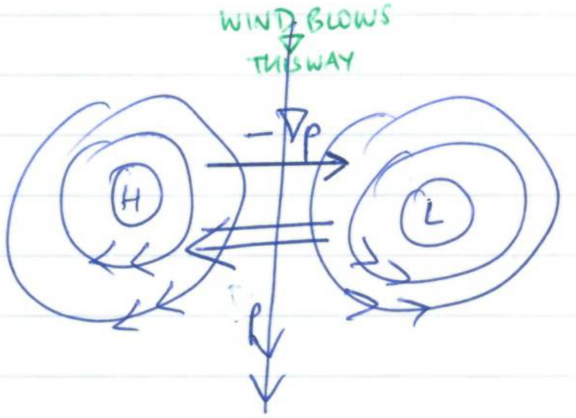
$$|\underline{u}\cdot\nabla\underline{u}| \ll |2\underline{\Omega}\times\underline{u}|$$ ← (in terms of order)

Then for 'slow', 'steady' flows, $2\underline{\Omega}\times\underline{u} = -\frac{1}{\rho}\nabla p_D$ (ie say Du/Dt disappears)

ie. pressure gradient is balanced by Coriolis force.
ie. geo|strophic balance
earth | turning (Greek!)

note $\underline{u} \perp^r \nabla p$,
i.e. $\underline{u}$ is // to lines of constant p,
ie. winds blow around isobars

$\left(\left|\frac{D\underline{u}}{Dt}\right| \ll |\underline{\Omega}\times\underline{u}|\right)$
we said this



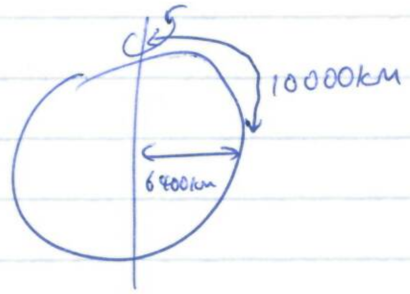
In the Northern Hemisphere, $\underline{\Omega}$ is +ve upwards
So by r.h. rule, $\Omega\times u$ is to the left ⇒ $-\nabla p$ left ⇒ ∇p right

So if the wind is blowing on your back, then high pressure lies to your right!!

This was discovered by a Dutchman, Buys-Baillot ~1660.

<u>Ocean depth:</u> Earth is 6400 km radius
Avg ocean depth is ~ <u>4 km</u>!

"skin of apple"



4 $\boxed{\text{PRETTY DAMN THIN}}$ / 10,000

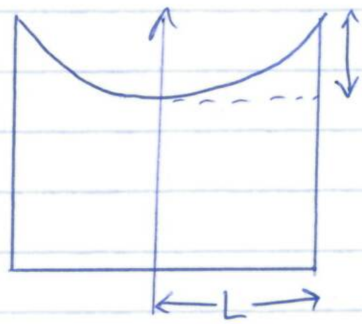
Shallow water equations

Consider a layer of fluid of average depth D and motions with a typical horizontal scale L. Consider the equations in the limit $D/L \to 0$.

We wish to take the undisturbed free surface to be 'horizontal' (ie. $\perp^r$ to local vertical)

For the Earth this is no problem: the undisturbed surface is an equipotential and the local vertical is $\perp^r$ to it.
For the whole Earth, use spherical polars but for the little bits we'll look at we'll use Cartesian.

In the lab we have a bit of a problem



$$\Delta D = \Omega^2 L^2 / 2g$$ from (✱)

We cannot spin the apparatus too fast, we require

$$\frac{\Delta D}{D} = \frac{\Omega^2 L^2}{2gD} \ll 1.$$

Alternative solution to the problem is shape the bottom like the top, although this only works for one rotation shape.

The mom^m equations: From the summary 3pgs ago. $\left[\underline{\Omega} = \begin{pmatrix} 0 \\ 0 \\ \Omega \end{pmatrix}\right]$

$$(1) \quad \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} - 2\Omega v = -\frac{1}{\rho}\frac{\partial p}{\partial x}$$

← dynamic pressure = p

$$(2) \quad \frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} + 2\Omega u = -\frac{1}{\rho}\frac{\partial p}{\partial y}$$

$$(3) \quad \frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial z}$$

$$(4) \quad \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$$

Let a typical value for z be D
" " scale for x, y be L
" " " (u, v) be U
" " " w be W
" " " time t be T
" " " pressure changes be P

We are interested in $\delta \ll 1$ where $\delta = D/L$.

Rewrite:

$$(1)' \quad \frac{U}{T} + \frac{U^2}{L} + \frac{U^2}{L} + \frac{UW}{D} - \Omega U = \frac{P}{\rho L}$$

$$(2)' \quad \frac{U}{T} + \frac{U^2}{L} + \frac{U^2}{L} + \frac{UW}{D} - \Omega U = \frac{P}{\rho L}$$

$$(3)' \quad \frac{W}{T} + \frac{UW}{L} + \frac{UW}{L} + \frac{W^2}{D} = \frac{P}{D\rho}$$

$$(4)' \quad \frac{U}{L} + \frac{U}{L} + \frac{W}{D} = 0$$

$\rightarrow \begin{cases} D \to 0 \\ L \to \infty \end{cases}$

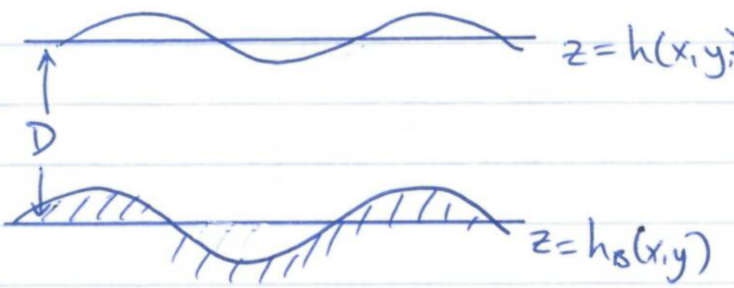
In (4) consider the limit $\delta \to 0$.

$\frac{W}{D}$ cannot be larger (in order, stays finite in limit) than $\frac{U}{L}$ otherwise the eq^n wouldn't equal zero, it needs the $\frac{U}{L}$ to balance it.

$$\Rightarrow \quad O\left(\frac{W}{U}\right) \leq \frac{D}{L} = \delta \quad - - - - - - - - \quad (*)$$

[So in a weather map then, rotation is dominant!]

The bottoms / surfaces of oceans are obviously not flat,



why?

Anyway, thus $\frac{P}{\rho L} \leq \max\left\{\frac{U}{T}, \frac{U^2}{L}, \underset{\text{small}}{\frac{UW}{D}}, 2\Omega U\right\}$

but $\frac{W}{D} \leq O\left(\frac{U}{L}\right)$ so $\frac{UW}{D} \leq O\left(\frac{U^2}{L}\right)$

i.e. $P \leq \max \rho U \left\{\frac{L}{T}, U, 2\Omega L\right\}$

Look at (3). Consider the ratio of the vertical acceleration to the pressure gradient.

$$\frac{O\left(\rho \frac{Dw}{Dt}\right) \;\leftarrow \text{RHS of (3)}}{O\left(\frac{\partial p}{\partial z}\right) \;\leftarrow \text{RHS of (3)}} = \frac{\rho \max\left\{\frac{W}{T}, \frac{UW}{L}\right\} \cdot D}{\rho U \max\left\{\frac{L}{T}, U, 2\Omega L\right\}}$$

$$= \frac{\frac{D\rho W}{L} \max\left\{\frac{L}{T}, U\right\}}{\rho U \max\left\{\frac{L}{T}, U, 2\Omega L\right\}}$$

$$= \frac{WD}{UL} \frac{\max\left\{\frac{L}{T}, U\right\}}{\max\left\{\frac{L}{T}, U, 2\Omega L\right\}}$$ non-dimensional.

but $\frac{D}{L} = \delta$ and $\frac{W}{U} \leq O(\delta)$ by (*)

$$\leq O(\delta^2) \underbrace{\frac{\max\left\{\frac{L}{T}, U\right\}}{\max\left\{\frac{L}{T}, U, 2\Omega L\right\}}}_{\leq 1 \text{ obviously}}$$ a very small term indeed!

Hence to a higher order ($O(\delta^2)$) the vertical momentum eq^n becomes simply $\frac{\partial p}{\partial z} = 0$, where p is the dynamic pressure

where from?

i.e. no vertical variation in the dynamic pressure
i.e. at any point, total pressure is hydrostatic.

ASIDE: if $2\Omega L > U$ i.e. $\varepsilon = U/2\Omega L < 1$ ("Rossby no.")
then in steady flow,
$$\frac{O\left(\rho \frac{Dw}{Dt}\right)}{O\left(\frac{\partial p}{\partial z}\right)} = \varepsilon \delta^2$$

which can be small even if $\delta \sim 1$ provided $\varepsilon \ll 1$
i.e. flow is sufficiently rapidly rotating.

e.g in a lab tank

$\delta \sim 1$, $D/L \sim 1$, $\varepsilon \ll 1$

on Earth $\delta \ll 1$

in Ocean $\varepsilon \ll 1$

in Atmosphere $\varepsilon \lessapprox O(1)$ Gin smells.

The total pressure is

$$p_T = p_0 - \rho g z + p_D$$

On the free surface, the pressure is constant, $p = p_{atm}$.

At $z = h(x,y,t)$, $\quad p_{atm} = p_0 - \rho g h + p_D$

$$\frac{\partial}{\partial x}: \quad \frac{\partial p_D}{\partial x} = \rho g \frac{\partial h}{\partial x}$$

$$\frac{\partial}{\partial y}: \quad \frac{\partial p_D}{\partial y} = \rho g \frac{\partial h}{\partial y}$$

So we can replace those p derivatives in our momentum eq^{n}s with h derivatives, and they become

can call this side the 'forcing'

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} - 2\Omega v = -g\frac{\partial h}{\partial x}$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} + 2\Omega u = -g\frac{\partial h}{\partial y}$$

NII says Accel = F/~~m~~ unit mass (LHS here is like accel.)

But RHS is the same at all depths z,

ie. the FORCE is the same at every depth

ie the ACCEL is the same at every depth !!

So if we start off a flow which is independent of depth, it will remain so. Thus we can consider flows where

$$u = u(x, y, t)$$
$$v = v(x, y, t)$$

ie. 2D flows.

So horizontal mom. eq^{n}s become

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} - 2\Omega v = -g\frac{\partial h}{\partial x}$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + 2\Omega u = -g\frac{\partial h}{\partial y}$$

or

$$\frac{Du}{Dt} - 2\Omega v = -g\frac{\partial h}{\partial x}$$

$$\frac{Dv}{Dt} + 2\Omega u = -g\frac{\partial h}{\partial y}$$

$$\left[\frac{D}{Dt} = \frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y}\right]$$

Vertical momentum: $p_T = p_0 - \rho g z + p_D$ <u>not</u> helpful.

We get our third eq^n using continuity

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$$

ie. $$\frac{\partial w}{\partial z} = \underbrace{-\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right)}$$

indpt of z because 'these guys' are indpt of z.

Integrate from bottom to top of channel:

$z = h(x,y,t)$

$z = h_B(x,y)$

$$\int_{h_B}^{h} \frac{\partial w}{\partial z}\,dz = -\int_{h_B}^{h} \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) dz$$

$$w\Big|_{h_B}^{h} = -\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right)(h - h_B)$$

$$= -\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) H \quad \text{local height}$$

$$= -H \nabla \cdot \underline{u} \qquad \left(\nabla \cdot \underline{u} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) \quad \cdots (\dagger)$$

Surface b.c.: (dynamic already done, ie $p = p_{atm}$)

Kinematic: particle on surface stays on surface

i.e. $z = h(x,y,t)$ on $z = h(x,y,t)$ $\forall\, x,y,t$

$$\text{i.e. } \frac{Dz}{Dt} = \left(\frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + w\frac{\partial}{\partial z}\right) z = \frac{Dh}{Dt}$$

i.e. $w = \dfrac{Dh}{Dt}$ on $z = h(x,y,t)$.

Same argument on bottom

$$w = \frac{Dh_B}{Dt} \text{ on } z = h_B(x,y)$$

Sub into $(\dagger)$: $\dfrac{Dh}{Dt} - \dfrac{Dh_B}{Dt} = -H \nabla \cdot \underline{u}$

$H = h - h_B$.

$$\text{i.e. } \frac{DH}{Dt} + H \nabla \cdot \underline{u} = 0,$$

the 3rd eq?

Thus we have a closed system, the rotating shallow water eq^{n}s (rSWE) are

rSWE

$$\boxed{\begin{cases} \dfrac{Du}{Dt} - 2\Omega v = -g\dfrac{\partial h}{\partial x} \\[2ex] \dfrac{Dv}{Dt} + 2\Omega u = -g\dfrac{\partial h}{\partial y} \\[2ex] \dfrac{DH}{Dt} + H\nabla\cdot\underline{u} = 0 \end{cases}}$$

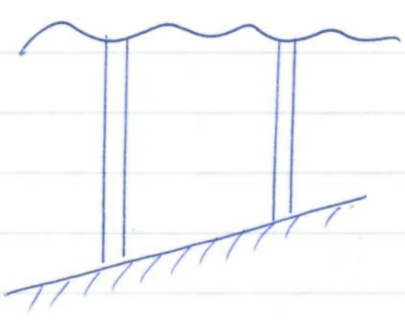
(Totally analogous w/o rotation to the 2D compressible Euler)

$H = h - h_B$

a lot of the weather is in here!
Extremely important!!

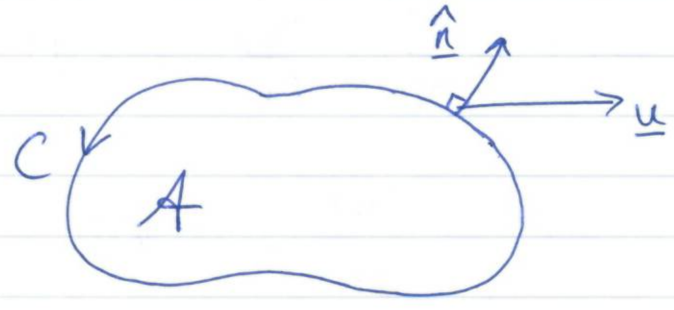
These are obviously non-linear: we've made no assumption (yet) of smallness.

Properties: (1)



Since $\dfrac{\partial u}{\partial z} = \dfrac{\partial v}{\partial z} = 0$, vertical columns of fluid remain vertical.

(2) Interpretation of eq^n 3 above.
Suppose we have an area A bounded by a curve C



The rate of increase of the area of A is

$$\frac{dA}{dt} = \oint_C \underline{u} \cdot \hat{\underline{n}} \, ds$$

$$= \int_A \nabla \cdot \underline{u} \, dA$$

$$\approx A \,.\, \nabla \cdot \underline{u}$$ provided A is sufficiently small.

ie. $\nabla \cdot \underline{u} \approx \frac{1}{A}\frac{dA}{dt}$

ie in 2D, $\nabla \cdot \underline{u}$ is the fractional rate of increase of an infinitessimal area (notice the coordinate-free def^n of div)

But the continuity eq^n in the SWE gives

$$\frac{DH}{Dt} + H \nabla \cdot \underline{u} = 0$$

ie. $\nabla \cdot \underline{u} = -\frac{1}{H}\frac{DH}{Dt}$

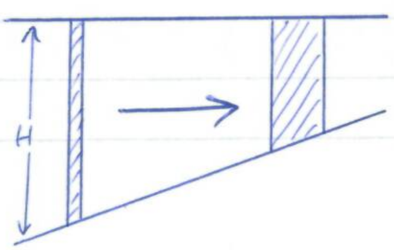
thus $\frac{1}{A}\frac{DA}{Dt} + \frac{1}{H}\frac{DH}{Dt} = 0$

ie. $A\frac{DH}{Dt} + H\frac{DA}{Dt} = 0$

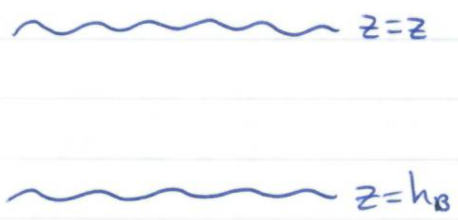
ie. $\frac{D(AH)}{Dt} = 0$

ie. AH following a column is conserved

ie. columns conserve volume



(3)



lower b.c.: particle on $z=h_B$ stays there

$$\frac{Dz}{Dt} = \frac{Dh_B}{Dt} \quad \text{on } z=h_B$$

at any level $z=z$,

$$\frac{Dz}{Dt} = w$$

Now consider $\displaystyle \frac{\partial w}{\partial z} = -\underbrace{\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right)}_{\text{indpt of } z}$

[Continuity: integrated from bottom to top]

Now integrate from bottom $z=h_B$ to $z=z$

$$\int_{z=h_B}^{z} \frac{\partial w}{\partial z}\, dz = -\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) \int_{z=h_B}^{z} dz$$

ie. $\displaystyle \Big[w\Big]_{h_B}^{z} = -(\nabla\cdot\underline{u})(z-h_B)$

ie. $\displaystyle \frac{Dz}{Dt} - \frac{Dh_B}{Dt} = -(z-h_B)\,\nabla\cdot\underline{u}$

ie $\displaystyle \frac{1}{z-h_B}\frac{D}{Dt}(z-h_B) = -\nabla\cdot\underline{u}$

before $\displaystyle \frac{1}{h-h_B}\frac{D}{Dt}(h-h_B) = -\nabla\cdot\underline{u}$

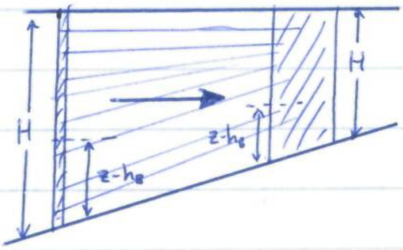
$H = h - h_B$

ie $$\frac{1}{z-h_B}\frac{D}{Dt}(z-h_B) - \frac{1}{H}\frac{DH}{Dt} = 0$$

ie. $$H\frac{D}{Dt}(z-h_B) - (z-h_B)\frac{DH}{Dt} = 0$$

ie. $$\frac{D}{Dt}\left[\frac{z-h_B}{H}\right] = 0$$

ie. $\frac{z-h_B}{H}$ is conserved following columns



ie. the fractional height of a particle remains constant during motion

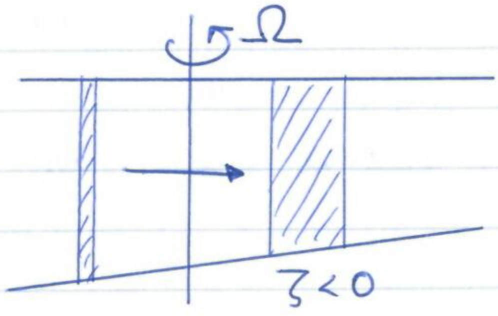
ie. particle at bottom stays there
" " top " "
" " $\frac{1}{2}$ way up " "
" " $\frac{1}{3}$ way up " "

$\Rightarrow$ vertical motion is trivial

So we can concentrate on looking at horizontal motion.

(4) Potential vorticity

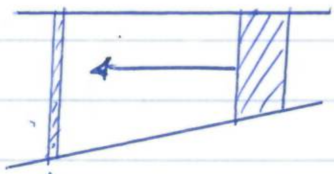
(a) The tall thin column moves into shallower water.



That's the end of the story if you're not in a rotating frame.

$\Rightarrow$ to conserve angular momentum, the column spins more slowly.
i.e. relative to rotating frame; negative relative vorticity.

Recall vorticity $\zeta = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}$ relative to rotating frame



stretched $\zeta > 0$

ie all motion where the bottom is not flat and $\Omega \neq 0$ generates vorticity, ie. cannot be irrotational

ie no velocity potentials, no complex velocities etc. but there is still a streamfunction.

(b) More rigorous argument:

Take a region A where the absolute vorticity ζ_{abs} (twice the local rate of rotation about the centre of mass of A).

The total vorticity in A is thus

$$\int_A \zeta_{abs}\, dA \approx A\zeta_{abs} \quad \text{for sufficiently small } A.$$

But we already have $\frac{D}{Dt}(AH) = 0$

ie. AH is const. following columns

But $A\zeta_{abs}$ is constant following columns (twice total angular momentum) so $\dfrac{\zeta_{abs}}{H}$ is constant following columns

But

$$\underset{\substack{\uparrow \\ \text{absolute} \\ \text{angular} \\ \text{velocity}}}{\frac{1}{2}\zeta_{abs}} = \underset{\substack{\uparrow \\ \text{frame} \\ \text{rotation}}}{\Omega} + \underset{\substack{\uparrow \\ \text{relative} \\ \text{angular} \\ \text{momentum}}}{\frac{1}{2}\zeta}$$

ie. $\zeta_{abs} = 2\Omega + \zeta$

Thus $q = \dfrac{\zeta + 2\Omega}{H}$ is conserved following columns.

q is called the potential vorticity ie. PV.

Thus if H doubles $\zeta + 2\Omega$ must double.

So if $\zeta = 0$ initially $\zeta = 2\Omega$ if H doubles
if $\zeta = 0$ and H halves then $\zeta = -\Omega$.

HOMEWORK: Prove this from the rSWE

$$\frac{Dq}{Dt} = 0$$

(1) $\dfrac{Du}{Dt} - 2\Omega v = -g\dfrac{\partial h}{\partial x}$

(2) $\dfrac{Dv}{Dt} + 2\Omega u = -g\dfrac{\partial h}{\partial y}$

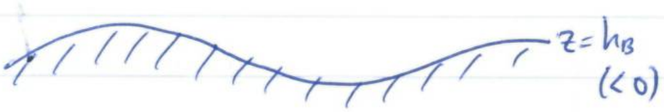
$\dfrac{\partial}{\partial y}(1) - \dfrac{\partial}{\partial x}(2)$: eq^n for ζ

use (3) to eliminate $\nabla \cdot \underline{u}$.

Linearised SWE

We will consider small amplitude wavelike sol^{n}s of the rSWE. We will find the rotationally-modified remnants of usual surface water waves. But also, a totally new wave: the Rossby wave.

$|\nabla h| \ll 1$



$|\nabla h| \sim \varepsilon \ll 1$

$u, v \sim \varepsilon$

$h_B \sim O(1)$

$$\underbrace{\frac{\partial u}{\partial t}}_{\varepsilon} + \underbrace{u\frac{\partial u}{\partial x}}_{\varepsilon^2} + \underbrace{v\frac{\partial u}{\partial y}}_{\varepsilon^2} - \underbrace{2\Omega v}_{\varepsilon} = \underbrace{-g\frac{\partial h}{\partial x}}_{\varepsilon}$$

For sufficiently small waves,

$$\frac{\partial u}{\partial t} - 2\Omega v = -g\frac{\partial h}{\partial x} \quad \cdots\cdots (1)$$

$$\frac{\partial v}{\partial t} + 2\Omega u = -g\frac{\partial h}{\partial y} \quad \cdots\cdots (2)$$

vol. cons. $\quad H(x,y,t) = \underbrace{h(x,y,t)}_{\varepsilon} - \underbrace{h_B(x,y)}_{1}$

rSWE. $\quad \frac{DH}{Dt} + H\nabla\cdot\underline{u} = 0$

Conservation of mass eq^n $\Longrightarrow$ $\quad \frac{\partial H}{\partial t} + (\underline{u}\cdot\nabla)H + H\nabla\cdot\underline{u} = 0$ $\rightarrow$ $\left[\text{alternative form } \quad \underbrace{\frac{\partial H}{\partial t}}_{\varepsilon} + \underbrace{\nabla\cdot(H\underline{u})}_{\varepsilon} = 0\right]$

$\Rightarrow$ to order ε, $\quad \frac{\partial H}{\partial t} + \nabla\cdot(\underline{u}H_0) = 0$

where $H_0(x,y)$ is the undisturbed depth $(-h_B(x,y))$

giving us:

$$\left\{\begin{aligned} &\frac{\partial u}{\partial t} - 2\Omega v = -g\frac{\partial h}{\partial x} \quad \cdots (1)\\ &\frac{\partial v}{\partial t} + 2\Omega u = -g\frac{\partial h}{\partial y} \quad \cdots (2)\\ &\frac{\partial h}{\partial t} + \underline{\nabla}\cdot(H_0 \underline{u}) = 0 \quad \cdots (3) \end{aligned}\right\}$$

lSWE when $H_0 = D = \text{const.}$

From the momentum eq^{n}s:

$$\frac{\partial \underline{u}}{\partial t} + 2\underline{\Omega}\times\underline{u} = -g\underline{\nabla}h \quad \cdots (4) \qquad \underline{\Omega} = \Omega\hat{\underline{z}}$$

$$2\Omega\hat{\underline{z}}\times(4): \quad \frac{\partial}{\partial t}\left(2\Omega\hat{\underline{z}}\times\underline{u}\right) + 4\Omega^2\left(\hat{\underline{z}}\times(\hat{\underline{z}}\times\underline{u})\right) = -2\Omega g\hat{\underline{z}}\times\underline{\nabla}h$$

$$\text{i.e.} \quad \frac{\partial}{\partial t}\left(2\underline{\Omega}\times\underline{u}\right) - 4\Omega^2\underline{u} = -2\Omega g\hat{\underline{z}}\times\underline{\nabla}h \quad \cdots (5)$$

$$\frac{\partial}{\partial t}(4): \quad \frac{\partial^2}{\partial t^2}\underline{u} + \frac{\partial}{\partial t}\left(2\underline{\Omega}\times\underline{u}\right) = -g\underline{\nabla}\frac{\partial h}{\partial t} \quad \cdots (6)$$

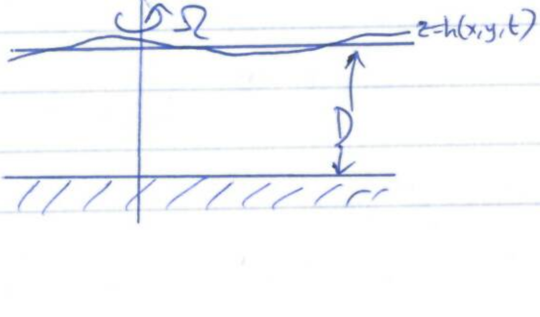
$$(6)-(5) \quad \boxed{\left(\frac{\partial^2}{\partial t^2} + 4\Omega^2\right)\underline{u} = -g\underline{\nabla}\frac{\partial h}{\partial t} + 2\Omega g\hat{\underline{z}}\times\underline{\nabla}h} \quad \cdots (7)$$

Mom^n eq^n

i.e. velocity in terms of the surface slope.

True for lSWE (linearised), always.

We will simplify our discussion by first considering only flow over a flat bottom, i.e. take $H_0(x,y) = D = \text{const}$



Then $\frac{\partial h}{\partial t} + D\underline{\nabla}\cdot\underline{u} = 0$ . . . . (8)

Operate on (8) with $\left(\frac{\partial^2}{\partial t^2} + 4\Omega^2\right)$ to turn $\underline{u}$'s into $\underline{\nabla}h$'s by (7)

$$\left(\frac{\partial^2}{\partial t^2} + 4\Omega^2\right)\frac{\partial h}{\partial t} + D\underline{\nabla}\cdot\left[-g\underline{\nabla}\frac{\partial h}{\partial t} + 2\Omega g\hat{\underline{z}}\times\underline{\nabla}h\right] = 0$$

$$\left(\frac{\partial^2}{\partial t^2} + 4\Omega^2\right)\frac{\partial h}{\partial t} - c^2\frac{\partial}{\partial t}\nabla^2 h = 0$$

where $c^2 = gD$, i.e. $c = \sqrt{gD}$ the speed of the longest wave

i.e. $$\boxed{\frac{\partial^2 h}{\partial t^2} + 4\Omega^2 h = c^2\nabla^2 h}$$ ← KLEIN–GORDON EQ^{N} for the surface elevation h, alone.

In non-rotating flow, $\Omega = 0$,

$$\frac{\partial^2 h}{\partial t^2} = c^2\nabla^2 h$$ 2D wave eq^{n} with wavespeed $c = \sqrt{gD}$

For motions indpt of y, 1D, $\frac{\partial^2 h}{\partial t^2} = c^2\frac{\partial^2 h}{\partial x^2}$

with sol^{n} of the form $h(x,t) = g_1(x+ct) + g_2(x-ct)$

2 waves: one to left with speed c, one to right with speed c.

HOMEWORK: Solve the non-rotating long wave 1D eq^{n} i.e.

$$\frac{\partial^2 h}{\partial t^2} = c^2\frac{\partial^2 h}{\partial x^2} \quad \text{subject to} \quad \frac{\partial h}{\partial t}(x,0) = 0$$

i.e. the surface is released from rest:

and $h(x,0) = -\eta_0\,\mathrm{sign}(x)$ (D'Alembert)

Sketch the sol^n of $t=1$, $t=10$, $t=\infty$.

General wave jargon

In (locally) constant coefficient eq^{n}s we can look for sol^{n}s of the form

$$\eta(x,y,t) = \text{Re}\left[A e^{i(kx+ly-\omega t)} \right]$$

where A, k, l, ω are constants

A	"amplitude" (possibly complex)
$\theta = kx+ly-\omega t$	argument of the complex n° "phase"
k	"x-wavenumber" (n° of waves in distance 2π)
l	"y-wavenumber"
ω	"frequency"

The sol^n repeats itself every time θ increases by 2π.
Thus the temporal period of motion $\tau = 2\pi/\omega$.

We can write this as

$$\eta = A e^{i(\underline{k}\cdot\underline{r} - \omega t)}$$

where $\underline{k} = \begin{pmatrix} k \\ l \\ m \end{pmatrix}$ "wave n° vector"

and $\underline{r} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ "position vector"

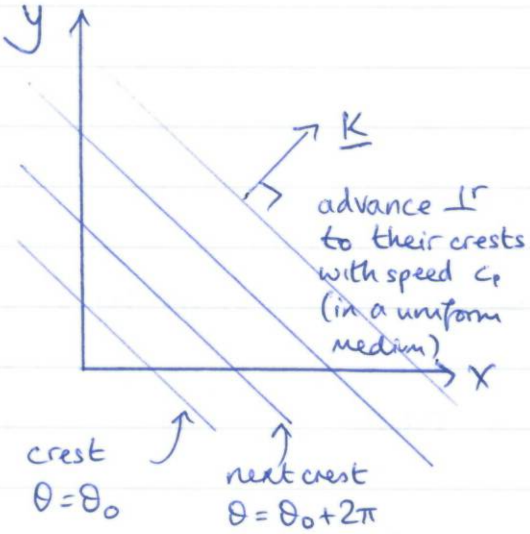
can be 2D as well.

Then the lines (in 2D) or surfaces (in 3D) of constant phase (at a given time) satisfy $\underline{k}\cdot\underline{r} = a$, a const.

ie. planes so we call them plane wave sol^{n}s

with normal $\underline{k}$ and dist from O, $a/|\underline{k}|$.

Lines of constant phase



The line with phase θ_0 has $\underline{k} \cdot \underline{r} = \theta_0 + \omega t$ and lies at a distance $\dfrac{\theta_0 + \omega t}{|\underline{k}|}$ from the origin, which increases at the constant rate $\dfrac{\omega}{|\underline{k}|}$ with time.

i.e. lines of constant phase advance in the dir^n $\underline{k}$ with speed $c_p = \omega/|\underline{k}|$ where c_p = phase speed.

We can define a velocity, the phase velocity,

$$\underline{c}_p = c_p \hat{\underline{k}} = \frac{\omega}{|\underline{k}|} \hat{\underline{k}} = \frac{\omega}{k^2} \underline{k}$$

The phase increases by 2π (and so f^n unchanged) when t increases by $2\pi/\omega$, i.e. motion at any point has period $\tau = 2\pi/\omega$

Distance from origin of crest $\theta = \theta_0$ is $\dfrac{\theta_0 + \omega t}{k}$

. . . . $\theta = \theta_0 + 2\pi$ is $\dfrac{\theta_0 + 2\pi + \omega t}{k}$.

Difference is $\dfrac{2\pi}{k} = \lambda$, the wavelength

$$c_p = \frac{\omega}{k} = \frac{2\pi}{\tau} \cdot \frac{\lambda}{2\pi} = \frac{\lambda}{\tau}$$

ie. crest has travelled a dist λ in time τ, ie. the crest speed is λ/τ.

Consider two waves of the same amplitude and similar wavenumber and frequency, ie.

$$\eta_1 = A\cos\left[(k+\delta)x - (\omega+\varepsilon)t\right]$$
$$\eta_2 = A\cos\left[(k-\delta)x - (\omega-\varepsilon)t\right]$$

Form the composite wave, $\eta = \eta_1 + \eta_2$.

HOMEWORK Sketch the wave when $\delta, \varepsilon \ll 1$. In this case, what is the phase speed? [approx. $\varepsilon \ll 1$, $\delta \ll 1$]. What is the speed of the envelope? (particularly if we have a relⁿ $\omega = F(k)$ relating frequency to wavenumber)

Return to Klein-Gordon eqⁿ

$$\frac{\partial^2 \eta}{\partial t^2} + f^2\eta = c^2\nabla^2\eta \qquad f = 2\Omega$$

Look for solⁿs of the form $\eta = \mathrm{Re}\left[Ae^{i(kx+ly-\omega t)}\right]$

This will be a real solⁿ iff $\eta = Ae^{i(kx+ly-\omega t)}$ is a solution.

