3304 Geophysical Fluid Dynamics Notes
Based on the spring 2013 lectures
by Prof E R Johnson.

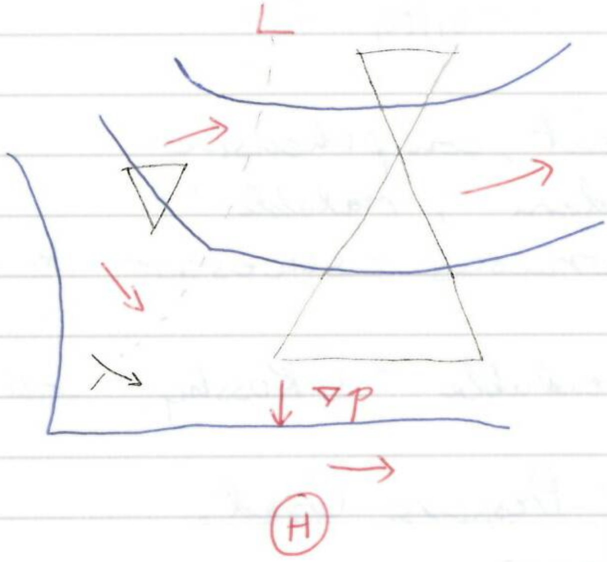
OUTDATED

To the times that we suffered for the sake of understanding mathematics and to which we shall be rewarded for it.

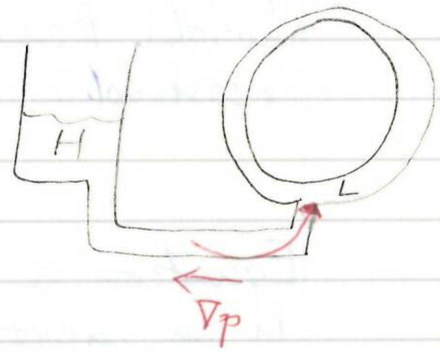
8/1/13

GFD
Tues 10
Fri 11 - 1

Office hour
2pm Tues

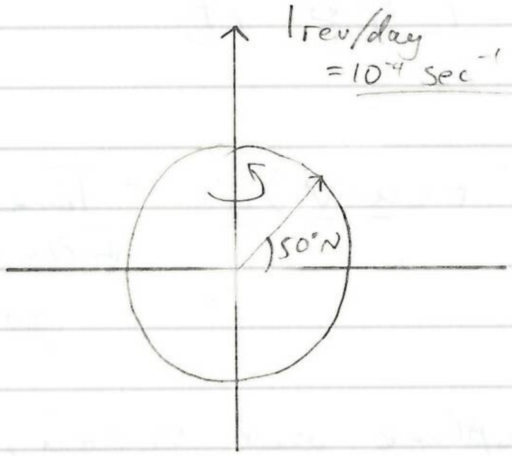


Isobar why?
- pressure acts as a streamfunction



$$\frac{Du}{Dt} \neq -\frac{1}{\rho}\nabla p$$

Does not work due to CORIOLIS



$R_e = 6000$ km

Speed $= \Omega R_e = 600$ msec^{-1}

Linear Waves (Moddle key: Rossby)
- DISPERSIVE -

Book : Pedlosky : Closest to course
- not a great book
- bitty -

Vallis : excellent, compehensive
modern ; readable :-
enormous - sucessor to Gill

Gill : nice ; readable - (Rossby - Gill adjustment).

Coriolis Force ~ Pressure Grad.
Geostrophic Balance

—/—

Equation of motion in a Rotating Frame.
In an inertial frame, we have the Euler equations

$$\frac{D\underline{u}}{Dt} = -\frac{1}{\rho}\underline{\nabla}p + \underset{\sim}{F}$$

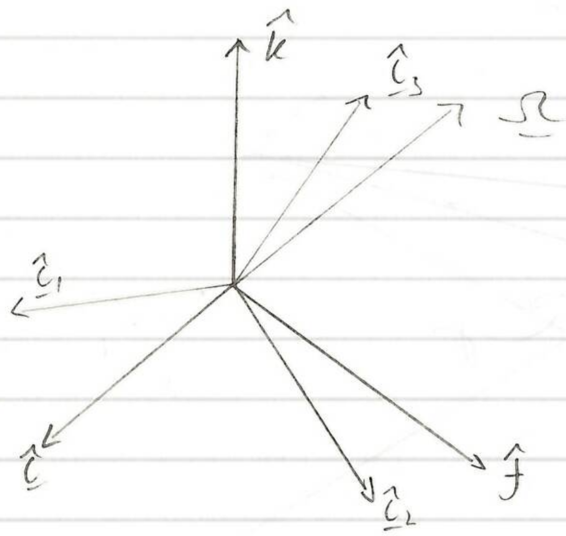
where $\frac{D\underline{u}}{Dt} = \frac{\partial \underline{u}}{\partial t} + (\underline{u}\cdot\underline{\nabla})\underline{u}$ - Time rate of change following a particle.

For both the atmosphere and ocean, the speeds we are interested in are extremely small compared to the speed of sound, [the Mach number $u/a \ll 1$] and so compressibility effects are negligable i.e.

$$\nabla \cdot \underline{u} = 0 .$$

We will also take the fluid to be homogenous i.e. $\underline{\rho = \text{constant}}$ initially (stratification at end).

$$\boxed{\frac{D\rho}{Dt} + \rho(\cancel{\underline{\nabla} \cdot \underline{u}}) = 0 \Rightarrow \boxed{\frac{D\rho}{Dt} = 0}}$$



Consider an inertial frame I with basis vectors $\hat{\underline{i}}, \hat{\underline{j}}, \hat{\underline{k}}$ [fixed in I]

Let there be a uniformly rotating frame R, rotating relative to I with angular velocity, $\underline{\Omega}$.

Let $\hat{\underline{i}}_1, \hat{\underline{i}}_2, \hat{\underline{i}}_3$ be the basis vector in R.

We can choose the orgin of R and I to coincide
$\hat{\underline{i}}_1, \hat{\underline{i}}_2, \hat{\underline{i}}_3$ fixed in R.

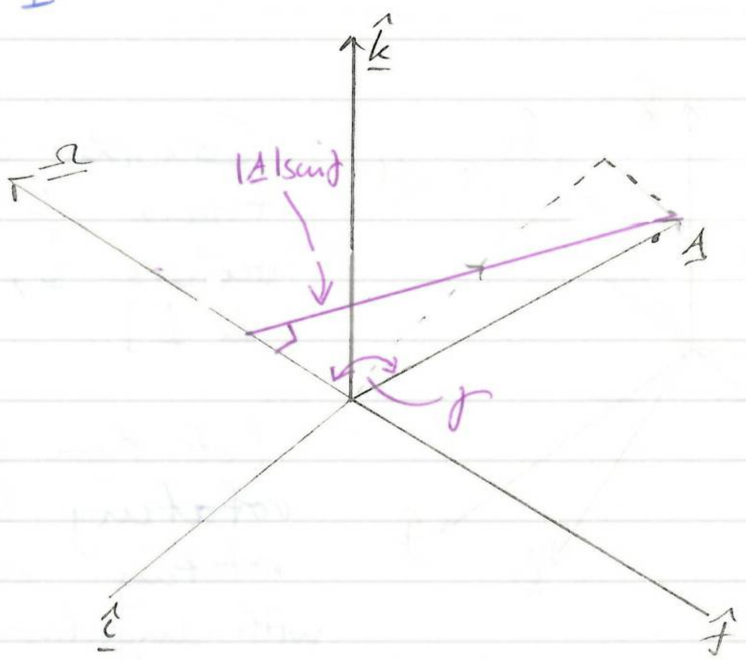
$$\left(\frac{d\hat{\underline{i}}_1}{dt}\right)_R = 0 \text{ (in frame } R) \qquad \left(\frac{d\hat{\underline{i}}}{dt}\right)_R \neq 0 .$$

$$\left(\frac{d\hat{\underline{i}}_1}{dt}\right)_I \neq 0 \qquad \left(\frac{d\hat{\underline{i}}}{dt}\right)_I = 0 .$$

Let $\underline{A}$ be any constant vector in frame $\underline{R}$.

What is $\left(\frac{d\underline{A}}{dt}\right)_I$?

$$\left(\frac{d\underline{A}}{dt}\right)_I = \lim_{\Delta t \to 0} \frac{\underline{A}(t+\Delta t) - \underline{A}(t)}{\Delta t}.$$



$\underline{A}$ moves on arm of length $|\underline{A}|\sin\gamma$ at angular speed $|\underline{\Omega}|$ for a time Δt, i.e it moves $|\underline{A}||\underline{\Omega}|\sin\gamma\,\Delta t\,\hat{\underline{n}}$

i.e $\left(\frac{d\underline{A}}{dt}\right)_I = \left(|\underline{A}||\underline{\Omega}|\sin\gamma\right)\hat{\underline{n}}$

where $[\underline{\Omega}, \underline{A}, \hat{\underline{n}}]$ is a RHanded system.

i.e $\left(\frac{d\underline{A}}{dt}\right)_I = \underline{\Omega} \wedge \underline{A}$ for any constant vector

How let $\underline{B}(t)$ be a time varying vetor in R.

i.e $\underline{B}(t) = B_1(t)\hat{\iota}_1 + B_2(t)\hat{\iota}_2 + B_3(t)\hat{\iota}_3$

Then :

$$\left(\frac{d\underline{B}}{dt}\right)_I = \left[\frac{d}{dt}\left(B_1(t)\hat{\iota}_1\right)\right]_I + \dots$$

$$= \frac{dB_1}{dt}\hat{\iota}_1 + B_1(t)\left(\frac{d\hat{\iota}_I}{dt}\right)_I + \dots$$

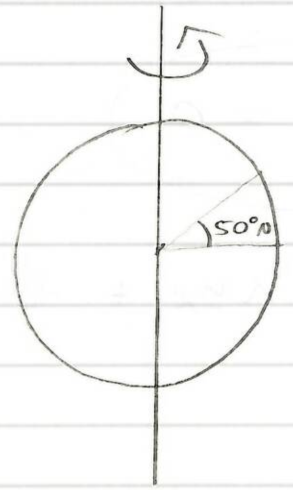
$$= \left(\frac{dB}{dt}\right)_R + B_1(\underline{\Omega} \wedge \hat{\iota}_1) + B_2(\Omega \wedge \hat{\iota}_2) + \dots$$

$$= \left(\frac{dB}{dt}\right)_R + \underline{\Omega} \wedge \underline{B}.$$

$\underline{r} \to \underline{u}_R,\ \underline{u}_I.$

$\underline{u}_R$
$\underline{u}_I$ $\Rightarrow \left(\frac{d\underline{u}_R}{dt}\right), \left(\frac{du_I}{dt}\right)$

11/1/13



E.E + rotation

$$\left(\frac{d\underline{B}}{dt}\right)_I = \left(\frac{d\underline{B}}{dt}\right)_R + \underline{\Omega} \wedge \underline{B}$$ for any vector $\underline{B}$.

e.g take $\underline{B} = \underline{r}$ (a particles displacement vector).

$$\left(\frac{d\underline{r}}{dt}\right)_I = \left(\frac{d\underline{r}}{dt}\right)_R + \underline{\Omega} \wedge \underline{r}.$$

$\underline{u}_I$ - velocity of the particle in the inertial frame
$\underline{u}_R$ - " " " " " " rotating frame

e.g. (2) Now take $\underline{B} = \underline{u}_I$

$$\left(\frac{d^2\underline{r}}{dt^2}\right)_I = \left(\frac{d\underline{u}_I}{dt}\right)_I = \left(\frac{d\underline{u}_I}{dt}\right)_R + \underline{\Omega} \wedge \underline{u}_I$$

Accel. in interial frame $= \left[\frac{d}{dt}(\underline{u}_R + \underline{\Omega} \wedge \underline{r})\right]_R + \underline{\Omega} \wedge \underline{u}_I.$

$$= \left(\frac{d\underline{u}_R}{dt}\right)_R + \underline{\Omega} \wedge \left(\frac{d\underline{r}}{dt}\right)_R + \underline{\Omega} \wedge \underline{u}_I$$

Let $\boxed{\dfrac{d\Omega}{dt} = 0}$

$$= \left(\frac{d\underline{u}_R}{dt}\right)_R + \underline{\Omega} \wedge \underline{u}_R + \underline{\Omega} \wedge \underline{u}_R + \underline{\Omega} \wedge (\underline{\Omega} \wedge \underline{r})$$

i.e

$$\underbrace{\left(\frac{d\underline{u}_I}{dt}\right)_I}_{\text{Accel in inertial frame}} = \underbrace{\left(\frac{d\underline{u}_R}{dt}\right)_R}_{\text{Accel in rot frame}} + \underbrace{2\,\underline{\Omega} \wedge \underline{u}_R}_{\text{Coriolis accel}} + \underbrace{\underline{\Omega} \wedge (\underline{\Omega} \wedge \underline{r})}_{\text{Centripetal accel}}$$

[Note in physics :-

$$m\left(\frac{d^2\underline{r}}{dt^2}\right)_I = \underline{F}.$$

$$m\left[\left(\frac{d^2\underline{r}}{dt^2}\right)_R + 2\,\underline{\Omega} \wedge \underline{u}_R + \underline{\Omega} \wedge (\underline{\Omega} \wedge \underline{r})\right] = \underline{F}.$$

Corilois force Centrifugal force

$$m\left(\frac{d^2\underline{r}}{dt^2}\right)_R = \underline{F} \underbrace{- 2m\,\underline{\Omega} \wedge \underline{u}_R - m\,\underline{\Omega} \wedge (\underline{\Omega} \wedge \underline{r})}_{\text{Fictictious force}}$$

Corolis force $\Rightarrow$ force to right on moving body observed in a rotating frame.]

In the inertial frame the Euler equations are:

$$\left(\frac{D\underline{u}_I}{Dt}\right)_I = -\frac{1}{\rho}\nabla p + \underline{F}.$$

So in the rotating frame

$$\left(\frac{D\underline{u}_R}{Dt}\right)_R + 2\underline{\Omega}\wedge\underline{u}_R + \underline{\Omega}\wedge(\underline{\Omega}\wedge\underline{r}) = -\frac{1}{\rho}\underline{\nabla}p + \underline{F}.$$

Since ρ is a scalar, the density equation remains

$$\left(\frac{D\rho}{Dt}\right)_R + \rho\underline{\nabla}\cdot\underline{u} = 0.$$

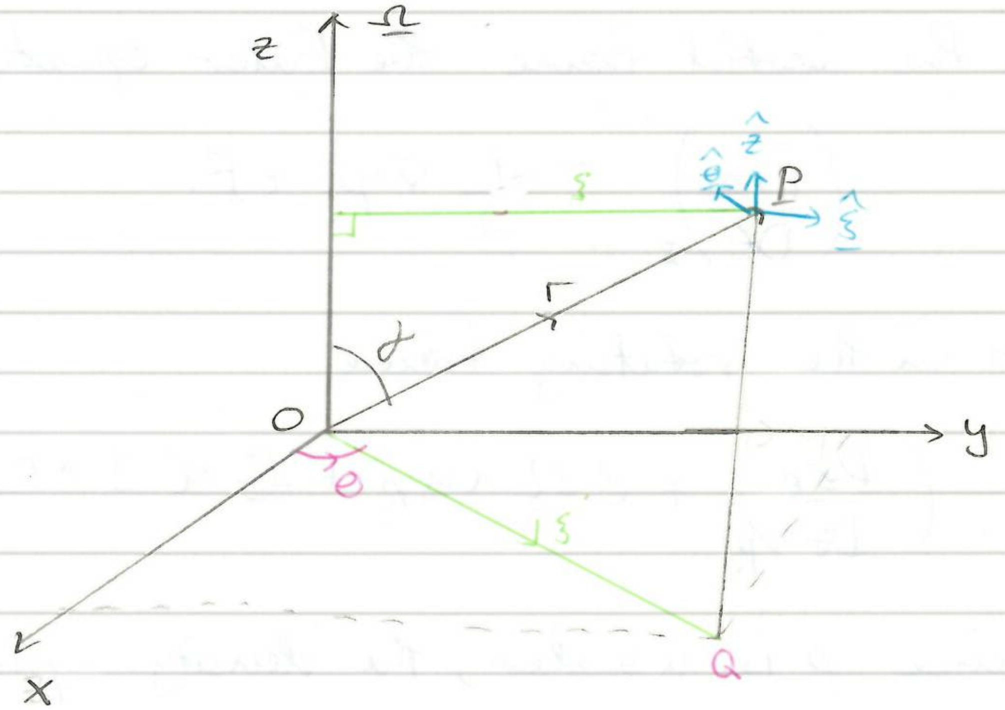
and when the flow is incompressible

$$\left(\frac{D\rho}{Dt}\right)_R = 0.$$

$$\Rightarrow \underline{\nabla}\cdot\underline{u} = 0. \text{ in all frames.}$$

<u>2.1 Centripetal acceleration as a potential.</u>

Introduce (very temporarily) cylindrical polar co-ords (ς, θ, z) with Oz aligned with $\underline{\Omega}$.



$$\underline{\Omega} \wedge \underline{r} = |\underline{\Omega}|\,|\underline{r}| \sin\gamma \, \hat{\underline{\theta}} .$$

$$= \Omega \xi \, \hat{\underline{\theta}} .$$

$$\underline{\Omega} \wedge (\underline{\Omega} \wedge \underline{r}) = -|\Omega| \Omega \xi \, \hat{\underline{\xi}}$$

$$= -\Omega^2 \xi \, \hat{\underline{\xi}}$$

$$\underline{\nabla} G_c = \frac{\partial G}{\partial \xi} \hat{\underline{\xi}} + \cancel{\frac{1}{\xi} \frac{\partial G_c}{\partial \theta} \hat{\underline{\theta}}} + \cancel{\frac{\partial G_c}{\partial z} \hat{\underline{z}}}$$

so $\dfrac{\partial G_c}{\partial \xi} = -\Omega^2 \xi$ so $G_c = -\dfrac{1}{2} \Omega^2 \xi^2$

$$= -\frac{1}{2} |\underline{\Omega} \wedge \underline{r}|^2$$

i.e we have shown that

$$\underline{\Omega} \wedge (\underline{\Omega} \wedge \underline{r}) = \nabla G_c .$$

where $G_c = -\frac{1}{2} |\underline{\Omega} \wedge \underline{r}|^2 .$

Thus we have

$$\left(\frac{D\underline{u}_R}{Dt}\right)_R + 2\underline{\Omega} \wedge \underline{u}_R = -\frac{1}{\rho} \nabla p + \underset{\sim}{F} - \nabla G_c .$$

Suppose the only external force is gravity so

$$\underset{\sim}{F} = -g\hat{\underline{z}} .$$
$$= -\underline{\nabla} G_g$$

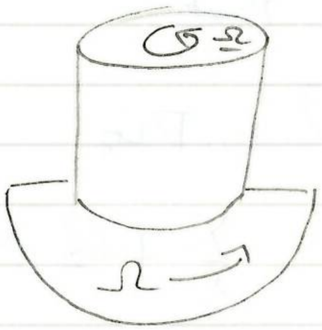
where $G_g = gz$ so

$$\left(\frac{D\underline{u}_R}{Dt}\right)_R + 2\underline{\Omega} \wedge \underline{u}_R = \underline{\nabla}\left[-\frac{1}{\rho} p - G_c - G_g\right]$$

(ρ constant).

Now take: $\underline{u}_R \equiv 0$ ($\underline{u}_R$ is the departure from solid rotation)

(i.e solid body rotation,
$$\underline{u}_I = \underline{u}_R + \underline{\Omega} \wedge \underline{r}$$
$$= \underline{\Omega} \wedge \underline{r}$$)



Then we find

$$\underline{\nabla}\left[-\frac{1}{\rho} p - G_c - G_g \right] = 0 .$$

i.e $\quad p + \rho G_c + \rho G_g = \text{const} .$

i.e $\quad p = \text{const} - \rho G_c - \rho G_g .$

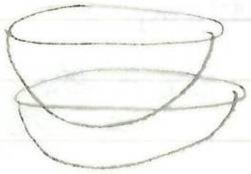
$\quad = p_e$, the equilibrium pressure.

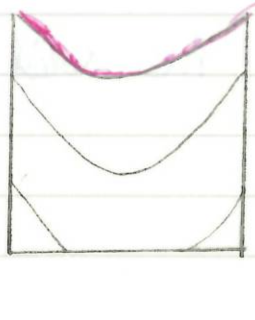
Thus $\quad p_e = p_0 + \frac{1}{2} \rho \Omega^2 \xi^2 - \rho g z .$ ✓

The surfaces of constant

$$\rho g z = \frac{1}{2} \rho \Omega^2 \xi^2 + \text{const}$$

i.e $\quad z = \left(\frac{1}{2} \frac{\Omega^2}{g} \right) (x^2 + y^2) + \text{const} .$





- surfaces of constant pressure are paraboloids
($\underline{u}_R = 0$)
- if there is a free surfaces; (where $p = p_a$, the constant then the free surface is one of these paraboloids.

Write $p = p_e + p_D$ in the equation $\underline{u}_R \neq 0$.

(c.f $p = p_h + p_D$
$\Omega = 0$, $p_e = p_0 - \rho g z$
$= p_h$)
hydrostatic.

—/—

Inertial ✓ Euler Equations.

Earth ≠ Inertial
Rotating

use E.E. ?

pressure $p = p_H + p_D$

∇G_c

E.E. + Coriolus + $\{\cancel{\text{Centrifugal}}\}$ = RHS

$p = p_e + p_D$

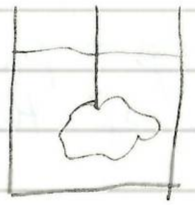
—/—

We now write

$$p = p_e + p_D$$

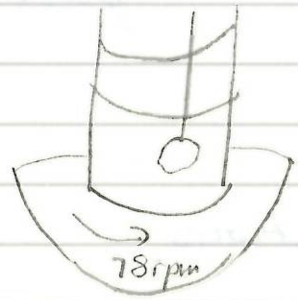
where $p_D = p - p_e$, i.e the deviation of p from its equilibrium value.

Homework 1: Archimedes.



experences
- upward frace evaluated weight of water displaced.

What happens for same experiment but rotating (solid body rot).



$$-\int p \hat{\underline{n}}\, ds = -\int \underline{\nabla} p \, dV.$$

We have:

$$\left(\frac{D\underline{u}_R}{Dt}\right)_R + 2\underline{\Omega} \wedge \underline{u}_R + \underline{\nabla} G_c = -\frac{1}{\rho}\underline{\nabla} p + \underline{\nabla} G_g$$

$\underline{\Omega} \wedge (\underline{\Omega} \wedge \underline{r})$ $\quad -g\hat{\underline{z}}$

$$p_e = p_0 + \frac{1}{2}\rho|\underline{\Omega} \wedge \underline{r}|^2 - \rho g z.$$

$$= p_0 - \rho G_c + \rho G_g.$$

$$\nabla p_e = -\rho \nabla G_c + \rho \nabla G_g$$

$$\underline{\nabla} p = \underline{\nabla} p_e + \underline{\nabla} p_D$$

$$-\frac{1}{\rho}\underline{\nabla} p = -\frac{1}{\rho}\underline{\nabla} p_0 + \underline{\nabla} G_c - \underline{\nabla} G_g.$$

Thus $\left(\frac{D\underline{u}_R}{Dt}\right)_R + 2\,\underline{\Omega} \wedge \underline{u}_R = -\frac{1}{\rho}\nabla p_D$

where p_D is, as before, the dynamic pressure.

Our equatcoins are thus, dropping the supscript "R", as we are now **always** in the rotating frame

$$\boxed{\frac{D\underline{u}}{Dt} + \underbrace{2\,\underline{\Omega} \wedge \underline{u}}_{\text{Coriolis Acceleration.}} = -\frac{1}{\rho}\nabla p_D \qquad \underline{\nabla}\cdot\underline{u} = 0.}$$

$$p = p_e + p_D.$$

If there are no free surface we can deal only with p_D. Only new term is the Coriolis accel.

[Geostrophic flow! –

$$\cancel{\frac{\partial \underline{u}}{\partial t}} + \cancel{(\underline{u}\cdot\underline{\nabla})\underline{u}} + 2\,\underline{\Omega} \wedge \underline{u} = \underline{-\frac{1}{\rho}\underline{\nabla} p_D}$$

$\left|\left(\frac{\partial \underline{u}}{\partial t}\right)\right| \ll 1$ if flow steady $\frac{\partial \underline{u}}{\partial t} = 0$.

$|(\underline{u}\cdot\underline{\nabla})\underline{u}| \sim |\underline{u}|^2 \ll 1$. flow is slow $[|\underline{u}| \ll 1)]$

$(\underline{u}\cdot\underline{\nabla})\underline{u} \sim 0$.

For slow, steady flow:

$$2 \underline{\Omega} \wedge \underline{u} = -\frac{1}{\rho} \underline{\nabla} p_D .$$

strophos : turn
peto : I seek
fuge : I flee
baros : weight

i.e Coriolis term balance the pressure gradient
geostrophic balance

$-\underline{\nabla} p$ is $\perp$'r to plane containing $\underline{u}$ and $\underline{\Omega}$

i.e $\underline{u}$ $\perp$'r to $\underline{\nabla} p$.

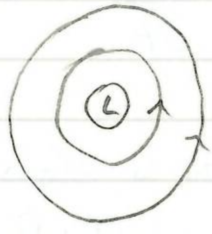
$\underline{u}$ is // to lines of constant p.

i.e $\underline{u}$ is // to isobars.

i.e. wind blow along isobars.

NOT from high pressure to low pressure.

If in NH, (what about SH?) + the wind is on your back then high pressure is to your right (Buys-Ballot 19th century).



anticlockwise in hurricane in NH

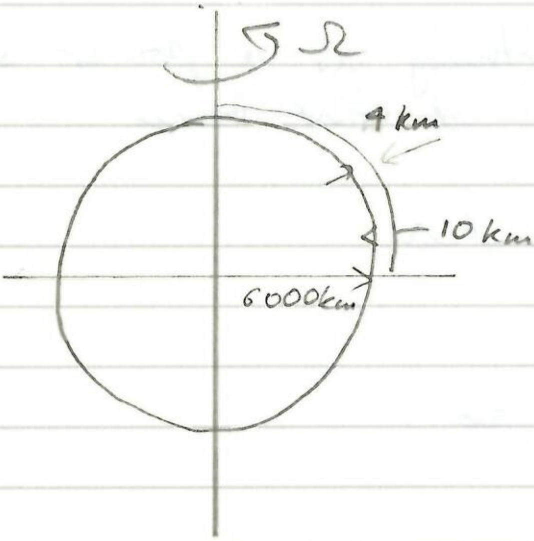
T.R.S

We have:

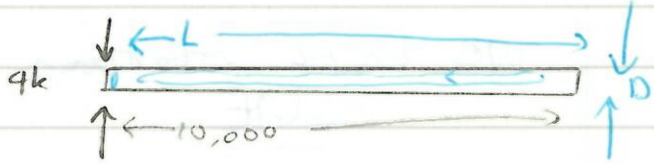
$$\frac{D\underline{u}}{Dt} + 2\underline{\Omega} \wedge \underline{u} = -\frac{1}{\rho}\underline{\nabla} p.$$

$$\underline{\nabla} \cdot \underline{u} = 0.$$

– this is 3D.



Rough idea:
"Oceans are to Earth as skin of an apple is the apple"

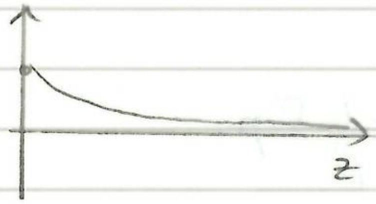


Oceans are extremely shallow.

For a basin-scale motion, $\frac{D}{L} \ll 1$

For the atmosphere the density decreases approximately exponentially.

$$\rho \simeq \rho_0 e^{-z/H_s}$$



We call H_s the e-folding scale - if we increase z by H_s we multiply ρ by $\frac{1}{e}$.

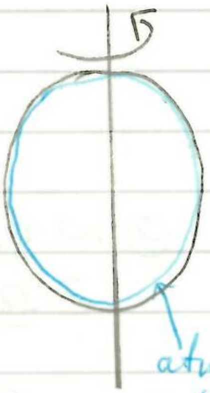
For the atmosphere we call H_s the scale height and $\underline{H_s \simeq 10\,km}$. (similar lab theory)

Use this Expand everthing as a power series in (D/L). We stop at first term

- Shallow Water Equation
  - SWE

we have rotation so
rSWE

15/1/13



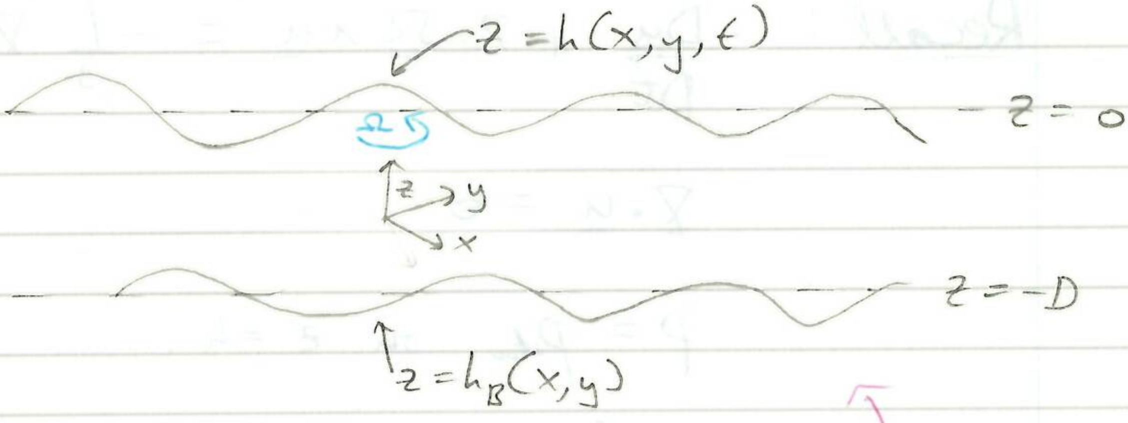
atmos + oceans
- thin layers ocean ~ 4km
atmos ~ 10km.

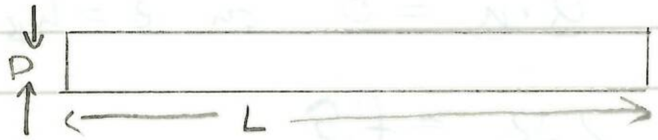
E.E

$\Big\downarrow$ + rotation

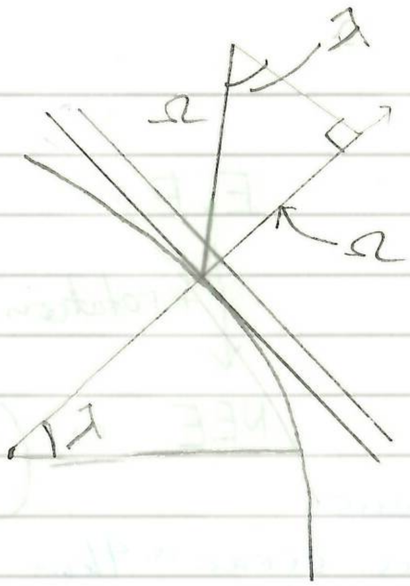
NEE (Coriolis and ~~Centripetal~~)

Shallow water Eqns. (E.E. + rotation using $D/L = \delta \ll 1$)





Cartesian co-ordinates.



$\odot \sim 50°N$

$2\underline{\Omega} \wedge \underline{u}$

$\underline{\Omega} = (\Omega \sin \lambda)\hat{\underline{z}}$

<u>Foucalt</u>

$$f = 2\underline{\Omega} \sin(\text{latitude})$$

<u>Recall</u>:

$$\frac{D\underline{u}}{Dt} + 2\underline{\Omega} \wedge \underline{u} = -\frac{1}{\rho}\nabla p_0$$

$$\nabla \cdot \underline{u} = 0$$

$$p = p_{atm} \quad \text{on } z = h.$$

$$\underline{u} \cdot \hat{\underline{n}} = 0 \quad \text{on } z = h_B$$

$$2\underline{\Omega} = f\hat{\underline{z}}$$

$$f = 2\Omega \sin \lambda.$$

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} - fv = -\frac{1}{\rho}\frac{\partial p_D}{\partial x} \qquad (1)$$

$$U/T \qquad U^2/L \qquad U^2/L \qquad UW/D \qquad fU \qquad P/\rho L$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} + fu = -\frac{1}{\rho}\frac{\partial p_D}{\partial y} \qquad (2)$$

$$U/T \qquad U^2/L \qquad U^2/L \qquad UW/D \qquad fU \qquad P/\rho L$$

$$\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z} = -\frac{1}{\rho}\frac{\partial p_D}{\partial z} \qquad (3)$$

$$W/T \qquad UW/L \qquad UW/L \qquad W^2/D \qquad P/\rho D$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \qquad (4)$$

$$\frac{U}{L} \qquad \frac{U}{L} \qquad \frac{W}{D}$$

—/—

$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{z} \\ 0 & 0 & f \\ u & v & w \end{vmatrix} = -fv\hat{i} + fu\hat{j} = f\underline{\hat{z}} \wedge \underline{u}$$

—/—

We need to estimate the size of each term in each equation, remind ourselves that $\delta << 1$, + so neglect small terms.

Let a typical horizontal scale be L
" " " vertical scale be D.

Let a typical horizontal velocity be U
" " " vertical velocity be W.

Let a typical change in the dynamic pressure be P.
and a typical time scale for the motion be T.

(4) says:

$$\frac{\partial w}{\partial z} = -\left(\frac{\partial u}{\partial x} + \frac{\partial y}{\partial y}\right)$$

so
$$\left|\frac{\partial w}{\partial z}\right| = \left|\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right|$$

$$\lesssim \left|\frac{\partial u}{\partial x}\right| + \left|\frac{\partial v}{\partial y}\right|$$

$$\lesssim 2 \max\left\{\left|\frac{\partial u}{\partial x}\right|, \left|\frac{\partial v}{\partial y}\right|\right\}$$

Hence:
$$\frac{W}{D} \lesssim \frac{U}{L}$$

i.e $\frac{W}{U} \lesssim \frac{D}{L} = \delta << 1$

i.e vertical velocities are small compared to horizontal velocities (in basin scale motions).

i.e we write:

$$\frac{W}{U} \lesssim O(\delta)$$

i.e $\lim_{\delta \to 0} \frac{W}{U\delta}$ remains bounded

or $W \sim \delta U$ i.e. W behaves as δU.

Now consider (1):
Then by same arguement:

$$|RHS'| = |LHS'|$$

$$\leq \begin{pmatrix} \text{no. of} \\ \text{terms} \end{pmatrix} \max \left\{ |\text{terms}| \right\}$$

Here:

$$\frac{P}{\rho L} \lesssim \max \left\{ fU, \frac{UW}{D}, \frac{U^2}{L}, \frac{U}{T} \right\}.$$

$$\frac{UW}{D} \sim \frac{U^2 \delta}{D} \qquad W \sim \delta U.$$

$$= \frac{U^2}{L} \qquad \delta = \frac{D}{L}.$$

So $$\frac{P}{\rho L} \lesssim U \max \left\{ f, \frac{U}{L}, \frac{1}{T} \right\}$$

$$P \lesssim \rho U \max \left\{ fL, U, \frac{L}{T} \right\}$$

Our deductions to date in fact show that the LHS' of (3) is negligable (even at first order in δ).

[In (3)] As before $\left| \frac{Dw}{Dt} \right| \ll \left| \frac{1}{\rho} \frac{\partial p}{\partial z} \right|$

$$= \max\left\{\frac{W}{T}, \frac{UW}{L}\right\} \rho D / P$$

$$= \frac{\cancel{\rho} D \max\left\{\frac{W}{T}, \frac{UW}{L}\right\}}{\cancel{\rho} U \max\left\{fL, U, \frac{L}{T}\right\}}, \quad W \sim \delta U$$

$$= \frac{D}{\cancel{U}} \cdot \frac{\delta \cancel{U}}{L} \max\left\{\frac{1}{T}, \frac{U}{L}\right\} / \max\left\{\frac{1}{T}, \frac{U}{L}, f\right\}$$

$$= \delta^2 \max\left\{\frac{1}{T}, \frac{U}{L}\right\} / \max\left\{\frac{1}{T}, \frac{U}{L}, f\right\}$$

$$\leq (\delta^2)$$

- i.e the LHS of (3) is negligible when $\delta^2 << 1$.

Hence (3) is simply $\frac{\partial p_P}{\partial z} = 0$.

(c.f. lubrication theory)

Remeber we briefly noted the Rossby number

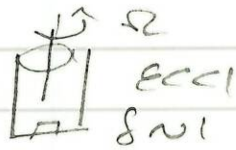
$$\varepsilon = \frac{U}{2\Omega L} = \frac{U}{fL}$$

We have $\left|\frac{Dw}{Dt}\right| / \left|\frac{1}{\rho}\frac{\partial p_P}{\partial z}\right| = \frac{\delta^2 \max\left\{\frac{L}{T}, U\right\}}{\max\left\{\frac{L}{T}, U, fL\right\}}$

For $\varepsilon \ll 1$, $fL \gg U$ then the ratio is $\delta^2 \varepsilon$.

In a rapidly rotating flow where $\varepsilon \ll 1$, we do not need such small δ.

Fast rotating "fat" fluids: lab model



Earth; slowly rotating "thin" fluids

18/1/13

— / —

SSW

⇒ Sudden Stratospheric Warming:

— / —

SWE (from rEE):

Vert mom'm eqn. $\frac{\partial p_D}{\partial z} = 0$.

Total pressure $p = p_H + p_D$

$= p_0 - \rho g z + p_D$

On surface, $z = h(x, y, t)$

Pressure $= p_a$, constant atmos pressure

Thus $p_a = p_0 - \rho g h + p_D$. (6).

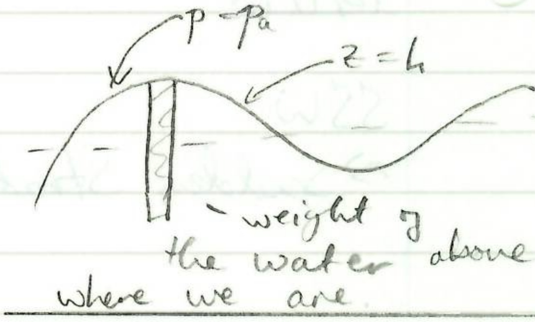
so $p_0 + p_D = p_a + \rho g h$.

Thus $\boxed{p = p_a + \rho g h - \rho g z = p_a + \rho g (h - z)}$

p_D gone - replaced by $h(x, y, t)$

- i.e hydrostatic.

Thus from (6), $\frac{\partial}{\partial x}$ gives:



$$-\rho g \frac{\partial h}{\partial x} + \frac{\partial p_D}{\partial x} = 0.$$

similarly.

$$\frac{\partial p_D}{\partial y} = \rho g \frac{\partial h}{\partial y}.$$

Then our mom'm eqns are:

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} - fv = -g\frac{\partial h}{\partial x}(x,y,t).$$

$$\underbrace{\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z}}_{\text{Accel.}} + fu = \underbrace{-g\frac{\partial h}{\partial y}(x,y,t)}_{\text{force}}.$$

The RHS is independant of depth z, i.e same at all depths and it is the forceing. Thus if the flow is depth independant at any instant it will remain depth indep. for all time.

Thus look for solutions $u(x,y,t)$ and $v(x,y,t)$ and get rSWE, horiz mom'm eqns.

$$\boxed{\begin{aligned} \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} - fv &= -g\frac{\partial h}{\partial x} \\ \frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + fu &= -g\frac{\partial h}{\partial y} \end{aligned}}$$

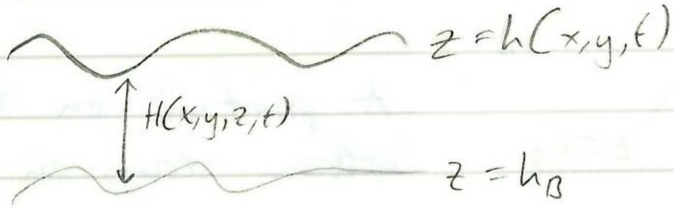
$$\left.\begin{aligned} u &= u(x,y,t) \\ v &= v(x,y,t) \\ h &= h(x,y,t) \end{aligned}\right\} \text{3 unknowns}$$

2D flow.

The remaining eqn is continuity:-

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0.$$

Integrate this from $z = h_B(x,y)$ to $z = h(x,y,t)$



Then $\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right)\int_{h_B}^{h} 1\, dz + \left[w\right]_{h_B}^{h} = 0.$

i.e $\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right)H + w(h) - w(h_B) = 0.$

where H = local total depth.
$= h - h_B = h(x,y,t) - h_B(x,y) = H(x,y,t).$

Differentiate, following a particle.

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y}.$$

Then $\frac{Dz}{Dt} = \frac{Dh}{Dt}$ on $z = h$

i.e. on $z = h$

$$w = \frac{Dh}{Dt} \left(= \frac{\partial h}{\partial t} + u\frac{\partial h}{\partial t} + v\frac{\partial h}{\partial t} \right)$$

Similiarly on $z = h_B$,

A particle on the bottom stays on bottom

On $z = h_B$, $\quad w = \frac{Dh_B}{Dt} = \left(= u\frac{\partial h_B}{\partial x} + v\frac{\partial h_B}{\partial y} \right)$

Thus:

$$\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) H + \frac{Dh}{Dt} - \frac{Dh_B}{Dt} = 0.$$

i.e. $\boxed{\frac{DH}{Dt} + H\, \underline{\nabla} \cdot \underline{u} = 0}$ $\qquad \underline{\nabla} = \frac{\partial}{\partial x}\hat{\underline{i}} + \frac{\partial}{\partial y}\hat{\underline{j}}$.