Here $\eta = Ae^{i\theta}$

$$\frac{\partial \eta}{\partial x} = ikAe^{i\theta}$$

$$\Rightarrow \quad (-i\omega)^2 Ae^{i\theta} + f^2 Ae^{i\theta} = c^2\left[(ik)^2 + (il)^2\right]Ae^{i\theta}$$

For non-trivial solⁿs, $\eta = Ae^{i\theta} \neq 0$.

$$f^2 - \omega^2 = c^2(-k^2 - l^2)$$
$$\Rightarrow \quad \omega^2 = c^2K^2 + f^2 \qquad \left[|\underline{K}| = \sqrt{k^2+l^2}\right]$$

This is the relⁿ between the frequency ω and wavenº vector $\underline{K} = k\hat{\underline{x}} + l\hat{\underline{y}}$.

Note (1) ω depends only on the magnitude of $\underline{k}$, not its direction.

(2) ~~The~~ If the flow is not rotating, $\omega^2 = c^2 k^2$

$$\frac{\omega}{k} = \pm c$$

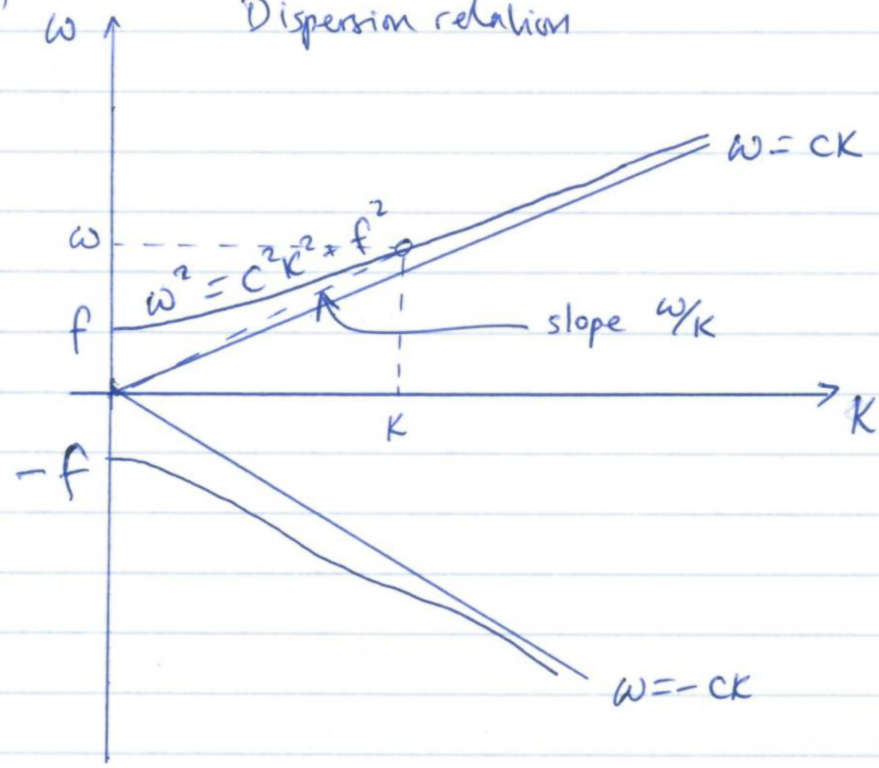
i.e. $c_p = \pm c$

i.e. phase speed is the usual long-wave speed $\sqrt{gD}$

(3) If the flow is rotating then $\frac{\omega}{k} = \sqrt{c^2 + \frac{f^2}{k^2}} > |c|$

i.e. waves travel faster

(4) Dispersion relation



Phase speed at wave $P(k)$ is slope of chord joining O to P.

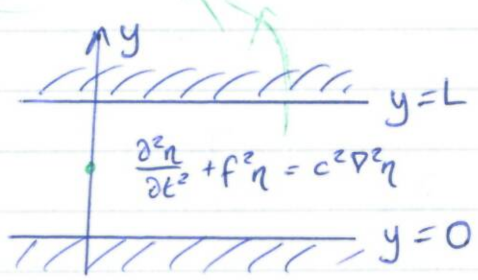
These waves are rotation-modified long surface waves aka tides.

They were studied initially by Kelvin but these waves are called Poincaré waves.

So far there are no boundaries: these are the open ocean tides. They are strongly affected by boundaries. Thus consider sol^{n}s of Klein-Gordon eq^n in the presence of boundaries.

Poincaré and Kelvin waves in a channel

Consider sol^{n}s of the KG eq^n in a channel of width L.



Plan view (ie. rotation axis comes out of page)

Governing eq^n: $$\frac{\partial^2\eta}{\partial t^2} + f^2\eta = c^2\left(\frac{\partial^2\eta}{\partial x^2} + \frac{\partial^2\eta}{\partial y^2}\right) \quad \ldots\ldots (1)$$

b.c.s: No normal flow at solid boundaries

ie. $\underline{u}\cdot\underline{\hat{n}} = 0$ at $y = 0, L$

ie. $v = 0$ at $y = 0, L$.

Now $$\left(\frac{\partial^2}{\partial t^2} + f^2\right)\underline{u} = -g\,\underline{\nabla}\frac{\partial\eta}{\partial t} + \underbrace{fg\,\underline{\hat{z}}\times\underline{\nabla}\eta} \qquad \left(\text{mom}^m \text{ eq}^n\right)$$

Take y component:

$$\left(\frac{\partial^2}{\partial t^2} + f^2\right)v = -g\frac{\partial^2\eta}{\partial t\,\partial y} + fg\frac{\partial\eta}{\partial x}$$

$\hat{y}\cdot\hat{z}\times\nabla\eta$
$= (\hat{y}\times\hat{z})\cdot\nabla\eta$
$= \hat{x}\cdot\nabla\eta$
$= \frac{\partial\eta}{\partial x}$

$v = 0 \;\forall t$ at $y = 0, L$

So LHS $\equiv 0$ at $y = 0, L$

so $$f\frac{\partial \eta}{\partial x} - \frac{\partial^2 \eta}{\partial t \partial y} = 0 \quad \text{at } y = 0, L. \quad \cdots (2)$$

So we will solve (1) subject to (2).

If it were unbounded, we'd look for sol^{n}s of the form

$$e^{ikx + ily - i\omega t}$$

ok ok

not OK because we're bounded in y

So look for sol^{n} of form

$$\eta(x,y,t) = \mathrm{Re}\left[\bar{\eta}(y)e^{ikx - i\omega t}\right]$$

Sub into dispersion relation to give

$$(-i\omega)^2\bar{\eta} + f^2\bar{\eta} = c^2(ik)^2\bar{\eta} + c^2\bar{\eta}''$$

i.e. $$(\omega^2 - f^2)\bar{\eta} = c^2k^2 - c^2\bar{\eta}'' \quad \cdots (3)$$

i.e. $$\bar{\eta}'' + \alpha^2\bar{\eta} = 0 \quad \text{where} \quad \alpha^2 = \frac{\omega^2 - f^2}{c^2} - k^2$$

The b.c.$^{(2)}$ becomes $ikf\bar{\eta} - (-i\omega)\eta' = 0 \quad y = 0, L.$

i.e. $$\bar{\eta}' + \frac{kf}{\omega}\bar{\eta} = 0 \qquad y = 0, L. \quad \cdots (3a)$$

The general sol^{n} of (3) is

$$\bar{\eta} = A\sin\alpha y + B\cos\alpha y$$

(3a) becomes ↓

b.c's

$$\Rightarrow \quad A\alpha + \frac{kf}{\omega}B = 0 \qquad \text{when } \underline{(y=0)}$$

and $$A\alpha\cos\alpha L - B\alpha\sin\alpha L + \frac{kf}{\omega}(A\sin\alpha L + B\cos\alpha L) = 0$$

when $\underline{(y=L)}$.

In matrix form:

$$\begin{pmatrix} \alpha\cos\alpha L + \frac{kf}{\omega}\sin\alpha L & -\alpha\sin\alpha L + \frac{kf}{\omega}\cos\alpha L \\ \alpha & \frac{kf}{\omega} \end{pmatrix}\begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

For a non-trivial sol^n, the determinant must vanish, ie.

$$\left(\alpha^2 + \frac{k^2 f^2}{\omega^2}\right)\sin\alpha L = 0$$

$$\Rightarrow \text{either} \quad \sin\alpha L = 0 \quad \ldots (i)$$

$$\text{or} \quad \alpha^2 + \frac{k^2f^2}{\omega^2} = 0 \quad \ldots (ii)$$

<u>Case (i)</u> $\sin\alpha L = 0 \Rightarrow \alpha L = n\pi \qquad n = \pm 1, \pm 2, \ldots$

$$\Rightarrow \alpha = \frac{n\pi}{L} \qquad n = 1, 2, \ldots$$

These are modes with frequencies:

$$\omega^2 = c^2(k^2 + \alpha^2) + f^2$$

$$= c^2\left(k^2 + \left(\frac{n\pi}{L}\right)^2\right) + f^2,$$ a discrete set of frequencies.

These are Poincaré waves with a quantised y-wavenumber.

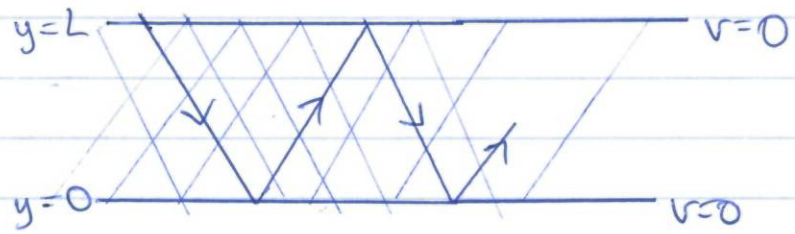
Our sol^n is

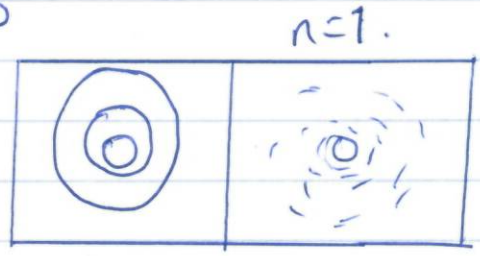
$$\eta(x,y,t) = \mathrm{Re}\left[\overbrace{\left(A\sin\frac{n\pi y}{L} + B\cos\frac{n\pi y}{L}\right)}^{\bar{\eta}} e^{(ikx - i\omega t)}\right].$$

Linear combinations of $e^{\pm i n\pi y/L}$ so our sol^n is a linear combination of

$$e^{i(kx \pm \frac{n\pi}{L}y - \omega t)},$$

two linked Poincaré waves with slopes $\pm \frac{Lk}{n\pi}$





$$c_p = \omega/k = c\sqrt{1 + \left(\frac{n\pi}{kL}\right)^2 + \frac{f^2}{c^2k^2}}$$

Dispersive.

<u>Case (ii)</u> $\quad \alpha^2 + \frac{f^2k^2}{\omega^2} = 0.$

<u>Notice</u> that if the flow is not rotating ($f=0$), this gives $\alpha = 0$ which gives $\bar{\eta} = B$(const), which gives sol^n

$$\eta = A\cos(kx - \omega t)$$

which is fine, it's 2D water waves where waves travel down channel with speed $\pm c = \sqrt{gD}$.

Where is the corresponding sol^n for rotation?

Remember α was defined through

$$\bar{\eta}'' + \alpha^2 \bar{\eta} = 0$$

and a sol^{n} of the form

$$\bar{\eta}(y) = A \sin \alpha y + B \cos \alpha y$$

In case (ii) take $\alpha^2 = -\frac{f^2 k^2}{\omega^2}$, ie. $\alpha = \pm \frac{ifk}{\omega}$

$$\Rightarrow \quad \bar{\eta}(y) = \tilde{A} \sinh\left(\tfrac{fk}{\omega}\right) y + \tilde{B} \cosh\left(\tfrac{fk}{\omega}\right) y$$

$$\text{or} \quad \bar{\eta}(y) = \tilde{\tilde{A}} e^{fky/\omega} + \tilde{\tilde{B}} e^{-fky/\omega}$$

OK, since domain bounded in y.

The waves have frequencies

$$\omega^2 = c^2 (k^2 + \alpha^2) + f^2$$

$$= c^2 \left(k^2 - \frac{f^2 k^2}{\omega^2} \right) + f^2$$

$$\Rightarrow \quad \omega^2 - f^2 = \frac{c^2 k^2}{\omega^2} (\omega^2 - f^2)$$

$$\Rightarrow \quad \omega^2 = c^2 k^2 \quad \text{or} \quad \omega^2 = f^2$$

↳ a part of $\omega^2 = c^2 k^2$, so consider only $\omega = \pm ck$.

- These waves are thus non-dispersive with constant phase speed $c_p = \frac{\omega}{k} = \pm c = \pm\sqrt{gD}$, precisely the non-rotating wave speed.

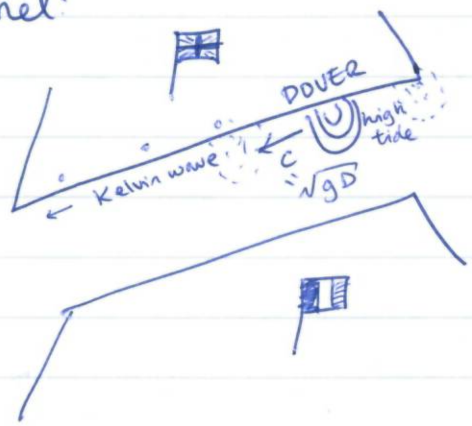
• Arbitrarily strong rotation does not affect the speed of the waves.

Kelvin waves

See sheet !!

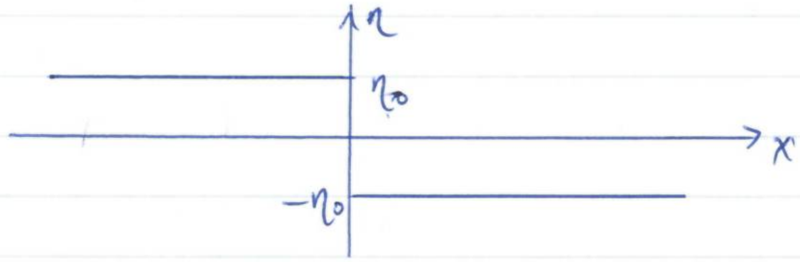
Then come back here ☺

English channel:



Tidal speed: 3–4 knots
Avg speed of boat: 4–6 knot

Rossby–Gill adjustment problem



Consider the initial value problem of a fluid surface released from rest, ie.

$$\frac{\partial \eta}{\partial t}(x,0) = 0 .$$

with initial step discontinuity in the surface

$$\eta(x,0) = -\eta_0 \, \mathrm{sgn}(x) .$$

The Kelvin Wave

The linearised shallow water equations are

$$u_t - fv = -g\eta_x, \tag{1}$$
$$v_t + fu = -g\eta_y, \tag{2}$$
$$\eta_t + D(u_x + v_y) = 0 \tag{3}$$

lSWE

D = const

The roots $\alpha = \pm \mathrm{i}fk/\sigma$ give the solution

$\sigma = \omega$

$$\eta = \Re\{\eta_0 e^{\pm fky/\sigma} e^{\mathrm{i}(kx-\sigma t)}\} = (Ae^{-fky/\sigma} + Be^{fky/\sigma})\cos(kx - \sigma t) \tag{4}$$

Now $\partial_t(2) - f(1)$ gives

$$v_{tt} + f^2 v = -g\eta_{yt} + fg\eta_x = -2Bfgke^{fky/\sigma}\sin(kx - \sigma t). \tag{5}$$

Notice that the coefficient of A vanishes identically. For v to vanish for all time, and so the left side of (5) to vanish for all t ($\sigma \neq \pm f$), at some fixed point (x, y), e.g. even a single point on the wall $y = 0$, equation (5) implies $B = 0$ and so $v \equiv 0$. A Kelvin wave has zero velocity normal to its supporting wall.

Using $v = 0$ in (1) and (3) and then eliminating u gives

$$\eta_{tt} = c^2 \eta_{xx}, \tag{6}$$

where $c = \sqrt{gD}$, the non-rotating wave equation. For this to have solutions of form (4), $\sigma^2 = c^2k^2$, i.e. $\sigma = \pm ck$. (This is precisely the same result as substituting for α in $\alpha = \pm \mathrm{i}fk/\sigma$.)

If $\sigma = +ck$, then (4) becomes

$$\eta = Ae^{-y/a}\cos[k(x - ct)], \tag{7}$$
$$u = (A/D)ce^{-y/a}\cos[k(x - ct)], \tag{8}$$

where $a = c/f$ is the Rossby radius and the form of u comes from (2) with $v = 0$:

faster rotation, smaller Rossby radius.

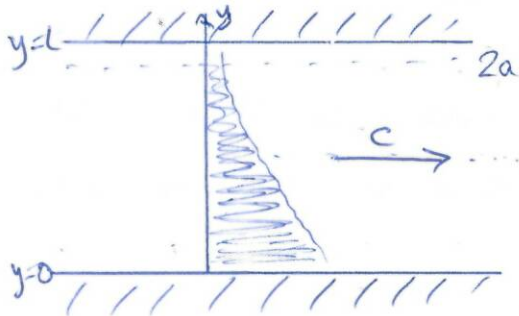
$$u = (gk/\sigma)Ae^{-fky/\sigma}\cos(kx - \sigma t). \tag{9}$$

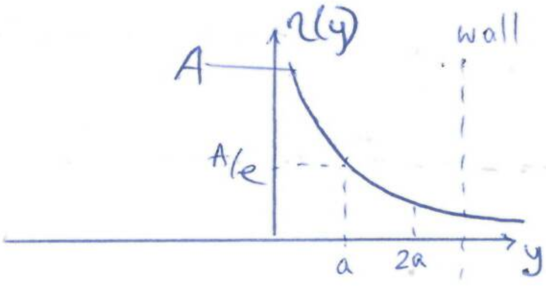
If $\sigma = -ck$, then

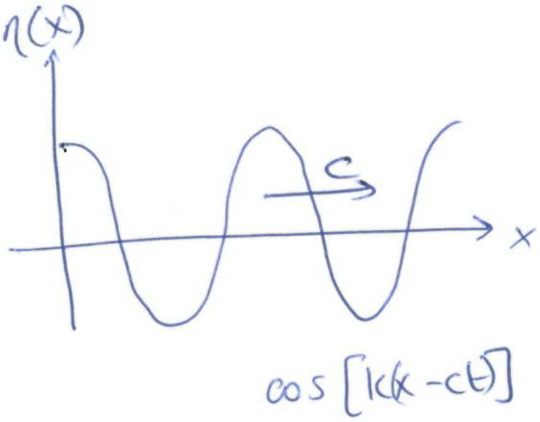
$$\eta = Ae^{y/a}\cos[k(x + ct)], \tag{10}$$
$$u = -(A/D)ce^{y/a}\cos[k(x + ct)]. \tag{11}$$

For both directions of propagation, the Kelvin wave propagates with the $y = 0$ wall to its right and decays exponentially away from the wall on the scale of the Rossby radius, with e-folding scale a, i.e. every time y increases by a, amplitude is multiplied by $1/e$.

$\omega = ck$:

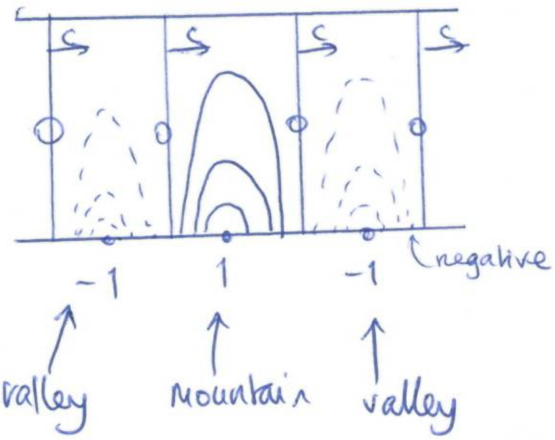






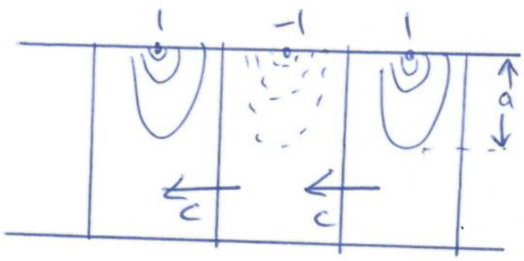
Want to plot $\eta(x,y)$

Contours of $\eta(x,y,t)$ at fixed t, i.e. of surface elevation.
From product of $e^{-y/a}$ and $\cos kx$



$\underline{\omega = -ck}$: $\eta = Ae^{y/a}\cos[k(x+ct)]$

decays exponentially as $y \to -\infty$ propagates with speed c in $-x$ dir^n



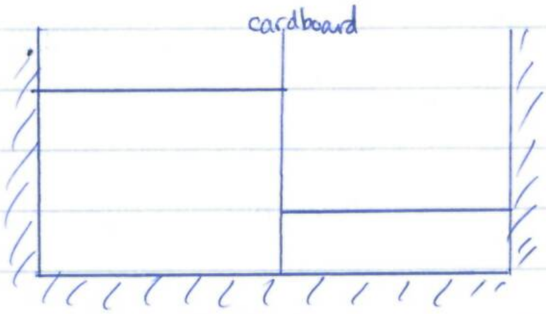
$\eta \neq 0$, $u \neq 0$, $\underline{\underline{v \equiv 0}}$.

Both waves propagate with supporting boundary to their right. (In fact each wave needs only its supporting boundary, it only needs its supporting one).

The governing equation is the lrSWE, ie. the Klein-Gordon eq^n (here $\partial_y \equiv 0$)

$$\frac{\partial^2 \eta}{\partial t} + f^2 \eta = c^2 \frac{\partial^2 \eta}{\partial x^2}$$

How to do this in experiment:



remove cardboard at $t=0$.

What to do with this equation? Well, we're mathematicians; we can solve it:

Look for soln of the form

$$\eta(x,t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{\eta}(k,t)\, e^{ikx}\, dk$$

Then
$$\frac{\partial \eta}{\partial x} = \frac{1}{2\pi} \int_{-\infty}^{\infty} ik\hat{\eta}(k,t)\, e^{ikx}\, dk$$
$$\frac{\partial \eta}{\partial t} = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{d\hat{\eta}}{dt}(k,t)\, e^{ikx}\, dk.$$

Sub into eq^n:

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} \left[\frac{d^2\hat{\eta}}{dt^2} + f^2\hat{\eta} + c^2k^2\hat{\eta} \right] e^{ikx}\, dk = 0$$

This is true if we find $\hat{\eta}(k,t)$ s.t.

$$\frac{d^2\hat{\eta}}{dt^2} + (f^2 + c^2k^2)\hat{\eta} = 0.$$

This has sol^{n}s

$$\hat{\eta}(k,t) = A(k)\cos\omega t + B(k)\sin\omega t$$

where $\omega^2 = f^2 + c^2k^2$

$\leftarrow$ (Poincaré waves!)

Apply b.c.: $\frac{d\eta}{dt}(x,0) = 0$

$$\Rightarrow \frac{d\hat{\eta}}{dt}(k,0) = 0$$

$$\Rightarrow B(k) \equiv 0 .$$

Thus we have

$$\eta(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} A(k)\cos\omega t \, e^{ikx} \, dk$$

At $t=0$, we have

$$\eta(x,0) = \frac{1}{2\pi}\int_{-\infty}^{\infty} A(k) e^{ikx} \, dk$$

$$\eta(x,0) = -\eta_0 \, \mathrm{sgn}(x) .$$

This can be inverted by the Fourier Integral Thm,

$$A(k) = \int_{-\infty}^{\infty} (-\eta_0 \, \mathrm{sgn}x) \, e^{-ikx} \, dx$$

$\left[\text{If } f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{f}(k) e^{ikx} dk \text{ then } \hat{f}(k) = \int_{-\infty}^{\infty} f(x) e^{-ikx} dx \right] \leftarrow$ F.I.T.

We continue by evaluating $A(k)$, and substituting in the integral for $\eta(x,t)$. Then consider this integral. But this is a lot of effort. We're interested in the large time behaviour, ie. $t \to \infty$.

$$\eta(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} A(k)\cos\omega t \, e^{ikx} \, dk$$

$$= \frac{1}{4\pi}\int_{-\infty}^{\infty} A(k)\, e^{i(kx-\omega(k)t)}\, dk$$

$$\left[\omega^2 = f^2 + c^2k^2\right]$$

$$+ \frac{1}{4\pi}\int_{-\infty}^{\infty} A(k)\, e^{i(kx+\omega(k)t)}\, dk,$$

This is a SUPERPOSITION of Poincaré waves

We are interested in the behaviour of these integrals at large time. More generally, of integrals of the form

$$\int_{-\infty}^{\infty} A(k)\, e^{i\Phi(k)t}\, dk \quad \text{as } t \to \infty .$$

Here, $\Phi(k) = k\frac{x}{t} + \omega(k)$.

[see handouts for method. Will come back to it later].

This approach is important because it explains exactly where and how fast energy is transmitted. However, it is possible to go straight to the final state (Rossby, 1953).

Take the lrSWE:

$$\left.\begin{aligned} u_t - fv &= -g\eta_x \quad (1)\\ v_t + fu &= -g\eta_y \quad (2)\\ \eta_t + D(u_x+v_y) &= 0 \quad (3) \end{aligned}\right\} \quad \text{and} \quad \zeta = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}$$

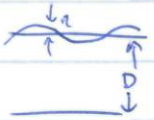
$\frac{\partial}{\partial x}(2) - \frac{\partial}{\partial y}(1)$: $\quad \frac{\partial \zeta}{\partial t} + f\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) = 0$

$$\Rightarrow \quad \zeta_t - \frac{f\eta_t}{D} = 0$$

$$\Rightarrow \quad \frac{\partial q}{\partial t} = 0 \quad \text{where } q = \zeta - f\frac{\eta}{D}.$$

i.e. q is conserved/constant throughout the motion (i.e. indpt of t).

[recall in H/W, $\dfrac{Dq}{Dt}=0$ where $q=\dfrac{\zeta+f}{H}$ for SWE
$f=2\Omega$



$$H = D+\eta = D\left(1+\frac{\eta}{D}\right)$$

$$H^{-1} = D^{-1}\left(1+\frac{\eta}{D}\right)^{-1} = D^{-1}\left(1-\frac{\eta}{D}\right)+O\left(\frac{\eta^2}{D^2}\right)$$

$$q \approx (\zeta+f)D^{-1}\left(1-\frac{\eta}{D}\right)$$

$$= \left(\zeta-\frac{\zeta\eta}{D}+f-f\frac{\eta}{D}\right)/D$$

$$= \cancel{\frac{f}{D}} + \frac{\zeta - f\eta/D}{D} - \cancel{\frac{\zeta\eta}{D^2}}$$

const | conserved | quadratic term

ie. $\dfrac{Dq}{Dt}=0 \Rightarrow \dfrac{\partial q}{\partial t}=0$ $\qquad \dfrac{D}{Dt}=\dfrac{\partial}{\partial t}+\cancel{u\cdot\nabla}$

in linear eq^{n}s
$q=\zeta-f\eta/D$.]

But at $t=0$, $\zeta\equiv 0$
and $\eta=-\eta_0\,\mathrm{sgn}\,x$

So at $t=0$, $q=\dfrac{f}{D}\eta_0\,\mathrm{sgn}\,x$

Hence at $t=\infty$, $q=\dfrac{f}{D}\eta_0\,\mathrm{sgn}\,x$

<u>Assume</u> the flow becomes steady eventually.

Then at $t=\infty$, the flow is geostrophic,
with $u=-\dfrac{g}{f}\eta_y$, $\quad v=\dfrac{g}{f}\eta_x$

Now $f(s) = 1/s$ and $\phi(s) = -s + \ln s$. Laplace's method applies directly to this transformed integral. The maximum of $\phi(s)$ occurs at $s = 1$ so (6.4.19c) gives

$$\Gamma(x) \sim x^x e^{-x}\sqrt{2\pi/x}, \qquad x \to +\infty, \tag{6.4.39}$$

in agreement with (5.4.1). To obtain the next term in the Stirling series we note that $\phi(1) = -1$, $\phi'(1) = 0$, $\phi''(1) = -1$, $\phi'''(1) = 2$, $(d^4\phi/ds^4)(1) = -6$, $f(1) = 1$, $f'(1) = -1$, $f''(1) = 2$. Substituting these coefficients into the formula (6.4.35), we obtain

$$\Gamma(x) \sim x^x e^{-x}\sqrt{\frac{2\pi}{x}}\left(1 + \frac{1}{12x}\right), \qquad x \to +\infty, \tag{6.4.40}$$

in agreement with (5.4.1).

The distinction between ordinary and movable maxima is examined in Probs. 6.45 to 6.47.

(I) 6.5 METHOD OF STATIONARY PHASE

There is an immediate generalization of the Laplace integrals studied in Sec. 6.4 which we obtain by allowing the function $\phi(t)$ in (6.4.1) to be complex. Note that, if we wish, we may assume that $f(t)$ is real; if it were complex, $f(t)$ could be decomposed into a sum of its real and imaginary parts. However, allowing $\phi(t)$ to be complex poses new and nontrivial problems. In this section we consider the special case in which $\phi(t)$ is pure imaginary: $\phi(t) = i\psi(t)$, where $\psi(t)$ is real. The resulting integral

$$I(x) = \int_a^b f(t)e^{ix\psi(t)}\,dt \tag{6.5.1}$$

with $f(t)$, $\psi(t)$, a, b, x all real is called a generalized Fourier integral. When $\psi(t) = t$, $I(x)$ is an ordinary Fourier integral. The general case in which $\phi(t)$ is complex is considered in Sec. 6.6.

To study the behavior of $I(x)$ in (6.5.1) as $x \to +\infty$, we can use integration by parts to develop an asymptotic expansion in inverse powers of x so long as the boundary terms are finite and the resulting integrals exist.

Example 1 *Asymptotic expansion of a Fourier integral as $x \to +\infty$.* We use integration by parts to find an asymptotic approximation to the Fourier integral

$$I(x) = \int_0^1 \frac{e^{ixt}}{1+t}\,dt.$$

After one integration by parts we obtain

$$I(x) = -\frac{i}{2x}e^{ix} + \frac{i}{x} - \frac{i}{x}\int_0^1 \frac{e^{ixt}}{(1+t)^2}\,dt. \tag{6.5.2}$$

The integral on the right side of (6.5.2) is negligible compared with the boundary terms as $x \to +\infty$; in fact, it vanishes like $1/x^2$ as $x \to +\infty$. To see this, we integrate by parts again:

$$-\frac{i}{x}\int_0^1 \frac{e^{ixt}}{(1+t)^2}\,dt = -\frac{1}{4x^2}e^{ix} + \frac{1}{x^2} - \frac{2}{x^2}\int_0^1 \frac{e^{ixt}}{(1+t)^3}\,dt.$$

The integral on the right is bounded because

$$\left|\int_0^1 \frac{e^{ixt}}{(1+t)^3}\,dt\right| \le \int_0^1 (1+t)^{-3}\,dt = \frac{3}{8}.$$

Since the integral on the right in (6.5.2) does vanish like $1/x^2$ as $x \to +\infty$, $I(x)$ is asymptotic to the boundary terms: $I(x) \sim -(i/2x)e^{ix} + i/x$ $(x \to +\infty)$.

Repeated application of integration by parts gives the complete asymptotic expansion of $I(x)$ as $x \to +\infty$: $I(x) = e^{ix}u(x) + v(x)$ where

$$u(x) \sim -\frac{i}{2x} - \frac{1}{4x^2} + \cdots + \frac{(-i)^n(n-1)!}{(2x)^n} + \cdots, \qquad x \to +\infty,$$

$$v(x) \sim \frac{i}{x} + \frac{1}{x^2} + \cdots - \frac{(-i)^n(n-1)!}{x^n} + \cdots, \qquad x \to +\infty.$$

Example 2 *Integration by parts applied to $\int_0^1 \sqrt{t}\,e^{ixt}\,dt$.* Integration by parts can be used just once for the Fourier integral $I(x) = \int_0^1 \sqrt{t}\,e^{ixt}\,dt$. One integration by parts gives

$$I(x) = -\frac{i}{x}e^{ix} + \frac{i}{2x}\int_0^1 \frac{e^{ixt}}{\sqrt{t}}\,dt. \tag{6.5.3}$$

The integral on the right side of (6.5.3) vanishes more rapidly than the boundary term as $x \to +\infty$. We cannot use integration by parts to verify this because the resulting integral does not exist. (Why?) However, we can use the following simple scaling argument. We let $s = xt$ and obtain

$$\frac{i}{2x}\int_0^1 \frac{e^{ixt}}{\sqrt{t}}\,dt = \frac{i}{2x^{3/2}}\int_0^x \frac{e^{is}}{\sqrt{s}}\,ds \sim \frac{i}{2x^{3/2}}\int_0^\infty \frac{e^{is}}{\sqrt{s}}\,ds, \qquad x \to +\infty.$$

To evaluate the last integral we rotate the contour of integration from the real-s axis to the positive imaginary-s axis in the complex-s plane and obtain

$$\int_0^\infty \frac{e^{is}}{\sqrt{s}}\,ds = \sqrt{\pi}\,e^{i\pi/4}. \tag{6.5.4}$$

(See Prob. 6.49 for the details of this calculation.) Therefore,

$$I(x) + \frac{i}{x}e^{ix} \sim \frac{i}{2x^{3/2}}\sqrt{\pi}\,e^{i\pi/4}, \qquad x \to +\infty. \tag{6.5.5}$$

Clearly, this result cannot be found by direct integration by parts of the integral on the right side of (6.5.3) because a fractional power of x has appeared. However, it is possible to find the full asymptotic expansion of $I(x)$ as $x \to +\infty$ by an indirect application of integration by parts (see Prob. 6.50).

In Example 1 we used integration by parts to argue that the integral on the right side of (6.5.2) vanishes more rapidly than the boundary terms as $x \to +\infty$. In Example 2 we used a scaling argument to show that the integral on the right side of (6.5.3) vanishes more rapidly than the boundary terms as $x \to +\infty$. There is, in fact, a very general result called the Riemann-Lebesgue lemma that guarantees that

$$\int_a^b f(t)e^{ixt}\,dt \to 0, \qquad x \to +\infty, \tag{6.5.6}$$

provided that $\int_a^b |f(t)|\,dt$ exists. This result is valid even when $f(t)$ is not differentiable and integration by parts or scaling do not work. We will cite the Riemann-Lebesgue lemma repeatedly throughout this section; we could have used it to justify neglecting the integrals on the right sides of (6.5.2) and (6.5.3).

We reserve a proof of the Riemann-Lebesgue lemma for Prob. 6.51. Although the proof of (6.5.6) is messy, it is easy to understand the result heuristically. When x becomes large, the integrand $f(t)e^{ixt}$ oscillates rapidly and contributions from adjacent subintervals nearly cancel.

The Riemann-Lebesgue lemma can be extended to cover generalized Fourier integrals of the form (6.5.1). It states that $I(x) \to 0$ as $x \to +\infty$ so long as $|f(t)|$ is integrable, $\psi(t)$ is continuously differentiable for $a \le t \le b$, and $\psi(t)$ is not constant on any subinterval of $a \le t \le b$ (see Prob. 6.52). The lemma implies that $\int_0^{10} t^3 e^{ix\sin^2 t}\,dt \to 0$ $(x \to +\infty)$, but it does not apply to $\int_0^{10} t^3 e^{2ix}\,dt$.

Integration by parts gives the leading asymptotic behavior as $x \to +\infty$ of generalized Fourier integrals of the form (6.5.1), provided that $f(t)/\psi'(t)$ is smooth for $a \le t \le b$ and nonvanishing at one of the endpoints a or b. Explicitly,

$$I(x) = \frac{f(t)}{ix\psi'(t)} e^{ix\psi(t)}\Bigg|_{t=a}^{t=b} - \frac{1}{ix}\int_a^b \frac{d}{dt}\frac{f(t)}{\psi'(t)} e^{ix\psi(t)}\,dt.$$

The Riemann-Lebesgue lemma shows that the integral on the right vanishes more rapidly than $1/x$ as $x \to +\infty$. Therefore, $I(x)$ is asymptotic to the boundary terms (assuming that they do not vanish):

$$I(x) \sim \frac{f(t)}{ix\psi'(t)} e^{ix\psi(t)}\Bigg|_{t=a}^{t=b}, \qquad x \to +\infty. \tag{6.5.7}$$

Observe that when integration by parts applies, $I(x)$ vanishes like $1/x$ as $x \to +\infty$.

Integration by parts may not work if $\psi'(t) = 0$ for some t in the interval $a \le t \le b$. Such a point is called a *stationary* point of ψ. When there are stationary points in the interval $a \le t \le b$, $I(x)$ must still vanish as $x \to +\infty$ by the Riemann-Lebesgue lemma, but $I(x)$ usually vanishes less rapidly than $1/x$ because the integrand $f(t)e^{ix\psi(t)}$ oscillates less rapidly near a stationary point than it does near a point where $\psi'(t) \ne 0$. Consequently, there is less cancellation between adjacent subintervals near the stationary point.

The method of stationary phase gives the *leading* asymptotic behavior of generalized Fourier integrals having stationary points. This method is very similar to Laplace's method in that the leading contribution to $I(x)$ comes from a small interval of width ε surrounding the stationary points of $\psi(t)$. We will show that if c is a stationary point and if $f(c) \ne 0$, then $I(x)$ goes to zero like $x^{-1/2}$ as $x \to +\infty$ if $\psi''(c) \ne 0$, like $x^{-1/3}$ if $\psi''(c) = 0$ but $\psi'''(c) \ne 0$, and so on; as $\psi(t)$ becomes flatter at $t = c$, $I(x)$ vanishes less rapidly as $x \to +\infty$.

Since any generalized Fourier integral can be written as a sum of integrals in which $\psi'(t)$ vanishes only at an endpoint, we can explain the method of stationary phase for the special integral (6.5.1) in which $\psi'(a) = 0$ and $\psi'(t) \ne 0$ for $a < t \le b$.

We decompose $I(x)$ into two terms:

$$I(x) = \int_a^{a+\varepsilon} f(t)e^{ix\psi(t)}\,dt + \int_{a+\varepsilon}^b f(t)e^{ix\psi(t)}\,dt, \tag{6.5.8}$$

where ε is a small positive number to be chosen later. The second integral on the right side of (6.5.8) vanishes like $1/x$ as $x \to +\infty$ because there are no stationary points in the interval $a + \varepsilon \le t \le b$.

To obtain the leading behavior of the first integral on the right side of (6.5.8), we replace $f(t)$ by $f(a)$ and $\psi(t)$ by $\psi(a) + \psi^{(p)}(a)(t-a)^p/p!$ where $\psi^{(p)}(a) \ne 0$ but $\psi'(a) = \cdots = \psi^{(p-1)}(a) = 0$:

$$I(x) \sim \int_a^{a+\varepsilon} f(a) \exp\left\{ix\left[\psi(a) + \frac{1}{p!}\psi^{(p)}(a)(t-a)^p\right]\right\} dt, \qquad x \to +\infty. \tag{6.5.9}$$

Next, we replace ε by ∞, which introduces error terms that vanish like $1/x$ as $x \to +\infty$ and thus may be disregarded, and let $s = (t - a)$:

$$I(x) \sim f(a)e^{ix\psi(a)} \int_0^\infty \exp\left[\frac{ix}{p!}\psi^{(p)}(a)s^p\right] ds, \qquad x \to +\infty. \tag{6.5.10}$$

To evaluate the integral on the right, we rotate the contour of integration from the real-s axis by an angle $\pi/2p$ if $\psi^{(p)}(a) > 0$ and make the substitution

$$s = e^{i\pi/2p}\left[\frac{p!\,u}{x\psi^{(p)}(a)}\right]^{1/p} \tag{6.5.11a}$$

with u real or rotate the contour by an angle $-\pi/2p$ if $\psi^{(p)}(a) < 0$ and make the substitution

$$s = e^{-i\pi/2p}\left[\frac{p!\,u}{x|\psi^{(p)}(a)|}\right]^{1/p}. \tag{6.5.11b}$$

Thus,

$$I(x) \sim f(a)e^{ix\psi(a) \pm i\pi/2p}\left[\frac{p!}{x|\psi^{(p)}(a)|}\right]^{1/p}\frac{\Gamma(1/p)}{p}, \qquad x \to +\infty, \tag{6.5.12}$$

where we use the factor $e^{i\pi/2p}$ if $\psi^{(p)}(a) > 0$ and the factor $e^{-i\pi/2p}$ if $\psi^{(p)}(a) < 0$.

The formula in (6.5.12) gives the leading behavior of $I(x)$ if $f(a) \ne 0$ but $\psi'(a) = 0$. If $f(a)$ vanishes, it is necessary to decide whether the contribution from the stationary point still dominates the leading behavior. When it does, the behavior is slightly more complicated than (6.5.12) (see Prob. 6.53).

Example 3 *Leading behavior of* $\int_0^{\pi/2} e^{ix\cos t}\,dt$ *as* $x \to +\infty$. The function $\psi(t) = \cos t$ has a stationary point at $t = 0$. Since $\psi''(0) = -1$, (6.5.12) with $p = 2$ gives $I(x) \sim \sqrt{\pi/2x}\, e^{i(x-\pi/4)}$ $(x \to +\infty)$.

Example 4 *Leading behavior of* $\int_0^\infty \cos(xt^2 - t)\,dt$ *as* $x \to +\infty$. To use the method of stationary phase, we write this integral as $\int_0^\infty \cos(xt^2 - t)\,dt = \mathrm{Re}\int_0^\infty e^{i(xt^2-t)}\,dt$. The function $\psi(t) = t^2$ has a stationary point at $t = 0$. Since $\psi''(0) = 2$, (6.5.12) with $p = 2$ gives $\int_0^\infty \cos(xt^2 - t)\,dt \sim \mathrm{Re}\,\frac{1}{2}\sqrt{\pi/x}\,e^{i\pi/4} = \frac{1}{2}\sqrt{\pi/2x}$ $(x \to +\infty)$.

Example 5 *Leading behavior of* $J_n(n)$ *as* $n \to \infty$. When n is an integer, the Bessel function $J_n(x)$ has the integral representation

$$J_n(x) = \frac{1}{\pi}\int_0^\pi \cos(x \sin t - nt)\,dt \tag{6.5.13}$$

(see Prob. 6.54). Therefore, $J_n(n) = \mathrm{Re}\int_0^\pi e^{in(\sin t - t)}\,dt/\pi$. The function $\psi(t) = \sin t - t$ has a stationary point at $t = 0$. Since $\psi''(0) = 0$, $\psi'''(0) = -1$, (6.5.12) with $p = 3$ gives

$$\begin{aligned} J_n(n) &\sim \frac{1}{\pi}\,\mathrm{Re}\left[\frac{1}{3}e^{-i\pi/6}\left(\frac{6}{n}\right)^{1/3}\Gamma\left(\frac{1}{3}\right)\right], \qquad x \to +\infty,\\ &= \frac{1}{\pi}2^{-2/3}3^{-1/6}\Gamma\left(\frac{1}{3}\right)n^{-1/3}, \qquad n \to \infty. \end{aligned} \tag{6.5.14}$$

Observe that because $\psi''(0) = 0$, $J_n(n)$ vanishes less rapidly than $n^{-1/2}$ as $n \to \infty$.

If n is not an integer, (6.5.14) still holds (see Prob. 6.55).

In this section we have obtained only the leading behavior of generalized Fourier integrals. Higher-order approximations can be complicated because nonstationary points may also contribute to the large-x behavior of the integral. Specifically, the second integral on the right in (6.5.8) must be taken into account when computing higher-order terms because the error incurred in neglecting this integral is usually algebraically small. By contrast, recall that the approximation in (6.4.2) for Laplace's method is valid to all orders because the errors are exponentially, rather than algebraically, small. To obtain the higher-order corrections to (6.5.12), one can either use the method of asymptotic matching (see Sec. 7.4) or the method of steepest descents (see Sec. 6.6).

(I) ## 6.6 METHOD OF STEEPEST DESCENTS

The method of steepest descents is a technique for finding the asymptotic behavior of integrals of the form

$$I(x) = \int_C h(t)e^{x\rho(t)}\,dt \tag{6.6.1}$$

as $x \to +\infty$, where C is an integration contour in the complex-t plane and $h(t)$ and $\rho(t)$ are analytic functions of t. The idea of the method is to use the analyticity of the integrand to justify deforming the contour C to a new contour C' on which $\rho(t)$ has a constant imaginary part. Once this has been done, $I(x)$ may be evaluated asymptotically as $x \to +\infty$ using Laplace's method. To see why, observe that on the contour C' we may write $\rho(t) = \phi(t) + i\psi$, where ψ is a real constant and $\phi(t)$ is a real function. Thus, $I(x)$ in (6.6.1) takes the form

$$I(x) = e^{ix\psi}\int_{C'} h(t)e^{x\phi(t)}\,dt. \tag{6.6.2}$$

Although t is complex, (6.6.2) can be treated by Laplace's method as $x \to +\infty$ because $\phi(t)$ is real.

Our motivation for deforming C into a path C' on which $\mathrm{Im}\,\rho(t)$ is a constant is to eliminate rapid oscillations of the integrand when x is large. Of course, one could also deform C into a path on which $\mathrm{Re}\,\rho(t)$ is a constant and then apply the method of stationary phase. However, we have seen that Laplace's method is a much better approximation scheme than the method of stationary phase because the full asymptotic expansion of a generalized Laplace integral is determined by the integrand in an arbitrarily small neighborhood of the point where $\mathrm{Re}\,\rho(t)$ is a maximum on the contour. By contrast, the full asymptotic expansion of a generalized Fourier integral typically depends on the behavior of the integrand along the entire contour. As a consequence, it is usually easier to obtain the full asymptotic expansion of a generalized Laplace integral than of a generalized Fourier integral.

Before giving a formal exposition of the method of steepest descents, we consider three preliminary examples which illustrate how shifting complex contours can greatly simplify asymptotic analysis. In the first example we consider a Fourier integral whose asymptotic expansion is difficult to find by the methods used in Sec. 6.5. However, deforming the contour reduces the integral to a pair of integrals that are easy to evaluate by Laplace's method.

Example 1 *Conversion of a Fourier integral into a Laplace integral by deforming the contour.* The behavior of the integral

$$I(x) = \int_0^1 \ln t\, e^{ixt}\,dt \tag{6.6.3}$$

as $x \to +\infty$ cannot be found directly by the methods of Sec. 6.5 because there is no stationary point. Also, integration by parts is useless because $\ln 0 = -\infty$. Integration by parts is doomed to fail because, as we will see, the leading asymptotic behavior of $I(x)$ contains the factor $\ln x$ which is not a power of $1/x$.

To approximate $I(x)$ we deform the integration contour C, which runs from 0 to 1 along the real-t axis, to one which consists of three line segments: C_1, which runs up the imaginary-t axis from 0 to iT; C_2, which runs parallel to the real-t axis from iT to $1 + iT$; and C_3, which runs down from $1 + iT$ to 1 along a straight line parallel to the imaginary-t axis (see Fig. 6.5). By Cauchy's theorem, $I(x) = \int_{C_1+C_2+C_3} \ln t\, e^{ixt}\,dt$. Next we let $T \to +\infty$. In this limit the contribution from C_2 approaches 0. (Why?) In the integral along C_1 we set $t = is$, and in the integral along C_3 we set $t = 1 + is$, where s is real in both integrals. This gives

$$I(x) = i\int_0^\infty \ln(is)\,e^{-xs}\,ds - i\int_0^\infty \ln(1 + is)\,e^{ix(1+is)}\,ds. \tag{6.6.4}$$

The sign of the second integral on the right is negative because C_3 is traversed downward.

Observe that both integrals in (6.6.4) are Laplace integrals. The first integral can be done exactly. We substitute $u = xs$ and use $\ln(is) = \ln s + i\pi/2$ and the identity $\int_0^\infty e^{-u}\ln u\,du = -\gamma$, where $\gamma = 0.5772\ldots$ is Euler's constant, and obtain

$$i\int_0^\infty \ln(is)\,e^{-xs}\,ds = -i(\ln x)/x - (i\gamma + \pi/2)/x.$$

We apply Watson's lemma to the second integral on the right in (6.6.4) using the Taylor expansion $\ln(1 + is) = -\sum_{n=1}^\infty (-is)^n/n$, and obtain

$$-i\int_0^\infty \ln(1 + is)\,e^{ix(1+is)}\,ds \sim ie^{ix}\sum_{n=1}^\infty \frac{(-i)^n(n-1)!}{x^{n+1}}, \qquad x \to +\infty.$$

Then $\zeta = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = \frac{g}{f}(\eta_{xx} + \eta_{yy}) = \frac{g}{f}\nabla^2\eta$

Thus at $t=\infty$,

$$q = \frac{g}{f}\nabla^2\eta - \frac{f\eta}{D}$$

But we know in this problem that

$$q = \frac{f\eta_0}{D}\,\mathrm{sgn}\,x$$

Hence $\frac{g}{f}\frac{d^2\eta}{dx^2} - \frac{f\eta}{D} = \frac{f\eta_0}{D}\,\mathrm{sgn}\,x$

This is a second order, linear, constant coefficient equation for the final state.

i.e. $\frac{d^2\eta}{dx^2} - \frac{1}{a^2}\eta = \underbrace{\frac{\eta_0}{a^2}\,\mathrm{sgn}\,x}_{\text{odd } f^n}$ $\left[\frac{f^2}{gD} = \frac{f^2}{c^2} = \frac{1}{a^2} \leftarrow \text{Rossby radius}\right]$

Look for odd sol^n, so for $x \geq 0$, $\eta(0) = 0$
and $\eta'' - \frac{1}{a^2}\eta = \frac{\eta_0}{a^2}$

Particular sol^n: $\eta_p = -\eta_0$

Complementary f^n: $\eta = Ae^{x/a} + Be^{-x/a}$

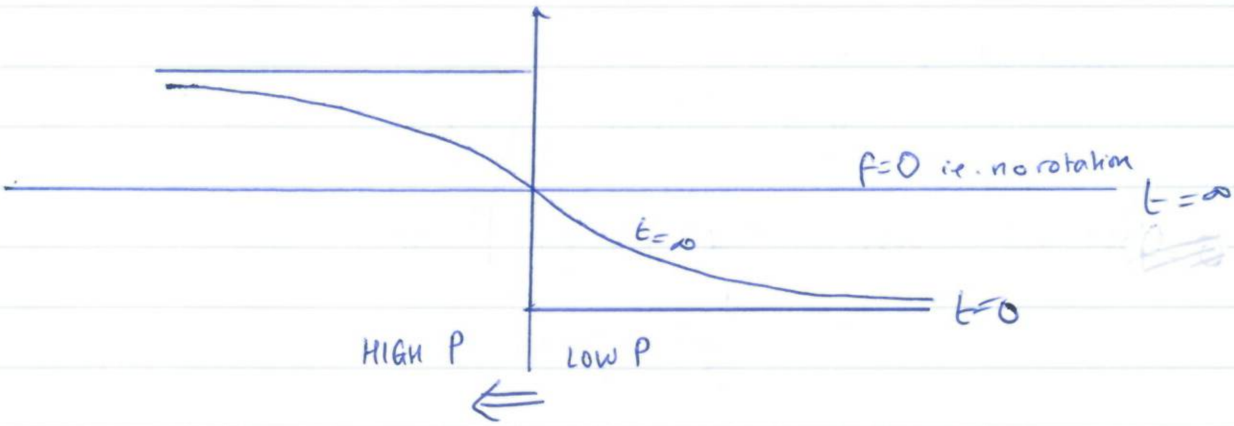
But required bounded as $x \to \infty$ so $A = 0$

General sol^n: $\eta = -\eta_0 + Be^{-x/a}$.

$\eta = 0$ at $x=0$ $\Rightarrow B = \eta_0$ $\Rightarrow$ $\overset{x\geq 0}{\eta} = -\eta_0(1-e^{-x/a})$

and $\underset{x<0}{\eta} = -\eta(-x) = \eta_0(1-e^{x/a})$

Combine to give $\eta(x) = -\eta_0 \operatorname{sgn} x \left(1 - e^{-|x|/a}\right)$



Does NOT collapse

So what do we have?

Initial state: ~~the~~ unbalanced $\quad 2\underline{\Omega} \times \underline{u} \neq -g\underline{\nabla}h$

Final state: balanced geostrophic $\quad 2\underline{\Omega} \times \underline{u} = -g\underline{\nabla}h.$

Two approaches:

(1) Linear problem: Just do it (FT's)

(2) Rossby – jump to final state
– assumed that flow becomes steady.

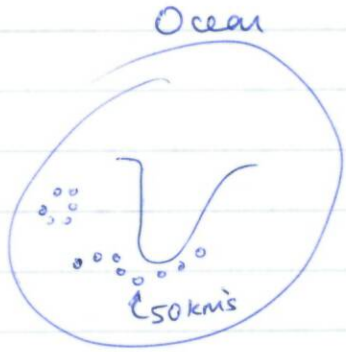
Use conservation of PV: $\quad PV|_{t=\infty} = PV|_{t=0}.$

Which gave us the final answer, at the top of this page.

$$\eta_s(x) = -\eta_0 \operatorname{sgn} x \left[1 - e^{-|x|/a}\right]$$

e-folding scale of $a = \frac{c}{f}$ = Rossby radius.

[In ocean, Rossby radius ~ 50 km
atmosphere, " " ~ 1000 km



Getting good models for the ocean is ∴ a lot harder

In the steady state,

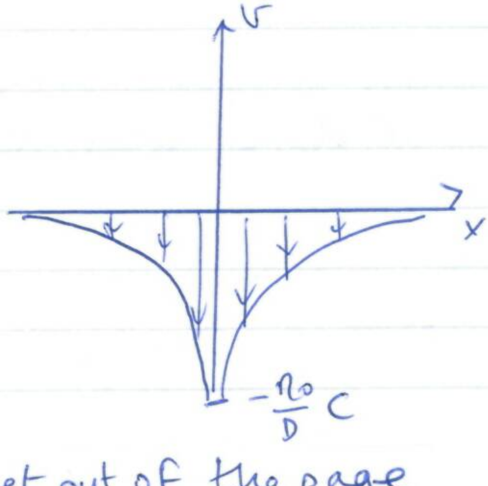
$$u = -\frac{g}{f}\frac{\partial \eta}{\partial y} = 0$$

$$v = \frac{g}{f}\frac{\partial \eta}{\partial x}$$

$$= -\frac{g\eta_0}{fa} e^{-|x|/a} \qquad fa = c$$

$$= -\frac{g\eta_0}{c} e^{-|x|/a}$$

$$= -\frac{\eta_0}{D} c\, e^{-|x|/a}$$

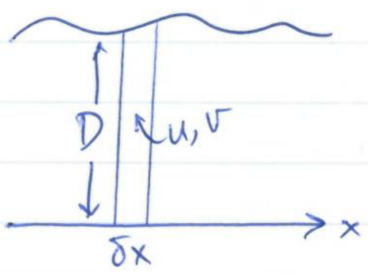


Much slower than the waves; Jet out of the page

Thus there is a Coriolis force to the right, i.e. in −ve x-dir^n and this opposes $-\nabla p$ (or $-\nabla h$) i.e. it holds the surface up.

Energetics of adjustment

The KE of a column of fluid (per unit width in y-dir^n) and length δx in x-dir^n is



$$\frac{1}{2}\rho\underbrace{(D\,\delta x\,1)}_{\text{volume}}(u^2+v^2)$$

$$= \frac{1}{2}\rho D\,\delta x\left(\frac{g}{f}\eta_x\right)^2 \qquad \text{(final KE)}$$

Final KE of the whole flow is

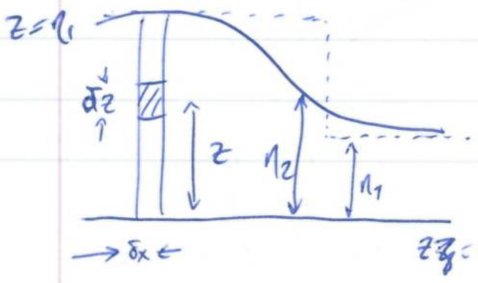
$$2\times\frac{1}{2}\rho\frac{g^2}{f^2}D\int_0^\infty \eta_x^{\,2}\,dx \qquad \text{(since } v \text{ even in } x\text{)}$$

$$= \frac{1}{2}\rho\left(\frac{g}{f}\right)D\eta_0^{\,2}a^{-2}\cdot 2\int_0^\infty e^{-2x/a}\,dx$$

$$= \frac{1}{2}\rho g a\eta_0^{\,2}.$$

i.e. increase in KE $= \frac{1}{2}\rho g a \eta_0^{\,2}$

No dissipation. Motion conserves energy. Thus increase in KE should equal decrease in PE.



The PE of a column, relative to the flat bottom follows

The PE of an element $\delta x \cdot \delta z \cdot 1$ at a height z is

$$mgz = \rho\,\delta x\,\delta z \cdot g \cdot z$$

Let the initial surface height be η_1.
and the final surface height be η_2.

Then the increase in PE as η moves from η_1 to η_2

$$= \int_{\eta_1}^{\eta_2} \rho\,\delta x\, g z\, dz \quad \text{for a column of width } \delta x.$$

$$= \frac{1}{2}\rho g\left[\eta_2^2 - \eta_1^2\right]\delta x$$

In our problem, $\eta_1 = -\eta_0 \operatorname{sgn} x$
$\eta_2 = -\eta_0 \operatorname{sgn} x\left(1 - e^{-|x|/a}\right)$

So total increase in PE for this whole flow is

$$\frac{1}{2}\rho g \eta_0^2 \cdot 2\int_0^\infty \left[\left(1 - e^{-x/a}\right)^2 - 1\right] dx$$

$$= -\frac{3}{2}\rho g a \eta_0^2$$

ie PE has decreased by amount $\frac{3}{2}\rho g a \eta_0^2$

KE has increased by amount $\frac{1}{2}\rho g a \eta_0^2$

Energy is conserved.

?!

Return to the full problem:

Now write $\eta(x,t) = \eta_s(x) + \bar{\eta}(x,t)$

final steady state

departure from steady state.

Then $\bar{\eta}$ satisfies the problem:

$$\text{b.c.s} \quad \bar{\eta}(x,0) = \eta(x,0) - \eta_s(x)$$
$$= -\eta_0 \operatorname{sgn} x - (-\eta_0 \operatorname{sgn} x)(1 - e^{-|x|/a})$$
$$= -\eta_0 \operatorname{sgn} x \, e^{-|x|/a}$$

And
$$\frac{\partial \bar{\eta}}{\partial t}(x,0) = \frac{\partial \eta}{\partial t}(x,0) - \frac{\partial \eta_s}{\partial t}$$
$$= 0 - 0 = 0.$$

$$\left(\frac{\partial^2}{\partial t^2} + f^2 - c^2 \frac{\partial^2}{\partial x^2}\right)\bar{\eta}$$
$$= \left(\frac{\partial^2}{\partial t^2} + f^2 - c^2 \frac{\partial^2}{\partial x^2}\right)\eta - \left(\frac{\partial^2}{\partial t^2} + f^2 - c^2 \frac{\partial^2}{\partial x^2}\right)\eta_s$$
$$= 0.$$

Observe that the b.c.'s are odd in x. Thus look for a sol.^{n} that is odd in x.

Now proceed as before:

$$\bar{\eta}(x,t) = \frac{2}{\pi}\int_0^\infty \hat{\eta}(k,t) \sin kx \, dk$$

We obtain:

$$\bar{\eta}(x,t) = \frac{2}{\pi}\int_0^\infty A(k) \sin kx \cos \omega t \, dk$$

$$\omega = \sqrt{f^2 + c^2k^2}.$$

But here, at $t=0$,

$$\frac{2}{\pi}\int_0^\infty A(k) \sin kx \, dk = -\eta_0 \operatorname{sgn} x \, e^{-|x|/a}$$

By the F.I.T., then,

$$A(k) = -\eta_0 \int_0^\infty \operatorname{sgn} x \, e^{-|x|/a} \sin kx \, dx$$

$$= -\eta_0 \int_0^\infty e^{-x/a} \sin kx \, dx$$

$$= \frac{-\eta_0 k}{k^2 + \frac{1}{a^2}}$$

Thus the full sol^{n} is

$$\eta(x,t) = \eta_s(x) - \frac{2\eta_0}{\pi}\int_0^\infty \underbrace{\frac{k}{k^2+\frac{1}{a^2}}}_{\text{Poincaré}} \underbrace{\cos \omega t}_{\text{amplitude}} \underbrace{\sin kx}_{\text{wave n}^{\text{o}}} \, dk$$

This is the complete answer:

the steady sol^{n} plus a superposition of Poincaré waves.

Stationary phase

$$\cos\omega t \sin kx = \frac{1}{2i}\left[e^{ikx} - e^{-ikx}\right]\frac{1}{2}\left[e^{i\omega t} + e^{-i\omega t}\right]$$

ie. $A(k)e^{i(kx \pm \omega(k)t)}$ $\qquad \omega^2 = c^2k^2 + f^2$

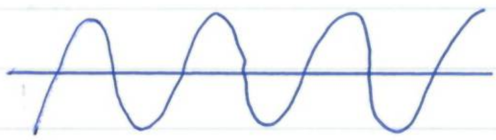
$$\int_{-\infty}^{\infty} A(k)e^{i\phi(k)t}\,dk$$

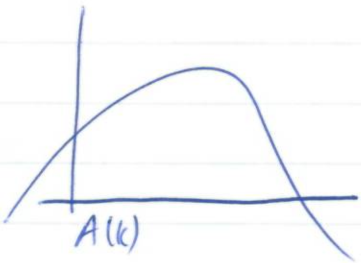
where $A(k)$ is the amplitude of wave of waven$^{\circ}$ k and $\phi(k)t = \left[k\frac{x}{t} - \omega(k)\right]t$ is the phase of the wave.

We are interested in these integrals at large times t

$e^{i\phi(k)t}$

$t=1$



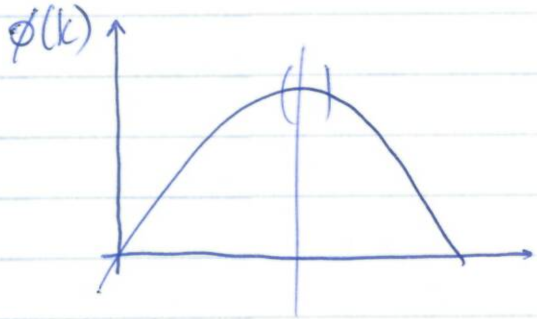


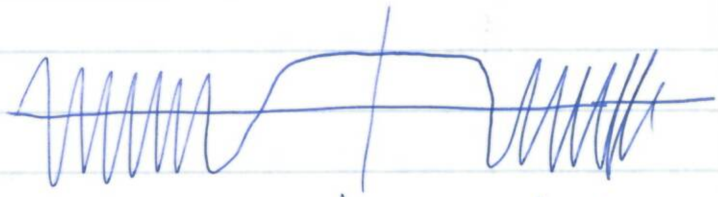
integrand $A(k)e^{i\phi(k)t}$

$t=1000$

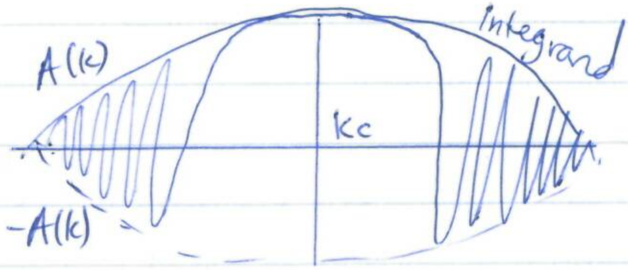
nothing when integrated

Riemann-Lebeque lemma guarantees that integrals of the form decay as $\frac{1}{t}$ when t is large.





pt of stationary phase



Dominant contribution comes from any points of stationary phase

Stationary phase occurs when $\dfrac{d\phi}{dk} = 0$

but in our wave problems

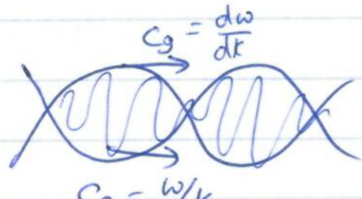
$$\phi(k) = k\frac{x}{t} - \omega(k)$$

$$\text{So } \frac{d\phi}{dk} = \frac{x}{t} - \frac{d\omega}{dk}$$

$$\text{So } \frac{d\phi}{dk} = 0 \Rightarrow \frac{x}{t} = \frac{d\omega}{dk}$$

ie. for a given wave no. k, the maximum contribution comes from points moving with speed

$$x = c_g t$$



where $c_g = \dfrac{d\omega}{dk}$, the group velocity

$c_p = c_g$ iff waves non-dispersive ie. $\omega = ck$

As it happens, no stationary pt $\Rightarrow \frac{1}{t}$ decay

and if $\phi'(k) = 0$, $\phi''(k) \neq 0 \Rightarrow \frac{1}{t^{1/2}}$ decay

and if $\phi'(k) = \phi''(k) = 0$, $\phi'''(k) \neq 0 \Rightarrow \frac{1}{t^{1/3}}$ decay.

The result $x = c_g t$ can be extended to any n° of dimensions

$$e^{i\phi(k,l,m)t} \qquad \phi(k,l,m) = k\frac{x}{t} + l\frac{y}{t} + m\frac{z}{t} - \omega(k,l,m)$$

Stationary pts are where $\nabla_{\underline{k}}\phi$ vanishes

ie. the points in waven° space $\underline{k} = k\hat{\underline{x}} + l\hat{\underline{y}} + m\hat{\underline{z}}$
where ϕ is ~~the field~~ flat

ie. $\nabla_{\underline{k}}\phi = 0$

ie. $$\frac{\partial\phi}{\partial k} = 0 \qquad \frac{\partial\phi}{\partial l} = 0 \qquad \frac{\partial\phi}{\partial m} = 0$$

$$\frac{x}{t} = \frac{\partial\omega}{\partial k} \qquad \frac{y}{t} = \frac{\partial\omega}{\partial l} \qquad \frac{z}{t} = \frac{\partial\omega}{\partial m}$$

Thus we have a 3D group velocity

$$\underline{c}_g = \nabla_{\underline{k}}\omega = \frac{\partial\omega}{\partial k}\hat{\underline{x}} + \frac{\partial\omega}{\partial l}\hat{\underline{y}} + \frac{\partial\omega}{\partial m}\hat{\underline{z}}$$

Let's apply this theory to Poincaré waves.