2D, horizontial divergence

$H = h - h_B$

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + (u \cdot \nabla)$$

2D.

Summary,

Mom'm: $\frac{Du}{Dt} - fv = -g\frac{\partial h}{\partial x}$

$$\frac{Dv}{Dt} + fu = -g\frac{\partial h}{\partial y}$$

Cty: $\frac{DH}{Dt} + H\underline{\nabla}\cdot\underline{u} = 0.$

3 equations in 3 unknowns $\longrightarrow$ $\begin{cases} u(x,y,t) \\ v(x,y,t) \\ h(x,y,t) \end{cases}$

as $H = h - h_B$

in 2D and time

If we need pressure we have:

$$p = p_a + \rho g(h-z).$$

Only remaining quantity is the vertical velocity, w.

Go back to $\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0.$

Integrate from $z = h_B$ to $z = z$.

Then as above

$$(\underline{\nabla}\cdot\underline{u})(z-h_B) + w(z) - w(h_B) = 0.$$

i.e. $w(x,y,z,t) = -(z - h_B)\, \underline{\nabla} \cdot \underline{u} + \dfrac{Dh_B}{Dt}$

linear in z.

$$= \underbrace{-(z-h_B)\, \underline{\nabla} \cdot \underline{u}}_{\text{linear in } z} + \underbrace{u\frac{\partial h_B}{\partial x} + v\frac{\partial h_B}{\partial y}}_{x,y,t}$$

Now we will look first at general properties that ALL solutions of the SWE satisfy without approximation.

Then look at <u>wavelike solution</u>.

Now our continuity eqn is:

$$\frac{DH}{Dt} + H\, \underline{\nabla} \cdot \underline{u} = 0$$

Note: $\dfrac{DH}{Dt} = \dfrac{\partial H}{\partial t} + (\underline{u} \cdot \nabla) H$

$$\Rightarrow \frac{\partial H}{\partial t} + (\underline{u} \cdot \underline{\nabla}) H + H(\underline{\nabla} \cdot \underline{u}) = 0$$

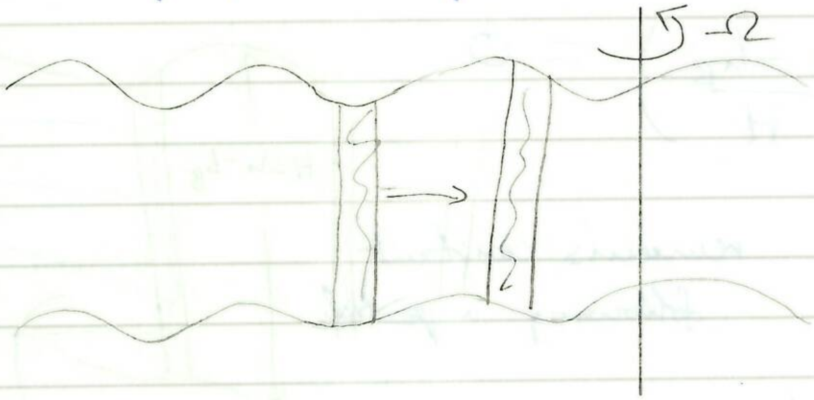
i.e $\dfrac{\partial H}{\partial t} + \nabla \cdot (H\underline{u}) = 0$.

c.f. conservation of mass equation, allowing for variable ρ,

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \underline{u}) = 0.$$

In the absense of topography (+ rotation) i.e $h_B = \text{const}$, the SWE are identical to the 2D compressible Euler eqns [$h \to \rho$] (the trouble-some waves in rSWE map to sound waves in compressible Euler.)

General properties of the rSWE



$u = u(x,y,t)$
$v = v(x,y,t)$.

2012 Q1(a) 1. A vertical column of fluid stays vertical throughout the motion.

2009 1(b) 2) $w = \frac{Dz}{Dt} = -(z - h_B)\,\underline{\nabla} \cdot \underline{u} + \frac{Dh_B}{Dt}$.

i.e $\frac{D}{Dt}(z - h_B) = -(z - h_B)\,\underline{\nabla} \cdot \underline{u}$.

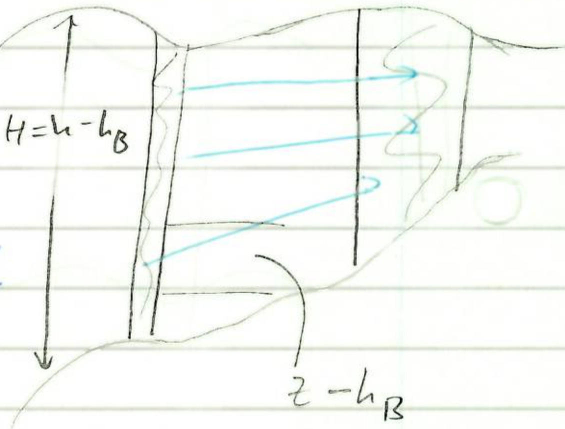
i.e $\frac{1}{z - h_B} \frac{D}{Dt}(z - h_B) = -\underline{\nabla} \cdot \underline{u}$.

But $\frac{DH}{Dt} + H\,\underline{\nabla} \cdot \underline{u} = 0$ i.e $\frac{1}{H}\frac{DH}{Dt} = -\underline{\nabla} \cdot \underline{u}$

Then $\frac{1}{(z - h_B)} \frac{D(z - h_B)}{Dt} = \frac{1}{H}\frac{DH}{Dt}$.

i.e $H \dfrac{D(z-h_B)}{Dt} - (z-h_B)\dfrac{DH}{Dt} = 0$

i.e $\dfrac{D}{Dt}\left(\dfrac{z-h_B}{H}\right) = 0$.

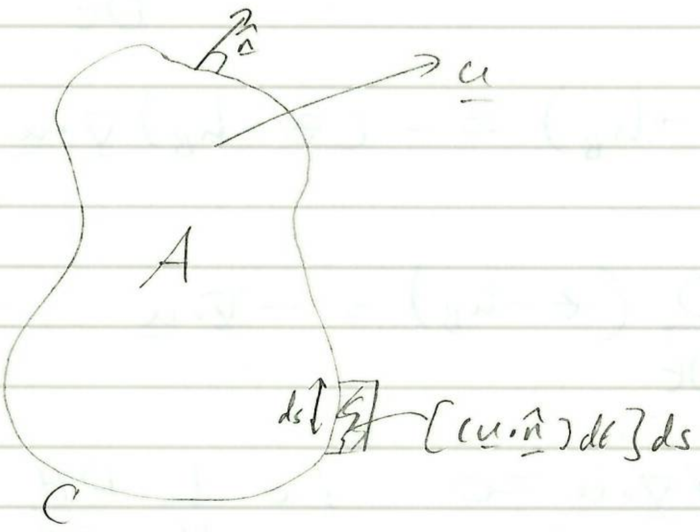


i.e $\dfrac{z-h_B}{h-h_B}$ remains constant following a particle

i.e Fractional height of particles following a particle.

e.g 1/3 way up always 1/3 way up

3. Look down on a column.



then

$$\frac{DA}{Dt} = \oint_C (\underline{u}\cdot\hat{\underline{n}})\,ds$$

$$= \int_A \underline{\nabla}\cdot\underline{u}\,dA$$

$$\simeq (\underline{\nabla}\cdot\underline{u})\,A$$

i.e. for sufficiently small A, + cts $\underline{\nabla} \cdot \underline{u}$.

$\underline{\nabla} \cdot \underline{u} = \frac{1}{A} \frac{DA}{Dt}$ - fractional rate of increase of area.

But we have $\frac{1}{H} \frac{DH}{Dt} = -\underline{\nabla} \cdot \underline{u}$.

i.e $\frac{1}{A} \frac{DA}{Dt} + \frac{1}{H} \frac{DH}{Dt} = 0$.

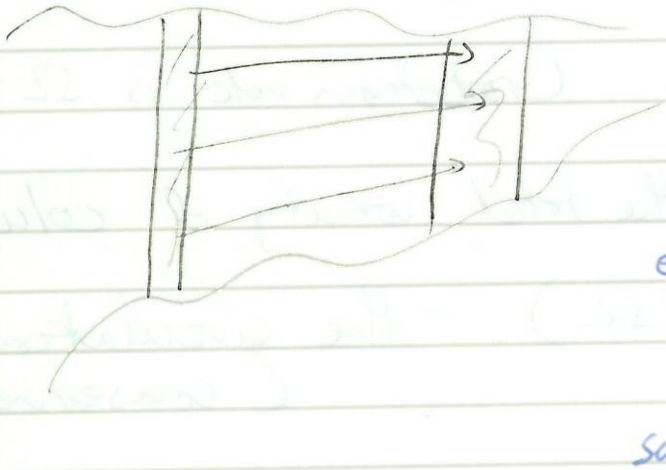
i.e $H \frac{DA}{Dt} + A \frac{DH}{Dt} = 0$

i.e $\frac{D}{Dt}(AH) = 0$.

i.e following a thin column, the column volume.

AH

is conserved.



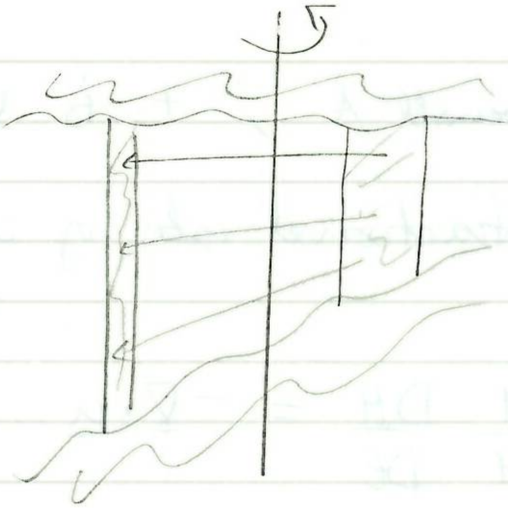
If a column shrinks it must get fatter to conserve volume.

This is really all that our eqn:

$$\frac{DH}{Dt} + H \underline{\nabla} \cdot \underline{u} = 0.$$

says:

4):

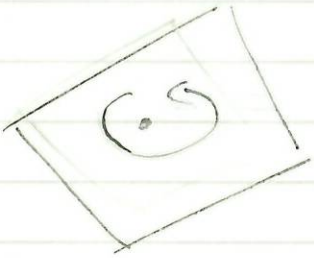


Ballerina Effect.

- columns stretch so must amplify their absolute rate of rotation

Element rotates at a rate:

$$\dot{\phi} = \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial y}{\partial y}\right)$$



$$= \frac{1}{2}\zeta \quad \underline{\text{VORTICITY}}$$

where ζ is the $\hat{\underline{z}}$ component of curl $\underline{u}$ i.e

$$\zeta = \hat{z}\cdot(\underline{\nabla}\wedge\underline{u})$$

Inviscid fluid $\Rightarrow$ no shear stress $\Rightarrow$ any momentum about C of M of elements (in 2D) is constant i.e $\dot{\phi}$ constant for an element.

This is in an inertial frame (Newton's laws). We thus need the absolute vorticity which is

$$\zeta + 2\Omega \quad (\text{rotation rate is } \Omega + \tfrac{1}{2}\zeta)$$

For our column, the total voricity of column is

$$A(\zeta + 2\Omega) \text{ - the circulation (conserved)}$$

But AH is conserved, so

$$q = \frac{\zeta + 2\Omega}{H}$$

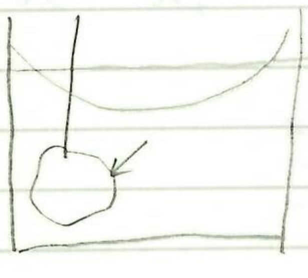
is conserved follow columns, q: potential vorticity
i.e if H increases ζ must increase
H decreases ζ must decrease (+ may go negative).

Conservation of q is called the conservation of potential velocity (PV)

Hw:2 Prove this from our rSWE.

—/—

Ans from HW 1:



$$\underset{\sim}{F} = \int_S -p\hat{\underline{n}}\, dS$$

$$= -\int_V \nabla p\, dV$$

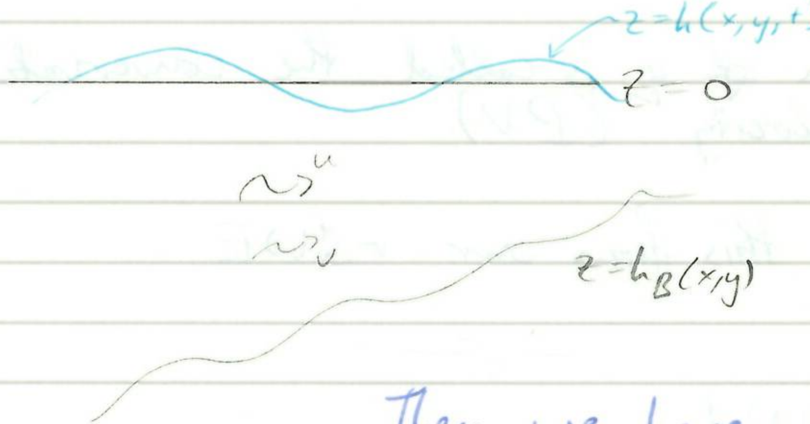
$$= +\rho g V\hat{z} - \rho\Omega^2 \int \xi\, dV\, \hat{\underline{\xi}}.$$

Body experiences two forces a buoyancy force equal and opposite to the weight of the water displaced and an inward centripetal force equal + opposite to the outward centrifugal force on the water displaced.

—//—

Linearised rSWE i.e lrSWE

Before discussing time depenant motions it is vital to understand the wave properties of the rSWE. Sufficent to consider infinitesimal waves, i.e to take all quantites to be of order $\delta << 1$ and neglect products of order δ^2.



We take the undistribed surface to be $z = 0$. and take $u, v, h \sim \delta$

Then we have $\frac{D}{Dt} =: \frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y}$

$$\frac{Du}{Dt} = \underbrace{\frac{\partial u}{\partial t}}_{\delta} + \underbrace{\cancel{u\frac{\partial u}{\partial x}}}_{\delta^2} + \underbrace{\cancel{v\frac{\partial u}{\partial y}}}_{\delta^2}$$

Then $$\frac{\partial u}{\partial t} - fv = -g\frac{\partial h}{\partial x}$$

$$\frac{\partial v}{\partial t} + fu = -g\frac{\partial h}{\partial y}$$

$H = h - h_B$, Thus we have $\frac{\partial h}{\partial t} + \underline{\nabla} \cdot (-h_B \underline{u}) = 0$.

$$\frac{DH}{Dt} = \frac{\partial h}{\partial t} - u\frac{\partial h_B}{\partial x} - v\frac{\partial h_B}{\partial y}$$ i.e $\frac{\partial h}{\partial t} + \underline{\nabla} \cdot (\underline{u} H_0) = 0$

hence $H_0(x,y) = -h_B(x,y) > 0$ is the undisturbed depth.

i.e the lr SWE

$$u_t - fv = -gh_x$$

$$v_t + fu = -gh_y$$

$$h_t + \underline{\nabla} \cdot (\underline{u} H_0) = 0.$$

- linear

2012 (Q2a)

Poincaré waves in a cylindrical domain

1. The linearised shallow water momentum equations are

2011 Q1a)

$$u_t - fv = -g\eta_x, \tag{1}$$
$$v_t + fu = -g\eta_y, \tag{2}$$

$\partial_t(1) + f(2)$ gives $u_{tt} + f^2 u = -g\eta_{xt} - fg\eta_y$.
$\partial_t(2) - f(1)$ gives $v_{tt} + f^2 v = -g\eta_{yt} + fg\eta_x$.
i.e. $(\partial_{tt} + f^2)\boldsymbol{u} = -g(\nabla\eta_t - f\hat{\boldsymbol{z}} \wedge \nabla\eta)$.

2016 1c)

2.
$$\eta_t + H_0(u_x + v_y) = 0 \tag{3}$$

$(\partial_{tt} + f^2)(3)$ gives

2008 1)

$$(\partial_{tt} + f^2)\eta_t - gH_0[\partial_x(\eta_{xt} + f\eta_y) + \partial_y(\eta_{yt} - f\eta_y)] = 0,$$

Hence η satisfies

$$[(\partial_{tt} + f^2)\eta - c^2(\eta_{xx} + \eta_{yy})]_t = 0,$$

where $c^2 = gH_0$.

3. Since govering equation has coefficients independent of θ and t, look for solutions of form $\eta = \Re\{R(r)\exp[\mathrm{i}(m\theta - \sigma t)]\}$. Note since η must be a single-valued function of position then m must be integral. Thus

$$(f^2 - \sigma^2)R - c^2[R'' + R'/r - m^2 R/r^2] = 0.$$

Introduce $\alpha = (\sigma^2 - f^2)^{1/2}/c$ so $r^2 R'' + rR' + (\alpha^2 r^2 - m^2)R = 0$. Then $R(r) = J_m(\alpha r)$ (requiring R finite at $r = 0$).

At $r = L$ we require $\boldsymbol{u}.\hat{\boldsymbol{n}} = 0$, i.e. $\boldsymbol{u}.\hat{\boldsymbol{r}} = 0$ thus

$$\eta_{rt} - f\hat{\boldsymbol{r}}.(\hat{\boldsymbol{z}} \times \nabla\eta) = 0, \qquad \text{i.e. } \eta_{rt} + (f/r)\eta_\theta = 0.$$

Hence $-\mathrm{i}\sigma\alpha r J_m'(\alpha r) + \mathrm{i}mfJ_m'(\alpha r) = 0$, at $r = L$, i.e.

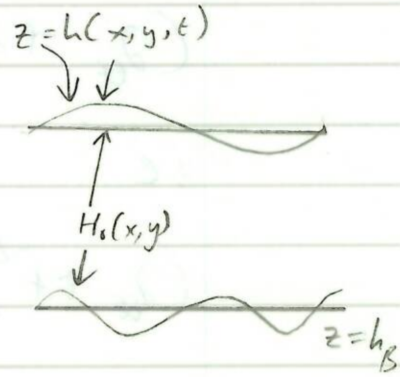
$$\frac{\sigma}{mf} = \frac{J_m(\alpha L)}{\alpha L J_m'(\alpha L)},$$

with $\alpha = (\sigma^2 - f^2)^{1/2}/c$ and $|\sigma| > f$.

4. If $|\sigma| < f$ then α is imaginary and J_m must be replaced by the modified Bessel function I_m. This gives the Kelvin waves.

22/1/13

G SWE



$$\frac{\partial \underline{u}}{\partial t} + 2\underline{\Omega} \wedge \underline{u} = -g\underline{\nabla} h$$

$$\frac{\partial h}{\partial t} + \underline{\nabla}\cdot(\underline{u} H_0(x,y)) = 0.$$

u, v, h unknown.

We can get a single equation in h.

$$\underline{\Omega} = \Omega\hat{\underline{z}}$$

$$\underline{u}_t = + 2\Omega\hat{\underline{z}} \wedge \underline{u} = -g\underline{\nabla} h \quad (1)$$

Cross with $\hat{\underline{z}}$, $\hat{z} \wedge (1)$;

(2); $(\hat{\underline{z}} \wedge \underline{u})_t - 2\Omega \underline{u} = -g\hat{\underline{z}} \wedge \underline{\nabla} h$

$\frac{\partial}{\partial t}(1)$

(3); $\underline{u}_{tt} + 2\Omega(\underline{z} \wedge \underline{u})_t = -g\underline{\nabla} h_t$

(3) $- 2\Omega$ (2)

$$\underline{u}_{tt} + 4\Omega^2 \underline{u} = -g\underline{\nabla} h_t + 2\Omega g\,\hat{\underline{z}} \wedge \underline{\nabla} h.$$

i.e $\boxed{(\partial_{tt} + f^2)\underline{u} = -g\underline{\nabla} h_t + fg\,\hat{\underline{z}} \wedge \underline{\nabla} h}$, $f = 2\Omega$.

(6)

Operate on (4) with $\partial_{tt} + f^2$:

$$(\partial_{tt} + f^2) h_t + \underline{\nabla} \cdot (H_0 (\partial_{tt} + f^2) \underline{u}) = 0.$$

i.e

$$(\partial_{tt} + f^2) h_t + \underline{\nabla} \cdot [H_0 (-g \nabla h_t + fg \hat{\underline{z}} \wedge \underline{\nabla} h)] = 0$$

– an equation in h alone.

Notice on a solid boundary $\underline{u} \cdot \hat{\underline{n}} = 0 \quad \forall t$
so dot (6) with $\hat{\underline{n}}$,

$$\underbrace{(\partial_{tt} + f^2) \underline{u} \cdot \hat{\underline{n}}}_{0} = -g \hat{\underline{n}} \cdot \underline{\nabla} h_t + fg \hat{\underline{n}} \cdot (\hat{\underline{z}} \wedge \underline{\nabla} h)$$

But LHS is zero for all time on stationary solid bdry, so:

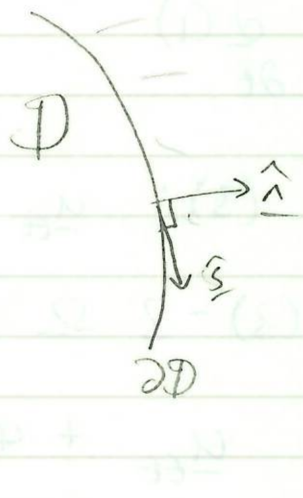
$$\boxed{\frac{\partial^2 h}{\partial n \partial t} = f \frac{\partial h}{\partial s}}$$

b.c on solid bdry for : Lr SWE

— / —

Note $\dfrac{\partial}{\partial n} \equiv \hat{\underline{n}} \cdot \underline{\nabla}$

i.e r.o.ch in normal direction.



— / —

$$\hat{\underline{n}} \cdot (\hat{z} \wedge \underline{\nabla} h) = (\hat{\underline{n}} \wedge \hat{\underline{z}}) \cdot \underline{\nabla} h.$$

$\ldots = \underline{\hat{s}} \cdot \underline{\nabla} h$ $\quad$ $\underline{\hat{s}}$, unit tangent vectored $90°$ clockwise to $\underline{\hat{n}}$.

<u>Notice</u>: If flow <u>not</u> rotating, $f \equiv 0$

Our b.c. is $\dfrac{\partial^2 h}{\partial n \partial t} = 0$.

Our b.c is $\dfrac{\partial^2 h}{\partial n \partial t} = 0$ on $\partial \mathcal{D}$ $\forall t$.

i.e $\dfrac{\partial h}{\partial n} = 0$ on $\partial \mathcal{D}$ $\forall t$,

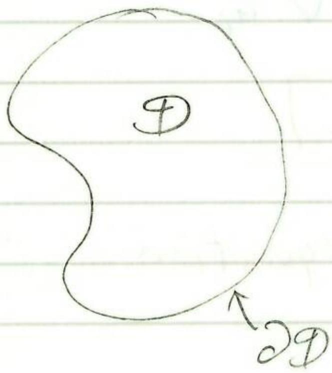
Governing equation:

$$\frac{\partial^3 h}{\partial t^3} + \underline{\nabla} \cdot \left[H_0 (-g \underline{\nabla} h_t) \right] = 0 \quad \forall t .$$

Define: $c^2 = g H_0(x,y)$. $\dfrac{\partial^2 h}{\partial t^2} = \underline{\nabla} \cdot (c^2 \underline{\nabla} h)$

- variable speed wave equation (e.g tsunamis)

For a flat-bottomed basin, $H_0 = \text{const}$ so $c^2 = \text{const}$.



$$\frac{\partial^2 h}{\partial t^2} = c^2 \nabla^2 h \text{ in } \mathcal{D}$$

$$\frac{\partial h}{\partial n} = 0 \text{ on } \partial \mathcal{D} .$$

$$\left(\frac{\partial h}{\partial r} = 0 \text{ on } r = a \right)$$

— 1 —

HW!

for circle radius a: $\frac{\partial^2 h}{\partial r \partial t} = -\frac{f}{a}\frac{\partial h}{\partial \theta}$

Drum

$h_{tt} = c^2 \nabla^2 h$

$h = 0$

Bessel

B.C:

$$\frac{\partial^2 h}{\partial n \partial t} = f \frac{\partial h}{\partial s}$$

HW: Eigenmode

$e^{in\theta}$, $J_n(kr)$

① Redo for water Fri week.

② Do it for Klein - Gordan equation

$f \neq 0$ We consider first, rotating ($f > 0$) flat-bottomed flow ($H_0 = $ const).

$$(\partial_{tt} + f^2) h_t - c^2 \underline{\nabla} \cdot \left[\underline{\nabla} h_t - f \hat{\underline{z}} \wedge \underline{\nabla} h \right] = 0$$

$$c^2 = g H_0$$

$$\hat{\underline{z}} \wedge \underline{\nabla} h = \underline{\nabla} \wedge (h \hat{z})$$

$$\text{div}(\text{curl } ?) = 0.$$

Thus:

$$(\partial_{tt} + f^2) h_t = c^2 \nabla^2 h_t$$

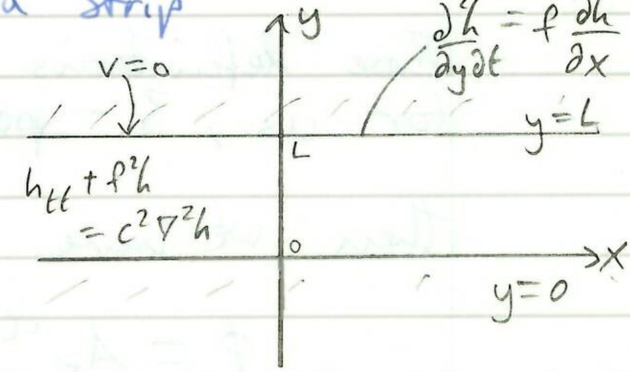
$$\boxed{h_t + f^2 h = c^2 \nabla^2 h}$$

Klein - Gordan equation.

$$\frac{\partial^2 h}{\partial n \partial t} = f \frac{\partial h}{\partial s}$$

for circle radius a $\quad \frac{\partial^2 h}{\partial r \partial t} = -\frac{f}{a}\frac{\partial h}{\partial \theta}$

We will solve the KG in a strip



Wave Jargon.

KG is an example of a linear eqn with constant coeff. Thus it possesses exponential solution of the form:

$$h(x,y,t) = Ae^{i(kx + ly - \omega t)}$$

where A, k, l, ω are fixed constants

Here $\quad h = Ae^{i\theta}$

Here A is the amplitute of h.
and θ is its phase.

i.e. $\theta = kx + ly - \omega t$ is the phase of the wave.

ω is the (radian) frequency, with $\tau = 2\pi/\omega$ being the period.

k is the wavenumber in the x-direction = number of wavecreasts in a distance 2π in the x-direction.

These definitions work in n dimensions. But for us, 3-space and one time are sufficient

Then we have

$$\phi = Ae^{i(kx + ly + mz - \omega t)}$$

$$= Ae^{i(\underline{k} \cdot \underline{r} - \omega t)} \quad \text{(Plane wave)}$$

(k - kappa)

$k = k\hat{\underline{x}} + l\hat{y} + m\hat{\underline{z}}$ is the wavenumber vector

25/1/13

Linear + Constant coefs
$\Rightarrow$ exponentials.

$h = Ae^{i\theta}$

A amplitude

θ phase

k, l, m, ω, A constants.

$\theta = kx + ly + my - \omega t$

$= \underline{k} \cdot \underline{r} - \omega t$

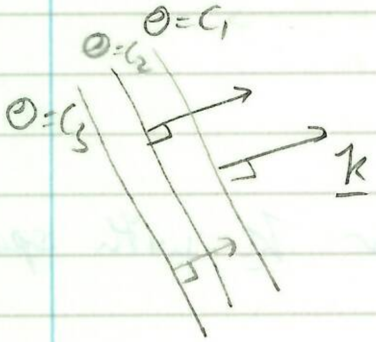
$\underline{k} = k\hat{\underline{x}} + l\hat{\underline{y}} + m\hat{\underline{z}}$.

$\underline{r} = x\hat{\underline{x}} + y\hat{\underline{y}} + z\hat{\underline{z}}$.

At fixed time, the surfaces of constant phase

$\underline{k} \cdot \underline{r} = d$, d a constant.

- the equation for a plane with normal $\underline{k}$, or unit normal $\hat{\underline{k}}$.



- one name for these type of solution is this "plane wave solution".

In fact:

$\theta = \underline{k} \cdot \hat{\underline{r}} - \omega t$.

A particular plane if constant of constant phase has:

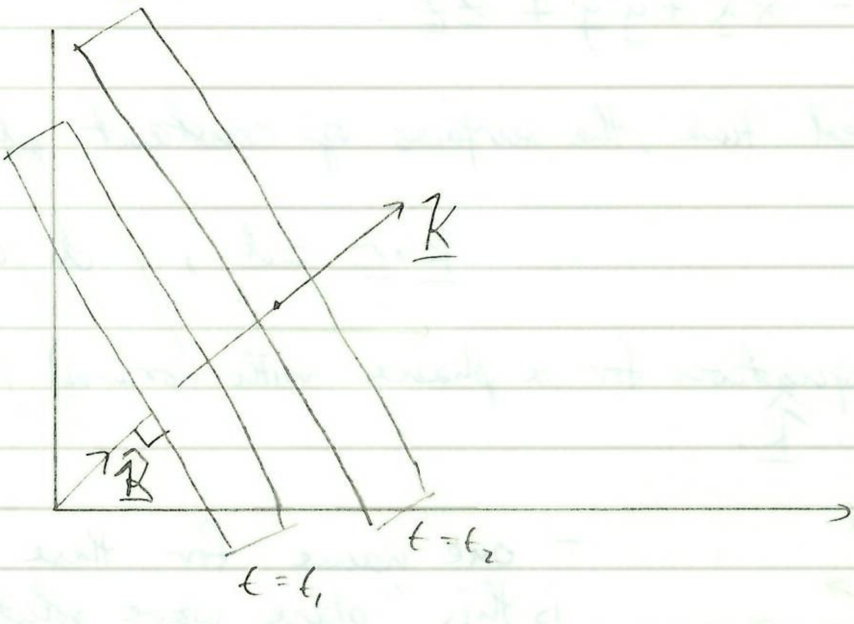
$$\underline{k} \cdot \underline{r} - \omega t = C_1$$

$$\text{or} \quad \underline{k} \cdot \underline{r} = C_1 + \omega t.$$

This plane is a distance,

$$\frac{C_1}{|\underline{k}|} + \frac{\omega}{|\underline{k}|} t.$$

form the orgin.



i.e the plane moves along the vector $\underline{k}$ with speed

$$C_p = \frac{\omega}{|\hat{\underline{k}}|} = \frac{\omega}{\sqrt{k^2 + \ell^2 + m^2}}$$

- the phase speed.

The phase θ increase by 2π between two consective planes with the same wave phase. The distance between these planes at any instant, is the wavelength of the wave, λ.

At some time t,

$$\underline{k} \cdot \underline{r} - \omega t = C_1 \quad \text{on first plane.}$$

$$\underline{k} \cdot \underline{r} - \omega t = C_1 + 2\pi \quad \text{on next plane.}$$

The difference in distance $\dfrac{2\pi}{|\underline{k}|} = \lambda$.

i.e $\lambda = 2\pi / |\underline{\hat{k}}| = 2\pi / \sqrt{k^2 + l^2 + m^2}$

- the generalisation to 3D of our 1D $(l=0, m=0)$ result.

Notice

2011 2c) $\quad C_p = \lambda / \tau = \dfrac{\omega}{|\underline{k}|} \quad (3D)$

as before.

1D: $\quad c_p = \dfrac{\lambda}{\tau} = \dfrac{\omega}{k}.$

$$(kx - \omega t) = \underline{k} \left(x - \frac{\omega}{k} t \right).$$

Now apply this to the Kelvin-Gordon eqn in an unbounded domain.

$$\eta_{tt} + f^2\eta = c^2\nabla^2\eta. \qquad (\eta = h)$$

Look for plane wave soln's:

$$\eta = Re\{Ae^{i\theta}\}, \quad \theta = kx + ly - \omega t.$$

$$(-\omega^2 + f^2)\cancel{Ae^{i\theta}} = c^2(-k^2 - l^2)\cancel{Ae^{i\theta}}$$

$$Ae^{i\theta} \not\equiv 0.$$

Then we get

in 2D

$$\omega^2 = f^2 + c^2(k^2 + l^2) \quad \text{-dispersion relation-}$$

$$= f^2 + c^2K^2 \quad \text{- frequency - wavenumber relation}$$

Isotropic.

[Not rotating, $f \equiv 0$, $\omega^2 = c^2K^2$

i.e $\omega = \pm cK$

in 1D, $\omega = \pm ck$.

- all waves have $c_p = \omega/k = c = \sqrt{gH}$
- non dispersive (all have same c_p).

- Long waves because we may assume SW limit.]

$$\delta = \frac{H}{\lambda} \longrightarrow 0$$

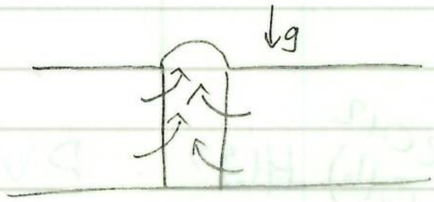
$$\lambda \to \left(\frac{D}{L}\right)$$

- the frequency depends on the magnitute of the wavenumber but NOT its direction.

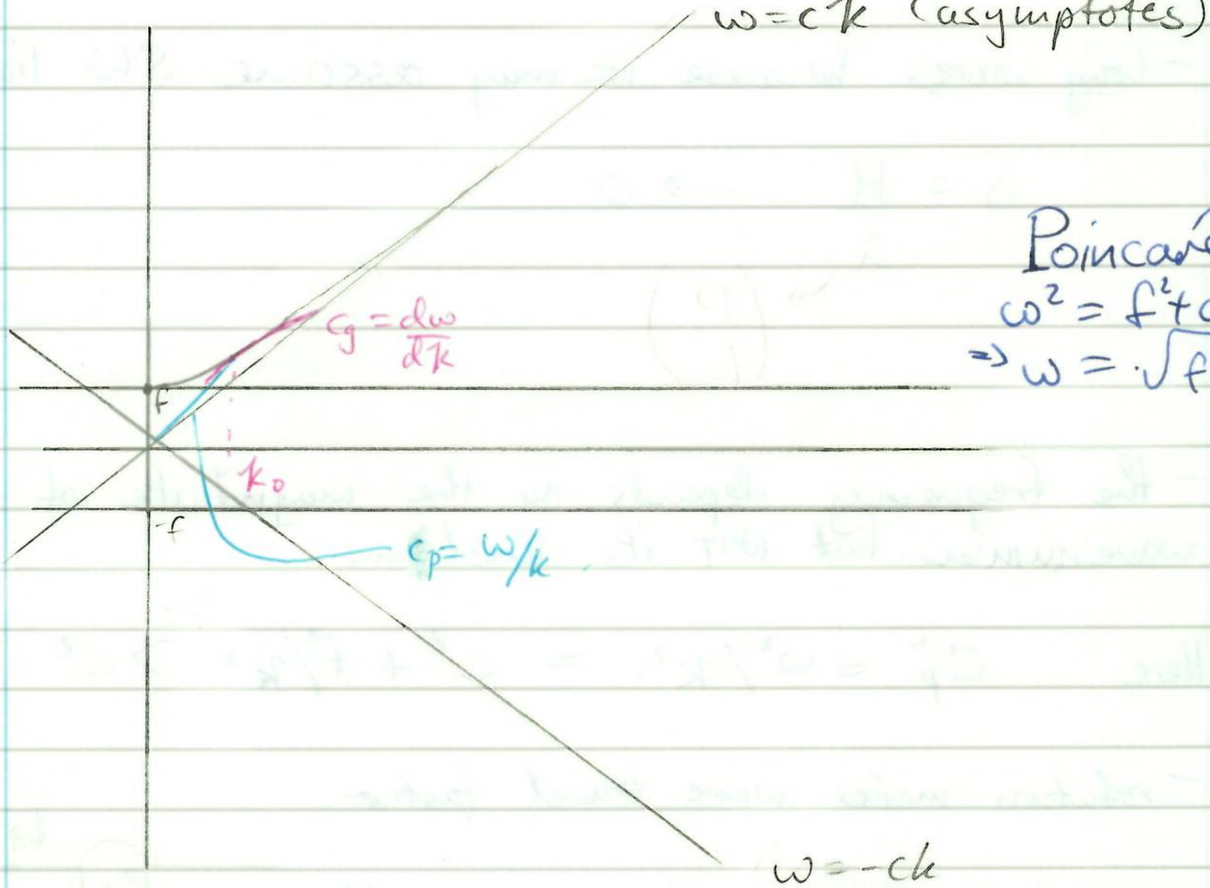
Here $\quad C_p^2 = \omega^2/k^2 = c^2 + f^2/k^2 > c^2$.

- rotation makes wave travel faster.

- ratation stiffens the surface, adds to restoring force - faster wave.



- notice long waves $k \ll 1$ that are most affected.
$\lambda \gg 1$



Poincaré wave

$\omega^2 = f^2 + c^2k^2$

$\Rightarrow \omega = \sqrt{f^2 + c^2k^2}$

2012 (Q1b) HW2 : PV derivation.

HW3 : Waves in circular domain - f = 0, f ≠ 0

drum :

HW4 : Consider two waves :

$$\eta_1 = A\cos[(k+\delta)x - (\omega+\varepsilon)t]$$

$$\eta_2 = A\cos[(k+\delta)x - (\omega-\varepsilon)t]$$

where $\delta/k \ll 1$, $\varepsilon/\omega \ll 1$

Sketch the compound wave $\eta = \eta_1 + \eta_2$

Show that an individual crest in your skech travels at the phase speed.

$$c_p = \omega/k.$$

but the envelope travel at speed

2011 2c)

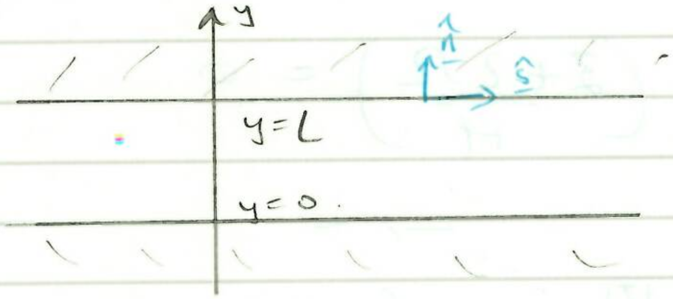
$$c_g = \frac{d\omega}{dk} = F'(k)$$

when the dispersive relation is $\omega = F(k)$

[Advance revison - method of steepest desent or stationary phase]

—/—

Poincaré waves in a channel.



$\hat{n} = \hat{y}$

$\frac{\partial}{\partial n} = \frac{\partial}{\partial y}$

$\hat{s} = \hat{x}$

$\frac{\partial}{\partial s} = \frac{\partial}{\partial x}$

—/—

2012 1b) 2010 d) Hw Ans for 2:

$$\frac{D\underline{u}}{Dt} + 2\underline{\Omega} \wedge \underline{u} = -g\underline{\nabla}h$$

Curl: $\frac{D\underline{u}}{Dt} = \frac{\partial \underline{u}}{\partial t} + (\underline{u}\cdot\underline{\nabla})\underline{u} = \frac{\partial \underline{u}}{\partial t} + \underline{\nabla}(\frac{1}{2}\underline{u}^2) + \underline{\omega} \wedge \underline{u}$.

so (Note $\underline{\omega} = \underline{\nabla} \wedge \underline{u}$)

$$\frac{\partial \underline{\omega}}{\partial t} + \underline{\nabla} \wedge (\underline{\omega} \wedge \underline{u}) = \frac{\partial \underline{\omega}}{\partial t} + (\underline{u} \cdot \underline{\nabla}) \omega + (\omega \cdot \underline{\nabla}) \underline{u}.$$

Thus:

$$\frac{D\underline{\omega}}{Dt} + (\underline{\omega} + 2\underline{\Omega}) \underline{\nabla} \cdot \underline{u} = 0.$$

$\underline{\omega} = \zeta \hat{\underline{z}}$ | ie $\frac{D\zeta}{Dt} + (\zeta + 2\Omega) \underline{\nabla} \cdot \underline{u} = 0.$

$\underline{\Omega} = \Omega \hat{\underline{z}}$

$$\frac{1}{\zeta + 2\Omega} \frac{D}{Dt} (\zeta + 2\Omega) = -\underline{\nabla} \cdot \underline{u} = \frac{1}{H} \frac{DH}{Dt}$$

$$\frac{D}{Dt}(\zeta + 2\Omega) \cdot H - \frac{DH}{Dt} \cdot (\zeta + 2\Omega) = 0$$

$$\text{i.e } \frac{D}{Dt} \left(\frac{\zeta + 2\Omega}{H} \right) = 0$$

—/—.