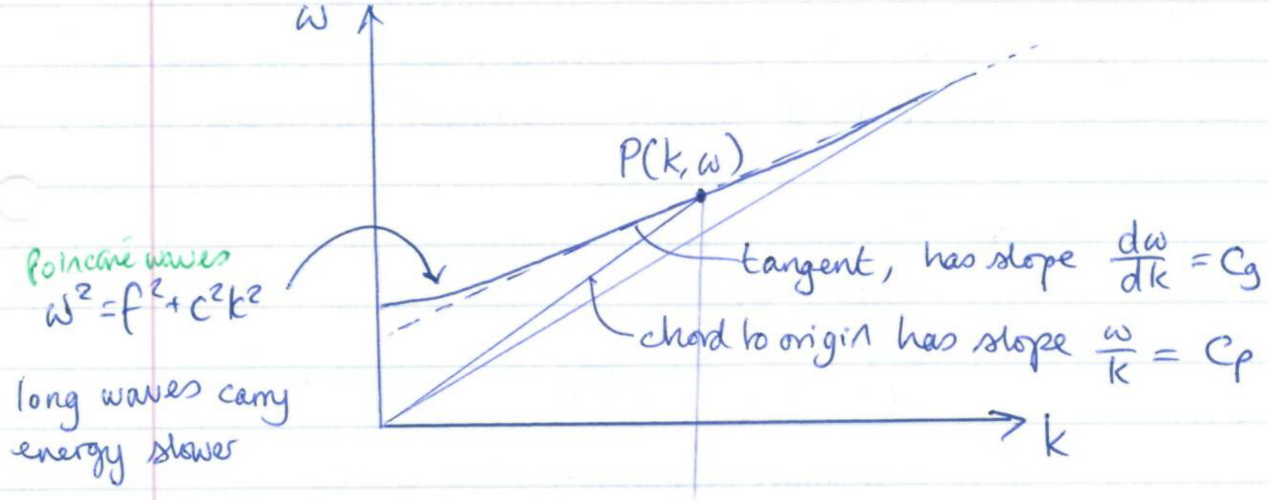
$$\omega^2 = c^2k^2 + f^2$$

Take $\frac{d}{dk}$:

$$2\omega \frac{d\omega}{dk} = 2c^2 k$$

$$\text{so} \quad \frac{d\omega}{dk} = c^2 \frac{k}{\omega}$$

$$\text{ie.} \quad c_p c_g = c^2$$



Notice $c_p > c \quad \forall k$
and so $c_g < c$

We can evaluate the FT for u by noting

$$u = \frac{2}{\pi}\frac{\eta_0}{D}\int_0^\infty \frac{\omega}{k}\frac{k}{k^2+\frac{1}{a^2}} \cos kx \sin \omega t \, dk \cdot \frac{\omega}{k}$$

from the LSWE.

End up with an exact sol^{n}:

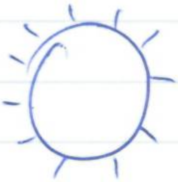
$$u = \begin{cases} \frac{\eta_0}{D} c \underset{\text{Bessel}}{J_0}\left[f\sqrt{t^2 - \frac{x^2}{c^2}} \right] & |x| < ct \\ 0 & |x| > ct \end{cases}$$

"Adrian (Gill) did this"

So where did Rossby's missing energy go?

The energy is lost to ∞ carried by Poincaré waves.
(never escapes $-ct < x < ct$)

HOMEWORK:



→ Kelvin too if you want

What are the Poincaré waves in a cylinder?
We've found them in a channel already.

Hints: Governing eq^n.
Klein-Gordon

$$\frac{\partial^2 \eta}{\partial t^2} + f^2 \eta = c^2 \nabla \eta \qquad \text{Wave eq}^n.$$

In a drum you solve $\frac{\partial^2 \eta}{\partial t^2} = c^2 \nabla \eta$
where $\eta = 0$ on $r = a$

Use polar coords $\eta = \eta(r, \theta, t)$

$$\frac{\partial^2 \eta}{\partial t^2} + f^2 \eta = c^2 \left(\frac{\partial^2 \eta}{\partial r^2} + \frac{1}{r}\frac{\partial \eta}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \eta}{\partial \theta^2} \right)$$

Our b.c. is $\underline{u} \cdot \hat{\underline{n}} = 0$ at $r = a$
i.e. $\underline{u} \cdot \hat{\underline{r}} = 0$ at $r = a$

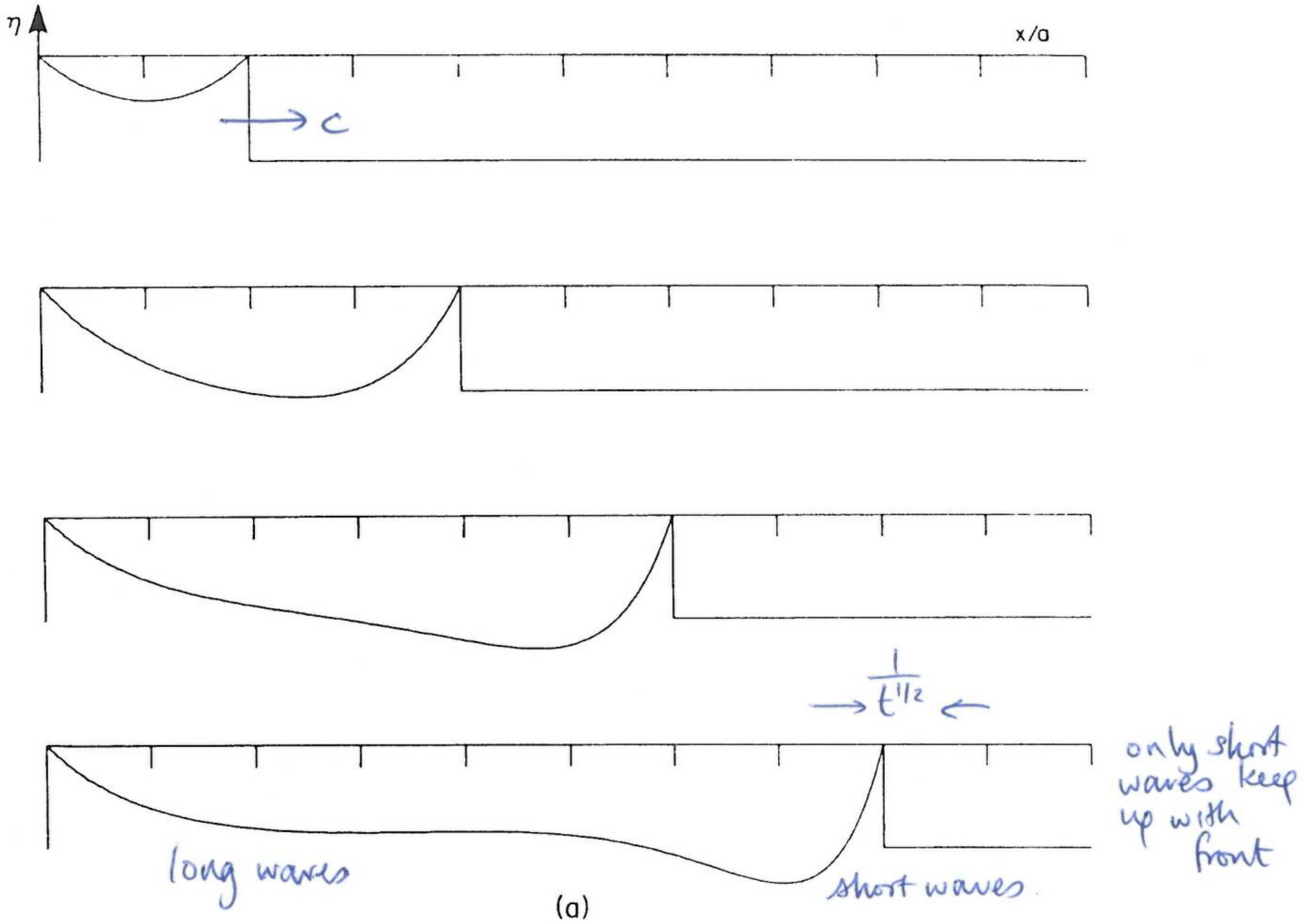
but $$\left(\frac{\partial^2}{\partial t^2} + f^2 \right) \underline{u} = -g \underline{\nabla} \frac{\partial \eta}{\partial t} + fg \hat{\underline{z}} \times \underline{\nabla} \eta$$

(momentum eq^{n}s)
for lrSW.

Dot it with $\hat{\underline{r}}$.

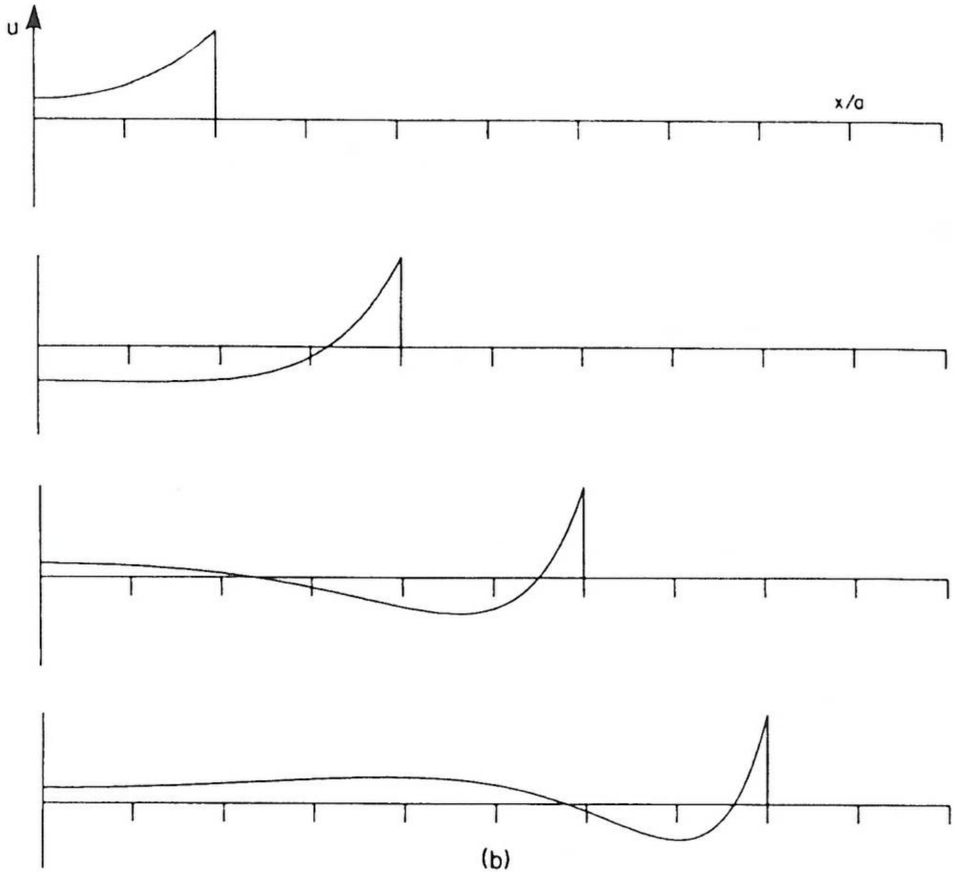
$$\left(\frac{\partial^2}{\partial t^2} + f^2 \right) \underline{u} \cdot \hat{\underline{r}} = -g \frac{\partial^2 \eta}{\partial r \partial t} + fg \hat{\underline{r}} \cdot \left(\hat{\underline{z}} \times \underline{\nabla} \eta \right)$$

$\therefore (1)$

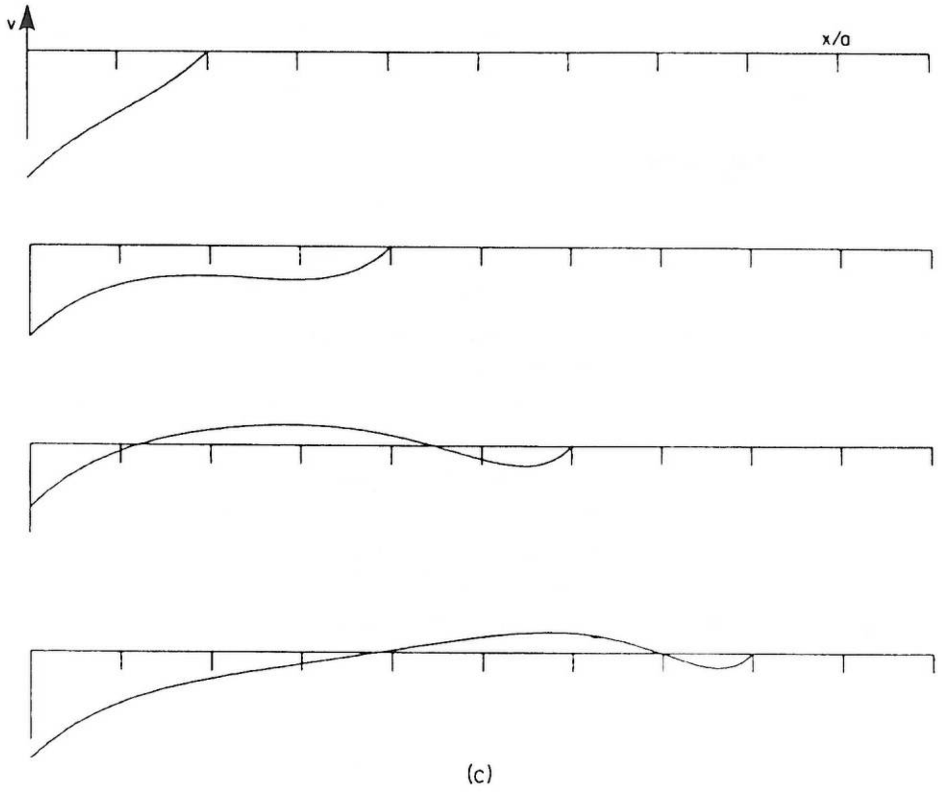


(a)

Fig. 7.3. Transient profiles for (a) η, (b) u, and (c) v for adjustment under gravity of a fluid with an initial infinitesimal discontinuity in level of $2\eta_0$ at $x = 0$. The solution is shown in the region $x > 0$, where the surface was initially depressed, at time intervals of $2f^{-1}$, where f is twice the rate of rotation of the system about a vertical axis. The marks on the x axis are at intervals of a Rossby radius, i.e., $(gH)^{1/2}/f$, where g is the acceleration due to gravity and H is the depth of fluid. The solutions retain their initial values until the arrival of a wave front that travels out from the position of the initial discontinuity at speed $(gH)^{1/2}$. When the front arrives, the surface elevation rises by η_0 and the u component of velocity rises by $(g/H)^{1/2}\eta_0$ just as in the nonrotating case depicted in Fig. 5.9a. This is because the first waves to arrive are the very short waves, which are unaffected by rotation. Behind the front, however, is a "wake" of waves produced by dispersion, which in the case of u, have the slope given by the Bessel function (7.3.14). This is the point impulse solution to the Klein–Gordon equation. The "width" of the front narrows in inverse proportion with time. Well behind the front, the solution adjusts to the geostrophic equilibrium solution depicted in Fig. 7.1.

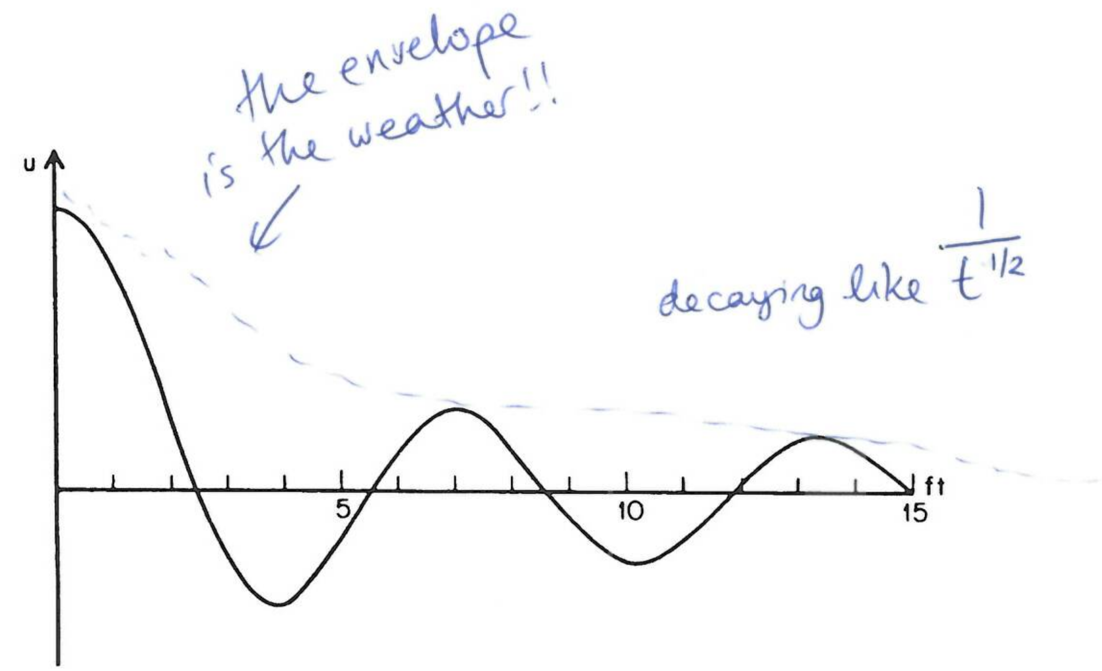


(b)

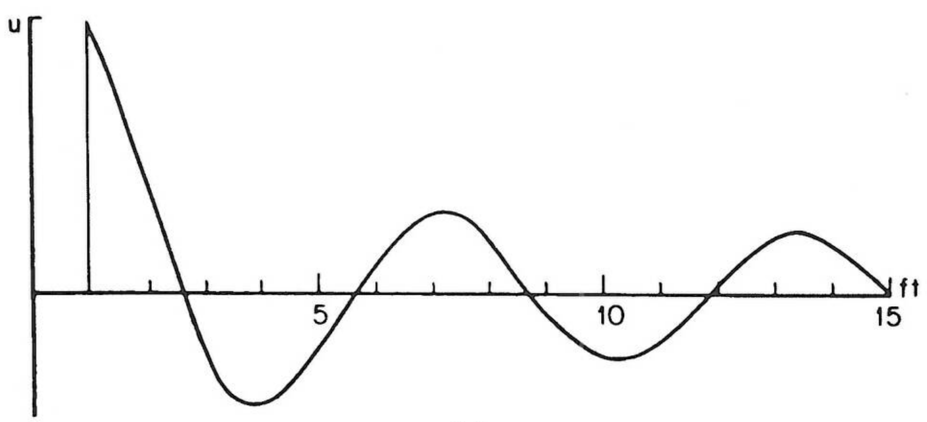


(c)

Fig. 7.3. *(continued)*



(a)



(b)

Fig. 7.4. The u velocity as a function of time t (a) at the position of the initial discontinuity in level and (b) one ssby radius away. The time axis is marked at intervals of f^{-1}, where f is the inertial frequency. The solutions show cillations with frequency near f, and these oscillations decay with time like $t^{-1/2}$ at large times.

out3.tiff

and $\hat{\underline{r}} \cdot (\hat{\underline{z}} \times \underline{\nabla}\eta)$

$$= (\hat{\underline{r}} \times \hat{\underline{z}}) \cdot \underline{\nabla}\eta$$

$$= -\hat{\underline{\theta}} \cdot \underline{\nabla}\eta$$

$$= -\frac{1}{r}\frac{\partial \eta}{\partial \theta}$$

So (1) becomes

$$\left(\frac{\partial^2}{\partial t^2} + f^2\right)(\underline{u} \cdot \hat{\underline{r}}) = -g\frac{\partial^2 \eta}{\partial r \partial t} - \frac{fg}{r}\frac{\partial \eta}{\partial \theta}$$

This is zero $\forall t, \theta$ on $r = a$ so

$$\frac{\partial^2 \eta}{\partial r \partial t} + \frac{f}{a}\frac{\partial \eta}{\partial \theta} = 0$$ [not Sturm-Louville btw]

Note: if $f = 0$, non-rotating

$$\frac{\partial \eta}{\partial r} = 0 \text{ on } r = a$$ (i.e. it's flat at the walls)

↑ the Neuman b.c.

Look for sol^{n}s of form

$$\eta(r, \theta, t) = \mathrm{Re}\left[\bar{\eta}(r)\, e^{i(n\theta - \omega t)}\right]$$

Sol^{n} is periodic with period 2π in θ, i.e. n is an integer.
(and we can take it to be 0 or positive)

Bessel's eq^{n}s: $J_n(\lambda r)$
$Y_n(\lambda r)$

Another b.c. is that sol^{n} is bounded at origin $\Rightarrow$ Y_n killed off.

End with $J_n(\lambda_{nm} a) = 0$ or $J_n'(\lambda_{nm} a) = 0$.
Mathematica can solve and plot some modes. [m = mode]
Try $n=0$, $m=20$

lrSWE with a sloping bottom

Our eq^{n}s are:

$$\boxed{\begin{aligned} u_t - fv &= -g\eta_x \\ v_t + fu &= -g\eta_y \\ \eta_t + \underline{\nabla}\cdot(\underline{u}H_0(x,y)) &= 0 \end{aligned}}$$

- - - (1)
- - - (2)
- - - (3)

Notice here $H_0 = H_0(x,y)$ (bottom not flat) so cannot be removed from divergence.

The mom^m eq^{n}s are unchanged so:

$$\boxed{\left(\frac{\partial^2}{\partial t^2} + f^2\right)\underline{u} = -g\underline{\nabla}\eta_t + gf\hat{\underline{z}}\times\underline{\nabla}\eta} \quad \text{- - - (4)}$$

As per Klein-Gordon, operate on (3) with $\left(\frac{\partial^2}{\partial t^2} + f^2\right)$ and use (4) to replace $\left(\frac{\partial^2}{\partial t^2} + f^2\right)\underline{u}$.

$$\left(\frac{\partial^2}{\partial t^2} + f^2\right)\eta_t + \underline{\nabla}\cdot\left[H_0(x,t)\left[-g\underline{\nabla}\eta_t + gf\hat{\underline{z}}\times\underline{\nabla}\eta\right]\right] = 0$$

$$\Rightarrow \left(\frac{\partial^2}{\partial t^2} + f^2\right)\eta_t - g\underline{\nabla}\cdot(H_0\underline{\nabla}\eta_t) + gf\underbrace{\underline{\nabla}\cdot\big[\underset{\text{scalar}}{H_0}\underbrace{\hat{\underline{z}}\times\underline{\nabla}\eta}_{\text{vector}}\big]} = 0$$

$$= (\hat{\underline{z}}\times\underline{\nabla}\eta)\cdot\underline{\nabla}H_0 + H_0\underbrace{\cancel{\underline{\nabla}\cdot(\hat{\underline{z}}\times\underline{\nabla}\eta)}}_{0 \text{ as before}}$$

using identity $\underline{\nabla}\cdot(\phi\underline{u}) = \phi\underline{\nabla}\cdot\underline{u} + (\underline{u}\cdot\underline{\nabla})\phi$.

$$= \hat{\underline{z}}\cdot(\underline{\nabla}\eta\times\underline{\nabla}H_0)$$

$$= \frac{\partial\eta}{\partial x}\frac{\partial H_0}{\partial y} - \frac{\partial\eta}{\partial y}\frac{\partial H_0}{\partial x}$$

$$= \frac{\partial(\eta, H_0)}{\partial(x,y)}, \quad \text{the Jacobian}$$

Thus we have

$$\left(\frac{\partial^2}{\partial t^2} + f^2\right)\eta_t - g\,\underline{\nabla}\cdot(H_0 \underline{\nabla}\eta_t) + gf\underbrace{\frac{\partial(\eta, H_0)}{\partial(x,y)}}_{\text{no } t} = 0$$

[this time we cannot integrate wrt t]
eq^n is fundamentally cubic in ∂_t.
ie. there are 3 waves.

Before ∂_{tt} ie. 2 similar wave

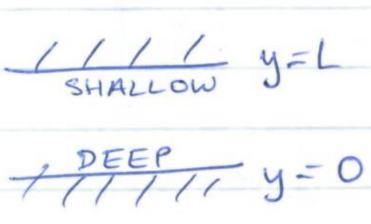
Now we have 2 similar waves plus a new special wave.

- non-constant coeffs in x and y : hard to solve.

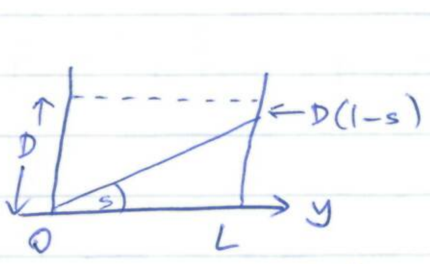
It is sufficient to consider a linearly sloping bottom, ie. take

$$H_0(x,y) = D\left(1 - \frac{sy}{L}\right)$$

in a channel of width L, where $v=0$ at $y=0, L$.



top view (plan)



side view (elevation)

Sub H_0 into big eq^n:

$$\left(\frac{\partial^2}{\partial t^2}+f^2\right)\eta_t - c^2\underline{\nabla}\cdot\left[\left(1-s\frac{y}{L}\right)\underline{\nabla}\eta_t\right] + c^2 f\frac{\partial \eta}{\partial x}\left(-\frac{s}{L}\right) = 0$$

b.c.s unchanged since mom^m eq^{n}s unchanged:

$$\frac{\partial^2\eta}{\partial y\partial t} - f\frac{\partial \eta}{\partial x} = 0 \quad \text{on } y=0, L$$

Problem remains homogeneous in x, t.
So look for sol^n:

$$\eta(x,y,t) = \mathrm{Re}\left[\bar{\eta}(y)\,e^{i(kx-\omega t)}\right]$$

$$\Rightarrow \quad -i\omega(f^2-\omega^2)\bar{\eta} - i\omega k^2c^2\left(1-s\frac{y}{L}\right)\bar{\eta}$$
$$+ i\omega c^2\frac{d}{dy}\left[\left(1-s\frac{y}{L}\right)\frac{d\bar{\eta}}{dy}\right]$$
$$+ ikf\bar{\eta}c^2\left(-\frac{s}{L}\right) = 0 \quad \cdots\cdots (\dagger)$$

and b.c. is

$$\Rightarrow -i\omega\frac{d\bar{\eta}}{dy} - ikf\bar{\eta} = 0 \quad \text{on} \quad y=0, L \ \text{(as before)}$$

It's sufficient to consider $s \ll 1$ (relevant too, as global shelves are shallow)

Then $\frac{sy}{L} \ll 1$

Then, $(\dagger)$ becomes $\bar{\eta}'' - \frac{s}{L}\bar{\eta}' + \bar{\eta}\left[\frac{\omega^2-f^2}{c^2} - k^2 - \frac{fsk}{L\omega}\right] = 0$

a constant coefficient eq^n!

And we can solve these in the usual auxiliary eq^n way

but we can also make an observation:

Introduce $\bar{\eta}(y) = e^{sy/2L}\phi(y)$

$$\bar{\eta}'(y) = \left(\frac{s}{2L}\phi + \phi'\right)e^{sy/2L}$$

$$\bar{\eta}''(y) = \left(\frac{s^2}{4L^2}\phi + \frac{s}{L}\phi' + \phi''\right)e^{sy/2L}$$

$$\bar{\eta}'' - \frac{s}{L}\bar{\eta}' = e^{sy/2L}\left(\phi'' - \frac{s^2}{4L^2}\phi\right)$$

Thus $\phi'' + \alpha^2\phi = 0$

where $\alpha = \left[\frac{\omega^2 - f^2}{c^2} - k^2 - \frac{fsk}{L} - \frac{s^2}{4L^2}\right]$

i.e. precisely as before but with slightly modified α.

[b.c.s changed too: $\phi' + \left(\frac{s}{2L} + \frac{fk}{\omega}\right)\phi = 0 \qquad y = 0, L$]

As before, look for sol^{n}s $\quad \phi = A\cos\alpha y + B\sin\alpha y$

substitute in b.c.s to get $\left(\quad\right)\begin{pmatrix}A\\B\end{pmatrix} = 0$

Non-Trivial sol^{n}s iff determinant vanishes, here

$$(\omega^2 - f^2)(\omega^2 - c^2k^2)\sin\alpha L = 0$$

Kelvin Waves

precisely as before.

To leading order in s, the Kelvin Waves are unaffected by small slope.

Thus it is the sol^n $\sin\alpha L = 0$

i.e. $\alpha L = n\pi, \qquad n = 0, 1, 2, \ldots$

that is affected.

Thus $\alpha^2 = \frac{n^2\pi^2}{L}$.

$$\omega^2 - \frac{fsk}{L\omega}c^2 - c^2\left(k^2 + \frac{f^2}{c^2} + \frac{n^2\pi^2}{L^2}\right) = 0$$

(dropping the term $\frac{s^2}{4L^2}$ which is small c.f. $\frac{n^2\pi^2}{L}$).

This is our new dispersion relation, $\omega = \omega(k)$.

Does it contain the old one?

$\underline{s \equiv 0}$: $\omega^2 = c^2\left(k^2 + \frac{f^2}{c^2} + \frac{n^2\pi^2}{L^2}\right)$

$= f^2 + c^2\left(k^2 + \frac{n^2\pi^2}{L^2}\right)$ exactly as before

$= c^2\left(k^2 + \frac{n^2\pi^2}{L^2} + \frac{1}{a^2}\right)$ $a = c/f$ Rossby rad.

Let us call these roots $\omega_n^{(0)}(k)$; $n = 1, 2$

For small s, i.e. $0 < s \ll 1$, then

$$\omega_n^2 = \left[\omega_n^{(0)}\right]^2 + \frac{c^2fk}{L}\frac{1}{\omega_n}s$$

$$= \left[\omega_n^{(0)}\right]^2 + \frac{c^2fk}{L}\frac{1}{\omega_n^{(0)}}s + O(s^2).$$

i.e. the frequency changes by an amount of order $s \ll 1$.

So this is not very interesting.

Have we missed something?

There are only two waves here, but our initial eq^n involved

$$\partial_{ttt} \leftarrow 3 \text{ waves!}$$

We have lost a wave!

Recall $\omega^2 - \dfrac{fskc^2}{L\omega} - c^2\left(k^2 + \dfrac{n^2\pi^2}{L^2} + \dfrac{1}{a^2}\right) = 0$

is a <u>cubic</u> for $\omega \Rightarrow 3$ roots

The third root behaves as s as $s \to 0$

So $\omega \sim s$ as $s \to 0$.

$\omega^2 \sim s^2 \ll 1$ is negligable; so first term goes ~~out~~ walkabout.

Thus $$\frac{\omega L}{fsk} = \frac{-1}{k^2 + \frac{n^2\pi^2}{L^2} + \frac{1}{a^2}}$$

ie. $$\omega = -\frac{\overset{\text{rotation}}{f}\,\overset{\text{slope}}{s}(kL)}{k^2L^2 + n^2\pi^2 + \frac{L^2}{a^2}}$$

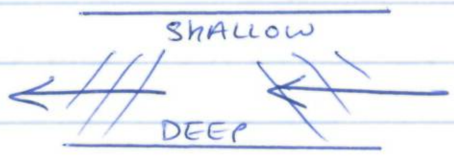
And note $\omega \sim s$ as expected.

Totally new wave. Needs both rotation (f) and slope (s); a (topographic) Rossby wave.

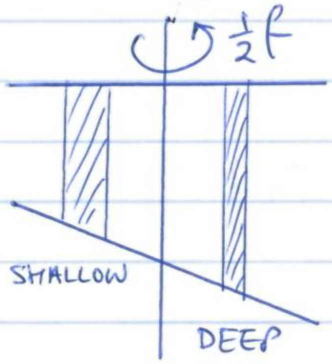
$$(C_p)_x = \frac{\omega}{k} = \frac{-fsL}{k^2L^2 + n^2\pi^2 + \frac{L^2}{a^2}} < 0$$

ie. wave crests always propogate with shallow water to the right (like Kelvin Waves)

NOT ISOTROPHIC

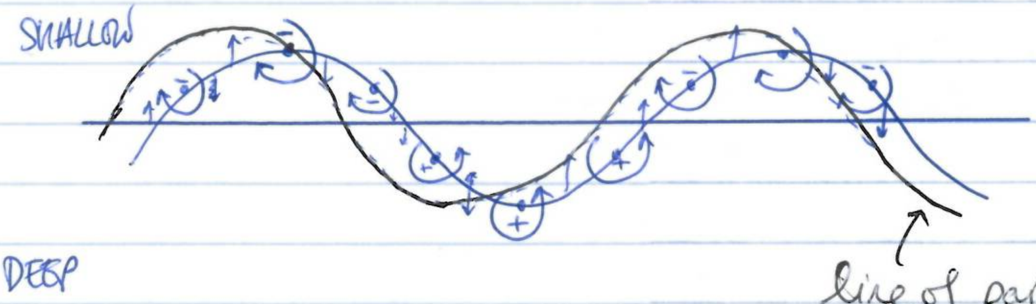


this is, in short, why Gulf Stream etc is always on the West of the Ocean Basin.



thinner
taller
spins faster
$\Rightarrow$ +ve ζ, rel. vort.

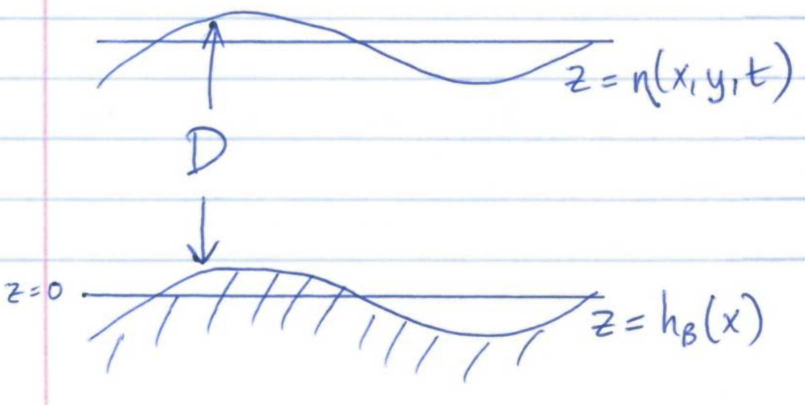
change in depth, plus rotation to give vortex amplification.



line of particles reproduced, displaced with shallow water to the right.

The Quasigeostrophic Limit of the SWE

- the low frequency, long period <u>nonlinear</u> approx. to SWE



average depth D
undisturbed depth $D - h_B = H_0(x,y)$

average depth D
undisturbed depth $D - h_B = H_0(x,y)$
instantaneous depth $H(x,y,t) = H_0(x,y) + \eta(x,y,t)$
$= D - h_B + \eta$

Scale $\underset{\substack{\uparrow \\ \text{dimensional}}}{(x,y)} = L\underset{\substack{\uparrow \\ \text{non-dimensional}}}{(x',y')}$

$$t = Tt'$$
$$(u,v) = U(u',v')$$
$$\eta = N_0 \eta'$$

$$H\left(\frac{x}{L}, \frac{y}{L}, \frac{t}{T}\right) = H_0(x',y') + \eta(x,y,t)$$
$$= D - h_B(x',y') + \eta(x,y,t)$$
$$= D\left[1 - \frac{h_B}{D} + \frac{N_0}{D}\eta'\right]$$

TIME DEP. NONLINEAR CORIOLIS

SWE

$$\frac{U}{T}\frac{\partial u'}{\partial t'} + \frac{U^2}{L}\left(u'\frac{\partial u'}{\partial x'} + v'\frac{\partial u'}{\partial y'}\right) - Ufv' = -\boxed{\frac{gN_0}{L}}\frac{\partial \eta'}{\partial x'} \quad \cdots (1)$$

$$\frac{U}{T}\frac{\partial v'}{\partial t'} + \frac{U^2}{L}\left(u'\frac{\partial v'}{\partial x'} + v'\frac{\partial v'}{\partial y'}\right) + Ufu' = -\boxed{\frac{gN_0}{L}}\frac{\partial \eta'}{\partial y'} \quad \cdot (2)$$

$$\frac{N_0}{T}\frac{\partial \eta'}{\partial t'} + \frac{U}{L}\left[u'\frac{\partial}{\partial x'}(N_0\eta' - h_B) + v'\frac{\partial}{\partial y}(N_0\eta' - h_B)\right]$$
$$+ \frac{U}{L}\left(D + N_0\eta' - h_B\right)\left(\frac{\partial u'}{\partial x'} + \frac{\partial v'}{\partial y'}\right) = 0 \quad \cdot (3)$$

We require that the leading order balance is geostrophy. Thus we take the scale for η s.t.

$$\boxed{N_0 \frac{g}{L}} = Uf$$

i.e. $N_0 = \dfrac{UfL}{g}$

The ratio of the nonlinear term to the Coriolis term is

$$\frac{U^2/L}{Uf} = \frac{U}{fL} = \varepsilon, \quad \text{Rossby n}^{\circ}$$

this measures the importance of nonlinearity (advection) to Coriolis.

For planetary flows, $\varepsilon \ll 1$ $\quad fL \sim 100\,m/s$, $\quad U \sim 1/10\ m/s$

The ratio of the time dependent terms to Coriolis is

$$\frac{U/T}{Uf} = \frac{1}{fT} = \varepsilon_T, \quad \text{a temporal Rossby n}^{\circ}$$

For long-period motions, $\varepsilon_T \ll 1$

(motions over $\frac{1}{2}$ day or more)
(cuts out PWs)

$\varepsilon_T \sim 1$ PW (+ KW)
$\varepsilon_T \ll 1$ RW (+ KW)

Now, $N_0 = \frac{UfL}{g}$ so

$$\frac{N_0}{D} = \frac{UfL}{gD} = \frac{UfL}{c^2}$$

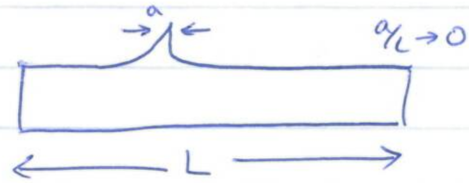
$$= \frac{UL}{f} \cdot \frac{f^2}{c^2} = \frac{UL}{f}\frac{1}{a^2} \qquad a = \frac{c}{f}$$

$$= \frac{U}{fL}\left(\frac{L}{a}\right)^2$$

$$\frac{N_0}{D} = \varepsilon\left(\frac{L}{a}\right)^2 \qquad a = \text{Rossby radius}$$

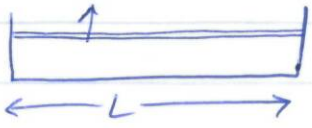
a: Rossby radius - scale over which a surface disturbance relaxes to eq$^{\underline{m}}$.

$\frac{L}{a} \gg 1$



Perturbations are localised

$\frac{L}{a} \ll 1$



Rigid lid $\quad \frac{a}{L} \to \infty$

$\frac{L}{a}$ measures the size of the domain in Rossby radii (i.e. the 'natural' horizontal scale)

It is traditional (and fairly American) to write $F = \frac{L^2}{a^2}$,

so $\quad \frac{N_0}{D} = \varepsilon F \ll 1 \qquad F \sim O(1)$

$$N_0 = \frac{UfL}{g} \qquad \underbrace{\varepsilon_T = \frac{1}{fT} \qquad \varepsilon = \frac{U}{fL}}_{\text{non-dimensional parameters}} \qquad \frac{N_0}{D} = \varepsilon F$$

dividing (1), (2), (3) by Uf

$$\left.\begin{aligned}
&\varepsilon_T \frac{\partial u}{\partial t} + \varepsilon\left(u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}\right) - v = -\frac{\partial n}{\partial x} && \text{(A)}\\
&\varepsilon_T \frac{\partial v}{\partial t} + \varepsilon\left(u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y}\right) + u = -\frac{\partial n}{\partial y} && \text{(B)}\\
&\varepsilon_T F\frac{\partial n}{\partial t} + \varepsilon F\left(u\frac{\partial n}{\partial x} + v\frac{\partial n}{\partial y}\right) - u\frac{\partial}{\partial x}\left(\frac{h_B}{D}\right) - v\frac{\partial}{\partial y}\left(\frac{h_B}{D}\right) && \\
&\quad + \left(1 + \varepsilon F n - \frac{h_B}{D}\right)\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) = 0 && \text{(C)}
\end{aligned}\right\}$$

Where we are using non-dimensional variables but have dropped dashes.

Sol^{n}s depend only on the values of the non-dimensional parameters.

Rossby n^{o} (importance of advection to Coriolis)

We want the behaviour of this system as $\varepsilon \to 0$
$\varepsilon_T \to 0$

Temporal Rossby n^{o} (importance of time-dep. to Coriolis)

Three possibilities:

(1) $1 \gg \varepsilon_T \gg \varepsilon$ linear system (done this) $1 \sim \varepsilon_T \quad \varepsilon \ll 1$

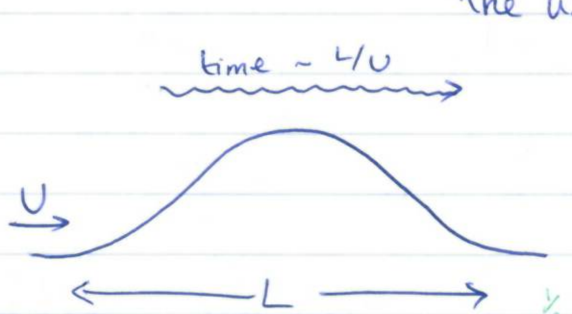
(2) $1 \gg \varepsilon_T \sim \varepsilon$ both time dependence and nonlinearity (advection), & no PWs

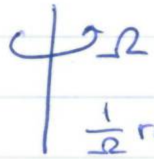
(3) $1 \gg \varepsilon \gg \varepsilon_T$ Steady version of (2)

Sufficient to consider (2).

Thus take $\varepsilon_T = \varepsilon$, i.e. $\dfrac{1}{fT} = \dfrac{U}{fL} \Rightarrow T = \dfrac{L}{U}$,

the usual advective time scale





$\frac{1}{\Omega}$ rotation period

$\dfrac{U}{fL} =: \varepsilon = \dfrac{\text{rotation period}}{\text{advection time}} = \dfrac{U}{fL}$

1/freq = period

Rossby n^{o}..

i.e. many rotations while particle crosses a distance L

" no tricks in exam – just know what's in the course
same format as every year.
no easier "

$\varepsilon \ll 1$: $u(x,y,t;\varepsilon) = u^{(0)} + \varepsilon u^{(1)} + \varepsilon^2 u^{(2)} + \dots$ (MacL. series around $\varepsilon = 0$)

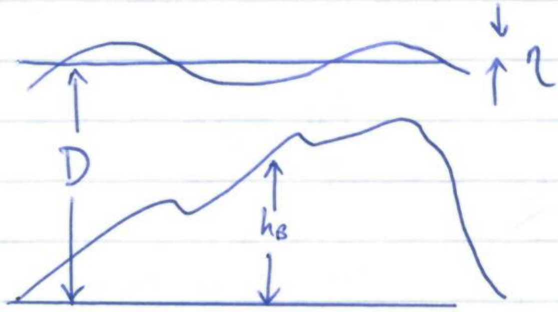
$$[\varepsilon^0] \quad \left.\begin{aligned} -v^{(0)} &= -\frac{\partial \eta^{(0)}}{\partial x} && \text{from (A)} \\ +u^{(0)} &= -\frac{\partial \eta^{(0)}}{\partial y} && \text{from (B)} \end{aligned}\right\}$$

the usual geostrophic relations

They satisfy
$$\frac{\partial u^{(0)}}{\partial x} + \frac{\partial v^{(0)}}{\partial y} = 0$$

Thus $(\vec{C})$ has different forms

If $\frac{h_B}{D} \sim \varepsilon^0$, $\quad -u^0 \frac{\partial}{\partial x}\left(\frac{h_B}{D}\right) - v^0 \frac{\partial}{\partial y}\left(\frac{h_B}{D}\right) = 0.$ $\quad (*)$



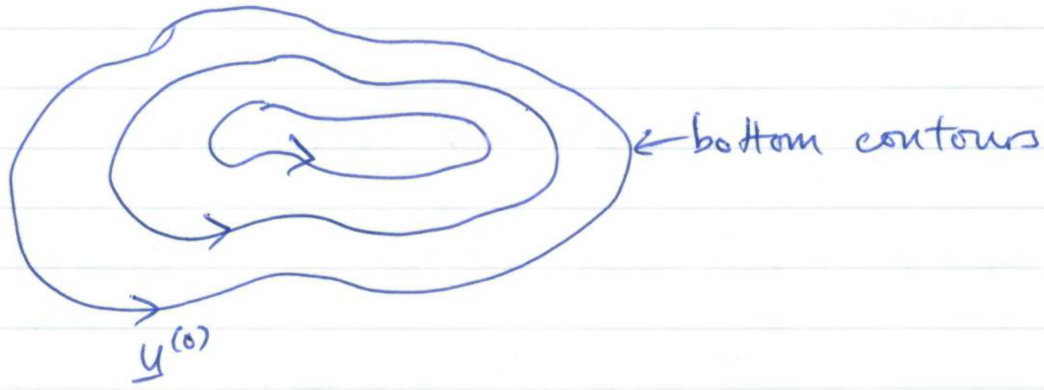
$h_B \sim D$, i.e. depth changes are of order the ocean depth, e.g. on coast around islands.

Thus $(*)$ is
$$\underline{u}^{(0)} \cdot \underline{\nabla}\left(\frac{h_B}{D}\right) = 0$$

i.e. $\underline{u}^{(0)}$ is $\perp$r to $\underline{\nabla}\left(\frac{h_B}{D}\right)$

i.e. $\underline{u}^{(0)}$ is // to lines of constant $\frac{h_B}{D}$
i.e. lines of constant depth

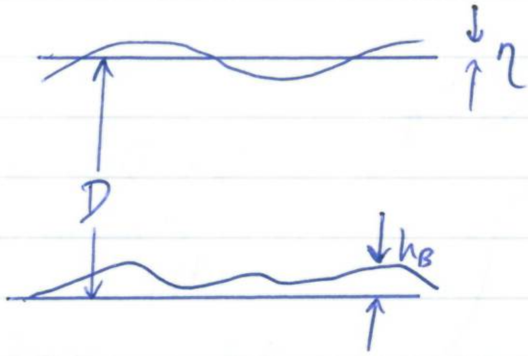
i.e. flow is around isobaths.



But in the <u>open ocean</u>, fractional depth changes are much smaller

i.e. $\dfrac{h_B}{D} \sim \varepsilon \ll 1$.

Mathematically say $\dfrac{h_B}{D} = \varepsilon \eta_B$ $\quad \eta_B = O(1)$



$$\frac{h_B}{D} \sim \varepsilon$$

$$\frac{h_B}{D\varepsilon} = \eta_B(x,y)$$

So (C) becomes

$$\varepsilon F\left(\frac{\partial \eta}{\partial t} + u\frac{\partial \eta}{\partial x} + v\frac{\partial \eta}{\partial y}\right) - \varepsilon u\frac{\partial \eta_B}{\partial x} - v\varepsilon\frac{\partial \eta_B}{\partial y} + (1+\varepsilon F\eta - \varepsilon\eta_B)\underbrace{\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right)}_{O(\varepsilon)} = 0$$

Now what is the leading order term ?

(5)

to order ε this is $\rightarrow$ $$F\frac{D_0\eta}{Dt} - u^{(0)}\frac{\partial\eta_B}{\partial x} - v^{(0)}\frac{\partial\eta_B}{\partial y} + \frac{\partial u^{(1)}}{\partial x} + \frac{\partial v^{(1)}}{\partial y} = 0$$

$$\frac{D_0}{Dt} = \frac{\partial}{\partial t} + u^{(0)}\frac{\partial}{\partial x} + v^{(0)}\frac{\partial}{\partial y}$$

Thus we must go to next order in (A) and (B):

$[\varepsilon^1]$ $$\frac{D_0 u^{(0)}}{Dt} - v^{(1)} = -\frac{\partial\eta^{(1)}}{\partial x}$$

$$\frac{D_0 v^{(0)}}{Dt} + u^{(1)} = -\frac{\partial\eta^{(1)}}{\partial y}$$

Eliminate $\eta^{(1)}$ by cross-differentiating, writing

$$\zeta^{(0)} = \frac{\partial v^{(0)}}{\partial x} - \frac{\partial u^{(0)}}{\partial y}$$

To get $$\frac{D_0}{Dt}\zeta^{(0)} + \frac{\partial u^{(1)}}{\partial x} + \frac{\partial v^{(1)}}{\partial y} = 0 \qquad (6)$$ $\left[\frac{\partial}{\partial x}(B) - \frac{\partial}{\partial y}(A)\right]$

Now, (6)-(5) gives

$$\frac{D_0}{Dt}\zeta^{(0)} - F\frac{D_0\eta^{(0)}}{Dt} + \frac{D_0\eta_B}{Dt} = 0 \qquad (7)$$

i.e. $\underline{\frac{D_0}{Dt}q^{(0)} = 0}$ where $q^{(0)} = \zeta^{(0)} - F\eta^{(0)} + \eta_B$

Conservation of QGPV

quasigeostrophic potential vorticity

Now, dropping superscript $^{(0)}$,

$$v = \frac{\partial \eta}{\partial x}$$

$$u = -\frac{\partial \eta}{\partial y}$$

[Recall, to use complex analysis, must use $\underline{u} = \nabla\phi = -\underline{k}\times\nabla\psi$

$u = +\frac{\partial \psi}{\partial y}$ $\quad v = -\frac{\partial \psi}{\partial x}$

NOT GFD, NO USE FOR CR eq^{n}s]

$$\zeta = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = \frac{\partial^2 \eta}{\partial x^2} + \frac{\partial^2 \eta}{\partial y^2} = \nabla^2 \eta$$

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + \frac{\partial \eta}{\partial y}\frac{\partial}{\partial x} + \frac{\partial \eta}{\partial x}\frac{\partial}{\partial y}$$

$$q = \nabla^2 \eta - F\eta + \eta_B$$

Everything in terms of η:

a CLOSED SYSTEM !!

η acts as a streamfunction (i.e. contours of surface elevation are streamlines).

Traditional to acknowledge this by writing $\eta = \psi$.

Then QGPV is

$$\left(\frac{\partial}{\partial t} + \frac{\partial \psi}{\partial x}\frac{\partial}{\partial y} + \frac{\partial \psi}{\partial y}\frac{\partial}{\partial x}\right)\left(\nabla^2\psi - F\psi + \eta_B\right) = 0$$

an equation in ψ alone

$$v = \frac{\partial \psi}{\partial x}, \quad u = -\frac{\partial \psi}{\partial y}, \quad \zeta = \nabla^2\psi, \quad q = \nabla^2\psi - F\psi + \eta_B$$

- non-linear set of eq^{n}s
- closed
- retains full advection
- first order in time and only 1 eq^n
  Hence at most, only one wave (Rossby)
- excellent model for ocean circulation
  (straightforward to integrate)

Remember the channel problem. The bottom had a linear slope s.

$$H_0(x,y) = D\left(1 - \frac{sy^*}{L}\right)$$ y^* = dimensional

ie. this is the case where $\eta_B = \beta y'$ $'$ = non-dimen'l

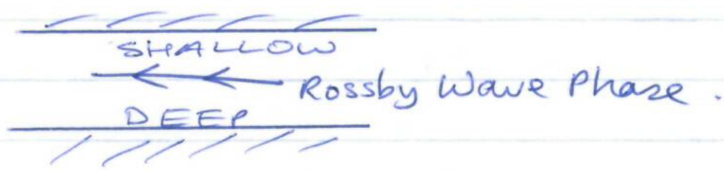
where $\beta = \frac{s}{\varepsilon}$ $y' = y^*/L$

(ie. require small slope $s \sim \varepsilon$)

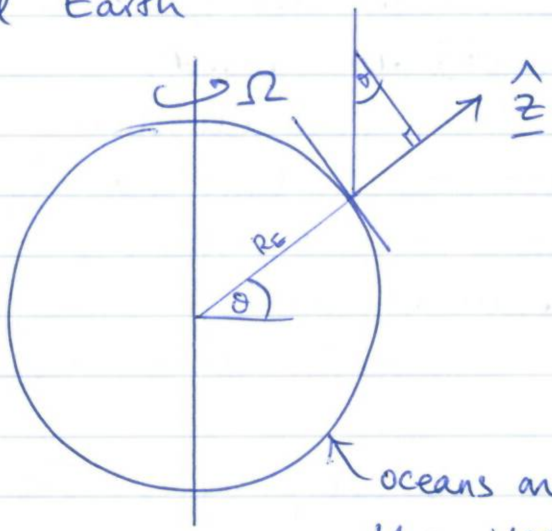
Then $q = \nabla^2\psi - F\psi + \beta y$

and QGPV is $\frac{D}{Dt}(\nabla^2\psi - F\psi) + \beta v = 0$.

Our previous work has shown this should have waves whose phase velocity always has component in –ve x-dir^n



for the spherical Earth



Thus in the Coriolis term, $2\underline{\Omega} \times \underline{u}$, only the component of $\underline{\Omega}$ $\perp$ to $\underline{u}$ contributes, i.e. only vertical component of $\underline{\Omega}$ important,

i.e. $\Omega \sin\theta \, \hat{\underline{z}}$ where θ = latitude

i.e. we can replace $\underline{\Omega}$ by $\Omega \sin\theta \, \hat{\underline{z}}$
(to order $\frac{D}{L}$, our approximation in the SWE)

the traditional approximation

Suppose we are interested in motion centred on latitude $\theta = \theta_0$

$$\begin{aligned} f &= 2\Omega \sin\theta \\ &= 2\Omega \sin(\theta_0 + \delta\theta) \qquad \text{where } \delta\theta \ll 1 \\ &= 2\Omega \sin\theta_0 + 2\Omega \cos\theta_0 (\delta\theta) + O((\delta\theta)^2) \end{aligned}$$

To the same order of approximation (i.e. $O((\delta\theta)^2)$), we can replace the sphere by its tangent plane at θ_0.

Introduce Cartesian coordinates on this plane with O_z vertical, O_y Northwards and O_x Eastwards

Now $\quad f = f_0 + \beta y + O((\delta\theta)^2)$

where
$$f_0 = 2\Omega \sin\theta_0$$
$$\beta = (2\Omega\cos\theta_0)/R_E \quad \longleftarrow$$
gradient of vertical component is single-signed (why no diff in N/S hemisp.)

i.e. we are on a Cartesian, tangent plane with a linearly varying Coriolis parameter

$$f = f_0 + \beta y$$
$$f_0 = 2\Omega\sin\theta_0 \quad \longleftarrow$$
Foucault (of Foucault's pendulum)

The governing eq^n is conservation of

$$q = \frac{\zeta + f}{H} = \frac{\zeta + f_0 + \beta y}{D(1 + \varepsilon F\eta)}$$

$\uparrow$ C const

$$= C + (\zeta - F\eta + \beta y)$$

This is the QGPV for the channel with linearly sloping bottom, i.e. channel displays the same dynamics as the spherical Earth.

In particular, Rossby Wave phase velocity always has a Westward component.

N

$\longleftarrow$ RW

S

β-plane approximation

So far, we only know that PHASE propagates to the West. But from Stationary Phase, we know it is Group Velocity that is important.

We will now show (1) Group velocity does determine energy propagation
(2) For RWs this does mean that energy ends up in the West.

Now the QGPV eq^n is

$$\frac{\partial q}{\partial t} + \frac{\partial(\psi, q)}{\partial(x,y)} = 0$$

On the β-plane, $\eta_B = \beta y$

$$q = \nabla^2\psi - F\psi + \beta y$$

So $$\frac{\partial}{\partial t}\left(\nabla^2\psi - F\psi\right) + \frac{\partial(\psi, \nabla^2\psi - F\psi)}{\partial(x,y)} + \beta\psi_x = 0 \qquad \cdots (1)$$

Now, remarkably, single RWs satisfy this eq^n.

Try $\psi = A\cos(kx + ly - \omega t)$ $\qquad \cdots\cdots (2)$

$$\nabla^2\psi = \psi_{xx} + \psi_{yy} = (-k^2 - l^2)\psi$$

So the nonlinear terms in (1) are identically zero for form (2).

$$\omega(-k^2-\ell^2-F)\cancel{A\sin()} - \beta k \cancel{A\sin()} = 0$$

i.e. $$\omega = \frac{-\beta k}{k^2+\ell^2+F}$$

dispersion relation for Rossby waves – exactly as in channel

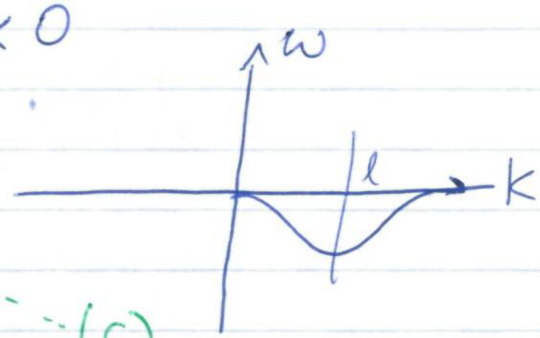
x-component of $\underline{c}_p$

$$\frac{\omega}{k} = (c_p)_x = \frac{-\beta}{k^2+\ell^2+F} < 0$$

$$\underline{c}_p = \left(\frac{\omega}{k}, \frac{\omega}{\ell}\right) = \left(-\frac{\beta}{k^2+\ell^2+F}, -\frac{\beta k/\ell}{k^2+\ell^2+F}\right)$$

always >0

(c)



<u>RW energy propagation</u>

The QGPV eq^n for linear waves is

$$\frac{\partial}{\partial t}\left(\nabla^2\psi - F\psi\right) + \beta\psi_x = 0$$

Multiply by ψ:

$$\psi\nabla^2\psi_t - F\psi\psi_t + \beta\psi\psi_x = 0$$

We want to turn this into a conservation relation.

A conservation relation for any quantity is an eq^n of the form

$$\frac{\partial E}{\partial t} + \underline{\nabla}\cdot\underline{F} = 0$$

Almost all physical laws are of this form. We call $\underline{E}$ the DENSITY of the quantity and $\underline{F}$ the FLUX of the quantity.

Consider some region V with surface S.

Integrating this relation over V gives

$$0 = \int_V \left(\frac{\partial E}{\partial t} + \underline{\nabla} \cdot \underline{F} \right) dV$$

$$= \int_V \frac{\partial E}{\partial t}\, dV + \int_V \underline{\nabla} \cdot \underline{F}\, dV$$

$$= \frac{d}{dt} \int_V E\, dV + \oint_S \underline{F} \cdot \hat{\underline{n}}\, dS$$

← our quantity

i.e. rate of increase of E in V equals minus the flux of E out of V.

$$\frac{d}{dt} \int_V E\, dV = - \oint_S \underline{F} \cdot \hat{\underline{n}}\, dS.$$

Meanwhile

from (c)

$$\underline{c}_g = \nabla_{\underline{k}}\, \omega = \frac{\partial \omega}{\partial k} \hat{\underline{x}} + \frac{\partial \omega}{\partial l} \hat{\underline{y}} = \frac{\beta}{(k^2+l^2+F)^2} \left[(k^2-l^2-F)\hat{\underline{x}} + 2kl\hat{\underline{y}} \right]$$

$\underline{c}_g$ is the gradient in wavenumber space of the frequency

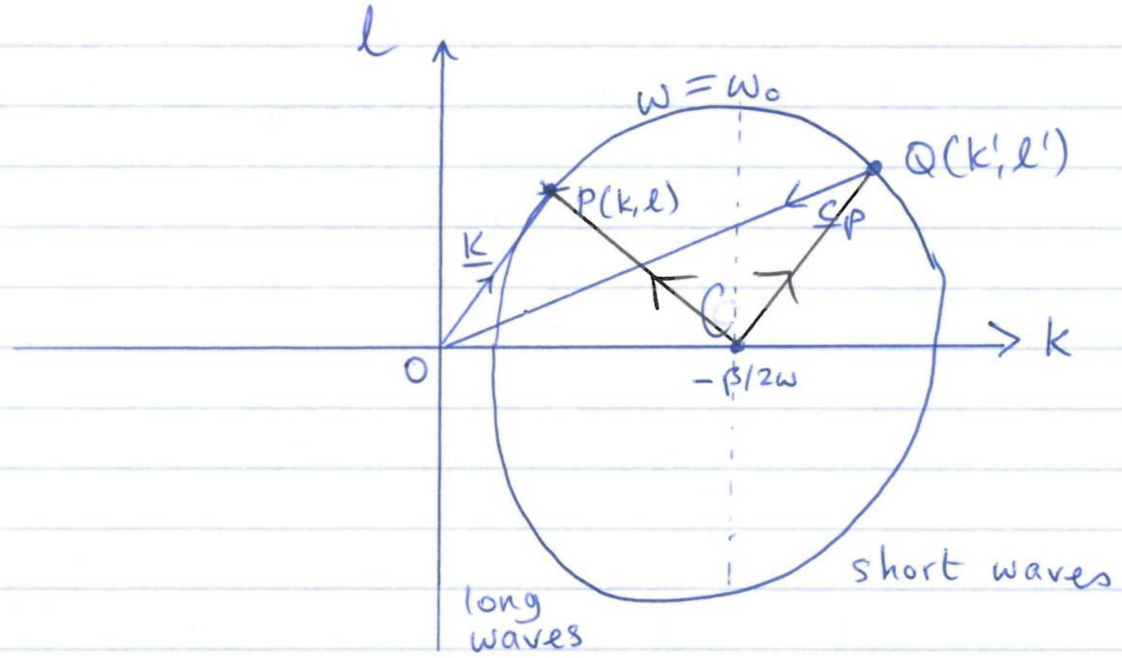
The dispersion relation is $k^2 + l^2 + F = -\frac{\beta}{\omega} k$

$$k^2 + \frac{\beta}{\omega} k + l^2 + F = 0$$

$$\left(k + \frac{\beta}{2\omega}\right)^2 + l^2 = \left(\frac{\beta}{2\omega}\right)^2 - F \qquad \text{completing the square}$$

a circle in (k, l) space, radius $\sqrt{\left(\frac{\beta}{2\omega}\right)^2 - F}$

centre $\left(-\frac{\beta}{2\omega}, 0\right)$.



Where to plot $\left(-\frac{\beta}{\omega}\right)$? $\quad k>0,\ \omega<0,\ \frac{\beta}{2\omega}<0.$

Contours $\omega = $ const. are called <u>SLOWNESS</u> surfaces. Here it's a circle

Consider a point P with wavenumber

$$\underline{K} = k\hat{\underline{x}} + l\hat{\underline{y}} \qquad \text{where} \quad \underline{K} = \overrightarrow{OP}.$$

$$\boxed{\underline{c}_p = \frac{\omega}{|\underline{K}|}\hat{\underline{K}}} = \text{phase velocity}$$

But $\omega < 0$ so $\underline{c}_p$ is in the opposite dir^n to $\underline{K}$

i.e. $\underline{c}_p$ is in the dir^n $\overrightarrow{PO}$.

note that the phase velocity always has its component to the West.

$$\underline{c}_g = \nabla_{\underline{K}}\,\omega$$

And since $\underline{c}_g$ is the gradient in waveno space of the frequency, we draw the $\underline{c}_g$ lines radially, since it's $\perp^r$ to slowness surfaces. Which dir^n?

∇f points in dir^n of increasing f. So $\underline{c}_g$ goes in the dir^n of increasing ω.

Recall $\omega < 0$, so ω increasing $\Rightarrow |\omega|$ decreasing

$\Rightarrow \left(\frac{\beta}{2\omega}\right)^2$ increasing

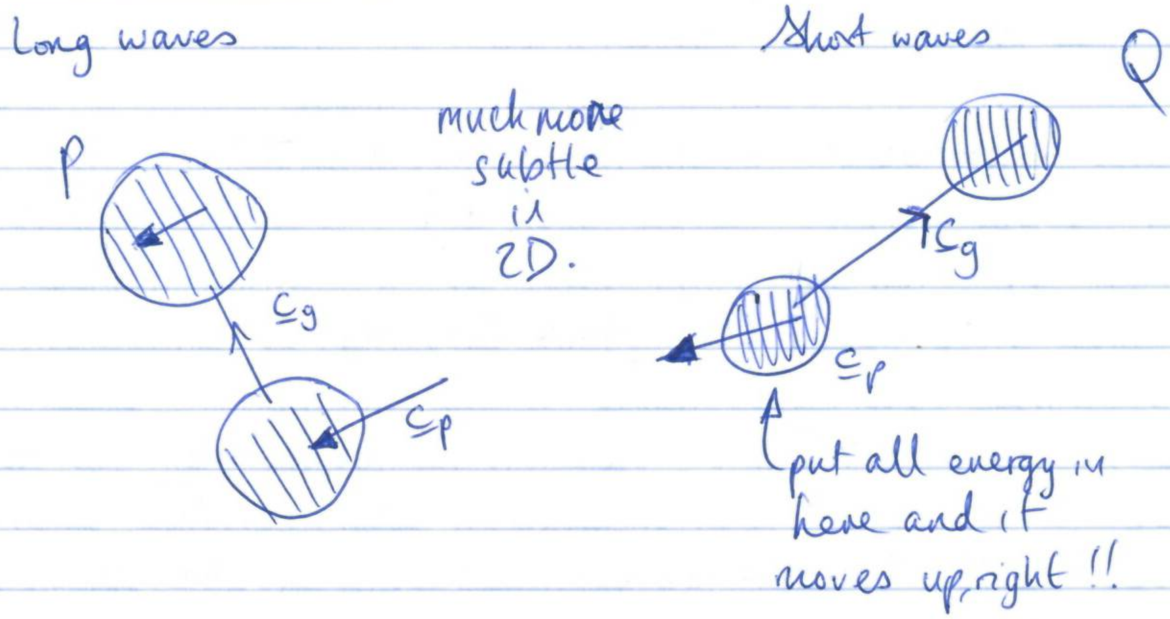
$\Rightarrow$ radius increases

$\Rightarrow$ circle lies outside.

$\Rightarrow \underline{c}_g$ points outwards, i.e. in dir^n $\overrightarrow{CP}$ or $\overrightarrow{CQ}$.

So in reality, only long waves ($k < -\frac{\beta}{2\omega}$) travel to the West, and short waves travel to the East.

they get destroyed by viscosity effects, etc.



Now to do this algebraically.

Conservation relations: $\frac{\partial E}{\partial t} + \underline{\nabla} \cdot \underline{S} = 0$ We're looking for this, remember

E density of quantity of interest
$\underline{S}$ is the flux " " "

Extremely important subclass: cases for which, in some rational sense, we can write

$$\underline{S} = \underline{V} E$$

for some velocity $\underline{V}$.

Then we say that our quantity travels at speed $\underline{V}$
ie. $\underset{\underline{S}}{\text{flux}} = \underline{V} \times \underset{E}{\text{density}}$.

e.g. $E = \rho$ the density of a fluid (mass/unit vol.)
conservation of mass eq^n is

$$\frac{\partial \rho}{\partial t} + \underline{\nabla} \cdot (\rho \underline{u}) = 0$$

$$E = \rho, \quad \underline{S} = \rho \underline{u}.$$

ie. ~~density~~ mass travels at speed $\underline{u}$.

Aim: use this with E, the energy density of RWs and show $\underline{S} = c_g E$, so that the energy travels at the group velocity.