2012 Q2b Governing eqn: Klein - Gordon.

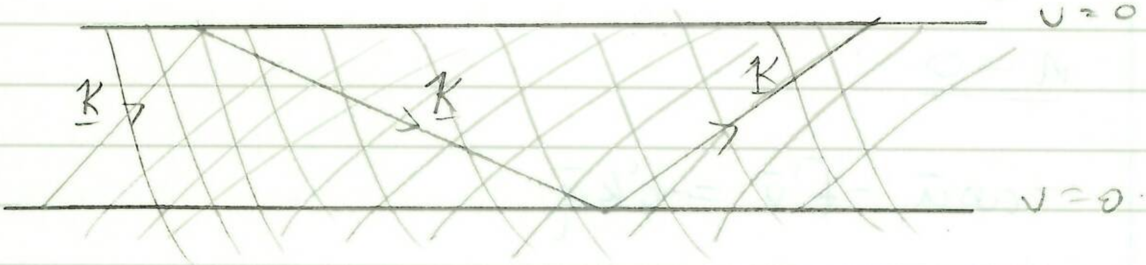
$$\eta_{tt} + f^2 \eta = c^2 \nabla^2 \eta$$

Boundary condition

$$\frac{\partial^2 \eta}{\partial t \partial n} = f \frac{\partial \eta}{\partial s}$$

i.e sum of two PWs of wavenumber $\underline{K} = k\hat{\underline{x}} \pm \alpha\hat{\underline{y}}$

- waves makes $\pm$ the same angle with x-axis, slopes $\pm k/\alpha$



Consider the $n=0$ solution then $\alpha = 0$, $\alpha = n\pi/L$. so

$$\bar{\eta} = \cancel{A\sin\alpha y} + B\cancel{\cos\alpha y}$$
$$= \text{constant}.$$

Thus

$$\eta = \cos(kx - \omega t).$$

u, v derived from η so u, v must have same form.

Write
$$\left.\begin{aligned} u &= \bar{u}(y)\, e^{i(kx-\omega t)} \\ v &= \bar{v}(y)\, e^{i(kx-\omega t)} \\ \eta &= \bar{\eta}(y)\, e^{i(kx-\omega t)} \end{aligned}\right\}$$ Taking the real parts.

$$u_t - fv = -g\eta_x$$

$$v_t + fu = -g\eta_y$$

$$\eta_t + H(u_x + v_y) = 0.$$

$\underline{n=0}$:

$$-i\omega\bar{u} - f\bar{v} = -ik\bar{\eta}$$

$$-i\omega\bar{v} + f\bar{u} = 0$$

$$-i\omega\bar{\eta} + H(ik\bar{u} + \bar{v}') = 0.$$

Thus $f\bar{u} = i\omega\bar{v}$.

29/1/13

E.E

$\downarrow$ + rotating

rE.E

$\downarrow$ $\delta = D/L \ll 1$

rSWE ← cannot solve in general. *

↖ cannot solve easily numerically * (Fast waves + slow evolution − Stiff problem −

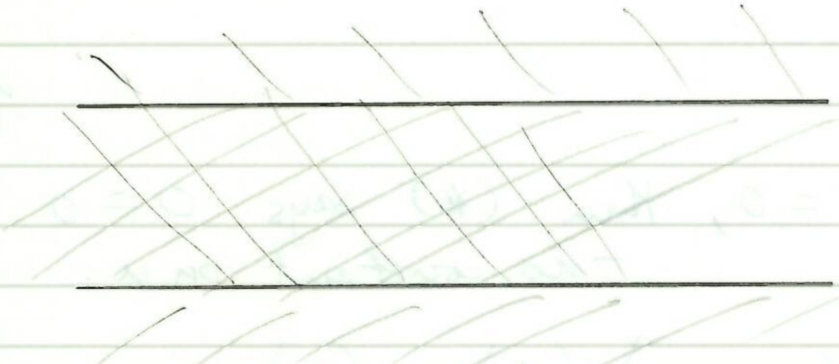
? $\downarrow$ Non geo − cannot use to predict climate (even weather really)

geo

$|\omega| < f$ i.e period less than a day ~ hours (PW)

Finish: Theory for ocean current + set of equs can deal with numerically (Adjust Geostroptic).

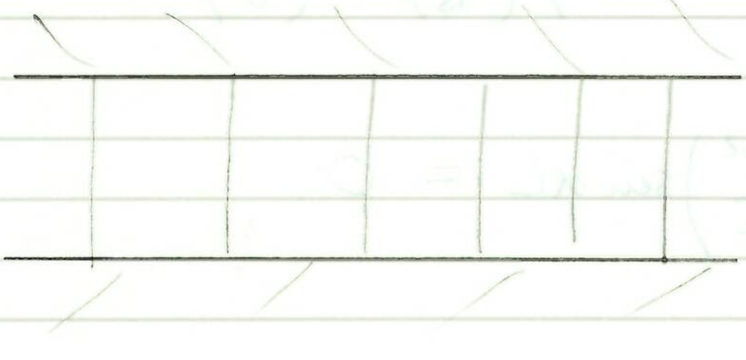
$2\underline{\Omega} \wedge \underline{u} = -\frac{1}{\rho} \nabla p$



v = 0

v = 0.

PWs



f = 0

2007
1)

<u>$n=0$ mode</u>

$$\Rightarrow \frac{\partial \eta}{\partial y} = 0$$

$$(\partial_{tt}^2 + f)v = -g\eta_y + f_g \eta_x$$

$\Rightarrow$ v indep of y.

But $v=0$ on $y=0, L \Rightarrow v=0$.

But we have

$$v_t + fu = \cancel{-g\eta_y} \overset{\to 0}{=} 0 .$$

But $v \equiv 0$, so $v_t \equiv 0$ so $u \equiv 0$ i.e nothing happens

i.e $n=0$ is not a non-trivial solution it is $u \equiv 0$,
$v \equiv 0$
$\eta \equiv 0$.

Notice if $f=0$, then $(*)$ says $0=0$
- no constant on u.

Recall : $\begin{pmatrix} & \end{pmatrix}\begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$$\left(\alpha^2 + \frac{k^2 f^2}{\omega^2}\right) \sin \alpha L = 0.$$

Here on $y = 0, L$.

$$\frac{\partial^2 \eta}{\partial y \partial t} = f \frac{\partial \eta}{\partial x} \qquad (v = 0).$$

Look for solutions:

$$\eta(x,y,t) = \mathrm{Re}\left\{ \bar{\eta}(y) e^{i(kx - \omega t)} \right\}.$$

$$-\omega^2 \bar{\eta} + f^2 \bar{\eta} = c^2 \left(-k^2 \bar{\eta} + \bar{\eta}'' \right)$$

i.e. $\bar{\eta}'' + \left(\dfrac{\omega^2 - f^2}{c^2} - k^2 \right) \bar{\eta} = 0.$

Write

$$\alpha^2 = \frac{\omega^2 - f^2}{c^2} - k^2.$$

then $\bar{\eta}'' + \alpha^2 \bar{\eta} = 0.$ (SHM)

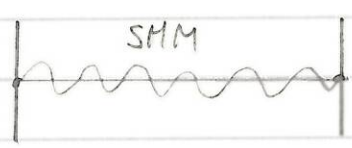
subject to:

$-\cancel{i}\omega \bar{\eta}' = f . \cancel{i} k \bar{\eta} \qquad y = 0, L. \rightarrow \left.\dfrac{\partial^2 \eta}{\partial y \partial t}\right|_{\substack{v=0 \\ y=0,L}} = f \left.\dfrac{\partial \eta}{\partial x}\right|_{\substack{v=0 \\ y=0,L}}$

i.e. $\omega \bar{\eta}' + k f \bar{\eta} = 0$ on $y = 0, L$.

drum 

$\sin\left(\dfrac{n\pi x}{L}\right)$

$\eta_x = 0$
under $f = 0$  $\eta_x = 0$ $\cos(n + \frac{1}{2}\pi)\dfrac{x}{L}$

Here try:

$\bar{\eta}(y) = A\sin\alpha y + B\cos\alpha y$. — solution to $\bar{\eta}'' + \alpha^2\bar{\eta} = 0$

$\bar{\eta}' = A\alpha\cos\alpha y - B\alpha\sin\alpha y$.

On $y=0$,

$$\omega A\alpha + kfB = 0.$$

On $y = L$.

- - - -

Thus:

$$\begin{array}{c} y=L \\ \\ y=0 \end{array} \begin{pmatrix} \alpha\cos\alpha L + \frac{kf}{\omega}\sin\alpha L & -\alpha\sin\alpha L + {kf}/{\omega}\cos\alpha L \\ \alpha & kf/\omega \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\underline{\underline{C}} \begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$\underline{\underline{C}}$ must <u>not</u> have an inverse (otherwise $A=B=0$)

i.e $\det \underline{\underline{C}} = 0$

i.e $\left(\alpha^2 + \dfrac{f^2k^2}{\omega^2}\right)\sin\alpha L = 0$.

2012 Q2(b)

This either:

$$\alpha^2 + \frac{f^2 k^2}{\omega^2} = 0.$$

or $\sin \alpha L = 0$.

—/—

If $\sin \alpha L = 0$.
then $\alpha L = n\pi$, $n = 0, 1, 2, 3,$
i.e $\alpha = \frac{n\pi}{L}$, $n = 0, 1, 2, 3, \ldots$

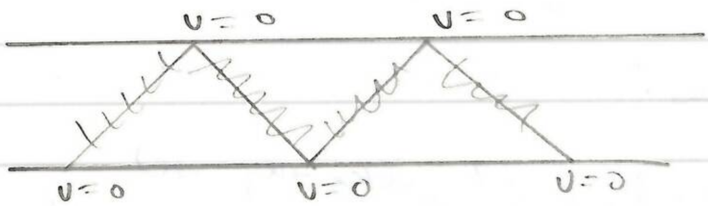
Reminder $\alpha^2 = \frac{\omega^2 - f^2}{c^2} - k^2$

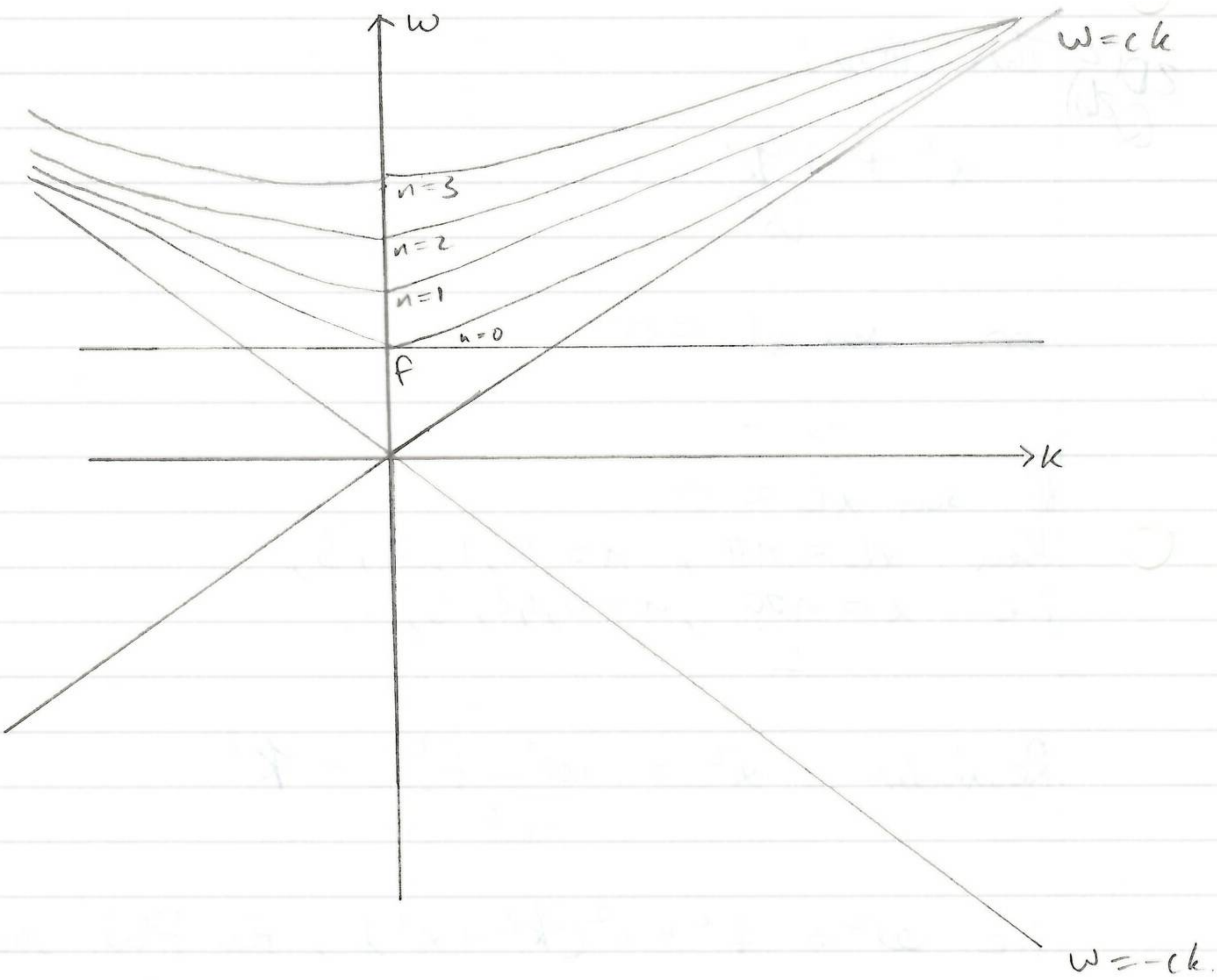
i.e $\omega^2 = f^2 + c^2(k^2 + \alpha^2)$, the P.W. dispersion relation.

- these are PWs.

with $\alpha = \frac{n\pi}{L}$, $n = 0, 1, 2, 3, 4, \ldots$

i.e the y-number, l, is quantised. ($l = \alpha$)





Now we have

$$\eta = Re\left\{(A\cos\alpha y + B\sin\alpha y)e^{i(kx-\omega t)}\right\}$$

$$\cos\alpha y = (e^{i\alpha y} + e^{-i\alpha y})/2$$

$$\sin\alpha y = (e^{i\alpha y} - e^{-i\alpha y})/2.$$

Thus:

$$\eta = Re\left\{C_1 e^{i(kx+\alpha y-\omega t)} + C_2 e^{i(kx-\alpha y-\omega t)}\right\}$$

$A \sin \alpha y + B \cos \alpha y$

$A^* \sinh \bar{\alpha} y + B^* \cosh \bar{\alpha} y.$

$$\bar{\eta}(y) = A^{**} e^{\bar{\alpha} y} + B^{**} e^{-\bar{\alpha} y}$$

$$\bar{\alpha} = i\alpha = \frac{kf}{\omega}.$$

$$\bar{\eta}(y) = A^{*} e^{-\frac{kf}{\omega} y} + B^{**} e^{-\frac{kf}{\omega} y}$$

See handout on the Kelvin wave.

If $\omega = ck$

$$\eta = A e^{-y/a} \cos\left[k(x - ct)\right] \qquad (7)$$

$a = \frac{c}{f}$, $c = \sqrt{gH_0} > 0$

If $\omega = -ck$

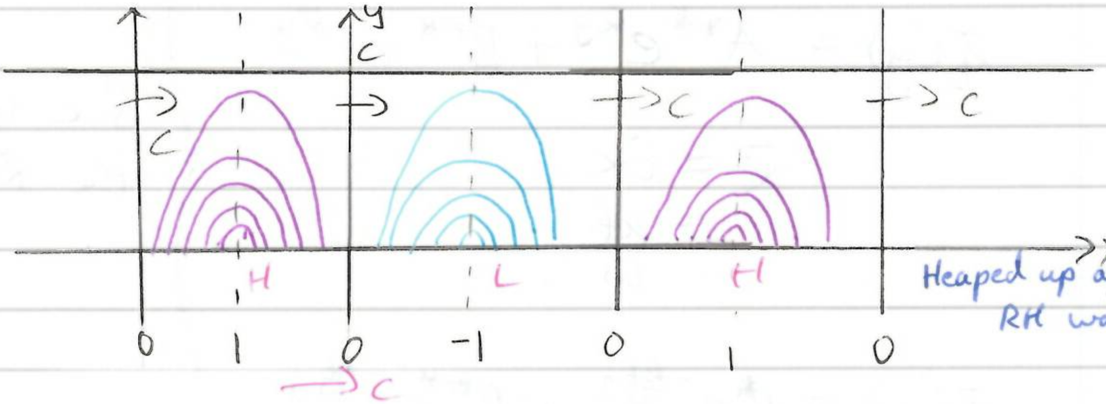
$$\eta = A e^{+y/a} \cos\left[k(x + ct)\right] \qquad (10)$$

These are the $n=0$ mode.
- Kelvin waves.

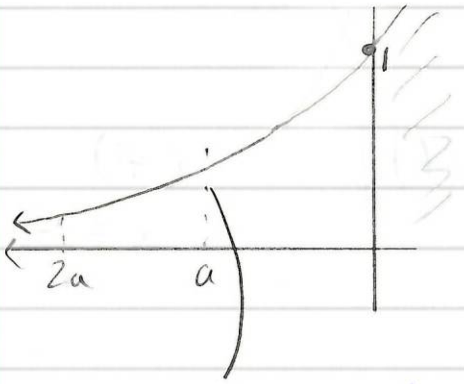
Note if $f \to 0$, $a \to \infty$, $\eta \Rightarrow \begin{cases} A\cos\left[k(x-ct)\right] \\ A\cos\left[k(x+ct)\right] \end{cases}$

precisely the y-indep non-rotating result.

$f > 0$: Contours of surface elevation:
- isobars.

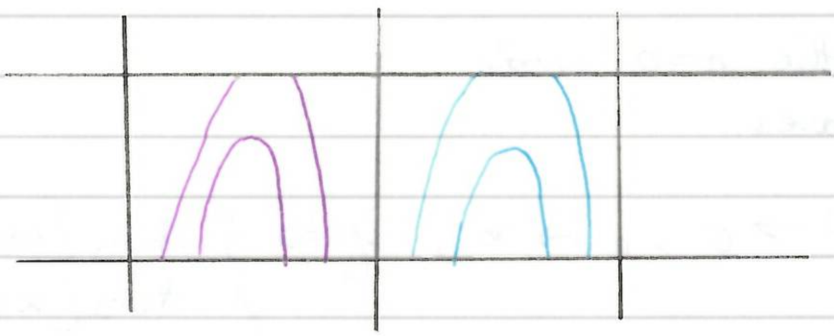


Heaped up against RH wall.

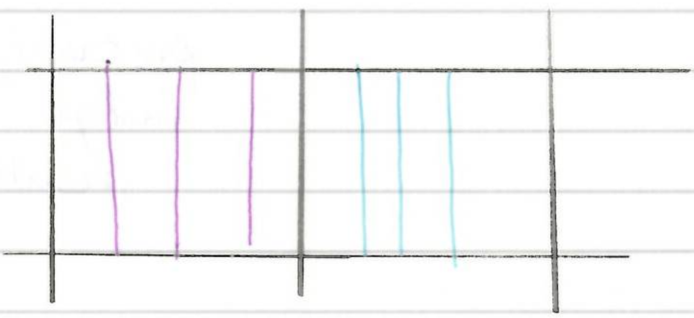


- nodal line are precisely those of non-rotating NW + move at same speed, c.

e-folding scale is a
- Rossby radius.



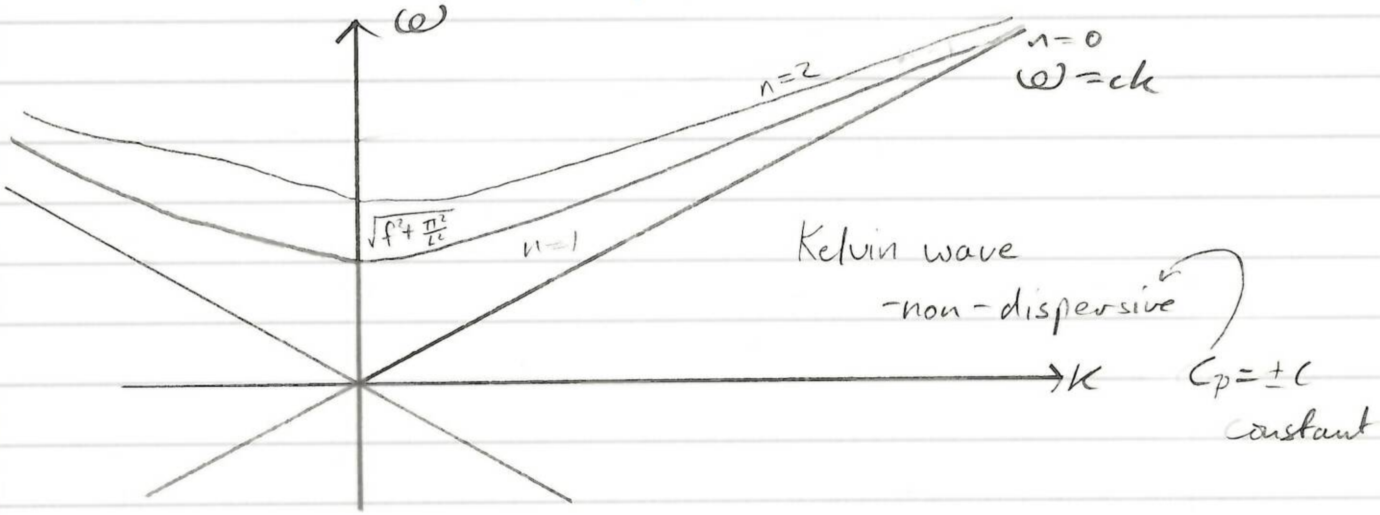
smaller f



$a = \infty, \ f = 0$

$$\frac{\partial}{\partial y} \equiv 0.$$

- Kelvin Tidal Analyser - Science Museum -

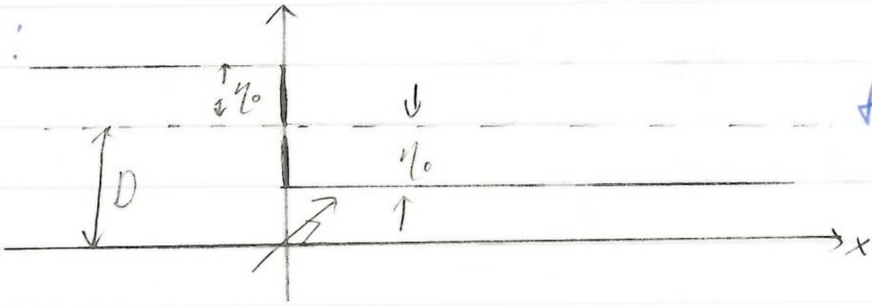


1/2/13

Geostrophic
how do we get there?

Rossyby - Grill Adjustment

HW:



Find solution for flow ($f=0$) when board is removed.

We use the lr SWE

$$u_t - fv = -g\eta_x$$

$$v_t + fu = -g\eta_y \qquad \eta/D << 1$$

$$\eta_t + D(u_x + v_y) = 0.$$

These give the KG eqns.

$$(\partial_{tt} + f^2)\eta = c^2 \nabla^2 \eta$$

The intial conditions are

$$\eta(x,0) = -\eta_0 \operatorname{sgn} x$$

Relase from rest $\frac{\partial \eta}{\partial t}(x,0) = 0.$

Look for solution indep of y.

i.e $\quad \eta = \eta(x,t) \quad , \quad c = \sqrt{gD}$.

Then $\quad (\partial_{tt} + f^2)\eta = c^2 \eta_{xx}$

when $f=0$, $\quad \eta_{tt} = c^2 \eta_{xx}$

1-D wave eqn.

— / —

HW: Solve $\quad \eta_{tt} = c^2 \eta_{xx}$

Subject to $\quad \eta(x,0) = -\eta_0 \operatorname{sgn} x$

$$\frac{\partial \eta}{\partial t}(x,0) = 0$$

— / —

Use a Fourier integral representation of η

$$\eta(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{\eta}(k,t)\, e^{ikx}\, dk.$$

Then $\quad \eta_x = \frac{1}{2\pi}\, ik \int_{-\infty}^{\infty} \hat{\eta}(k,t)\, e^{ikx}\, dk.$

$$\eta_t = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{d\hat{\eta}}{dt}(k,t)\, e^{ikx}\, dk.$$

Put into 1D KG $\quad (\partial_{tt} + f^2)\eta - c^2 \eta_{xx} = 0.$

This satisfies the KG provided

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\left(\frac{d^2\hat{\eta}}{dt^2}+f^2\hat{\eta}+c^2k^2\hat{\eta}\right)e^{ikx}\,dk=0.$$

This is true provided

$$\frac{d^2\hat{\eta}}{dt^2}+(f^2+c^2k^2)\hat{\eta}=0.$$

This solution will satisfies the b.c's provided

$$\frac{d\hat{\eta}}{dt}(k,0)=0.$$

and $-\eta_0\operatorname{sgn}x=\frac{1}{2\pi}\int_{-\infty}^{\infty}\hat{\eta}(k,0)e^{ikx}\,dk.$

Using the F.I.T

$$\hat{\eta}(k,0)=\int_{-\infty}^{\infty}(-\eta_0\operatorname{sgn}x)e^{-ikx}\,dx.$$

$$\left(=2\eta_0 i\int_0^{\infty}\sin kx\,dx\right)$$ [Generalised Function: Lighthill]

$= N(k)$

This is SHM

$$\hat{\eta}=A\sin\omega t+B\cos\omega t$$

where $\omega^2=\sqrt{f^2+c^2k^2}$, $\omega^2=f^2+c^2k^2$ (Poincaré Waves)

Note here $\frac{d\hat{\eta}}{dt}(k,0) = A(k)\omega(k)$

For this to be zero $\forall k$, $A=0$ here $\hat{\eta}(k,0) = B(k)$ so $B(k) = N(k)$.

Thus we have finished

$$\hat{\eta} = N(k)\cos\omega t.$$

i.e $\eta(x,t) = \int_{-\infty}^{\infty} N(k)\cos\omega(k)t\, e^{ikx}\, dk.$

Finished.

– a superposition of PWs (with a singularity sometimes in N).

—/—

2011 2a) <u>Rossby</u>: Do not forget PV is conserved throughout the motion. So if you know the initial PV you know PV for all time and particularly at the end.

$u_t - fv = -g\eta_x$ (1) (Potential vorticity – PV).

$v_t + fu = -g\eta_y$ (2)

$\frac{\partial}{\partial x}(2) - \frac{\partial}{\partial y}(1)$:-

2012

1b) **From the rotating Shallow Water Equations to the Conservation of Potential Vorticity**

The vector form of the shallow water momentum equation is

$$\frac{\mathrm{D}\boldsymbol{u}}{\mathrm{D}t} + 2\boldsymbol{\Omega} \times \boldsymbol{u} = -g\nabla h, \tag{1}$$

where $\boldsymbol{\Omega} = f\hat{\boldsymbol{z}}$ is the constant rotation rate of the frame about the vertical and

$$\frac{\mathrm{D}\boldsymbol{u}}{\mathrm{D}t} = \frac{\partial \boldsymbol{u}}{\partial t} + (\boldsymbol{u}.\nabla)\boldsymbol{u} \tag{2}$$

$$= \frac{\partial \boldsymbol{u}}{\partial t} + \nabla(\boldsymbol{u}^2/2) + \boldsymbol{\omega} \times \boldsymbol{u}, \tag{3}$$

is the usual advective derivative and $\boldsymbol{\omega} = \nabla \times \boldsymbol{u}$ is the vorticity. Since $\boldsymbol{u}$ is horizontal and independent of z,

$$\boldsymbol{\omega} = \nabla \times \boldsymbol{u} = \zeta\hat{\boldsymbol{z}}. \tag{4}$$

Thus (1) is

$$\frac{\partial \boldsymbol{u}}{\partial t} + (\boldsymbol{\omega} + 2\boldsymbol{\Omega}) \times \boldsymbol{u} = -\nabla(\boldsymbol{u}^2/2 + gh). \tag{5}$$

Then, since the curl of a gradient vanishes identically, taking the curl of (5) gives

$$\frac{\partial \boldsymbol{\omega}}{\partial t} + \nabla \times [(\boldsymbol{\omega} + 2\boldsymbol{\Omega}) \times \boldsymbol{u}] = 0. \tag{6}$$

Now use the identity

$$\nabla \times (\boldsymbol{A} \times \boldsymbol{B}) = \boldsymbol{A}(\nabla.\boldsymbol{B}) - \boldsymbol{B}(\nabla.\boldsymbol{A}) + (\boldsymbol{B}.\nabla)\boldsymbol{A} - (\boldsymbol{A}.\nabla)\boldsymbol{B}, \tag{7}$$

with $\boldsymbol{A} = \boldsymbol{\omega} + 2\boldsymbol{\Omega} = (\zeta + f)\hat{\boldsymbol{z}}$ and $\boldsymbol{B} = \boldsymbol{u}$. First note that

$$\nabla.(\boldsymbol{\omega} + 2\boldsymbol{\Omega}) = \nabla.\boldsymbol{\omega} = 0, \tag{8}$$

since $\boldsymbol{\Omega}$ is constant and curl of a divergence is zero. Then note that

$$[(\boldsymbol{\omega} + 2\boldsymbol{\Omega}).\nabla]\boldsymbol{u} = (\zeta + f)(\hat{\boldsymbol{z}}.\nabla)\boldsymbol{u} = (\zeta + f)\frac{\partial \boldsymbol{u}}{\partial z} = 0, \tag{9}$$

since $\boldsymbol{u}$ is independent of z. Thus (6) becomes

$$\frac{\partial \omega}{\partial t} + (\boldsymbol{u}.\nabla)(\omega + 2\Omega) + (\omega + 2\Omega)\nabla.\boldsymbol{u} = 0. \tag{10}$$

which in fact has only the z component

$$\frac{\partial \zeta}{\partial t} + (\boldsymbol{u}.\nabla)(\zeta + f) + (\zeta + f)\nabla.\boldsymbol{u} = 0, \tag{11}$$

i.e.

$$\frac{\mathrm{D}\zeta}{\mathrm{D}t} + (\zeta + f)\nabla.\boldsymbol{u} = 0. \tag{12}$$

Since f is a constant this can be written

$$\frac{\mathrm{D}(\zeta + f)}{\mathrm{D}t} + (\zeta + f)\nabla.\boldsymbol{u} = 0. \tag{13}$$

But the remaining shallow water equation can be written

$$\frac{\mathrm{D}H}{\mathrm{D}t} + H\nabla.\boldsymbol{u} = 0. \tag{14}$$

Taking H times (13) and subtracting $(\zeta + f)$ times (14) gives

$$H\frac{\mathrm{D}(\zeta + f)}{\mathrm{D}t} - (\zeta + f)\frac{\mathrm{D}H}{\mathrm{D}t} = 0, \tag{15}$$

i.e.

$$\frac{\mathrm{D}}{\mathrm{D}t}\left(\frac{\zeta + f}{H}\right) = 0. \tag{16}$$

$$v_{xt} - u_{yt} + f u_x + f v_y = 0.$$

$$\zeta_t + f \underline{\nabla} \cdot \underline{u} = 0.$$

$$\zeta = v_x - u_y.$$

Also $\eta_t + D(u_x + v_y) = 0.$

$$\eta_t = -D \underline{\nabla} \cdot \underline{u}.$$

Thus $\zeta_t - \frac{f}{D} \eta_t = 0.$

i.e $\frac{\partial}{\partial t}\left[\zeta - f \, \eta/D\right] = 0.$

– linearised conservation PV.

—/—

In fact this follows from

$$\frac{Dq}{Dt} = 0.$$

$$q = \frac{\zeta + f}{H}$$

Write

$$H = D + \eta \qquad \eta/D \ll 1$$

and neglect quadratic terms.

—/—

Thus $\zeta - \frac{f\eta}{D} = \left(\zeta - \frac{f\eta}{D}\right)\Big|_{t=0}$

$= \frac{f}{D}\eta_0 \operatorname{sgn} x. \quad \forall t.$

2011 2(b) Suppose the flow eventually becomes steady. i.e $\partial/\partial t \equiv 0$ as $t \to \infty$

- the flow becomes geostropic with

$$\zeta = v_x - u_y = \frac{g}{f}(\eta_{xx} + \eta_{yy})$$

i.e. $\zeta = \frac{g}{f}\nabla^2\eta \qquad (t \to \infty)$

$= \frac{g}{f}\eta_{xx}$ in our y-indp problem.

Thus we have the fluid state:

$$\frac{g}{f}\eta_{xx} - \frac{f}{D}\eta = \frac{f}{D}\eta_0 \operatorname{sgn} x$$

i.e $\eta_{xx} - \frac{f^2}{gD}\eta = \frac{f^2}{gD}\eta_0 \operatorname{sgn} x$

i.e $\eta_{xx} - \frac{f^2}{c^2}\eta = \frac{f^2}{c^2}\eta_0 \operatorname{sgn} x. \quad (c^2 = gD)$

But $c/f = a$, the Rossby radius (from KWs)

i.e $\eta_{xx} - \frac{1}{a^2}\eta = +\frac{1}{a^2}\eta_0 \operatorname{sgn} x$.

- final state

RHS is an odd function of x. Thus take η to be odd in x i.e for cty on η, $\eta(0) = 0$.

In $x > 0$,

$$\eta_{xx} - \frac{1}{a^2}\eta = \frac{1}{a^2}\eta_0$$

PI soln:

$$\eta = -\eta_0$$

Complemtry Fun.

$$\eta = A_1 e^{+\frac{1}{a}x} + B_1 e^{-\frac{1}{a}x}$$

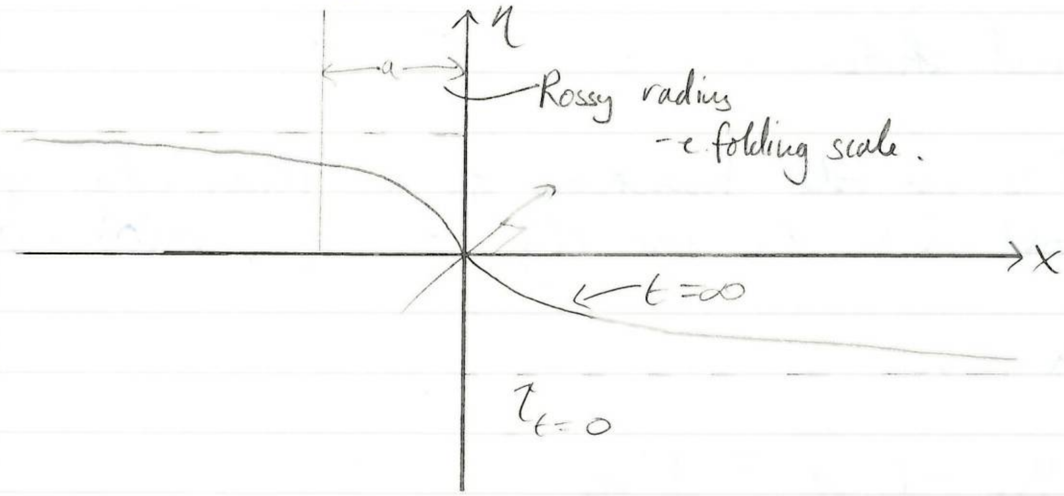
Bdd as $x \longrightarrow \infty$ so $A_1 = 0$.

Thus we have

$$\eta = -\eta_0 + B_1 e^{-x/a}$$

But $\eta(0) = 0$, so $B_1 = \eta_0$

i.e $$\eta = \begin{cases} -\eta_0 \left(1 - e^{-x/a}\right) & x > 0 \\ +\eta_0 \left(1 - e^{+x/a}\right) & x < 0 . \end{cases}$$



$$\eta = -\eta_0 \operatorname{sgn} x \left[1 - e^{-|x|/a}\right]$$

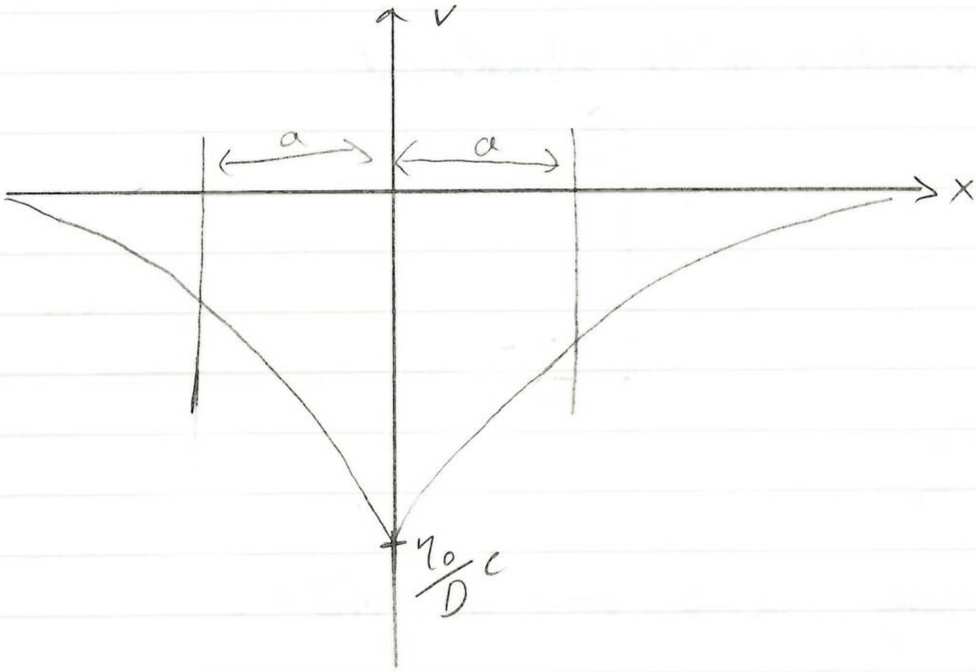
$$u = -\frac{g}{f}\eta_y \equiv 0 \qquad (t = \infty)$$

$$v = \frac{g}{f}\eta_x = \frac{g}{af} \cdot -\eta_0 (\operatorname{sgn} x)^2 e^{-|x|/a}.$$

$$= \frac{gD}{c} \cdot -\frac{\eta_0}{D} e^{-|x|/a}$$

$$= -\left(\frac{\eta_0}{D}\right) c e^{-|x|/a}.$$

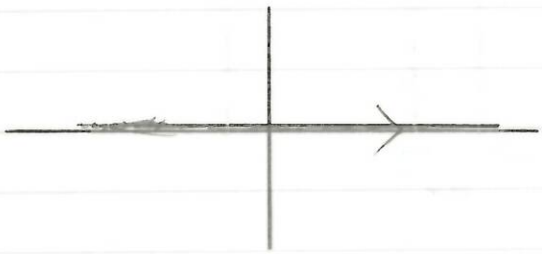
- a small speed compared to longwave speed c
$\eta/D \ll 1$.



Maximum of $x=0$, decays away over distance of Rossby radius. Out of board.

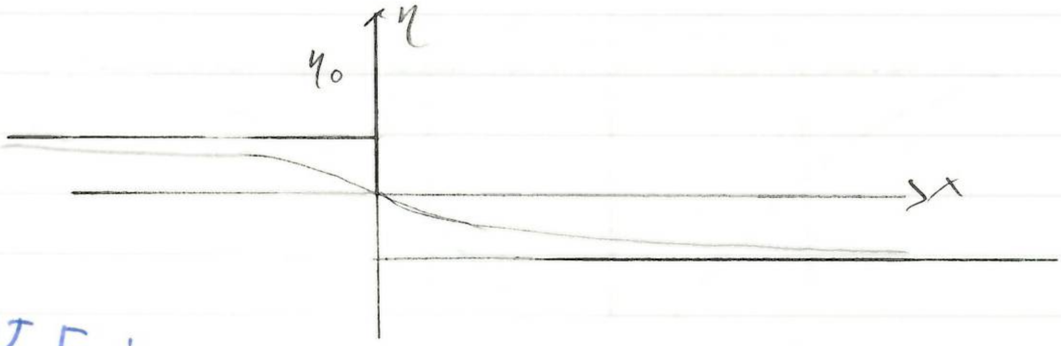
Thus current experiences a Coriolis force to right, i.e along x-axis in the negative $-x$ direction.
- precisely balancing the pressure gradient - force (from H to L $-\nabla p$) along x-axis in the positive $-x$ direction.
- geostrophic balance.

[$f \to 0$, $\frac{c}{f} \to \infty$ i.e $a \to \infty$
$t \to \infty$, $y \to 0$



]

Energetics of the adjustment



K.E :

Initially $u=0, v=0$ so $KE=0$.

Finally $u=0,\ v=-\left(\frac{\eta_0}{D}\right)ce^{-|x|/a}$

$$v^2=\left(\frac{\eta_0}{D}\right)^2c^2e^{-2|x|/a}$$

—/—

Consider a column of width $\delta x \times 1$ in the x-y direction (i.e per unit width in y-direction) and height $D+\eta$ comprised of fluid of density ρ.

the KE of this column is

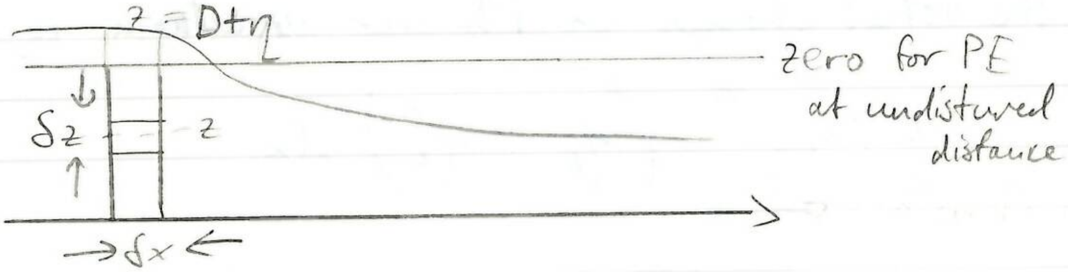
$$\frac{1}{2}\underbrace{\rho_0\,(\underbrace{D.\ \delta x\ 1}_{\text{volume}})}_{\text{mass}}\ \underbrace{\left(\frac{\eta_0}{D}\right)^2 c^2 e^{-2|x|/a}}_{\text{speed}^2}$$

i.e Total K.E of this final state is

$$\frac{1}{2}\rho_0\frac{D\eta_0^2}{D^2}c^2\int_{-\infty}^{\infty} e^{-2|x|/a}$$

$$= \frac{1}{2}\rho_0\frac{\eta_0^2 c^2}{D}a = \frac{1}{2}\rho g a \eta_0^2$$

<u>PE</u>:



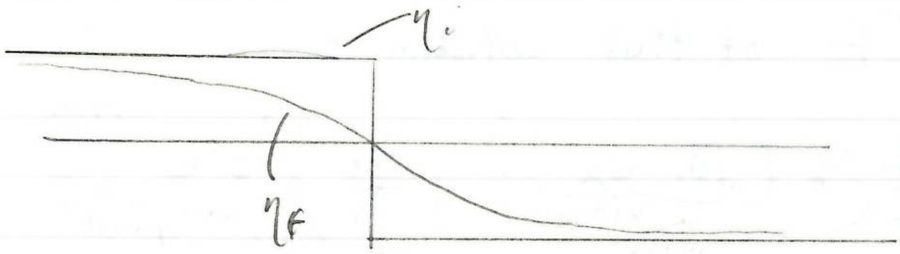
Consider a cube of fluid at height z

Thus has PE:

$$\underbrace{\rho\,(\underbrace{\delta z\,\delta x\ 1}_{\text{volume}})}_{\text{mass}}\ z$$

Thus a column of height η has PE

$$\rho\,\delta x\int_0^{\eta} z\,dz = \frac{1}{2}\rho\eta^2\delta x$$



Change in PE for a column is final PE − initial PE.

$$= \frac{1}{2} g \rho \eta_f^2 \delta x - \frac{1}{2} g \rho \eta_i^2 \delta x$$

$$= \frac{1}{2} \rho g \left(\eta_f^2 - \eta_i^2 \right) \delta x$$

The total change in PE over all column is

$$\frac{1}{2} \rho g \int_{-\infty}^{\infty} \left(\eta_f^2 - \eta_i^2 \right) dx$$

$$= \frac{1}{2} \rho g \eta_0^2 \int_{-\infty}^{\infty} \left[\left(1 - e^{-|x|/a}\right)^2 - 1 \right] dx$$

$$= -\frac{3}{2} g \rho \eta_0^2 a .$$

– given up an amount $3/2 \rho g a \eta_0^2$ during the adjustment

But we have only gained

$$\frac{1}{2} \rho g a \eta_0^2$$

in KE.

Surely energy is conserved?

The remaining PE release must be in the waves, the PW field. ∴ ✓

They do not stay near the origin.

- Remember we have:

$$\eta(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} N(k)\cos\omega t \, e^{ikx} \, dk.$$

Now note that $\cos \omega t = \frac{1}{2}\left(e^{i\omega t} + e^{-i\omega t}\right)$

so $$\eta(x,t) = \frac{1}{4\pi}\int_{-\infty}^{\infty} N(k)\, e^{i(kx-\omega(k)t)} \, dk$$

$$+ \frac{1}{4\pi}\int_{-\infty}^{\infty} N(k)\, e^{i(kx+\omega(k)t)} \, dk.$$

This is an integral part of the form:

2011 2d) $$\eta_1 = \frac{1}{4\pi}\int_{-\infty}^{\infty} N(k)\, e^{i\Phi(k)t} \, dk.$$

where $\Phi(k) = \frac{kx}{t} - \omega(k)$

$$\left[= \frac{kx}{t} - \sqrt{f^2 + c^2k^2} \right]$$

we are interested in this integral as $t \to \infty$.

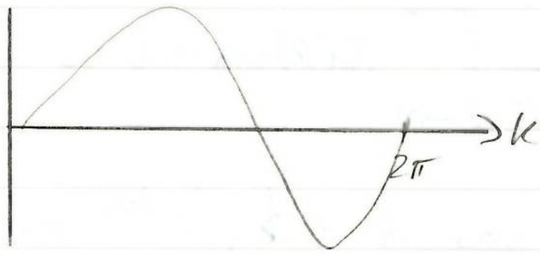
$$= \ldots \qquad + O\left(\tfrac{1}{t}\right) \qquad \underline{t \to \infty}$$

—/—

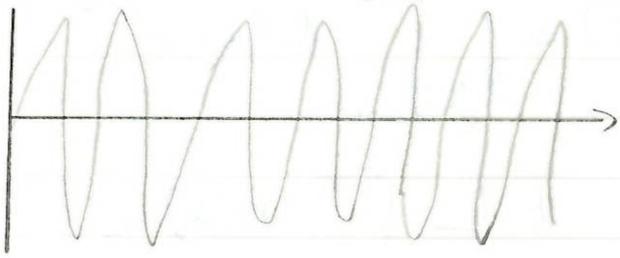
Use method of stationary phase:-

$\sin kt$.

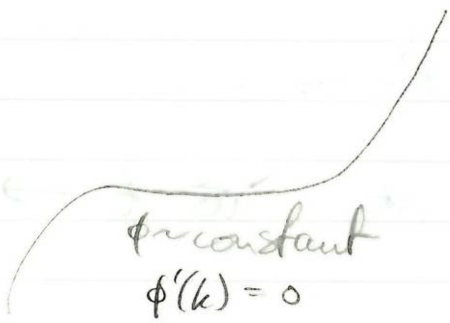
$t=1$

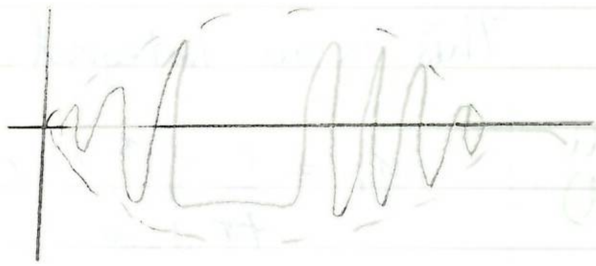


$t=100$



$\sin[\phi(k)t]$ $\qquad\qquad$ $t = 100$.





$$\int f(k) \sin[\phi(k)t] \, dk$$

- maximum contribution where $\phi'(k)=0$
- point of stationary phase

Now $f(s) = 1/s$ and $\phi(s) = -s + \ln s$. Laplace's method applies directly to this transformed integral. The maximum of $\phi(s)$ occurs at $s = 1$ so (6.4.19c) gives

$$\Gamma(x) \sim x^x e^{-x}\sqrt{2\pi/x}, \qquad x \to +\infty, \tag{6.4.39}$$

in agreement with (5.4.1). To obtain the next term in the Stirling series we note that $\phi(1) = -1$, $\phi'(1) = 0$, $\phi''(1) = -1$, $\phi'''(1) = 2$, $(d^4\phi/ds^4)(1) = -6$, $f(1) = 1$, $f'(1) = -1$, $f''(1) = 2$. Substituting these coefficients into the formula (6.4.35), we obtain

$$\Gamma(x) \sim x^x e^{-x}\sqrt{\frac{2\pi}{x}}\left(1 + \frac{1}{12x}\right), \qquad x \to +\infty, \tag{6.4.40}$$

in agreement with (5.4.1).

The distinction between ordinary and movable maxima is examined in Probs. 6.45 to 6.47.

(I) 6.5 METHOD OF STATIONARY PHASE

There is an immediate generalization of the Laplace integrals studied in Sec. 6.4 which we obtain by allowing the function $\phi(t)$ in (6.4.1) to be complex. Note that, if we wish, we may assume that $f(t)$ is real; if it were complex, $f(t)$ could be decomposed into a sum of its real and imaginary parts. However, allowing $\phi(t)$ to be complex poses new and nontrivial problems. In this section we consider the special case in which $\phi(t)$ is pure imaginary: $\phi(t) = i\psi(t)$, where $\psi(t)$ is real. The resulting integral

$$I(x) = \int_a^b f(t)e^{ix\psi(t)}\,dt \tag{6.5.1}$$

with $f(t)$, $\psi(t)$, a, b, x all real is called a generalized Fourier integral. When $\psi(t) = t$, $I(x)$ is an ordinary Fourier integral. The general case in which $\phi(t)$ is complex is considered in Sec. 6.6.

To study the behavior of $I(x)$ in (6.5.1) as $x \to +\infty$, we can use integration by parts to develop an asymptotic expansion in inverse powers of x so long as the boundary terms are finite and the resulting integrals exist.

Example 1 *Asymptotic expansion of a Fourier integral as $x \to +\infty$.* We use integration by parts to find an asymptotic approximation to the Fourier integral

$$I(x) = \int_0^1 \frac{e^{ixt}}{1+t}\,dt.$$

After one integration by parts we obtain

$$I(x) = -\frac{i}{2x}e^{ix} + \frac{i}{x} - \frac{i}{x}\int_0^1 \frac{e^{ixt}}{(1+t)^2}\,dt. \tag{6.5.2}$$

The integral on the right side of (6.5.2) is negligible compared with the boundary terms as $x \to +\infty$; in fact, it vanishes like $1/x^2$ as $x \to +\infty$. To see this, we integrate by parts again:

$$-\frac{i}{x}\int_0^1 \frac{e^{ixt}}{(1+t)^2}\,dt = -\frac{1}{4x^2}e^{ix} + \frac{1}{x^2} - \frac{2}{x^2}\int_0^1 \frac{e^{ixt}}{(1+t)^3}\,dt.$$

The integral on the right is bounded because

$$\left|\int_0^1 \frac{e^{ixt}}{(1+t)^3}\,dt\right| \le \int_0^1 (1+t)^{-3}\,dt = \frac{3}{8}.$$

Since the integral on the right in (6.5.2) does vanish like $1/x^2$ as $x \to +\infty$, $I(x)$ is asymptotic to the boundary terms: $I(x) \sim -(i/2x)e^{ix} + i/x$ $(x \to +\infty)$.

Repeated application of integration by parts gives the complete asymptotic expansion of $I(x)$ as $x \to +\infty$: $I(x) = e^{ix}u(x) + v(x)$ where

$$u(x) \sim -\frac{i}{2x} - \frac{1}{4x^2} + \cdots + \frac{(-i)^n(n-1)!}{(2x)^n} + \cdots, \qquad x \to +\infty,$$

$$v(x) \sim \frac{i}{x} + \frac{1}{x^2} + \cdots - \frac{(-i)^n(n-1)!}{x^n} + \cdots, \qquad x \to +\infty.$$

Example 2 *Integration by parts applied to $\int_0^1 \sqrt{t}\, e^{ixt}\,dt$.* Integration by parts can be used just once for the Fourier integral $I(x) = \int_0^1 \sqrt{t}\, e^{ixt}\,dt$. One integration by parts gives

$$I(x) = -\frac{i}{x}e^{ix} + \frac{i}{2x}\int_0^1 \frac{e^{ixt}}{\sqrt{t}}\,dt. \tag{6.5.3}$$

The integral on the right side of (6.5.3) vanishes more rapidly than the boundary term as $x \to +\infty$. We cannot use integration by parts to verify this because the resulting integral does not exist. (Why?) However, we can use the following simple scaling argument. We let $s = xt$ and obtain

$$\frac{i}{2x}\int_0^1 \frac{e^{ixt}}{\sqrt{t}}\,dt = \frac{i}{2x^{3/2}}\int_0^x \frac{e^{is}}{\sqrt{s}}\,ds \sim \frac{i}{2x^{3/2}}\int_0^\infty \frac{e^{is}}{\sqrt{s}}\,ds, \qquad x \to +\infty.$$

To evaluate the last integral we rotate the contour of integration from the real-s axis to the positive imaginary-s axis in the complex-s plane and obtain

$$\int_0^\infty \frac{e^{is}}{\sqrt{s}}\,ds = \sqrt{\pi}\, e^{i\pi/4}. \tag{6.5.4}$$

(See Prob. 6.49 for the details of this calculation.) Therefore,

$$I(x) + \frac{i}{x}e^{ix} \sim \frac{i}{2x^{3/2}}\sqrt{\pi}\, e^{i\pi/4}, \qquad x \to +\infty. \tag{6.5.5}$$

Clearly, this result cannot be found by direct integration by parts of the integral on the right side of (6.5.3) because a fractional power of x has appeared. However, it is possible to find the full asymptotic expansion of $I(x)$ as $x \to +\infty$ by an indirect application of integration by parts (see Prob. 6.50).

In Example 1 we used integration by parts to argue that the integral on the right side of (6.5.2) vanishes more rapidly than the boundary terms as $x \to +\infty$. In Example 2 we used a scaling argument to show that the integral on the right side of (6.5.3) vanishes more rapidly than the boundary terms as $x \to +\infty$. There is, in fact, a very general result called the Riemann-Lebesgue lemma that guarantees that

$$\int_a^b f(t)e^{ixt}\,dt \to 0, \qquad x \to +\infty, \tag{6.5 6}$$

provided that $\int_a^b |f(t)|\,dt$ exists. This result is valid even when $f(t)$ is not differentiable and integration by parts or scaling do not work. We will cite the Riemann-Lebesgue lemma repeatedly throughout this section; we could have used it to justify neglecting the integrals on the right sides of (6.5.2) and (6.5.3).

We reserve a proof of the Riemann-Lebesgue lemma for Prob. 6.51. Although the proof of (6.5.6) is messy, it is easy to understand the result heuristically. When x becomes large, the integrand $f(t)e^{ixt}$ oscillates rapidly and contributions from adjacent subintervals nearly cancel.

The Riemann-Lebesgue lemma can be extended to cover generalized Fourier integrals of the form (6.5.1). It states that $I(x) \to 0$ as $x \to +\infty$ so long as $|f(t)|$ is integrable, $\psi(t)$ is continuously differentiable for $a \le t \le b$, and $\psi(t)$ is not constant on any subinterval of $a \le t \le b$ (see Prob. 6.52). The lemma implies that $\int_0^{10} t^3 e^{ix\sin^2 t}\,dt \to 0$ $(x \to +\infty)$, but it does not apply to $\int_0^{10} t^3 e^{2ix}\,dt$.

Integration by parts gives the leading asymptotic behavior as $x \to +\infty$ of generalized Fourier integrals of the form (6.5.1), provided that $f(t)/\psi'(t)$ is smooth for $a \le t \le b$ and nonvanishing at one of the endpoints a or b. Explicitly,

$$I(x) = \frac{f(t)}{ix\psi'(t)} e^{ix\psi(t)}\bigg|_{t=a}^{t=b} - \frac{1}{ix}\int_a^b \frac{d}{dt}\frac{f(t)}{\psi'(t)} e^{ix\psi(t)}\,dt.$$

The Riemann-Lebesgue lemma shows that the integral on the right vanishes more rapidly than $1/x$ as $x \to +\infty$. Therefore, $I(x)$ is asymptotic to the boundary terms (assuming that they do not vanish):

$$I(x) \sim \frac{f(t)}{ix\psi'(t)} e^{ix\psi(t)}\bigg|_{t=a}^{t=b}, \qquad x \to +\infty. \tag{6.5.7}$$

Observe that when integration by parts applies, $I(x)$ vanishes like $1/x$ as $x \to +\infty$.

Integration by parts may not work if $\psi'(t) = 0$ for some t in the interval $a \le t \le b$. Such a point is called a *stationary* point of ψ. When there are stationary points in the interval $a \le t \le b$, $I(x)$ must still vanish as $x \to +\infty$ by the Riemann-Lebesgue lemma, but $I(x)$ usually vanishes less rapidly than $1/x$ because the integrand $f(t)e^{ix\psi(t)}$ oscillates less rapidly near a stationary point than it does near a point where $\psi'(t) \ne 0$. Consequently, there is less cancellation between adjacent subintervals near the stationary point.

The method of stationary phase gives the *leading* asymptotic behavior of generalized Fourier integrals having stationary points. This method is very similar to Laplace's method in that the leading contribution to $I(x)$ comes from a small interval of width ε surrounding the stationary points of $\psi(t)$. We will show that if c is a stationary point and if $f(c) \ne 0$, then $I(x)$ goes to zero like $x^{-1/2}$ as $x \to +\infty$ if $\psi''(c) \ne 0$, like $x^{-1/3}$ if $\psi''(c) = 0$ but $\psi'''(c) \ne 0$, and so on; as $\psi(t)$ becomes flatter at $t = c$, $I(x)$ vanishes less rapidly as $x \to +\infty$.

Since any generalized Fourier integral can be written as a sum of integrals in which $\psi'(t)$ vanishes only at an endpoint, we can explain the method of stationary phase for the special integral (6.5.1) in which $\psi'(a) = 0$ and $\psi'(t) \ne 0$ for $a < t \le b$.

We decompose $I(x)$ into two terms:

$$I(x) = \int_a^{a+\varepsilon} f(t)e^{ix\psi(t)}\,dt + \int_{a+\varepsilon}^b f(t)e^{ix\psi(t)}\,dt, \tag{6.5.8}$$

where ε is a small positive number to be chosen later. The second integral on the right side of (6.5.8) vanishes like $1/x$ as $x \to +\infty$ because there are no stationary points in the interval $a + \varepsilon \le t \le b$.

To obtain the leading behavior of the first integral on the right side of (6.5.8), we replace $f(t)$ by $f(a)$ and $\psi(t)$ by $\psi(a) + \psi^{(p)}(a)(t-a)^p/p!$ where $\psi^{(p)}(a) \ne 0$ but $\psi'(a) = \cdots = \psi^{(p-1)}(a) = 0$:

$$I(x) \sim \int_a^{a+\varepsilon} f(a)\exp\left\{ix\left[\psi(a) + \frac{1}{p!}\psi^{(p)}(a)(t-a)^p\right]\right\}dt, \qquad x \to +\infty. \tag{6.5.9}$$

Next, we replace ε by ∞, which introduces error terms that vanish like $1/x$ as $x \to +\infty$ and thus may be disregarded, and let $s = (t-a)$:

$$I(x) \sim f(a)e^{ix\psi(a)}\int_0^\infty \exp\left[\frac{ix}{p!}\psi^{(p)}(a)s^p\right]ds, \qquad x \to +\infty. \tag{6.5.10}$$

To evaluate the integral on the right, we rotate the contour of integration from the real-s axis by an angle $\pi/2p$ if $\psi^{(p)}(a) > 0$ and make the substitution

$$s = e^{i\pi/2p}\left[\frac{p!\,u}{x\psi^{(p)}(a)}\right]^{1/p} \tag{6.5.11a}$$

with u real or rotate the contour by an angle $-\pi/2p$ if $\psi^{(p)}(a) < 0$ and make the substitution

$$s = e^{-i\pi/2p}\left[\frac{p!\,u}{x|\psi^{(p)}(a)|}\right]^{1/p}. \tag{6.5.11b}$$

Thus,

$$I(x) \sim f(a)e^{ix\psi(a) \pm i\pi/2p}\left[\frac{p!}{x|\psi^{(p)}(a)|}\right]^{1/p}\frac{\Gamma(1/p)}{p}, \qquad x \to +\infty, \tag{6.5.12}$$

where we use the factor $e^{i\pi/2p}$ if $\psi^{(p)}(a) > 0$ and the factor $e^{-i\pi/2p}$ if $\psi^{(p)}(a) < 0$.

The formula in (6.5.12) gives the leading behavior of $I(x)$ if $f(a) \ne 0$ but $\psi'(a) = 0$. If $f(a)$ vanishes, it is necessary to decide whether the contribution from the stationary point still dominates the leading behavior. When it does, the behavior is slightly more complicated than (6.5.12) (see Prob. 6.53).

Example 3 *Leading behavior of* $\int_0^{\pi/2} e^{ix\cos t}\,dt$ *as* $x \to +\infty$. The function $\psi(t) = \cos t$ has a stationary point at $t = 0$. Since $\psi''(0) = -1$, (6.5.12) with $p = 2$ gives $I(x) \sim \sqrt{\pi/2x}\, e^{i(x-\pi/4)}$ $(x \to +\infty)$.

Example 4 *Leading behavior of* $\int_0^\infty \cos(xt^2 - t)\,dt$ *as* $x \to +\infty$. To use the method of stationary phase, we write this integral as $\int_0^\infty \cos(xt^2 - t)\,dt = \operatorname{Re}\int_0^\infty e^{i(xt^2 - t)}\,dt$. The function $\psi(t) = t^2$ has a stationary point at $t = 0$. Since $\psi''(0) = 2$, (6.5.12) with $p = 2$ gives $\int_0^\infty \cos(xt^2 - t)\,dt \sim \operatorname{Re}\frac{1}{2}\sqrt{\pi/x}\,e^{i\pi/4} = \frac{1}{2}\sqrt{\pi/2x}$ $(x \to +\infty)$.

Example 5 *Leading behavior of* $J_n(n)$ *as* $n \to \infty$. When n is an integer, the Bessel function $J_n(x)$ has the integral representation

$$J_n(x) = \frac{1}{\pi}\int_0^\pi \cos(x \sin t - nt)\,dt \tag{6.5.13}$$

(see Prob. 6.54). Therefore, $J_n(n) = \operatorname{Re}\int_0^\pi e^{in(\sin t - t)}\,dt/\pi$. The function $\psi(t) = \sin t - t$ has a stationary point at $t = 0$. Since $\psi''(0) = 0$, $\psi'''(0) = -1$, (6.5.12) with $p = 3$ gives

$$\begin{aligned} J_n(n) &\sim \frac{1}{\pi}\operatorname{Re}\left[\frac{1}{3}e^{-i\pi/6}\left(\frac{6}{n}\right)^{1/3}\Gamma\left(\frac{1}{3}\right)\right], \qquad x \to +\infty, \\ &= \frac{1}{\pi}2^{-2/3}3^{-1/6}\Gamma\left(\frac{1}{3}\right)n^{-1/3}, \qquad n \to \infty. \end{aligned} \tag{6.5.14}$$

Observe that because $\psi''(0) = 0$, $J_n(n)$ vanishes less rapidly than $n^{-1/2}$ as $n \to \infty$.

If n is not an integer, (6.5.14) still holds (see Prob. 6.55).

In this section we have obtained only the leading behavior of generalized Fourier integrals. Higher-order approximations can be complicated because nonstationary points may also contribute to the large-x behavior of the integral. Specifically, the second integral on the right in (6.5.8) must be taken into account when computing higher-order terms because the error incurred in neglecting this integral is usually algebraically small. By contrast, recall that the approximation in (6.4.2) for Laplace's method is valid to all orders because the errors are exponentially, rather than algebraically, small. To obtain the higher-order corrections to (6.5.12), one can either use the method of asymptotic matching (see Sec. 7.4) or the method of steepest descents (see Sec. 6.6).

(I) 6.6 METHOD OF STEEPEST DESCENTS

The method of steepest descents is a technique for finding the asymptotic behavior of integrals of the form

$$I(x) = \int_C h(t)e^{x\rho(t)}\,dt \tag{6.6.1}$$

as $x \to +\infty$, where C is an integration contour in the complex-t plane and $h(t)$ and $\rho(t)$ are analytic functions of t. The idea of the method is to use the analyticity of the integrand to justify deforming the contour C to a new contour C' on which $\rho(t)$ has a constant imaginary part. Once this has been done, $I(x)$ may be evaluated asymptotically as $x \to +\infty$ using Laplace's method. To see why, observe that on the contour C' we may write $\rho(t) = \phi(t) + i\psi$, where ψ is a real constant and $\phi(t)$ is a real function. Thus, $I(x)$ in (6.6.1) takes the form

$$I(x) = e^{ix\psi}\int_{C'} h(t)e^{x\phi(t)}\,dt. \tag{6.6.2}$$

Although t is complex, (6.6.2) can be treated by Laplace's method as $x \to +\infty$ because $\phi(t)$ is real.

Our motivation for deforming C into a path C' on which $\operatorname{Im}\rho(t)$ is a constant is to eliminate rapid oscillations of the integrand when x is large. Of course, one could also deform C into a path on which $\operatorname{Re}\rho(t)$ is a constant and then apply the method of stationary phase. However, we have seen that Laplace's method is a much better approximation scheme than the method of stationary phase because the full asymptotic expansion of a generalized Laplace integral is determined by the integrand in an arbitrarily small neighborhood of the point where $\operatorname{Re}\rho(t)$ is a maximum on the contour. By contrast, the full asymptotic expansion of a generalized Fourier integral typically depends on the behavior of the integrand along the entire contour. As a consequence, it is usually easier to obtain the full asymptotic expansion of a generalized Laplace integral than of a generalized Fourier integral.

Before giving a formal exposition of the method of steepest descents, we consider three preliminary examples which illustrate how shifting complex contours can greatly simplify asymptotic analysis. In the first example we consider a Fourier integral whose asymptotic expansion is difficult to find by the methods used in Sec. 6.5. However, deforming the contour reduces the integral to a pair of integrals that are easy to evaluate by Laplace's method.

Example 1 *Conversion of a Fourier integral into a Laplace integral by deforming the contour.* The behavior of the integral

$$I(x) = \int_0^1 \ln t\, e^{ixt}\,dt \tag{6.6.3}$$

as $x \to +\infty$ cannot be found directly by the methods of Sec. 6.5 because there is no stationary point. Also, integration by parts is useless because $\ln 0 = -\infty$. Integration by parts is doomed to fail because, as we will see, the leading asymptotic behavior of $I(x)$ contains the factor $\ln x$ which is not a power of $1/x$.

To approximate $I(x)$ we deform the integration contour C, which runs from 0 to 1 along the real-t axis, to one which consists of three line segments: C_1, which runs up the imaginary-t axis from 0 to iT; C_2, which runs parallel to the real-t axis from iT to $1 + iT$; and C_3, which runs down from $1 + iT$ to 1 along a straight line parallel to the imaginary-t axis (see Fig. 6.5). By Cauchy's theorem, $I(x) = \int_{C_1+C_2+C_3} \ln t\, e^{ixt}\,dt$. Next we let $T \to +\infty$. In this limit the contribution from C_2 approaches 0. (Why?) In the integral along C_1 we set $t = is$, and in the integral along C_3 we set $t = 1 + is$, where s is real in both integrals. This gives

$$I(x) = i\int_0^\infty \ln(is)\,e^{-xs}\,ds - i\int_0^\infty \ln(1 + is)\,e^{ix(1+is)}\,ds. \tag{6.6.4}$$

The sign of the second integral on the right is negative because C_3 is traversed downward.

Observe that both integrals in (6.6.4) are Laplace integrals. The first integral can be done exactly. We substitute $u = xs$ and use $\ln(is) = \ln s + i\pi/2$ and the identity $\int_0^\infty e^{-u}\ln u\,du = -\gamma$, where $\gamma = 0.5772\ldots$ is Euler's constant, and obtain

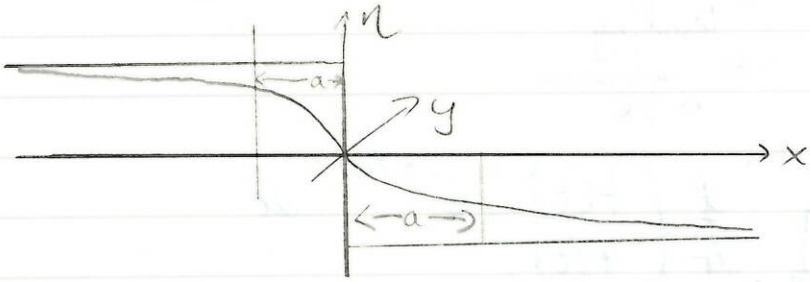
$$i\int_0^\infty \ln(is)\,e^{-xs}\,ds = -i(\ln x)/x - (i\gamma + \pi/2)/x.$$

We apply Watson's lemma to the second integral on the right in (6.6.4) using the Taylor expansion $\ln(1 + is) = -\sum_{n=1}^\infty (-is)^n/n$, and obtain

$$-i\int_0^\infty \ln(1 + is)\,e^{ix(1+is)}\,ds \sim ie^{ix}\sum_{n=1}^\infty \frac{(-i)^n(n-1)!}{x^{n+1}}, \qquad x \to +\infty.$$

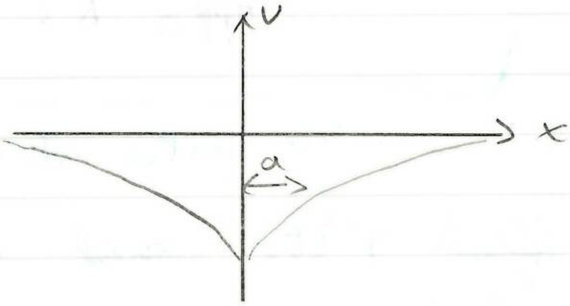
5/2/12

Rossby – Grill Adjustment



a – Rossby radius

$= \frac{c}{f}$, $c = \sqrt{gD}$



—/—

$$\eta(x,t) = \int_{-\infty}^{\infty} A(k) e^{i\Phi(k)t} \, dk.$$

$$\Phi(k) = k\left(\frac{x}{t}\right) - \omega(k) \qquad t \gg 1$$

$x \gg 1$, $I(x) = \int_a^b f(t) e^{ix\Psi(t)} \, dt$ his dummy variable t, ours is k.

his x, our t.

$$\int_a^b f(t) e^{ixt} \, dt \to 0 \quad \text{as } x \to \infty$$

provided $\int_a^b |f(t)| \, dt$ exist

(Note $T = \Psi(t)$, $dT = \Psi'(t) \, dt$)

$$I(x) = \left.\frac{f(t)}{i x \psi'(t)} e^{ix\psi(t)}\right]_a^b$$

leading order behaviour is $1/x$.

$$\underbrace{- \frac{1}{ix}\int_a^b \underbrace{\frac{d}{dt}\left(\frac{f(t)}{\psi(t)}\right) e^{ix\psi(t)}\, dt}_{\text{apply R.L to this}}}_{\to 0 \text{ faster then } 1/x}$$

But $\psi'(t)$ could vanish somewhere in the integral $[a,b]$.

If $\psi'(a) = 0,\ \psi''(a) = 0, \dots, \psi^{(p)}(a) \neq 0$.

(6.5.12)

$$I(x) \sim f(a) e^{ix\psi(a) \pm i\pi/2p} \left[\frac{p!}{x|\psi^{(p)}(a)|}\right]^{1/p} \frac{\Gamma(\frac{1}{p})}{p}$$

where $\Gamma(p) = \int_0^\infty z^{p-1} e^{-z}\, dz$.

2011 2d). In our case the phase is :

$$\Phi(k) = k\left(\frac{x}{t}\right) - \omega(k)$$

The phase is stationary when $\Phi'(k) = 0$.

i.e $\frac{x}{t} - \frac{d\omega}{dk} = 0$.

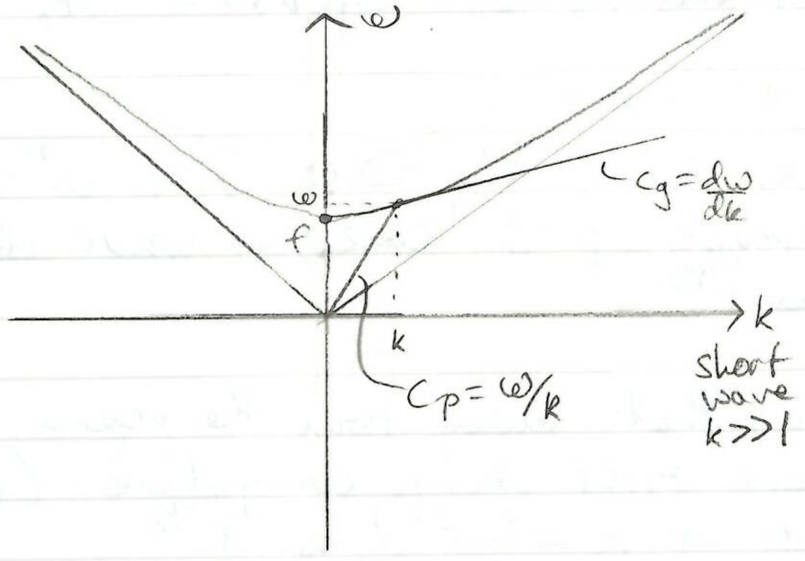
i.e $x = C_g \cdot t$ (1)

where $C_g = d\omega/dk$, the group velocity.

First, if we do NOT move at the group velocity, i.e we do not satisfy (1), then the phase is not stationary so we can integrate by parts and our solutcoin decays at least as fast as $1/t$.

If we DO travel at the group velocity, then the phase is statcoinary and provided $C_g'(k) = 0$ then we use the formula with $p=2$. $[\Phi'(k) = 0, \Phi''(k) = 0]$.

2011
2c)



$C_p^2 = c^2 + f^2/k^2$

$\omega^2 = f^2 + c^2k^2$.

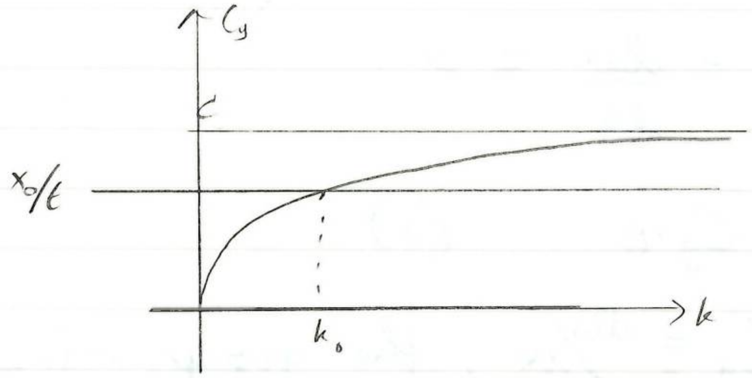
$2\omega \frac{d\omega}{dk} = 2c^2 k$.

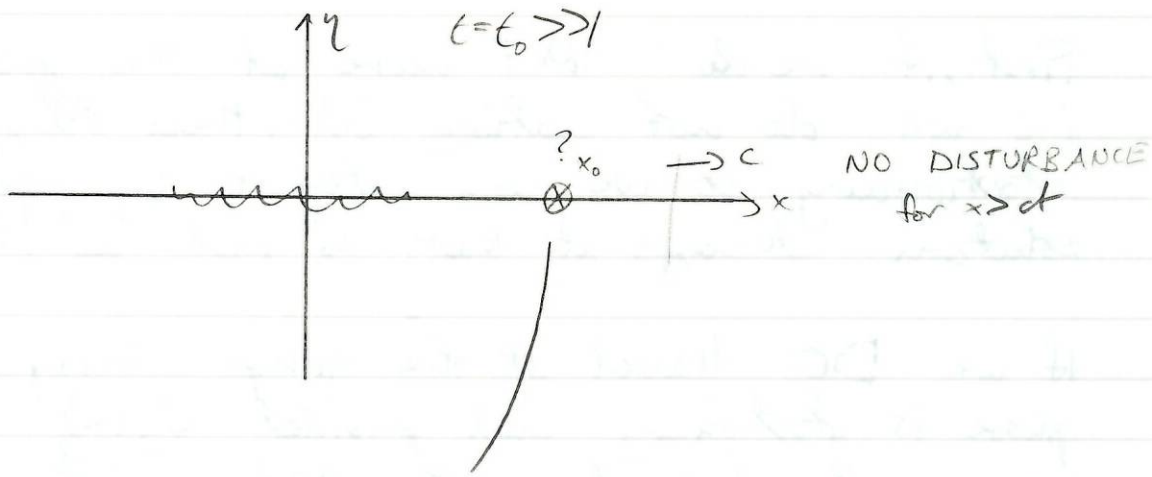
i.e $\frac{\omega}{k}\frac{d\omega}{dk} = c^2$

i.e $C_p \cdot C_g = c^2$.

But $C_p > c$, $C_g < c$

As $k \to \infty$, $C_p \to c^+$, $C_g \to c^-$



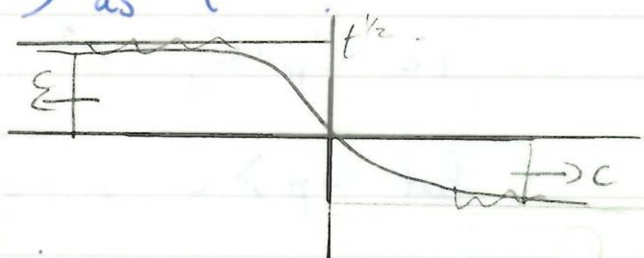


Imporant contributcon, of order $t^{-1/2}$ at a part x_0 at time t_0 has k_0 st $C_g(k_0) = x_0/t_0$.

—/—

Aside, As $x_0 \to ct$, $C_g' \to 0$, i.e $\omega'' \to 0$ and we approach $p=3$ case, so wave decays more steady as $t^{-1/3}$

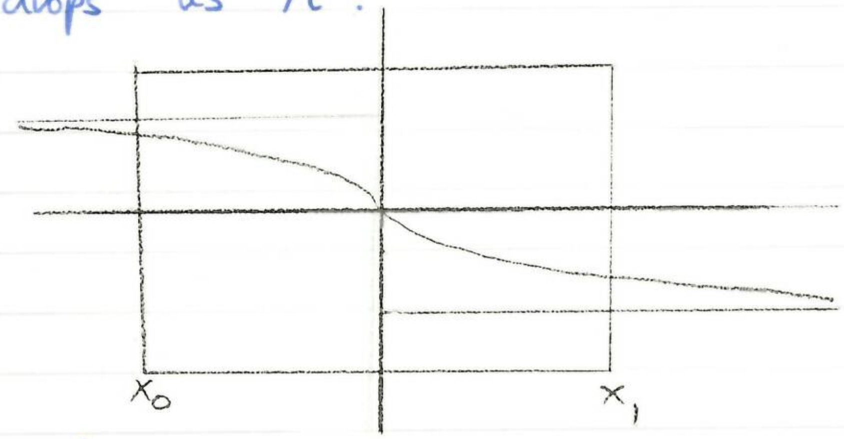
We have shown, that away from the region of the orgin, the wave field decays everywhere (except near wavefronts at $x = \pm ct$) as $t^{-1/2}$



Near wavefronts, decay as $t^{-1/3}$

Energy: Integrate out $\pm ct$
then PE released = KE in geostrophic flow + KE in PWs

But in any finite domain as $t \to \infty$, energy in PWs drops as $1/t$.



x_0, x_1 fixed $t \to \infty$
then $x_0 \to -\infty$, $x_1 \to \infty$

Energy radiated to infinity at group velocity and approach $p=3$.

Energy travel at group velocity.

2011) 1b)

The Kelvin Wave

The linearised shallow water equations are

$$u_t - fv = -g\eta_x, \tag{1}$$
$$v_t + fu = -g\eta_y, \tag{2}$$
$$\eta_t + D(u_x + v_y) = 0 \tag{3}$$

The roots $\alpha = \pm \mathrm{i}fk/\sigma$ give the solution

$$\eta = \Re\{\eta_0 e^{\pm fky/\sigma} e^{\mathrm{i}(kx-\sigma t)}\} = (Ae^{-fky/\sigma} + Be^{fky/\sigma})\cos(kx - \sigma t) \tag{4}$$

Now $\partial_t(2) - f(1)$ gives

$$v_{tt} + f^2 v = -g\eta_{yt} + fg\eta_x = -2Bfgke^{fky/\sigma}\sin(kx - \sigma t). \tag{5}$$

Notice that the coefficient of A vanishes identically. For v to vanish for all time, and so the left side of (5) to vanish for all t ($\sigma \neq \pm f$), at some fixed point (x, y), e.g. even a single point on the wall $y = 0$, equation (5) implies $B = 0$ and so $v \equiv 0$. A Kelvin wave has zero velocity normal to its supporting wall.

Using $v = 0$ in (1) and (3) and then eliminating u gives

$$\eta_{tt} = c^2 \eta_{xx}, \tag{6}$$

where $c = \sqrt{gD}$, the non-rotating wave equation. For this to have solutions of form (4), $\sigma^2 = c^2k^2$, i.e. $\sigma = \pm ck$. (This is precisely the same result as substituting for α in $\alpha = \pm \mathrm{i}fk/\sigma$.)

If $\sigma = +ck$, then

$$\eta = Ae^{-y/a}\cos[k(x - ct)], \tag{7}$$
$$u = (A/D)ce^{-y/a}\cos[k(x - ct)], \tag{8}$$

where $a = c/f$ is the Rossby radius and the form of u comes from (2) with $v = 0$:

$$u = (gk/\sigma)Ae^{-fky/\sigma}\cos(kx - \sigma t). \tag{9}$$

If $\sigma = -ck$, then

$$\eta = Ae^{y/a}\cos[k(x + ct)], \tag{10}$$
$$u = -(A/D)ce^{y/a}\cos[k(x + ct)]. \tag{11}$$

For both directions of propagation, the Kelvin wave propagates with it supporting wall to its right and decays exponentially away from the wall on the scale of the Rossby radius.

8/2/13

$\left.\begin{array}{l}\underline{PW}\\ \underline{KW}\end{array}\right\}$ surface waves + effect rotation.

$\left(\frac{\partial}{\partial y} \equiv 0\right)$

— ✓ —

Channel with variable $\left\{\begin{array}{l}\text{topography}\\ \text{orography}\end{array}\right.$

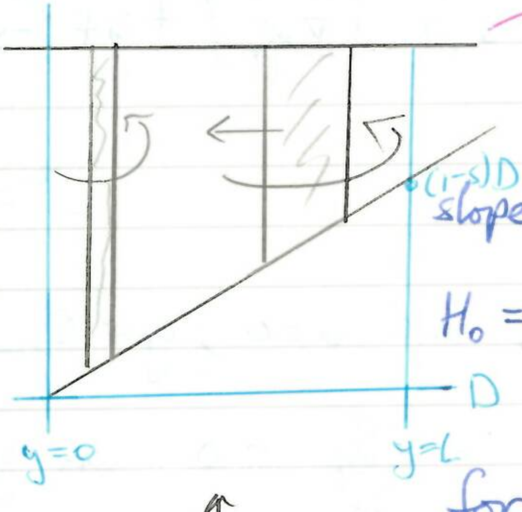
$$(\partial_{tt} + f^2)\eta_t - g\nabla\cdot[H_0\nabla\eta] + gf\nabla\cdot[H_0(\hat{z}\wedge\nabla\eta)] = 0$$

- linearised rSWE -

Previously, took $H_0 = $ constant, so final term disappeared and $c^2 = gH_0$ - Klein-Gordon eqn.