Write out the energy eq^n and manipulate it to

$$\frac{\partial E}{\partial t} + \underline{\nabla} \cdot \underline{S} = 0.$$

For QGPV,

$$\frac{\partial q}{\partial t} + \frac{\partial(\psi, \nabla^2\psi - F\psi + \beta y)}{\partial(x,y)} = 0$$

$\downarrow$ linearise (infinitessimal waves)

$$(\nabla^2\psi - F\psi)_t + \beta\psi_x = 0$$

Multiply by ψ: $\psi\nabla^2\psi_t - F\psi\psi_t + \beta\psi\psi_x = 0$

$$\psi\nabla^2\psi_t = \psi\,\underline{\nabla}\cdot(\underline{\nabla}\psi_t)$$

$$= \underline{\nabla}\cdot[\psi\underline{\nabla}\psi_t] - \underline{\nabla}\psi\cdot\underline{\nabla}\psi_t$$

so $\underline{-\underline{\nabla}\psi\cdot\underline{\nabla}\psi_t} + \underline{\nabla}\cdot[\psi\underline{\nabla}\psi_t] \underline{- F\psi\psi_t} + \beta\psi\psi_x = 0$

Hence $E = \frac{1}{2}\underline{\nabla}\psi\cdot\underline{\nabla}\psi + \frac{1}{2}F\psi^2$

$$\underline{S} = -\psi\underline{\nabla}\psi_t - \frac{1}{2}\beta\psi^2\underline{\hat{x}}$$

Thus, conservation of RW energy is

$$\frac{\partial E}{\partial t} + \underline{\nabla}\cdot\underline{S} = 0 \quad \text{(D)}$$

where $E = \frac{1}{2}|\underline{\nabla}\psi|^2 + \frac{1}{2}F\psi^2 > 0$ scalar

$\underline{S} = -\psi\underline{\nabla}\psi_t - \frac{1}{2}\beta\psi^2\underline{\hat{x}}$ vector

<u>Comments</u>: (1) $\underline{S}$ is not unique. We can add on any quantity q s.t. $\underline{\nabla}\cdot q = 0$. So we can choose S to be convenient.

(2) $\underline{u} = \underline{k} \times \underline{\nabla}\psi$, so $|\underline{\nabla}\psi|^2 = |\underline{u}|^2$

so $\frac{1}{2}|\nabla\psi|^2 = \frac{1}{2}|\underline{u}|^2$,

the KE density, i.e. $\frac{1}{2}m\underline{u}^2$ per mass m.

and $\frac{1}{2}F\psi^2$ is the PE density

$E = KE + PE$ per unit mass as expected.

F large $\Rightarrow$ lot of PE stored in surface perturbations
F small $\Rightarrow$ lot of KE stored in horizontal motions
(rigid lid case)

So, our wave is $\psi = A\cos(kx + ly - \omega t)$

$$\frac{\partial\psi}{\partial x} = -kA\sin\theta \qquad \theta = kx + ly - \omega t$$

$$\frac{\partial\psi}{\partial y} = -lA\sin\theta$$

(1) $\Rightarrow$ $$E = \frac{1}{2}A^2\sin^2\theta\,(k^2+l^2) + \frac{1}{2}FA^2\cos^2\theta$$

We introduce the concept of the average of a periodic quantity. Here, our wave is periodic in x, y and t, so we could average over any of these. For definiteness, choose t, i.e. for any quantity $h(t)$ of period T, write

$$\langle h\rangle = \frac{1}{T}\int_{t_0}^{t_0+T} h(t)\,dt \qquad \text{indpt of } t_0.$$

Integration is linear so so is averaging.

$$\langle \alpha h_1 + \beta h_2 \rangle = \alpha \langle h_1 \rangle + \beta \langle h_2 \rangle \qquad \forall\, h_1, h_2, \overbrace{\alpha, \beta}^{\text{const.}}$$

In particular, $\langle \alpha \rangle = \alpha$.

$$\langle \sin t \rangle = 0$$

$$\langle \cos t \rangle = 0$$

$$\langle \sin^2 t \rangle = \langle \cos^2 t \rangle = ?$$

$$\langle f(\sin t) \rangle = \langle f(\cos t) \rangle$$

$\sin^2 t + \cos^2 t = 1$ though

$\langle \sin^2 t \rangle + \langle \cos^2 t \rangle = 1$

$\Rightarrow \langle \sin^2 t \rangle = \langle \cos^2 t \rangle = 1/2.$

$$E = \frac{1}{2} A^2 \sin^2\theta \,(k^2 + l^2) + \frac{1}{2} F A^2 \cos^2\theta$$

$$\Rightarrow \langle E \rangle = \frac{1}{4} A^2 (k^2 + l^2 + F)$$

$\psi_x \rightarrow -k \ldots$

$\psi_t \rightarrow +\omega \ldots$

$$S = -\psi \underline{\nabla} \psi_t - \frac{1}{2} \beta \psi^2 \hat{\underline{x}}$$

$$= -A\cos\theta \left[-k\omega \hat{\underline{x}} - l\omega \hat{\underline{y}} \right] A \cos\theta$$

$$- \frac{1}{2} \beta A^2 \cos^2\theta \, \hat{\underline{x}}$$

$$= A^2 \left[-\omega \underline{k} - \frac{1}{2} \beta \hat{\underline{x}} \right] \cos^2\theta \qquad \underline{k} = k\hat{\underline{x}} + l\hat{\underline{y}}$$

$$\Rightarrow \langle S \rangle = \frac{1}{2} A^2 \left(-\omega \underline{k} - \frac{1}{2} \beta \hat{\underline{x}} \right). \qquad \left[= \frac{\overbrace{A^2(-\omega)}^{>0}}{2} \left(\underline{k} + \frac{\beta}{2\omega} \hat{\underline{x}} \right) \right]$$

$$= \frac{1}{2} A^2 \left(\frac{\beta k}{k^2 + l^2 + F} (k\hat{\underline{x}} + l\hat{\underline{y}}) - \frac{1}{2} \beta \hat{\underline{x}} \right)$$

$$= \frac{2\langle E \rangle}{(k^2 + l^2 + F)^2} \left[\beta k^2 \hat{\underline{x}} - \frac{1}{2} \beta (k^2 + l^2 + F) \hat{\underline{x}} + \beta k l \hat{\underline{y}} \right]$$

$$= \frac{\langle E \rangle}{(k^2+l^2+F)^2} \left[\beta(k^2-l^2-F)\hat{\underline{x}} + 2\beta k l \hat{\underline{y}} \right]$$

$$= \langle E \rangle \underline{c}_g .$$

$$\Longrightarrow \quad \langle \underline{S} \rangle = \langle E \rangle \underline{c}_g .$$

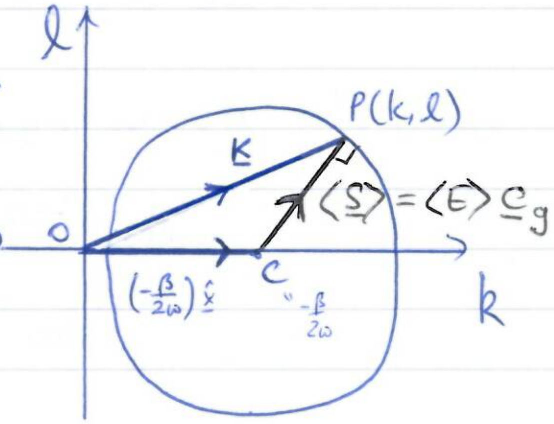
i.e. the Rossby wave energy travels with the group velocity.

$\langle \underline{S} \rangle$ has dir^{n} $\underline{k} + \frac{\beta}{2\omega}\hat{\underline{x}}$ (by black bit on prev. page)

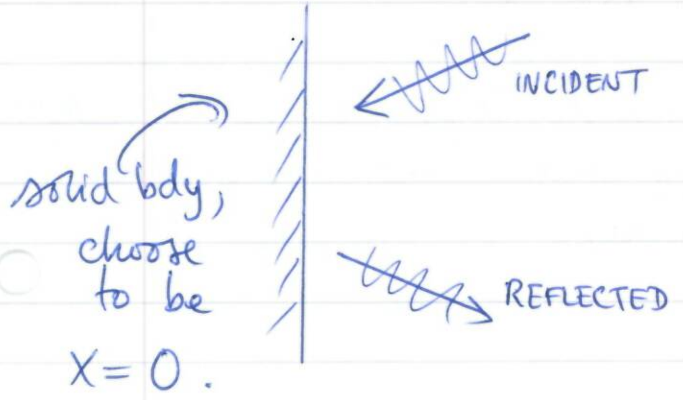
$$= \underline{k} - \left(\frac{\beta}{-2\omega}\right)\hat{\underline{x}}$$

as before, $\underline{c}_g$ lies along $\overrightarrow{CP}$

i.e. stationary phase really does show energy propagation.



Example: RW reflection from a Western boundary



Let the incident wave be

$$\Psi_I = \mathrm{Re}\left[A_I e^{i(k_I x + l_I y - \omega_I t)} \right]$$

$$\Psi_R = \mathrm{Re}\left[A_R e^{i(k_R x + l_R y - \omega_R t)} \right]$$

The b.c. is no flow through $x=0$ $\forall y, t$

The total field is $\Psi = \Psi_I + \Psi_R$
and so

$$\Psi = \text{const on } x = 0 \quad \forall y, t$$

w.l.o.g. $\Psi = 0$ on $x=0$
(since there's only one boundary)

So we now have to go find all 8 constants!!

- At the origin, $x=0, y=0$, it is sufficient that

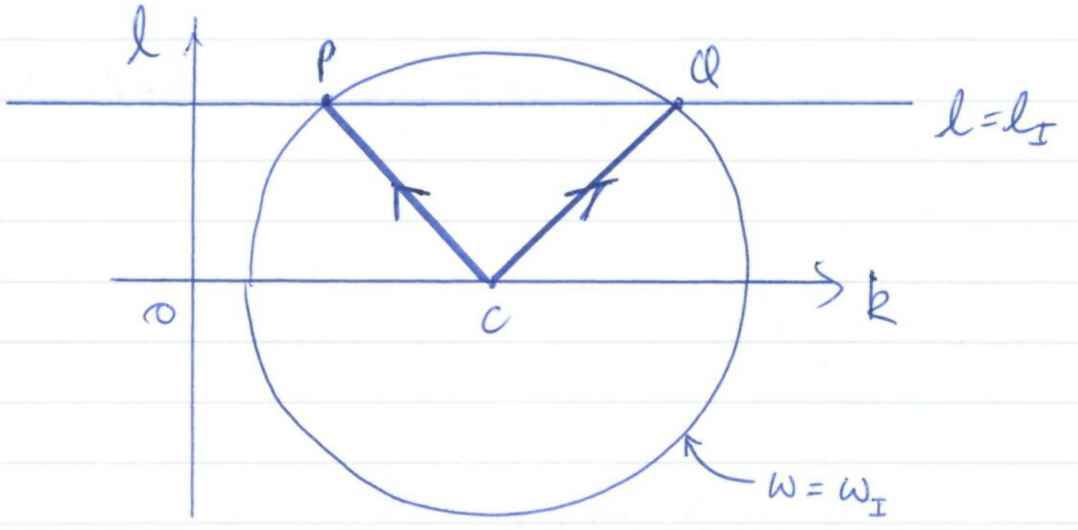
$$A_I e^{-i\omega_I t} + A_R e^{-i\omega_R t} = 0 \quad \forall t$$

ie. $$e^{\underbrace{-i(\omega_I - \omega_R)}t} = \underbrace{-\frac{A_R}{A_I}}_{\text{just a const.}} \quad \forall t$$

ie. $\omega_I - \omega_R = 0$
$\omega_I = \omega_R$ and $A_R = -A_I$

as expected, as constraints on problem are t-indpt
ie. the incident and reflected waves lie on the same slowness circle,

$$\omega = \omega_I .$$



b.c. sufficient that $A_I e^{i(k_I x + l_I y - \omega_I t)} + A_R e^{i(k_R x + l_R y - \omega_R t)} = 0$ on $x = 0$ $\forall y, t$.

But ~~suffice~~ , $\omega_I = \omega_R$ and $x = 0$

$$e^{il_I y} = e^{il_R y} \quad \forall y$$

i.e. $e^{i(l_I - l_R)y} = 1 \quad \forall y \quad \Rightarrow \quad l_I = l_R$

i.e. the y wavenumber of the incident and reflected waves both lie on the line $l = l_I$ [no surprise as problem is inhomogenous (translation invariant) in y].

There are only 2 waves that lie on the slowness circle $\omega = \omega_I$ and the line $l = l_I$, i.e. P and Q here. Thus one is the <u>incident</u> wave and one is the <u>reflected</u> wave.

Which is which? Well the incident waves <u>'go'</u> to the West.

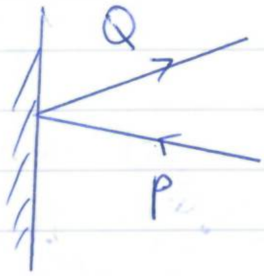
<u>'go'</u> = carries its energy, i.e. the <u>group velocity</u> $\underline{c}_g$.

So join to centre points P and Q on the diagram and point the arrows outward.

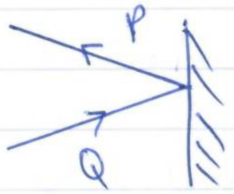
Then it's obvious that the wave P carries energy to be incident on $x=0$. This energy is reflected into wave Q.

So $P = (k_I, l_I)$ LONG WAVE
$Q = (k_R, l_R)$ SHORT WAVE

And $A_R = -A_I$
$\omega_R = \omega_I$
$l_R = l_I$
k_R satisfies the dispersion relation

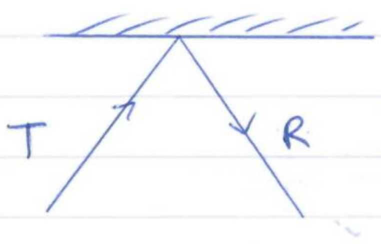


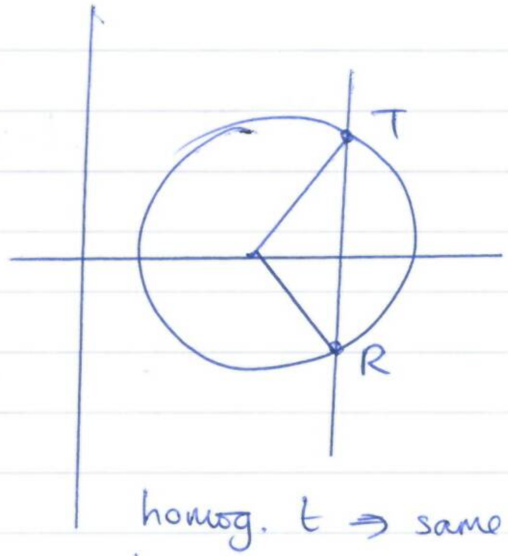
Example: Reflection from an Eastern boundary



Same waves, but
Q is incident
P is reflected.

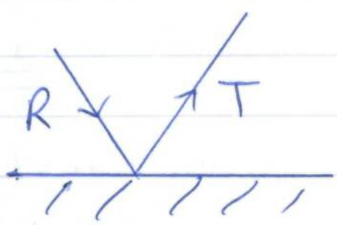
Example: Reflection from Northern boundary





homog. $t \Rightarrow$ same circle
homog. $x \Rightarrow$ same k

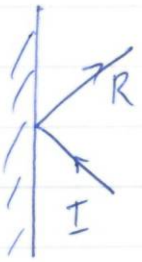
Example: Reflection from Southern boundary

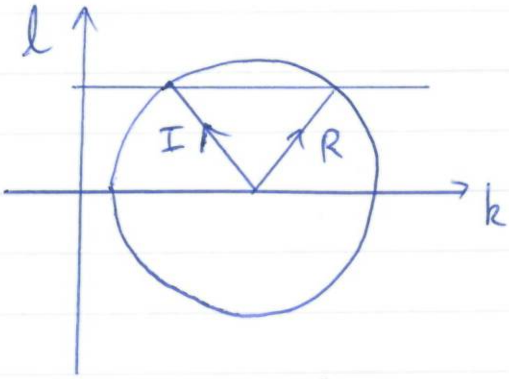


H/W: Example: Arbitrarily-oriented linear boundary

$ax + by = 0$

$$\langle E \rangle = \frac{1}{4} A^2 (k^2 + l^2 + F)$$





$$l_I = l_R$$
$$k_I < -\frac{\beta}{2\omega} < k_R$$

$$\langle E \rangle_I < \langle E \rangle_R$$

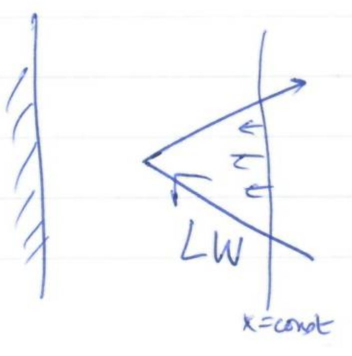
so have we invented energy?!
Where has it come from?

Trick question: E is energy <u>density</u> !!!

The group velocity varies exactly inversely to the energy density so that the energy flux across any line $x = \text{const}$ is zero in total

i.e. reflected = incident

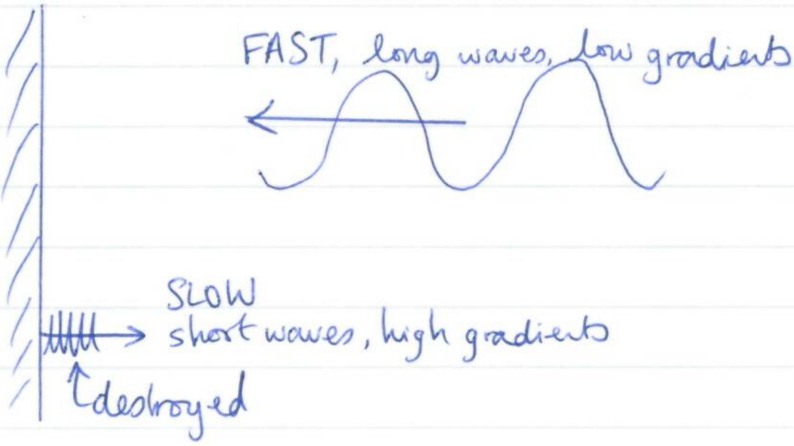
long wave is FAST but LOW ENERGY DENSITY
it turns into a ~~short~~
short wave, SLOW HIGH ENERGY DENSITY



reflected flux $\langle \underline{S}_R \rangle \cdot \hat{\underline{x}}$

= minus incident flux $\langle \underline{S}_I \rangle \cdot \hat{\underline{x}}$

Any form of dissipation destroys the slow, high energy (highest gradient) short waves but does not affect the fast, low energy long waves.



energy piles up on West.

end of Rossby Waves.

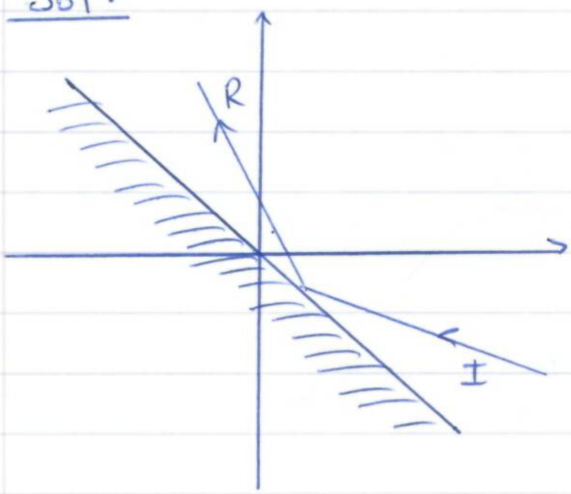
Question: Consider an arbitrary boundary

$$ax + by = 0$$

for a, b positive constants.

Discuss RW reflection

Soln



boundary condition

$$\Psi = \Psi_I + \Psi_R$$
$$= 0 \quad \text{when } ax + by = 0$$

Consider pt $x=0, y=0$: As before, $\omega_I = \omega_R$

$$A_R = -A_I$$

ie. waves lie on the same slowness circle $\omega = \omega_I$.

This leaves

$$e^{i(k_I x + l_I y)} = e^{i(k_R x + l_R y)} \qquad \forall x, y$$

where $ax + by = 0$

ie. for x when $y = -\frac{ax}{b}$

ie. $$e^{ix(k_I - k_R - \frac{a}{b} l_I + \frac{a}{b} l_R)} = 1 \qquad \forall x$$

ie. $bk_I - bk_R - al_I + al_R = 0$

ie. $bk_I - al_I = bk_R + al_R$

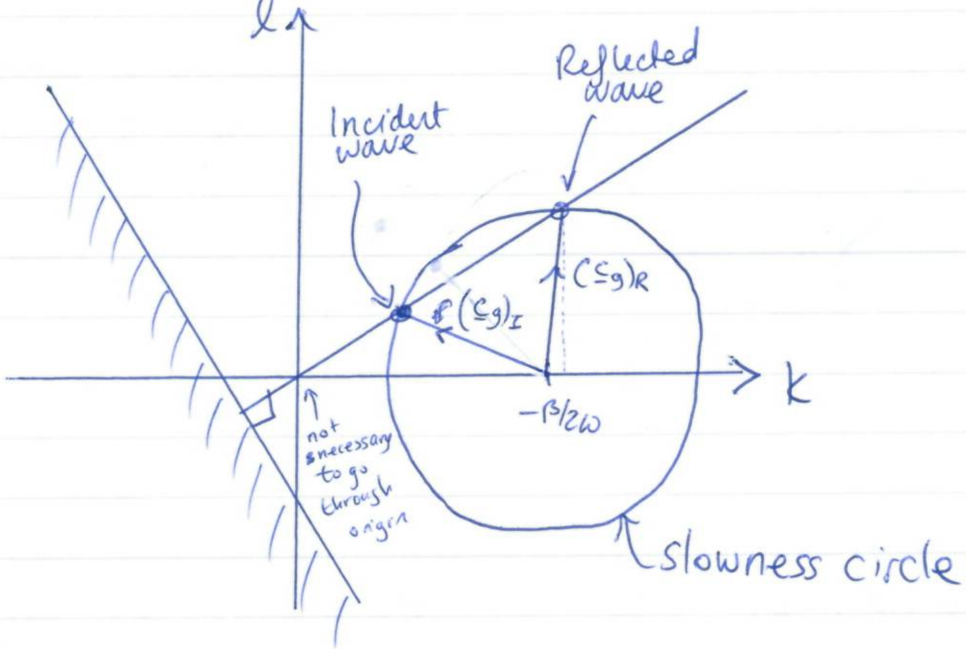
ie both waves lie on the straight line

$$bk - al = bk_I - al_I$$

This line has slope b/a

boundary has slope $-a/b$

Product is -1, i.e. perpendicular.



DISSIPATION & VISCOSITY
EKMAN LAYERS

The first and only discussion on viscous effects.

- Why does the ocean move?
  → WIND.

- Must require viscosity

The relevant eq.ⁿ is the Navier-Stokes eq.ⁿ, which relative to our rotating frame is

$$\boxed{\frac{D\underline{u}}{Dt} + 2\underline{\Omega} \times \underline{u} = -\frac{1}{\rho}\underline{\nabla} p + \underbrace{\nu \nabla^2 \underline{u}}_{\text{Stokes term}}}$$

ν = kinematic viscosity

$\nu = \frac{\mu}{\rho}$, μ = coefficient of viscosity

μ is the constant of proportionality relating 'rate of strain' to 'stress', i.e. the resulting motion to the force.

$\mu = 0$ inviscid

$0 < \mu \ll 1$ air, water.

We also still have continuity $\boxed{\underline{\nabla} \cdot \underline{u} = 0}$

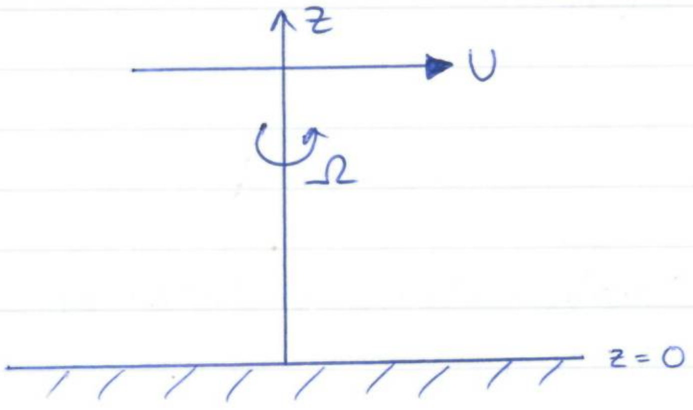
But we have an extra boundary condition on solid boundaries (Stokes)

$\underline{u} \cdot \underline{t} = 0$ as well as $\underline{u} \cdot \hat{\underline{n}} = 0$

i.e. $\boxed{\underline{u} = 0 \text{ on solid boundaries}}$

He is taking this really slowly: exam fodder?

<u>Example</u>: Consider the flow in the half-plane $z>0$ of a viscous fluid. Let the flow far above the boundary (i.e. $z \to \infty$) have uniform horizontal speed U relative to the rotating frame



The boundary condition on $z=0$ is

$$\underline{u} = 0 \quad \text{on } z=0.$$

Choose the x-axis in the dir^n of the flow at $z=\infty$. Then the far-field b.c. is

$$\underline{u} \to U\hat{\underline{x}} \quad \text{as } z \to \infty.$$

Notice that b.c.'s are homogeneous (translation invariant) in x, y and t.

Hence look for sol^n's independent of x, y, t, i.e. take $\underline{u} = \underline{u}(z)$, i.e.

$$u = u(z)$$
$$v = v(z)$$
$$w = w(z).$$

Now consider first continuity

$$\underbrace{\cancel{\frac{\partial u}{\partial x}}}_{u=u(z)} + \underbrace{\cancel{\frac{\partial v}{\partial y}}}_{v=v(z)} + \frac{\partial w}{\partial z} = 0$$

$$\Rightarrow \frac{\partial w}{\partial z} = 0.$$

But $w = 0$ on $z = 0$
$\Rightarrow w \equiv 0 \ \forall z.$ ie. no vertical motion.

Now consider $\frac{D\underline{u}}{Dt} = \cancel{\frac{\partial \underline{u}}{\partial t}} + u \cancel{\frac{\partial \underline{u}}{\partial x}} + v \cancel{\frac{\partial \underline{u}}{\partial y}} + \cancel{w} \frac{\partial \underline{u}}{\partial z}$

$\underline{u} = \underline{u}(z)$ $\quad \underline{u} = \underline{u}(z)$ $\quad \underline{u} = \underline{u}(z)$

$w \equiv 0$ by above

$$\Rightarrow \frac{D\underline{u}}{Dt} \equiv 0.$$

(True without any approximation).

We are left with

(1) $\quad -2\Omega v = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu \frac{\partial^2 u}{\partial z^2}$ $\quad$ x-momentum

(2) $\quad 2\Omega u = -\frac{1}{\rho}\frac{\partial p}{\partial y} + \nu \frac{\partial^2 v}{\partial z^2}$ $\quad$ y-momentum

(3) $\quad 0 = -\frac{1}{\rho}\frac{\partial p}{\partial z}$ $\quad$ z-momentum

Now (3) says that the pressure p is the same at all z. But we know the flow at $z = \infty$.

At $z = \infty$, (1) gives $-2\Omega 0 = -\frac{1}{\rho}\frac{\partial p}{\partial x} + 0$

$\underline{u} \to U\hat{\underline{x}}$
$u \to U$
$v \to 0.$

i.e. $\dfrac{\partial p}{\partial x} = 0$ at $z = \infty$.

But p is the same at all z. Thus
$$\frac{\partial p}{\partial x} \equiv 0 \quad \forall z.$$

At $z = \infty$, (2) gives
$$2\Omega U = -\frac{1}{\rho}\frac{\partial p}{\partial y} + 0$$

i.e. at $z = \infty$, $\dfrac{\partial p}{\partial y} = -2\Omega\rho U$

But p same $\forall z \Rightarrow \dfrac{\partial p}{\partial y} = -2\Omega\rho U \quad \forall z$

Hence (1) and (2) become

$$-2\Omega v = \nu u'' \qquad \text{- - - - (4)}$$
$$2\Omega u = 2\Omega U + \nu v'' \qquad \text{- - - - (5)}$$

with b.c.'s, $u \to U \quad v \to 0 \quad$ as $z \to \infty$
$u = 0 \quad v = 0 \quad$ on $z = 0$

2 simultaneous 2nd order ODEs with constant coefficients and 4 b.c.'s (2 at each end).

Introduce the complex velocity

$$\alpha = u + iv$$ (not $u - iv$ as in inviscid fluid dyn's)

$$(4) + i(5) \Rightarrow \quad 2\Omega(-v + iu) = 2\Omega U i + \nu(u'' + iv'')$$

$$\Rightarrow 2\Omega i\alpha - 2\Omega U i = \nu\alpha''$$

$$\Rightarrow \alpha'' - \left(\frac{2\Omega}{\nu} i\right)\alpha = -\frac{2\Omega}{\nu} U i$$

with $\alpha = 0$ on $z = 0$

$\alpha \to U$ as $z \to \infty$.

Second order linear ODE with constant coefficients

Particular solⁿ: $\alpha_p = U$ (far-field solⁿ)

Complementary fⁿ: Auxiliary eqⁿ:

$$m^2 - \left(\frac{2\Omega}{\nu} i\right) = 0.$$

$$\Rightarrow m = \sqrt{\frac{\Omega}{\nu}}\sqrt{2i}$$

$$= \frac{\pm(1+i)}{\delta}, \quad \delta = \left(\frac{\nu}{\Omega}\right)^{1/2}$$

$$[\delta] = \left(\frac{L^2T^{-1}}{T^{-1}}\right)^{1/2} = L.$$

General solⁿ: CF + PS

i.e.

$$\alpha = Ae^{(1+i)z/\delta} + Be^{-(1+i)z/\delta} + U.$$

(A, B unknown constants) found from b.c.s.

α bounded as $z \to \infty$ so $A = 0$.

this gives $\alpha \to U$ as $z \to \infty$ ✓

$\alpha = 0$ on $z = 0$ so $B = -U$

$$\Rightarrow \quad \alpha = U\left[1 - e^{-(1+i)z/\delta}\right]$$

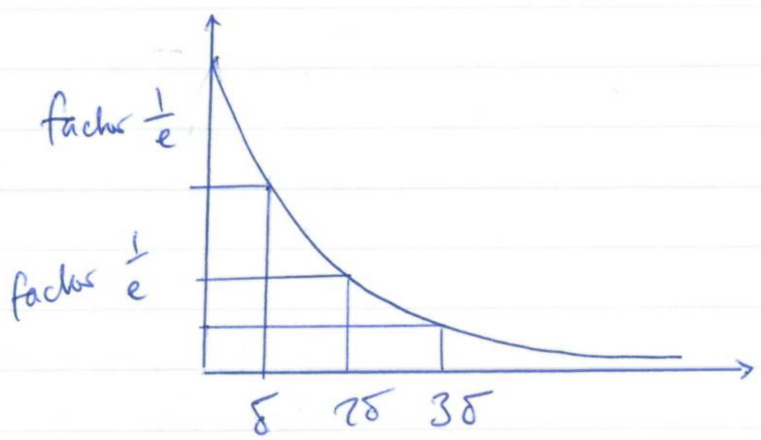
$$= u + iv$$

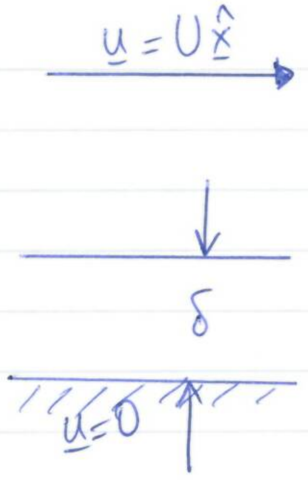
$$\Rightarrow \quad u = U\left[1 - e^{-z/\delta}\cos\left(\frac{z}{\delta}\right)\right]$$

$$v = U e^{-z/\delta}\sin\left(z/\delta\right).$$

Note that $|\alpha - U| \leq e^{-z/\delta}$

i.e. $\alpha \to U$ with exponential decay on vertical scale of
$$\delta = \sqrt{\nu/\Omega}$$
i.e. the vertical e-folding scale





Ekman boundary layer

Typical laboratory values : 1 revolution / sec
$= 2\pi$ radians / sec

for water $\nu = 10^{-2}\ \text{cm}^2/\text{sec}$

$$\sqrt{\frac{\nu}{\Omega}} = \delta \approx \left(\frac{10^{-2}\ \text{cm}^2/\text{sec}}{6\ \ /\text{sec}}\right)^{1/2}$$

$$\approx \frac{1}{25}\ \text{cm}.$$

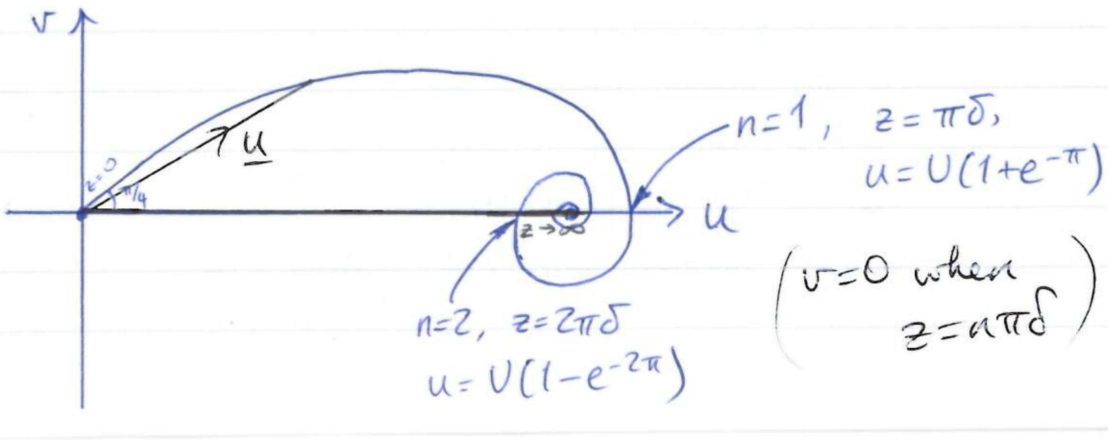
But note as $\Omega \to 0$, $\delta \to \infty$, i.e. we no longer have a sol^n. There is no constant thickness layer, in fact

$u = u(x, z)$
$v = v(x, z)$
$w = w(x, z)$.

u, v as on previous page

What does our flow look like?