Now we allow for variable depth.

→ elevated view

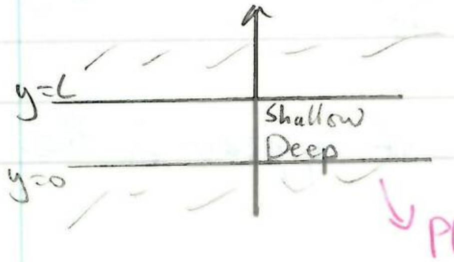


We can obtain the new dynamics by simply consider a small linear slope i.e take

$$H_0 = D\left(1 - \frac{sy}{L}\right)$$

for $0 \le y \le L$, i.e channel of width L.

$\underline{s \ll 1}$



↘ Plain view

$$(\partial_{tt} + f^2)\eta_t - g\underline{\nabla}\cdot[H_0 \underline{\nabla}\eta_t] + gf\,\underline{\nabla}\cdot[H_0(\hat{\underline{z}} \wedge \underline{\nabla}\eta)] = 0$$

$$(\partial_{tt} + f^2)\eta_t + g\,\underline{\nabla}\cdot[H_0 \underline{\nabla}\eta_t] + gf\frac{\partial(\eta, H)}{\partial(x,y)} = 0$$

Notice $\eta_{ttt} \Rightarrow$ 3 waves
(2 PWs + another)

$$\underline{\nabla}\cdot(\phi\underline{u}) = \underline{u}\cdot\underline{\nabla}\phi + \phi\underline{\nabla}\cdot\underline{u}.$$

$$\underline{\nabla}\cdot[H_0(\hat{\underline{z}} \wedge \underline{\nabla}\eta)] = (\hat{\underline{z}} \wedge \underline{\nabla}\eta)\cdot\underline{\nabla}H_0 + H_0\,\cancel{\underline{\nabla}\cdot[\hat{\underline{z}} \wedge \underline{\nabla}\eta]}^{=0}$$

$$= \hat{\underline{z}}\cdot(\nabla\eta \wedge \nabla H_0)$$

$$= \frac{\partial\eta}{\partial x}\frac{\partial H_0}{\partial y} - \frac{\partial\eta}{\partial y}\frac{\partial H_0}{\partial x}$$

$$= \frac{\partial(\eta, H_0)}{\partial(x,y)}$$

For $H_0 = D(1 - sy/L)$

$$(\partial_{tt} + f^2)\eta_t - c^2\,\underline{\nabla}\cdot[(1 - s\,y/L)\underline{\nabla}\eta_t] + gf\left(-s\frac{D}{L}\right)\frac{\partial\eta}{\partial x} = 0$$

$$c^2 = gD.$$

[$s = 0 \to$ KG - PWs.] The bc's are $v = 0$, $y = 0, L$

i.e $\dfrac{\partial^2\eta}{\partial y\partial t} - f\dfrac{\partial\eta}{\partial t} = 0$. on $y = 0, L$

as before since variables H_0 does not appear in the mom'm eqns.

Look for solutions of the form; (as before):

$$\eta(x,y,t) = Re\left\{\bar{\eta}(y)e^{i(kx-\omega t)}\right\}$$

$s \ll 1$. Replace $1-\frac{sy}{L}$ by 1 except if differentiated.

$$\text{Then } -i\omega(f^2-\omega^2)\bar{\eta} - c^2\left\{ (1-\cancel{\tfrac{sy}{L}})(-i\omega)(-k^2)\bar{\eta} + \frac{\partial}{\partial y}\left[(1-s\frac{y}{L})\frac{\partial\bar{\eta}}{\partial y}(-i\omega)\right]\right\}$$

Cubic in ω.

$$- c^2\frac{fs}{L}(ik)\bar{\eta} = 0.$$

BCs (as for $s=0$),

$$(-i\omega)\bar{\eta}' - f(ik)\bar{\eta} = 0 \quad \text{on } y = 0, L.$$

- This can now be solved for arbitrary slope s, elegantly using Laplace transform (converts to first order ode for transform, (NFE)

$$\mathcal{L}(tf(t)) = \frac{d}{d\lambda}\hat{f}(\lambda)$$ [Le Blond + Mysak Wave in Fluids]

- This is not necessary to get the underlying dynamics. It is sufficient to consider $0 < s \ll 1$ slopes.

We want to retain the leading order term in S.

In $1 - \frac{sy}{L}$ the leading term is 1

In $\frac{\partial}{\partial y}\left(1 - \frac{sy}{L}\right) = -\frac{s}{L}$ the leading term is $-\frac{s}{L}$

Then $-i\omega(f^2 - \omega^2)\bar{\eta} - (-i\omega)k^2c^2\bar{\eta} +$

$$(-i\omega)\bar{\eta}'' + (-i\omega)\left(\frac{-s}{L}\right)\bar{\eta}'$$

$$- \frac{c^2 fs}{L}(ik)\bar{\eta} = 0.$$

[note if $s=0$
this is quadratic
in ω : $s \neq 0$ cubic]

Then

$$\bar{\eta}'' - \frac{s}{L}\bar{\eta}' + \bar{\eta}\left[\underbrace{\frac{\omega^2 - f^2}{c^2} - k^2}_{\alpha^2} - \frac{fsk}{L\omega}\right] = 0$$

$s=0 \longrightarrow$ KG, our channel flow.

Write $\bar{\eta} = e^{sy/2L}\phi(y)$

$$\bar{\eta}' = \frac{s}{2L}\bar{\eta} + e^{2y/2L}\phi'$$

$$\bar{\eta}'' = \frac{s}{2L}\bar{\eta}' + \frac{s}{2L}e^{sy/2L}\phi' + e^{sy/2L}\phi''.$$

$$\bar{\eta}'' - \frac{s}{L}\bar{\eta}' = \frac{s}{2L}\bar{\eta}' + \frac{s}{2L}e^{sy/2L}\phi' + e^{sy/2L}\phi'' - \frac{s}{L}\bar{\eta}'$$

$$= -\frac{s}{2L}\left(\frac{s}{2L}\bar{\eta} + e^{sy/2L}\phi'\right) + \frac{s}{2L}e^{sy/2L}\phi'$$
$$+ e^{sy/2L}\phi''$$

$$= -\frac{s^2}{4L^2}\bar{\eta} - e^{sy/2L}\phi''$$

$$= e^{sy/2L}\left[\phi'' - \frac{s^2}{4L^2}\phi\right]$$

Our equatcoin is thus

$$\phi'' + \phi\left[\frac{\omega^2 - f^2}{c^2} - k^2 - \frac{fsk}{L\omega} - \frac{s^2}{4L^2}\right] = 0.$$

Drop the final term as it is if order s^2

And since $e^{sy/2L} = 1 + O(s^2)$ is to leading order in s,

$$\bar{\eta}'' + \alpha^2\bar{\eta} = 0$$

where $\alpha^2 = \frac{\omega^2 - f^2}{c^2} - k^2 - \frac{fsk}{L\omega}$

Notice this is identical to our previous problem and our previous α when $s=0$. The only change for $s \neq 0$ is an extra term in α. Bring over all old results.

— / —

Ans to Hw:

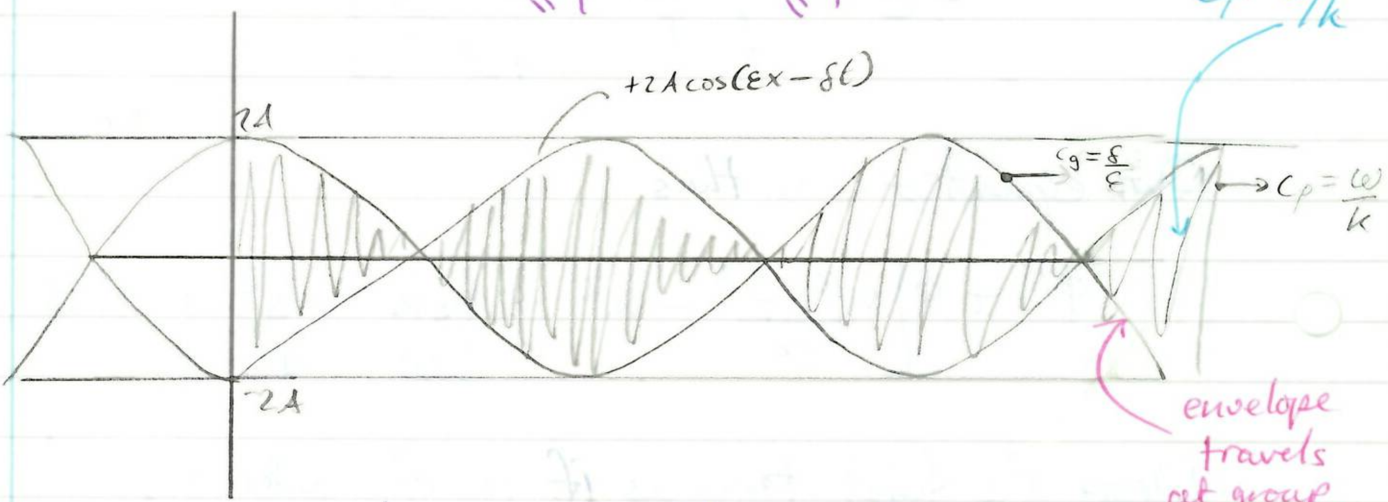
$$\eta_1 = A\cos((k+\epsilon)x - (\omega+\delta)t) \qquad \epsilon/k \ll 1$$

$$\eta_2 = A\cos((k-\epsilon)x - (\omega-\delta)t) \qquad \delta/\omega \ll 1$$

$$\eta = \eta_1 + \eta_2 = 2A\cos(kx-\omega t)\cos(\epsilon x - \delta t)$$

$$|\eta| = |2A|\,\underbrace{|\cos(\ldots)|}_{\leq 1}\,\underbrace{|\cos(\ldots)|}_{\leq 1}.$$

crest travels at phase velocity $C_p = \omega/k$



envelope travels at group speed $C_g = \frac{d\omega}{dk}$

Disprese relation: $\omega = \omega(k) = F(k)$

$$\cancel{\omega} + \delta = F(k+\epsilon) = \cancel{F(k)} + \epsilon F'(k) + \ldots$$

$$C_g = \frac{\delta}{\epsilon} = F'(k) = \frac{d\omega}{dk}.$$

— / —

There no work. (Sloping bottom)

For the PWs, (using matrix) $\begin{pmatrix} & \end{pmatrix}\begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$$\sin \alpha L = 0.$$

i.e $\alpha L = n\pi \qquad n = 1, 2, 3, \ldots$

i.e $\alpha = \frac{n\pi}{L}$

$$\alpha^2 = \frac{\omega^2 - f^2}{c^2} - k^2 - \underline{\frac{fsk}{L\omega}}.$$

only new thing today

Thus our waves satisfy

$$\frac{\omega^2 - f^2}{c^2} - k^2 - \frac{fsk}{L\omega} = \frac{n^2\pi^2}{L^2}$$

i.e. $\omega^2 = f^2 + c^2\left(k^2 + \frac{n^2\pi^2}{L^2} - \frac{fsk}{L\omega}\right)$ <u>CUBIC</u>

c.f flat $(s = 0)$

$\omega^2 = f^2 + c^2\left(k^2 + \frac{n^2\pi^2}{L^2}\right) \leftarrow$ quantised PWs <u>QUADRATIC</u>

c.f PWs

$$\omega^2 = f^2 + c^2(k^2 + l^2)$$

For small s: —

$s=0$ Flat bottomed modes ω_n with

$$\omega_n = \sqrt{f^2 + c^2\left(k^2 + \frac{n^2\pi^2}{L^2}\right)} \qquad n = 1, 2, 3, \ldots$$

$0 < s \ll 1$: Expect ω to move away from ω_n by an amount of order s.

Thus to order s, we get

$$\omega = \sqrt{\omega_n^2 + c^2\frac{fsk}{L\omega_n}} + O(s^2)$$

– only a small change.

$$\omega = \sqrt{\omega_n^2 + \frac{c^2 fsk}{L\omega_n}} = \omega_n\left[1 + c^2\frac{fsk}{\omega_n^2 L}\right]^{1/2}$$

$$= \omega_n\left[1 + \left(\frac{c^2 fk}{2\omega_n^2 L}\right)s + \ldots\right]$$

– small slope $\Rightarrow$ small change
– not imporant.

To get the thrid root we need to keep the final term as $s \to 0$. Our new root must be of order s^1. i.e $\omega \sim s$, so $\omega^2 \sim s^2$ and so the left side is negligible (we have lost the PWs)

$$k^2 + \frac{n^2\pi^2}{L^2} + \frac{fsk}{L\omega} = -\frac{f^2}{c^2}$$

i.e $\frac{fsk}{L\omega} = -\left(k^2 + \frac{n^2\pi^2}{L^2} + \frac{1}{a^2}\right)$ $\qquad a = c/f$.

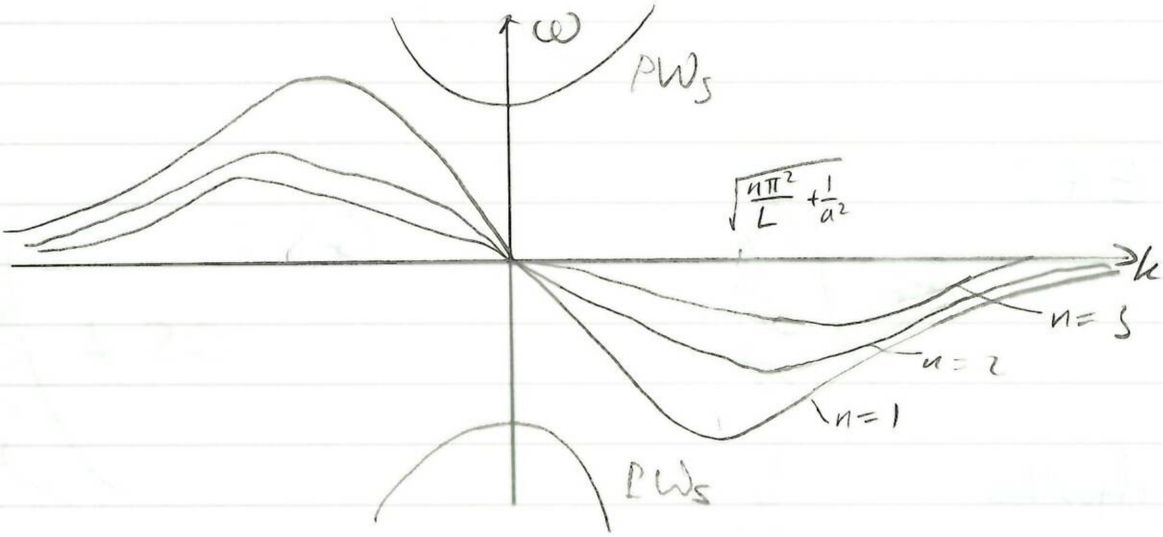
Thus:

$$\omega = -\frac{fsk/L}{k^2 + \frac{n\pi^2}{L^2} + \frac{1}{a^2}}$$

– a Rossby wave –

Notice $\omega = 0$, i.e a steady current, no wave, if $f = 0$ <u>OR</u> $s = 0$.

– needs a slope <u>and</u> rotation.



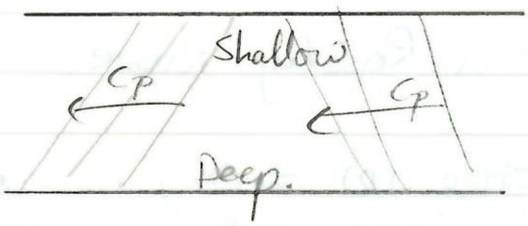
$$C_p = \frac{\omega}{k} = -\frac{fs/L}{k^2 + \frac{n^2\pi^2}{L^2} + \frac{1}{a^2}}$$

$< 0 \quad \forall k$.

Thus $\omega = - \dfrac{fsk/L}{k^2 + \dfrac{n^2\pi^2}{L^2} + \dfrac{1}{a^2}} \sim O(sf)$ low frequency waves

Creast travel in only one directcoi, the negative-x directcoi.

(Creasts propagate with shallow water to their right - c.f. Kelvin wave and its boundary)

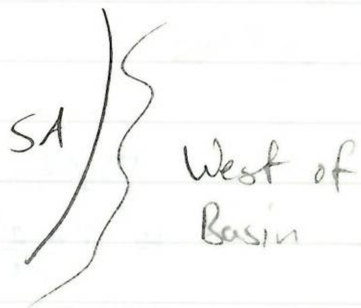


e.g: Kuroshio

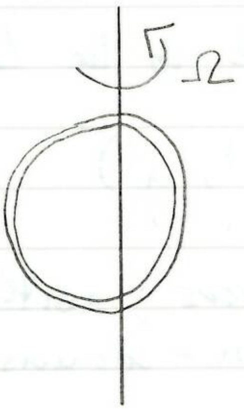
EAC

GS:

Agulhas



19/2/13



E.E + rotation
(+ stratification)

r SWE

+slope.

PW
(KWs)
(RG adjustment
non geo $\Rightarrow$ geo)

RWs

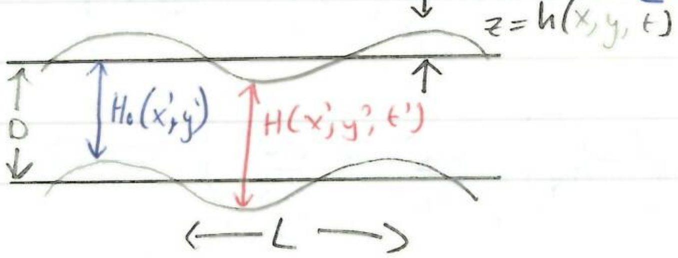
$$\omega = -\frac{ksf/L}{K + \frac{n^2\pi^2}{L^2} + \frac{1}{a^2}}$$

1) Low-frequency motions
- Quasi-geostrophic Equations.
(Non-linear)
- RW in Q6.

1½) Viscous flow - Ekman layer

2) Stratification.

<u>The quasi-geostrophic approximation</u> (of the r SWE)
[no linearisation]



Take a typical horizonal scale to be L.

Write $(x, y) = L(x', y')$

where x', y' are simple numbers, without dimensions, i.e. dimension variables – non-dimensionalisation –

i.e. $x' = \frac{x}{L}$, $y' = \frac{y}{L}$ Similarly write $t = Tt'$.

$(u, v) = U(u', v')$ $\qquad \eta = N_0 \eta'$

The layer depth is $H\left(\frac{x}{L}, \frac{y}{L}, \frac{t}{T}\right) = H_0(x', y') + \eta$

undisturbed depth.

Total instantaneous depth.

i.e $H(x', y', t') = H_0(x', y') + \eta(x', y', t')$

average depth $= D - h_B(x', y') + \eta(x', y', t')$

$$\underbrace{\underbrace{\frac{U}{T}\frac{\partial u'}{\partial t'}}_{\text{time depe}} + \underbrace{\frac{U^2}{L}\left(u'\frac{\partial u'}{\partial x'} + v'\frac{\partial u'}{\partial x'}\right)}_{\text{non linear}}}_{\text{smaller}} \underbrace{- \underbrace{Ufv'}_{\text{Corolis}} = -\frac{gN_0}{L}\underbrace{\frac{\partial \eta'}{\partial x'}}_{\text{pressure grad}}}_{\text{main}}$$

$$\frac{U}{T}\frac{\partial v'}{\partial t'} + \frac{U^2}{L}\left(u'\frac{\partial v'}{\partial x'} + v'\frac{\partial v'}{\partial y'}\right) + Ufu' = -\frac{gN_0}{L}\frac{\partial \eta'}{\partial y'}$$

$$\frac{N_0}{T}\frac{\partial \eta'}{\partial t} + \frac{U}{L}\left(u'\frac{\partial}{\partial x}(N_0\eta' - h_B) + v'\frac{\partial}{\partial y'}(N_0\eta' - h_B)\right\}$$

$$+ \frac{U}{L}\left(D + N_0\eta' - h_B\right)\left(\frac{\partial u'}{\partial x'} + \frac{\partial v'}{\partial y'}\right) = 0.$$

the rSWE

For the leading order balance to be geostrophic, we need

$\frac{N_0 g}{L}$ is of **same** order as Uf.

Guarrantee thus by taking $\boxed{N_0 = \frac{UfL}{g}}$

The ratio of the non-linear terms to Corislois in horiz mom'm is

$$\frac{\text{Non linear}}{\text{Coriolis}} = \frac{U^2/L}{Uf} = \frac{U}{fL} = \epsilon$$

the Rossby number (a dimensionless parameter) (cf Reynolds No)

$$\frac{\text{Time-dep}}{\text{Coriolis}} = \frac{U/T}{Uf} = \frac{1}{fT} = \epsilon_T$$

a temporal Rossby number (a dimensionless parameter)

The final non-dimensional parameter is

$$\frac{N_0}{D} = \frac{UfL/g}{D}$$

$$= UfL/gD$$

$= UfL/c^2$

$= \frac{UL}{f} \cdot f/c^2$

$= \frac{U}{fL} \cdot \frac{L^2}{a^2} = \varepsilon F.$, $a = c/f$, Rossby radius $[a] = L$

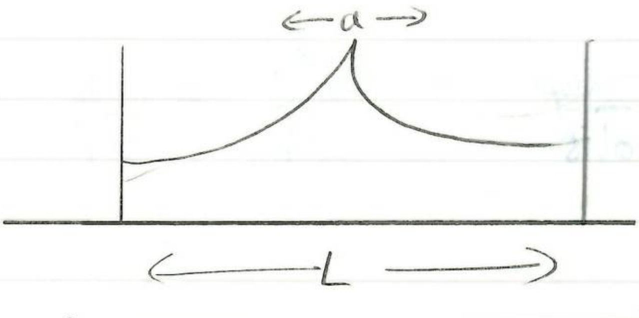
where $F = L^2/a^2$

- ratio of scale of our motion to the Rossby radius.

What does F mean?

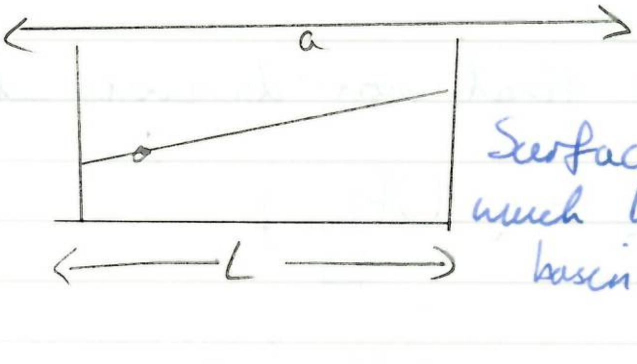
F>>1 Basin large compared to R.R.

$a << L$



F<<1

$a >> L$



Surface relaxation much larger than basin width.

$$\epsilon_T \frac{\partial u'}{\partial t'} + \epsilon\left(u'\frac{\partial u'}{\partial x'} + v'\frac{\partial u'}{\partial y'}\right) - v' = \frac{\partial \eta'}{\partial x'}$$ Divide by Uf

$$\epsilon_T \frac{\partial v}{\partial t'} + \epsilon\left(u'\frac{\partial v'}{\partial x'} + v'\frac{\partial v}{\partial u'}\right) + u' = -\frac{\partial \eta'}{\partial y'}$$

$$\epsilon_T F \frac{\partial \eta'}{\partial t'} + \epsilon F\left(u'\frac{\partial \eta'}{\partial x'} + v'\frac{\partial \eta'}{\partial y'}\right) - u'\frac{\partial}{\partial x'}\left(\frac{h_B}{D}\right) - v'\frac{\partial}{\partial y'}\left(\frac{h_B}{D}\right)$$

$$+\left(1 + \epsilon F\eta' - \frac{h_B}{D}\right)\left(\frac{\partial u'}{\partial x'} + \frac{\partial v'}{\partial y'}\right) = 0.$$

From now on, we drop the dashes. i.e. we understand that we always mean non-dimensional variables

3 non-d parameters $\underbrace{\epsilon, \epsilon_T}_{\text{small}}$ and F
small to give geostrophy

Firstly we will keep F as order unity.

Thus we need to consider

1) $\epsilon \ll \epsilon_T \sim 1$ $\boxed{\lim_{\substack{\epsilon \to \\ \epsilon_T \text{ fixed}}}}$

→ lrSWE Done
PWs + etc.

2) $\epsilon_T \ll \epsilon \ll 1$ Steady, non linear.

3) $\epsilon_T \ll \epsilon \ll 1$ Unsteady + non linear

3) includes 2)
- what we will do.

$\varepsilon = \varepsilon_T$.

$\frac{1}{fT} = \frac{U}{fL}$ i.e choose $T = U/L$

22/2/13.

$$\varepsilon_T \frac{\partial u}{\partial t} + \varepsilon\left(u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}\right) - v = -\frac{\partial \eta}{\partial x} \quad (1)$$

$$\varepsilon_T \frac{\partial v}{\partial t} + \varepsilon\left(u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial x}\right) + u = -\frac{\partial \eta}{\partial y} \quad (2)$$

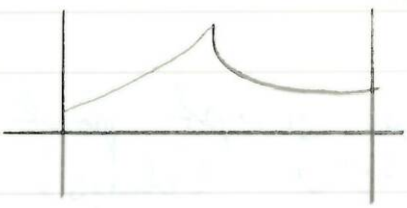
$$\varepsilon_T F\frac{\partial \eta}{\partial t} + \varepsilon F\left(u\frac{\partial \eta}{\partial x} + v\frac{\partial \eta}{\partial y}\right) - u\frac{\partial}{\partial x}\left(\frac{h_B}{D}\right) - v\frac{\partial}{\partial y}\left(\frac{h_B}{D}\right)$$

$$+\left(1 + \varepsilon F\eta - \frac{h_B}{D}\right)\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) = 0.$$

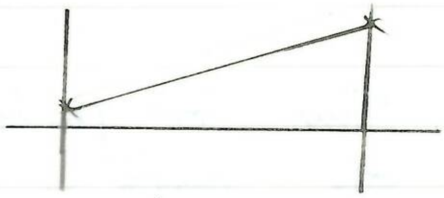
$\varepsilon = \frac{U}{fL}$ Rossby number $\varepsilon \ll 1$

$\varepsilon_T = \frac{1}{fT}$ temporal Rossby number $\varepsilon_T \ll 1$

$F = \frac{L^2}{a^2}$ - Basin scale to Rossby radius, squared
- measure of imporantence of free surface (arbitrary). $0 \leq F \leq \infty$



$a \ll L$



$L \ll a$

$(\varepsilon \ll 1)$

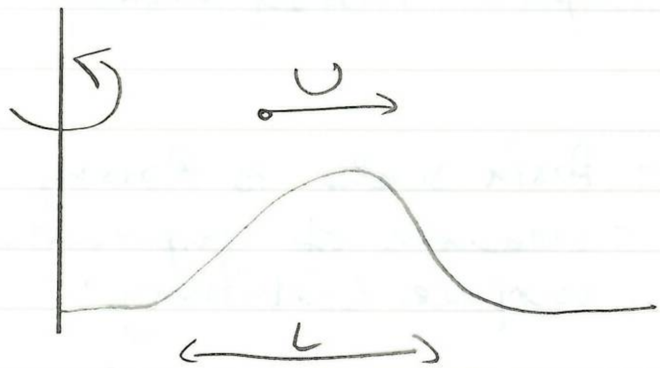
(1) $\varepsilon \ll \varepsilon_T \ll 1$ Linear: can then in fact have $\varepsilon_T \sim 1$, (r SWE — already done.

(2) $\varepsilon_T \ll \varepsilon \ll 1$ Steady but non-linear (new)

(3) $\varepsilon_T \sim \varepsilon \ll 1$ Unsteady but non-linear (includes case (2)) — [turn to (2) by putting $\frac{\partial}{\partial t} = 0$]

Ensure this by taking
$\frac{1}{fT} = \frac{U}{fL}$ i.e $T = \frac{L}{U}$ then $\varepsilon_T = \varepsilon$.

i.e. in these motions the important time scale is the advection time.



Time for particle to be swept past a feature of length L is $T = L/U$ (days).

For PWs, $T = 1/f$ i.e the rotals periods (12 hours).

Ratio $\frac{\text{Inertial period}}{}$: Advection time

$$= \frac{1/f}{L/U} = \frac{U}{fL} = \varepsilon << 1$$

— / —

Since $\varepsilon << 1$ look for a power series solution in ε.

$$u(x,y,t;\varepsilon,F) = u_0(x,y,t;F) + \varepsilon u_1(x,y,t;F) + \varepsilon^2 u_2(x,y,t;F) + \dots$$

Similarly: $v = v_0 + \varepsilon v_1 + \varepsilon^2 v_2 + \dots$
$\eta = \eta_0 + \varepsilon \eta_1 + \varepsilon^2 \eta_2 + \dots$

We are going to insist that every coefficent of ε^n is identically zero.

$[\varepsilon^0]$ (1) gives $-v_0 = -\frac{\partial \eta_0}{\partial x}$

(2) gives $u_0 = -\frac{\partial \eta_0}{\partial y}$

i.e η_0 is a streamfunction for (u_0, v_0)

$$u_0 = -\frac{\partial \eta_0}{\partial y}, \quad v_0 = \frac{\partial \eta_0}{\partial x}$$

i.e surface elevation is a streamfunction $\frac{\partial u_0}{\partial x} + \frac{\partial v_0}{\partial y} = 0$.

(Remember opposite sign to classical streamfunction).

(3) Since $\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = \epsilon\left(\frac{\partial u_1}{\partial x} + \frac{\partial v_1}{\partial y}\right) + \ldots$

the leading order term in (3) is (because $\frac{\partial u_0}{\partial x} + \frac{\partial v_0}{\partial y} = 0$)

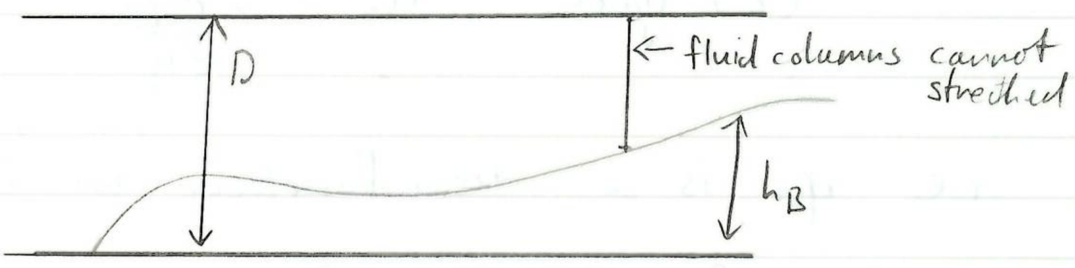
$$u_0 \frac{\partial}{\partial x}\left(\frac{h_B}{D}\right) + v_0 \frac{\partial}{\partial y}\left(\frac{h_B}{D}\right) = 0.$$

i.e. $\underline{u}_0 \cdot \underline{\nabla}\left(\frac{h_B}{D}\right) = 0$

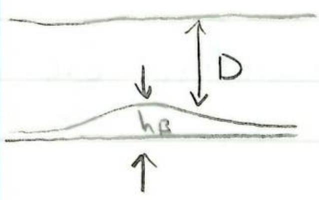
i.e $\underline{u}_0$ is $\perp$'r to $\underline{\nabla}\left(\frac{h_B}{D}\right)$

i.e $\underline{u}_0$ is // to lines of constant h_B/D, i.e. lines of constant undisturbed depth. - isobaths.

- Flow is along depth contours.



However for weaker topography i.e $\frac{h_B}{D} \sim \epsilon$



$\frac{h_B}{D} \sim \epsilon$. $\frac{h_B}{\epsilon D} = \eta_B \sim 1$ or $\frac{h_B}{\epsilon D} \sim 1$ or $h_B \sim \epsilon D$

$$H = D[1 - \varepsilon[F\eta - \eta_B]]$$

$$\frac{D}{Dt} \to \frac{\partial}{\partial t} + u_0 \frac{\partial}{\partial x} + v_0 \frac{\partial}{\partial y} .$$

$$q = \frac{\varepsilon \zeta_0 + 1}{D[1+\varepsilon(F\eta - \eta_B)]} = \frac{1}{D}\left(1+\varepsilon\eta_0\right)\left[1+\varepsilon\left(F\eta - \eta_B\right)\right]^{-1}$$

$$= \frac{1}{D}\left(1 - \varepsilon\zeta_0\right)\left[1 - \varepsilon\left(F\eta - \eta_B\right)\right] + \ldots$$

$$= \frac{1}{D}\left[1 + \varepsilon\left(\zeta_0 - F\eta_0 + \eta_B\right)\right] + \ldots$$

Thus $\frac{Dq}{Dt} = 0$ because $\frac{D_0 q_0}{Dt} = 0$ as $\varepsilon \to 0$.

Write $\psi = \eta_0$

Then $u_0 = -\frac{\partial \psi}{\partial y}$, $v_0 = \frac{\partial \psi}{\partial x}$

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + u_0 \frac{\partial}{\partial x} + v_0 \frac{\partial}{\partial y}$$

$$= \frac{\partial}{\partial t} + \frac{\partial \psi}{\partial x}\frac{\partial}{\partial y} - \frac{\partial \psi}{\partial y}\frac{\partial}{\partial x}$$

$$= \frac{\partial}{\partial t} + \frac{\partial(\psi, \cdot)}{\partial(x,y)}$$ Jacobian.., $\psi(x,y,t)$ alone..

Write $q_0 = \zeta_0 - F\eta_0 + \eta_B$

$$= \frac{\partial v_0}{\partial x} - \frac{\partial u_0}{\partial y} - F\eta_0 + \eta_B$$

$$= \frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} - F\psi + \eta_B$$

$$= \nabla^2\psi - F\psi + \eta_B$$ - depends on ψ alone.

Thus $\frac{D_0 q_0}{Dt} = 0$ because

$$\frac{\partial q_0}{\partial t} + \frac{\partial(\psi, q_0)}{\partial(x,y)} = 0.$$

$$q_0 = \nabla^2\psi - F\psi + \eta_B$$

Problem in ψ only: QG equation.

— / —

rSWE $\quad u(x,y,t), v(x,y,t), h(x,y,t)$

3 equs, 3 unknowns (PWs)

$\varepsilon \ll 1$

$\varepsilon_T = \varepsilon$

$\varepsilon_T \ll 1$ Low frequency → No PWs!!

QG $\quad$ Slow evolutions, Rotating RWs.

Weaker Topog: $\eta_B = \dfrac{h_B}{\epsilon D} \sim 1$

$$\frac{h_B}{D} = \epsilon \eta_B$$

In this case the leading order term in (3) is

$$[\epsilon^1] \quad F\left[\frac{\partial \eta_0}{\partial t} + u_0 \frac{\partial \eta_0}{\partial x} + v_0 \frac{\partial \eta_0}{\partial y}\right] - u_0 \frac{\partial \eta_B}{\partial x} - v_0 \frac{\partial \eta_B}{\partial y}$$

$$+ \frac{\partial u_1}{\partial x} + \frac{\partial v_1}{\partial y} = 0.$$

Write $\dfrac{D_0}{Dt} = \dfrac{\partial}{\partial t} + u_0 \dfrac{\partial}{\partial x} + v_0 \dfrac{\partial}{\partial y}$

Then we have $\dfrac{D_0}{Dt}\left(F\eta_0 - \eta_B\right) + \dfrac{\partial u_1}{\partial x} + \dfrac{\partial v_1}{\partial y} = 0.$ (4)

To get equation for u, v go back to (1), (2) and look at $[\epsilon^1]$

$$\frac{D_0 u_0}{Dt} - v_1 = -\frac{\partial \eta_1}{\partial x} \qquad (5)$$

$$\frac{D_0 v_0}{Dt} + u_1 = -\frac{\partial \eta_1}{\partial y} \qquad (6)$$

Take $\frac{\partial}{\partial x}(6) - \frac{\partial}{\partial y}(5)$. This introduce

$$\zeta_0 = \frac{\partial v_0}{\partial x} - \frac{\partial u_0}{\partial y}$$

the vertical component of relative vorticity:

Giving $\frac{D_0 \zeta_0}{Dt} + \left(\frac{\partial u_1}{\partial x} + \frac{\partial v_1}{\partial y} \right) = 0$. (7)

Take (7) − (4):

$$\frac{D_0}{Dt}\left[\zeta_0 - F\eta_0 + \eta_B \right] = 0.$$

- everything here is zero-order.

Write $q_0 = \zeta_0 - F\eta_0 + \eta_B$

then we have $\frac{D_0 q_0}{Dt} = 0$. QG PV equation

i.e q_0 is conserved by particles moving with velocity (u_0, v_0).

Remember in SWE $q = \frac{\zeta + f}{H}$

$$\frac{Dq}{Dt} = 0.$$

What waves do the QG eqn support?

$$\frac{Dq}{Dt} = 0 \qquad , q = \nabla^2 \psi - F\psi + \eta_B$$

$$\frac{D\cdot}{Dt} = \frac{\partial \cdot}{\partial t} + \frac{\partial(\psi, \cdot)}{\partial(x,y)}$$

2011 3a) Choose simplest $\eta_B = \beta y$ i.e a simple linear slope.

$$\frac{\partial}{\partial t}\left(\nabla^2 \psi - F\psi + \beta y\right) + \frac{\partial(\psi, \nabla^2\psi - F\psi + \beta y)}{\partial(x,y)} = 0.$$

i.e $$\nabla^2 \psi_t - F\psi_t + \frac{\partial(\psi, \nabla^2 \psi - F\psi)}{\partial(x,y)} + \beta \frac{\partial \psi}{\partial x} = 0.$$

Linearise to find waves:

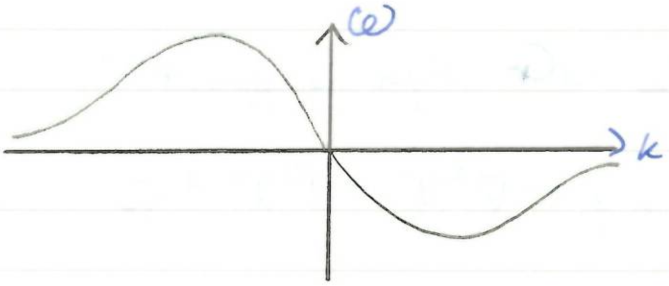
$$\nabla^2 \psi_t - F\psi_t + \beta \psi_x = 0.$$

Look for plane wave solution:

$$\psi = Re\left\{ Ae^{i(kx+ly-\omega t)} \right\}.$$

$$-i\omega(-k^2 - l^2) + i\omega F + ik\beta = 0.$$

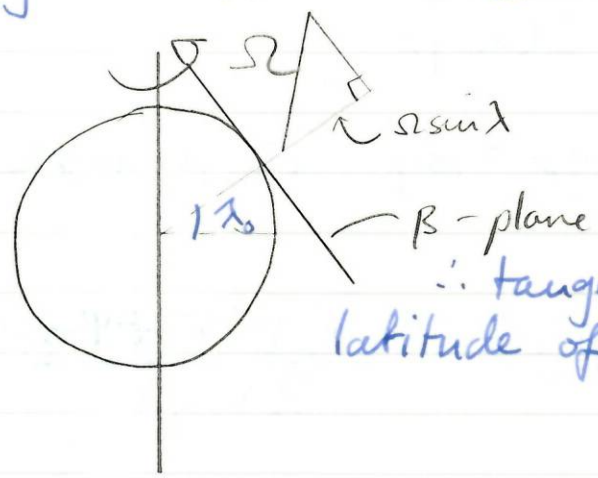
$$\omega = \frac{-\beta k}{k^2 + l^2 + F}$$ - precisely the Rossby waves



($\beta = 0$: no waves)

– no PWs.

Rossby (1939) : The Beta-Plane.



∴ tangent to the Earth at the latitude of interest, λ_0, eg 45°N.

Replacing the sphere by its tangent makes an error of order the square of distance moved from λ_0.

$$f(x_0+\epsilon) = f(x_0) + \epsilon f'(x_0) + \frac{1}{2}\epsilon^2 f''(x_0) + \dots$$

Here, error is of order $[(\delta\lambda)R_E]^2$ i.e of order $(\delta\lambda)^2$

The relevant component of the Earths rotation is

$$\Omega \sin\lambda$$

Write : $y = \delta\lambda R_E$ so the geometric error of order y^2.

Now $\Omega \sin\lambda = \Omega \sin[\lambda_0 + \delta\lambda]$.

$$= \Omega[\sin\lambda_0 + (\cos\lambda_0)\delta\lambda - (\sin\lambda_0)(\delta\lambda)^2 + \dots$$

Thus with error of order $(\delta\lambda)^2$, we have

$$\Omega \sin\lambda = \Omega_0 + 2\beta y$$

where $\beta = \dfrac{\Omega \cos\lambda_0}{2R_E}$

Thus on the β-plane

SW PV is $\dfrac{\zeta + 2\Omega \sin\lambda}{H}$.

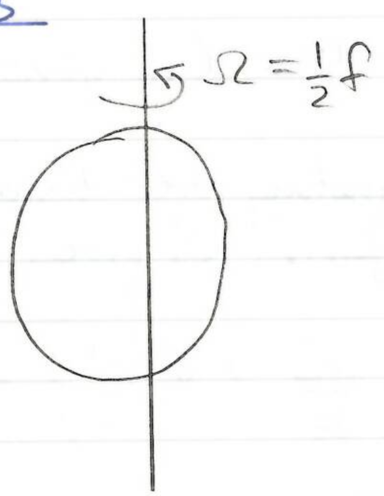
$$= \frac{\zeta + f_0 + \beta y}{H}$$

c.f QGPV on a soloping bottom

$$\zeta - F\eta + \beta y.$$

i.e the effect of variable vertical component of the Earth's rotating in the β-plane approximation (in the QG limit) is identical to the effect of a linearly sloping lower bdry.

26/2/13



E.E + r

↓

rSWF - (conservation of PV)

low frequency, $\epsilon_T \ll 1$, $\epsilon \ll 1$ ↙

QG - non linear
- conserves PW

$q = \nabla^2 \psi - F\psi + \eta_B$

waves ↘ (✳)

PWs, KWs, RWs

PWs: $\omega \sim f$ high frequency

↓

Rossby-Gill adjustment
- "things to geostrophic" (steady)

RWs: low frequency (small slope)

(✳)

linearly conserved : PV

$$\frac{Dq}{Dt} = 0 \,, \quad q = \frac{\zeta + f}{H}$$

$$\zeta_t + \frac{f}{H} \eta_t = 0$$

—/—

Energy conservation (in QG)

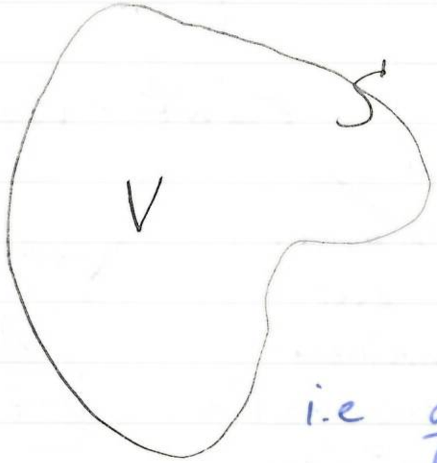
We define a conservation relation as an equation

$$\frac{\partial E}{\partial t} + \underset{\sim}{\nabla} \cdot \underset{\sim}{S} = 0 .$$

E - the density of conserved quantity - (scalar)

$\underset{\sim}{S}$ - the flux " " " - vector.

Physics is conservation laws (in general)



Integrating over fixed volume V, surface S gives

$$\int_V \left(\frac{\partial E}{\partial t} + \nabla \cdot \underline{S} \right) dV = 0.$$

i.e $$\frac{d}{dt}\int_V E \, dV + \int_S (\underline{S} \cdot \underline{\hat{n}}) \, dS = 0.$$

$\underline{S}$ tells us the rate at which our quantity is leaving V flux out dV.

i.e $$\frac{d}{dt}\int_V E \, dV = -\int_S \underline{S} \cdot \underline{\hat{n}} \, dS.$$

e.g. decrease of heat in a region = heat flux out of the region.

There is an imporant SUBCLASS, when

$$\underset{\sim}{S} = \underline{u}_E E.$$

i.e the flux is given by a velocity times the density.

Then we say that E travels with speed $\underline{u}_E$.

e.g our most fundamental law, conservation of mass,

$E = \rho$, the fluid density

$\underline{S} = \underline{u}\rho$, $\underline{u}$ fluid velocity

Conservation of mass:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \underline{u}) = 0 .$$

$$\frac{\partial E}{\partial t} + \nabla \cdot \underline{S} = 0 .$$

i.e. mass travels at speed $\underline{u}$.

Apply this to a Rossby wave on the infinite β - plane.

2011 3(c)

QG PV : $\frac{\partial q}{\partial t} + \frac{\partial(\psi, q)}{\partial(x,y)} = 0$.

Intrested in waves so linearise:

$$\frac{\partial}{\partial t}\left(\nabla^2 \psi - F\psi\right) + \beta \frac{\partial \psi}{\partial x} = 0 .$$

(QG PV : Rossby wave equ)

Multiply by ψ

Then $\psi \nabla^2 \psi_t - F\psi\psi_t + \beta \psi \psi_x = 0$.

$$\psi \nabla^2 \psi_t = \psi \underline{\nabla} \cdot [\underline{\nabla} \psi_t] = \underline{\nabla} \cdot (\psi \underline{\nabla} \psi_t) - \underline{\nabla} \psi \cdot \underline{\nabla} \psi_t .$$

Thus we have
$$\underbrace{-\underline{\nabla} \psi \cdot \underline{\nabla} \psi_t}_{(*)} + \underbrace{F \psi \psi_t}_{(*)} + \underbrace{\underline{\nabla} \cdot (\psi \underline{\nabla} \psi_t)}_{(*)} - \underbrace{\beta \frac{\partial}{\partial x} \left(\frac{1}{2} \psi^2 \right)}_{(*)} = 0.$$

i.e. $$\frac{\partial E}{\partial t} + \underline{\nabla} \cdot \underline{S} = 0 .$$

where

$$E = \underbrace{\frac{1}{2} |\underline{\nabla} \psi|^2}_{(*)} + \underbrace{\frac{1}{2} F \psi^2}_{(*)}.$$ Energy density for a RW.

$$\underline{S} = \underbrace{- \psi \underline{\nabla} \psi_t}_{(*)} \underbrace{- \frac{1}{2} \beta \psi^2 \hat{\underline{x}}}_{(*)}$$ Energy flux for a RW – <u>not isotropic</u>.

Remember, ψ is a s'function:

so $\underline{u} = \underline{k} \wedge \underline{\nabla} \psi$. $\quad \underline{u} = u \hat{\underline{x}} + v \hat{\underline{y}}$

so $|\underline{u}| = |\nabla \psi|$

so $|\nabla \psi|^2 = |\underline{u}|^2$

other side from last year
$$\left\{ \begin{aligned} u &= -\frac{\partial \psi}{\partial y} \\ v &= \frac{\partial \psi}{\partial x} \end{aligned} \right.$$

The first term in E is the KE/unit mass (non-dimensionalised)

The second term is $\frac{1}{2}F\Psi^2$, the PE/unit mass (PE proportional to η^2. Here $\psi = \Psi$.)

E total energy / unit mass.

Thus we have the **interpretation** for F that if $F \ll 1$ most of energy is in KE. If $F \gg 1$ most energy is the PE, i.e distortion of free surface.

Notice always add a divergence-free quatity to any flux w/o changing equ. The flux is never unique. Choose what is most **convenient** for a given problem.

Now we have

$$\partial_t(\nabla^2\Psi - F\Psi) + \beta\Psi_x = 0.$$

Look for a solution of the form:

$$\Psi = A\cos(kx + ly - \omega t)$$

$$\Psi_{xx} = -k^2\Psi$$

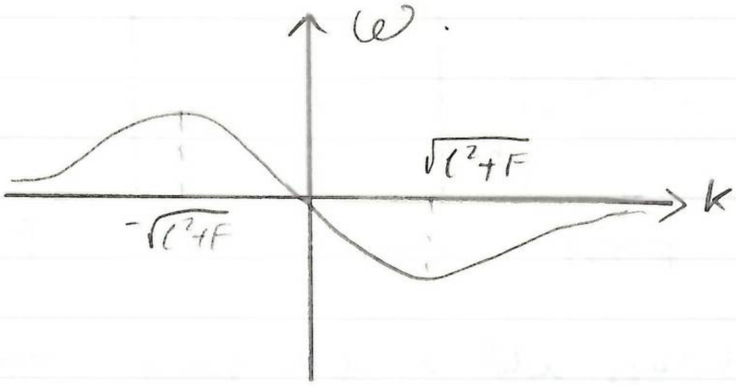
$$\Psi_{yy} = -l^2\Psi$$

$$\psi_t = A\omega \sin\theta.$$

$$-A\omega k^2 \sin\theta - A\omega l^2 \sin\theta - \omega F A \sin\theta - \beta k A \sin\theta = 0.$$

For non-trivial solns, $A\sin\theta \neq 0$.

so $\omega = -\beta k / (k^2 + l^2 + F)$ dispersion relation



It happens that a monochromatic RW satisfies the full non-linear QG eqns.

monochromatic: i.e one k and one l only present.

i.e if $\psi = A\cos(kx + ly - \omega t)$ for some k, l (fixed)

$$\psi_{xx} = -k^2\psi$$

$$\psi_{yy} = -l^2\psi$$

so $\nabla^2\psi = -(k^2 + l^2)\psi$

so $$\frac{\partial(\psi, \nabla^2\psi)}{\partial(x,y)} = -(k^2+l^2)\frac{\partial(\psi,\psi)}{\partial(x,y)} = 0.$$

3/1/13

March 5th 10am

Chadwick G08.

Gordon (25) Rm 107 ?

$E_t + \nabla \cdot \underline{S} = 0$ Conservation relation.

E - density

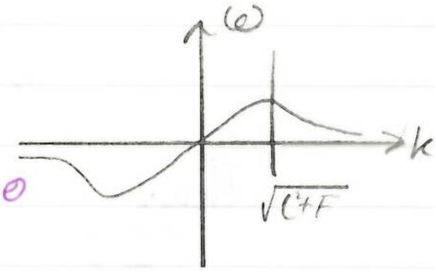
$\underline{S}$ - flux.

Special subclass : $\underline{S} = \underline{u}_E E$.

say "E" travels at speed $\underline{u}_E$.

linear RW equs

$$(\nabla \psi - F\psi)_t + \beta \psi_x = 0.$$



$$\psi = A\cos(kx + ly - \omega t) = A\cos\theta$$

$$\omega = -\beta k / (k^2 + l^2 + F).$$

$$E = \frac{1}{2}|\underline{\nabla}\psi|^2 + \frac{1}{2}F\psi$$

$$\underline{S} = -\psi \underline{\nabla} \psi_t - \frac{1}{2}\beta\psi^2 \hat{\underline{x}}.$$

Then

2011 3c) $$E = \frac{1}{2}A^2(k^2 + l^2)\sin^2\theta + \frac{1}{2}A^2 F\cos^2\theta.$$

Introduce the average of any time-varying quantity of period T as

$$\langle h \rangle = \frac{1}{T}\int_0^T h(t)\,dt \qquad \left(T = \frac{2\pi}{\omega}\right)$$

$\langle \sin \omega t \rangle = 0 \qquad \langle c \rangle = c.$

$\langle \cos \omega t \rangle = 0 \qquad \langle f+g \rangle = \langle f \rangle + \langle g \rangle.$

$\langle \sin^2 \omega t \rangle = \langle \cos^2 \omega t \rangle$ (same graph)

$\sin^2 \omega t + \cos^2 \omega t = 1.$

$\langle \sin^2 \omega t \rangle + \langle \cos^2 \omega t \rangle = 1$

So $\langle \sin^2 \omega t \rangle = 1/2$

$\langle \cos^2 \omega t \rangle = 1/2.$

Thus

$\langle E \rangle = \frac{1}{4} A^2 (k^2 + l^2 + F)$ nw average energy density.

[shortest waves $k \gg 1$, $l \gg 1$ have highest energy density]

Similarly

$$\underset{\sim}{S} = -A\cos\theta \left[\omega A \cos\theta \, (k\hat{\underline{x}} + l\hat{\underline{y}}) \right] - \frac{1}{2} \beta A^2 \cos^2\theta \, \hat{\underline{x}}.$$

$$\langle \underline{S} \rangle = -\frac{1}{2} A^2 \omega (k\hat{\underline{x}} + l\hat{\underline{y}}) - \frac{1}{4} \beta A^2 \hat{\underline{x}}$$

i.e. $\langle \underline{S} \rangle = -\frac{1}{2} A^2 \omega \, \underline{\mathcal{K}} - \frac{1}{4} A^2 \beta \, \hat{\underline{x}}.$

$$= +\frac{1}{2}A^2 \underline{K}\,\beta k/(k^2+l^2+F) - \frac{1}{4}A^2\beta\,\hat{\underline{x}}.$$

$$= \frac{\langle E\rangle}{(k^2+l^2+F)^2}\left\{2\beta k\ \underline{K} - \beta(k^2+l^2+F)\hat{\underline{x}}\right\}$$

$$= \frac{\langle E\rangle}{(k^2+l^2+F)^2}\left\{\hat{\underline{x}}\,\beta(l^2+F-k^2) + \hat{\underline{y}}\cdot 2\beta k l\right\}$$

Note: $\underline{K} = k\hat{\underline{x}} + l\hat{\underline{y}}$.

But the group velocity is:

$$\underset{\sim}{C}_g = \frac{\partial\omega}{\partial k}\hat{\underline{x}} + \frac{\partial\omega}{\partial l}\hat{\underline{y}} = (\nabla_{\underline{K}}\,\omega)$$

$$= \boxed{\frac{\beta(k^2-l^2-F)}{(k^2+l^2+F^2)}}\hat{\underline{x}} + \boxed{\frac{2\beta k l}{(k^2+l^2+F)^2}}\hat{\underline{y}}.$$

RW Energy density

i.e. $\langle \underset{\sim}{S} \rangle = \underset{\sim}{C}_g \langle E \rangle$ RW energy flux

Special Subclass velocity

i.e. the Rossby wave energy travels at the group velocity.

$\underline{C}_g = \underline{\nabla}_{\underline{K}}\,\omega.$

i.e $\underline{C}_g$ is $\perp$'r to lines of constant ω.

We call the lines (or surface of constant ω, slowness surfaces or slowness curves. Lines of constant ω in (k, l) space :- $k^2 + l^2 + F = -\frac{\beta k}{\omega}$.

$$k^2 + \frac{\beta}{\omega} k + l^2 = -F.$$

$$\left(k + \frac{\beta}{2\omega}\right)^2 + l^2 = \left(\frac{\beta}{2\omega}\right)^2 - F.$$

Circles, centre $\left(-\frac{\beta}{2\omega}, 0\right)$
radius $\left(\left(\frac{\beta}{2\omega}\right)^2 - F\right)^{1/2}$.

$\omega = -\beta k / (k^2 + l^2 + F)$ $\quad \omega < 0$ when $k > 0$.

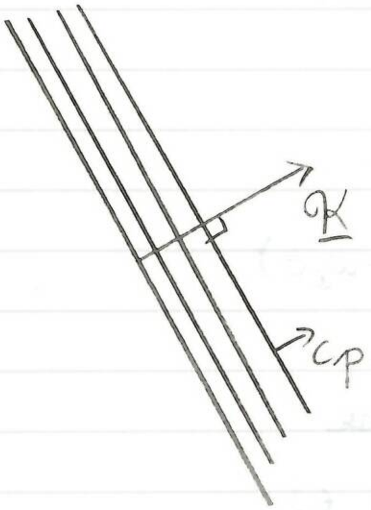
$$-\beta/2\omega > 0.$$

$\underline{c}_g$ is $\perp$'r to this circle i.e it is a radius for any wave $P(k,l)$ on the circle. $\underline{\nabla}$ is in direction of the function increasing.
(i.e $|\omega|$ decreases)
So if ω increases from ω_0 what happens

to the circle? [Group velocity : Lighthill]

-slowness circle lies outside $\omega = \omega_0$.

For completeness, note that the phase velocity is



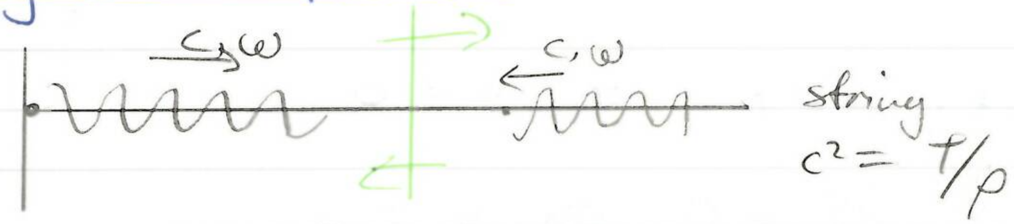
$$\underline{c}_p = \frac{\omega}{|\underline{K}|} \cdot \hat{\underline{K}}.$$

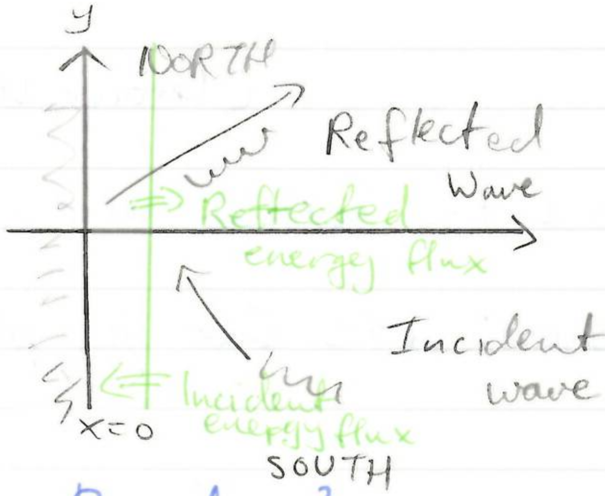
For RWs, $\omega < 0$.
so $\underline{c}_p$ is in the oppsite direction to $\hat{\underline{K}}$ or $\underline{K}$.

Notice the phase velocity always has a positive westward component.

The group velocity has an eastward component for short waves $(k > -\beta/2\omega)$ and a westbound component for long waves $(k < -\beta/2\omega)$

<u>Rossby wave reflection</u>.





Western Boundary?

Let the incident wave be

$$\psi_I = A_I e^{i(k_I x + l_I y - \omega_I t)}$$

and the reflected wave be

$$\Psi_R = A_R e^{i(k_R x + l_R y - \omega_R t)}$$

Find A_R, k_R, l_R, ω_R.

The combined wave has

$$\psi = \Psi_I + \Psi_R$$

There is a solid wall at $x=0$.

i.e. $u=0$ on $x=0$.

i.e. $\dfrac{\partial \psi}{\partial x} = 0$ on $x=0$.

i.e $\psi = \text{const}$ on $x=0$.

i.e $\psi = 0$ on $x = 0$ (aly one bdry)

i.e $\psi = 0$ on $x = 0$ $\forall y, t$.

In particular, $\psi = 0$ at $x = 0$, $y = 0$, $\forall t$.

i.e $A_I e^{-i\omega_I t} + A_R e^{-i\omega_R t} = 0$.

$$e^{i(\omega_R - \omega_I)t} = -A_R / A_I \quad \forall t.$$

But RHS is a constant. So LHS is constant
i.e. $\omega_R = \omega_I$ $\forall t$.

and $A_R = -A_I$.

Remains to obtain k_I, l_I

Now we have, on $x = 0$.

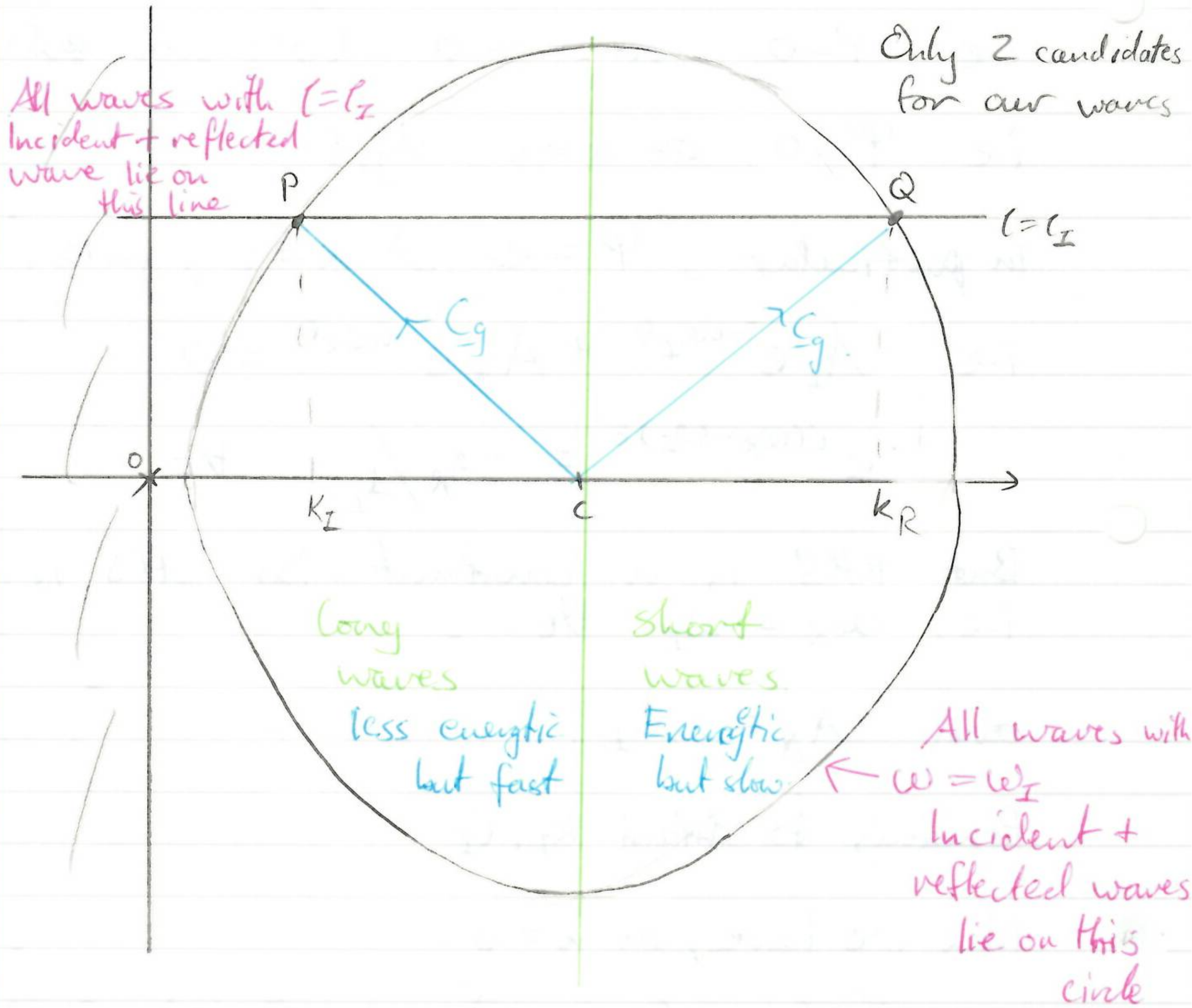
$$\cancel{A_I} e^{i(\ldots + l_I y - \cancel{\omega_I t})} - \cancel{A_I} e^{i(\ldots + ky - \cancel{\omega_I t})} = 0$$

$\forall y, t$.

$$e^{i(l_I - l_R)y} = 1 \quad \forall y$$

i.e $l_I = l_R$.

Remains to find k_R.



The incident waves, carries energy towards the wall, i.e. has group velocity with component towards the wall.

This here the incident wave is P with group vel. towards wall. And the reflected wave with group velocity component away from wall is Q.

$$\langle E \rangle = \frac{1}{4} A^2 (k^2 + l^2 + F)$$

$$k_R \ggg k_I, \quad \langle E \rangle_R \ggg \langle E_I \rangle .$$

It is the energy <u>**fluxes**</u> that must balance: the amount of energy crosses a lie $x = \text{constant}$ inwards must equal the current crossing per unit outwards.

$$\langle \underline{S}' \rangle = \underline{C}_g \langle E \rangle .$$

$$\underline{C}_g' = \frac{(\hat{x} - 2\beta k l \hat{y})}{(k^2 + l^2 + F)^2}$$

The short wave travel portorially slower.

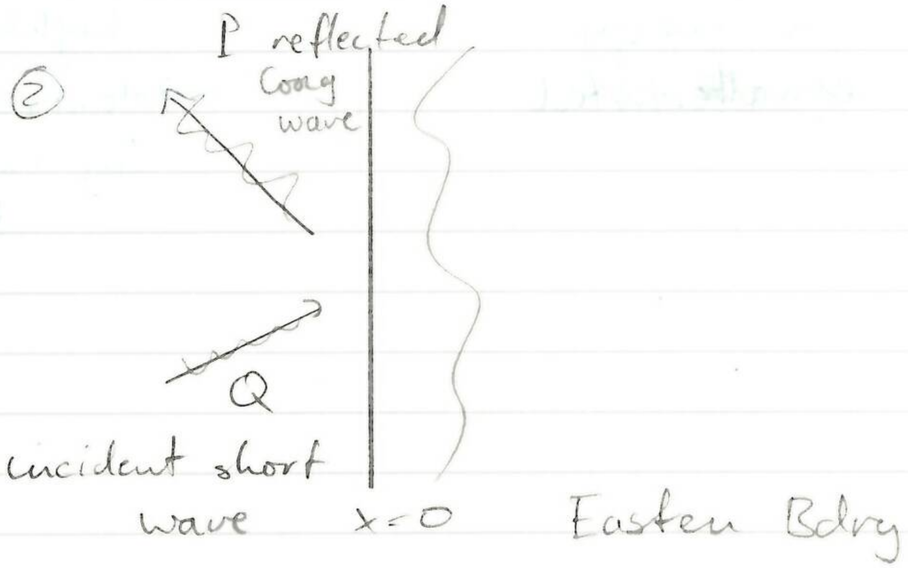
$$\langle E \rangle = \frac{1}{4} A^2 (k^2 + l^2 + F)$$

$$k_R \gg\gg k_I .$$

$$\langle E \rangle_R \gg\gg \langle E \rangle_I$$

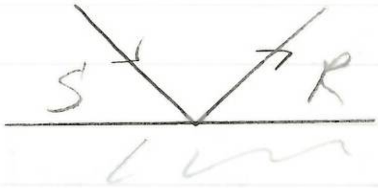
But $(|\underline{C}_g|)_R \ll\ll (|\underline{C}_g|)_I$

So $\langle \underline{S} \rangle_R = \langle \underline{S} \rangle_I$

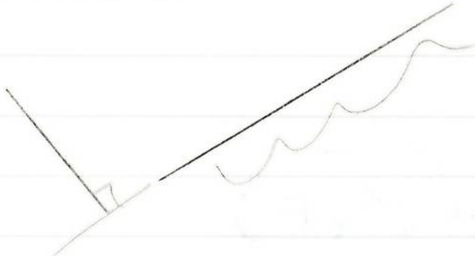


③

④



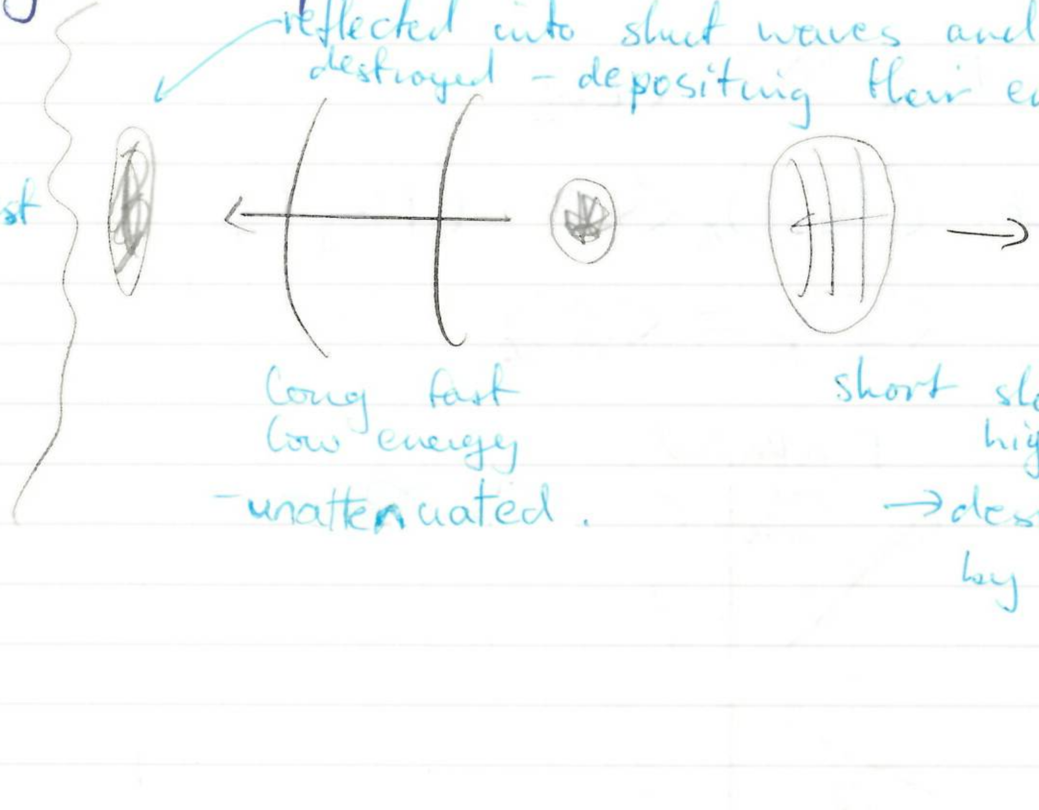
⑤



Physical interpr.

reflected into short waves and are destroyed - depositing their energy

West



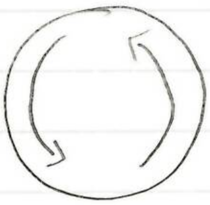
Long fast
Low energy
-unattenuated.

short slow
highly energetic
→destroyed
by random
processes

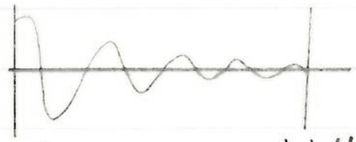
5/3/13

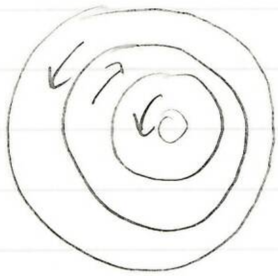
$R_o \gg 1$

$\frac{U}{fL}$ $f \ll 1$.

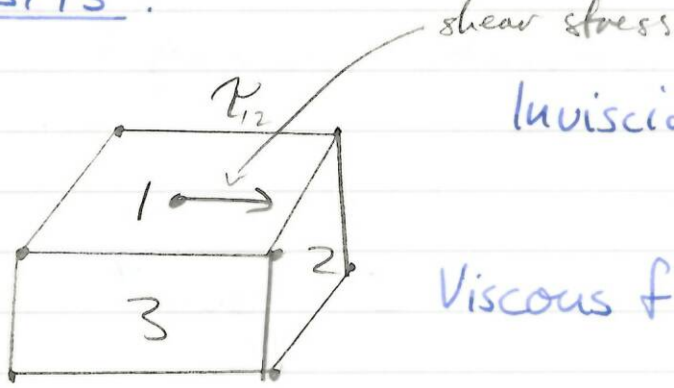


$J_n(\lambda r)\, e^{i(n\theta - \omega t)}$





8/3/13.



Inviscid $\Rightarrow$ cannot support a shear stress

Viscous fluid $\Rightarrow$ support a shear stress.

τ_{ij} – tensor.
i = which force.
j = which direction.

τ_{ii} pressure. (normal).

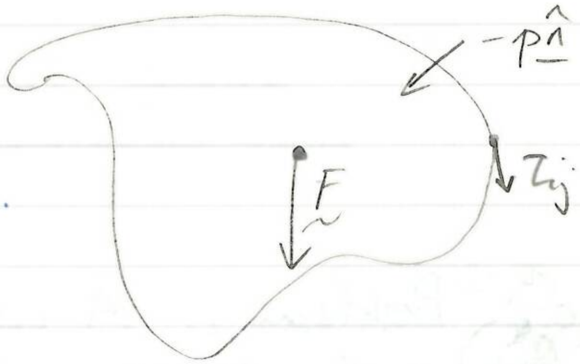
Newtonian fluid

$$\tau_{ij} = \mu\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$$

Stress is propotional to the rate of strain.

μ – coefficient of viscosity.

Obtain the NS eqns:



$$\rho\frac{D\underline{u}}{Dt} = -\underline{\nabla}p + \mu\nabla^2\underline{u}.$$

Or, in our rotating frame.

$$\frac{D\underline{u}}{Dt} + 2\underline{\Omega}\wedge\underline{u} = -\frac{1}{\rho}\underline{\nabla}p + \nu\underline{\nabla}^2 u, \quad \nu = \frac{\mu}{\rho}.$$

Here ν is the kinematic viscosity:

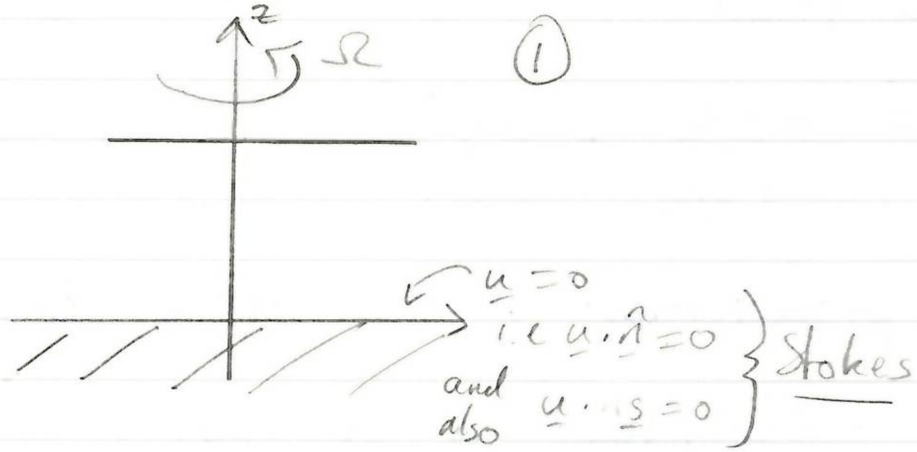
$$[\nu] = L^2 T^{-1}.$$

Water $\nu = 0.01 \text{ cm}^2 \text{sec}^{-1}$.

Our governing eqns are the NS relative to a rotating frame.

$$\frac{D\underline{u}}{Dt} + 2\underline{\Omega} \wedge \underline{u} = -\frac{1}{\rho}\nabla p + \nu \nabla^2 \underline{u}.$$

Continuity $\nabla \cdot \underline{u} = 0$.



②

2010 4a) <u>Problem 1</u>. A uniform flow at infinity (relative to the rotating frame) above a fixed (relative to rot. frame) flat frame.

The governing equations are ⓪ and ⓒ.

The b.c.'s are $\underline{u} = 0$ on $z = 0$. (3)

$\underline{u} \to U\hat{\underline{x}}$ as $z \to \infty$. (4)

(where we have chosen the x-dirn to be in direction of the far field flow)

Our governing eqns are the NS' relative to a rotating frame.

$$\frac{D\underline{u}}{Dt} + 2\underline{\Omega} \wedge \underline{u} = -\frac{1}{\rho}\underline{\nabla} p + \nu \nabla^2 \underline{u} \qquad (1)$$

$$\nabla \cdot \underline{u} = 0. \qquad (2)$$

The inhomogeneous bc, (4) drives the flow. It is the same $\forall t$, so look for a steady solution: $\frac{\partial \underline{u}}{\partial t} = 0$.

It is the same $\forall y$ look for a solution indep of y,

$$\frac{\partial \underline{u}}{\partial y} = 0.$$

It is the same $\forall x$ " " " " " " of x

$$\frac{\partial \underline{u}}{\partial x} = 0$$

Hence $\underline{u} = \underline{u}(z)$, a function of z alone.

Cty, ② gives

$$\cancel{\frac{\partial u}{\partial x}} + \cancel{\frac{\partial v}{\partial y}} + \frac{\partial w}{\partial z} = 0$$

i.e $\frac{\partial w}{\partial z} = 0$, i.e. w same $\forall z$.

But $\underline{u} = 0$ on $z = 0$, so $w = 0$ on $z = 0$
so $w = 0$ $\forall z$.

Then:

$$\frac{D}{Dt} = \cancel{\frac{\partial}{\partial t}} + \cancel{u\frac{\partial}{\partial x}} + \cancel{v\frac{\partial}{\partial y}} + \cancel{w}\frac{\partial}{\partial z}$$

not dep   no x dep   no y dep   There is z-dep BUT $w = 0$ $\forall z$.

$\Rightarrow \frac{D\underline{u}}{Dt} = 0$. (Why not do this when $\Omega = 0$?)

Thus we have

$$2\underline{\Omega} \wedge \underline{u} = -\frac{1}{\rho}\underline{\nabla} p + \nu \nabla^2 \underline{u}.$$

Vert. comp of (4).

$$0 = -\frac{1}{\rho}\frac{\partial p}{\partial z} + \cancel{\nu \nabla^2 w}$$

$w \equiv 0$

Horiz comps of (4):

$$-2\Omega v = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu \frac{\partial^2 u}{\partial z^2} \quad (5)$$

$$\begin{pmatrix} u = u(z) \\ v = v(z) \end{pmatrix}$$

$$2\Omega u = -\frac{1}{\rho}\frac{\partial p}{\partial y} + \nu \frac{\partial^2 v}{\partial z^2} \quad (6)$$

Consider $z \to \infty$,

(5) says $\quad 0 = -\frac{1}{\rho}\frac{\partial p}{\partial x} + 0$.

(6) says $\quad 2\Omega U = -\frac{1}{\rho}\frac{\partial p}{\partial y} + 0$.

Now p is same $\forall z$, so $\partial p/\partial x$, $\partial p/\partial y$ same $\forall z$, so $\partial p/\partial x$, $\partial p/\partial y$ everywhere.

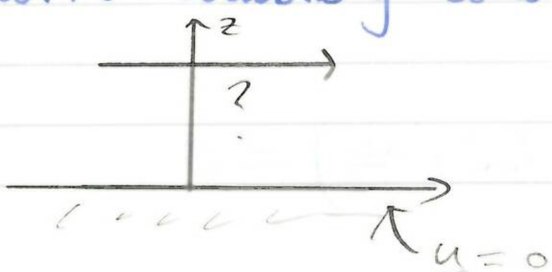
Thus (5) becomes

$$-2\Omega v = \nu u'' \quad (7), \qquad ' \equiv \frac{d}{dz}$$

(6) becomes

$$2\Omega u = 2\Omega U + \nu v'' \quad (8)$$

with boundary condition:



$u = 0,\ v = 0$ on $z = 0$

$u = U,\ v = 0$ on $z \to \infty$.

2 coupled, linear, constant coef ode's with 4 bc's - exponential solns.

It is convenient to introduce the complex velocity:

$$q = u + iv.$$

(7) + i(8).

$$-2\Omega v + i2\Omega u = \nu u'' + 2\Omega U_i + i\nu v''$$

i.e $\nu q'' + 2\Omega U i = 2i\Omega q.$

i.e. $q'' - 2i\left(\frac{\Omega}{\nu}\right)q = -2\frac{\Omega}{\nu}U i$

with boundary condition

$$q = 0 \quad \text{on } z = 0.$$

$$q \to U \quad \text{as } z \to \infty.$$

- 2nd order ode with constant coefs + 2 bc's

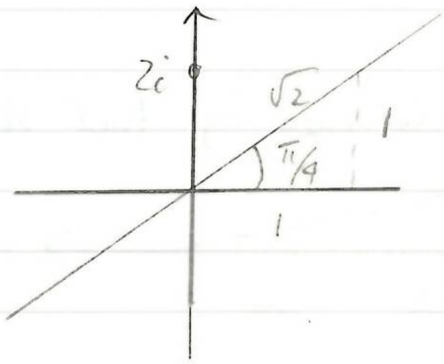
- exponential solns.

GS = PI + CF.

PI, $q_p = U$.

CF, A.E. $\lambda^2 - 2i\left(\frac{\Omega}{\nu}\right) = 0.$

$$\lambda = \pm\sqrt{\frac{\Omega}{\nu}}\sqrt{2i} = \pm\sqrt{\frac{\Omega}{\nu}}(1+i)$$



G.S.

$$q = U + Ae^{\sqrt{\frac{\Omega}{\nu}}(1+i)z} + Be^{\sqrt{\frac{\Omega}{\nu}}(1+i)z}$$

A, B to be determined.

For q bdd as $z \to 0$, $A = 0$.
But $q = 0$ on $z = 0$ so $B = -U$.

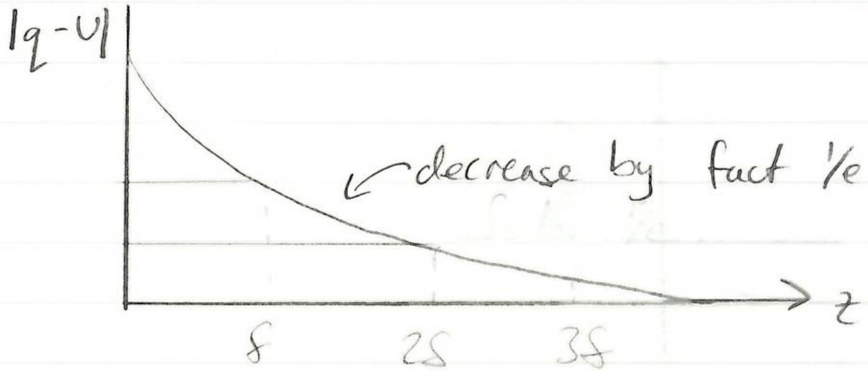
Thus $q = U\left[1 - e^{-(1+i)(z/\delta)}\right]$ Coriolis - Viscous balance.

$$\delta = \sqrt{\nu/\Omega}, \quad [\delta] = \left(\frac{L^2T^{-1}}{T^{-1}}\right)^{-\frac{1}{2}} = L$$

- a length (not surprising as it non-dimensionalise z).