'Hodograph plane'



$\underline{z \ll 1}$

$\alpha \sim U(1+i)\frac{z}{\delta}$ (first term of series expansion of $e^{-(1+i)z/\delta}$)

i.e. $u \sim U\frac{z}{\delta}$, $v \sim U\frac{z}{\delta}$ } since $\underline{u}=0$ on $z=0$

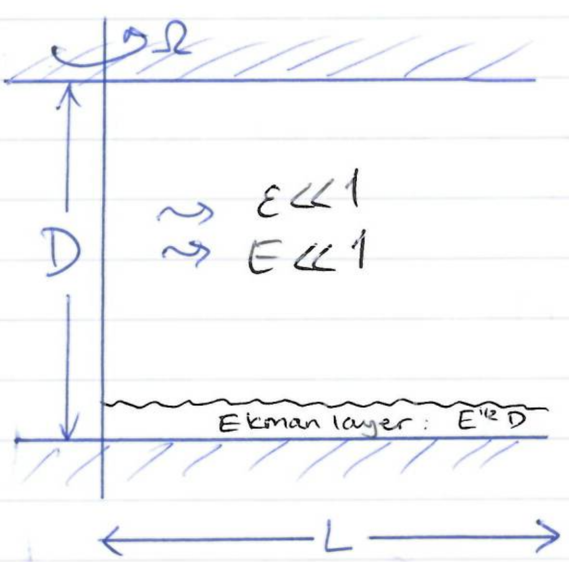
$\frac{v}{u} = 1$

Note also $v = 0$ when $\frac{z}{\delta} = n\pi$ $\quad n = 0, 1, 2, \ldots$

so $z = n\pi\delta$

at these points $u = U\left[1 - e^{-n\pi}(-1)^n\right]$

Ekman compatibility condition



$\varepsilon = \frac{U}{2\Omega L} \ll 1$

bulk of flow to be geostrophic. Viscous effects confined to an Ekman Layer of thickness $\delta = \left(\nu/\Omega\right)^{1/2}$

The ratio of Ekman layer thickness to fluid depth is a new nondimensional parameter,

$$\frac{\text{Ekman layer thickness}}{\text{depth}} = \frac{(\nu/\Omega)^{1/2}}{D} = \left[\frac{\nu}{\Omega D^2}\right]^{1/2}$$

Traditional these days to write the parameter as

$$E = \frac{\nu}{2\Omega D^2} \quad \text{... the Ekman nº.}$$

Thus thickness ratio is of order $E^{1/2}$.

We approach this problem mathematically using matched asymptotic expansions.

Our eqⁿs are

$$\frac{\partial \underline{u}}{\partial t} + (\underline{u}\cdot\underline{\nabla})\underline{u} - 2\underline{\Omega}\times\underline{u} = -\frac{1}{\rho}\underline{\nabla}p + \nu\nabla^2\underline{u} \qquad \text{N-S}$$

$$\underline{\nabla}\cdot\underline{u} = 0 \qquad \text{Cont.}$$

We non-dimensionalise,

x, y on L	$x' = \frac{x}{L}$	$y' = \frac{y}{L}$
z on D	$z' = \frac{z}{D}$	
u, v on U	$u' = u/U$	$v' = \frac{v}{U}$
w on UD/L		
p on $2\Omega U \rho L$	(guess to balance $2\underline{\Omega}\times\underline{u}$)	

$\varepsilon = \frac{U}{2\Omega L}$

Horizontal: $$\varepsilon\left(\frac{\partial \underline{u}'}{\partial t'} + (\underline{u}'\cdot\underline{\nabla}')\underline{u}'\right) + \hat{\underline{z}}\times\underline{u}' = -\underline{\nabla}'p + E\underline{\nabla}^{2\prime}\underline{u}'$$

where $$\nabla^{2\prime} = \left(\frac{D}{L}\right)^2\left[\frac{\partial^2}{\partial x'^2} + \frac{\partial^2}{\partial y'^2}\right] + \frac{\partial^2}{\partial z'^2}$$

$$E = \frac{\nu}{2\Omega D^2}.$$

Vertical: $\left(\frac{D}{L}\right)^2 \varepsilon\left(\frac{\partial w'}{\partial t'} + (\underline{u}'.\underline{\nabla}')w'\right)$

$$= -\frac{\partial p'}{\partial z'} + \left(\frac{D}{L}\right)^2 E \nabla'^2 w' \quad \ldots\ldots (\dagger)$$

Cont: $\underline{\nabla}'.\underline{u}' = 0$

We three non-dimensional ratios

$$\varepsilon = \frac{U}{2\Omega L}$$

$$E = \frac{\nu}{2\Omega D^2}$$

and $\frac{D}{L}$.

Now consider the limit $\left.\begin{matrix}\varepsilon \to 0\\ E \to 0\end{matrix}\right\}$ with $\frac{D}{L}$ fixed

'this is the important bit'

Provided everything remains order unity, i.e. we remain in the bulk of the flow, outside the boundary layer (i.e. z fixed), we are in the outer flow, denoted by a superscript (0). Dropping dashes,

Horiz: $\hat{\underline{z}} \times \underline{u}^{(0)} = -\underline{\nabla} p^{(0)}$

Vert: $0 = -\frac{\partial p^{(0)}}{\partial z}$

Cont: $\frac{\partial u^{(0)}}{\partial x} + \frac{\partial v^{(0)}}{\partial y} + \frac{\partial w^{(0)}}{\partial z} = 0$.

Hence the outer flow is geostrophic (as expected – zero Rossby n°. plus no viscosity).

Pressure is depth-independent. Since $\frac{\partial u^{(0)}}{\partial x} + \frac{\partial v^{(0)}}{\partial y} = 0$

then $\frac{\partial w^{(0)}}{\partial z} = 0$

i.e. vertical velocity in outer flow is depth independent.
otherwise $\underline{u}^{(0)}$ is arbitrary.

Here comes the nice bit. How could this possibly satisfy the boundary condition?

To discuss the boundary layer we need a different limit ($\varepsilon \to 0$, $E \to 0$, $\frac{D}{L}$ fixed).

NS x comp: $$\underset{0}{\varepsilon}\left(\frac{\partial u}{\partial t} + (\underline{u}\cdot\underline{\nabla})u\right) - v = -\frac{\partial p}{\partial x} + E\left[\left(\frac{D}{L}\right)^2\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) + \frac{\partial^2 u}{\partial z^2}\right]$$

Introduce $Z = \frac{z}{E^{1/2}}$ i.e. $z = E^{1/2} Z$

Consider the limit ($\varepsilon \to 0$, $E \to 0$, $\frac{D}{L}$ fixed) with Z fixed.
Then we stay inside the Ekman layer during this limiting process.

$$= -\frac{\partial p}{\partial x} + E\left[\left(\frac{D}{L}\right)^2\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) + E\frac{\partial^2 u}{\partial Z^2}\right]$$

NS x-comp: $\varepsilon \to 0, E \to 0,$ $$-v = -\frac{\partial p}{\partial x} + \frac{\partial^2 u}{\partial Z^2}$$

y-comp: $$u = -\frac{\partial p}{\partial y} + \frac{\partial^2 v}{\partial Z^2}$$

Momentum eq^{n}s, precisely the Ekman layer eq^{n}s

Continuity: $\dfrac{\partial u}{\partial x} + \dfrac{\partial v}{\partial y} + \dfrac{\partial w}{\partial z} = 0$

$$\Rightarrow \quad \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + E^{-1/2}\frac{\partial w}{\partial Z} = 0$$

ie $\dfrac{\partial w}{\partial Z} = -E^{1/2}\left(\dfrac{\partial u}{\partial x} + \dfrac{\partial v}{\partial y}\right)$

ie. w is of order $E^{1/2}$,
so write $w = E^{1/2} W$ (W of order 1)

In our new layer the vertical velocities are order $E^{1/2} U \ll U$ but not zero as in the first problem.

Recall the vertical eq^n (†) is

$$\left(\frac{D}{L}\right)^2 \varepsilon\left(\frac{\partial w}{\partial t} + (\underline{u}\cdot\underline{\nabla})w\right) = \underbrace{-\frac{\partial p}{\partial z}}_{\sim E^{-1/2}} + \underbrace{E\left(\frac{D}{L}\right)^2 \nabla^2 w}_{\sim E^{1/2}}$$

becomes $\dfrac{\partial p}{\partial Z} = 0$ in limit $\varepsilon \to 0$
$E \to 0$

since $\dfrac{\partial p}{\partial Z} \sim E^{-1/2}$

and $w \sim E^{1/2}$

Summary: Outer flow $z \sim 1$
Geostrophic $\hat{z} \times \underline{u}^{(0)} = -\underline{\nabla} p^{(0)}$

$(z$ fixed$)$

$$\frac{\partial w^{(0)}}{\partial z} = 0$$

$$\frac{\partial p^{(0)}}{\partial z} = 0$$

In Ekman layer, $z \sim E^{1/2}$

i.e. $Z = z/E^{1/2} \sim 1$

(Z fixed)

$$(1) \quad -v = -\frac{\partial p}{\partial x} + \frac{\partial^2 u}{\partial Z^2}$$

$$(2) \quad u = -\frac{\partial p}{\partial y} + \frac{\partial^2 v}{\partial Z^2}$$

$$(3) \quad \frac{\partial p}{\partial Z} = 0$$

now need to solve these.

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial W}{\partial Z} = 0,$$

$$W = \frac{w}{E^{1/2}}, \text{ i.e. } w = E^{1/2} W$$

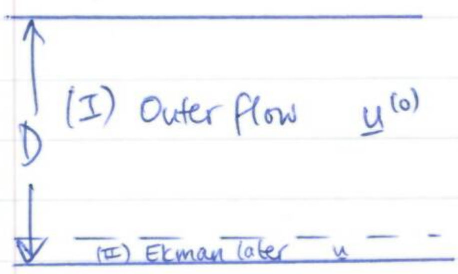
In the limit of small viscosity, i.e. $E \ll 1$,

$\nu \ll \Omega D^2$

$[\nu] = L^2 T^{-1}$ ($0.01\,\text{cm}^2\text{sec}^{-1}$ water)

($Re = \nu/UL$ as an aside)

Summary:



(I) $E \to 0$, $z' = z/D$ fixed

$$u^{(0)} = -\frac{\partial p^{(0)}}{\partial y} \qquad v^{(0)} = \frac{\partial p^{(0)}}{\partial x} \quad \text{geostrophic}$$

$$\frac{\partial p^{(0)}}{\partial z} = 0.$$

(II) $E \to 0$, $Z = \underset{\text{dimensional}}{\overset{\text{non-dim.}}{z}}/E^{1/2}$

$$= \frac{z^*/D}{E^{1/2}}$$

$$= \frac{z^*}{DE^{1/2}}$$

Observation : (3): pressure field same at all Z.

Hence look as $Z \to \infty$. We require the flow at the top of the Ekman layer to match flow at the bottom of interior

$(Z \to \infty)$ $(z \to 0)$

ie $\quad u \to U^{(0)}$, $v \to V^{(0)}$ } which are both indpt of z

At $Z = \infty$ (1) becomes $\quad -V^{(0)} = -\dfrac{\partial p}{\partial x} \quad \cdots (4)$

(2) becomes $\quad U^{(0)} = -\dfrac{\partial p}{\partial y} \quad \cdots (5)$

This gives $\dfrac{\partial p}{\partial x}$ and $\dfrac{\partial p}{\partial y}$ at $Z = \infty$.

But p is same $\forall Z$

So we have $\dfrac{\partial p}{\partial x}$ and $\dfrac{\partial p}{\partial y}$ $\forall Z$

ie
$$-v = -V^{(0)} + \frac{\partial^2 u}{\partial Z^2}$$
$$u = U^{(0)} + \frac{\partial^2 v}{\partial Z^2}$$

<u>b.c.s</u>: as $Z \to \infty$, $(u, v) \to (U^{(0)}, V^{(0)})$

On $Z = 0$, $\quad u = 0$, $v = 0$

Introduce $q = u + iv$

(only for Ekman layers does q have this defn!
↑ 'overloaded symbol' C++ joke)

$(6) + i(7)$

$$\frac{\partial^2}{\partial Z^2}(u+iv) - v^{(0)} + iu^{(0)} = -v + iu$$

i.e. $$\frac{\partial^2 q}{\partial Z^2} + iq^{(0)} = iq$$

$q \to q^{(0)}$ as $Z \to \infty$
$q = 0$ on $Z = 0$.

$\therefore$ $$\frac{\partial^2 q}{\partial Z^2} - iq = -iq^{(0)}$$

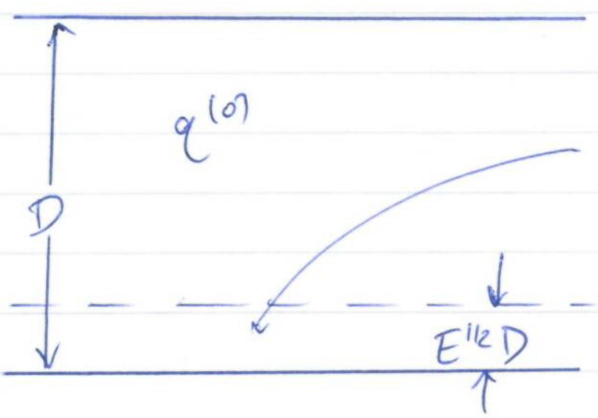
P.S.: $q_p = q^{(0)}$

AE: $\lambda^2 = i$ i.e. $\lambda = \pm(1+i)/\sqrt{2}$

G.S.: $$q = q^{(0)} + Ae^{(1+i)Z/\sqrt{2}} + Be^{-(1+i)Z/\sqrt{2}}$$

Bounded as $Z \to \infty$ so $A = 0$
Vanishes on $Z = 0$ so $B = -q^{(0)}$

Hence $$q = q^{(0)}\left[1 - e^{-(1+i)Z/\sqrt{2}}\right]$$



$$q = q^{(0)}\left[1 - e^{-(1+i)Z/\sqrt{2}}\right] \text{ inner}$$
$$= q^{(0)}\left[1 - e^{-(1+i)z/\sqrt{2E}}\right] \text{ outer}$$
$$= q^{(0)}\left[1 - e^{-(1+i)z^*/\delta}\right] \text{ dimensional}$$

$(\delta = \sqrt{\nu/\Omega})$

i.e. the outer flow matches to the non-slip condition on $z=0$ through an Ekman layer of thickness $\sqrt{\nu/\Omega}$, as before

[$q^{(0)}(x,y)$ is different at each x,y but at each point the velocity vector spirals up as in a simple Ekman layer]

Now, $q = \left[u^{(0)} + i v^{(0)}\right]\left[1 - e^{-(1+i)Z/\sqrt{2}}\right]$

$$u = u^{(0)}\left[1 - e^{-Z/\sqrt{2}} \cos\frac{Z}{\sqrt{2}}\right] - v^{(0)} e^{-Z/\sqrt{2}} \sin\frac{Z}{\sqrt{2}} \qquad \text{- - - (8)}$$

$$v = u^{(0)} e^{-Z/\sqrt{2}} \sin\frac{Z}{\sqrt{2}} + v^{(0)}\left[1 - e^{-Z/\sqrt{2}} \cos\frac{Z}{\sqrt{2}}\right] \qquad \text{- - - (9)}$$

Remember $\dfrac{\partial W}{\partial Z} = -\left(\dfrac{\partial u}{\partial x} + \dfrac{\partial v}{\partial y}\right)$

So $\dfrac{\partial W}{\partial Z} = \left[\dfrac{\partial v^{(0)}}{\partial x} - \dfrac{\partial u^{(0)}}{\partial y}\right] e^{-Z/\sqrt{2}} \sin\dfrac{Z}{\sqrt{2}}$

using $\dfrac{\partial u^{(0)}}{\partial x} + \dfrac{\partial v^{(0)}}{\partial y} = 0$

$= \zeta^{(0)} e^{-Z/\sqrt{2}} \sin\dfrac{Z}{\sqrt{2}}$

where $\zeta^{(0)} = \dfrac{\partial v^{(0)}}{\partial x} - \dfrac{\partial u^{(0)}}{\partial y}$ is the vorticity in the outer flow integrating over layer: $\int_0^\infty dZ$

$$\int_0^\infty \frac{\partial W}{\partial Z}\, dZ = \zeta^{(0)}(x,y) \underbrace{\int_0^\infty e^{-Z/\sqrt{2}} \sin\left(\frac{Z}{\sqrt{2}}\right) dZ}$$

$$W(\infty) - W(0) = \zeta^{(0)}(x,y) \cdot \frac{1}{\sqrt{2}}$$

taking real and imaginary parts, apparently.

But $W(0) = 0$ (no normal flow at $Z=0$)

and $W(\infty) = \frac{w}{E^{1/2}}(\infty) = \frac{w^{(0)}(0)}{E^{1/2}}$

def^n of W

matching with outer layer

hence $w^{(0)} = \frac{1}{\sqrt{2}} E^{1/2} \zeta^{(0)}(x,y)$

entirely in terms of the outer region variables.

i.e. presence of Ekman layer forces outer region to satisfy

$$w^{(0)} = \frac{1}{\sqrt{2}} E^{1/2} \zeta^{(0)},$$

the Ekman Compatibility Condition.

– the outer flow is <u>controlled</u> by the Ekman layer.

NOT PASSIVE

ACTIVE!!

Poincaré waves in a cylindrical domain

1. The linearised shallow water momentum equations are

$$u_t - fv = -g\eta_x, \tag{1}$$
$$v_t + fu = -g\eta_y, \tag{2}$$

$\partial_t(1) + f(2)$ gives $u_{tt} + f^2 u = -g\eta_{xt} - fg\eta_y$.
$\partial_t(2) - f(1)$ gives $v_{tt} + f^2 v = -g\eta_{yt} + fg\eta_x$.
i.e. $(\partial_{tt} + f^2)\boldsymbol{u} = -g(\nabla\eta_t - f\hat{\boldsymbol{z}} \wedge \nabla\eta)$.

2.

$$\eta_t + H_0(u_x + v_y) = 0 \tag{3}$$

$(\partial_{tt} + f^2)(3)$ gives

$$(\partial_{tt} + f^2)\eta_t - gH_0[\partial_x(\eta_{xt} + f\eta_y) + \partial_y(\eta_{yt} - f\eta_y)] = 0,$$

Hence η satisfies

$$[(\partial_{tt} + f^2)\eta - c^2(\eta_{xx} + \eta_{yy})]_t = 0,$$

where $c^2 = gH_0$.

3. Since govering equation has coefficients independent of θ and t, look for solutions of form $\eta = \Re\{R(r)\exp[\mathrm{i}(m\theta - \sigma t)]\}$. Note since η must be a single-valued function of position then m must be integral. Thus

$$(f^2 - \sigma^2)R - c^2[R'' + R'/r - m^2R/r^2] = 0.$$

Introduce $\alpha = (\sigma^2 - f^2)^{1/2}/c$ so $r^2R'' + rR' + (\alpha^2r^2 - m^2)R = 0$. Then $R(r) = J_m(\alpha r)$ (requiring R finite at $r = 0$).

At $r = L$ we require $\boldsymbol{u}.\hat{\boldsymbol{n}} = 0$, i.e. $\boldsymbol{u}.\hat{\boldsymbol{r}} = 0$ thus

$$\eta_{rt} - f\hat{\boldsymbol{r}}.(\hat{\boldsymbol{z}} \times \nabla\eta) = 0, \qquad \text{i.e. } \eta_{rt} + (f/r)\eta_\theta = 0.$$

Hence $-\mathrm{i}\sigma\alpha r J'_m(\alpha r) + \mathrm{i}mfJ'_m(\alpha r) = 0$, at $r = L$, i.e.

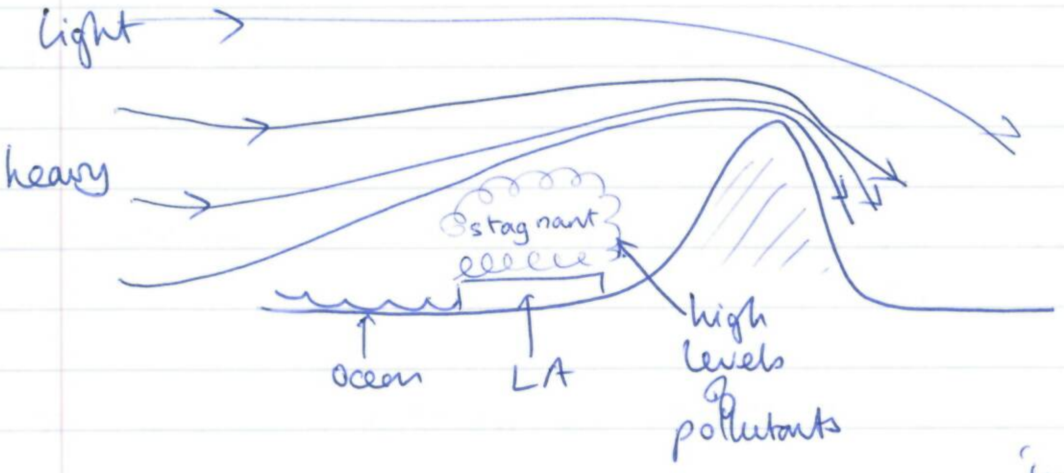
$$\frac{\sigma}{mf} = \frac{J_m(\alpha L)}{\alpha L J'_m(\alpha L)},$$

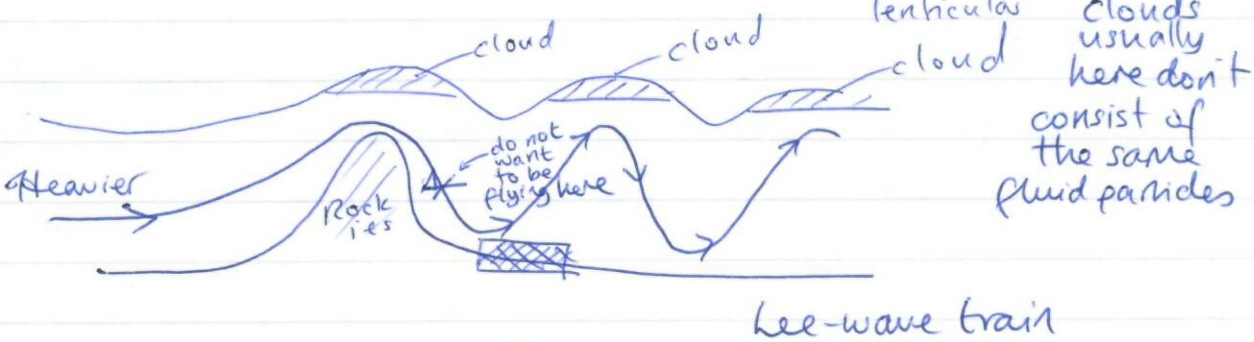
with $\alpha = (\sigma^2 - f^2)^{1/2}/c$ and $|\sigma| > f$.

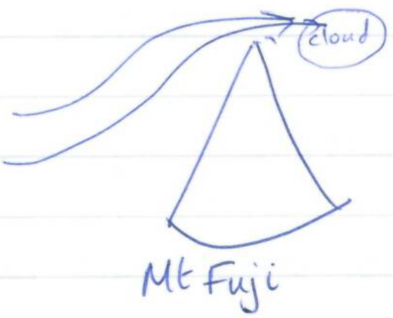
4. If $|\sigma| < f$ then α is imaginary and J_m must be replaced by the modified Bessel function I_m. This gives the Kelvin waves.

STRATIFICATION

Smallish scale, neglect Earth's rotation
50 - 100 km
Mountains, Mountain Ridges - Rockies







Equations of motion:

$$\rho \frac{D\underline{u}}{Dt} = -\underline{\nabla} p + \rho \underline{F}$$

Euler with arbitrary density and external force gravity,

$\underline{F} = -g\hat{\underline{z}}$ ($\underline{F}$ = force per unit mass)

$$\Rightarrow \quad \rho \frac{D\underline{u}}{Dt} = -\underline{\nabla} p - \rho g \hat{\underline{z}}$$ Mom'm eq^n.

Conservation of mass

$$\frac{D\rho}{Dt} + \rho \underline{\nabla} \cdot \underline{u} = 0$$ Full, variable ρ eq^n.

or $$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \underline{u}) = 0$$ equivalent by def^n of $\frac{D}{Dt}$

5 unknowns, 4 eq^{n}s so we need something else.

But for 'slow' flows, i.e. small velocities relative to the speed of sound, then we can take the flow to be incompressible. (so winds less than 300m/s ~ 700mph).

gales are ~30 mph so we're safe with this approximation

$\Rightarrow$ air is incompressible for our purposes

So an infinitesimal particle of fluid cannot change its volume during motion. But by conservation of mass, its mass does not change. Hence the density of an infinitesimal fluid element is constant during motion.

ie. $$\frac{D\rho}{Dt} = 0$$ ie. rate of change of ρ following a particle is zero

Combined with conservation of mass, this gives

$$\underline{\nabla} \cdot \underline{u} = 0$$

i.e. we have

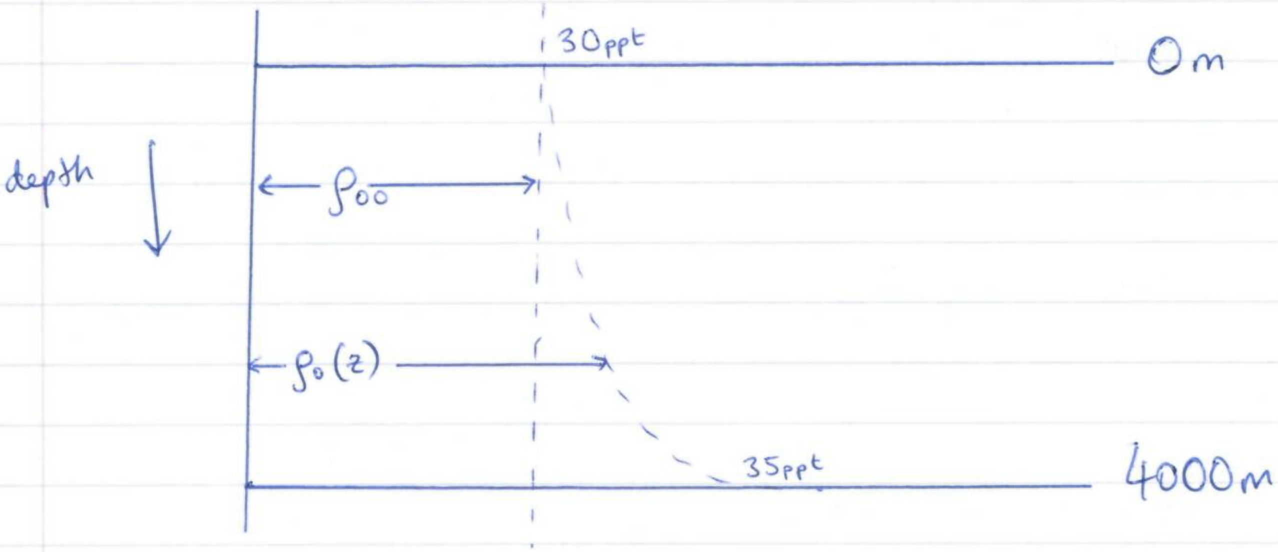
$$\frac{D\rho}{Dt} = 0$$

$$\rho \frac{D\underline{u}}{Dt} = -\underline{\nabla} p - \rho g \hat{\underline{z}}$$

$$\underline{\nabla} \cdot \underline{u} = 0$$

ie 5 eq^{n}s, 5 unknowns.

Density of the ocean: for 1l water, 30g salt



In the absence of motion, we take the fluid to have an undisturbed vertical density profile

$$\rho_0(z)$$

A typical value for the density is ρ_{00}.

Let the weight of this fluid be balanced by a hydrostatic pressure $p_0(z)$.

$$\underline{u} \equiv 0 \Rightarrow \qquad 0 = -\underline{\nabla} p - \rho g \hat{\underline{z}}$$

$$\text{ie} \quad \frac{\partial p}{\partial z} = -\rho g$$

so we require $\dfrac{\partial p_0}{\partial z} = -g \rho_0(z)$ [hydrostatic pressure]

Thus express all pressure as deviation from hydrostatic.

$$p(x,y,z,t) = p_0(z) + p'(x,y,z,t)$$

and similarly for the density

$$\rho(x,y,z,t) = \rho_0(z) + \rho'(x,y,z,t)$$

Then
$$(\rho_0 + \rho') \frac{D\underline{u}}{Dt} = -\underline{\nabla}(p_0 + p') - (\rho_0 + \rho') g \hat{\underline{z}}$$

$$= -\underline{\nabla} p' - \rho' g \hat{\underline{z}}$$

i.e.
$$\frac{D\underline{u}}{Dt} = -\frac{1}{\rho_0 + \rho'} \underline{\nabla} p' - \underbrace{\frac{\rho'}{\rho_0 + \rho'} g \hat{\underline{z}}}_{\text{buoyancy}}$$

Now we follow Boussinesq (1903), by taking the limit

$$\frac{\rho'}{\rho_0} \to 0 \quad \text{and} \quad g \to \infty$$

s.t. $\dfrac{\rho' g}{\rho_0}$ is fixed.

ie density variations are small so they do not affect inertia, but gravity is strong so buoyancy effects remain.

$$\frac{D\underline{u}}{Dt} = -\frac{1}{\rho_0}\underline{\nabla}p' - \frac{\rho' g}{\rho_0}\underline{\hat{z}}$$

[ie. drop ρ' when compared to ρ_0]

Now assume that the changes in $\rho_0(z)$ are small over the depths we are interested in, ie. replace ρ_0 by ρ_{00}.

So $$\frac{D\underline{u}}{Dt} = -\frac{1}{\rho_{00}}\underline{\nabla}p' \underbrace{- \frac{\rho'}{\rho_{00}} g\underline{\hat{z}}}_{\text{buoyancy}}$$

- traditional constant density Euler eq^n plus a buoyancy term (which we investigate)

Write $\sigma = -\frac{\rho'}{\rho_{00}} g$, is an acceleration
- buoyancy acceleration

$\rho' > 0$ ie. element has higher density than surroundings
$\sigma < 0$ ie downward accel.

$$\Rightarrow \frac{D\underline{u}}{Dt} = -\frac{1}{\rho_{00}}\underline{\nabla}p' + \sigma\underline{\hat{z}} \qquad (\underline{\hat{z}} \text{ up})$$

Density eq^n: $$\frac{D[\rho_0(z) + \rho']}{Dt} = 0$$
$\hookrightarrow \frac{D\rho}{Dt} = 0 \Rightarrow$

$$w \frac{\partial \rho_0}{\partial z} + \frac{D\rho'}{Dt} = 0$$

Multiply by $-\frac{g}{\rho_{00}}$ to get

$$\frac{D\sigma}{Dt} + \left(\frac{-g}{\rho_{00}} \frac{d\rho_0}{dz}\right) w = 0$$

In a stable environment (light above heavy)

$$\frac{d\rho_0}{dz} < 0$$

write

$$N^2 = -\frac{g}{\rho_{00}} \frac{d\rho_0}{dz} > 0 \quad \text{in a stable environment}$$

ie $N = \left(-\frac{g}{\rho_{00}} \frac{d\rho_0}{dz}\right)^{1/2}$

Our density eq^n is

$$\frac{D\sigma}{Dt} + N^2 w = 0$$

what is N^2?

$$[N^2] = \frac{[g]}{\cancel{[\rho_{00}]}} \frac{\cancel{[\rho_0]}}{[z]} = \frac{LT^{-2}}{L} = T^{-2}$$

so

$[N] = T^{-1}$, ie. it's a buoyancy frequency.

Justification: Linearise ~~Linewise~~ buoyancy eq^{n}

$$\frac{\partial \sigma}{\partial t} + N^2 w = 0$$

Vertical mom^{m} eq^{n} and drop pressure term and linearise:

$$\frac{\partial w}{\partial t} = \cancel{-\frac{1}{\rho_{00}}\frac{\partial p}{\partial z}} + \sigma$$

so $\frac{\partial^2 w}{\partial t^2} = \frac{\partial \sigma}{\partial t}$

ie $\frac{\partial^2 w}{\partial t^2} + N^2 w = 0$

ah we meet again, old friend

SHM with frequency N

The equations of motion for a Boussinesq fluid in a stratified environment

$$\begin{cases} \frac{D\underline{u}}{Dt} = -\frac{1}{\rho_{00}}\underline{\nabla} p' + \sigma \hat{\underline{z}} & \text{mom}^{m} \\ \frac{D\sigma}{Dt} + N^2 w = 0 & \text{density/ buoyancy} \\ \underline{\nabla}\cdot\underline{u} = 0 & \text{incompressibility} \end{cases}$$

5 eq^{n}s, 5 unknowns: $\underline{u}, \sigma, p$.