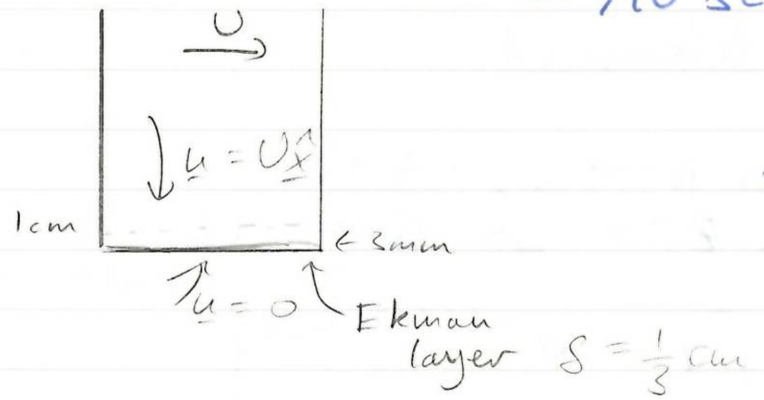
$|q - U|$

Exact solution of NS with a B.L.



decrease by fact $1/e$, i.e. δ is the e-folding scale – thickness of the boundary layer.

Suppose $\Omega = 1$ rev/min
$= 2\pi$ rad / 60 secs
$= 1/10 \text{ sec}^{-1}$.



$$\frac{\nu}{\Omega} = \frac{10^{-2}\,\text{cm}^2\text{sec}^{-1}}{10^{-1}\text{sec}^{-1}} = 10^{-1}\,\text{cm}^2.$$

(Benthic BL)

$$\underline{q} = U\left[1 - e^{-(1+i)(z/\delta)}\right]$$

$$u = U\left[1 - e^{-z/\delta}\cos(z/\delta)\right].$$

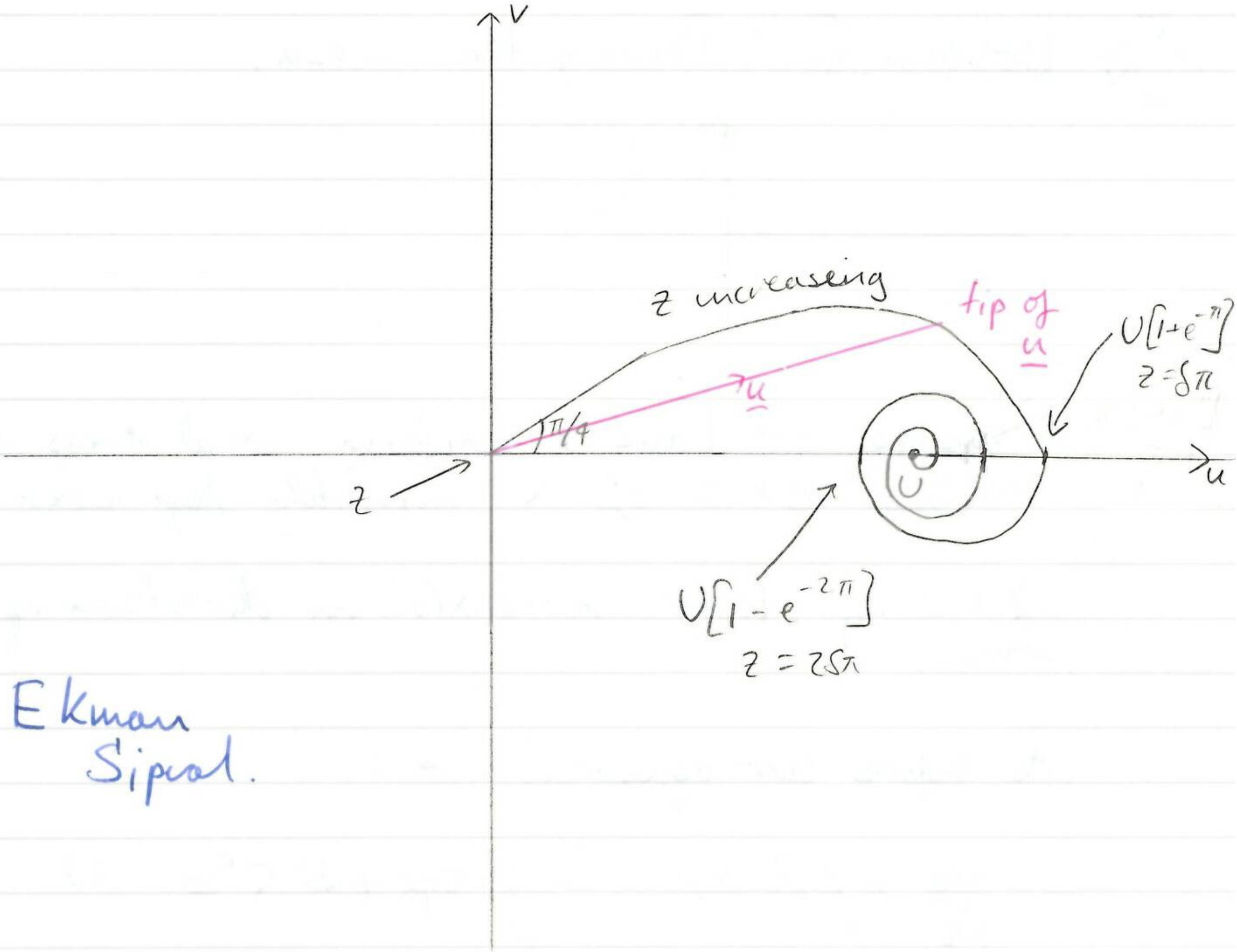
$$v = U\left[e^{-z/\delta}\sin(z/\delta)\right]$$

$z \ll 1$, $\quad \underline{q} = U(1+i)\,z/\delta + O(z^2)$

$v = 0$ when $\frac{z}{\delta} = n\pi$ and then $u = U\left[1 - e^{-n\pi}(-1)^n\right]$

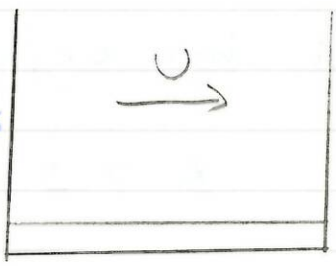
2010
4b)

Ekman Sipral.



Hodograph Plane (u,v)

The total amount of fluid carried by the Ekmann layer is



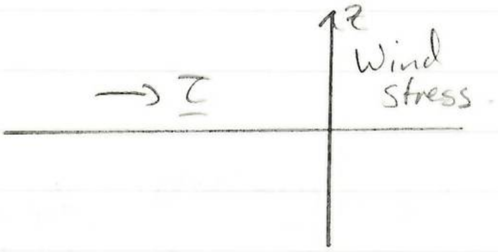
$$M = \int_0^\infty \rho q \, dz$$ at each point.

$= ??$

$$q'' = \frac{2\Omega i}{\nu}(q-U).$$

2012 Q4)

Problem ② : Driving the Ocean.



Suppose we have a uniform wind stress at the surface $z=0$ of a infinitely deep ocean, $z<0$.

W.l.o.g take x-axis in direction of the stress.

As before our equation are:

$$\frac{D\underline{u}}{Dt} + 2\underline{\Omega} \wedge \underline{u} = -\frac{1}{\rho}\underline{\nabla} p + \nu \nabla^2 \underline{u} \quad (1)$$

$$\underline{\nabla} \cdot \underline{u} = 0 \quad (2)$$

with boundary conditions

$\underline{u} \to 0$ as $z \to -\infty$ (no motion at great depth)

Otherwise the surface stress is continuous (otherwise an element of zero thickness, and thus zero mass, would suffer a non-zero force and so have infinite accel) i.e stress in interior $\to \tau_0 \hat{\underline{x}}$.

$$\tau_{zx} = \tau_0 .$$

surface $\tau_{zy} = 0 .$ direction

i.e $\mu\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right) = \tau_0$ on $z=0$.

$\mu\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right) = 0$ on $z=0$.

Inhomog bc is on $z=0$. It is steady so look for a soln with $\frac{\partial \underline{u}}{\partial t} = 0$.

It is the same $\forall y$ so look for a soln with $\frac{\partial \underline{u}}{\partial y} = 0$

" " " " $\forall x$ " " " " " " $\frac{\partial \underline{u}}{\partial x} = 0$

Thus $\underline{u} = \underline{u}(z)$, a function of z alone.

Then (2) (cty) is $\cancel{\frac{\partial u}{\partial x}} + \cancel{\frac{\partial v}{\partial y}} + \frac{\partial w}{\partial z} = 0$.

i.e. w is the same $\forall z$.
But $w \to 0$ as $z \to -\infty$ so $w = 0 \ \forall z$.

Now consider $\frac{D}{Dt} = \underbrace{\cancel{\frac{\partial}{\partial t}}}_{\text{not } t \text{ dep}} + u\underbrace{\cancel{\frac{\partial}{\partial x}}}_{\text{no } x \text{ dep}} + v\underbrace{\cancel{\frac{\partial}{\partial y}}}_{\text{no } y \text{ dep}} + \underbrace{\cancel{w}\frac{\partial}{\partial z}}_{w \equiv 0.}$

Then (1) is

$$2\underline{\Omega} \wedge \underline{u} = -\frac{1}{\rho}\underline{\nabla} p + \nu \nabla^2 \underline{u}.$$

Vertical comp gives:

$$0 = -\frac{1}{\rho}\frac{\partial p}{\partial z} + 0.$$

i.e. p is the same $\forall z$

so $\frac{\partial p}{\partial x}$ and $\frac{\partial p}{\partial y}$ are the same $\forall z$.

Use the horiz comp of (1),

$$-2\Omega v = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\frac{\partial^2 u}{\partial z^2} \qquad (3)$$

$$+2\Omega u = -\frac{1}{\rho}\frac{\partial p}{\partial y} + \nu\frac{\partial^2 u}{\partial z^2} \qquad (4)$$

Look at $z \to -\infty$, (3) gives

$$0 = -\frac{1}{\rho}\frac{\partial p}{\partial x} + 0.$$

so $\frac{\partial p}{\partial x} = 0 \quad \forall z$.

(4) gives $0 = -\frac{1}{\rho}\frac{\partial p}{\partial y} + 0.$

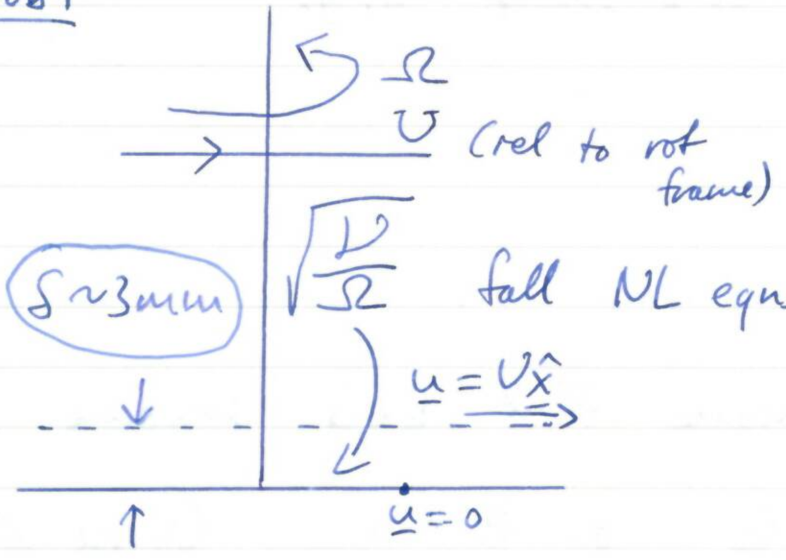
Thus $\frac{\partial p}{\partial y} = 0 \quad \forall z$.

Hence
$$-2\Omega v = \nu u'' \qquad ' \equiv \frac{d}{dz}.$$
$$+2\Omega u = \nu v''$$

slove as before.

12/3/13

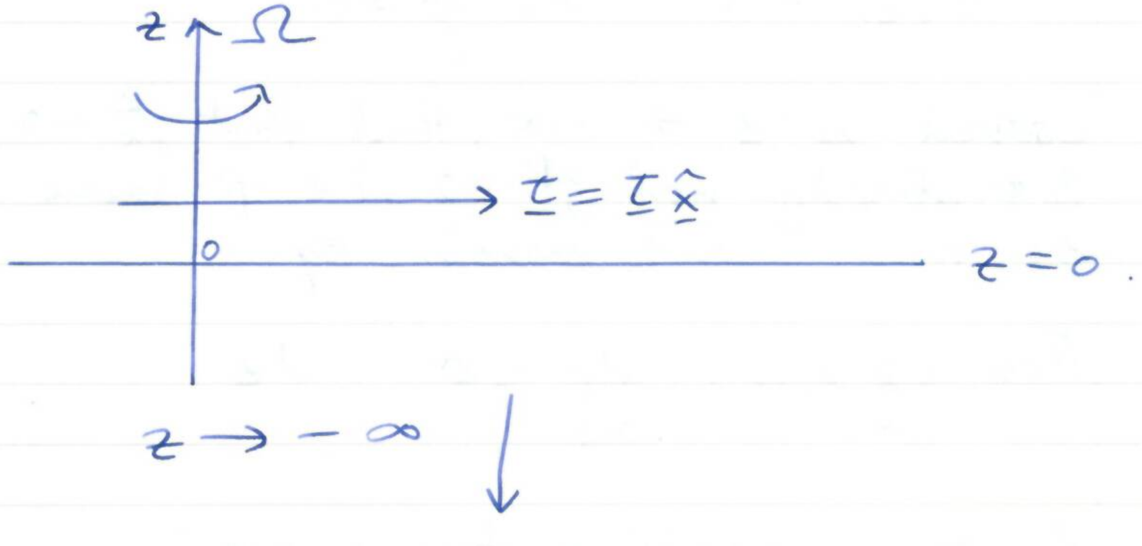
Prob 1



full NL eqns without approx

$$\frac{D}{Dt} \equiv 0 .$$

Prob 2



BC's cty of stress at $z = 0$.

$$\tau = \mu \left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} \right)$$

$$0 = \mu \left(\frac{\partial v}{\partial z} + \frac{\partial}{\partial y} \right)$$

$\underline{u} = \underline{u}(z)$ bc's indep x, y, t.

$w \equiv 0$

$\frac{D}{Dt} \equiv 0$.

Notice also means that bc's become

$u'(z) = \frac{\tau}{\mu}$ on $z=0$ (since $w \equiv 0$)

$v'(z) = 0$ on $z = 0$.

Looked as $z \to -\infty$, find that $\frac{\partial p}{\partial x} = 0$, $\frac{\partial p}{\partial y} = 0$.
But already had $\frac{\partial p}{\partial z} = 0$, i.e p same at all z, from vert mom'm eqn.

Thus $\frac{\partial p}{\partial x} \equiv 0$, $\frac{\partial p}{\partial y} \equiv 0$ $\forall z$.

The only surviving terms are:

$$-2\Omega v = \nu u'' \quad (1)$$

$$+2\Omega u = \nu v'' \quad (2)$$

Introduce the complex velocity

$$q = u + iv.$$

$(1) + i(2)$

$$2\Omega(iu - v) = \nu(u'' + iv'')$$

$$\nu q'' = 2\Omega i q.$$

A.E.

$$\nu\lambda^2 = 2\Omega i.$$

$$\lambda^2 = \frac{\Omega}{\nu} \cdot 2i.$$

i.e $\lambda = \pm\sqrt{\frac{\Omega}{\nu}}\,(1+i)$

$$= \pm\frac{(1+i)}{\delta}$$

$$\delta = \sqrt{\frac{\nu}{\Omega}}.$$

Thus

$$q = Ae^{+(1+i)(z/\delta)} + Be^{-(1+i)(z/\delta)}$$

But q bdd as $z \to -\infty$ so $B = 0$

By the bc's $q' = u' + iv' = \tau/\mu$ on $z = 0$.

But soln gives:

$$q' = \frac{A}{\delta}(1+i) \quad \text{on } z=0.$$

So $$\frac{\tau}{\mu} = \frac{A}{\delta}(1+i)$$

So $$A = \frac{\delta\tau}{\mu(1+i)}$$

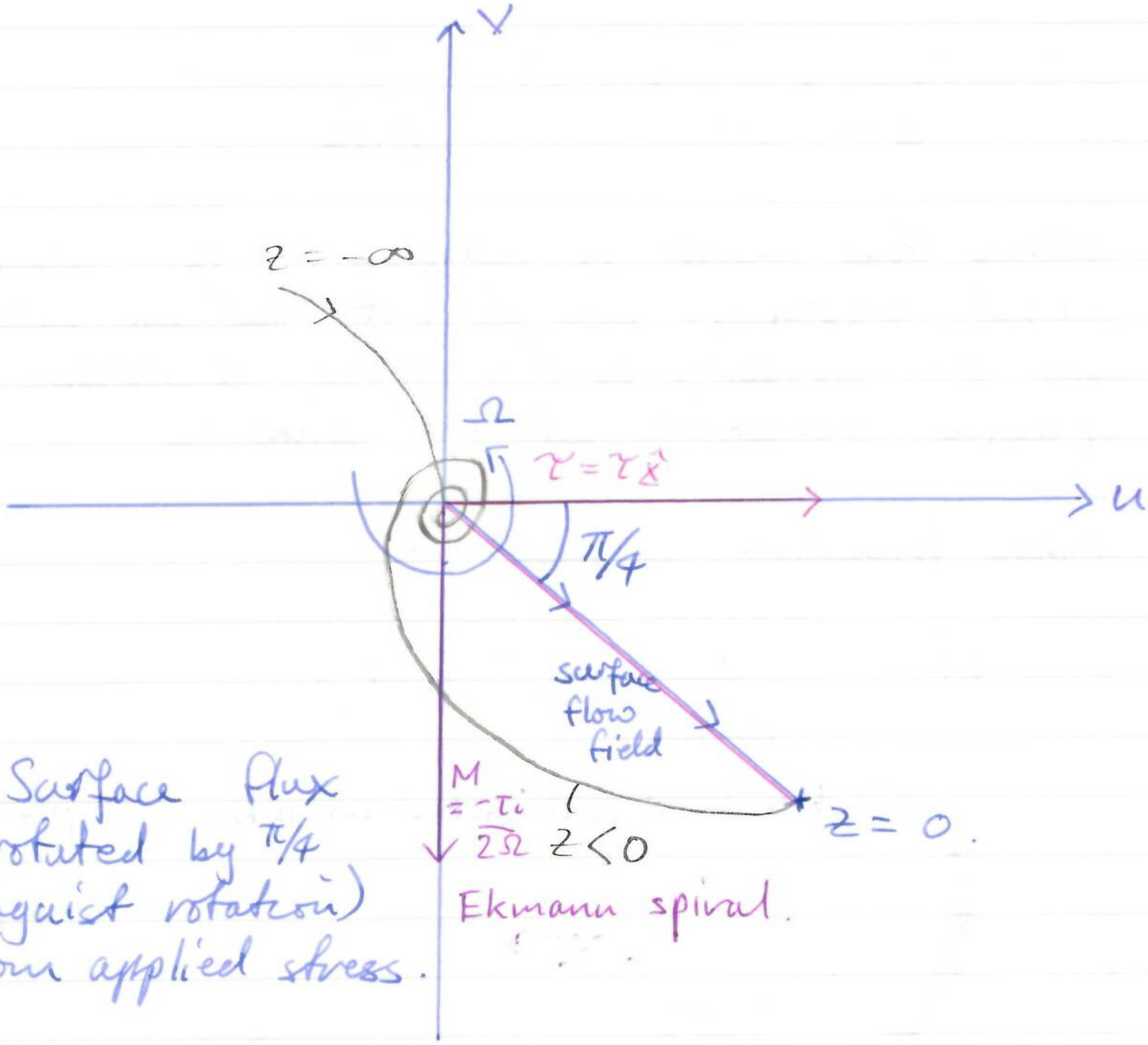
$$= \frac{\delta\tau(1-i)}{2\mu}$$

Hence $$q = \frac{\delta\tau(1+i)}{2\mu}e^{+(1+i)(\frac{z}{\delta})}.$$

$$= u + iv.$$

As the bottom Ekman b.l., the layer thickness is:

$$\delta = \sqrt{\frac{\nu}{\Omega}}.$$



Surface flux rotated by $\pi/4$ (aguist rotation) from applied stress.

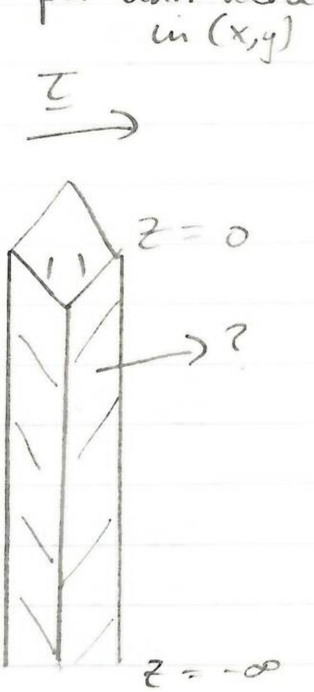
Mass flux associded this stress

$$M = \int_{z=-\infty\ \text{(bottom column)}}^{z=0\ \text{(top column)}} \rho q \, dz.$$

$$= \rho \int_{-\infty}^{0} \frac{\nu}{2\Omega}(-1) q'' \, dz.$$

i.e. $M = -\frac{\rho\nu}{2\Omega} i \int_{-\infty}^{0} q'' \, dz.$

$$= -\frac{\mu}{2\Omega}\left[q'\right]_{-\infty}^{0}.$$

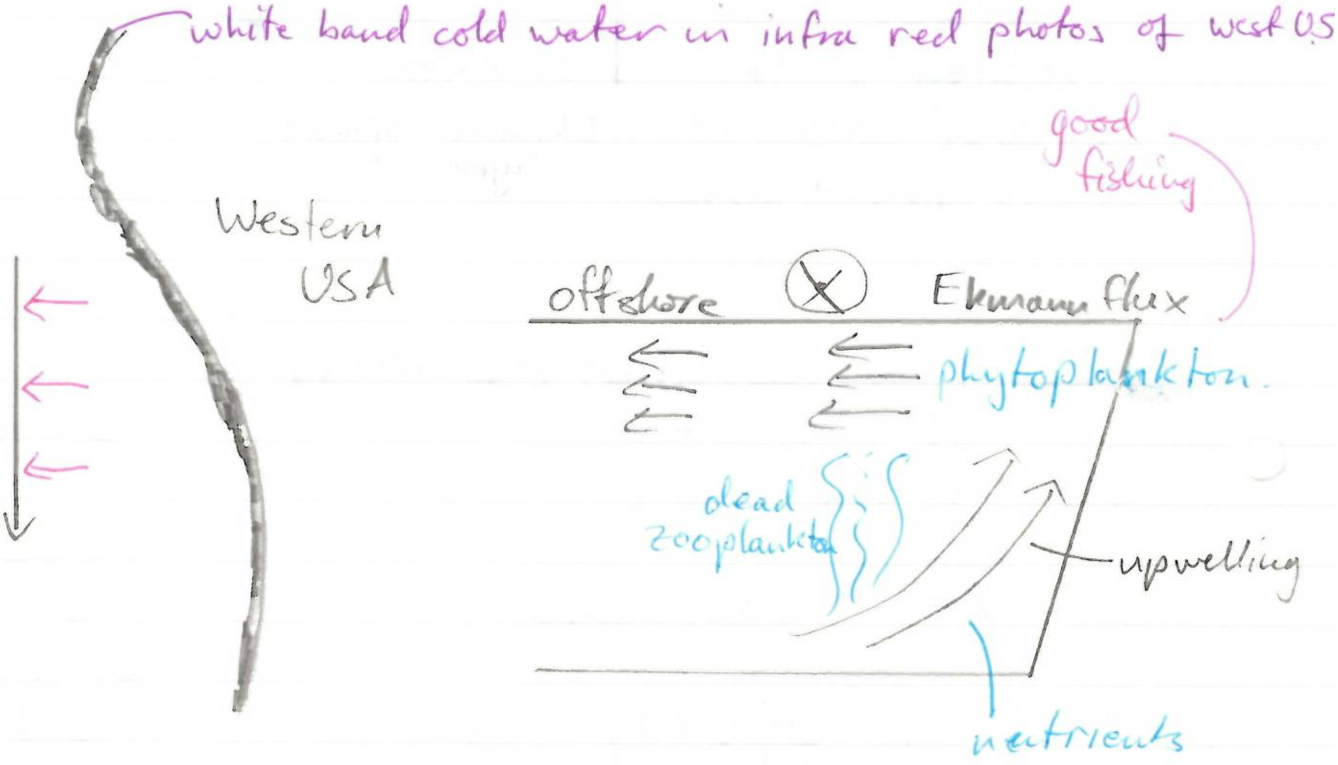


$$= -\frac{\mu}{2\Omega} \cdot \frac{\tau}{\mu} i = -\frac{\tau}{2\Omega} i$$

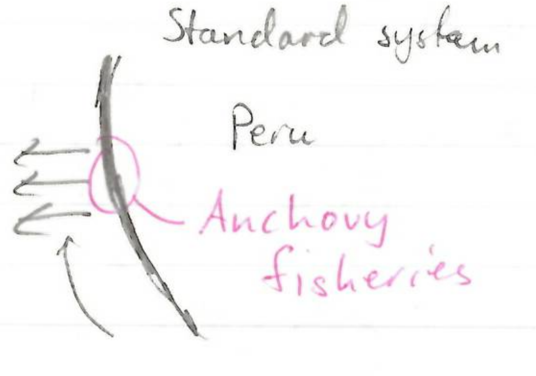
Notice this result is independant of μ, the fluid viscosity: we do not need an estimate for the viscosity of the ocean to obtain the Ekman transport at the surface.

Mass transport in layer is

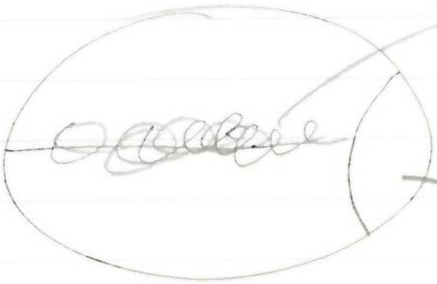
$$\underset{\sim}{M} = -\frac{1}{2\Omega}\left(\hat{\underline{z}} \wedge \underline{\tau}\right).$$



El Nino



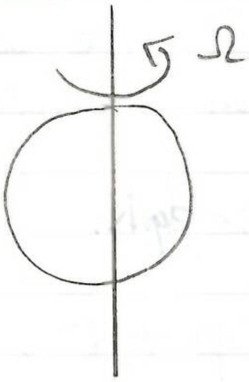
Appox every 4-7 years. Wind relax Upwelling stops No fishing ← collapose.



KW
9 months to cross Pacific

Peru.

<u>15/3/13</u>.



<u>Rotation</u>.
(2D)

rEE

rSWE — nonlin coris (PV)

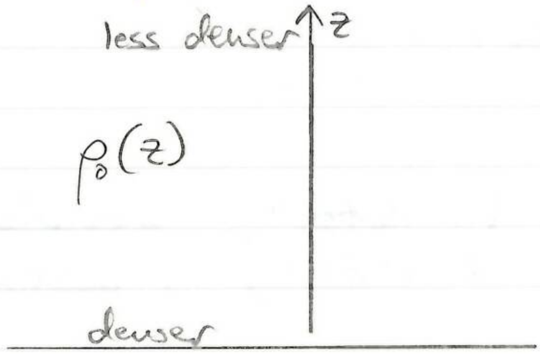
— waves PW, KW, RW ($\omega \ll 1$) (Rossby - Gill Adjustment) motion go to geostrophy.

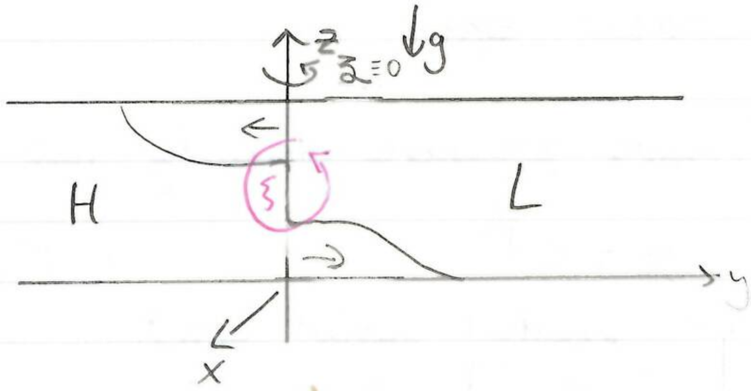
$\omega \ll 1$
$\epsilon = \frac{U}{fL} \ll 1$

QG, PV (RWs) (geostrophically balance)

Viscous : Ekman layers ; top & bottom.

<u>Stratification</u>.





$\boxed{\text{baroclinic}}$. generator of vorticity (inviscid)

We will neglect rotation $\Omega \equiv 0$, i.e consider scales up to 50km to 100km. (small compared to the Rossby radius.

The equations are the 3D Euler eqns.

$$\frac{\partial \underline{u}}{\partial t} + (\underline{u} \cdot \underline{\nabla})\underline{u} = -\frac{1}{\rho} \nabla p + \underline{F}.$$

$\underline{F}$ - external force per unit mass (accel).

Here it is simply gravity $\underline{F} = -g\hat{\underline{z}}$.

i.e. $\rho \frac{D\underline{u}}{Dt} = -\underline{\nabla} p - \rho g \hat{\underline{z}}$.

Conservation of Mass:

$$\frac{\partial \rho}{\partial t} + \nabla . (\rho \underline{u}) = 0.$$

Unknowns $\underline{u}, p, \rho$ 1 vector, 2 scalars.
Eqns. Mom, mass 1 vector, 1 scalar.

Missing 1 scalar eqn.

① In a gas, have pressure density law

$$p = \rho^{\gamma} \qquad (\gamma = \tfrac{5}{3} \text{ in air})$$

(Our needed scalar eqn).

– Sound waves –

(Gas dynamics – Hyperbolic systems).

② Atmosphere and ocean are effectively incompressible (speed law compared to speed of sound, i.e Mach number is small).

Incompressible means a fluid element cannot change volume. But it cannot change its mass. So no volume change. Already cannot change mass.

$$\Rightarrow \frac{\partial \rho}{\partial t} = 0.$$

Thus mass eqn becomes:

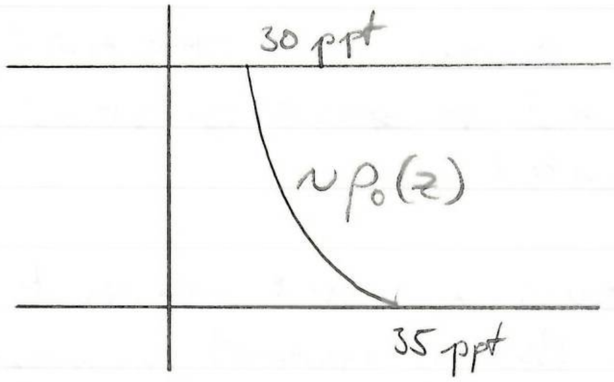
$$\nabla \cdot \underline{u} = 0.$$

Hence we have

$$\rho \frac{D\underline{u}}{Dt} = -\nabla p - \rho g \hat{\underline{z}}.$$

$$\frac{D\rho}{Dt} = 0.$$

$$\nabla \cdot \underline{u} = 0 \qquad \text{(2 scalar, 1 vector eqn as required).}$$



Take $\rho(x,y,z,t) = \rho_0(z) + \rho'(x,y,z,t)$.

Then the hydrostatic $(\underline{u} \equiv 0)$ pressure satisfies:

$$-\underline{\nabla} p_0 - \rho_0 g \hat{\underline{z}} = 0.$$

i.e. $p_0 = C - g\int \rho_0(z)\, dz$.

i.e. $\dfrac{\partial p_0}{\partial z} = -g\rho_0(z)$. Hydrostatic pressure.

Write $p(x,y,z,t) = p_0(z) + p'(x,y,z,t)$.

Perturbation pressure p'

Perturbation density ρ'.

Then we have:

$$(\rho_0 + \rho')\frac{D\underline{u}}{Dt} = -\underline{\nabla}(p_0 + p) - g(\rho_0 + \rho')\hat{\underline{z}}.$$

$$= -g\rho_0\hat{\underline{z}} + g\rho_0\hat{\underline{z}} - \underline{\nabla} p' - g\rho'\hat{\underline{z}}.$$

$$\ldots = -\underline{\nabla} p' - g\rho'\underline{\hat{z}}.$$

2008, 2011, 5) i.e. $$\frac{D\underline{u}}{Dt} = -\frac{1}{(\rho_0+\rho')}\underline{\nabla} p' - \frac{g\rho'}{(\rho_0+\rho')}\underline{\hat{z}}.$$

Boussinesq (1903).

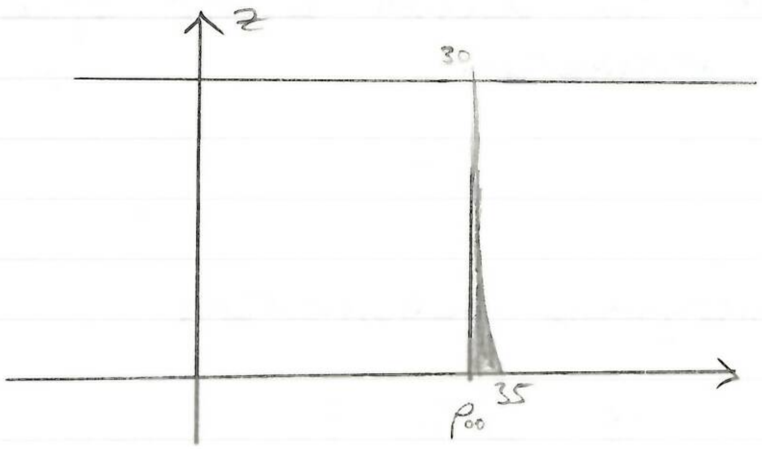
Suppose density perturbation are small,

$$\frac{\rho'}{\rho_0} \ll 1$$

but gravity is strong $g \to \infty$.

$\lim \left(\frac{\rho'}{\rho_0}\right) \to 0$ with $\frac{g\rho'}{\rho_0}$ remains constant.

Then $$\frac{D\underline{u}}{Dt} = -\frac{1}{\rho_0}\nabla p' - \frac{g\rho'}{\rho_0}\underline{\hat{z}}.$$



Then

$$\frac{D\underline{u}}{Dt} = -\frac{1}{\rho_0}\underline{\nabla} p' - \frac{g\rho'}{\rho_0}\underline{\hat{z}}$$

Now make the further approximation that over the heights we are interested in, $\rho_0(z)$ does not change much (e.g 10% in oceans) and replace $\rho_0(z)$ by its average value ρ_{00}.

Then

$$\frac{D\underline{u}}{Dt} = -\frac{1}{\rho_{00}} \underline{\nabla} p' - \frac{g\rho'}{\rho_{00}} \hat{\underline{z}} .$$

If we write $\sigma = -\dfrac{g\rho'}{\rho_{00}}$,

then we have:

$$\frac{D\underline{u}}{Dt} = -\frac{1}{\rho_{00}} \underline{\nabla} p' + \sigma \hat{\underline{z}} .$$

These are the usual Euler equations with an external acceleration.

$$\sigma .$$

- the buoyancy acceleration

Check: If $\rho' > 0$
$\sigma < 0$.

[Equation of state from seawater $\rho(S', T) = \rho_{00}(1 + \alpha S' + \beta T + \ldots)$].

<u>Density eqn</u>:

$$\frac{D\rho}{Dt} = 0$$

i.e $\frac{D}{Dt}(\rho_0(z) + \rho') = 0.$

i.e $\frac{D\rho'}{Dt} + w\rho_0'(z) = 0.$

Multiply by $-g/\rho_{00}$.

$$\frac{D\sigma}{Dt} - \left[\frac{g\rho_0'(z)}{\rho_{00}}\right] w = 0$$

In a stable flow, $\rho_0'(z) < 0$ - density decreases with height

Thus $-\frac{g\rho_0(z)}{\rho_{00}} > 0.$

Thus write.

$$N^2 = -\frac{g\rho_0'(z)}{\rho_{00}}.$$

Then we have:

$$\frac{D\sigma}{Dt} + N^2 w = 0.$$

Our equations in the Boussinesq approximation are thus:

$$\boxed{\begin{aligned} \frac{D\underline{u}}{Dt} &= -\frac{1}{\rho_{00}}\nabla p' + \sigma\hat{\underline{z}}. \\ &\quad \text{given constant.} \\ \frac{D\sigma}{Dt} + N^2 w &= 0. \\ \nabla\cdot\underline{u} &= 0. \end{aligned}}$$

1 vector, 2 scalar eqns.

Unknown $\underline{u}, p', \sigma$.

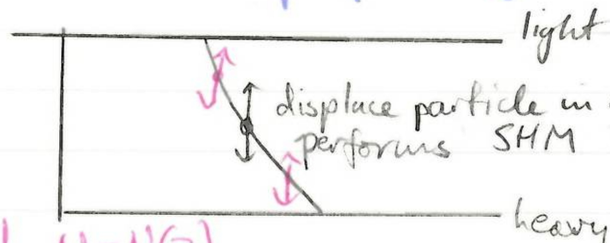
We have a known parameter N^2 in our problem (N^2 is a function of given environment)

$$[N^2] = \left[\frac{g\,\rho_0'(z)}{\rho_{00}}\right]$$

$$= \frac{\cancel{L}\,T^{-2}\cdot(\cancel{\text{density}})\,\cancel{L}^{-1}}{(\cancel{\text{density}})}$$

$$= T^{-2}$$

Thus N is a "frequency".



displace particle in a stable density profile performs SHM with frequency N – the buoyancy frequency –

In general $N = N(z)$.

For simplicity we will take N constant (linear density distribution)

Operate on (*) $(\partial_{tt} + N^2)$:-

$$-\frac{1}{\rho_{00}}(\partial_{tt} + N^2)(\partial_{xx} + \partial_{yy})\, p'$$
$$- \frac{1}{\rho_{00}} \frac{\partial^4 p'}{\partial^2 t\, \partial^2 z} = 0.$$

- in terms of p' alone.

i.e: $$\partial_{tt}\underbrace{(\partial_{xx} + \partial_{yy} + \partial_{zz})}_{\nabla_3^2}\, p'$$
$$+ N^2 \underbrace{(\partial_{xx} + \partial_{yy})}_{\nabla_2^2}\, p' = 0. \quad (\Delta)$$

∇_3^2 : Laplacian in 3D $\quad \partial_{xx} + \partial_{yy} + \partial_{zz}$

∇_2^2 : " " 2D $\quad \partial_{xx} + \partial_{yy}$.

$$\nabla_3^2 p'_{tt} + N^2 \nabla_2^2 p' = 0.$$

INTERNAL wave equation. 3D equation.

Look for plane waves:

$$p = Ae^{i(kx + ly + mz - \omega t)}$$

Using (Δ)

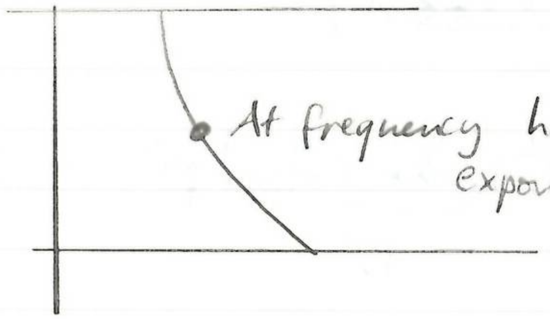
$$[-\omega^2(-k^2 - l^2 - m^2) + N^2(-k^2 - l^2)]\cancel{Ae^{\cdots}} = 0.$$

$$\Rightarrow \omega^2(k^2+l^2+m^2) = N^2(k^2+l^2).$$

i.e. $$\omega^2 = \frac{N^2(k^2+l^2)}{(k^2+l^2+m^2)}$$

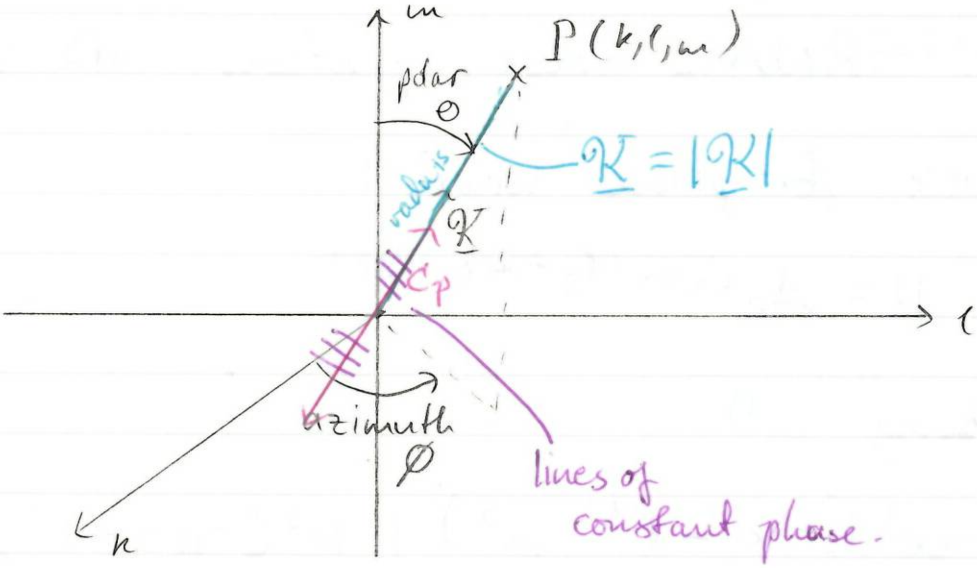
$|\omega| \leq N$ since $k^2+l^2+m^2 \geq k^2+l^2$.

Propagating waves all have frequency less than N.



Introduce spherical polar co-ordinates for the wave number $\underline{K} = k\hat{\underline{x}} + l\hat{\underline{y}} + m\hat{\underline{z}}$.

$$p' = Ae^{i(\underline{K}\cdot\underline{r} - \omega t)}.$$



The linearised density eqn is

$$\sigma_t + N^2 w = 0.$$

The vertical mom'm eqn is (linearised)

$$w_t = \sigma \quad \text{(ignore } p'\text{)}$$

i.e. $\sigma_{tt} + N^2 \sigma = 0$

SHM frequency N.

— / —

To understand our equations we sould examine their wave properties, i.e. we should look at plane-wave solutions of the linear equations. The linear equations (for small perturbation) are

$$\underline{u}_t = -\frac{1}{\rho_{00}} \underline{\nabla} p' + \sigma \hat{\underline{z}}.$$

$$\sigma_t + N^2 w = 0.$$

$$\underline{\nabla} \cdot \underline{u} = 0.$$

The vertical mom'm eqn is

$$w_t = -\frac{1}{\rho_{00}} \frac{\partial p'}{\partial z} + \sigma.$$

So

$$w_{tt} = -\frac{1}{\rho_{00}} \frac{\partial^2 p'}{\partial z \partial t} + \sigma_t .$$

i.e.

$$w_{tt} + N^2 w = -\frac{1}{\rho_{00}} \frac{\partial^2 p'}{\partial z \partial t}$$

or $$(\partial_{tt} + N^2) w = -\frac{1}{\rho_{00}} \frac{\partial^2 p'}{\partial z \partial t} .$$

Eliminded σ.

x - mom'm.

$$u_t = -\frac{1}{\rho_{00}} \frac{\partial p'}{\partial x}$$

$$v_t = -\frac{1}{\rho_{00}} \frac{\partial p'}{\partial y}$$

cty:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 .$$

i.e. $$\frac{\partial^2 u}{\partial x \partial t} + \frac{\partial^2 v}{\partial y \partial t} + \frac{\partial^2 w}{\partial z \partial t} = 0 .$$

i.e. $$-\frac{1}{\rho_{00}} \frac{\partial^2 p'}{\partial x^2} - \frac{1}{\rho_{00}} \frac{\partial^2 p'}{\partial y^2} + \frac{\partial^2 w}{\partial z \partial t} = 0 . \quad (*)$$

i.e Write

$$k = (\mathcal{K} \sin\theta)\cos\phi.$$

$$l = (\mathcal{K} \sin\theta)\sin\phi.$$

$$m = \mathcal{K}\cos\theta.$$

$0 < \theta \leq \pi$, $-\pi \leq \phi < \pi$, $k > 0$

$$\omega^2 = N^2 \frac{k^2 + l^2}{k^2 + l^2 + m^2}$$

$$= \frac{N^2}{\mathcal{K}^2} \mathcal{K}^2 \sin^2\theta.$$

$$= N^2 \sin^2\theta.$$

$$\omega = \pm N \sin\theta.$$

(frequency indep of wavelength).

Phase velocity.

$$\underline{c}_p = \frac{\omega}{|\underline{\mathcal{K}}|} \underline{\mathcal{K}}$$

$$= \pm \frac{N\sin\theta}{\mathcal{K}} \hat{\underline{\mathcal{K}}}.$$

Group velocity is $\underline{C}_g = \underline{\nabla}_{\underline{k}}\omega$.

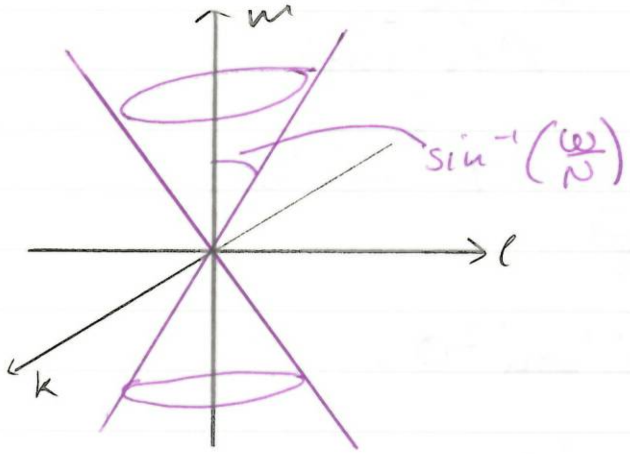
Slowness surface: $\omega = \text{const}$.

$$N \sin\theta = \text{const}.$$

$$\sin\theta = \text{const}.$$

$$\theta = \text{const}.$$

$$\text{i.e} = \text{cones}.$$



$$\sin\theta = \frac{\omega}{N}$$

$$\theta = \sin^{-1}\left(\frac{\omega}{N}\right)$$

$\underline{\nabla}_k\, \omega$ $\perp$'r to slowness curve.

19/3/13

Stratifacation

Boussinesq Appox $\quad \frac{\rho'}{\rho_{00}} << 1 \qquad \frac{g}{\omega_t} \to 0.$

$$\frac{D\underline{u}}{Dt} = -\frac{1}{\rho_{00}} \underline{\nabla} p' + \underbrace{\sigma \hat{\underline{z}}}_{\text{only in z momentum eqn}} , \qquad \sigma = \frac{-g\rho'}{\rho_{00}}$$

σ bouyancy accel.

$$\frac{D\rho}{Dt} + \rho \underline{\nabla} \cdot \underline{u} = 0$$

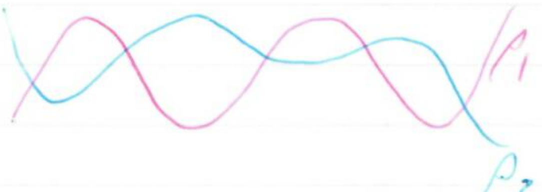
Incompressible : $\Rightarrow \frac{D\rho}{Dt} = 0$

elements conserve their density But different els. can conserve different density.

get

$$\nabla . \underline{u} = 0.$$

(sound waves travel infinitely fast)



$\underline{u}, \sigma, p'.$

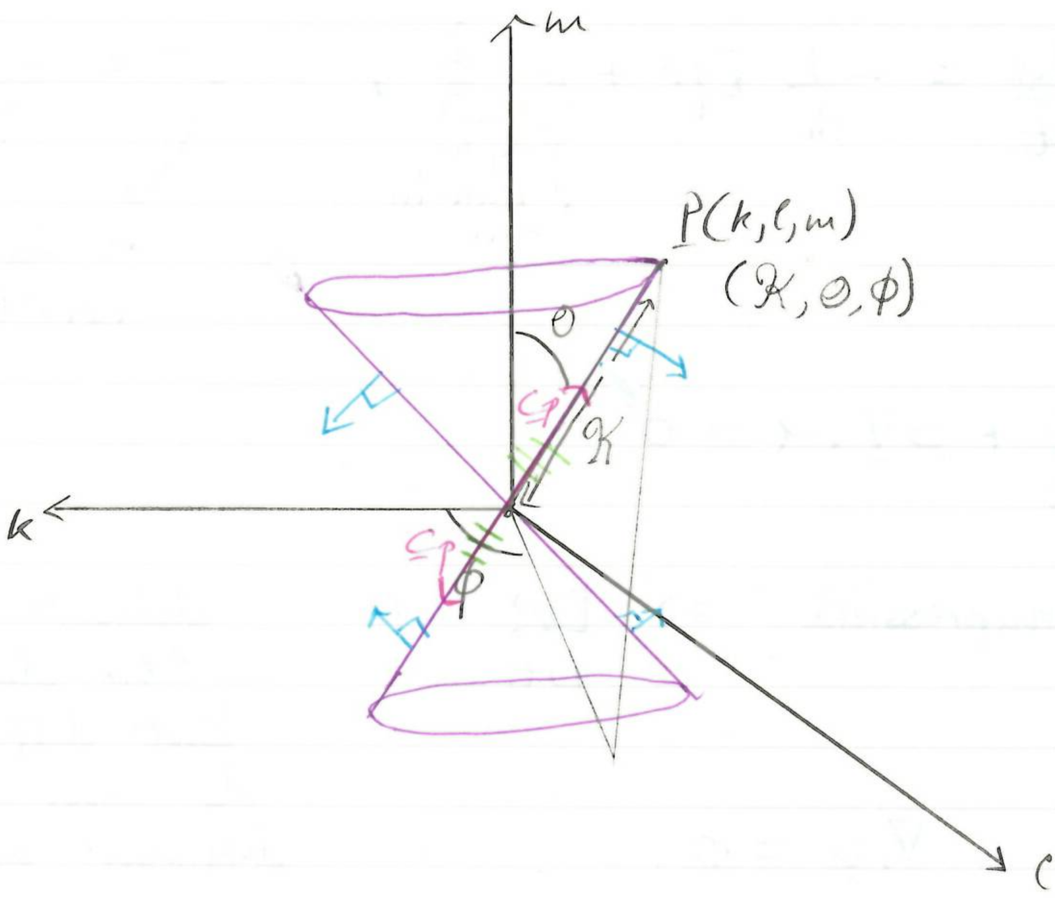
Internal Wave Equation:

$$\frac{\partial^2}{\partial t^2} \nabla_3^2 p' + N^2 \nabla_2^2 p' = 0.$$

Look for plane wave solns:

$$-\omega^2(-k^2-l^2-m^2)+N^2(-k^2-l^2)=0.$$

dispersion relation.



$$m=\mathcal{K}\cos\theta.$$
$$k=\mathcal{K}\sin\theta\cos\phi$$
$$l=\mathcal{K}\sin\theta\sin\phi.$$

Then:

$$\omega^2=N^2\sin^2\theta.$$
$$\omega=\pm N\sin\theta.$$

$$\underset{\sim}{c}_p = \frac{\omega}{\kappa}\hat{\underset{\sim}{\kappa}}$$

For waves of frequency ω, the phase velocity lies along the cones.

$$\sin\theta = \pm\frac{\omega}{N}$$

$$\text{or} \quad \theta = \pm\sin^{-1}\left(\frac{\omega}{N}\right)$$ gives direction of $\underset{\sim}{c}_p$.

Note $|\omega| < N$.

For the group velocity;

$$\underset{\sim}{c}_g = \nabla_{\underset{\sim}{k}}\,\omega. \quad , \quad \omega = \omega(k,l,m) \overset{\text{or}}{=} \omega(\kappa,\theta,\phi)$$

- the gradient in wavenumber space of ω

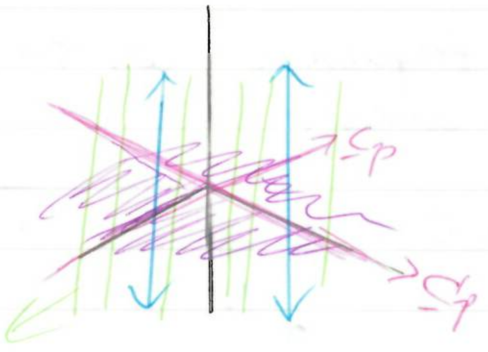
- $\perp$'r to the level surfaces of ω, slowness surface

- these are the cones $\theta = \pm\sin^{-1}\left(\frac{\omega}{N}\right)$

- If $0 < \theta < \pi/2$, then θ increases with $\sin\theta$ if $\pi/2 < \theta < \pi$, θ decreases with increasing $\sin\theta$, i.e $\underset{\sim}{c}_g$ directed towards plane $m = 0$.

limits:

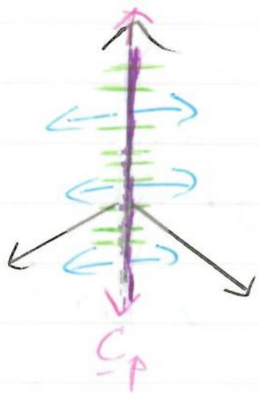
$\omega \to N$.



cones $\to$ plane $m = 0$.

energy travels vertically.

$\omega \to 0$.



energy travels horizontally.

$$\omega = \pm N \sin\theta .$$

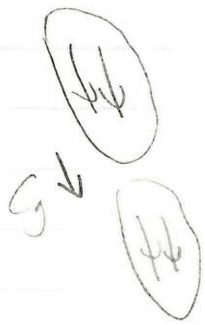
$$\underline{c}_p = \frac{\omega}{\kappa} \hat{\underline{\kappa}} .$$

$$= \pm \frac{N}{\kappa} \sin\theta \; \hat{\underline{\kappa}} .$$

$$\underline{c}_g = \underline{\nabla}_{\underline{\kappa}} \omega$$

$$= \cancel{\frac{\partial\omega}{\partial\kappa}} \hat{\underline{\kappa}} + \frac{1}{\kappa} \frac{\partial\omega}{\partial\theta} \hat{\underline{\theta}} + \frac{1}{\kappa\sin\theta} \cancel{\frac{\partial\omega}{\partial\phi}} \hat{\underline{\phi}} .$$

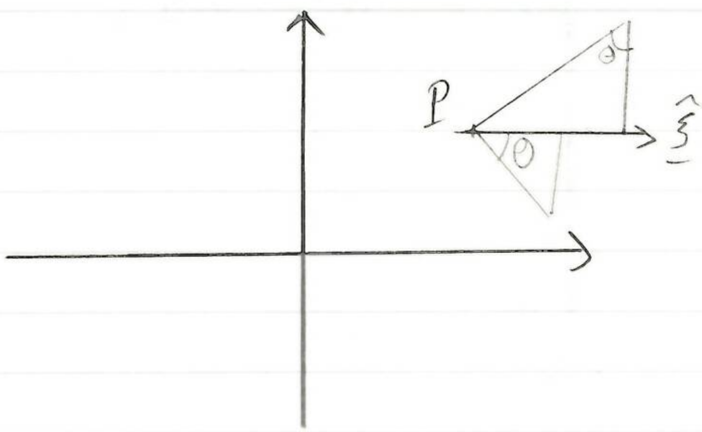
$$= \pm \frac{N}{K} \cos\theta \, \underline{\hat{\theta}}.$$



$$\underline{c}_p \cdot \underline{c}_g = 0.$$

as before $\quad \underline{\hat{\theta}} \cdot \underline{\hat{K}} = 0.$

Notice group velocity + so the energy propagation always lies along the wavecrests.



$$\underline{\hat{\theta}} = \cos\theta \, \underline{\hat{\xi}} - \sin\theta \, \underline{\hat{m}}.$$

$$\underline{\hat{K}} = \sin\theta \, \underline{\hat{\xi}} + \cos\theta \, \underline{\hat{m}}.$$

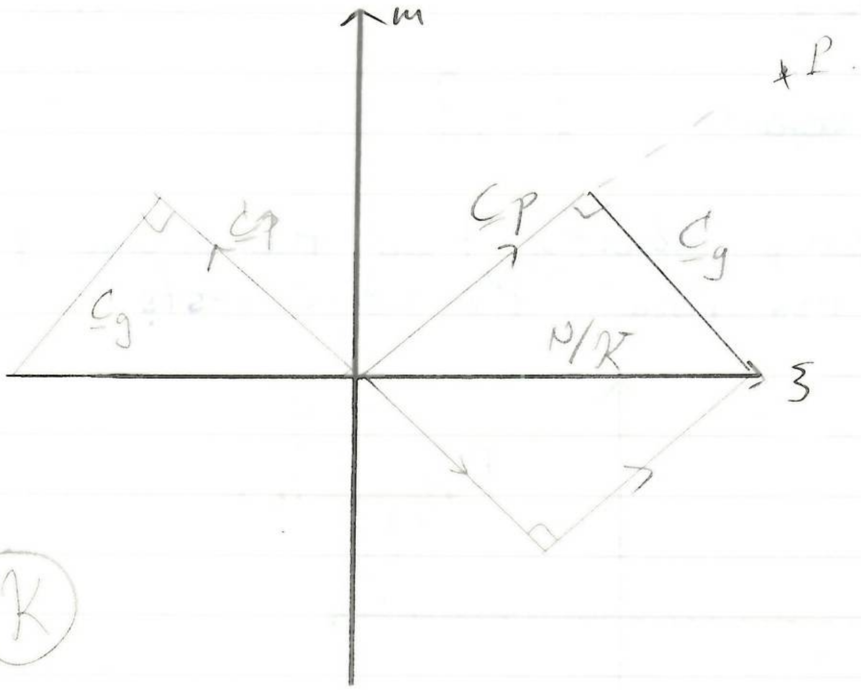
$$\underline{c}_p = \pm \frac{N}{K} \sin\theta \, \underline{\hat{K}} = \pm \frac{N}{K}\left[\sin^2\theta \, \underline{\hat{\xi}} + \sin\theta\cos\theta \, \underline{\hat{m}}\right]$$

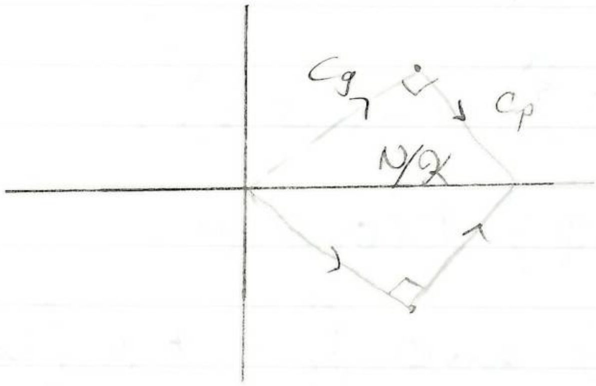
$$\underline{c}_g = \pm \frac{N}{K} \sin\theta \, \underline{\hat{\theta}} = \pm \frac{N}{K}\left[\cos^2\theta \, \underline{\hat{\xi}} - \sin\theta\cos\theta \, \underline{\hat{m}}\right]$$

$$\underline{c}_p \cdot \underline{c}_g = 0.$$

$$\underline{c}_p + \underline{c}_g = \pm \frac{N}{K} \underline{\hat{\xi}}.$$

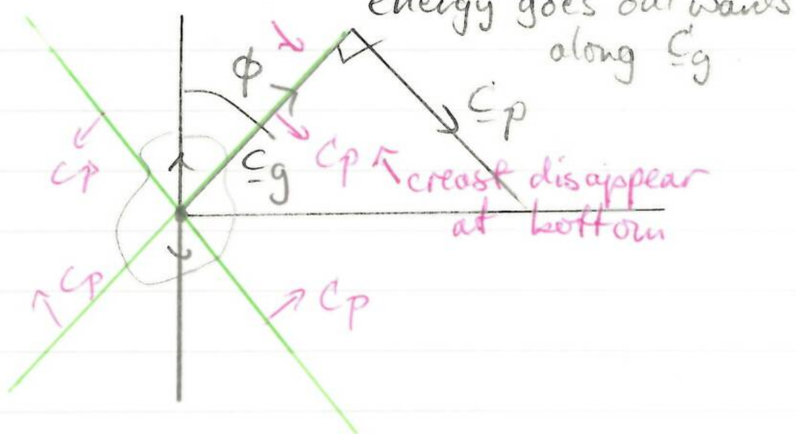
i.e. $\underline{c}_p$, $\underline{c}_g$ are $\perp$'r and add to give the horizontal vector $\pm \frac{N}{K} \hat{\underline{\xi}}$.





— c —

N=constant :



$|\omega| > N$: a very localised distrubance
- everything decays exponentically - no waves.

$\theta = \sin^{-1}\left(\frac{\omega}{N}\right)$

Cone - St. Andrew Cross

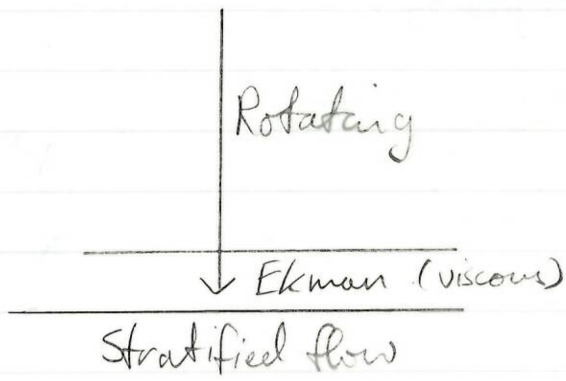
$\phi = \frac{\pi}{2} - \sin^{-1}\left(\frac{\omega}{N}\right)$

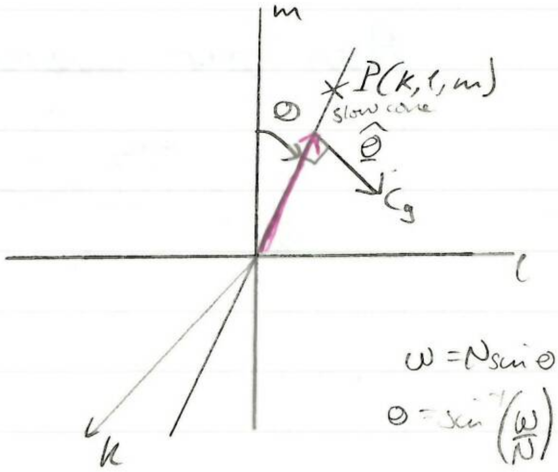
$= \cos^{-1}\left(\frac{\omega}{N}\right)$

Note : limits $\omega \to 0$, $\omega \to N$ gives above.

$|\omega| < N$: Internal Wave Beams.

22/3/13.





Internal Wave Eqn (Boussinesq approx) to 3D Euler with density variations

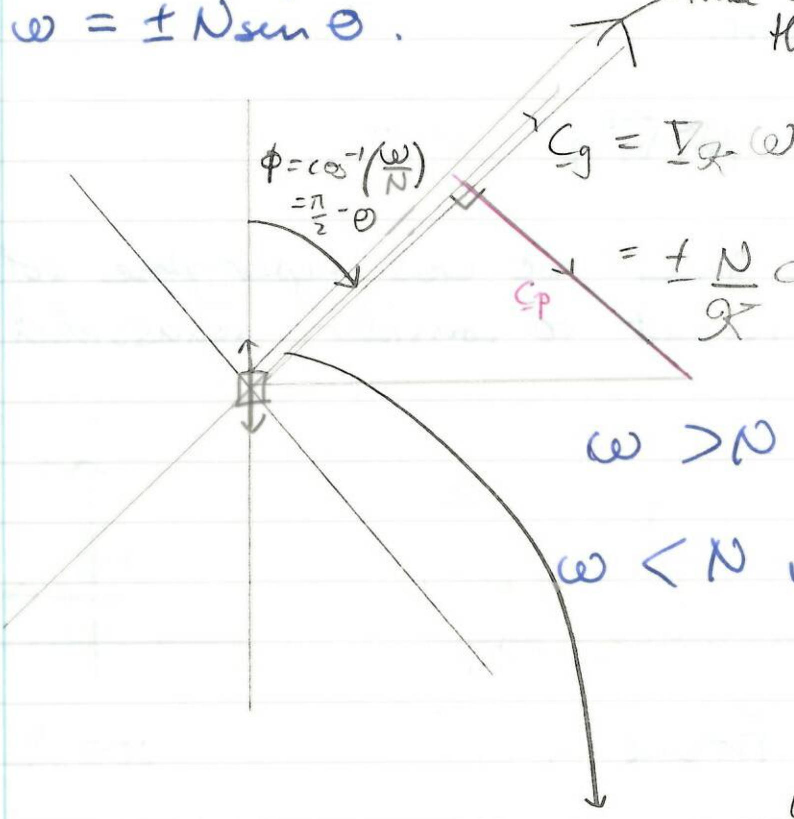
$$\partial_{tt}\nabla_3^2 p' + N^2 \nabla_2^2 p' = 0.$$

$\omega = \pm N\sin\theta$.

tube (the envelope) that lengths with time

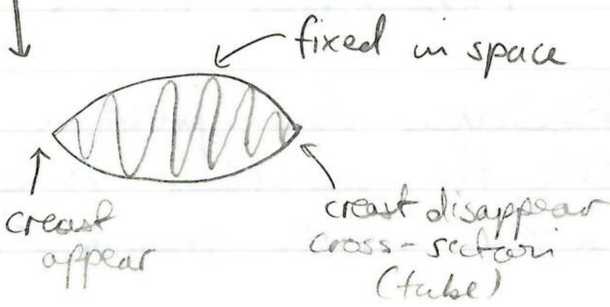
$$\underline{c}_g = \nabla_{\underline{K}}\,\omega$$

$$= \pm \frac{N}{\underline{K}}\cos\theta\,\hat{\underline{\theta}}$$



$\omega > N$ no waves.

$\omega < N$ waves : energy travels with group velocity.



Stephenson (1973)

Flow over uneven orography.

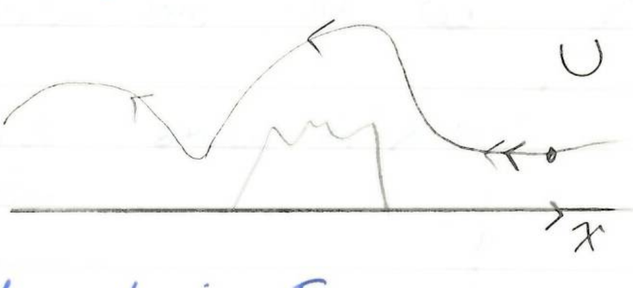
$$X = x - Ut$$
$$x = X + Ut.$$

Assume the orography is low, so that disturbances to the oncoming stream are small (of order $0 < \varepsilon < 1$).

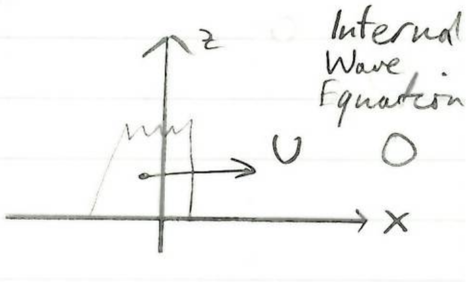
Then the problem is linear, governed (relative to stationary air, i.e $\underline{u} = 0$) by the (linear) internal wave eqn.

$$\partial_{tt} \nabla_3^2 p' + N^2 \nabla_2^2 p' = 0.$$

Because this is linear we can superpose solutions. Thus it is sufficient to consider sinusoidal topography.



Mountain Frame.



$x = X + Ut.$

Because the problem is linear, it is sufficient to take the mountain to be $z = \sin[k(x - Ut)]$ in the (x, z) frame, + any other shape follows.

by Fourier superposition : k arbitrary fixed wavenumber.

$(\underline{u}-\underline{U})\cdot\hat{\underline{n}} = 0$ on $z = \varepsilon \sin(k(x-Ut))$

$\underline{U} = U\hat{\underline{x}}$

Relative normal velocity must vanish.

Surface is $F(x, z) = 0$.

where $F = z - \varepsilon \sin[k(x-Ut)]$

and so has normal

$$\underline{\nabla} F = -\varepsilon k \cos[k(x-\omega t)]\,\hat{\underline{x}} + \hat{\underline{z}}.$$

On $F=0$ we require

$$(\underline{u} - \underline{U})\cdot\underline{\nabla} F = 0.$$

i.e. $$(\underset{\varepsilon}{u} - \overset{1}{U})\left\{-\varepsilon k \cos[k(x-Ut)]\right\} + \underset{\varepsilon}{w} = 0$$

on $z = F$.

The leading order (in ε) give

$$w = -\varepsilon U k \cos[k(x-Ut)] \text{ on } z = \varepsilon \sin[k(x-Ut)]$$

The leading order (in ε) give:

$$w = -\varepsilon U k \cos[k(x-\omega t)] \quad \text{on } z = 0$$

(Taylor) (with error order ε^2)

We required w bdd as $z \to +\infty$.

$$\left(\frac{\partial^2}{\partial t^2} + N^2\right) w = -\frac{1}{\rho_{00}} \frac{\partial^2 \rho'}{\partial z \partial t}$$

Hence we have (by operating on the IWE with $\partial^2/\partial z \partial t$ that w satisfies the IWE., i.e. in 2D.

$$\partial_{tt}(\partial_{xx} + \partial_{zz}) w + N^2 \partial_{xx} w = 0 . \quad (1) \quad z > 0.$$

$$(\partial_y \equiv 0) \quad w = -\epsilon U k \cos[k(x - Ut)] \quad (2) \quad z = 0.$$

$$w \text{ bdd as } z \to \infty . \quad (3)$$

Notice the lower bc is:

$$w = -\epsilon U k \,\mathrm{Re}\left\{ e^{i(k(x-Ut))} \right\} \quad \text{on } z = 0.$$

Try $w = -\varepsilon U k \,\mathrm{Re}\left\{ \tilde{w}(z) e^{i[k(z-Ut)]} \right\}$ for $z \geq 0$

where $\tilde{w}(0) = 1$ to satisfy bc (2) on $z = 0$
$\tilde{w}(z)$ bdd as $z \to \infty$ satisfies (3).

$$[-k^2 U^2(-k^2 + \partial_{zz}) - k^2 N^2]\, \tilde{w} = 0 . \text{ from (1)}$$

$$\tilde{w}'' + \left[\frac{N^2}{U^2} - k^2\right]\tilde{w} = 0.$$

Case 1.

$\frac{N^2}{U^2} < k^2$.

i.e. $\frac{Uk}{N} > 1$ $\quad k \gg 1$

$\Rightarrow$ wavelength $\ll 1$

i.e. fast flow over a short obstacle when the stratification is weak ($N=0$: no stratification)

$$\tilde{w} = Ae^{\sqrt{k^2-\frac{N^2}{U^2}}z} + Be^{\sqrt{k^2-\frac{N^2}{U^2}}z}.$$

(If $N=0$, $\tilde{w} = Ae^{bz} + Be^{-bz}$ soln of Laplace's eqn).

But $\tilde{w}$ bdd as $z \to \infty$ as $A = 0$.

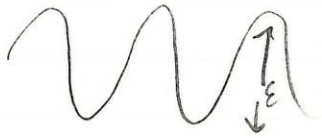
But $\tilde{w}(0) = 1$ so $B = 1$.

i.e $\tilde{w}(z) = e^{-\sqrt{k^2-\frac{N^2}{U^2}}z}$

So

$$w = -\epsilon Uk \, Re\left\{e^{-\sqrt{k^2-\frac{N^2}{U^2}}z} e^{i[k(x-Ut)]}\right\}$$

$$= -\epsilon Uke^{\sqrt{k^2-\frac{N^2}{U^2}}z} \cos[k(x-Ut)].$$



Consider a particle whose average height is z_0 (constant)

Then its height at any instant is

$$z = z_0 + \varepsilon z'$$

(i.e. an order ε perturbation as a bump is order ε).

Then $\dfrac{Dz}{Dt} = \varepsilon \dfrac{Dz'}{Dt}$.

i.e. $w = \varepsilon\left[\dfrac{\partial z'}{\partial t} + (\underset{\uparrow\,O(\varepsilon)}{\underline{u}} \cdot \nabla) z'\right]$

$$= \varepsilon \frac{\partial z'}{\partial t} \quad \text{to } O(\varepsilon^2)$$

i.e. $\dfrac{\partial z'}{\partial t} = -Uk\cos\left[k(x - Ut)\right] e^{-\sqrt{k^2 - \frac{N^2}{U^2}}\, z_0}$.

i.e. $z' = \sin\left[k(x - Ut)\right] e^{-\sqrt{k^2 - \frac{N^2}{U^2}}\, z_0}$

On $z_0 = 0$ (i.e. the bottom)

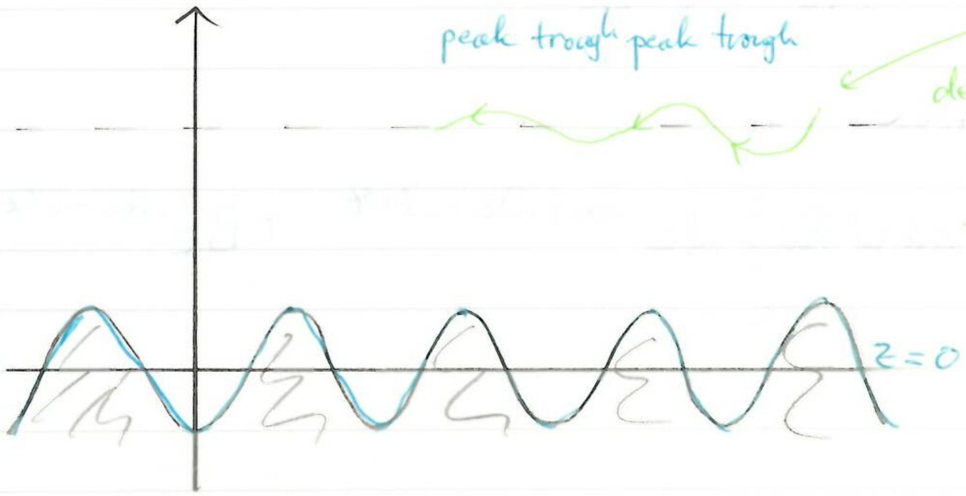
$$z' = \sin\left[k(x - Ut)\right]$$

[As required since $z = \varepsilon \sin\left[k(x - Ut)\right]$ on $z_0 = 0$]

Hence:

$$z = z_0 + \epsilon \sin\left[k(x - Ut)\right] e^{-\sqrt{k^2 - \frac{N^2}{U^2}}\, z_0}$$

In mountain frame: $z = z_0 + \epsilon \sin kx \, e^{-\sqrt{k^2 - \frac{N^2}{U^2}}\, z_0}$.



same shape
decreased amplitude

- e folding lenght

$$\frac{1}{\sqrt{k^2 - \frac{N^2}{U^2}}}$$

$N > 0$: Slower decay
- pseudo-potental flow

- for $N = 0$ this is $1/k$ (as expected from potential flow).
- Laplace's eqn (cf water waves)

<u>Case 2</u>. $\frac{N^2}{U^2} > k^2$.

$$\bar{w}'' + \left(\frac{N^2}{U^2} - k^2\right)\bar{w} = 0.$$

General solution: $\bar{w}(z) = A \cos \lambda z + B \sin \lambda z$
where $\lambda = \sqrt{N^2/U^2 - k^2}$, a vertial wavenumber.

1) $\tilde{w}(0) = A$ so $A = 1$

2) $\tilde{w}$ bdd $z \to \infty$ $B = ?$

$\Big\}$ No conclusion.

General soln:

$$\tilde{w}(z) = Ae^{i\lambda z} + Be^{-i\lambda z}.$$

This gives

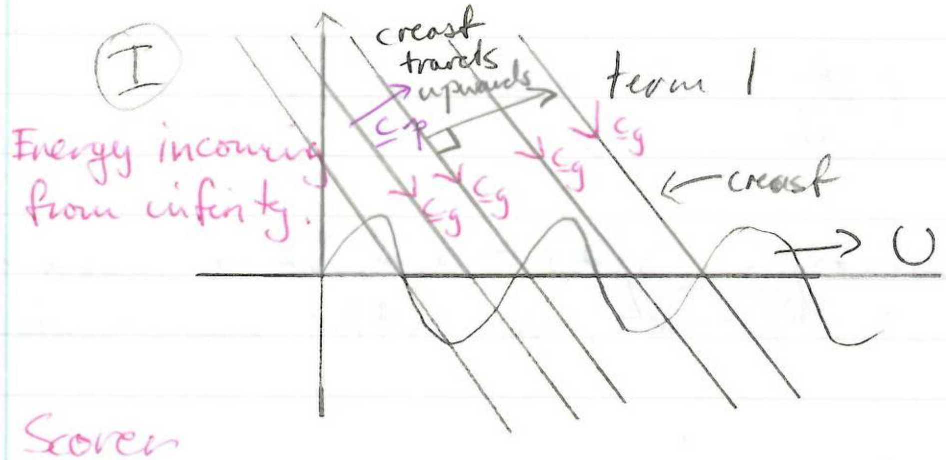
$$w = -\varepsilon k U \,\mathrm{Re}\left\{ Ae^{ikx + i\lambda z - ikUt} + Be^{ikx - i\lambda z - ikUt} \right\}$$

$k > 0$
$\lambda > 0$.

i.e. $$w = -\varepsilon k U \left\{ A\cos[\underbrace{kx + \lambda z - kUt}_{\mathrm{I}}] \right.$$

$$\left. + B\cos[\underbrace{kx - \lambda z - kUt}_{\mathrm{II}}] \right\}.$$

These are internal wave terms.



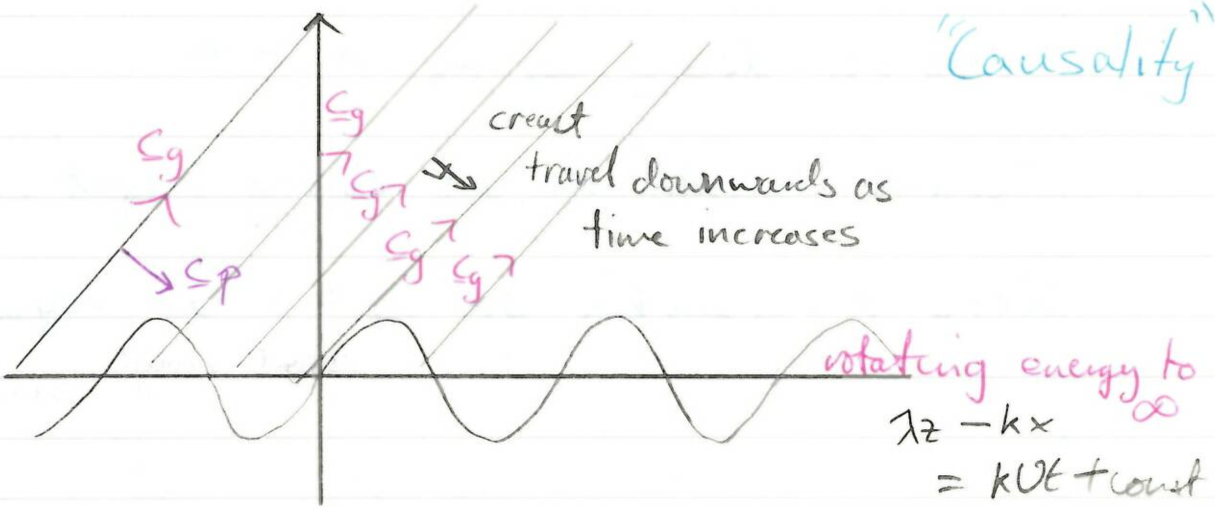
$\lambda z + kx = kUt + \text{const}.$

Scorer

Correct one

II

"Causality"



Note: creasts : lines of constant energy

c_p - nothing to do with energy.

By causality

$$w = -\epsilon k U B \cos[kx - \lambda z - kUt]$$

But $\tilde{w}(0) = 1$ so $B = 1$

i.e. $w = -\epsilon k U \cos[kx - \lambda z - kUt]$ ← Only term II waves.

As before, write $z = z_0 + \epsilon z'$

Then

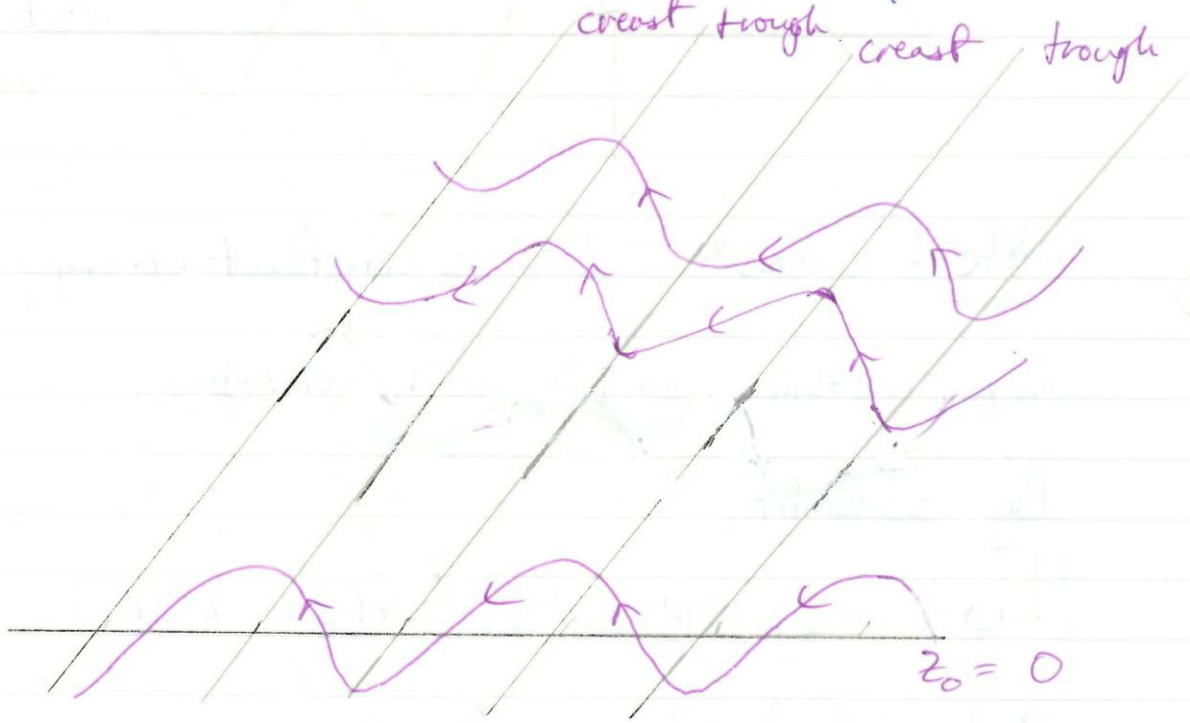
$$z = z_0 + \epsilon \sin[kx - \lambda z_0 - kUt]$$

In the frame of the ridges

$$z = z_0 + \epsilon \sin(kX - \lambda z_0)$$

- does not decay as $z_0 \to +\infty$
- Same amplitude at every height.

Note $z = \varepsilon \sin kx$ on $z_0 = 0$ as required.



mountain wave crest lean into the wind.

—/—