To discuss these equations, first consider plane wave solutions,

Linearise:

$$\frac{\partial u}{\partial t} = -\frac{1}{\rho_{00}}\frac{\partial p'}{\partial x} \quad \dots \quad (1)$$

$$\frac{\partial v}{\partial t} = -\frac{1}{\rho_{00}}\frac{\partial p'}{\partial y} \quad \dots \quad (2)$$

vertical accel. still present

$$\frac{\partial w}{\partial t} = -\frac{1}{\rho_{00}}\frac{\partial p'}{\partial z} + \sigma \quad \dots \quad (3)$$

NOT hydrostatic.

$$\frac{\partial \sigma}{\partial t} + N^2 w = 0 \quad \dots \quad (4)$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \quad \dots \quad (5)$$

Use (3), (4) to get σ in terms of w

$\frac{\partial}{\partial t}(3) + (4)$ gives

$$\frac{\partial^2 w}{\partial t^2} + N^2 w = -\frac{1}{\rho_{00}}\frac{\partial^2 p}{\partial z \partial t} \quad \dots \quad (6)$$

Operate on (5) with $\left(\frac{\partial^2}{\partial t^2} + N^2\right)\frac{\partial}{\partial t}$:→

$$\left(\frac{\partial^2}{\partial t^2} + N^2\right)\left(-\frac{1}{\rho_{00}}\frac{\partial^2 p'}{\partial x^2} - \frac{1}{\rho_{00}}\frac{\partial^2 p'}{\partial y^2}\right) - \frac{1}{\rho_{00}}\frac{\partial^4 p}{\partial^2 z \partial^2 t} = 0 \quad (7)$$

having used (1), (2), (6) to introduce p'

3D Lap. 2D Lap.

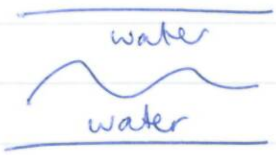
ie. $$\frac{\partial^2}{\partial t^2} \nabla_3^2 p' + N^2 \nabla_2^2 p' = 0$$

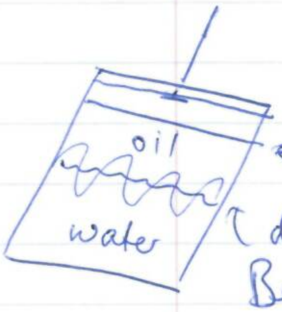
INTERNAL WAVE EQ$^{\underline{N}}$

$$\nabla_3^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

$$\nabla_2^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$$

$$\left[\underline{\text{or}} \quad \left(\frac{\partial^2}{\partial t^2} + N^2\right) \nabla_2^2 p' + \frac{\partial^4 p'}{\partial t^2 \partial z^2} = 0 \right]$$





← flat

discovered by Benjamin Franklin!

Plane wave sol$^{\underline{n}}$s:
$$p' = \mathrm{Re}\left[A e^{i(kx + ly + mz - \omega t)} \right]$$

$$-\omega^2(-k^2 - l^2 - m^2) + N^2(-k^2 - l^2) = 0$$

$$\omega^2 = \frac{N^2(k^2 + l^2)}{k^2 + l^2 + m^2}$$

ie. $$\omega = \pm N \frac{\sqrt{k^2 + l^2}}{\sqrt{k^2 + l^2 + m^2}}$$

$$\rightarrow |\omega| \leq |N|$$

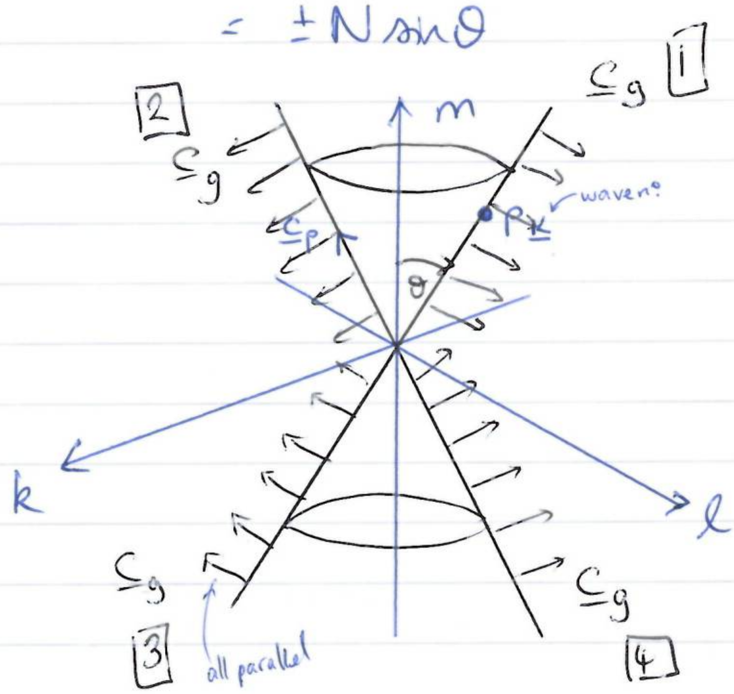
ie propagating waves have frequency less than N.

We want to consider the slowness surfaces in wavenumber space. Introduce spherical polar coords in $\underline{k}$-space

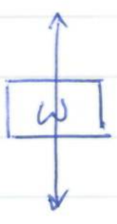
$$k = K\sin\theta\cos\phi \qquad K \to \text{radial}$$
$$l = K\sin\theta\sin\phi \qquad \phi \to \text{azimuth}$$
$$m = K\cos\theta \qquad \theta \to \text{colatitude} \quad (\pi/2 - \text{lat.})$$

$$\omega = \pm N \frac{K\sin\theta}{K}$$

$$= \pm N\sin\theta$$



Slowness surface: $\omega = \text{const}$.
$\to \theta = \text{const} \to$ cones.



oscillate upwards and downwards with freq. ω

For P, $\underline{c}_p = \frac{\omega}{|\underline{k}|}\hat{\underline{k}} = \frac{\pm N\sin\theta}{K}\hat{\underline{k}}$

Ref. 01/01
04457

DO NOT WRITE IN THIS SHADED AREA

DEPARTMENT OF MATHEMATICS - Course Assessment

Date:______________________________ Lecturer:______________________________

Course code number:__________

Term:__________

Year:__________

Please complete the following survey by answering the various questions by marking boldly appropriate boxes like this ▬
Do NOT tick, cross or ring boxes

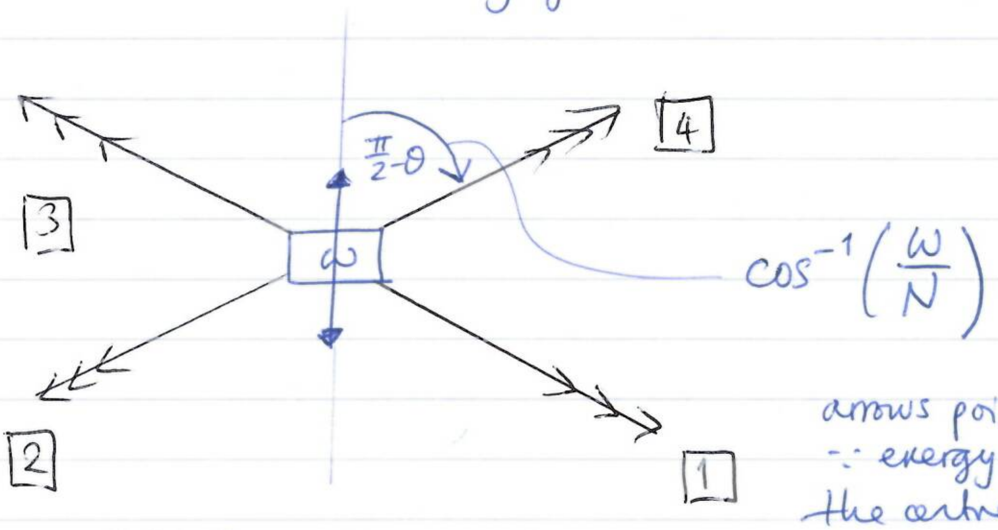
	-2	-1	0	1	2		-2	-1	0	1	2
1. Roughly, what percentage of the lectures did you attend (-2 =Less than 70%; 0 =70-90%; 2 =more than 90%)?						11. How did you find the starting standard of the course (-2 = too low; 2 = too high)?					
2. Do you feel your pre-University education, or previous course units, prepared you for this course (-2 = No; 2 = Yes)?						12. How much material was there in the course (-2 = too little; 2 = too much)?					
3. Were the aims of the course made clear (-2 = not at all; 2 = very)?						13. How difficult was the course material (-2 = too easy; 2 = too hard)?					
4. How was the verbal presentation of lectures (-2 = very poor; 2 = very good)?						14. How much associated course-work was there (-2 = too little; 2 = too much)? Please select "0" if there was no course-work.					
5. How was the visual presentation of lectures (-2 = very poor; 2 = very good)?						15. How difficult was the course-work (-2 = too easy; 2 = too hard)? Please select "0" if there was no course-work.					
6. How stimulating were the lectures (-2 = not at all; 2 = very)?						16. How was the feedback from course-work (-2 = very poor; 2 = very good)? Please select "0" if there was no course-work.					
7. How sympathetic was the lecturer to questions (-2 = not at all; 2 = very)?						17. How many problem classes were there (-2 = too few; 2 = too many)? Please select "0" if there were no problem classes.					
8. How easy was it to get good notes from the lectures (-2 = very hard; 2 = very easy)?						18. How effective were the problem classes (-2 = not at all; 2 = very)? Please select "0" if there were no problem classes.					
9. Was the lecturer available for consultation (-2 = not at all; 2 = readily)?						19. **What is your view of the course overall (-2 = very poor; 2 = very good)?**					
10. How helpful were the recommended texts (-2 = not at all; 2 = very)? (Please select "0" if no text was recommended.)						**GENERAL COMMENTS** **Please write overleaf any further comments that you feel would be helpful in improving the course.**					

Group velocity is perpendicular to slowness surface, but which way does it point?

$$\underline{c}_g = \nabla_{\underline{k}}\, \omega$$ $\perp^r$ to level surfaces of ω points in dir^n ω increasing.

On the positive side, if we increase ω we increase θ (by def^n of ω).
On the negative side, if we increase ω we decrease θ.

If $\omega > N$, b.t.w, there are no waves since $\omega = \pm N \sin\theta$ breaks down. Where do they go?

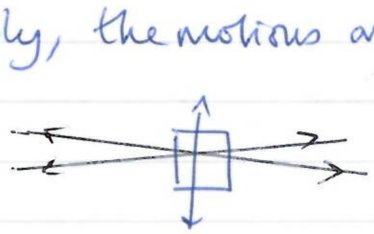


arrows point out $\therefore$ energy is leaving the centre

'St Andrew's Cross'
(Stevenson, 1960s)

As $\frac{\omega}{N} \to 0$, $\quad \Phi = \cos^{-1}\left(\frac{\omega}{N}\right) \longrightarrow \frac{\pi}{2}$.

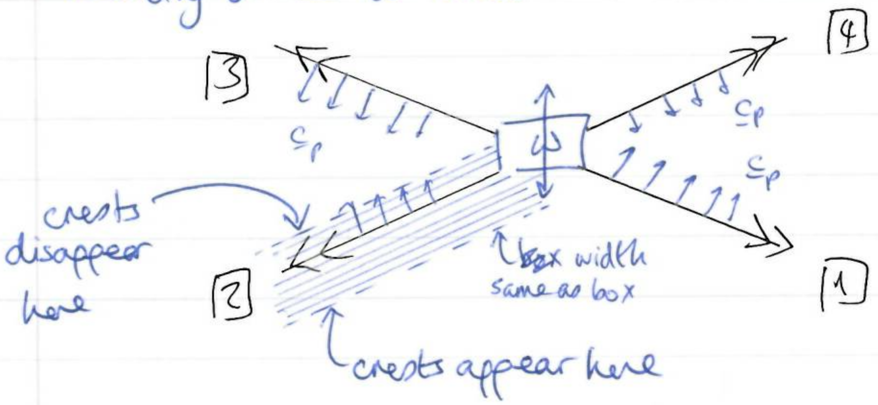
So if we move this thing slowly, the motions are almost horizontal — as we'd expect.



$\omega \ll N$

This is strong stratification. The fluid is 'heavy': doesn't want to move up and down.

So what do we see? Look at $\boxed{1}$. The phase velocity is $\perp^r$ to that



Let's calculate $\underline{c}_g = \underline{\nabla}_k \omega$.

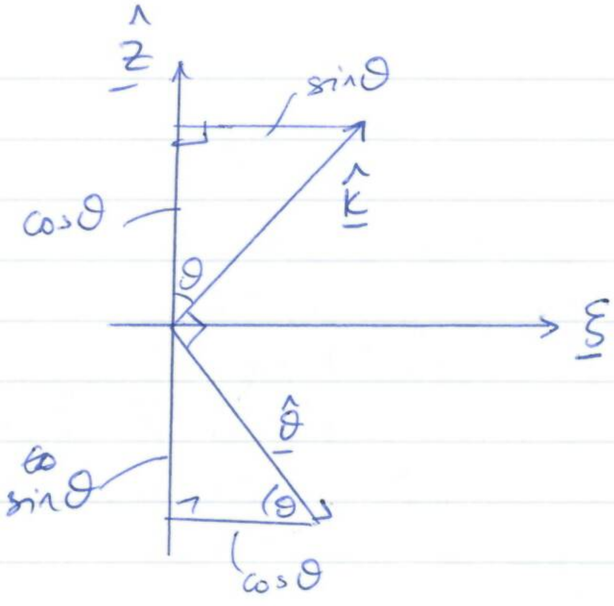
$$\omega = \pm N \sin\theta$$
$$\underline{\nabla}_k = \frac{\partial}{\partial k}\hat{\underline{k}} + \frac{1}{k}\frac{\partial}{\partial\theta}\hat{\underline{\theta}} + \frac{1}{k\sin\theta}\frac{\partial}{\partial\phi}\hat{\underline{\phi}} \quad \text{in sphericals}$$

So $\underline{c}_g = \underline{\nabla}_k \omega = \dfrac{\pm N\cos\theta}{k}\hat{\underline{\theta}}$

$$\underline{c}_p = \frac{\pm N\sin\theta}{k}\hat{\underline{k}}$$

$\rightarrow \quad \underline{c}_p \cdot \underline{c}_g = 0$, i.e. $\underline{c}_p \perp^r \underline{c}_g$, which we knew, but now have shown mathematically.

Letting $\underline{\xi}$ be a horizontal radius vector (as in cylindricals)



Consider $\underline{c}_p + \underline{c}_g$

$$= \pm \frac{N}{K}\left[\cos\theta\,\hat{\underline{\theta}} + \sin\theta\,\hat{\underline{k}}\right]$$

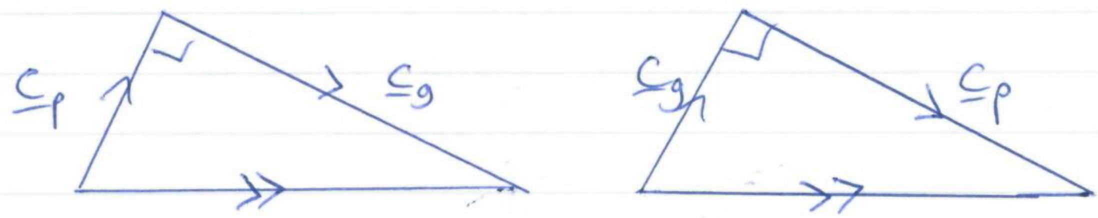
$$= \pm \frac{N}{K}\Big[\cos\theta\left(\cos\theta\,\hat{\underline{\xi}} - \sin\theta\,\hat{\underline{z}}\right)$$

$$+ \sin\theta\left(\sin\theta\,\hat{\underline{\xi}} + \cos\theta\,\hat{\underline{z}}\right)\Big] \quad \text{(in Cartesians)}$$

$$= \pm \frac{N}{K}\left[(\cos^2 + \sin^2)\,\hat{\underline{\xi}}\right]$$

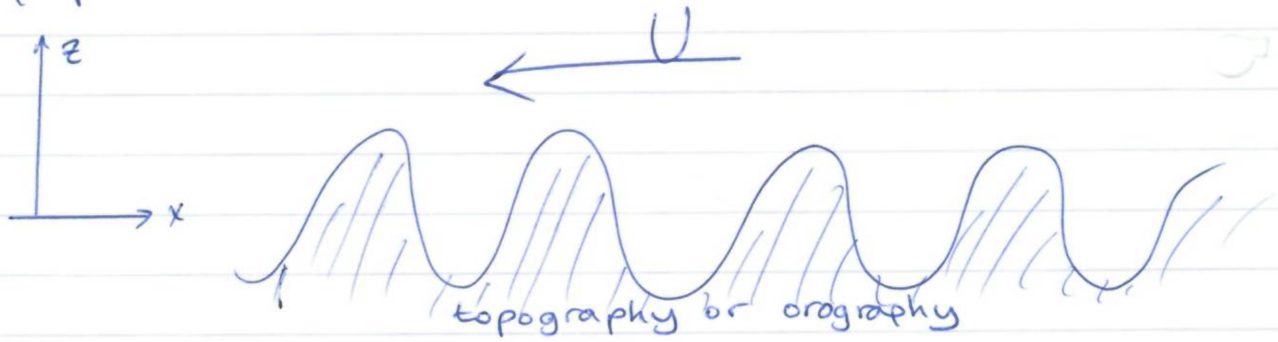
$$= \pm \frac{N}{K}\,\hat{\underline{\xi}}$$

ie. $\underline{c}_g$ adds to $\underline{c}_p$ to give a horizontal vector



Flow over uneven ground

We consider small bumps so that motions are weak, and we can use linear eq^{n}s. Because flow is linear we can decompose any shape into its Fourier components. Hence it is sufficient to consider a single sinusoidal ridge in 2D. More complicated shapes follow by superposition.



In a frame of reference moving with the wind, i.e. moving to the left with speed U, we see a mountain moving to the right with speed U.

i.e. our boundary is $$z = \underset{\substack{\text{small}\\ \text{amplitude}}}{\varepsilon} \sin\left[\underset{\text{wave n}^{\circ}}{k}\,(x - \underset{\text{speed}}{U}t)\right]$$

Above this boundary we take the flow to be uniformly stratified with constant buoyancy frequency N, so that is governed by the internal wave eq^n.

What do we see?

(Sufficient to consider sine waves only since we can take the Fourier Transform)

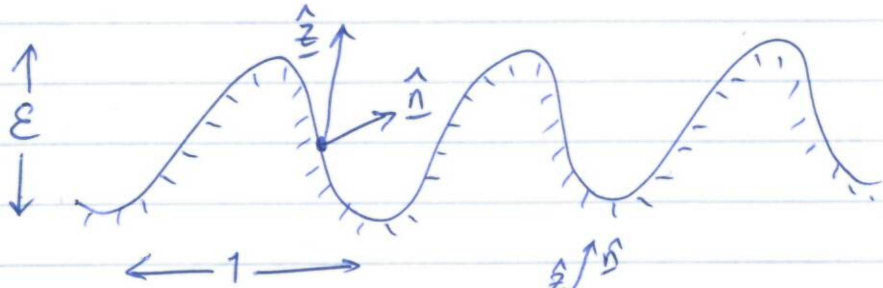
The governing equation for small disturbances is the Internal Wave Equation:

$$\frac{\partial^2}{\partial t^2} \nabla_3^2 p + N^2 \nabla_2^2 p = 0$$

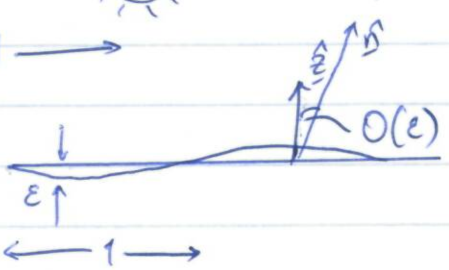
We need boundary conditions:

On the orography (wave surface), the boundary is impermeable.
ie. normal component of velocity relative to bdy is zero.
ie.

$$(\underline{u} - \underline{U}) \cdot \hat{\underline{n}} = 0 \quad \text{on} \quad z = \varepsilon \sin[k(x - Ut)]$$



But



possibly rubbish

ie the lower boundary never slopes by more than ε
so $\hat{\underline{n}}$ differs from $\hat{\underline{z}}$ by order ε.

Hence on the boundary $z = \varepsilon \sin[k(x - Ut)]$, to leading order,

$$(\underline{u} - \underline{U}) \cdot \hat{\underline{z}} = 0$$

ie $\underline{u} \cdot \hat{\underline{z}} = 0$ since $\underline{U} \cdot \hat{\underline{z}} = 0$

ie. $w = 0$ on $z = \varepsilon \sin[k(x - Ut)]$?!?!

Surface is $F(x,z)=0$ where $F = z - \varepsilon \sin[\ ]$

normal $\underline{\nabla} F = -\varepsilon k \cos[\ ]\,\hat{\underline{x}} + \hat{\underline{z}}$

Thus on $z = \varepsilon \sin[\ ]$, $-\overset{\varepsilon}{\varepsilon} k\cos[\ ]\, \overset{\text{order } \varepsilon}{u} + \overset{\text{order } \varepsilon}{w}$
$+ \varepsilon k \cos[\ ] U = 0$

So to leading order, $w + \varepsilon k \cos[\ ] U = 0$

ie

$$w = -\varepsilon U k \cos[k(x - Ut)]$$

on $z=0$ to leading order, making an error of order ε

We need a second b.c. in z (as the eq^n is 2nd order in z).
Definitely$^{(?)}$ must satisfy

$$\underline{u} \to 0 \quad \text{as } z \to \infty.$$

Since
$$\left(\frac{\partial^2}{\partial t^2} + N^2\right) w = -\frac{1}{\rho_{00}} \frac{\partial^2 p}{\partial z \partial t}$$

Take $\dfrac{\partial^2}{\partial z \partial t}$ of IWE replace p by $\left(\dfrac{\partial^2}{\partial t^2} + N^2\right) w$.

Then w satisfies the IWE.

$$\Rightarrow \quad \frac{\partial^2}{\partial t^2} \nabla_3^2 w + N^2 \nabla_2^2 w = 0 \qquad \ldots\ldots (1)$$

$$w = -\varepsilon U k \cos[k(x-Ut)] \quad \text{on } z=0 \qquad \ldots\ldots (2)$$

$$(?)\ w \to 0 \quad \text{as } z \to \infty \qquad \ldots\ldots (3)$$

Write $w = -\varepsilon U k \, \mathrm{Re}\left[e^{ik(x-Ut)} \right]$ on $z=0$.

Look for sol^{n}s of the form

$$w = -\varepsilon U k \, \mathrm{Re}\left[\tilde{w}(z) \, e^{ik(x-Ut)} \right] \quad \cdots (4)$$

$\Rightarrow \tilde{w}(0) = 1$, satisfying (2). $\quad \cdots \cdots (5)$

Substitute (4) into (1) $\Rightarrow$

$$-k^2U^2\left[-k^2\tilde{w} + \tilde{w}''\right] + N^2\left[-k^2\tilde{w}\right] = 0$$

$$\Rightarrow \quad \tilde{w}'' + \left[\frac{N^2}{U^2} - k^2\right]\tilde{w} = 0 \quad \cdots\cdots (6)$$

Solve (6) subject to (5):

Notice (6) looks like SHM.

<u>Case 1</u>: $k^2 > \frac{N^2}{U^2}$ (Fast flows $U \gg 1$
Weak stratification $N \ll 1$
Short obstacles $k \gg 1$)

$\Downarrow$

$\frac{Uk}{N} > 1$

$\uparrow$ Froude number: supercritical!

Then $\tilde{w} = A e^{-\sqrt{k^2 - \frac{N^2}{U^2}}\, z} + B e^{+\sqrt{k^2 - \frac{N^2}{U^2}}\, z}$

$B = 0$ to satisfy (3)

$A = 1$ to satisfy (5)

$\Rightarrow \quad w = \varepsilon U k \, \mathrm{Re}\left[e^{-\sqrt{k^2 - \frac{N^2}{U^2}}\, z} \, e^{ik(x-Ut)} \right]$

$$\Rightarrow \quad w = -\varepsilon U k e^{-\sqrt{k^2 - \frac{N^2}{U^2}}\, z} \cos\left[k(x-Ut)\right]$$

and $\quad w = \dfrac{Dz}{Dt} = \dfrac{\partial z}{\partial t}$ to order ε

So the particles follow lines in the wind frame given by

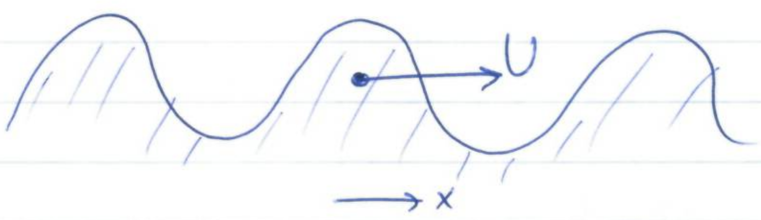
$$z^* = \varepsilon e^{-\sqrt{k^2 - \frac{N^2}{U^2}}\, z_0} \sin\left[k(x-Ut)\right]$$

where z_0 is upstream height.

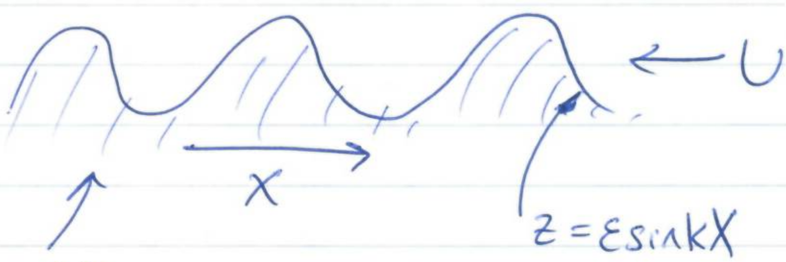
In the frame of the orography, $X = x - Ut$,

$$z = \varepsilon e^{-\sqrt{k^2 - \frac{N^2}{U^2}}\, z_0} \sin(kX)$$

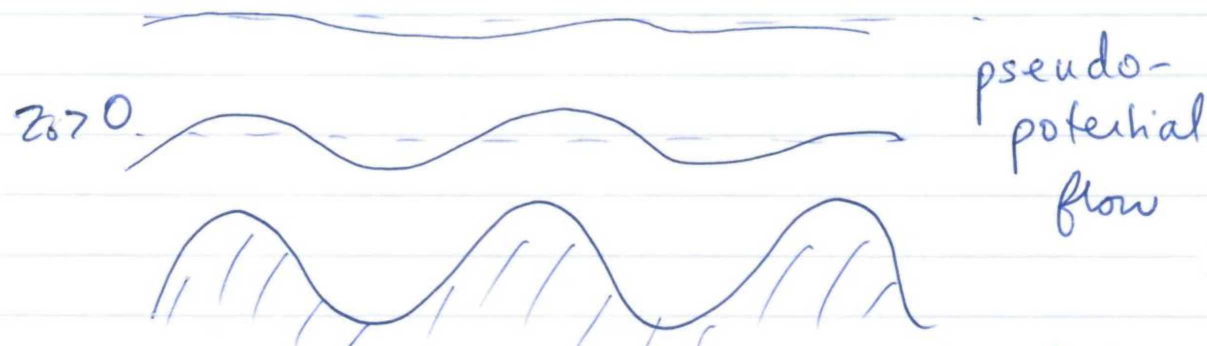
In wind frame:



In orography frame:



On $z_0 = 0$, $\quad z = \varepsilon \sin kX$



it decays exponentially from the boundary with e-folding height $\dfrac{1}{\sqrt{k^2 - \frac{N^2}{U^2}}}$.

[simply a sol^{n} that decays with distance above mountain.

No waves.]

Note if $N = 0$, $z = \underbrace{\varepsilon e^{-kz_0} \sin kX}_{\text{a sol}^{n}\text{ to Laplace's eq}^{n}\text{.}}$, this is precisely potential flow.

In this case $\frac{Uk}{N} > 1$, the flow looks just like potential (unstratified) flow with instead of decay scale $1/k$, the slower decay scale of $\dfrac{1}{\sqrt{k^2 - \frac{N^2}{U^2}}}$.

But still with crests lying directly above crests and troughs lying directly above troughs.

Case 2: $\frac{Uk}{N} < 1$.

Write $\lambda = \left(\frac{N^2}{U^2} - k^2\right)^{1/2} > 0$.

$\tilde{w}'' + \lambda^2 \tilde{w} = 0$. SHM!!

$$\tilde{w} = \underbrace{Ae^{i\lambda z} + Be^{-i\lambda z}}_{\text{Both terms are bounded at } \infty.}$$

$$\text{Then } w = -\varepsilon kU \,\mathrm{Re}\left[Ae^{ikx + i\lambda z - ikUt} + Be^{ikx - i\lambda z - ikUt}\right]$$

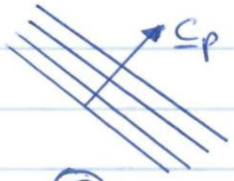
$$= -\varepsilon kU \left[A \cos(kx + \lambda z - kUt) + B\cos(kx - \lambda z - kUt)\right]$$

I

II

while $\tilde{\omega}(0) = 1$.

These are internal waves!



I

In term I, as time t increases, to keep phase constant, z must increase. (at rate $\frac{kU}{\lambda}$)

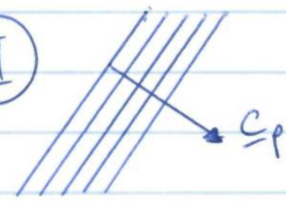
Thus the lines of constant phase propagate upwards

So wave crests move up.

Crests lean backwards.

In term II,

II



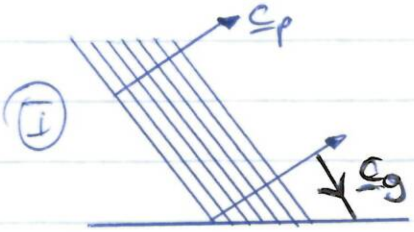
Wave leans into flow. Crests height in wave ahead of crest in bump.

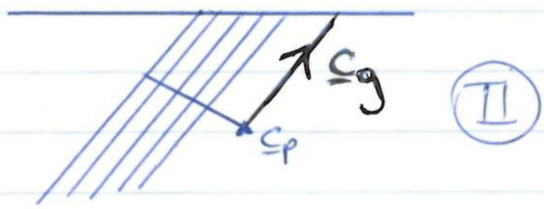
As t increases, z must decrease (at rate $\frac{kU}{\lambda}$) to keep phase constant.

The motion is driven by the fact that we move the corrugated flow. Hence all waves must carry <u>energy</u> away from the inner boundary.

One of these sets carries <u>energy</u> away from the boundary (as req^d) and one requires an <u>energy</u> source at infinity which does not exist.

Which is which? Well, what about the <u>group velocity</u>?





So I energy source at ∞

II generated a lower boundary

Term I is absent so we can say $A=0$

$$\Rightarrow B=1 \quad \text{to satisfy} \quad \tilde{w}(0)=1$$

$$\Rightarrow w = -\varepsilon k U \cos\left[k(x-Ut)-\lambda z\right]$$

end of course →

<u>Aim</u>: find particle paths in the orography frame.

The height z of any particle satisfies

$$\frac{Dz}{Dt}=w$$

If we have a particle that follows a path $(x(t), y(t), z(t))$ then

$$\frac{dz}{dt}=w$$

Here $\dfrac{dz}{dt} = -\varepsilon k U \cos\left[k(x-Ut) - \lambda z\right]$

$$z = z_0 + \varepsilon \tilde{z}$$

Let the deviation of our particle from its upstream value z_0, be small (of order ε).

Then $\varepsilon \dfrac{d\tilde{z}}{dt} = -\varepsilon k U \cos\left[k(x-Ut) - \lambda z_0\right]$

error of order ε

to order ε^2.

ie $$\frac{d\tilde{z}}{dt} = -kU\cos\left[k(x-Ut) - \lambda z_0\right]$$

Integrate wrt t, $\tilde{z} = \sin\left[k(x-Ut) - \lambda z_0\right]$

giving $z = z_0 + \varepsilon \sin\left[k(x-Ut) - \lambda z_0\right]$

in the wind frame.

In the frame of the orography,

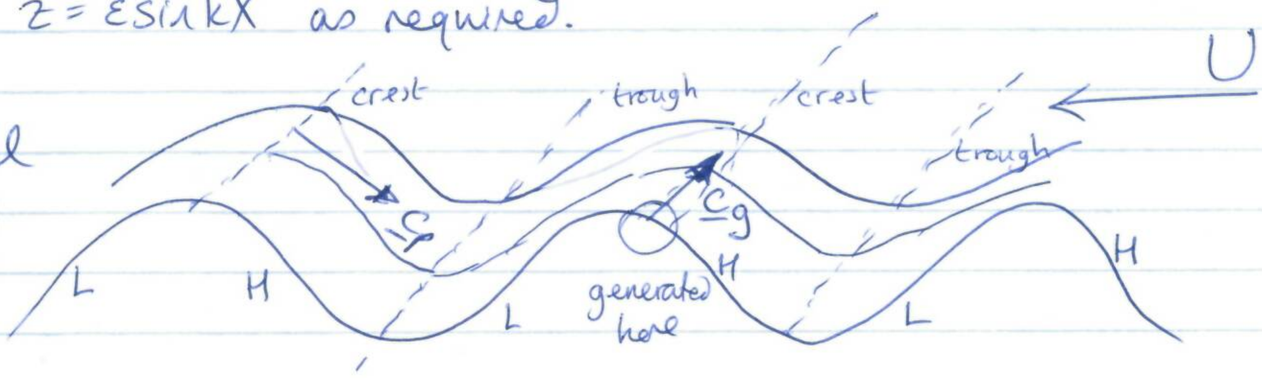
$$X = x - Ut$$

$$z = z_0 + \varepsilon \sin\left[kX - \lambda z_0\right]$$

<u>Check</u>: for a particle on the bottom, $z_0 = 0$ and

$z = \varepsilon \sin kX$ as required.

Internal Gravity Waves



amplitude does not decrease with height; steady.

Fixed pattern in space. Continually putting in energy due to the drag on the lower body.

A very counterintuitive problem, so much so, the first time it was done it was done wrong.

Fixed pattern in space. Pressure higher on right than left so there is a DRAG which slows the wind down!